Analysis of Loops
Solutions to Exercises

1. The inner loop iterates $(2i\lfloor\log_5(i)\rfloor - 7 + 1)$ times and takes $ci\lfloor\log_2(i)\rfloor$ time.

   Running time is $T(n) = \sum_{i=3}^{n^2} ci\lfloor\log_2(i)\rfloor$.

   $$T(n) = \sum_{i=3}^{n^2} ci\lfloor\log_2(i)\rfloor \le \sum_{i=1}^{n^2} cn^2\lfloor\log_2(n^2)\rfloor = n^2cn^2\log_2(n^2) = 2cn^4\log_2(n) \in O(n^4\log_2(n)).$$

   $$\begin{aligned} T(n) &= \sum_{i=3}^{n^2} ci\lfloor\log_2(i)\rfloor \ge \sum_{i=\lceil n^2/2\rceil}^{n^2} ci(\log_2(i)-1) \ge \sum_{i=\lceil n^2/2\rceil}^{n^2} c(n^2/2)(\log_2(n^2/2)-1) \\ &\ge (n^2-(n^2/2))c(n^2/2)(\log_2(n^2)-\log_2(2)-1) \ge c(n^2/2)(n^2/2)(2\log_2(n)-2) \\ &= cn^4\log_2(n)/2 - cn^2/2 \in \Omega(n^4\log_2(n)). \end{aligned}$$

   Since $T(n) \in O(n^4\log_2(n))$ and $T(n) \in \Omega(n^4\log_2(n))$, running time $T(n) \in \Theta(n^4\log_2(n))$.

2. Inner while loop (steps 3-7) iterates $(i^3 - i)/4$ times and takes ci^3 time for some $c > 0$.

   Runing time is $T(n) = \sum_{i=3}^{\lfloor\sqrt{n}\rfloor} ci^3$.

   $$T(n) = \sum_{i=3}^{\lfloor\sqrt{n}\rfloor} ci^3 \le \sum_{i=1}^{\lfloor\sqrt{n}\rfloor} c(\sqrt{n})^3 = \sqrt{n}cn^{1.5} \le cn^2 \in O(n^2).$$

   $$T(n) = \sum_{i=3}^{\lfloor\sqrt{n}\rfloor} ci^3 \ge \sum_{i=\lceil\sqrt{n}/2\rceil}^{\lfloor\sqrt{n}\rfloor} ci^3 \ge \sum_{i=\lceil\sqrt{n}/2\rceil}^{\lfloor\sqrt{n}\rfloor} c(\sqrt{n}/2)^3 \ge (\sqrt{n}-\sqrt{n}/2)cn^{1.5}/2^3 \ge c(n^2)/2^4 \in \Omega(n^2).$$

   Since $T(n) \in O(n^2)$ and $T(n) \in \Omega(n^2)$, running time $T(n) \in \Theta(n^2)$.

3. Inner for loop (steps 4-6) iterates $(6i + 21 - 6i + 1) = 22$ times or takes time c.

   Running time is $T(n) = \sum_{i=2n}^{n^2}\sum_{j=i}^{n^2} c = \sum_{i=2n}^{n^2} c(n^2 - i + 1)$.

   $$T(n) = \sum_{i=2n}^{n^2} c(n^2-i+1) \le \sum_{i=1}^{n^2} c(n^2) = n^2c(n^2) = cn^4 \in O(n^4).$$

   $$\begin{aligned} T(n) &= \sum_{i=2n}^{n^2} c(n^2-i+1) \ge \sum_{i=2n}^{\lfloor n^2/2\rfloor} c(n^2-i+1) \ge \sum_{i=2n}^{\lfloor n^2/2\rfloor} c(n^2-\lfloor n^2/2\rfloor+1) \\ &\ge \sum_{i=2n}^{\lfloor n^2/2\rfloor} c(n^2/2) \ge (n^2/2-2n)c(n^2/2) = c(n^4)/4 - cn^3 \in \Omega(n^4). \end{aligned}$$

   Since $T(n) \in O(n^4)$ and $T(n) \in \Omega(n^4)$, running time $T(n) \in \Theta(n^4)$.

4. Running time is:

$$T(n) = \sum_{i=\lfloor n/2 \rfloor}^{\lfloor 4n\sqrt{n} \rfloor} \sum_{j=3}^{i} \sum_{k=j}^{i} c = \sum_{i=\lfloor n/2 \rfloor}^{\lfloor 4n\sqrt{n} \rfloor} \sum_{j=3}^{i} (i-j+1)c.$$

$$\begin{aligned}
T(n) &= \sum_{i=\lfloor n/2 \rfloor}^{\lfloor 4n\sqrt{n} \rfloor} \sum_{j=3}^{i} (i-j+1)c \le \sum_{i=\lfloor n/2 \rfloor}^{\lfloor 4n\sqrt{n} \rfloor} \sum_{j=1}^{i} c(4n\sqrt{n}) = \sum_{i=\lfloor n/2 \rfloor}^{\lfloor 4n\sqrt{n} \rfloor} c(4n\sqrt{n})i \\
&\le \sum_{i=1}^{\lfloor 4n\sqrt{n} \rfloor} c(4n\sqrt{n})^2 \le (4n\sqrt{n})16cn^3 = 64cn^{4.5} \in O(n^{4.5}).
\end{aligned}$$

$$\begin{aligned}
\sum_{j=3}^{i}(i-j+1)c &\ge \sum_{j=1}^{\lceil i/2 \rceil} (i-j+1)c \ge \sum_{j=1}^{\lceil i/2 \rceil} (i - \lceil i/2 \rceil + 1)c \\
&\ge \lceil i/2 \rceil (i - \lceil i/2 \rceil + 1)c \ge (i/2)^2 c = ci^2/4.
\end{aligned}$$

$$\begin{aligned}
T(n) &= \sum_{i=\lfloor n/2 \rfloor}^{\lfloor 4n\sqrt{n} \rfloor} \sum_{j=3}^{i} (i-j+1)c \ge \sum_{i=\lfloor n/2 \rfloor}^{\lfloor 4n\sqrt{n} \rfloor} ci^2/4 \ge \sum_{i=\lfloor 2n\sqrt{n} \rfloor}^{\lfloor 4n\sqrt{n} \rfloor} ci^2/4 \ge \sum_{i=\lfloor 2n\sqrt{n} \rfloor}^{\lfloor 4n\sqrt{n} \rfloor} c(2n\sqrt{n})^2 \\
&= (\lfloor 4n\sqrt{n} \rfloor - \lfloor 2n\sqrt{n} \rfloor + 1)c(2n\sqrt{n})^2 \ge c(2n\sqrt{n})^3 = 8cn^{4.5} \in \Omega(n^{4.5}).
\end{aligned}$$

Since $T(n) \in O(n^{4.5})$ and $T(n) \in \Omega(n^{4.5})$, running time $T(n) \in \Theta(n^{4.5})$.

5. Inner while loop (steps 5-8) iterates $\lfloor (n-8)/\lceil \log_2(n) \rceil \rfloor$ times and takes $c_1 n/\log_2(n)$ time.
   Outer while loop iterates $\lfloor (n^2-4)/\lceil \sqrt{n} \rceil \rfloor$ times or about $c_2 n^{3/2}$ times for some constant c_2.
   Total running time is $c_2 n^{3/2} c_1 n/\log_2(n) = c_1 c_2 n^{5/2}/\log_2(n) \in \Theta(n^{5/2}/\log_2(n))$.

6. Inner while loop (steps 5-8) iterates $\lfloor (i^3-6)/i \rfloor$ times and takes ci^2 time.
   Outer while loop iterates $\lfloor (n^{3/2}-4)/n \rfloor \approx \sqrt{n}$ times.

$$\begin{aligned}
T(n) &= c5^2 + c(5+n)^2 + c(5+2n)^2 + \ldots + c(n^{3/2})^2 \\
&\le \underbrace{c(n^{3/2})^2 + c(n^{3/2})^2 + \ldots + c(n^{3/2})^2}_{\sqrt{n}} \le c\sqrt{n}(n^{3/2})^2 = cn^{7/2}.
\end{aligned}$$

$$\begin{aligned}
T(n) &= c5^2 + c(5+n)^2 + c(5+2n)^2 + \ldots + c(n^{3/2})^2 \\
&\ge c(n^{3/2}/2)^2 + c(n^{3/2}/2+n)^2 + c(n^{3/2}/2+2n)^2 + \ldots + c(n^{3/2})^2 \\
&\ge \underbrace{c(n^{3/2}/2)^2 + c(n^{3/2}/2)^2 + \ldots + c(n^{3/2}/2)^2}_{\sqrt{n}/2} \\
&= (\sqrt{n}/2)c(n^{3/2})^2/4 = cn^{7/2}/8.
\end{aligned}$$

   Since $cn^{7/2}/8 \le T(n) \le cn^{7/2}$, running time $T(n) \in \Theta(n^{7/2})$.

7. Inner for loop takes ci time.
   Running time is:

$$\begin{aligned}
T(n) &= c + 7c + 7^2c + 7^3c + \ldots + \lfloor 6n^{3/2} \rfloor c/7^2 + \lfloor 6n^{3/2} \rfloor c/7 + \lfloor 6n^{3/2} \rfloor c \\
&= \lfloor 6n^{3/2} \rfloor c + \lfloor 6n^{3/2} \rfloor c/7 + \lfloor 6n^{3/2} \rfloor c/7^2 + \ldots 7^2c + 7c + c \\
&= \lfloor 6n^{3/2} \rfloor c(1 + 1/7 + 1/7^2 + 1/7^3 + \ldots + 1/\lfloor 6n^{3/2} \rfloor) \\
&\le 6n^{3/2}c(1 + 1/7 + 1/7^2 + 1/7^3 + \ldots) \\
&= 6n^{3/2}c\frac{1}{1-1/7} = (7/6)6n^{3/2}c \in O(n^{3/2}). \\
T(n) &= \lfloor 6n^{3/2} \rfloor c(1 + 1/7 + 1/7^2 + 1/7^3 + \ldots + 1/\lfloor 6n^{3/2} \rfloor) \ge \lfloor 6n^{3/2} \rfloor c \in \Omega(n^{3/2}).
\end{aligned}$$

   Therefore $T(n) \in \Theta(n^{3/2})$.

8. At the end of the k'th iteration of the inner while loop (steps 4-7), variable j equals $7 * 3^k$. While loop terminates when:

$$\begin{aligned} 7 * 3^k &= 3i, \text{ or} \\ 3^k &= 3i/7, \text{ or} \\ k &= \log_3(3i/7) = \log_3(i) + \log_3(3/7). \end{aligned}$$

Thus the inner while loop takes $c \log_2(i)$ times for some constant c.

Running time is $T(n) = \sum_{i=n}^{2n^2} c \log_2(i)$.

$$T(n) = \sum_{i=n}^{2n^2} c \log_2(i) \leq \sum_{i=1}^{2n^2} c \log_2(2n^2) = 2n^2 c(2\log_2(n) + \log_2(2)) = 4cn^2 \log_2(n) + cn^2 \in O(n^2 \log_2(n)).$$

$$\begin{aligned} T(n) &= \sum_{i=n}^{2n^2} c \log_2(i) \geq \sum_{i=n^2}^{2n^2} c \log_2(i) \geq \sum_{i=n^2}^{2n^2} c \log_2(n^2) \\ &= (2n^2 - n^2 + 1)2c \log_2(n) \geq 2cn^2 \log_2(n) \in \Omega(n^2 \log_2(n)). \end{aligned}$$

Since $T(n) \in O(n^2 \log_2(n))$ and $T(n) \in \Omega(n^2 \log_2(n))$, running time $T(n) \in \Theta(n^2 \log_2(n))$.

9. At the end of the k'th iteration of the inner while loop (steps 4-8), variable j equals $3n^3/4^k$. Inner while loop terminates when:

$$\begin{aligned} 3n^3/4^k &= 18, \text{ or} \\ 3n^3/18 &= 4^k, \text{ or} \\ k &= \log_4(3n^3/18) = 3\log_4(n) + \log_4(3/18). \end{aligned}$$

Thus the inner while loop takes $c \log_2(n)$ times for some constant c.

At the end of the k'th iteration of the outer while loop, variable i equals $n4^k$. Outer while loop terminates when:

$$\begin{aligned} n4^k &= 5n^3, \text{ or} \\ 4^k &= 5n^3/n = 5n^2, \text{ or} \\ k &= \log_4(5n^2) = 2\log_4(n) + \log_4(5). \end{aligned}$$

Thus the outer while loop takes $c_2 \log_2(n)$ times for some constant c_2.

Since the running time of the inner while loop does not depend upon the running time of the outer while loop, the total running time is $(c \log_2(n) * c_2 \log_2(n)) \in \Theta((\log_2(n))^2)$.

10. At the end of the k'th iteration of the inner while loop (steps 4-8), variable j equals $9*3^k$. Inner while loop terminates when:

$$\begin{aligned} 9*3^k &= i^2, \text{ or} \\ 3^k &= i^2/9, \text{ or} \\ k &= \log_3(i^2/9) = 2\log_3(i) - \log_3(9). \end{aligned}$$

Thus the inner while loop takes $c\log_2(n)$ times for some constant c.

Total running time is $T(n) = c\log_2(n) + c\log_2(n+4) + c\log_2(n+8) + \ldots + c\log_2(\lfloor n\log_5(n)\rfloor)$.

$$\log_2(\lfloor n\log_5(n)\rfloor) \leq \log_2(n\log_5(n)) = \log_2(n) + \log_2(\log_5(n)) \leq 2\log_2(n).$$

$$\begin{aligned} T(n) &= \underbrace{c\log_2(n) + c\log_2(n+4) + c\log_2(n+8) + \ldots + c\log_2(\lfloor n\log_5(n)\rfloor)}_{(\lfloor n\log_5(n)\rfloor - n)/4} \\ &\leq c(\underbrace{2\log_2(n) + 2\log_2(n) + \ldots + 2\log_2(n)}_{(\lfloor n\log_5(n)\rfloor - n)/4}) \\ &\leq c(n\log_5(n)/4)2\log_2(n) = (c/2)n(\log_2(n)/\log_2(5))\log_2(n) \\ &= (c/2)(1/\log_2(5))n(\log_2(n))^2 \in O(n(\log_2(n))^2). \end{aligned}$$

$$\begin{aligned} T(n) &= \underbrace{c\log_2(n) + c\log_2(n+4) + c\log_2(n+8) + \ldots + c\log_2(\lfloor n\log_5(n)\rfloor)}_{(\lfloor n\log_5(n)\rfloor - n)/4} \\ &\geq \underbrace{c\log_2(n) + c\log_2(n) + c\log_2(n) + \ldots + c\log_2(n)}_{(\lfloor n\log_5(n)\rfloor - n)/4} \\ &= ((\lfloor n\log_5(n)\rfloor - n)/4)c\log_2(n) \geq (n\log_5(n)/8)c\log_2(n) \\ &= (cn/8)(\log_2(n)/\log_2(5))\log_2(n) = (cn/8)(1/\log_2(5))(\log_2(n))^2 \in \Omega(n(\log_2(n))^2). \end{aligned}$$

Since $T(n) \in O(n(\log_2(n))^2)$ and $T(n) \in \Omega(n(\log_2(n))^2)$, running time $T(n) \in \Theta(n(\log_2(n))^2)$.

11. Inner while loop iterates $(n^2-5)/i$ times and takes cn^2/i time.

$$\begin{aligned} T(n) &= cn^2 + cn^2/2.5 + cn^2/(2.5)^2 + cn^2/(2.5)^3 + \ldots + cn^2/3n \\ &= cn^2(1 + 1/2.5 + (1/2.5)^2 + (1/2.5)^3 + \ldots + 1/3n) \\ &\leq cn^2(1 + 1/2.5 + (1/2.5)^2 + (1/2.5)^3 + \ldots) \\ &= cn^2\frac{1}{1-(1/2.5)} = cn^2(2.5/1.5) \in O(n^2). \\ T(n) &= cn^2 + cn^2/2.5 + cn^2/(2.5)^2 + \ldots + cn^2/3n \geq cn^2 \in \Omega(n^2). \end{aligned}$$

Therefore, $T(n) \in \Theta(n^2)$.

12. Inner while loop iterates $(2n^3 - n)/3$ times and takes cn^3 time.

At the end of the k'th iteration of the outer while loop, variable i equals n^2. Outer while loop terminates when:

$$\begin{aligned} 4^k &= n^2, \text{ or} \\ k &= \log_4(n^2) = 2\log_4(n). \end{aligned}$$

Thus the outer while loop takes $c_2\log_2(n)$ times for some constant c_2.

Since the running time of the inner while loop does not depend upon the outer while loop, the total running time is $(cn^3 c_2\log_2(n)) \in \Theta(n^3\log_2(n))$.

13. The inner for loop (steps 4-6) iterate j times and take cj time.
Running time $T_2(i)$ of steps 2-8 is:

$$\begin{aligned}
T_2(i) &= 5c + (5*3)c + (5*3^2)c + (5*3^3)c + \ldots + (i/3^2)c + (i/3)c + ic \\
&= ic + (i/3)c + (i/3^2)c + (i/3^3)c + \ldots + (5*3)c + 5c \\
&= ic(1 + (1/3) + (1/3)^2 + (1/3)^3 + \ldots + (5*3)/i + 5/i) \\
&\le ic(1 + (1/3) + (1/3)^2 + (1/3)^3 + \ldots) = ic\frac{1}{1-(1/3)} = (3/2)ic. \\
T_2(i) &= 5c + (5*3)c + (5*3^2)c + \ldots + (i/3)c + ic \ge ic.
\end{aligned}$$

Since $ic \le T_2(i) \le (3/2)ic$, running time $T_2(i)$ of steps 2-8 is $c'i$ for some constant c'.

Total running time is:

$$\begin{aligned}
T(n) &= \sum_{i=3}^{n^2} c'i \le \sum_{i=3}^{n^2} c'n^2 \le n^2 c' n^2 = c'n^4 \in O(n^4). \\
T(n) &= \sum_{i=3}^{n^2} c'i \ge \sum_{i=\lceil n^2/2\rceil}^{n^2} c'i \ge \sum_{i=\lceil n^2/2\rceil}^{n^2} c'n^2/2 = (n^2 - \lceil n^2/2\rceil)c'n^2/2 \\
&\ge (n^2/2 - 1)c'n^2/2 = c'n^4/4 - c'n^2/2 \in \Omega(n^4).
\end{aligned}$$

Since $T(n) \in \Omega(n^4)$ and $T(n) \in O(n^4)$, total running time $T(n) \in \Theta(n^4)$.

14. The inner while loop (steps 5-11) iterates $(2j - 9 + 1)/5$ times and takes cj time.
Running time $T_2(i)$ of steps 3-11 is:

$$\begin{aligned}
T_2(i) &= 6c + (6*5)c + (6*5^2)c + (6*5^3)c + \ldots + \frac{\lfloor i\log_2(i)\rfloor c}{5^2} + \frac{\lfloor i\log_2(i)\rfloor c}{5} + \lfloor i\log_2(i)\rfloor c \\
&= \lfloor i\log_2(i)\rfloor c + \frac{\lfloor i\log_2(i)\rfloor c}{5} + \frac{\lfloor i\log_2(i)\rfloor c}{5^2} + \ldots + (6*5)c + 6c \\
&= \lfloor i\log_2(i)\rfloor c(1 + (1/5) + (1/5)^2 + (1/5)^3 + \ldots + \frac{6*5}{\lfloor i\log_2(i)\rfloor} + \frac{6}{\lfloor i\log_2(i)\rfloor}) \\
&\le i\log_2(i)c(1 + (1/5) + (1/5)^2 + (1/5)^3 + \ldots) \\
&= i\log_2(i)c\frac{1}{1-(1/5)} = i\log_2(i)c(5/4). \\
T_2(i) &= 6c + (6*5)c + (6*5^2)c + \ldots + \frac{\lfloor i\log_2(i)\rfloor c}{5} + \lfloor i\log_2(i)\rfloor c \ge \lfloor i\log_2(i)\rfloor c.
\end{aligned}$$

Since $\lfloor i\log_2(i)\rfloor c \le T_2(i) \le i\log_2(i)c(5/4)$, running time $T(n) = c'i\log_2(i)$ for some constant c'.

Total running time is:

$$\begin{aligned}
T(n) &= \underbrace{c'n^3\log_2(n^3) + c'(n-5)^3\log_2((n-5)^3) + c'(n-10)^3\log_2((n-10)^3 + \ldots + c'6\log_2(6)}_{n^3/5} \\
&\le \underbrace{c'n^3\log_2(n^3) + c'n^3\log_2(n^3) + \ldots + c'n^3\log_2(n^3)}_{n^3/5} \\
&= c'n^3\log_2(n^3)(n^3/5) = c'n^3(3\log_2(n))n^3/5 = c'(3/5)n^6\log_2(n) \in O(n^6\log_2(n)). \\
T(n) &= \underbrace{c'n^3\log_2(n^3) + c'(n-5)^3\log_2((n-5)^3) + c'(n-10)^3\log_2((n-10)^3 + \ldots + c'6\log_2(6)}_{n^3/5} \\
&\ge \underbrace{c'n^3\log_2(n^3) + c'(n-5)^3\log_2((n-5)^3) + c'(n-10)^3\log_2((n-10)^3 + \ldots + c'(n^3/2)\log_2(n^3/2)}_{n^3/10} \\
&\ge \underbrace{c'(n^3/2)\log_2(n^3/2) + c'(n^3/2)\log_2(n^3/2) + +c'(n^3/2)\log_2(n^3/2) + \ldots + c'(n^3/2)\log_2(n^3/2)}_{n^3/10} \\
&= (n^3/10)c'(n^3/2)\log_2(n^3/2) = (c'/20)n^6(3\log_2(n) - \log_2(2)) \\
&= (3c'/20)n^6\log_2(n) - (3c'/20)n^6 \in \Omega(n^6\log_2(n)).
\end{aligned}$$

Since $T(n) \in \Omega(n^6\log_2(n))$ and $T(n) \in O(n^6\log_2(n))$, total running time $T(n) \in \Theta(n^6\log_2(n))$.