Asymptotic Notation

Asymptotic Running Time

What is the asymptotic running time of the following code fragment?

```
for i ← 1 to 3n^2 + 5n + 8 do
    for j ← 1 to ⌊√(7n + 6)⌋ do
        x ← x + i − j;
    end
end
```

Asymptotic Notation

$f(n) \in O(n^2)$ if there exists $c, n_0 > 0$ such that:

$$f(n) \leq cn^2 \quad \text{for all } n \geq n_0.$$

$f(n) \in \Omega(n^2)$ if there exists $c, n_0 > 0$ such that:

$$f(n) \geq cn^2 \quad \text{for all } n \geq n_0.$$

$f(n) \in \Theta(n^2)$ if there exists $c_1, c_2, n_0 > 0$ such that:

$$c_1 n^2 \leq f(n) \leq c_2 n^2 \quad \text{for all } n \geq n_0.$$

Asymptotic Notation

"$f(n) \in O(g(n))$" means:

$f(n)$ grows at most as fast as $g(n)$.

"$f(n) \in \Omega(g(n))$" means:

$f(n)$ grows at least as fast as $g(n)$.

"$f(n) \in \Theta(g(n))$" means:

$f(n)$ grows at the same rate as $g(n)$.

Asymptotic Notation

$f(n) \in O(g(n))$ if there exists $c, n_0 > 0$ such that:

$$f(n) \leq cg(n) \quad \text{for all } n \geq n_0.$$

$f(n) \in \Omega(g(n))$ if there exists $c, n_0 > 0$ such that:

$$f(n) \geq cg(n) \quad \text{for all } n \geq n_0.$$

$f(n) \in \Theta(g(n))$ if there exists $c_1, c_2, n_0 > 0$ such that:

$$c_1 g(n) \leq f(n) \leq c_2 g(n) \quad \text{for all } n \geq n_0.$$

Asymptotic Notation: Examples

- $5n^2 + 6n + 8 \in O(n^3)$;
- $2^n \in \Omega(n^2)$;
- $\sqrt{6n^3 + 7n^2 + 3n + 5} \in \Theta(n^{1.5})$;
- $\sqrt{6n^3 - 7n^2 + 3n + 5} \in \Theta(n^{1.5})$;

Asymptotic Notation: Example

Claim: $\sqrt{6n^3 + 7n^2 + 3n + 5} \in \Theta(n^{1.5})$.

Proof:

$$\sqrt{6n^3 + 7n^2 + 3n + 5} \geq \sqrt{6n^3} = \sqrt{6}n^{1.5}.$$

$$\begin{aligned} \sqrt{6n^3 + 7n^2 + 3n + 5} &\leq \sqrt{6n^3 + 7n^3 + 3n^3 + 5n^3} \text{ for } n \geq 1 \\ &\leq \sqrt{21n^3} = \sqrt{21}n^{1.5}. \end{aligned}$$

Thus, $\sqrt{6}n^{1.5} \leq \sqrt{6n^3 + 7n^2 + 3n + 5} \leq \sqrt{21}n^{1.5}$,
and $\sqrt{6n^3 + 7n^2 + 3n + 5} \in \Theta(n^{1.5})$.

Asymptotic Notation: Example

Claim: $\sqrt{6n^3 - 7n^2 + 3n + 5} \in \Theta(n^{1.5})$.

Proof:

$$\begin{aligned}
\sqrt{6n^3 - 7n^2 + 3n + 5} &\leq \sqrt{6n^3 + 3n + 5} \\
&\leq \sqrt{6n^3 + 3n^3 + 5n^3} \text{ for } n \geq 1 \\
&\leq \sqrt{14n^3} = \sqrt{14}n^{1.5}. \\
\sqrt{6n^3 - 7n^2 + 3n + 5} &\geq \sqrt{6n^3 - 7n^2} \\
&\geq \sqrt{6n^3 - n^3} \text{ for } n \geq 7 \\
&= \sqrt{5n^3} = \sqrt{5}n^{1.5}.
\end{aligned}$$

Thus, $\sqrt{5}n^{1.5} \leq \sqrt{6n^3 - 7n^2 + 3n + 5} \leq \sqrt{14}n^{1.5}$,
and $\sqrt{6n^3 - 7n^2 + 3n + 5} \in \Theta(n^{1.5})$.

Asymptotic Notation

$f(n) \in O(g(n))$ if there exists $c, n_0 > 0$ such that:

$$f(n) \leq cg(n) \quad \text{for all } n \geq n_0.$$

$f(n) \in \Omega(g(n))$ if there exists $c, n_0 > 0$ such that:

$$f(n) \geq cg(n) \quad \text{for all } n \geq n_0.$$

$f(n) \in \Theta(g(n))$ if there exists $c_1, c_2, n_0 > 0$ such that:

$$c_1 g(n) \leq f(n) \leq c_2 g(n) \quad \text{for all } n \geq n_0.$$

Asymptotic Notation

$f(n) \in O(g(n))$ if there exists $c > 0$ such that:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} \leq c.$$

$f(n) \in \Omega(g(n))$ if there exists $c > 0$ such that:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} \geq c.$$

$f(n) \in \Theta(g(n))$ if there exists $c_1, c_2 > 0$ such that:

$$c_1 \leq \lim_{n\to\infty} \frac{f(n)}{g(n)} \leq c_2.$$

Asymptotic Notation

If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, then

$$f(n) \in O(g(n)) \text{ but } f(n) \notin \Theta(g(n)).$$

If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$, then

$$f(n) \in \Omega(g(n)) \text{ but } f(n) \notin \Theta(g(n)).$$

If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c > 0$ $(c \neq \infty)$, then

$$f(n) \in \Theta(g(n)).$$

Asymptotic Notation: Examples

Compare:

- $2n^4 + 4n^3 + n^2$ and $9n^3 + 7n^2 + 6n$;
- $n^{1/2}$ and $n^{1/4}$;
- $\log_2(n)$ and $\log_3(n)$;
- $\log_2(n)$ and $\log_2(n^2)$;
- $\log_2(n)$ and $(\log_2(n))^2)$;
- $\log_2(n)$ and $n^{0.1}$;
- n^3 and 3^n;
- 2^n and 3^n;
- 2^{2n} and 3^n;
- n and $n \log_2(n)$;
- n^2 and $n \log_2(n)$;
- n and $\log_2(3^n)$.

Math Equalities

Logarithms:

$$\log_a(n) = \frac{\log_2(n)}{\log_2(a)}.$$

l'Hopital's rule: If $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$ and $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

Example A of using limits

Compare $\sqrt{5n^3 + 6n^2 + n + 1}$ and $n^{1.5}$.

$$\begin{aligned}\lim_{n\to\infty} \frac{\sqrt{5n^3 + 6n^2 + n + 1}}{n^{1.5}} &= \lim_{n\to\infty} \frac{\sqrt{n^3} \times \sqrt{5 + (6/n) + (1/n^2) + (1/n^3)}}{n^{1.5}} \\ &= \lim_{n\to\infty} \frac{n^{1.5} \times \sqrt{5}}{n^{1.5}} \\ &= \sqrt{5}.\end{aligned}$$

Therefore, $\sqrt{5n^3 + 6n^2 + n + 1} \in \Theta(n^{1.5})$.

Example B of using limits

Compare 7^n and 5^n.

$$\lim_{n\to\infty} \frac{7^n}{5^n} = \lim_{n\to\infty} \left(\frac{7}{5}\right)^n = \infty.$$

Therefore, $7^n \in \Omega(5^n)$ but $7^n \notin \Theta(5^n)$.

Asymptotic Notation

- $f(n) \in O(g(n))$ if and only if $g(n) \in \Omega(f(n))$.
- $f(n) \in \Theta(g(n))$ if and only if
$$f(n) \in O(g(n)) \text{ and } f(n) \in \Omega(g(n)).$$
- Some (older) books use the (bad) notation "$f(n) = O(g(n))$" in place of "$f(n) \in O(g(n))$".
- Text uses $\overset{\infty}{\forall}\, n$ in place of "for all $n \geq n_0$".

The Hierarchy

- $\Theta(n^n)$
- $\Theta(3^n)$
- $\Theta(2^n)$
- $\Theta(n^3)$
- $\Theta(n^2)$
- $\Theta(n\log(n))$
- $\Theta(n)$
- $\Theta(n^{0.5})$
- $\Theta(n^{0.1})$
- $\Theta((\log(n))^2)$
- $\Theta(\log(n))$
- $\Theta(1)$

Sample "for" loop

```
function func(n)
x ← 0;
for i ← 1 to n do
    for j ← 1 to i do
        x ← x + (i − j);
    end
end
return (x);
```

Sample "for" loop

```
function func(n)
if (n > 100000) then return (0);
x ← 0;
for i ← 1 to n do
    for j ← 1 to n do
        x ← x + (i − j);
    end
end
return (x);
```

Sample “for” loop

```
function func(n)
if (n < 100000) then return (0);
x ← 0;
for i ← 1 to n do
    for j ← 1 to n do
        x ← x + (i − j);
    end
end
return (x);
```