Lecture 11: Functions

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Outline

- Lectures 9-10 review
- Functions
  - Notation, domain, co-domain, image/preimage
  - 1-1 (injective), onto (surjective), bijective functions
  - Inverse functions
  - Compositions
  - Partial functions

- A repeating note: **make sure you read the textbook**

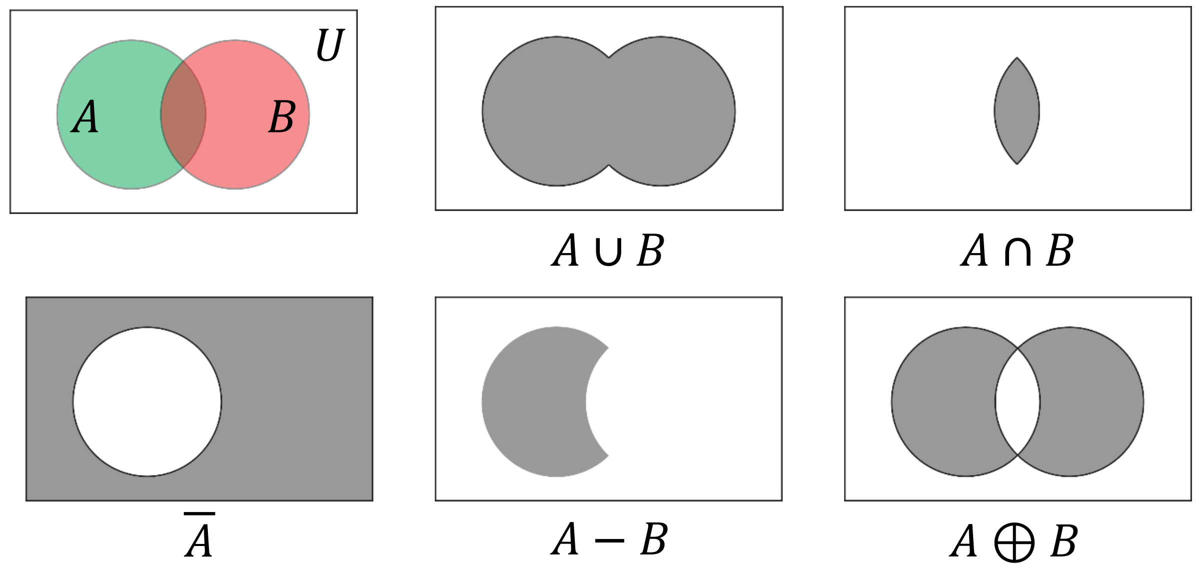
L09-10: Sets and Set Operations

- A set is a collection of elements
  - Roster representation
    - $A = \{1, a, cup, \pi, apple\}$ – elements do need not be of the same type
    - Natural numbers (an infinite set), $\mathbb{N} = \{0, 1, 2, \dots\}$
  - Set builder notation — such that
    - $\mathbb{R}^3 = \{\,(x, y, z) \mid x, y, z \in \mathbb{R}\}$
    - $A = \{x \mid x\ is\ a\ student\ at\ Rutgers\}$

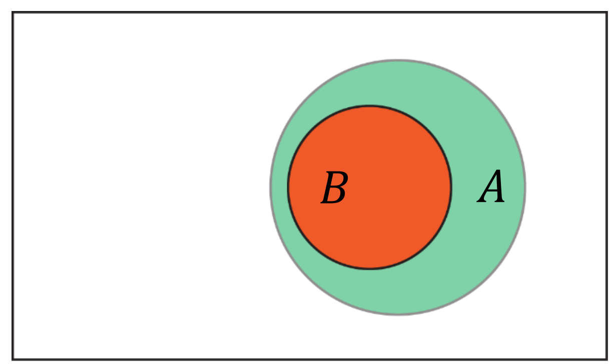
- Set operations
  - Union: $A \cup B = \{x \mid x \in A \vee x \in B\}$
  - Intersection: $A \cap B = \{x \mid x \in A \wedge x \in B\}$
  - Complement: $\bar{A} = \{x \mid x \in U \wedge x \notin A\}$
  - Difference: $A - B = \{x \mid x \in A \wedge x \notin B\}\ (or\ A \backslash \mathrm{B})$
  - Symmetric difference: $A \oplus B = A \cup B\ - A \cap B$

L09-10: Venn Diagram, Subset, Power Set

- Venn diagram



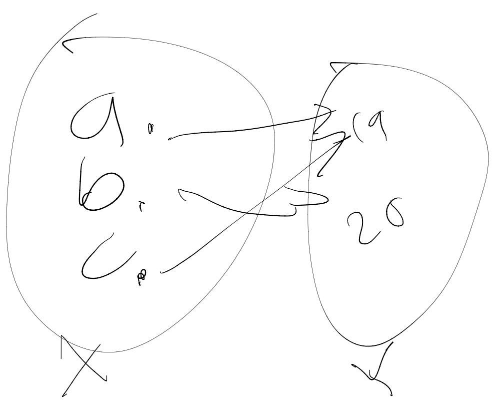
- Subset ($\subseteq$): $B \subseteq A \Leftrightarrow \forall x \in B, x \in A$
- Superset ($\supseteq$): $A \supseteq B \Leftrightarrow B \subseteq A$
- Powerset: $\mathcal{P}(S) = \{A \mid A \subseteq S\}$, example:
  - $S = \{1,2\}$
  - $\mathcal{P}(S) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$
- Cardinality (size)
  - $|\emptyset|=0$
  - $|\{1, 2\}|=2$



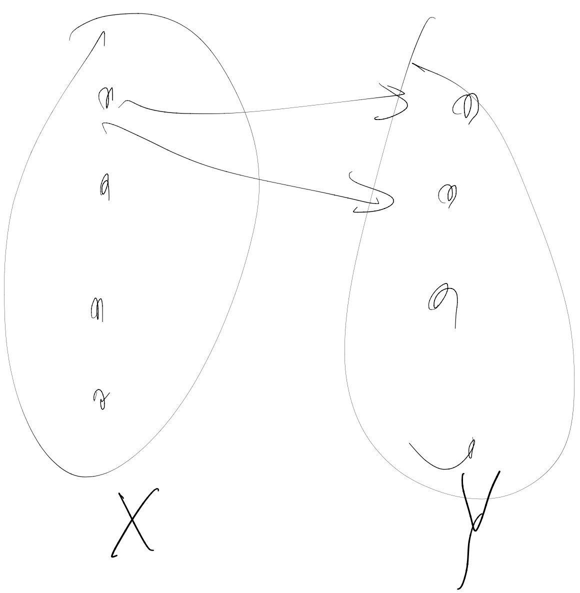
$B \subseteq A$

Functions

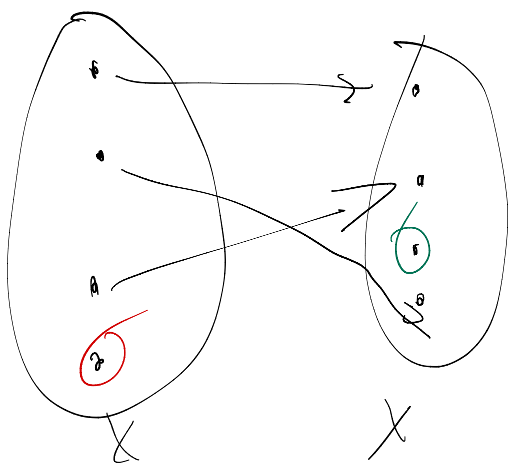
- **Def$^{\boldsymbol{n}}$**: A **function**, or a **mapping**, f, denoted $f: X \to Y$, assigns exactly one element $y \in Y$ to each and every element $x \in X$, written as $f(x) = y$ or $x \mapsto y$.
  - X: **domain**
  - Y: **co-domain**
  - If $f(a) = b$, then b is the **image** of a under f, a is the **preimage** of b
  - $f(X) = \{f(x) | x \in X\}$ is the **range** or **image** of X under f
  - **Ex**: People $X = \{a, b, c\}$, age $Y = N$. $f(a) = 19$, $f(b) = 20$, $f(c) = 19$.
    - Domain? Co-domain?



Two "Non"-Functions



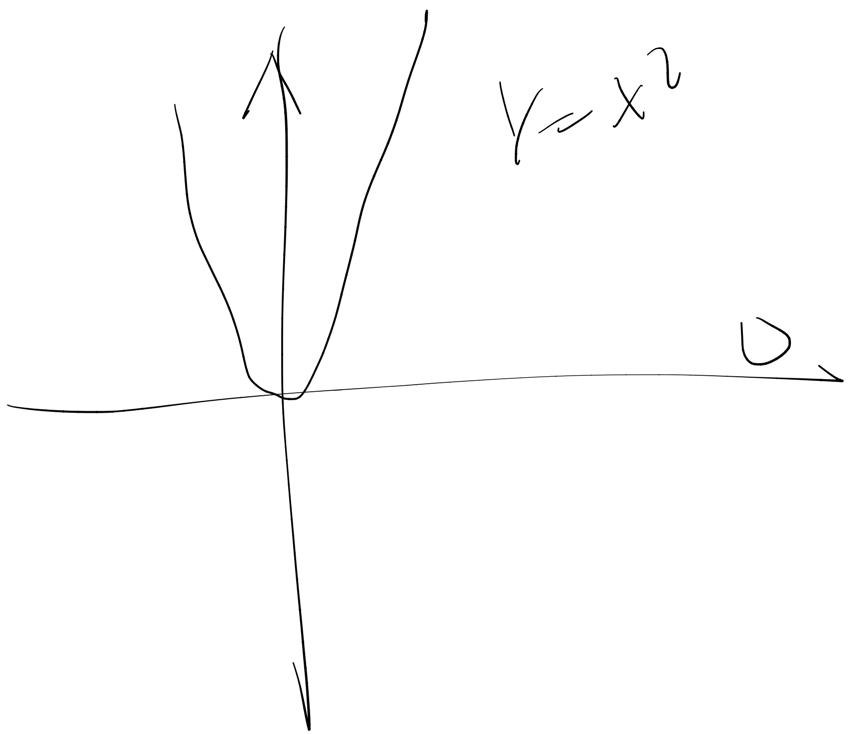
Not a function

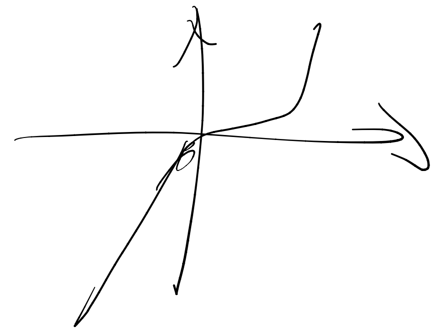


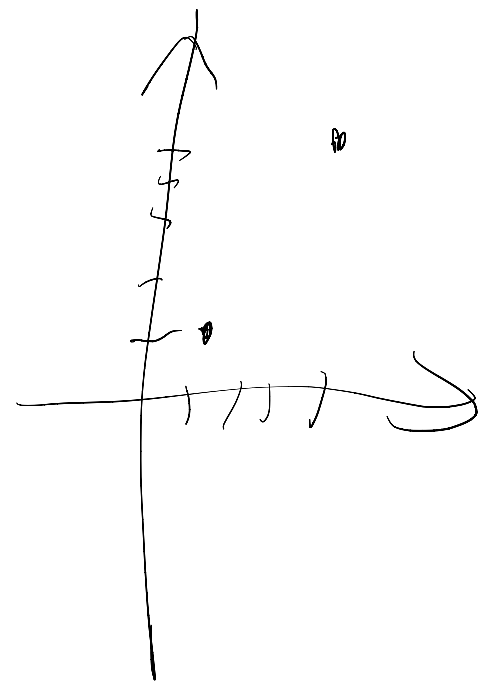
Not a total function

Examples

- **Ex**: $f: \mathbb{R} \to \mathbb{R}, x \mapsto x^2$
- **Ex**: $f: \mathbb{N} \to \mathbb{N}, x \mapsto x^2$
- **Ex**: $f: \mathbb{R} \to \mathbb{R}, x \mapsto x^3$







Examples, Cont.

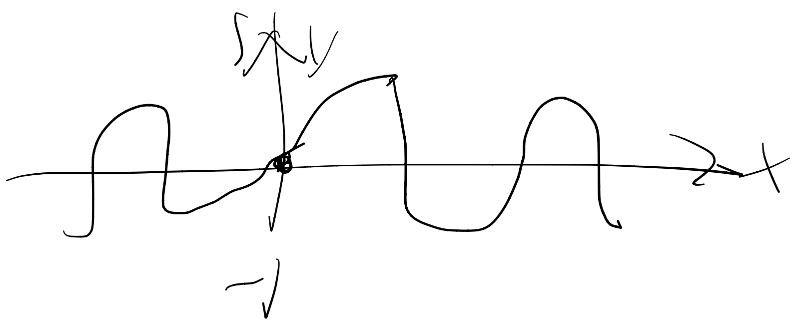
- **Ex**: sine: $\mathbb{R} \to [-1,1]$, $x \mapsto \sin x$
- **Ex**: floor: $\mathbb{R} \to \mathbb{Z}$, $x \mapsto$ largest integer that is no larger than x
  - This is also written as $\lfloor x \rfloor$
- **Ex**: ceiling: $\mathbb{R} \to \mathbb{Z}$, $x \mapsto$ smallest integer that is no smaller than x
  - This is also written as $\lceil x \rceil$
- What is $\lfloor \pi \rfloor$ and $\lceil -\pi \rceil$?

$x = 1.1$

$\text{floor}(1.1) = 1$

$x = 1.1$ $\lceil x \rceil = 2$

$x = 1$ $\lfloor x \rfloor = \lceil x \rceil = x$



sin x

$x = -1.1$ $\lfloor x \rfloor = -2$

$\lceil x \rceil = -1$

Adding & Multiplying Function

- Multiple functions can be added/multiplied
  - Addition/subtraction
  - Multiplications
  - The domain should be the same and the co-domain should be compatible

DOMAIN MUST BE THE SAME

$f_1(x) = e^x, \; f_2(x) = x, \; f_3(x) = 1$

$f(x) = (f_1 + f_2 + f_3)(x) = e^x + x + 1$

$g(x) = (f_1 f_2)(x) = x e^x$

$f: \mathbb{R} \to \mathbb{N}, \; x \to \boxed{x}$

$g: \mathbb{N} \to \mathbb{N}, \; x \to x+1$

$g: S \to S_e$

Can't add student names

One-to-One and Onto Functions

- **Def^n**: A function f is **one-to-one** (or an **injection**) iff $f(x_1) \neq f(x_2)$ for $x_1 \neq x_2$. f is **onto** (or a **surjection**) iff $f(X) = Y$. f is a **bijection** iff it is **one-to-one** and **onto** (**injection** and **surjection**).
  - **Ex**: $f: \mathbb{N} \to \mathbb{N}, x \mapsto x + 1$

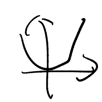
    $x_1 \neq x_2 \Rightarrow f(x_1) = x_1 + 1 \neq x_2 + 1 = f(x_2)$

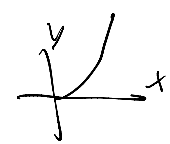
    if $y = 0$, then no $x \in \mathbb{N}$ satisfies $x + 1 = y$
  - **Ex**: $f: \mathbb{R} \to \mathbb{R}, x \mapsto x + 1$

    $x = y - 1$
  - **Ex**: $f: \mathbb{R} \to \mathbb{R}, x \mapsto x^2$

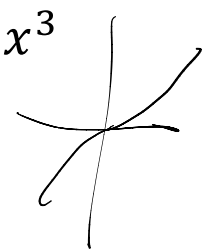
    $f(x) = 1 \Rightarrow x = \pm 1$

    $x_1 = 1, x_2 = -1$ $f(x_1) = f(x_2)$

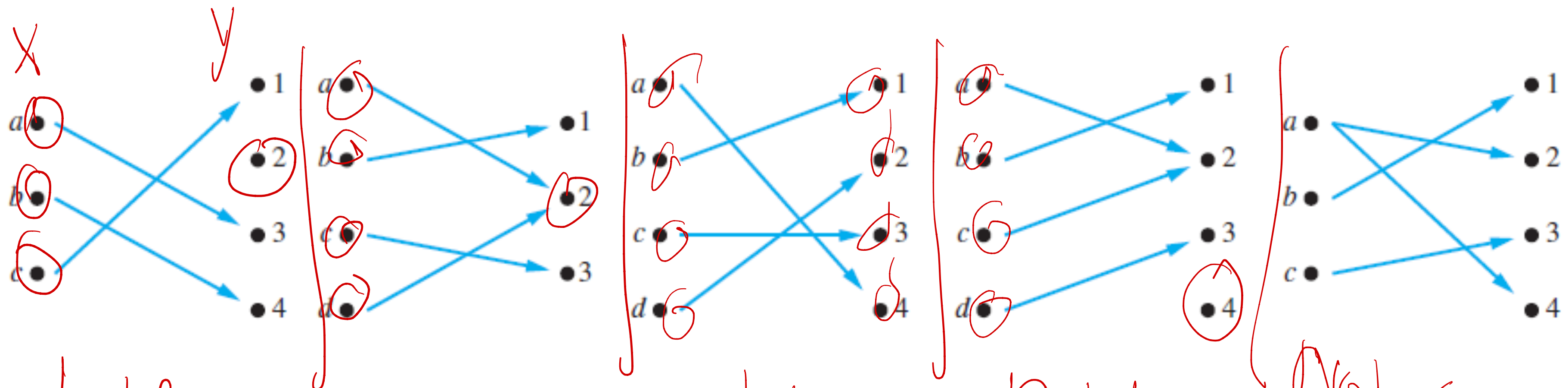
    
  - **Ex**: $f: \mathbb{R}_+ \to \mathbb{R}_+, x \mapsto x^2$

  - **Ex**: $f: \mathbb{R} \to \mathbb{R}, x \mapsto x^3$

    
  - **Ex**: Identity: $f: \mathbb{R} \to \mathbb{R}, x \mapsto x$

One-to-One and Onto Functions



1-1 function
Not onto

not 1-1
onto

1-1
Onto

not 1-1
not onto

Not a
function

Proving 1-1 and Onto Properties

$x \neq y \rightarrow f(x) \neq f(y)$

- Showing f is 1-1 (injectiveness): $f(x) = f(y) \rightarrow x = y$
  - **Ex**: $f: \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x + 1$

$f(x) = f(y) \Rightarrow x + 1 = y + 1 \Rightarrow x = y$

- Showing f is not 1-1: $\exists x, y\Big((x \neq y) \rightarrow \big(f(x) = f(y)\big)\Big)$
  - **Ex**: $f: \mathbb{R} \rightarrow \mathbb{R}, x \mapsto \sin x$

$x = 0, y = \pi \quad x \neq y$

$\sin 0 = \sin \pi = 0$

Proving 1-1 and Onto Properties

- Showing f is onto (surjectiveness): $\forall y \exists x (f(x) = y)$
  - **Ex**: $f : \mathbb{R} \to \mathbb{R}, x \mapsto x + 1$

$y = x + 1 \Rightarrow x = y_0 - 1$

- Showing f is not onto: $\exists y \forall x (f(x) \neq y)$
  - **Ex**: $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$

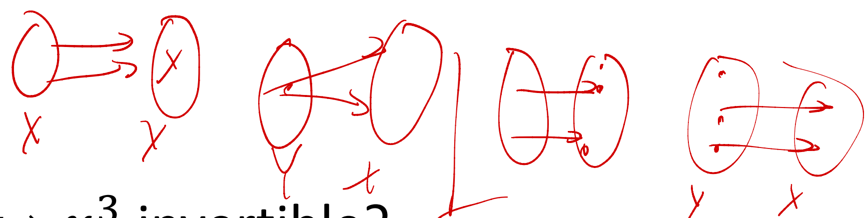
Let $y = -1$, $x^2 = y = -1 \Rightarrow x = \pm\sqrt{-1}$

Inverse Functions

- **Def^n**: A bijective function f has an **inverse** function, often denoted as f^{-1}, such that $f^{-1}(y) = x$ if $f(x) = y$.
  - Domains/co-domain flipped

    $f: X \to Y,\ x \mapsto f(x),\ f^{-1}: Y \to X,\ y \mapsto f^{-1}(y)$

  - Only bijective functions have inverses.



  - **Ex**: Is $f: \mathbb{R} \to \mathbb{R}, x \mapsto x^3$ invertible?

    Yes it is

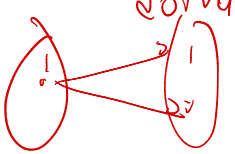
  - **Ex**: Is $f: \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ invertible?

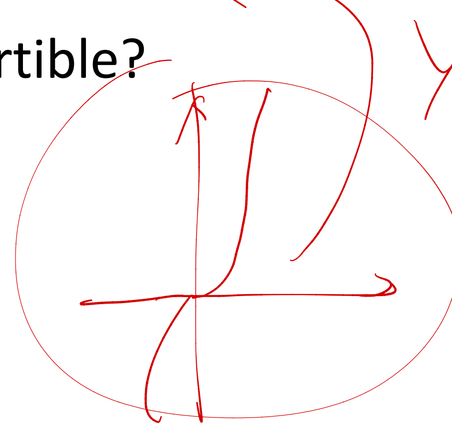
No

$x^2 = 1$

$x = \pm 1$

We have to restrict the domain to make a bijection





Function Composition

- **Def^{n}**: $g : X \to Y$ and $f : Y \to Z$ are two functions, then the **composition** of f and g, denoted $f \circ g$, is the function $f \circ g: X \to Z, x \mapsto f(g(x))$.
  - For invertible function, we always have $f^{-1}(f(x)) = x$.
  - In general, $f \circ g \neq g \circ f$
  - **Ex**: $X = Y = Z = \mathbb{R}$. $f(x) = x + 2$, $g(x) = 3x$

$(f \cdot g)(x) = f(g(x)) = 3x+2$

$(g \cdot f)(x) = g(f(x)) = 3(x+2) = 3x+6$