M1J1 Summary Notes

JMC Year 1, 2017/2018 syllabus

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UNDER CONSTRUCTION

The structure of this document is split since the two parts were taught by different lecturers.

Note that the exam will probably require you to PROVE some of these theorems, so you should refer back to the original notes for the proofs.

Boxes cover content in more detail. Titles of some theorems are given in italics.

Contents

Part I.
Applied Methods

1. Definitions

Order (of derivative) An n^{th} derivative has order n.

Order (of ODE) The order of the highest derivative present in an ODE.

Degree (of ODE) The highest power to which a term is raised in an ODE (excluding fractional powers).

Linear (ODE) An ODE which has no terms raised to more than the 1^{st} power, and with no y, x or other derivative terms multiplied by each other.

System of diff. equations A set of simultaneous equations of derivatives, where derivatives of y, x etc. are given w.r.t. a parameter t

Order (of system) The order of the highest derivative present in the system.

Degree (of system) The highest power to which a term is raised in an ODE (excluding fractional powers).

Linear (system) A system which has no terms raised to more than the 1^{st} power, and with no y or other derivative terms multiplied by each other.

Homogeneous (system) A system with no explicit functions of t (i.e. $f(t)$) present.

2. 1st order non-linear ODEs

2.1. Exact equations

Let us say we have an ODE of the form:

$$P(x,y) + Q(x,y)\frac{dy}{dx} = 0 \tag{1}$$

(note the coefficients are multi-variable functions). This can be rewritten as:

$$P(x,y)dx + Q(x,y)dy = 0 \tag{2}$$

We can try the exact equations method. We say an equation is exact iff:

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x} \tag{3}$$

This simple condition implies some important results. It can be shown that an exact equation implies the LHS of equation 2 is an exact (total) differential). This means it can be written as df, where f is some function of x and y. But the equation of this total differential is:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{4}$$

Comparing to equation 2 we can note 3 things:

$$\begin{aligned} P(x,y) &= \frac{\partial f}{\partial x} \\ Q(x,y) &= \frac{\partial f}{\partial y} \\ df &= 0 \end{aligned} \tag{5}$$

We integrate $P(x,y)$ w.r.t x and $Q(x,y)$ w.r.t y and 'merge' the two expressions together (i.e. for any matching terms, write them down only once) to give us an expression for $f(x,y)$. Ignore constants of integration. $df = 0$ tells us that $f(x,y) = c$ by integration. Therefore the general solution is given by:

$$f(x,y) = c \tag{6}$$

for some arbitrary constant c.

3. 2nd order ODEs

3.1. Special case - y missing

If we can write the 2^{nd} derivative in the form:

$$\frac{d^2y}{dx^2} = f(x, \frac{dy}{dx}) \tag{7}$$

(i.e. no y terms present), then we can make a substitution. Let $P = \frac{dy}{dx}$. This means $\frac{d^2y}{dx^2} = \frac{dP}{dx}$, the refore we have:

$$\frac{dP}{dx} = f(x,P) \tag{8}$$

This is 1st order w.r.t P and can be solved by appropriate 1st order methods.

3.2. Special case - x missing

If we can write the 2^{nd} derivative as:

$$\frac{d^2y}{dx^2} = f(y, \frac{dy}{dx}) \tag{9}$$

(i.e. no x terms present), then we can make the same substitution. Let $P = \frac{dy}{dx}$. This means $\frac{d^2y}{dx^2} = \frac{dP}{dx}$, therefore we have:

$$\frac{dP}{dx} = f(y, P) \tag{10}$$

However, this is not yet a 1st order equation since the derivative is w.r.t. x, but we only have y terms on the RHS.

DIFFERENT TO LAST TIME: we must rewrite $\frac{dP}{dx}$ as a derivative with respect to y. Luckily, we can see that:

$$\frac{dP}{dx} = \frac{dP}{dy}\frac{dy}{dx} = P\frac{dP}{dy} \tag{11}$$

Therefore:

$$P\frac{dP}{dy} = f(y, P) \tag{12}$$

This is 1st order w.r.t P and can be solved by appropriate 1st order methods.

3.3. General case - finding the CF

The general solution (GS) of a 2nd order ODE can be expressed as the sum of two other functions, called the 'complementary function' (CF) and a 'particular integral' (PI).

$$y_{GS} = y_{CF} + y_{PI} \tag{13}$$

A 2nd order ODE will usually be presented to us in the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c = f(x) \tag{14}$$

It can be shown that the CF can be calculated from the LHS of the above equation. We write down the *auxiliary equation*, which is simply the equation:

$$a\lambda^2 + b\lambda + c = 0 \tag{15}$$

using a, b, c from above. Solving this gives us two values, λ_1 and λ_2.

Case 1: $\lambda_1 \neq \lambda_2$, both real

We can express the CF as:

$$y_{CF} = A_1 e^{\lambda_1 x} + A_2 e^{\lambda_2 x} \tag{16}$$

where A_1 and A_2 are arbitrary constants.

Case 2: $\lambda_1 = \lambda_2$, both real

Same as above, but we stick an x in front of one of the clashing parts of the solution.

$$y_{CF} = A_1 e^{\lambda_1 x} + A_2 x e^{\lambda_2 x} \tag{17}$$

Case 3: λ_1, λ_2 are complex

If the auxiliary equation has complex roots, λ_1 and λ_2 will be complex conjugates. The CF can be expressed as:

$$\begin{aligned} y_{CF} &= A_1 e^{(a+bi)x} + A_2 e^{(a-bi)x} \\ &= e^a (A_1 e^{i(bx)} + A_2 e^{-i(bx)}) \\ &= e^a (C_1 cos(bx) + C_2 sin(bx)) \end{aligned} \tag{18}$$

where $C_1 = A_1 + A_2$ and $C_2 = (A_1 - A_2)i$. Note that even though A_1 and A_2 may have been complex, C_1 and C_2 are necessarily real.

3.4. General case - finding the PI

The particular integral is *any function y_{PI} that satisfies the ENTIRE differential equation.* The particular integral can be calculated depending on the form of the RHS of equation 14. We will refer to the RHS as simply $f(x)$ and the particular integral (as before) as y_{PI}. We can follow some basic rules:

Case 1: $f(x)$ is a polynomial

Try setting y_{PI} as a general polynomial of the same degree. e.g. if $f(x)$ is a quadratic, try setting $y_{PI} = ax^2 + bx + c$ and substituting into the ODE. We will solve for a, b, c, and this will give us y_{PI}.

Case 2: $f(x)$ is a multiple of e^{bx}, e^{bx} NOT in CF

Choose $y_{PI} = Ae^{bx}$ for some real number A.

Case 3: $f(x)$ **is a multiple of** e^{bx}**,** e^{bx} **IS in CF**

We now have a clash between the PI and the CF. We can try $y_{PI} = Axe^{bx}$, i.e. sticking an x in the PI to avoid the clash. If this doesn't work, we can choose $y_{PI} = A(x)e^{bx}$ for some real FUNCTION A. Remember to use the CHAIN RULE to differentiate A this time.

At the end remove any clashing terms, i.e. terms of the form $Be^{\lambda x}$ where $e^{\lambda x}$ is already present in the CF. Other terms with more x's included are allowed, e.g. $xe^{\lambda x}$ would not count as a clashing term.

Case 4: $f(x) = A(x)e^{bx}$ **where** $A(x)$ **is a polynomial**

Choose $y_{PI} = C(x)e^{bx}$ for some polynomial $C(x)$.

Case 5: $f(x)$ **is trigonometric (e.g. sin, cos, sinh etc.)**

Look for a pattern in $f(x)$. A good tip for an $f(x)$ with only sines/cosines is to use $y_{PI} = A\cos(x) + B\sin(x)$ and solve for A and B. A similar story for sinh and cosh. CAUTION: sinh, cosh and tanh are actually exponential functions in disguise, so make sure they do not clash with any $e^{\lambda x}$ terms in the CF.

Other cases

If $f(x)$ has a term of the form $e^x\cos(x)$ or $e^x\sin(x)$ then we can rewrite it as the real/imaginary part of a complex function (in this case $e^{(1+i)x}$ would be appropriate, since it expands to $e^x(\cos(x) + i\sin(x))$.

If $f(x)$ is more complicated, we may have to be imaginative with the choice of y_{PI}. e.g. for $f(x) = Ae^{ax} + Be^{bx}$ we could choose $y_{PI} = Ce^{ax} + De^{bx}$ for some constants C, D. Again be careful of terms that clash with the CF.

4. Solving systems of diff. equations

A homogeneous 1st order system of equations can be written as:

$$\begin{aligned} \frac{dx}{dt} &= F(x, y) \\ \frac{dy}{dt} &= G(x, y) \end{aligned} \tag{19}$$

Let us choose an example coupled system:

$$\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy \end{aligned} \tag{20}$$

We can rewrite this in matrix form:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{21}$$

The system is now of the form

$$\frac{d}{dt}v = Mv \tag{22}$$

If we set $v = Ve^{\lambda t}$, where V is a constant vector independent of x, y or t, then we get

$$\begin{aligned} \lambda V &= MV \\ (M - \lambda I_n)V &= 0_v \\ \det(M - \lambda I_n) &= 0 \end{aligned} \tag{23}$$

Predictably, we find two eigenvalues λ_1, λ_2 and (any) two eigenvectors v_1, v_2. The solution to the system is given by:

$$\begin{pmatrix} x \\ y \end{pmatrix} = A_1 v_1 e^{\lambda_1 t} + A_2 v_2 e^{\lambda_2 t} \tag{24}$$

The dimension of the eigenvectors will always match the number of variables being dealt with, for example a possible scenario is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = A_1 \begin{pmatrix} 3 \\ -5 \end{pmatrix} e^{-3t} + A_2 \begin{pmatrix} 7 \\ -2 \end{pmatrix} e^{2t} \tag{25}$$

The values of the individual derivatives can be found by reading off the rows of the matrices.

$$\begin{aligned} x &= 3A_1 e^{-3t} + 7A_2 e^{2t} \\ y &= -5A_1 e^{-3t} + -2A_2 e^{2t} \end{aligned} \tag{26}$$

4.0.1. Complex eigenvalues

If the eigenvalues turn out to be complex conjugates, the solution can be written as:

$$\begin{pmatrix} x \\ y \end{pmatrix} = A_1 v_1 e^{(a+bi)t} + A_2 v_2 e^{(a-bi)t} \tag{27}$$

(Note that A_1 and A_2 may be complex). We can do some rearranging like before to tidy up the solution:

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= A_1 v_1 e^{(a+bi)t} + A_2 v_2 e^{(a-bi)t} \\ &= e^a (A_1 v_1 e^{i(bt)} + A_2 v_2 e^{-i(bt)}) \\ &= e^a (C_1 \cos(bt) + C_2 \sin(bt)) \end{aligned} \tag{28}$$

where $C_1 = A_1 v_1 + A_2 v_2$ and $C_2 = (A_1 v_1 - A_2 v_2) i$. Note that C_1 and C_2 are vectors.

Part II.
Linear Algebra

5. Definitions

$\mathbb{R}^n$ The set of all column vectors with height n.

Zero vector A column vector with all n entries 0. Denoted by $\mathbf{0}_n$.

Standard basis vectors of $\mathbb{R}^n$ The set of column vectors with n entries, with a single entry as 1 and the rest 0.

Linear combination A linear combination of vectors $\mathbf{v}_1 ... \mathbf{v}_n$ is an expression:

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + ... + \lambda_n \mathbf{v}_n \tag{29}$$

where $\lambda_1 ... \lambda_n \in \mathbb{R}$.

Span The span of of a set of vectors is the set of all linear combinations of those vectors.

Dot product The dot product of $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ is defined as:

$$\mathbf{v} \cdot \mathbf{u} = v_1 u_1 + v_2 u_2 + ... + v_n u_n \tag{30}$$

Norm The norm (aka length) of a vector $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ in $\mathbb{R}^n$ is defined by:

$$||\mathbf{v}|| = \sqrt{\sum_{i=1}^{n} v_i^2} \tag{31}$$

Unit vector Any $\mathbf{v}$ such that $||\mathbf{v}|| = 1$.

Zero matrix An $m \times n$ matrix with all entries 0. Denoted by $\mathbf{0}_{m \times n}$.

Transpose The transpose of a matrix $\mathbf{A}$ is the result of flipping all rows with columns and vice versa. Denoted by $\mathbf{A}^T$.

(a_{ij}) notation For an $m \times n$ matrix $\mathbf{A}$, we can say $\mathbf{A} = (a_{ij})$ if we represent the matrix as:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} \tag{32}$$

i denotes the row number and j denotes the column number. Hence we have $1 \leq i \leq m$ and $1 \leq j \leq n$.

Leading diagonal Let $\mathbf{A} = (a_{ij})$. The leading diagonal is made up of all entries of the form a_{ii}, e.g. a_{11}, a_{22}, etc.

Identity matrix A square $n \times n$ matrix where the leading diagonal is all 1s and the rest of the entries are 0. Denoted by I_n.

Linear equation A linear equation of variables $x_1 ... x_n \in \mathbb{R}$ is given by:

$$\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n = b \tag{33}$$

where $\lambda_1 ... \lambda_n, b \in R$.

System of linear equations A system of linear equations is a set of simultaneous linear equations.

Free variable A variable in a system of linear equations that can be set to any value, without affecting the validity of the solution.

Basic variable Any variable that is not a free variable (i.e. a solution only exists for particular values of this variable).

Augmented matrix For a system of linear equations represented as $\mathbf{Ax} = \mathbf{b}$, the augmented matrix is $(\mathbf{A} \mid \mathbf{b})$. This is the matrix $\mathbf{A}$ but with $\mathbf{b}$ added in as an extra column on the right.

Row operation There are 3 types of row operation:

- Swapping rows
- Multiplying a row by a scalar value
- Adding a multiple of one row to another

Leading entry The leading entry of a row is the first non-zero entry in that row.

Row echelon form A matrix is in row echelon form if:

- All non-zero rows are above rows of all zeroes
- Each leading entry is 1, and is in a column to the right of the leading entry in the row above
- All entries in the column below a leading entry are 0

Note that any matrix in row echelon form is upper triangular.

Reduced row echelon form (RREF) A matrix is in reduced row echelon form if:

- The matrix is in reduced echelon form
- Each leading entry is the only non-zero entry in its column

Any matrix in RREF is upper AND lower triangular.

Upper triangular A matrix is upper triangular if $a_{ij} = 0$ for $i > j$.
i.e. only the top-right half contains non-zero entries.

Strictly upper triangular A matrix is strictly upper triangular if $a_{ij} = 0$ for $i \geq j$.

Lower triangular A matrix is lower triangular if $a_{ij} = 0$ for $i < j$.
i.e. only the bottom-left half contains non-zero entries.

Strictly lower triangular A matrix is strictly lower triangular if $a_{ij} = 0$ for $i \leq j$.

Diagonal matrix A diagonal matrix is such that $a_{ij} = 0$ if $i \neq j$.
i.e. Only the leading diagonal contains non-zero entries.

Inverse The inverse of an $n \times n$ matrix $\mathbf{A}$ is such that:

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = I_n \tag{34}$$

Note that this is one of the only occasions where matrix multiplication is commutative. Not all matrices have inverses.

Singular A matrix is called singular if it is non-invertible.

Determinant A specific number calculated from the entries in a matrix. Used in computing inverses. Denoted by $\det(A)$.

Elementary matrix A matrix that differs from the identity matrix by one row operation.

Eigenvalue An eigenvalue of a matrix $\mathbf{A}$ is some $\lambda \in R$ that satisfies:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \tag{35}$$

for some $\mathbf{v} \in \mathbb{R}^n$, called the eigenvector.

Eigenvalue An eigenvector of a matrix $\mathbf{A}$ is some $\mathbf{v} \in R^n$ that satisfies:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \tag{36}$$

for some $\lambda \in \mathbb{R}$, called the eigenvalue.

6. Theorems