

# Differential Equations Cheatsheet (JMC Year 1, 2017/2018 syllabus)

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Items not covered in this summary: phase portraits, similarity transformations.

## Contents

|          |   |          |
|----------|---|----------|
| <b>I</b> | <b>Individual differential equations</b>                        | <b>3</b> |
| <b>1</b> | <b>Definitions</b>  | <b>3</b> |
| <b>2</b> | <b>1st order linear ODEs</b>                                    | <b>3</b> |
| <b>3</b> | <b>1st order non-linear ODEs</b>                                | <b>3</b> |
| 3.1      | Exact equations . . . . .                                       | 3        |
| 3.2      | Separable ODEs . . . . .  | 4        |
| 3.3      | Homogenous ODEs . . . . .                                       | 5        |
| 3.4      | Bernoulli type ODEs . . . . .                                   | 5        |
| <b>4</b> | <b>2nd order ODEs</b>   | <b>5</b> |
| 4.1      | Special case - y missing . . . . .                              | 5        |
| 4.2      | Special case - x missing . . . . .                              | 6        |
| 4.3      | General case - finding the CF . . . . .                         | 6        |
| 4.3.1    | $\lambda_1 \neq \lambda_2$ . . . . .                            | 7        |
| 4.3.2    | $\lambda_1 = \lambda_2$ . . . . .                               | 7        |
| 4.3.3    | $\lambda_1$ and $\lambda_2$ are complex-valued . . . . .        | 7        |
| 4.4      | General case - finding the PI . . . . .                         | 8        |
| 4.4.1    | $f(x)$ is a polynomial . . . . .                                | 8        |
| 4.4.2    | $f(x)$ is a multiple of $e^{bx}$ , $e^{bx}$ NOT in CF . . . . . | 8        |
| 4.4.3    | $f(x)$ is a multiple of $e^{bx}$ , $e^{bx}$ IS in CF . . . . .  | 8        |

|           |  |          |
|-----------|--|----------|
| 4.4.4     | $f(x) = A(x)e^{bx}$ where $A(x)$ is a polynomial . . | 8        |
| 4.4.5     | $f(x)$ is trigonometric (e.g. sin, cos, sinh etc.) . | 8        |
| 4.4.6     | Special cases . . . . .                              | 8        |
| <b>II</b> | <b>Systems of differential equations</b>             | <b>9</b> |
| <b>5</b>  | <b>Definitions</b>                                   | <b>9</b> |
| <b>6</b>  | <b>Solving systems of diff. equations</b>            | <b>9</b> |
| 6.0.1     | Complex eigenvalues . . . . .                        | 10       |

## Part I

# Individual differential equations

## 1 Definitions

**Order (of derivative)** An  $n^{th}$  derivative has order  $n$ .

**Order (of ODE)** The order of the highest derivative present in an ODE.

**Degree (of ODE)** The highest power to which a term is raised in an ODE (excluding fractional powers).

**Linear** An ODE which has no terms raised to more than the  $1^{st}$  power, and with no  $y$ ,  $x$  or other derivative terms multiplied by each other.

## 2 1st order linear ODEs

Every 1st order linear ODE can be expressed as:

$$\frac{dy}{dx} + p(x)y = q(x) \quad (1)$$

These can ALL be solved by the *integrating factor* method:

1. Multiply both sides by  $\exp(\int p(x)dx)$
2. Use the reverse product rule to express the LHS as a single derivative (of a function of  $y$ ).
3. Integrate both sides and rearrange.

## 3 1st order non-linear ODEs

### 3.1 Exact equations

Let us say we have an ODE of the form:

$$P(x, y) + Q(x, y)\frac{dy}{dx} = 0 \quad (2)$$

(note the coefficients are multi-variable functions). This can be rewritten as:

$$P(x, y)dx + Q(x, y)dy = 0 \quad (3)$$

We can try the exact equations method. We say an equation is exact iff:

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x} \quad (4)$$

This simple condition implies some important results. It can be shown that an exact equation implies the LHS of equation 3 is an exact (total) differential). This means it can be written as  $df$ , where  $f$  is some function of  $x$  and  $y$ . But the equation of this total differential is:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \quad (5)$$

Comparing to equation 3 we can note 3 things:

$$\begin{aligned} P(x, y) &= \frac{\partial f}{\partial x} \\ Q(x, y) &= \frac{\partial f}{\partial y} \\ df &= 0 \end{aligned} \quad (6)$$

We integrate  $P(x, y)$  w.r.t  $x$  and  $Q(x, y)$  w.r.t  $y$  and 'merge' the two expressions together (i.e. for any matching terms, write them down only once) to give us an expression for  $f(x, y)$ . Ignore constants of integration.  $df = 0$  tells us that  $f(x, y) = c$  by integration. Therefore the general solution is given by:

$$f(x, y) = c \quad (7)$$

for some arbitrary constant  $c$ .

### 3.2 Separable ODEs

Separable equations can be written in the form:

$$\frac{dy}{dx} = f(x)g(y) \quad (8)$$

These can be rearranged and integrated on both sides, with respect to the different variables.

### 3.3 Homogenous ODEs

Homogenous equations can be written in the form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \quad (9)$$

To solve, set  $v = \frac{y}{x}$ , so that  $y = xv$ . Note that  $v$  is still a single-variable function of  $x$ , since  $y$  is a function of  $x$ . Now we can differentiate both sides to get:

$$\frac{dy}{dx} = v + x \frac{dv}{dx} \quad (10)$$

We now have simultaneous equations for  $\frac{dy}{dx}$ . Equate and solve for  $\frac{dv}{dx}$ , and then solve this 1st order linear ODE in  $\frac{dv}{dx}$  to find  $v$  (and then  $y$ ).

### 3.4 Bernoulli type ODEs

A Bernoulli type ODE is of the form:

$$\frac{dy}{dx} + p(x)y = q(x)y^n \quad (11)$$

To solve:

1. Multiply both sides by  $(1 - n)y^{-n}$
2. Let  $z = y^{1-n}$  and substitute into equation, including rewriting one of the terms as  $\frac{dz}{dx}$
3. The resulting equation is 1st order linear in  $z$ , so solve for  $z$  (and then  $y$ ).

## 4 2nd order ODEs

### 4.1 Special case - $y$ missing

If we can write the 2<sup>nd</sup> derivative in the form:

$$\frac{d^2y}{dx^2} = f\left(x, \frac{dy}{dx}\right) \quad (12)$$

(i.e. no  $y$  terms present), then we can make a substitution. Let  $P = \frac{dy}{dx}$ . This means  $\frac{d^2y}{dx^2} = \frac{dP}{dx}$ , the refore we have:

$$\frac{dP}{dx} = f(x, P) \quad (13)$$

This is 1st order w.r.t  $P$  and can be solved by appropriate 1st order methods.

## 4.2 Special case - x missing

If we can write the  $2^{nd}$  derivative as:

$$\frac{d^2y}{dx^2} = f(y, \frac{dy}{dx}) \quad (14)$$

(i.e. no  $x$  terms present), then we can make the same substitution. Let  $P = \frac{dy}{dx}$ . This means  $\frac{d^2y}{dx^2} = \frac{dP}{dx}$ , therefore we have:

$$\frac{dP}{dx} = f(y, P) \quad (15)$$

DIFFERENT TO LAST TIME: if we want a 1st order equation, we must rewrite  $\frac{dP}{dx}$  as a derivative with respect to  $y$ , since the RHS contains  $y$  terms. Luckily, we can see that:

$$\frac{dP}{dx} = \frac{dP}{dy} \frac{dy}{dx} = P \frac{dP}{dy} \quad (16)$$

Therefore:

$$P \frac{dP}{dy} = f(y, P) \quad (17)$$

This is 1st order w.r.t  $P$  and can be solved by appropriate 1st order methods.

## 4.3 General case - finding the CF

The general solution (GS) of a 2nd order ODE can be expressed as the sum of two other functions, called the 'complementary function' (CF) and a 'particular integral' (PI).

$$y_{GS} = y_{CF} + y_{PI} \quad (18)$$

A 2nd order ODE will usually be presented to us in the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c = f(x) \quad (19)$$

It can be shown that the CF can be calculated from the LHS of the above equation. We write down the *auxiliary equation*, which is simply the equation:

$$a\lambda^2 + b\lambda + c = 0 \quad (20)$$

using a, b, c from above. Solving this gives us two values,  $\lambda_1$  and  $\lambda_2$ .

#### 4.3.1 $\lambda_1 \neq \lambda_2$

We can express the CF as:

$$y_{CF} = A_1e^{\lambda_1x} + A_2e^{\lambda_2x} \quad (21)$$

where  $A_1$  and  $A_2$  are arbitrary constants.

#### 4.3.2 $\lambda_1 = \lambda_2$

Same as above, but we stick an  $x$  in front of one of the clashing parts of the solution.

$$y_{CF} = A_1e^{\lambda_1x} + A_2xe^{\lambda_2x} \quad (22)$$

#### 4.3.3 $\lambda_1$ and $\lambda_2$ are complex-valued

If the auxiliary equation has complex roots,  $\lambda_1$  and  $\lambda_2$  will be complex conjugates. The CF can be expressed as:

$$\begin{aligned} y_{CF} &= A_1e^{(a+bi)x} + A_2e^{(a-bi)x} \\ &= e^a(A_1e^{i(bx)} + A_2e^{-i(bx)}) \\ &= e^a(C_1\cos(bx) + C_2\sin(bx)) \end{aligned} \quad (23)$$

where  $C_1 = A_1 + A_2$  and  $C_2 = (A_1 - A_2)i$ . Note that even though  $A_1$  and  $A_2$  may have been complex,  $C_1$  and  $C_2$  are necessarily real.

## 4.4 General case - finding the PI

The particular integral is *any function*  $y_{PI}$  that satisfies the *ENTIRE differential equation*. The particular integral can be calculated depending on the form of the RHS of equation 19. We will refer to the RHS as simply  $f(x)$  and the particular integral (as before) as  $y_{PI}$ .

### 4.4.1 $f(x)$ is a polynomial

Try setting  $y_{PI}$  as a general polynomial of the same degree. e.g. if  $f(x)$  is a quadratic, try setting  $y_{PI} = ax^2 + bx + c$  and substituting into the ODE. We will solve for a, b, c, and this will give us  $y_{PI}$ .

### 4.4.2 $f(x)$ is a multiple of $e^{bx}$ , $e^{bx}$ NOT in CF

Choose  $y_{PI} = Ae^{bx}$  for some real number A.

### 4.4.3 $f(x)$ is a multiple of $e^{bx}$ , $e^{bx}$ IS in CF

We now have a clash between the PI and the CF. Choose  $y_{PI} = A(x)e^{bx}$  for some real FUNCTION A. At the end remove any clashing terms (i.e. terms with a component of  $e^{(something)x}$  which is already present in the CF).

### 4.4.4 $f(x) = A(x)e^{bx}$ where $A(x)$ is a polynomial

Choose  $y_{PI} = C(x)e^{bx}$  for some polynomial  $C(x)$ .

### 4.4.5 $f(x)$ is trigonometric (e.g. sin, cos, sinh etc.)

Look for a pattern in  $f(x)$ . A good tip for an  $f(x)$  with only sines/cosines is to use  $y_{PI} = A\cos(x) + B\sin(x)$  and solve for A and B. A similar story for sinh and cosh. CAUTION: sinh, cosh and tanh are actually exponential functions in disguise, so make sure they do not clash with any  $e^{\lambda x}$  terms in the CF.

### 4.4.6 Special cases

If  $f(x)$  has a term of the form  $e^x \cos(x)$  or  $e^x \sin(x)$  then we can rewrite it as the real/imaginary part of a complex function (in this case  $e^{(1+i)x}$  would be appropriate, since it expands to  $e^x(\cos(x) + i\sin(x))$ ).



## Part II

# Systems of differential equations

## 5 Definitions

**System of diff. equations** A set of simultaneous equations of derivatives, where derivatives of  $y, x$  etc. are given w.r.t. a parameter  $t$

**Order (of system)** The order of the highest derivative present in the system.

**Degree (of system)** The highest power to which a term is raised in an ODE (excluding fractional powers).

**Linear** A system which has no terms raised to more than the 1<sup>st</sup> power, and with no  $y$  or other derivative terms multiplied by each other.

**Homogeneous** A system with no explicit functions of  $t$  (i.e.  $f(t)$ ) present.

## 6 Solving systems of diff. equations

A homogeneous 1st order system of equations can be written as:

$$\begin{aligned}\frac{dx}{dt} &= F(x, y) \\ \frac{dy}{dt} &= G(x, y)\end{aligned}\tag{24}$$

Let us choose an example coupled system:

$$\begin{aligned}\frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy\end{aligned}\tag{25}$$

We can rewrite this in matrix form:

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\tag{26}$$

The system is now of the form

$$\frac{d}{dt}v = Mv \quad (27)$$

If we set  $v = Ve^{\lambda t}$ , where  $V$  is a constant vector independent of  $x, y$  or  $t$ , then we get

$$\begin{aligned} \lambda V &= MV \\ (M - \lambda I_n)V &= 0_v \\ \det(M - \lambda I_n) &= 0 \end{aligned} \quad (28)$$

Predictably, we find two eigenvalues  $\lambda_1, \lambda_2$  and (any) two eigenvectors  $v_1, v_2$ . The solution to the system is given by:

$$\begin{pmatrix} x \\ y \end{pmatrix} = A_1 v_1 e^{\lambda_1 t} + A_2 v_2 e^{\lambda_2 t} \quad (29)$$

The dimension of the eigenvectors will always match the number of variables being dealt with, for example a possible scenario is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = A_1 \begin{pmatrix} 3 \\ -5 \end{pmatrix} e^{-3t} + A_2 \begin{pmatrix} 7 \\ -2 \end{pmatrix} e^{2t} \quad (30)$$

The values of the individual derivatives can be found by reading off the rows of the matrices.

$$\begin{aligned} x &= 3A_1 e^{-3t} + 7A_2 e^{2t} \\ y &= -5A_1 e^{-3t} - 2A_2 e^{2t} \end{aligned} \quad (31)$$

### 6.0.1 Complex eigenvalues

If the eigenvalues turn out to be complex conjugates, the solution can be written as:

$$\begin{pmatrix} x \\ y \end{pmatrix} = A_1 v_1 e^{(a+bi)t} + A_2 v_2 e^{(a-bi)t} \quad (32)$$

(Note that  $A_1$  AND  $A_2$  ARE COMPLEX CONSTANTS HERE). We can do some rearranging like before to tidy up the solution:

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= A_1 v_1 e^{(a+bi)t} + A_2 v_2 e^{(a-bi)t} \\ &= e^a (A_1 v_1 e^{i(bt)} + A_2 v_2 e^{-i(bt)}) \\ &= e^a (C_1 \cos(bt) + C_2 \sin(bt)) \end{aligned} \quad (33)$$

where  $C_1 = A_1v_1 + A_2v_2$  and  $C_2 = (A_1v_1 - A_2v_2)i$ . Note that  $C_1$  and  $C_2$  are vectors.