Asymptotic Analysis

Alfred Rossi, Ph.D.

Why Asymptotic Analysis?

Algorithm A is ***2x faster*** on My Computer than *Algorithm B* is ***on yours***.

Why Asymptotic Analysis?

What about this?

- *Algorithm A* is 2x faster than *Algorithm B* on *Hardware 1*

Why Asymptotic Analysis?

What about this?

- *Algorithm A* is 2x faster than *Algorithm B* on *Hardware 1*
- *Algorithm B* is 5x faster than *Algorithm A* on *Hardware 2*

Why Asymptotic Analysis?

Which algorithm is faster?

- The ***method*** behind *Algorithm C* is ***universally slower*** than the method of *Algorithm D*.

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- So the authors of *Algorithm C* have precomputed a lookup table for all results up to size 10000.

Why Asymptotic Analysis?

Which algorithm is faster?

- The ***method*** behind *Algorithm C* is ***universally slower*** than the method of *Algorithm D*.
- So the authors of *Algorithm C* have precomputed a lookup table for all results up to size 10000.
- This size is big enough (and therefore instant) for almost all applications.

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- In other words, ***we want to count steps*** and not worry about how long each step takes.
- Because your $10n$ steps are comparable to my $5n$ steps if your computer is twice as fast as mine we need to ignore the constants and, instead, focus on the ***rate of growth of the number of steps as the input size grows.***

Why Asymptotic Analysis?

- We want to compare algorithms directly without reference to hardware, so we ***should not be sensitive to multiplicative constants***.
- In other words, ***we want to count steps*** and not worry about how long each step takes.
- Because your $10n$ steps are comparable to my $5n$ steps if your computer is twice as fast as mine we need to ignore the constants and, instead, focus on the ***rate of growth of the number of steps as the input size grows.***
- We need to ***rule out the lookup table trick***, so we choose to compare them on very large inputs ($n \to \infty$).

In other words, in the asymptotic analysis of algorithms, we don’t care about how long individual steps take, we ***care about how the number of steps blows up as the input size goes to infinity.***

Let’s leave algorithms behind until next week and refresh on the math.

Notation

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- the set of integers, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- the set of rational numbers, $\mathbb{Q} = \{p/q : p, q \in \mathbb{Z},\ q \neq 0\}$
- the set of real numbers, $\mathbb{R}$
- the set of positive real numbers, $\mathbb{R}^+ = \{x : x \in \mathbb{R},\ x > 0\}$

Notation

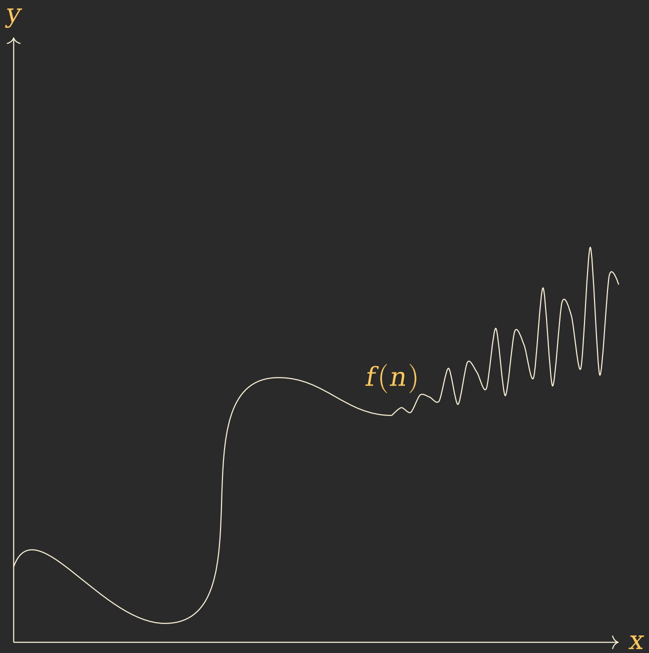
We will also use $\log(n)$ in here to mean $\log_2(n)$.

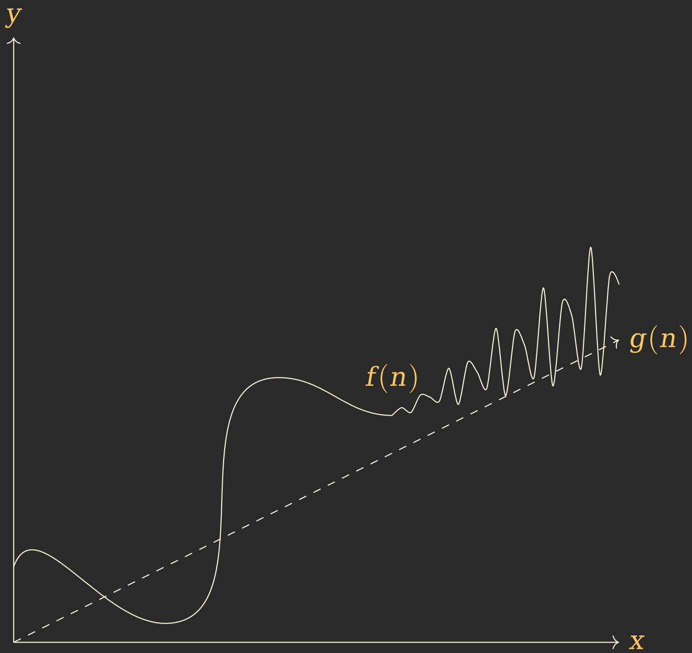
Big O

Let f and g be a pair of functions defined on an unbounded subset of positive real numbers.

Definition

We say $f \in O(g)$ if there exists $c \in \mathbb{R}^+$, $n_0 \in \mathbb{N}_0$, such that for all $n \geq n_0$, $0 \leq f(n) \leq c \cdot g(n)$.


y
f(n)
x



y
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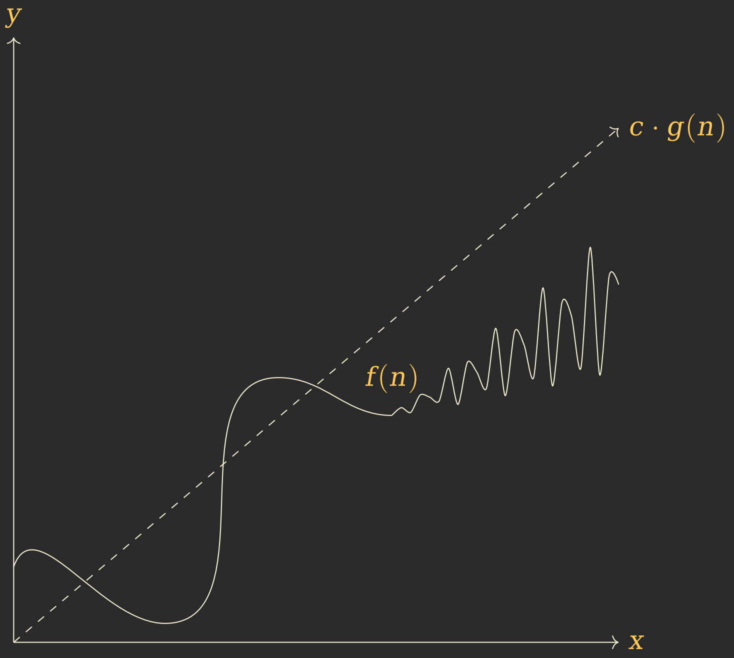
y
$c \cdot g(n)$
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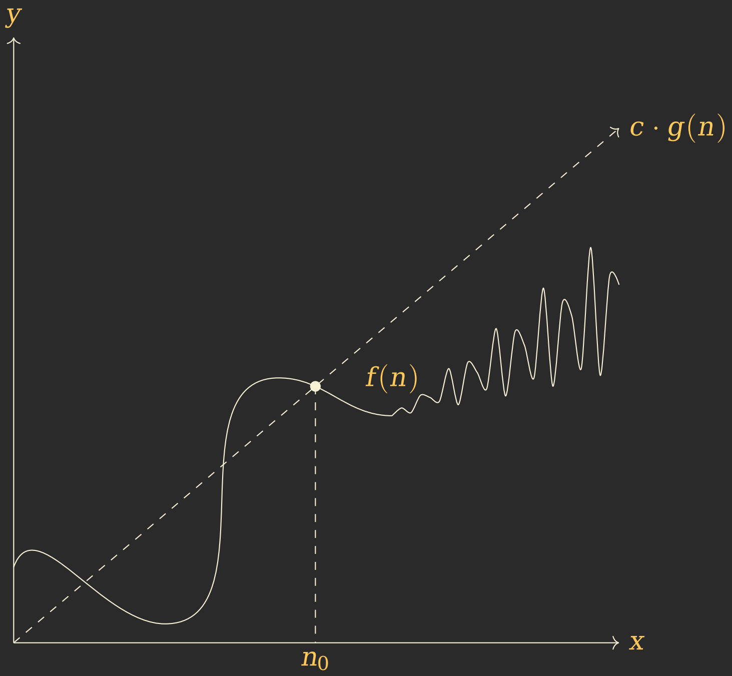
y
$c \cdot g(n)$
$f(n)$
x
n_0

Set Notation

Note that the use of set notation in $f \in O(g)$ is not accidental. We can think of $O(g)$ as the ***set of functions*** whose tail behavior is asymptotically dominated by a ***rescaling*** of g.

Transitivity of O

Theorem (Transitivity of O)

Let f, g, and h be functions defined on an unbounded subset of positive real numbers. If $f \in O(g)$ and $g \in O(h)$ then $f \in O(h)$.

Recall, our goal is to show $f \in O(h)$.

Proof.

- Since $f \in O(g)$ there exists constants $c_g \in \mathbb{R}^+$, $n_g \in \mathbb{N}_0$ such that, for all $n \geq n_g$, $0 \leq f(n) \leq c_g g(n)$.

$\square$

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- Since $g \in O(h)$ there exists constants $c_h \in \mathbb{R}^+$, $n_h \in \mathbb{N}_0$ such that, for all $n \geq n_h$, $0 \leq g(n) \leq c_h h(n)$.

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- Since $g \in O(h)$ there exists constants $c_h \in \mathbb{R}^+$, $n_h \in \mathbb{N}_0$ such that, for all $n \geq n_h$, $0 \leq g(n) \leq c_h h(n)$.
- So, provided $n \geq n_h$, it follows that $0 \leq c_g g(n) \leq c_g c_h h(n)$.

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Proof.

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- Since $g \in O(h)$ there exists constants $c_h \in \mathbb{R}^+$, $n_h \in \mathbb{N}_0$ such that, for all $n \geq n_h$, $0 \leq g(n) \leq c_h h(n)$.
- So, provided $n \geq n_h$, it follows that $0 \leq c_g g(n) \leq c_g c_h h(n)$.
- It follows that $0 \leq f(n) \leq c_g g(n) \leq c_g c_h h(n)$, whenever n is *also* greater than n_g.

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- Since $g \in O(h)$ there exists constants $c_h \in \mathbb{R}^+$, $n_h \in \mathbb{N}_0$ such that, for all $n \geq n_h$, $0 \leq g(n) \leq c_h h(n)$.
- So, provided $n \geq n_h$, it follows that $0 \leq c_g g(n) \leq c_g c_h h(n)$.
- It follows that $0 \leq f(n) \leq c_g g(n) \leq c_g c_h h(n)$, whenever n is *also* greater than n_g.
- Thus, picking $c = c_g c_h$ and $n_0 = \max(n_g, n_h)$ gives the desired result.

$\square$

Big O: Problem 1

Show that $5n^2 \in O(n^2)$.

- Note that $f(n) = 5n^2$ and $g(n) = n^2$.

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- We need to find $c \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}_0$ such that $5n^2 \leq cn^2$ for all $n \geq n_0$.
- Observe we may simply pick $c = 5$.

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- Plugging in for c, it holds that $5n^2 \leq 5n^2$, for any n.

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- We need to find $c \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}_0$ such that $5n^2 \leq cn^2$ for all $n \geq n_0$.
- Observe we may simply pick $c = 5$.
- Plugging in for c, it holds that $5n^2 \leq 5n^2$, for any n.
- So we can just pick $n_0 = 1$.

Big O: Problem 2

Show that $5n^2 \notin O(n)$.

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Proof.

Suppose $5n^2 \in O(n)$.

Show that $5n^2 \notin O(n)$.

Proof.

Suppose $5n^2 \in O(n)$. Then there exists a $c \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}_0$ such that $5n^2 \leq cn$ for all $n \geq n_0$.

Show that $5n^2 \notin O(n)$.

Proof.

Suppose $5n^2 \in O(n)$. Then there exists a $c \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}_0$ such that $5n^2 \leq cn$ for all $n \geq n_0$. Dividing through by n we see that $5n \leq c$.

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Big O: Problem 3

Show that $\sqrt{8n^{10} + 3n^3 + 8} \in O(n^5)$.

- Observe that for $n \geq 1$,
$\sqrt{8n^{10} + 3n^3 + 8} \leq \sqrt{8n^{10} + 3n^{10} + 8n^{10}} = \sqrt{19}n^5$

Show that $\sqrt{8n^{10} + 3n^3 + 8} \in O(n^5)$.

- Observe that for $n \geq 1$,
  $\sqrt{8n^{10} + 3n^3 + 8} \leq \sqrt{8n^{10} + 3n^{10} + 8n^{10}} = \sqrt{19}n^5$
- Recall, we need to find $c \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}_0$ such that
  $\sqrt{8n^{10} + 3n^3 + 8} \leq cn^5$ for all $n \geq n_0$.

Show that $\sqrt{8n^{10} + 3n^3 + 8} \in O(n^5)$.

- Observe that for $n \geq 1$,
  $\sqrt{8n^{10} + 3n^3 + 8} \leq \sqrt{8n^{10} + 3n^{10} + 8n^{10}} = \sqrt{19}n^5$
- Recall, we need to find $c \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}_0$ such that $\sqrt{8n^{10} + 3n^3 + 8} \leq cn^5$ for all $n \geq n_0$.
- Thus if we pick $c = \sqrt{19}$ and $n_0 = 1$ we are done.

Big O: Problem 4

Show that $\log(n^5) \in O(\log(n))$.

- By the property of logs that $\log_b(n^a) = a \log_b(n)$, we have $\log(n^5) = 5 \log(n)$.

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- By the property of logs that $\log_b(n^a) = a \log_b(n)$, we have $\log(n^5) = 5 \log(n)$.
- Thus if we pick $c = 5$ and $n_0 = 1$ we are done.

Big O: Problem 5

Show that $\log_b(n) \in O(\log_c(n))$.

- By the change of base formula for logs, $\log_b(n) = \log_c(n) / \log_c(b)$.

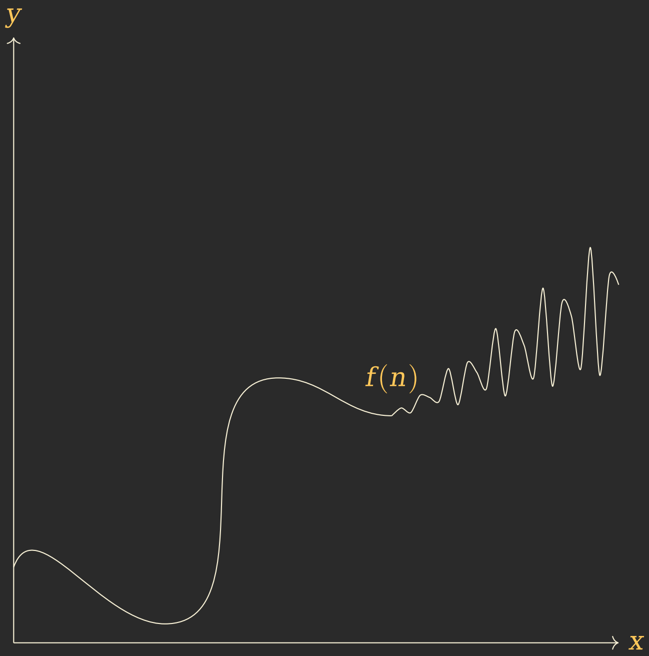
Show that $\log_b(n) \in O(\log_c(n))$.

- By the change of base formula for logs, $\log_b(n) = \log_c(n) / \log_c(b)$.
- Thus if we pick $c = 1/\log_c(b)$ and $n_0 = 1$ we are done.

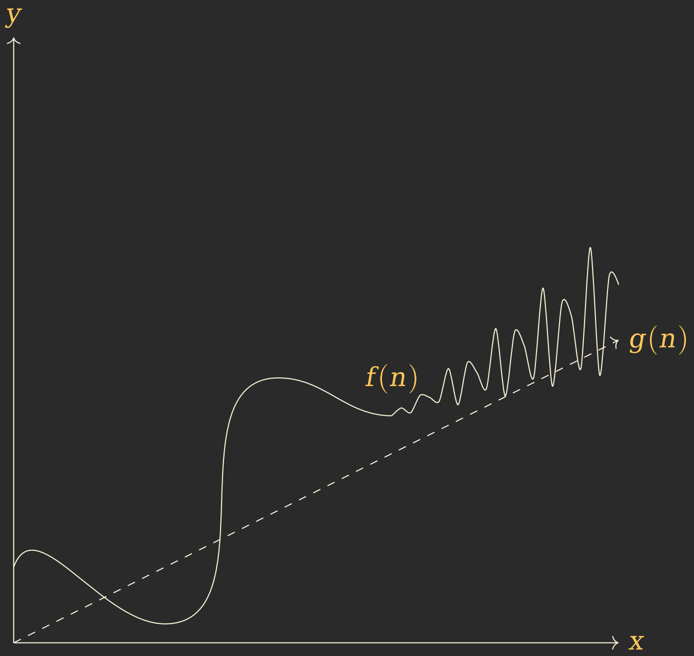
Big Ω

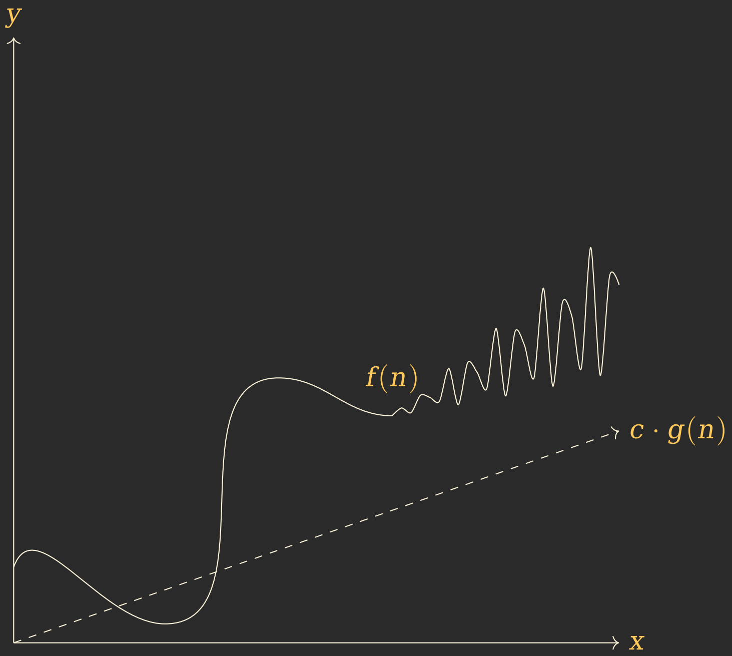
Definition

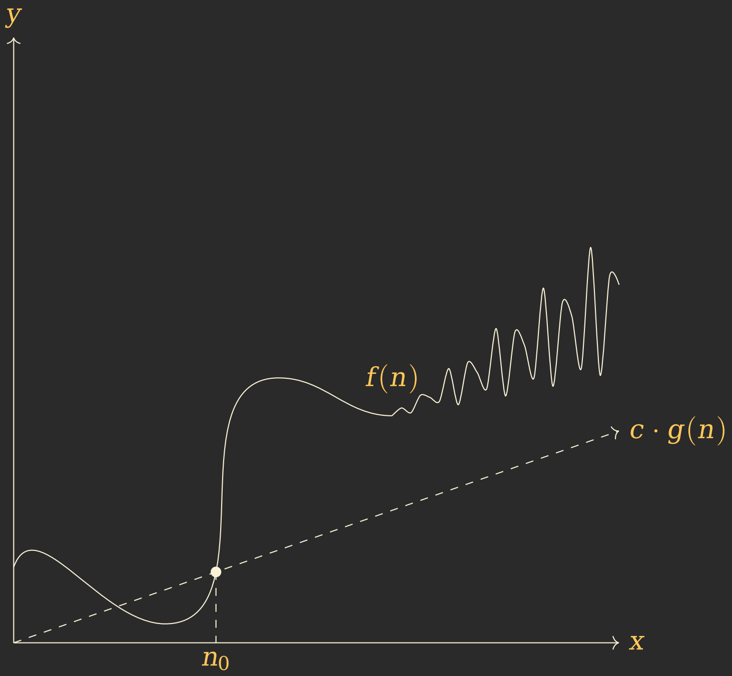
We say $f \in \Omega(g)$ if there exists $c \in \mathbb{R}^+$, $n_0 \in \mathbb{N}_0$, such that for all $n \geq n_0$, $0 \leq c \cdot g(n) \leq f(n)$.



y
f(n)
x


y
x
f(n)
g(n)


y
$f(n)$
$c \cdot g(n)$
x



y
$f(n)$
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Set Notation

Again, the use of set notation in $f \in \Omega(g)$ is not accidental. We can think of $\Omega(g)$ as the ***set of functions*** whose tail behavior is asymptotically dominates g.

Transitivity of Ω

Theorem (Transitivity of Ω)

Let f, g, and h be functions defined on an unbounded subset of positive real numbers. If $f \in \Omega(g)$ and $g \in \Omega(h)$ then $f \in \Omega(h)$.

Relationship Between O and Ω

Theorem

Let f and g be a pair of functions defined on an unbounded subset of positive real numbers, then $f \in O(g)$ if and only if $g \in \Omega(f)$.

Proof.

- Since $f \in O(g)$ there exists constants $c_g \in \mathbb{R}^+$, $n_g \in \mathbb{N}_0$ such that, for all $n \geq n_g$, $0 \leq f(n) \leq c_g g(n)$.

$\square$

Proof.

- Since $f \in O(g)$ there exists constants $c_g \in \mathbb{R}^+$, $n_g \in \mathbb{N}_0$ such that, for all $n \geq n_g$, $0 \leq f(n) \leq c_g g(n)$.
- That is, for all $n \geq n_g$, $0 \leq (1/c_g) f(n) \leq g(n)$.

$\square$

Proof.

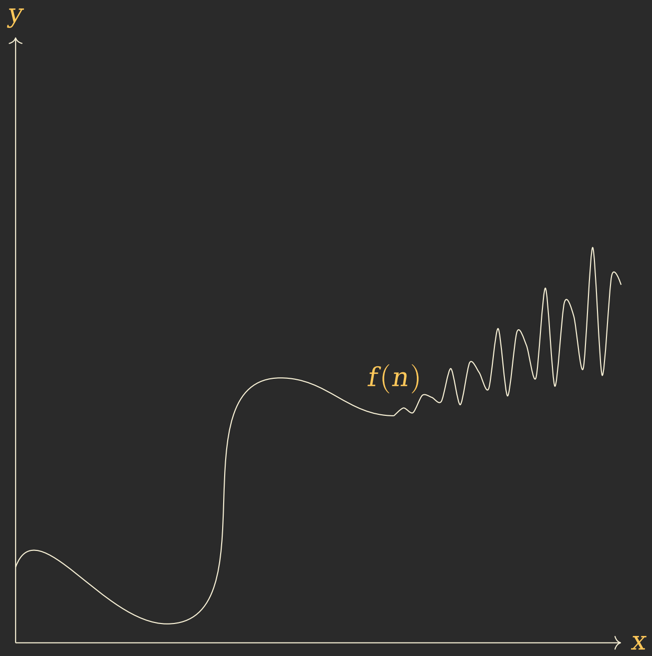
- Since $f \in O(g)$ there exists constants $c_g \in \mathbb{R}^+$, $n_g \in \mathbb{N}_0$ such that, for all $n \geq n_g$, $0 \leq f(n) \leq c_g g(n)$.
- That is, for all $n \geq n_g$, $0 \leq (1/c_g)\, f(n) \leq g(n)$.
- But this is just the definition of $g \in \Omega(f)$ with $c = 1/c_g \in \mathbb{R}^+$, and $n_0 = n_g \in N_0$.

$\square$

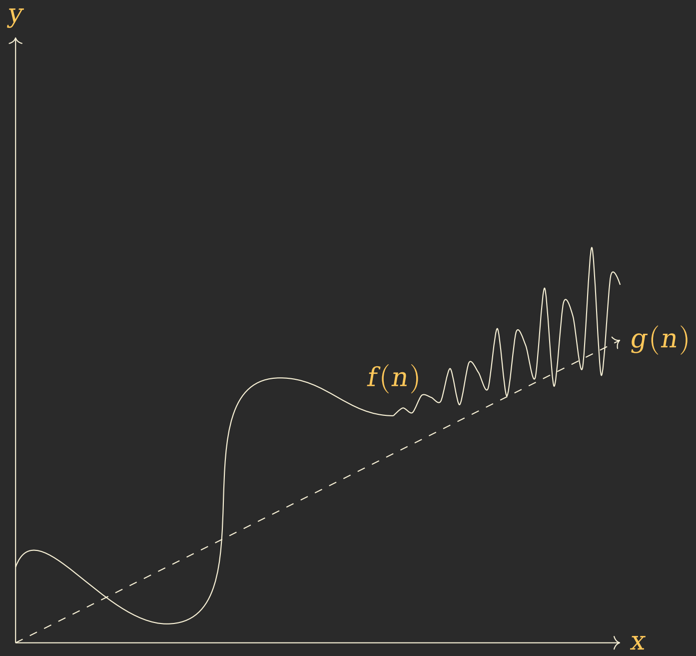
Big Θ

Definition

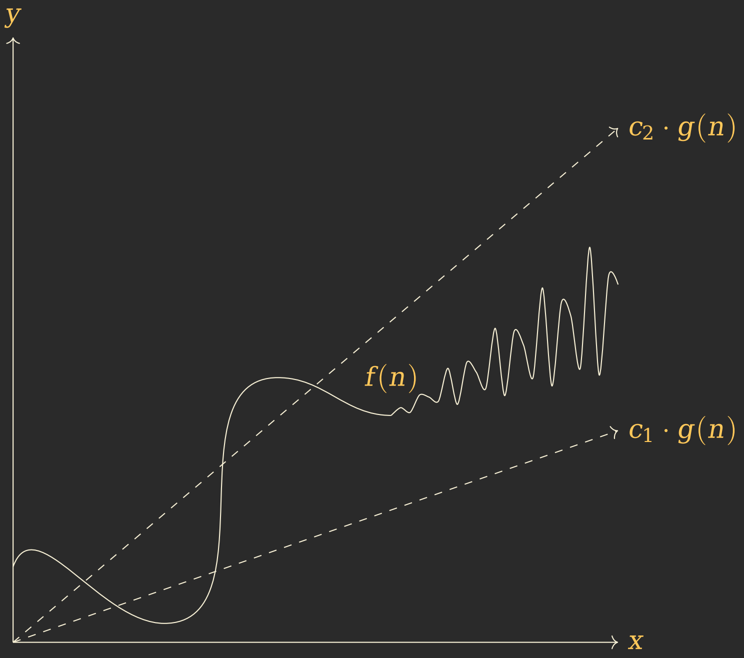
We say $f \in \Theta(g)$ if there exists $c_1, c_2 \in \mathbb{R}^+$, $n_0 \in \mathbb{N}_0$, such that for all $n \geq n_0$, $0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$.



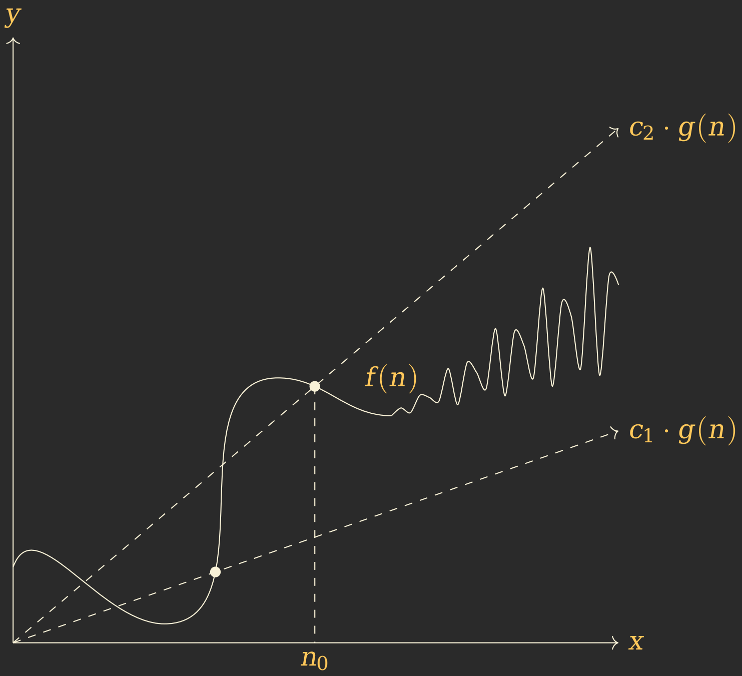
y
$f(n)$
x



y
$f(n)$
$g(n)$
x



y
$c_2 \cdot g(n)$
$f(n)$
$c_1 \cdot g(n)$
x



y
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$c_1 \cdot g(n)$
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Relationship with O and Ω

Theorem

Let f and g be a pair of functions defined on an unbounded subset of positive real numbers, then $f \in \Theta(g)$ if and only if $f \in O(g)$ and $g \in O(f)$.

Other Asymptotic Relations

Little o

Let f and g be a pair of functions defined on an unbounded subset of positive real numbers.

Definition (Little o)

We say $f \in o(g)$ whenever $f \in O(g)$ but $f \notin \Theta(g)$.

Little o

Let f and g be a pair of functions defined on an unbounded subset of positive real numbers.

Definition (Alternative Definition of Little o)

We say $f \in o(g)$ if, for every $c \in \mathbb{R}^+$, there exists $n_0 \in \mathbb{N}_0$ such that for all $n \geq n_0$, $0 \leq f(n) < c \cdot g(n)$.

Little o

In other words, $f \in o(g)$ if and only if, for any constant $c > 0$, and all large enough n,

$$0 \leq \frac{f(n)}{g(n)} < c.$$

Little o

In *other* other words, $f \in o(g)$ is equivalent to saying

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

Little ω

Let f and g be a pair of functions defined on an unbounded subset of positive real numbers.

Definition (Little ω)

We say $f \in \omega(g)$ whenever $f \in \Omega(g)$ but $f \notin \Theta(g)$.

Little ω

Analogously, $f \in \omega(g)$ if and only if, for all $c \in \mathbb{R}^+$, and all large enough n, we have that

$$\frac{f(n)}{g(n)} > c.$$

Little ω

Analogously, $f \in \omega(g)$ if and only if, for all $c \in \mathbb{R}^+$, and all large enough n, we have that

$$\frac{f(n)}{g(n)} > c.$$

In other words, $f \in \omega(g)$ is equivalent to saying

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty.$$

Comparator Analogy

In other words, we can think of the relationships as:

- $f \in o(g)$ is like "$f < g$" (f blows up strictly slower than g)
- $f \in O(g)$ is like "$f \leq g$" (f blows up at most as fast as g)
- $f \in \Theta(g)$ is like "$f = g$" (f blows up at the same rate as g)
- $f \in \Omega(g)$ is like "$f \geq g$" (f blows up at least as fast as g)
- $f \in \omega(g)$ is like "$f > g$" (f blows up strictly faster than g)

where the middle three states hold up to a constant factor.

Comparator Analogy

Example $f(n) = n^2$ and $g(n) = n^2 + n$, then $f \in \Theta(g)$ and f is blowing up at the same rate as g (and vice versa), while $f(n) = n^2$ and $g(n) = n^3$ then $f \in o(g)$ and f is blowing up strictly slower than g.

Limit Theorems

Theorem (Limit Theorems)

1. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ *if and only if* $f \in o(g)$.
2. *If* $\lim_{n\to\infty} \frac{f(n)}{g(n)} \in \mathbb{R}^+$, *then* $f \in \Theta(g)$.
3. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$ *if and only if* $f \in \omega(g)$.

Theorem (Limit Theorems)

1. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ *if and only if* $f \in o(g)$.
2. *If* $\lim_{n\to\infty} \frac{f(n)}{g(n)} \in \mathbb{R}^+$, *then* $f \in \Theta(g)$.
3. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$ *if and only if* $f \in \omega(g)$.

Please note that the middle theorem is a bit more nuanced than the other two. If the limit fails to exist (say because the ratio oscillates), then the theorem does not apply. This also means the converse does not hold.

Advanced Topics

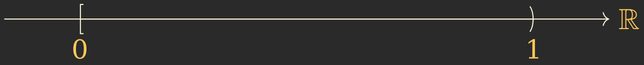
What about limit theorems that test for big O and Ω?

Supremum

Informally, the supremum of a set is a kind of natural maximum.

Supremum

For example, what is the maximum value of the set $[0, 1)$?



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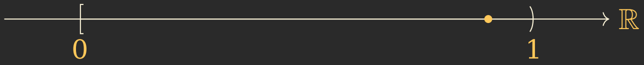

Is it 0.9?

Supremum

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Is it 0.9? 0.99?

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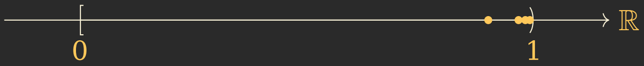

We want to say it's 1, but $1 \notin [0, 1)$.

Supremum

Let P be an ordered set, and $S \subset P$.

A value $x \in P$ is an ***upper bound*** of S if $s \leq x$ for all $s \in S$.

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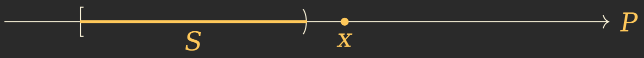

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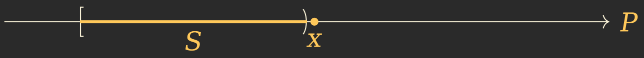

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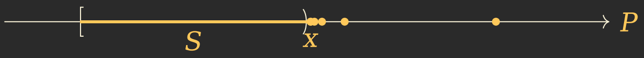

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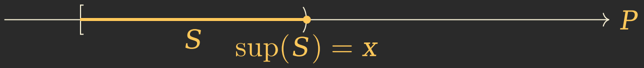

We write $\sup_P(S)$ or $\sup(S)$ when P is clear from context.

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Note! When the S has a maximum element, $\sup(S) = \max(S)$.

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Let P be an ordered set, and $S \subset P$.

Similarly, the ***infimum*** of S in P is the ***greatest*** *lower bound* of S in P.

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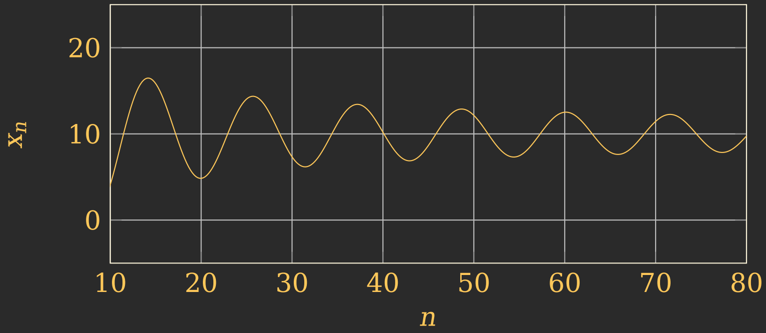
Note! When the S has a ***minimum element***, $\inf(S) = \min(S)$.

Notation

We will somtimes write $\{x_i\}_{i=1}^{\infty}$ to denote a sequence of numbers $x_1, x_2, \ldots$

Question?



How can we capture the idea that the upper envelope remains bounded, even though the sequence $\{x_i\}_{i=1}^{\infty}$ doesn't have a limit?

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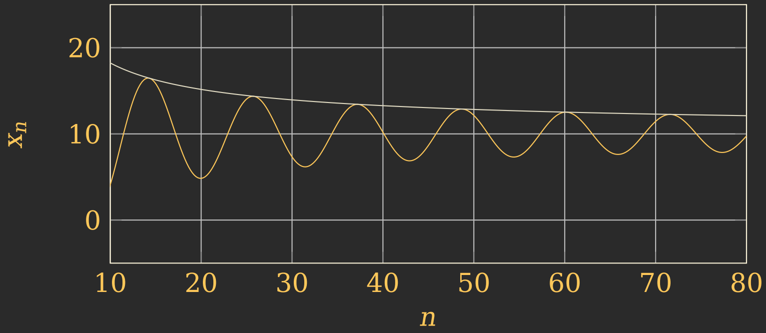
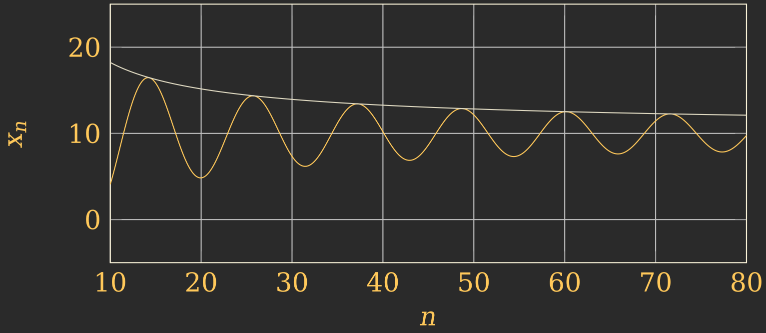


How can we capture the idea that the upper envelope remains bounded, even though the sequence $\{x_i\}_{i=1}^{\infty}$ doesn't have a limit?

Answer!

Consider the "maximum value" (***supremum***) ***of the tail*** region as we slide the start of the tail further and further to the right.



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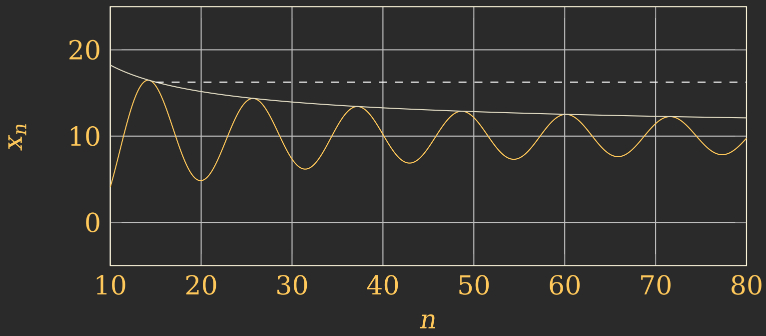

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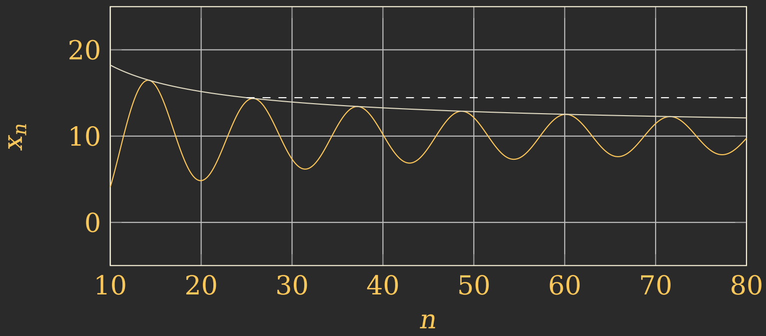

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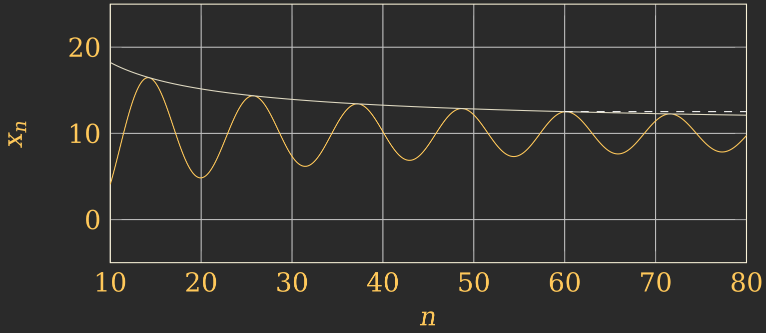

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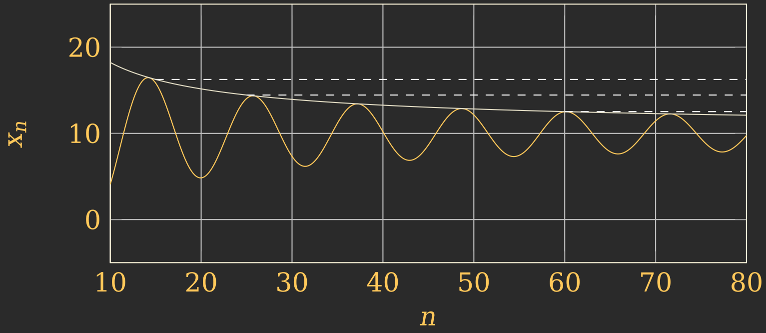

Limit Superior

Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of real numbers.

The ***limit superior*** of $\{x_i\}_{i=1}^{\infty}$ (denoted $\limsup_{n\to\infty} x_n$) is

$$\lim_{n\to\infty} \sup(\{x_i : i \geq n\}),$$

and the ***limit inferior*** of $\{x_i\}_{i=1}^{\infty}$ (denoted $\liminf_{n\to\infty} x_n$) is

$$\lim_{n\to\infty} \inf(\{x_i : i \geq n\}).$$

Theorem (Limit Theorems)

- *If* $\limsup_{n\to\infty} \frac{f(n)}{g(n)} \in \mathbb{R}^+$, *then* $f \in O(g)$.
- *If* $\liminf_{n\to\infty} \frac{f(n)}{g(n)} \in \mathbb{R}^+$, *then* $f \in \Omega(g)$.