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Wireless Channel

2G (~ 1990's)

Gen.	Technology	$\approx$ Data Rate
2G	GSM	10 Kbps / user
2G	CDMA	10 Kbps
2.5G	GPRS	≈ 50 Kbps
2.5G	EDGE	≈ 200 Kbps

GSM : Global System for Mobile
CDMA : Code Division for Multiple Access
GPRS : General Packet Radio Service
EDGE : Enhanced Data for GSM Evolution

3G (~ 2000's)

Gen.	Technology	$\approx$ Data Rate
3G	WCDMA/UMTS	384 Kbps
3G	CDMA 2000	384 Kbps
3.5G	HSDPA / HSUPA	$5-30$ Mbps
3.5G	1 EVDO (Rev A,B,C)	$5-30$ Mbps

WCMD : Wideband CDMA
UMTS : Universal Mobile Telecomm. Standard
HSDPA : High Speed Downlink Packet Access
EVDO : Evolution Data Optimized

4G (~ 2010's)

Gen.	Technology	$\approx$ Data Rate
4G	LTE	$100-200$ Mbps
4G	WiMax 2000	100 Kbps
4G	LTE-Adv	$0.5-1$ Gbps

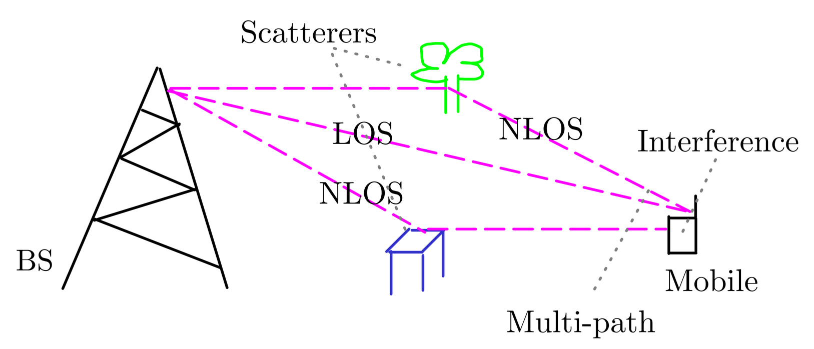
LTE : Long Term Evolution
WiMax : Worldwide Interoperability for Microwave Access

Evolution of Applications in 3G/4G

Gen.	$\approx$ Data Rate	Enablers	Applications
2G / 2.5G	$10-100$ Kbps		Voice + Basic data
3G / 3.5G	300 Kbps - 30 Mbps	CDMA, MIMO	Voice, High speed Data, Video calling
4G	>100 Mbps	MIMO, OFDM	Online Gaming, HDTV

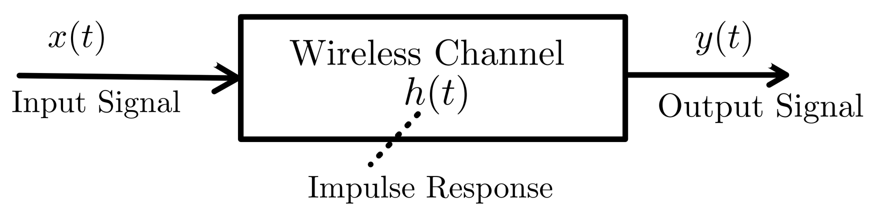
$\rightarrow$ Increase in data rates and realibility made possible by
MIMO: Multiple Input Multiple Output
CDMA: Code Division for Multiple Access
OFDM: Orthogonal Frequency Division Multiplexing

Challenges for Wireless



i) Presense of **scatterers** lead to **multipath propagation**
$\rightarrow$ Leads to **superposition** of multiple signals
$\rightarrow$ Further leads to **constructive** or **destructive** interference
ii) Constructive interf. **amplifies** received signal amplitude
$\rightarrow$ Destructive interf. **attenuates** received signal amplitude
iii) Develop a **model** for multipath propagation

Wireless Channel Model



i) i^{th} **path** of wireless env. is characterized by **delay** τ_i and **attenuation** a_i
$\implies$ can be **modeled** as $a_i\delta(t-\tau_i)$
ii) The **impulse response** of a system with **multipath** components is $h(t)=\sum_{i=0}^{L-1} a_i\delta(t-\tau_i)$ ∎

iii) **Transmitted Signal**
$\rightarrow$ $S_p(t)=\Re\left\{S(t)e^{j2\pi f_c t}\right\}$, where
$S_p(t)$ is **Passband** signal,
$S(t)$ is the complex **Baseband** signal and f_c is **carrier freq.**
$\rightarrow$ For GSM $f_c\approx 900$ MHz, for 3G $f_c\approx 2.1$ GHz and for 4G $f_c\approx 2.5$ GHz

Received Signal

i) **Received signal** $y_p(t)$ = sum of the various multipath components is

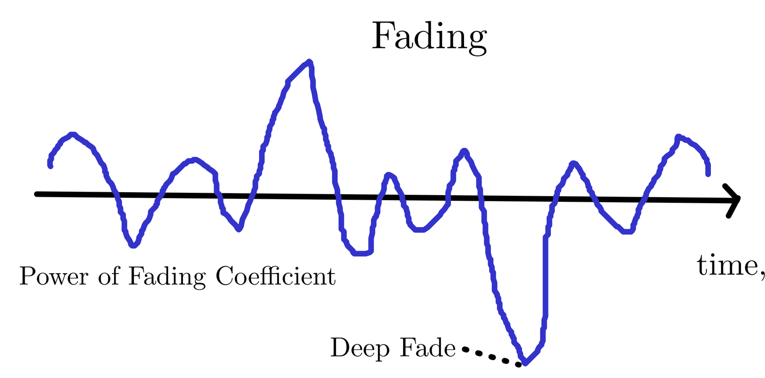
$$\begin{aligned} y_p(t) &= \sum_{i=0}^{L-1}\Re\left\{a_iS(t-\tau_i)e^{j2\pi f_c(t-\tau_i)}\right\} \\ &= \Re\left\{\left(\sum_{i=0}^{L-1} a_iS(t-\tau_i)e^{-j2\pi f_c\tau_i}\right)e^{j2\pi f_c t}\right\}, \text{ where} \end{aligned}$$

$$\underbrace{y(t)}_{\text{Complex Baseband Received Signal}} = \sum_{i=0}^{L-1} a_iS(t-\tau_i)\,\underbrace{e^{-j2\pi f_c\tau_i}}_{\text{Complex Phase Factor}}$$

ii) Under **Narrowband** assumption i.e., $f_m\le f_c$, we have $S(t-\tau_i)\approx S(t)$
$\implies y(t)=h\times S(t)$, where **channel coefficient** is $h\triangleq\left(\sum_{i=0}^{L-1} a_ie^{-j2\pi f_c\tau_i}\right)$ ∎
depends on the **attenuation** and **delay** of different **multipath** components

Example: h with $L=2$

i) Then $h=a_0e^{-j2\pi f_c\tau_0}+a_1e^{-j2\pi f_c\tau_1}$
ii) Let $a_0=a_1=1$ and $\tau_0=0,\tau_1=\frac{1}{2f_c}$, then
$h=1e^0+1e^{-j\pi}=0$ (**Destructive** interference)
$y(t)=h\times s(t)=0$
iii) Let $a_0=a_1=1$ and $\tau_0=0,\tau_1=\frac{1}{f_c}$, then
$h=1e^0+1e^{-j2\pi}=2$ (**Constructive** interference)
$y(t)=h\times s(t)=2s(t)$ (Enhanced signal amplitude)
iv) The power of channel coefficient h varies with time



v) This **Fading** process causes the received power to vary and is a key aspect of wireless channels

Fading Channel Coefficient (Exponential Model)

i) We have $h=x+j\,y=ae^{j\phi}$, where
Amplitude $a=\sqrt{x^2+y^2}$ and **phase** $\phi=\tan^{-1}\frac{y}{x}$
ii) Then **joint distribution** $F_{A,\Phi}(a,\phi)=\frac{1}{\pi}e^{-(x^2+y^2)}|J_{XY}|$ where

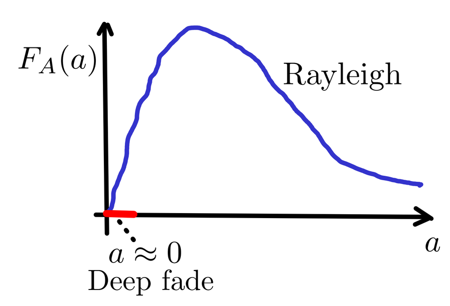
Det. of Jacobian $|J_{XY}|=\begin{vmatrix}\frac{\partial x}{\partial a} & \frac{\partial y}{\partial a}\\ \frac{\partial x}{\partial\phi} & \frac{\partial y}{\partial\phi}\end{vmatrix}=\begin{vmatrix}\cos\phi & \sin\phi\\ -a\sin\phi & a\cos\phi\end{vmatrix}=a.$

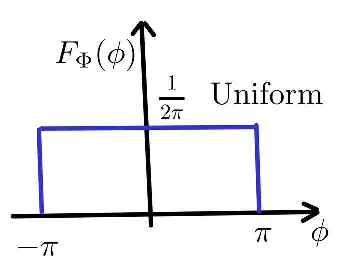
$\therefore\quad F_{A,\Phi}(a,\phi)=\frac{a}{\pi}e^{-a^2}$ ∎

iii) **Marginal** distribution of **amplitude** A is

$$\begin{aligned} F_A(a) &= \int_{-\pi}^{\pi}F_{A,\Phi}(a,\phi)d\phi \\ &= \frac{a}{\pi}e^{-a^2}\int_{-\pi}^{\pi}d\phi = 2ae^{-a^2} \text{ for } 0\le a\le\infty \end{aligned}$$ ∎

is a **Rayleigh distribution**, and as result h is also termed as rayleigh fading channel.





iv) **Marginal** distribution of **phase** Φ is

$$\begin{aligned} F_\Phi(\phi) &= \int_{-\infty}^{\infty}F_{A,\Phi}(a,\phi)da \\ &= \frac{1}{2\pi}\int_0^{\infty}2ae^{-a^2}da = \frac{1}{2\pi} \text{ for } -\pi<\phi\le\pi \end{aligned}$$ ∎

is a **Uniform distribution**.

v) Note that $F_{A,\Phi}(a,\phi)=F_A(a)F_\Phi(\phi)$,
$\implies A$ and Φ are **independent** RVs.

Fading Channel Coefficient (Real and Img. Model)

i) We have real and imaginary parts $h=x+j\,y$, where
$x=\sum_{i=0}^{L-1}a_i\cos(2\pi f_c\tau_i)$ and $y=-\sum_{i=0}^{L-1}a_i\sin(2\pi f_c\tau_i)$
ii) Note a_i,τ_i are **random variables** and
x,y are **sum** of large number of random variables
$\implies$ by CLT x,y are assumed to be **Gaussian** distributed
iii) Assuming x,y to be i.i.d $\sim\mathcal{N}(0,\frac{1}{2})$, we have

$$\begin{aligned} F_{X,Y}(x,y) &= F_X(x)F_Y(y) \\ &= \frac{1}{\sqrt{\pi}}e^{-x^2}\frac{1}{\sqrt{\pi}}e^{-y^2} \\ &= \frac{1}{\sqrt{\pi}}e^{-(x^2+y^2)} \end{aligned}$$ ∎

Example:

i) To compute the probability that the channel attenuation is worse than $20\ dB$
ii) We have channel gain $=a^2$
$\implies 10\log_{10}\ a^2<-20\ dB,$
$\implies a<0.1$, then

$$\begin{aligned} Pr(a<0.1) &= \int_0^{0.1}2ae^{-a^2}da \\ &= -e^{-a^2}\Big|_0^{0.1} = 1-e^{-0.01}\approx 0.01 \end{aligned}$$

Using $1-e^{-x}\approx x$, for small x ∎

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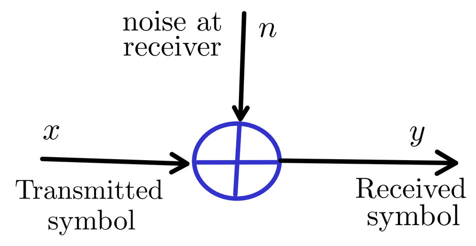
Bit-Error Rate, BER

i) **BER** is a metric to characterize the channel performance
ii) Bits of information undergo errors when transmitted
iii) BER is the average number of bits that are in error
iv) As bits are RV, BER is frequently expressed as **Probability of Bit error** P_e, where $0 \leq P_e \leq 0.5$

i) Bits are **modulated** prior to transmission over channel
ii) Binary Phase shift keying (**BPSK**) is one such scheme, where two phases are used $0 \rightarrow x = \sqrt{P}$ and $1 \rightarrow x = -\sqrt{P}$
$\implies$ Avg. power of BPSK is P
iii) At the receiver, if
received symbol $y \geq 0$, map $\rightarrow 0$ and
received symbol $y < 0$, map $\rightarrow 1$
(which is also called threshold-based detection)

Wireline Communication system (AWGN)

i) Consider Additive White Gaussian Noise wireline system



AWGN Wireline system

we have, $y = x + n$

where n is Gaussian noise with $F_N(n) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{n^2}{2\sigma^2}}$

ii) Consider $x = -\sqrt{P}$, then at receiver $y = x + n = -\sqrt{P} + n$

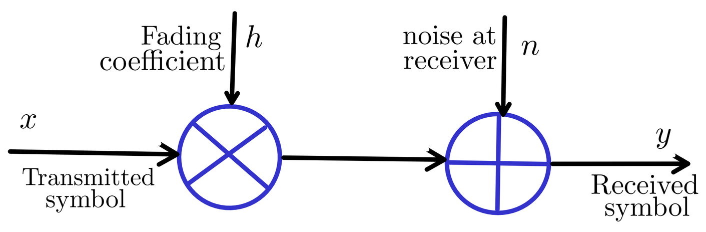
$\implies$ **error occurs** if $y \geq 0$, $\implies n \geq \sqrt{P}$

$$P_e = Pr(n > \sqrt{P})$$
$$= \int_{\sqrt{P}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{n^2}{2\sigma^2}} dn = \int_{\sqrt{\frac{P}{\sigma^2}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$
$$\implies P_e = Pr\left(\mathcal{N}(0,1) \geq \sqrt{\frac{P}{\sigma^2}}\right) = Q\left(\sqrt{\frac{P}{\sigma^2}}\right) = Q\left(\sqrt{SNR}\right) \blacksquare$$

Single antenna wireless communication system

i) Considering **wireless** system

Wireless System



we have, $y = h\,x + n$ where
h is **fading coefficient** and n is Gaussian noise

ii) Then, received power $= |h|^2 P = a^2 P$

$\implies$ Fading SNR, $SNR_F = \frac{a^2 P}{\sigma^2} = a^2\ SNR$ $\blacksquare$

iii) From BER derived for BPSK modulation,
BER $= Q\left(\sqrt{SNR_F}\right) = Q\left(\sqrt{a^2 SNR}\right)$

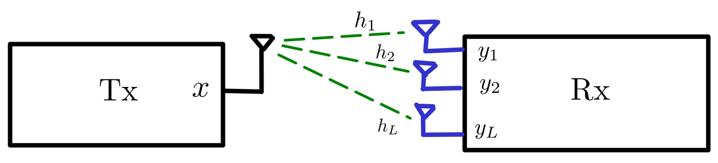
$\rightarrow$ a depends on the fading coefficient h which is a RV,
$\rightarrow$ $\implies$ BER in wireless is a **random quantity**

iv) We then have **average** BER in a **Rayleigh fading** channel as

$$BER_{avg} = \int_0^\infty Q\left(\sqrt{a^2 SNR}\right) F_A(a)\, da$$
$$= \int_0^\infty Q\left(\sqrt{a^2 SNR}\right) 2ae^{-a^2}\, da$$
$$= \frac{1}{2}\left(1 - \sqrt{\frac{SNR}{2 + SNR}}\right) \blacksquare$$

Multiple receiver antenna wireless communication system

i) Considering **wireless** system with antenna **diversity** and **optimal MRC** combining



Then the received SNR after **MRC** is given by,

$$SNR_m = \frac{P}{\sigma^2}\{|h_1|^2 + |h_2|^2 + \cdots + |h_L|^2\} = \frac{P}{\sigma^2} \parallel \mathbf{h} \parallel^2 = \frac{P}{\sigma^2} g\ ,$$

ii) Assuming $h_i, \forall i$ are **IID** Rayleigh fading channel coefficients with $E\{|h_i|^2\} = 1, \forall i$, we have

$\rightarrow$ g is also a **RV** with **Chi-Squared distribution** χ^2_{2L} with $2L$ degrees of freedom and $F_G(g) = \frac{1}{(L-1)!} g^{L-1} e^{-g}$ $\blacksquare$

iii) We now have $BER = Q(\sqrt{SNR_m}) = Q(\sqrt{g\ SNR})$ $\blacksquare$

iv) We then have **average** BER in a **Rayleigh fading** channel as

$$BER_{avg} = \int_0^\infty Q\left(\sqrt{gSNR}\right) F_G(g)\, dg$$
$$= \int_0^\infty Q\left(\sqrt{gSNR}\right) \frac{1}{(L-1)!} g^{L-1} e^{-g}\, dg$$
$$= \left(\frac{1-\lambda}{2}\right)^L \Sigma_{l=0}^{L-1} \binom{L+l-1}{l} \left(\frac{1+\lambda}{2}\right)^l, \text{ where}$$
$$\lambda = \sqrt{\frac{SNR}{2+SNR}} \text{ and } \binom{n}{k} = \frac{n!}{k!(n-k)!} \blacksquare$$

v) At **high SNR** using **approx.** we have

$$\frac{1-\lambda}{2} \approx \frac{1}{2SNR} \text{ and } \frac{1+\lambda}{2} \approx 1;$$
$$\implies \text{BER} \approx \left(\frac{1}{2SNR}\right)^L \Sigma_{l=0}^{L-1} \binom{L+l-1}{l}$$
$$\approx \binom{2L-1}{L} \left(\frac{1}{2SNR}\right)^L \blacksquare$$

BER of different communication systems

i) SNR required to achieve BER$= 10^{-6}$ for same noise power

Channel Type	SNR Required
AWGN	$14.19\ dB$
Rayleigh Fading with $L = 1$	$57\ dB$
Rayleigh Fading with $L = 2$	$29.37\ dB$
Saving in SNR due to diversity	$(57 - 29.37) = 27.63\ dB$

Numerical Example: Exact BER

i) To compute the BER at $SNR = 10dB$
$\rightarrow$ $10\log_{10} SNR = 10dB$, $\implies SNR = 10$
$\rightarrow$ $P_e = Q\left(\sqrt{SNR}\right) = Q\left(\sqrt{10}\right) \approx 7.82 \times 10^{-4}$

Approximate BER

i) Using the **approximation** $Q(x) \leq \frac{1}{2} e^{\frac{-x^2}{2}}$,
$\rightarrow$ $P_e = Q\left(\sqrt{SNR}\right) \leq \frac{1}{2} e^{-\frac{SNR}{2}}$ $\blacksquare$

Numerical Example: Approx SNR

i) To compute the SNR required for $P_e = 10^{-6}$
$\rightarrow$ $\frac{1}{2} e^{-\frac{SNR}{2}} = 10^{-6}$, $\implies SNR = 26.24$
$\rightarrow$ $SNR_{dB} = 10\log_{10}(26.24) = 14.19dB$

Numerical example: Exact BER

i) To compute the BER at $SNR = 20\ dB$
$\rightarrow$ $10\log_{10} SNR = 20dB$, $\implies SNR = 100$
$\rightarrow$ $BER = \frac{1}{2}\left(1 - \sqrt{\frac{100}{2+100}}\right) \approx 4.92 \times 10^{-3}$

BER Approximation

i) BER on SNR using **approximation**, (at high SNR)

$$BER = \frac{1}{2}\left(1 - \sqrt{\frac{1}{1+\frac{2}{SNR}}}\right) = \frac{1}{2}\left(1 - (1 + \frac{2}{SNR})^{-\frac{1}{2}}\right)$$
$$\approx \frac{1}{2}\left(1 - (1 - \frac{1}{2}\frac{2}{SNR})\right) = \frac{1}{2\ SNR} \blacksquare$$

i) Note that while the BER is **decreasing exponentially** (i.e., $\frac{1}{2}e^{-\frac{1}{2}SNR}$) in a wireline system,
it is **decreasing only** by $\frac{1}{2\ SNR}$ in case of **wireless system**.

Numerical example: Exact and Approx. SNR

i) To compute the SNR required for $BER = 10^{-6}$
$\rightarrow$ $\frac{1}{2}\left(1 - \sqrt{\frac{SNR}{2+SNR}}\right) = 10^{-6}$, $\implies SNR = 4.99 \times 10^5$
$\rightarrow$ $SNR_{dB} = 10\log_{10}(4.99 \times 10^5) = 56.98\ dB$
ii) Using **approx formulae**, we have $SNR = \frac{1}{2\ BER}$,
further we get SNR $= 57dB$ for BER $= 10^{-6}$
iii) $\implies$ a wireless channel **requires an extra** SNR of 43 dB (i.e., $10^{4.3}$ **times more**) to acheive the same BER when compared to AWGN

Numerical example: Exact BER for the given SNR and $L = 2$

i) The average BER can be obtained as,

$$BER = \frac{(1-\lambda)^2}{2} \Sigma_{l=0}^{1} \binom{2+l-1}{l} \left(\frac{1+\lambda}{2}\right)^l,$$
$$= \left(\frac{1-\lambda}{2}\right)^2 \{2 + \lambda\}$$

ii) To compute BER when $SNR_{dB} = 20dB$, we have

$$SNR = 10^{\frac{SNR_{dB}}{10}} = 100 \text{ and } \lambda = \sqrt{\frac{100}{2+100}} = 0.9901$$
$$BER = 7.2564 \times 10^{-5}$$

Numerical example: Approx BER for the given SNR and $L = 2$

i) The approx average BER for $L = 2$ can be obtained as,

$$BER = \binom{3}{2}\left(\frac{1}{2SNR}\right)^2 = \frac{3}{4}\frac{1}{SNR^2}$$

ii) To compute approx BER when $SNR_{dB} = 20dB$, we have

$$SNR = 10^{\frac{SNR_{dB}}{10}} = 100$$
$$BER = \frac{3}{4}\frac{1}{100^2} = 7.5 \times 10^{-5}$$

Numerical example: Approx SNR for the given BER and $L = 2$

i) Given BER $= 10^{-6}$, the SNR can be obtained as,

$$SNR = \sqrt{\frac{3}{4}\frac{1}{10^{-6}}}, \implies SNR = 866.0254 = 29.37dB,$$

Conclusions

i) **Huge** amount of **additional power** is required to obtained the desired BER in a single antenna wireless system when compared to AWGN
ii) As the antennas are increased to $L = 2$, the required transmit power P is **decreased** by $27.63\ dB$
iii) **Increase in antennas** leads to **significant decrease** in BER. This is the advantage of **Diversity**.

Intuition behind the decrease in BER

i) For L rx antennas , we have $BER \approx \binom{2L-1}{L}\left(\frac{1}{2SNR}\right)^L$

$\implies$
for $L = 1$, $BER \approx \frac{1}{2SNR}$ $\propto \frac{1}{SNR}$
for $L = 2$, $BER \approx \frac{3}{4}\frac{1}{SNR^2}$ $\propto \frac{1}{SNR^2}$
for $L = 3$, $BER \approx \frac{5}{4}\frac{1}{SNR^3}$ $\propto \frac{1}{SNR^3}$
in general for L rx antennas, BER $\propto \frac{1}{SNR^L}$

ii) **Deep fade** analysis will help understand this behaviour

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Performance Analysis (Deep Fade)

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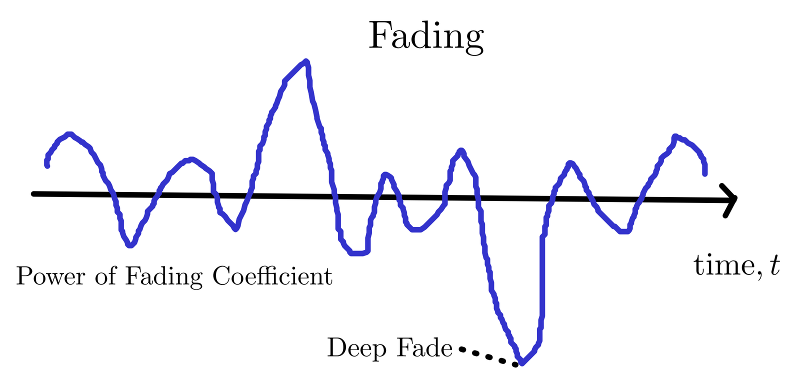
Deep Fade

i) The power of each channel coefficient $h_i, \forall i$ **varies with time**

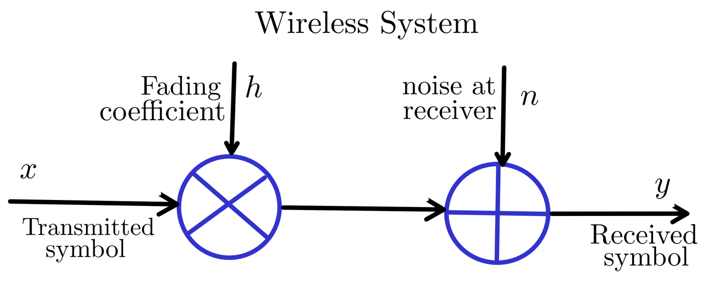


ii) This **Fading** process causes the received power to **vary** and is a key aspect of wireless channels

iii) The **poor performance** of a wireless comm. system can be explained based on the **Deep Fade** events

Single antenna wireless system

i) Recall, for a single antenna system , we have, $y = h\,x + n$

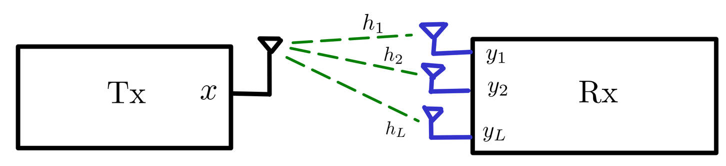


ii) And received power $= |h|^2 P = a^2 P$

$\implies$ Fading SNR, $SNR_F = \dfrac{a^2 P}{\sigma^2} = a^2\, SNR$

Multiple receive antenna Wireless system

i) Recall that for a system with multi rx antenna



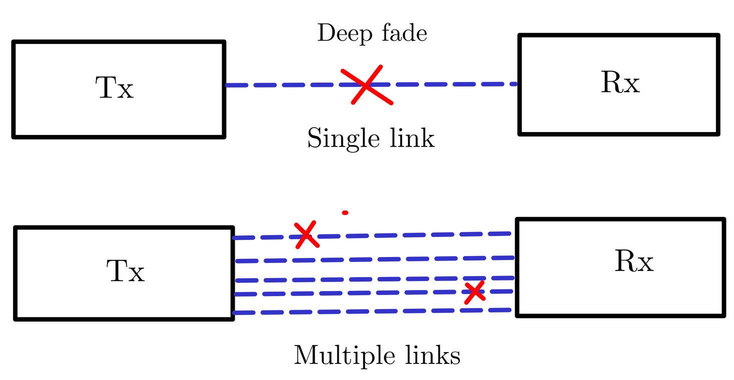
we have the received SNR after **MRC** as,

$SNR_m = \dfrac{P}{\sigma^2}\{|h_1|^2 + |h_2|^2 + \cdots + |h_L|^2\} = \dfrac{P}{\sigma^2} \parallel \mathbf{h} \parallel^2 = \dfrac{P}{\sigma^2}\, g$,

ii) Assuming $h_i, \forall i$ are **IID** Rayleigh fading channel coefficients with $E\{|h_i|^2\} = 1, \forall i$, we have

$\rightarrow$ g is also a **RV** with **Chi-Squared distribution** χ^2_{2L} with $2L$ degrees of freedom and $F_G(g) = \dfrac{1}{(L-1)!} g^{L-1} e^{-g}$ ■

Intuitive explanation



i) In a system with single tx and single rx antenna, we have a **single link**, and the **probability of Deep Fade** $\propto \dfrac{1}{SNR}$

ii) In a system with single tx antenna and L rx antennas we have L **links** between the txer and rxer

$\rightarrow$ Let E_i denote the **event** that link i is in deep fade we know $Pr(E_i) = \dfrac{1}{SNR}, \forall i$

$\rightarrow$ For a **system** to be in **deep fade**, **all links** have to be in deep fade, i.e.,

$Pr(\text{Deep fade}) = Pr(E_1 \cap E_2 \cap \cdots \cap E_L)$
$= Pr(E_1) \times Pr(E_2) \times \cdots \times Pr(E_L)$,
for **IID** channel coefficient $h_i, \forall i$
$= \dfrac{1}{SNR^L}$

Deep Fade Analysis

i) We have **received power** $= a^2 P$ and **noise power** $= \sigma^2$

$\rightarrow$ System is in **deep fade**, if the received signal power is **lower than** noise power

$\implies a^2 P < \sigma^2, \implies a^2 < \frac{\sigma^2}{P} = \frac{1}{SNR}, \implies a < \sqrt{\frac{1}{SNR}}$

ii) **Probability of deep fade**, $P_{DF} = Pr\left(a < \sqrt{\frac{1}{SNR}}\right)$

$$\implies P_{DF} = \int_0^{\sqrt{\frac{1}{SNR}}} F_A(a)\, da = \int_0^{\sqrt{\frac{1}{SNR}}} 2ae^{-a^2}\, da = 1 - e^{-\frac{1}{SNR}} \quad ■$$
$$\approx 1 - \left(1 - \frac{1}{SNR}\right) \text{ at high SNR } = \frac{1}{SNR} \quad ■$$

iii) Note, that **Bit Error Rate** $(\approx \frac{1}{2SNR}) \propto$ **Deep Fade**

$\rightarrow$ The **destructive interference** in wireless channel results in the deep fades

$\rightarrow$ The performance degrade due to deep fades can be **overcome** by the **principles** of **diversity**

Deep Fade Analysis with diversity

i) We have **received power** $= g\,P$ and **noise power** $= \sigma^2$

$\rightarrow$ System is in **deep fade**, if the received signal power is **lower than** noise power

$\implies g\,P < \sigma^2, \implies g < \frac{\sigma^2}{P} = \frac{1}{SNR}$

ii) **Probability of deep fade**, $P_{DF} = Pr\left(g < \frac{1}{SNR}\right)$

$$\implies P_{DF} = \int_0^{\frac{1}{SNR}} F_G(g)\, dg = \int_0^{\frac{1}{SNR}} \frac{1}{(L-1)!} g^{L-1} e^{-g}\, dg$$

At high SNR, we have $\frac{1}{SNR} \approx 0$,
and note that considered limit is $0 \le g \le \frac{1}{SNR}, \implies e^{-g} \approx 1$

$$\implies P_{DF} = \int_0^{\frac{1}{SNR}} \frac{1}{(L-1)!} g^{L-1}\, dg = \frac{1}{(L-1)!} \frac{g^L}{L}\Big|_0^{\frac{1}{SNR}} = \frac{1}{L!}\frac{1}{SNR^L} \propto \frac{1}{SNR^L} \quad ■$$

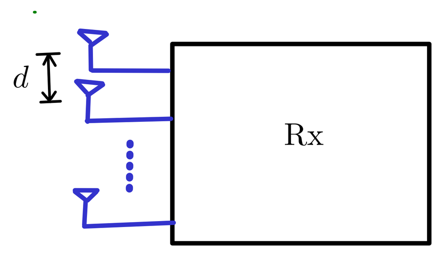
iii) With **increasing** number of antennas L, **probability of Deep fade decreases** and this leads to **significant decrease in BER**

Inference

i) This form of diversity is called **spatial diversity**

ii) One key assumption is that all the coefficients $h_i, \forall i$ are **Independently** fading

$\rightarrow$ For this to hold true, antennas have to be placed **sufficiently far** apart



iii) Let d denote the spacing between the **uniformly** spaced antennas

$\rightarrow$ For independently fading channels, the condition is given by $d_{min} = \frac{\lambda}{2}$, where d_{min} is the min separation between antennas and $\lambda (= \frac{C}{f_c})$ is the wavelength of the carrier

Numerical example (GSM)

Q) To compute the d_{min} for a given $f_c = 900MHz$ (**GSM**)

Sol) We have, $\lambda = \dfrac{3 \times 10^8 m/s}{900 \times 10^6/s} = 33.33cm$

$\therefore d_{min} = 16.66cm$, i.e., distance for independent fading

$\rightarrow$ Note, that this distance is **not possible** in a regular phone, however, possible in large devices like routers etc.

Numerical example (3G)

Q) To compute the d_{min} for a given $f_c = 2.3GHz$ (**3G**)

Sol) We have, $\lambda = \dfrac{3 \times 10^8 m/s}{2.3 \times 10^9/s} = 13.04cm$

$\therefore d_{min} = 6.5cm$, i.e., distance for independent fading

$\rightarrow$ Note that this distance is within dimensions of a smart phone (3G / 4G)

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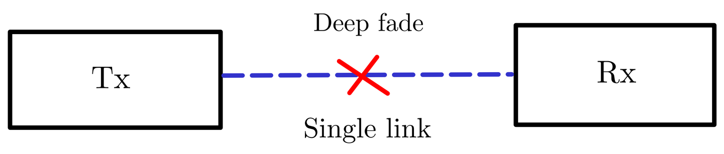
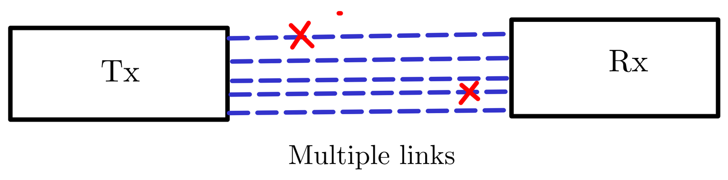
Diversity

Principle of Diversity

i) **Diversity** is used to **combat** the effects of **Deep Fades**

ii) *Single link system*: Communication fails if the single link is in deep fade





iii) *Multiple link system*: Communication is **not disrupted** even if two links are deep fade because of many alternative paths

iv) Ex: **Antenna diversity**

→ Multiple receive antennas introduces **receive diversity**

→ Other forms: Transmit and receive diversity, time diversity, freq. diversity, user diversity etc.

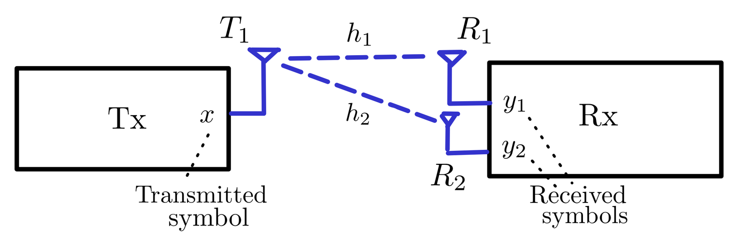
Diversity Order, d

i) Is a **measure** to **characterize** the diversity of a system

$$d = -\lim_{SNR\to\infty} \frac{\log P_e(SNR)}{log\ SNR},$$

where $P_e(SNR)$ denotes BER as a function of SNR

Multiple Antenna System



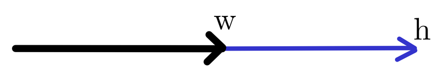
i) **System model**:
$$y_1 = h_1\,x + n_1$$
$$y_2 = h_2\,x + n_2,$$ where n_1, n_2 are independent zero-mean Gaussian noise, each with power σ^2.
i.e., $E\{|n_1|^2\} = E\{|n_2|^2\} = \sigma^2$ and $E\{n_1 n_2\} = 0$

ii) **Vector model** (assuming h_1, h_2 as real)

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} x + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}; \quad \mathbf{y} = \mathbf{h}\,x + \mathbf{n}$$

Optimal Receiver (MRC)

i) **SNR is maximum** when $\theta = 0$,
i.e., $\mathbf{w}$ is in the same direction of $\mathbf{h}$ (or $\mathbf{w} \propto \mathbf{h}$)



$$\implies SNR_{max} = \frac{P(h_1^2 + h_2^2)}{\sigma^2},\ \text{when}\ \mathbf{w} = \frac{1}{\sqrt{h_1^2 + h_2^2}} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

→ When channel coefficients are complex, we use

$$\tilde{y} = \mathbf{w}^H \mathbf{y} = \begin{bmatrix} w_1^* & w_2^* \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\ \text{and}\ \mathbf{w} = \frac{1}{\sqrt{|h_1|^2 + |h_2|^2}} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \blacksquare$$

→ This receiver is called **Maximal Ratio Combiner** (MRC)

Analytical Example with $L = 1$

Q) Compute the diversity order of a wireless system with $L = 1$ antenna

Sol) In this case, we have $P_e(SNR) \approx \frac{1}{2SNR}$ under high SNR

$$d = -\lim_{SNR\to\infty} \frac{\log \frac{1}{2SNR}}{\log SNR} = \lim_{SNR\to\infty} 1 + \frac{\log 2}{\log SNR} = 1 \blacksquare$$

Analytical Example with multiple antennas

Q) Compute the diversity order of a wireless system with L antennas

Sol) In this case, we have

$$\begin{aligned} d &= -\lim_{SNR\to\infty} \frac{\log \binom{2L-1}{L}\left(\frac{1}{2SNR}\right)^L}{\log SNR}, \\ &= -\lim_{SNR\to\infty} \frac{-L\log SNR + \log \binom{2L-1}{L}\frac{1}{2L}}{\log SNR} \\ &= \lim_{SNR\to\infty} L - \frac{\log \binom{2L-1}{L}\frac{1}{2L}}{\log SNR} = L \quad \blacksquare \end{aligned}$$

Analytical Example for wireline system

Q) Compute the diversity order of a wireline system

Sol) In this case, we have

$$\begin{aligned} d &= -\lim_{SNR\to\infty} \frac{\log \frac{1}{2} e^{-\frac{1}{2}SNR}}{\log SNR} = -\lim_{SNR\to\infty} \frac{\log \frac{1}{2} - \frac{1}{2}SNR}{\log SNR}, \\ &= \lim_{SNR\to\infty} \frac{\frac{1}{2}SNR}{\log SNR} = \infty,\ \text{using L'Hospital's rule} \quad \square \end{aligned}$$

$\implies$ A AWGN channel is a combination of ∞ independently fading links

Receiver

i) **Processing** at the receiver using **weighted** combination

$$\tilde{y} = w_1\,y_1 + w_2\,y_2 = \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{w^T y},\ \text{where}$$

w_1, w_2 are combining weights,

$$\implies \tilde{y} = \mathbf{w}^T(\mathbf{h}\,x + \mathbf{n}) = \mathbf{w}^T\mathbf{h}\,x + \mathbf{w}^T\mathbf{n} \quad \blacksquare$$

ii) **SNR** at the receiver $SNR = \frac{P\,|\mathbf{w}^T\mathbf{h}|^2}{|\mathbf{w}^T\mathbf{n}|^2}$ where

→ **expected value** of noise power:

$$\begin{aligned} E\{|\mathbf{w}^T\mathbf{n}|^2\} &= E\{(w_1 n_1 + w_2 n_2)^2\} \\ &= w_1^2 E\{n_1^2\} + w_2^2 E\{n_2^2\} + 2w_1 w_2 E\{n_1 n_2\} \\ &= \sigma^2(w_1^2 + w_2^2) = \sigma^2 \parallel \mathbf{w} \parallel^2 \quad \blacksquare \end{aligned}$$

→ signal power at receiver:

$$\begin{aligned} \mathbf{w}^T\mathbf{h} &= \mathbf{w} \cdot \mathbf{h} \\ |\mathbf{w} \cdot \mathbf{h}|^2 &= \parallel \mathbf{w} \parallel^2 \parallel \mathbf{h} \parallel^2 \cos^2\theta \ \blacksquare \end{aligned}$$

→ $\therefore\ SNR = \frac{P\ \parallel \mathbf{w} \parallel^2 \parallel \mathbf{h} \parallel^2 \cos^2\theta}{\sigma^2 \parallel \mathbf{w} \parallel^2} = \frac{P}{\sigma^2} \parallel \mathbf{h} \parallel^2 \cos^2\theta \quad \blacksquare$

Example of a MRC

i) Given, $h_1 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j$ and $h_2 = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j$ and noise power $\sigma^2 = \frac{1}{2} = -3dB$

Sol) We have $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j \\ \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j \end{bmatrix} x + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$

→ then, optimal MRC vector $\mathbf{w} = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j \\ \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j \end{bmatrix}$

→ Futher we obtain, $|\mathbf{w}^H\mathbf{h}|^2 = 2$ and $\parallel \mathbf{w} \parallel^2 = 1$,

→ $\therefore SNR = \frac{P\,|\mathbf{w}^T\mathbf{h}|^2}{|\mathbf{w}^T\mathbf{n}|^2} = \frac{2P}{\frac{1}{2}} = 4P$

Generalizing the MRC for L receiver antennas

i) Given $\begin{matrix} E\{|n_i|^2\} = \sigma^2, \forall i \\ E\{n_i n_j\} = 0, \forall i \neq j \end{matrix}$, $\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_L \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_L \end{bmatrix} x + \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_L \end{bmatrix}$

and $\tilde{y} = w_1^* y_1 + w_2^* y_2 + \cdots + w_L^* y_L$

$$\implies \tilde{y} = \mathbf{w}^H(\mathbf{h}\,x + \mathbf{n}) = \mathbf{w}^H\mathbf{h}\,x + \mathbf{w}^H\mathbf{n}$$

ii) For maximum SNR, we choose MRC with $\mathbf{w} = \frac{\mathbf{h}}{\parallel \mathbf{h} \parallel}$

$$\therefore SNR = \frac{\parallel \mathbf{h} \parallel^2 P}{\sigma^2} = \frac{\left(|h_1|^2 + |h_2|^2 + \cdots + |h_L|^2\right) P}{\sigma^2} \ \blacksquare$$

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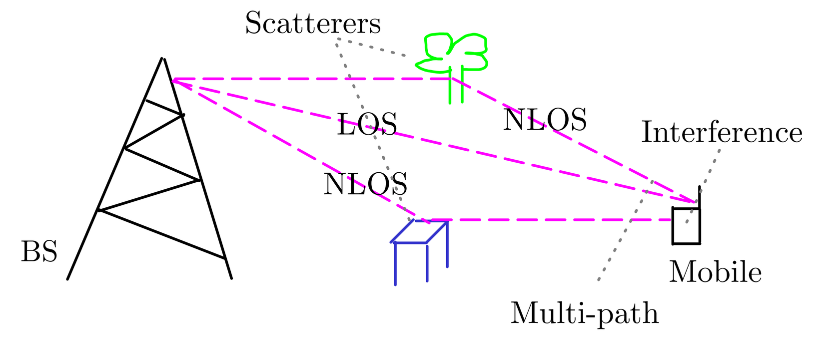
Wireless Channel Characterization

Channel

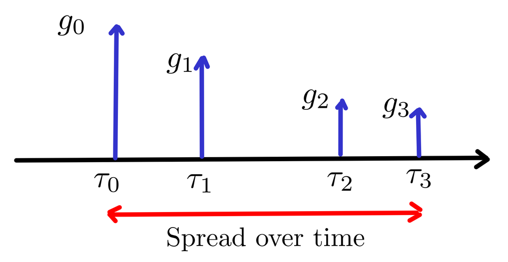
Power Profile



i) **Channel response** is
$h(t) = \sum_{i=0}^{L-1} a_i\,\delta(t-\tau_i)$, where a_i is attenuation and τ_i is the delay of i^{th} multipath

ii) Then **Power profile** of the i^{th} multipath is
$\phi(t) = \sum_{i=0}^{L-1} |a_i|^2\,\delta(t-\tau_i)$
$= \sum_{i=0}^{L-1} g_i\,\delta(t-\tau_i)$, where $g_i = |a_i|^2$ is **gain** of i^{th} path

Example: Consider an $L=4$ multipath channel



multipath, i	gain, g_i	delay, τ_i
0	g_0	τ_0
1	g_1	τ_1
2	g_2	τ_2
3	g_3	τ_3

i) The spread of delay over which different signal components are arriving is called **Delay spread**

ii) **Maximum Delay spread** σ_τ^{max} characterizes the delay spread,
$\sigma_\tau^{max} = \tau_{L-1} - \tau_0$, where
τ_0 is the delay of the first arriving component and
τ_{L-1} is the delay of the last arriving component

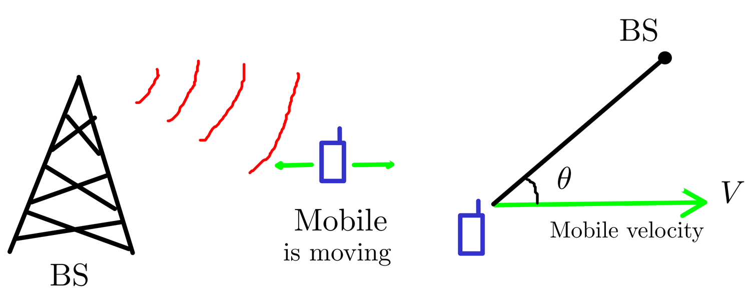
iii) **Ex**: For $L=4$, let $\tau_0 = 0\mu s$ and $\tau_3 = 5\mu s$, then $\sigma_\tau^{max} = 5\mu s$

Frequency Domain Analysis

i) We have the response of the channel in **time domain** as
$y(t) = h(t) * x(t)$, where $x(t)$ is transmitted signal,
$y(t)$ is received signal and $h(t)$ is the impulse response

ii) Then in the **frequency domain**, we have response as
$Y(f) = H(f)\,X(f)$,
where $X(f)$, $Y(f)$ and $H(f)$ are the Fourier transforms

Doppler Effect



i) In a scenario where the mobile is **moving**, will causes the **freq.** of the received signal to **change**

→ This is known as the **Doppler effect**

→ It is the change in freq. of the received signal due to motion between Txer and the Rxer

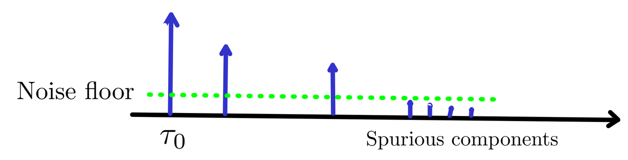
ii) Assuming the mobile is moving at an angle θ with the BS,
we have the doppler freq. $f_d = \dfrac{V\cos\theta}{C} \times f_c$, where ■
V is velocity of mobile, C is speed of light and f_C is carrier freq.

iii) Frequency of rxed signal $f_r = \left(1 + \frac{V\cos\theta}{C}\right) f_C$. ■
If mobile is

→ moving towards BS (i.e., $0 < \theta \le \frac{\pi}{2}$), then $f_r \ge f_c$

→ moving away from BS (i.e., $\frac{\pi}{2} < \theta \le \pi$), then $f_r \le f_c$

RMS Delay Spread

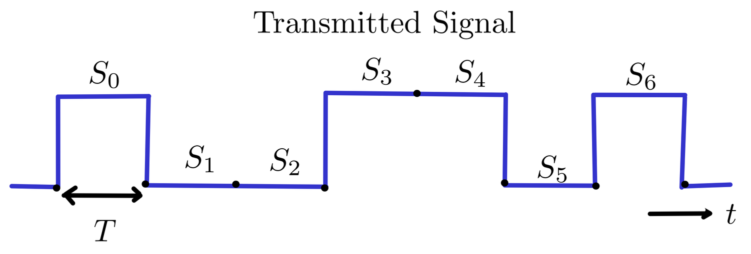


i) Practically, there might be large number of spurious paths with negligible power below the noise floor
In such case, max. delay spread is not an **appropriate** metric

ii) **RMS Delay spread** (root mean square) is considered in such scenario. We now consider
$b_i = \dfrac{g_i}{\sum_{j=0}^{L-1} g_j}$, the fraction of power in i^{th} path
Average delay by weighing each delay by fraction of power
$$\overline{\tau} = \sum_{i=0}^{L-1} b_i\,\tau_i = \frac{\sum_{i=0}^{L-1} g_i\tau_i}{\sum_{i=0}^{L-1} g_i} \quad ■$$
Root Mean square delay spread
$$\sigma_\tau = \sqrt{b_0(\tau_0-\overline{\tau})^2 + \cdots + b_{L-1}(\tau_{L-1}-\overline{\tau})^2}$$
$$= \sqrt{\frac{\sum_{i=0}^{L-1} g_i(\tau_i-\overline{\tau})^2}{\sum_{i=0}^{L-1} g_i}} = \sqrt{\frac{\sum_{i=0}^{L-1} |a_i|^2(\tau_i-\overline{\tau})^2}{\sum_{i=0}^{L-1} |a_i|^2}} \quad ■$$

Impact of channel power profile

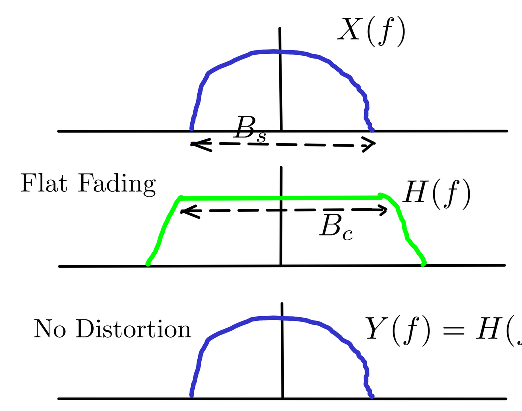
i) Consider a transmitted signal with symbols $\{S_0, S_1, S_2, \cdots\}$



where T is the symbol time

ii) Futher consider 2-path wireless channel (i.e., $L=2$) where
$h(t) = a_0\delta(t-\tau_0) + a_1\delta(t-\tau_1)$
$= \delta(t-\tau_0) + \delta(t-\tau_1)$, where $a_0 = a_1 = 1$

Signal Bandwidth B_s < Channel Coherence Bandwidth B_c



i) Consider that the bandwidth of the signal B_s, is less than the coherence bandwidth of the channel B_c. I.e., $B_s < B_c$

→ In this scenario, there is **no distortion** in the received signal $Y(f)$

→ Note that the flat portion of channel freq. response $H(f)$ is called the **Coherence bandwidth**

ii) This scenario is known as a **Flat fading** channel, as the signal BW is within the **flat portion** of the channel freq. response

Channel frequency response

i) To obtain the freq. response of the channel, we have
$h(t) = \sum_{i=0}^{L-1} a_i\delta(t-\tau_i)$, and
$\delta(t-\tau_i) \leftrightarrow e^{-j2\pi f\tau_i}$
$\implies \sum_{i=0}^{L-1} a_i\delta(t-\tau_i) \leftrightarrow \sum_{i=0}^{L-1} a_i e^{-j2\pi f\tau_i} = H(f)$

Example

Q) Given $f_c = 1.85GHz$, $V = 60mph$ and $\theta = 30°$, compute the doppler shift and received freq.

Sol) We have $V = 60 \times 1.6 \times \frac{5}{16} = 26.8m/s$
$f_d = \frac{26.8\times\cos 30°}{3\times 10^8} \times 1.85 \times 10^9 = 143Hz$ and
$f_r = 1.85 \times 10^9 + 143Hz$

Impact of Dopper on Wireless channel

i) Recall fading coefficient is $h = \sum_{i=0}^{i=L-1} a_i e^{-j2\pi f_c\tau_i}$

ii) When the mobile is moving, the delay τ_i changes with time
$\implies \tau_i(t) = \tau_i \pm \dfrac{V\cos\theta t}{C}$ and
$h = \sum_{i=0}^{i=L-1} a_i e^{-j2\pi f_c\left(\tau_i \pm \frac{V\cos\theta}{C}t\right)}$
$= \sum_{i=0}^{i=L-1} a_i e^{-j2\pi f_c\tau_i}\, e^{\mp j2\pi f_d t} = h(t)$ ■
Note, the channel coefficient is time-varying and hence known as time selective channnel

iii) Mobility → Dopper → Time-varying phase → Time selective

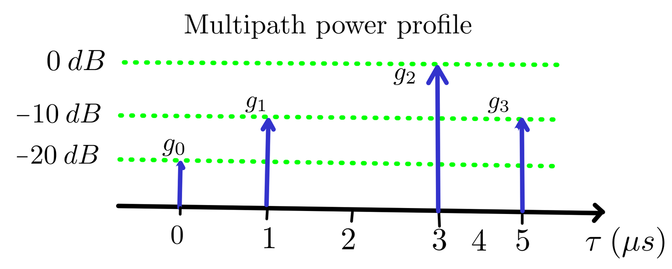
iv) Considering the phase factor
at $t=0$ the phase $=0$, and at $t = \frac{1}{4f_d}$ the phase $= \frac{\pi}{2}$
has changed significatly

→ **Coherence time** $T_c (= \frac{1}{4f_d})$ is defined as the time over which channel is approx. constant ■

Example:

Q) Compute RMS delay, given $L=4$, $\tau_i = \{0,1,3,5\}\ \mu s\,\forall i$,
$g_i = \{0.01, 0.1, 1, 0.1\} = \{-20, -10, 0, -10\} dB\,\forall i$,

Sol) Given,



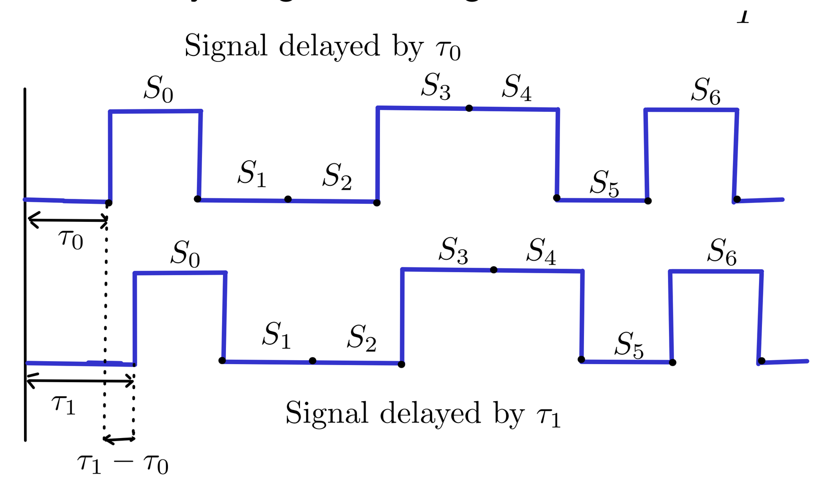
→ $\overline{\tau} = \dfrac{0.01\times 0 + 0.1\times 1 + 1\times 3 + 0.1\times 5}{0.01 + 0.1 + 1 + 0.1} = 2.9752\mu s$

→ RMS Delay $\sigma_\tau = \sqrt{\dfrac{0.01\times(0-2.98)^2 + \cdots}{0.01+0.1+1+0.1}} = 0.8573\mu s$

→ $\tau_{max} = 5 - 0 = 5\,\mu s$

Inter Symbol Interference (ISI)

i) The two delayed signals arriving at the receiver add-up,



Because of the difference in delays the adjacent symbols interfere

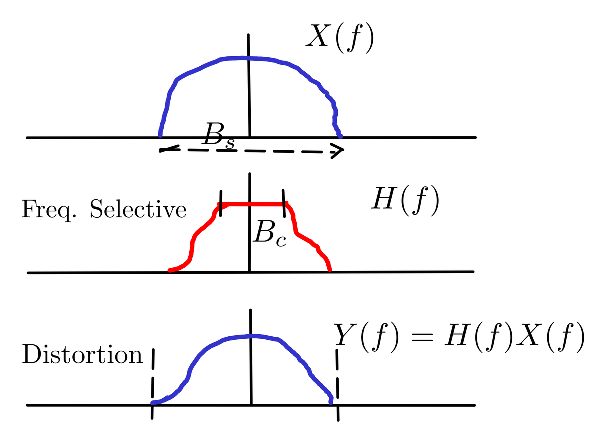
ii) Inter symbol inteference (ISI) occurs when $(\tau_1 - \tau_0) > T$
i.e, **delay spread** is greater than **symbol time** or $\sigma_\tau > T$

→ ISI is undesirable since it leads to distortion of the original signal

→ To **avoid** ISI, we need the symbol time T to be larger than delay spread i.e., $T > \sigma_\tau$

iii) **Typical outdoor** delay spread in a wireless channels is $2\mu s$
(difference in LOS and NLOS path in a cell is $1KM$,)

Signal Bandwidth B_s > Channel Coherence Bandwidth B_c



i) Consider that the bandwidth of the signal B_s, is greater than the coherence bandwidth of the channel B_c. I.e., $B_s > B_c$ ■

→ In this scenario, there is **distortion** in the received signal $Y(f)$

ii) This scenario is termed as a **Frequency selective fading**

Typical Component

i) Focusing on a typical channel component $e^{-j2\pi f\tau_i}$,
At $f=0$ the phase $=0$ and at $f = \frac{1}{2\tau_i}$ the phase $=\pi$

→ Note that the phase (freq. response) has changed **significantly** within $f = \frac{1}{2\tau_i}$, i.e., within a bandwidth $= 2 \times \frac{1}{2\tau_i} = \frac{1}{\tau_i}$ ■
∴In general, approximately the BW of the channel $= \frac{1}{\sigma_\tau} = B_c$

ii) $\implies$ The **larger** the **delay spread**, the **smaller** is the **coherence BW**

iii) Typically $\sigma_\tau = 2\mu s$, $\implies B_c = 500KHz$

Interpretation from freq. and time domain analysis

Freq. domain	**Time** domain	**Implication**
If $B_s > B_c$	$\implies \sigma_\tau > T$	**Freq. Selective Fading** and **ISI**
If $B_s < B_c$	$\implies \sigma_\tau < T$	**Freq. Flat Fading** and **no ISI**

Example:

i) Considering $f_d = 143Hz$,
the coherence time $T_c = \frac{1}{4\times 143} = 1.7ms$

→ The channel is approx. constant for (or changing after) $1.7ms$

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Code Division Multiple Access

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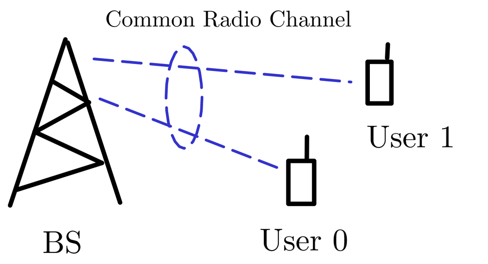
Code Division Multiple Access

i) CDMA is a key technology used for multiple access in various $3G$ standards like WCDMA, HSDPA / HSUPA, CDMA 2000 and $1\times$EVDO



ii) Unlike wireline where each user is allocated a dedicated channel, in multiple access, users share a common radio channel

iii) TDMA and FDMA are other examples of multiple access technologies

→ TDMA: In time division multiple access, different users are allocated different time slots

→ FDMA: In frequency division multiple access, different users are allocated different bands. FDMA was used in $1G$ comm. systems

iv) Similarly, in CDMA different users are allocated different codes

→ CDMA uses the concept of codes for multiple access

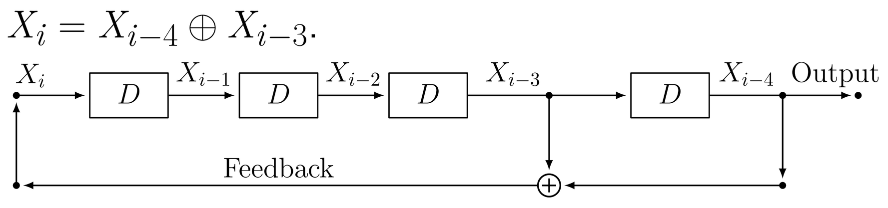
Code generation of CDMA

i) The code is typically a Pseudo Noise (PN) sequence (random). A PN seq. is generated by a linear feedback shift register (LFSR)

ii) A LFSR with D states can go through 2^D-1 states (except all zeros $[0\cdots 0]$)

→ A LFSR which goes through 2^D-1 states is called maximal length LFSR

iii) Example: Consider a LFSR with 4 state variables and $X_i = X_{i-4}\oplus X_{i-3}$.



This has $15(=2^4-1)$ states, as a result is a maximal length LFSR

→ The code is obtained by mapping the output X_{i-4} as, $0\to 1$ and $1\to -1$

System model of a CDMA (single-user) system

i) Consider a single-user CDMA system, i.e., User-0 transmits symbol a_0, uses $C_0(n)\in\{-1,+1\}\,\forall n$ with length N, and let

$$\begin{aligned} \text{Transmitted symbol } x(n) &= a_0 C_0(n)\\ \text{Received symbol } y(n) &= hx(n)+w(n), \text{ where}\\ h &\text{ is Rayleigh fading coeff}\\ w(n) &\text{ is IID Gaussian noise } \mathcal{N}(0,\sigma^2)\end{aligned}$$

$\implies E\{w(n)\}=0 \quad \forall n$ and

$$E\{w(n)w^*(m)\} = \begin{cases}\sigma^2 & \text{if } n=m,\\ 0 & \text{if } n\neq m\end{cases}$$

ii) At receiver, $y(n)$ is correlated with $C_0(n)$

$$\begin{aligned} r_{y0} &= \tfrac{1}{N}\textstyle\sum_{n=0}^{N} y(n)C_0(n) = \tfrac{1}{N}\sum_{n=0}^{N}[hx(n)+w(n)]\,C_0(n)\\ &= \tfrac{1}{N}\textstyle\sum_{n=0}^{N} ha_0C_0^2(n) + \tfrac{1}{N}\sum_{n=0}^{N} w(n)C_0(n)\\ &= ha_0+\widetilde{w}, \text{ where } \widetilde{w} = \tfrac{1}{N}\textstyle\sum_{n=0}^{N} w(n)C_0(n)\end{aligned}$$

CDMA

System model of CDMA (Multi-user) system

i) Consider a system with {user-0, user-1} transmitting symbols $\{a_0,\, a_1\}$, using $\{C_0(n),\, C_1(n)\}$ with length N, and let

$$\begin{aligned} \text{Transmitted symbol } x(n) &= a_0C_0(n)+a_1C_1(n)\\ \text{Received symbol } y(n) &= hx(n)+w(n)\\ &= ha_0C_0(n)+ha_1C_1(n)+w(n)\end{aligned}$$

ii) At receiver, $y(n)$ is correlated with $C_0(n)$

$$\begin{aligned} r_{y0} &= \tfrac{1}{N}\textstyle\sum_{n=0}^{N} y(n)C_0(n)\\ &= \tfrac{1}{N}\textstyle\sum_{n=0}^{N}[ha_0C_0(n)+ha_1C_1(n)+w(n)]\,C_0(n)\\ &= ha_0 + ha_1\tfrac{1}{N}\textstyle\sum_{n=0}^{N} C_0(n)C_1(n)+\widetilde{w}\end{aligned}$$

Note, the multi-user interference (MUI) component above $\left[= ha_1\frac{1}{N}\sum_{n=0}^{N} C_0(n)C_1(n)\right]$ arises because of simultaneous transmission in CDMA

Example: 2-user multiple access system

i) Consider codes C_0 and C_1, given by $C_0 = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}$ and $C_1 = \begin{bmatrix}1 & -1 & -1 & 1\end{bmatrix}$ and transmitted symbol of user-0 and user-1 as a_0 and a_1

ii) Before transmission each symbol is multiplied with resp. code as

$$a_0\,C_0 = \begin{bmatrix}a_0 & a_0 & a_0 & a_0\end{bmatrix},$$
$$a_1\,C_1 = \begin{bmatrix}a_1 & -a_1 & -a_1 & a_1\end{bmatrix}$$

and the combined sum is transmitted

$$a_0\,C_0+a_1\,C_1 = \begin{bmatrix}a_0+a_1 & a_0-a_1 & a_0-a_1 & a_0+a_1\end{bmatrix}$$

As a result 4 chips are transmitted in a symbol duration.

iii) To extract the individual symbols at the receiver, each user correlates the received symbol with its code i.e.,

user-0: $1(a_0+a_1)+1(a_0-a_1)+1(a_0-a_1)+1(a_0+a_1) = 4a_0$

user-1: $1(a_0+a_1)-1(a_0-a_1)-1(a_0-a_1)+1(a_0+a_1) = 4a_1$

There by multiple users are able to simultaneously receive signals on a common channel

Example: LFSR for $N=4$

State Variables →	X_{i-1}	X_{i-2}	X_{i-3}	X_{i-4}	$X_i(=X_{i-4}\oplus X_{i-3})$
States ↓ 1	1	1	1	1	0
2	0	1	1	1	0
3	0	0	1	1	0
4	0	0	0	1	1
5	1	0	0	0	0
6	0	1	0	0	0
7	0	0	1	0	1
8	1	0	0	1	1
9	1	1	0	0	0
10	0	1	1	0	1
11	1	0	1	1	0
12	0	1	0	1	1
13	1	0	1	0	1
14	1	1	0	1	1
15	1	1	1	0	1
All 1's 16	1	1	1	1	0

The CDMA PN sequence for this case is,

$$C(n) = \begin{bmatrix}-1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1\end{bmatrix}$$

SNR of a single-user CDMA system

Further, analyzing the noise component

$$\begin{aligned} E\{\widetilde{w}\} &= \tfrac{1}{N}\textstyle\sum_{n=0}^{N} E\{w(n)\}C_0(n) = 0\\ E\{\widetilde{w}\widetilde{w}^*\} &= E\Big\{\Big(\tfrac{1}{N}\textstyle\sum_n w(n)C_0(n)\Big)\Big(\tfrac{1}{N}\sum_m w^*(m)C_0^*(m)\Big)\Big\}\\ &= \tfrac{1}{N^2}\textstyle\sum_n\sum_m E\{w(n)w^*(m)\}E\{C_0(n)C_0^*(m)\}\\ &= \tfrac{1}{N^2}\textstyle\sum_n E\{|w(n)|^2\}1 = \dfrac{\sigma^2}{N} \quad \blacksquare\end{aligned}$$

Note that, Noise power decreased by N

iii) $SNR_{CDMA} = \dfrac{\text{Signal Power}}{\text{Noise Power}} = \dfrac{E\{|h|^2|a_0|^2\}}{\sigma^2/N} = N\dfrac{|h|^2P}{\sigma^2} \quad \blacksquare$

This important factor N is called spreading gain or processing gain of CDMA system

SINR of multi-user CDMA system

Further to analyze the interference component, define $r_{01} \triangleq \frac{1}{N}\sum_{n=0}^{N} C_0(n)C_1(n)$

$$\begin{aligned} E\{r_{01}\} &= \tfrac{1}{N}\textstyle\sum_n E\{C_0(n)\}\,E\{C_1(n)\} = 0\\ E\{r_{01}^2\} &= E\Big\{\Big(\tfrac{1}{N}\textstyle\sum_n C_0(n)C_1(n)\Big)\Big(\tfrac{1}{N}\sum_m C_0(m)C_1(m)\Big)\Big\}\\ &= \tfrac{1}{N^2}\textstyle\sum_n\sum_m E\{C_0(n)C_1(m)\}E\{C_0(m)C_1(n)\}\\ &= \tfrac{1}{N^2}\textstyle\sum_n E\{C_0(n)^2\}E\{C_1(n)^2\} = \dfrac{1}{N} \quad \blacksquare\end{aligned}$$

As $E\{C_0(n)C_1(m)\} = \begin{cases}1 & \text{if } n=m,\\ 0 & \text{if } n\neq m\end{cases}$

i) Signal to interference plus noise ratio in this case is

$$\begin{aligned} SINR &= \frac{\text{Signal Power}}{\text{Interference Power} + \text{Noise Power}}\\ &= \frac{|h|^2|P_0|^2}{\frac{\sigma^2}{N}+\frac{|h|^2|P_1|^2}{N}} = \frac{N|h|^2P_0}{\sigma^2+|h|^2P_1} \quad \blacksquare\end{aligned}$$

ii) Generalizing for $K+1$ users, $SINR = \dfrac{N|h|^2P_0}{\sigma^2+\sum_{k=1}^{K}|h|^2P_k} \quad \blacksquare$

Example: 2-user CDMA system

Q) Consider $P_0=P_1=15dB$, $\sigma^2=3dB$, $N=256$ and $h=\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}$. Compute the SINR for user-0.

Sol. $SINR = \dfrac{256\times(\frac{1}{2}+\frac{1}{2})\times 10^{1.5}}{2+(\frac{1}{2}+\frac{1}{2})\times 10^{1.5}} = 233.81 \approx 23.68dB$

Near-Far effect

i) The interference of User-1 who is close to the BS than the desired user-0 by $d_1=\frac{d_0}{\sqrt{N}}$ equals the signal power. This leads to disruption

ii) Thereby the concept of power control is used to avoid this.

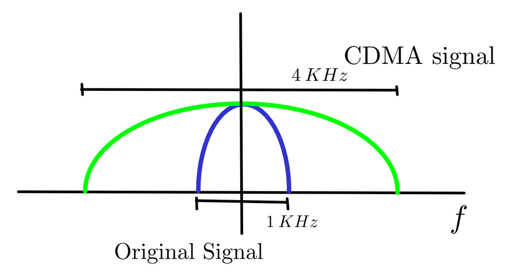
Chip Time and Bandwidth expansion in CDMA

i) Considering an example with symbol rate of $1\,kbps$, i.e., $1000\ sym/sec$, we have time per symbol $T_s=\frac{1}{1000}\,sec$. i.e., Bandwidth required $BW=\frac{1}{T_s}=1\,KHz$.

ii) However in CDMA (with 4 $chips/sym$), chip time $T_c=\frac{T_s}{N}=\frac{1\,ms}{4}=0.25\,ms$

$\implies$ new bandwidth $=\frac{1}{T_c}=4\,KHz$.

Original bandwidth has spread in CDMA.



→ Spectrum of the original signal is spread by a factor of N, where N is the length of the code. As a result, it is also called spread spectrum system

→ CDMA codes are also known as the spreading codes. And N is termed as spreading factor.

Properties of the CDMA PN sequence

i) **Balance Property:**

→ The number of (-1)'s $(=8)\approxeq$ the number of (1)'s $(=7)$

ii) **Run length Property:**

→ A Run is defined as the string of continuous values

This property states, that $\frac{1}{2^n}$ of total runs are of length n

There are a total of 8 runs in the above example and

1 run of $n=4$ (Note $\#$ of state variables $=4$)

$1=(\frac{1}{2^3}\times 8)$ run of $n=3$, $2=(\frac{1}{2^2}\times 8)$ run of $n=2$

$4=(\frac{1}{2^1}\times 8)$ run of $n=1$

ii) **Auto Correlation Property:**

→ Auto correlation $r(d)\triangleq \frac{1}{N}\sum_{n=0}^{N} C(n)C(n-d)$

Note that, $r(d=0)=\frac{1}{N}\sum_{n=0}^{N} C(n)^2 = \frac{N}{N}=1$

However, $r(d=2)=\frac{1}{N}\sum_{n=0}^{N} C(n)C(n-2) = \frac{-1}{N}=\frac{-1}{15}$

$$\begin{aligned} C(n) &= [\ -1\ \ -1\ \ -1\ \ -1\ \ 1\ \ 1\ \ 1\ \ -1\ \ 1\ \ 1\ \ -1\ \ -1\ \ 1\ \ -1\ \ 1\ ]\\ C(n-2) &= [\ -1\ \ 1\ \ -1\ \ -1\ \ -1\ \ -1\ \ 1\ \ 1\ \ 1\ \ -1\ \ 1\ \ 1\ \ -1\ \ -1\ \ 1\ ]\\ C(n)\times C(n-2) &= [\ 1\ \ -1\ \ 1\ \ 1\ \ -1\ \ -1\ \ 1\ \ -1\ \ 1\ \ -1\ \ -1\ \ -1\ \ -1\ \ 1\ \ 1\ ]\end{aligned}$$

$$\implies r(d) = \begin{cases}1 & \text{if } d=0,\\ \frac{-1}{N}\ (\to 0) & \text{if } d\neq 0 \text{ (and large} N).\end{cases}$$

BER of a single-user CDMA system

i) Using $SNR=SNR_{BPSK}=\frac{P}{\sigma^2}$, we have $SNR_{CDMA}=N|h|^2SNR$

$$\implies BER = \frac{1}{2}\left(1-\sqrt{\frac{N\ SNR}{2+N\ SNR}}\right) \text{ (Exact)} \ \blacksquare$$

$$\approx \frac{1}{2N\ SNR} \text{ (Approx. for high SNR)} \ \blacksquare$$

Q) Consider CDMA system with code length $N=256$ and $SNR=\frac{P}{\sigma^2}=15dB$, compute BER over Rayleigh fading channel

Sol. $BER=\frac{1}{2}\left(1-\sqrt{\dfrac{256\times 10^{1.5}}{2+256\times 10^{1.5}}}\right) = 6.175\times 10^{-5}$

CDMA system with Frequency selective channel (or ISI)

i) Consider an $L-$tap channel with $\{h(0),h(1),\cdots,h(L-1)\}$ and the transmitted symbols as $\{x(0),\cdots,x(n)\}$. Then,

$$\begin{aligned} y(n) &= h(0)x(n)+h(1)x(n-1)+\cdots+w(n)\\ &= \textstyle\sum_{l=0}^{L-1} h(l)x(n-l)+w(n)\end{aligned}$$

i.e., a clear case of inter symbol interference

ii) Considering a single-user CDMA system $x(n)=a_0C_0(n)$, then $y(n)=\sum_{l=0}^{L-1} h(l)a_0C_0(n-l)+w(n)$

iii) Performing correlation operation at the receiver,

$$\begin{aligned} r(0) &= \tfrac{1}{N}\textstyle\sum_{n=0}^{N-1} y(n)C_0(n)\\ &= \tfrac{1}{N}\textstyle\sum_n\sum_l h(l)a_0C_0(n-l)C_0(n)+\tfrac{1}{N}\sum_n w(n)C_0(n)\\ &= \textstyle\sum_l h(l)a_0\tfrac{1}{N}\sum_n C_0(n-l)C_0(n)+\widetilde{w}, \text{ we have}\end{aligned}$$

autocorrelation, $\frac{1}{N}\sum_n C_0(n-l)C_0(n) = \begin{cases}1 & \text{if } l=0,\\ \frac{-1}{N}\approx 0 & \text{if } l\neq 0\end{cases}$

$\implies r(0)=h(0)a_0+\widetilde{w} \quad \blacksquare$

iv) Similarly, performing correlation with $C_0(n-1)$, we get $r(1)=h(1)a_0+\widetilde{w} \quad \blacksquare$

v) Vector model can be written as, $\mathbf{r}=\mathbf{h}a_0+\widetilde{\mathbf{w}}$,

$$\begin{bmatrix}r(0)\\ \vdots\\ r(L-1)\end{bmatrix} = \begin{bmatrix}h(0)\\ \vdots\\ h(L-1)\end{bmatrix}a_0 + \begin{bmatrix}\widetilde{w}(0)\\ \vdots\\ \widetilde{w}(L-1)\end{bmatrix}$$

Rake Receiver: Carrying out Optimal processing using MRC on $\mathbf{r}$, i.e., $\dfrac{\mathbf{h}^H}{\|\mathbf{h}\|^2}\mathbf{r}$, we obtain

i) $SNR_{rake}=\dfrac{\|\mathbf{h}\|^2\,P}{\sigma^2/N} = N\,\|\mathbf{h}\|^2\,SNR \quad \blacksquare$

$\|\mathbf{h}\|^2$ is also known as multi-path diversity in CDMA

ii) As a result $BER=\binom{2L-1}{L}\left(\dfrac{1}{2NSNR}\right)^L$, where L is the $\#$ of taps of freq. selective channel and coefficients $h(0)\cdots h(L-1)$ are IID rayleigh with unity avg power

Example: Given $L=3$ taps for FSC, $N=256$ and $SNR=15dB$.

Then $BER=\binom{5}{3}\left(\dfrac{1}{2\times 256\times 10^{1.5}}\right)^3 = 2.36\times 10^{-12}$

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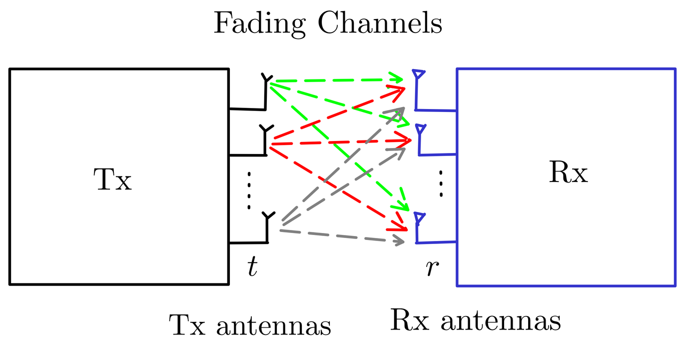
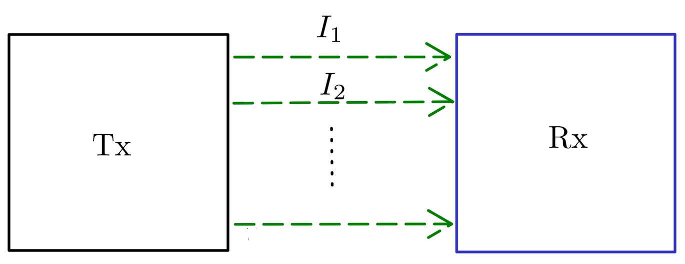
Multiple Input Multiple Output

Multiple Input Multiple Output

i) This is a key technology for 3G/4G wireless systems
Multiple Inputs $\implies$ Multiple Transmit Antennas (t)
Multiple Outputs $\implies$ Multiple Receive Antennas (r)
MIMO $\implies$ System with $r \times t$ antennas



ii) MIMO leads to
$\rightarrow$ Increase in Data Rates by transmitting several information streams in parallel also termed **Spatial Multiplexing**.



$\rightarrow$ Increase in Reliability due to **increased diversity**

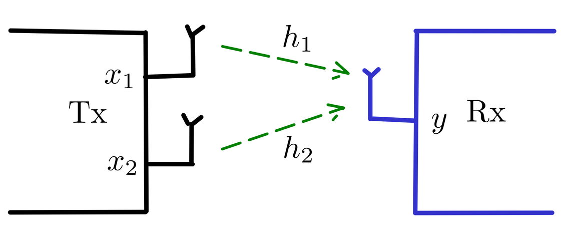
MIMO receiver (when $r = t$)

i) Objective is to recover the transmit vector $\mathbf{x}$ at the receiver from the receive vector, $\mathbf{y} = \mathbf{Hx} + \mathbf{w}$
ii) In a simple scenario, where $r = t$, $\mathbf{H}_{r\times r}$ is a square matrix (i.e., # of Rx antennas = # of Tx antennas).
$\rightarrow$ Then, if $\mathbf{H}$ is invertible, we get $\hat{\mathbf{x}} = \mathbf{H}^{-1}\mathbf{y}$ using approx. that $\mathbf{y} \approx \mathbf{Hx}$, i.e., system of linear equations

MIMO

MISO: Transmit Beamforming ($r = 1$)

i) Considering a 2×1 MISO system, we have



$y = \begin{bmatrix} h_1 & h_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + w$, then What is the optimal tx scheme x?

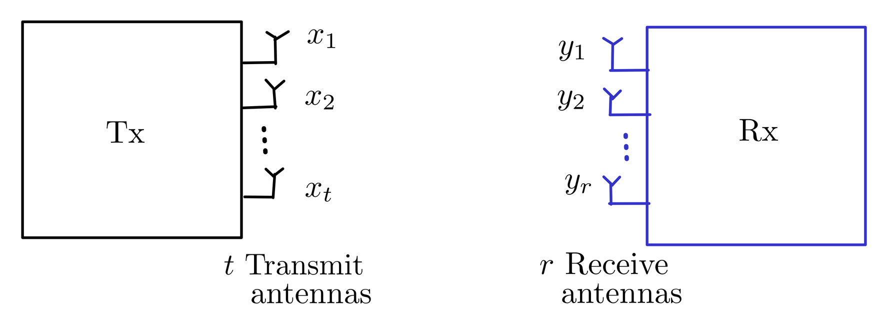
ii) Transmit Beamformer uses, $\mathbf{x} = \frac{1}{\|\mathbf{h}\|}\begin{bmatrix} h_1^* \\ h_2^* \end{bmatrix} x$,
where x is the transmit symbol and the unit vector $\frac{1}{\|\mathbf{h}\|}\begin{bmatrix} h_1^* \\ h_2^* \end{bmatrix}$ is the transmit beamformer
iii) At the receiver, we then have
$$y = \begin{bmatrix} h_1 & h_2 \end{bmatrix}\begin{bmatrix} h_1^* \\ h_2^* \end{bmatrix}\frac{1}{\|\mathbf{h}\|}x + w \quad = \|\mathbf{h}\|\, x + w$$

ALAMOUTI Space Time Block Code

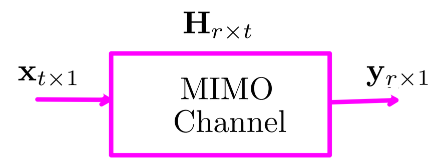
i) Does NOT require knowledge of CSI at Transmitter. Very useful from practical viewpoint
ii) Consider a 2×1 MISO system transmit symbols x_1 and x_2 using transmit vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\begin{bmatrix} -x_2^* \\ x_1^* \end{bmatrix}$ at time instants 1 and 2 respectively.
iii) The, the symbol y that we receive at time instant 1 and 2 is
$$y_1 = \begin{bmatrix} h_1 & h_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + w_1 \text{ and } y_2 = \begin{bmatrix} h_1 & h_2 \end{bmatrix}\begin{bmatrix} -x_2^* \\ x_1^* \end{bmatrix} + w_2$$

System Model

i) **Transmitter:** t symbols, $\mathbf{x}^T = \begin{bmatrix} x_1, x_2, \cdots, x_t \end{bmatrix}$ can be transmitted on the t transmit antennas. Where $\mathbf{x}$ is the t dimensional transmit vector.



ii) **Receiver:** r samples, $\mathbf{y}^T = \begin{bmatrix} y_1, y_2, \cdots, y_r \end{bmatrix}$ are received across r receive antennas. $\mathbf{y}$ is the r dimensional receive vector.
iii) Channel is represented by matrix $\mathbf{H}_{r\times t}$, which is defined by
$$\mathbf{y}_{r\times 1} = \mathbf{H}_{r\times t}\,\mathbf{x}_{t\times 1} + \mathbf{w}_{r\times 1}$$
$$\mathbf{H} \triangleq \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1t} \\ h_{21} & h_{22} & \cdots & h_{2t} \\ \vdots & & & \\ h_{r1} & h_{r2} & \cdots & h_{rt} \end{bmatrix}_{r\times t},$$



where h_{ij} is the channel coefficient between i^{th} receive and j^{th} transmit antenna
iv) In general, in a $r \times t$ system we have $y_l = h_{l1}x_1 + h_{l2}x_2 + \cdots + h_{lt}x_t + w_l$, where y_l is the sample received at l^{th} antenna.

Example: 3×2 MIMO system

3 receive antennas and 2 transmit antennas
$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}_{3\times 1} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \\ h_{31} & h_{32} \end{bmatrix}_{3\times 2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}_{3\times 1}$$
$$\begin{aligned} y_1 &= h_{11}\,x_1 + h_{12}\,x_2 + w_1 \\ \implies y_2 &= h_{21}\,x_1 + h_{22}\,x_2 + w_2 \\ y_3 &= h_{31}\,x_1 + h_{32}\,x_2 + w_3 \end{aligned}$$

Least Square problem (when $r > t$)

i) However, if $r > t$ (i.e., # of Rx antennas > # of Tx antennas), then $\mathbf{H}$ is not a square matrix, $\implies$ inverse of $\mathbf{H}$ does not exist.
$\rightarrow$ $r > t \implies$ # of Equations > # of unknowns
ii) Alternatively, define $\mathbf{e} \triangleq \mathbf{y} - \mathbf{H}\,\mathbf{x}$, now find $\mathbf{x}$ that minimizes the error, i.e.,
$$min \;\|\, e \,\|^2 = min \;\|\, \mathbf{y} - \mathbf{H}\,\mathbf{x} \,\|^2$$
iii) To find the $\hat{\mathbf{x}}$, differentiate the RHS w.r.t vector $\mathbf{x}$ and set equal to zero. Further we use the below two properties,
a) $\frac{\partial(\mathbf{c}^T\mathbf{x})}{\partial\mathbf{x}} = \frac{\partial(\mathbf{x}^T\mathbf{c})}{\partial\mathbf{x}} = \mathbf{c}$ and
b) $\frac{\partial(\mathbf{x}^T\mathbf{P}\mathbf{x})}{\partial\mathbf{x}} = 2\mathbf{P}\mathbf{x}$, where $\mathbf{P} = \mathbf{P}^T$

BER for BPSK with ZF receiver

i) Considering transmit symbol power $E\{|x_i|^2\} = P$ and each h_{ij} as IID complex Gaussian with unity avg. power,
$$BER = \binom{2L-1}{L}\left(\frac{1}{2SNR}\right)^L, \text{ where } L = r - t + 1$$
i.e., equivalent to MRC system with $r - t + 1$ receive antennas

Q) Given, 3×2 MIMO system, $SNR = 25dB$ and ZF receiver, compute the BER.
Sol) Avg BER $= \binom{3}{2}\left(\frac{1}{2\times 10^{2.5}}\right)^2 = 7.5 \times 10^{-6}$

SNR due to Transmit Beamforming

i) We then have $SNR = \frac{\|\mathbf{h}\|^2\; E\{|x|^2\}}{\sigma^2} = \frac{\|\mathbf{h}\|^2\; P}{\sigma^2}$
$\implies$ Trasmit Beamforming is able to acheive SAME SNR as the MRC
ii) However, major CHALLENGE is that Channel State Information (CSI) is to be known as the transmitter

Q.) Given a 1×2 MISO system, with CSI $h_1 = 2 + j$ and $h_2 = 1 - 2j$ available at the transmitter, compute the optimal beamforming vector.
Sol) We have $\mathbf{x} = \frac{1}{\|\mathbf{h}\|}\begin{bmatrix} h_1^* \\ h_2^* \end{bmatrix} x = \frac{1}{\sqrt{10}}\begin{bmatrix} 2-j \\ 1+2j \end{bmatrix} x$ and
$$y = \begin{bmatrix} 2+j & 1-2j \end{bmatrix}\begin{bmatrix} 2-j \\ 1+2j \end{bmatrix}\frac{1}{\sqrt{10}}x + w \quad = \sqrt{10}x + w$$

Alamouti Receiver (Orthogonal STBC)

ii) By rearranging y_1 and y_2^* and process them jointly, we have
$$\begin{bmatrix} y_1 \\ y_2^* \end{bmatrix} = \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2^* \end{bmatrix} \quad = \begin{bmatrix} h_1 \\ h_2^* \end{bmatrix} x_1 + \begin{bmatrix} h_2 \\ -h_1^* \end{bmatrix} x_2 + \begin{bmatrix} w_1 \\ w_2^* \end{bmatrix}$$
$\implies \mathbf{y} = \mathbf{c}_1\,x_1 + \mathbf{c}_2\,x_2 + \mathbf{w}$, where $\mathbf{c}_1 \triangleq \begin{bmatrix} h_1 \\ h_2^* \end{bmatrix}$ and $\mathbf{c}_2 \triangleq \begin{bmatrix} h_2 \\ -h_1^* \end{bmatrix}$
Note that $\mathbf{c}_1$ and $\mathbf{c}_2$ are ORTHOGONAL !. i.e., $\mathbf{c}_1^H\mathbf{c}_2 = 0$
iii) To recover x_1 from $\mathbf{y}$, we process as
$$\frac{\mathbf{c}_1^H}{\|\mathbf{c}_1\|}\mathbf{y} = \frac{\mathbf{c}_1^H}{\|\mathbf{c}_1\|}\mathbf{c}_1\,x_1 + \frac{\mathbf{c}_1^H}{\|\mathbf{c}_1\|}\mathbf{c}_2\,x_2 + \frac{\mathbf{c}_1^H}{\|\mathbf{c}_1\|}\mathbf{w} \quad = \|\,\mathbf{c}_1\,\|\, x_1 + 0 + \widetilde{\mathbf{w}} \text{ where,}$$
$$E\left\{\widetilde{\mathbf{w}}\widetilde{\mathbf{w}}^*\right\} = E\left\{\frac{\mathbf{c}_1^H}{\|\mathbf{c}_1\|}\mathbf{w}\mathbf{w}^H\frac{\mathbf{c}_1}{\|\mathbf{c}_1\|}\right\} = \frac{\mathbf{c}_1^H}{\|\mathbf{c}_1\|}E\left\{\mathbf{w}\mathbf{w}^H\right\}\frac{\mathbf{c}_1}{\|\mathbf{c}_1\|} = \sigma^2$$
iv) Then, $SNR = \frac{\|\,\mathbf{c}_1\,\|^2\; E\{|x_1|^2\}}{\sigma^2} = \frac{\|\,\mathbf{h}\,\|^2\,\frac{P}{2}}{\sigma^2} = \frac{1}{2}$SNR of MRC !

Special Cases

SIMO: Single Input Multiple Output system ,
i.e., $t = 1$ and $\begin{bmatrix} y_1 \\ \vdots \\ y_r \end{bmatrix}_{r\times 1} = \begin{bmatrix} h_1 \\ \vdots \\ h_r \end{bmatrix}_{r\times 1} x + \begin{bmatrix} w_1 \\ \vdots \\ w_r \end{bmatrix}_{r\times 1}$

MISO: Multiple Input Single Output system ,
i.e., $r = 1$ and $y = \begin{bmatrix} h_1 & \cdots & h_t \end{bmatrix}_{1\times t}\begin{bmatrix} x_1 \\ \vdots \\ x_t \end{bmatrix}_{t\times 1} + w$

SISO: Single Input Single Output system ,
i.e., $r = t = 1$ and $y = hx + w$

Noise Vector analysis

i) Each w_i is Gaussian with zero-mean and σ^2 variance ,
$E\{w_i\} = 0$ and $E\{|w_i|^2\} = E\{w_i w_i^*\} = \sigma^2$
ii) Further, w_i are IID, i.e.,
$$E\left\{w_i w_j^*\right\} = \begin{cases} \sigma^2 & \text{if } i = j, \\ E\{w_i\}\,E\left\{w_j^*\right\} = 0 & \text{if } i \neq j \end{cases}$$
$\rightarrow$ As a result noise Covariance matrix $\mathbf{R}$ can be obtained as
$$E\left\{\mathbf{w}\mathbf{w}^H\right\} = E\left\{\begin{bmatrix} w_1 \\ \vdots \\ w_r \end{bmatrix}\begin{bmatrix} w_1^* & \cdots & w_r^* \end{bmatrix}\right\} = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & & \sigma^2 \end{bmatrix}$$
$\mathbf{R} = \sigma^2\mathbf{I}_{r\times r}$, where $\mathbf{I}$ is an identity matrix

Least Square Cost Function

$$\begin{aligned} f(\mathbf{x}) &\triangleq \|\,\mathbf{y} - \mathbf{H}\,\mathbf{x}\,\|^2 = (\mathbf{y} - \mathbf{H}\,\mathbf{x})^T(\mathbf{y} - \mathbf{H}\,\mathbf{x}) \\ &= (\mathbf{y}^T - \mathbf{x}^T\mathbf{H}^T)(\mathbf{y} - \mathbf{H}\,\mathbf{x}) \\ &= \mathbf{y}^T\mathbf{y} - \mathbf{x}^T\mathbf{H}^T\mathbf{y} - \mathbf{y}^T\mathbf{H}\,\mathbf{x} + \mathbf{x}^T\mathbf{H}^T\mathbf{H}\,\mathbf{x} \\ &= \mathbf{y}^T\mathbf{y} - 2\mathbf{x}^T\mathbf{H}^T\mathbf{y} + \mathbf{x}^T\mathbf{H}^T\mathbf{H}\,\mathbf{x} \end{aligned}$$
Differentiating $f(\mathbf{x})$ and equating to 0, we have
$$\begin{aligned} \frac{\partial f(\mathbf{x})}{\partial\mathbf{x}} = \quad 0 - 2\mathbf{H}^T\mathbf{y} + 2\mathbf{H}^T\mathbf{H}\,\mathbf{x} &= 0 \\ \implies \qquad \mathbf{H}^T\mathbf{H}\,\mathbf{x} &= \mathbf{H}^T\mathbf{y} \\ \implies \qquad \hat{\mathbf{x}} &= (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{y} \quad \blacksquare \end{aligned}$$

Zero Forcing Receiver

i) $\hat{\mathbf{x}} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{y}$, is called as the Zero Forcing receiver
$\rightarrow$ Generalizing it to the case where $\mathbf{H}$ is complex, we have
$\hat{\mathbf{x}} = (\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H\mathbf{y} \quad \blacksquare$
ii) Note that $\mathbf{H}^H\mathbf{H}$ is a $t \times t$ square matrix and $(\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H$ is the Pseudo (also called left) inverse of $\mathbf{H}$ i.e., $(\mathbf{H}^H\mathbf{H})^{-1}(\mathbf{H}^H\mathbf{H}) = \mathbf{I}$

Example: ZF receiver for 3×2 MIMO system

i) Consider a MIMO system with $r = 3$ and $t = 2$
$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}_{3\times 1} = \begin{bmatrix} 2 & 3 \\ 1 & 3 \\ 4 & 2 \end{bmatrix}_{3\times 2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}_{3\times 1}$$
ii) ZF receiver $= (\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H = \begin{bmatrix} -0.0405 & -0.1676 & 0.3121 \\ 0.1676 & 0.2659 & -0.1503 \end{bmatrix}$
$$\begin{aligned} \implies \hat{x}_1 &= -0.0405\;y_1 - 0.1676\;y_2 + 0.3121\;y_3 \\ \hat{x}_2 &= 0.1676\;y_1 + 0.2659\;y_2 - 0.1503\;y_3 \end{aligned}$$

BER of Transmit Beamforming

i) Trasmit beamforming can be extended to a general scenario with $1 \times L$ MISO system, i.e., $\mathbf{h} = \begin{bmatrix} h_1 & \cdots & h_L \end{bmatrix}$
$$y = \begin{bmatrix} h_1 & \cdots & h_L \end{bmatrix}\begin{bmatrix} h_1^* \\ \vdots \\ h_L^* \end{bmatrix}\frac{1}{\|\mathbf{h}\|}x + w \quad \text{and at high SNR}$$
$$BER = \binom{2L-1}{L}\left(\frac{1}{2SNR}\right)^L, \text{ where } L = \# \text{ of Tx antennas}$$

Q.) Given 1×3 MISO with SNR$= 25dB$, compute avg. BER.
Sol) We have $L = 3$ and $BER = \binom{5}{3}\left(\frac{1}{2\times 10^{2.5}}\right)^3 = 6.3 \times 10^{-8}$

Alamouti Code BER

i) We have $L = 2$ and $BER = \binom{3}{2}\left(\frac{1}{2\times\frac{1}{2}SNR}\right)^2 = \frac{3}{SNR^2}$

Q.) Alamouti processing given CSI $h_1 = 2 + j$ and $h_2 = 1 - 2j$, We have
Sol) $y_1 = \begin{bmatrix} 2+j & 1-2j \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} x + w_1$, $y_2 = \begin{bmatrix} 2+j & 1-2j \end{bmatrix}\begin{bmatrix} -x_2^* \\ x_1^* \end{bmatrix} x + w_2$
$$\begin{bmatrix} y_1 \\ y_2^* \end{bmatrix} = \begin{bmatrix} 2+j & 1-2j \\ 1+2j & -2+j \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2^* \end{bmatrix} \quad = \begin{bmatrix} 2+j \\ 1+2j \end{bmatrix} x_1 + \begin{bmatrix} 1-2j \\ -2+j \end{bmatrix} x_2 + \begin{bmatrix} w_1 \\ w_2^* \end{bmatrix}$$
Extract x_1 using , $\frac{\mathbf{c}_1^H}{\|\mathbf{c}_1\|}\mathbf{y} = \frac{1}{\sqrt{10}}\begin{bmatrix} 2-j & 1-2j \end{bmatrix}\begin{bmatrix} y_1 \\ y_2^* \end{bmatrix} = \|\,\mathbf{c}_1\,\|\, x_1 + \widetilde{\mathbf{w}_1}$, similarly
Extract x_2 using , $\frac{\mathbf{c}_2^H}{\|\mathbf{c}_2\|}\mathbf{y} = \frac{1}{\sqrt{10}}\begin{bmatrix} 1+2j & -2-j \end{bmatrix}\begin{bmatrix} y_1 \\ y_2^* \end{bmatrix} = \|\,\mathbf{c}_2\,\|\, x_1 + \widetilde{\mathbf{w}_2}$

Q.) Given $SNR = 25dB$, compute the BER
Sol) $BER = \frac{3}{SNR^2} = \frac{3}{(10^2.5)^2} = 3 \times 10^5$

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Characterizing the MIMO Channel

MIMO-II

Singular Value Decomposition (SVD)

i) The channel matrix $\mathbf{H}_{r\times t}$ is SV decomposed for $r \geq t$ when

$$\mathbf{H}_{r\times t} = \mathbf{U}_{r\times t}\,\mathbf{\Sigma}_{t\times t}\,\mathbf{V}^H_{t\times t}$$

$$\mathbf{H}_{r\times t} = \left[\mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_t\right]_{r\times t} \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \sigma_t \end{bmatrix}_{t\times t} \begin{bmatrix} \mathbf{v}_1^H \\ \mathbf{v}_2^H \\ \vdots \\ \mathbf{v}_t^H \end{bmatrix}_{t\times t}$$

ii) Columns of $\mathbf{U}$ are Orthonormal

i.e., $\mathbf{u}_i^H \mathbf{u}_j = \begin{cases} \| \mathbf{u}_i \|^2 = 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$

$$\mathbf{U}^H\mathbf{U} = \begin{bmatrix} \mathbf{u}_1^H \\ \mathbf{u}_2^H \\ \vdots \\ \mathbf{u}_t^H \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_t \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & 0 & 1 \end{bmatrix} = \mathbf{I}_{t\times t}$$

→ Similarly $\mathbf{V}^H\mathbf{V} = \mathbf{V}\mathbf{V}^H = \mathbf{I}_{t\times t}$, i.e. $\mathbf{V}$ is UNITARY matrix

iii) $\mathbf{\Sigma}_{t\times t}$ is a diagonal matrix with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_t \geq 0$,

→ Singular values are non-negative and are arranged in decreasing order

→ The $\#$ of non-zero singular values indicate the rank of the channel matrix

Capacity of a MIMO wireless system

i) Considereing spatial multiplexing in MIMO i.e., $\widetilde{y}_i = \sigma_i\,\widetilde{x}_i + \widetilde{w}_i$ where σ_i is gain of i^{th} channel, $E\{|\widetilde{x}_i|^2\} = P_i$ is power of i^{th} transmitted symbol and $\widetilde{w}_i$ is Gaussian with var σ^2, we have $SNR_i = \dfrac{\sigma_i^2 P_i}{\sigma^2}$

ii) Then we have *Shannon rate* $= \log_2\left(1 + \dfrac{\sigma_i^2 P_i}{\sigma^2}\right)$

The sum rate $= \sum_{i=0}^{t} \log_2\left(1 + SNR_i\right)$

iii) Considering transmit power as P, the capacity of MIMO

$$C = \max_{\mathbf{P}}. \quad \sum_{i=0}^{t} \log_2\left(1 + \frac{\sigma_i^2 P_i}{\sigma^2}\right)$$
$$sub.to \quad \sum_{i=0}^{t} P_i = P$$

which is a constrained optimization problem

SVD in MIMO Systems

i) SVD can be employed in MIMO comm. systems as a tool. Consider the MIMO model

$$\mathbf{y}_{r\times 1} = \mathbf{H}_{r\times t}\,\mathbf{x}_{t\times 1} + \mathbf{w}_{r\times 1}, \text{ now substituting } \mathbf{H} = \mathbf{U}\,\mathbf{\Sigma}\,\mathbf{V}^H$$
$$\mathbf{y} = \mathbf{U}\,\mathbf{\Sigma}\,\mathbf{V}^H\,\mathbf{x} + \mathbf{w}$$

ii) At the receiver, multiply the received vector with $\mathbf{U}^H$, we have

$$\mathbf{U}^H\mathbf{y} = \mathbf{U}^H\mathbf{U}\,\mathbf{\Sigma}\,\mathbf{V}^H\,\mathbf{x} + \mathbf{U}^H\mathbf{w}$$
$$\widetilde{\mathbf{y}} = \mathbf{\Sigma}\,\mathbf{V}^H\,\mathbf{x} + \widetilde{\mathbf{w}} \quad \text{, where } \widetilde{\mathbf{y}} \triangleq \mathbf{U}^H\mathbf{y} \text{ and } \widetilde{\mathbf{w}} \triangleq \mathbf{U}^H\mathbf{w}$$

Examples:

I) SVD of $\mathbf{H} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is $= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}$

where $\mathbf{U}^H\mathbf{U} = 1$, $\sigma_1 = \sqrt{2} > 0$ and $\mathbf{V} = [1]$ is unitary matrix

II) $\mathbf{H} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{5} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \sqrt{5} \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{5} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

where $\mathbf{U}^H\mathbf{U} = \mathbf{I}$, $\quad \mathbf{V}^H\mathbf{V} = \mathbf{V}\mathbf{V}^H = \mathbf{I}$

and $\sigma_1(=\sqrt{5}) > \sigma_2(=\sqrt{1}) > 0$

III) $\mathbf{H} = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Solution using Lagrange multipliers

i) Need to maximize the below equation

$$\sum_{i=0}^{t} \log_2\left(1 + \frac{\sigma_i^2 P_i}{\sigma^2}\right) + \lambda\left(P - \sum_{i=0}^{t} P_i\right)$$

ii) From partial differentiation w.r.t P_i and equating to 0

$$\frac{\partial}{\partial P_i} = \frac{\sigma_i^2/\sigma^2}{1 + \sigma_i^2 P_i/\sigma^2} + \lambda(0-1) = 0$$
$$\implies \lambda = \frac{\sigma_i^2/\sigma^2}{1 + \sigma_i^2 P_i/\sigma^2} \quad , \implies P_i = \left(\frac{1}{\lambda} - \frac{\sigma^2}{\sigma_i^2}\right)^+$$
$$P_i = \begin{cases} \frac{1}{\lambda} - \frac{\sigma^2}{\sigma_i^2} & \text{if } \frac{1}{\lambda} - \frac{\sigma^2}{\sigma_i^2} \geq 0, \\ 0 & \text{otherwise} \end{cases} \qquad \blacksquare$$

iii) To obtain the value of Lagrange multiplier λ, need to solve

$$\sum_{i=0}^{t} P_i = P, \qquad \implies \sum_{i=0}^{t}\left(\frac{1}{\lambda} - \frac{\sigma^2}{\sigma_i^2}\right) = P$$

→ This algorithm is termed as the Water Filling Algorithm

Spatial Multiplexing

i) At the tx, by performing precoding i.e., $\mathbf{x} = \mathbf{V}\widetilde{\mathbf{x}}$, we have

$$\widetilde{\mathbf{y}} = \mathbf{\Sigma}\,\mathbf{V}^H\,\mathbf{V}\widetilde{\mathbf{x}} + \widetilde{\mathbf{w}} \quad = \mathbf{\Sigma}\,\widetilde{\mathbf{x}} + \widetilde{\mathbf{w}}$$

$$\begin{bmatrix} \widetilde{y}_1 \\ \widetilde{y}_2 \\ \vdots \\ \widetilde{y}_t \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & & \\ \vdots & & & \\ 0 & \cdots & 0 & \sigma_t \end{bmatrix} \begin{bmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \vdots \\ \widetilde{x}_t \end{bmatrix} + \begin{bmatrix} \widetilde{w}_1 \\ \widetilde{w}_2 \\ \vdots \\ \widetilde{w}_t \end{bmatrix} ,$$

$$\implies \widetilde{y}_i = \sigma_i\,\widetilde{x}_i + \widetilde{w}_i \quad \text{, for } 1 \leq i \leq t$$

ii) This indicates, there are t parallel channels and t information symbols can be transmitted in parallel

Using SVD, SPATIAL MULTIPLEXING is obtained !

Example: Consider 3×3 MIMO Channel

i) Given H, and observing that its columns are orthogonal

$$\mathbf{H} = \begin{bmatrix} 2 & -6 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{13}} & \frac{-6}{\sqrt{52}} & 0 \\ \frac{3}{\sqrt{13}} & \frac{4}{\sqrt{52}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{13} & 0 & 0 \\ 0 & \sqrt{52} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

To get the singular values in decreasing order

$$= \begin{bmatrix} \frac{-6}{\sqrt{52}} & \frac{2}{\sqrt{13}} & 0 \\ \frac{4}{\sqrt{52}} & \frac{3}{\sqrt{13}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \sqrt{52} & 0 \\ \sqrt{13} & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-6}{\sqrt{52}} & \frac{2}{\sqrt{13}} & 0 \\ \frac{4}{\sqrt{52}} & \frac{3}{\sqrt{13}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{52} & 0 & 0 \\ 0 & \sqrt{13} & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{U}\,\mathbf{\Sigma}\,\mathbf{V}^H$$

→ Note, $\mathbf{U}^H\mathbf{U} = \mathbf{I}$, $\quad \mathbf{V}^H\mathbf{V} = \mathbf{V}\mathbf{V}^H = \mathbf{I}$ and $\sigma_1(=\sqrt{52}) > \sigma_2(=\sqrt{13}) > \sigma_3(=2) > 0$

ii) Applying transmit preprocessing (Precoding) at the transmitter,

$$\mathbf{x} = \mathbf{V}\widetilde{\mathbf{x}}, \quad \text{i.e.,} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \widetilde{x}_3 \end{bmatrix}$$

iii) At the receiver, $\widetilde{\mathbf{y}} = \mathbf{\Sigma}\widetilde{\mathbf{x}} + \widetilde{\mathbf{w}}$

$$\text{i.e.,} \begin{bmatrix} \widetilde{y}_1 \\ \widetilde{y}_2 \\ \widetilde{y}_3 \end{bmatrix} = \begin{bmatrix} \sqrt{52} & 0 & 0 \\ 0 & \sqrt{13} & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \widetilde{x}_3 \end{bmatrix} + \begin{bmatrix} \widetilde{w}_1 \\ \widetilde{w}_2 \\ \widetilde{w}_3 \end{bmatrix}$$

→ 3 decoupled channels resulting in spatial multiplexing.

Example: Optimal Power Allocation

i) To achieve Shannon capacity

$$P_1 = \left(\frac{1}{\lambda} - \frac{\sigma^2}{\sigma_1^2}\right)^+ = \left(\frac{1}{\lambda} - \frac{\sigma^2}{52}\right)^+, P_2 = \left(\frac{1}{\lambda} - \frac{\sigma^2}{\sigma_2^2}\right)^+ = \left(\frac{1}{\lambda} - \frac{\sigma^2}{13}\right)^+$$

and $P_3 = \left(\frac{1}{\lambda} - \frac{\sigma^2}{\sigma_3^2}\right)^+ = \left(\frac{1}{\lambda} - \frac{\sigma^2}{4}\right)^+$

ii) Considering $\sigma^2 = 0dB = 1$ and total Tx power $P = 3dB \approx 2$

→ $P_1 + P_2 + P_3 = 2, \implies \frac{1}{\lambda} = 0.7821$

$P_1 = -1.1755dB$, $P_2 = -1.517dB$ and $P_3 = -2.74dB$

→ Power allocation decreases as the channel gain σ_i^2 decreases

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Orthogonal Frequency Division Multiplexing

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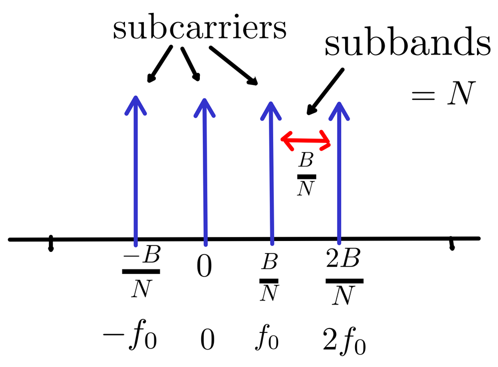
Orthogonal Frequency Division Multiplexing

i) Key enabler of high data rates (BROADBAND Technology) in 4G wireless systems and WiFi (WLAN) systems
→ LTE: Long Term Evolution
→ LTE-A: LTE Advanced
→ WiMAX: Worldwide Interoperability for Microwave Access
→ WLAN: 802.11a, 802.11g, 802.11n, 802.11ac

Transmission in MCM System



i) Consdier each subcarrier is transmitted with spacing of $f_0 = \frac{B}{N}$, i.e., at frequencies $\{\cdots, -2f_0, -f_0, 0, f_0, 2f_0, \cdots\}$
ii) k^{th} subcarrier frequency is $kf_0 = k\frac{B}{N}$
→ Carrier is $e^{j2\pi kf_0}$ and let symbol be denoted by X_k
→ The signal transmitted on k^{th} subcarrier is
$s_k(t) = X_k e^{j2\pi kf_0 t}$
iii) Net transmitted MCM signal
$x(t) = \sum_k X_k e^{j2\pi kf_0 t}$
Note, this is similar to Fourier series

OFDM

Frequency Selective Channel scenario

i) Model for Freq. selective channel is given by
$y(n) = h(0)x(n) + h(1)x(n-1) + \cdots + h(L-1)x(n-L+1)$
ii) Consider 2 consecutive OFDM blocks, then

$\widetilde{X}_0, \widetilde{X}_1, \cdots, \widetilde{X}_{N-1}$ $\xrightarrow{\text{N-pt IFFT}}$ $\widetilde{x}(0), \widetilde{x}(1), \cdots, \widetilde{x}(N-1)$ (Transmit samples of OFDM sym 1)

$X_0, X_1, \cdots, X_{N-1}$ $\xrightarrow{\text{N-pt IFFT}}$ $x(0), x(1), \cdots, x(N-1)$ (Transmit samples of OFDM sym 2)

Channel

output for $x(0)$
$y(0) = h(0)x(0) + h(1)\widetilde{x}(N-1) + \cdots + h(L-1)\widetilde{x}(N-L+1)$

→ Results in Inter Block Interference (IBI)

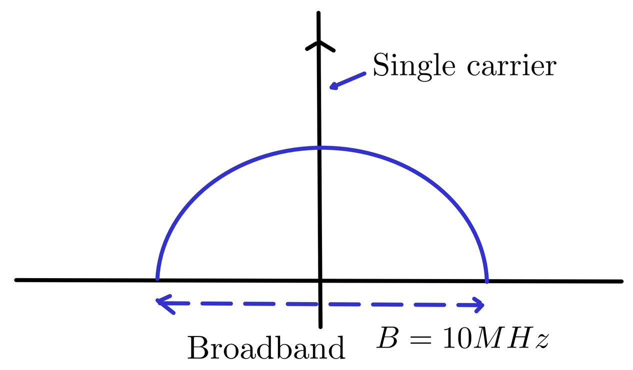
BER of OFDM

i) Considering the Gaussian noise at the receiver we have, $y = h \circledast x + w$, and $Y(k) = H(k)\,X(k) + W(k)$, where
→ $W(k)$ is the k^{th} FFT pt of $[w(0), w(1), \cdots, w(N-1)]$, i.e., $W(k) = \sum_{l=0}^{N-1} w(l) e^{-j2\pi k\frac{l}{N}}$ and
→ $H(k) = \sum_{l=0}^{L-1} h(l) e^{-j2\pi k\frac{l}{N}}$
ii) Assuming that noise samples $w(l)$ are IID Gaussian with, $E\{w(l)\} = 0$, $E\{|w(l)|^2\} = \sigma^2$ and $E\{w(l)w^*(\tilde{l})\} = 0$
→ We then have

$$
\begin{aligned}
E\{W(k)\} &= \sum_{l=0}^{N-1} E\{w(l)\}\, e^{-j2\pi k\frac{l}{N}} = 0 \\
E\{|W(k)|^2\} &= E\{W(k)W(k)^*\} \\
&= E\left\{\left(\sum_{l=0}^{N-1} w(l) e^{-j2\pi k\frac{l}{N}}\right)\left(\sum_{\tilde{l}=0}^{N-1} w(\tilde{l}) e^{-j2\pi k\frac{\tilde{l}}{N}}\right)^*\right\} \\
&= E\left\{\sum_{l=0}^{N-1}\sum_{\tilde{l}=0}^{N-1} w(l)^* w(\tilde{l}) e^{-j2\pi k\frac{(l-\tilde{l})}{N}}\right\} \\
&= \sum_{l=0}^{N-1}\sum_{\tilde{l}=0}^{N-1} E\left\{w(l)w^*(\tilde{l})\right\} e^{-j2\pi k\frac{(l-\tilde{l})}{N}} \\
&= \sum_{l=0}^{N-1} \sigma^2 = N\sigma^2 \quad \blacksquare
\end{aligned}
$$

iii) Similarly, assuming the channel coeffs $h(l)$ as IID complex Gaussian with $E\{h(l)\} = 0$, $E\{|h(l)|^2\} = 1$ and $E\{h(l)h^*(\tilde{l})\} = 0$ (Uncorrelated Scattering)
→ $E\{H(k)\} = 0$ and $E\{|H(k)|^2\} = L$ ■ (Rayleigh fading with avg. power L)
iv) With $E\{|X(k)|^2\} = P$, we have SNR at receiver as
$SNR_r = \frac{|H(k)|^2 P}{N\sigma^2}$ and Avg$\{SNR_r\} = \frac{LP}{N\sigma^2}$
$\implies BER_{OFDM} = \frac{1}{2}\left(1 - \sqrt{\frac{\frac{L}{N}SNR}{2 + \frac{L}{N}SNR}}\right)$, where $SNR = \frac{P}{\sigma^2}$ (Transmit SNR)

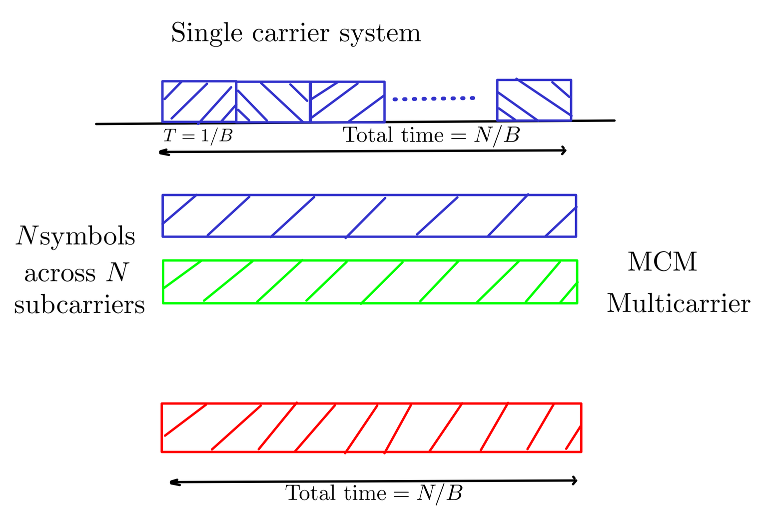
Basic Principle of OFDM



i) Consider a broadband channel with single-carrier and bandwidth $B = 10$ MHz,
→ Symbol duration $T = \frac{1}{B} = \frac{1}{10MHz} = 0.1\mu s$
Typical delay spread of a channel $T_d \approx 2 - 3\mu s$
$\implies T \ll T_d$ (i.e., Symbol Time ≪ Delay spread), results in ISI or Inter symbol interference
→ As bandwidth B increases symbol time decreases and leads to ISI
ii) It is a significant challenge in a broadband wireless system to overcome ISI

Extraction of symbols at receiver

i) To extract symbol X_l on the l^{th} subcarrier
$f_0 \int_0^{\frac{1}{f_0}} e^{-j2\pi lf_0 t} x(t)\,dt = f_0 \int_0^{\frac{1}{f_0}} e^{-j2\pi lf_0 t} \sum_k X_k e^{j2\pi kf_0 t}\,dt$
$= \sum_k X_k f_0 \int_0^{\frac{1}{f_0}} e^{-j2\pi(k-l)f_0 t} = \sum_k X_k \delta(k-l) = X_l$
ii) At receiver by correlating (coherent demodulation) with $e^{-j2\pi lf_0 t}$ or matched filtering with $e^{-j2\pi lf_0(T-t)}$, one can extract symbol on l^{th} subcarrier

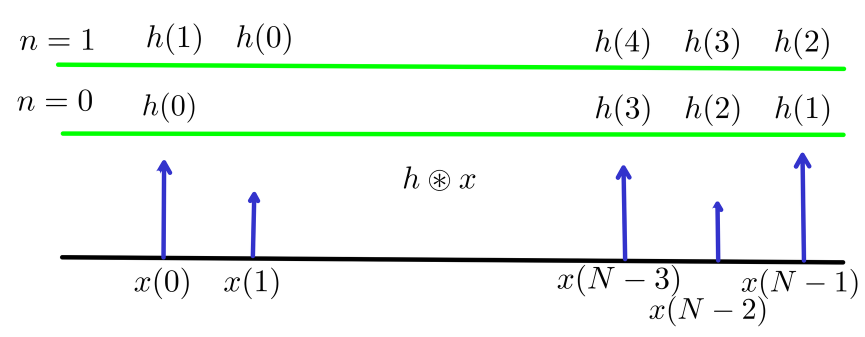


Practical difficulties of MCM system:
i) Generating a MCM signal is difficult because of the large number of subcarriers
$N(=1000)$ number of local oscillators required, filters etc

Cyclic Prefix to avoid IBI

Cyclic Prefix: $X_0, X_1, \cdots, X_{N-1}$ $\xrightarrow{\text{N-pt IDFT}}$ $\underbrace{x(N-\bar{L}), \cdots, x(N-1)}_{\text{Prefix of } \bar{L} \text{ samples from tail}}$ $\underbrace{x(0), x(1), \cdots, x(N-1)}_{\bar{L} \text{ Samples}}$

i) In a block of transmitted sysmbols, the $\bar{L}$ samples from the tail are prefixed at the head
ii) The received symbols $y(0), y(1)$ corresponding to $x(0), x(1)$ are given by
$y(0) = h(0)x(0) + h(1)x(N-1) + \cdots + h(L-1)x(N-L+1)$
$y(1) = h(0)x(1) + h(1)x(0) + \cdots + h(L-1)x(N-L)$
Circular Convolution



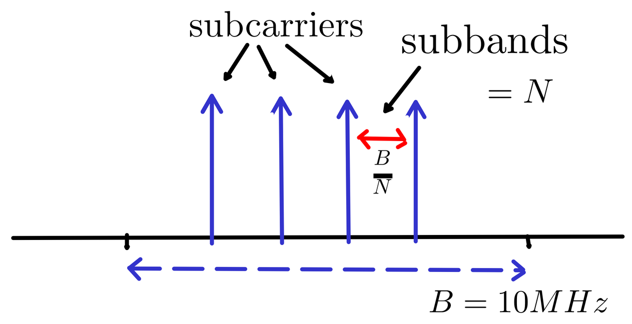
iii) Addition of cyclic prefix (CP) has resulted in a circular convolution at the output of the OFDM system, i.e., $y = h \circledast x$
$\implies$ applying FFT, $Y(k) = H(k)\,X(k)$, where
→ $Y(k)$ is the k^{th} DFT/FFT coefficient of OFDM received signal $[y(0)\ y(1)\ \cdots y(N-1)]$
→ $H(k)$ is the k^{th} DFT/FFT of $[h(0)\ h(1)\ \cdots\ h(L-1)\ 0 \cdots 0]$

Example:

Q) Given $L = 16$ channel taps, $N = 256$ subcarriers and $SNR = 35\ dB$, compute OFDM BER.

Sol) $BER_{OFDM} = \frac{1}{2}\left(1 - \sqrt{\frac{\frac{16}{256}10^{3.5}}{2 + \frac{16}{256}10^{3.5}}}\right) = 2.5 \times 10^{-3}$

Principle to Overcome ISI



i) Divide the bandwidth B into N subbands of bandwidth $\frac{B}{N}$ each
Use a sub-carrier for each subband
ii) Example: $B = 10MHz$ and $N = 1000$
$\implies$ BW of each subband $= \frac{10MHz}{1000} = 10KHz$
Symbol time in each subband $T = \frac{1}{10KHz} = 100\mu s \gg T_d (= 2\mu s)$
$\implies$ NO ISI in new system with Multi-Carrier Modulated (MCM) system

Alleviate the need for large number of subcarriers

i) Since the signal is bandlimited to $\{-f_{max}, f_{max}\}$ i.e., $\{-\frac{B}{2}, \frac{B}{2}\}$, it can be sampled at the Nyquist rate $= 2f_{max} = B$
ii) Sampling interval $= \frac{1}{B} = T$
l^{th} sampling instant $= lT = \frac{l}{B}$
iii) we have

$$
\begin{aligned}
x(t) &= \sum_k X_k e^{j2\pi kf_0 t} \\
x(l) = x(lT) &= \sum_k X_k e^{j2\pi kf_0 lT} = \sum_k X_k e^{j2\pi k\frac{B}{N}\frac{l}{B}} \\
&= \sum_k X_k e^{j2\pi\frac{kl}{N}} \text{ (which is IDFT or IFFT)}
\end{aligned}
$$

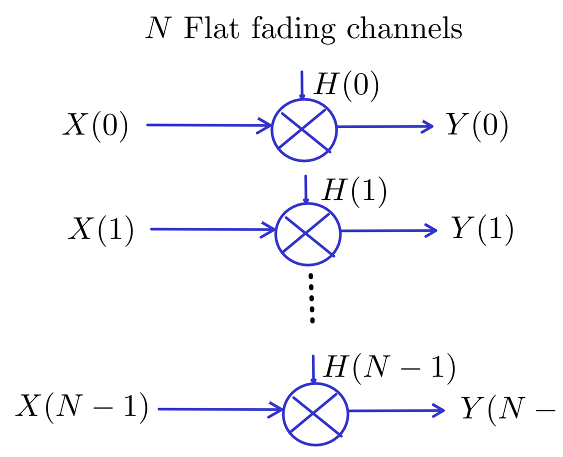
→ where $x(l)$ is the l^{th} IDFT point of symbols transmitted on N subcarriers $\{X_0, X_1, \cdots, X_{N-1}\}$
iv) The principle of OFDM is to transmit the sample points i.e., IFFT at Transmitter and FFT at Receiver

IFFT at Transmitter and FFT at Receiver

Transmit symbols: $\{X_0, X_1, \cdots, X_{N-1}\}$
↓ N-pt IFFT
Trasmitted Samples: $\{x(0), x(1), \cdots, x(N-1)\}$
Sampled version of MCM signal
↓ Channel
$\{y(0), y(1), \cdots, y(N-1)\}$
Reciver: ↓ N-pt FFT
$\{X_0, X_1, \cdots, X_{N-1}\}$

Equivalent N Flat fading channels

i) Equivalently, $Y(k)$ can be considered received symbol on k^{th} subcarrier, $H(k)$ channel coefficient of k^{th} subcarrier and $X(k)$ as the transmitted symbol on the k^{th} subcarrier



$\implies$ OFDM has converted Frequency Selective channel into equivalent N Flat fading channels

Tx and Rx Schematic

Transmit Symbols: $X(0)\ X(1)\ \cdots X(N-1)$
↓ N-pt IFFT
Symbols to be Transmitted: $x(0)\ x(1)\ \cdots\ x(N-1)$
↓ Cyclic Prefix
Symbols Transmitted with Cyclic Prefix: $\underbrace{x(N-\bar{L})\ \cdots\ x(N-1)}_{\text{Prefix}}\ x(0)\ \cdots\ x(N-1)$
↓ Channel
Received Symbols: $y(N-\bar{L})\ \cdots\ y(N-1)\ y(0)\ \cdots\ y(N-1)$
↓ Remove Prefix
Symbols to be processed: $y(0)\ y(1)\ \cdots\ y(N-1)$
↓ N-pt FFT
Symbols with Flat fading: $Y(0)\ Y(1)\ \cdots\ Y(N-1)$
↓ Detect symbols
Detected Symbols: $\hat{X}(0)\ \hat{X}(1)\ \cdots\ \hat{X}(N-1)$

References

[1] NOC: Principles of Modern CDMA MIMO OFDM Wireless Communications By Prof. Aditya K. Jagannatham, IIT Kanpur

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