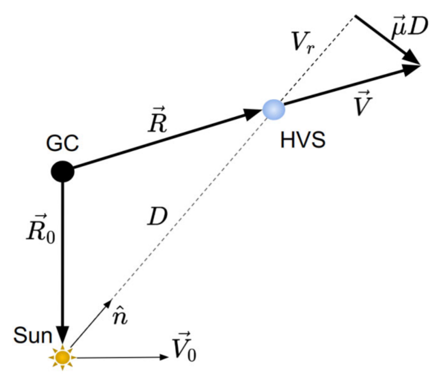
Derivation of the Distance D

Given Equations



The position vector:

$$\vec{R} = \vec{R}_0 + D\,\hat{n}$$

where:

- $\vec{R}_0$: Vector from the Galactic Center (GC) to the Sun.
- D: Heliocentric distance to the source.
- $\hat{n}$: Unit vector from the Sun to the source.

The velocity vector:

$$\vec{V} = \vec{V}_0 + V_r\,\hat{n} + D\,\vec{\mu}$$

where:

- $\vec{V}_0$: Velocity of the Sun in the galactic rest frame.
- V_r: Radial velocity of the source relative to the Sun.
- $\vec{\mu}$: Proper motion of the source in the heliocentric frame (perpendicular to $\hat{n}$).

Assuming the star is coming from the Galactic Center, we have:

$$\vec{V} \times \vec{R} = \vec{0}$$

which implies that $\vec{V}$ and $\vec{R}$ are parallel vectors.

Derivation

Starting with the cross product:

$$\vec{V} \times \vec{R} = \vec{0}$$

Substitute the expressions for $\vec{V}$ and $\vec{R}$:

$$\left(\vec{V}_0 + V_r\,\hat{n} + D\,\vec{\mu}\right) \times \left(\vec{R}_0 + D\,\hat{n}\right) = \vec{0}$$

Expand the cross product using the distributive property:

$$\begin{aligned}\vec{V}_0 \times \vec{R}_0 + \vec{V}_0 \times D\,\hat{n} + V_r\,\hat{n} \times \vec{R}_0 + V_r\,\hat{n} \times D\,\hat{n}\\ + D\,\vec{\mu} \times \vec{R}_0 + D\,\vec{\mu} \times D\,\hat{n} = \vec{0}\end{aligned}$$

Simplify each term:

$$\begin{aligned}\vec{V}_0 \times D\,\hat{n} &= D\,(\vec{V}_0 \times \hat{n})\\ V_r\,\hat{n} \times \vec{R}_0 &= -V_r\,(\vec{R}_0 \times \hat{n})\\ V_r\,\hat{n} \times D\,\hat{n} &= V_r D\,(\hat{n} \times \hat{n}) = \vec{0}\\ D\,\vec{\mu} \times \vec{R}_0 &= D\,(\vec{\mu} \times \vec{R}_0)\\ D\,\vec{\mu} \times D\,\hat{n} &= D^2\,(\vec{\mu} \times \hat{n})\end{aligned}$$

Substitute back into the expanded expression:

$$\vec{V}_0 \times \vec{R}_0 + D\,(\vec{V}_0 \times \hat{n}) - V_r\,(\vec{R}_0 \times \hat{n}) + D\,(\vec{\mu} \times \vec{R}_0) + D^2\,(\vec{\mu} \times \hat{n}) = \vec{0}$$

Taking the Dot Product with $\hat{n}$

Take the dot product of both sides with $\hat{n}$:

$$\hat{n} \cdot \left(\vec{V}_0 \times \vec{R}_0 + D\,(\vec{V}_0 \times \hat{n}) - V_r\,(\vec{R}_0 \times \hat{n}) + D\,(\vec{\mu} \times \vec{R}_0) + D^2\,(\vec{\mu} \times \hat{n})\right) = \hat{n} \cdot \vec{0}$$

Simplify each term using the scalar triple product identity $\hat{n} \cdot (\vec{A} \times \vec{B}) = (\hat{n} \times \vec{A}) \cdot \vec{B}$:

1. $\hat{n} \cdot (\vec{V}_0 \times \vec{R}_0) = (\hat{n} \times \vec{V}_0) \cdot \vec{R}_0$
2. $\hat{n} \cdot \left(D\,(\vec{V}_0 \times \hat{n})\right) = D\,\hat{n} \cdot (\vec{V}_0 \times \hat{n}) = 0$ (since $\vec{V}_0 \times \hat{n}$ is perpendicular to $\hat{n}$)
3. $\hat{n} \cdot \left(-V_r\,(\vec{R}_0 \times \hat{n})\right) = -V_r\,\hat{n} \cdot (\vec{R}_0 \times \hat{n}) = 0$
4. $\hat{n} \cdot \left(D\,(\vec{\mu} \times \vec{R}_0)\right) = D\,(\hat{n} \times \vec{\mu}) \cdot \vec{R}_0$
5. $\hat{n} \cdot \left(D^2\,(\vec{\mu} \times \hat{n})\right) = D^2\,\hat{n} \cdot (\vec{\mu} \times \hat{n}) = 0$

Therefore, the equation simplifies to:

$$(\hat{n} \times \vec{V}_0) \cdot \vec{R}_0 + D\,(\hat{n} \times \vec{\mu}) \cdot \vec{R}_0 = 0$$

Solving for D

Rewriting the equation:

$$(\hat{n} \times \vec{V}_0) \cdot \vec{R}_0 + D\,(\hat{n} \times \vec{\mu}) \cdot \vec{R}_0 = 0$$

Isolate D:

$$D\,(\hat{n} \times \vec{\mu}) \cdot \vec{R}_0 = -(\hat{n} \times \vec{V}_0) \cdot \vec{R}_0$$

Therefore, the expression for D is:

$$D = -\frac{(\hat{n} \times \vec{V}_0) \cdot \vec{R}_0}{(\hat{n} \times \vec{\mu}) \cdot \vec{R}_0}$$

If we rearange in order to eliminate the minus:

$$D = \frac{\hat{n} \cdot (\vec{R}_0 \times \vec{V}_0)}{\vec{R}_0 \cdot (\hat{n} \times \vec{\mu})}$$

Derivation of V_r

Starting from the equation:

$$\vec{V}_0 \times \vec{R}_0 + D\,(\vec{V}_0 \times \hat{n}) - V_r\,(\vec{R}_0 \times \hat{n}) + D\,(\vec{\mu} \times \vec{R}_0) + D^2\,(\vec{\mu} \times \hat{n}) = \vec{0}$$

Take the dot product with $\vec{\mu}$:

$$\vec{\mu} \cdot \Big(\vec{V}_0 \times \vec{R}_0 + D\,(\vec{V}_0 \times \hat{n}) - V_r\,(\vec{R}_0 \times \hat{n}) + D\,(\vec{\mu} \times \vec{R}_0) + D^2\,(\vec{\mu} \times \hat{n})\Big) = \vec{\mu} \cdot \vec{0}$$

Simplify each term:

1. **First Term:**
$$\vec{\mu} \cdot (\vec{V}_0 \times \vec{R}_0)$$
2. **Second Term:**
$$D\,\vec{\mu} \cdot (\vec{V}_0 \times \hat{n})$$
3. **Third Term:**
$$-V_r\,\vec{\mu} \cdot (\vec{R}_0 \times \hat{n})$$
4. **Fourth Term:**
$$D\,\vec{\mu} \cdot (\vec{\mu} \times \vec{R}_0) = 0$$
(Because $\vec{\mu} \cdot (\vec{\mu} \times \vec{R}_0) = 0$)
5. **Fifth Term:**
$$D^2\,\vec{\mu} \cdot (\vec{\mu} \times \hat{n}) = 0$$
(Because $\vec{\mu} \cdot (\vec{\mu} \times \hat{n}) = 0$)

Therefore, the equation simplifies to:

$$\vec{\mu} \cdot (\vec{V}_0 \times \vec{R}_0) + D\,\vec{\mu} \cdot (\vec{V}_0 \times \hat{n}) - V_r\,\vec{\mu} \cdot (\vec{R}_0 \times \hat{n}) = 0$$

Rewriting:

$$\vec{\mu} \cdot (\vec{V}_0 \times \vec{R}_0) + D\,\vec{\mu} \cdot (\vec{V}_0 \times \hat{n}) = V_r\,\vec{\mu} \cdot (\vec{R}_0 \times \hat{n})$$

Solving for V_r:

$$V_r = \frac{\vec{\mu} \cdot (\vec{V}_0 \times \vec{R}_0) + D\,\vec{\mu} \cdot (\vec{V}_0 \times \hat{n})}{\vec{\mu} \cdot (\vec{R}_0 \times \hat{n})}$$

Derivation of the Errors in D and V_r

Expressions for D and V_r

The expressions are:

$$D = -\frac{(\hat{n} \times \vec{V}_0) \cdot \vec{R}_0}{(\hat{n} \times \vec{\mu}) \cdot \vec{R}_0} \tag{1}$$

$$V_r = \frac{\vec{\mu} \cdot (\vec{V}_0 \times \vec{R}_0) + D\,\vec{\mu} \cdot (\vec{V}_0 \times \hat{n})}{\vec{\mu} \cdot (\vec{R}_0 \times \hat{n})} \tag{2}$$

Uncertainty in D

Let:

$$N = (\hat{n} \times \vec{V}_0) \cdot \vec{R}_0$$
$$D_{\text{den}} = (\hat{n} \times \vec{\mu}) \cdot \vec{R}_0$$

The uncertainty in D is:

$$(\delta D)^2 = \left(\frac{\delta N}{D_{\text{den}}}\right)^2 + \left(D \cdot \frac{\delta D_{\text{den}}}{D_{\text{den}}}\right)^2$$

Where:

$$(\delta N)^2 = \sum_i \left(\left((\hat{n} \times \vec{e}_i) \cdot \vec{R}_0\right) \delta V_{0i}\right)^2 + \sum_i \left(\left(\left[\hat{n} \times \vec{V}_0\right]_i\right) \delta R_{0i}\right)^2$$

$$(\delta D_{\text{den}})^2 = \sum_i \left(\left((\hat{n} \times \vec{e}_i) \cdot \vec{R}_0\right) \delta \mu_i\right)^2 + \sum_i \left(\left([\hat{n} \times \vec{\mu}]_i\right) \delta R_{0i}\right)^2$$

Uncertainty in V_r

Let:

$$N' = \vec{\mu} \cdot (\vec{V}_0 \times \vec{R}_0) + D\,\vec{\mu} \cdot (\vec{V}_0 \times \hat{n})$$
$$D' = \vec{\mu} \cdot (\vec{R}_0 \times \hat{n})$$

The uncertainty in V_r is:

$$(\delta V_r)^2 = \left(\frac{\delta N'}{D'}\right)^2 + \left(V_r \cdot \frac{\delta D'}{D'}\right)^2$$

Where:

$$(\delta N')^2 = \sum_i \left(\frac{\partial N'}{\partial \mu_i}\delta\mu_i\right)^2 + \sum_i \left(\frac{\partial N'}{\partial V_{0i}}\delta V_{0i}\right)^2 + \sum_i \left(\frac{\partial N'}{\partial R_{0i}}\delta R_{0i}\right)^2 + \left(\frac{\partial N'}{\partial D}\delta D\right)^2$$
$$(\delta D')^2 = \sum_i \left(\left(\vec{e}_i \cdot (\vec{R}_0 \times \hat{n})\right)\delta\mu_i\right)^2 + \sum_i \left((\vec{\mu} \cdot (\vec{e}_i \times \hat{n}))\,\delta R_{0i}\right)^2$$

Notes

- $\vec{e}_i$ is the unit vector in the i-th coordinate direction. - The sums are over the vector components $i = x, y, z$. - The uncertainties δV_{0i}, δR_{0i}, $\delta\mu_i$, and δD are the standard deviations of the respective quantities.

Uncertainties for uncorrelated errors

Uncertainty in D

Given:

$$D = -\frac{N}{D_{\text{den}}}$$

where:

$$N = (\hat{n} \times \vec{V}_0) \cdot \vec{R}_0$$
$$D_{\text{den}} = (\hat{n} \times \vec{\mu}) \cdot \vec{R}_0$$

Assuming uncorrelated errors, the relative uncertainty in D is:

$$\left(\frac{\delta D}{D}\right)^2 = \left(\frac{\delta N}{N}\right)^2 + \left(\frac{\delta D_{\text{den}}}{D_{\text{den}}}\right)^2$$

Computing δN and δD_{den}: For δN:

$$(\delta N)^2 = \left[(\hat{n} \times \delta\vec{V}_0) \cdot \vec{R}_0\right]^2 + \left[(\hat{n} \times \vec{V}_0) \cdot \delta\vec{R}_0\right]^2$$

For δD_{den}:

$$(\delta D_{\text{den}})^2 = \left[(\hat{n} \times \delta\vec{\mu}) \cdot \vec{R}_0\right]^2 + \left[(\hat{n} \times \vec{\mu}) \cdot \delta\vec{R}_0\right]^2$$

Final Expression for δD:

$$\delta D = D\sqrt{\left(\frac{\delta N}{N}\right)^2 + \left(\frac{\delta D_{\text{den}}}{D_{\text{den}}}\right)^2}$$

Uncertainty in V_r

Given:

$$V_r = \frac{N'}{D'}$$

where:

$$N' = \vec{\mu} \cdot (\vec{V}_0 \times \vec{R}_0) + D\,\vec{\mu} \cdot (\vec{V}_0 \times \hat{n})$$
$$D' = \vec{\mu} \cdot (\vec{R}_0 \times \hat{n})$$

Assuming uncorrelated errors, the relative uncertainty in V_r is:

$$\left(\frac{\delta V_r}{V_r}\right)^2 = \left(\frac{\delta N'}{N'}\right)^2 + \left(\frac{\delta D'}{D'}\right)^2$$

Computing $\delta N'$ and $\delta D'$: For $\delta N'$:

$$\begin{aligned}(\delta N')^2 = \; & \left[\delta\vec{\mu} \cdot (\vec{V}_0 \times \vec{R}_0 + D\,\vec{V}_0 \times \hat{n})\right]^2 \\ & + \left[\vec{\mu} \cdot (\delta\vec{V}_0 \times \vec{R}_0 + D\,\delta\vec{V}_0 \times \hat{n})\right]^2 \\ & + \left[\vec{\mu} \cdot (\vec{V}_0 \times \delta\vec{R}_0)\right]^2 \\ & + \left[\vec{\mu} \cdot (\vec{V}_0 \times \hat{n})\delta D\right]^2\end{aligned}$$

For $\delta D'$:

$$(\delta D')^2 = \left[\delta\vec{\mu} \cdot (\vec{R}_0 \times \hat{n})\right]^2 + \left[\vec{\mu} \cdot (\delta\vec{R}_0 \times \hat{n})\right]^2$$

Final Expression for δV_r:

$$\delta V_r = V_r\sqrt{\left(\frac{\delta N'}{N'}\right)^2 + \left(\frac{\delta D'}{D'}\right)^2}$$

Fully correlated variables

We consider the following variables, each with associated uncertainties and possible correlations:

- Right Ascension: α, with uncertainty $\delta\alpha$
- Declination: δ, with uncertainty $\delta\delta$

- Proper motion in RA: μ_α, with uncertainty $\delta\mu_\alpha$
- Proper motion in Dec: μ_δ, with uncertainty $\delta\mu_\delta$
- Distance: D, with uncertainty δD

The covariance between these variables is represented by the covariance matrix $\mathbf{C}$, where each element $C_{ij} = \mathrm{Cov}(x_i, x_j)$.

Derivation of Uncertainty in D

Total Differential of D

The total differential of D is:

$$\delta D = \frac{\partial D}{\partial \alpha}\delta\alpha + \frac{\partial D}{\partial \delta}\delta\delta + \frac{\partial D}{\partial \mu_\alpha}\delta\mu_\alpha + \frac{\partial D}{\partial \mu_\delta}\delta\mu_\delta$$

Variance of D

The variance δD^2 considering correlations is:

$$\delta D^2 = \sum_{i,j} \frac{\partial D}{\partial x_i}\frac{\partial D}{\partial x_j} C_{ij}$$

where x_i and x_j are the variables $\alpha, \delta, \mu_\alpha, \mu_\delta$.

Computing Partial Derivatives

We compute the partial derivatives of D with respect to each variable.

Partial Derivative with Respect to α

$$\frac{\partial D}{\partial \alpha} = -\frac{1}{D_{\text{den}}}\left(\frac{\partial N}{\partial \alpha} - D\frac{\partial D_{\text{den}}}{\partial \alpha}\right)$$

Similarly for $\delta, \mu_\alpha, \mu_\delta$.

Expressions for Partial Derivatives

Computing $\frac{\partial N}{\partial x_i}$ and $\frac{\partial D_{\mathbf{den}}}{\partial x_i}$ The expressions for N and D_{den} involve vector cross products and dot products. The partial derivatives require computing the derivatives of these vector operations with respect to α, δ.

Let's denote:

$$\hat{n} = \begin{pmatrix} \cos\delta\cos\alpha \\ \cos\delta\sin\alpha \\ \sin\delta \end{pmatrix}$$

$$\vec{\mu} = \mu_\alpha \hat{e}_\alpha + \mu_\delta \hat{e}_\delta$$

where $\hat{e}_\alpha$ and $\hat{e}_\delta$ are the unit vectors in the directions of increasing α and δ, respectively.

The partial derivatives involve computing:

$$\frac{\partial \hat{n}}{\partial \alpha}, \quad \frac{\partial \hat{n}}{\partial \delta}$$

$$\frac{\partial \vec{\mu}}{\partial \alpha}, \quad \frac{\partial \vec{\mu}}{\partial \delta}$$

Covariance Matrix

The covariance matrix $\mathbf{C}$ is:

$$\mathbf{C} = \begin{pmatrix} \mathrm{Var}(\alpha) & \mathrm{Cov}(\alpha,\delta) & \mathrm{Cov}(\alpha,\mu_\alpha) & \mathrm{Cov}(\alpha,\mu_\delta) \\ \mathrm{Cov}(\delta,\alpha) & \mathrm{Var}(\delta) & \mathrm{Cov}(\delta,\mu_\alpha) & \mathrm{Cov}(\delta,\mu_\delta) \\ \mathrm{Cov}(\mu_\alpha,\alpha) & \mathrm{Cov}(\mu_\alpha,\delta) & \mathrm{Var}(\mu_\alpha) & \mathrm{Cov}(\mu_\alpha,\mu_\delta) \\ \mathrm{Cov}(\mu_\delta,\alpha) & \mathrm{Cov}(\mu_\delta,\delta) & \mathrm{Cov}(\mu_\delta,\mu_\alpha) & \mathrm{Var}(\mu_\delta) \end{pmatrix}$$

Final Expression for δD^2

Combining the partial derivatives and the covariance matrix, the variance of D is:

$$\delta D^2 = \sum_{i,j} \left(\frac{\partial D}{\partial x_i} \frac{\partial D}{\partial x_j} \right) C_{ij}$$

Derivation of Uncertainty in V_r

Similarly, the variance δV_r^2 is given by:

$$\delta V_r^2 = \sum_{i,j} \frac{\partial V_r}{\partial x_i} \frac{\partial V_r}{\partial x_j} C_{ij}$$

where the variables x_i include $\alpha, \delta, \mu_\alpha, \mu_\delta, D$.

Partial Derivatives of V_r

The partial derivatives $\frac{\partial V_r}{\partial x_i}$ can be computed using:

$$\frac{\partial V_r}{\partial x_i} = \frac{1}{D'} \left(\frac{\partial N'}{\partial x_i} - V_r \frac{\partial D'}{\partial x_i} \right)$$

Expressions for Partial Derivatives

We need to compute $\frac{\partial N'}{\partial x_i}$ and $\frac{\partial D'}{\partial x_i}$, taking into account the dependence of N' and D' on the variables and their correlations.

Including Correlations with D

Since D is a function of $\alpha, \delta, \mu_\alpha, \mu_\delta$ and has uncertainties correlated with these variables, the covariance terms involving D must be included in δV_r^2.

Implementation Using Covariance Matrices

Combining All Variables

Define the vector of variables:

$$\vec{x} = \begin{pmatrix} \alpha \\ \delta \\ \mu_\alpha \\ \mu_\delta \\ D \end{pmatrix}$$

and the corresponding covariance matrix $\mathbf{C}$ of size 5×5.

Variance Expressions

The variance of D:

$$\delta D^2 = \sum_{i,j=1}^{4} \frac{\partial D}{\partial x_i} \frac{\partial D}{\partial x_j} C_{ij}$$

The variance of V_r:

$$\delta V_r^2 = \sum_{i,j=1}^{5} \frac{\partial V_r}{\partial x_i} \frac{\partial V_r}{\partial x_j} C_{ij}$$

Note that D is included in the variables for V_r.

By incorporating the covariance matrix $\mathbf{C}$, we account for the correlations between proper motion errors, sky positions, and the distance D itself. The derived expressions for δD^2 and δV_r^2 provide a comprehensive method to calculate the uncertainties in D and V_r, considering all relevant correlations.

Improved Error Propagation Using Gaia Formalism

In this section, we improve the previous derivation of the uncertainties in distance D and radial velocity V_r by incorporating the formalism used in Gaia data processing for error propagation, as described in Section 3.1.7 of the Gaia Data Release documentation (Butkevich and Lindegren).

Gaia Astrometric Model and Error Propagation

Gaia's astrometric measurements involve the five parameters: right ascension α, declination δ, parallax ϖ, and proper motions $\mu_{\alpha*} = \mu_\alpha \cos\delta$ and μ_δ. These parameters are correlated, and their uncertainties are quantified by a 5×5 covariance matrix $\mathbf{C}_0$.

The transformation of these parameters from an initial epoch T_0 to an arbitrary epoch $T = T_0 + t$ involves considering the uniform rectilinear motion of the source relative to the solar-system barycentre. The Gaia astrometric model assumes that the barycentric position vector $\mathbf{b}$ of a source at epoch T is given by:

$$\mathbf{b} = \mathbf{b}_0 + t\,\mathbf{v}, \tag{3}$$

where $\mathbf{b}_0$ is the barycentric position at the initial epoch T_0, and $\mathbf{v}$ is the constant space velocity vector. The space velocity $\mathbf{v}$ can be expressed in terms of the astrometric parameters as:

$$\mathbf{v} = A_V \varpi_0 (\mu_{\alpha*0}\,\mathbf{p}_0 + \mu_{\delta 0}\,\mathbf{q}_0 + \mu_{r0}\,\mathbf{r}_0), \tag{4}$$

where:

- $A_V = 4.740470446\,\mathrm{km\,yr\,s^{-1}}$ is the astronomical unit expressed in $\mathrm{km\,yr^{-1}}$.
- ϖ_0 is the parallax at T_0.
- $\mu_{\alpha*0}$ and $\mu_{\delta 0}$ are the proper motions at T_0.
- μ_{r0} is the radial proper motion at T_0.
- $\mathbf{p}_0, \mathbf{q}_0, \mathbf{r}_0$ form an orthonormal basis associated with the source direction at T_0.

Propagation of Astrometric Parameters

The propagation of the astrometric parameters and their covariances involves calculating the Jacobian matrix $\mathbf{J}$ of partial derivatives of the parameters at epoch T with respect to those at T_0:

$$\mathbf{J} = \frac{\partial(\alpha, \delta, \varpi, \mu_{\alpha*}, \mu_\delta, \mu_r)}{\partial(\alpha_0, \delta_0, \varpi_0, \mu_{\alpha*0}, \mu_{\delta 0}, \mu_{r0})}. \tag{5}$$

The covariance matrix $\mathbf{C}$ of the propagated parameters is then given by:

$$\mathbf{C} = \mathbf{J}\,\mathbf{C}_0\,\mathbf{J}^\top. \tag{6}$$

Application to Distance D and Radial Velocity V_r

To compute the uncertainties in D and V_r considering the correlations, we extend the covariance matrix to include D and V_r. Assuming that D and V_r are functions of the astrometric parameters, their uncertainties can be propagated using the extended Jacobian.

Distance D

The distance D is related to the parallax ϖ by:

$$D = \frac{1}{\varpi}, \tag{7}$$

with the uncertainty in D given by:

$$\sigma_D = \left|\frac{\partial D}{\partial \varpi}\right| \sigma_\varpi = \frac{\sigma_\varpi}{\varpi^2}. \tag{8}$$

However, since ϖ is correlated with the other astrometric parameters, we need to consider the full covariance matrix when propagating the uncertainty.

Radial Velocity V_r

The radial velocity V_r can be expressed in terms of the proper motions, parallax, and radial proper motion μ_r:

$$V_r = A_V D \mu_r = A_V \frac{\mu_r}{\varpi}. \tag{9}$$

The uncertainty in V_r is then:

$$\sigma_{V_r}^2 = \left(\frac{\partial V_r}{\partial \varpi}\right)^2 \sigma_\varpi^2 + \left(\frac{\partial V_r}{\partial \mu_r}\right)^2 \sigma_{\mu_r}^2 + 2\frac{\partial V_r}{\partial \varpi}\frac{\partial V_r}{\partial \mu_r}\,\mathrm{Cov}(\varpi, \mu_r) + \sum_{i,j} \frac{\partial V_r}{\partial x_i}\frac{\partial V_r}{\partial x_j}\,\mathrm{Cov}(x_i, x_j), \tag{10}$$

where x_i and x_j run over all correlated parameters $(\alpha, \delta, \mu_{\alpha *}, \mu_\delta)$.

Extended Covariance Matrix

We construct an extended covariance matrix $\mathbf{C}_{\mathrm{ext}}$ that includes D and V_r along with the astrometric parameters:

$$\mathbf{C}_{\mathrm{ext}} = \begin{pmatrix} \mathbf{C} & \mathbf{C}_{\mathrm{cross}} \\ \mathbf{C}_{\mathrm{cross}}^\top & \mathbf{C}_{DV_r} \end{pmatrix},$$

where:

- $\mathbf{C}$ is the 6×6 covariance matrix of $(\alpha, \delta, \varpi, \mu_{\alpha*}, \mu_\delta, \mu_r)$.
- $\mathbf{C}_{\text{cross}}$ contains the covariances between D, V_r and the astrometric parameters.
- $\mathbf{C}_{DV_r}$ is the 2×2 covariance matrix of (D, V_r).

Calculating the Jacobian for D and V_r

The Jacobian matrix $\mathbf{J}_{\text{ext}}$ for D and V_r with respect to the astrometric parameters is:

$$\mathbf{J}_{\text{ext}} = \begin{pmatrix} \frac{\partial D}{\partial \alpha} & \frac{\partial D}{\partial \delta} & \frac{\partial D}{\partial \varpi} & \frac{\partial D}{\partial \mu_{\alpha*}} & \frac{\partial D}{\partial \mu_\delta} & \frac{\partial D}{\partial \mu_r} \\ \frac{\partial V_r}{\partial \alpha} & \frac{\partial V_r}{\partial \delta} & \frac{\partial V_r}{\partial \varpi} & \frac{\partial V_r}{\partial \mu_{\alpha*}} & \frac{\partial V_r}{\partial \mu_\delta} & \frac{\partial V_r}{\partial \mu_r} \end{pmatrix}.$$

The partial derivatives are computed as:

$$\frac{\partial D}{\partial \varpi} = -\frac{1}{\varpi^2}, \tag{11}$$

$$\frac{\partial D}{\partial x_i} = 0 \quad \text{for} \quad x_i \neq \varpi, \tag{12}$$

$$\frac{\partial V_r}{\partial \varpi} = -A_V \frac{\mu_r}{\varpi^2}, \tag{13}$$

$$\frac{\partial V_r}{\partial \mu_r} = A_V \frac{1}{\varpi}, \tag{14}$$

$$\frac{\partial V_r}{\partial x_i} = A_V D \frac{\partial \mu_r}{\partial x_i} \quad \text{for} \quad x_i = \alpha, \delta, \mu_{\alpha*}, \mu_\delta. \tag{15}$$

Note that μ_r depends on the astrometric parameters through the perspective acceleration. However, for nearby stars or when the radial velocity is small, the dependence may be negligible.

Propagating Uncertainties

The uncertainties in D and V_r are then calculated using the covariance matrix and the Jacobian:

$$\sigma_D^2 = \mathbf{J}_D \, \mathbf{C} \, \mathbf{J}_D^\top, \tag{16}$$

$$\sigma_{V_r}^2 = \mathbf{J}_{V_r} \, \mathbf{C} \, \mathbf{J}_{V_r}^\top, \tag{17}$$

where $\mathbf{J}_D$ and $\mathbf{J}_{V_r}$ are the rows of $\mathbf{J}_{\text{ext}}$ corresponding to D and V_r, respectively.

Including Correlations from Gaia Data

Gaia provides the full covariance matrix $\mathbf{C}_0$ for the astrometric parameters, including correlations between $\alpha, \delta, \varpi, \mu_{\alpha*}, \mu_\delta$. The radial proper motion μ_r is not directly measured but can be derived if radial velocities are available.

When propagating errors, we include the correlations provided by Gaia. The error propagation formulas take into account these correlations, ensuring accurate estimation of uncertainties in D and V_r.

Practical Implementation

In practice, the steps are:

1. Obtain the astrometric parameters $(\alpha, \delta, \varpi, \mu_{\alpha*}, \mu_\delta)$ and their covariance matrix $\mathbf{C}_0$ from Gaia data.
2. Compute $D = 1/\varpi$ and $V_r = A_V D \mu_r$.
3. Construct the Jacobian $\mathbf{J}_{\text{ext}}$ using the partial derivatives, considering any dependencies of μ_r on the astrometric parameters.
4. Propagate the uncertainties using $\mathbf{C} = \mathbf{J}\,\mathbf{C}_0\,\mathbf{J}^\top$.
5. Extract σ_D^2 and $\sigma_{V_r}^2$ from the extended covariance matrix.

Example of Covariance Matrix

An example of the covariance matrix $\mathbf{C}_0$ from Gaia is:

$$\mathbf{C}_0 = \begin{pmatrix} \sigma_\alpha^2 & \sigma_{\alpha\delta} & \sigma_{\alpha\varpi} & \sigma_{\alpha\mu_{\alpha*}} & \sigma_{\alpha\mu_\delta} \\ \sigma_{\alpha\delta} & \sigma_\delta^2 & \sigma_{\delta\varpi} & \sigma_{\delta\mu_{\alpha*}} & \sigma_{\delta\mu_\delta} \\ \sigma_{\alpha\varpi} & \sigma_{\delta\varpi} & \sigma_\varpi^2 & \sigma_{\varpi\mu_{\alpha*}} & \sigma_{\varpi\mu_\delta} \\ \sigma_{\alpha\mu_{\alpha*}} & \sigma_{\delta\mu_{\alpha*}} & \sigma_{\varpi\mu_{\alpha*}} & \sigma_{\mu_{\alpha*}}^2 & \sigma_{\mu_{\alpha*}\mu_\delta} \\ \sigma_{\alpha\mu_\delta} & \sigma_{\delta\mu_\delta} & \sigma_{\varpi\mu_\delta} & \sigma_{\mu_{\alpha*}\mu_\delta} & \sigma_{\mu_\delta}^2 \end{pmatrix}.$$

This matrix can be extended to include μ_r if radial velocities are available.

By incorporating the Gaia formalism for error propagation, we account for the correlations between astrometric parameters when calculating the uncertainties in distance D and radial velocity V_r. Using the Jacobian matrix and the covariance matrix provided by Gaia, we ensure accurate and reliable estimates of these uncertainties, crucial for precise astrophysical analyses.