Differential Equations Cheatsheet

JMC Year 1, 2017/2018 syllabus

Fawaz Shah

Topics not covered in this summary: phase portraits, similarity transformations.

Contents

1 Definitions

Order (of derivative) An n^{th} derivative has order n.

Order (of ODE) The order of the highest derivative present in an ODE.

Degree (of ODE) The highest power to which a term is raised in an ODE (excluding fractional powers).

Linear (ODE) An ODE which has no terms raised to more than the 1^{st} power, and with no y, x or other derivative terms multiplied by each other.

System of diff. equations A set of simultaneous equations of derivatives, where derivatives of y, x etc. are given w.r.t. a parameter t

Order (of system) The order of the highest derivative present in the system.

Degree (of system) The highest power to which a term is raised in an ODE (excluding fractional powers).

Linear (system) A system which has no terms raised to more than the 1^{st} power, and with no y or other derivative terms multiplied by each other.

Homogeneous (system) A system with no explicit functions of t (i.e. $f(t)$) present.

2 1st order linear ODEs

Every 1st order linear ODE can be expressed as:

$$\frac{dy}{dx} + p(x)y = q(x) \tag{1}$$

These can ALL be solved by the *integrating factor* method:

1. Multiply both sides by $exp(\int p(x)dx)$
2. Use the reverse product rule to express the LHS as a single derivative (of a function of y).
3. Integrate both sides and rearrange.

3 1st order non-linear ODEs

3.1 Exact equations

Let us say we have an ODE of the form:

$$P(x,y) + Q(x,y)\frac{dy}{dx} = 0 \tag{2}$$

(note the coefficients are multi-variable functions). This can be rewritten as:

$$P(x,y)dx + Q(x,y)dy = 0 \tag{3}$$

We can try the exact equations method. We say an equation is exact iff:

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x} \tag{4}$$

This simple condition implies some important results. It can be shown that an exact equation implies the LHS of equation 3 is an exact (total) differential). This means it can be written as df, where f is some function of x and y. But the equation of this total differential is:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{5}$$

Comparing to equation 3 we can note 3 things:

$$\begin{aligned} P(x,y) &= \frac{\partial f}{\partial x} \\ Q(x,y) &= \frac{\partial f}{\partial y} \\ df &= 0 \end{aligned} \tag{6}$$

We integrate $P(x,y)$ w.r.t x and $Q(x,y)$ w.r.t y and 'merge' the two expressions together (i.e. for any matching terms, write them down only once) to give us an expression for $f(x,y)$. Ignore constants of integration. $df = 0$ tells us that $f(x,y) = c$ by integration. Therefore the general solution is given by:

$$f(x,y) = c \tag{7}$$

for some arbitrary constant c.

3.2 Separable ODEs

Separable equations can be written in the form:

$$\frac{dy}{dx} = f(x)g(y) \tag{8}$$

These can be rearranged and integrated on both sides, with respect to the different variables.

3.3 Homogenous ODEs

Homogenous equations can be written in the form:

$$\frac{dy}{dx} = f(\frac{y}{x}) \tag{9}$$

To solve, set $v = \frac{y}{x}$, so that $y = xv$. Note that v is still a single-variable function of x, since y is a function of x. Now we can differentiate both sides to get:

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \tag{10}$$

We now have simultaneous equations for $\frac{dy}{dx}$. Equate and solve for $\frac{dv}{dx}$, and then solve this 1st order linear ODE in $\frac{dv}{dx}$ to find v (and then y).

3.4 Bernoulli type ODEs

A Bernoulli type ODE is of the form:

$$\frac{dy}{dx} + p(x)y = q(x)y^n \tag{11}$$

To solve:

1. Multiply both sides by $(1-n)y^{-n}$
2. Let $z = y^{1-n}$ and substitute into equation, including rewriting one of the terms as $\frac{dz}{dx}$
3. The resulting equation is 1st order linear in z, so solve for z (and then y).

4 2nd order ODEs

4.1 Special case - y missing

If we can write the 2^{nd} derivative in the form:

$$\frac{d^2y}{dx^2} = f(x, \frac{dy}{dx}) \tag{12}$$

(i.e. no y terms present), then we can make a substitution. Let $P = \frac{dy}{dx}$. This means $\frac{d^2y}{dx^2} = \frac{dP}{dx}$, the refore we have:

$$\frac{dP}{dx} = f(x, P) \tag{13}$$

This is 1st order w.r.t P and can be solved by appropriate 1st order methods.

4.2 Special case - x missing

If we can write the 2^{nd} derivative as:

$$\frac{d^2y}{dx^2} = f(y, \frac{dy}{dx}) \tag{14}$$

(i.e. no x terms present), then we can make the same substitution. Let $P = \frac{dy}{dx}$. This means $\frac{d^2y}{dx^2} = \frac{dP}{dx}$, therefore we have:

$$\frac{dP}{dx} = f(y, P) \tag{15}$$

However, this is not yet a 1st order equation since the derivative is w.r.t. x, but we only have y terms on the RHS.

DIFFERENT TO LAST TIME: we must rewrite $\frac{dP}{dx}$ as a derivative with respect to y. Luckily, we can see that:

$$\frac{dP}{dx} = \frac{dP}{dy}\frac{dy}{dx} = P\frac{dP}{dy} \tag{16}$$

Therefore:

$$P\frac{dP}{dy} = f(y, P) \tag{17}$$

This is 1st order w.r.t P and can be solved by appropriate 1st order methods.

4.3 General case - finding the CF

The general solution (GS) of a 2nd order ODE can be expressed as the sum of two other functions, called the 'complementary function' (CF) and a 'particular integral' (PI).

$$y_{GS} = y_{CF} + y_{PI} \tag{18}$$

A 2nd order ODE will usually be presented to us in the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c = f(x) \tag{19}$$

It can be shown that the CF can be calculated from the LHS of the above equation. We write down the *auxiliary equation*, which is simply the equation:

$$a\lambda^2 + b\lambda + c = 0 \tag{20}$$

using a, b, c from above. Solving this gives us two values, λ_1 and λ_2.

Case 1: $\lambda_1 \neq \lambda_2$, both real

We can express the CF as:

$$y_{CF} = A_1 e^{\lambda_1 x} + A_2 e^{\lambda_2 x} \tag{21}$$

where A_1 and A_2 are arbitrary constants.

Case 2: $\lambda_1 = \lambda_2$, both real

Same as above, but we stick an x in front of one of the clashing parts of the solution.

$$y_{CF} = A_1 e^{\lambda_1 x} + A_2 x e^{\lambda_2 x} \tag{22}$$

Case 3: λ_1, λ_2 are complex

If the auxiliary equation has complex roots, λ_1 and λ_2 will be complex conjugates. The CF can be expressed as:

$$\begin{aligned} y_{CF} &= A_1 e^{(a+bi)x} + A_2 e^{(a-bi)x} \\ &= e^a (A_1 e^{i(bx)} + A_2 e^{-i(bx)}) \\ &= e^a (C_1 cos(bx) + C_2 sin(bx)) \end{aligned} \tag{23}$$

where $C_1 = A_1 + A_2$ and $C_2 = (A_1 - A_2)i$. Note that even though A_1 and A_2 may have been complex, C_1 and C_2 are necessarily real.

4.4 General case - finding the PI

The particular integral is *any function y_{PI} that satisfies the ENTIRE differential equation.* The particular integral can be calculated depending on the form of the RHS of equation 19. We will refer to the RHS as simply $f(x)$ and the particular integral (as before) as y_{PI}. We can follow some basic rules:

Case 1: $f(x)$ is a polynomial

Try setting y_{PI} as a general polynomial of the same degree. e.g. if $f(x)$ is a quadratic, try setting $y_{PI} = ax^2 + bx + c$ and substituting into the ODE. We will solve for a, b, c, and this will give us y_{PI}.

Case 2: $f(x)$ is a multiple of e^{bx}, e^{bx} NOT in CF

Choose $y_{PI} = Ae^{bx}$ for some real number A.

Case 3: $f(x)$ is a multiple of e^{bx}, e^{bx} IS in CF

We now have a clash between the PI and the CF. We can try $y_{PI} = Axe^{bx}$, i.e. sticking an x in the PI to avoid the clash. If this doesn't work, we can choose $y_{PI} = A(x)e^{bx}$ for some real FUNCTION A. Remember to use the CHAIN RULE to differentiate A this time.

At the end remove any clashing terms, i.e. terms of the form $Be^{\lambda x}$ where $e^{\lambda x}$ is already present in the CF. Other terms with more x's included are allowed, e.g. $xe^{\lambda x}$ would not count as a clashing term.

Case 4: $f(x) = A(x)e^{bx}$ where $A(x)$ is a polynomial

Choose $y_{PI} = C(x)e^{bx}$ for some polynomial $C(x)$.

Case 5: $f(x)$ is trigonometric (e.g. sin, cos, sinh etc.)

Look for a pattern in $f(x)$. A good tip for an $f(x)$ with only sines/cosines is to use $y_{PI} = A\cos(x) + B\sin(x)$ and solve for A and B. A similar story for sinh and cosh. CAUTION: sinh, cosh and tanh are actually exponential functions in disguise, so make sure they do not clash with any $e^{\lambda x}$ terms in the CF.

Other cases

If $f(x)$ has a term of the form $e^x\cos(x)$ or $e^x\sin(x)$ then we can rewrite it as the real/imaginary part of a complex function (in this case $e^{(1+i)x}$ would be appropriate, since it expands to $e^x(\cos(x) + i\sin(x))$.

If $f(x)$ is more complicated, we may have to be imaginative with the choice of y_{PI}. e.g. for $f(x) = Ae^{ax} + Be^{bx}$ we could choose $y_{PI} = Ce^{ax} + De^{bx}$ for some constants C, D. Again be careful of terms that clash with the CF.

5 Solving systems of diff. equations

A homogeneous 1st order system of equations can be written as:

$$\begin{aligned} \frac{dx}{dt} &= F(x,y) \\ \frac{dy}{dt} &= G(x,y) \end{aligned} \tag{24}$$

Let us choose an example coupled system:

$$\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy \end{aligned} \tag{25}$$

We can rewrite this in matrix form:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{26}$$

The system is now of the form

$$\frac{d}{dt}v = Mv \tag{27}$$

If we set $v = Ve^{\lambda t}$, where V is a constant vector independent of x, y or t, then we get

$$\begin{aligned} \lambda V &= MV \\ (M - \lambda I_n)V &= 0_v \\ \det(M - \lambda I_n) &= 0 \end{aligned} \tag{28}$$

Predictably, we find two eigenvalues λ_1, λ_2 and (any) two eigenvectors v_1, v_2. The solution to the system is given by:

$$\begin{pmatrix} x \\ y \end{pmatrix} = A_1 v_1 e^{\lambda_1 t} + A_2 v_2 e^{\lambda_2 t} \tag{29}$$

The dimension of the eigenvectors will always match the number of variables being dealt with, for example a possible scenario is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = A_1 \begin{pmatrix} 3 \\ -5 \end{pmatrix} e^{-3t} + A_2 \begin{pmatrix} 7 \\ -2 \end{pmatrix} e^{2t} \tag{30}$$

The values of the individual derivatives can be found by reading off the rows of the matrices.

$$\begin{aligned} x &= 3A_1 e^{-3t} + 7A_2 e^{2t} \\ y &= -5A_1 e^{-3t} + -2A_2 e^{2t} \end{aligned} \tag{31}$$

Complex eigenvalues

If the eigenvalues turn out to be complex conjugates, the solution can be written as:

$$\begin{pmatrix} x \\ y \end{pmatrix} = A_1 v_1 e^{(a+bi)t} + A_2 v_2 e^{(a-bi)t} \tag{32}$$

(Note that A_1 and A_2 may be complex). We can do some rearranging like before to tidy up the solution:

$$\begin{aligned}\begin{pmatrix} x \\ y \end{pmatrix} &= A_1 v_1 e^{(a+bi)t} + A_2 v_2 e^{(a-bi)t} \\ &= e^a (A_1 v_1 e^{i(bt)} + A_2 v_2 e^{-i(bt)}) \\ &= e^a (C_1 \cos(bt) + C_2 \sin(bt))\end{aligned} \tag{33}$$

where $C_1 = A_1 v_1 + A_2 v_2$ and $C_2 = (A_1 v_1 - A_2 v_2) i$. Note that C_1 and C_2 are vectors.