

Differential Equations Cheatsheet (JMC Year 1, 2017/2018)

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Part I

Differential Equations

1 Definitions

Order (of derivative) An n^{th} derivative has order n .

Order (of ODE) The order of the highest derivative present in an ODE.

Degree (of ODE) The highest power to which a term is raised in an ODE (excluding fractional powers).

Linear An ODE which has no terms raised to more than the 1^{st} power, and with no y or other derivative terms multiplied by each other.

2 1st order linear ODEs

Every 1st order linear ODE can be expressed as:

$$\frac{dy}{dx} + p(x)y = q(x) \quad (1)$$

These can ALL be solved by the *integrating factor* method:

1. Multiply both sides by $\exp(\int p(x)dx)$
2. Use the reverse product rule to express the LHS as a single derivative (of a function of y).
3. Integrate both sides and rearrange.

3 1st order non-linear ODEs

3.1 Separable ODEs

Separable equations can be written in the form:

$$\frac{dy}{dx} = f(x)g(y) \quad (2)$$

These can be rearranged and integrated on both sides, with respect to the different variables.

3.2 Homogenous ODEs

Homogenous equations can be written in the form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \quad (3)$$

To solve, set $v = \frac{y}{x}$, so that $y = xv$. Note that v is still a single-variable function of x , since y is a function of x . Now we can differentiate both sides to get:

$$\frac{dy}{dx} = v + x \frac{dv}{dx} \quad (4)$$

We now have simultaneous equations for $\frac{dy}{dx}$. Equate and solve for $\frac{dv}{dx}$, and then solve this 1st order linear ODE in $\frac{dv}{dx}$ to find v (and then y).

3.3 Bernoulli type ODEs

A Bernoulli type ODE is of the form:

$$\frac{dy}{dx} + p(x)y = q(x)y^n \quad (5)$$

To solve:

1. Multiply both sides by $(1 - n)y^{-n}$
2. Let $z = y^{1-n}$ and substitute into equation, including rewriting one of the terms as $\frac{dz}{dx}$
3. The resulting equation is 1st order linear in z , so solve for z (and then y).

4 2nd order ODEs

4.1 Special case - y missing

If we can write the 2nd derivative as:

$$\frac{d^2y}{dx^2} = f\left(x, \frac{dy}{dx}\right) \quad (6)$$

(i.e. y is missing), then we can make a substitution. Let $P = \frac{dy}{dx}$. This means $\frac{d^2y}{dx^2} = \frac{dP}{dx}$, therefore we have:

$$\frac{dP}{dx} = f(x, P) \quad (7)$$

This is 1st order w.r.t P and can be solved by appropriate 1st order methods.

4.2 Special case - x missing

If we can write the 2^{nd} derivative as:

$$\frac{d^2y}{dx^2} = f(y, \frac{dy}{dx}) \quad (8)$$

(i.e. x is missing), then we can make the same substitution. Let $P = \frac{dy}{dx}$. This means $\frac{d^2y}{dx^2} = \frac{dP}{dy}$, therefore we have:

$$\frac{dP}{dy} = f(y, P) \quad (9)$$

DIFFERENT TO LAST TIME: we must rewrite $\frac{dP}{dx}$ as a derivative with respect to y . Luckily, we can see that:

$$\frac{dP}{dx} = \frac{dP}{dy} \frac{dy}{dx} = P \frac{dP}{dy} \quad (10)$$

Therefore:

$$P \frac{dP}{dy} = f(y, P) \quad (11)$$

This is 1st order w.r.t P and can be solved by appropriate 1st order methods.

4.3 General case - finding the CF

The general solution (GS) of a 2nd order ODE can be expressed as the sum of two other functions, called the 'complementary function' (CF) and a 'particular integral' (PI).

$$y_{GS} = y_{CF} + y_{PI} \quad (12)$$

A 2nd order ODE will usually be presented to us in the form:

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + c = f(x) \quad (13)$$

It can be shown that the CF can be calculated from the LHS of the above equation. We write down the *auxiliary equation*, which is simply the equation:

$$a\lambda^2 + b\lambda + c = 0 \quad (14)$$

using a, b, c from above. Solving this gives us two values, λ_1 and λ_2 .

4.3.1 $\lambda_1 \neq \lambda_2$

We can express the CF as:

$$y_{CF} = A_1 e^{\lambda_1 x} + A_2 e^{\lambda_2 x} \quad (15)$$

where A_1 and A_2 are arbitrary constants.

4.3.2 $\lambda_1 = \lambda_2$

Same as above, but we stick an x in front of one of the clashing parts of the solution.

$$y_{CF} = A_1 e^{\lambda_1 x} + A_2 x e^{\lambda_2 x} \quad (16)$$

4.3.3 λ_1 and λ_2 are complex-valued

If the auxiliary equation has complex roots, λ_1 and λ_2 will be complex conjugates. It is also important to note A_1 and A_2 are now complex-valued. The CF can be expressed as:

$$\begin{aligned} y_{CF} &= A_1 e^{(a+bi)x} + A_2 e^{(a-bi)x} \\ &= e^a (A_1 e^{i(bx)} + A_2 e^{-i(bx)}) \\ &= e^a (C_1 \cos(bx) + C_2 \sin(bx)) \end{aligned} \quad (17)$$

where $C_1 = A_1 + A_2$ and $C_2 = (A_1 - A_2)i$. Note that A_1 and A_2 would have been complex, however C_1 and C_2 are necessarily real.

4.4 General case - finding the PI

The particular integral is *any function* y_{PI} that satisfies the *ENTIRE differential equation*. The particular integral can be calculated depending on the form of the RHS of equation 13. We will refer to the RHS as simply $f(x)$ and the particular integral (as before) as y_{PI} .

4.4.1 $f(x)$ is a polynomial

Try setting y_{PI} as a general polynomial of the same degree. e.g. if $f(x)$ is a quadratic, try setting $y_{PI} = ax^2 + bx + c$ and substituting into the ODE. We will solve for a, b, c, and this will give us y_{PI} .

4.4.2 $f(x)$ is a multiple of e^{bx} , e^{bx} NOT in CF

Choose $y_{PI} = Ae^{bx}$ for some real number A.

4.4.3 $f(x)$ is a multiple of e^{bx} , e^{bx} IS in CF

We now have a clash between the PI and the CF. Choose $y_{PI} = A(x)e^{bx}$ for some real FUNCTION A. At the end remove any clashing terms (i.e. terms with a component of $e^{(something)x}$ which is already present in the CF).

4.4.4 $f(x) = A(x)e^{bx}$ where $A(x)$ is a polynomial

Choose $y_{PI} = C(x)e^{bx}$ for some polynomial $C(x)$.

4.4.5 $f(x)$ is trigonometric (e.g. sin, cos, sinh etc.)

Look for a pattern in $f(x)$. A good tip for an $f(x)$ with only sines/cosines is to use $y_{PI} = A \cos(x) + B \sin(x)$ and solve for A and B. A similar story for sinh and cosh. CAUTION: sinh, cosh and tanh are actually exponential functions in disguise, so make sure they do not clash with any $e^{\lambda x}$ terms in the CF.

4.4.6 Special cases

If $f(x)$ has a term of the form $e^x \cos(x)$ or $e^x \sin(x)$ then we can rewrite it as the real/imaginary part of a complex function (in this case $e^{(1+i)x}$ would be appropriate, since it expands to $e^x(\cos(x) + i \sin(x))$).

Part II

Systems of differential equations

5 Definitions

Order (of system) The order of the highest derivative present in the system.

Degree (of system) The highest power to which a term is raised in an ODE (excluding fractional powers).

Linear An ODE which has no terms raised to more than the 1st power, and with no y or other derivative terms multiplied by each other.

Homogenous

Equilibrium point