Lecture 02: Propositional Equivalences

Jingjin Yu | Computer Science @ Rutgers

September 4, 2020

RUTGERS
THE STATE UNIVERSITY OF NEW JERSEY

Algorithmic Robotics & Control Lab

Outline

- Tautologies, contradictions, & contingencies
- Logical equivalence
- Equivalences involving conjunctions and disjunctions
- Equivalences involving conditionals
- Equivalences involving bidirectionals

- Note: there will be lots of **proofs**! Make sure you understand and can reproduce the proofs that we will cover in the class! It will take some effort but will be worth it as the effort will help build a strong foundation

Tautologies, Contradictions & Contingencies

- **Def^n**: A compound proposition that is always true regardless of the truth value of the propositional variables is a **tautology**. A proposition that is always false is a **contradiction**. Otherwise, a proposition is a **contingency**.
- **Ex**:
  - Tautology — always true
    - $1 + 1 = 2$
    - $p \vee \neg p$, e.g., "John is a student or John is not a student"
  - Contradiction — always false
    - The sky is always blue
    - $p \wedge \neg p$
  - Contingency
    - Today is sunny
    - p

Logical Equivalences

- **Def^n**: Propositions p and q are **logically equivalent**, denoted by $p \equiv q$, $\boxed{\text{if}}$ $p \leftrightarrow q$ is a tautology (always true)
  - Note that $\equiv$ is not a logical operator/connective
  - To check $p \equiv q$, use truth table or rules built from truth table

- **Ex**: De Morgan's laws, basic form
  - $\neg(p \land q) \equiv \neg p \lor \neg q$
  - $\neg(p \lor q) \equiv \neg p \land \neg q$

p	q	$\neg p$	$\neg q$	$p \land q$	$\neg(p \land q)$	$\neg p \lor \neg q$	$p \lor q$	$\neg(p \lor q)$	$\neg p \land \neg q$
T	T								
T	F								
F	T								
F	F								

Logical Equivalences, cont.

- **Ex**: $p \rightarrow q \equiv \neg p \vee q$

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T			
T	F			
F	T			
F	F			

  - Very useful if you forget truth values for $p \rightarrow q$

Logical Equivalences, cont.

- **Ex**: Distributive law $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

p	q	r	$q \wedge r$	$p \vee q$	$p \vee r$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T					
T	F	T					
F	T	T					
F	F	T					
T	T	F					
T	F	F					
F	T	F					
F	F	F					

Common Equivalences

Name	Equivalence
Identity laws	$p \wedge T \equiv p, \qquad p \vee F \equiv p$
Domination laws	$p \vee T \equiv T, \qquad p \wedge F \equiv F$
Idempotent laws	$p \vee p \equiv p, \qquad p \wedge p \equiv p$
Double negation law	$\neg(\neg p) \equiv p$
Commutative laws	$p \vee q \equiv q \vee p, \qquad p \wedge q \equiv q \wedge p$
Associative laws	$(p \vee q) \vee r \equiv p \vee (q \vee r), \qquad (p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Distributive laws	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
De Morgan's laws	$\neg(p \vee q) \equiv \neg p \wedge \neg q, \qquad \neg(p \wedge q) \equiv \neg p \vee \neg q$
Absorption laws	$p \vee (p \wedge q) \equiv p, \qquad p \wedge (p \vee q) \equiv p$
Negation laws	$p \vee \neg p \equiv T, \qquad p \wedge \neg p \equiv F$

De Morgan's Laws, General Version

- General versions of De Morgan's laws

$$\neg(p_1 \vee \cdots \vee p_n) \equiv \neg p_1 \wedge \cdots \wedge \neg p_n,$$
$$\neg(p_1 \wedge \cdots \wedge p_n) \equiv \neg p_1 \vee \cdots \vee \neg p_n.$$

- How do we prove these?
  - Truth table would be too big: 2^n rows for a single n
  - We need more powerful tools, e.g., induction

- We can also write De Morgan's laws as

$$\neg\left(\bigvee_{i=1}^{n} p_i\right) \equiv \bigwedge_{i=1}^{n} \neg p_i\,, \qquad \neg\left(\bigwedge_{i=1}^{n} p_i\right) \equiv \bigvee_{i=1}^{n} \neg p_i$$

Equivalences with Conditionals

Equivalence Containing Conditionals
$p \to q \equiv \neg p \lor q$
$p \to q \equiv \neg q \to \neg p$
$p \lor q \equiv \neg p \to q$
$p \land q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \land \neg q$
$(p \to q) \land (p \to r) \equiv p \to (q \land r)$
$(p \to r) \land (q \to r) \equiv (p \lor q) \to r$
$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$
$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$

Proving $p \wedge q \equiv \neg(p \rightarrow \neg q)$

$$\neg(p \wedge q) \equiv p \rightarrow \neg q$$

$$\neg p \vee \neg q \equiv \neg p \vee \neg q$$

$$p \rightarrow q \equiv \neg p \vee q$$

$$\neg p \vee q \equiv \neg p \vee q$$

$$\neg p \rightarrow \neg q$$

$$\neg(\neg p \wedge q) \quad p \wedge \neg q$$

- Try yourself: $\neg(p \rightarrow q) \equiv p \wedge \neg q$

Proving $(p \to q) \land (p \to r) \equiv p \to (q \land r)$

$\neg p \lor (q \land r)$

$(\neg p \lor q) \land (\neg p \lor r)$

$(p \to q) \land (p \to r)$

$\neg (p \lor q) \lor r$

$(\neg p \land \neg q) \lor r$

$(\neg p \lor r) \land (\neg q \lor r)$

$(p \to r) \land (q \to r)$

- Try yourself: $(p \to r) \land (q \to r) \equiv (p \lor q) \to r$

Proving $(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$

$\neg p \lor (q \lor r)$

$(\neg p \lor q) \lor (\neg p \lor r)$

~~$(p \to q) \lor (p \to r)$~~

$\neg(p \land q) \lor r$

$(\neg p \lor \neg q) \lor r$

$(\neg p \lor r) \lor (\neg q \lor r)$

$(p \to r) \lor (q \to r)$

- Try yourself: $(p \to r) \lor (q \to r) \equiv (p \land q) \to r$

Equivalences with Bidirectionals

Equivalence Containing Bidirectionals
$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

p	q	$\neg p$	$\neg q$	$p \leftrightarrow q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$\neg p \leftrightarrow \neg q$
T	T							
T	F							
F	T							
F	F							

Proving $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$$(\underbrace{p \vee \neg p}_{T} \wedge \neg q) \vee (\underbrace{q \vee \neg p}_{T} \wedge \neg q)$$

$$(T \wedge \neg q) \vee (T \wedge \neg q)$$

done

Proving $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Equivalence Containing Bidirectionals
$p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$