



Lecture 18: Relations and Representing Relations

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November 6, 2020



Outline

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- ▶ Lecture 17 review
- ▶ Relations
 - ▷ Set operations over relations
 - ▷ Composition of relations
 - ▷ Representations
 - ▣ Matrix
 - ▣ Directed graph

- ▶ A repeating note: **make sure you read the textbook**



L17: Binary Relation

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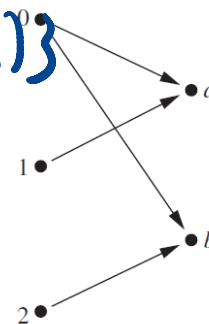
- ▶ **Defⁿ:** Let A and B be sets. A **binary relation** from A to B is a subset of $A \times B$.
- ▷ Notation: $R \subset A \times B$ is a set of relations. E.g. aRb means $(a, b) \in R$. This reads a is related to b by R . $a \not R b$ means $(a, b) \notin R$. Sometimes relation is also written as $a \sim b$.

- ▶ **Ex:** A : US cities, B : US states

$$R = \{(NB, NJ), (Chicago, IL)\}$$
$$\{ \text{City} \times \text{State} \}$$

- ▶ **Ex:** $A = \{0; 1; 2\}; B = \{a; b\}, R = \{(0, a), (0, b), (1, a), (2, b)\}$.

$$A \times B = \{(0, a), (0, b), (1, a), (1, b), (2, a), (2, b)\}$$
$$R \subset A \times B$$



R	a	b
0	×	×
1	×	
2		×

L17: Binary Rations on a Set

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- ▶ **Defⁿ**: A (binary) relation R on a set A is a relation from $A \rightarrow A$.
- ▶ **Ex**: A : students, R : “classmates”

- ▶ If $|A| = n$, how many possible relations R ?
 - ▷ Total # pairs: n^2
 - ▣ E.g., $A = \{1, 2\}$, pairs $P = \{(1,1), (1,2), (2,1), (2,2)\}$
 - ▷ Any subset of P defines a relation, so total of 2^{n^2}

- ▶ Types of binary relations on a set
 - ▷ Reflexive
 - ▷ Symmetric, antisymmetric
 - ▷ Transitive



L17: Reflexivity – Symmetry – Transitivity

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- ▶ **Defⁿ:** A relation R on A is **reflexive** if $(a, a) \in R$ for all $a \in A$

- ▶ **Ex:** A : Cities, R : “Road between”

(a, a)

- ▶ **Defⁿ:** A relation R on A is **symmetric** if $(a, b) \in R \leftrightarrow (b, a) \in R$ for all $a, b \in A$

- ▶ **Ex:** R : Friendship

(a, b) (b, a)

- ▶ **Defⁿ:** A relation R on A is **antisymmetric** if $(a, b) \in R$ and $(b, a) \in R$ implies $a = b$ for $a, b \in A$

- ▶ **Ex:** R : Parent-of

$(a, b) \in R, (b, a) \notin R$

- ▶ **Defⁿ:** A relation R on A is **transitive** if $(a, b) \in R$ and $(b, c) \in R$ implies that $(a, c) \in R$

- ▶ **Ex:** R : Ancestor-of

$(a, b), (b, c) \rightarrow (a, c)$

L17: One More Example

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- Is the “divides” relation reflexive, symmetric, antisymmetric, transitive on positive integers?

$a|b$

$\{(2,4), (4,8), \dots\}$

✓ Reflexivity $a|a$

✓ Symmetry $a|b \Rightarrow a \leq b$

✓ Antisymmetry $a|b \Rightarrow \begin{cases} b|a \\ a=b \end{cases}$

✓ Transitivity $a|b \quad b|c \Rightarrow a|c$

- Note: what if we use integers instead of positive integers?

Set Operations over Relations

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- ▶ Relations are sets – so we can do union, intersection, difference over them.
- ▶ **Ex:** $A = \{1,2,3\}$, $B = \{1,2,3,4\}$, $R_1 = \{(1,1), (2,2), (3,3)\}$, $R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$
- ▶ Compute $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$

The screenshot shows a video player interface. At the top, the title 'Set Operations over Relations' is displayed in a red font. Below the title, the text 'Copyrighted Material – Do Not Distribute' is visible. The main content of the slide is a list of bullet points: 'Relations are sets – so we can do union, intersection, difference over them.', 'Ex: A = {1,2,3}, B = {1,2,3,4}, R1 = {(1,1), (2,2), (3,3)}, R2 = {(1,1), (1,2), (1,3), (1,4)}', and 'Compute R1 ∪ R2, R1 ∩ R2, R1 - R2, R2 - R1'. Below these points, four set operations are written in a handwritten style: R1 ∪ R2 = {(1,1), (2,2), (3,3), (1,2), (1,3), (1,4)}, R1 ∩ R2 = {(1,1)}, R1 - R2 = {(2,2), (3,3)}, and R2 - R1 = {(1,2), (1,3), (1,4)}. The video player controls at the bottom show a progress bar at 02:45 / 10:37 and a volume icon.

Set Operations over Relations

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- ▶ Relations are sets – so we can do union, intersection, difference over them.
- ▶ **Ex:** $A = \{1,2,3\}$, $B = \{1,2,3,4\}$, $R_1 = \{(1,1), (2,2), (3,3)\}$, $R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$
- ▶ Compute $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$

$$R_1 \cup R_2 = \{(1,1), (2,2), (3,3), (1,2), (1,3), (1,4)\}$$
$$R_1 \cap R_2 = \{(1,1)\}$$
$$R_1 - R_2 = \{(2,2), (3,3)\}$$
$$R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$$

Composition of Relations

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- Given sets A, B, C and relations $R \subset A \times B$ and $S \subset B \times C$, the composition $S \circ R$ is

$$S \circ R = \{(a, c) \mid (a, b) \in R \text{ and } (b, c) \in S\}$$

- Ex:** $A = \{1, 2, 3\}, B = \{1, 2, 3, 4\}, C = \{0, 1, 2\}, R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}, S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$
- To compute $S \circ R$, do it for each pair of $r \in R, s \in S$

$$\begin{matrix} a & b & c \\ (1, 1) & (1, 4) & (1, 0) \end{matrix} \rightarrow (1, 0)$$

Composition of Relations

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- Given sets A, B, C and relations $R \subset A \times B$ and $S \subset B \times C$, the composition $S \circ R$ is

$$S \circ R = \{(a, c) \mid (a, b) \in R \text{ and } (b, c) \in S\}$$
- Ex:** $A = \{1, 2, 3\}, B = \{1, 2, 3, 4\}, C = \{0, 1, 2\}, R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}, S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$
- To compute $S \circ R$, do it for each pair of $r \in R, s \in S$
 - $(1, 1), (1, 0) \rightarrow (1, 0)$
 - $(1, 4), (4, 1) \rightarrow (1, 1)$
 - $(2, 3), (3, 1) \rightarrow (2, 1)$
 - $(2, 3), (3, 2) \rightarrow (2, 2)$
 - $(3, 1), (1, 0) \rightarrow (3, 0)$
 - $(3, 4), (4, 1) \rightarrow (3, 1)$

$S \circ R = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}$

$$S \circ R = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}$$

Matrix Representation

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- ▶ **Ex:** $A = \{1,2,3\}, B = \{1,2\}, R = \{(2,1), (3,1), (3,2)\}$
- ▶ **Ex:** $A = \{a,b,c,d\}, R = \{(a,b), (a,d), (b,b), (b,d), (c,a), (c,b), (d,b)\}$

$$\begin{array}{ccc}
 & 1 & 2 \\
 1 & 0 & 0 \\
 2 & 1 & 0 \\
 3 & 1 & 1
 \end{array}$$

$$\begin{array}{cccc}
 & a & b & c & d \\
 a & 0 & 0 & 0 & 0 \\
 b & 0 & 1 & 0 & 0 \\
 c & 1 & 1 & 0 & 0 \\
 d & 0 & 1 & 0 & 1
 \end{array}$$

Representation

$|A|=3$ $|B|=2$

▶ **Ex:** $A = \{1,2,3\}, B = \{1,2\}, R = \{(2,1), (3,1), (3,2)\}$

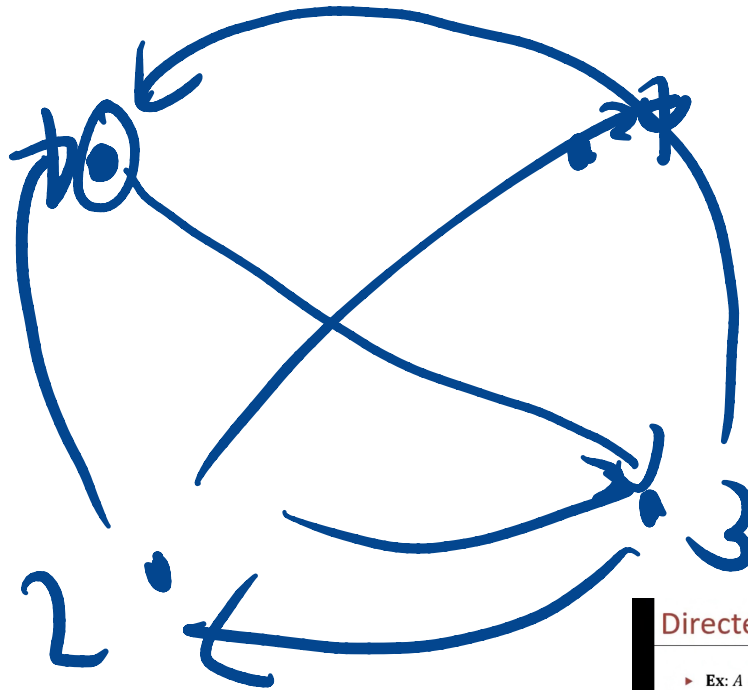
▶ **Ex:** $A = \{a,b,c,d\}, R = \{(a,b), (a,d), (b,b), (b,d), (c,a), (c,b), (d,b)\}$

$$\begin{array}{ccc}
 1 & 2 \\
 2 & \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \\
 3 & \begin{bmatrix} 1 & 1 \end{bmatrix}
 \end{array}
 \quad
 \begin{array}{cccc}
 a & b & c & d \\
 a & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}
 \end{array}$$

Directed Graph (Digraph) Representation

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- **Ex:** $A = \{1,2,3,4\}$, $R = \{(1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1)\}$



Directed Graph (Digraph) Representation

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- **Ex:** $A = \{1,2,3,4\}$, $R = \{(\underline{1},1), (\underline{1},3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1)\}$

