# CSE 2231: Notes on Asymptotic Analysis 1

**$O$-notation:**
O-notation is used to indicate that a function $g(n)$ is an asymptotic upper bound of another function $f(n)$. This is written $f(n) = O(g(n))$ and is read as "$f(n)$ is big-oh of $g(n)$". Informally, $O$ acts like a a $\leq$ sign. That is, $f(n) = O(g(n))$ is analogous to $f(n) \leq g(n)$.

Here is the precise definition, which you should memorize:
$f(n) = O(g(n))$ means that that there exist positive constants $c$ and $n_0$ such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$.

Note: Many books write $f(n) \in O(g(n))$ instead of $f(n) = O(g(n))$. That is, they use the set inclusion symbol rather than an equals sign.

**Examples:**

1. Show that $3n^2 = O(n^2)$.

   In this case, $f(n) = 3n^2$ and $g(n) = n^2$. We need to find positive constants $c$ and $n_0$ such that $3n^2 \leq cn^2$ for all $n \leq n_0$.

   Let $c = 3$ and $n_0 = 1$. Then clearly $3x^2 \leq 3x^2$ for all $n \geq 1$. Therefore $3n^2 \in O(n^2)$.

   Note that $c$ could be any number greater than or equal to 3 and $n_0$ could be any number greater than or equal to 1.

2. Show that $n^2 + n = O(n^2)$.

   In this case, $f(n) = n^2 + n$ and $g(n) = n^2$.

   Since $n \leq n^2$ for all $n \geq 1$, it follows that $n^2 + n \leq n^2 + n^2 = 2n^2$, for all $n \geq 1$.

   Therefore, $n^2 + n = O(n^2)$ [with $c = 2$ and $n_0 = 1$]

3. Show that $n^3 + 2n + 5 = O(n^3)$.

   For $n \geq 1$, $n^3 + 2n + 5 \leq n^3 + 2n^3 + 5n^3 = 8n^3$.

   Therefore, $n^3 + 2n + 5 = O(n^3)$ [with $c = 8$ and $n_0 = 1$]
   .

4. Show that $\sqrt{6n^4 + 2n^3 + 5} = O(n^2)$.

   For $n \geq 1, \sqrt{6n^4 + 2n^3 + 5} \leq \sqrt{6n^4 + 2n^4 + 5n^4} = \sqrt{13n^4} = \sqrt{13}\sqrt{n^4} = \sqrt{13}n^2$

   So $\sqrt{6n^4 + 2n^3 + 5} \leq \sqrt{13}n^2$

   Therefore, $\sqrt{6n^4 + 2n^3 + 5} = O(n^2)$ $[c = \sqrt{13};\ n_0 = 1]$

5. Show that $n^3 \neq O(n^2)$.

   Proof by contradiction: Assume to the contrary that $n^3 = O(n^2)$. By definition, there then exist positive constants $c$ and $n_0$ such that $n^3 \leq cn^2$ for all $n \geq n_0$. If we divide both sides of the inequality by $n^2$, this implies that $n \leq c$ for all $n \geq n_0$. This means though that $n$, which can be any number, is bounded above by the constant $c$. In other words, there is a constant $c$ that is greater than any other number! Clearly this is impossible. Our assumption that $n^3 = O(n^2)$ must be incorrect, so $n^3 \neq O(n^2)$.

6. Show that $log_2(n^2) = O(log_2(n))$.

   Recall that $log_c(a^n) = nlog_c(a)$. Therefore, $log_2(n^2) = 2log_2(n)$. So we need to show that $2log_2(n) = O(log_2(n))$. This is clearly true, since $2log_2(n) \leq 2log_2(n)$ for all $n \geq 1$. $[c = 2;$ $n_0 = 1]$

7. Show that $3^n \neq O(2^n)$.

   Assume to the contrary that $3^n = O(2^n)$. Then, by definition, there exist positive constants $c$ and $n_0$ such that $3^n \leq c2^n$, for all $n \geq n_0$. If we divide both sides of the inequality by $2^n$ though, we get $\frac{3^n}{2^n} \leq c$. This is equivalent to $(\frac{3}{2})^n \leq c$. This is not possible though, since $lim_{n\to\infty}(\frac{3}{2})^n = \infty$, and can't be bounded above by a constant $c$. Thus, our assumption is incorrect, and $3^n \neq O(2^n)$.

8. Show that $log_2(n) = O(log_3(n))$.

   Recall the change of base formula:

   $$log_b(a) = \frac{log_c(a)}{log_c(b)}$$

   Therefore,

   $$log_3(n) = \frac{log_2(n)}{log_2(3)} = \frac{1}{log_2(3)} \times log_2(n)$$

So it suffices to show that there exist positive constants $c$ and $n_0$ such that:

$$log_2(n) \leq c \times \frac{1}{log_2(3)} \times log_2(n), \forall n \geq n_0.$$

Clearly this is true for $c = log_2(3)$ and $n_0 = 1$.

# CSE 2231: Notes on Asymptotic Analysis 2

In part 1 of Notes on Asymptotic Analysis, we covered the definition of Big-O:
$f(n) = O(g(n))$ means that that there exist positive constants $c$ and $n_0$ such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$.

Here are some other definitions that you should memorize:

1. $f(n) = \Omega(g(n))$ means that that there exist positive constants $c$ and $n_0$ such that $0 \leq cg(n) \leq f(n)$ for all $n \geq n_0$.

2. $f(n) = \Theta(g(n))$ means that that there exist positive constants $c_1, c_2$ and $n_0$ such that $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0$.

Here are some theorems that you should know. These are easy to memorize if you keep in mind that $\Theta$ is analogous to $=$, $O$ is analogous to $\leq$ and $\Omega$ is analogous to $\geq$.

We will prove the first theorem. The proofs of the others are similar.

**Theorem:** $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $g(n) = O(f(n))$.

Proof:
($\Rightarrow$) If $f(n) = \Theta(g(n))$, then, by definition, there exist positive constants $c_1, c_2$ and $n_0$ such that $0 \leq c_1 f(n) \leq g(n) \leq c_2 f(n)$ for all $n \geq n_0$. Since there exists a constant $c_2$ such that $0 \leq g(n) \leq c_2 f(n)$ for all $n \geq n_0$, $g(n) = O(f(n))$. Likewise, $f(n) \leq \frac{1}{c_1} g(n)$ for all $n \geq n_0$, so $f(n) = O(g(n))$.

($\Leftarrow$) If $f(n) = O(g(n))$ and $g(n) = O(f(n))$. Then, by definition:

1. there exist positive constants $c_0$ and $n_0$ such that $f(n) \leq c_0 g(n)$ for all $n \geq n_0$, and

2. there exist positive constants $c_1$and $n_1$ such that $g(n) \leq c_1 f(n)$ for all $n \geq n_1$.

Letting $n_2$ be the maximum of $n_0$ and $n_1$, it follows from the above that $\frac{1}{c_1} g(n) \leq f(n) \leq c_0 g(n)$ for all $n \geq n_2$, so $f(n) = \Theta(g(n))$.

**Theorem:** $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$.

**Theorem:** If $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) = O(h(n))$.

**Theorem:** If $f(n) = \Omega(g(n))$ and $g(n) = \Omega(h(n))$, then $f(n) = \Omega(h(n))$.

# CSE 2321: Notes on Asymptotic Analysis 3

In the previous notes, we have worked directly with the definitions of $O$, $\Omega$, and $\Theta$. In these notes, we'll look at some theorems that are often easier to apply.

Let $f(n)$ and $g(n)$ be two positive, monotonically increasing functions on the natural numbers. Then:

$$\text{If } \lim_{n\to\infty} \frac{f(n)}{g(n)} = \text{ a positive constant, then } f(n) = \Theta(g(n)).$$

$$\text{If } \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0, \text{ then } f(n) = O(g(n)) \text{ but } f(n) \neq \Theta(g(n)).$$

$$\text{If } \lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty, \text{ then } f(n) = \Omega(g(n)) \text{ but } f(n) \neq \Theta(g(n)).$$

**Examples:**
(1) Let $f(n) = x^2 - 2x$ and $g(n) = 3x^2$. Then:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{x^2 - 2x}{3x^2} = \lim_{n\to\infty} \frac{\frac{x^2}{x^2} - \frac{2x}{x^2}}{\frac{3x^2}{x^2}} = \lim_{n\to\infty} \frac{1 - \frac{2}{x}}{3} = \frac{1}{3}$$

Since $\lim_{n\to\infty} \frac{f(n)}{g(n)} =$ a positive constant, $x^2 - 2x = \Theta(3x^2)$.

(2) Let $f(n) = x^2$ and $g(n) = 2^x$. Then:

$$\lim_{n\to\infty} \frac{x^2}{2^x} = \lim_{n\to\infty} \frac{2x}{ln2 \times 2^x} = \lim_{n\to\infty} \frac{2}{lin2 \times ln2 \times 2^x} = 0 \text{ [Note: we are using L'Hospital's rule]}$$

Since $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, $x^2 = O(2^x)$, but $x^2 \neq \Theta(2^x)$.

(3) Let $f(n) = 3^n$ and $g(n) = 2^n$. Then:

$$\lim_{n\to\infty} \frac{3^n}{2^n} = \lim_{n\to\infty} \left(\frac{3}{2}\right)^n = \infty$$

Since $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$, $3^n = \Omega(2^n)$, but $3^n \neq \Theta(2^n)$.

(4) Let $f(n) = log_2(n)$ and $g(n) = n^2$. Then:

$$\lim_{n\to\infty} \frac{log_2(n)}{n^2} = \lim_{n\to\infty} \frac{\frac{1}{nln2}}{2n} = \lim_{n\to\infty} \frac{1}{2n^2 ln2} = 0$$

Note that we use L'Hospital's Rule in the second step.

Since $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, $log_2(n) = O(n^2)$, but $log_2(n) \neq \Theta(n^2)$.