# Public key cryptography - basic concepts. Encryption and key transport

Applied Cryptography – Spring 2024

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March 11, 2024

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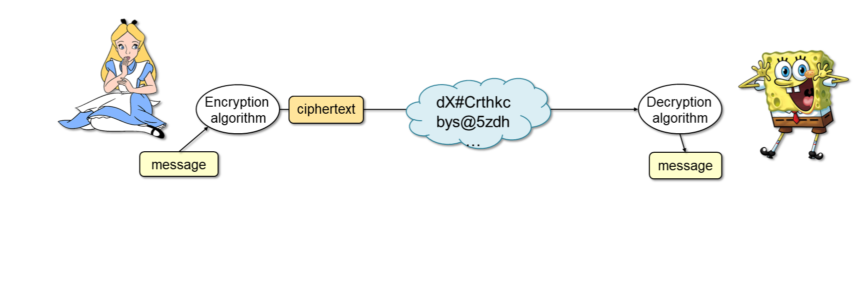
Radboud University

# Outline

# Public Key Cryptography

# Recall our everyday scenario

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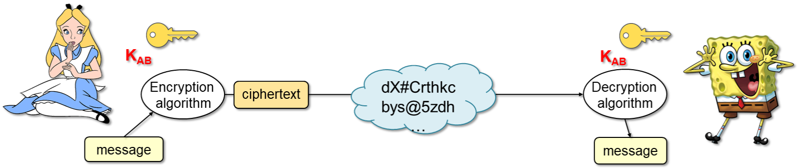


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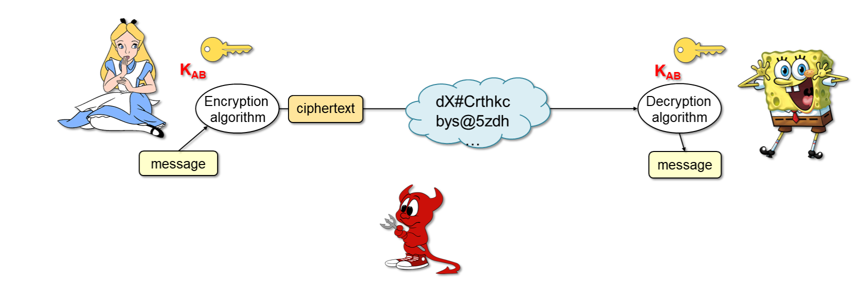


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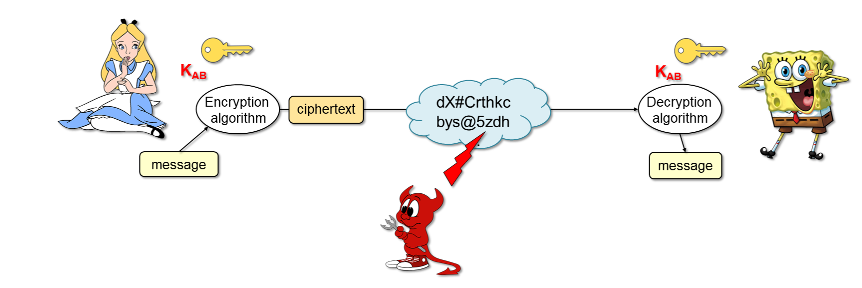


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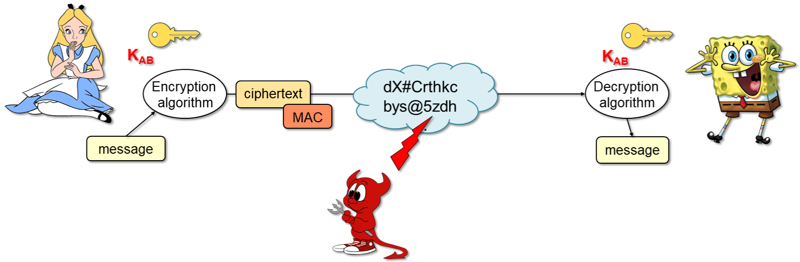


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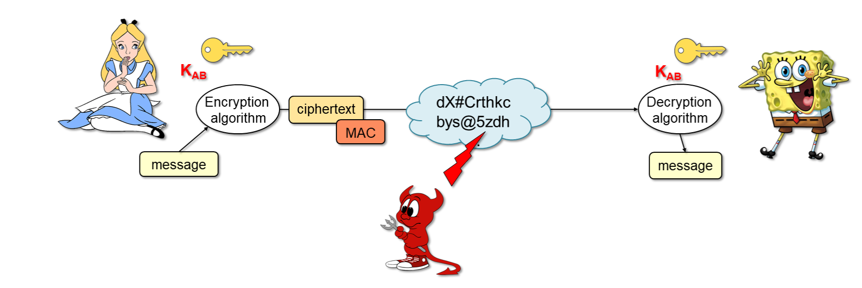


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- **What can be a problem in this scenario?**

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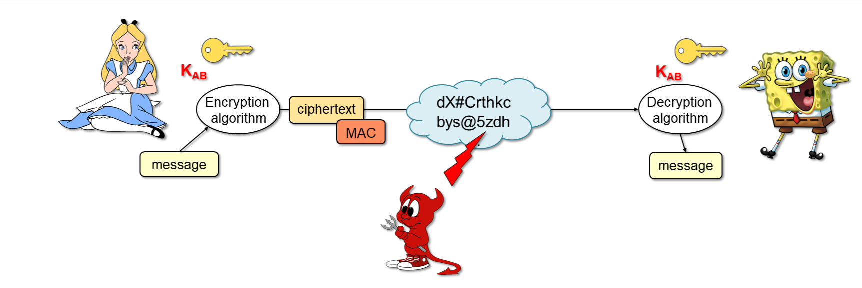


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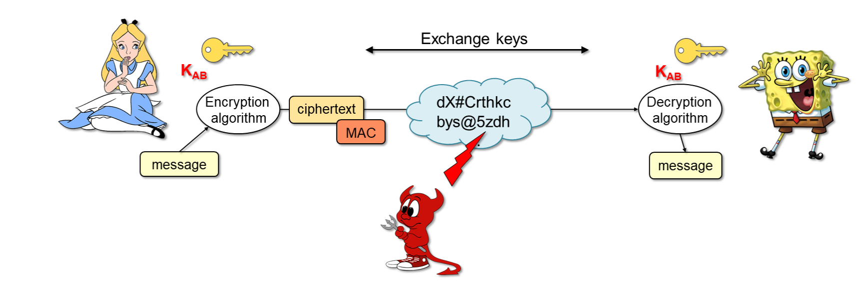


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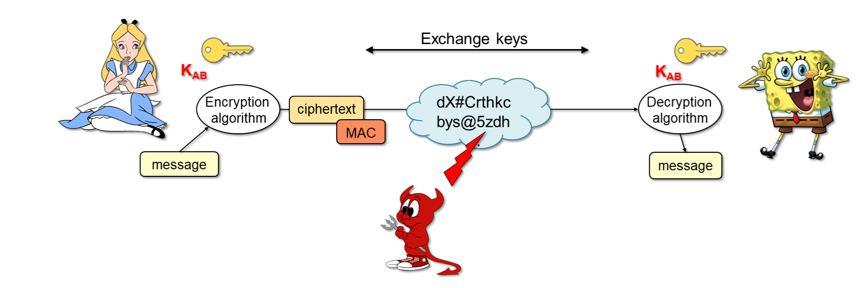


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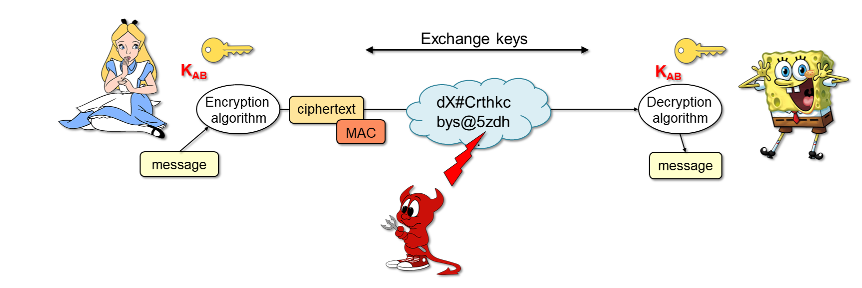


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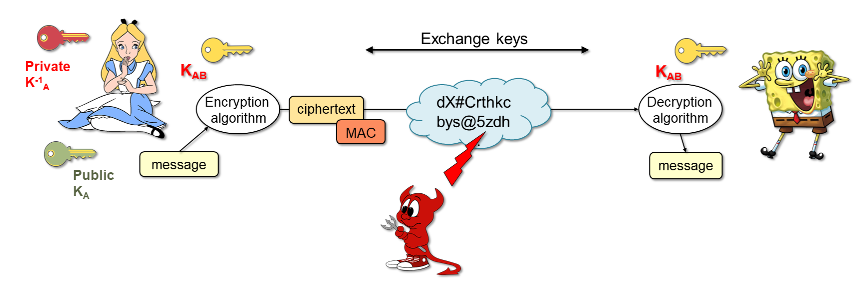


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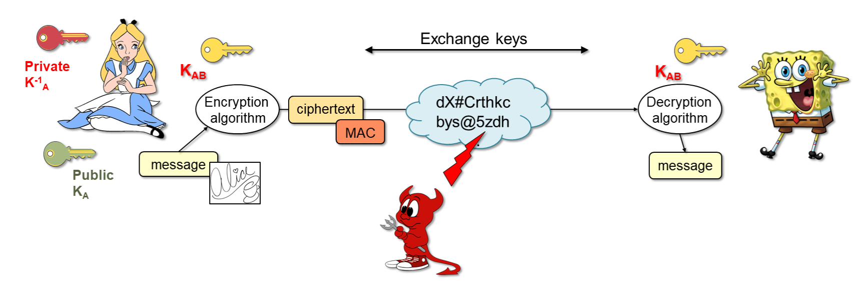


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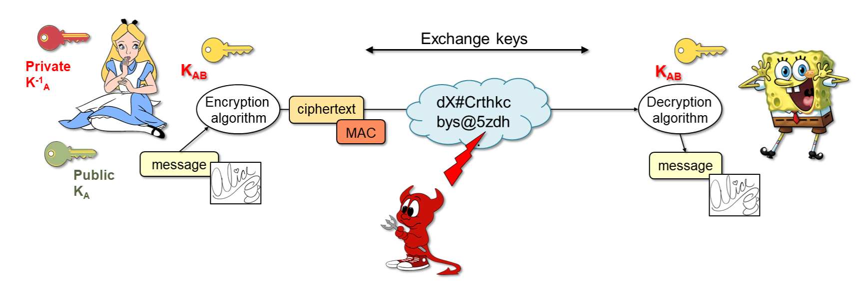


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  - **Key Exchange:** Eve can not learn the key
  - **Entity Authentication:** Eve cannot impersonate the parties
  - **Non-repudiation:** The parties can not repudiate the messages

# Core Functionalities

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## Key Encapsulation Mechanism (KEM) and Key Exchange (KEX)

- Goal is to obtain a shared symmetric key
- KEM (simplified)
  - encrypt symmetric key with public key of receiver
  - receiver decrypts symmetric key with his private key
- KEX - a protocol to agree on a shared symmetric key
  - comes in different flavors and constructions (Diffie-Hellman-style, from KEMs, etc.)

# Versatility of Public Key Cryptography

## Examples of other, more subtle flavors of Public Key Cryptography

- Group/ring, blind signatures
- Commitments
- Identification schemes
- Secret Sharing schemes
- Threshold encryption
- Homomorphic Encryption
- Identity-based cryptography
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- Credential schemes
- Functional Encryption
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## Examples of real-world protocols employing Public Key Cryptography

- Secure messaging protocols
- SSL/TLS (https, ftps)
- SSH (sftp, scp)
- IPsec (IKE)
- OpenVPN, Wireguard
- IEEE 802.11
- DNSSEC
- EMV
- Electronic voting
- ...

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- complexity of availability, versioning, updates, following standards etc.
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### Prudent practices for future deployment?

- reflections on mistakes made
- how not to repeat them in the future

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+ $\boxed{\text{Secure implementation}}$

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# Hardness assumptions

## Easy problems

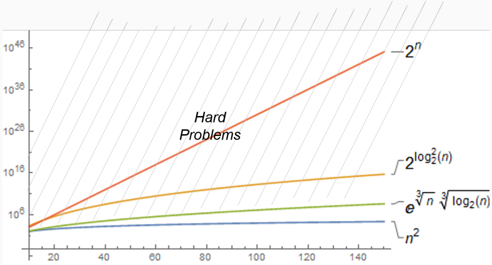
vs

## Hard problems

$\mathcal{O}(n)$

$$\underbrace{\begin{bmatrix}1 & 2 & 3\end{bmatrix}}_{1\times 3} \cdot \underbrace{\begin{bmatrix}2 & 1 & 3\\ 3 & 3 & 2\\ 4 & 1 & 2\end{bmatrix}}_{3\times 3} = \begin{bmatrix}1\cdot 2+2\cdot 3+3\cdot 4\\ 1\cdot 1+2\cdot 3+3\cdot 1\\ 1\cdot 3+2\cdot 2+3\cdot 2\end{bmatrix} = \underbrace{\begin{bmatrix}20\\ 10\\ 13\end{bmatrix}}_{1\times 3}$$

$\mathcal{O}(n^2)$




(x OR y OR z) AND (x OR $\bar{y}$ OR z) AND

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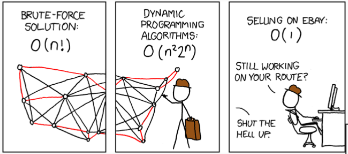
$\mathcal{O}(2^n)$




$\mathcal{O}(poly(n)2^n)$

Hard problems:
No efficient (polynomial time) algorithm exists

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**Remark**: A security reduction does not show that a scheme is secure, but only **as secure as** the hardness assumption!

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- **The virtual "players"** that interact with the adversary
  - **Challenger** - creates an instance of the **real** cryptographic scheme, following its algorithms, and interacts with the adversary by answering queries about the scheme
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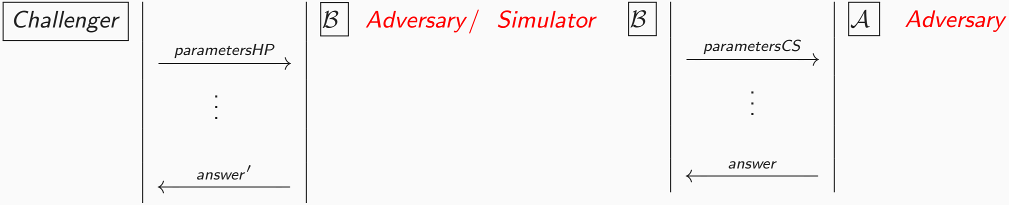
  - **Decisional attack** - the adversary will spit out a decision that in a reduction can be used to solve a hard decisional problem

# A high level view of security reduction

Hard-problem (HP) security game

Cryptographic scheme (CS) security game




real game

simulated game

# Security reduction - reduction cost and reduction lost

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  - ***k*-bit security loss** - if $L = 2^k$
  - the parameters need to be increased to add additional $k$ bits of security
  - Example:
    - The underlying problem has 128 bits of security, the reduction has loss of 12 bits,
    - $\Rightarrow$ the scheme can be claimed to have only 116 bits of security
  - A vastly overlooked/ignored issue in public-key cryptography

# Public Key Encryption (PKE)

# Public Key Encryption (PKE) – definition

Given security parameter $\lambda \in \mathbb{N}$ and two finite sets $\mathcal{M}, \mathcal{R} \subseteq \{0,1\}^*$, a Public Key Encryption $\Pi = (\mathsf{KGen}, \mathsf{Enc}, \mathsf{Dec})$ consists of three algorithms:

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    - encryption only requires the public key!
    - What more could the attacker do?

# Security of Public Key Encryption (PKE)

**Baseline security**: **indistinguishability under chosen-plaintext attacks (IND-CPA)**

A PKE scheme $\Pi$ is called IND-CPA-secure if any PPT adversary $\mathcal{A}$ has only negligible advantage

$$Adv = \mathbf{Pr}\left(\mathrm{Exp}^{\text{ind-cpa}}_{\Pi(1^k)}(\mathcal{A}) = 1\right) - 1/2 = negl(k)\,.$$

in the following $\mathrm{Exp}^{\textbf{ind-cpa}}_{\Pi(1^k)}(\mathcal{A})$ **game (experiment)**:

| **Challenger** | | **Adversary** |
|---|---|---|
| $(\mathrm{pk}, \mathrm{sk}) \leftarrow \mathrm{KGen}()$ | $\xrightarrow{\mathrm{pk}}$ | |
| | $\xleftarrow{M_i}$ | $M_i$ for number of $i$-s |
| | $\xrightarrow{\mathrm{Enc}(\mathrm{pk}, M_i)}$ | |
| $b \xleftarrow{\$} \{0,1\}$ | $\xleftarrow{(M_0^*, M_1^*)}$ | $M_0^*, M_1^*$ |
| $C \leftarrow \mathrm{Enc}(\mathrm{pk}, M_b^*)$ | $\xrightarrow{C}$ | |
| | $\xleftarrow{b'}$ | $b'$ |

Return 1 iff $b = b'$ otherwise 0.

- **A deterministic PKE can never be IND-CPA! Why?**

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- **Active attacker**
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  - **can craft ciphertexts** and use the decryption algorithm as an **oracle** to obtain the plaintexts
    - access switched off **before** target ciphertext is given to the attacker
    - unlimited access, **before and after** the target ciphertext is made available (of course the target ciphertext can not be queried)

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A PKE scheme $\Pi$ is called IND-CCA-secure (IND-CCA2-secure) if any PPT adversary $\mathcal{A}$ has only negligible advantage

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Recall textbook RSA (for more info see I2C slides):

**Textbook RSA:**

**KeyGen**:

1. Choose two primes $p, q$ s.t. $|p| \approx |q|$
2. Compute $N = pq$ and $\phi(N) = (p-1)(q-1)$
3. Choose a random $e < \phi(N)$, s.t. $\gcd(e, \phi(N)) = 1$
4. Compute $d$ such that $ed = 1 \pmod{\phi(N)}$
5. Output public key $\mathrm{pk} = (N, e)$ and private key $\mathrm{sk} = d$

**Encrypt**:

Compute ciphertext as $C \leftarrow M^e \pmod{N}$

**Decrypt**:

Decrypt ciphertext as $M \leftarrow C^d \pmod{N}$

**Passive (Meet-in-the-middle) attack:**

- Let $C = M^e \pmod{N}$, and Eve knows that $M < 2^\ell$ (for example PIN, short password)

# An example walkthrough - Insecurity of textbook RSA

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- $\Rightarrow$ Message $M = i \cdot j$ recovered!
- $\Rightarrow$ **Message recovery in time and space cost of $\tilde{\mathcal{O}}(2^{\ell/2})$ (factors polynomial in $\ell$ neglected)**

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- $\Rightarrow$ **Message recovery in 1 oracle query!**

**Conclussion:**

- We need some sort of randomization of the message! (we need IND-CPA)
- The adversary should not be able to construct valid ciphertexts! (we need IND-CCA)

# Summary

**Today:**

- Public Key Cryptography - a Recap
- Security of PKC
- Security of Public Key Encryption and Key Encapsulation

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**Next time:**

- Security of Public Key Encryption and Key Encapsulation (contd.)
- Security of Digital Signatures