M1F Summary Notes

JMC Year 1, 2017/2018 syllabus

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This document contains a list of definitions and a list of theorems.

Note that the exam will probably require you to PROVE some of these theorems, so you should refer back to the original notes for the proofs.

Boxes cover content in more detail. Titles of some theorems are given in italics.

Contents

1 Definitions

Arbitrary union We can define the arbitrary union

$$\bigcup_{i \in I} X_i \tag{1}$$

to be the union of all sets X_i.

Arbitrary intersection We can define the arbitrary intersection

$$\bigcap_{i \in I} X_i \tag{2}$$

to be the intersection of all sets X_i.

Modulus (complex number) The modulus of $z = x + iy$ is defined as:

$$|z| = \sqrt{x^2 + y^2} \tag{3}$$

Argument (complex number) The argument of $z = x + iy$ is defined as:

$$\arg(z) = \arctan(\frac{y}{x}) \tag{4}$$

Fundamental Theorem of Algebra If a polynomial is of degree n then it has n roots in the complex plane.

Root of unity Any z that is a solution to $z^n = 1$.

Supremum A real number b is the sup (least upper bound) of a set S if:

- every $s \in S$ is less than b (i.e. b is an upper bound)
- b is less than every other upper bound of S

Infimum A real number b is the inf (greatest lower bound) of a set S if:

- every $s \in S$ is greater than b (i.e. b is a lower bound)
- b is greater than every other lower bound of S

Bounded above A set is bounded above if it has an upper bound.

Bounded below A set is bounded below if it has a lower bound.

Coprime We say two integers a and b are coprime if $\gcd(a, b) = 1$.

Fundamental Theorem of Arithmetic Every integer greater than 1 can be expressed as a unique product of primes.

$\equiv$ (congruence operator) We say $a \equiv b \mod m$ if $(a - b)$ is a multiple of m.

Binary relation A binary relation $\sim$ on a set S is a mapping satisfying $S \times S \to$ Bool.

Reflexive A binary relation $\sim$ is reflexive if for any $a \in S$:

$$a \sim a \tag{5}$$

Symmetric A binary relation $\sim$ is symmetric if:

$$a \sim b \Leftrightarrow b \sim a \tag{6}$$

Transitive A binary relation $\sim$ is transitive if:

$$a \sim b \wedge b \sim c \Rightarrow a \sim c \tag{7}$$

Equivalence relation A binary relation $\sim$ is an equivalence relation iff it is reflexive, symmetric and transitive.

Equivalence class The equivalence class of $b \in S$ is the set of all $s \in S$ such that $b \sim s$.

Injective A function $f : A \to B$ is injective if every $b \in B$ has at MOST one $a \in A$ that maps to it. Mathematically:

$$\forall x, y \in A \quad f(x) = f(y) \Rightarrow x = y \tag{8}$$

Surjective A function $f : A \to B$ is surjective if every $b \in B$ has at LEAST one $a \in A$ that maps to it. Mathematically:

$$\forall b \in B \quad \exists a \in A \quad s.t. \quad f(a) = b \tag{9}$$

Bijective A function $f : A \to B$ is bijective iff it is both surjective and injective.

(Two-sided) inverse function The inverse of a function $f : A \to B$ is defined as $f^{-1} : B \to A$, and satisfies:

$$f^{-1} \circ f \equiv f \circ f^{-1} \equiv Id_A \tag{10}$$

Power set The power set of S is the set of all subsets of S.

Countably infinite An infinitely-sized set S is countably infinite if there exists a bijection $f : \mathbb{Z}_{>0} \to S$.

2 Theorems

2.1 Logic and Sets

Contrapositive law
For any Boolean statements P, Q:

$$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P) \tag{11}$$

To negate any logical statements, we can flip the $\forall$ and $\exists$ signs and negate the predicate (the mathematical statement at the end).

2.2 Complex numbers

Complex numbers can be added, subtracted, multiplied and divided.

For any complex number z,

$$z\overline{z} = |z|^2 \tag{12}$$

For any complex numbers z_1, z_2:

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \tag{13}$$
$$\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2} \tag{14}$$
$$\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2} \tag{15}$$
$$\overline{(\frac{z_1}{z_2})} = \frac{\overline{(z_1)}}{\overline{(z_2)}} \tag{16}$$
$$\overline{(z_1)^n} = (\overline{z_1})^n \tag{17}$$

The inverse of a complex number z is such that:

$$zz^{-1} = 1 \tag{18}$$

Any complex number z can be represented in exponential form:

$$z = re^{i\theta} \tag{19}$$

where r is the modulus of z and θ is the argument.

Euler's formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{20}$$

This can be proven using power series.

De Moivre's Theorem
For any complex number z:

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)) \tag{21}$$

This can be proven by induction or using Euler's formula.

Fundamental Theorem of Algebra
Any polynomial of degree n has n roots in the complex plane.

Roots of unity
Any root of unity (i.e. a solution to $z^n = 1$) can be expressed in the form:

$$z = e^{\frac{2\pi k i}{n}}, \quad k \in \mathbb{Z}_{\geq 0} \tag{22}$$

2.3 Proofs and Induction

Counterexample
This involves disproving a statement by providing an example situation where the statement is false.

Proof by contradiction
Assume the statement is true and reach a contradiction. This means the statement must be false.

Weak induction
If we take P as a predicate:

$$P(0) \wedge (P(k) \Rightarrow P(k+1)) \Rightarrow P(n) \quad (\forall n) \tag{23}$$

Note that we do not have to start at 0:

$$P(m) \wedge (P(k) \Rightarrow P(k+1)) \Rightarrow P(n) \quad (\forall n \geq m) \tag{24}$$

Strong induction
If we take P as a predicate:

$$P(0) \land ((\forall j \leq k \; P(j)) \Rightarrow P(k+1)) \Rightarrow P(n) \quad (\forall n) \tag{25}$$

Note that we do not have to start at 0:

$$P(m) \land ((\forall j \leq k \; P(j)) \Rightarrow P(k+1)) \Rightarrow P(n) \quad (\forall n \geq m) \tag{26}$$

> We must prove $P(k) \Rightarrow P(k+1)$ for weak induction.
> We must prove $(\forall j \leq k \; P(j)) \Rightarrow P(k+1)$ for strong induction.
>
> Strong induction is mathematically equivalent to weak induction.

2.4 Number Theory

Completeness Axiom
A set S has a least upper bound (sup) iff:

- S is non-empty
- S is bounded above

Corresponding statement for greatest lower bound (inf).

Every real number has a decimal expansion.

A number is rational $\Leftrightarrow$ it has a periodic decimal expansion.

For any $x \in \mathbb{Z}_{\geq 0}$, $(x \mid a)$ and $(x \mid b) \Rightarrow (x \mid \gcd(a,b))$.
($x \mid a$ means "x divides a")

If $(x \mid ab)$ but $\gcd(a,x) = 1$ then $(x \mid b)$.

> Note: In general, $\gcd(a,b) = 1$ doesn't mean that $(a \nmid b)$.
>
> The only time that $\gcd(a,b) = 1$ implies $(a \nmid b)$ is if one of a, b are prime. This is because (take p prime) the only way for $\gcd(a,p) \neq 1$ would be if a is a multiple of p, in which case $(p \mid a)$.

Euclid's Algorithm
To find the *gcd* of a and b where $a > b$. Write:

$$\begin{aligned} a &= q_1 b + r_1 && (27) \\ b &= q_2 r_1 + r_2 && (28) \\ r_1 &= q_3 r_2 + r_3 && (29) \\ &\vdots && (30) \\ r_{n-1} &= q_n r_{n-1} + r_n && (31) \end{aligned}$$

Continue until $r_n = 0$, return r_{n-1}.

In Haskell notation, Euclid's algorithm is specified by:

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gcd(a, 0) = 0
gcd(a, b) = gcd(b, a mod b)
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Bezout's Theorem

$$\gcd(a, b) = \lambda a + \mu b \quad (\text{for some } \lambda, \mu \in \mathbb{R}) \tag{32}$$

We can write a and b as:

$$\begin{aligned} a &= \alpha \gcd(a, b) && (33) \\ b &= \beta \gcd(a, b) && (34) \end{aligned}$$

In general the solution to equation 32 is given by:

$$\gcd(a, b) = (\lambda + \beta n)a + (\mu - \alpha n)b \tag{35}$$

noting that the extra terms will always cancel out. So we have a set of solutions (λ_n, μ_n), where:

$$\lambda_n = \lambda + \beta n, \quad \mu_n = \mu - \alpha n \tag{36}$$

Every integer larger than 1 can be written as a product of primes.
(use strong induction to prove)

Fundamental Theorem of Arithmetic (proof not needed)
Every integer larger than 1 can be written UNIQUELY as a product of primes.

2.5 Modular arithmetic

$(\equiv \mod m)$ is an equivalence relation.

If $a \equiv b \mod m$ and $c \equiv d \mod m$:

$$a + c \equiv b + d \mod m \tag{37}$$
$$a - c \equiv b - d \mod m \tag{38}$$
$$ac \equiv bd \mod m \tag{39}$$

If $a \equiv b \mod m$, then for any n :

$$a^n \equiv b^n \mod m \tag{40}$$

Fermat's Little Theorem
For any prime p and $a \in \mathbb{Z}$:

$$a^p \equiv a \mod p \tag{41}$$

Also, if a is not a multiple of p:

$$a^{p-1} \equiv 1 \mod p \tag{42}$$

In the case a is a multiple of p, we can easily see $a^{p-1} \equiv 0 \mod p$.

2.6 Equivalence relations and functions

A binary relation $\sim$ is an equivalence relation iff it is reflexive, symmetric and transitive.

The equivalence class of $b \in S$ is the set of all $s \in S$ such that $b \sim s$.

The set of equivalence classes of a set S form a partition of S. In other words, each equivalence class is a disjoint subset of S.

A function has an inverse iff it is a bijection.

If f and g are injective, so is $g \circ f$.
If f and g are surjective, so is $g \circ f$.
If f and g are bijective, so is $g \circ f$.

If we define $X \sim Y$ as "there exists a bijection $f : X \to Y$", then $\sim$ is an equivalence relation.

Let $P(A)$ denote the power set of a set A.

There exists no bijection $A \to P(A)$.

Examples of countably finite/infinite sets
$\mathbb{Z}, \mathbb{Q}$ are countably infinite.
$\mathbb{R}, \mathbb{C}, \mathbb{R} \setminus \mathbb{Q}$ (set of all irrationals) are uncountably infinite.

2.7 Combinatorics

Multiplicative principle
In any n-stage process, where the r^{th} stage can be completed in a_r ways, the total number of was of completing the process is:

$$\prod_{r=0}^{n} a_r \tag{43}$$

i.e. just multiply the number of options together.

Multinomials
When expanding $(a + b + c + ...)^n$, with r terms inside the brackets, the general term of any coefficient is $a^\alpha b^\beta c^\gamma ...$, where $\alpha + \beta + \gamma + ... = n$.
The coeffient in the expansion for any term $a^\alpha b^\beta c^\gamma ...$ is given by:

$$\binom{n}{\alpha, \beta, \gamma ...} = \frac{n!}{\alpha! \; \beta! \; \gamma! \; ...} \tag{44}$$

Note if we set $r = 2$ in the above equation we get the familiar binomial coefficient:

$$\binom{n}{\alpha, \beta} = \frac{n!}{\alpha! \; \beta!} \tag{45}$$

$$= \frac{n!}{\alpha! \; (n - \alpha)!} \tag{46}$$

since $\beta = n - \alpha$.