Derivation of the Distance D

Given Equations

The position vector:

$$\vec{R} = \vec{R}_0 + D\,\hat{n}$$

where:

- $\vec{R}_0$: Vector from the Galactic Center (GC) to the Sun.
- D: Heliocentric distance to the source.
- $\hat{n}$: Unit vector from the Sun to the source.

The velocity vector:

$$\vec{V} = \vec{V}_0 + V_r\,\hat{n} + D\,\vec{\mu}$$

where:

- $\vec{V}_0$: Velocity of the Sun in the galactic rest frame.
- V_r: Radial velocity of the source relative to the Sun.
- $\vec{\mu}$: Proper motion of the source in the heliocentric frame (perpendicular to $\hat{n}$).

Assumption

Assuming the star is coming from the Galactic Center, we have:

$$\vec{V} \times \vec{R} = \vec{0}$$

which implies that $\vec{V}$ and $\vec{R}$ are parallel vectors.

Derivation

Starting with the cross product:

$$\vec{V} \times \vec{R} = \vec{0}$$

Substitute the expressions for $\vec{V}$ and $\vec{R}$:

$$\left(\vec{V}_0 + V_r\,\hat{n} + D\,\vec{\mu}\right) \times \left(\vec{R}_0 + D\,\hat{n}\right) = \vec{0}$$

Expand the cross product using the distributive property:

$$\begin{aligned}\vec{V}_0 \times \vec{R}_0 + \vec{V}_0 \times D\,\hat{n} + V_r\,\hat{n} \times \vec{R}_0 + V_r\,\hat{n} \times D\,\hat{n}\\ + D\,\vec{\mu} \times \vec{R}_0 + D\,\vec{\mu} \times D\,\hat{n} = \vec{0}\end{aligned}$$

Simplify each term:

$$\vec{V}_0 \times D\,\hat{n} = D\,(\vec{V}_0 \times \hat{n})$$
$$V_r\,\hat{n} \times \vec{R}_0 = -V_r\,(\vec{R}_0 \times \hat{n})$$
$$V_r\,\hat{n} \times D\,\hat{n} = V_r D\,(\hat{n} \times \hat{n}) = \vec{0}$$
$$D\,\vec{\mu} \times \vec{R}_0 = D\,(\vec{\mu} \times \vec{R}_0)$$
$$D\,\vec{\mu} \times D\,\hat{n} = D^2\,(\vec{\mu} \times \hat{n})$$

Substitute back into the expanded expression:

$$\vec{V}_0 \times \vec{R}_0 + D\,(\vec{V}_0 \times \hat{n}) - V_r\,(\vec{R}_0 \times \hat{n}) + D\,(\vec{\mu} \times \vec{R}_0) + D^2\,(\vec{\mu} \times \hat{n}) = \vec{0}$$

Taking the Dot Product with $\hat{n}$

Take the dot product of both sides with $\hat{n}$:

$$\hat{n} \cdot \Big(\vec{V}_0 \times \vec{R}_0 + D\,(\vec{V}_0 \times \hat{n}) - V_r\,(\vec{R}_0 \times \hat{n}) + D\,(\vec{\mu} \times \vec{R}_0) + D^2\,(\vec{\mu} \times \hat{n})\Big) = \hat{n} \cdot \vec{0}$$

Simplify each term using the scalar triple product identity $\hat{n} \cdot (\vec{A} \times \vec{B}) = (\hat{n} \times \vec{A}) \cdot \vec{B}$:

1. $\hat{n} \cdot (\vec{V}_0 \times \vec{R}_0) = (\hat{n} \times \vec{V}_0) \cdot \vec{R}_0$
2. $\hat{n} \cdot \Big(D\,(\vec{V}_0 \times \hat{n})\Big) = D\,\hat{n} \cdot (\vec{V}_0 \times \hat{n}) = 0$ (since $\vec{V}_0 \times \hat{n}$ is perpendicular to $\hat{n}$)
3. $\hat{n} \cdot \Big(-V_r\,(\vec{R}_0 \times \hat{n})\Big) = -V_r\,\hat{n} \cdot (\vec{R}_0 \times \hat{n}) = 0$
4. $\hat{n} \cdot \Big(D\,(\vec{\mu} \times \vec{R}_0)\Big) = D\,(\hat{n} \times \vec{\mu}) \cdot \vec{R}_0$
5. $\hat{n} \cdot \big(D^2\,(\vec{\mu} \times \hat{n})\big) = D^2\,\hat{n} \cdot (\vec{\mu} \times \hat{n}) = 0$

Therefore, the equation simplifies to:

$$(\hat{n} \times \vec{V}_0) \cdot \vec{R}_0 + D\,(\hat{n} \times \vec{\mu}) \cdot \vec{R}_0 = 0$$

Solving for D

Rewriting the equation:

$$(\hat{n} \times \vec{V}_0) \cdot \vec{R}_0 + D\,(\hat{n} \times \vec{\mu}) \cdot \vec{R}_0 = 0$$

Isolate D:

$$D\,(\hat{n} \times \vec{\mu}) \cdot \vec{R}_0 = -(\hat{n} \times \vec{V}_0) \cdot \vec{R}_0$$

Therefore, the expression for D is:

$$D = -\frac{(\hat{n} \times \vec{V}_0) \cdot \vec{R}_0}{(\hat{n} \times \vec{\mu}) \cdot \vec{R}_0}$$

If we rearange in order to eliminate the minus:

$$D = \frac{\hat{n} \cdot (\vec{R}_0 \times \vec{V}_0)}{\vec{R}_0 \cdot (\hat{n} \times \vec{\mu})}$$

Derivation of V_r

Starting from the equation:

$$\vec{V}_0 \times \vec{R}_0 + D\,(\vec{V}_0 \times \hat{n}) - V_r\,(\vec{R}_0 \times \hat{n}) + D\,(\vec{\mu} \times \vec{R}_0) + D^2\,(\vec{\mu} \times \hat{n}) = \vec{0}$$

Take the dot product with $\vec{\mu}$:

$$\vec{\mu} \cdot \Big(\vec{V}_0 \times \vec{R}_0 + D\,(\vec{V}_0 \times \hat{n}) - V_r\,(\vec{R}_0 \times \hat{n}) + D\,(\vec{\mu} \times \vec{R}_0) + D^2\,(\vec{\mu} \times \hat{n})\Big) = \vec{\mu} \cdot \vec{0}$$

Simplify each term:

1. **First Term:**
$$\vec{\mu} \cdot (\vec{V}_0 \times \vec{R}_0)$$
2. **Second Term:**
$$D\,\vec{\mu} \cdot (\vec{V}_0 \times \hat{n})$$
3. **Third Term:**
$$-V_r\,\vec{\mu} \cdot (\vec{R}_0 \times \hat{n})$$
4. **Fourth Term:**
$$D\,\vec{\mu} \cdot (\vec{\mu} \times \vec{R}_0) = 0$$
(Because $\vec{\mu} \cdot (\vec{\mu} \times \vec{R}_0) = 0$)
5. **Fifth Term:**
$$D^2\,\vec{\mu} \cdot (\vec{\mu} \times \hat{n}) = 0$$
(Because $\vec{\mu} \cdot (\vec{\mu} \times \hat{n}) = 0$)

Therefore, the equation simplifies to:

$$\vec{\mu} \cdot (\vec{V}_0 \times \vec{R}_0) + D\,\vec{\mu} \cdot (\vec{V}_0 \times \hat{n}) - V_r\,\vec{\mu} \cdot (\vec{R}_0 \times \hat{n}) = 0$$

Rewriting:

$$\vec{\mu} \cdot (\vec{V}_0 \times \vec{R}_0) + D\,\vec{\mu} \cdot (\vec{V}_0 \times \hat{n}) = V_r\,\vec{\mu} \cdot (\vec{R}_0 \times \hat{n})$$

Solving for V_r:

$$V_r = \frac{\vec{\mu} \cdot (\vec{V}_0 \times \vec{R}_0) + D\,\vec{\mu} \cdot (\vec{V}_0 \times \hat{n})}{\vec{\mu} \cdot (\vec{R}_0 \times \hat{n})}$$

Derivation of the Errors in D and V_r

Expressions for D and V_r

The expressions are:

$$D = -\frac{(\hat{n} \times \vec{V}_0) \cdot \vec{R}_0}{(\hat{n} \times \vec{\mu}) \cdot \vec{R}_0} \tag{1}$$

$$V_r = \frac{\vec{\mu} \cdot (\vec{V}_0 \times \vec{R}_0) + D\, \vec{\mu} \cdot (\vec{V}_0 \times \hat{n})}{\vec{\mu} \cdot (\vec{R}_0 \times \hat{n})} \tag{2}$$

Uncertainty in D

Let:

$$N = (\hat{n} \times \vec{V}_0) \cdot \vec{R}_0$$
$$D_{\text{den}} = (\hat{n} \times \vec{\mu}) \cdot \vec{R}_0$$

The uncertainty in D is:

$$(\delta D)^2 = \left(\frac{\delta N}{D_{\text{den}}}\right)^2 + \left(D \cdot \frac{\delta D_{\text{den}}}{D_{\text{den}}}\right)^2$$

Where:

$$(\delta N)^2 = \sum_i \left(\left((\hat{n} \times \vec{e}_i) \cdot \vec{R}_0\right) \delta V_{0i}\right)^2 + \sum_i \left(\left(\left[\hat{n} \times \vec{V}_0\right]_i\right) \delta R_{0i}\right)^2$$

$$(\delta D_{\text{den}})^2 = \sum_i \left(\left((\hat{n} \times \vec{e}_i) \cdot \vec{R}_0\right) \delta \mu_i\right)^2 + \sum_i \left(\left([\hat{n} \times \vec{\mu}]_i\right) \delta R_{0i}\right)^2$$

Uncertainty in V_r

Let:

$$N' = \vec{\mu} \cdot (\vec{V}_0 \times \vec{R}_0) + D\, \vec{\mu} \cdot (\vec{V}_0 \times \hat{n})$$
$$D' = \vec{\mu} \cdot (\vec{R}_0 \times \hat{n})$$

The uncertainty in V_r is:

$$(\delta V_r)^2 = \left(\frac{\delta N'}{D'}\right)^2 + \left(V_r \cdot \frac{\delta D'}{D'}\right)^2$$

Where:

$$(\delta N')^2 = \sum_i \left(\frac{\partial N'}{\partial \mu_i} \delta \mu_i\right)^2 + \sum_i \left(\frac{\partial N'}{\partial V_{0i}} \delta V_{0i}\right)^2 + \sum_i \left(\frac{\partial N'}{\partial R_{0i}} \delta R_{0i}\right)^2 + \left(\frac{\partial N'}{\partial D} \delta D\right)^2$$

$$(\delta D')^2 = \sum_i \left(\left(\vec{e}_i \cdot (\vec{R}_0 \times \hat{n})\right) \delta \mu_i\right)^2 + \sum_i \left(\left(\vec{\mu} \cdot (\vec{e}_i \times \hat{n})\right) \delta R_{0i}\right)^2$$

Notes

- $\vec{e}_i$ is the unit vector in the i-th coordinate direction. - The sums are over the vector components $i = x, y, z$. - The uncertainties δV_{0i}, δR_{0i}, $\delta \mu_i$, and δD are the standard deviations of the respective quantities.

Uncertainties for uncorrelated errors

Uncertainty in D

Given:

$$D = -\frac{N}{D_{\text{den}}}$$

where:

$$N = (\hat{n} \times \vec{V}_0) \cdot \vec{R}_0$$

$$D_{\text{den}} = (\hat{n} \times \vec{\mu}) \cdot \vec{R}_0$$

Assuming uncorrelated errors, the relative uncertainty in D is:

$$\left(\frac{\delta D}{D}\right)^2 = \left(\frac{\delta N}{N}\right)^2 + \left(\frac{\delta D_{\text{den}}}{D_{\text{den}}}\right)^2$$

Computing δN and δD_{den}: For δN:

$$(\delta N)^2 = \left[(\hat{n} \times \delta\vec{V}_0) \cdot \vec{R}_0\right]^2 + \left[(\hat{n} \times \vec{V}_0) \cdot \delta\vec{R}_0\right]^2$$

For δD_{den}:

$$(\delta D_{\text{den}})^2 = \left[(\hat{n} \times \delta\vec{\mu}) \cdot \vec{R}_0\right]^2 + \left[(\hat{n} \times \vec{\mu}) \cdot \delta\vec{R}_0\right]^2$$

Final Expression for δD:

$$\delta D = D\sqrt{\left(\frac{\delta N}{N}\right)^2 + \left(\frac{\delta D_{\text{den}}}{D_{\text{den}}}\right)^2}$$

Uncertainty in V_r

Given:

$$V_r = \frac{N'}{D'}$$

where:

$$N' = \vec{\mu} \cdot (\vec{V}_0 \times \vec{R}_0) + D\,\vec{\mu} \cdot (\vec{V}_0 \times \hat{n})$$

$$D' = \vec{\mu} \cdot (\vec{R}_0 \times \hat{n})$$

Assuming uncorrelated errors, the relative uncertainty in V_r is:

$$\left(\frac{\delta V_r}{V_r}\right)^2 = \left(\frac{\delta N'}{N'}\right)^2 + \left(\frac{\delta D'}{D'}\right)^2$$

Computing $\delta N'$ and $\delta D'$: For $\delta N'$:

$$\begin{aligned}(\delta N')^2 = &\left[\delta\vec{\mu}\cdot(\vec{V}_0\times\vec{R}_0 + D\,\vec{V}_0\times\hat{n})\right]^2\\ &+\left[\vec{\mu}\cdot(\delta\vec{V}_0\times\vec{R}_0 + D\,\delta\vec{V}_0\times\hat{n})\right]^2\\ &+\left[\vec{\mu}\cdot(\vec{V}_0\times\delta\vec{R}_0)\right]^2\\ &+\left[\vec{\mu}\cdot(\vec{V}_0\times\hat{n})\delta D\right]^2\end{aligned}$$

For $\delta D'$:

$$(\delta D')^2 = \left[\delta\vec{\mu}\cdot(\vec{R}_0\times\hat{n})\right]^2 + \left[\vec{\mu}\cdot(\delta\vec{R}_0\times\hat{n})\right]^2$$

Final Expression for δV_r:

$$\delta V_r = V_r\sqrt{\left(\frac{\delta N'}{N'}\right)^2 + \left(\frac{\delta D'}{D'}\right)^2}$$