Fundamentals of Probability

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Probability

Basics

i) Sample space: set of all possible outcomes
Ex: M-ary PAM with $M = 4$ symbols,
$S = \{-3\alpha, -\alpha, \alpha, 3\alpha\}$

ii) Event: Is any subset of S
Ex: Event $A = \{-3\alpha, \alpha\}$
Event complement $\bar{A} = \{-\alpha, 3\alpha\}$

iii) Set theory: Union and Intersection
$A \cup B$ and $A \cap B$

iv) NULL Event: $\phi = \{\}$

v) Mutually Exclusive events: If $A \cap \bar{A} = \{\}$
Contain no common sample points

Axioms of Probability

Ax.1: $P(A) \geq 0$
Ax.2: $P(S) = 1$
Ax.3: $P(A \cup B) = P(A) + P(B)$
Iff $A \cap B = \phi$

Properties

i) $P(A) \leq 1$
ii) $P(\phi) = 0$
iii) $P(A) < P(B)$ if $A \subset B$
iv) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Example:

i) For 4-PAM, let $P = \{\frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}\}$ then
ii) For $A = \{-3\alpha, \alpha\}$, $P(A) = \frac{3}{8}$
and $B = \{\alpha, 3\alpha\}$, $P(B) = \frac{3}{4}$
iii) We have $P(A \cup B) = P(\{-3\alpha, \alpha, 3\alpha\}) = \frac{7}{8}$
$= P(A) + P(B) - P(A \cap B) = \frac{7}{8}$

Conditional Probability

i) $P(A|B) \triangleq \dfrac{P(A \cap B)}{P(B)}$
$\implies P(A \cap B) = P(A|B)P(B)$

ii) $P(B|A) \triangleq \dfrac{P(B \cap A)}{P(A)}$
$\implies P(B \cap A) = P(B|A)P(A)$

iii) **Bayes Rule**:
$P(A|B)P(B) = P(B|A)P(A)$
Note $P(A|B) \neq P(B|A)$

Example

i) In 4-PAM, let $A = \{-3\alpha, \alpha\}$ and $B = \{\alpha\}$
ii) $P(B|A) = \dfrac{P(B \cap A)}{P(A)} = \frac{1}{4}/\frac{3}{8} = \frac{2}{3}$
iii) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \frac{1}{4}/\frac{1}{4} = 1$
iv) Note $P(A|B) \neq P(B|A)$

Statistically Independent Events

i) $P(A|B) = P(A)$
i.e., Occurence of B has no-effect on A
$\implies \dfrac{P(A \cap B)}{P(B)} = P(A)$
$\implies P(A \cap B) = P(A)P(B)$

ii) $\implies \dfrac{P(A \cap B)}{P(A)} = P(B)$
$P(B|A) = P(B)$
Similarly, A has no-effect on B

iii) Note: If $P(A \cap B) = P(A)P(B)$
then $P(A \cap \bar{B}) = P(A)P(\bar{B})$

Example: Block Transmission

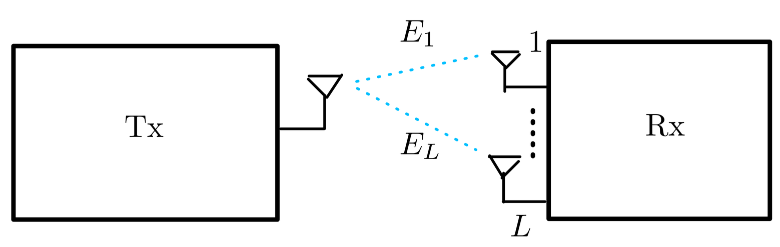
Symbols generated at different time instances are independent

i) $[x_1\ x_2 \cdots x_i]$ where $x_i \in S$ and is IID
Each x_i is generated at different time instance

ii) IID - Independent Identically
Distributed from 4-PAM constellation S

iii) $P(x_1 = 3\alpha,\ x_2 = \alpha)$
$= P(x_1 = 3\alpha)P(x_2 = \alpha) = \frac{1}{4}\frac{1}{2} = \frac{1}{8}$

Example: Wireless Systems (Deep Fade)

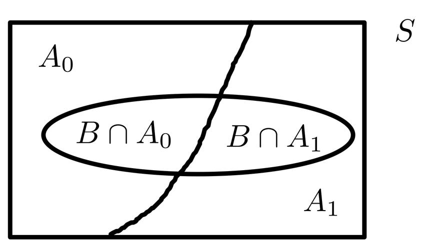


i) SIMO: Single Input (Tx) - Multiple Output (Rx) System

ii) E_i denotes the event that link i is in deep-fade
We have, $P(E_i) = \dfrac{1}{SNR} = \dfrac{1}{P/\sigma^2}$
where P is Tx Power and σ^2 is Noise Power

iii) System is in deep-fade when all links are in deep-fade
P(Deep-fade) $= P(E_1 \cap E_2 \cap \cdots \cap E_L)$
$P_{DF} = P(E_1)P(E_2) \cdots P(E_L) = \dfrac{1}{SNR^L}$
This **Diversity** leads to significant decrease in BER

Bayes Theorem



i) Let $A_0 \cup A_1 = S$ and $A_0 \cap A_1 = \phi$
Mutually exclusive and exhaustive

ii) Note, for any event B,
$(B \cap A_0)$ and $(B \cap A_1)$ are mutually exclusive
and $(B \cap A_0) \cup (B \cap A_1) = B$

iii) $\implies P(B) = P(B \cap A_0) + P(B \cap A_1)$
$= P(B|A_0)\,P(A_0) + P(B|A_1)\,P(A_1)$

iv) Using **Bayes rule** we get **Bayes Theorem** as,

$$P(A_0|B) = \frac{P(B|A_0)P(A_0)}{P(B)} = \frac{P(B|A_0)P(A_0)}{P(B|A_0)\,P(A_0) + P(B|A_1)\,P(A_1)}$$

$$P(A_1|B) = \frac{P(B|A_1)P(A_1)}{P(B|A_0)\,P(A_0) + P(B|A_1)\,P(A_1)}$$

Key Terms

i) $P(A_0|B)$ and $P(A_1|B)$ are called **Aposteriori** probabilities
ii) $P(A_0)$ and $P(A_1)$ are called **Prior** probabilities
iii) $P(B|A_0)$ and $P(B|A_1)$ are called **Likelihoods**

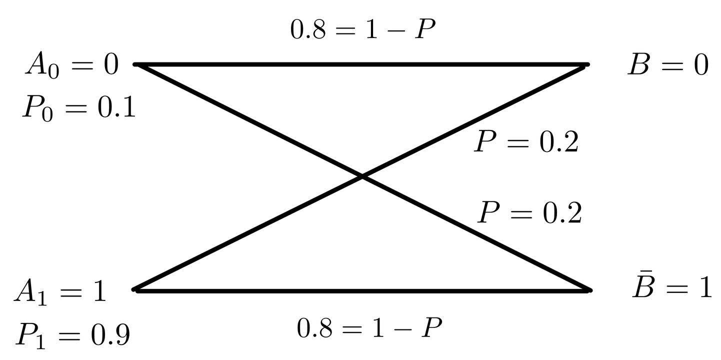
Generalized Bayes Theorem

For N mutually exclusive and exhaustive events

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{j=0}^{N-1} P(B|A_j)P(A_j)}$$

Also note, $\sum_{i=0}^{N-1} P(A_i|B) = 1$

Example: MAP Receiver (Maximum Aposteriori Probability)



i) Prior prob.: $P(A_0) = 0.1$ and $P(A_1) = 0.9$

ii) Likelihood: $P(B|A_1) = P(\bar{B}|A_0) = 0.2$
$P(B|A_0) = P(\bar{B}|A_1) = 0.8$

iii) Compute Aposteriori probs.
$P(A_0|B) = \dfrac{0.8 \times 0.1}{0.8 \times 0.1 + 0.2 \times 0.9} = \dfrac{8}{26}$
$P(A_1|B) = \dfrac{0.2 \times 0.9}{0.8 \times 0.1 + 0.2 \times 0.9} = \dfrac{18}{26}$

iv) $\therefore$ Given B, MAP receiver decides A_1 was transmitted

v) Whereas the ML receiver decides A_0 was transmitted

vi) MAP is optimal and minimizes the prob. of error

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Random Variables

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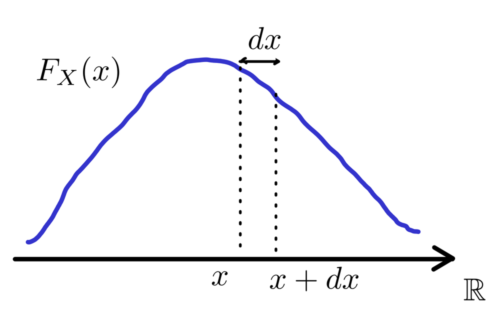
Random Variables

i) RV X, takes values randomly from $\mathbb{R}$ or a subset of $\mathbb{R}$

Characteristics

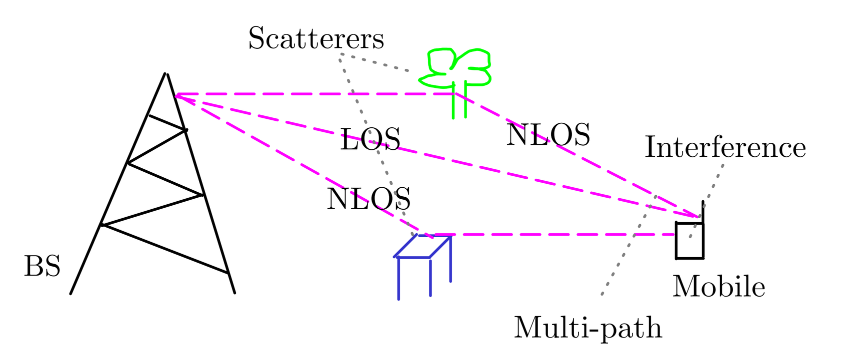


i) Characterized by PDF $F_X(x)$
ii) Quantity $F_X(x)dx$ represents the probability that X takes value in the infinitesimal interval $[x, x+dx]$
iii) $F_X(x) \geq 0$ for all values of x
iv) $P(X \in [a,b]) = \int_a^b F_X(x)dx$
v) Naturally, $\int_{-\infty}^{\infty} F_X(x)dx = 1$

Example: PDF

i) Consider $F_x(x) = Ke^{-ax}$
ii) $\implies \int_{-\infty}^{\infty} Ke^{-ax} = 1$
$-\frac{K}{a}e^{-ax}\,|_0^{\infty} = 1$
$\implies K = a$
iii) $F_X(x) = ae^{-ax}$, is known as the Exponential PDF

Application: Wireless Channel



i) Multipath propagation leads to 'Fading' Signal
ii) Power of Fading channel g, is random
iii) $F_G(g) \triangleq e^{-g}$ where $g > 0$
iv) This is known as **Rayleigh** Fading channel where power g is distributed as Exponential and the amplitude of the channel coefficient h is distributed as Rayleigh

Example:

Probability that the attenuation of the channel is worse than $20dB$.

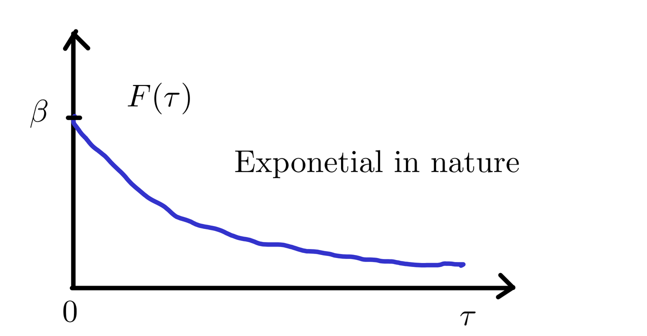
$\implies 10\log_{10} g \leq -20dB$
$\implies P(g \in [0, 0.01]) = \int_0^{0.01} e^{-g}dg$
$= 1 - e^{-0.01}$, where $e^{-0.01} \approxeq (1-0.01)$
$\implies P(g \in [0, 0.01]) = 0.01$

Statistical Parameters (Mean μ and Variance σ^2)

i) Mean $\mu \triangleq$ Average value of X
ii) $\mu = E\{X\}$, the expected value of X
iii) $\mu = \int_{-\infty}^{\infty} x\, F_X(x)dx$
i.e., integration of the values that X takes weighed with the corresponding probability

i) Variance $\sigma_X^2 \triangleq$ Average square deviation of X about mean μ
ii) $\sigma_X^2 = E\{(X-\mu)^2\}$
$= \int_{-\infty}^{\infty} (x-\mu)^2\, F_X(x)dx$
$= \int_{-\infty}^{\infty} x^2\, F_X(x)dx - \mu^2$
$= E\{X^2\} - E\{X\}^2$

Application: (Power Profile of wireless channel)



i) At receiver, multiple signal copies with different delays
ii) Power arriving with different delays
- Hence **power profile** of the channel
- Used to compute **Average delay (μ_τ)** and **variance delay** (σ_τ^2)

iii) Typical power profile is $F(\tau) = \beta e^{-\beta\tau}$

Average Delay (μ_τ)

i) $\mu_\tau = \int_0^{\infty} \tau e^{-\beta\tau} d\tau$
$= -\tau e^{-\beta\tau}|_0^{\infty} + \int_0^{\infty} e^{-\beta\tau} d\tau$
$= -\frac{1}{\beta}e^{-\beta\tau}|_0^{\infty} = \dfrac{1}{\beta}$

Variance Delay (σ_τ^2)

i) $\sigma_\tau^2 = E\{(\tau - \mu_\tau)^2\}$
$= E\{\tau^2\} - \mu_\tau^2$
$= \frac{1}{\beta^2}$
ii) **Root Mean square** delay spread
$\sigma_\tau = \dfrac{1}{\beta}$

Transformation of RVs (Function of RV)

i) Consider RV $Y = \Psi(X)$ where $\Psi(\cdot)$ is 1-to-1
$\implies \Psi(\cdot)$ is invertible and $x = \Psi^{-1}(y)$
ii) If $X \in [x, x+dx] \implies Y \in [y, y+dy]$
$\therefore P\left(X \in [x, x+dx]\right) = P\left(Y \in [y, x+dy]\right)$
$F_X(x)dx = F_Y(y)dy$
$F_Y(y) = \dfrac{F_X(x)|_{x=\Psi^{-1}(y)}}{|\frac{dy}{dx}|_{x=\Psi^{-1}(y)}}$

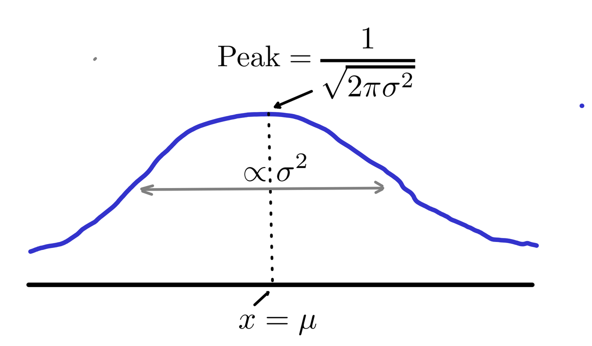
Example:

i) For a Rayleigh fading channel, we have $F_X(x) = e^{-x}$, where X denotes the power
ii) Amplitude Y is then given by $Y = \sqrt{X}$
$y = \Psi(x) = \sqrt{x}$, where $x \geq 0$ and $y \geq 0$
$\therefore \Psi(\cdot)$ is 1-to-1 and an invertible fn
we have, $x = \Psi^{-1}(y) = y^2$

$$\implies F_Y(y) = \frac{e^{-x}|_{x=y^2}}{|\frac{1}{2\sqrt{x}}|_{x=y^2}} = \frac{e^{-y^2}}{\frac{1}{2\sqrt{y^2}}} = 2ye^{-y^2}$$

which is known as the **Rayleigh** Fading channel.

Gaussian Random Variable X, (Normal RV)



i) $F_X(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-\dfrac{(x-\mu)^2}{2\sigma^2}}$ where,
$E\{X\} = \mu$ is the Mean
$E\{(X-\mu)^2\} = \sigma^2$ is the Variance
ii) Notation $X \sim \mathcal{N}(\mu, \sigma^2)$

Applications:

i) **Noise** at an instant in time in comm. system is modeled as Gaussian RV
ii) Wireless **fading channel coefficient** is modeled as complex Gaussian RV which leads to Rayleigh channel (i.e., amplitude is Rayleigh)

Property of Gaussian RV:

i) Linear combination of Gaussian RV results in another Gaussian RV
ii) Consider $X_1, X_2, \cdots, X_L$ Gaussian RVs
$E\{X_i\} = \mu_i$ and $E\{(X_i - \mu_i)^2\} = \sigma_i$, for $i \in \mathbb{N}_1^L$
Covariance of X_i, X_j is $E\{(X_i - \mu_i)(X_j - \mu_j)\} = \sigma_{ij}$
ii) Let linear combination of Gaussian RVs be
$X \triangleq a_1X_1 + a_2X_2 + \cdots + a_LX_L$

$$\implies X = \begin{bmatrix} a_1 & a_2 & \cdots & a_L \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_L \end{bmatrix} = \mathbf{a}^T\mathbf{X}$$

X is another Gaussian RV

Mean and Variance of X

i) $\mu = E\{X\} = E\{\sum_i a_iX_i\}$
$= \sum_i a_iE\{X_i\} = \sum_i a_i\mu_i = \mathbf{a}^T\boldsymbol{\mu}$
ii) $\sigma^2 = E\left\{(X-\mu)^2\right\}$
$= E\left\{\left(\sum_i a_iX_i - \mu\right)^2\right\} = E\left\{\left(\sum_i a_iX_i - \sum_i a_i\mu_i\right)^2\right\}$
$= E\left\{\left(\sum_i a_i(X_i - \mu_i)\right)^2\right\}$
$= E\left\{\sum_i\sum_j a_ia_j(X_i-\mu_i))(X_j-\mu_j)\right\}$
$= \sum_i a_i^2\sigma_i^2 + \sum_i\sum_{j,j\neq i} a_ia_j\sigma_{ij}$
iii) $X \sim \mathcal{N}\left(\sum_i a_i\mu_i, \quad \sum_i a_i^2\sigma_i^2 + \sum_i\sum_{j,j\neq i} a_ia_j\sigma_{ij}\right)$

Special Case:

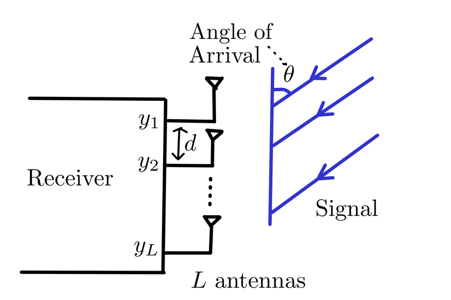
Consider $X_i, \forall i$ are **zero-mean**, **equal variance** and **uncorrelated**

i) Zero Mean : $E\{X_i\} = \mu_i = 0$
ii) Same Variance: $E\{(X_i-\mu_i)^2\} = \sigma_i^2 = \sigma^2$
iii) Uncorrelated: $E\{(X_i-\mu_i)(X_j-\mu_j)\} = \sigma_{ij} = 0$

iv) Note that **uncorrelated** Gaussian RVs are **independent** as well
v) $\implies$ Considered group of X_i's are **Idependent** and **Identically Distributed** (IID) RVs

vi) $X = \mathbf{a}^T\mathbf{X}$ is Guassian
vii) $E\{X\} = \mathbf{a}^T\boldsymbol{\mu} = 0$
viii) $E\{(X-\mu)^2\} = E\{X^2\} = \sum_i a_i^2\sigma^2 = \sigma^2\,\|\mathbf{a}\|^2$
ix) $X \sim \mathcal{N}\left(0, \sigma^2\,\|\mathbf{a}\|^2\right)$

Application: Uniform Linear Antenna Array



i) $y_i = x\, e^{-j\,(i-1)\phi} + w_i,\ \forall i$, where x is the transmitted signal, y_i is the received signal at i^{th} receiver antenna and w_i is the Gaussian noise.

$$\implies \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_L \end{bmatrix} = \begin{bmatrix} 1 \\ e^{-j\phi} \\ \vdots \\ e^{-j(L-1)\phi} \end{bmatrix} x + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix},\ \mathbf{y} = \mathbf{h}(\phi)x + \mathbf{w}$$

where $\mathbf{h}(\phi)$ is known as Array Steering Vector.
ii) Received signal at each successive antenna is delayed by an additional phase of ϕ, hence named Linear Phased array where $\phi = 2\pi f_c \frac{d\cos(\theta)}{C} = \frac{2\pi d}{\lambda}\cos(\theta)$

Application: (continuation ...)

iii) Assuming the Gaussian noise samples to be zero-mean IID, i.e., $E\{w_i\} = 0,\ E\{|w_i|^2\} = \sigma^2$ and $E\{w_iw_j^*\} = 0$ (uncorrelated)
iv) Considering beamforming vector $\mathbf{a}^T = \begin{bmatrix} a_1 & a_2 & \cdots a_L \end{bmatrix}$

$$\widetilde{y} = a_1^*y_1 + a_2^*y_2 + \cdots + a_L^*y_L = \begin{bmatrix} a_1^* & a_2^* & \cdots a_L^* \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_L \end{bmatrix} = \mathbf{a}^H\mathbf{y}$$

This combining process is called **beam forming**.
v) We have, $\widetilde{y} = \mathbf{a}^H\mathbf{y} = \mathbf{a}^H\mathbf{h}(\phi)x + \mathbf{a}^H\mathbf{w}$ and considering trasmitted signal power $E\{|x|^2\} = P$

We have, $\text{SNR} = \dfrac{\text{Signal Power}}{\text{Noise Power}} = \dfrac{|\mathbf{a}^H\mathbf{h}(\phi)|^2E\{|x|^2\}}{E\{|\mathbf{a}^H\mathbf{w}|^2\}} = \dfrac{|\mathbf{a}^H\mathbf{h}(\phi)|^2P}{\sigma^2\,\|\mathbf{a}\|^2}$

vi) To find the maximum SNR possible, we use the Cauchy-schwartz Inequality $|\mathbf{a}^H\mathbf{h}(\phi)|^2 \leq \|\mathbf{a}\|^2\|\mathbf{h}(\phi)\|^2$,

$$\implies \text{SNR} \leq \frac{\|\mathbf{a}\|^2\|\mathbf{h}(\phi)\|^2P}{\sigma^2\,\|\mathbf{a}\|^2} = \frac{\|\mathbf{h}(\phi)\|^2P}{\sigma^2} = \text{SNR}_{\text{max}}$$

Note $\|\mathbf{h}(\phi)\|^2 = 1 + |e^{-j\phi}|^2 + \cdots + |e^{-j(L-1)\phi}|^2 = 1 + 1 + \cdots + 1 = L$

vii) $\therefore \text{SNR}_{\text{max}} = \dfrac{LP}{\sigma^2}$, i.e., L times the initial SNR. (i.e., Array gain $= L$)
viii) To obtain $\text{SNR} = \text{SNR}_{\text{max}}$, **spatially matched filter** (**matched beam former**) is used, i.e., $\mathbf{a} = \mathbf{h}(\phi)$ (also called as **Maximum Ratio Combining** and coherent combining)

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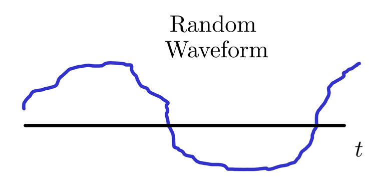
Random Processes

Random Processes

i) A **random process** $X(t)$, is a random variable X which is a function of time t

ii) Is a RV at every time instant or is a RV **indexed** by time

Characteristics and Statistical Parameters



i) Can be used to model random signal or random noise process

ii) Is characterized by $F_X(x,t)$, a PDF at each instance of time

iii) Statistical parameters are also function of time
Mean : $\mu_X(t) = E\{X(t)\} = \int_{-\infty}^{\infty} x\, F_X(x,t)\, dx$

iv) Auto-Correlation (or self-correlation) at time instances t_1 and t_2, is

$$R_{XX}(t_1,t_2) \triangleq E\{X(t_1)\,X(t_2)\} = \int_{-\infty}^{\infty} x_1 x_2 F(x_1,x_2,t_1,t_2)\, dx_1\, dx_2$$

where $F(x_1,x_2,t_1,t_2)$ is the joint PDF for t_1 and t_2

Special Class of Random Process

Special Random Process: (Wide Sense Stationary, WSS)

Satisfies two properties:

I) $E\{X(t)\} = \mu_X =$ Constant
i.e., mean does **NOT depend** on time (is stationary)

II) Let $t_1 = t$ and $t_2 = t+\tau$
$E\{X(t)X(t+\tau)\} = R_{XX}(t,t+\tau) = R_{XX}(\tau)$
i.e., Auto-correlation **depends only** on the **shift** τ and NOT on the specific time instance t.

a) Note, when $\tau = 0$
$E\{X(t)X(t)\} = E\{X(t)^2\} = R_{XX}(0)$
i.e., $R_{XX}(0) =$ **Avg. power** in the process $X(t)$

Gaussian Random Process

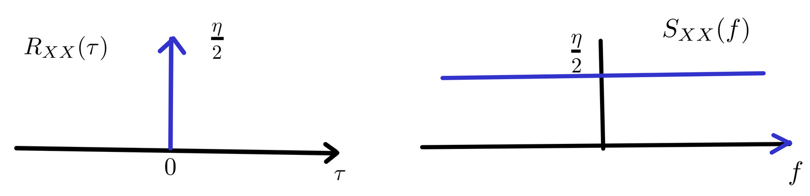
i) Random Process $X(t)$ is a **Gaussian** random process, if $X(t_1), X(t_2), \cdots, X(t_n)$ are **jointly Gaussian** at any time instance
i.e., Multivariate Density $F_{X(t_1)X(t_2)\cdots X(T_n)}(x_1,x_2,\cdots,x_n)$ is Guassian $\forall n$ and $\forall t_1, t_2, \cdots t_n$

ii) Considering $n=1$, the PDF of $X(t)$ at any time instance must be Gaussian
$\implies F_{X(t)}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$ where μ and σ^2 are function of time

iii) Considering a **WSS Guassian Process**, we have
$E\{X(t)\} = \mu_X$ and $E\{X(t)X(t+\tau)\} = R_{XX}(\tau)$ are stationary. Further with $\tau = 0$ we have $E\{X(t)^2\} = R_{XX}(0)$ and $\sigma^2 = R_{XX}(0) - \mu_X^2$

$$\implies F_{X(t)}(x) = \frac{1}{\sqrt{2\pi(R_{XX}(0)-\mu_X^2)}} e^{\frac{-(x-\mu)^2}{2(R_{XX}(0)-\mu_X^2)}}$$

White Noise (White Random Process)



I) A Random Process is **white** if it is

a) **WSS**

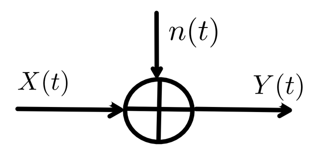
b) and its **autocorrelation** $R_{XX}(\tau) = \frac{\eta}{2}\delta(\tau)$, i.e., $R_{XX}(\tau) = 0$ if $\tau \neq 0$.
For any non-zero shift τ, the correlation
$E\{X(t)X(t+\tau)\} = 0$, i.e., $X(t)$ and $X(t+\tau)$ are **un-correlated**.

II) **PSD** of White Noise is obtained by Fourier Transform of $R_{XX}(\tau)$, i.e.,
$S_{XX}(f) = \frac{\eta}{2}$ for $-\infty < f < \infty$ i.e., **constant over entire freq. range**.
Uniform power distribution over the entire freq. range similar to white light

Example: Additive White Gaussian Noise

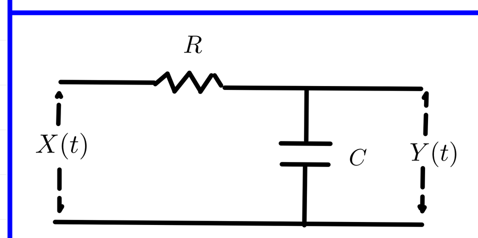
i) Random process $n(t)$, is both **Gaussian** and **White**, i.e.,
$S_{nn}(f) = \dfrac{\eta}{2}$

ii) Ex: Additive White Gaussian Noise (**AWGN**) adds to the transmitted signal



$Y(t) = X(t) + n(t)$
Such a comm. channel that adds WGN to transmit signal is termed as **AWGN channel** and is a typical model for **wireline** channel.
(Note that there is no fading coefficient h as in wireless channel)

White Gaussian Noise when passed through RC (low-pass) filter



I) Consider $X(t)$ is zero-mean White Gaussian noise, i.e.,
$\mu_X = 0$, $R_{XX}(\tau) = \frac{\eta}{2}\delta(\tau)$ and $S_{XX}(f) = \frac{\eta}{2}$ for entire freq. range

II) Characteristics of RP $Y(t)$ at the ouput of RC filter (LTI system)

a) $Y(t)$ is also zero-mean **Gaussian** as the filter is LTI system

b) We have, $H(f) = \dfrac{1}{1+j2\pi fRC}$;
$\implies S_{YY}(f) = \dfrac{\eta}{2}\dfrac{1}{(1+4\pi^2f^2R^2C^2)}$ **(no longer White)**

c) Power $E\{|Y(t)|^2\} = \frac{\eta}{2}\int_{-\infty}^{\infty}\frac{1}{(1+4\pi^2f^2R^2C^2)}df = \dfrac{\eta}{4RC} = R_{YY}(0) = \sigma_Y^2$

d) $\implies$ PDF $F_{Y(t)}(y) = \sqrt{\dfrac{2RC}{\pi\eta}}\, e^{-\dfrac{2RCy^2}{\eta}}$

Example: In Wireless Comm. System

i) Consider a signal $X(t) = \alpha\cos(2\pi f_c t + \theta)$, where Phase θ, is RANDOM and uniformly distributed in $[-\pi,\pi)$
i.e., $F_\Theta(\theta) = \frac{1}{2\pi}$, for $-\pi \leq \theta < \theta$ and the parameters α, f_c are constant with time.

ii) To examine if $X(t)$ is WSS

a)
$$\begin{aligned} E\{X(t)\} &= E\{\alpha\cos(2\pi f_c t+\theta)\} = \int_{-\pi}^{\pi}\alpha\cos(2\pi f_c t+\theta)\,F_\Theta(\theta)d\theta \\ &= \tfrac{\alpha}{2\pi}\int_{-\pi}^{\pi}\cos(2\pi f_c t+\theta)d\theta = 0 \end{aligned}$$
$\mu_X(t) = 0 = \mu_x$ is a constant and is **stationary** in the **mean**.

b)
$$\begin{aligned} E\{X(t)X(t+\tau)\} &= E\{\alpha\cos(2\pi f_c t+\theta)\times\alpha\cos(2\pi f_c(t+\tau)+\theta)\} \\ &= \tfrac{\alpha^2}{2\pi}\int_{-\pi}^{\pi}\cos(2\pi f_c t+\theta)\times\alpha\cos(2\pi f_c(t+\tau)+\theta)d\theta \\ &= \tfrac{\alpha^2}{2}\cos(2\pi f_c\tau) = R_{XX}(\tau) \end{aligned}$$
does NOT depend on t
$\therefore$**stationary** in the **autocorrelation**

$\implies$ Random process $X(t)$ is **WSS** with avg. power $R_{XX}(0) = \frac{\alpha^2}{2}$

Power Spectral Density (PSD)

i) PSD $S_{XX}(f)$, of a WSS random process $\triangleq$ Fourier Transform of $R_{XX}(\tau)$
$\implies S_{XX}(f) \triangleq \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j2\pi f\tau}d\tau$
$\implies R_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(f)e^{j2\pi f\tau}df$ (Inverse Fourier Transform)

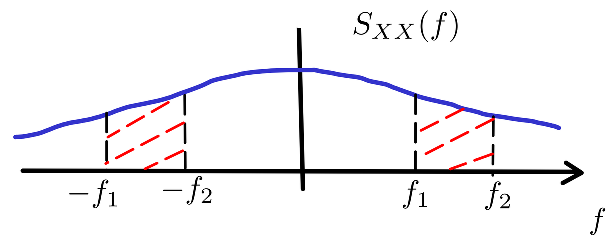
Properties of PSD

a) $S_{XX}(f)$ is a real quantity and $S_{XX}(f) \geq 0$, as it related to power

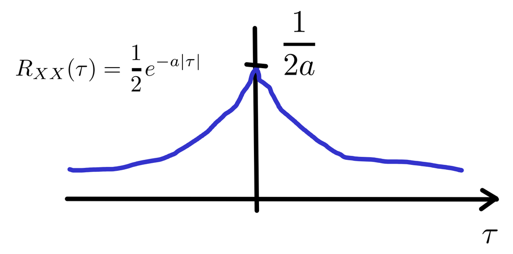
b) $S_{XX}(f) = S_{XX}(-f)$ is symmetric

c) Setting $\tau = 0$, $R_{XX}(0) = \int_{-\infty}^{\infty} S_{XX}(f)df$
$\implies$ Power of WSS RP $P_X =$ Integral of PSD over entire freq.

d) Power in WSS RP in the band $[f_1, f_2]$ is
$P_X[f_1,f_2] = \int_{-f_1}^{-f_2} S_{XX}(f)df + \int_{f_1}^{f_2} S_{XX}(f)df = 2\int_{f_1}^{f_2} S_{XX}(f)df$



Example: Wireless context

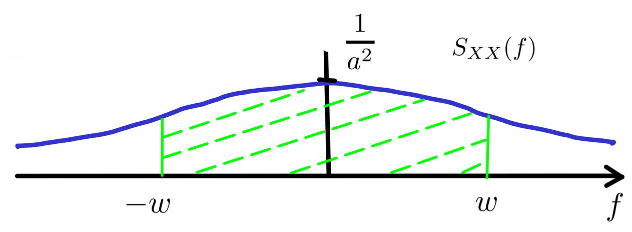


i) Consider Wireless signal $X(t)$ with $R_{XX}(\tau) = \frac{1}{2a}e^{-a|\tau|}$, and $a = 5KHz$

ii) Then, power in $X(t)$ is $E\{X^2(t)\} = R_{XX}(0) = \frac{1}{2a}$

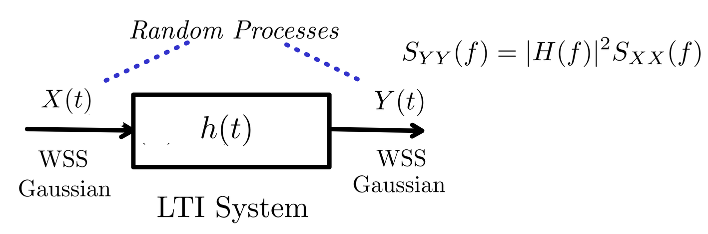
iii) Power spectral density, PSD is

$$\begin{aligned} S_{XX}(f) &= \int_{-\infty}^{\infty}\tfrac{1}{2a}e^{-a|\tau|}e^{-j2\pi f\tau}d\tau \\ &= \tfrac{1}{2a}\left\{\int_0^{\infty} e^{-a\tau}e^{-j2\pi f\tau}d\tau + \int_{-\infty}^{0}\tfrac{1}{2a}e^{a\tau}e^{-j2\pi f\tau}d\tau\right\} = \frac{1}{a^2+4\pi^2f^2} \end{aligned}$$



iv) Band $[-w, w]$ with 90% of the energy can be obtained as
$$\frac{0.9}{2a} = \int_{-w}^{w}\frac{1}{a^2+4\pi^2f^2}df = \frac{1}{4\pi^2}\int_{-w}^{w}\frac{1}{f^2+\frac{a^2}{4\pi^2}}df$$
Further solving we get, $w = 1.005a$
$\implies$ Required bandwidth $= [-1.005a, 1.005a]$,
i.e., $[-5$ KHz, 5 KHz$]$ and passband BW is 10 KHz.

Transmission of Random Process through LTI



i) Consider $X(t)$ is input to LTI system with impulse response $h(t)$, then
$Y(t) = X(t) * h(t) = \int_{-\infty}^{\infty} X(t-\alpha)h(\alpha)d\alpha$, where $Y(t)$ is the output

i) $Y(t)$ is a random process

ii) We have Mean of $Y(t)$ as
$$\begin{aligned} \mu_Y(t) &= E\{Y(t)\} = E\{\textstyle\int_{-\infty}^{\infty} X(t-\alpha)h(\alpha)d\alpha\} \\ &= \textstyle\int_{-\infty}^{\infty} E\{X(t-\alpha)\}h(\alpha)d\alpha = \mu_X\int_{-\infty}^{\infty} h(\alpha)d\alpha = \mu_Y \end{aligned}$$
$\therefore Y(t)$ is **stationary** in the mean

iii) We have Auto-correlation of $Y(t)$
$$\begin{aligned} R_{YY}(t) &= E\{Y(t)Y(t+\tau)\} \\ &= E\{\textstyle\int_{-\infty}^{\infty} X(t-\alpha)h(\alpha)d\alpha \times \int_{-\infty}^{\infty} X(t+\tau-\beta)h(\beta)d\beta\} \\ &= \textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\{X(t-\alpha)X(t+\tau-\beta)\}h(\alpha)h(\beta)d\alpha d\beta \\ &= \textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau-\beta+\alpha)h(\alpha)h(\beta)d\alpha d\beta \end{aligned}$$
$\implies$ depends only on τ
$\implies$ $Y(t)$ is **stationary** in the auto-correlation

$\implies$ $Y(t)$ is also **WSS**

iv) Further $R_{YY}(\tau) = R_{XX}(\tau) * h(\tau) * h(-\tau)$ (in τ domain)

v) And $S_{YY}(f) = S_{XX}(f)\,|H(f)|^2$, where $H(f) \leftrightarrow h(t)$

References:

[1] NPTEL Course: Probability and Random Variables for Wireless Communications by Prof. Aditya Jaganathan

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