M1J2 Summary Notes (JMC Year 1, 2017/2018 syllabus)

Fawaz Shah

STILL UNDER CONSTRUCTION

Dr Lawn refers to propositions, theorems, corollaries and lemmas. In this document I will refer to them all as 'theorems'.

This document only contains a list of definitions and a list of theorems.

Contents

Part I

Abstract Linear Algebra

1 Definitions

Vector space

2 Theorems

Part II
Group Theory

3 Definitions

4 Theorems

Part III
Analysis

5 Definitions

Sequence A sequence is simply a map $f : \mathbb{N} \to \mathbb{R}$, denoted by a_n

Convergence (as $n \to \infty$) A sequence a_n converges to a limit L if for all real numbers $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$ we have $|a_n - L| < \epsilon$.

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad s.t \quad \forall n > N \quad |a_n - L| < \epsilon \tag{1}$$

Tends to infinity (sequence) We say a sequence tends to infinity if for all $R \in \mathbb{R}$, the sequence a_n is eventually bigger than R.

$$\forall R \in \mathbb{R} \quad \exists N \in \mathbb{N} \quad s.t. \quad \forall n > N \quad a_n > R \tag{2}$$

Shift The shift of a sequence by say, k, is the sequence $b_n = a_{n+k}$

Triangle inequality The general triangle inequality is:

$$|x - y| < |x - z| + |z - y| \tag{3}$$

Setting $z = 0$ gives us:

$$|x - y| > |x| - |y| \tag{4}$$

Then setting $y = -y$ gives us the familiar case:

$$|x + y| < |x| + |y| \tag{5}$$

Bounded above A sequence a_n is bounded above if there's a real number A such that $a_n < A$ for all n.

Bounded below A sequence a_n is bounded below if there's a real number A such that $a_n > A$ for all n.

Bounded A sequence a_n is bounded if there's a real number A such that $|a_n| < A$ for all n.

Increasing A sequence is increasing if $a_{n+1} \geq a_n$ for all n.

Strictly increasing A sequence is strictly increasing if $a_{n+1} > a_n$ for all n.

Decreasing A sequence is decreasing if $a_{n+1} \leq a_n$ for all n.

Strictly decreasing A sequence is strictly decreasing if $a_{n+1} < a_n$ for all n.

Monotonic A sequence is monotonic if it is increasing or decreasing.

Supremum The supremum A of a set S is the least upper bound of that set i.e. the smallest number such that $\forall s \in S \quad s \leq A$

Infimum The infimum B of a set S is the greatest lower bound of that set i.e. the largest number such that $\forall s \in S \quad s \geq B$

Subsequence A subsequence of a_n is a sequence $a_{f(n)}$, where $f(n)$ is a strictly increasing function.

Cauchy sequence A sequence is Cauchy if the terms get arbitrarily close to one another. To put it mathematically:

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad s.t \quad \forall m, n \geq N \quad |a_n - a_m| < \epsilon \tag{6}$$

Partial sum The n^{th} partial sum S_n of a sequence a_n is the sum of terms up to that point:

$$S_n = \sum_{i=1}^{n} a_n \tag{7}$$

Summable A sequence is summable if the sequence of its partial sums converges. The limit of the sequence of partial sums will be:

$$L = \sum_{i=1}^{\infty} a_n \tag{8}$$

Absolutely summable A sequence a_n is absolutely summable if $|a_n|$ is summable.

Conditionally summable A sequence is conditionally summable if it is summable but not absolutely summable.

Power series The power series associated with a sequence a_n is the sequence of partial sums:

$$\sum_{i=1}^{n} a_i x^i \tag{9}$$

Radius of convergence The radius of convergence R of a power series $P(x)$ is defined as the largest x for which $P(x)$ is convergent.

$$R = sup\{x \in \mathbb{R} | P(x) \text{ convergent}\} \tag{10}$$

Limit as $x \to \infty$ (function) A function $f(x)$ tends to a limit L as $x \to \infty$ if for all real numbers $\epsilon > 0$, there exists an $R \in \mathbb{R}$ such that for all $x \geq R$ we have $|f(x) - L| < \epsilon$.

$$\forall \epsilon > 0 \quad \exists R \in \mathbb{R} \quad s.t \quad \forall x > R \quad |f(x) - L| < \epsilon \tag{11}$$

Tends to infinity (function) A function $f(x)$ tends to infinity as $x \to \infty$ if for any $M \in \mathbb{R}$ there exists an $R \in \mathbb{R}$ such that if $x > M$ then $f(x) > R$.

$$\forall M \in \mathbb{R} \quad \exists R \in \mathbb{R} \quad s.t. \quad x > M \implies f(x) > R \tag{12}$$

One-sided limit (function) A function $f(x)$ tends to a limit L as $x \to a^-$ if for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $x \in (a-\delta, a)$ then $|f(x) - L| < \epsilon$.

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad s.t. \quad x \in (a - \delta, a) \implies |f(x) - L| < \epsilon \tag{13}$$

Same format for the other sided limit $(x \to a^+)$
(Note that $\epsilon - \delta$ definition is only used for limits as x tends to a finite number a, not infinity)

Limit as $x \to a$ (function) A function $f(x)$ tends to a limit L as $x \to a$ if we have both:

$$\lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L \tag{14}$$

Continuous (ver. 1) A function $f(x)$ is continuous at a if:

$$\lim_{x \to a} f(x) = f(a) \tag{15}$$

Continuous (ver. 2) A function $f(x)$ is continuous at a if for all $\epsilon > 0$ there is a $\delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$.

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad s.t. \quad |x - a| < \delta \implies |f(x) - f(a)| < \epsilon \tag{16}$$

Continuous everywhere A function $f(x)$ is continuous everywhere if it is continuous at a for all $a \in \text{dom}(f)$

Open interval An open inteval I is a set $I \subseteq \mathbb{R}$ of the form:

1. $I = (a, b)$ for some $a, b \in \mathbb{R}$, or
2. $I = (-\infty, b)$, or
3. $I = (a, +\infty)$, or
4. $I = \mathbb{R}$

Discontinuity Discontinuity is the negation of continuity. Hence a function $f(x)$ is discontinuous at a if there exists $\epsilon > 0$ such that for all $\delta > 0$, $|x - a| < \delta$ AND $|f(x) - f(a)| > \epsilon$.

$$\exists \epsilon > 0 \quad s.t. \quad \forall \delta > 0 \quad |x - a| < \delta \text{ AND } |f(x) - f(a)| > \epsilon \tag{17}$$

Bounded (function) A function $f(x)$ is bounded if the set of all possible values of f(x) is bounded.

Differentiable (ver. 1) A function $f(x)$ is differentiable at a if:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{18}$$

exists.

Differentiable (ver. 2) A function $f(x)$ is differentiable at a if:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{19}$$

exists.

Differentiable everywhere A function $f(x)$ is differentiable everywhere if it is differentiable at a for all $a \in \mathrm{dom}(f)$

Global maximum A function $f(x)$ has a global maximum at a if $f(a) \geq f(x)$ for all other values of $f(x)$.

Similar definition for global minimum.

Local maximum A function $f(x)$ has a local maximum at a if $f(a) \geq f(x)$ for all x in the set $(a - \epsilon, a + \epsilon)$, for some ϵ.

Similar definition for local minimum.

6 Theorems