Lecture 17: Relations

Jingjin Yu | Computer Science @ Rutgers

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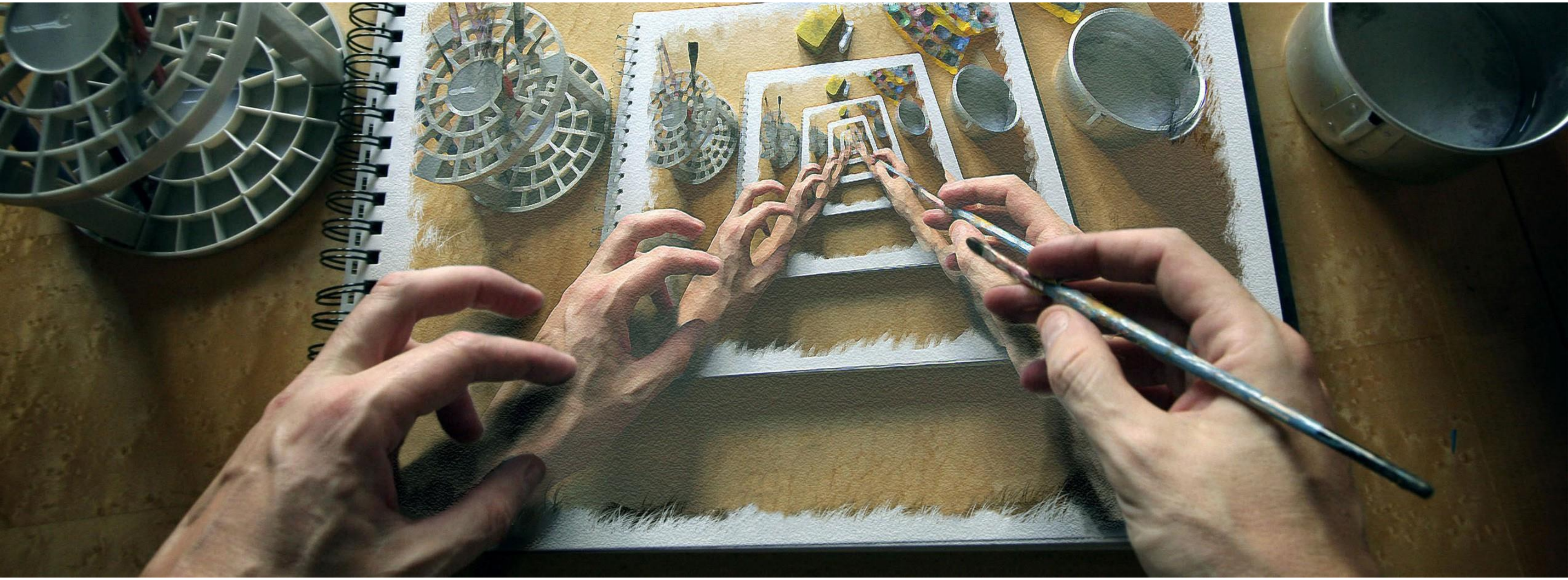
Outline

- Lecture 16 review
- Relations
  - Examples & overview
  - Definition of binary relation
  - Relations on a set
    - Reflexivity
    - Symmetry and antisymmetry
    - Transitivity
    - One more example

- A repeating note: **make sure you read the textbook**

L16: Recursive Definitions: Intuition



Picture source: https://engineering.tripping.com/building-an-recursive-nested-dropdown-component-in-react-b1c883e06ac4

- Sometimes, it is difficult to define certain things explicitly
- A better way may be recursive or inductive:
  - Define a "base case" (iteration 1)
  - Define iteration $k+1$ based on iteration k

L16: Recursive Definitions

- Specifying a recursive definition $f(n), n \geq 0$
  - (1) Basis step: specify $f(0)$
  - (2) Specify $f(n+1)$ in terms of $f(n)$

- **Ex**: $f(0) = 1, f(n+1) = 2f(n)$

- Variation:
  - (1) Basis step: specify $f(0), f(1), \dots, f(k)$ for some $k \geq 1$
  - (2) Specify $f(n+1)$ in terms of $f(n-k), \dots, f(n)$

- **Ex**: $f(0) = 0, f(1) = 1, f(n+1) = f(n) + f(n-1)$.
  - This is the famous Fibonacci sequence

Relations: Example and Overview

- Relations are everywhere

TABLE 2 GPAs.	
Student_name	*GPA*
Ackermann	3.88
Adams	3.45
Chou	3.49
Goodfriend	3.45
Rao	3.90
Stevens	2.99

TABLE 4 Majors.	
Student	*Major*
Glauser	Biology
Marcus	Mathematics
Miller	Computer Science

Binary relation

n-ary relation

TABLE 7 Teaching_schedule.				
Professor	*Department*	*Course_number*	*Room*	*Time*
Cruz	Zoology	335	A100	9:00 A.M.
Cruz	Zoology	412	A100	8:00 A.M.
Farber	Psychology	501	A100	3:00 P.M.
Farber	Psychology	617	A110	11:00 A.M.
Grammer	Physics	544	B505	4:00 P.M.
Rosen	Computer Science	518	N521	2:00 P.M.
Rosen	Mathematics	575	N502	3:00 P.M.

TABLE 1 Students.			
Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

TABLE 8 Flights.				
Airline	*Flight_number*	*Gate*	*Destination*	*Departure_time*
Nadir	122	34	Detroit	08:10
Acme	221	22	Denver	08:17
Acme	122	33	Anchorage	08:22
Acme	323	34	Honolulu	08:30
Nadir	199	13	Detroit	08:47
Acme	222	22	Denver	09:10
Nadir	322	34	Detroit	09:44

Relations: Example and Overview

- Equivalence classes
- Types of relations on a set
  - Transitivity
  - Symmetric
  - Anti-symmetric
  - Reflexive

$(a,b) \wedge (b,c) \rightarrow (a,c)$

$(a,b) \leftrightarrow (b,a)$

$(a,b) \wedge (b,a) \rightarrow a=b$

A: $a \in A$ (a,a)

Binary Relation

- Some examples of binary relations
  - A person and their address (p_1, a)
  - A person and their phone number (p, ph)
  - Integers a and b, whether $a \mid b$ $\quad a \nmid b \quad b = a \cdot r \quad |r| \geq 1$
  - A function, x and $f(x)$ $\quad (x, f(x))$

- **Def^n**: Let A and B be sets. A **binary relation** from A to B is a subset of $A \times B$. $= \{(a,b) \mid a \in A, b \in B\}$

- Notation: Notation: $R \subseteq A \times B$ is a set of relations. E.g. aRb means $(a, b) \in R$. This reads a is related to b by R. $a \not R\, b$ means $(a, b) \notin R$. Sometimes relation is also written as $a \sim b$. $\quad (a,b) \notin R$ ✓

Binary Relation: Formal Examples

Induction memo

- **Ex**: A: students, B: courses

$R \subseteq A \times B$

(Stuname, CS200)

(s_1, c_1)
(s_1, c_2)
(s_1, c_3)
(s_1, c_4)

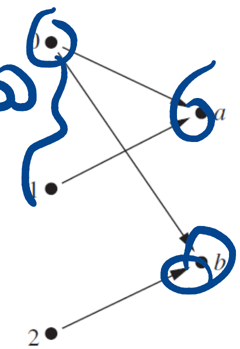
- **Ex**: A: US cities, B: US states

$R \subseteq A \times B$

$(a, b) \in R$ a is a city in state B

- **Ex**: $A = \{0;\ 1;\ 2\}; B = \{a; b\}, R = \{(0, a), (0, b), (1, a), (2, b)\}$.

$R \subseteq A \times B$ $A \times B = \{(0,a), (0,b), (1,b),$
$(2,a), (2,b)\}$



R	a	b
0	×	×
1	×	
2		×

Binary Rations on a Set

- **Def^n**: A (binary) relation R on a set A is a relation from $A \rightarrow A$.
- **Ex**: A: students, R: “classmates”

- If $|A| = n$, how many possible relations R?
  - Total # pairs: n^2
    - E.g., $A = \{1, 2\}$, pairs $P = \{(1,1), (1,2), (2,1), (2,2)\}$ 2^2
  - Any subset of P defines a relation, so total of 2^{n^2}

$A = \{1, 2, 3\}$, $P = \{(1,1), (1,2), (1,3), (2,1)$ ... 3 3^2

- Types of binary relations on a set
  - Reflexive
  - Symmetric, antisymmetric
  - Transitive

Reflexivity

- **Def^n**: A relation R on A is reflexive if $(a, a) \in R$ for all $a \in A$
- **Ex**: A: cities, R: "road between"

- **Ex**: $A = \{1\}, R = \{(1,1)\}$. Reflexive?
- **Ex**: $A = \{1,2\}, R = \{(1,1)\}$. Reflexive?

Consider the following relations on $\{1, 2, 3, 4\}$: $\{(1,1), (2,2), (3,3), (4,4)\}$

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$, ✗

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$, ✗

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$, ✓

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$, ✗

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$, ✓

$R_6 = \{(3, 4)\}$. ✗

Which of these relations are reflexive?

Reflexivity, Cont.

Consider these relations on the set of integers:

$R_1 = \{(a, b) \mid a \le b\}$,	Yes. Because $a \le a$ **always** holds!
$R_2 = \{(a, b) \mid a > b\}$,	No. Because $a > a$ does not hold, $(a, a) \notin R$
$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$,	Yes. Because $a = a$ **always** holds!
$R_4 = \{(a, b) \mid a = b\}$,	Yes. Because $a = a$ **always** holds!
$R_5 = \{(a, b) \mid a = b + 1\}$,	No. Because $a = a + 1$ does not hold
$R_6 = \{(a, b) \mid a + b \le 3\}$.	No. Because $a + a \le 3$ does not always hold

Which of these relations are reflexive?

To check, we replace b with a and see whether $(a, a) \in R$ is true

Symmetry and Antisymmetry

- $\mathbf{Def}^{n}$: A relation R on A is symmetric if $(a, b) \in R \leftrightarrow (b, a) \in R$ for all $a, b \in A$
  - Equivalently, $R\ symmetric \Leftrightarrow \forall a \forall b((a, b) \in R \leftrightarrow (b, a) \in R)$.
- $\mathbf{Def}^{n}$: A relation R on A is antisymmetric if $(a, b) \in R$ and $(b, a) \in R$ implies $a = b$ for $a, b \in A$
  - Equivalently, $R\ antisymmetric \Leftrightarrow \forall a \forall b((((a, b) \in R) \wedge ((b, a) \in R)) \rightarrow (a = b))$
  - Same as $\forall a \forall b\big(\big((a \neq b) \wedge (a, b) \in R\big) \rightarrow (b, a) \notin R\big)$.
- A set can be both!
  - $\mathbf{Ex}$: $A = \{1,2\}, R = \{(1,1)\}$. Symmetric? Antisymmetric?

Symmetry and Antisymmetry, Cont.

Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$
$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$
$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$
$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$
$$R_6 = \{(3, 4)\}.$$

Which of the relations are symmetric and which are antisymmetric?

To check, you look at every element and see whether the flipped one is also in there.

R_1: No. $(3,4) \in R_1$ but $(4,3) \notin R_1$
R_2: Yes.
R_3: Yes.
R_4: No. $(2,1) \in R_4$ but $(1,2) \notin R_4$
R_5: No. $(1,2) \in R_5$ but $(2,1) \notin R_5$
R_6: No. $(3,4) \in R_6$ but $(4,3) \notin R_6$

Symmetry and Antisymmetry, Cont.

Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$
$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$
$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$
$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$
$$R_6 = \{(3, 4)\}.$$

Which of the relations are symmetric and which are antisymmetric?

To check, if $a \neq b$, then we cannot have both (a, b) and (b, a)

R_1: No. Has both (1,2) and (2,1)
R_2: No. Has both (1,2) and (2,1)
R_3: No. Has both (1,2) and (2,1)
R_4: Yes. In this case, for all (a, b) pairs where $a \neq b$, $a > b$
R_5: Yes. In this case, for all (a, b) pairs where $a \neq b$, $a < b$
R_6: Yes.

Symmetry and Antisymmetry, Cont.

Consider these relations on the set of integers:

$$R_1 = \{(a, b) \mid a \leq b\},$$
$$R_2 = \{(a, b) \mid a > b\},$$
$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\},$$
$$R_4 = \{(a, b) \mid a = b\},$$
$$R_5 = \{(a, b) \mid a = b + 1\},$$
$$R_6 = \{(a, b) \mid a + b \leq 3\}.$$

Which of the relations are symmetric and which are antisymmetric?

To check, see whether the condition holds when a, b are flipped

R_1: No. $a \leq b$ does not imply $b \leq a$
R_2: No. $a > b$ does not imply $b > a$
R_3: Yes.
R_4: Yes.
R_5: No. $a = b + 1$ does not imply $b = a + 1$
R_6: Yes. $a + b \leq 3 \leftrightarrow b + a \leq 3$

Symmetry and Antisymmetry, Cont.

Consider these relations on the set of integers:

$R_1 = \{(a, b) \mid a \leq b\}$,

$R_2 = \{(a, b) \mid a > b\}$,

$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a, b) \mid a = b\}$,

$R_5 = \{(a, b) \mid a = b + 1\}$,

$R_6 = \{(a, b) \mid a + b \leq 3\}$.

Which of the relations are symmetric and which are antisymmetric?

To check, if $a \neq b$, then we cannot have both (a, b) and (b, a)

R_1: Yes. When $a \neq b$, $a < b$
R_2: Yes. When $a \neq b, a > b$
R_3: No. When $a \neq b$, it's possible that $a = -b$, e.g., $(1, -1) \in R_3$
R_4: Yes. $a \neq b$ is false, so it trivially holds
R_5: Yes. When $a \neq b$, $a = b + 1 > b$
R_6: No. E.g., $(1,2) \in R_6$ and $(2,1) \in R_6$

Transitivity

- **Def^{n}**: A relation R on A is transitive if $(a, b) \in R$ and $(b, c) \in R$ implies that $(a, c) \in R$
  - Equivalently, R *transitive* $\Leftrightarrow \forall a, b, c(((a, b) \in R \wedge (b, c) \in R) \rightarrow (a, c) \in R)$.

- **Ex**: $A = \{1,2\}, R = \{(1,1), (1,2), (2,1)\}$. Transitive?

$(1,1) \wedge (1,2)$

$(1,1)$

$(1,2)$

$(1,2) \wedge (2,1)$

$(1,1)$

Transitivity, Cont.

Consider the following relations on $\{1, 2, 3, 4\}$:

NO

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

Yes

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$R_6 = \{(3, 4)\}$.

Which of these relations are transitive?

To check, for each $(a, b) \in R \wedge (b, c) \in R$, check that $(a, c) \in R$

a b b c

R_1: No. $(3,4) \in R$, $(4,1) \in R$, but $(3,1) \notin R$

R_2: No. $(2,1) \in R$, $(1,2) \in R$, but $(2,2) \notin R$

R_3: No. $(2,1) \in R$, $(1,4) \in R$, but $(2,4) \notin R$

R_4: Yes. All decreasing pairs, i.e., $a > b$

R_5: Yes. All non-decreasing pairs, i.e., $a \leq b$

R_6: Yes. Trivially holds

Transitivity, Cont.

Consider these relations on the set of integers:

$$R_1 = \{(a, b) \mid a \le b\},$$
$$R_2 = \{(a, b) \mid a > b\},$$
$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\},$$
$$R_4 = \{(a, b) \mid a = b\},$$
$$R_5 = \{(a, b) \mid a = b + 1\},$$
$$R_6 = \{(a, b) \mid a + b \le 3\}.$$

Which of these relations are transitive?

Check directly whether the relation is transitive

R_1: Yes. $a \le b \wedge b \le c \to a \le c$
R_2: Yes. $a > b \wedge b > c \to a > c$
R_3: Yes. 4 possibilities, $a = b \wedge b = c \to a = c$, $a = b \wedge b = -c \to a = -c$, …
R_4: Yes. $a = b \wedge b = c \to a = c$
R_5: No. Counterexample: $a = 2, b = 1, c = 0$. $a = b + 1, b = c + 1, a \ne c + 1$
R_6: No. $a = c = 3, b = 0$. $a + b \le 3, b + c \le 3, a + c > 3$

One More Example

- Is the "divides" relation reflexive, symmetric, antisymmetric, transtive on positive integers?

✓ Reflexive $(a,a) \in R$ for all $a \in A$ $a|a$

× symmetric $a|b \to b|a$

✓ antisymmetric $(a,b) \wedge (b,a) \to a = b$

$(a|b) \wedge (b|a) \Rightarrow a = b$

$a \le b \quad b \le a$

✓ transitive $(a|b) \wedge (b|c) \to a|c$

- Note: what if we use integers instead of positive integers?