Homework 8

Question 1

The height of the tree can be reduced by up to half.
The maximum number of children of any node can be 4.

Question 2

min nodes:
$2^{(h_b-1)+1}-1=2^{h_b}-1$
max nodes:
$2^{(2h_b-1)+1}-1=4^{h_b}-1$

Question 3

1.

```
1B
```

1.

```
1B
  \
  2R
```

1.

```
1B \
  \ \
  2R \ CASE IV
    \ \
    3R \

  2B
 /  \
1R  3R
```

4.

```
  2B
 /  \
1R  3R
      \
      4R

   2 B
  /   \
1 B   3 B
         \
         4 R
```

5.

```
  2B
 /  \
1B  3B \
      \ \
      4R \ CASE III
        \ \
        5R \

  2B
 /  \
1B  4B
   /  \
  3R  5R
```

6.

```
  2B
 /  \
1B  4B
   /  \
  3R  5R
        \
        6R

  2B
 /  \
1B  4R
   /  \
  3B  5B
        \
        6R
```

7.

```
  2B
 /  \
1B  4R
   /  \
  3B  5B \
        \ \
        6R \
          \ \
          7R \
```

```
  2B
 /  \
1B  4R
   /  \
  3B  6B
     /  \
    5R  7R
```

8.

```
  2B
 /  \
1B  4R
   /  \
  3B  6B
     /  \
    5R  7R
          \
          8R

  2B
 /  \
1B  4R
   /  \
  3B  6R
     /  \
    5B  7B
          \
          8R

     4R
   /     \
  2B     6R
 /  \   /  \
1B  3B 5B  7B
             \
             8R

     4B
   /     \
  2R     6R
 /  \   /  \
1B  3B 5B  7B
             \
             8R

BST
1B
  \
  2B
    \
    3B
      \
      4B
        \
        5B
          \
          6B
            \
            7B
              \
              8B
```

Question 4

For any cycle in G let e denote its max weight edge. If T uses e_M then e can be swapped with any cycle-edge and the tree will remain connected. So e can be swapped for a cheaper tree so T does not use e. Thus T is a minimum spanning tree of G.

Question 5

Consider graph $U, V(G) - U$.
Assume many possible MSTs and fix one, T. An edge of T must cross a cut or the graph is disconnected. Since T spans $V(G)$ it spans U and $V(G) - U$. Every node on both sides of the cut is in the tree. Any edge crossing the cut results in a spanning tree. If any edge other than the minimum is selected the tree will not have minimal cost, since its cost could be reduced by using this edge.There is a unique choice of edge T across any cut.

Question 6

No.
Suppose the shortest path from s to t contains a cycle. Then some vertex v appears at least twice from s to t. Deleting a portion of the walk which loops back to v decreases the length since the graph has positive edge weights.

Question 7

```
           ----- 4 -----
          /             \
         E - 3 - A - 2 - B
         |       |\     /
         5       | 5   3
         |       |  \ /
 H - 1 - F       4   C
  \     /        |  /
   3   2         | 6
    \ /          |/
     G --- 1 --- D
```