

M1M1 Summary Notes

JMC Year 1, 2017/2018 syllabus

Fawaz Shah

UNDER CONSTRUCTION

This document contains a bunch of definitions and techniques, sorted by category.

M1M1 is more about applying mathematical methods rather than proving theorems, so sly manipulation of mathematical techniques is crucial. Most of the content is A Level, so I have only included relatively non-A-Level content.

Boxes cover content in more detail.

Contents

1	Trigonometric identities	3
2	Even and odd functions	3
3	Power series	3
3.1	Maclaurin vs. Taylor series	4
3.2	Complex power series	5
4	Limits and continuity	5
5	Differentiation	6
6	Conic sections	7
7	Series	7
8	Complex numbers	8
9	Fundamental Theorem of Calculus	9

10 Integrating functions of two variables	9
11 Lengths, areas and volumes	10
12 Ordinary Differential Equations (ODEs)	10
12.1 1st order linear ODEs	10
12.2 Separable ODEs	11
12.3 Homogenous ODEs	11
12.4 Bernoulli type ODEs	11

1 Trigonometric identities

Several useful trig identities (on top of the ones from A Level):

$$\sin\left(x + \frac{\pi}{2}\right) = \cos(x) \quad (1)$$

$$\cos\left(x + \frac{\pi}{2}\right) = \sin(x) \quad (2)$$

$$(3)$$

$$\sin(x + \pi) = -\sin(x) \quad (4)$$

$$\cos(x + \pi) = -\cos(x) \quad (5)$$

$$(6)$$

$$\sin(x + 2\pi) = \sin(x) \quad (7)$$

$$\cos(x + 2\pi) = \cos(x) \quad (8)$$

$$(9)$$

2 Even and odd functions

An even function is such that, for all x :

$$f(x) = f(-x) \quad (10)$$

An odd function is such that, for all x :

$$f(x) = -f(-x) \quad (11)$$

Any function can be expressed as the sum of an even and an odd function. We can write any function as follows:

$$f(x) = \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x)) \quad (12)$$

The left part of the function is even, the right part is odd.

3 Power series

The general form of a power series is:

$$\sum_{n=0}^{\infty} a_n x^n \quad (13)$$

for some sequence a_n .

There are several power series which must be memorized:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad (14)$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \quad (15)$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \quad (16)$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \quad (17)$$

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \quad (18)$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \quad |x| < 1 \quad (19)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \quad |x| < 1 \quad (20)$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad |x| < 1 \quad (21)$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots \quad |x| < 1 \quad (22)$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots \quad |x| < 1 \quad (23)$$

Every power series has a corresponding radius of convergence R , where the series converges for $|x| < R$ and diverges otherwise. This can be found using the ratio test (see section 7).

If a power series converges for all x , then we say $R = \infty$.

Power series can be added, subtracted, multiplied, divided, differentiated, integrated and also substituted into one another within their radius of convergence.

3.1 Maclaurin vs. Taylor series

The general form of a Maclaurin series is:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots \quad (24)$$

The general form for a Taylor series is:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots \quad (25)$$

So a Maclaurin series is a special case of a Taylor series, just centred about $x = 0$.

The entire point of power series in general is that we can approximate the value of a function at any point x , given we know all the derivatives at a certain point a . Realistically x must be close to a .

The Taylor series for a polynomial is finite, therefore we can calculate the actual values of polynomial functions. The Taylor series for non-polynomials are infinite series, so we must truncate the series at some point to calculate a value. This is why this method only gives an approximation to the actual value.

3.2 Complex power series

We have to know how to calculate the radius of convergence for a complex power series, i.e. a series of the form:

$$\sum_{i=0}^n c_n z^n \quad (26)$$

This involves a result from 2nd year maths. Note that the radius of convergence of a complex power series is the radius of a disk in the Argand diagram, centered about $(0,0)$. If a complex function has no singularities (i.e. nowhere where it 'blows up') in a disk of radius R centered about $\omega \in \mathbb{C}$, then it converges absolutely for $|z - \omega| < R$.

Therefore the radius of convergence about any point $\omega \in \mathbb{C}$ is the distance between ω and the nearest singularity.

4 Limits and continuity

Here we provide the $\epsilon - \delta$ definition of a limit L at $x = a$:

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \text{s.t.} \quad |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon \quad (27)$$

The limit at a only exists if:

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) \quad (28)$$

Properties of limits

$$\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x) \quad (29)$$

$$\lim_{x \rightarrow a} (f(x) g(x)) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x) \quad (30)$$

L'Hôpital's Rule

If we have an indeterminate limit (that is, a limit that evaluates to $\frac{0}{0}$ or $\frac{\infty}{\infty}$) then we can employ L'Hôpital's Rule:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} \quad (31)$$

A function is continuous at a if:

$$\lim_{x \rightarrow a} f(x) = f(a) \quad (32)$$

If the limit of a function f does not exist at a , then f is not continuous at a (note that the opposite is NOT necessarily true).

5 Differentiation

Derivatives of the main trigonometric and hyperbolic functions should be derivable. These are:

sin	cos	tan
cosec	sec	cot
arcsin	arccos	arctan
sinh	cosh	tanh
cosech	sech	coth
arcsinh	arccosh	arctanh

Leibniz' Rule

To take the n^{th} derivative of a product of functions $f(x) \cdot g(x)$, we can use Leibniz' rule:

$$\frac{d^n}{dx^n} (f(x) \cdot g(x)) = \sum_{r=0}^n \binom{n}{r} f^{(r)}(x) \cdot g^{(n-r)}(x) \quad (33)$$

6 Conic sections

There are 3 types of conic sections:

- ellipses
- hyperbolas
- parabolas

Any conic section can be expressed in polar form as:

$$r = \frac{L}{1 + e \cos(\theta)} \quad (34)$$

where e varies per type. The main differences are illustrated in the following table:

conic section	cartesian form	value of e for polar form
ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$e < 1$
parabola	$y = ax^2$	$e = 1$
hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$e > 1$

Special case: if $e = 0$ in polar form, we get a circle.

7 Series

A series is the sum of elements of a sequence a_n . We say the series is convergent (i.e. a_n is 'summable') if the sum to infinity exists. The sum to infinity is:

$$\sum_{i=0}^{\infty} a_i = \lim_{n \rightarrow \infty} \sum_{i=0}^n a_i \quad (35)$$

We say a series $\sum_{i=0}^n a_i$ is absolutely convergent if $\sum_{i=0}^n |a_i|$ is convergent.

A series that is convergent but not absolutely convergent is called conditionally convergent.

To evaluate whether a series is convergent or divergent, we can use some standard tests.

Preliminary test

If $a_n \not\rightarrow 0$ then $\sum_{i=0}^n a_n$ diverges.

Alternating series test

If a_n is alternating, and $|a_n|$ is decreasing, and $a_n \rightarrow 0$, then $\sum_{i=0}^n a_n$ converges.

Comparison test

If $|a_n| < |b_n|$ for all n :

- If $\sum_{i=0}^n b_i$ converges absolutely, then so does $\sum_{i=0}^n a_i$
- If $\sum_{i=0}^n a_i$ diverges, then so does $\sum_{i=0}^n b_i$

Integral test

If a_n is +ve and decreasing, and:

$$\int_N^\infty a_n \, dn \quad (36)$$

exists for some $N \in \mathbb{N}$, then a_n converges.

Ratio test

Consider the following limit:

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \quad (37)$$

- If $L < 1$, the series is absolutely convergent (in particular the sequence itself converges to 0).
- If $L > 1$ the series diverges.
- $L = 1$ is an indeterminate case.

If a series is absolutely convergent then the convergent sum does not depend on the order of summation.

If a series is conditionally convergent then the summed terms can be rearranged to produce any convergent sum you want.

8 Complex numbers

Mostly the same as M1F. See M1F notes.

Some useful identities:

$$\cos(z) = \cosh(iz) \quad (38)$$

$$\sin(z) = i \sinh(iz) \quad (39)$$

For calculating the radius of convergence of a complex power series, see section 3.2.

9 Fundamental Theorem of Calculus

The fundamental theorem of calculus defines the antiderivative $F(x)$ and shows the relation between differentiation and integration. It can be expressed in several different forms:

$$\frac{d}{dx} \int_a^x f(t) dt = f(x) \quad (40)$$

$$\int_a^b f(x) dx = F(b) - F(a) \quad (41)$$

Note that in equation 40, any lower bound a and dummy variable t can be picked, without affecting the validity of the theorem.

10 Integrating functions of two variables

Similar to how a single integral represents the **area** underneath a 2D curve $y = f(x)$, a double integral represents the **volume** underneath a 3D curve $z = f(x, y)$. The double integral over a region S is denoted as:

$$\iint_S f(x, y) dx dy \quad (42)$$

If we say S ranges from say, $x = a$ to $x = b$ and from $y = c$ to $y = d$, we can write the double integral as:

$$\int_c^d \int_a^b f(x, y) dx dy \quad (43)$$

To evaluate the double integral, simply evaluate the inner integral w.r.t x and then integrate the result w.r.t y .

Changing into polar form

Suppose in the region S , $f(x, y) = g(r, \theta)$ for some function g , where $0 \leq \theta \leq 2\pi$ and $r = h(\theta)$ for some function h . We can convert the double integral into polar coordinates (this might perhaps make it easier to evaluate). HOWEVER, we must

put an extra r term in the part of the integral w.r.t r . This extra bit is called the **Jacobian** (not covered fully in M1M1).

$$\iint_S f(x, y) \, dx \, dy = \int_0^{2\pi} \int_0^{h(\theta)} g(r, \theta) \cdot r \, dr \, d\theta \quad (44)$$

11 Lengths, areas and volumes

The length of a curve $y = f(x)$ between $x = a$ and $x = b$ is given by:

$$L = \int_a^b \sqrt{1 + (f'(x))^2} \, dx \quad (45)$$

For parametric curves defined by $x(t)$ and $y(t)$, the length is given by:

$$L = \int_{t_a}^{t_b} \sqrt{(x'(t))^2 + (y'(t))^2} \, dt \quad (46)$$

For polar curves defined by $r = f(\theta)$, the length is given by:

$$L = \int_{\theta_a}^{\theta_b} \sqrt{(f(\theta))^2 + (f'(\theta))^2} \, d\theta \quad (47)$$

The surface area of revolution obtained by rotating a curve $f(x)$ about the x axis is given by:

$$A = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} \, dx \quad (48)$$

The volume of revolution obtained by rotating a curve $f(x)$ about the x axis is given by:

$$V = \pi \int_a^b (f(x))^2 \, dx \quad (49)$$

12 Ordinary Differential Equations (ODEs)

12.1 1st order linear ODEs

Every 1st order linear ODE can be expressed as:

$$\frac{dy}{dx} + p(x)y = q(x) \quad (50)$$

These can ALL be solved by the *integrating factor* method:

1. Multiply both sides by $\exp(\int p(x) \, dx)$
2. Use the reverse product rule to express the LHS as a single derivative (of a function of y).
3. Integrate both sides and rearrange.

12.2 Separable ODEs

Separable equations can be written in the form:

$$\frac{dy}{dx} = f(x)g(y) \quad (51)$$

These can be rearranged and integrated on both sides, with respect to the different variables.

12.3 Homogenous ODEs

Homogenous equations can be written in the form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \quad (52)$$

To solve, set $v = \frac{y}{x}$, so that $y = xv$. Note that v is still a single-variable function of x , since y is a function of x . Now we can differentiate both sides to get:

$$\frac{dy}{dx} = v + x \frac{dv}{dx} \quad (53)$$

We now have simultaneous equations for $\frac{dy}{dx}$. Equate and solve for $\frac{dv}{dx}$, and then solve this 1st order linear ODE in $\frac{dv}{dx}$ to find v (and then y).

12.4 Bernoulli type ODEs

A Bernoulli type ODE is of the form:

$$\frac{dy}{dx} + p(x)y = q(x)y^n \quad (54)$$

To solve:

1. Multiply both sides by $(1 - n)y^{-n}$
2. Let $z = y^{1-n}$ and substitute into equation, including rewriting one of the terms as $\frac{dz}{dx}$
3. The resulting equation is 1st order linear in z , so solve for z (and then y).