CSE 2321: Notes on Summations

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(2n+1)(n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

You should memorize (and know how to derive) the first formula above. You do not need to memorize the closed forms for the sum of squares and cubes. Here is how to derive the first formula:
Set S equal to the sum.

$$S = 1 + 2 + 3 + \cdots + n$$

Then reverse the series and write this below S:

$$S = 1 + 2 + 3 + \cdots + n \tag{1}$$
$$S = n + (n-1) + (n-2) + \cdots + 1 \tag{2}$$

If we then add down, adding corresponding terms in the equations above we get:

$$2S = (n+1) + (n+1) + (n+1) + \cdots + (n+1)$$

Since there are n terms, this becomes:

$$2S = n(n+1)$$

$$S = \frac{n(n+1)}{2}$$

Some Summation Rules:

If c is a constant, then

$$\sum_{i=1}^{n} c = cn$$

Note that the summation does not need to start at 1.

$$\sum_{k=10}^{20} 4 = (20 - 10 + 1) \times 4 = 11 \times 4 = 44$$

In the example above, we added 1 to the difference of 20 and 10, since there are 11 numbers from 10 to 20 inclusive. In general:

$$\sum_{i=a}^{b} c = (b - a + 1)c \text{ (assuming that } b > a)$$

Constants can be pulled-through a summation symbol. For example:

$$\sum_{i=1}^{n} 3i^2 = 3\sum_{i=1}^{n} i^2 = 3 \times \frac{n(2n+1)(n+1)}{2}$$

Summations can be split over addition and subtraction. That is:

$$\sum_{k=1}^{n}(b_k + c_k) = \sum_{k=1}^{n} b_k + \sum_{k=1}^{n} c_k$$

Example: What is the value of $\sum_{i=1}^{20}(2i - 4)$?

$$\sum_{i=1}^{20}(2i - 4) = \tag{3}$$
$$= \sum_{i=1}^{20} 2i - \sum_{i=1}^{20} 4 \tag{4}$$
$$= 2\sum_{i=1}^{20} i - \sum_{i=1}^{20} 4 \tag{5}$$
$$= 2 \times \frac{20 \times 21}{2} - 20 \times 4 \tag{6}$$
$$= 340 \tag{7}$$

Example: What is the value of $\sum_{j=3}^{5}\sum_{i=2}^{4}(2ij)$?

$$\begin{aligned}
\sum_{j=3}^{5}\sum_{i=2}^{4}(2ij) = & \qquad (8)\\
&= \sum_{j=3}^{5}(4j+6j+8j) \qquad (9)\\
&= \sum_{j=3}^{5}(18j) \qquad (10)\\
&= 18\times 3 + 18\times 4 + 18\times 5 \qquad (11)\\
&= 18\times 12 \qquad (12)\\
&= 216 \qquad (13)
\end{aligned}$$

If r is a real number not equal to 0, then the following is a geometric series:

$$\sum_{i=0}^{n} r^n = 1 + r + r^2 + r^3 + \cdots + r^n$$

Its closed form is:

$$\sum_{i=0}^{n} r^n = \frac{1-r^{n+1}}{1-r}$$

Here is how to derive the closed form:

Set S equal to the sum.

$$S = 1 + r + r^2 + r^3 + \cdots + r^{n-1} + r^n$$

Next, write rS below:

$$\begin{aligned}
S &= 1 + r + r^2 + r^3 + \cdots + r^{n-1} + r^n\\
rS &= r + r^2 + r^3 + \cdots + r^n + r^{n+1}
\end{aligned}$$

If we then subtract the bottom equation from the top equation, we get:

$$S - rS = 1 - r^{n+1}$$

$$S(1-r) = 1 - r^{n+1}$$

$$S = \frac{1 - r^{n+1}}{1 - r}$$

Geometric series do not need to begin with 1. For example $3 + 6 + 12 + 24 + 48 + 96$ is a geometric series with multiplier $r = 2$ and initial term $a_0 = 3$. The general form of of a geometric series is: $a_0 + a_0 r^1 + a_0 r^2 + ... + a_0 r^n$. Using sigma notation, this is $\sum_{i=0}^{n} a_0 r^n$. The closed form can be derived easily from our previous result as follows:

$$\sum_{i=0}^{n} a_0 r^n = a_0 \sum_{i=0}^{n} r^n = a_0 \frac{1 - r^{n+1}}{1 - r}$$

If the summation is infinite and $|r| < 1$, then r^{n+1} tends toward 0, so

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1 - r}$$

and

$$\sum_{i=0}^{\infty} a_0 r^i = \frac{a_0}{1 - r}$$

The two summation formulas above should be memorized.

Some examples:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{1}{1 - \frac{1}{2}} = 2$$

And:

$$3 + 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \frac{3}{1 - \frac{1}{3}} = \frac{9}{2}$$

The n^{th} harmonic number is:

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + + \frac{1}{5} \cdots + \frac{1}{n}$$

Note carefully that the series above is **not** geometric.

Example: What is the value of $\sum_{i=1}^{\infty} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots$

Perhaps surprisingly, this sum is unbounded. We can show this by grouping terms so that the sum of each group is greater than $\frac{1}{2}$

$$H_\infty = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>\frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{>\frac{1}{2}} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}_{>\frac{1}{2}} + \cdots$$

CSE 2321: Geometric Series

When analyzing loops, you will sometimes need to work with geometric series. Here are some formulas you should know.

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}$$

If $|r| < 1$ then:

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$$

So, for example:

$$\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = \frac{1}{1-\frac{1}{2}} = 2$$

We can break the asymptotic analysis of geometric series into 3 cases:

If $0 < r < 1$, then:

$$\sum_{i=0}^{n} r^i = \Theta(1)$$

If $r = 1$, then:

$$\sum_{i=0}^{n} r^i = \Theta(n)$$

If $r > 1$, then:

$$\sum_{i=0}^{n} r^i = \Theta(r^n)$$

So, for example:

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \left(\frac{1}{3}\right)^n = \Theta(1)$$

and,

$$1 + 3 + 3^3 + +3^3 + \cdots + 3^n = \Theta(3^n)$$

CSE 2321: Bounding Summations

Example 1: We have previously proven that:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

.

Therefore,

$$\sum_{i=1}^{n} i = \Theta(n^2)$$

We will derive this result in another way, in order to demonstrate a useful method for bounding summations.

We'll start by finding an upper bound on the summation. We'll write out $\sum_{i=1}^{n} i$ as $1+2+\cdots+n$ so that the technique is clearer.

$1+2+3+\cdots+n \leq \underbrace{n+n+n+\cdots+n}_{n} = n^2$.

Therefore, $1+2+3+\cdots+n = O(n^2)$.

Next, we'll attempt to find a lower bound. Here is how NOT to do it:

$1+2+3+\cdots+n \geq \underbrace{1+1+1+\cdots+1}_{n} = n$

Therefore $1+2+3+\cdots+n = \Omega(n)$.

While this is correct, the lower and upper bounds are not tight.

Now we'll see how to find a better lower bound. First throw away all terms less than $n/2+1$, keep the largest $n/2$ terms:

$$\underbrace{1+2+3+\ldots\frac{n}{2}}_{\text{throw these away}} + \underbrace{(\frac{n}{2}+1)\cdots+(n-1)+n}_{\text{n/2 terms}} \geq \underbrace{(\frac{n}{2}+1)\cdots+(n-1)+n}_{\text{n/2 terms}}$$

Then decrease each of the remaining terms to $\frac{n}{2}$:

$$\underbrace{(\frac{n}{2}+1)\cdots+(n-1)+n}_{\text{n/2 terms}} \geq \underbrace{\frac{n}{2}\cdots+\frac{n}{2}+\frac{n}{2}}_{\text{n/2 terms}} = \tfrac{n}{2}\times\tfrac{n}{2} = \tfrac{n^2}{4}$$

This is what we want, since the result is quadratic.

Therefore $1 + 2 + 3 + \cdots + n = \Omega(n^2)$.

We have now found tight lower and upper bounds. Since $1 + 2 + 3 + \cdots + n = O(n^2)$ and $1 + 2 + 3 + \cdots + n = \Omega(n^2)$, we can conclude that $1 + 2 + 3 + \cdots + n = \Theta(n^2)$.

Example 2:

$$\sum_{i=1}^{n} \log(i) = \Theta(n \log(n))$$

Proof

We will first find an upper bound. Since the logarithm function is increasing,

$$\begin{aligned}
\sum_{i=1}^{n} \log(i) &= \log(1) + \log(2) + \log(3) + \cdots + \log(n) && (1)\\
&< \underbrace{\log(n) + \log(n) + \log(n) + \cdots \log(n)}_{\text{n terms}} && (2)\\
&= n \log(n) && (3)
\end{aligned}$$

Therefore:

$$\sum_{i=1}^{n} \log(i) = O(n \log(n))$$

Next, we will find a lower bound; we will do this by keeping the largest $\frac{n}{2}$ terms and throwing away the other terms, and then decreasing the argument of the log function of each term to $\frac{n}{2}$.

$$\begin{aligned}
\sum_{i=1}^{n} \log(i) &= \log(1) + \log(2) + \log(3) + \cdots + \log(\frac{n}{2}) + \cdots + \log(n) && (4)\\
&\geq \underbrace{\log(\frac{n}{2} + 1) + \log(\frac{n}{2} + 2) + \cdots + \log(n)}_{\frac{n}{2} \text{ terms}} && (5)\\
&\geq \underbrace{\log(\frac{n}{2}) + \log(\frac{n}{2}) + \cdots + \log(\frac{n}{2})}_{\frac{n}{2} \text{ terms}} && (6)\\
&= \frac{n}{2} \log(\frac{n}{2}) && (7)\\
&= \frac{n}{2} \log(n) - \frac{n}{2} \log(2) && (8)
\end{aligned}$$

Therefore:

$$\sum_{i=1}^{n} \log(n) = \Omega(n \log(n))$$

Since $\sum_{i=1}^{n} \log(i) = \Omega(n \log(n))$ and $\sum_{i=1}^{n} \log(i) = O(n \log(n))$, it follows that $\sum_{i=1}^{n} \log(n) = \Theta(n \log(n))$

Example 3:

$$\sum_{i=4}^{n^2} i \log(i) = \Theta(n^4 \log(n))$$

Proof
We will first find an upper bound. Since the logarithm function is increasing,

$$\begin{align}
\sum_{i=4}^{n^2} i \log(i) &= \underbrace{4 \log(4) + 5 \log(5) + 6 \log(6) + \cdots + n^2 \log(n^2)}_{n^2 - 3 \text{ terms}} \tag{9} \\
&\leq \underbrace{n^2 \log(n^2) + n^2 \log(n^2) + n^2 \log(n^2) + \cdots n^2 \log(n^2)}_{n^2 \text{ terms}} \tag{10} \\
&= n^4 \log(n^2) \tag{11}
\end{align}$$

Therefore:

$$\sum_{i=4}^{n^2} i \log(i) = O(n^4 \log(n))$$

Next, we will find a lower bound, we will do this by keeping the largest $\frac{n^2}{2}$ terms and throwing away the smaller terms, and then decreasing the argument of the log function of each term to $\frac{n^2}{2}$.

$$
\begin{aligned}
\sum_{i=4}^{n^2} i\log(i) &= \underbrace{4\log(4) + 5\log(5) + \cdots + \frac{n^2}{2}\log(\frac{n^2}{2}) + (\frac{n^2}{2}+1)\log(\frac{n^2}{2}+1) + \cdots + n^2\log(n^2)}_{n^2-3 \text{ terms}} && (12)\\
&> \underbrace{\left(\frac{n^2}{2}+1\right)\log(\frac{n^2}{2}+1) + \left(\frac{n^2}{2}+2\right)\log(\frac{n^2}{2}+2) + \cdots + n^2\log n^2}_{\frac{n^2}{2} \text{ terms}} && (13)\\
&> \underbrace{\frac{n^2}{2}\log(\frac{n^2}{2}) + \frac{n^2}{2}\log(\frac{n^2}{2}) + \cdots + \frac{n^2}{2}\log(\frac{n^2}{2})}_{\frac{n^2}{2} \text{ terms}} && (14)\\
&= \frac{n^2}{2} \times \frac{n^2}{2}\log(\frac{n^2}{2}) && (15)\\
& && (16)
\end{aligned}
$$

Therefore:

$$\sum_{i=4}^{n^2} i\log(i) = \Omega(n^4\log(n))$$

Since $\sum_{i=4}^{n^2} i\log(i) = \Omega(n^4\log(n))$ and $\sum_{i=4}^{n^2} i\log(i) = O(n^4\log(n))$, it follows that $\sum_{i=4}^{n^2} i\log(i) = \Theta(n^4\log(n))$

Example 4:

$$\sum_{i=1}^{n} \sqrt{i} = \Theta(n^{\frac{3}{2}})$$

Proof
We will first find an upper bound. Since the root function is increasing,

$$
\begin{aligned}
\sum_{i=1}^{n} \sqrt{i} &= \sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} && (17)\\
&< \underbrace{\sqrt{n} + \sqrt{n} + \sqrt{n} + \cdots + \sqrt{n}}_{\text{n terms}} && (18)\\
&= n\sqrt{n} && (19)
\end{aligned}
$$

Therefore:

$$\sum_{i=1}^{n} \sqrt{i} = O(n^{\frac{3}{2}})$$

Next, we will find a lower bound, we will do this by throwing away the lower half of the terms, and then decreasing the argument of each term to $\frac{n}{2}$.

$$\sum_{i=1}^{n} \sqrt{i} = \sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} \tag{20}$$

$$> \underbrace{\sqrt{\frac{n}{2} + 1} + \sqrt{\frac{n}{2} + 2} + \cdots + \sqrt{n}}_{\frac{n}{2}\text{terms}} \tag{21}$$

$$> \underbrace{\sqrt{\frac{n}{2}} + \sqrt{\frac{n}{2}} + \cdots + \sqrt{\frac{n}{2}}}_{\frac{n}{2}\text{terms}} \tag{22}$$

$$= \frac{n}{2}\sqrt{\frac{n}{2}} \tag{23}$$

$$= \frac{n}{2\sqrt{2}}\sqrt{n} \tag{24}$$

Therefore:

$$\sum_{i=1}^{n} \log(i) = \Omega(n^{\frac{3}{2}})$$

So $\sum_{i=1}^{n} \sqrt{i} = \Theta(n^{\frac{3}{2}})$

Example 5:
In the example below, a and b are constants where $a > 0$ and $b \geq 1$

$$\sum_{i=b}^{n} i^a = \Theta(n^{a+1})$$

In the proof below, we will use sigma notation, rather than expanding out the sum:

Proof
We will first find an upper bound.

$$\sum_{i=b}^{n} i^a \leq \sum_{i=1}^{n} i^a \leq \sum_{i=1}^{n} n^a = n \times n^a = n^{a+1}$$

Therefore:

$$\sum_{i=b}^{n} n^a = O(n^{a+1})$$

Next, we will find a lower bound.

$$\sum_{i=b}^{n} i^a \geq \sum_{i=n/2+1}^{n} i^a \geq \sum_{i=n/2+1}^{n} \left(\frac{n}{2}\right)^a = \frac{n}{2}\left(\frac{n}{2}\right)^a$$

Therefore:

$$\sum_{i=1}^{n} n^a = \Omega(n^{a+1})$$

So

$$\sum_{i=1}^{n} n^a = \Theta(n^{a+1})$$

CSE 2231: Bounding Summations using Integrals

When a summation has the form $\sum_{k=m}^{n} f(k)$ where $f(k)$ is a monotonically increasing function, we can approximate it by integrals:

$$\int_{m-1}^{n} f(x)\,dx \le \sum_{k=m}^{n} f(k) \le \int_{m}^{n+1} f(x)\,dx$$

When $f(k)$ is a monotonically decreasing function, we can use a similar method to provide the bounds:

$$\int_{m}^{n+1} f(x)\,dx \le \sum_{k=m}^{n} f(k) \le \int_{m-1}^{n} f(x)\,dx$$

Example 1: We will find an upper bound on the following summation:

$$\sum_{i=n}^{n^2} i^3$$

$$\sum_{i=n}^{n^2} i^3 \le \int_{n}^{n^2+1} x^3\,dx = \frac{x^4}{4}\Big|_{n}^{n^2+1} = \frac{(n^2+1)^4}{4} - \frac{n^4}{4} = O(n^8)$$

Integration can also be used to show that n^8 is a lower bound, so

$$\sum_{i=n}^{n^2} i^3 = \Theta(n^8)$$

Example 2 (Optional):

The method below uses integration by parts, which is typically covered in Calculus 2. If you are not familiar with integration by parts, you can also bound the summation below using the other method shown in class.

$$\sum_{i=1}^{n} \lg(i) = \Theta(n \lg(n))$$

Proof:
$\lg(i)$ is shorthand for $\log_2(i)$. We will first use the change of base formula to convert to $\ln(i)$:

$$\log_2(i) = \frac{\ln(i)}{\ln(2)}$$

So we want to bound the following summation:

$$c\sum_{i=1}^{n} \ln(i), \text{ where c} = \frac{1}{\ln 2}$$

Since $\ln(i)$ is a monotonically increasing function, the summation can be approximated by integrals. To avoid improper integrals, we will use the fact that, since $\ln(1) = 0$, $\sum_{i=1}^{n} \ln(i) = \sum_{i=2}^{n} \ln(i)$. We will bound $\sum_{i=2}^{n} \ln(i)$ instead.

For the lower bound, we have

$$\int_1^n \ln(x)\,dx \le \sum_{i=2}^{n} \ln(i)$$

We can use integration by parts to evaluate $\int_1^n \ln(x)\,dx$. See the appropriate review notes.

$$\sum_{i=2}^{n} \ln(i) \ge \int_1^n \ln(x)\,dx = xln(x)\Big|_1^n - \int_1^n 1\,dx = x\ln(x) - x\Big|_1^n = n\ln(n) - n = \Omega(n\lg(n))$$

For the upper-bound, we have

$$\sum_{i=2}^{n} \ln(i) \le \int_2^{n+1} \ln(x)\,dx$$

Again, using integration by parts:

$$\sum_{i=2}^{n} \ln(i) \ge \int_2^{n+1} \ln(x)\,dx = xln(x)\Big|_1^{n+1} - \int_2^{n+1} 1\,dx$$

$$= x\ln(x) - x\Big|_2^{n+1} = (n+1)\ln(n+1) - (n+1) - (2\ln(2) - 2) = O(n\lg(n))$$