M1J2 Summary Notes

JMC Year 1, 2017/2018 syllabus

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(Dr Lawn refers to propositions, theorems, corollaries and lemmas. In this document I will refer to them all as 'theorems'.)

This document contains a list of definitions and a list of theorems.

Note that the exam will probably require you to PROVE some of these theorems, so you should refer back to the original notes for the proofs.

Boxes cover content in more detail. Titles of some theorems are given in italics.

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Part I.
Abstract Linear Algebra

1. Definitions

Vector space A vector space is a set V coupled with:

- a function $+ : V \times V \to V$ (addition)
- a function $\cdot : \mathbb{R} \times V \to V$ (scalar multiplication)

(For the rest of this part, we will assume V is a vector space)

Subspace A subset $U \subseteq V$ is a subspace if:

- $\mathbf{0}_V \in U$
- If $\mathbf{x}, \mathbf{y} \in U$ then $\mathbf{x} + \mathbf{y} \in U$ (closure under addition)
- If $\mathbf{x} \in U$ then for all $\lambda \in \mathbb{R}$, $\lambda\mathbf{x} \in U$ (closure under scalar multiplication)

Linear combination A linear combination of a set of vectors $\{\mathbf{v}_1 ... \mathbf{v}_n\}$ is any vector $\mathbf{x}$ of the form:

$$\mathbf{x} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + ... + \lambda_n\mathbf{v}_n \tag{1}$$

for some real numbers $\lambda_1 ... \lambda_n$

Span The span of a set $S \subseteq V$ is the set of all linear combinations of elements of S. We define $\text{span}(\emptyset) = \{\mathbf{0}_V\}$.

Spanning set A subset $S \subseteq V$ is called a spanning set of V if $\text{span}(S) = V$.

Linear dependence A subset of vectors $\{\mathbf{v}_1 ... \mathbf{v}_n\} \subseteq V$ is linearly dependent if there exists some real numbers $\lambda_1 ... \lambda_n$ (which are not just all 0s) such that:

$$\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + ... + \lambda_n\mathbf{v}_n = \mathbf{0}_V \tag{2}$$

Basis A basis of a vector space is a linearly independent spanning set.

We can also think of a basis as a spanning set of minimum possible size, or a linearly independent set of maximum possible size (theorems to show this later).

Standard basis of $\mathbb{R}^n$ We define the standard basis elements of any $\mathbb{R}^n$ to be:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \; ... \; e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \; e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{3}$$

The standard basis of $\mathbb{R}^n$ is therefore $\{e_1, e_2 \; ... \; e_n\}$.

Dimension The dimension of a vector space is the size of any basis of that vector space.

Linear map Let U and V be vector spaces. A linear map is a function $f : U \to V$ such that:

- for all $\mathbf{x}, \mathbf{y} \in U$, $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$
- for all $\mathbf{x} \in U$ and $\lambda \in \mathbb{R}$, $f(\lambda\mathbf{x}) = \lambda f(\mathbf{x})$

Image The image of a linear map $f : U \to V$ is the set of all $f(\mathbf{u}) \in V$ where $\mathbf{u} \in U$.

$$\text{image}(f) = \{f(\mathbf{u}) \mid u \in U\} \tag{4}$$

Kernel The kernel of a linear map $f : U \to V$ is the set of all $\mathbf{u} \in U$ such that $f(\mathbf{u}) = \mathbf{0}_V$.

$$\text{kernel}(f) = \{\mathbf{u} \mid u \in U, f(\mathbf{u}) = \mathbf{0}_V\} \tag{5}$$

Isomorphism A linear map $f : U \to V$ is an isomorphism if it is bijective. We say $U \simeq V$.

Rank The rank of f is defined as dim(image(f)).

Nullity The rank of f is defined as dim(kernel(f)).

T_A We define a function T_A that pre-multiplies a vector by a matrix $\mathbf{A}$:

$$T_A : \mathbb{R}^n \to \mathbb{R}^m, \; \mathbf{v} \mapsto \mathbf{A}\mathbf{v}, \; \mathbf{A} \in \text{Mat}_{m \times n}(\mathbb{R}) \tag{6}$$

where $\text{Mat}_{m \times n}(\mathbb{R})$ denotes the set of all $m \times n$ matrices with real entries.
Note that if $\mathbf{A}$ is an $m \times n$ matrix, then T_A transforms a vector in $\mathbb{R}^n$ to a vector in $\mathbb{R}^m$.

Matrix representing f Following from the previous definition, if we have:

- $f : U \to V$
- B is a basis of U
- C is a basis of V
- There is an isomorphism $f_B : \mathbb{R}^n \to U$
- There is an isomorphism $f_C : \mathbb{R}^m \to V$

We can define a version of T_A that is equivalent to f. The matrix $\mathbf{A}$ in this case is called the matrix representing f with respect to B and C. This is denoted by:

$$\mathbf{A} = \left[f\right]_B^C \tag{7}$$

Change-of-basis matrix Let B and C be two bases for V. The matrix:

$$\mathbf{A} = \left[\mathrm{Id}_V\right]_B^C \tag{8}$$

is called the change-of-basis matrix from B to C. Id_V denotes the identity function in the vector space V (maps every vector to itself).

In this case the linear map T_A will convert a vector given with respect to the basis B into a vector with respect to the basis C.

'Vector with respect to a basis' If we have an n-dimensional vector space V and a basis $B = \{\mathbf{b}_1 ... \mathbf{b}_n\}$, then we say any $\mathbf{v} \in V$ is given with respect to B if:

$$\mathbf{v} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}, \ \mathbf{v} = \lambda_1 \mathbf{b}_1 + \lambda_2 \mathbf{b}_2 + ... + \lambda_n \mathbf{b}_n \tag{9}$$

2. Theorems

2.1. Vector spaces

Vector space axioms

- $(V, +)$ is an Abelian group (the identity element being $\mathbf{0}_V$)
- for any $\mathbf{v} \in V$, $1\mathbf{v} = \mathbf{v}$
- for any $\mathbf{v} \in V, \lambda, \mu \in \mathbb{R}, \lambda(\mu\mathbf{v}) = (\lambda\mu)\mathbf{v}$ (commutative w.r.t. scalar multiplication)

- for any $\mathbf{u}, \mathbf{v} \in V, \lambda \in \mathbb{R}, \lambda(\mathbf{u}+\mathbf{v}) = \lambda\mathbf{u}+\lambda\mathbf{v}$ (scalar multiplication distributes over addition)
- for any $\mathbf{v} \in V, \lambda, \mu \in \mathbb{R}, (\lambda+\mu)\mathbf{v} = \lambda\mathbf{v}+\mu\mathbf{v}$ (scalar multiplication distributes over scalar addition)

For any $\mathbf{v} \in V$:

- For any $n \in \mathbb{Z}$, $n\mathbf{v} = \mathbf{v} + \mathbf{v} + ... + \mathbf{v}$ (n times)
- $0\mathbf{v} = \mathbf{0}_V$
- $(-1)\mathbf{v}$ is the additive inverse of $\mathbf{v}$

2.2. Subspaces

Every vector space V has two trivial subspaces, itself and $\{\mathbf{0}_V\}$.

For any subspaces $U, W \subseteq V$:

- $U \cap W$ is a subspace
- $U \cup W$ is NOT a subspace

Any $U \subseteq V$ is a subspace iff every linear combination of vectors in U is again in U (i.e. $\mathrm{span}(U) \subseteq U$).

For any $S \subseteq V$, $\mathrm{span}(S)$ is a subspace.

If $U \subset V$ is a subspace and $S \subset U$ then $\mathrm{span}(S) \subset U$.

2.3. Spanning sets, linear independence, bases, dimension

Every element of a vector space V can be written as a unique linear combination of basis vectors (for any basis).

For any set $S \subseteq V$:

- If $\mathbf{v}_1 = \lambda\mathbf{v}_2$ for any $\mathbf{v}_1, \mathbf{v}_2 \in S$ then S is linearly dependent
- If $\mathbf{0}_V \in S$ then S is linearly dependent

If a set S is linearly independent/dependent then any subset of S is also linearly independent/dependent respectively.

A vector space is finite dimensional if it contains a finite spanning set.

Every finite spanning set contains a basis.

Therefore, a vector space is finite dimensional if it has a finite basis.

If a finite dimensional vector space has a basis, then there exists a finite dimensional spanning set.

If $S \subseteq V$ is a linearly DEPENDENT spanning set, there exists some $\mathbf{s} \in S$ such that $S - \{\mathbf{s}\}$ is still a spanning set.

> In other words, we can keep removing elements from a spanning set until it is linearly independent. At this point the spanning set is now a basis, by definition. This gives us our alternate definition of a basis as a spanning set of minimum size.

Steinitz exchange lemma - base case
Let $S \subset V$ be a spanning set, and let $\mathbf{v} \in V$. There always exists an $\mathbf{s} \in S$ such that

$$(S \setminus \{\mathbf{s}\}) \cup \{\mathbf{v}\} \tag{10}$$

is still a spanning set.

Steinitz exchange lemma - in full
Let $S \subset V$ be a spanning set, and let $\mathbf{v}_1 ... \mathbf{v}_n \in V$ be a linearly independent set. There always exists some $\mathbf{s}_1 ... \mathbf{s}_n \in S$ such that

$$(S \setminus \{\mathbf{s}_1 ... \mathbf{s}_n\}) \cup \{\mathbf{v}_1 ... \mathbf{v}_n\} \tag{11}$$

is still a spanning set.

> In other words, we can substitute in any linearly independent set, and S will still be a spanning set.

Any linearly independent set is smaller than or equal to any spanning set. (Consequence of Steinitz exchange lemma).

If $L \subset V$ linearly independent and $\mathbf{v} \notin \text{span}(L)$ then $L \cup \mathbf{v}$ is linearly independent.

In other words, we can keep adding elements to a linearly independent set until it is a spanning set. At this point the linearly independent set is a basis, by definition. This gives us our alternate definition of a basis as a linearly independent set of maximum size.

If $\dim(V) = n$ then every basis of V has size n.

If V is infinite-dimensional, we can always find a linearly independent subset of V with size n, for any n.

Any linearly independent set is contained in a basis.

Any linearly independent set L where $\#L = \dim(V)$ is a basis.

If V is finite dimensional and $U \in V$:

- U is finite dimensional
- $\dim(U) \leq \dim(V)$
- if $\dim(U) = \dim(V)$ then $U = V$

2.4. Linear maps

(For the rest of this subsection assume f, g are linear maps, and let $f : U \to V$)

$g \circ f$ is also a linear map.

$f(\mathbf{0}_U) = f(\mathbf{0}_V)$.

$\text{image}(f)$ is a subspace of V.
$\text{kernel}(f)$ is a subspace of U.

If f surjective then $\text{image}(f) = V$.
If f injective then $\text{kernel}(f) = \{\mathbf{0}_U\}$.

If $f(\mathbf{x}) = \mathbf{y}$ then $f^{-1}(\mathbf{y}) = \{\mathbf{x} + \mathbf{w} \mid \mathbf{w} \in \text{kernel}(f)\}$.

If $f : \mathbb{R}^n \to \mathbb{R}^m$ then $f \equiv T_A$ for some matrix $\mathbf{A} \in \text{Mat}_{m \times n}(\mathbb{R})$.

Specifically $f : \lambda_1 \mathbf{e}_1 + ... + \lambda_n \mathbf{e}_n \mapsto \lambda_1 f(\mathbf{e}_1) + ... + \lambda_n f(\mathbf{e}_n)$

Therefore we can set:

$$\mathbf{A} = [f(\mathbf{e}_1) \mid f(\mathbf{e}_2) \mid ... \mid f(\mathbf{e}_n)] \tag{12}$$

so that for any $\mathbf{v} \in U$:

$$T_A(\mathbf{v}) = \mathbf{A} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} = \lambda_1 f(\mathbf{e}_1) + ... + \lambda_n f(\mathbf{e}_n) \tag{13}$$

Let $g : U \to V$, let $B = \{\mathbf{b}_1 ... \mathbf{b}_n\}$ be a basis of U.
If $f(\mathbf{b}_i) = g(\mathbf{b}_i)$ for all $\mathbf{b}_i$ then $f \equiv g$.

There is always a linear map between a basis of U and any set of vectors in V.

If $U \simeq V$ then $\dim(U) = \dim(V)$

If $\dim(V) = n$ then $f \simeq \mathbb{R}^n$.

Let $B = \{\mathbf{b}_1 ... \mathbf{b}_n\}$ be a basis of U and $C = \{f(\mathbf{b}_1) ... f(\mathbf{b}_n)\}$ a subset of V:

- $\text{span}(C) = \text{image}(f)$
- C is a spanning set $\Leftrightarrow f$ is surjective
- C is linearly independent $\Leftrightarrow f$ is injective
- C is a basis $\Leftrightarrow f$ is bijective (aka an isomorphism)

If $\dim(U) = \dim(V)$ then f bijective $\Leftrightarrow f$ surjective $\Leftrightarrow f$ injective

Rank-Nullity Theorem
$\text{rank}(f) + \text{nullity}(f) = \dim(U)$

Any $f : U \to V$ can be represented as T_A for some matrix $\mathbf{A}$.
We denote $\mathbf{A}$ by $\left[f\right]_B^C$, also known as the matrix representing f with respect to B and C.

Steps for computing $\mathbf{A}$*:*

Let $B = \{\mathbf{b}_1 ... \mathbf{b}_n\}$ be a basis of U
Let $C = \{\mathbf{c}_1 ... \mathbf{c}_m\}$ be a basis of V

We have isomorphisms:
$f_B : \mathbb{R}^n \to U,\ \lambda_1 \mathbf{e}_1 + ... + \lambda_n \mathbf{e}_n \mapsto \lambda_1 \mathbf{b}_1 + ... + \lambda_n \mathbf{b}_n$
$f_C : \mathbb{R}^m \to V,\ \lambda_1 \mathbf{e}_1 + ... + \lambda_m \mathbf{e}_m \mapsto \lambda_1 \mathbf{c}_1 + ... + \lambda_m \mathbf{c}_m$

Note that the linear map $(f_C)^{-1} \circ f \circ f_B$ sends vectors from $\mathbb{R}^n \to \mathbb{R}^m$, therefore we can define:

$$T_A \equiv (f_C)^{-1} \circ f \circ f_B \tag{14}$$

since, from earlier, $T_A : \mathbb{R}^n \to \mathbb{R}^m$.

1. Take basis vectors of U ($\mathbf{b}_j$) in some order. Compute $f(\mathbf{b}_j)$. We have just applied f_B, followed by f.
2. Express each $f(\mathbf{b}_j)$ as a linear combination of basis vectors of V ($\mathbf{c}_i$).
3. Applying $(f_C)^{-1}$ sends vectors in V to their coefficients w.r.t the basis vectors $\mathbf{c}_i$.

The matrix A is such that the j^{th} column of A is the vector $(f_C)^{-1} \circ f \circ f_B(\mathbf{e}_j) = (f_C)^{-1} \circ f(\mathbf{b}_j)$

Change-of-basis matrix
Let B and C both be bases of V. To convert a vector $\mathbf{v} \in V$ given with respect to a basis B to a different basis C, we can premultiply by $A = \left[Id_V\right]_B^C$. If we denote the two representations as $\mathbf{v}_B$ and $\mathbf{v}_C$:

$$\mathbf{v}_C = \left[Id_V\right]_B^C \mathbf{v}_B \tag{15}$$

Let B and B' be two bases of U, C and C' two bases of V, and $f : U \to V$. The following equation holds:

$$\left[f\right]_{B'}^{C'} = \left[Id_V\right]_C^{C'} \left[f\right]_B^C \left[Id_U\right]_{B'}^B \tag{16}$$

Part II.
Group Theory

3. Definitions

Binary operation A binary operation on a set G is a any function $f : G \times G \to G$

Associative A binary operation $\star$ on a set G is associative if it satisfies:

$$(a \star b) \star c = a \star (b \star c) \tag{17}$$

for all $a, b, c \in G$.

Commutative A binary operation $\star$ on a set G is commutative if it satisfies:

$$a \star b = b \star a \tag{18}$$

for all $a, b \in G$.

Left/right identity An element $e \in G$ is called the left identity if:

$$e \star g = g \tag{19}$$

for all $g \in G$. Similar statement for right identity.

(Two sided) Identity element An element $e \in G$ is a two-sided identity element if it is both a left identity and a right identity.

From now on the two-sided identity element will be referred to as e.

Left/right inverse An element $h \in G$ is called the left inverse of $g \in G$ if:

$$h \star g = e \tag{20}$$

Similar statement for right inverse.

Two sided inverse A two sided inverse of an element $g \in G$ is both a left inverse and a right inverse of g.

From now on the two-sided inverse of g will be referred to as g^{-1}.

Group A group $(G, \star)$ is a set G equipped with a binary operation $\star$ such that:

- $\star$ is associative
- $\star$ has an identity element $e \in G$
- Every $g \in G$ has an inverse $g^{-1} \in G$

> The above three suffice for the exam, however there is technically a fourth requirement:
>
> - G is closed under $\star$, i.e. for all $g, h, \in G, g \star h \in G$

(For the rest of this part, we will assume $(G, \star)$ is a group)

Order (group) The order of a group $(G, \star)$ is the size of G.

Abelian group An Abelian group is a group with a commutative binary operation $\star$.

Powers of g We can define the powers of any $g \in G$ to be:

$$g^n = \begin{cases} g \star g \star ... g & n > 0 \\ g^{-1} \star g^{-1} \star ... g^{-1} & n < 0 \\ e & n = 0 \end{cases} \tag{21}$$

where in the first cases there are n copies of g, and in the second case there are $-n$ copies of g^{-1}.

Definition of $[a]_n$ and $\mathbb{Z}_n$ For any $a \in \mathbb{Z}$:

$$[a]_n = \{b \in \mathbb{Z} \mid b \equiv a \bmod n\} \tag{22}$$

Note that $[a]_n$ forms an equivalence class, and there are exactly n of these equivalence classes. $\mathbb{Z}_n$ is the set of all these equivalence classes.

$$\mathbb{Z}_n = \{[a]_n \mid a \in \mathbb{Z}\} \tag{23}$$

Definition of $\mathbb{Z}_n^*$ $\mathbb{Z}_n^*$ is the set of all invertible $[a]_n$. Note in this case the identity element is $[1]_n$.

$$\mathbb{Z}_n^* = \{[a]_n \mid \exists [b]_n \in Z_n \quad s.t. \quad [a]_n[b]_n = [1]\} \tag{24}$$

Note that $[a]_n[b]_n = 1 \Leftrightarrow gcd(a, n) = 1$.

Order (element) The order of any $g \in G$ is the smallest positive integer such that:

$$g^n = e \tag{25}$$

Cyclic group + generator A group $(G, \star)$ is cyclic if:

$$G = \{g^n \mid n \in \mathbb{Z}\} \tag{26}$$

g is called the generator of the group.

Permutation A permutation σ on n symbols is a bijection:

$$\sigma : \{1...n\} \to \{1...n\} \tag{27}$$

Symmetric group The symmetric group S_n on n symbols is the set of all permutations of n symbols.

$$S_n = \{\sigma : \{1...n\} \to \{1...n\}\} \tag{28}$$

Note that S_n is a set of functions. Therefore the identity element is the identity function.

k-cycle A permutation $\sigma \in S_n$ is a k-cycle if there exists some $a_1...ak \in \{1...n\}$ such that:

$$\sigma(a_1) = a_2, \quad \sigma(a_2) = a_3 \quad ... \quad \sigma(a_k) = a_1 \tag{29}$$

and $\sigma(i) = i$ for all $i \notin \{1...n\}$. k is called the length of the cycle. The notation for a cycle is $(a_1...a_k)$.

Disjoint cycles Two cycles $(a_1...a_m)$ and $(b_1...b_n)$ are disjoint if no a_i is equal to any b_j.

Subgroup Let $(G, \star)$ be a group, and $H \subseteq G$. $(H, \star)$ is a subgroup of G if:

- $e \in H$
- For any $g, h \in H$, $g \star h \in H$
- For any $g \in H$, $g^{-1} \in H$

Cyclic subgroup Let $(G, \star)$ be a group. For any $g \in G$, the cyclic subgroup $\langle g \rangle$ generated by g is defined as:

$$\langle g \rangle = (\{g^i \mid i \in \mathbb{Z}\}, \star) \tag{30}$$

Note that order of g = size of cyclic subgroup $\langle g \rangle$.

Left/right cosets Let $(G, \star)$ be a group and $(H, \star)$ a subgroup. For any $g \in G$, the left coset of H by g (denoted by gH) is defined as:

$$gH = \{g \star h \mid h \in H\} \tag{31}$$

Similar definition for right coset of H by g (denoted by Hg).

The set of all left cosets of H by g is denoted by $G : H$.
The set of all right cosets of H by g is denoted by $H : G$.

4. Theorems

4.1. Groups

Any identity element e is unique for that group.

Any two-sided inverse g^{-1} of an element $g \in G$ is unique.

For any $g, h \in G$

$$(g \star h)^{-1} = h^{-1} \star g^{-1} \tag{32}$$

The normal exponent rules apply within groups, e.g.

$$g^n \star g^m = g^{n+m} \tag{33}$$
$$(g^n)^{-1} = g^{-n} \tag{34}$$
$$(g^n)^m = g^{nm} \tag{35}$$

Some examples of groups: $(\mathbb{R}, +)$, $(\mathbb{Z}, +)$, $(\mathbb{Z}^*, \times)$

4.2. Modular arithmetic and $\mathbb{Z}_n$

$(\mathbb{Z}_n, +)$ is an Abelian group.

$(\mathbb{Z}_n^*, \ \cdot\ )$ is an Abelian group.

4.3. Cyclic groups

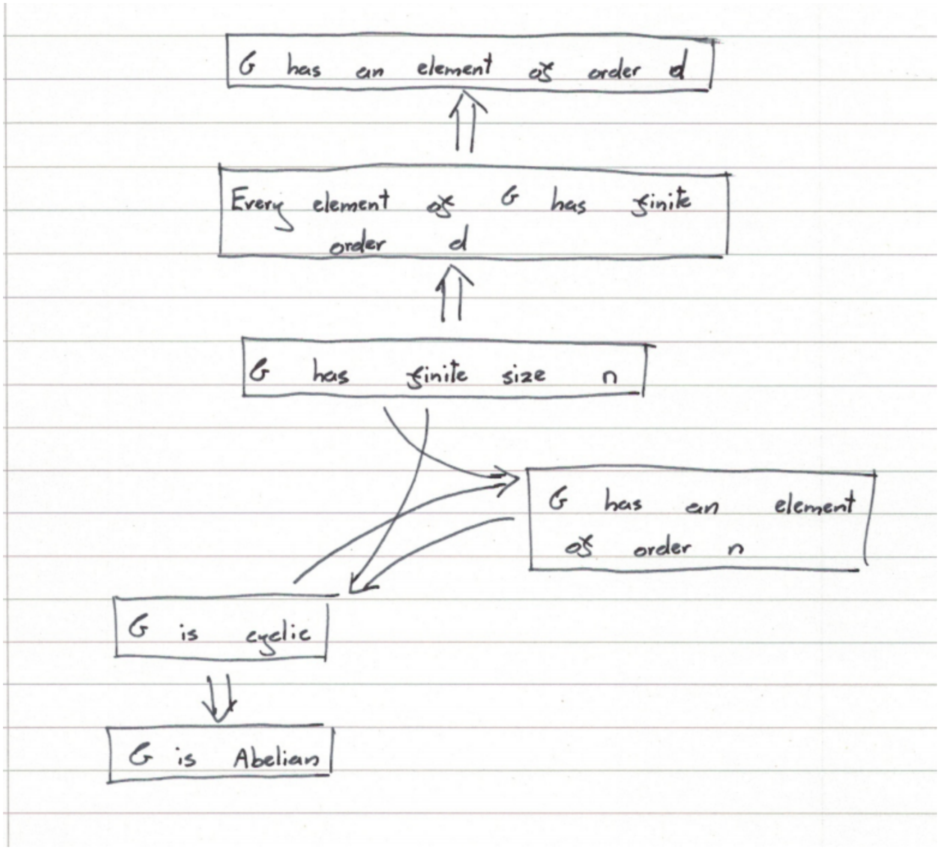
If $(G, \star)$ is a finite group then every $g \in G$ has finite order.

Any $g \in G$ with order n has distinct powers $g^0, g^1, g^2 ... g^{n-1}$.

All cyclic groups are Abelian.

Assume G is finite with size n.

G is cyclic $\Leftrightarrow$ G contains an element of order n.



4.4. Symmetric groups

$(S_n, \circ)$ is a group.

The size of any S_n is $n!$

The order of a k-cycle is k.

For any $\sigma \in S_n$:

- for any $i \in \{0...n\}$ there exists a $d > 0$ such that $\sigma^d(i) = i$ (i.e. $\sigma^d \equiv Id = e$)
- if d is the smallest integer such that $\sigma^d(i) = i$ then the numbers $i, \sigma^1(i), \sigma^2(i)...\sigma^{d-1}(i)$ are distinct
- If j is not in the set $\{i, \sigma(i), \sigma^2(i)...\sigma^{d-1}(i)\}$ then neither is $\sigma(j)$

Any permutation σ can be expressed as the product of disjoint k-cycles.

4.5. Subgroups

Any group $(G, \star)$ has two trivial subgroups, $(e, \star)$ and itself.

Subgroup test
Any $H \subseteq G$ is a subgroup if:

- $H \neq \emptyset$
- for all $x, y \in H, x \star y^{-1} \in H$

4.6. Cosets and Lagrange's Theorem

For any $g_1, g_2 \in G$ and subgroup H:

$$g_1 H = g_2 H \Leftrightarrow g_1 \in g_2 H \tag{36}$$

The left cosets of H form a partition of G. This means any $g \in G$ is in exactly one left coset of H. The right cosets also form a (different) partition.

For any $g \in G$:

$$\#gH = \#hG = \#H \tag{37}$$

Lagrange's Theorem
For any subgroup $(H, \star)$ where $H \subseteq G$:

$$\#G = \#H \cdot \#(G : H) \tag{38}$$

For any subgroup $(H, \star)$, $\#H$ divides $\#G$. (Consequence of Lagrange's Theorem).

For any $g \in G$, the order of g divides $\#G$. (Since the order of g is the size of the subgroup $\langle g \rangle$).

If $\#G = p$, where p is prime, then G is cyclic.

Part III.
Analysis

5. Definitions

Sequence A sequence is simply a map $f : \mathbb{N} \to \mathbb{R}$, denoted by a_n.

Limit (sequence) A sequence a_n converges to a limit L if for all real numbers $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$ we have $|a_n - L| < \epsilon$.

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad s.t \quad \forall n > N \quad |a_n - L| < \epsilon \tag{39}$$

Tends to infinity (sequence) We say a sequence tends to infinity if for all $R \in \mathbb{R}$, the sequence a_n is eventually bigger than R.

$$\forall R \in \mathbb{R} \quad \exists N \in \mathbb{N} \quad s.t. \quad \forall n > N \quad a_n > R \tag{40}$$

Shift The shift of a sequence by say, k, is the sequence $b_n = a_{n+k}$.

Triangle inequality The general triangle inequality is:

$$|x - y| < |x - z| + |z - y| \tag{41}$$

Setting $z = 0$ gives us:

$$|x - y| > |x| - |y| \tag{42}$$

Then setting $y = -y$ gives us the familiar case:

$$|x + y| < |x| + |y| \tag{43}$$

Bounded above A sequence a_n is bounded above if there's a real number A such that $a_n < A$ for all n.

Bounded below A sequence a_n is bounded below if there's a real number A such that $a_n > A$ for all n.

Bounded A sequence a_n is bounded if there's a real number A such that $|a_n| < A$ for all n.

Increasing A sequence is increasing if $a_{n+1} \geq a_n$ for all n.

Strictly increasing A sequence is strictly increasing if $a_{n+1} > a_n$ for all n.

Decreasing A sequence is decreasing if $a_{n+1} \leq a_n$ for all n.

Strictly decreasing A sequence is strictly decreasing if $a_{n+1} < a_n$ for all n.

Monotonic A sequence is monotonic if it is increasing or decreasing.

Supremum (set) The supremum A of a set S is the least upper bound of that set i.e. the smallest number such that $s \leq A$ for all $s \in S$.

Supremum (function) The supremum of a function f is the sup of $\{f(x) \mid x \in \mathrm{dom}(f)\}$.

Infimum (set) The infimum B of a set S is the greatest lower bound of that set i.e. the largest number such that $s \geq B$ for all $s \in S$.

Infimum (function) The infimum of a function f is the inf of $\{f(x) \mid x \in \mathrm{dom}(f)\}$.

Subsequence A subsequence of a_n is a sequence $a_{f(n)}$, where $f(n)$ is a strictly increasing function.

Cauchy sequence A sequence is Cauchy if all the terms get arbitrarily close to one another. To put it mathematically:

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad s.t \quad \forall m, n \geq N \quad |a_n - a_m| < \epsilon \tag{44}$$

Partial sum The n^{th} partial sum S_n of a sequence a_n is the sum of terms up to that point:

$$S_n = \sum_{i=1}^{n} a_n \tag{45}$$

Summable A sequence a_n is summable if the sequence of its partial sums converges. The limit of the sequence of partial sums will be:

$$L = \lim_{n \to \infty} (\sum_{i=1}^{n} a_n) = \sum_{i=1}^{\infty} a_n \tag{46}$$

Absolutely summable A sequence a_n is absolutely summable if $|a_n|$ is summable.

Conditionally summable A sequence is conditionally summable if it is summable but not absolutely summable.

Power series The power series associated with a sequence a_n is the sequence of partial sums:

$$\sum_{i=1}^{n} a_i x^i \tag{47}$$

Radius of convergence The radius of convergence R of a power series $P(x)$ is defined as the largest x for which $P(x)$ is convergent.

$$R = sup\{x \in \mathbb{R} \mid P(x) \text{ convergent}\} \tag{48}$$

Limit as $x \to \infty$ (function) A function $f(x)$ tends to a limit L as $x \to \infty$ if for all real numbers $\epsilon > 0$, there exists an $R \in \mathbb{R}$ such that for all $x \geq R$ we have $|f(x) - L| < \epsilon$.

$$\forall \epsilon > 0 \quad \exists R \in \mathbb{R} \quad s.t \quad \forall x > R \quad |f(x) - L| < \epsilon \tag{49}$$

Tends to infinity (function) A function $f(x)$ tends to infinity as $x \to \infty$ if for any $M \in \mathbb{R}$ there exists an $R \in \mathbb{R}$ such that if $x > M$ then $f(x) > R$.

$$\forall M \in \mathbb{R} \quad \exists R \in \mathbb{R} \quad s.t. \quad x > M \Rightarrow f(x) > R \tag{50}$$

One-sided limit A function $f(x)$ tends to a limit L as $x \to a^-$ if for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $x \in (a - \delta, a)$ then $|f(x) - L| < \epsilon$.

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad s.t. \quad x \in (a - \delta, a) \Rightarrow |f(x) - L| < \epsilon \tag{51}$$

Same format for the other sided limit ($x \to a^+$)
(Note that $\epsilon - \delta$ definition is only used for limits as x tends to a finite number a, not infinity)

Limit as $x \to a$ A function $f(x)$ tends to a limit L as $x \to a$ if we have both:

$$\lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L \tag{52}$$

Limit as $x \to a$ (ϵ - δ def.) A function $f(x)$ tends to a limit L as $x \to a$ if:

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad s.t. \quad |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon \tag{53}$$

Continuous A function $f(x)$ is continuous at a if:

$$\lim_{x \to a} f(x) = f(a) \tag{54}$$

Continuous (ϵ - δ def.) A function $f(x)$ is continuous at a if for all $\epsilon > 0$ there is a $\delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$.

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad s.t. \quad |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon \tag{55}$$

Continuous everywhere A function $f(x)$ is continuous everywhere if it is continuous at a for all $a \in \text{dom}(f)$.

Open interval An open inteval I is a set $I \subseteq \mathbb{R}$ of the form:

- $I = (a, b)$ for some $a, b \in \mathbb{R}$, or
- $I = (-\infty, b)$, or
- $I = (a, +\infty)$, or
- $I = \mathbb{R}$

Discontinuity Discontinuity is the negation of continuity. Hence a function $f(x)$ is discontinuous at a if there exists $\epsilon > 0$ such that for all $\delta > 0$, $|x - a| < \delta$ AND $|f(x) - f(a)| > \epsilon$.

$$\exists \epsilon > 0 \quad s.t. \quad \forall \delta > 0 \quad |x - a| < \delta \text{ AND } |f(x) - f(a)| > \epsilon \tag{56}$$

Bounded (function) A function $f(x)$ is bounded if the set of all possible values of f(x) is bounded.

Differentiable (ver. 1) A function $f(x)$ is differentiable at a if:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{57}$$

exists.

Differentiable (ver. 2) A function $f(x)$ is differentiable at a if:

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \tag{58}$$

exists.

Differentiable everywhere A function $f(x)$ is differentiable everywhere if it is differentiable at a for all $a \in \mathrm{dom}(f)$.

Global maximum A function $f(x)$ has a global maximum at a if $f(a) \geq f(x)$ for all other values of $f(x)$.

Similar definition for global minimum.

Local maximum A function $f(x)$ has a local maximum at a if $f(a) \geq f(x)$ for all x in the set $(a - \epsilon, a + \epsilon)$, for some ϵ.

Similar definition for local minimum.

Lipschitz continuous A function is Lipschitz continuous if:

$$|f'(x)| \leq L \Rightarrow |f(x_1) - f(x_2)| \leq L|x_1 - x_2| \tag{59}$$

6. Theorems

6.1. Sequences

Every convergent sequence has a unique limit.

Every convergent sequence is bounded.

If all terms of a convergent sequence are larger than a number B, then so is its limit.

Some properties of limits:

$$\lim_{x\to\infty} (a_n + b_n) = \lim_{x\to\infty} a_n + \lim_{x\to\infty} b_n \tag{60}$$

$$\lim_{x\to\infty} (\lambda a_n) = \lambda \lim_{x\to\infty} a_n \tag{61}$$

$$\lim_{x\to\infty} (a_n b_n) = \lim_{x\to\infty} a_n \lim_{x\to\infty} b_n \tag{62}$$

$$\lim_{x\to\infty} (\frac{a_n}{b_n}) = \frac{\lim_{x\to\infty} a_n}{\lim_{x\to\infty} b_n} \tag{63}$$

where λ is any real number.

If $a_n \to \infty$ and b_n is bounded below, $a_n + b_n \to \infty$.

If $a_n \to \infty$ and b_n is bounded below by a positive number, $a_n b_n \to \infty$.

If a_n is bounded and $b_n \to \infty$, then $\frac{a_n}{b_n} \to 0$.

If $a_n \to \infty$, for any real number λ:

- $\lambda < 0 \Rightarrow \lambda a_n \to -\infty$
- $\lambda = 0 \Rightarrow \lambda a_n \to 0$
- $\lambda > 0 \Rightarrow \lambda a_n \to \infty$

If $a_n \to a$ and $b_n \to b$, and for all n $a_n < b_n$, then $a < b$.

Sandwich Theorem
If $a_n \leq b_n \leq c_n$ for all n, and a_n and c_n tend to the same limit L, then $b_n \to L$.

Every bounded monotonic sequence is convergent.

Completeness Axiom
Every non-empty subset of the real numbers which is bounded above has a supremum. Corresponding statement for infimum.

Useful results for sequences:

$$\lim_{n\to\infty} \lambda^n = \begin{cases} \infty & \lambda > 1 \\ 1 & \lambda = 1 \\ 0 & -1 < \lambda < 1 \end{cases} \tag{64}$$

λ^n diverges if $\lambda = -1$.

If $m > 0$ and $\lambda > 1$ then $\frac{\lambda^n}{n^m} \to \infty$ (exponentials beat powers).

If $m > 0$ then $\frac{\log(n)}{n^m} \to 0$ (powers beat logs).

6.2. Subsequences

If $a_n \to L$ then any subsequence $a_{f(n)} \to L$.

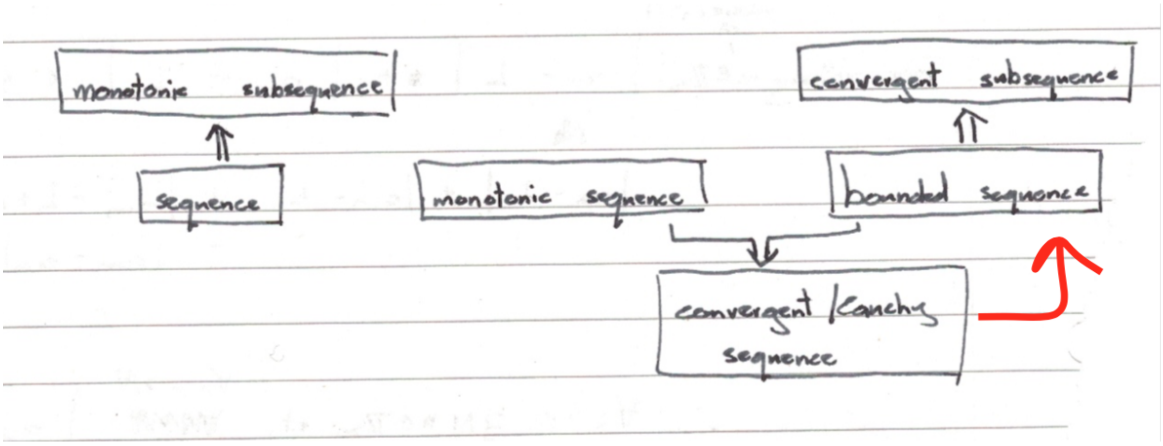
If two subsequences of a_n converge to different limits, a_n doesn't converge to a limit.

Every sequence has a monotonic subsequence.

Bolzano-Weierstrass Theorem
Every bounded sequence has a convergent subsequence.

Every Cauchy sequence is bounded.

Cauchy sequence $\Leftrightarrow$ convergent sequence (for real numbers).



6.3. Summability

A sequence is summable iff the sequence of its partial sums converges.

If two subsequences of a sequence a_n converge to two different limits, a_n is not summable.

If a_n and b_n are summable with $\sum_{i=0}^{\infty} a_i = a$ and $\sum_{i=0}^{\infty} b_i = b$:

- $a_n + b_n$ is summable with $\sum_{i=0}^{\infty}(a_i + b_i) = a + b$.
- λa_n is summable with $\sum_{i=0}^{\infty} \lambda a_i = \lambda a$ (for any real number λ)

If $b_n = a_{n+k}$ then a_n summable $\Leftrightarrow$ b_n summable.

a_n is summable $\Rightarrow a_n \to 0$.

> Note that the converse is not necessarily true, take $a_n = \frac{1}{n}$ as an example.

Let S_n denote the sequence of partial sums of a_n ($S_n = \sum_{i=0}^{n} a_n$). A sequence of non-negative numbers a_n is summable iff S_n is bounded above. Similar statement for sequences of non-positive numbers.

Every absolutely summable sequence is summable.

Comparison test
If $b_n > a_n$ for all n then b_n summable $\Rightarrow a_n$ summable.

Alternating series test
If a_n is a decreasing sequence AND $a_n \geq 0$ for all n AND $a_n \to 0$ then $(-1)^{n+1}a_n$ is summable.

Ratio test for sequences
Let $r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$:

- $r < 1 \Rightarrow a_n$ is absolutely summable
- $r > 1 \Rightarrow a_n$ is not summable
- $r = 1$ is an indeterminate case

6.4. Power series

The power series associated with a sequence a_n converges iff the sequence of partial sums of $a_n x^n$ converges (i.e. if $\sum_{i=0}^{n} a_i x^i$ converges).

Let $P(x)$ be a power series. If $P(a)$ converges absolutely for some a, then $P(x)$ converges absolutely for all x such that $|x| < |a|$

Let R be the radius of convergence of $P(x)$. For all real numbers a:

- $|a| < R \Rightarrow P(a)$ converges absolutely
- $|a| > R \Rightarrow P(a)$ diverges

Ratio test for power series
Let $r = \frac{a_{n+1}}{a_n}$. Let $P(x) = \sum_{i=0}^{n} a_i x^i$ (i.e. the power series associated with a_n):

- $r \to 0 \Rightarrow R = \infty$
- $r \to L$ for some $L \Rightarrow R = \frac{1}{L}$
- $r \to \infty \Rightarrow R = 0$

Note: if $r = 1$ here then $R = 1$. This is DIFFERENT to the ratio test for sequences, where $r = 1$ is an indeterminate case.

6.5. Continuity

The limit of a function at any specific point is unique.

If functions f and g are continuous at a:

- $(f+g)$ is continuous at a
- fg is continuous at a
- $\frac{1}{f(x)} and \frac{1}{g(x)}$ are continuous at a
- $g \circ f$ is continuous at a

Any polynomial in $\mathbb{R}$ is continuous

Any rational function in $\mathbb{R}$ is continuous

Sequential continuity
A function f is continuous at a iff $f(a_n) \to f(a)$ for all sequences a_n such that $a_n \to a$.

Any continuous function on a closed bounded interval is bounded.

Intermediate Value Theorem
If f continuous and $f(a) \leq f(b)$ for some a, b, then there exists some $c \in [a, b]$ such that $f(a) \leq f(c) \leq f(b)$.

Fixed Point Theorem
If f continuous and $f : [a, b] \to [a, b]$, then there exists some $c \in [a, b]$ such that $f(c) = c$.

Polynomials of odd degree have at least 1 root.

f differentiable $\Rightarrow f$ continuous.

6.6. Differentiable functions

If functions f and g are differentiable at a:

- $(f+g)$ is differentiable at a
- fg is differentiable at a
- $\frac{1}{f(x)} and \frac{1}{g(x)}$ are differentiable at a
- $g \circ f$ is differentiable at a

- g^{-1} and f^{-1} are differentiable at a

Let f be continuous and differentiable. If f has a local extremum at a then $f'(a) = 0$ (except at endpoints of the interval).

Let f be continuous and differentiable. If f has a local extremum at c (say in the interval $[a, b]$), there are 3 possiblities:

- c is an endpoint of $[a, b]$
- $f'(c) = 0$
- c is a non-differentiable point

Mean Value Theorem
Let f be continuous on $[a, b]$ and differentiable on (a, b). There exists a point $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{65}$$

Rolle's Theorem
Let f be continuous and differentiable on (a, b). If $f(a) = f(b)$ then there exists some $c \in (a, b)$ such that $f'(c) = 0$. This is a special case of the Mean Value Theorem.