Lecture 12: Cardinality of Sets

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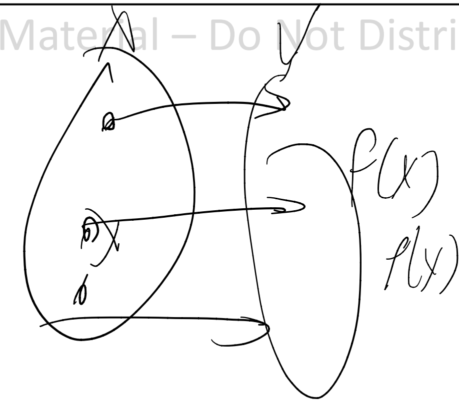
Outline

- Lectures 11 review
- Cardinality
- Countable sets
- Uncountable sets

- A repeating note: **make sure you read the textbook**

L11: Functions

- A function is specified as



$$f: X \to Y, x \mapsto f(x)$$

  - For each $x \in X$, there must be exactly one $f(x) \in Y$
  - X: **domain**, Y: **co-domain**
  - $f(a) = b$, b is the **image** of a, a is the **preimage** of b
  - **Ex**: $\lfloor x \rfloor$, $\lceil x \rceil$, $\sin x$, …

- Functions can be added and multiplied
  - The domain must be the same and the co-domain must be compatible
  - **Ex**: $f_1(x) = e^x$, $f_2(x) = x$, $f_3 = 1$

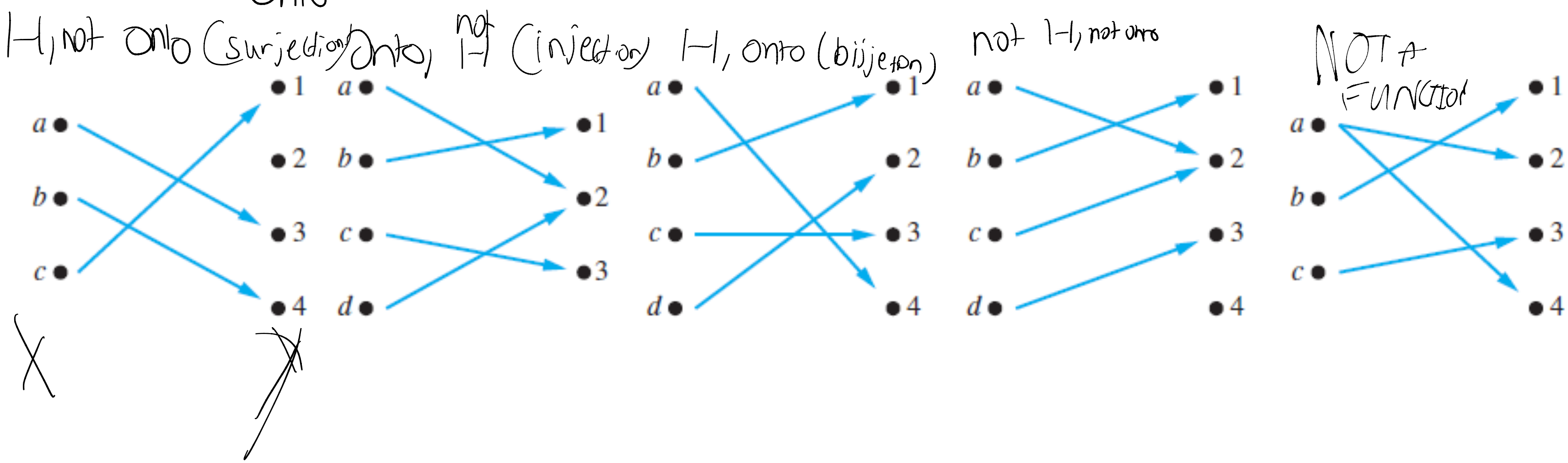
$f(x) = (f_1 + f_2 + f_3)(x) = e^x + x + 1$

L11: Injection, Surjection, Bijection

- **Def^n**: A function f is **one-to-one** (or an **injection**) iff $f(x_1) \neq f(x_2)$ for $x_1 \neq x_2$. f is **onto** (or a **surjection**) iff $f(X) = Y$. f is a **bijection** iff it is **one-to-one** and **onto** (**injection** and **surjection**).

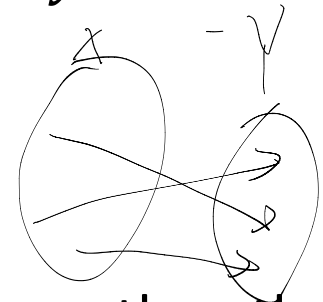
f is bijection
- one to one
- onto

$\forall y \in Y$
$\exists x\ f(x) = y$



L11: Inverse & Function Composition

- **Def^{n}**: A bijective function f has an **inverse** function, often denoted as f^{-1}, such that $f^{-1}(y) = x$ if $f(x) = y$.
  - Domains/co-domain flipped
  - Only bijective functions have inverses.



- **Def^{n}**: $g : X \to Y$ and $f : Y \to Z$ are two functions, then the **composition** of f and g, denoted $f \circ g$, is the function $f \circ g: X \to Z, x \mapsto f(g(x))$.
  - For invertible function, we always have $f^{-1}(f(x)) = x$.
  - In general, $f \circ g \neq g \circ f$

$f \circ f^{-1} = Y \to Y$

$Y \to Y$

$f^{-1} \circ f = X \to X$

$X \to f^{-1}(f(x))$

$\Downarrow$

X

$f: X \to Y,\ x \to f(x),\ f^{-1} = Y \to X,\ f(x) \to x$

$Y \to f^{-1}(x)$

Cardinality

same size

- **Def^{n}**: The sets X and Y have the same **cardinality**, denoted as $|X| = |Y|$, if there is a bijection between X and Y.

- **Ex**: $X = \{apple, orange, pecan\}$, $Y = \{1, 2, 3\}$

$f(apple) = 1$
$f(orange) = 2$
$f(pecan) = 3$

$f_1(a) = 1,\ f_1(o) = 1,\ f_1(p) = 1$
$f_2(a) = 3,\ f_2(o) = 1,\ f_2(p) = 2$ $\quad |X| = |Y|$

- For finite X and Y, $|X| = |Y| \Leftrightarrow \exists f: X \to Y$ that is a bijection

Cardinality, Cont.

{0, 1, 2, ... } {1, 2, 3, ... } Nonzero

- **Ex:** $X = \mathbb{N}, Y = \mathbb{N}^{+} = \mathbb{N} - \{0\}. |X| = |Y|?$

f is a bijection

X: 0 1 2 3 ......
Y: 1 2 3 4 ......

$|X| = |Y|$

$f: X \to Y,\ x \mapsto x+1$

10 members 26 members

- **Ex:** $X = \{0, \ldots, 9\}, Y = \{a, \ldots, z\}$

Can't do a bijection

So $|X| \neq |Y|$

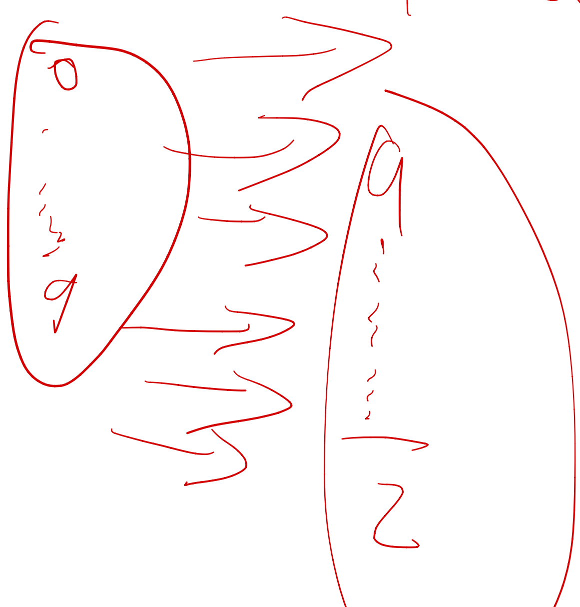
Cardinality, Cont.

- **Def^n**: If there is an injection from X to Y , then the cardinality of X is no more than that of Y, denoted as $|X| \leq |Y|$. If $|X| \leq |Y|$ and $|X| \neq |Y|$, then $|X| < |Y|$.
- **Ex**: $X = \{0, \dots, 9\}, Y = \{a, \dots, z\}$

$|X| < |Y|$

$|X| \leq |Y|$

f is an injection



Countable Sets

- **Def^n**: The **cardinality of the set of positive integers** is denoted as $\aleph_0$ (aleph-zero), that is, $|\mathbb{N}^+| = \aleph_0$.
- **Def^n**: A set X is **countably infinite** if it has the same cardinality as the set of positive integers. In this case, $|X| = \aleph_0$. A set is **countable** if it is either finite or countably infinite.
- **Ex**: Show that the set of positive odd integers is countable.

$|N^+ = 1 \quad 2 \quad 3 \quad 4 \quad 5$

$O = 1 \quad 3 \quad 5 \quad 7 \quad 9$

$|N^+| = |O|$

$1 \quad 2-1=1$

$2 \quad 2(2)-1=3$

$3 \quad 2(3)-1=5$

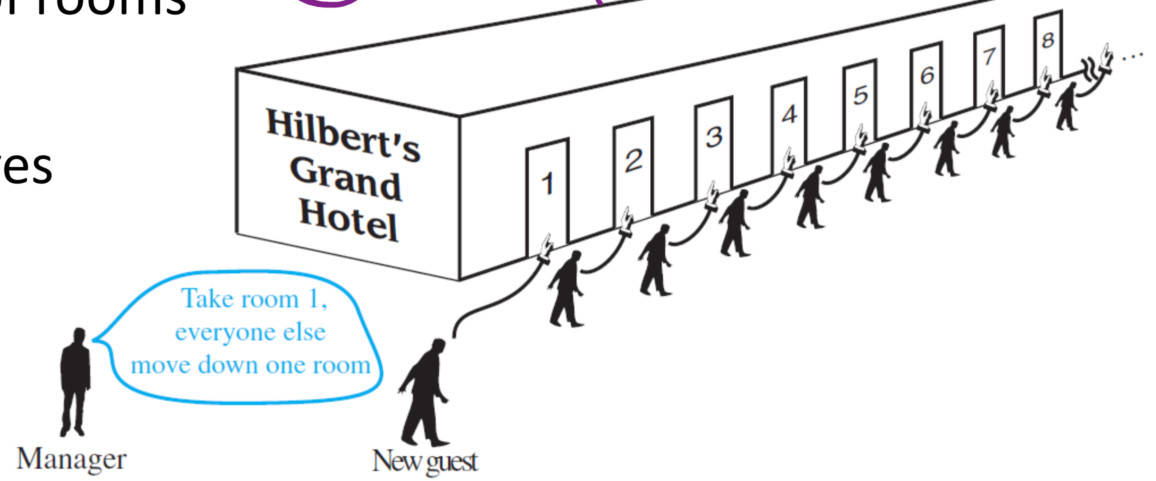
$f: |N^+ \to O, \quad x \to 2x-1$

O is countably infinite, so countable.

Hilbert’s Grand Hotel

- Hilbert’s Grand Hotel
  - A hotel with infinite # of rooms
  - Filled to capacity
  - Then, a new guest arrives
  - What now?

Countably infinite



- Important: this does not cause problems in reality
  - Physical entities are finite
  - The number of atoms in the observable universe is $\sim 10^{80}$

Some Countably Infinite Sets

- **Ex:** $\mathbb{Z}$ is countable.

Need to show $|\mathbb{Z}| = |\mathbb{N}^+|$

$\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2 \ldots\}$

$n \in \mathbb{Z}^+$

Need to provide f:

$\mathbb{N}^+ = \{1, 2, 3, \ldots\}$

$2n \to n$

$2n+1 \to -n$

$x \in \mathbb{N}^+$ $\quad f(x) = \lfloor \frac{x}{2} \rfloor \cdot (-1)^x$

$x = 4$, $f(4) = \lfloor \frac{4}{2} \rfloor \cdot (-1)^4 = 2 \cdot 1 = 2$

$4 \to 2$
$5 \to -2$
$6 \to 3$
$7 \to -3$

f is 1–1 and onto, so f is a bijection

$x = 1$, $f(1) = \lfloor \frac{1}{2} \rfloor \cdot (-1)^1 = (0)(-1) = 0$

$x = 2$, $f(2) = \lfloor 1 \rfloor \cdot (-1)^2 = 1 \cdot 1 = 1$

$x = 3$, $f(3) = \lfloor \frac{3}{2} \rfloor \cdot (-1)^3 = 1 \cdot (-1) = -1$

Some Countably Infinite Sets, Cont.

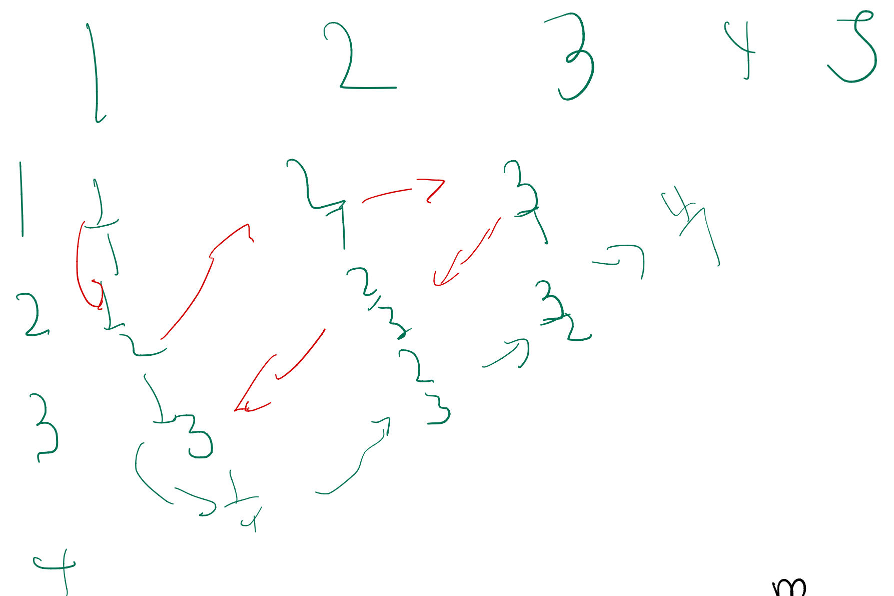
- **Ex**: $\mathbb{Q}^+$ is countable.

$(0,1)$

f is a bijec, $|\mathbb{Q}^+| = |\mathbb{N}|$

$\forall r \in \mathbb{Q}^+$, r can be represented as (m,n)

$r = \frac{m}{n}$



This show f is 1-1
for any $x \in \mathbb{N}^+$
$f(x)$ will appear
f is 1-1 by constru

$f(1) = (1,1)$
$f(2) = (1,2)$
$f(3) = (2,\ )$
$f(4) = (3,\ )$

$\frac{m}{n}$

f is also onto

- **Corollary**: $\mathbb{Q}$ is countable.

Uncountable Sets

there is no bijection $f: \mathbb{N}^+ \to x$

- **Def^n**: A set X is **uncountable** if it is not countable.
- **Ex**: The real numbers in [0,1] is uncountable. no $f: \mathbb{N}^+ \to [0,1]$

$f(1)$ 0 . a_1 b_1 c_1 ...

2 . a_2 b_2 c_2

3 . a_3 b_3 c_3

$f(n)$

$\bar{a} \neq a_1$, $\bar{a} \neq a$

$\bar{b} \neq b_2$, $b \neq 9$

$0.\bar{a}\bar{b}\bar{c}\bar{d}\ldots$

$x \in [0,1]$
there is no $n \in \mathbb{N}$
so $f(n) = y$
there are bijection
f done

nth digit $\neq$ nth digit of $f(n)$

Schroder-Bernstein Theorem

- **Theorem**: $(|X| \leq |Y|) \wedge (|Y| \leq |X|) \leftrightarrow |X| = |Y|$.
  - That is, if we can find $f: X \to Y$ and $g: Y \to X$ that are both injections, or 1-1, then $|X| = |Y|$
  - **Ex**: Prove $|(0,1)| = |(0,1]|$

$$f: \overset{X}{(0,1)} \to \overset{Y}{[0,1]}, \quad x \mapsto x$$

$$g: \underset{Y}{[0,1]} \to \underset{X}{(0,1)}, \quad x \to \frac{x}{2}$$

Computable and Uncomputable Functions

- **Def^n**: A function f is computable if all its values $f(n), n \in \mathbb{N}^+$, can be computed by some computer program in some programming language. A function that is not computable is then uncomputable.
- **Theorem**. There exists an uncomputable function.

	1	2	3	4	5	...
f_1	Y	N	N	Y	Y	...
f_2	N	N	N	N	Y	
f_3	N	Y	Y	Y	N	

$f(1) = \neg f_1(1)$

$f(2) = \neg f_2(2)$

$f(3) = \neg f_3(3)$

f cannot be one of the computable functions