Probabilistic Analysis
Exercises

(a) What is the asymptotic worst case running time of each of the following functions? Justify your solution.

(b) What is the asymptotic expected running time of each of the following functions? Justify your solution.

Simplify and express your answer as $\Theta(n^k)$ or $\Theta(n^k(\log n))$ wherever possible. If the asymptotic running time is exponential, then just give exponential lower bounds.

Random(n) generates a random number between 1 and n with uniform distribution (every integer between 1 and n is equally likely.) **CoinFlip**() returns **heads** or **tails** with equal probability.

1.

```
   func1(A, n)
   /* A is an array of integers                                    */
 1 s ← 0;
 2 k ← Random(n);
 3 for i ← 1 to k do
 4 |   j ← 1;
 5 |   while (j < k) do
 6 |   |   s ← s + A[i] * A[j];
 7 |   |   j ← 2 * j;
 8 |   end
 9 end
10 return (s);
```

2.

```
   func2(A, n)
   /* A is an array of integers                                    */
 1 s ← 0;
 2 k ← Random(n);
 3 for i ← 1 to √k do
 4 |   s ← s + A[i] * A[j];
 5 end
 6 return (s);
```

3.

```
   func3(A, n)
   /* A is an array of integers                                    */
 1 s ← 0;
 2 k ← Random(⌊log2(n)⌋);
 3 for i ← 1 to 2^k do
 4 |   for j ← 1 to 2^k do
 5 |   |   s ← s + A[i] * A[j];
 6 |   end
 7 end
 8 return (s);
```

4.

```
   func4(A, n)
   /* A is an array of integers                                    */
 1 s ← A[1];
 2 k ← Random(n);
 3 if (k < log_2(n)) then
 4 |   for i ← 1 to n do
 5 |   |   for j ← 1 to ⌊log_2(n)⌋ do
 6 |   |   |   s ← s + A[i] * A[j];
 7 |   |   end
 8 |   end
 9 else
10 |   s ← s + A[1] * A[n];
11 end
12 return (s);
```

5.

```
   func5(A, n)
   /* A is an array of integers                                    */
 1 s ← A[1];
 2 k ← Random(n);
 3 if (k < √n) then
 4 |   for i ← 1 to n do
 5 |   |   for j ← 1 to n do
 6 |   |   |   s ← s + A[i] * A[j];
 7 |   |   end
 8 |   end
 9 else
10 |   for i ← 1 to n do
11 |   |   s ← s + A[i] * A[n − i + 1];
12 |   end
13 end
14 return (s);
```

6.

```
   func6(A, n)
   /* A is an array of integers                                    */
 1 if (n ≤ 10) then return (A[1]);
 2 for i ← 1 to ⌊√n⌋ do
 3 |   A[i] ← A[i] − A[⌊i * √n⌋];
 4 end
 5 s ← A[1];
 6 k ← Random(n);
 7 if (k < 2n/3) then
 8 |   s ← s + func6(A, n − 5);
 9 end
10 return (s);
```

7.

```
   func7(A, n)
   /* A is an array of integers                                    */
 1 if (n ≤ 10) then return (A[1]);
 2 for i ← 1 to n do
 3 |   for j ← 1 to n do
 4 |   |   A[i] ← A[i] + A[i] * A[j];
 5 |   end
 6 end
 7 s ← A[1];
 8 k1 ← Random(n);
 9 k2 ← Random(n);
10 if (k1 ≤ k2) then
11 |   s ← s + func7(A, n − 3);
12 end
13 return (s);
```

8.

```
   func8(A, n)
   /* A is an array of integers                                    */
 1 if (n ≤ 10) then return (A[1]);
 2 c1 ← CoinFlip();
 3 c2 ← CoinFlip();
 4 s ← A[k];
 5 if (c1 = c2) then
 6 |   s ← s + func8(A, n − 4) + func8(A, n − 7);
 7 end
 8 return (s);
```

9.

```
   func9(A, n)
   /* A is an array of integers                                    */
 1 if (n ≤ 10) then return (A[1]);
 2 s ← A[1];
 3 for i ← 1 to ⌊n/2⌋ do
 4 |   for j ← 1 to ⌊n/2⌋ do
 5 |   |   s ← s + A[2 * i] * A[i + j];
 6 |   end
 7 end
 8 c1 ← CoinFlip();
 9 if c1 = heads then s ← s + func9(A, n − 2);
10 c2 ← CoinFlip();
11 if (c1 = c2) then s ← s + func9(A, n − 9);
12 return (s);
```

10.

```
   func10(A, n)
   /* A is an array of integers                                    */
 1 if (n ≤ 10) then return (A[1]);
 2 s ← 0;
 3 for i ← 1 to n do
 4 |   A[i] ← A[i] + A[n − i + 1];
 5 |   s ← s + A[i];
 6 end
 7 for i ← 1 to 4 do
 8 |   c ← CoinFlip();
 9 |   if (c = heads) then
10 |   |   s ← s + func10(A, ⌊n/4⌋);
11 |   end
12 end
13 return (s);
```

11.

```
   func11(A, n)
   /* A is an array of integers                                    */
 1 if (n ≤ 10) then return (A[1]);
 2 s ← A[n];
 3 for i ← 1 to n do
 4 |   A[i] ← A[i] + A[n − i + 1];
 5 end
 6 k1 ← Random(n);
 7 if (k1 ≤ n/3) then s ← s + func11(A, ⌊n/2⌋);
 8 k2 ← Random(n);
 9 if (k2 ≤ 2n/3) then s ← s + func11(A, ⌊n/2⌋);
10 return (s);
```

12.

```
   func12(A, n)
   /* A is an array of integers                                    */
 1 if (n ≤ 20) then return (A[1]);
 2 for i ← 2 to n − 3 do
 3 |   A[i]s ← A[i] − A[i + 3];
 4 end
 5 s ← A[1];
 6 k1 ← Random(n);
 7 if (k1 ≤ n/2) then s ← s + func12(A, n − 5);
 8 k2 ← Random(n);
 9 if (k2 ≤ n/3) then s ← s + func12(A, n − 7);
10 k3 ← Random(n);
11 if (k3 ≤ n/6) then s ← s + func12(A, n − 11);
12 return (s);
```

13.

```
   func13(A, n)
   /* A is an array of integers                                    */
 1 if (n ≤ 20) then return (A[1]);
 2 for i ← 2 to n − 3 do
 3 |   A[i] ← A[i] + A[i] − A[i + 3];
 4 end
 5 s ← A[1];
 6 c1 ← CoinFlip();
 7 if (c1 = heads) then s ← s + func13(A, n − 4);
 8 c2 ← CoinFlip();
 9 if (c2 = heads) then s ← s + func13(A, n − 6);
10 c3 ← CoinFlip();
11 if (c3 = heads) then s ← s + func13(A, n − 10);
12 return (s);
```