Lecture 18: Relations and Representing Relations

Jingjin Yu | Computer Science @ Rutgers

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Outline

- Lecture 17 review
- Relations
  - Set operations over relations
  - Composition of relations
  - Representations
    - Matrix
    - Directed graph

- A repeating note: **make sure you read the textbook**

L17: Binary Relation

- **Def^n**: Let A and B be sets. A **binary relation** from A to B is a subset of $A \times B$.
  - Notation: $R \subset A \times B$ is a set of relations. E.g. aRb means $(a, b) \in R$. This reads a is related to b by R. $a \not R\, b$ means $(a, b) \notin R$. Sometimes relation is also written as $a \sim b$.

$\subset R = \{(NB, NJ), (Chicago, IL), \ldots\}$

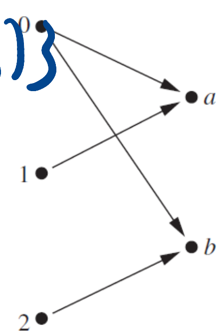
- **Ex**: A: US cities, B: US states

$\{\text{cities} \times \text{states}\}$

- **Ex**: $A = \{0;\ 1;\ 2\}; B = \{a; b\}, R = \{(0, a), (0, b), (1, a), (2, b)\}$.

$A \times B = \{(0,a), (0,b), (1,a), (1,b), (2,a), (2,b)\}$

$R \subset A \times B$



R	a	b
0	×	×
1	×	
2		×

L17: Binary Rations on a Set

- **Def^n**: A (binary) relation R on a set A is a relation from $A \rightarrow A$.
- **Ex**: A: students, R: "classmates"

- If $|A| = n$, how many possible relations R?
  - Total # pairs: n^2
    - E.g., $A = \{1, 2\}$, pairs $P = \{(1,1), (1,2), (2,1), (2,2)\}$
  - Any subset of P defines a relation, so total of 2^{n^2}

- Types of binary relations on a set
  - Reflexive
  - Symmetric, antisymmetric
  - Transitive

L17: Reflexivity – Symmetry – Transitivity

- **Def^n**: A relation R on A is **reflexive** if $(a, a) \in R$ for all $a \in A$
  - **Ex**: A: Cities, R: "Road between"

  (a, a)

- **Def^n**: A relation R on A is **symmetric** if $(a, b) \in R \leftrightarrow (b, a) \in R$ for all $a, b \in A$
  - **Ex**: R: Friendship

  Siblings (a, b) (b, a)

- **Def^n**: A relation R on A is **antisymmetric** if $(a, b) \in R$ and $(b, a) \in R$ implies $a = b$ for $a, b \in A$
  - **Ex**: R: Parent-of

  $(a, b) \in R$, $(b, a) \notin R$

- **Def^n**: A relation R on A is **transitive** if $(a, b) \in R$ and $(b, c) \in R$ implies that $(a, c) \in R$
  - **Ex**: R: Ancestor-of

  (a, b), $(b, c) \rightarrow (a, c)$

L17: One More Example

- Is the "divides" relation reflexive, symmetric, antisymmetric, transtive on positive integers?

$\{(2,8), (4,8), \ldots\}$

$a|b$

√ Reflexivity $a|a$

√ Symmetry $a|b \Rightarrow a \leq b$ $a=3$ $b=6$

$(3,6), (6,3)$

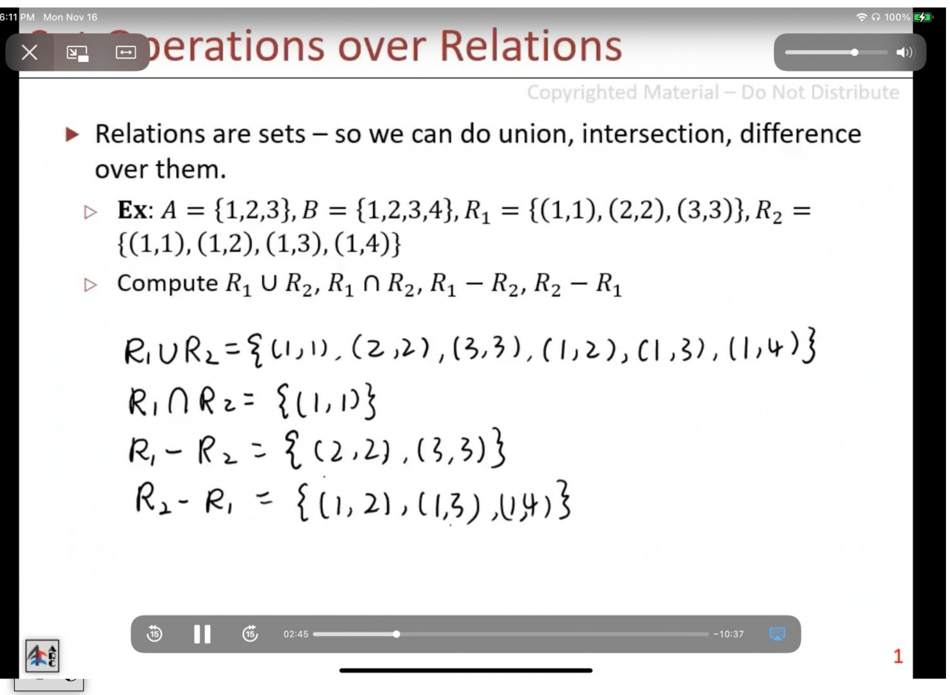
√ Antisymmetry $a|b \Rightarrow \{ b \times a, a=b$

√ Transitivity $a|b$ $b|c$ $a|c$

- Note: what if we use integers instead of positive integers?

Set Operations over Relations

- Relations are sets – so we can do union, intersection, difference over them.
  - **Ex**: $A = \{1,2,3\}, B = \{1,2,3,4\}, R_1 = \{(1,1),(2,2),(3,3)\}, R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$
  - Compute $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$



Composition of Relations

- Given sets A, B, C and relations $R \subset A \times B$ and $S \subset B \times C$, the composition $S \circ R$ is

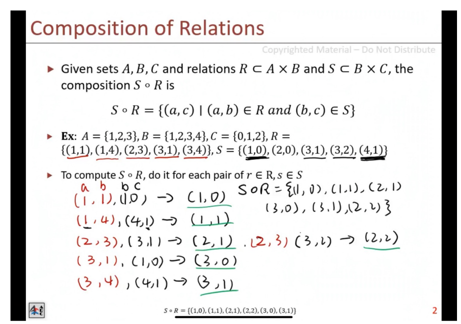
$$S \circ R = \{(a, c) \mid (a, b) \in R \; and \; (b, c) \in S\}$$

- **Ex**: $A = \{1,2,3\}, B = \{1,2,3,4\}, C = \{0,1,2\}, R = \{(1,1), (1,4), (2,3), (3,1), (3,4)\}, S = \{(1,0), (2,0), (3,1), (3,2), (4,1)\}$

- To compute $S \circ R$, do it for each pair of $r \in \mathrm{R}, s \in S$

a b b c

$(1,1), (1,0) \to (1,0)$



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a b b c

$(1,1), (1,0) \to (1,0)$ $S \circ R = \{(1,0), (1,1), (2,1), (3,0), (3,1), (2,2)\}$

$(1,4), (4,1) \to (1,1)$

$(2,3), (3,1) \to (2,1)$. $(2,3)$ $(3,2) \to (2,2)$

$(3,1), (1,0) \to (3,0)$

$(3,4), (4,1) \to (3,1)$

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2

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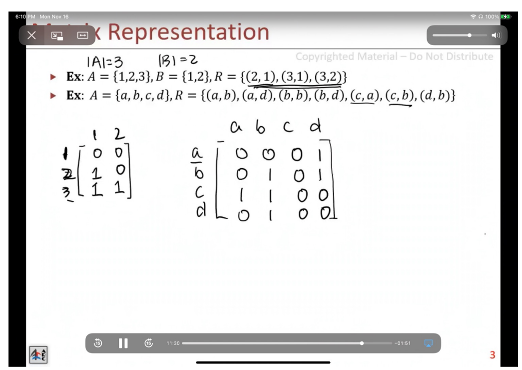
Matrix Representation

- **Ex:** $A = \{1,2,3\}, B = \{1,2\}, R = \{(2,1),(3,1),(3,2)\}$
- **Ex:** $A = \{a,b,c,d\}, R = \{(a,b),(a,d),(b,b),(b,d),(c,a),(c,b),(d,b)\}$

$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{array}$$

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 0 & 0 & 1 \\ b & 0 & 1 & 0 & 1 \\ c & 1 & 1 & 0 & 0 \\ d & 0 & 1 & 0 & 0 \end{array}$$



Directed Graph (Digraph) Representation

- **Ex:** $A = \{1,2,3,4\}, R = \{(1,1),(1,3),(2,1),(2,3),(2,4),(3,1),(3,2),(4,1)\}$