Asymptotic Analysis
Solutions to Exercises

1. (a) $6\log_5(n^5+3n^3)+3n^{0.2}\in\Theta(n^{0.2})$;
   (b) $\log_3(6n+7)\times\log_2(5n^{0.3}+21)+\log_4((3n+2)(2n+1)(5n+6))\in\Theta((\log_2(n))^2)$;
   (c) $5n^{0.6}+3n^{0.7}\in\Theta(n^{0.7})$;
   (d) $\sqrt{3n^2+2n+74}\in\Theta(n)$;
   (e) $2\log_4(4n+17)+9\log_5(6n+8)\in\Theta(\log_2(n))$;
   (f) $6^{13}+2^6\times 7\log_4(62)\Theta(1)$;
   (g) $5\log_2(3n^2+n+8)+7\log_6(3n^3+4n^2+75)\Theta(\log_2(n))$;
   (h) $\sqrt{2\log_2(n)+3+7n}\Theta(\sqrt{n})$;
   (i) $2n\log_3(2n^3+17n+1)+5n\Theta(n\log_2(n))$;
   (j) $2\log_3(n)+\sqrt{2n}+3n\in\Theta(n)$;
   (k) $5\log_2(3^n+n^3+1)\in\Theta(n)$;
   (l) $3(n+17)\log_5(2n^2+17n+1)+4n+64\in\Theta(n\log_2(n))$;
   (m) $5^n+10^n+15^n\in\Theta(15^n)$;
   (n) $3^{2n}+2\times 3^n\in\Theta(3^{2n})$ or $\Theta(9^n)$;
   (o) $7n^2+2^{n+5}+2^{n+9}\in\Theta(2^n)$;
   (p) $3\times 5^{n+9}+6\times 3^{n+9}\in\Theta(5^n)$;
   (q) $\sqrt{2n^3+3n^2}\in\Theta(n^{1.5})$;
   (r) $3n^3+2^{3n}+7\times 5^n\in\Theta(2^{3n})$ or $\Theta(8^n)$;
   (s) $9\times 2^{\log_2(n^2+2n)}\in\Theta(n^2)$;
   (t) $(3\log_4(n^2+8)+6\sqrt{n})\times(7\log_5(2n+9)+4\log_3(6n+7))\in\Theta(\sqrt{n}\log_2(n))$;

2. Give an example of a function $f(n)$ such that:

   - $f(n)\in O(n^2)$ and $f(n)\in\Omega(n)$ but $f(n)\notin\Theta(n^2)$ and $f(n)\notin\Theta(n)$.

   Possible solutions:

   - $f(n)=n\log_2(n)$;
   - $f(n)=n^{1.5}$;
   - $f(n)=n^2/\log_2(n)$.

3. Give an example of a function $f(n)$ such that:

   - $f(n)\in O(n)$ and $f(n)\in\Omega(\log_2(n))$ but $f(n)\notin\Theta(n)$ and $f(n)\notin\Theta(\log_2(n))$.

   Possible solutions:

   - $f(n)=(\log_2(n))^2$;
   - $f(n)=\sqrt{n}$;
   - $f(n)=n/\log_2(n)$.

4. Prove that $7\sqrt{2n^4+6n^3+5n^2+9} \in \Theta(n^2)$ using the definition of $\Theta(n^2)$ as functions $f(n)$ such that $c_1 n^2 \le f(n) \le c_2 n^2$ for constants $c_1, c_2 \ge 0$ for all large n.

$$7\sqrt{2n^4+6n^3+5n^2+9} \ge 7\sqrt{2n^4} = 7\sqrt{2}\sqrt{n^4} = 7\sqrt{2}n^2.$$
$$7\sqrt{2n^4+6n^3+5n^2+9} \le 7\sqrt{2n^4+6n^4+5n^4+9n^4} = 7\sqrt{22n^4} = 7\sqrt{22}\sqrt{n^4} = 7\sqrt{22}n^2.$$

Thus, $7\sqrt{2}n^2 \le 7\sqrt{2n^4+6n^3+5n^2+9} \le 7\sqrt{22}n^2$.
By definition, $7\sqrt{2n^4+6n^3+5n^2+9} \in \Theta(n^2)$.

5. Let $f(n) = 8\log_5(n^3+6) \times 7\log_3(n^2+n)$ and $g(n) = 5(\log_6(5n+8))^3$. Prove that $f(n) \in O(g(n))$ using $\lim_{n\to\infty} f(n)/g(n)$.

$$\begin{aligned}
\lim_{n\to\infty}\frac{f(n)}{g(n)} &= \lim_{n\to\infty}\frac{8\log_5(n^3+6)\times 7\log_3(n^2+n)}{5(\log_6(5n+8))^3}\\
&\le \lim_{n\to\infty}\frac{8\log_5(n^3+6n^3)\times 7\log_3(n^2+n^2)}{5(\log_6(5n))^3}\\
&= \lim_{n\to\infty}\frac{8\log_5(7n^3)\times 7\log_3(2n^2)}{5(\log_6(5n))^3}\\
&= \frac{(8)(7)}{5}\lim_{n\to\infty}\frac{(\log_3(7n^3)/\log_3(5))\times\log_3(2n^2)}{(\log_3(5n)/\log_3(6))^3}\\
&= \frac{56(\log_3(6))^3}{5\log_3(5)}\lim_{n\to\infty}\frac{(3\log_3(n)+\log_3(7))(2\log_3(n)+\log_3(2))}{(\log_3(n)+\log_3(5))^3}\\
&= c_2\lim_{n\to\infty}\frac{1/(\log_3(n))^2}{1/(\log_3(n))^2}\frac{(3\log_3(n)+\log_3(7))(2\log_3(n)+\log_3(2))}{(\log_3(n)+\log_3(5))^3} \text{ for } c_2 = \frac{56(\log_3(6))^3}{5\log_3(5)}\\
&= c_2\lim_{n\to\infty}\frac{\left(3\frac{\log_3(n)}{\log_3(n)}+\frac{\log_3(7)}{\log_3(n)}\right)\left(2\frac{\log_3(n)}{\log_3(n)}+\frac{\log_3(2)}{\log_3(n)}\right)}{\left(\frac{\log_3(n)}{\log_3(n)}+\frac{\log_3(5)}{\log_3(n)}\right)^2(\log_3(n)+\log_3(5))}\\
&= c_2\lim_{n\to\infty}\frac{(3+0)(2+0)}{1^2(\log_3(n)+\log_3(5))} = c_2\lim_{n\to\infty}\frac{6}{\log_3(n)} = 0
\end{aligned}$$

Since $\lim_{n\to\infty}\frac{f(n)}{g(n)}$ is upper bounded by 0, the limit must equal 0.
Since $\lim_{n\to\infty}\frac{f(n)}{g(n)} = 0 < c$ for some constant c, $f(n) \in O(g(n))$.