CSE 2331 Homework 1 Solutions

First, don't panic! There is a lot of dense formalism here but things are not as complicated as they seem. We will work some of these out together!

Unless otherwise stated, let f and g be a pair of asymptotically non-negative functions defined on an unbounded subset of positive real numbers.

1. Break the ice! Create an introductory post on the forum that includes your name, major, specialization areas, and hobbies. Please reply to at least two other posts. (Self-explanatory)

2. Using the definition of Θ, argue $3n\log_2(30n^3 - 10n + 1) \in \Theta(n \log n)$.

   Proof. To show $3n\log_2(30n^3 - 10n + 1) \in \Theta(n \log n)$ we need to show that there exists constants $c_1, c_2 \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}_0$ such that, for all $n \geq n_0$, it holds that

   $$c_1 n \log n \leq 3n\log_2(30n^3 - 10n + 1) \leq c_2 n \log n.$$

   We first prove the "big-O" side. In other words, we establish that there exists some $c_2 \in \mathbb{R}^+$, such that, for all sufficiently large n, $3n\log_2(30n^3 - 10n + 1) \leq c_2 n \log n$. Observe that,

   $$\begin{aligned} 3n\log_2(30n^3 - 10n + 1) &\leq 3n\log_2(30n^3 + 1) \leq 3n\log_2(31n^3) \\ &\leq 3n\log_2(n^4) = 12n\log_2(n), \end{aligned}$$

   where the first inequality drops the negative term $-10n$, the second inequality holds provided $n \geq 1$, and the third inequality holds provided $n \geq 31$.

   Next, we argue the "big-Ω" side. Namely, that there exists some $c_1 \in \mathbb{R}^+$ such that $c_1 n \log n \leq 3n\log_2(30n^3 - 10n + 1)$ for all sufficiently large n. To see this, observe that $30n^3 - 10n + 1 \geq 30n^3 - 10n \geq 30n - 10n = 20n$, where the last inequality holds for all $n \geq 1$. Therefore, since $\log(n)$ is an increasing function, we have

   $$3n\log_2(30n^3 - 10n + 1) \geq n\log_2(20n) \geq n\log_2(n).$$

By picking $c_1 = 1$, $c_2 = 12$ and $n_0 = \max(1, 31) = 31$ we have, for all $n \geq n_0$,

$$c_1 n \log n \leq 3n \log_2(30n^3 - 10n + 1) \leq c_2 n \log n.$$

Therefore, $3n \log_2(30n^3 - 10n + 1) \in \Theta(n \log n)$. $\square$

3. Using the definition of Θ, argue $7\sqrt{n^3 - 8n^2 - 11} + 1 \in \Theta(n\sqrt{n})$.

Proof. To show $7\sqrt{n^3 - 8n^2 - 11} + 1 \in \Theta(n\sqrt{n})$, we need to show that there exists constants $c_1, c_2 \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}_0$ such that, for all $n \geq n_0$, it holds that

$$c_1 n\sqrt{n} \leq 7\sqrt{n^3 - 8n^2 - 11} + 1 \leq c_2 n\sqrt{n}.$$

We first prove the "big-O" side. In other words, we establish that there exists some $c_2 \in \mathbb{R}^+$, such that, for all sufficiently large n, $7\sqrt{n^3 - 8n^2 - 11} + 1 \in \Theta(n\sqrt{n})$. Observe that,

$$7\sqrt{n^3 - 8n^2 - 11} + 1 \leq 7\sqrt{n^3} + 1 = 7n\sqrt{n} + 1 \leq 8n\sqrt{n},$$

where the first inequality simply drops the negative term $-8n^2 - 11$ and the second inequality holds provided $n \geq 1$.

Next, we argue the "big-Ω" side. Namely, that there exists some $c_1 \in \mathbb{R}^+$ such that $c_1 n\sqrt{n} \leq 7\sqrt{n^3 - 8n^2 - 11} + 1$ for all sufficiently large n. To see this, observe that

$$7\sqrt{n^3 - 8n^2 - 11} + 1 \geq \sqrt{n^3 - 8n^2 - 11}.$$

Now, for sufficiently large n, it must be the case that $n^3 - 8n^2 - 11 \geq n^3/2$. To see this, consider

$$n^3 - 8n^2 - 11 \geq n^3 - 8n^2 - 11n^2 = n^3 - 19n^2 \geq n^3(1 - 19/n) \geq n^3/2,$$

provided $n \geq 38$. Since $\sqrt{\cdot}$ is an increasing function, it follows that

$$7\sqrt{n^3 - 8n^2 - 11} + 1 \geq \sqrt{n^3/2} = n\sqrt{n/2} = n\sqrt{n}/\sqrt{2}.$$

Therefore, picking $c_1 = \frac{1}{\sqrt{2}}$, $c_2 = 8$, and $n_0 = 38$ we have, for all $n \geq n_0$,

$$c_1 n\sqrt{n} \leq 7\sqrt{n^3 - 8n^2 - 11} + 1 \leq c_2 n\sqrt{n}.$$

That is, $7\sqrt{n^3 - 8n^2 - 11} + 1 \in \Theta(n\sqrt{n})$. $\square$

4. Prove $O(f) \cap \Omega(f) \subset \Theta(f)$. We give two possible arguments. The first is directly from the definition, while the second uses a theorem from the lecture slides.

   Proof. We first argue that $O(f) \cap \Omega(f)$ is a subset of $\Theta(f)$ by arguing that any $g \in O(f) \cap \Omega(f)$ must also be in $\Theta(f)$. To see this, suppose $g \in O(f) \cap \Omega(f)$. Since $g \in O(f)$, there exists a pair of constants $c_O \in \mathbb{R}^+$, $n_O \in \mathbb{N}_0$, such that, for all $n \geq n_O$, we have

   $$g(n) \leq c_O f(n).$$

   At the same time, since $g \in \Omega(f)$, so there exists constants $c_\Omega \in \mathbb{R}^+$, $n_\Omega \in \mathbb{N}_0$, such that, for all $n \geq n_\Omega$, we have

   $$c_\Omega f(n) \leq g(n).$$

   Therefore, for all $n \geq \max(n_O, n_\Omega)$, it holds that

   $$c_\Omega f(n) \leq g(n) \leq c_O f(n).$$

   That is, the definition of $g \in \Theta(f)$ is satisfied by picking $c_1 = c_\Omega$, $c_2 = c_O$, and $n = \max(n_O, n_\Omega)$. □

   You may have used the following argument instead, which uses a theorem from the lecture slides:

   Proof. Suppose $g \in O(f) \cap \Omega(f)$. Since $g \in \Omega(f)$, we know that $f \in O(g)$. Because $g \in O(f)$, and $f \in O(g)$, and it follows by a Theorem on the lecture slides that $g \in \Theta(f)$. Therefore, any g in $O(f) \cap \Omega(f)$ is also in $\Theta(f)$. □

5. Prove $\Theta(f) \subset O(f) \cap \Omega(f)$.[1]

   We again give two possible arguments. The first is directly from the definition, while the second uses a theorem from the lecture slides.

[1] This, together with the previous question, shows that $O(f) \cap \Omega(f) = \Theta(f)$.

Proof. We argue that $\Theta(f)$ is a subset of $O(f) \cap \Omega(f)$. Suppose $g \in \Theta(f)$. Then, by definition of Θ, there must be constants $c_1, c_2 \in \mathbb{R}^+$, and $n_\Theta \in \mathbb{N}_0$ such that, for all $n \geq n_\Theta$,

$$c_1 f(n) \leq g(n) \leq c_2 f(n).$$

Thus, we see that $g \in O(f)$ by picking $n_0 = n_\Theta$ and $c = c_2$ in the definition of O. Further, we see that $g \in \Omega(f)$ by picking $n_0 = n_\Theta$ and $c = c_1$ in the definition of Ω. Therefore, $g \in O(f) \cap \Omega(f)$. □

You may have used the following argument instead, which uses a theorem from the lecture slides:

Proof. Suppose $g \in \Theta(f)$. Then, it follows by a Theorem on the lecture slides that $f \in O(g)$ and $g \in O(f)$. Since $f \in O(g)$, we know that $g \in \Omega(f)$. Therefore, $g \in O(f) \cap \Omega(f)$. □

Note that, taken together, this result and the previous problem prove $(O(f) \cap \Omega(f)) = \Theta(f)$.

6. Suppose $f \in O(g)$. Prove or disprove $2^{f(n)} \in O(2^{g(n)})$.

   Proof. We will argue that this is false by giving one possible counterexample. Let $f(n) = 2n$ and $g(n) = n$. By our limit test, $f(n) \in \Theta(g(n))$. Yet,

   $$\lim_{n\to\infty} \frac{2^{f(n)}}{2^{g(n)}} = \lim_{n\to\infty} \frac{2^{2n}}{2^n} = \frac{4^n}{2^n} = \lim_{n\to\infty} 2^n = \infty.$$

   It follows that $2^{f(n)} \in \omega(2^{g(n)})$, and so $2^{f(n)} \notin O(2^{g(n)})$. □

7. Compute

   $$o(f) \cap \omega(f).$$

   (Hint: Suppose $g(n)$ is in the intersection and use limit tests to compare with $f(n)$. Could such a $g(n)$ exist?)

Proof. We claim that $o(f) \cap \omega(f) = \varnothing$. To see why, suppose there exists a $g(n) \in o(f(n)) \cap \omega(f(n))$. Since $g(n) \in o(f(n))$, any such $g(n)$ must satisfy

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty.$$

On the other hand, since $g(n) \in \omega(f(n))$, any such $g(n)$ must satisfy

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

This is a contradiction since the limit cannot be both 0 and ∞. Therefore, no such $g(n)$ exists. $\square$

8. Let $c \in \mathbb{R}^+$, $f(n) = 2^{n+c}$, $g(n) = 2^{n\cdot c}$ Prove that $f(n) \in \Theta(2^n)$ and, for $c \neq 1$, that $g \notin \Theta(2^n)$.

   Proof. Observe,

   $$\lim_{n\to\infty} \frac{f(n)}{2^n} = \lim_{n\to\infty} \frac{2^{n+c}}{2^n} = \lim_{n\to\infty} 2^c = 2^c.$$

   Therefore, $f(n) \in \Theta(2^n)$.

   Suppose $c \neq 1$ and consider,

   $$\lim_{n\to\infty} \frac{g(n)}{2^n} = \lim_{n\to\infty} \frac{2^{n\cdot c}}{2^n} = \lim_{n\to\infty} 2^{n\cdot(c-1)} = \lim_{n\to\infty} \left(2^{c-1}\right)^n = \begin{cases} \infty & c > 1 \\ 0 & c < 1 \end{cases}.$$

   Note that neither outcome is a positive constant and so $g(n) \notin \Theta(2^n)$. $\square$

9. Prove that for any $\varepsilon > 0$, no matter how small, and $k \geq 1$, no matter how large, it holds that $n \log(n)^k \in o(n^{1+\varepsilon})$. Hint[2]

[2]Use a ratio test and L'Hopital's rule. One possibly helpful trick to simplify the work is to make everything to the k power and then use the fact that raising to the k power is continuous to "pull" the limit inside. i.e. $\lim_{x\to\infty}(\cdots)^k = (\lim_{x\to\infty} \cdots)^k$

Proof.

$$\lim_{n\to\infty} \frac{n\log^k(n)}{n^{1+\varepsilon}} = \lim_{n\to\infty} \frac{\log^k(n)}{n^{\varepsilon}} = \lim_{n\to\infty} \left(\frac{\log(n)}{n^{\varepsilon/k}}\right)^k.$$

Now, by continuity of $(\dots)^k$,

$$\lim_{n\to\infty} \left(\frac{\log(n)}{n^{\varepsilon/k}}\right)^k = \left(\lim_{n\to\infty} \frac{\log(n)}{n^{\varepsilon/k}}\right)^k.$$

Applying L'Hopital's rule and noting that $\varepsilon/k > 0$,

$$\left(\frac{k}{\varepsilon\ln(2)} \lim_{n\to\infty} \frac{1}{n^{\varepsilon/k}}\right)^k = \left(\frac{k}{\varepsilon\ln(2)}\right)^k \cdot \left(\lim_{n\to\infty} \frac{1}{n^{\varepsilon/k}}\right)^k = \left(\frac{k}{\varepsilon\ln(2)}\right)^k \cdot 0^k = 0.$$

□

10. Show $n! \in o(n^n)$ Hint[3] and $n! \in \omega(2^n)$.

    Proof. Note that,

    $$0 \le \frac{n!}{n^n} = \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{1}{n} \le \frac{1}{n},$$

    where the second inequality follows from the fact that every term in the product is between 0 and 1. By taking a limit,

    $$0 \le \lim_{n\to\infty} \frac{n!}{n^n} \le \lim_{n\to\infty} \frac{1}{n} = 0.$$

    Thus,

    $$\lim_{n\to\infty} \frac{n!}{n^n} = 0,$$

    and so $n! \in o(n^n)$ by the ratio test.

    Now consider

    $$\frac{n!}{2^n} = \frac{n}{2} \cdot \frac{n-1}{2} \cdots \frac{3}{2} \cdot \frac{2}{2} \cdot \frac{1}{2} \ge \left(\frac{3}{2}\right)^{n-2} \cdot \frac{2}{2} \cdot \frac{1}{2},$$

[3] Use ratio test. If you expand the ratio, you get $\frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{1}{n}$. Note that the whole product is bounded from above by from the smallest term $(1/n)$ since every positive term in the product is at most 1.

and note

$$\lim_{n\to\infty} \frac{n!}{2^n} \geq \frac{1}{2} \lim_{n\to\infty} \left(\frac{3}{2}\right)^{n-2} = \infty,$$

where the first inequality is obtained by replacing all but the last two factors in the product 3/2. Thus $n! \in \omega(2^n)$.

$\square$