# Differential Equations Cheatsheet (JMC Year 1, 2017/2018)

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## Contents

# Part I
# Differential Equations

## 1 Definitions

**Order (of derivative)** An $n^{th}$ derivative has order $n$.

**Order (of ODE)** The order of the highest derivative present in an ODE.

**Degree (of ODE)** The highest power to which a term is raised in an ODE (excluding fractional powers).

**Linear** An ODE which has no terms raised to more than the $1^{st}$ power, and with no $y, x$ or other derivative terms multiplied by each other.

## 2 1st order linear ODEs

Every 1st order linear ODE can be expressed as:

$$\frac{dy}{dx} + p(x)y = q(x) \tag{1}$$

These can ALL be solved by the *integrating factor* method:

1. Multiply both sides by $exp(\int p(x)dx)$
2. Use the reverse product rule to express the LHS as a single derivative (of a function of y).
3. Integrate both sides and rearrange.

## 3 1st order non-linear ODEs

### 3.1 Separable ODEs

Separable equations can be written in the form:

$$\frac{dy}{dx} = f(x)g(y) \tag{2}$$

These can be rearranged and integrated on both sides, with respect to the different variables.

### 3.2 Homogenous ODEs

Homogenous equations can be written in the form:

$$\frac{dy}{dx} = f(\frac{y}{x}) \tag{3}$$

To solve, set $v = \frac{y}{x}$, so that $y = xv$. Note that v is still a single-variable function of x, since y is a function of x. Now we can differentiate both sides to get:

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \tag{4}$$

We now have simultaneous equations for $\frac{dy}{dx}$. Equate and solve for $\frac{dv}{dx}$, and then solve this 1st order linear ODE in $\frac{dv}{dx}$ to find v (and then y).

### 3.3 Bernoulli type ODEs

A Bernoulli type ODE is of the form:

$$\frac{dy}{dx} + p(x)y = q(x)y^n \tag{5}$$

To solve:

1. Multiply both sides by $(1-n)y^{-n}$
2. Let $z = y^{1-n}$ and substitute into equation, including rewriting one of the terms as $\frac{dz}{dx}$
3. The resulting equation is 1st order linear in z, so solve for z (and then y).

## 4 2nd order ODEs

### 4.1 Special case - y missing

If we can write the $2^{nd}$ derivative in the form:

$$\frac{d^2y}{dx^2} = f(x, \frac{dy}{dx}) \tag{6}$$

(i.e. no $y$ terms present), then we can make a substitution. Let $P = \frac{dy}{dx}$. This means $\frac{d^2y}{dx^2} = \frac{dP}{dx}$, therefore we have:

$$\frac{dP}{dx} = f(x, P) \tag{7}$$

This is 1st order w.r.t P and can be solved by appropriate 1st order methods.

### 4.2 Special case - x missing

If we can write the $2^{nd}$ derivative as:

$$\frac{d^2y}{dx^2} = f(y, \frac{dy}{dx}) \tag{8}$$

(i.e. x is missing), then we can make the same substitution. Let $P = \frac{dy}{dx}$. This means $\frac{d^2y}{dx^2} = \frac{dP}{dx}$, therefore we have:

$$\frac{dP}{dx} = f(y, P) \tag{9}$$

DIFFERENT TO LAST TIME: if we want a 1st order equation, we must rewrite $\frac{dP}{dx}$ as a derivative with respect to y, since the RHS contains y terms. Luckily, we can see that:

$$\frac{dP}{dx} = \frac{dP}{dy}\frac{dy}{dx} = P\frac{dP}{dy} \tag{10}$$

Therefore:

$$P\frac{dP}{dy} = f(y, P) \tag{11}$$

This is 1st order w.r.t P and can be solved by appropriate 1st order methods.

### 4.3 General case - finding the CF

The general solution (GS) of a 2nd order ODE can be expressed as the sum of two other functions, called the 'complementary function' (CF) and a 'particular integral' (PI).

$$y_{GS} = y_{CF} + y_{PI} \tag{12}$$

A 2nd order ODE will usually be presented to us in the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c = f(x) \tag{13}$$

It can be shown that the CF can be calculated from the LHS of the above equation. We write down the *auxiliary equation*, which is simply the equation:

$$a\lambda^2 + b\lambda + c = 0 \tag{14}$$

using a, b, c from above. Solving this gives us two values, $\lambda_1$ and $\lambda_2$.

#### 4.3.1 $\lambda_1 \neq \lambda_2$

We can express the CF as:

$$y_{CF} = A_1e^{\lambda_1 x} + A_2e^{\lambda_2 x} \tag{15}$$

where $A_1$ and $A_2$ are arbitrary constants.

#### 4.3.2 $\lambda_1 = \lambda_2$

Same as above, but we stick an $x$ in front of one of the clashing parts of the solution.

$$y_{CF} = A_1e^{\lambda_1 x} + A_2xe^{\lambda_2 x} \tag{16}$$

#### 4.3.3 $\lambda_1$ and $\lambda_2$ are complex-valued

If the auxiliary equation has complex roots, $\lambda_1$ and $\lambda_2$ will be complex conjugates. It is also important to note $A_1$ and $A_2$ are now complex-valued. The CF can be expressed as:

$$\begin{aligned} y_{CF} &= A_1e^{(a+bi)x} + A_2e^{(a-bi)x} \\ &= e^a(A_1e^{i(bx)} + A_2e^{-i(bx)}) \\ &= e^a(C_1cos(bx) + C_2sin(bx)) \end{aligned} \tag{17}$$

where $C_1 = A_1 + A_2$ and $C_2 = (A_1 - A_2)i$. Note that $A_1$ and $A_2$ would have been complex, however $C_1$ and $C_2$ are necessarily real.

## 4.4 General case - finding the PI

The particular integral is *any function $y_{PI}$ that satisfies the ENTIRE differential equation*. The particular integral can be calculated depending on the form of the RHS of equation 13. We will refer to the RHS as simply $f(x)$ and the particular integral (as before) as $y_{PI}$.

### 4.4.1 $f(x)$ is a polynomial

Try setting $y_{PI}$ as a general polynomial of the same degree. e.g. if $f(x)$ is a quadratic, try setting $y_{PI} = ax^2 + bx + c$ and substituting into the ODE. We will solve for a, b, c, and this will give us $y_{PI}$.

### 4.4.2 $f(x)$ is a multiple of $e^{bx}$, $e^{bx}$ NOT in CF

Choose $y_{PI} = Ae^{bx}$ for some real number A.

### 4.4.3 $f(x)$ is a multiple of $e^{bx}$, $e^{bx}$ IS in CF

We now have a clash between the PI and the CF. Choose $y_{PI} = A(x)e^{bx}$ for some real FUNCTION A. At the end remove any clashing terms (i.e. terms with a component of $e^{(something)x}$ which is already present in the CF.

### 4.4.4 $f(x) = A(x)e^{bx}$ where $A(x)$ is a polynomial

Choose $y_{PI} = C(x)e^{bx}$ for some polynomial $C(x)$.

### 4.4.5 $f(x)$ is trigonometric (e.g. sin, cos, sinh etc.)

Look for a pattern in $f(x)$. A good tip for an $f(x)$ with only sines/cosines is to use $y_{PI} = A\cos(x) + B\sin(x)$ and solve for A and B. A similar story for sinh and cosh. CAUTION: sinh, cosh and tanh are actually exponential functions in disguise, so make sure they do not clash with any $e^{\lambda x}$ terms in the CF.

### 4.4.6 Special cases

If $f(x)$ has a term of the form $e^x\cos(x)$ or $e^x\sin(x)$ then we can rewrite it as the real/imaginary part of a complex function (in this case $e^{(1+i)x}$ would be appropriate, since it expands to $e^x(\cos(x) + i\sin(x))$.

# Part II
# Systems of differential equations

## 5 Definitions

**System of diff. equations** A set of simultaneous equations of derivatives, where derivatives of $y, x$ etc. are given w.r.t. a parameter $t$

**Order (of system)** The order of the highest derivative present in the system.

**Degree (of system)** The highest power to which a term is raised in an ODE (excluding fractional powers).

**Linear** A system which has no terms raised to more than the $1^{st}$ power, and with no $y$ or other derivative terms multiplied by each other.

**Homogenous** A system with no explicit functions of $t$ (i.e. $f(t)$) present.

**Equilibrium point** A point at which all the derivatives in the sytem equate to 0.

## 6 Solving systems of diff. equations

A 1st order system of equations can be written as:

$$\begin{aligned} \frac{dx}{dt} &= F(x,y) \\ \frac{dy}{dt} &= G(x,y) \end{aligned} \tag{18}$$

To solve these, we want to *decouple* the equations. This means $\frac{dx}{dt}$ should be expressed as a function of $x$ alone, and similarly for $\frac{dy}{dt}$

This is best explained, with an example, so let us choose a coupled system:

$$\begin{aligned} \frac{dx}{dt} &= -4x - 3y \\ \frac{dy}{dt} &= 2x + 3y \end{aligned} \tag{19}$$

We can rewrite this in matrix form:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 & -3 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{20}$$