Lectures 13-14: Mathematical Induction

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Algorithmic Robotics & Control Lab

Outline

- Lectures 12 review
- Principles of mathematical induction
- (many) Examples

- A repeating note: **make sure you read the textbook**

L12: Cardinality & Countable Sets

- $\mathbf{Def}^n$: The sets X and Y have the same **cardinality**, denoted as $|X| = |Y|$, if there is a bijection between X and Y.
- $\mathbf{Def}^n$: If there is an injection from X to Y , then the cardinality of X is no more than that of Y, denoted as $|X| \leq |Y|$. If $|X| \leq |Y|$ and $|X| \neq |Y|$, then $|X| < |Y|$.
- $\mathbf{Def}^n$: The **cardinality of the set of positive integers** is denoted as $\aleph_0$ (aleph-zero), that is, $|\mathbb{N}^+| = \aleph_0$.
- $\mathbf{Def}^n$: A set X is **countably infinite** if it has the same cardinality as the set of positive integers. In this case, $|X| = \aleph_0$. A set is **countable** if it is either finite or countably infinite.
- $\mathbf{Def}^n$: A set X is **uncountable** if it is not countable.

L12: Examples: Countable Sets

- **Ex:** $\mathbb{Z}$ is countable.

$f: \mathbb{N}^+ \to \mathbb{Z}, \quad x \to \left\lceil \frac{x}{2} \right\rceil \cdot (-1)^x$

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 1 & -1 & 2 & -2 \end{array}$$

- **Ex:** $\mathbb{Q}^+$ is countable.

(m, n) $\quad n/m$

$f(1) = (1, 1)$

1 2 3 4 ...

1
2
3
4
⋮

$f(2) = (1, 2)$

$f(3) = (2, 1)$

$f(4) = (3, 1)$

L12: Examples: Uncountable Sets

- **Ex**: The real numbers in $[0,1]$ is uncountable.

$f: \mathbb{N}^+ \to [0,1]$

$\mathbb{N}^+ \quad [0,1]$

$f(1) \quad 0.\,a_1\,a_2\,a_3 \ldots \qquad x_1 \neq a_1,\ x_1 = c$

$f(2) \quad 0.\,b_1\,b_2\,b_3 \ldots \qquad x_2 \neq b_2,\ x_2 \neq a$

$f(3) \quad 0.\,c_1\,c_2\,c_3 \ldots \qquad x_3 \neq c_3,\ x_3 \neq a$

$f(\ldots)$

$x = 0.\,x_1\,x_2\,x_3 \ldots$

L12: Schroder-Bernstein Theorem

- **Theorem**: $(|X| \leq |Y|) \wedge (|Y| \leq |X|) \leftrightarrow |X| = |Y|$.
  - That is, if we can find $f: X \to Y$ and $g: Y \to X$ that are both injections, or 1-1, then $|X| = |Y|$
- **Ex**: Prove $|(0,1)| = |(0,1]|$

$X \qquad Y$

$f: X \to Y, \; x \to x$

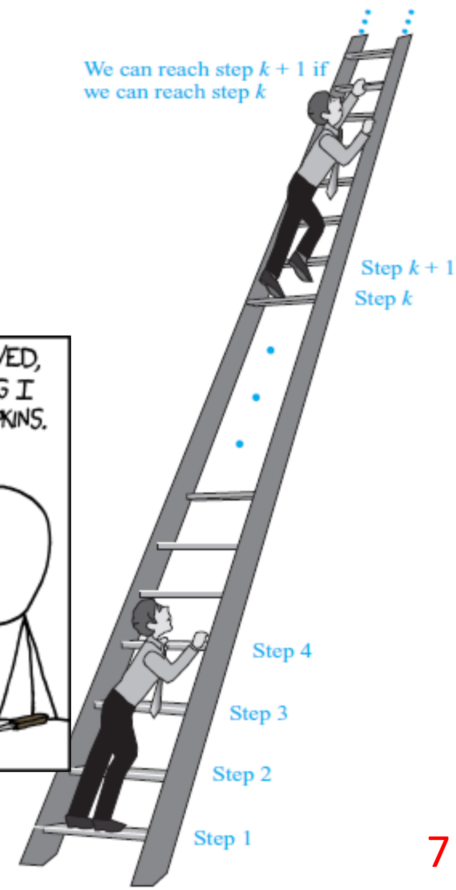
$g: Y \to X, \; y \to y/2 \qquad (0,1] \to (0,\tfrac{1}{2}]$

$\subseteq (0,1)$

Mathematical Induction: the Statement

- Let $P(n)$ be a predicate depending on a positive integer $n \geq 1$. To show $P(n)$ is true for all n, two steps are needed:
  - (1) Basis step (or base case): verify that $P(1)$ is true.
  - (2) Inductive step: establish $P(k) \rightarrow P(k+1)$ for all $k \geq 1$.
- In (2), $P(k)$ is called the **inductive hypothesis**.

$P(1)$ $P(k) \to P(k+1)$ $P(2)$ $P(k) \to P(k+1)$ $P(3)$ … $\to P(n)$



Basic Example – 1

Summation of numbers

- **Ex:** $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. $= P(n)$

(1) Basis Step: $P(1) = 1 = \frac{1(1+1)}{2}$ ✓

(2) Inductive Step: assume $1+2+\dots+k = \frac{k(k+1)}{2}$

need to show $1+2+\dots+k+(k+1) = \frac{(k+1)(k+2)}{2}$

$1+2+\dots+k = \frac{k(k+1)}{2}$

add $k+1$ to both side

$1+2+\dots+k+k+1 = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k)+(2k+2)}{2}$

$= \frac{k^2+k+2k+2}{2} = \frac{(k+1)(k+2)}{2}$

Basic Example – 2

- **Ex**: Conjecture a formula for $1+3+\cdots+(2n-1)$ and prove.

1

$1+3=4$

$1+3+5=9$

$1+3+5+7=16$

1^2

2^2

3^2

4^2

guess: $1+3+\cdots+(2n-1)=n^2$

$P(n)$

Proof (1) Basic Step $P(1) = $ "$1=1^2$" ✓

(2) Assume $1+3+\cdots+2k-1=k^2$ $(P(k))$

Adding $2(k+1)-1$ to both sides

$1+3+\cdots+(2k-1)+(2(k+1)-1)=k^2+2(k+1)-1$

$=k^2+2k+1$

$=(k+1)^2$

$(2k+1)^2$

$2k+2-1$

$k^2+2k+2-1$

Basic Example – 3

- **Ex:** Conjecture a formula for $2 + 4 + \cdots + 2n$ and prove.

2

$2 + 4 = 6$

$2 + 4 + 6 = 12$

$2(1 + 2 + \cdots + n)$

$\frac{n(n+1)}{2}$

~~Alternative, 2 + 4 + ... + 2n~~

$= (1 + 2 + \cdots + 2n) - (1 + 3 + \cdots + (2n-1))$

$= \frac{2n(n+1)}{2} - n^2$

Basic Example – 4

- **Ex:** $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1.$

Proof (1) Basic Step $P(1) = $ "$1+2 = 2^{1+1} - 1$" $3=3$

(2) Assume $1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$ $P(k)$

$1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$

$= 2^{(k+1)+1} - 1$

Basic Example – 5

- **Ex**: Geometric sum: $1 + r + r^2 + \cdots + r^n = \frac{r^{n+1}-1}{r-1}$ for $r \neq 1$.

Proof of (1) Basic step: $(n=1)$ $1 + r = \frac{r^2-1}{r-1} = \frac{(r-1)(r+1)}{r-1} = r+1$

(2) Assume $1 + r + r^2 + \ldots r^k = \frac{r^{k+1}-1}{r-1}$

$\Rightarrow (1 + r + r^2 + \cdots + r^k + r^{k+1}) = \frac{r^{k+1}-1}{r-1} + r^{k+1}$

$= \frac{\cancel{r^{k+1}} - 1 + r^{k+2} - \cancel{r^{k+1}}}{r-1}$

$\Rightarrow \frac{r^{k+2}-1}{r-1}$

Basic Example – 6

- **Ex**: $2^n < n!$ for $n \geq 4$.

Basic Example – 7

- **Ex**: Harmonic series: $H_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}$. Show $H_{2^n} \geq 1 + \frac{n}{2}$.

Harmonic Series: $H_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}$

Show $H_{2^n} \geq 1 + \frac{n}{2}$

Proof (1) Basis Step $P(1) =$ "$H_{2^1} \geq 1 + \frac{1}{2}$"

(2) Assume $H_{2^k} \geq 1 + \frac{k}{2}$

We know $\frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^{k+1}}$

$P(k+1) =$ "$H_{2^{k+1}} \geq 1 + \frac{k+1}{2}$"

$\underbrace{1 + \cdots + \frac{1}{2^k}}_{H_{2^k}} + \underline{\frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^{k+1}}} \geq 1 + \frac{k}{2} + \frac{1}{2}$

Basic Example – 8

- **Ex:** $n^3 - n$ is divisible by 3.

Proof (1) Basis Step $P(1) =$ "$1^3 - 1$ is divisible by 3"

(2) Assume $k^3 - k$ is divisible by 3

$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1$

$= \underline{(k^3 - k)} + \underline{3(k^2 + k)}$ divisible by 3

Basic Example – 9

- **Ex**: The power set of a set with n elements has 2^n elements.

Proof. Let S_n be the set $\{1, 2, \ldots, n\}$

$P(S_2) = \{\{\}, \{1\}, \{2\}, \{1, 2\}\}$

$P(S_3) = \{\{\}, \{1\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

(1) Basis case: $P(1)$: $|P(S_1)| = 2^1$

$\{\{\}, \{1\}\}$

(2) Assume $|P(S_k)| = 2^k$

To get $P(S_{k+1})$ from $P(S_k)$ we make two copies of $P(S_k)$ and then add n to each element of one of the copies. This shows $|P(S_{k+1})| = 2^{k+1}$

More Complex Examples: De Morgan's Law

- **Ex**: Generalized De Morgan's law:

$\overline{A \cap B} = \overline{A} \cup \overline{B}$

$$\overline{\bigcap_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A_i}$$

Proof

(1) Basis step

$P(1) = \overline{A_1} = \overline{A_1}$

(2) Assume $\overline{\bigcap_{i=1}^{k} A_i} = \bigcup_{i=1}^{k} \overline{A_i}$

For $k+1$ LHS $= \overline{\bigcap_{i=1}^{k+1} A_i} = \overline{\bigcap_{i=1}^{k} A_i \cap A_{k+1}}$

$= \overline{\bigcap_{i=1}^{k} A_i} \cup \overline{A_{k+1}}$

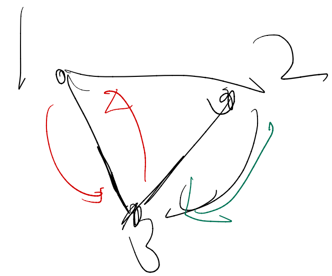
$= \bigcup_{i=1}^{k} \overline{A_i} \cup \overline{A_{k+1}} = \bigcup_{i=1}^{k+1} \overline{A_i} = RHS$

$P(k+1)$ ∎

More Complex Examples: Odd Pie Fights

- **Ex**: $2n + 1$ people stand at locations such that the pair wise distances are all different. Each has a pie and on the mark will throw the pie and hit the person who is closest to him/her. Someone will not be hit.

Proof (1) Basis Step



WLOG

(2) Assume the statement holds for $2k+1$ people $P(k)$

1 $k+1$ 3 … $2k+3$

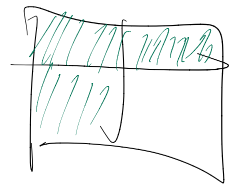
$P(k+1)$

1 2

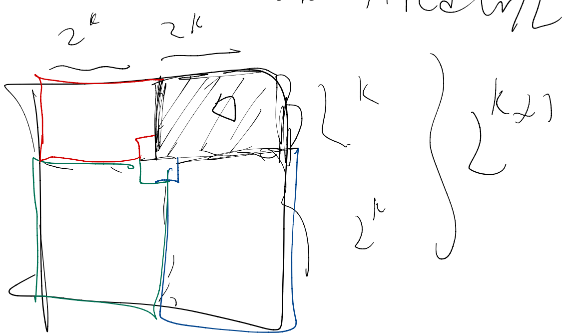
More Complex Examples: L-Trimino Tiling

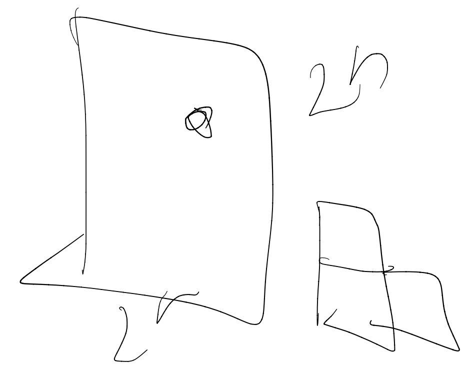
- **Ex**: A $2^n \times 2^n$ grid with one square removed can be tiled using L-shaped triminos.

Proof: On n Basis Step



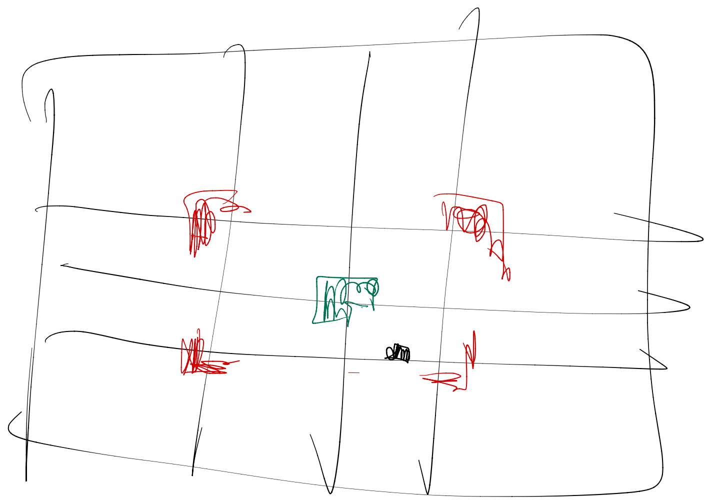
can be tiled w/ L triminos





$P(k) \to P(k+1)$

Algorithms for L-Trimino Tiling



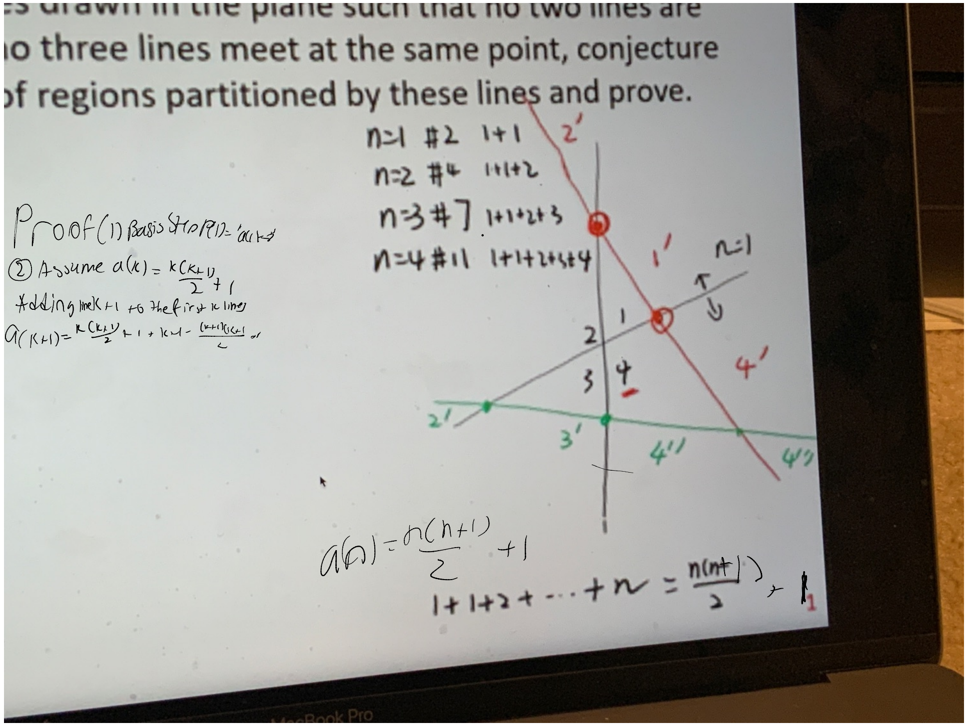
$2^k \times 2^k \to 4\ 2^{k-1} \times 2^{k-1}$

$\to 16\ 2^{k-2} \times 2^{k-2}$

$\to 4^{k-1}\ 2 \times 2$

More Complex Examples: Lines in 2D

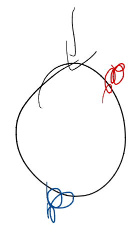
- **Ex**: For n lines drawn in the plane such that no two lines are parallel and no three lines meet at the same point, conjecture the number of regions partitioned by these lines and prove.



More Complex Examples: Dots on a Circle

- **Ex**: n red dots and n blue dots are placed around a circle; no two dots overlap. Show that if we start with some place on the circle to travel clockwise along the circle, the total number of red dots that we pass by is always no less than the total number of blue dots that we pass by.

Proof: (1) Basis Step
$n=1$



Assume it holds for $n=k$

For $n=k+1$