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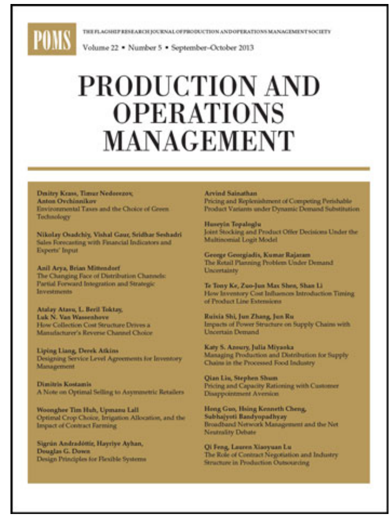

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Capacity Allocation with Multiple Suppliers and Multiple Demand Classes*

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We consider the capacity allocation problem for a retailer with multiple suppliers and multiple demand classes. The retailer offers one seasonal product and reserves capacity from multiple suppliers. Customers in different classes are charged with different selling prices for the same product. We analyze the optimal capacity allocation policies with the following three types of customers: (1) patient customers, (2) impatient customers, and (3) customers with limited patience. To empower our analysis, we propose a new preservation property of decomposition under a maximization operator. Based on the preservation property, we show that the value function in each period is decomposable for each type of customers. We then characterize the optimal capacity allocation policy for each type of customers and develop an efficient algorithm to obtain the respective optimal policy by exploiting decomposition. We also numerically investigate the optimal policy and show its value against a counterpart static heuristic policy. Finally, we extend our results to systems with multiple products, new capacity additions, etc.

Key words: multiple suppliers; multiple demand classes; nested protection level policy; class-specific protection level policy; customer waiting behavior

1. Introduction

Dynamically matching suppliers with customers is common in industry nowadays due to the fast development of e-commerce, online platforms, and supply chain networks. On the supply side, firms usually source from multiple suppliers with potentially different costs due to the considerations of supplier diversification and capacity constraints (Simchi-Levi et al. 2008). A firm may face with different ordering costs even from the same supplier. This happens when the supplier has both regular and emergent capacity (e.g., through overtime work) or has both long-term and short-term supply contracts with different prices (Cohen and Agrawal 1999). On the demand side, a firm may segment customers into different priority levels in order to charge personalized prices (Baker et al. 2001) or adopt target promotions (e.g., personalized catalogs in Simester et al. (2006)). Customer

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segmentation can be achieved by, e.g., clickstream tracking techniques that are commonly used by e-commerce firms. Customers with different priority levels may be associated with different selling prices and waiting costs (e.g., customers have heterogeneous sensitivities toward the delay of fulfillment). To maximize its profit in a dynamic matching environment, a firm has to allocate different classes of capacity to different segments of customers. This is challenging for firms in complex business environments.

Our paper is partially motivated by a cross-border e-commerce firm in China facing the above challenge. This firm fulfills customer demand with drop shipping, a system in which the firm does not hold inventory but offers suppliers' inventory for sale. It sources from two suppliers: a global supplier in Canada and a local supplier with warehouses in the foreign-trade zone in Southern China. The delivery times of the two suppliers are almost the same[1]. The local supplier offers a low unit usage cost (e.g., the unit procurement cost) but incurs a high unit holding cost because it is capacity-constrained. In contrast, the global supplier offers a high unit usage cost (e.g., the unit procurement cost plus the unit global shipping cost) but a low unit holding cost. The firm has both offline and online stores with possibly different selling prices as, e.g., some customers may receive coupons and/or personalized discounts while others may not. The firm also provides free shipping for most goods in its online store, but the shipping costs may be different for various destinations. Hence, even with the same nominal price, the actual price (i.e., the nominal price minus the shipping cost) varies for online customers. In fact, this practice — offering different prices for the same product to customers — is also adopted by other firms[2]. The firm has to make capacity allocation decisions for demand associated with different prices in each period in order to maximize its profit.

To address the challenge of dynamically matching capacity with multi-class demand, we consider the capacity allocation problem for a retailer offering one seasonal product. The retailer sources from multiple suppliers with different unit usage and holding costs. It reserves limited capacity from the suppliers before the selling season because the capacity reservation is costly and the suppliers may be capacity-constrained. The units from different suppliers are the same. The retailer can charge different prices for customers through personalized pricing. Moreover, customers may have different types of waiting behavior when their orders cannot be fulfilled immediately. To capture customers' rich variety of waiting behavior, we consider the following three types of customers: (1)

[1] The firm uses air freight if the goods are drop-shipped from the global supplier.

[2] As in Chu et al. (2008), a grocery retail chain in Spain has a high-low promotion policy and practices zone pricing for its offline stores.

patient customers who can wait for their orders to be fulfilled; (2) impatient customers who leave if their orders cannot be fulfilled immediately; and (3) customers with limited patience who can wait for a limited time period. The research question is how to dynamically allocate limited capacity from suppliers to different segments of customers under each of the above three types of customer waiting behavior.

Note that patient customers are typically assumed in the inventory management literature, e.g., the backordering setting. While impatient customers are common in the revenue management literature. Unlike traditional channels, in e-business, customers usually have limited patience. For example, online retailers usually communicate with customers via email, instant messenger, etc. Customers also frequently receive emails or notifications from large retailers, e.g., Amazon and Walmart, when there is a price reduction for a product. The product typically has been searched by those customers in their platforms. These customers usually have limited patience as they have updated information from multiple sellers.

However, how to manage customers with limited patience receives less attention in the existing literature. To capture customers with limited patience, we build the following customer behavioral model. By exploiting past transaction data, the retailer chooses personalized prices (e.g., by issuing discounts and coupons to individual customers) based on customers' valuations. The valuation declines as a customer is waiting because the waiting customer may search for alternative products or sellers. Once the valuation of the customer is smaller than the lowest price the retailer is willing to offer, the customer leaves.

We formulate the capacity allocation problem as an Markov decision process under each of the three types of customers. To empower our analysis, we establish a new preservation property of decomposition under a maximization operator. Exploiting the preservation property of decomposition, we find that the value function under each of the three types of customers is decomposable in each period. Thus, the multi-dimensional value function in each period can be decomposed as the sum of single-variable component functions.

By leveraging the decomposition of the value functions, we show that for either patient customers or customers with limited patience, the optimal capacity allocation policy can be described by a nested protection level (NPL) policy: There exists a nested protection level depending on the system state; it is optimal to reserve the total remaining capacity to the nested protection level, if it is feasible. For impatient customers, we show that a class-specific protection level (CSPL) policy is optimal: There exists a fixed protection level for each demand class; it is optimal to accept a customer if the total remaining capacity is larger than the corresponding protection level, and to reject the customer otherwise. Based on the decomposition property of the value functions, we develop efficient algorithms to obtain the optimal policies. Finally, we also numerically show the impacts of system parameters on the optimal policies and the value of the optimal policy against a simple heuristic policy based on the deterministic linear programming (DLP). We find that under our settings, the heuristic policy may not be effective.

Our main contributions are summarized below.

- We provide a modeling framework to the capacity allocation problems under three types of customers. We allow multiple demand classes and multiple suppliers.
- We show a new result on the preservation of decomposition under a maximization operator.
- We show that the optimal capacity allocation policy for each type of customers can be described by either the NPL policy or the CSPL policy. Note that the model in Topkis (1968) is a special case of our model. However, he does not show the decomposition of value functions.
- Based on the decomposition of value functions, we develop efficient algorithms to obtain the optimal policies.

The rest of our paper is organized as follows. We review the related literature in Section 2. In Section 3, we analyze the system with patient customers. In Section 4, we present our results for the other two types of customers. In Section 5, we present the numerical studies. In Section 6, we consider several extensions. Finally, we provide concluding remarks in Section 7. All proofs are relegated to the Appendix.

2. Related Literature

Our paper is related to the literature on inventory rationing. With inventory rationing, firms may delay the fulfillment of lower-priority demand and reserve capacity for higher-priority demand in the future. The objective is to maximize the expected total profit over multiple periods. The capacity reservation in revenue management is also a form of inventory rationing. There are two streams of literature on inventory rationing: inventory rationing with a myopic ordering policy or without ordering, and priority inventory models with ordering.

In the stream of literature on inventory rationing with a myopic ordering policy or without ordering, Veinott (1965) proposes a fixed rationing level policy and shows the optimality of the myopic ordering policy. Evans (1968) and Kaplan (1969) analyze the optimal rationing policies with two demand classes. In particular, Topkis (1968) shows that the state-independent rationing level policy is optimal for an inventory system with a single product, multiple demand classes under both the lost-sales and backordering settings when there is no ordering during the horizon.

Even with this result, it is unclear how to obtain the optimal rationing levels efficiently because he does not show the decomposition of value functions. Bao et al. (2018) show the decomposition property of the value functions for the Topkis's model. Unlike the Topkis's model, we assume that capacity can be reserved from multiple suppliers. We derive the decomposition property for the value functions in our problems. Note that the decomposition approach is first proposed by Clark and Scarf (1960). However, their method is not applicable to our settings as we have multiple demand classes.

Our paper is also related to the literature on periodic-review priority inventory models with ordering. Cohen et al. (1988) consider an (s, S)-type ordering policy and a strict priority rule of stock rationing for an inventory system with two demand classes and lost sales. They present effective approximate solutions for their model. Sobel and Zhang (2001) investigate a priority inventory system with two demand classes (deterministic demand and stochastic demand), and show that a modified (s, S) policy with a fixed cost is optimal. Frank et al. (2003) also explore a priority inventory system with two demand classes but assume that the deterministic demand must be satisfied immediately and that any unfulfilled stochastic demand is lost. They depict the structure of the optimal policy and provide a simple heuristic policy. Chen et al. (2010) consider an inventory system with a setup cost and two stochastic demand classes. They show that the state-dependent (s, S) policy is optimal for ordering and partially characterize the rationing structure. Ding et al. (2006) consider the joint pricing and demand allocation problem with multiple demand classes for a limited inventory. Zhou et al. (2011) consider an inventory system with limited production capacity and multiple demand classes. They demonstrate that a modified base stock policy is optimal for ordering and that a multi-level rationing policy is optimal for inventory allocation. Unlike the above papers, we consider the optimal capacity allocation decisions with multiple suppliers for different types of customers and characterize the optimal policies based on the decomposition of value functions. We also develop efficient algorithms to obtain the optimal policies.

Finally, our paper is related to the revenue management literature, in particular the network revenue management (NRM) literature. In this literature, the focus is to dynamically maximize the revenue with multiple demand classes and multiple products. One stream of the NRM literature, e.g., Curry (1990), Wong et al. (1993), Gallego and van Ryzin (1997), Bertsimas and Boer (2005), and van Ryzin and Vulcano (2008), assumes that demand for each product is a stochastic process that is unaffected by the availability of other products. Cooper and Homem-de Mello (2007) provide an MDP formulation for a network revenue management problem. They consider a "time-decomposition approach" to approximate the optimal reservation policy. Another stream of the NRM literature incorporates customer choice models, see Zhang and Cooper (2005), Liu and van Ryzin (2008), Zhang and Adelman (2009), and Zhang (2011). Typically, it is challenging to solve the NRM problems due to the high dimensionality. As a result, the researchers resort to effective heuristics. The main heuristics include deterministic linear programming (DLP) (Talluri and van Ryzin 1998, Cooper 2002) and approximate dynamic programming methods (Bertsimas and Popescu 2003, Talluri and van Ryzin 2004, Zhang 2011). For more details on revenue management, see Bitran and Caldentey (2003) and Talluri and van Ryzin (2004) for surveys of the revenue management literature.

Consistent with the revenue management literature, in our models we allow the retailer to order only once before the planning horizon. Although we consider a single product, we allow multiple suppliers with heterogeneous costs. Compared with the NRM literature, by leveraging decomposition, we are able to fully characterize the structure of the optimal policy with multiple demand classes for each type of customer waiting behavior. We also develop efficient algorithms to obtain the optimal policies.

3. Model and Results for Patient Customers

In this section, we consider patient customers who can wait for their orders to be fulfilled. This type of customers includes those who sign supply contracts with their sellers or customers with online orders. We first present the model and the Markov Decision Process (MDP) formulation of our problem, and then provide the analytical results.

3.1. Model and Formulation

We consider that a retailer sells a seasonal product to customers by reserving units from m suppliers under a finite planning horizon with T periods. The units from different suppliers are the same. Following the literature on revenue management, the retailer reserves the capacity $c_{j,1}$ from supplier j, $j=1,\cdots,m$, before the planning horizon. The retailer is not allowed to reserve any more capacity during the rest of the horizon. That is, the retailer can only use the reserved capacity to fulfill demand during the planning horizon. For the available capacity from supplier j, there is a unit holding cost h_j per period and a unit usage cost u_j if a unit is delivered. We refer to u_j-h_j as the marginal usage cost of supplier j because using a unit of the capacity from supplier j incurs a usage cost u_j but also reduces a holding cost h_j. Without loss of generality, we assume that the marginal usage costs satisfy the following property: $u_1-h_1\le u_2-h_2\le\cdots\le u_m-h_m$. In the terminal period $T+1$, we assume that the leftover units have no salvage value.

Though the units reserved from different suppliers are the same to customers, the selling prices for different customers can be different (due to, e.g., personalized pricing). We thus segment demand into n classes based on the selling prices. Let q_i be the selling price for class i demand, $i=1,\cdots,n$. For patient customers, any unfulfilled demand is backlogged, incurring a unit waiting cost per period. Let b_i be the unit waiting cost of demand class i, $i=1,\cdots,n$. We refer to q_i+b_i as the marginal revenue of demand class i because fulfilling a unit of demand class i earns q_i and also reduces a waiting cost b_i. Without loss of generality, we assume that the marginal revenues satisfy the following property: $q_1+b_1 \le q_2+b_2 \le \cdots \le q_n+b_n$.

We assume that the duration of each time period is infinitesimal such that at most one unit of demand is realized in period t, $t=1,\cdots,T$. This setting is commonly adopted in the literature of capacity management (Cooper and Homem-de-Mello, 2007). Let λ_t be the probability that there is one unit of demand in period t, and $p_{i,t}$ be the probability that the demand is of class i for $i=1,\cdots,n$. Then, $1-\lambda_t$ is the probability that no demand is realized in period t and $\sum_{i=1}^n p_{i,t}=1$ must hold. Let $\lambda_{i,t}=\lambda_t p_{i,t}$ be the probability that there is one unit of class i demand and we must have $\sum_{i=1}^n \lambda_{i,t}<1$. Let $D_{i,t}\in\{0,1\}$ be the indicator random variable for class i demand realization in period t. Then, $\Pr(D_{i,t}=1)=\lambda_{i,t}$, $\Pr(D_{i,t}=0)=1-\lambda_{i,t}$, and $\Pr(D_{i_1,t}=1, D_{i_2,t}=1)=0$ for any $i_1\neq i_2$ because at most one unit of demand can be realized in each period.

In each period, the retailer observes the demand realization and then makes the capacity allocation decision. Our objective is to maximize the expected profit of the retailer during the entire planning horizon by optimally allocating the capacity reserved from different suppliers to fulfill the demand of different classes in each period. Let $w_{i,t}$ be the quantity of the backorders of demand class i and $\mathbf{w}_t=(w_{i,t})_{i=1,\cdots,n}$, $c_{j,t}$ be the quantity of the remaining capacity reserved from supplier j and $\mathbf{c}_t=(c_{j,t})_{j=1,\cdots,m}$. Also let $a_{ij,t}$ be the quantity of the class i demand that is fulfilled by the capacity reserved from supplier j and $\mathbf{a}_t=(a_{ij,t})_{i=1,\cdots,n;\,j=1,\cdots,m}$. Define $\mathbf{e}_r$ as the unit vector with the r-th element being 1 and all the others being 0.

The MDP formulation for patient customers is then provided as follows:

$$\bar{v}_t(\mathbf{c}_t,\mathbf{w}_t)=\sum_{r=1}^n \lambda_{r,t}\bar{g}_t(\mathbf{c}_t,\mathbf{w}_t+\mathbf{e}_r)+(1-\lambda_t)\bar{g}_t(\mathbf{c}_t,\mathbf{w}_t), \tag{1}$$

where

$$\begin{aligned}\bar{g}_t(\mathbf{c}_t,\mathbf{w}_t)=\max_{\mathbf{a}_t\in\mathcal{B}(\mathbf{c}_t,\mathbf{w}_t)}\Bigg[&\bar{v}_{t+1}\left(\mathbf{c}_t-\sum_{j=1}^m\sum_{i=1}^n a_{ij,t}\mathbf{e}_j,\mathbf{w}_t-\sum_{j=1}^m\sum_{i=1}^n a_{ij,t}\mathbf{e}_i\right)+\sum_{j=1}^m\sum_{i=1}^n a_{ij,t}(q_i-u_j)\\&-\sum_{j=1}^m\left(c_{j,t}-\sum_{i=1}^n a_{ij,t}\right)h_j-\sum_{i=1}^n b_i\left(w_{i,t}-\sum_{j=1}^m a_{ij,t}\right)\Bigg],\end{aligned} \tag{2}$$

and $\mathcal{B}(\mathbf{c}_t,\mathbf{w}_t)=\{a_{ij,t}\ge 0, \sum_{j=1}^m a_{ij,t}\le w_{i,t}, \sum_{i=1}^n a_{ij,t}\le c_{j,t}, j=1,\cdots,m, i=1,\cdots,n\}$. The value function $\bar{v}_t(\mathbf{c}_t,\mathbf{w}_t)$ in Eq. (1) is the expected maximum profit from period t and onward. After demand realization, the retailer makes the optimal capacity allocation decision as in Eq. (2). The constraints in $\mathcal{B}(\mathbf{c}_t, \mathbf{w}_t)$ imply that in each period, the total quantity of the capacity allocated to the demand class i should be no more than $w_{i,t}$ and the total quantity of the allocated capacity from supplier j should not exceed $c_{j,t}$. In the terminal period, we let $\bar{v}_{T+1}(\mathbf{c}_{T+1}, \mathbf{w}_{T+1}) \equiv 0$ for any $(\mathbf{c}_{T+1}, \mathbf{w}_{T+1})$, i.e., there is no salvage value for leftover capacity.

The MDP above is multi-dimensional and has $m \times n$ decision variables, which is difficult to analyze and compute. In the following, we aim to simplify the MDP based on a transformation of the state variables.

Notice that the units from different suppliers are the same from customers' perspective and each unit of demand requires one unit of capacity to be fulfilled. In addition, the marginal usage costs have the following sequential property $u_1 - h_1 \leq \cdots \leq u_m - h_m$. Intuitively, this implies that it is optimal for the firm to use a unit from a supplier with a smaller marginal usage cost first. Similarly, because the sequential property $q_1 + b_1 \leq \cdots \leq q_n + b_n$, intuitively it is optimal for the firm to allocate a unit of capacity to a customer with a higher marginal revenue first. These intuitions lead to the following results.

Lemma 1. *With patient customers, it is optimal to allocate the capacity from suppliers with smaller marginal usage costs first and fulfill demand classes with larger marginal revenues first in each period.*

Lemma 1 indicates that it is optimal to allocate the capacity from supplier j only if the capacity reserved from suppliers $1, \cdots, j-1$ is depleted. Similarly, it is optimal to fulfill the class i demand only if the demand of classes $i+1, \cdots, n$ is fully fulfilled.

Based on Lemma 1, we propose the following state transformation. Let $\mathbf{z}_{c,t} = (z_{j,t})_{j=1,\cdots,m}$ and $\mathbf{z}_{w,t} = (z_{m+i,t})_{i=1,\cdots,n}$ such that

$$\begin{cases} z_{j,t} = \sum_{k=j}^{m} c_{k,t}, & j = 1, \cdots, m, \\ z_{m+i,t} = z_{1,t} - \sum_{k=i}^{n} w_{k,t}, & i = 1, \cdots, n. \end{cases} \tag{3}$$

We refer to $z_{j,t}$ as the total capacity reserved from suppliers $j, \cdots, m$ at the beginning of period t and $z_{m+i,t}$ as the total capacity reserved from m suppliers minus the total demand of classes $i, \cdots, n$ at the beginning of period t, for $i \in \{1, \cdots, n\}$, $j \in \{1, \cdots, m\}$. Accordingly, $\mathbf{z}_{c,t}$ and $\mathbf{z}_{w,t}$ are referred to as the *echelon capacity state* and the *echelon demand state*, respectively.

We adopt the state transformation in Eq. (3) so that if we allocate the capacity up to supplier j and fulfill demand up to class i in period t, then $z_{1,t+1} = \cdots = z_{j,t+1} = z_{m+n,t+1} = \cdots = z_{m+i+1,t+1}$. Let $\bar{z}_{m+i,t} = z_{m+i,t} - \sum_{k=i}^{n} D_{k,t}$. Then, $\bar{\mathbf{z}}_{w,t} = (\bar{z}_{m+i,t})_{i=1,\cdots,n}$ is the echelon demand state after demand realization in each period t, i.e., $\bar{\mathbf{z}}_{w,t} = \mathbf{z}_{w,t}$ if no demand is realized and $\bar{\mathbf{z}}_{w,t} = z_{w,t} - \mathbf{e}_{[1,i]}$ if a unit of class i demand is realized, where $\mathbf{e}_{[k,r]} = \mathbf{e}_k + \mathbf{e}_{k+1} + \cdots + \mathbf{e}_r$ for integers $r \geq k$. Under this definition of the state variables, we must have $z_{m,t} \leq \cdots \leq z_{1,t}$, $z_{m+1,t} \leq \cdots \leq z_{m+n,t}$, $\bar{z}_{m+1,t} \leq \cdots \leq \bar{z}_{m+n,t}$ and $z_{1,t} \geq z_{m+n,t} \geq \bar{z}_{m+n,t}$.

Notice that in period t the system states are $(\mathbf{z}_{c,t}, \mathbf{z}_{w,t})$ and $(\mathbf{z}_{c,t}, \bar{\mathbf{z}}_{w,t})$ before and after demand realization, respectively. In fact, the system state can be further simplified as follows. Define $\Theta = (\theta_j)_{j=1,\cdots,m}$ for $\theta_j \equiv \sum_{k=j}^{m} c_{k,1}$ and $\theta_{m+1} \equiv 0$. Each θ_j, $j = 1, \cdots, m$, is a constant which is independent of demand realizations and allocations in different periods (note that the reserved capacity $c_{j,1}$ from supplier j is a fixed value). Then, based on the priority properties in Lemma 1 and the definitions of $\mathbf{z}_{c,t}$, $\mathbf{z}_{w,t}$ and Θ, we have the following lemma. Note that we define $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$, and $x \wedge \mathbf{y} = (x \wedge y_1, \cdots, x \wedge y_n)$ for $\mathbf{y} = (y_1, \cdots, y_n)$.

LEMMA 2. *Let z_t be the total remaining capacity of m suppliers at the beginning of period t. Then, for $t = 1, \cdots, T$, we have*

(1) $\mathbf{z}_{c,t} = z_t \wedge \Theta = (z_t \wedge \theta_1, \cdots, z_t \wedge \theta_m)$, and $z_t \in [0, \theta_1]$. Moreover, $\bar{z}_{m+i,t} \leq z_{m+i,t} \leq z_t$ for any $i = 1, \cdots, n$.

(2) $\mathbf{z}_{w,t+1} = z_{t+1} \wedge \bar{\mathbf{z}}_{w,t} = (z_{t+1} \wedge \bar{z}_{m+1,t}, \cdots, z_{t+1} \wedge \bar{z}_{m+n,t})$.

The results in Lemma 2 can be explained as follows. By Lemma 1, if $\theta_{j+1} \leq z_t \leq \theta_j$, the reserved capacity from suppliers $1, \cdots, j-1$ (resp., $j+1, \cdots, m$) must be depleted (resp., have not been allocated yet) and the reserved capacity from supplier j is allocated before period t, i.e., $z_t \wedge \theta_k = z_t$ for $k = 1, \cdots, j$ and $z_t \wedge \theta_k = \theta_k$ for $k = j+1, \cdots, m$. Hence, we can use $z_t \wedge \Theta$ to represent the state $\mathbf{z}_{c,t}$. Similarly, if $\bar{z}_{m+i,t} \leq z_{t+1} \leq \bar{z}_{m+i+1,t}$, then Lemma 1 implies that in period t no demand of classes $1, \cdots, i-1$ is fulfilled, some demand of class i is fulfilled, and demand of classes $i+1, \cdots, n$ is fully fulfilled. Therefore, from period t to period $t+1$, the echelon demand state is updated as $\mathbf{z}_{w,t+1} = z_{t+1} \wedge \bar{z}_{w,t}$.

Based on the above analysis, we then provide a simplified MDP of our problem for patient customers as follows. For each period t, $t = 1, \cdots, T$,

$$v_t(z_t \wedge \Theta, \mathbf{z}_{w,t}) = \sum_{r=1}^{n} \lambda_{r,t} g_t(z_t \wedge \Theta, \mathbf{z}_{w,t} - \mathbf{e}_{[1,r]}) + (1 - \lambda_t) g_t(z_t \wedge \Theta, \mathbf{z}_{w,t}), \tag{4}$$

where

$$\begin{aligned} g_t(z_t \wedge \Theta, \bar{\mathbf{z}}_{w,t}) = \max_{0 \vee \bar{z}_{m+1,t} \leq z_{t+1} \leq z_t} \Bigg[& v_{t+1}\left(z_{t+1} \wedge \Theta, z_{t+1} \wedge \bar{\mathbf{z}}_{w,t}\right) - \sum_{j=1}^{m} u_j (z_t \wedge \theta_j - z_t \wedge \theta_{j+1}) \\ & + \sum_{j=1}^{m} (u_j - h_j)(z_{t+1} \wedge \theta_j - z_{t+1} \wedge \theta_{j+1}) + \sum_{i=1}^{n} q_i (\bar{z}_{m+i+1,t} - (\bar{z}_{m+i,t} \vee z_{t+1}) \wedge \bar{z}_{m+i+1,t}) \end{aligned}$$

$$-\sum_{i=1}^{n} b_i((\bar{z}_{m+i,t} \vee z_{t+1}) \wedge \bar{z}_{m+i+1,t} - \bar{z}_{m+i,t})\Bigg]. \tag{5}$$

The terminal condition is that $v_{T+1}(z_{T+1} \wedge \Theta, \mathbf{z}_{w,T+1}) \equiv 0$ for any $(z_{T+1}, \mathbf{z}_{w,T+1})$ as we assume zero salvage value. We explain the equivalence between the MDP in Eq. (4) - (5) and the MDP in Eq. (1) - (2) in the Appendix.

In this simplified MDP, there is only one decision variable z_{t+1}. Once z_{t+1} is determined, we know exactly the echelon capacity state $\mathbf{z}_{c,t+1} = z_{t+1} \wedge \Theta$ and the echelon demand state $\mathbf{z}_{w,t+1} = z_{t+1} \wedge \bar{\mathbf{z}}_{w,t}$ at the beginning of period $t+1$.

3.2. Analytical Results

In this section, we first introduce a new preservation property of decomposition under a maximization operator to facilitate the subsequent analysis. Then, we analyze the MDP in Eq. (4) - Eq. (5) and characterize the optimal capacity allocation policy for patient customers by exploiting the decomposition.

3.2.1. Preliminary Results We first introduce the definitions of *discrete concavity* and *decomposition* as follows.

Definition 1. A function $f: \mathbb{Z} \to \mathbb{R}$ is discrete concave if, $\forall x \in \mathbb{Z}$, $f(x+2) - f(x+1) \le f(x+1) - f(x)$.

Definition 2. A function $f: \mathbb{Z}^m \to \mathbb{R}$ is decomposable if $f(\mathbf{x}) = \sum_{j=1}^{m} f_j(x_j)$ for $f_j: \mathbb{Z} \to \mathbb{R}$, $j = 1, \cdots, m$.

Decomposition is a functional property proposed by Clark and Scarf (1960). If a multivariate function is decomposable, then the computational complexity of solving the multivariate optimization problem can be reduced significantly because each variable can be optimizated independently.

We then show that under a maximization operator, the decomposition and the discrete concavity can be preserved. Denote $\mathbb{I}_{\{A\}}$ as the indicator function such that $\mathbb{I}_{\{A\}} = 1$ if the condition A holds and 0 otherwise, and define $\sum_a^b x = 0$ for any $a > b$.

Lemma 3. *Suppose that* $\mathbf{y} = (y_1, \cdots, y_n)$ *is an integer vector such that* $y_0 < y_1 \le \cdots \le y_n \le y_{n+1}$ *and* $y_{n+1} \ge 0$. *Consider the maximization problem* $g(y_0, y_{n+1}, \mathbf{y}) = \max_{0 \vee y_0 \le x \le y_{n+1}} F(x, \mathbf{y})$, *where* $F(x, \mathbf{y}) = f(x) + \sum_{j=1}^{n} f_j(x \wedge y_j)$, $f(\cdot), f_1(\cdot), \cdots, f_n(\cdot): \mathbb{Z} \to \mathbb{R}$ *are discrete concave and* $f_j(\cdot)$*'s for* $j = 1, \cdots, n$ *are nondecreasing. Define* $\mathbf{s} = (s_0, \cdots, s_n)$ *where*

$$s_j = \min \arg\max_{x \ge 0} \left(f(x) + \sum_{k=j+1}^{n} f_k(x) \right), \quad j = 0, \cdots, n. \tag{6}$$

Then, we have the following results:

(1) $s_0 \geq \cdots \geq s_n \geq 0$ *and the optimal solution of* x*, denoted by* x^**, is* $x^* = (y_0 \vee S(\mathbf{y}|\mathbf{s})) \wedge y_{n+1}$ *where* $S(\mathbf{y}|\mathbf{s}) = \mathbb{I}_{\{s_0<y_1\}}s_0 + \sum_{j=1}^{n-1}\mathbb{I}_{\{y_j\leq s_j<y_{j+1}\}}s_j + \mathbb{I}_{\{s_n\geq y_n\}}s_n + \sum_{j=1}^{n}\mathbb{I}_{\{s_j<y_j\leq s_{j-1}\}}y_j$.

(2) The function $g(y_0, y_{n+1}, \mathbf{y})$ *is discrete concave and decomposable, i.e.,* $g(y_0, y_{n+1}, \mathbf{y}) = \sum_{j=0}^{n+1} g_j(y_j)$*, where each component function* $g_j(y_j)$ *is discrete concave and*

$$g_j(y_j) = \begin{cases} f(y_0) + \sum_{k=1}^{n} f_k(y_0), & j = 0, \\ f(s_{j-1}\wedge y_j) + \sum_{k=j}^{n} f_k(s_{j-1}\wedge y_j) - f(s_j \wedge y_j) - \sum_{k=j+1}^{n} f_k(s_j\wedge y_j), & j = 1, \cdots, n, \\ f(s_n \wedge y_{n+1}) & j = n+1. \end{cases}$$

Lemma 3 (1) presents the optimal solution to the maximization problem $\max_{0\vee y_0\leq x\leq y_{n+1}} F(x, \mathbf{y})$, which is a function of $(y_0, y_{n+1}, \mathbf{y})$. As $s_0 \geq \cdots \geq s_n$ while $y_1 \leq \cdots \leq y_n$, there is one and only one indicator function equal to 1 in the expression of $S(\mathbf{y}|\mathbf{s})$, which is explicitly determined by $\mathbf{s}$ and $\mathbf{y}$. Each s_j, $j = 0, \cdots, n$, is the global maximum of $f(x) + \sum_{k=j+1}^{n} f_k(x)$ per Eq. (6). Lemma 3 (2) indicates that under the maximization operator, the decomposition can be preserved. It explicitly shows how to update the component functions of the post-optimization function $g(y_0, y_{n+1}, \mathbf{y})$ based on the component functions of the objective function $F(x, \mathbf{y})$.

Lemma 3 provides a new way to show the decomposition of value functions for dynamic optimization problems through backward induction. Suppose that the value function in period $t+1$ is decomposable. Based on Lemma 3, the value function in period t is also decomposable and we can obtain the optimal solution as in Lemma 3 (1). In the subsequent analysis, we adopt this backward inductive approach to tackle our problems.

Note that the maximization operator in Lemma 3 is different from the counterpart in Clark and Scarf (1960) because we have the minimum operator "$\wedge$" between x and y_j for $j = 1, \cdots, n$. To the best of our knowledge, Lemma 3 is a new result in the existing literature.

3.2.2. Results In this section, we analyze the MDP in Eq. (4) - (5). We first show that the value function $v_t(z_t \wedge \Theta, \mathbf{z}_{w,t})$ in Eq. (4) has some monotone properties due to Lemma 1. These monotone properties are important in deriving our main results.

Lemma 4. *$v_t(z_t \wedge \Theta, \mathbf{z}_{w,t}) + (z_t \wedge \theta_j)[(u_j - h_j) - (u_{j-1} - h_{j-1})]$ is nondecreasing in $z_t \wedge \theta_j$ for $j = 2, \cdots, m$; and $v_t(z_t \wedge \Theta, \mathbf{z}_{w,t}) + z_{m+i,t}[(q_i + b_i) - (q_{i-1} + b_{i-1})]$ is nondecreasing in $z_{m+i,t}$ for $i = 2, \cdots, n$.*

We then characterize the optimal capacity allocation policy for the system with patient customers and show the discrete concavity and the decomposition of the function $v_t(z_t \wedge \Theta, \mathbf{z}_{w,t})$ based on Lemmas 2, 3 and 4 as follows.

Theorem 1. *For patient customers, in period t, $t = 1, \cdots, T$, we have the following results:*

(1) The function $v_t(z_t\wedge\Theta,\mathbf{z}_{w,t})$ *in Eq. (4) is discrete concave in* $(z_t\wedge\Theta,\mathbf{z}_{w,t})$*, and decomposable, i.e., there exist discrete concave functions* $\bar{v}_{i,t}(\cdot)$*'s and* $\hat{v}_{j,t}(\cdot)$*'s for* $i=1,\cdots,n$ *and* $j=1,\cdots,m$ *such that* $v_t(z_t\wedge\Theta,\mathbf{z}_{w,t})=\sum_{j=1}^m\hat{v}_{j,t}(z_t\wedge\theta_j)+\sum_{i=1}^n\bar{v}_{i,t}(z_{m+i,t})$.

(2) A nested protection level (NPL) policy is optimal: There exists a nested protection level $\mathcal{R}_t(\bar{\mathbf{z}}_{w,t}|\mathbf{R}_t)$*, which is defined in Eq. (7); the optimal solution of* z_{t+1} *in Eq. (5), denoted by* z_{t+1}^**, is* $z_{t+1}^*=(\bar{z}_{m+1,t}\vee\mathcal{R}_t(\bar{\mathbf{z}}_{w,t}|\mathbf{R}_t))\wedge z_t$*, i.e., it is optimal to sequentially fulfill demand of classes* $n,n-1,\cdots,1$ *whenever the total remaining capacity of all suppliers is higher than the nested protection level* $\mathcal{R}_t(\bar{\mathbf{z}}_{w,t}|\mathbf{R}_t)$.

(3) The nested protection level is defined as

$$\begin{aligned}\mathcal{R}_t(\bar{\mathbf{z}}_{w,t}|\mathbf{R}_t)=&\mathbb{I}_{\{R_{1,t}<\bar{z}_{m+2,t}\}}R_{1,t}+\sum_{i=2}^{n-1}\mathbb{I}_{\{\bar{z}_{m+i,t}\le R_{i,t}<\bar{z}_{m+i+1,t}\}}R_{i,t}+\mathbb{I}_{\{\bar{z}_{m+n,t}\le R_{n,t}\}}R_{n,t}\\&+\sum_{i=1}^{n-1}\mathbb{I}_{\{R_{i+1,t}<\bar{z}_{m+i+1,t}\le R_{i,t}\}}\bar{z}_{m+i+1,t},\end{aligned}\tag{7}$$

where $\mathbf{R}_t=(R_{1,t},\cdots,R_{n,t})$ *is a vector of constants such that* $R_{1,t}\ge\cdots\ge R_{n,t}\ge 0$*, and*

$$R_{i,t}=\min\arg\max_{z\ge 0}\left(f_t(z)+\sum_{k=i+1}^n\hat{f}_{k,t}(z)\right),\quad i=1,\cdots,n,\quad z\in\mathbb{Z},\tag{8}$$

where

$$\begin{cases}f_t(z)=\sum_{j=1}^m\hat{v}_{j,t+1}(z\wedge\theta_j)+(u_1-h_1-q_n-b_n)z+\sum_{j=2}^m(u_j-h_j-u_{j-1}+h_{j-1})(z\wedge\theta_j),\\ \hat{f}_{i,t}(z)=\bar{v}_{i,t+1}(z)+(q_i+b_i-q_{i-1}-b_{i-1})z,\quad i=1,\cdots,n.\end{cases}$$

There is one and only one indicator function equal to 1 in the expression of $\mathcal{R}_t(\bar{\mathbf{z}}_{w,t}|\mathbf{R}_t)$ as $R_{1,t}\ge\cdots\ge R_{n,t}$ while $\bar{z}_{m+1,t}\le\cdots\le\bar{z}_{m+n,t}$. The NPL policy indicates that we sequentially allocate $z_t-\mathcal{R}_t(\bar{\mathbf{z}}_{w,t}|\mathbf{R}_t)$ (resp., $z_t-\bar{z}_{m+1,t}$) units of reserved capacity from suppliers $1,\cdots,m$ to sequentially fulfill demand of classes $n,n-1,\cdots,1$ when $\mathcal{R}_t(\bar{\mathbf{z}}_{w,t}|\mathbf{R}_t)\ge\bar{z}_{m+1,t}$ (resp., $\mathcal{R}_t(\bar{\mathbf{z}}_{w,t}|\mathbf{R}_t)<\bar{z}_{m+1,t}$).

The nested protection level $\mathcal{R}_t(\bar{\mathbf{z}}_{w,t}|\mathbf{R}_t)$ is a function of the echelon demand state $\bar{\mathbf{z}}_{w,t}$. Given $\bar{\mathbf{z}}_{w,t}$, $\mathcal{R}_t(\bar{\mathbf{z}}_{w,t}|\mathbf{R}_t)$ can be directly determined as long as we know $\mathbf{R}_t$. Due to the decomposition of the function $v_t(z_t\wedge\Theta,\mathbf{z}_{w,t})$ in each period, we can develop an efficient algorithm, i.e., Algorithm 1, to calculate these state-independent constants. In Algorithm 1, we provide the procedure on how to sequentially update the component functions of $v_t(z_t\wedge\Theta,\mathbf{z}_{w,t})$ and to calculate $R_{i,t}$'s for $i=1,\cdots,n$ based on these component functions in each period. See the proof of Theorem 1 for more details.

Algorithm 1 is developed based on the decomposition of the function $v_t(z_t\wedge\Theta,\mathbf{z}_{w,t})$ in each period. The computational complexity of this algorithm is $\mathcal{O}((m+n)T)$ because in each period we only need to update the $m+n$ component functions of $v_t(z_t\wedge\Theta,\mathbf{z}_{w,t})$ based on the component

Algorithm 1 The decomposition-based backward procedure.

In the terminal period, $v_{T+1}(z_{T+1}\wedge\Theta,\mathbf{z}_{w,T+1})=\sum_{j=1}^{m}\hat{v}_{j+1,T+1}(z_{T+1}\wedge\theta_j)+\sum_{i=1}^{n}\bar{v}_{i,T+1}(z_{m+i,T+1})$ with $\hat{v}_{j+1,T+1}(z_{T+1}\wedge\theta_j)=0$ for $j=1,\cdots,m$ and $\bar{v}_{i,T+1}(z_{m+i,T+1})=0$ for $i=1,\cdots,n$;

for $t=T,T-1,\cdots,1$ **do**

for $i=1,\cdots,n$ **do**

Calculate $R_{i,t}$ as in Eq. (8) with $v_{t+1}(z_{t+1}\wedge\Theta,\mathbf{z}_{w,t+1})=\sum_{j=1}^{m}\hat{v}_{j+1,t+1}(z_{t+1}\wedge\theta_j)+\sum_{i=1}^{n}\bar{v}_{i,t+1}(z_{m+i,t+1})$;

end for

The function $v_t(z_t\wedge\Theta,\mathbf{z}_{w,t})=\sum_{j=1}^{m}\hat{v}_{j,t}(z_t\wedge\theta_j)+\sum_{i=1}^{n}\bar{v}_{i,t}(\bar{z}_{m+i,t})$, where

$$\begin{cases}\hat{v}_{j,t}(z)=\hat{g}_{j,t}(z), & j=1,\cdots,m,\\ \bar{v}_{i,t}(z)=(1-\lambda_t)\bar{g}_{i,t}(z)+\sum_{r=i}^{n}\lambda_{r,t}\bar{g}_{i,t}(z-1)+\sum_{r=2}^{i}\lambda_{r-1,t}\bar{g}_{i,t}(z), & i=1,\cdots,n,\end{cases}$$

and

$$\begin{cases}\hat{g}_{1,t}(z) &= (q_n-u_1)z+\hat{v}_{1,t+1}(R_{n,t}\wedge z)+(u_1-h_1-q_n-b_n)(R_{n,t}\wedge z),\\ \hat{g}_{j,t}(z) &= -(u_j-u_{j-1})z+\hat{v}_{j,t+1}(R_{n,t}\wedge z)+(u_j-h_j-u_{j-1}+h_{j-1})(R_{n,t}\wedge z),\quad j=2,\cdots,m,\\ \bar{g}_{1,t}(z) &= \sum_{i=1}^{n}\bar{v}_{i,t+1}(z)+\hat{v}_{1,t+1}(z)+(u_1-h_1-q_1)z\\ &\quad+\sum_{j=2}^{m}\left[\hat{v}_{j,t+1}(z\wedge\theta_j)+(u_j-h_j-u_{j-1}+h_{j-1})(z\wedge\theta_j)\right],\\ \bar{g}_{i,t}(z) &= (q_{i-1}-q_i)z+[\hat{v}_{1,t+1}(R_{i-1,t}\wedge z)+(u_1-h_1-q_{i-1}-b_{i-1})(R_{i-1,t}\wedge z)]-[\hat{v}_{1,t+1}(R_{i,t}\wedge z)\\ &\quad+(u_1-h_1-q_i-b_i)(R_{i,t}\wedge z)]+\sum_{j=2}^{m}[\hat{v}_{j,t+1}(R_{i-1,t}\wedge z\wedge\theta_j)-\hat{v}_{j,t+1}(R_{i,t}\wedge z\wedge\theta_j)\\ &\quad+(u_j-h_j-u_{j-1}+h_{j-1})(R_{i-1,t}\wedge z\wedge\theta_j-R_{i,t}\wedge z\wedge\theta_j)]+\sum_{k=i}^{n}\bar{v}_{k,t+1}(R_{i-1,t}\wedge z)\\ &\quad-\sum_{k=i+1}^{n}\bar{v}_{k,t+1}(R_{i,t}\wedge z),\quad i=2,\cdots,n.\end{cases}$$

end for

functions of $v_{t+1}(z_{t+1}\wedge\Theta,\mathbf{z}_{w,t+1})$. However, without the decomposition property, the computational complexity grows exponentially in $m+n$ as in general we have to compute the $(m+n)$-dimensional functions to make optimal capacity allocation decisions.

4. Analysis for the Other Types of Customers

In this section, we consider the systems with impatient customers, and customers with limited patience, respectively. We analyze each of these systems based on a similar approach as that in Section 3. Notice that the model settings in this section are similar to the counterpart in Section 3 except the customer waiting behavior. Thus, in this section, we only illustrate the main idea on how to apply the approach in Section 3 to these two systems, respectively.

4.1. Impatient Customers

For the system with impatient customers, we can simplify the capacity allocation decision for any individual customer to an “accept-or-reject” decision in each period. Specifically, a customer is *accepted* by the retailer if a unit of capacity is allocated to fulfill her demand. In this case the retailer receives the revenue q_i for $i \in \{1, \cdots, n\}$. In contrast, a customer is *rejected* if the retailer does not fulfill her demand. In this case, no revenue is received and the demand is immediately lost. In each period, once demand is realized, the retailer decides whether or not to accept the demand based on the remaining capacity of suppliers $1, \cdots, m$, i.e., $c_{1,t}, \cdots, c_{m,t}$. We assume that the acceptance or rejection of any demand does not affect the exogenous arrival process.

Similar to the system with patient customers, the following property still holds.

LEMMA 5. *With impatient customers, it is optimal to allocate the capacity reserved from suppliers with smaller marginal usage costs first in each period.*

Lemma 5 is consistent with our intuition that allocating the capacity reserved from a supplier with a smaller marginal usage cost first is optimal as $u_1 - h_1 \le \cdots \le u_m - h_m$. This result implies that if $0 < c_{j,t} < c_{j,1}$ (i.e., some units of the capacity from supplier j are allocated), then $c_{1,t} = \cdots = c_{j-1,t} = 0$ (i.e., the capacity from suppliers $1, \cdots, j-1$ is depleted) and $c_{k,t} = c_{k,1}$ for $k = j+1, \cdots, m$ (i.e., the capacity from suppliers $j+1, \cdots, m$ has not yet been used).

Based on the property in Lemma 5 and the definition of θ_j's for $j = 1, \cdots, m+1$, we present the MDP for impatient customers as follows. In period t, $t = 1, \cdots, T$,

$$v_t(z_t \wedge \Theta) = \sum_{r=1}^{n} \lambda_{r,t} g_{r,t}(z_t \wedge \Theta) + (1 - \lambda_t) \left(v_{t+1}(z_t \wedge \Theta) - \sum_{j=1}^{m} h_j (z_t \wedge \theta_j - z_t \wedge \theta_{j+1}) \right), \tag{9}$$

where for $r = 1, \cdots, n$,

$$\begin{aligned} g_{r,t}(z_t \wedge \Theta) = \max_{0 \vee (z_t - 1) \le z_{t+1} \le z_t} \Bigg[& v_{t+1}(z_{t+1} \wedge \Theta) - \sum_{j=1}^{m} u_j (z_t \wedge \theta_j - z_t \wedge \theta_{j+1}) \\ & + \sum_{j=1}^{m} (u_j - h_j)(z_{t+1} \wedge \theta_j - z_{t+1} \wedge \theta_{j+1}) + q_r (z_t - z_{t+1}) \Bigg]. \end{aligned} \tag{10}$$

The terminal condition is $v_{T+1}(z_{T+1} \wedge \Theta) \equiv 0$ for any z_{T+1}. See the detailed explanations of the MDP in Eq. (9) - (10) in the Appendix.

For the system with impatient customers, we characterize the structure of the optimal policy for the accept-or-reject decision in the following theorem.

THEOREM 2. *For impatient customers, in period t, $t = 1, \cdots, T$, we have the following results:*

(1) The function $v_t(z_t \wedge \Theta)$ is discrete concave in $z_t \wedge \Theta$ and decomposable, i.e., there exist discrete concave functions $\hat{v}_{j,t}(\cdot)$'s for $j = 1, \cdots, m$ such that $v_t(z_t \wedge \Theta) = \sum_{j=1}^{m} \hat{v}_{j,t}(z_t \wedge \theta_j)$.

(2) A class-specific protection level (CSPL) policy is optimal for the accept-or-reject decision: For each demand class i, $i = 1, \cdots, n$, there is a fixed protection level $R_{i,t}$, defined in Eq. (11) ; it is optimal to accept a unit of class i demand if $z_t > R_{i,t}$ and to reject it otherwise.

(3) The fixed protection level $R_{i,t}$, $i=1,\cdots,n$, is defined as

$$R_{i,t} = \min \arg\max_{z\geq 0} \left[\sum_{j=1}^{m} \hat{v}_{j,t+1}(z\wedge\theta_j) + (u_1 - h_1 - q_i)z \right.$$
$$\left. + \sum_{j=2}^{m} (u_j - h_j - u_{j-1} + h_{j-1})(z\wedge\theta_j) \right], \; z\in\mathbb{Z}. \tag{11}$$

The fixed protection level $R_{i,t}$, $i=1,\cdots,n$, is indeed the global optimum of the objective function in Eq. (10). Theorem 2 (2) results from the discrete concavity of the function $v_t(z_t\wedge\Theta)$.

Similar to the system with patient customers, we can develop an efficient algorithm to obtain the optimal fixed protection levels for the CSPL policy. The logic of this algorithm is the same as that of Algorithm 1. Hence, we only show how to update the component functions of $v_t(z_t\wedge\Theta)$ based on the component functions of $v_{t+1}(z_{t+1}\wedge\Theta)$ below. More details can be found in the proof of Theorem 2.

Given $v_{t+1}(z_{t+1}\wedge\Theta) = \sum_{j=1}^{m} \hat{v}_{j,t+1}(z_{t+1}\wedge\theta_j)$, we have $v_t(z_t\wedge\Theta) = \sum_{j=1}^{m} \hat{v}_{j,t}(z_t\wedge\theta_j)$, where

$$\begin{cases} \hat{v}_{1,t}(z) = \sum_{r=1}^{n} \lambda_{r,t}\hat{g}_{r1,t}(z) + (1-\lambda_t)\left[\hat{v}_{1,t+1}(z) - h_1(z)\right], \\ \hat{v}_{j,t}(z) = \sum_{r=1}^{n} \lambda_{r,t}\hat{g}_{rj,t}(z) + (1-\lambda_t)\left[\hat{v}_{j,t+1}(z) + (h_{j-1}-h_j)z\right], \quad j=2,\cdots,m, \end{cases}$$

and

$$\begin{cases} \hat{g}_{i1,t}(z) = \sum_{j=1}^{m} \hat{v}_{1,t+1}(z\wedge R_{i,t}\wedge\theta_j) + \sum_{j=1}^{m} \hat{v}_{j,t+1}((z-1)\wedge\theta_j) - \sum_{j=1}^{m} \hat{v}_{j,t+1}((z-1)\wedge R_{i,t}\wedge\theta_j) \\ \qquad +(u_1-h_1-q_i)(z\wedge R_{i,t}) - h_1 z - (u_1-h_1-q_i) - (u_1-h_1-q_i)((z-1)\wedge R_{i,t}) \\ \qquad +\sum_{j=2}^{m}(u_j-h_j)\left(z\wedge R_{i,t}\wedge\theta_j + (z-1)\wedge\theta_j - (z-1)\wedge R_{i,t}\wedge\theta_j\right) \\ \qquad -\sum_{j=1}^{m-1}(u_j-h_j)\left(z\wedge R_{i,t}\wedge\theta_{j+1} + (z-1)\wedge\theta_{j+1} - (z-1)\wedge R_{i,t}\wedge\theta_{j+1}\right) \\ \hat{g}_{ij,t}(z) = -(u_j-u_{j-1})z, \quad j=2,\cdots,m. \end{cases}$$

4.2. Customers with Limited Patience

In this section, we consider customers with limited patience. We assume that the retailer charges the price for each arriving customer from a menu of prices $\{q_1,\cdots,q_n\}$. Without loss of generality, we assume that $0\equiv q_0 < q_1 < \cdots < q_n$. If a customer is not fulfilled upon arrival, she can wait for the fulfillment but her valuation decreases when she waits. Once her valuation is lower than q_1 (i.e., the smallest price that the retailer offers), this customer leaves. For a customer with limited patience, the behavior of downgrading valuation over time represents her waiting cost for the retailer. Such customer behavior is applicable to fashion, technology, and seasonal products, etc.

We assume that the evolution of waiting customers' valuations can be anticipated by, e.g., tracking individual customers' purchasing behavior through advanced information technology. For instance, it is common that customers frequently receive emails or notifications from large retailers such as Amazon and Walmart when there is a price reduction or discount of a product. The product usually has been searched for by those customers. The evolution of customers' valuations can then be anticipated by tracking the searching and purchasing behavior. This assumption has also been adopted by the existing literature to characterize the evolution of customers' valuations, such as Su (2007) and Aviv and Pazgal (2008).

We refer to customers whose valuations are within the range $[q_i, q_{i+1})$ as class i customers, and impose the following two assumptions for the dynamics of the waiting customers' valuations.

ASSUMPTION 1. *Class i customers always have higher valuations than class i' customers for $i > i'$ in any future period if they wait for the fulfillment of their orders.*

Assumption 1 ensures that a high-valuation customer is always willing to pay more than a low-valuation customer.

ASSUMPTION 2. *A high-valuation customer is less patient and hence downgrades her valuation faster than a low-valuation customer in each period.*

Assumption 2 implies that it is optimal to fulfill a high-valuation customer first.

We illustrate two models used in the existing literature that satisfy our two assumptions. Let w_t be the valuation of a customer after waiting t periods. Su (2007) considers a linear model $w_t = w_0 - \gamma t$, where w_0 is the initial valuation and γ is the decreasing rate. In his paper, customers with different valuations have different degrees of patience. Aviv and Pazgal (2008) consider that customers' valuations decline over the selling season and use a multiplicative model such that $w_t = w_0 e^{-\alpha t}$ to reflect that, where w_0 is the initial valuation and $\alpha \geq 0$ is an exponential decline factor.

Let $\psi(i) \in \{0, \cdots, i\}$ be the index of the new class that the class i customers downgrade to after waiting one period, where $i = 1, \cdots, n$. The assumptions 1 and 2 imply that for $1 \leq j < i \leq n$, we have $q_{\psi(i)} \geq q_{\psi(j)}$ and $q_i - q_{\psi(i)} \geq q_j - q_{\psi(j)}$. In particular, if $\psi(i) = 0$, then customers in class i cannot wait any longer. Under these two assumptions, the following properties still hold in this case.

LEMMA 6. *For customers with limited patience, it is optimal to allocate the capacity reserved from suppliers with smaller marginal usage costs first and fulfill customer classes with larger marginal revenues first in each period.*

Due to the properties in Lemma 6, the MDP for this case is similar to the counterpart in Section 3 for the system with patient customers. Specifically, based on the MDP in Eq. (4) - Eq. (5), we can rewrite the MDP for customers with limited patience by setting $b_i = 0$, $i = 1, \cdots, n$, and replacing $v_{t+1}(z_{t+1} \wedge \Theta, z_{t+1} \wedge \mathbf{z}_{w,t})$ with $v_{t+1}(z_{t+1} \wedge \Theta, z_{t+1} \wedge \tilde{\mathbf{z}}_{w,t})$ in Eq. (5), where $\tilde{\mathbf{z}}_{w,t} = (\tilde{z}_{1,t}, \cdots, \tilde{z}_{n,t})$ and

$$\tilde{z}_{i,t} = \begin{cases} z_{\zeta(i),t}, & \zeta(i) = \min\{r : i \leq r \leq n, \psi(r) \geq i\} \text{ if } \{r : i \leq r \leq n, \psi(r) \geq i\} \neq \emptyset \ , \\ z_{1,t}, & \text{if } \{r : i \leq r \leq n, \psi(r) \geq i\} = \emptyset. \end{cases}$$

The rest of the analysis of this MDP is similar to that in Section 3. By a similar argument, we can show that for customers with limited patience, $v_t(z_t \wedge \Theta, z_t \wedge \mathbf{z}_{w,t})$ is discrete concave and decomposable, and again an NPL policy is optimal in each period. Moreover, we can also use Algorithm 1 to obtain the NPL policy under our assumptions.

5. Numerical Studies

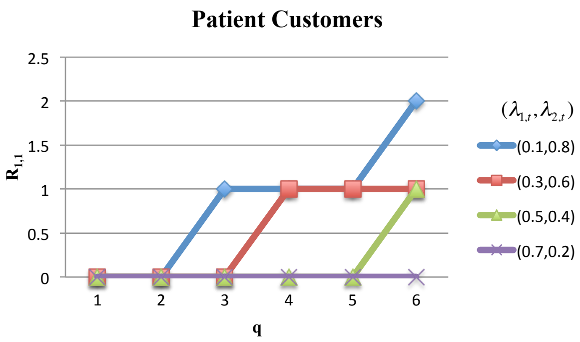
In this section, we numerically illustrate the optimal capacity allocation policy and show its value by comparing it with a simple heuristic policy.

5.1. Illustration of the Optimal Policy

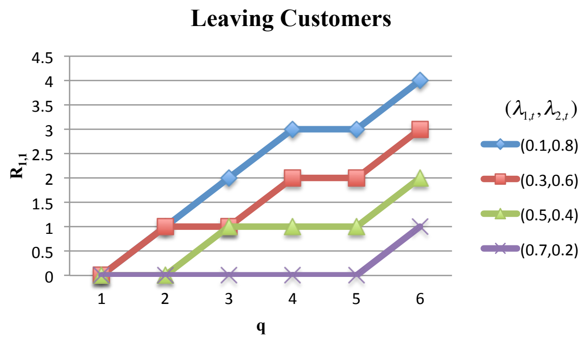
Note that the NPL and CSPL policies are specified by the constant parameters $R_{i,t}$'s. In this section we numerically illustrate the optimal capacity allocation policies by providing these constants under different scenarios. Consider a system with two suppliers and two demand classes. By Algorithm 1 or its variants, we can obtain $R_{1,t}$ and $R_{2,t}$ for period $t = 1, \cdots, T$ under each type of customer waiting behavior. For customers with limited patience, we assume that in each period demand of class 2 downgrades to class 1 and demand of class 1 downgrades to class 0. As there are only two demand classes and class 2 demand has a higher priority, $R_{2,t} = 0$ for $t = 1, \cdots, T$ under our setting.

We first investigate how $R_{1,t}$ for $t = 1$ is affected by the demand characteristics such as the price difference and arrival probability difference of the two demand classes. We set $T = 20$, $(\theta_1, \theta_2) = (10, 5)$, $(b_1, b_2) = (1, 1)$, $(h_1, h_2) = (1, 1)$, $(u_1, u_2) = (1, 1)$, $(q_1, q_2) = (q, 2q)$ with $q \in \{1, 2, 3, 4, 5, 6\}$, $\lambda_{1,t} + \lambda_{2,t} = 0.9$ for any t, and $(\lambda_{1,t}, \lambda_{2,t}) \in \{(0.1, 0.8), (0.3, 0.6), (0.5, 0.4), (0.7, 0.2)\}$, i.e., the arrival probability difference $\Delta\lambda = \lambda_{2,t} - \lambda_{1,t}$ decreases from 0.7 to -0.5 with a step of 0.4. Similar to Zhang and Cooper (2005), these instances are selected to clearly capture how the system parameters affect $R_{1,t}$. Figure 1 depicts the change of $R_{1,1}$ with respect to the price difference and the arrival probability difference under each type of customers.

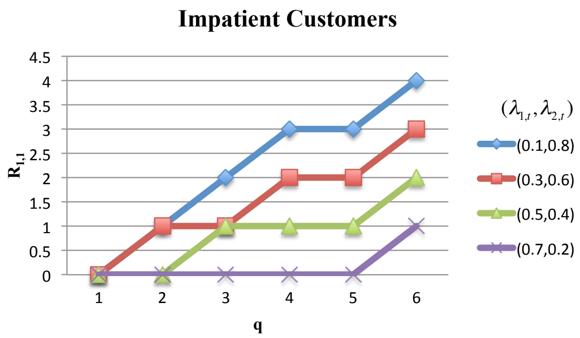
In Figure 1 (b) and (c), we can see that the values of $R_{1,1}$ for the system with customers with limited patience and the system with impatient customers are the same because for these systems class 1 demand cannot wait. We thus focus on $R_{1,1}$'s for the systems with patient customers and impatient customers.



(a) Patient customers, $(q_1, q_2) = (q, 2q), q \in \{1, 2, 3, 4, 5, 6\}$



(b) Customers with limited patience, $(q_1, q_2) = (q, 2q), q \in \{1, 2, 3, 4, 5, 6\}$



(c) Impatient customers, $(q_1, q_2) = (q, 2q), q \in \{1, 2, 3, 4, 5, 6\}$

Figure 1 **The impacts of price differences and arrival probability differences on $R_{1,1}$ under three types of customers ($T = 20, \theta_1 = 10, \theta_2 = 5, b_1 = b_2 = h_1 = h_2 = u_1 = u_2 = 1$).**

Recall that in our setting the ratio $q_2/q_1 = 2$ and $\Delta q = q_2 - q_1 = q$. Under each type of customers, $R_{1,1}$ is increasing in the price difference q. That is, with a higher price difference, we reserve more capacity for class 2 demand. For each combination of $\lambda_{1,t}$ and $\lambda_{2,t}$, $R_{1,1}$ for impatient customers is always no smaller than that for patient customers and the difference is nondecreasing in q. Intuitively, for impatient customers, we fulfill more class 1 demand as it cannot wait and hence reserve less capacity for class 2 demand, especially when q_1 is large. In contrast, our numerical results indicate that no matter how large q_1 is, we reserve more capacity for class 2 demand for the system with impatient customers than the system with patient customers. This is because for patient customers there is a unit waiting cost b_1 if class 1 demand is not fulfilled.

For the arrival probability difference $\Delta\lambda$ under the condition $\lambda_{1,t} + \lambda_{2,t} = 0.9$, Figure 1 shows that a larger $\Delta\lambda$ leads to a larger $R_{1,1}$. This is consistent with our intuition that more capacity is reserved for class 2 demand when $\lambda_{2,t}$ increases while $\lambda_{1,t}$ decreases. We also observe that under a smaller $\Delta\lambda$ (e.g., $\Delta\lambda = -0.5$), $R_{1,1}$ is strictly positive only when q is sufficiently large. However, with a larger $\Delta\lambda$, $R_{1,1}$ increases significantly in q even when q is small. Hence, when there is a higher (resp. lower) arrival probability for class 2 (resp. class 1) demand and a larger q, we reserve more capacity for future demand of class 2. Essentially, for a system with two demand classes, we only need to consider the allocation problem for class 1 demand because a state-independent rationing level policy is optimal in this case.

We then investigate how $R_{1,t}$ is affected by the supply characteristics for patient customers. Specifically, we numerically analyze how $R_{1,t}$ changes with the supplier usage cost difference, the holding cost difference, the waiting cost difference, and the capacity level difference (our results are robust for the other two types of customers).

We set $T=20$ and $(\lambda_{1,t},\lambda_{2,t})=(0.5,0.4)$ for $t=1,\cdots,T$. First, by fixing $(\theta_1,\theta_2)=(12,8)$ and $(u_1,u_2,h_1,h_2,\theta_1,\theta_2)=(1,1,1,1,12,8)$, we investigate how $R_{1,t}$'s for $t=1,\cdots,6$ change with the waiting cost difference $\Delta b=b_1-b_2$ in Table 1. We consider three settings under which $q_2+b_2=20$ and $q_1+b_1\in\{12,14,16,18,20\}$ so that $q_1+b_1\le q_2+b_2$ is satisfied. Table 1 indicates that $R_{1,t}$ is time-dependent and decreasing in t and Δb. Combining the results in Figure 1 (a) and Table 1, we find that $R_{1,t}$ decreases when both the price difference Δq and the waiting cost difference Δb increase, and $R_{1,t}$ tends to be 0 when Δq is small while $\Delta b\ge 0$.

Table 1 **The constant $R_{1,t}$ for $t=1,\cdots,6$ with $(\lambda_{1,t},\lambda_{2,t},u_1,u_2,h_1,h_2,\theta_1,\theta_2)=(0.5,0.4,1,1,1,1,12,8)$.**

Setting (1) with $(q_1,q_2)=(8,12)$ and $b_2=8$:							
$\Delta_b=b_1-b_2$	b_1	$R_{1,1}$	$R_{1,2}$	$R_{1,3}$	$R_{1,4}$	$R_{1,5}$	$R_{1,6}$
-4	4	4	3	3	3	3	3
-2	6	2	2	1	1	1	1
0	8	0	0	0	0	0	0
2	10	0	0	0	0	0	0
4	12	0	0	0	0	0	0
Setting (2) with $(q_1,q_2)=(10,10)$ and $b_2=10$:							
$\Delta_b=b_1-b_2$	b_1	$R_{1,1}$	$R_{1,2}$	$R_{1,3}$	$R_{1,4}$	$R_{1,5}$	$R_{1,6}$
-8	2	6	6	6	5	5	5
-6	4	4	4	4	4	3	3
-4	6	3	2	2	2	2	2
-2	8	1	1	1	1	1	1
0	10	0	0	0	0	0	0
Setting (3) with $(q_1,q_2)=(12,8)$ and $b_2=12$:							
$\Delta_b=b_1-b_2$	b_1	$R_{1,1}$	$R_{1,2}$	$R_{1,3}$	$R_{1,4}$	$R_{1,5}$	$R_{1,6}$
-12	0	9	8	8	8	7	7
-10	2	6	6	6	6	5	5
-8	4	5	5	4	4	4	4
-6	6	3	3	3	3	3	2
-4	8	2	2	2	2	2	1

Second, by fixing $(\theta_1,\theta_2)=(12,8)$ and $(q_1,q_2,b_1,b_2,\theta_1,\theta_2)=(4,16,1,1,12,8)$, we analyze how

$R_{1,t}$'s for $t = 1, \cdots, 6$ change with the supplier usage cost difference $\Delta u = u_1 - u_2$ and the holding cost difference $\Delta h = h_2 - h_1$ in Table 2. We consider four settings under which $u_1 - h_1 \leq u_2 - h_2$ is satisfied and $h_2 \in \{0,1,2,3,4\}$. In general, $R_{1,t}$ decreases with the unit holding cost h_2 but is independent of the holding cost difference Δh in our numerical results, which implies that h_1 plays a lesser role in determining the value of $R_{1,t}$. Moreover, by comparing the four settings, we observe that the supplier usage costs u_1 and u_2, and the supplier usage cost difference Δu, also have a limited impact on $R_{1,t}$. Table 2 reveals that given the prices and unit waiting costs (q_1, q_2, b_1, b_2), $R_{1,t}$ is highly dependent on the unit holding cost of the high-priority demand class.

Table 2 The constant $R_{1,t}$ for $t = 1, \cdots, 6$ with $(\lambda_{1,t}, \lambda_{2,t}, q_1, q_2, b_1, b_2, \theta_1, \theta_2) = (0.5, 0.4, 4, 16, 1, 1, 12, 8)$.

Setting (1) with $(u_1, u_2) = (2, 10)$ and $h_1 = 0$:							
$\Delta_h = h_2 - h_1$	h_2	$R_{1,1}$	$R_{1,2}$	$R_{1,3}$	$R_{1,4}$	$R_{1,5}$	$R_{1,6}$
0	0	4	4	4	4	4	4
1	1	2	2	2	2	2	2
2	2	1	1	1	1	1	1
3	3	1	1	1	1	1	1
4	4	0	0	0	0	0	0
Setting (2) with $(u_1, u_2) = (6, 6)$ and $h_1 = 6$:							
$\Delta_h = h_2 - h_1$	h_2	$R_{1,1}$	$R_{1,2}$	$R_{1,3}$	$R_{1,4}$	$R_{1,5}$	$R_{1,6}$
-6	0	4	4	4	4	4	4
-5	1	2	2	2	2	2	2
-4	2	1	1	1	1	1	1
-3	3	1	1	1	1	1	1
-2	4	0	0	0	0	0	0
Setting (3) with $(u_1, u_2) = (10, 2)$ and $h_1 = 12$:							
$\Delta_h = h_2 - h_1$	h_2	$R_{1,1}$	$R_{1,2}$	$R_{1,3}$	$R_{1,4}$	$R_{1,5}$	$R_{1,6}$
-12	0	4	4	4	4	4	4
-11	1	2	2	2	2	2	2
-10	2	1	1	1	1	1	1
-9	3	1	1	1	1	1	1
-8	4	0	0	0	0	0	0
Setting (4) with $(u_1, u_2) = (4, 2)$ and $h_1 = 6$:							
$\Delta_h = h_2 - h_1$	h_2	$R_{1,1}$	$R_{1,2}$	$R_{1,3}$	$R_{1,4}$	$R_{1,5}$	$R_{1,6}$
-6	0	4	4	4	4	4	4
-5	1	2	2	2	2	2	2
-4	2	1	1	1	1	1	1
-3	3	1	1	1	1	1	1
-2	4	0	0	0	0	0	0

Finally, we investigate the relationship between $R_{1,t}$ and the capacity level difference $\Delta\theta = \theta_1 - \theta_2$ by fixing $(q_1, q_2, b_1, b_2, u_1, u_2, h_1, h_2) = (10, 10, 2, 10, 1, 1, 1, 1)$ in Table 3. When θ_1 is fixed, $R_{1,t}$ is independent of the capacity level difference $\Delta\theta$ because by intuition the rationing level only depends on the total capacity level θ_1. When θ_2 is fixed, $R_{1,t}$ is increasing in $\Delta\theta$ but the rate of increase is limited. With the increasing of θ_2, the increase rate is decreasing and eventually becomes 0 when $\theta_2 = 8$.

Table 3 **The constant $R_{1,t}$ for $t = 1, \cdots, 6$ with $(q_1, q_2, b_1, b_2, u_1, u_2, h_1, h_2) = (10, 10, 2, 10, 1, 1, 1, 1)$ and $(\lambda_{1,t}, \lambda_{2,t}) = (0.5, 0.4)$.**

Setting (1) with $\theta_1 = 20$:								
$\Delta_\theta = \theta_1 - \theta_2$	θ_1	θ_2	$R_{1,1}$	$R_{1,2}$	$R_{1,3}$	$R_{1,4}$	$R_{1,5}$	$R_{1,6}$
16	20	4	6	6	6	5	5	5
12	20	8	6	6	6	5	5	5
8	20	12	6	6	6	5	5	5
4	20	16	6	6	6	5	5	5
Setting (2) with $\theta_2 = 8$:								
$\Delta_\theta = \theta_1 - \theta_2$	θ_1	θ_2	$R_{1,1}$	$R_{1,2}$	$R_{1,3}$	$R_{1,4}$	$R_{1,5}$	$R_{1,6}$
0	8	8	6	6	6	5	5	5
4	12	8	6	6	6	5	5	5
8	16	8	6	6	6	5	5	5
12	20	8	6	6	6	5	5	5
Setting (2) with $\theta_2 = 4$:								
$\Delta_\theta = \theta_1 - \theta_2$	θ_1	θ_2	$R_{1,1}$	$R_{1,2}$	$R_{1,3}$	$R_{1,4}$	$R_{1,5}$	$R_{1,6}$
0	4	4	5	5	4	4	4	4
4	8	4	6	6	6	5	5	5
8	12	4	6	6	6	5	5	5
12	16	4	6	6	6	5	5	5

5.2. Comparison with A Simple Heuristic Policy

To show the value of the optimal capacity allocation policy, we compare its performance against a static heuristic policy based on the deterministic linear program (DLP) (see the discussion in Section 2). In this numerical study, we let $q_1 < q_2$, $u_1 \le u_2$ and $b_1 = b_2 = h_1 = h_2 = 0$. During the planning horizon T, given the arrival probabilities of two demand classes as $(\lambda_{1,t}, \lambda_{2,t})$, the expected amounts of class i demand for $i = 1, 2$ are $\lambda_{1,t}T$ and $\lambda_{2,t}T$, respectively. Then, we solve the following optimization problem in the heuristic policy:

$$\begin{aligned}
\max_{y_1, y_2} \quad & q_1 y_1 + q_2 y_2 - u_1 \min\{\theta_1 - \theta_2, y_1 + y_2\} - u_2 \max\{0, y_1 + y_2 - \theta_1 + \theta_2\} \\
s.t. \quad & y_1 + y_2 \le \theta_1, \\
& 0 \le y_i \le \lambda_{i,t} T, \quad i = 1, 2, \\
& y_1, y_2 \in \mathbb{Z}.
\end{aligned}$$

We determine the optimal values of y_1 and y_2 at the beginning of period 1. Then, we fulfill demand in the planning horizon based on the following policy. Let $\gamma_{i,t}$ be the quantity of the fulfilled demand of class i, $i \in \{1,2\}$, from period 1 to period t, $t \in \{1,\cdots,T\}$. Then, for the newly realized demand of class i we fulfill it as long as $\gamma_{i,t}+1 \le y_i$, i.e., y_i is the maximum quantity of demand of class i to be fulfilled during the entire planning horizon.

We set $T=20$ and $(\lambda_{1,t},\lambda_{2,t})=(0.5,0.4)$ for any t, and there is no backorder at the beginning of period 1. The average total profit under the optimal policy for patient customers, i.e., V_{opt}, and the average total profit under the simple heuristic policy, V_{heu}, are provided in Table 4. The average total profit is calculated by randomly generating 100000 scenarios of the demand set $(D_{i,t})_{i=1,2,\ t=1,\cdots,T}$, where $D_{i,t} \in \{0,1\}$ and $D_{1,t}+D_{2,t} \le 1$. We also present the percentage of the profit increase under the optimal policy as $\Delta V = \frac{100\times\left(V_{opt}-V_{heu}\right)}{V_{heu}}\%$ and the computational time of the two policies as T_{opt} and T_{heu} (we provide the CPU time of the program written by Fortran). Table 4 indicates that the computational time of the optimal policy is similar to that of the simple heuristic policy and even smaller under certain cases. Moreover, the optimal policy significantly outperforms the heuristic policy under various parameter settings. In particular, when the unit usage cost u_2 is sufficiently large and/or the total capacity level θ_1 is limited, an appropriate rationing strategy is required to increase the total profit. In these cases, the static heuristic policy leads to poor performance. Thus, the simple heuristic policy is not recommended for our settings.

Table 4 **The value of the optimal policy against the heuristic policy with $T=20$ and $(\lambda_{1,t},\lambda_{2,t})=(0.5,0.4)$ for $t=1,\cdots,T$.**

		(q_1,q_2)					
$(\theta_1,\theta_2)=(16,8),\ (u_1,u_2)=(1,1)$		$(2,6)$	$(6,10)$	$(10,14)$	$(10,16)$	$(10,22)$	$(10,26)$
	V_{opt} (T_{opt})	48.0 (0.422)	111.7 (0.469)	175.5 (0.438)	207.5 (0.422)	239.4 (0.453)	271.6 (0.422)
	V_{heu} (T_{heu})	42.9 (0.438)	105.1 (0.422)	167.4 (0.328)	194.7 (0.469)	221.9 (0.407)	249.2 (0.438)
	ΔV (%)	11.9	6.28	4.84	6.57	7.89	8.99
		(u_1,u_2)					
$(\theta_1,\theta_2)=(16,8),\ (q_1,q_2)=(6,20)$		$(1,3)$	$(3,5)$	$(5,7)$	$(5,10)$	$(5,13)$	$(5,16)$
	V_{opt} (T_{opt})	175.8 (0.422)	143.7 (0.453)	119.1 (0.438)	116.6 (0.422)	113.9 (0.438)	111.5 (0.406)
	V_{heu} (T_{heu})	158.4 (0.422)	127.1 (0.406)	96.1 (0.391)	73.3 (0.422)	50.7 (0.438)	27.9 (0.407)
	ΔV (%)	11.0	13.1	23.9	59.1	124.7	299.6
		(θ_1,θ_2)					
$(u_1,u_2)=(1,3),\ (q_1,q_2)=(4,12)$		$(4,4)$	$(10,4)$	$(16,4)$	$(16,7)$	$(16,10)$	$(16,13)$
	V_{opt} (T_{opt})	35.8 (0.421)	84.2 (0.421)	103.9 (0.421)	97.9 (0.421)	91.9 (0.421)	85.9 (0.406)
	V_{heu} (T_{heu})	18.2 (0.421)	57.5 (0.421)	94.2 (0.406)	88.3 (0.406)	82.2 (0.406)	76.2 (0.437)
	ΔV (%)	96.7	46.4	10.3	10.9	11.8	12.7

6. Extensions

In this section, we consider several extensions of the model for patient customers: (1) multiple products; (2) new capacity additions; and (3) Markov modulated demand. Similar extensions can be made for impatient customers and customers with limited patience. We refer to the model for patient customers described in Section 4.1 as the basic model.

6.1. Multiple Products

We generalize the basic model with a single product to the model with multiple products. Following Akçay et al. (2010), we consider N substitutable products that are only different in their qualities. We denote by s_j the quality index of product j, $j=1,\cdots,N$, and assume that $s_1 > s_2 > \cdots > s_N$ without loss of generality. An arriving customer can either choose an available product from the retailer or purchase from others. We normalize the value of the outside option to 0 for convenience. The retailer starts the selling horizon with an initial inventory $c_{j,1}$ for product j, $j=1,\cdots,N$, and cannot replenish inventory during the horizon. Similar to the basic model, we assume the following property for the unit holding costs and unit usage costs of the N products: $u_1 - h_1 \le u_2 - h_2 \le \cdots \le u_N - h_N$.

We consider two types of customers for simplicity: the high-valuation (H) customers and the low-valuation (L) customers with $H > L$. The high-valuation customers have a higher unit waiting cost than the low-valuation customers, i.e., $b_H \ge b_L$. We denote by λ_t the probability that demand is realized in period t and $p_{H,t}$ (resp., $p_{L,t}$) the probability that the realized demand has a high (resp., low) valuation. For the customers with valuation K, $K \in \{H, L\}$, the retailer sets the selling prices $q_{K1}, q_{K2}, \cdots, q_{KN}$ for the N products.

Note that, with multiple types of customers, deciding on the optimal pricing scheme is complicated and out of the scope of this paper. Hence, we consider an applicable pricing scheme that has been thoroughly studied in the literature. To ensure the priority properties as in Lemma 1, we adopt the pricing scheme in Akçay et al. (2010) for each type of customer. This pricing scheme is optimal if there is only one demand class.

We then introduce the pricing scheme in Akçay et al. (2010) and also the customer choice model below. For each type of customer, the prices $q_{K1}, q_{K2}, \cdots, q_{KN}$ for each $K \in \{H, L\}$ satisfy the following constraints:

$$\begin{cases} 1 \ge \frac{q_{K1}}{Ks_1} \ge \frac{q_{K2}}{Ks_2} \ge \cdots \ge \frac{q_{KN}}{Ks_N} > 0, \\ 1 \ge \frac{q_{K1}-q_{K2}}{K(s_1-s_2)} \ge \frac{q_{K2}-q_{K3}}{K(s_2-s_3)} \ge \cdots \ge \frac{q_{K(N-1)}-q_{KN}}{K(s_{N-1}-s_N)} \ge 0, \end{cases} \qquad K \in \{H, L\}.$$

Let $\alpha_j(\mathbf{q}_K)$ be the choice probability for product j and $\alpha_0(\mathbf{q}_K)$ be the probability of no purchase from the retailer. Then, the choice probability of a consumer with valuation K under this pricing scheme is

$$\alpha_j(\mathbf{q}_K) = \begin{cases} 1 - \frac{q_{K1}-q_{K2}}{K(s_1-s_2)} & j=1 \\ \frac{q_{K(j-1)}-q_{Kj}}{K(s_{j-1}-s_j)} - \frac{q_{Kj}-q_{K(j+1)}}{K(s_j-s_{j+1})} & j=2,\cdots,N-1 \\ \frac{q_{K(N-1)}-q_{KN}}{K(s_{N-1}-s_N)} - \frac{q_{KN}}{Ks_N} & j=N \\ \frac{q_{KN}}{Ks_N} & j=0. \end{cases} \tag{12}$$

Akçay et al. (2010) also indicate that under this pricing scheme, the highest-quality product alone has a positive choice probability. That is, if the products $1,\cdots,j-1$ are out of stock, then the pricing scheme must satisfy the constraint

$$\frac{q_{Kj}-q_{K(j+1)}}{K(s_j-s_{j+1})} = \cdots = \frac{q_{K(N-1)}-q_{KN}}{K(s_{N-1}-s_N)} = \frac{q_{KN}}{Ks_N}, \quad K \in \{H,L\}.$$

With the pricing scheme in Akçay et al. (2010), we sell the products sequentially in descending order of product quality. That is, if product j is the highest-quality product available for sales, then products $1,\cdots,j-1$ are out of stock and the capacity of products $j+1,\cdots,N$ does not affect the selling strategy as customers only choose product j. In this sense, the basic model can be used to analyze the capacity allocation problem for each product j independently, where the product j of interest has the highest quality among the available products. Note that under the pricing scheme in Akçay et al. (2010), q_{Kj} is proportional to K and hence $q_{Hj} \geq q_{Lj}$ for any $j=1,\cdots,N$. We thus have the properties $u_1-h_1 \leq u_2-h_2 \leq \cdots \leq u_N-h_N$ and $q_{Hj}+b_H \geq q_{Lj}+b_L$ for any j. Therefore, all of the results of the basic model still hold.

We can also generalize the basic model to capacity allocation problems with multiple products when the customer choice is assortment based, i.e., unaffected by the availability of products and depend only on the specific assortment of products (Goyal et al. 2016). Here, if customers do not switch to other products when their preferred products are out of stock, then the demand for each product is independent of the availability of products and hence the capacity allocation problem of each product can be considered individually. Essentially, the results of the basic model can be preserved for multiple products only when a sequential selling property holds among different products. However, they may not hold for more general scenarios.

It is worth noting that the assortment planning problem studied by Goyal et al. (2016) is related to our model for impatient customers with multiple products. However, our focus is different from theirs. We focus on the capacity allocation problem during the selling season by fixing the capacity of different suppliers at the beginning. Due to the price scheme in Akçay et al. (2010), the optimal capacity allocation policy can be obtained by an efficient algorithm. In contrast, Goyal et al. (2016) consider the joint assortment and inventory ordering problem, i.e., they decide which products to offer and how many units to stock for each offered product. Under the general customer choice models in Goyal et al. (2016), they show that the problem is NP hard and hence provide an approximation scheme for several interesting and practical customer choice models.

6.2. New Capacity Additions

In the basic model, we assume that the capacity reserved from different suppliers is fixed before the selling season. In practice, new capacity may be added over time. In this subsection, we show that the results of the basic model still hold if there is a fixed schedule of capacity additions. Specifically, we assume that in each period t, $t=1,\cdots,T$, a fixed amount of capacity of supplier j, denoted by c_j^t, is added to the system for $j=1,\cdots,m$. The MDP for this case is presented below.

Define $\Theta_t=(\theta_{j,t})_{j=1,\cdots,m}$ and $\hat{\Theta}_t=(\hat{\theta}_{j,t})_{j=1,\cdots,m}$, where

$$\begin{cases}\theta_{j,t}=\sum_{k=j}^m c_{k,t}+\sum_{s=1}^{t-1}\sum_{k=j}^m c_k^s,\\ \hat{\theta}_{j,t}=\sum_{k=j}^m c_k^t.\end{cases}$$

Moreover, $\theta_{j,t+1}=\theta_{j,t}+\hat{\theta}_{j,t}$ for any $j=1,\cdots,m$ and $t=1,\cdots,T$. Let $(z_t\wedge\Theta_t,\mathbf{z}_{w,t})$ be the system state at the beginning of period t. Then the state transits to $(z_t\wedge\Theta_t+\hat{\Theta}_t,\mathbf{z}_{w,t}+\hat{\theta}_{1,t}\mathbf{e})$, where $\mathbf{e}$ is the unit vector with all of the elements being 1, before the demand realization in period t. Then, the MDP is given as

$$v_t(z_t\wedge\Theta_t,\mathbf{z}_{w,t})=\sum_{r=1}^n\lambda_{r,t}g_t(z_t\wedge\Theta_t,\mathbf{z}_{w,t}-\mathbf{e}_{[1,r]})+(1-\lambda_t)g_t(z_t\wedge\Theta_t,\mathbf{z}_{w,t}),$$

where

$$\begin{aligned}g_t(z_t\wedge\Theta,\bar{\mathbf{z}}_{w,t})=&\max_{0\vee(\bar{z}_{m+1,t}+\hat{\theta}_{1,t})\le z_{t+1}\le(z_t+\hat{\theta}_{1,t})}\Bigg[v_{t+1}\left(z_{t+1}\wedge\Theta_{t+1},z_{t+1}\wedge(\bar{\mathbf{z}}_{w,t}+\hat{\theta}_{1,t}\mathbf{e})\right)\\&-\sum_{j=1}^m u_j(z_t\wedge\theta_{j,t}+\hat{\theta}_{j,t}-z_t\wedge\theta_{j+1,t}-\hat{\theta}_{j+1,t})+\sum_{j=1}^m(u_j-h_j)(z_{t+1}\wedge\theta_{j,t+1}\\&-z_{t+1}\wedge\theta_{j+1,t+1})+\sum_{i=1}^n q_i(\bar{z}_{m+i+1,t}-(\bar{z}_{m+i,t}\vee(z_{t+1}-\hat{\theta}_{1,t}))\wedge\bar{z}_{m+i+1,t})\\&-\sum_{i=1}^n b_i((\bar{z}_{m+i,t}\vee(z_{t+1}-\hat{\theta}_{1,t}))\wedge\bar{z}_{m+i+1,t}-\bar{z}_{m+i,t})\Bigg].\end{aligned}$$

The terminal condition is $v_{T+1}(z_{T+1}\wedge\Theta,\mathbf{z}_{w,T+1})\equiv 0$ for any $(z_{T+1},\mathbf{z}_{w,T+1})$.

As $\hat{\theta}_{j,t}$ is a constant for any $j=1,\cdots,m$ and $t=1,\cdots,T$, the analysis of this model is the same as the basic model and the new capacity additions do not change the structure of the optimal capacity allocation policy as in the basic model.

6.3. Markov Modulated Demand

In this section, we show how to extend the results of the basic model to Markov modulated demand. There is a Markov chain ω_t, named the *world* as in Zipkin (2008). The arrival probability $\lambda_t(\omega_t)$ of a customer in period t depends on the current world state. Let ω_{t+1} be the world state in the next period given the current state ω_t. The MDP is given as follows:

$$v_t(z_t \wedge \Theta, \mathbf{z}_{w,t}, \omega_t) = \mathbb{E}_{\omega_{t+1}|\omega_t}\left[\sum_{r=1}^{n} \lambda_{r,t}(\omega_t) g_t(z_t \wedge \Theta, \mathbf{z}_{w,t} - \mathbf{e}_{[1,r]}, \omega_{t+1}) + (1-\lambda_t(\omega_t)) g_t(z_t \wedge \Theta, \mathbf{z}_{w,t}, \omega_{t+1})\right],$$

where

$$\begin{aligned} g_t(z_t \wedge \Theta, \bar{\mathbf{z}}_{w,t}, \omega_{t+1}) = \max_{0 \vee \bar{z}_{m+1,t} \le z_{t+1} \le z_t} \Bigg[& v_{t+1}(z_{t+1} \wedge \Theta, z_{t+1} \wedge \bar{\mathbf{z}}_{w,t}, \omega_{t+1}) - \sum_{j=1}^{m} u_j (z_t \wedge \theta_j - z_t \wedge \theta_{j+1}) \\ & + \sum_{j=1}^{m} (u_j - h_j)(z_{t+1} \wedge \theta_j - z_{t+1} \wedge \theta_{j+1}) + \sum_{i=1}^{n} q_i (\bar{z}_{m+i+1,t} \\ & - (\bar{z}_{m+i,t} \vee z_{t+1}) \wedge \bar{z}_{m+i+1,t}) - \sum_{i=1}^{n} b_i ((\bar{z}_{m+i,t} \vee z_{t+1}) \wedge \bar{z}_{m+i+1,t} - \bar{z}_{m+i,t}) \Bigg]. \end{aligned}$$

The remainder of the analysis of this MDP is similar to that of the basic model, except that now the optimal capacity allocation policy also depends on the current state of the world.

7. Concluding Remarks

In this paper, we analyze the capacity allocation problem with a single product, multiple suppliers and multiple demand classes. The units reserved from different suppliers are identical to customers but incur different unit usage costs and different unit holding costs for the retailer. We consider three types of customers: patient customers, impatient customers, and customers with limited patience. We also discuss how to incorporate multiple products, new capacity additions and Markov modulated demand into the models in the extension.

To analyze our problems, we derive a new result for the preservation of decomposition and, based on this result, show that the value functions are decomposable for three types of customers. We then characterize the optimal capacity allocation policy as the NPL policy for patient customers and customers with limited patience. For impatient customers, we show that the CSPL policy is optimal for the capacity allocation. We also develop efficient algorithms to obtain these optimal policies based on the decomposition of the value functions.

In future studies, we may incorporate more customer choice and/or dynamic pricing into the basic models in this study. However, those models are quite different from the models in this study. They shall be addressed by some other techniques and hence deserve separate research. We conjecture that for those models, value functions may no longer be decomposable and we need to use a different framework (such as a dynamic game), and perhaps develop new methodologies for their analysis.

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Appendix

Supplementary

We present some lemmas here to show the preservation of discrete concavity. They shall use them to prove the results shown in the main body.

Lemma 7. *For a discrete concave function $f:\mathbb{Z}\to\mathbb{R}$, (1) if $f(x)$ is nondecreasing in x for $x\leq s$, where s is an integer constant, then $f(x\wedge s)$ must be discrete concave in x; (2) if $s=\min\arg\max_{x\geq 0} f(x)$, then $g(x)=f(x)-f(x\wedge s)$ is discrete concave in x.*

Proof of Lemma 7.

(1) As $f(x)$ is discrete concave and nondecreasing for $x\leq s$,

$$f((x+1)\wedge s)-f(x\wedge s)=\begin{cases} f(x+1)-f(x), & x+1\leq s,\\ 0, & s\leq x,\end{cases}$$

is nonincreasing in x. Hence, $f(x\wedge s)$ must be discrete concave in x.

(2) In this case, $f(x)$ is discrete concave and nonincreasing in x for $x\geq s$. Then,

$$g(x+1)-g(x)=\begin{cases} 0, & x+1\leq s,\\ f(x+1)-f(x), & s\leq x,\end{cases}$$

is nonincreasing in x. Hence, $g(x)=f(x)-f(x\wedge s)$ must be discrete concave in x. ■

Lemma 8. *The function $F(y)=f_1(a\wedge y)-f_2(b\wedge y)$ is discrete concave in y if the following two conditions hold: (1) $f_1(\cdot),f_2(\cdot),f(\cdot):\mathbb{Z}\to\mathbb{R}$ are discrete concave, where $f(x)=f_1(x)-f_2(x)$; (2) $a=\min\arg\max_{x\geq 0} f_1(x)$, $b=\min\arg\max_{x\geq 0} f_2(x)$ and $a\geq b$.*

Proof of Lemma 8.

If $a=b$, then $F(y)$ is clearly a discrete concave function. If $a>b$, $F(y)$ is discrete concave in y inside the following three disjoint regions: (i) $y\leq b$ and $F(y)=f(y)$, (ii) $b<y\leq a$ and $F(y)=f_1(y)-f_2(b)$, and (iii) $y>a$ and $F(y)=f_1(a)-f_2(b)$. Then, to show that $F(y)$ is discrete concave in its whole domain, we shall show that $F(y)$ is discrete concave on the boundaries of the three disjoint regions. Specifically, we shall show that $F(y)-F(y-1)$ is nonincreasing in y when increasing y from b to $b+1$ and from a to $a+1$.

Note that

$$F(y)-F(y-1)=\begin{cases} f_1(b)-f_2(b)-f_1(b-1)+f_2(b-1), & y=b,\\ f_1(b+1)-f_2(b)-f_1(b)+f_2(b), & y=b+1.\end{cases}$$

Moreover, $f_1(b+1)-f_2(b+1)-f_1(b)+f_2(b)\leq f_1(b)-f_2(b)-f_1(b-1)+f_2(b-1)$ as $f(y)$ is discrete concave, and $f_1(b+1)-f_2(b)-f_1(b)+f_2(b)\leq f_1(b+1)-f_2(b+1)-f_1(b)+f_2(b)$ as $f_2(b+1)\leq f_2(b)$.

Then, $f_1(b+1) - f_2(b) - f_1(b) + f_2(b) \le f_1(b) - f_2(b) - f_1(b-1) + f_2(b-1)$, i.e., $F(y) - F(y-1)$ is nonincreasing when we increase y from b to $b+1$.

Similarly, note that

$$F(y) - F(y-1) = \begin{cases} f_1(a) - f_2(b) - f_1(a-1) + f_2(b), & y = a, \\ f_1(a) - f_2(b) - f_1(a) + f_2(b), & y = a+1. \end{cases}$$

Moreover, $f_1(x)$ is nondecreasing in $x \le a$ based on the conditions (1) and (2). Then, $f_1(a) \ge f_1(a-1)$ and hence $F(y) - F(y-1)$ is nonincreasing when we increase y from a to $a+1$.

As a result, $F(y)$ is discrete concave in the whole domain. ■

Main Proofs

Proof of Lemma 1.

To mathematically show the results of this lemma, we shall prove that (1) $\bar{v}_t(\mathbf{c}_t + \delta(\mathbf{e}_j - \mathbf{e}_k), \mathbf{w}_t) + \delta[(u_j - h_j) - (u_k - h_k)] \ge \bar{v}_t(\mathbf{c}_t, \mathbf{w}_t)$ for all $1 \le k < j \le m$ and $0 \le \delta \le c_{k,t}$, i.e., in period $t-1$ it is optimal to use the capacity in supplier j to fill demand only if there is no capacity in suppliers $1, \cdots, j-1$; (2) $\bar{v}_t(\mathbf{c}_t, \mathbf{w}_t + \delta(\mathbf{e}_l - \mathbf{e}_i)) + \delta[(q_i + b_i) - (q_l + b_l)] \ge \bar{v}_t(\mathbf{c}_t, \mathbf{w}_t)$ for all $1 \le l < i \le n$ and $0 \le \delta \le w_{i,t}$, i.e., in period $t-1$ it is optimal to fulfill the class l demand only if the demands in classes $l+1, \cdots, n$ are fully fulfilled. Note that, if the system state at the beginning of period t is $(\mathbf{c}_t + \delta(\mathbf{e}_j - \mathbf{e}_k), \mathbf{w}_t)$ instead of $(\mathbf{c}_t, \mathbf{w}_t)$, then we must have used δ units of the capacity in supplier k to replace the same quantity of the capacity in supplier j for fulfilling demand in period $t-1$, which reduces the ordering cost by $\delta(u_j - u_k)$ and the holding cost of $\delta(h_k - h_j)$. This fact explains why we have the item $\delta[(u_j - h_j) - (u_k - h_k)]$ in inequality (1). Due to a similar reason, we have the item $\delta[(q_i + b_i) - (q_l + b_l)]$ in the inequality (2).

In the following, we show the two inequalities by induction. The inequalities hold in period $T+1$. Suppose that they hold in period $t+1$, i.e., (1) $\bar{v}_{t+1}(\mathbf{c}_t + \delta(\mathbf{e}_j - \mathbf{e}_k), \mathbf{w}_t) + \delta[(u_j - h_j) - (u_k - h_k)] \ge \bar{v}_{t+1}(\mathbf{c}_t, \mathbf{w}_t)$ for all $1 \le k < j \le m$ and $0 \le \delta \le c_{k,t}$; (2) $\bar{v}_{t+1}(\mathbf{c}_t, \mathbf{w}_t + \delta(\mathbf{e}_l - \mathbf{e}_i)) + \delta[(q_i + b_i) - (q_l + b_l)] \ge \bar{v}_{t+1}(\mathbf{c}_t, \mathbf{w}_t)$ for all $1 \le l < i \le n$ and $0 \le \delta \le w_{i,t}$. It suffices to show that the inequalities hold for period t as well.

(1) Suppose that, under $(\mathbf{c}_t, \mathbf{w}_t)$ in period t, the optimal allocation policy is $\mathbf{a}_t^*$. Consider another state $(\mathbf{c}_t + \delta(\mathbf{e}_j - \mathbf{e}_k), \mathbf{w}_t)$, a feasible allocation policy is that $\tilde{a}_{il,t} = a_{il,t}^*$ for $i = 1, \cdots, n$, $l = 1, \cdots, m, l \ne k, j$, and $\tilde{a}_{ik,t} = a_{ik,t}^* - \bar{\delta}$, $\tilde{a}_{ij,t} = a_{ij,t}^* + \bar{\delta}$ for $\bar{\delta} = \min\{a_{ij,t}^*, \delta\}$. Due to the inductive assumption and $0 \le \bar{\delta} \le \delta$,

$$\bar{g}_t(\mathbf{c}_t, \mathbf{w}_t) - \{\bar{g}_t(\mathbf{c}_t + \delta(\mathbf{e}_j - \mathbf{e}_k), \mathbf{w}_t) + \delta[(u_j - h_j) - (u_k - h_k)]\}$$

$$
\begin{aligned}
&\le \bar{v}_{t+1}\left(\mathbf{c}_t - \sum_{l=1}^{m}\sum_{i=1}^{n} a^*_{il,t}\mathbf{e}_l, \mathbf{w}_t - \sum_{l=1}^{m}\sum_{i=1}^{n} a^*_{il,t}\mathbf{e}_i\right) - \sum_{l=1}^{m}\sum_{i=1}^{n} a^*_{il,t}u_l - \sum_{l=1}^{m}\left(c_{l,t} - \sum_{i=1}^{m} a^*_{il,t}\right)h_l \\
&\quad - \bar{v}_{t+1}\left(\mathbf{c}_t + \delta(\mathbf{e}_j - \mathbf{e}_k) - \sum_{l=1}^{m}\sum_{i=1}^{n} \tilde{a}_{il,t}\mathbf{e}_l, \mathbf{w}_t - \sum_{l=1}^{m}\sum_{i=1}^{n} \tilde{a}_{il,t}\mathbf{e}_i\right) + \sum_{l=1}^{m}\sum_{i=1}^{n} \tilde{a}_{il,t}u_l + \sum_{l=1}^{m}\left(c_{l,t} - \sum_{i=1}^{m} a^*_{il,t}\right)h_l \\
&\quad - \delta[(u_j - h_j) - (u_k - h_k)] \\
&= \bar{v}_{t+1}\left(\mathbf{c}_t - \sum_{l=1}^{m}\sum_{i=1}^{n} a^*_{il,t}\mathbf{e}_l, \mathbf{w}_t - \sum_{l=1}^{m}\sum_{i=1}^{n} a^*_{il,t}\mathbf{e}_i\right) - \bar{v}_{t+1}\left(\mathbf{c}_t + \delta(\mathbf{e}_j - \mathbf{e}_k) - \sum_{l=1}^{m}\sum_{i=1}^{n} \tilde{a}_{il,t}\mathbf{e}_l, \mathbf{w}_t\right. \\
&\quad \left. - \sum_{l=1}^{m}\sum_{i=1}^{n} \tilde{a}_{il,t}\mathbf{e}_i\right) - (\delta - \bar{\delta})[(u_j - h_j) - (u_k - h_k)] \le 0,
\end{aligned}
$$

where the first inequality holds as $\tilde{\mathbf{a}}_t$ is only a feasible policy.

Since $\bar{v}_t(\mathbf{c}_t, \mathbf{w}_t) = \sum_{k=1}^{n} \lambda_{k,t}\bar{g}_t(\mathbf{c}_t, \mathbf{w}_t + \mathbf{e}_k) + (1 - \lambda_t)\bar{g}_t(\mathbf{c}_t, \mathbf{w}_t)$, the above inequality leads directly to the result $\bar{v}_t(\mathbf{c}_t, \mathbf{w}_t) \le \bar{v}_t(\mathbf{c}_t + \delta(\mathbf{e}_j - \mathbf{e}_k), \mathbf{w}_t) + \delta[(u_j - h_j) - (u_k - h_k)]$.

(2). Suppose that, under the state $(\mathbf{c}_t, \mathbf{w}_t)$ in period t, the optimal allocation policy is $\mathbf{a}^*_t$. Then under the state $(\mathbf{c}_t, \mathbf{w}_t + \delta(\mathbf{e}_l - \mathbf{e}_i))$ in period t, a feasible policy is that $\tilde{a}_{hj,t} = a^*_{hj,t}$ for $j = 1, \cdots, m$, $h = 1, \cdots, n$ but $h \ne i, l$, $\tilde{a}_{ij,t} = a^*_{ij,t} - \bar{\delta}$, and $\tilde{a}_{lj,t} = a^*_{lj,t} + \bar{\delta}$ for $\bar{\delta} = \min\{\delta, a^*_{ij,t}\}$. Then we have

$$
\begin{aligned}
&\bar{g}_t(\mathbf{c}_t, \mathbf{w}_t) - \{\bar{g}_t(\mathbf{c}_t, \mathbf{w}_t + \delta(\mathbf{e}_l - \mathbf{e}_i)) + \delta[(q_i + b_i) - (q_l + b_l)]\} \\
&\le \bar{v}_{t+1}\left(\mathbf{c}_t - \sum_{j=1}^{m}\sum_{h=1}^{n} a^*_{hj,t}\mathbf{e}_j, \mathbf{w}_t - \sum_{j=1}^{m}\sum_{h=1}^{n} a^*_{hj,t}\mathbf{e}_h\right) - \bar{v}_{t+1}\left(\mathbf{c}_t - \sum_{j=1}^{m}\sum_{h=1}^{n} a^*_{hj,t}\mathbf{e}_j, \mathbf{w}_t + (\delta - \bar{\delta})(\mathbf{e}_l - \mathbf{e}_i)\right. \\
&\quad \left. - \sum_{j=1}^{m}\sum_{h=1}^{n} a^*_{hj,t}\mathbf{e}_h\right) - (\delta - \bar{\delta})[(q_i + b_i) - (q_l + b_l)] \le 0
\end{aligned}
$$

due to the inductive assumption and $0 \le \bar{\delta} \le \delta$. It follows that $\bar{v}_t(\mathbf{c}_t, \mathbf{w}_t) \le \bar{v}_t(\mathbf{c}_t, \mathbf{w}_t + \delta(\mathbf{e}_l - \mathbf{e}_i)) + \delta[(q_i + b_i) - (q_l + b_l)]$. ■

Proof of Lemma 2.

At the beginning of the selling season, i.e. the period 1, it is easy to verify that $\mathbf{z}_{c,1} = z_t \wedge \Theta$ for $z_t = \theta_1$. Suppose that in period $t-1$ we use a part of the capacity in supplier j and in period t we have $z_{j,t} = z$. Based on Lemma 1, there is no capacity in suppliers $1, \cdots, j-1$ and the capacity in suppliers $j+1, \cdots, m$ has not been used yet. Then, at the beginning of period t, $z_{1,t} = \cdots = z_{j-1,t} = z_{j,t} = z$, $\theta_{j+1} \le z < \theta_j$, and $z_{k,t} = \theta_k$ for $k = j+1, \cdots, m$. Hence, $\mathbf{z}_{c,t} = z_t \wedge \Theta$ for $z_t = z$. As a result, for every period t, $\mathbf{z}_{c,t} = z_t \wedge \Theta$ for some variable $z_t \in [0, \theta_1]$. By definitions of $z_{m+i,t}$ and $\bar{z}_{m+i,t}$, $\bar{z}_{m+i,t} \le z_{m+i,t} \le z_{1,t}$ in every period t. Since $z_{1,t} = z_t \wedge \theta_1$, we must have $z_{m+i,t} \le z_t \wedge \theta_1 = z_t$ as $z_t \le \theta_1$.

Recall that z_{t+1} is the total quantity of the reserved capacity in all suppliers at the beginning of period $t+1$, or equivalently at the end of period t. Then, based on Lemma 1, $z_{m+k,t+1} = z_{t+1}$

for $k=i,\cdots,n$ and $z_{m+k,t+1}=\bar{z}_{m+k,t}$ for $k=1,\cdots,i-1$ if it is optimal to fulfill a part of the class $i-1$ demand in period t. Hence, $\mathbf{z}_{w,t+1}=z_{t+1}\wedge\mathbf{z}_{w,t}$. ■

The equivalence between the MDP in Eq. (4) - (5) and the MDP in Eq. (1) - Eq. (2). First of all, we use $(z_t\wedge\Theta,\mathbf{z}_{w,t})$ to denote the system state based on Lemma 2. In fact, once we know the state $(z_t\wedge\Theta,\mathbf{z}_{w,t})$, we know exactly the state $(\mathbf{c}_t,\mathbf{w}_t)$, and vice versa. Eq. (4) is simply resulted from our assumption that at most one unit of demand realizes in each period. It is exactly the same with Eq. (1) as long as Eq. (5) is equivalent to Eq. (2). Below, we show that indeed Eq. (5) is equivalent to Eq. (2).

Eq. (5) is resulted from Lemma 1. Specifically, the quantity of the used capacity in supplier j is $(z_t\wedge\theta_j-z_t\wedge\theta_{j+1})-(z_{t+1}\wedge\theta_j-z_{t+1}\wedge\theta_{j+1})$ as $\theta_j-\theta_{j+1}$ is the capacity if supplier j before the selling horizon. Hence, $\sum_{j=1}^m u_j[(z_t\wedge\theta_j-z_t\wedge\theta_{j+1})-(z_{t+1}\wedge\theta_j-z_{t+1}\wedge\theta_{j+1})]$ denotes the usage cost in period t, which equals to $\sum_{j=1}^m\sum_{i=1}^n a_{ij,t}u_j$. Accordingly, $\sum_{j=1}^m h_j(z_{t+1}\wedge\theta_j-z_{t+1}\wedge\theta_{j+1})$ denotes the holding cost in period t, which equals to $\sum_{j=1}^m(c_{j,t}-\sum_{i=1}^n a_{ij,t})h_j$ in Eq. (2). Moreover, the quantity of the fulfilled demand in class i is $\bar{z}_{m+i+1,t}-(\bar{z}_{m+i,t}\vee z_{t+1})\wedge\bar{z}_{m+i+1,t}$ as we fulfill demand class i only if $\bar{z}_{m+i,t}\le z_{t+1}<\bar{z}_{m+i+1,t}$ and accordingly the quantity of the backlogged demand in class i at the end of period t is $(\bar{z}_{m+i,t}\vee z_{t+1})\wedge\bar{z}_{m+i+1,t}-\bar{z}_{m+i,t}$. Hence, in period t, the revenue is $\sum_{i=1}^n q_i(\bar{z}_{m+i+1,t}-(\bar{z}_{m+i,t}\vee z_{t+1})\wedge\bar{z}_{m+i+1,t})$ and the waiting cost is $\sum_{i=1}^n b_i((\bar{z}_{m+i,t}\vee z_{t+1})\wedge\bar{z}_{m+i+1,t}-\bar{z}_{m+i,t})$, which are equal to $\sum_{j=1}^m\sum_{i=1}^n a_{ij,t}q_i$ and $\sum_{i=1}^n b_i(w_{i,t}-\sum_{j=1}^m a_{ij,t})$ respectively in Eq. (2). Hence, the objective function in Eq. (5) is equivalent to that in Eq. (2).

For the feasible region of z_{t+1}, we explain it as follows. The echelon capacity state is $z_t\wedge\Theta$ at the beginning of period t and it transits to $z_{t+1}\wedge\Theta$ at the end of period t after the capacity allocation decision. Then, $0\vee\bar{z}_{m+1,t}\le z_{t+1}\le z_t$, where $0\le z_{t+1}\le z_t$ holds due to the definitions of z_{t+1} and z_t while $\bar{z}_{m+1,t}\le z_{t+1}$ holds as $\bar{z}_{m+1,t}$ is the total remaining capacity of m suppliers if we fulfill all the demands. Based on Lemma 1, the optimal solution of the maximization problem in Eq. (5) implies the optimal solution of the maximization problem in Eq. (2). Hence, through the above discussion, we have shown that the MDP in Eq. (4) - Eq. (5) is equivalent to the MDP in Eq. (1) - Eq. (2) as they lead to the same optimal capacity allocation strategy.

Proof of Lemma 3.

Part (1): By definition of s_j, $s_j\ge 0$ holds for any $j=0,\cdots,n$. The property $s_0\ge\cdots\ge s_n$ results from the fact that $f_j(\cdot)$'s are nondecreasing and also the result in Therorem 2.7.7 of Topkis (1998).

We then shall show that $S(\mathbf{y}|\mathbf{s})$ is the global maximizer of $F(x,\mathbf{y})$. Note that there is one and only one indicator function equal to 1 in the expression of $S(\mathbf{y}|\mathbf{s})$ due to $s_0\ge\cdots\ge s_n$ while $y_1\le\cdots\le y_n$. Suppose that there exists some j, $j\in\{1,\cdots,n-1\}$, such that $y_j\le s_j<y_{j+1}$. Then, we must have

$s_k \geq y_{k+1}$ for $k=0,\cdots,j-1$ and $s_k < y_k$ for $k=j+1,\cdots,n$. As $f(\cdot)$ and $f_j(\cdot)$'s for $j=1,\cdots,n$ are discrete concave, $F(x,\mathbf{y})$ is nondecreasing in x for $x \leq s_j$ while it is nonincreasing in x for $x \geq s_j$. Hence, in this case, s_j is the global maximizer of $F(x,\mathbf{y})$. Similar results can be obtained when $s_0 < y_1$ and when $s_n \geq y_n$. If there exists some j such that $s_j < y_j \leq s_{j-1}$, $j=1,\cdots,n$, we must have that $s_k \geq y_{k+1}$ for $k=0,\cdots,j-1$ and $s_k < y_k$ for $k=j,\cdots,n$. As $f(\cdot)$ and $f_j(\cdot)$'s for $j=1,\cdots,n$ are discrete concave, $F(x,\mathbf{y})$ must be nondecreasing in x for $x \leq y_j$ but nonincreasing in x for $x \geq y_j$. Hence, in this case, y_j is the global maximizer of $F(x,\mathbf{y})$.

We thus have shown that $S(\mathbf{y}|\mathbf{s})$ is the global maximizer of $F(x,\mathbf{y})$. Then the optimal solution is $x^* = ((0 \vee y_0) \vee S(\mathbf{y}|\mathbf{s})) \wedge y_{n+1} = (y_0 \vee S(\mathbf{y}|\mathbf{s})) \wedge y_{n+1}$ as $S(\mathbf{y}|\mathbf{s}) \geq 0$ holds due to $s_0 \geq \cdots \geq s_n \geq 0$.

Part (2): We first show the decomposition of $g(y_0, y_{n+1}, \mathbf{y})$. Based on the result in part (1),

$$\begin{aligned} g(y_0,y_{n+1},\mathbf{y}) =& \mathbb{I}_{\{s_0<y_1\}} F((y_0 \vee s_0) \wedge y_{n+1}, \mathbf{y}) + \sum_{j=1}^{n-1} \mathbb{I}_{\{y_j \leq s_j < y_{j+1}\}} F((y_0 \vee s_j) \wedge y_{n+1}, \mathbf{y}) \\ &+ \mathbb{I}_{\{s_n \geq y_n\}} F((y_0 \vee s_n) \wedge y_{n+1}, \mathbf{y}) + \sum_{j=1}^{n} \mathbb{I}_{\{s_j < y_j \leq s_{j-1}\}} F(y_j, \mathbf{y}). \end{aligned} \tag{13}$$

Since $y_0 \leq y_j \leq y_{n+1}$ for any $j=1,\cdots,n$, we have

$$\begin{cases} \mathbb{I}_{\{s_0<y_1\}} F((y_0 \vee s_0) \wedge y_{n+1}, \mathbf{y}) = F((y_0 \vee s_0) \wedge y_1, \mathbf{y}) - \mathbb{I}_{\{s_0 \geq y_1\}} F(y_1, \mathbf{y}) \\ \mathbb{I}_{\{y_j \leq s_j < y_{j+1}\}} F((y_0 \vee s_j) \wedge y_{n+1}, \mathbf{y}) = F((y_j \vee s_j) \wedge y_{j+1}, \mathbf{y}) - \mathbb{I}_{\{y_j > s_j\}} F(y_j, \mathbf{y}) - \mathbb{I}_{\{s_j \geq y_{j+1}\}} F(y_{j+1}, \mathbf{y}), \\ \mathbb{I}_{\{s_n \geq y_n\}} F((y_0 \vee s_n) \wedge y_{n+1}, \mathbf{y}) = F((y_n \vee s_n) \wedge y_{n+1}, \mathbf{y}) - \mathbb{I}_{\{s_n < y_n\}} F(y_n, \mathbf{y}) \\ \mathbb{I}_{\{s_j < y_j \leq s_{j-1}\}} F(y_j, \mathbf{y}) = F(y_j, \mathbf{y}) - \mathbb{I}_{\{s_j \geq y_j\}} F(y_j, \mathbf{y}) - \mathbb{I}_{\{y_j > s_{j-1}\}} F(y_j, \mathbf{y}). \end{cases}$$

Then, $g(y_0, y_{n+1}, \mathbf{y})$ can be rewritten as

$$\begin{aligned} g(y_0,y_{n+1},\mathbf{y}) =& \sum_{j=0}^{n} F((y_j \vee s_j) \wedge y_{j+1}, \mathbf{y}) - \sum_{j=1}^{n} F(y_j, \mathbf{y}) \\ =& F(y_0, \mathbf{y}) + \sum_{j=0}^{n} F(s_j \wedge y_{j+1}, \mathbf{y}) - \sum_{j=0}^{n} F(y_j \wedge s_j, \mathbf{y}). \end{aligned} \tag{14}$$

The second equation holds since $F((y_j \vee s_j) \wedge y_{j+1}, \mathbf{y}) = F(y_j, \mathbf{y}) + F(s_j \wedge y_{j+1}, \mathbf{y}) - F(y_j \wedge s_j, \mathbf{y})$.

Recall that $F(x,\mathbf{y}) = f(x) + \sum_{j=1}^{n} f_j(x \wedge y_j)$. Based on the equation (14),

$$g(y_0, y_{n+1}, \mathbf{y}) = \sum_{j=0}^{n+1} g_j(y_j),$$

where

$$g_j(y_j) = \begin{cases} f(y_0) + \sum_{k=1}^{n} f_k(y_0), & j=0, \\ f(s_{j-1} \wedge y_j) + \sum_{k=j}^{n} f_k(s_{j-1} \wedge y_j) - f(s_j \wedge y_j) - \sum_{k=j+1}^{n} f_k(s_j \wedge y_j), & j=1,\cdots,n-1, \\ f(s_{n-1} \wedge y_n) + f_n(s_{n-1} \wedge y_n) - f(s_n \wedge y_n), & j=n, \\ f(s_n \wedge y_{n+1}) & j=n+1. \end{cases}$$

We thus have shown the decomposition of $g(y_0, y_{n+1}, \mathbf{y})$.

For the discrete concavity of $g(y_0, y_{n+1}, \mathbf{y})$, as it is decomposable, we only need to show that each $g_j(y_j)$ is discrete concave. For $j=1,\cdots,n$, $g_j(y_j)$ must be discrete concave due to Lemma 8; for $j=0$, $g_j(y_j)$ is discrete concave as $f(\cdot)$ and $f_k(\cdot)$ for $k=1,\cdots,n$ are all discrete concave; for $j=n+1$, $g_j(y_j)$ is discrete concave based on Lemma 7. ■

Proof of Lemma 4

(1) To show that $v_t(z_t\wedge\Theta, \mathbf{z}_{w,t}) + (z_t\wedge\theta_j)[(u_j-h_j)-(u_{j-1}-h_{j-1})]$ is nondecreasing in $z_t\wedge\theta_j$, we shall show that $v_t(z_t\wedge\Theta+\delta\mathbf{e}_j, \mathbf{z}_{w,t}) + (z_t\wedge\theta_j+\delta)[(u_j-h_j)-(u_{j-1}-h_{j-1})] \geq v_t(z_t\wedge\Theta, \mathbf{z}_{w,t}) + (z_t\wedge\theta_j)[(u_j-h_j)-(u_{j-1}-h_{j-1})]$, i.e.,

$$v_t(z_t\wedge\Theta+\delta\mathbf{e}_j, \mathbf{z}_{w,t}) + \delta[(u_j-h_j)-(u_{j-1}-h_{j-1})] \geq v_t(z_t\wedge\Theta, \mathbf{z}_{w,t}). \tag{15}$$

Increasing $z_t\wedge\theta_j$ by δ is equivalent to increasing $c_{j,t}$ and decreasing $c_{j-1,t}$ by δ. Then, based on the inequality (1) in the proof of Lemma 1, the inequality (15) must hold for any $2\leq j\leq m$.

(2) Similarly, to show that $v_t(z_t\wedge\Theta, \mathbf{z}_{w,t}) + z_{m+i,t}[(q_i+b_i)-(q_{i-1}+b_{i-1})]$ is nondecreasing in $z_{m+i,t}$, we shall show that

$$v_t(z_t\wedge\Theta, \mathbf{z}_{w,t}+\delta\mathbf{e}_i) + \delta[(q_i+b_i)-(q_{i-1}+b_{i-1})] \geq v_t(z_t\wedge\Theta, \mathbf{z}_{w,t}). \tag{16}$$

Increasing $z_{m+i,t}$ by δ is equivalent to decreasing $w_{i,t}$ and increasing $w_{i-1,t}$ by δ. Then, based on the inequality (2) in the proof of Lemma 1, the inequality (16) must hold for any $2\leq i\leq n$. ■

Proof of Theorem 1.

We show the results by induction. In the terminal period, v_{T+1} is clearly discrete concave and decomposable. Suppose that $v_{t+1}(z_{t+1}\wedge\Theta, \mathbf{z}_{w,t+1})$ is discrete concave in $(z_{t+1}\wedge\Theta, \mathbf{z}_{w,t+1})$ and can be decomposed to $\sum_{j=1}^m \hat{v}_{j,t+1}(z_{t+1}\wedge\theta_j) + \sum_{i=1}^n \bar{v}_{i,t+1}(z_{m+i,t+1})$, it suffices to show that $v_t(z_t\wedge\Theta, \mathbf{z}_{w,t})$ is also discrete concave and decomposable. When showing the discrete concavity and decomposition, we also prove the details of the optimal policy.

The decomposition of $v_{t+1}(z_{t+1}\wedge\Theta, \mathbf{z}_{w,t+1})$ implies that $g_t(z_t\wedge\Theta, \bar{\mathbf{z}}_{w,t}) = \max_{0\vee\bar{z}_{m+1,t}\leq z_{t+1}\leq z_t} G_t(z_t\wedge\Theta, z_{t+1}, \bar{\mathbf{z}}_{w,t})$, where

$$\begin{aligned} G_t(z_t\wedge\Theta, z_{t+1}, \bar{\mathbf{z}}_{w,t}) = &\sum_{j=1}^m \hat{v}_{j,t+1}(z_{t+1}\wedge\theta_j) + \bar{v}_{1,t+1}(\bar{z}_{m+1,t}) + \sum_{i=2}^n \bar{v}_{i,t+1}(z_{t+1}\wedge\bar{z}_{m+i,t}) - u_1(z_t\wedge\theta_1)\\ &-\sum_{j=2}^m (u_j-u_{j-1})(z_t\wedge\theta_j) + (u_1-h_1)z_{t+1} + \sum_{j=2}^m (u_j-h_j-u_{j-1}+h_{j-1})(z_{t+1}\wedge\theta_j)\\ &+(q_1+b_1)(\bar{z}_{m+1,t}-z_{t+1}\wedge\bar{z}_{m+2,t}) + \sum_{i=1}^n q_i(\bar{z}_{m+i+1,t}-\bar{z}_{m+i,t}) \end{aligned}$$

$$+\sum_{i=2}^{n-1}(q_i+b_i)(z_{t+1}\wedge\bar{z}_{m+i,t}-z_{t+1}\wedge\bar{z}_{m+i+1,t})+(q_n+b_n)(z_{t+1}\wedge\bar{z}_{m+n,t}-z_{t+1}).$$

The expression of G_t is derived based on the decomposition of v_{t+1}, the equation $(a\vee x)\wedge b = a+x\wedge b-a\wedge x$, and the inequalities $0\vee\bar{z}_{m+1,t}\le z_{t+1}\le z_t\le\theta_1$. Note that we define $\bar{z}_{m+n+1,t}\equiv z_{1,t}=z_t\wedge\theta_1=z_t$.

By collecting the items depending on $z_{t+1}\wedge\bar{z}_{m+i,t}$ and z_{t+1}, respectively, in the expression of G_t, we define

$$\hat{f}_{i,t}(z_{t+1}\wedge\bar{z}_{m+i,t})=\bar{v}_{i,t+1}(z_{t+1}\wedge\bar{z}_{m+i,t})+(q_i+b_i-q_{i-1}-b_{i-1})(z_{t+1}\wedge\bar{z}_{m+i,t}),\quad i=2,\cdots,n,$$
$$f_t(z_{t+1})=\sum_{j=1}^{m}\hat{v}_{j,t+1}(z_{t+1}\wedge\theta_j)+(u_1-h_1-q_n-b_n)z_{t+1}+\sum_{j=2}^{m}(u_j-h_j-u_{j-1}+h_{j-1})(z_{t+1}\wedge\theta_j).$$

Similarly, by collecting the items depending on z_t and $\bar{z}_{m+i,t}$ for $i=1,\cdots,n$, we define

$$\tilde{f}_t(z_t\wedge\Theta,\bar{\mathbf{z}}_{w,t})=\bar{v}_{1,t+1}(\bar{z}_{m+1,t})+b_1\bar{z}_{m+1,t}+\sum_{i=1}^{n-1}(q_i-q_{i+1})\bar{z}_{m+i+1,t}+(q_n-u_1)(z_t\wedge\theta_1)$$
$$-\sum_{j=2}^{m}(u_j-u_{j-1})(z_t\wedge\theta_j).$$

Then, $G_t(z_t\wedge\Theta,z_{t+1},\bar{\mathbf{z}}_{w,t})=f_t(z_{t+1})+\sum_{i=2}^{n}\hat{f}_{i,t}(z_{t+1}\wedge\bar{z}_{m+i,t})+\tilde{f}_t(z_t\wedge\Theta,\bar{\mathbf{z}}_{w,t})$. Further define

$$\tilde{g}_t(z_t,\bar{\mathbf{z}}_{w,t})=\max_{0\vee\bar{z}_{m+1,t}\le z_{t+1}\le z_t}\left[f_t(z_{t+1})+\sum_{i=2}^{n}\hat{f}_{i,t}(z_{t+1}\wedge\bar{z}_{m+i,t})\right].\tag{17}$$

Then, $g_t(z_t\wedge\Theta,\bar{\mathbf{z}}_{w,t})$ in Eq. (5) can be equivalently expressed as

$$g_t(z_t\wedge\Theta,\bar{\mathbf{z}}_{w,t})=\tilde{g}_t(z_t,\bar{\mathbf{z}}_{w,t})+\tilde{f}_t(z_t\wedge\Theta,\bar{\mathbf{z}}_{w,t}).$$

Eq. (8) indicates that $R_{i,t}$, $i=1,\cdots,n$, is the global optimum of the decision variable z_{t+1} in Eq. (17) or equivalently Eq. (5) when $z_{t+1}<\bar{z}_{m+2,t}$ for $i=1$, $\bar{z}_{m+i,t}\le z_{t+1}<\bar{z}_{m+i+1,t}$ for $i=2,\cdots,n-1$ and $z_{t+1}\ge\bar{z}_{m+n,t}$ for $i=n$.

Note that the function $f_t(z_{t+1})$ is discrete concave in z_{t+1} because (1) $\hat{v}_{1,t+1}(z_{t+1}\wedge\theta_1)=\hat{v}_{1,t+1}(z_{t+1})$, which is discrete concave in z_{t+1}; (2) $\hat{v}_{j,t+1}(z_{t+1}\wedge\theta_j)+(u_j-h_j-u_{j-1}+h_{j-1})(z_{t+1}\wedge\theta_j)$ is discrete concave in z_{t+1} due to the inductive assumption and Lemmas 4 and 7; (3) $(u_1-h_1-q_n-b_n)z_{t+1}$ is linear in z_{t+1}. Moreover, the function $\hat{f}_{i,t}(z_{t+1}\wedge\bar{z}_{m+i,t})$ is discrete concave and nondecreasing in $z_{t+1}\wedge\bar{z}_{m+i,t}$ due to the inductive assumption and Lemma 4.

Based on Lemma 3 and Eq. (8), the discrete concavity and/or monotonicity of the functions $f_t(\cdot)$ and $\hat{f}_{i,t}(\cdot)$ discussed above lead to the following results: (1) $R_{1,t}\ge\cdots\ge R_{n,t}\ge 0$ and (2) the optimal solution of z_{t+1} in Eq. (17) or equivalently Eq. (5) is $z^*_{t+1}=$

$(\bar{z}_{m+1,t} \vee \mathcal{R}_t(\bar{\mathbf{z}}_{w,t}|\mathbf{R}_t)) \wedge z_t$ for $\mathcal{R}_t(\bar{\mathbf{z}}_{w,t}|\mathbf{R}_t) = \mathbb{I}_{\{R_{1,t}<\bar{z}_{m+2,t}\}} R_{1,t} + \sum_{i=2}^{n-1} \mathbb{I}_{\{\bar{z}_{m+i,t}\le R_{i,t}<\bar{z}_{m+i+1,t}\}} R_{i,t} + \mathbb{I}_{\{\bar{z}_{m+n,t}\le R_{i,t}\}} R_{n,t} + \sum_{i=1}^{n-1} \mathbb{I}_{\{R_{i+1,t}<\bar{z}_{m+i+1,t}\le R_{i,t}\}} \bar{z}_{m+i+1,t}$.

Lemma 3 also indicates that $\tilde{g}_t(z_t, \bar{\mathbf{z}}_{w,t})$ is discrete concave in $(z_t, \bar{\mathbf{z}}_{w,t})$ (or equivalently $(z_t \wedge \theta_1, \bar{\mathbf{z}}_{w,t})$ as $z_t = z_t \wedge \theta_1$) and decomposable. Moreover, $\tilde{f}_t(z_t \wedge \Theta, \bar{\mathbf{z}}_{w,t})$ is discrete concave in $(z_t \wedge \Theta, \bar{\mathbf{z}}_{w,t})$ and decomposable, which can be directly observed from its expression. Hence, $g_t(z_t \wedge \Theta, \bar{\mathbf{z}}_{w,t}) = \tilde{g}_t(z_t, \bar{\mathbf{z}}_{w,t}) + \tilde{f}_t(z_t \wedge \Theta, \bar{\mathbf{z}}_{w,t})$ must be discrete concave in $(z_t \wedge \Theta, \bar{\mathbf{z}}_{w,t})$ and decomposable, i.e., $g_t(z_t \wedge \Theta, \bar{\mathbf{z}}_{w,t}) = \sum_{j=1}^{m} \hat{g}_{j,t}(z_t \wedge \theta_j) + \sum_{i=1}^{n} \bar{g}_{i,t}(\bar{z}_{m+i,t})$. Based on Lemma 3, the component functions of $g_t(z_t \wedge \Theta, \bar{\mathbf{z}}_{w,t})$ are listed below (recall that $z_t \wedge \theta_1 = z_t$):

$$
\begin{aligned}
\hat{g}_{1,t}(z) =& (q_n - u_1)z + \hat{v}_{1,t+1}(R_{n,t} \wedge z) + (u_1 - h_1 - q_n - b_n)(R_{n,t} \wedge z),\\
\hat{g}_{j,t}(z) =& -(u_j - u_{j-1})z + \hat{v}_{j,t+1}(R_{n,t} \wedge z) + (u_j - h_j - u_{j-1} + h_{j-1})(R_{n,t} \wedge z), \quad j = 2, \cdots, m,\\
\bar{g}_{1,t}(z) =& \sum_{i=1}^{n} \bar{v}_{i,t+1}(z) + \hat{v}_{1,t+1}(z) + (u_1 - h_1 - q_1)z\\
&+ \sum_{j=2}^{m} \left[\hat{v}_{j,t+1}(z \wedge \theta_j) + (u_j - h_j - u_{j-1} + h_{j-1})(z \wedge \theta_j)\right],\\
\bar{g}_{i,t}(z) =& (q_{i-1} - q_i)z + \left[\hat{v}_{1,t+1}(R_{i-1,t} \wedge z) + (u_1 - h_1 - q_{i-1} - b_{i-1})(R_{i-1,t} \wedge z)\right]\\
&- \left[\hat{v}_{1,t+1}(R_{i,t} \wedge z) + (u_1 - h_1 - q_i - b_i)(R_{i,t} \wedge z)\right]\\
&+ \sum_{j=2}^{m} \left[\hat{v}_{j,t+1}(R_{i-1,t} \wedge z \wedge \theta_j) + (u_j - h_j - u_{j-1} + h_{j-1})(R_{i-1,t} \wedge z \wedge \theta_j)\right]\\
&- \sum_{j=2}^{m} \left[\hat{v}_{j,t+1}(R_{i,t} \wedge z \wedge \theta_j) + (u_j - h_j - u_{j-1} + h_{j-1})(R_{i,t} \wedge z \wedge \theta_j)\right]\\
&+ \sum_{k=i}^{n} \bar{v}_{k,t+1}(R_{i-1,t} \wedge z) - \sum_{k=i+1}^{n} \bar{v}_{k,t+1}(R_{i,t} \wedge z), \qquad i = 2, \cdots, n.
\end{aligned}
$$

As $g_t(z_t \wedge \Theta, \bar{\mathbf{z}}_{w,t})$ is discrete concave and decomposable, $v_t(z_t \wedge \Theta, \mathbf{z}_{w,t})$ must be discrete concave and decomposable since discrete concavity and decomposition preserves under summation. This completes the inductive proof. The component function of v_t is obtained based on the component functions of g_t as

$$
\begin{aligned}
v_t(z_t \wedge \Theta, \mathbf{z}_{w,t}) =& \sum_{r=1}^{n} \lambda_{r,t} \left[\sum_{j=1}^{m} \hat{g}_{j,t}(z_t \wedge \theta_j) + \sum_{i=1}^{r} \bar{g}_{i,t}(z_{m+i,t} - 1) + \sum_{i=r+1}^{n} \bar{g}_{i,t}(z_{m+i,t})\right]\\
&+ (1 - \lambda_t) \left[\sum_{j=1}^{m} \hat{g}_{j,t}(z_t \wedge \theta_j) + \sum_{i=1}^{n} \bar{g}_{i,t}(z_{m+i,t})\right],
\end{aligned}
$$

where the functions $\hat{g}_{j,t}(\cdot)$ for $j = 1, \cdots, m$ and $\bar{g}_{i,t}(\cdot)$ for $i = 1, \cdots, n$ have been shown above. ■

Proof of Lemma 5.

Let $\mathbf{c}_t = (c_{1,t}, \cdots, c_{m,t})$ be the capacity state at the beginning of period t. Then, we have the following MDP for impatient customers:

$$
\bar{v}_t(\mathbf{c}_t) = \sum_{i=1}^{n} \lambda_{i,t} \max_{j=1,\cdots,m} \bar{g}_{ij,t}(\mathbf{c}_t) + (1 - \lambda_t)\left(\bar{v}_{t+1}(\mathbf{c}_t) - \sum_{k=1}^{m} c_{k,t} h_k\right), \tag{18}
$$

where

$$\bar{g}_{ij,t}(\mathbf{c}_t) = \begin{cases} \max\left\{\bar{v}_{t+1}(\mathbf{c}_t - \mathbf{e}_j) + q_i - u_j + h_j - \sum_{k=1}^m c_{k,t}h_k, \bar{v}_{t+1}(\mathbf{c}_t) - \sum_{k=1}^m c_{k,t}h_k\right\}, & c_{j,t} > 0, \\ \bar{v}_{t+1}(\mathbf{c}_t) - \sum_{k=1}^m c_{k,t}h_k, & c_{j,t} = 0. \end{cases}$$

In addition, $\bar{v}_{T+1}(\mathbf{c}_{T+1}) \equiv 0$ for any $\mathbf{c}_{T+1}$.

To prove this lemma, we shall show that $\bar{v}_t(\mathbf{c}_t - \mathbf{e}_k) - u_k + h_k \geq \bar{v}_t(\mathbf{c}_t - \mathbf{e}_j) - u_j + h_j$ for all $k < j$ by induction. The result holds in period $T+1$. Suppose that the result holds in period $t+1$. It suffices to show that it holds in period t as well.

Under the state $\mathbf{c}_t - \mathbf{e}_j$, we denote by $\mathbf{o}^*_{i,t} = (o^*_{i1,t}, \cdots, o^*_{im,t})$ the optimal allocation decision for demand class i, where $o^*_{ir,t} \in \{0,1\}$ and $\sum_{r=1}^m o^*_{ir,t} \leq 1$. Under another state $\mathbf{c}_t - \mathbf{e}_k$, a feasible allocation decision is $\tilde{\mathbf{o}}_{i,t}$, which is the same with $\mathbf{o}^*_{i,t}$ except that $\tilde{o}_{ij,t} = 1$ whenever $o^*_{ik,t} = 1$. Then,

$$\begin{aligned} &\bar{v}_t(\mathbf{c}_t - \mathbf{e}_k) - u_k + h_k \\ \geq &\sum_{i=1}^n \lambda_{i,t} \sum_{r=1}^m \left[\tilde{o}_{ir}(\bar{v}_{t+1}(\mathbf{c} - \mathbf{e}_k - \mathbf{e}_r) - u_k + h_k + q_i - u_r + h_r) + (1 - \tilde{o}_{ir})(\bar{v}_{t+1}(\mathbf{c} - \mathbf{e}_k) - u_k + h_k)\right] \\ &+ (1 - \lambda_t)\left(\bar{v}_{t+1}(\mathbf{c} - \mathbf{e}_k) - u_k + h_k\right). \end{aligned}$$

By inductive assumption,

$$\begin{cases} \bar{v}_{t+1}(\mathbf{c}_t - \mathbf{e}_k - \mathbf{e}_r) - u_k + h_k + q_i - u_r + h_r \geq \bar{v}_{t+1}(\mathbf{c}_t - \mathbf{e}_j - \mathbf{e}_r) - u_j + h_j + q_i - u_r + h_r, \\ \bar{v}_{t+1}(\mathbf{c}_t - \mathbf{e}_k) - u_k + h_k \geq \bar{v}_{t+1}(\mathbf{c}_t - \mathbf{e}_j) - u_j + h_j. \end{cases}$$

If $o^*_{ik,t} = 0$, then $\tilde{\mathbf{o}}_{i,t} = \mathbf{o}^*_{i,t}$ and hence $\bar{v}_t(\mathbf{c}_t - \mathbf{e}_k) - u_k + h_k \geq \bar{v}_t(\mathbf{c}_t - \mathbf{e}_j) - u_j + h_j$. If $o^*_{ik,t} = 1$, though $\tilde{\mathbf{o}}_{i,t} \neq \mathbf{o}^*_{i,t}$, we still have $\bar{v}_t(\mathbf{c}_t - \mathbf{e}_k) - u_k + h_k \geq \bar{v}_t(\mathbf{c}_t - \mathbf{e}_j) - u_j + h_j$ since $\bar{v}_{t+1}(\mathbf{c}_t - \mathbf{e}_k - \mathbf{e}_j) - u_k + h_k + q_i - u_j + h_j = \bar{v}_{t+1}(\mathbf{c}_t - \mathbf{e}_j - \mathbf{e}_k) - u_j + h_j + q_i - u_k + h_k$. ∎

Explanation of the MDP in Eq. (9) - Eq. (10).

As unfulfilled demand is lost immediately, with impatient customers, the system state is denoted by the echelon capacity state $z_t \wedge \Theta$ at the beginning of each period t. If no demand realizes in period t, the echelon capacity state at the beginning of period $t+1$ is still $z_t \wedge \Theta$ and there is a holding cost $\sum_{j=1}^m h_j(z_t \wedge \theta_j - z_t \wedge \theta_{j+1})$. If a unit of class r demand realizes, $r = 1, \cdots, n$, we make the optimal capacity allocation decision as in (10), where the decision variable z_{t+1} equals to z_t or $z_t - 1$. In period t, the retailer receives a revenue $q_r(z_t - z_{t+1})$, but pays a holding cost $h_j(z_{t+1} \wedge \theta_j - z_{t+1} \wedge \theta_{j+1})$ and a usage cost $u_j[(z_t \wedge \theta_j - z_t \wedge \theta_{j+1}) - (z_{t+1} \wedge \theta_j - z_{t+1} \wedge \theta_{j+1})]$ to each supplier j, $j = 1, \cdots, m$. To see how we obtain the expressions of the holding and usage costs, one may refer to the "the equivalence between the MDP in Eq. (4) - Eq. (5) and the MDP in Eq. (1) - Eq. (2)" shown above. After making the capacity allocation decision, the echelon capacity state updates to $z_{t+1} \wedge \Theta$ at the beginning of period $t+1$. Based on these facts, we derive the MDP formulae in (9) and (10) directly by intuition.

Proof of Theorem 2.

(1) Similar to the proof of Theorem 1, we shall show the results by induction. Suppose that v_{t+1} is discrete concave and decomposable. Then $g_{i,t}(z_t\wedge\Theta)=\max_{0\vee(z_t-1)\le z_{t+1}\le z_t} G_{i,t}(z_t\wedge\Theta, z_{t+1})$, where

$$\begin{aligned}G_{i,t}(z_t\wedge\Theta, z_{t+1}) = &\sum_{j=1}^{m}\hat{v}_{j,t+1}(z_{t+1}\wedge\theta_j) - u_1(z_t\wedge\theta_1) - \sum_{j=2}^{m}(u_j-u_{j-1})(z_t\wedge\theta_j)\\ &+(u_1-h_1)z_{t+1}+\sum_{j=2}^{m}(u_j-h_j-u_{j-1}+h_{j-1})(z_{t+1}\wedge\theta_j)+q_i(z_t-z_{t+1}).\end{aligned}$$

Note that $\hat{v}_{1,t+1}(z_{t+1}\wedge\theta_1)=\hat{v}_{1,t+1}(z_{t+1})$ as $z_{t+1}\le z_t\le\theta_1$, and $\hat{v}_{j,t+1}(z_{t+1}\wedge\theta_j)+(u_j-h_j-u_{j-1}+h_{j-1})(z_{t+1}\wedge\theta_j)$ is discrete concave in z_{t+1} for $j=2,\cdots,n$ due to the inductive assumption and Lemmas 5 and 7.

By collecting the functions depending on z_{t+1} in the expression of $G_{i,t}$, we define

$$F_{i,t}(z_{t+1})=\sum_{j=1}^{m}\hat{v}_{j,t+1}(z_{t+1}\wedge\theta_j)+(u_1-h_1-q_i)z_{t+1}+\sum_{j=2}^{m}(u_j-h_j-u_{j-1}+h_{j-1})(z_{t+1}\wedge\theta_j),$$

which is discrete concave in z_{t+1}, and $R_{i,t}=\min\arg\max_{z\ge 0}F_{i,t}(z)$, for $i=1,\cdots,n$. Then, the optimal solution to Eq. (10), denoted by z^*_{t+1}, is $z^*_{t+1}=((z_t-1)\vee R_{i,t})\wedge z_t$ as $R_{i,t}\ge 0$.

After the optimization, $g_{i,t}(z_t\wedge\Theta)$ is expressed as

$$\begin{aligned}g_{i,t}(z_t\wedge\Theta)=&F_{i,t}(((z_t-1)\vee R_{i,t})\wedge z_t)-(u_1-q_i)(z_t\wedge\theta_1)-\sum_{j=2}^{m}(u_j-u_{j-1})(z_t\wedge\theta_j)\\ =&F_{i,t}(R_{i,t}\wedge z_t)+F_{i,t}(z_t-1)-F_{i,t}((z_t-1)\wedge R_{i,t})-(u_1-q_i)(z_t\wedge\theta_1)\\ &-\sum_{j=2}^{m}(u_j-u_{j-1})(z_t\wedge\theta_j).\end{aligned}$$

Clearly, $g_{i,t}(z_t\wedge\Theta)$ is decomposable as $F_{i,t}(z)$ is decomposable.

For the discrete concavity of $g_{i,t}(z_t\wedge\Theta)$, we show it as follows. Due to Lemma 7, $F_{i,t}(z_t-1)-F_{i,t}((z_t-1)\wedge R_{i,t})$ is discrete concave in z_t and $F_{i,t}(R_{i,t}\wedge z_t)$ is discrete concave in z_t. Hence, $F_{i,t}(R_{i,t}\wedge z_t)+F_{i,t}(z_t-1)-F_{i,t}((z_t-1)\wedge R_{i,t})$ must be discrete concave in $z_t\wedge\theta_1$ as $z_t=z_t\wedge\theta_1$. Moreover, the linear function $(u_1-q_i)(z_t\wedge\theta_1)$ is discrete concave in $z_t\wedge\theta_1$ and the linear function $(u_j-u_{j-1})(z_t\wedge\theta_j)$ is discrete concave in $z_t\wedge\theta_j$ for $j=2,\cdots,m$. Hence, $g_{i,t}(z_t\wedge\Theta)$ is discrete concave in $z_t\wedge\Theta$.

By plugging the expression of $F_{i,t}(z)$ into the expression of $g_{i,t}(z_t\wedge\Theta)$ above, we can obtain the component functions of $g_{i,t}(z_t\wedge\Theta)$ as follows:

$$g_{i,t}(z_t\wedge\Theta)=\sum_{j=1}^{m}\hat{g}_{ij,t}(z_t\wedge\theta_j)$$

where

$$
\begin{aligned}
\hat{g}_{i1,t}(z) =& \sum_{j=1}^{m} \hat{v}_{1,t+1}(R_{i,t} \wedge \theta_j \wedge z) + (u_1 - h_1 - q_i)(R_{i,t} \wedge z) + \sum_{j=2}^{m} (u_j - h_j - u_{j-1} + h_{j-1})(R_{i,t} \wedge \theta_j \wedge z) \\
&+ \sum_{j=1}^{m} \hat{v}_{j,t+1}((z-1) \wedge \theta_j) - h_1 z - (u_1 - h_1 - q_i) + \sum_{j=2}^{m} (u_j - h_j - u_{j-1} + h_{j-1})((z-1) \wedge \theta_j) \\
&- \sum_{j=1}^{m} \hat{v}_{j,t+1}((z-1) \wedge R_{i,t} \wedge \theta_j) - (u_1 - h_1 - q_i)((z-1) \wedge R_{i,t}) \\
&- \sum_{j=2}^{m} (u_j - h_j - u_{j-1} + h_{j-1})((z-1) \wedge R_{i,t} \wedge \theta_j) \\
\hat{g}_{ij,t}(z) =& -(u_j - u_{j-1})z, \quad j = 2, \cdots, m.
\end{aligned}
$$

Accordingly $v_t(z_t \wedge \Theta)$ is discrete concave in $z_t \wedge \Theta$ and decomposable, i.e., $v_t(z_t \wedge \Theta) = \sum_{j=1}^{m} \hat{v}_{j,t}(z_t \wedge \theta_j)$. The component functions of $v_t(z_t \wedge \Theta)$ are obtained based on the expression of $g_{i,t}(z_t \wedge \Theta)$ shown above. Specifically,

$$
\begin{cases}
\hat{v}_{1,t}(z) = \sum_{r=1}^{n} \lambda_{r,t} \hat{g}_{r1,t}(z) + (1 - \lambda_t) \left[\hat{v}_{j,t+1}(z) - h_1(z) \right], \\
\hat{v}_{j,t}(z) = \sum_{r=1}^{n} \lambda_{r,t} \hat{g}_{rj,t}(z) + (1 - \lambda_t) \left[\hat{v}_{j,t+1}(z) + (h_{j-1} - h_j) z \right], \quad j = 2, \cdots, m.
\end{cases}
$$

(2) Once a unit of class i demand realizes, $z^*_{t+1} = ((z_t - 1) \vee R_{i,t}) \wedge z_t$. It implies that it is optimal to accept the class i demand if $z_t > R_{i,t}$. Otherwise, it is optimal to reject the demand. ■

Proof of Lemma 6.

We first show the MDP of this case as follows. We denote by $\mathbf{w}_t$ the state of unfulfilled demand at the beginning of period t and $\mathbf{a}_t$ the capacity allocation decision in period t. We also use $\bar{\mathbf{w}}_t$ to denote the demand state after observing the demand realization. Specifically, $\bar{\mathbf{w}}_t = \mathbf{w}_t + \mathbf{e}_k$ for $k \in \{1, \cdots, n\}$ if a class k demand realizes while $\bar{\mathbf{w}}_t = \mathbf{w}_t$ if no demand realizes. At the beginning of period $t+1$, the demand state transits to $\mathbf{w}_{t+1} = (w_{i,t+1})_{i=1,\cdots,n}$ after the downgrading of customers' valuations, where

$$
w_{i,t+1} = \sum_{r=i}^{n} \mathbb{I}_{\{\psi(r)=i\}} \left(\bar{w}_{r,t} - \sum_{j=1}^{m} a_{jr,t} \right) = \sum_{r=i}^{n} \mathbb{I}_{\{\psi(r)=i\}} \left(w_{r,t} + \mathbb{I}_{\{r=k\}} 1 - \sum_{j=1}^{m} a_{jr,t} \right), \quad i = 1, \cdots, n.
$$

Then, the MDP for this model is given as

$$
\bar{v}_t(\mathbf{c}_t, \mathbf{w}_t) = \sum_{k=1}^{n} \lambda_{k,t} \bar{g}_t(\mathbf{c}_t, \mathbf{w}_t + \mathbf{e}_k) + (1 - \lambda_k) \bar{g}_t(\mathbf{c}_t, \mathbf{w}_t),
$$

where

$$
\bar{g}_t(\mathbf{c}_t, \bar{\mathbf{w}}_t) = \max_{\mathbf{a} \in \mathcal{B}(\mathbf{c}_t, \bar{\mathbf{w}}_t)} \left[\bar{v}_{t+1} \left(\mathbf{c}_t - \sum_{j=1}^{m} \sum_{i=1}^{n} a_{ji,t} \mathbf{e}_j, \mathbf{w}_{t+1} \right) + \sum_{j=1}^{m} \sum_{i=1}^{n} a_{ji,t} (q_i - u_j + h_j) - \sum_{j=1}^{m} c_{j,t} h_j \right] \tag{19}
$$

with $\mathcal{B}(\mathbf{c}_t, \bar{\mathbf{w}}_t) = \{0 \le a_{ji,t}, \sum_{j=1}^m a_{ji,t} \le \bar{w}_{i,t}, \sum_{i=1}^n a_{ji,t} \le c_{j,t}, j = 1, \cdots, m, i = 1, \cdots, n\}$. In the terminal period $T+1$, $\bar{v}_{T+1}(\mathbf{c}_{T+1}, \mathbf{w}_{T+1}) \equiv 0$ for any $(\mathbf{c}_{T+1}, \mathbf{w}_{T+1})$.

To show this lemma, following the proof of Lemma 1, we then shall prove that (1) $\bar{v}_t(\mathbf{c}_t + \delta(\mathbf{e}_j - \mathbf{e}_k), \mathbf{w}_t) + \delta(u_j - h_j - u_k + h_k) \ge \bar{v}_t(\mathbf{c}_t, \mathbf{w}_t)$ for any $1 \le k < j \le m$ and $0 \le \delta \le c_{k,t}$; and (2) $\bar{v}_t(\mathbf{c}_t, \mathbf{w}_t + \delta(\mathbf{e}_l - \mathbf{e}_i)) + \delta(q_{\tilde{i}} - q_{\tilde{l}}) \ge \bar{v}_t(\mathbf{c}_t, \mathbf{w}_t)$ for any $1 \le l < i \le m$ and $0 \le \delta \le w_{i,t}$, where $\tilde{i}$ and $\tilde{l}$ denote the customer classes downgrading to the classes i and l, respectively, in the last period.

The property (1) can be similarly shown by the proof of Lemma 1. We thus omit the details. For the property (2), we show it by induction. The result holds for period $T+1$. Suppose that it holds for period $t+1$. It suffices to show that the result holds for period t as well.

Under the state $(\mathbf{c}_t, \bar{\mathbf{w}}_t)$, the optimal allocation policy in period t is assumed to be $\{a^*_{jh,t}, h = 1, \cdots, n, j = 1, \cdots, m\}$. Consider another state $(\mathbf{c}_t, \bar{\mathbf{w}}_t + \delta(\mathbf{e}_l - \mathbf{e}_i))$ for $l < i$ in period t, a feasible allocation policy is $\tilde{a}_{jh,t} = a^*_{jh,t}$ for $h \in \{1, \cdots, n\}$ but $h \ne i, l$, $\tilde{a}_{ji,t} = a^*_{ji,t} - \bar{\delta}$, and $\tilde{a}_{jl,t} = a^*_{jl,t} + \bar{\delta}$, where $\bar{\delta} = \min\{\delta, a^*_{ji,t}\}$. Let $\mathbf{w}^*_{t+1}$ and $\mathbf{w}_{t+1}$ be the customer states after the downgrading behaviour under the states $(\mathbf{c}_t, \bar{\mathbf{w}}_t)$ and $(\mathbf{c}_t, \bar{\mathbf{w}}_t + \delta(\mathbf{e}_l - \mathbf{e}_i))$, respectively. We have that $w^*_{h,t+1} = \sum_{r=h}^n \mathbb{I}_{\{\psi(r)=h\}}(\bar{w}_{r,t} - \sum_{j=1}^m a^*_{jr,t})$ and $w_{h,t+1} = \sum_{r=h}^n \mathbb{I}_{\{\psi(r)=h\}}(\bar{w}_{r,t} + \mathbb{I}_{\{r=l\}}(\delta) - \mathbb{I}_{\{r=i\}}(\delta) - \sum_{j=1}^m \tilde{a}_{jr,t})$.

Note that the state $\bar{\mathbf{w}}_t + \delta(\mathbf{e}_l - \mathbf{e}_i)$ implies that in period $t-1$ we serve δ more demands in class $\tilde{i}$ but δ fewer in class $\tilde{l}$, where $\tilde{i}$ and $\tilde{l}$ denote some classes downgrade to classes i and l, respectively. Such an action earns the addition profit $\delta(q_{\tilde{i}} - q_{\tilde{l}})$. Let $\hat{i}$ and $\hat{l}$ denote the classes the demands in classes i and l will downgrade to if waiting one more period. Then,

$$
\begin{aligned}
&\bar{g}_t(\mathbf{c}_t, \bar{\mathbf{w}}_t) - [\bar{g}_t(\mathbf{c}_t, \bar{\mathbf{w}}_t + \delta(\mathbf{e}_l - \mathbf{e}_i)) + \delta(q_{\tilde{i}} - q_{\tilde{l}})] \\
\le& \bar{v}_{t+1}\left(\mathbf{c}_t - \sum_{j=1}^m \sum_{h=1}^n a^*_{jh,t}\mathbf{e}_j, \mathbf{w}^*_{t+1}\right) + \sum_{j=1}^m \sum_{h=1}^n a^*_{jh,t} q_h - \bar{v}_{t+1}\left(\mathbf{c}_t - \sum_{j=1}^m \sum_{h=1}^n \tilde{a}_{jh,t}\mathbf{e}_j, \mathbf{w}_{t+1}\right) \\
&- \sum_{j=1}^m \sum_{h=1}^n \tilde{a}_{jh,t} q_h - \delta(q_{\tilde{i}} - q_{\tilde{l}}) \\
=& \bar{v}_{t+1}\left(\mathbf{c}_t - \sum_{j=1}^m \sum_{h=1}^n a^*_{jh,t}\mathbf{e}_j, \mathbf{w}^*_{t+1}\right) - \bar{v}_{t+1}\left(\mathbf{c}_t - \sum_{j=1}^m \sum_{h=1}^n a^*_{jh,t}\mathbf{e}_j, \mathbf{w}^*_{t+1} + (\delta - \bar{\delta})(\mathbf{e}_{\hat{l}} - \mathbf{e}_{\hat{i}})\right) - (\delta - \bar{\delta})(q_{\tilde{i}} - q_{\tilde{l}}) \\
\le& (\delta - \bar{\delta})[(q_{\hat{i}} - q_{\hat{l}}) - (q_{\tilde{i}} - q_{\tilde{l}})] \le 0
\end{aligned}
$$

due to the inductive proof, the inequality $0 \le \bar{\delta} \le \delta$, and also the fact that $q_{\hat{i}} - q_{\hat{l}} \le q_{\tilde{i}} - q_{\tilde{l}}$ based on Assumption 2. It follows that $\bar{v}_t(\mathbf{c}_t, \mathbf{w}_t) \le \bar{v}_t(\mathbf{c}_t, \mathbf{w}_t + \delta(\mathbf{e}_l - \mathbf{e}_i)) + \delta(q_{\tilde{i}} - q_{\tilde{l}})$, where $\tilde{i}$ and $\tilde{l}$ denote the classes of δ customers downgrade to the classes i and l, respectively, in the last period. ■