Lecture 04: Nested Quantifiers

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Outline

- A brief review of lecture 03
- Nested quantifiers
  - Understanding statements with nested quantifiers
  - Order of quantifiers
  - Translations
  - Negating nested quantifiers

- A repeating note: **make sure you read the textbook**

L03: Predicates & Quantifiers

- **Def^n**: A **predicate** is a partial proposition where one or more variables replace part of a propositional statement.
  - **Ex**: $P(x) =$ "$x > 3$"
  - Usually use $P, Q, R, \ldots$ to denote predicates
  - Can be unary ($P(x)$), binary ($Q(x, y)$), and so on
  - **Bound** if variables have fixed values, become propositions

- Quantifiers
  - Universal quantification: $\forall x\ P(x)$
    - True if only if for **every** x in the domain, $P(x)$ holds
  - Existential quantification: $\exists x\ P(x)$
    - True if for **any** x in the domain, $P(x)$ holds

L03: Predicates & Quantifiers

- Quantifiers with restricted domains, e.g., $\forall x \neq 0\ (x^3 \neq 0)$
- Precedence: $\forall, \exists > \neg > \wedge > \vee > \rightarrow > \leftrightarrow$
  - **Ex**: $\forall x P(x) \wedge Q(x) \equiv \big(\forall x P(x)\big) \wedge Q(x)$
- Bound and free variables, e.g., $\exists x\ (x + y = 1)$
- Equivalence involving quantifiers
  - **Ex**: show $\forall x\big(P(x) \wedge Q(x)\big) \equiv \forall x P(x) \wedge \forall x Q(x)$
- Negating statements with quantifiers
  - $\neg\big(\forall x\ P(x)\big) \equiv \exists x\ (\neg P(x))$
  - $\neg\big(\exists x\ P(x)\big) \equiv \forall x\ (\neg P(x))$

Interpreting Nested Quantifiers

- **Ex**: $\forall x \exists y (x + y = 0), x, y \in \mathbb{R}$
  - This is equivalent to $\forall x (\exists y (x + y = 0))$
  - That is, for all x, there exists a y such that $x + y = 0$
  - We can show this is true
    - Let $x = a$ be arbitrary $a \in \mathbb{R}$
    - Then we can let $y = -a$ to satisfy $x + y = 0$
    - Therefore $\forall x \exists y (x + y = 0)$ is a true quantified statement
- **Ex**: $\forall x \forall y (x + y = y + x)$
  - For all x, for all y, $x + y = y + x$
  - This is the commutative law for additions
- In general, we do this layer by layer, **Ex**:
  - $\forall x \exists y P(x, y) = \forall x Q(x)$ with $Q(x) = \exists y P(x, y)$

Interpreting Nested Quantifiers

- **Ex**: $\forall x \forall y \forall z\ (x + (y + z) = (x + y) + z)$
  - This is the associative law of addition of real numbers

- Quantification as loops
  - $\forall x \forall y\ P(x, y)$: "Loop" through all x and then in an inner loop go through all y, for all combinations of x, y $P(x, y) = true$
  - $\forall x \exists y\ P(x, y)$: In this case, for a given x, the inner loop can stop as soon as $P(x, y)$ becomes true for some y
  - $\exists x \forall y\ P(x, y)$: In this case, the outer loop can stop as soon as one inner loop goes through completely
  - $\exists x \exists y\ P(x, y)$: We only need to have a single $P(x, y) = \text{true}$ to hold

Ordering of Quantifiers

- Quantifier orders do not matter for same type
  - $\forall x \forall y\ P(x, y) \equiv \forall y \forall x\ P(x, y)$,
  - $\exists x \exists y\ P(x, y) \equiv \exists y \exists x\ P(x, y)$

- Ordering matters for mixed quantifier types. In general,
  - $\forall x \exists y\ P(x, y) \neq \exists y \forall x\ P(x, y)$
  - Note that $\exists x \forall y\ P(x, y) \neq \forall y \exists x\ P(x, y)$ is the basically same statement
- **Ex**: x, y persons, $P(x, y)$: x and y are best friends
  - What is $\forall x \exists y\ P(x, y)$?
  - What is $\exists y \forall x\ P(x, y)$?

Ordering of Quantifiers, cont.

- Let $P(x, y) =$ "$x + y = 0$", $x, y \in \mathbb{R}$
  - What is $\forall x \exists y\ P(x, y)$?
    - For every (real number) x, there is some y such that $x + y = 0$
    - For every fixed $x = a$, we can let $y = -a$, so $x + y = 0$
    - $\forall x \exists y\ P(x, y)$ is true!
  - What about $\exists y \forall x\ P(x, y)$?
    - There is some y such that for any (real number) x, $x + y = 0$
    - This is not posible!
    - $\exists y \forall x\ P(x, y)$ is false
- What about $Q(x, y) =$ "$xy = 0$"?
  - In this case, $\forall x \exists y\ Q(x, y) = \exists y \forall x\ Q(x, y)$
- Read Example 5 in the textbook and pay attention to Table 1

Translating Statements using Quantifiers

- **Ex**: The sum of two positive integers is always positive"
  - Note that the translation is "domain dependent"
  - Let x, y be integers, then the translation can be
    - $\forall x \forall y\ (((x > 0) \land (y > 0)) \rightarrow (x + y > 0))$
  - Let x, y be in the **domain of positive integers**, then
    - $\forall x \forall y\ (x + y > 0)$
- **Ex**: Every non-zero real number has a multiplicative inverse

- Read Example 8 from the textbook

Translating into English

- **Ex**: $\forall x\left(C(x) \vee \exists y(C(y) \wedge F(x,y))\right)$
  - x and y are in the domain of students
  - $C(a)$: a has a computer, $F(a,b)$: a and b are friends
  - Every student has a computer or has a friend who has

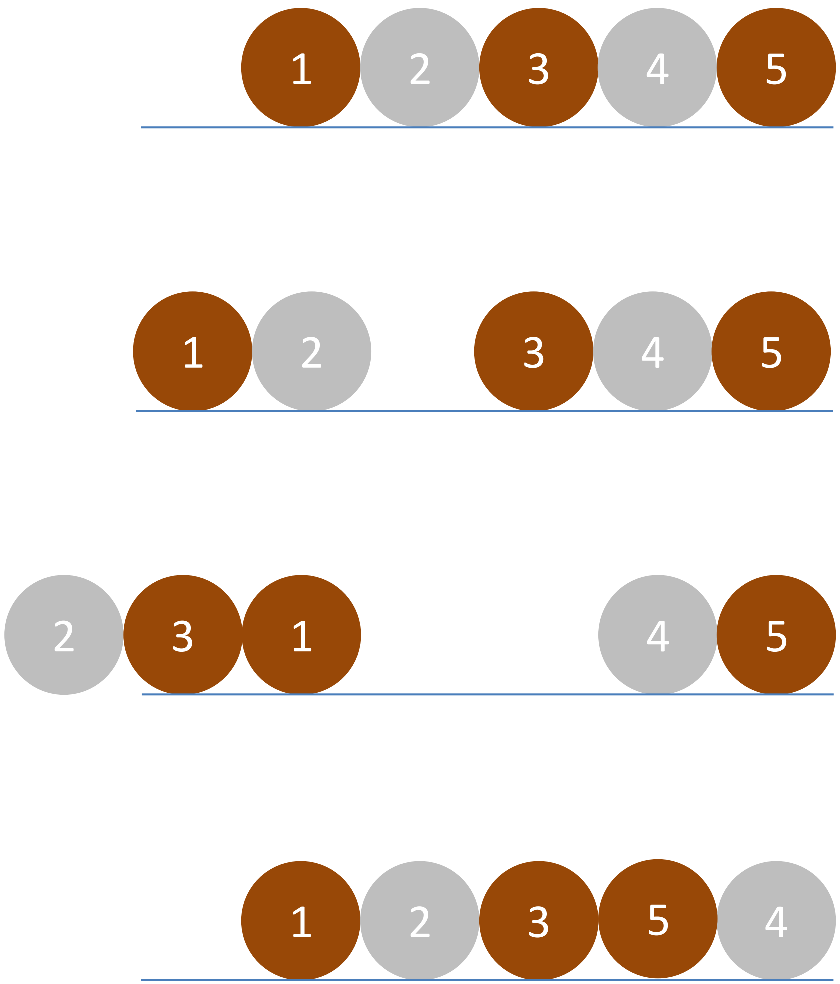
- **Ex**: $\exists x \forall y \forall z\left((F(x,y) \wedge F(x,z) \wedge (y \neq z)) \rightarrow \neg F(y,z)\right)$
  - $\exists x \forall y \forall z \rightarrow$ There is a student x and for all students y, z
  - $F(x,y) \wedge F(x,z) \wedge (y \neq z) \rightarrow x$ is friend with both y and z
  - $\neg F(y,z) \rightarrow y$ and z are not friends
  - There is a student none of whose friends are friends with each other

Negating Nested Quantifiers

- Recall: $\neg\forall x P(x) = \exists x(\neg P(x))$ and $\neg\exists x P(x) = \forall x(\neg P(x))$
  - Negating nested quantifiers is just doing this one at a time
  - $\neg\forall x\forall y P(x,y) = \exists x\neg(\forall y P(x,y)) = \exists x\exists y(\neg P(x,y))$
  - $\neg\exists x\exists y P(x,y) = \forall x\neg(\exists y P(x,y)) = \forall x\forall y(\neg P(x,y))$
  - $\neg\forall x\exists y P(x,y) = \exists x\neg(\exists y P(x,y)) = \exists x\forall y(\neg P(x,y))$
  - $\neg\exists x\forall y P(x,y) = \forall x\neg(\forall y P(x,y)) = \forall x\exists y(\neg P(x,y))$

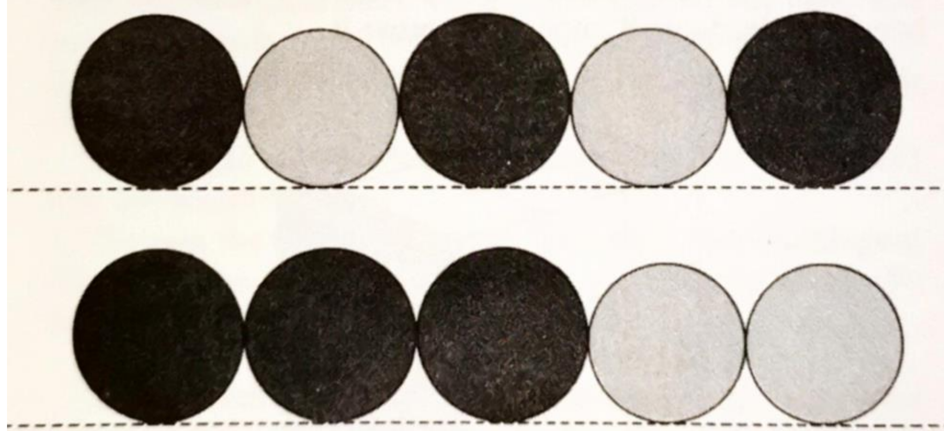
- We can now more easily state when a nested quantification is a true or false

Coin Rearrangement



ARRANGE THREE pennies and two dimes in a row, alternating the coins as shown. The problem is to change their positions to those shown at the bottom of the illustration in the shortest possible number of moves.

A move consists of placing the tips of the first and second fingers on any two touching coins, *one of which must be a penny and the other a dime,* then sliding the pair to another spot along the imaginary line shown in the illustration. The two coins in the pair must touch at all times. The coin at left in the pair must remain at left; the coin at right must remain at right. Gaps in the chain are allowed at the end of any move except the final one. After the last move the coins need not be at the same spot on the imaginary line that they occupied at the start.



If it were permissible to shift two coins of the same kind, the puzzle could be solved easily in three moves: slide 1, 2 to left, fill the gap with 4, 5, then move 5, 3 from right to left end. But with the proviso that each shifted pair must include a dime and penny it is a baffling and pretty problem.

Martin Gardner: My best mathematical and logic puzzles