Probabilistic Analysis
Solutions to Exercises

1. Steps 3-9 take $ck\log_2(k)$ time.

   (a) In the worst case, $k = n$, so $T(n) \in \Theta(n\log_2(n))$.

   (b) $\text{Prob}(k = j) = 1/n$ for $1 \le j \le n$.

$$\begin{aligned}
ET(n) &= \sum_{j=1}^{n} \text{Prob}(k=j)\text{Time}(k=j) = \sum_{j=1}^{n}(1/n)ck\log_2(k) = (1/n)\sum_{j=1}^{n} ck\log_2(k) \\
&\le (c/n)\sum_{j=1}^{n} n\log_2(n) = (c/n)n(n\log_2(n)) = cn\log_2(n) \in O(n\log_2(n)). \\
ET(n) &= (1/n)\sum_{j=1}^{n} ck\log_2(k) \ge (c/n)\sum_{j=n/2}^{n} k\log_2(k) \ge (c/n)\sum_{j=n/2}^{n}(n/2)\log_2(n/2) \\
&= (c/n)(n/2)(n/2)(\log_2(n)-1) = cn\log_2(n)/4 - cn/4 \in \Omega(n\log_2(n)).
\end{aligned}$$

   Thus, $ET(n) \in \Theta(n\log_2(n))$.

2. Steps 3-5 take cn time.

   (a) In the worst case, $k = n$, so $T(n) \in \Theta(\sqrt{n})$.

   (b) $\text{Prob}(k = j) = 1/n$ for $1 \le j \le n$.

$$\begin{aligned}
ET(n) &= \sum_{j=1}^{n} \text{Prob}(k=j)\text{Time}(k=j) = \sum_{j=1}^{n}(1/n)c\sqrt{k} = (1/n)\sum_{j=1}^{n} c\sqrt{k} \\
&\le (c/n)\sum_{j=1}^{n}\sqrt{n} = (c/n)n\sqrt{n} = c\sqrt{n} \in O(\sqrt{n}). \\
ET(n) &= (1/n)\sum_{j=1}^{n} c\sqrt{k} \ge (c/n)\sum_{j=n/2}^{n}\sqrt{k} \ge (c/n)\sum_{j=n/2}^{n}\sqrt{n/2} \\
&= (c/n)(n/2)\sqrt{n}/\sqrt{2} = c\sqrt{n}/\sqrt{2} \in \Omega(\sqrt{n}).
\end{aligned}$$

   Thus, $ET(n) \in \Theta(\sqrt{n})$.

3. Steps 3-7 take $c2^k * 2^k$ time.

   (a) In the worst case, $k = \lfloor \log_2(n) \rfloor$, so

$$\begin{aligned}
T(n) &= c2^{\lfloor \log_2(n) \rfloor} \times 2^{\lfloor \log_2(n) \rfloor} \\
&\approx cn \times n = cn^2 \in \Theta(n^2).
\end{aligned}$$

   (b)

$$\begin{aligned}
ET(n) &= \sum_{q=1}^{\log_2(n)} \text{Prob}(k=q)\text{Time}(k=q) = \sum_{q=1}^{\log_2(n)}(1/\log_2(n))c2^q * 2^q = (c/\log_2(n))\sum_{q=1}^{\log_2(n)} c2^q * 2^q \\
&= (c/\log_2(n))(4 + 4^2 + 4^3 + \ldots + n^2/4^2 + n^2/4 + n^2) \qquad \text{since } 2^{\log_2(n)} = n \\
&= (c/\log_2(n))(n^2 + n^2/4 + n^2/4^2 + \ldots + 4^2 + 4) \\
&= (cn^2/\log_2(n))(1 + 1/4 + 1/4^2 + 1/4^3 + \ldots + 4/n^2) \\
&\le (cn^2/\log_2(n))(1 + 1/4 + 1/4^2 + 1/4^3 + \ldots) = (cn^2/\log_2(n))\frac{1}{1-(1/4)} = (4/3)c(n^2/\log_2(n)).
\end{aligned}$$

$$ET(n) = (c/\log_2(n))(n^2 + n^2/4 + n^2/4^2 + \ldots + 4^2 + 4) \ge (c/\log_2(n))n^2 = c(n^2/\log_2(n).$$

   Since $(c(n^2/\log_2(n)) \le ET(n) \le (4/3)c(n^2/\log_2(n))$, expected running time $ET(n) \in \Theta(n^2/\log_2(n))$.

4. Steps 4-10 take $cn\log_2(n)$ time.

   (a) In the worst case, $k < \log_2(n)$, so $T(n) \in \Theta(n\log_2(n))$.

   (b) $\text{Prob}(k < \log_2(n)) = \lfloor \log_2(n) \rfloor / n \approx \log_2(n)/n$.

$$\begin{aligned} ET(n) &= \text{Prob}(k < \log_2(n))\text{Time}(k < \log_2(n)) + \text{Prob}(k \geq \log_2(n))\text{Time}(k \geq \log_2(n)) \\ &= \frac{\log_2(n)}{n}(cn\log_2(n)) + \left(1 - \frac{\log_2(n)}{n}\right)c \\ &= (c(\log_2(n))^2 + c - \frac{\log_2(n)}{n}c) \approx c(\log_2(n))^2 \in \Theta((\log_2(n))^2). \end{aligned}$$

   Thus, $ET(n) \in \Theta((\log_2(n))^2)$.

5. Steps 4-8 take cn^2 time.

   (a) In the worst case, $k < \sqrt{n}$, so $T(n) \in \Theta(n^2)$.

   (b) $\text{Prob}(k < \sqrt{n}) = \lfloor \sqrt{n} \rfloor / n \approx 1/\sqrt{n}$.

$$\begin{aligned} ET(n) &= \text{Prob}(k < \sqrt{n})\text{Time}(k < \sqrt{n}) + \text{Prob}(k \geq \sqrt{n})\text{Time}(k \geq \sqrt{n}) \\ &= (1/\sqrt{n})(cn^2) + (1 - (1/\sqrt{n}))cn = cn^2/\sqrt{n} + cn - c\sqrt{n} \approx cn^{3/2}. \end{aligned}$$

   Thus, $ET(n) \in \Theta(cn^{3/2})$.

6. Steps 2-4 take $c\sqrt{n}$ time.

   (a) In the worst case, k is less than $2n/3$.

$$\begin{aligned} T(n) &= c\sqrt{n} + T(n-5) = c\sqrt{n} + c\sqrt{n-5} + c\sqrt{n-10} + \ldots + c \\ &\leq \underbrace{c\sqrt{n} + c\sqrt{n} + \ldots + c\sqrt{n}}_{n/5} = c\sqrt{n}(n/5) \in O(n^{1.5}). \\ T(n) &= c\sqrt{n} + c\sqrt{n-5} + c\sqrt{n-10} + \ldots + c \\ &\geq c\sqrt{n} + c\sqrt{n-5} + c\sqrt{n-10} + \ldots + c\sqrt{n/2} \geq \underbrace{c\sqrt{n/2} + c\sqrt{n/2} + \ldots + c\sqrt{n/2}}_{n/10} \\ &= c\sqrt{n/2}(n/10) = cn^{1.5}/(10\sqrt{2}) \in \Omega(n^{1.5}). \end{aligned}$$

   Thus, $T(n) \in \Theta(n^{1.5})$.

   (b) $\text{Prob}(k < 2n/3) = (2n/3)/n = 2/3$.

$$\begin{aligned} ET(n) &= \text{Prob}(k < 2n/3)ET(k < 2n/3) + \text{Prob}(k \geq 2n/3)ET(k < 2n/3) \\ &= (2/3)(c\sqrt{n} + ET(n-5)) + (1/3)c\sqrt{n} = c\sqrt{n} + (2/3)ET(n-5) \geq c\sqrt{n}. \\ ET(n) &= c\sqrt{n} + (2/3)ET(n-5) \\ &= c\sqrt{n} + (2/3)c\sqrt{n-5} + (2/3)^2c\sqrt{n-10} + (2/3)^3c\sqrt{n-15} + \ldots + (2/3)^{n/5}c \\ &\leq c\sqrt{n} + (2/3)c\sqrt{n} + (2/3)^2c\sqrt{n} + (2/3)^3c\sqrt{n} + \ldots + (2/3)^{n/5}c\sqrt{n} \\ &\leq c\sqrt{n}(1 + 2/3 + (2/3)^2 + (2/3)^3 + \ldots) = c\sqrt{n}\frac{1}{1-(2/3)} = 3c\sqrt{n}. \end{aligned}$$

   Since $c\sqrt{n} \leq ET(n) \leq 3c\sqrt{n}$, expected running time $ET(n) \in \Theta(\sqrt{n})$.

7. Steps 2-6 take cn^2 time.

   (a) In the worst case, k_1 is less than k_2.

$$\begin{aligned}
T(n) &= cn^2 + T(n-3) = \underbrace{cn^2 + c(n-3)^2 + c(n-6)^2 + \ldots + c}_{n/3} \\
&\leq \underbrace{cn^2 + cn^2 + cn^2 + \ldots + cn^2}_{n/3} = cn^3/3 \in O(n^3). \\
T(n) &= \underbrace{cn^2 + c(n-3)^2 + c(n-6)^2 + \ldots + c}_{n/3} \\
&\geq \underbrace{cn^2 + c(n-3)^2 + c(n-6)^2 + \ldots + c(n/2)^2}_{n/6} \\
&\geq \underbrace{c(n/2)^2 + c(n/2)^2 + c(n/2)^2 + \ldots + c(n/2)^2}_{n/6} \\
&= c(n^2/4)(n/6) = cn^3/24 \in \Omega(n^3).
\end{aligned}$$

   Thus, $T(n) \in \Theta(n^3)$.

   (b) $\text{Prob}(k_1 \leq k_2) \approx (1/2)$ (when n is large).

$$\begin{aligned}
ET(n) &= \text{Prob}(k_1 \leq k_2)ET(k_1 \leq k_2) + \text{Prob}(k_1 > k_2)ET(k_1 > k_2) \\
&= (1/2)(cn^2 + ET(n-3)) + (1/2)cn^2 = cn^2 + (1/2)ET(n-3) \geq cn^2. \\
ET(n) &= cn^2 + (1/2)ET(n-3) \\
&= cn^2 + (1/2)c(n-3)^2 + (1/2)^2c(n-6)^2 + \ldots + (1/2)^{n/3}c \\
&\leq cn^2 + (1/2)cn^2 + (1/2)^2cn^2 + \ldots + (1/2)^{n/3}cn^2 \\
&\leq cn^2 + (1/2)cn^2 + (1/2)^2cn^2 + \ldots = cn^2(1 + (1/2) + (1/2)^2 + \ldots) = 2cn^2.
\end{aligned}$$

   Since $cn^2 \leq ET(n) \leq 2cn^2$, expected running time $ET(n) \in \Theta(n^2)$.

8. Steps 2-4 take c time.

   (a) In the worst case $c_1 = c_2$ so statement 6 is executed.

$$\begin{aligned}
T(n) &= c + T(n-4) + T(n-7) \geq T(n-4) + T(n-7) \geq 2T(n-7) \\
&\geq 2 * 2T(n-14) \geq 2 * 2 * 2T(n-21) \geq \underbrace{2 * 2 * 2 * \ldots * 2}_{n/7} * T(1) \\
&\geq 2^{n/7}c \in \Omega(2^{n/7}).
\end{aligned}$$

   Since $T(n) \in \Omega(2^{n/7})$, worst case running time $T(n)$ has an exponential lower bound.

   (b) $\text{Prob}(c_1 = c_2)$ is $1/2$.

$$\begin{aligned}
ET(n) &= \text{Prob}(c_1 = c_2)ET(c_1 = c_2) + \text{Prob}(c_1 \neq c_2)ET(c_1 \neq c_2) \\
&= (1/2)(c + T(n-4) + T(n-7)) + (1/2)c \\
&= c + (1/2)T(n-4) + (1/2)T(n-7). \\
ET(n) &= c + (1/2)T(n-4) + (1/2)T(n-7) \\
&\leq c + (1/2)T(n-4) + (1/2)T(n-4) = c + T(n-4) = c + c + T(n-8) \\
&= \underbrace{c + c + \ldots + T(1)}_{n/4} = \underbrace{c + c + \ldots + c}_{n/4} = cn/4. \\
ET(n) &= c + (1/2)T(n-4) + (1/2)T(n-7) \\
&\geq c + (1/2)T(n-7) + (1/2)T(n-7) = c + T(n-7) = c + c + T(n-14) \\
&= \underbrace{c + c + \ldots + T(1)}_{n/7} = \underbrace{c + c + \ldots + c}_{n/7} = cn/7.
\end{aligned}$$

   Since $cn/7 \leq ET(n) \leq cn/4$, expected running time $ET(n) \in \Theta(n)$.

9. Steps 2-5 take cn^2 time.

   (a) In the worst case, c_1 is always heads and c_2 always equals c_1, so statements 9 and 11 are always executed.

   $$\begin{aligned} T(n) &= cn^2 + T(n-2) + T(n-9) \geq T(n-2) + T(n-9) \geq 2T(n-9) \\ &\geq 2 * 2T(n-18) \geq 2*2*2T(n-27) \geq \underbrace{2*2*2*\ldots*2}_{n/9} * T(1) \geq 2^{n/9}c \in \Omega(2^{n/9}). \end{aligned}$$

   Since $T(n) \in \Omega(2^{n/9})$, worst case running time $T(n)$ has an exponential lower bound.

   (b) Prob(c_1 = heads) is 1/2.
   Prob($c_2 = c_1$) is 1/2. (Coin c_2 equals c_1 if both c_1 and c_2 are heads or both c_1 and c_2 are tails. Thus, in two out of four cases, c_1 equals c_2.)
   Let $ET_1(n)$ be the expected time for steps 8-9.
   Let $ET_2(n)$ be the expected time for steps 10-11.
   Let $ET_3(n)$ be the expected time for steps 1-7.
   By linearity of expectation, $ET(n) = ET_1(n) + ET_2(n) + ET_3(n)$.

   $$\begin{aligned} ET_1(n) &= \text{Prob}(c_1 = \text{heads})ET_1(c_1 = \text{heads}) + \text{Prob}(c_1 \neq \text{heads})ET_1(c_1 \neq \text{heads}) \\ &= (1/2)(c + ET(n-2)) + (1/2)c = c + (1/2)ET(n-2). \\ ET_2(n) &= \text{Prob}(c_2 = c_1)ET_2(c_2 = c_1) + \text{Prob}(c_2 \neq c_1)ET_2(c_2 \neq c_1) \\ &= (1/2)(c + ET(n-9)) + (1/2)c = c + (1/2)ET(n-9). \\ ET_3(n) &= cn^2. \\ ET(n) &= ET_1(n) + ET_2(n) + ET_3(n) = c + (1/2)ET(n-2) + c + (1/2)ET(n-9) + cn^2 \\ &= 2c + cn^2 + (1/2)ET(n-2) + (1/2)ET(n-9) \\ &\approx cn^2 + (1/2)ET(n-2) + (1/2)ET(n-9). \end{aligned}$$

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   (continued)

   $$\begin{aligned} ET(n) &= cn^2 + (1/2)ET(n-2) + (1/2)ET(n-9) \\ &\leq n^2 + (1/2)ET(n-2) + (1/2)ET(n-2) = cn^2 + ET(n-2) \\ &= cn^2 + c(n-2)^2 + c(n-4)^2 + \ldots + c!l \leq \underbrace{cn^2 + cn^2 + \ldots + cn^2}_{n/2} = (n/2)cn^2 = cn^3. \end{aligned}$$

   $$\begin{aligned} ET(n) &= cn^2 + (1/2)ET(n-2) + (1/2)ET(n-9) \\ &\geq cn^2 + (1/2)ET(n-9) + (1/2)ET(n-9) = cn^2 + ET(n-9) \\ &= cn^2 + c(n-9)^2 + c(n-18)^2 + \ldots + c \\ &\geq cn^2 + c(n-9)^2 + c(n-18)^2 + \ldots + c(n/2)^2 \\ &\geq \underbrace{c(n/2)^2 + c(n/2)^2 + c(n/2)^2 + \ldots + c(n/2)^2}_{n/9} = (n/9)c(n/2)^2 = cn^3/36. \end{aligned}$$

   Since $cn^3/36 \leq ET(n) \leq cn^3$, expected running time $ET(n) \in \Theta(n^3)$.

10. Steps 1-6 take cn time.

    (a) In the worst case c is always heads.

    $$\begin{aligned} T(n) &= cn + 4T(n/4) = cn + 4(cn/4 + 4T(n/4^2)) = cn + cn + 4^2T(n/4^2) \\ &= \underbrace{cn + cn + cn + \ldots + cn + 4^{\log_4(n)}T(1)}_{\log_4(n)} \\ &= \underbrace{cn + cn + cn + \ldots + cn + cn}_{\log_4(n)} = cn\log_4(n). \end{aligned}$$

    Thus, $T(n) \in \Theta(n\log_2(n))$.

(b)

$$
\begin{aligned}
ET(n) &= cn + 4 * ET(\text{steps } 8-11) = cn + 4((1/2)ET(n/4)) = cn + 2ET(n/4) \\
&= cn + 2(cn/4 + 2ET(n/4^2)) = cn + cn/2 + 2^2 ET(n/4^2) \\
&= \underbrace{cn + cn/2 + cn/4 + cn/8 + \ldots + 2^{\log_4(n)} ET(1)}_{\log_4(n)} \\
&= \underbrace{cn + cn/2 + cn/4 + cn/8 + \ldots + c\sqrt{n}}_{\log_4(n)} \\
&= \underbrace{cn + cn/2 + cn/4 + cn/8 + \ldots + cn/\sqrt{n}}_{\log_4(n)} \\
&\leq cn + cn/2 + cn/4 + cn/8 + \ldots \leq cn(1 + 1/2 + 1/4 + 1/8 + \ldots) \\
&\leq cn\frac{1}{1-(1/2)} = 2cn. \\
ET(n) &= cn + 2ET(n/4) \geq cn.
\end{aligned}
$$

Since $cn \leq ET(n) \leq 2cn$, expected running time $ET(n) \in \Theta(n)$.

11. Steps 3-5 take cn time.

(a) In the worst case, $k_1 \leq n/3$ and $k_2 \leq 2n/3$.

$$
\begin{aligned}
T(n) &= cn + 2T(n/2) = cn + 2(cn/2 + 2T(n/2^2)) = cn + cn + 2^2 T(n/2^2) \\
&= \underbrace{cn + cn + \ldots + cn + 2^{\log_2(n)} T(1)}_{\log_2(n)} = \underbrace{cn + cn + \ldots + cn + cn}_{\log_2(n)} = cn \log_2(n).
\end{aligned}
$$

Thus, $T(n) \in \Theta(n \log_2(n))$.

(b) Let $ET_1(n)$ be the expected time for steps 6-7.
Let $ET_2(n)$ be the expected time for steps 8-10.
Let $ET_3(n)$ be the expected time for steps 1-5.
By linearity of expectation, $ET(n) = ET_1(n) + ET_2(n) + ET_3(n)$.

$$
\begin{aligned}
ET_1(n) &= \text{Prob}(k_1 \leq n/3) ET_1(k_1 \leq n/3) + \text{Prob}(k_1 > n/3) ET_1(k_1 > n/3) \\
&= (1/3)(c + ET(n/2)) + (2/3)c = c + (1/3)ET(n/2). \\
ET_2(n) &= \text{Prob}(k_2 \leq 2n/3) ET_2(k_2 \leq 2n/3) + \text{Prob}(k_2 > 2n/3) ET_2(k_2 > 2n/3) \\
&= (2/3)(c + ET(n/2)) + (1/3)c = c + (2/3)ET(n/2). \\
ET_3(n) &= cn. \\
ET(n) &= ET_1(n) + ET_2(n) + ET_3(n) = c + (1/3)ET(n/2) + c + (2/3)ET(n/2) + cn \\
&= 2c + cn + ET(n/2) \approx cn + ET(n/2). \\
ET(n) &= cn + ET(n/2) = cn + cn/2 + cn/4 + \ldots + c = cn(1 + 1/2 + 1/4 + \ldots + 1/n) \leq 2cn. \\
ET(n) &= cn + ET(n/2) \geq cn.
\end{aligned}
$$

Since $cn \leq ET(n) \leq 2cn$, expected running time $ET(n) \in \Theta(n)$.

12. Steps 2-4 take cn time.

(a) In the worst case, $k_1 \leq n/2$ and $k_2 \leq n/3$ and $k_3 \leq n/6$, so all three recursive calls are executed.

$$
\begin{aligned}
T(n) &= cn + T(n-5) + T(n-7) + T(n-11) \geq cn + T(n-7) + T(n-7) \\
&\geq 2T(n-7) \geq 2 \times 2T(n-14) \geq 2 \times 2 \times 2T(n-21) \geq \underbrace{2 \times 2 \times 2 \times \ldots \times 2}_{n/7} T(1) \\
&= 2^{n/7} c \in \Omega(2^{n/7}).
\end{aligned}
$$

Since $T(n) \in \Omega(2^{n/7})$, worst case running time $T(n)$ has an exponential lower bound.

(b) Let $ET_1(n)$ be the expected time for steps 6-7.
Let $ET_2(n)$ be the expected time for steps 8-9.
Let $ET_3(n)$ be the expected time for steps 10-12.
Let $ET_4(n)$ be the expected time for steps 1-5.
By linearity of expectation, $ET(n) = ET_1(n) + ET_2(n) + ET_3(n) + ET_4(n)$.

$$\begin{aligned}
ET_1(n) &= \text{Prob}(k_1 \le n/2)ET_1(k_1 \le n/2) + \text{Prob}(k_1 > n/2)ET_1(k_1 > n/2) \\
&= (1/2)(c + ET(n-5)) + (1/2)c = c + (1/2)ET(n-5). \\
ET_2(n) &= \text{Prob}(k_2 \le n/3)ET_2(k_2 \le n/3) + \text{Prob}(k_2 > n/3)ET_2(k_2 > n/3) \\
&= (1/3)(c + ET(n-7)) + (2/3)c = c + (1/3)ET(n-7). \\
ET_3(n) &= \text{Prob}(k_3 \le n/6)ET_3(k_3 \le n/6) + \text{Prob}(k_3 > n/6)ET_3(k_3 > n/6) \\
&= (1/6)(c + ET(n-11)) + (5/6)c = c + (1/6)ET(n-11). \\
ET_4(n) &= cn.
\end{aligned}$$

$$\begin{aligned}
ET(n) &= ET_1(n) + ET_2(n) + ET_3(n) + ET_4(n) \\
&= c + (1/2)ET(n-5) + c + (1/3)ET(n-7) + c + (1/6)ET(n-11) + cn \\
&= 3c + cn + (1/2)ET(n-5) + (1/3)ET(n-7) + (1/6)ET(n-11) \\
&\approx cn + (1/2)ET(n-5) + (1/3)ET(n-7) + (1/6)ET(n-11). \\
ET(n) &= cn + (1/2)ET(n-5) + (1/3)ET(n-7) + (1/6)ET(n-11) \\
&\le cn + (1/2)ET(n-5) + (1/3)ET(n-5) + (1/6)ET(n-5) \\
&= cn + (3/6)ET(n-5) + (2/6)ET(n-5) + (1/6)ET(n-5) \\
&= cn + ET(n-5) \le cn + c(n-5) + ET(n-10) \le cn + c(n-5) + c(n-10) + ET(n-15) \\
&\le \underbrace{cn + c(n-5) + c(n-10) + c(n-15) + \ldots + c}_{n/5} \\
&\le \underbrace{cn + cn + cn + cn + \ldots + cn}_{n/5} = cn(n/5) = cn^2/5. \\
ET(n) &= cn + (1/2)ET(n-5) + (1/3)ET(n-7) + (1/6)ET(n-11) \\
&\ge cn + (1/2)ET(n-11) + (1/3)ET(n-11) + (1/6)ET(n-11) \\
&= cn + ET(n-11) \ge \underbrace{cn + c(n-11) + c(n-22) + \ldots + c}_{n/11} \\
&\ge \underbrace{cn + c(n-11) + c(n-22) + \ldots + c(n/2)}_{n/22} \\
&\ge \underbrace{c(n/2) + c(n/2) + c(n/2) + \ldots + c(n/2)}_{n/22} = c(n/2)(n/22) = cn^2/44.
\end{aligned}$$

Since $cn^2/44 \le ET(n) \le cn^2/5$, expected running time $ET(n) \in \Theta(n^2)$.

13. Steps 2-4 take cn time.

(a) In the worst case, c_1, c_2 and c_3 are heads, so all three recursive calls are executed.

$$\begin{aligned}
T(n) &= cn + T(n-4) + T(n-6) + T(n-10) \ge cn + T(n-6) + T(n-6) \\
&\ge 2T(n-6) \ge 2 \times 2T(n-12) \ge 2 \times 2 \times 2T(n-18) \ge \underbrace{2 \times 2 \times 2 \times \ldots \times 2}_{n/6} T(1) \\
&= 2^{n/6}c \in \Omega(2^{n/6}).
\end{aligned}$$

Since $T(n) \in \Omega(2^{n/6})$, worst case running time $T(n)$ has an exponential lower bound.

(b) Let $ET_1(n)$ be the expected time for steps 6-7.
Let $ET_2(n)$ be the expected time for steps 8-9.
Let $ET_3(n)$ be the expected time for steps 10-12.
Let $ET_4(n)$ be the expected time for steps 1-5.
By linearity of expectation, $ET(n) = ET_1(n) + ET_2(n) + ET_3(n) + ET_4(n)$.

$$
\begin{aligned}
ET_1(n) &= \text{Prob}(c_1 = \text{heads})ET_1(c_1 = \text{heads}) + \text{Prob}(c_1 = \text{tails})ET_1(c_1 = \text{tails}) \\
&= (1/2)(c + ET(n-4)) + (1/2)c = c + (1/2)ET(n-4). \\
ET_2(n) &= \text{Prob}(c_2 = \text{heads})ET_2(c_2 = \text{heads}) + \text{Prob}(c_2 = \text{tails})ET_2(c_2 = \text{tails}) \\
&= (1/2)(c + ET(n-6)) + (1/2)c = c + (1/2)ET(n-6). \\
ET_3(n) &= \text{Prob}(c_3 = \text{heads})ET_3(c_3 = \text{heads}) + \text{Prob}(c_3 = \text{tails})ET_3(c_3 = \text{tails}) \\
&= (1/2)(c + ET(n-10)) + (1/2)c = c + (1/2)ET(n-10). \\
ET_4(n) &= cn. \\
ET(n) &= ET_1(n) + ET_2(n) + ET_3(n) + ET_4(n) \\
&= c + (1/2)ET(n-4) + c + (1/2)ET(n-6) + c + (1/2)ET(n-10) + cn \\
&= 3c + cn + (1/2)ET(n-4) + (1/2)ET(n-6) + (1/2)ET(n-10) \\
&\geq (1/2)ET(n-4) + (1/2)ET(n-6) + (1/2)ET(n-10) \\
&\geq (1/2)ET(n-10) + (1/2)ET(n-10) + (1/2)ET(n-10) \\
&= (3/2)ET(n-10). \\
ET(n) &\geq (3/2)ET(n-10) \geq (3/2)^2 ET(n-20) \geq (3/2)^3 ET(n-30) \geq (3/2)^{n/10} ET(1) \\
&= (3/2)^{n/10} c \in \Omega((3/2)^{n/10}).
\end{aligned}
$$

Since $ET(n) \in \Omega((3/2)^{n/10})$, expected running time $ET(n)$ has an exponential lower bound.