Lecture 02: Propositional Equivalences

Jingjin Yu | Computer Science @ Rutgers





Outline

- ► Tautologies, contradictions, & contingencies
- Logical equivalence
- Equivalences involving conjunctions and disjunctions
- Equivalences involving conditionals
- Equivalences involving bidirectionals
- Note: there will be lots of **proofs**! Make sure you understand and can reproduce the proofs that we will cover in the class! It will take some effort but will be worth it as the effort will help build a strong foundation



Tautologies, Contradictions & Contingencies

- ▶ **Def**ⁿ: A compound proposition that is always true regardless of the truth value of the propositional variables is a tautology. A proposition that is always false is a contradiction. Otherwise, a proposition is a contingency.
- **Ex**:
 - Tautology 1+1=2

 - $p \lor \neg p$, e.g., "John is a student or John is not a student"
 - > Contradiction
 - The sky is always blue
 - $p \land \neg p$
 - Contingency
 - Today is sunny



Logical Equivalences

- ▶ **Def**ⁿ: Propositions p and q are **logically equivalent**, denoted by $p \equiv q$, if $p \leftrightarrow q$ is a tautology (always true)
 - \triangleright Note that \equiv is not a logical operator/connective
 - \triangleright To check $p \equiv q$, use truth table or rules built from truth table
- **Ex**: De Morgan's laws, basic form

$$\neg (p \land q) \equiv \neg p \lor \neg q$$

$$\neg (p \lor q) \equiv \neg p \land \neg q$$

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \land q)$	$\neg p \lor \neg q$	$p \lor q$	$\neg(p \lor q)$	$\neg p \land \neg q$
T	T								
T	F								
F	T								
F	F								



Logical Equivalences, cont.

Copyrighted Material – Do not Distribute

ightharpoonup **Ex**: $p \rightarrow q \equiv \neg p \lor q$

p	q	$\neg p$	$\neg p \lor q$	$p \rightarrow q$
T	T			
T	F			
F	T			
F	F			

 \triangleright Very useful if you forget truth values for $p \rightarrow q$

Logical Equivalences, cont.

Copyrighted Material – Do not Distribute

Ex: Distributive law $p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$

p	q	r	$q \wedge r$	$p \lor q$	$p \lor r$	$p \lor (q \land r)$	$(p \lor q) \land (p \lor r)$
T	T	T					
T	F	T					
F	T	T					
F	F	T					
T	T	F					
T	F	F					
F	T	F					
F	F	F					



Common Equivalences

Name	Equiva	lence
Identity laws	$p \wedge T \equiv p$,	$p \lor F \equiv p$
Domination laws	$p \lor T \equiv T$,	$p \wedge F \equiv F$
Idempotent laws	$p \lor p \equiv p$,	$p \wedge p \equiv p$
Double negation law	$\neg(\neg p)$	$)\equiv p$
Commutative laws	$p \vee q \equiv q \vee p,$	$p \wedge q \equiv q \wedge p$
Associative laws	$(p \lor q) \lor r \equiv p \lor (q \lor r),$	$(p \land q) \land r \equiv p \land (q \land r)$
Distributive laws	$p \lor (q \land r) \equiv (p \\ p \land (q \lor r) \equiv (p \\ p \land (q \lor r) \equiv (p \\ p \land (q \lor r) \Rightarrow (q \lor r) \Rightarrow (q \lor r)$	
De Morgan's laws	$\neg(p \lor q) \equiv \neg p \land \neg q,$	$\neg(p \land q) \equiv \neg p \lor \neg q$
Absorption laws	$p \lor (p \land q) \equiv p,$	$p \wedge (p \vee q) \equiv p$
Negation laws	$p \vee \neg p \equiv T,$	$p \wedge \neg p \equiv F$



De Morgan's Laws, General Version

Copyrighted Material – Do not Distribute

General versions of De Morgan's laws

$$\neg(p_1 \lor \cdots \lor p_n) \equiv \neg p_1 \land \cdots \land \neg p_n, \\ \neg(p_1 \land \cdots \land p_n) \equiv \neg p_1 \lor \cdots \lor \neg p_n.$$

- How do we prove these?
 - \triangleright Truth table would be too big: 2^n rows for a single n
 - ▶ We need more powerful tools, e.g., induction
- We can also write De Morgan's laws as

$$\neg \left(\bigvee_{i=1}^{n} p_{i}\right) \equiv \bigwedge_{i=1}^{n} \neg p_{i}, \qquad \neg \left(\bigwedge_{i=1}^{n} p_{i}\right) \equiv \bigvee_{i=1}^{n} \neg p_{i}$$



Equivalences with Conditionals

Equivalence Containing Conditionals
$p \to q \equiv \neg p \lor q$
$p \to q \equiv \neg q \to \neg p$
$p \lor q \equiv \neg p \to q$
$p \land q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \land \neg q$
$(p \to q) \land (p \to r) \equiv p \to (q \land r)$
$(p \to r) \land (q \to r) \equiv (p \lor q) \to r$
$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$
$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$



Proving $p \land q \equiv \neg(p \rightarrow \neg q)$

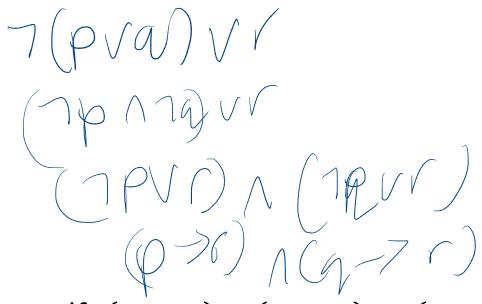
Copyrighted Material – Do not Distribute



Try yourself: $\neg(p \rightarrow q) \equiv p \land \neg q$

Proving $(p \rightarrow q) \land (p \rightarrow r) \equiv p \rightarrow (q \land r)$

 $\begin{array}{c} \text{Copyrighted Material - Do not Distribute} \\ (p > q) \wedge (p > r) \\ (p > q) \wedge (p > r) \end{array}$





Try yourself: $(p \rightarrow r) \land (q \rightarrow r) \equiv (p \lor q) \rightarrow r$

Proving $(p \rightarrow q) \lor (p \rightarrow r) \equiv p \rightarrow (q \lor r)$ Copyrighted Material – Do not Distribute (P) V (P) V) 7(p/a)// MPV 79)V (pv) v (ndvr) P > 0 V (q > r)



► Try yourself: $(p \to r) \lor (q \to r) \equiv (p \land q) \to r$

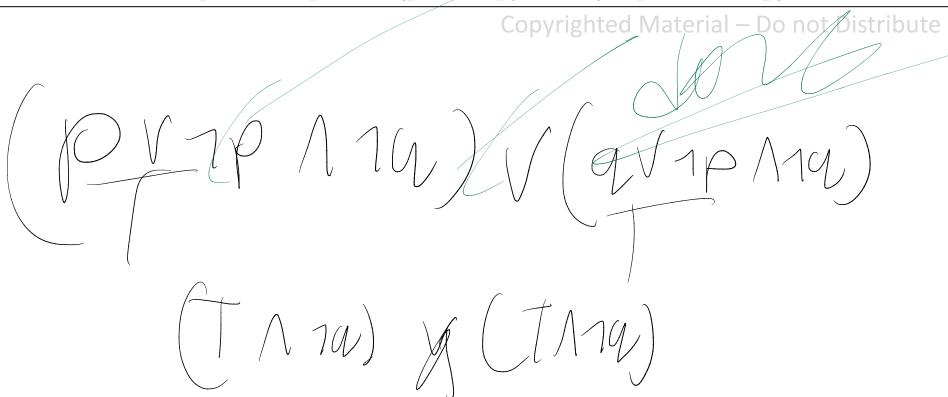
Equivalences with Bidirectionals

Equivalence Containing Bidirectionals
$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

p	q	$\neg p$	$\neg q$	$p \leftrightarrow q$	$p \rightarrow q$	$q \rightarrow p$	$(p \to q) \land (q \to p)$	$\neg p \leftrightarrow \neg q$
T	T							
T	F							
F	T							
F	F							



Proving $p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$





Proving $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Copyri	Equivalence Containing Bidirectionals				
	$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$				
	$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$				
	$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$				
	$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$				

