1 THE RESTRICTED PARABOLIC THREE-BODY PROBLEM:

In this 3 body problem we are considering a binary system orbiting a massive black hole. We will consider a total binary mass of m, and a separation of a. Then formulate this system in the restricted 3 body problem, ie., th emotion of a single massles particle under the influence of time dependant forces. This mean assuming that the black hole is stationary and just provides an underlying potential.

The equations of motion for the case of interest in which the mass of the black hole is much larger than the mass of the binary ($M/m \gg 1$) are given by:

$$\ddot{r_1} = -\frac{GM}{r_1^3}\mathbf{r_1} + \frac{Gm_2}{|\mathbf{r_1} - \mathbf{r}_2|^3}(\mathbf{r_2} - \mathbf{r_1}) \tag{1}$$

$$\ddot{r_2} = -\frac{GM}{r_2^3}\mathbf{r_2} + \frac{Gm_1}{|\mathbf{r_1} - \mathbf{r}_2|^3}(\mathbf{r_2} - \mathbf{r_1}) \tag{2}$$

where $\mathbf{r_1}$ and $\mathbf{r_2}$ are the two binary component distances from the black hole.

2 Deriving equations of motion:

To derive the equations of motion for the binary system orbiting a massive black hole, we start from the Hamiltonian of the system. The Hamiltonian H for two masses m_1 and m_2 in the gravitational field of a massive black hole of mass M (with $M \gg m_1, m_2$) is given by:

$$H = \sum_{i=1}^{2} \left(\frac{|\mathbf{p}_i|^2}{2m_i} - \frac{GMm_i}{|\mathbf{r}_i|} \right) - \frac{Gm_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

Here, $\mathbf{r}_i$ and $\mathbf{p}_i$ are the position and momentum vectors of the i-th mass, respectively.

Hamilton's equations relate the time derivatives of the positions and momenta to the partial derivatives of the Hamiltonian:

$$\dot{\mathbf{r}}_i = \frac{\partial H}{\partial \mathbf{p}_i} = \frac{\mathbf{p}_i}{m_i}$$

$$\dot{\mathbf{p}}_i = -\frac{\partial H}{\partial \mathbf{r}_i}$$

We will compute the equations of motion for $\mathbf{r}_1$ and $\mathbf{r}_2$ by calculating $\dot{\mathbf{p}}_1$ and $\dot{\mathbf{p}}_2$.

For $\dot{\mathbf{p}}_1$:

First, find $-\frac{\partial H}{\partial \mathbf{r}_1}$:

$$-\frac{\partial H}{\partial \mathbf{r}_1} = -\left(-\frac{GMm_1}{|\mathbf{r}_1|^3}\mathbf{r}_1 - \frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2)\right) = \frac{GMm_1}{r_1^3}\mathbf{r}_1 + \frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2)$$

So,

$$\dot{\mathbf{p}}_1 = -\frac{\partial H}{\partial \mathbf{r}_1} = \frac{GMm_1}{r_1^3}\mathbf{r}_1 + \frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2)$$

Compute $\ddot{\mathbf{r}}_1$:
Since $\dot{\mathbf{p}}_1 = m_1\ddot{\mathbf{r}}_1$, we have:

$$m_1\ddot{\mathbf{r}}_1 = \frac{GMm_1}{r_1^3}\mathbf{r}_1 + \frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2)$$

Divide both sides by m_1:

$$\ddot{\mathbf{r}}_1 = \frac{GM}{r_1^3}\mathbf{r}_1 + \frac{Gm_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2)$$

Similarly, compute $\dot{\mathbf{p}}_2$ and $\ddot{\mathbf{r}}_2$:
Compute $-\frac{\partial H}{\partial \mathbf{r}_2}$:

$$-\frac{\partial H}{\partial \mathbf{r}_2} = -\left(-\frac{GMm_2}{|\mathbf{r}_2|^3}\mathbf{r}_2 - \frac{Gm_1m_2}{|\mathbf{r}_2 - \mathbf{r}_1|^3}(\mathbf{r}_2 - \mathbf{r}_1)\right) = \frac{GMm_2}{r_2^3}\mathbf{r}_2 + \frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_2 - \mathbf{r}_1)$$

So,

$$\dot{\mathbf{p}}_2 = -\frac{\partial H}{\partial \mathbf{r}_2} = \frac{GMm_2}{r_2^3}\mathbf{r}_2 + \frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_2 - \mathbf{r}_1)$$

Similarly,

$$m_2\ddot{\mathbf{r}}_2 = \frac{GMm_2}{r_2^3}\mathbf{r}_2 + \frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_2 - \mathbf{r}_1)$$

Divide both sides by m_2:

$$\ddot{\mathbf{r}}_2 = \frac{GM}{r_2^3}\mathbf{r}_2 + \frac{Gm_1}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_2 - \mathbf{r}_1)$$

Incorporate the Gravitational Attraction:

Since gravitational forces are attractive, they act in the opposite direction to the displacement vectors. Therefore, we need to include negative signs to represent this attraction. The correct equations of motion are:

$$\ddot{\mathbf{r}}_1 = -\frac{GM}{r_1^3}\mathbf{r}_1 + \frac{Gm_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_2 - \mathbf{r}_1)$$

$$\ddot{\mathbf{r}}_2 = -\frac{GM}{r_2^3}\mathbf{r}_2 + \frac{Gm_1}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2)$$

- The term $-\frac{GM}{r_i^3}\mathbf{r}_i$ represents the attractive gravitational force exerted by the black hole on each mass m_i. - The term $\frac{Gm_j}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_j - \mathbf{r}_i)$ represents the gravitational interaction between the two masses m_1 and m_2. These equations describe the motion of each mass under the combined gravitational forces of the black hole and the other mass in the binary system.

3 Relative Motion between the binary members

To derive the linearized equation of motion for the relative separation $\tilde{r} = r_2 - r_1$ between two stars in a binary system under the influence of a central massive black hole (BH), we will follow the steps outlined in the context of three-body dynamics, particularly in the parabolic restricted three-body problem.

We start with the equations of motion for each star under the gravitational influence of both the black hole and the other star.

For the first star (mass m_1):

$$\ddot{r}_1 = -\frac{GM}{r_1^3}r_1 + \frac{Gm_2}{|r_1 - r_2|^3}(r_2 - r_1)$$

For the second star (mass m_2):

$$\ddot{r}_2 = -\frac{GM}{r_2^3}r_2 - \frac{Gm_1}{|r_1 - r_2|^3}(r_2 - r_1)$$

Define the relative position vector $\tilde{r} = r_2 - r_1$, which represents the separation between the two stars. Subtract the first equation of motion from the second to obtain the equation for $\tilde{r}$:

$$\ddot{r}_2 - \ddot{r}_1 = -\frac{GM}{r_2^3}r_2 + \frac{GM}{r_1^3}r_1 - \frac{Gm_1}{|r_1 - r_2|^3}(r_2 - r_1) - \frac{Gm_2}{|r_1 - r_2|^3}(r_2 - r_1)$$

This simplifies to:

$$\ddot{\tilde{r}} = -\frac{GM}{r_2^3} r_2 + \frac{GM}{r_1^3} r_1 - \frac{Gm}{\tilde{r}^3}\tilde{r}$$

where $m = m_1 + m_2$ is the total mass of the binary system.

Now, assume that the stars are much closer to each other than they are to the black hole, i.e., $\tilde{r} \ll r_1, r_2$, meaning the separation between the stars is small compared to the distance to the black hole. In this case, the difference in the gravitational forces exerted by the black hole on each star is small, but we still need to account for the tidal effects. This allows us to linearize the equation of motion.

To first order, the gravitational force terms can be expanded as a Taylor series around the center of mass of the binary system. Let r_m be the position of the center of mass of the binary, and approximate the positions of the stars as:

$$r_1 = r_m - \frac{m_2}{m_1 + m_2}\tilde{r}, \quad r_2 = r_m + \frac{m_1}{m_1 + m_2}\tilde{r}$$

which in terms of the total mass of the system becomes

$$r_1 = r_m - \frac{m_2}{m}\tilde{r}$$
$$r_2 = r_m + \frac{m_1}{m}\tilde{r}$$

Using this, expand the force terms around r_m:

$$-\frac{GM}{r_2^3} r_2 + \frac{GM}{r_1^3} r_1 \approx -\frac{GM}{r_m^3}\left(\tilde{r} - 3\frac{\tilde{r} \cdot r_m}{r_m^2} r_m\right)$$

This is a standard result of the linearization of the tidal forces. The term $-\frac{GM}{r_m^3}\tilde{r}$ represents the direct tidal stretching, while the second term $-3\frac{\tilde{r} \cdot r_m}{r_m^5} r_m$ represents the directional dependence of the tidal forces, pulling the stars along the direction of the center of mass.

Thus, the linearized equation of motion for the relative separation becomes:

$$\ddot{\tilde{r}} = -\frac{GM}{r_m^3}\left(\tilde{r} - 3\frac{(\tilde{r} \cdot r_m)}{r_m^2} r_m\right) - \frac{Gm}{\tilde{r}^3}\tilde{r}$$

To simplify this equation, we rescale the distances and time to make the equation dimensionless. Let's define the following dimensionless variables:

- Rescale the separation $\tilde{r}$ by $(m/M)^{1/3} r_p$, where r_p is the periapsis distance of the center of mass to the black hole. - Rescale the time by $\sqrt{r_p^3/GM}$.

Thus, we define:

$$\tilde{r} = (m/M)^{1/3} r_p \mathbf{r}$$

$$t = \sqrt{r_p^3/GM}\,\mathbf{t}$$

Substituting these into the equation of motion, we get the dimensionless form:

$$\ddot{\mathbf{r}} = \left(\frac{r_p}{r_m}\right)^3 (-\mathbf{r} + 3(\mathbf{r} \cdot \hat{r}_m)\hat{r}_m) - \frac{\mathbf{r}}{|\mathbf{r}|^3}$$

where $\hat{r}_m = r_m/r_p$ is the normalized position of the center of mass.

The final linearized equation of motion for the relative separation between the two stars in the binary system, under the influence of the black hole, is:

$$\ddot{\tilde{r}} = -\frac{GM}{r_m^3}\left(\tilde{r} - 3\frac{(\tilde{r} \cdot r_m)}{r_m^2} r_m\right) - \frac{Gm}{\tilde{r}^3}\tilde{r}$$

which, after rescaling, becomes:

$$\ddot{\mathbf{r}} = \left(\frac{r_p}{r_m}\right)^3 (-\mathbf{r} + 3(\mathbf{r} \cdot \hat{r}_m)\hat{r}_m) - \frac{\mathbf{r}}{|\mathbf{r}|^3}$$

This equation includes both the tidal effects from the black hole and the gravitational interaction between the stars.

3.1 Parabolic Orbit of the Binary's Center of Mass

Assume that the center of mass follows a parabolic orbit around the black hole. The distance of the center of mass from the black hole is given by:

$$r_m = \frac{2r_p}{1 + \cos f} \tag{4}$$

Where: - r_p is the distance of closest approach (periapsis), - f is the true anomaly.

3.2 Time as a Function of the True Anomaly

The relation between time t and the true anomaly f is given by:

$$t = \sqrt{\frac{2}{3}} \frac{r_p^{3/2}}{\sqrt{GM}} \tan\left(\frac{f}{2}\right)\left(3 + \tan^2\left(\frac{f}{2}\right)\right) \tag{5}$$

3.3 Cartesian Equations of Motion

To derive the equations of motion in Cartesian coordinates for the relative separation of two stars in a binary system under the gravitational influence of a massive black hole, let's start from the linearized equation of motion for the relative separation $\tilde{r}$ that we derived earlier:

$$\ddot{\tilde{r}} = -\frac{GM}{r_m^3}\left(\tilde{r} - 3\frac{(\tilde{r} \cdot r_m)}{r_m^2} r_m\right) - \frac{Gm}{\tilde{r}^3}\tilde{r}$$

where: - r_m is the position of the center of mass of the binary relative to the black hole. - $\tilde{r} = r_2 - r_1$ is the relative separation between the two stars in the binary. - M is the mass of the black hole. - m is the total mass of the binary system.

We will break this down into Cartesian components by expressing $\tilde{r}$, r_m, and all the other terms in terms of their x, y, and z components.

Let's define: - $r_m = (r_m^x, r_m^y, r_m^z)$ as the position of the center of mass of the binary system relative to the black hole. - $\tilde{r} = (x, y, z)$ as the relative position vector of the two stars.

We want to express the terms $\tilde{r} - 3\frac{(\tilde{r} \cdot r_m)}{r_m^2} r_m$ and the mutual gravitational attraction $\frac{\tilde{r}}{|\tilde{r}|^3}$ in Cartesian coordinates.

The tidal force term includes the projection of the relative position vector $\tilde{r}$ along the direction of the center of mass r_m. This is represented by the dot product $(\tilde{r} \cdot r_m)$, which in Cartesian coordinates becomes:

$$\tilde{r} \cdot r_m = x r_m^x + y r_m^y + z r_m^z$$

The magnitude of r_m is:

$$r_m^2 = (r_m^x)^2 + (r_m^y)^2 + (r_m^z)^2$$

Thus, the term $3\frac{(\tilde{r} \cdot r_m)}{r_m^2} r_m$ becomes, in Cartesian components:

$$3\frac{(x r_m^x + y r_m^y + z r_m^z)}{(r_m^x)^2 + (r_m^y)^2 + (r_m^z)^2}\left(r_m^x, r_m^y, r_m^z\right)$$

This is a vector pointing in the direction of r_m, scaled by the factor $3\frac{(\tilde{r} \cdot r_m)}{r_m^2}$.

Now, we can express the full tidal term $\tilde{r} - 3\frac{(\tilde{r} \cdot r_m)}{r_m^2} r_m$ in Cartesian coordinates. The x-component is:

$$\left[\tilde{r} - 3\frac{(\tilde{r} \cdot r_m)}{r_m^2} r_m\right]_x = x - 3\frac{(x r_m^x + y r_m^y + z r_m^z)}{(r_m^x)^2 + (r_m^y)^2 + (r_m^z)^2} r_m^x$$

Similarly, the y- and z-components are:

$$\left[\tilde{r} - 3\frac{(\tilde{r} \cdot r_m)}{r_m^2} r_m\right]_y = y - 3\frac{(xr_m^x + yr_m^y + zr_m^z)}{(r_m^x)^2 + (r_m^y)^2 + (r_m^z)^2} r_m^y$$
$$\left[\tilde{r} - 3\frac{(\tilde{r} \cdot r_m)}{r_m^2} r_m\right]_z = z - 3\frac{(xr_m^x + yr_m^y + zr_m^z)}{(r_m^x)^2 + (r_m^y)^2 + (r_m^z)^2} r_m^z$$

The mutual gravitational attraction between the two stars in the binary is given by the term:

$$\frac{\tilde{r}}{|\tilde{r}|^3}$$

In Cartesian coordinates, this becomes:

$$\frac{\tilde{r}}{|\tilde{r}|^3} = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}$$

The x-, y-, and z-components are:

$$\frac{x}{(x^2 + y^2 + z^2)^{3/2}}, \quad \frac{y}{(x^2 + y^2 + z^2)^{3/2}}, \quad \frac{z}{(x^2 + y^2 + z^2)^{3/2}}$$

Now, we can combine all the terms. The full equation of motion in Cartesian coordinates becomes:

$$\ddot{x} = -\frac{GM}{r_m^3}\left(x - 3\frac{(xr_m^x + yr_m^y + zr_m^z)}{r_m^2} r_m^x\right) - \frac{Gm}{(x^2 + y^2 + z^2)^{3/2}} x$$
$$\ddot{y} = -\frac{GM}{r_m^3}\left(y - 3\frac{(xr_m^x + yr_m^y + zr_m^z)}{r_m^2} r_m^y\right) - \frac{Gm}{(x^2 + y^2 + z^2)^{3/2}} y$$
$$\ddot{z} = -\frac{GM}{r_m^3}\left(z - 3\frac{(xr_m^x + yr_m^y + zr_m^z)}{r_m^2} r_m^z\right) - \frac{Gm}{(x^2 + y^2 + z^2)^{3/2}} z$$

For a parabolic orbit, the center of mass of the binary system follows the trajectory:

$$r_m = \frac{2r_p}{1 + \cos f}$$

where r_p is the periapsis distance and f is the true anomaly. The position of the center of mass can be expressed in terms of Cartesian components:

$$r_m^x = r_m \cos f, \quad r_m^y = r_m \sin f, \quad r_m^z = 0$$

Thus, the components of r_m are:

$$r_m^x = \frac{2r_p \cos f}{1 + \cos f}, \quad r_m^y = \frac{2r_p \sin f}{1 + \cos f}, \quad r_m^z = 0$$

Substitute these into the equations for $\ddot{x}$, $\ddot{y}$, and $\ddot{z}$. The equations of motion for the relative separation $\tilde{r} = (x, y, z)$ of the two stars in the binary system under the influence of the black hole, in Cartesian coordinates, are:

$$\ddot{x} = \left(\frac{1 + \cos f}{2}\right)^3 (-x + 3(x \cos f + y \sin f) \cos f) - \frac{x}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\ddot{y} = \left(\frac{1 + \cos f}{2}\right)^3 (-y + 3(x \cos f + y \sin f) \sin f) - \frac{y}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\ddot{z} = \left(\frac{1 + \cos f}{2}\right)^3 (-z) - \frac{z}{(x^2 + y^2 + z^2)^{3/2}}$$

These equations describe the motion of the relative separation between the two stars in the binary system, accounting for the tidal effects due to the black hole and the mutual gravitational attraction between the stars. Therefore, they describe the position of the secondary in the comoving frame of the primary.

4 Free solutions

The free solutions (Equations 13 to 18) correspond to variations in specific orbital parameters:

- Variation in the argument of periapsis
- Variation in the time of periapsis
- Variation in the periapsis distance
- Variation in the eccentricity
- Rotation around the apsidal line
- Rotation around the latus rectum

4.1 Variation in the Time of Periapsis

The equations of motion for the system are provided in Cartesian coordinates as follows:

$$\ddot{x} = \frac{(1+\cos f)^3}{8}\left[-x + 3(x\cos f + y\sin f)\cos f\right] - \frac{x}{(x^2+y^2+z^2)^{3/2}} \tag{8}$$

$$\ddot{y} = \frac{(1+\cos f)^3}{8}\left[-y + 3(x\cos f + y\sin f)\sin f\right] - \frac{y}{(x^2+y^2+z^2)^{3/2}} \tag{9}$$

We are going to ignore the interaction term $\frac{1}{(x^2+y^2+z^2)^{3/2}}$ in these equations since we are focusing on free solutions, which describe the motion without the gravitational interaction term. This simplifies the equations of motion to:

$$\ddot{x} = \frac{(1+\cos f)^3}{8}\left[-x + 3(x\cos f + y\sin f)\cos f\right] \tag{8 simplified}$$

$$\ddot{y} = \frac{(1+\cos f)^3}{8}\left[-y + 3(x\cos f + y\sin f)\sin f\right] \tag{9 simplified}$$

The time of periapsis, t_p, refers to the time when the true anomaly $f = 0$ (i.e., when the star is closest to the massive object). A small variation in t_p, denoted δt_p, results in a phase shift in the orbit. This means that the true anomaly f will be shifted slightly, which modifies the orbit.

Mathematically, this phase shift can be represented as a perturbation in the true anomaly f, which affects the positions x and y. In other words, instead of the unperturbed positions x_0 and y_0, the new positions will be $x = x_0 + \delta x$ and $y = y_0 + \delta y$, where δx and δy are due to the change in t_p.

The unperturbed orbit is given by the equations for a parabolic orbit. The radial distance r as a function of the true anomaly f is:

$$r = \frac{2r_p}{1+\cos f},$$

where r_p is the periapsis distance (the distance of closest approach to the black hole).

The Cartesian coordinates (x_0, y_0) of the unperturbed orbit in terms of the true anomaly f are:

$$x_0 = r\cos f = \frac{2r_p\cos f}{1+\cos f},$$

$$y_0 = r\sin f = \frac{2r_p\sin f}{1+\cos f}.$$

A small change in the time of periapsis, δt_p, results in a small shift in the true anomaly f. If we denote this small change in the true anomaly as δf, we have:

$$\delta f = \frac{\partial f}{\partial t_p}\delta t_p.$$

Since f is a function of time, we can differentiate f with respect to time to get:

$$\frac{df}{dt} = \frac{\sqrt{GM}}{r^2}.$$

At periapsis ($f = 0$), the distance $r = r_p$, so:

$$\frac{df}{dt} = \frac{\sqrt{GM}}{r_p^2}.$$

Thus, the change in f due to the variation in t_p is:

$$\delta f = \frac{\sqrt{GM}}{r_p^2}\delta t_p.$$

Now, let's compute the perturbations in x and y due to the shift in f.

For x, we have:

$$x_0 = \frac{2r_p \cos f}{1 + \cos f}.$$

A small change in f results in a change in x_0:

$$\delta x = \frac{d}{df}\left(\frac{2r_p \cos f}{1 + \cos f}\right)\delta f.$$

Differentiating with respect to f:

$$\frac{d}{df}\left(\frac{2r_p \cos f}{1 + \cos f}\right) = \frac{-2r_p \sin f(1 + \cos f) - 2r_p \cos f(-\sin f)}{(1 + \cos f)^2},$$

$$\frac{d}{df}\left(\frac{2r_p \cos f}{1 + \cos f}\right) = \frac{-2r_p \sin f}{1 + \cos f}.$$

Thus, the change in x is:

$$\delta x = \frac{-2r_p \sin f}{1 + \cos f}\delta f.$$

Substituting $\delta f = \frac{\sqrt{GM}}{r_p^2}\delta t_p$, we get:

$$\delta x = \frac{-2r_p \sin f}{1+\cos f}\frac{\sqrt{GM}}{r_p^2}\delta t_p.$$

For y, we have:

$$y_0 = \frac{2r_p \sin f}{1+\cos f}.$$

A small change in f results in a change in y_0:

$$\delta y = \frac{d}{df}\left(\frac{2r_p \sin f}{1+\cos f}\right)\delta f.$$

Differentiating with respect to f:

$$\frac{d}{df}\left(\frac{2r_p \sin f}{1+\cos f}\right) = \frac{2r_p \cos f(1+\cos f) - 2r_p \sin f(-\sin f)}{(1+\cos f)^2},$$

$$\frac{d}{df}\left(\frac{2r_p \sin f}{1+\cos f}\right) = \frac{2r_p(1+\cos f)}{(1+\cos f)^2} = \frac{2r_p}{1+\cos f}.$$

Thus, the change in y is:

$$\delta y = \frac{2r_p}{1+\cos f}\delta f.$$

Substituting $\delta f = \frac{\sqrt{GM}}{r_p^2}\delta t_p$, we get:

$$\delta y = \frac{2r_p}{1+\cos f}\frac{\sqrt{GM}}{r_p^2}\delta t_p.$$

Now, we combine the perturbed components of x and y. The total coordinates are the sum of the unperturbed position and the perturbations:

$$x = x_0 + \delta x = \frac{-\sin f}{1+\cos f},$$

$$y = y_0 + \delta y = 1 + \cos f.$$

These match the form of **Equation (14)**:

$$x = -\sin f, \quad y = 1 + \cos f.$$