

# M1J2 Summary Notes (JMC Year 1, 2017/2018 syllabus)

Fawaz Shah

(STILL UNDER CONSTRUCTION)

Dr Lawn refers to propositions, theorems, corollaries and lemmas. In this document I will refer to them all as 'theorems'.

This document contains a list of definitions and a list of theorems.

## Contents

<b>I</b>	<b>Abstract Linear Algebra</b>	<b>3</b>
<b>1</b>	<b>Definitions</b>	<b>3</b>
<b>2</b>	<b>Theorems</b>	<b>3</b>
2.1	Vector spaces . . . . .	3
2.2	Subspaces . . . . .	3
2.3	Spanning sets and linear independence . . . . .	3
2.4	Bases and dimension . . . . .	3
2.5	Linear maps . . . . .	3
<b>II</b>	<b>Group Theory</b>	<b>4</b>
<b>3</b>	<b>Definitions</b>	<b>4</b>
<b>4</b>	<b>Theorems</b>	<b>7</b>
4.1	Groups . . . . .	7
4.2	Modular arithmetic and $\mathbb{Z}_n$ . . . . .	7

4.3	Cyclic groups . . . . .	8
4.4	Symmetric groups . . . . .	8
4.5	Subgroups . . . . .	9
4.6	Cosets and Lagrange's Theorem . . . . .	9
<b>III</b>	<b>Analysis</b>	<b>10</b>
<b>5</b>	<b>Definitions</b>	<b>10</b>
<b>6</b>	<b>Theorems</b>	<b>15</b>
6.1	Sequences . . . . .	15
6.2	Subsequences . . . . .	16
6.3	Summability . . . . .	17
6.4	Power series . . . . .	18
6.5	Continuity . . . . .	19
6.6	Differentiable functions . . . . .	19

## Part I

# Abstract Linear Algebra

## 1 Definitions

## 2 Theorems

### 2.1 Vector spaces

### 2.2 Subspaces

### 2.3 Spanning sets and linear independence

### 2.4 Bases and dimension

### 2.5 Linear maps

## Part II

# Group Theory

### 3 Definitions

**Binary operation** A binary operation on a set  $G$  is a any function  $f : G \times G \rightarrow G$

**Associative** A binary operation  $\star$  on a set  $G$  is associative if it satisfies:

$$(a \star b) \star c = a \star (b \star c) \quad (1)$$

for all  $a, b, c \in G$ .

**Commutative** A binary operation  $\star$  on a set  $G$  is commutative if it satisfies:

$$a \star b = b \star a \quad (2)$$

for all  $a, b \in G$ .

**Left/right identity** An element  $e \in G$  is called the left identity if:

$$e \star g = g \quad (3)$$

for all  $g \in G$ . Similar statement for right identity.

**(Two sided) Identity element** An element  $e \in G$  is a two-sided identity element if it is both a left identity and a right identity.

From now on the two-sided identity element will be referred to as  $e$ .

**Left/right inverse** An element  $h \in G$  is called the left inverse of  $g \in G$  if:

$$h \star g = e \quad (4)$$

Similar statement for right inverse.

**Two sided inverse** A two sided inverse of an element  $g \in G$  is both a left inverse and a right inverse of  $g$ .

From now on the two-sided inverse of  $g$  will be referred to as  $g^{-1}$ .

**Group** A group  $(G, \star)$  is a set  $G$  equipped with a binary operation  $\star$  such that:

- $\star$  is associative
- $\star$  has an identity element  $e \in G$
- Every  $g \in G$  has an inverse  $g^{-1} \in G$

The above three suffice for the exam, however there is technically a fourth requirement:

- $G$  is closed under  $\star$ , i.e. for all  $g, h \in G, g \star h \in G$

**Order (group)** The order of a group  $(G, \star)$  is the size of  $G$ .

**Abelian group** An Abelian group is a group with a commutative binary operation  $\star$ .

**Powers of  $g$**  We can define the powers of any  $g \in G$  to be:

$$g^n = \begin{cases} g \star g \star \dots g & n > 0 \\ g^{-1} \star g^{-1} \star \dots g^{-1} & n < 0 \\ e & n = 0 \end{cases} \quad (5)$$

where in the first cases there are  $n$  copies of  $g$ , and in the second case there are  $-n$  copies of  $g^{-1}$ .

**Definition of  $[a]_n$  and  $\mathbb{Z}_n$**  For any  $a \in \mathbb{Z}$ :

$$[a]_n = \{b \in \mathbb{Z} \mid b \equiv a \pmod{n}\} \quad (6)$$

Note that  $[a]_n$  forms an equivalence class, and there are exactly  $n$  of these equivalence classes.  $\mathbb{Z}_n$  is the set of all these equivalence classes.

$$\mathbb{Z}_n = \{[a]_n \mid a \in \mathbb{Z}\} \quad (7)$$

**Definition of  $\mathbb{Z}_n^*$**   $\mathbb{Z}_n^*$  is the set of all invertible  $[a]_n$ . Note in this case the identity element is  $[1]_n$ .

$$\mathbb{Z}_n^* = \{[a]_n \mid \exists [b]_n \in \mathbb{Z}_n \text{ s.t. } [a]_n [b]_n = [1]_n\} \quad (8)$$

Note that  $[a]_n [b]_n = 1 \equiv \gcd(a, n) = 1$ .

**Order (element)** The order of any  $g \in G$  is the smallest positive integer such that:

$$g^n = e \quad (9)$$

**Cyclic group + generator** A group  $(G, \star)$  is cyclic if:

$$G = \{g^n \mid n \in \mathbb{Z}\} \quad (10)$$

$g$  is called the generator of the group.

**Permutation** A permutation  $\sigma$  on  $n$  symbols is a bijection:

$$\sigma : \{1 \dots n\} \rightarrow \{1 \dots n\} \quad (11)$$

**Symmetric group** The symmetric group  $S_n$  on  $n$  symbols is the set of all permutations of  $n$  symbols.

$$S_n = \{\sigma : \{1 \dots n\} \rightarrow \{1 \dots n\}\} \quad (12)$$

Note that  $S_n$  is a set of functions. Therefore the identity element is the identity function.

**$k$ -cycle** A permutation  $\sigma \in S_n$  is a  $k$ -cycle if there exists some  $a_1 \dots a_k \in \{1 \dots n\}$  such that:

$$\sigma(a_1) = a_2, \quad \sigma(a_2) = a_3 \quad \dots \quad \sigma(a_k) = a_1 \quad (13)$$

and  $\sigma(i) = i$  for all  $i \notin \{1 \dots n\}$ .  $k$  is called the length of the cycle. The notation for a cycle is  $(a_1 \dots a_k)$ .

**Disjoint cycles** Two cycles  $(a_1 \dots a_m)$  and  $(b_1 \dots b_n)$  are disjoint if no  $a_i$  is equal to any  $b_j$ .

**Subgroup** Let  $(G, \star)$  be a group, and  $H \subseteq G$ .  $(H, \star)$  is a subgroup of  $G$  if:

- $e \in H$
- For any  $g, h \in H$ ,  $g \star h \in H$
- For any  $g \in H$ ,  $g^{-1} \in H$

**Cyclic subgroup** Let  $(G, \star)$  be a group. For any  $g \in G$ , the cyclic subgroup  $\langle g \rangle$  generated by  $g$  is defined as:

$$\langle g \rangle = (\{g^i \mid i \in \mathbb{Z}\}, \star) \quad (14)$$

Note that order of  $g$  = size of cyclic subgroup  $\langle g \rangle$ .

**Left/right cosets** Let  $(G, \star)$  be a group and  $(H, \star)$  a subgroup. For any  $g \in G$ , the left coset of  $H$  by  $g$  (denoted by  $gH$ ) is defined as:

$$gH = \{g \star h \mid h \in H\} \quad (15)$$

Similar definition for right coset of  $H$  by  $g$  (denoted by  $Hg$ ).

The set of all left cosets of  $H$  by  $g$  is denoted by  $G : H$ .

The set of all right cosets of  $H$  by  $g$  is denoted by  $H : G$ .

## 4 Theorems

### 4.1 Groups

(For the rest of this section, assume  $(G, \star)$  is a group)

Any identity element  $e$  is unique for that group.

Any two-sided inverse  $g^{-1}$  of an element  $g \in G$  is unique.

For any  $g, h \in G$

$$(g \star h)^{-1} = h^{-1} \star g^{-1} \quad (16)$$

The normal exponent rules apply within groups, e.g.

$$g^n \star g^m = g^{n+m} \quad (17)$$

$$(g^n)^{-1} = g^{-n} \quad (18)$$

$$(g^n)^m = g^{nm} \quad (19)$$

Some examples of groups:  $(\mathbb{R}, +)$ ,  $(\mathbb{Z}, +)$ ,  $(\mathbb{Z}^*, \times)$

### 4.2 Modular arithmetic and $\mathbb{Z}_n$

$(\mathbb{Z}_n, +)$  is an Abelian group.

$(\mathbb{Z}_n^*, \cdot)$  is an Abelian group.

### 4.3 Cyclic groups

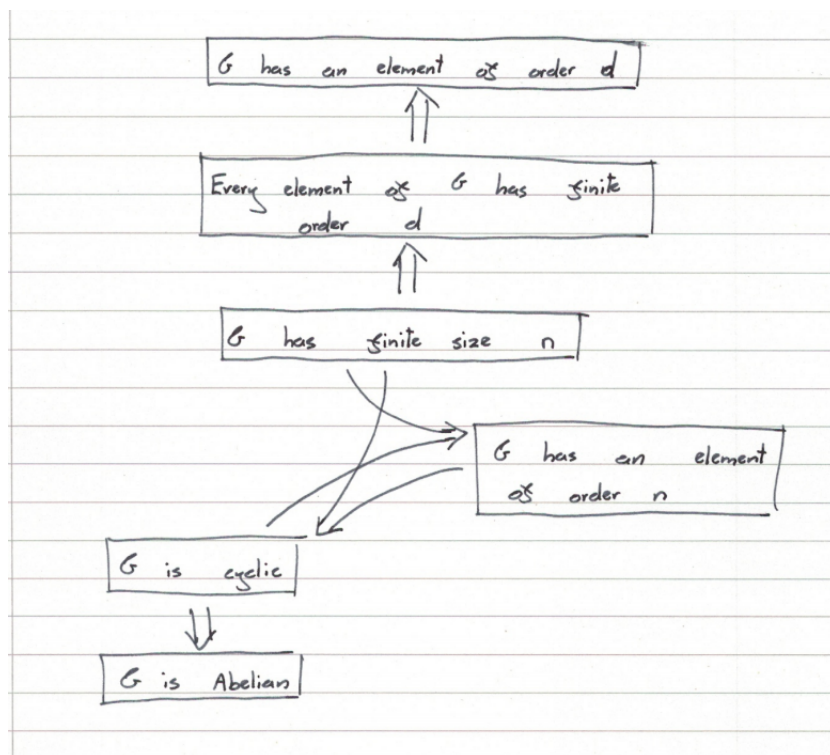
If  $(G, \star)$  is a finite group then every  $g \in G$  has finite order.

Any  $g \in G$  with order  $n$  has distinct powers  $g^0, g^1, g^2 \dots g^{n-1}$ .

All cyclic groups are Abelian.

Assume  $G$  is finite with size  $n$ .

$G$  is cyclic  $\Leftrightarrow G$  contains an element of order  $n$ .



### 4.4 Symmetric groups

$(S_n, \circ)$  is a group.

The size of any  $S_n$  is  $n!$

The order of a  $k$ -cycle is  $k$ .



For any  $\sigma \in S_n$ :

- for any  $i \in \{0 \dots n\}$  there exists a  $d > 0$  such that  $\sigma^d(i) = i$  (i.e.  $\sigma^d \equiv Id = e$ )
- if  $d$  is the smallest integer such that  $\sigma^d(i) = i$  then the numbers  $i, \sigma^1(i), \sigma^2(i) \dots \sigma^{d-1}(i)$  are distinct
- If  $j$  is not in the set  $\{i, \sigma(i), \sigma^2(i) \dots \sigma^{d-1}(i)\}$  then neither is  $\sigma(j)$

Any permutation  $\sigma$  can be expressed as the product of disjoint  $k$ -cycles.

## 4.5 Subgroups

Any group  $(G, \star)$  has two trivial subgroups,  $(e, \star)$  and itself.

*Subgroup test*

Any  $H \subseteq G$  is a subgroup if:

- $H \neq \emptyset$
- for all  $x, y \in H, x \star y^{-1} \in H$

## 4.6 Cosets and Lagrange's Theorem

For any  $g_1, g_2 \in G$  and subgroup  $H$ :

$$g_1 H = g_2 H \Leftrightarrow g_1 \in g_2 H \quad (20)$$

The left cosets of  $H$  form a partition of  $G$ . This means any  $g \in G$  is in exactly one left coset of  $H$ . The right cosets also form a (different) partition.

For any  $g \in G$ :

$$\#gH = \#hG = \#H \quad (21)$$

*Lagrange's Theorem*

For any subgroup  $(H, \star)$  where  $H \subseteq G$ :

$$\#G = \#H \cdot \#(G : H) \quad (22)$$

For any  $g \in G$ , the order of  $g$  divides  $\#G$ .

If  $\#G = p$ , where  $p$  is prime, then  $G$  is cyclic.

## Part III

# Analysis

### 5 Definitions

**Sequence** A sequence is simply a map  $f : \mathbb{N} \rightarrow \mathbb{R}$ , denoted by  $a_n$

**Convergence (as  $n \rightarrow \infty$ )** A sequence  $a_n$  converges to a limit  $L$  if for all real numbers  $\epsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that for all  $n > N$  we have  $|a_n - L| < \epsilon$ .

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad s.t. \quad \forall n > N \quad |a_n - L| < \epsilon \quad (23)$$

**Tends to infinity (sequence)** We say a sequence tends to infinity if for all  $R \in \mathbb{R}$ , the sequence  $a_n$  is eventually bigger than  $R$ .

$$\forall R \in \mathbb{R} \quad \exists N \in \mathbb{N} \quad s.t. \quad \forall n > N \quad a_n > R \quad (24)$$

**Shift** The shift of a sequence by say,  $k$ , is the sequence  $b_n = a_{n+k}$

**Triangle inequality** The general triangle inequality is:

$$|x - y| < |x - z| + |z - y| \quad (25)$$

Setting  $z = 0$  gives us:

$$|x - y| > |x| - |y| \quad (26)$$

Then setting  $y = -y$  gives us the familiar case:

$$|x + y| < |x| + |y| \quad (27)$$

**Bounded above** A sequence  $a_n$  is bounded above if there's a real number  $A$  such that  $a_n < A$  for all  $n$ .

**Bounded below** A sequence  $a_n$  is bounded below if there's a real number  $A$  such that  $a_n > A$  for all  $n$ .

**Bounded** A sequence  $a_n$  is bounded if there's a real number  $A$  such that  $|a_n| < A$  for all  $n$ .

**Increasing** A sequence is increasing if  $a_{n+1} \geq a_n$  for all  $n$ .

**Strictly increasing** A sequence is strictly increasing if  $a_{n+1} > a_n$  for all  $n$ .

**Decreasing** A sequence is decreasing if  $a_{n+1} \leq a_n$  for all  $n$ .

**Strictly decreasing** A sequence is strictly decreasing if  $a_{n+1} < a_n$  for all  $n$ .

**Monotonic** A sequence is monotonic if it is increasing or decreasing.

**Supremum** The supremum  $A$  of a set  $S$  is the least upper bound of that set i.e. the smallest number such that  $s \leq A$  for all  $s \in S$ .

**Supremum (function)** The supremum of a function  $f$  is the sup of  $\{f(x) \mid x \in \text{dom}(f)\}$ .

**Infimum** The infimum  $B$  of a set  $S$  is the greatest lower bound of that set i.e. the largest number such that  $s \geq B$  for all  $s \in S$ .

**Infimum (function)** The infimum of a function  $f$  is the inf of  $\{f(x) \mid x \in \text{dom}(f)\}$ .

**Subsequence** A subsequence of  $a_n$  is a sequence  $a_{f(n)}$ , where  $f(n)$  is a strictly increasing function.

**Cauchy sequence** A sequence is Cauchy if all the terms get arbitrarily close to one another. To put it mathematically:

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad \text{s.t.} \quad \forall m, n \geq N \quad |a_n - a_m| < \epsilon \quad (28)$$

**Partial sum** The  $n^{\text{th}}$  partial sum  $S_n$  of a sequence  $a_n$  is the sum of terms up to that point:

$$S_n = \sum_{i=1}^n a_i \quad (29)$$

**Summable** A sequence is summable if the sequence of its partial sums converges. The limit of the sequence of partial sums will be:

$$L = \sum_{i=1}^{\infty} a_n \quad (30)$$

**Absolutely summable** A sequence  $a_n$  is absolutely summable if  $|a_n|$  is summable.

**Conditionally summable** A sequence is conditionally summable if it is summable but not absolutely summable.

**Power series** The power series associated with a sequence  $a_n$  is the sequence of partial sums:

$$\sum_{i=1}^n a_i x^i \quad (31)$$

**Radius of convergence** The radius of convergence  $R$  of a power series  $P(x)$  is defined as the largest  $x$  for which  $P(x)$  is convergent.

$$R = \sup\{x \in \mathbb{R} \mid P(x) \text{ convergent}\} \quad (32)$$

**Limit as  $x \rightarrow \infty$  (function)** A function  $f(x)$  tends to a limit  $L$  as  $x \rightarrow \infty$  if for all real numbers  $\epsilon > 0$ , there exists an  $R \in \mathbb{R}$  such that for all  $x \geq R$  we have  $|f(x) - L| < \epsilon$ .

$$\forall \epsilon > 0 \quad \exists R \in \mathbb{R} \quad s.t. \quad \forall x > R \quad |f(x) - L| < \epsilon \quad (33)$$

**Tends to infinity (function)** A function  $f(x)$  tends to infinity as  $x \rightarrow \infty$  if for any  $M \in \mathbb{R}$  there exists an  $R \in \mathbb{R}$  such that if  $x > M$  then  $f(x) > R$ .

$$\forall M \in \mathbb{R} \quad \exists R \in \mathbb{R} \quad s.t. \quad x > M \Rightarrow f(x) > R \quad (34)$$

**One-sided limit** A function  $f(x)$  tends to a limit  $L$  as  $x \rightarrow a^-$  if for any  $\epsilon > 0$  there exists a  $\delta > 0$  such that if  $x \in (a - \delta, a)$  then  $|f(x) - L| < \epsilon$ .

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \text{s.t.} \quad x \in (a - \delta, a) \Rightarrow |f(x) - L| < \epsilon \quad (35)$$

Same format for the other sided limit ( $x \rightarrow a^+$ )  
(Note that  $\epsilon - \delta$  definition is only used for limits as  $x$  tends to a finite number  $a$ , not infinity)

**Limit as  $x \rightarrow a$**  A function  $f(x)$  tends to a limit  $L$  as  $x \rightarrow a$  if we have both:

$$\lim_{x \rightarrow a^-} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow a^+} f(x) = L \quad (36)$$

**Limit as  $x \rightarrow a$  ( $\epsilon - \delta$  def.)** A function  $f(x)$  tends to a limit  $L$  as  $x \rightarrow a$  if:

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \text{s.t.} \quad |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon \quad (37)$$

**Continuous** A function  $f(x)$  is continuous at  $a$  if:

$$\lim_{x \rightarrow a} f(x) = f(a) \quad (38)$$

**Continuous ( $\epsilon - \delta$  def.)** A function  $f(x)$  is continuous at  $a$  if for all  $\epsilon > 0$  there is a  $\delta > 0$  such that if  $|x - a| < \delta$  then  $|f(x) - f(a)| < \epsilon$ .

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \text{s.t.} \quad |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon \quad (39)$$

**Continuous everywhere** A function  $f(x)$  is continuous everywhere if it is continuous at  $a$  for all  $a \in \text{dom}(f)$ .

**Open interval** An open interval  $I$  is a set  $I \subseteq \mathbb{R}$  of the form:

- $I = (a, b)$  for some  $a, b \in \mathbb{R}$ , or
- $I = (-\infty, b)$ , or
- $I = (a, +\infty)$ , or
- $I = \mathbb{R}$

**Discontinuity** Discontinuity is the negation of continuity. Hence a function  $f(x)$  is discontinuous at  $a$  if there exists  $\epsilon > 0$  such that for all  $\delta > 0$ ,  $|x - a| < \delta$  AND  $|f(x) - f(a)| > \epsilon$ .

$$\exists \epsilon > 0 \quad s.t. \quad \forall \delta > 0 \quad |x - a| < \delta \text{ AND } |f(x) - f(a)| > \epsilon \quad (40)$$

**Bounded (function)** A function  $f(x)$  is bounded if the set of all possible values of  $f(x)$  is bounded.

**Differentiable (ver. 1)** A function  $f(x)$  is differentiable at  $a$  if:

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \quad (41)$$

exists.

**Differentiable (ver. 2)** A function  $f(x)$  is differentiable at  $a$  if:

$$\lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} \quad (42)$$

exists.

**Differentiable everywhere** A function  $f(x)$  is differentiable everywhere if it is differentiable at  $a$  for all  $a \in \text{dom}(f)$ .

**Global maximum** A function  $f(x)$  has a global maximum at  $a$  if  $f(a) \geq f(x)$  for all other values of  $f(x)$ .

Similar definition for global minimum.

**Local maximum** A function  $f(x)$  has a local maximum at  $a$  if  $f(a) \geq f(x)$  for all  $x$  in the set  $(a - \epsilon, a + \epsilon)$ , for some  $\epsilon$ .

Similar definition for local minimum.

**Lipschitz continuous** A function is Lipschitz continuous if:

$$|f'(x)| \leq L \Rightarrow |f(x_1) - f(x_2)| \leq L|x_1 - x_2| \quad (43)$$

## 6 Theorems

### 6.1 Sequences

Every convergent sequence has a unique limit.

Every convergent sequence is bounded.

If all terms of a convergent sequence are larger than a number  $B$ , then so is its limit.

Some properties of limits:

$$\lim_{x \rightarrow \infty} (a_n + b_n) = \lim_{x \rightarrow \infty} a_n + \lim_{x \rightarrow \infty} b_n \quad (44)$$

$$\lim_{x \rightarrow \infty} (\lambda a_n) = \lambda \lim_{x \rightarrow \infty} a_n \quad (45)$$

$$\lim_{x \rightarrow \infty} (a_n b_n) = \lim_{x \rightarrow \infty} a_n \lim_{x \rightarrow \infty} b_n \quad (46)$$

$$\lim_{x \rightarrow \infty} \left( \frac{a_n}{b_n} \right) = \frac{\lim_{x \rightarrow \infty} a_n}{\lim_{x \rightarrow \infty} b_n} \quad (47)$$

where  $\lambda$  is any real number.

If  $a_n \rightarrow \infty$  and  $b_n$  is bounded below,  $a_n + b_n \rightarrow \infty$ .

If  $a_n \rightarrow \infty$  and  $b_n$  is bounded below by a positive number,  $a_n b_n \rightarrow \infty$ .

If  $a_n$  is bounded and  $b_n \rightarrow \infty$ , then  $\frac{a_n}{b_n} \rightarrow 0$ .

If  $a_n \rightarrow \infty$ , for any real number  $\lambda$ :

- $\lambda < 0 \Rightarrow \lambda a_n \rightarrow -\infty$
- $\lambda = 0 \Rightarrow \lambda a_n \rightarrow 0$
- $\lambda > 0 \Rightarrow \lambda a_n \rightarrow \infty$

If  $a_n \rightarrow a$  and  $b_n \rightarrow b$ , and for all  $n$   $a_n < b_n$ , then  $a < b$ .

*Sandwich Theorem*

If  $a_n \leq b_n \leq c_n$  for all  $n$ , and  $a_n$  and  $c_n$  tend to the same limit  $L$ , then  $b_n \rightarrow L$ .

Every bounded monotonic sequence is convergent.

*Completeness Axiom*

Every non-empty subset of the real numbers which is bounded above has a supremum. Similar statement for infimum.

Useful results for sequences:

$$\lim_{n \rightarrow \infty} \lambda^n = \begin{cases} \infty & \lambda > 1 \\ 1 & \lambda = 1 \\ 0 & -1 < \lambda < 1 \end{cases} \quad (48)$$

$\lambda^n$  diverges if  $\lambda = -1$ .

If  $m > 0$  and  $\lambda > 1$  then  $\frac{\lambda^n}{n^m} \rightarrow \infty$  (exponentials beat powers).

If  $m > 0$  then  $\frac{\log(n)}{n^m} \rightarrow 0$  (powers beat logs).

## 6.2 Subsequences

If  $a_n \rightarrow L$  then any subsequence  $a_{f(n)} \rightarrow L$ .

If two subsequences of  $a_n$  converge to different limits,  $a_n$  doesn't converge to a limit.

Every sequence has a monotonic subsequence.

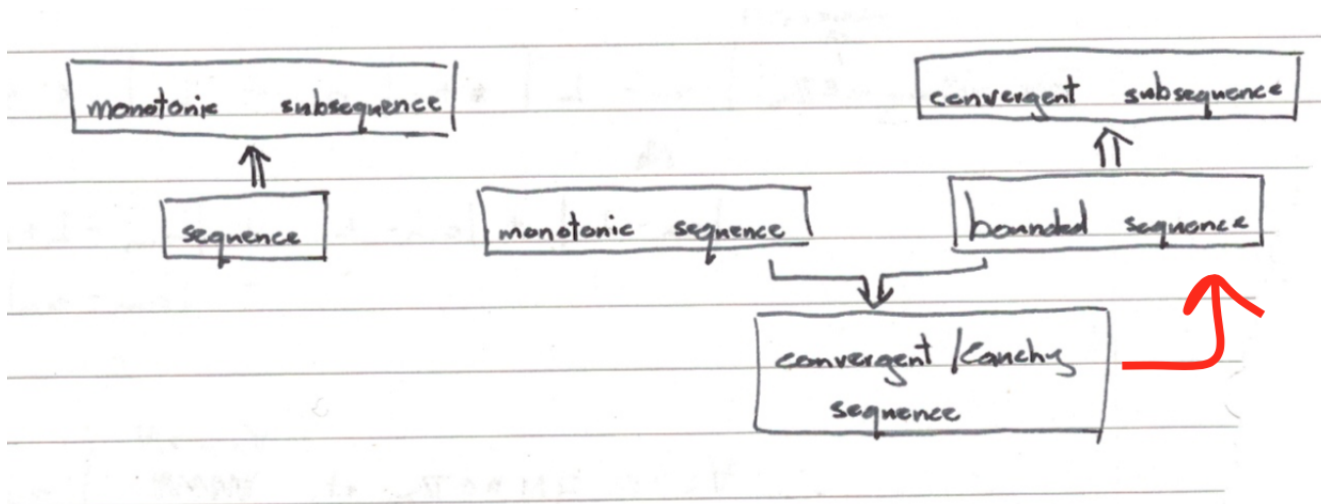
*Bolzano-Weierstrass Theorem*

Every bounded sequence has a convergent subsequence.

Every Cauchy sequence is bounded.

Cauchy sequence  $\Leftrightarrow$  convergent sequence (for real numbers).





### 6.3 Summability

A sequence is summable iff the sequence of its partial sums converges.

If two subsequences of a sequence  $a_n$  converge to two different limits,  $a_n$  is not summable.

If  $a_n$  and  $b_n$  are summable with  $\sum_{i=0}^{\infty} a_i = a$  and  $\sum_{i=0}^{\infty} b_i = b$ :

- $a_n + b_n$  is summable with  $\sum_{i=0}^{\infty} (a_i + b_i) = a + b$ .
- $\lambda a_n$  is summable with  $\sum_{i=0}^{\infty} \lambda a_i = \lambda a$  (for any real number  $\lambda$ )

If  $b_n = a_{n+k}$  then  $a_n$  summable  $\Leftrightarrow b_n$  summable.

$a_n$  is summable  $\Rightarrow a_n \rightarrow 0$ .

Let  $S_n$  denote the sequence of partial sums of  $a_n$  ( $S_n = \sum_{i=0}^n a_i$ ). A sequence of non-negative numbers  $a_n$  is summable iff  $S_n$  is bounded above. Similar statement for sequences of non-positive numbers.

Every absolutely summable sequence is summable.

*Comparison test*

If  $b_n > a_n$  for all  $n$  then  $b_n$  summable  $\Rightarrow a_n$  summable.

*Alternating series test*

If  $a_n$  is a decreasing sequence AND  $a_n \geq 0$  for all  $n$  AND  $a_n \rightarrow 0$  then  $(-1)^{n+1}a_n$  is a convergent sequence.

*Ratio test for sequences*

Let  $r = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$ :

- $r < 1 \Rightarrow a_n$  is absolutely summable
- $r > 1 \Rightarrow a_n$  is not summable
- $r = 1$  is an indeterminate case

## 6.4 Power series

The power series associated with a sequence  $a_n$  converges iff the sequence of partial sums of  $a_n x^n$  converges (i.e. if  $\sum_{i=0}^n a_i x^i$  converges).

Let  $P(x)$  be a power series. If  $P(a)$  converges absolutely for some  $a$ , then  $P(x)$  converges absolutely for all  $x$  such that  $|x| < |a|$

Let  $R$  be the radius of convergence of  $P(x)$ . For all real numbers  $a$ :

- $|a| < R \Rightarrow P(a)$  converges absolutely
- $|a| > R \Rightarrow P(a)$  diverges

*Ratio test for power series*

Let  $r = \frac{a_{n+1}}{a_n}$ . Let  $P(x) = \sum_{i=0}^n a_i x^i$  (i.e. the power series associated with  $a_n$ ):

- $r \rightarrow 0 \Rightarrow R = \infty$
- $r \rightarrow L$  for some  $L \Rightarrow R = \frac{1}{L}$
- $r \rightarrow \infty \Rightarrow R = 0$

Note: if  $r = 1$  here then  $R = 1$ . This is DIFFERENT to the ratio test for sequences, where  $r = 1$  is an indeterminate case.

## 6.5 Continuity

The limit of a function at any specific point is unique.

If functions  $f$  and  $g$  are continuous at  $a$ :

- $(f + g)$  is continuous at  $a$
- $fg$  is continuous at  $a$
- $\frac{1}{f(x)}$  and  $\frac{1}{g(x)}$  are continuous at  $a$
- $g \circ f$  is continuous at  $a$

Any polynomial in  $\mathbb{R}$  is continuous

Any rational function in  $\mathbb{R}$  is continuous

*Sequential continuity*

A function  $f$  is continuous at  $a$  iff  $f(a_n) \rightarrow f(a)$  for all sequences  $a_n$  such that  $a_n \rightarrow a$ .

Any continuous function on a closed bounded interval is bounded.

*Intermediate Value Theorem*

If  $f$  continuous and  $f(a) \leq f(b)$  for some  $a, b$ , then there exists some  $c \in [a, b]$  such that  $f(a) \leq f(c) \leq f(b)$ .

*Fixed Point Theorem*

If  $f$  continuous and  $f : [a, b] \rightarrow [a, b]$ , then there exists some  $c \in [a, b]$  such that  $f(c) = c$ .

Polynomials of odd degree have at least 1 root.

$f$  differentiable  $\Rightarrow f$  continuous.

## 6.6 Differentiable functions

If functions  $f$  and  $g$  are differentiable at  $a$ :

- $(f + g)$  is differentiable at  $a$
- $fg$  is differentiable at  $a$

- $\frac{1}{f(x)}$  and  $\frac{1}{g(x)}$  are differentiable at  $a$
- $g \circ f$  is differentiable at  $a$
- $g^{-1}$  and  $f^{-1}$  are differentiable at  $a$

Let  $f$  be continuous and differentiable. If  $f$  has a local extremum at  $a$  then  $f'(a) = 0$  (except at endpoints of the interval).

Let  $f$  be continuous and differentiable. If  $f$  has a local extremum at  $c$  (say in the interval  $[a, b]$ ), there are 3 possibilities:

- $c$  is an endpoint of  $[a, b]$
- $f'(c) = 0$
- $c$  is a non-differentiable point

#### *Mean Value Theorem*

Let  $f$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . There exists a point  $c \in (a, b)$  such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad (49)$$

#### *Rolle's Theorem*

Let  $f$  be continuous and differentiable on  $(a, b)$ . If  $f(a) = f(b)$  then there exists some  $c \in (a, b)$  such that  $f'(c) = 0$ . This is a special case of the Mean Value Theorem.