Max-Flow Min-Cut

Outline for Today

Max-Flow Min-Cut

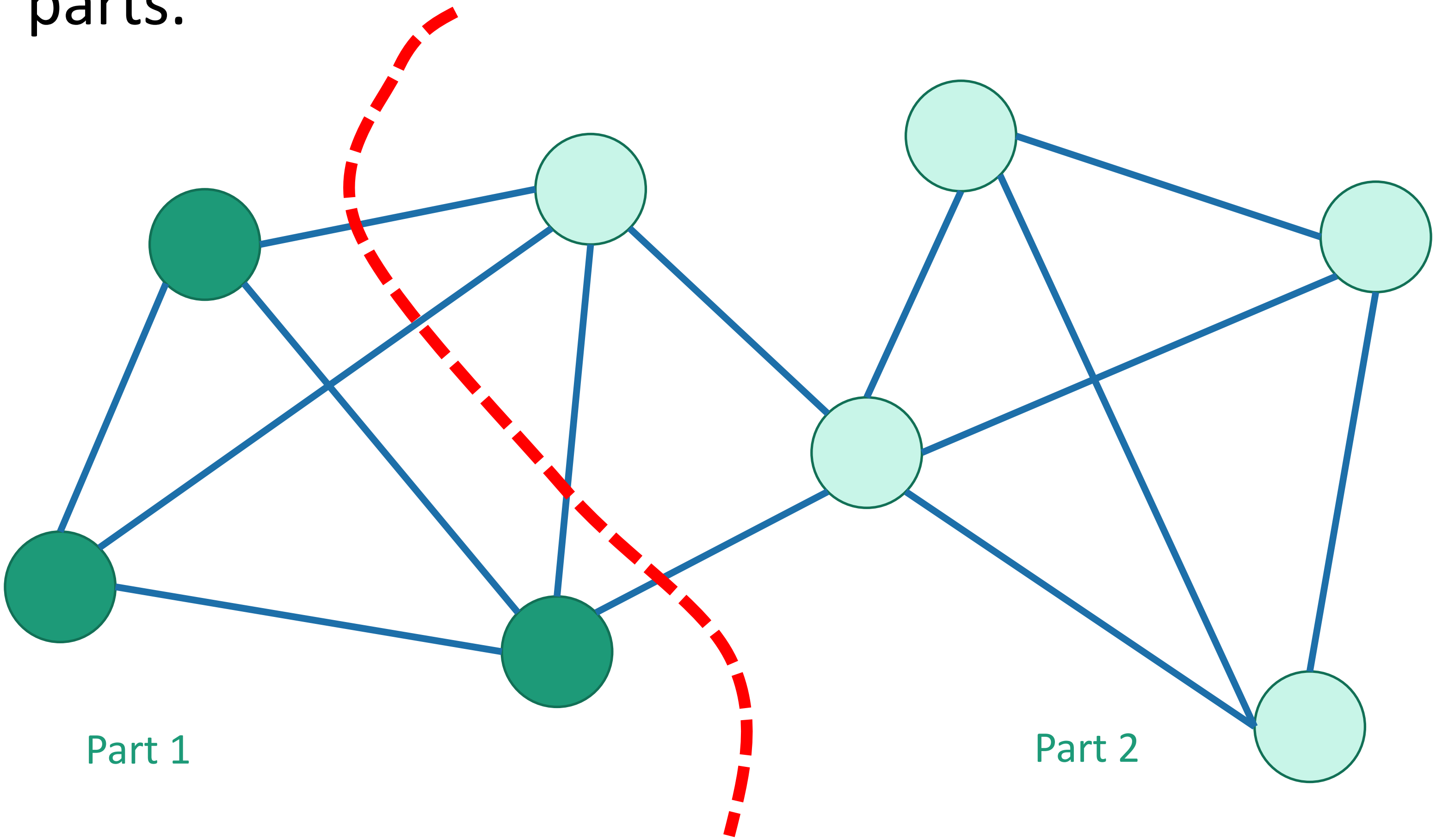
- Background
- Ford-Fulkerson Algorithm

Max-Flow Min-Cut

Last time

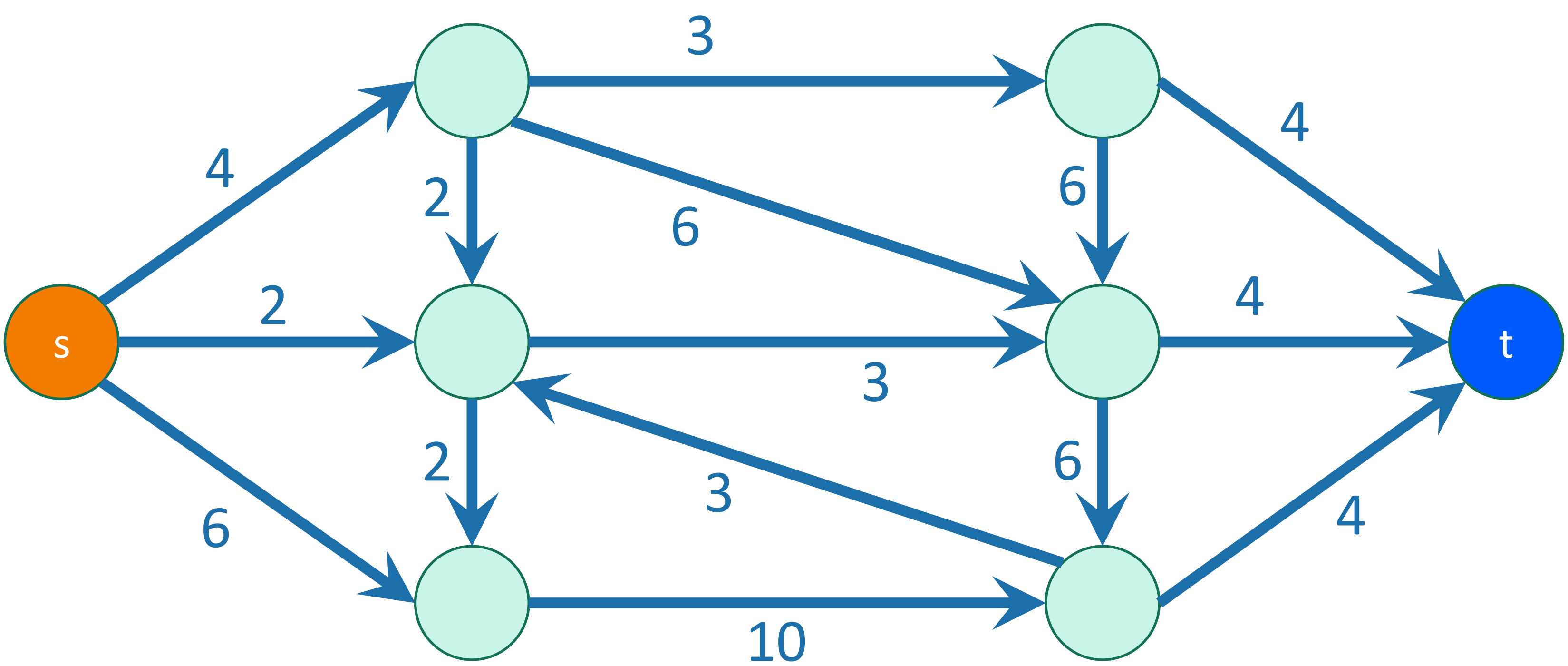
Last time graphs were undirected and unweighted.

- We talked about global min-cuts by Karger's Algorithm
- A cut is a partition of the vertices into two nonempty parts.



Today

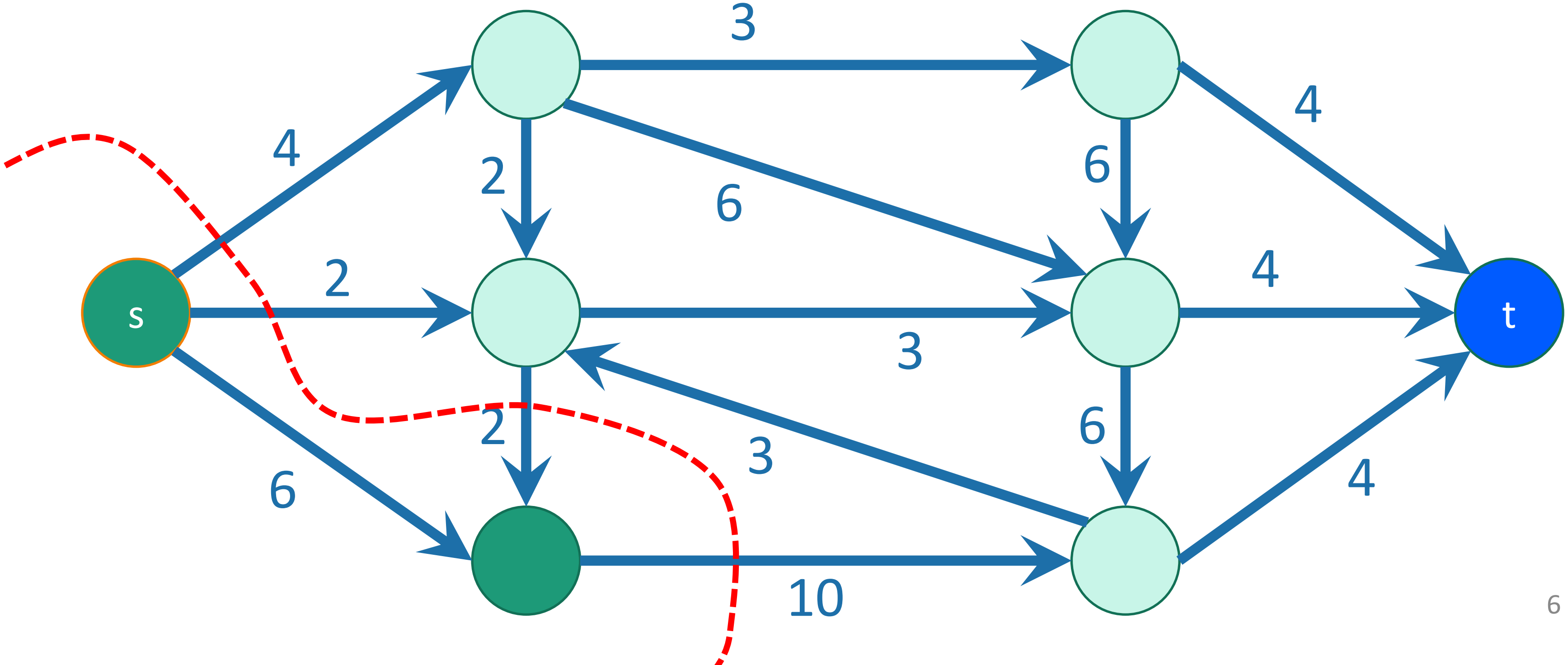
- Graphs are directed and edges have "capacities" (weights)
- We have a special "source" vertex s and "sink" vertex t.
  - s has only outgoing edges*
  - t has only incoming edges*



*at least for this class

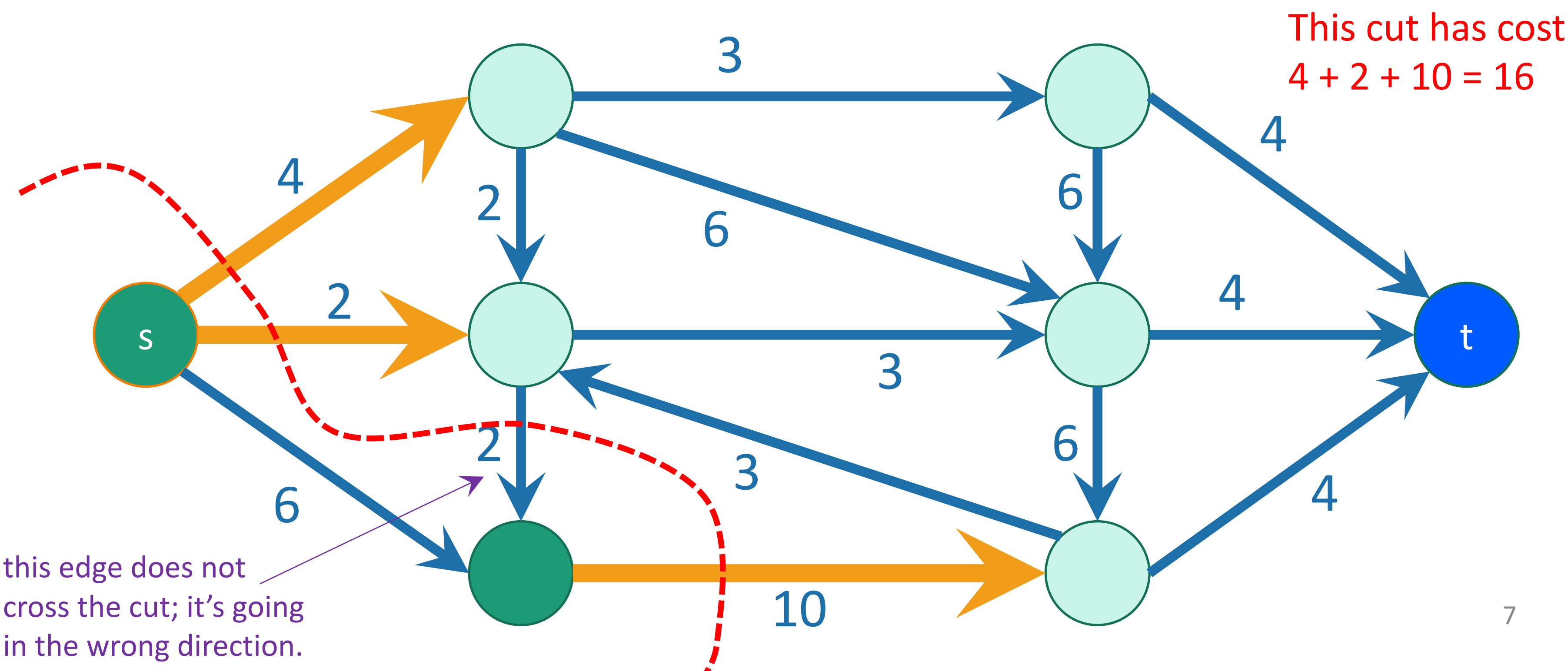
An **s-t cut**

is a cut which separates s from t



An **s-t cut**
is a cut which separates s from t

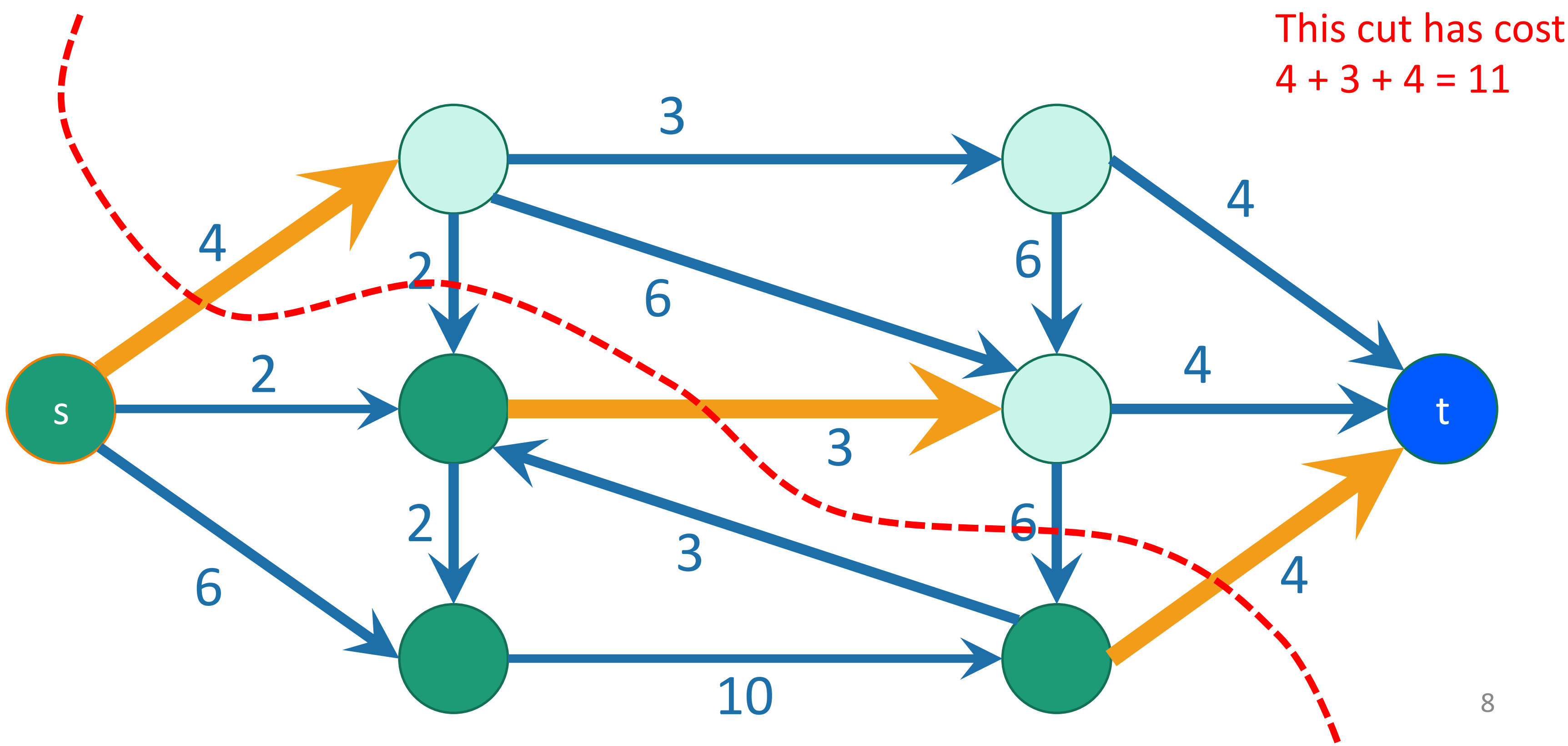
- An edge **crosses the cut** if it goes from s's side to t's side.
- The **cost** (or capacity) of a cut is the sum of the capacities of the edges that cross the cut.



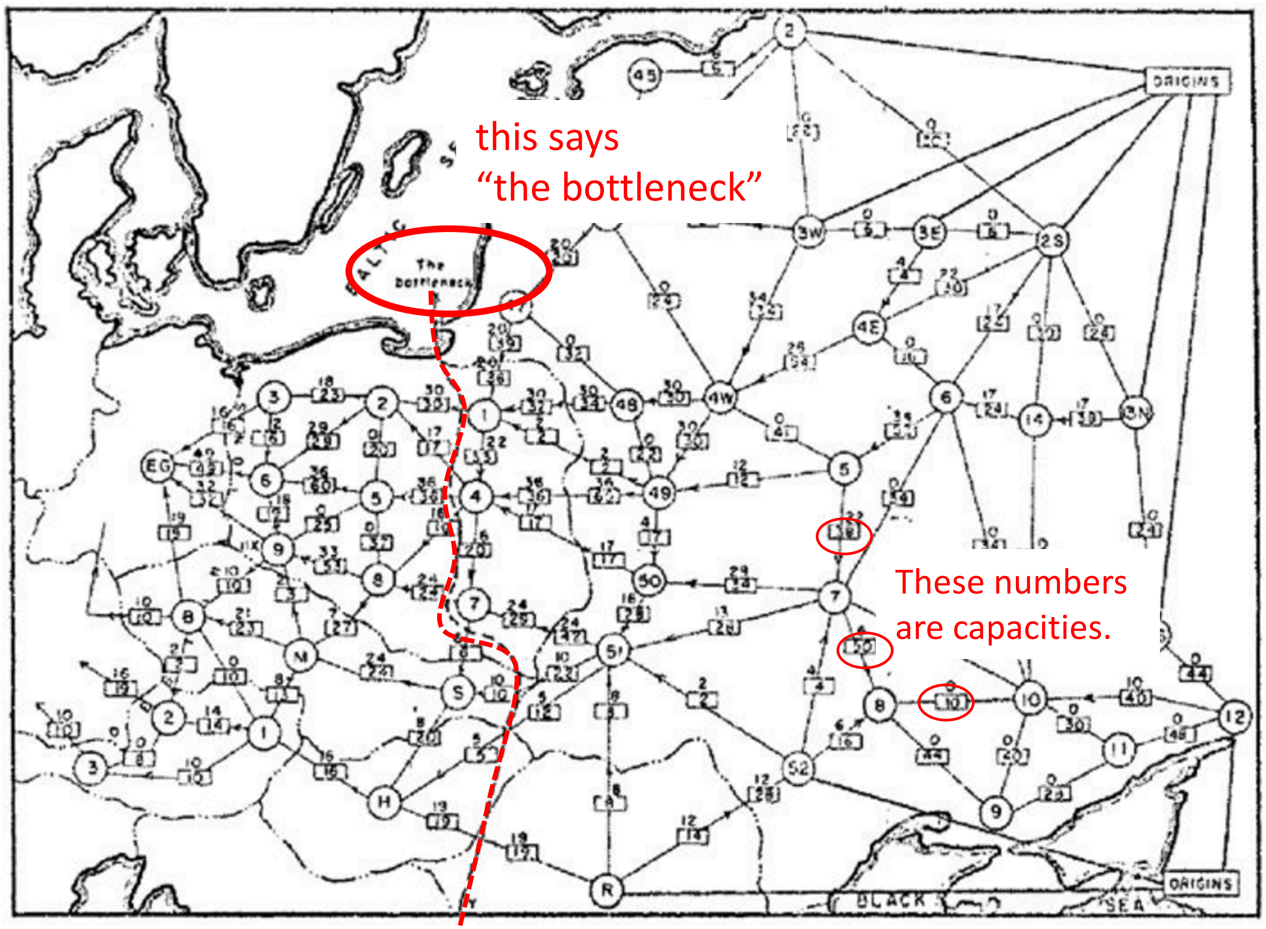
A minimum **s-t cut**

is a cut which separates s from t
with minimum capacity.

- Question: how do we find a minimum s-t cut?



Example where this comes up

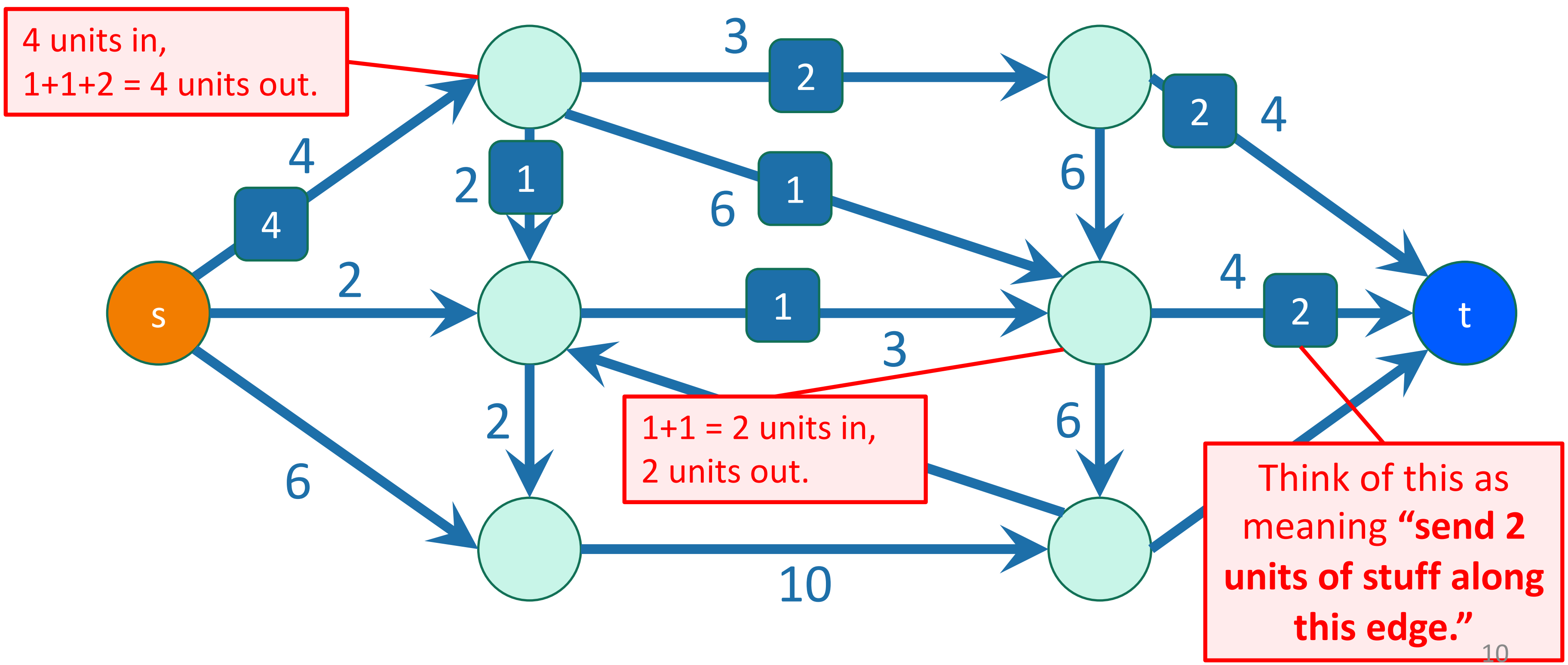


Schriver 2002

- 1955 map of rail networks from the Soviet Union to Eastern Europe.
  - Declassified in 1999.
  - 44 edges, 105 vertices
- The US wanted to cut off routes from suppliers in Russia to Eastern Europe as efficiently as possible.
- In 1955, Ford and Fulkerson at the RAND corporation gave an algorithm which finds the optimal s-t cut.

Flows

- In addition to a capacity, each edge has a flow
  - (unmarked edges in the picture have flow 0)
- The flow on an edge must be less that its capacity.
- At each vertex, the incoming flows must equal the outgoing flows.



Flows

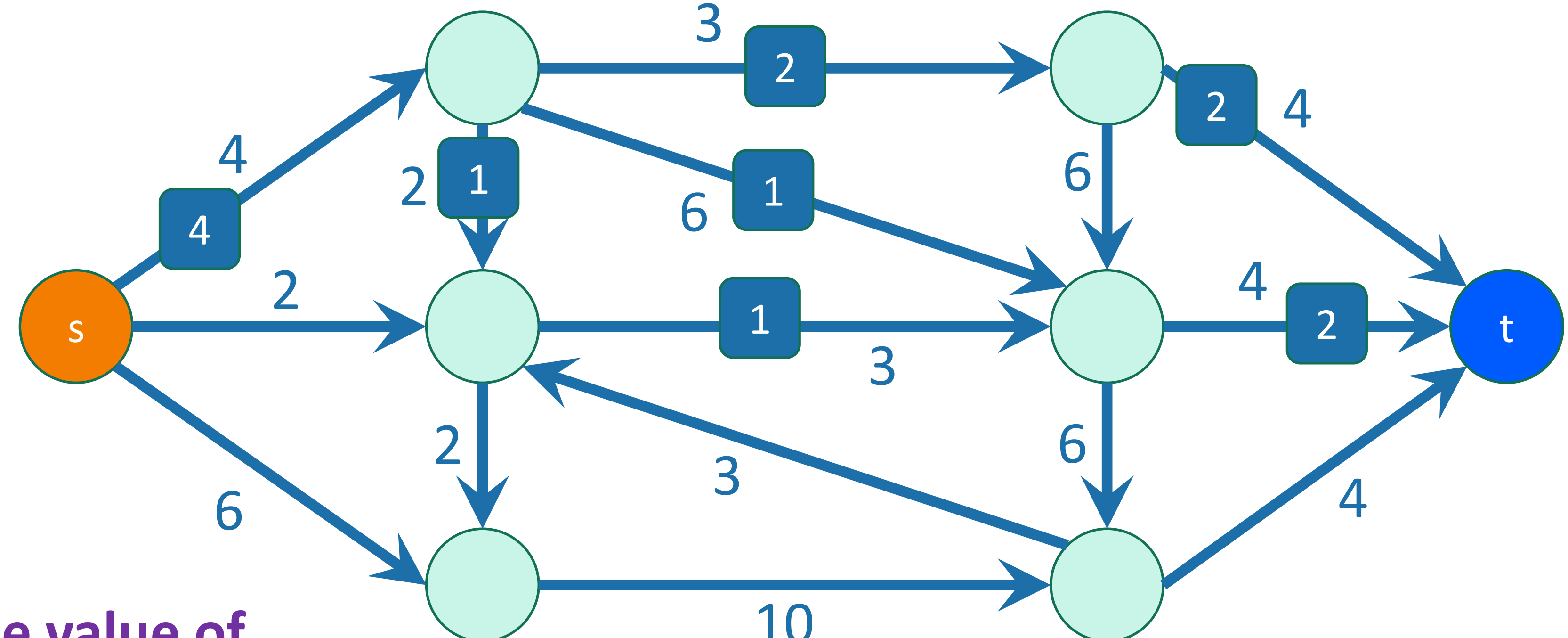
- The **value of a flow** is:
  - The amount of stuff coming out of s
  - The amount of stuff flowing into t
  - These are the same!

Because of conservation of flows at vertices,

stuff you put in

=

stuff you take out.

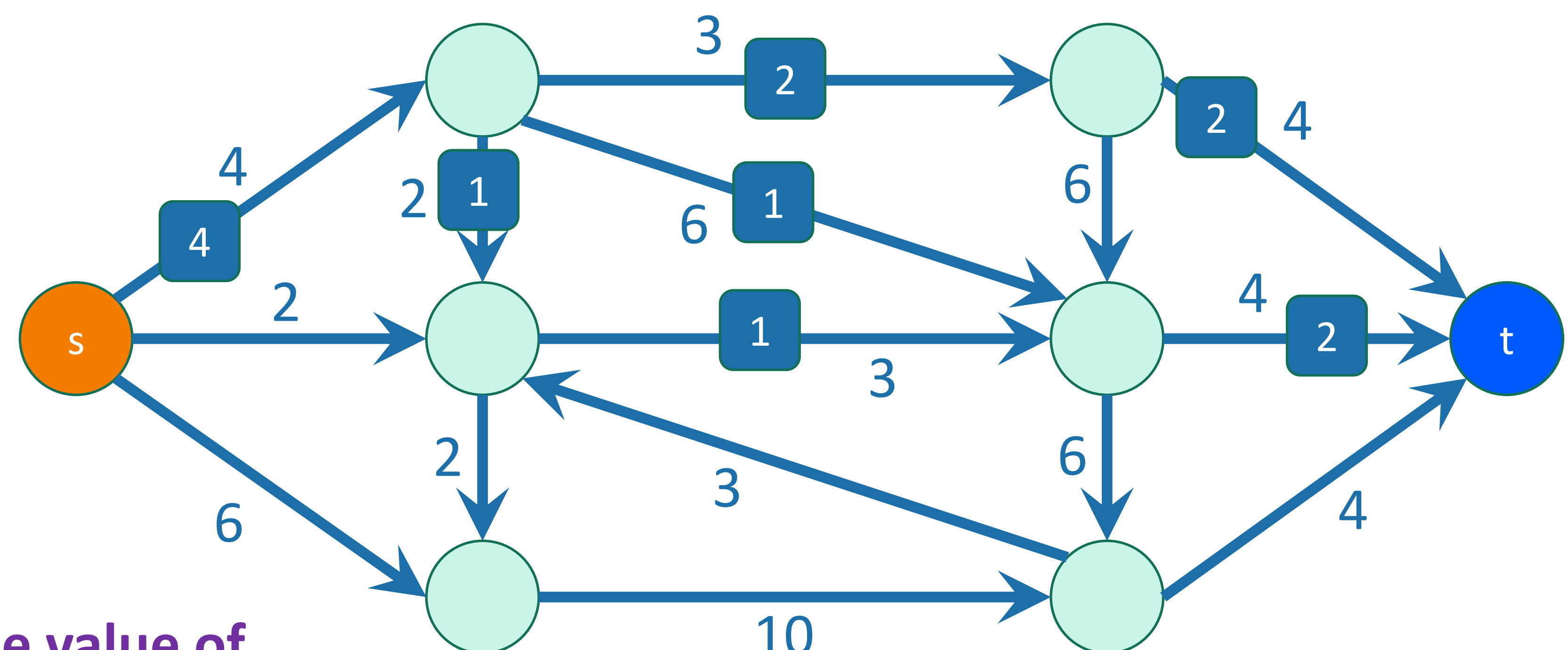


The value of this flow is 4.

A maximum flow

is a flow of maximum value.

- This example flow is pretty wasteful, I'm not utilizing the capacities very well.



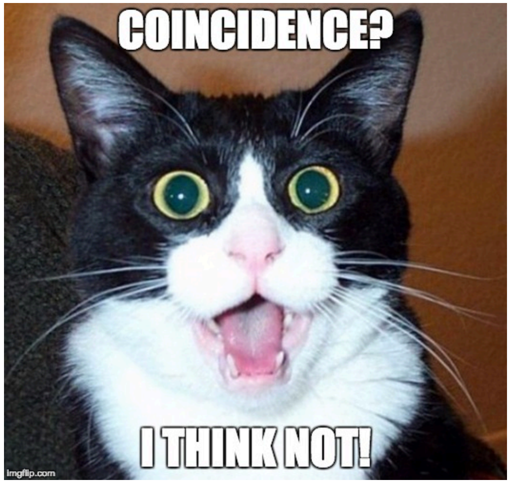
The value of this flow is 4.

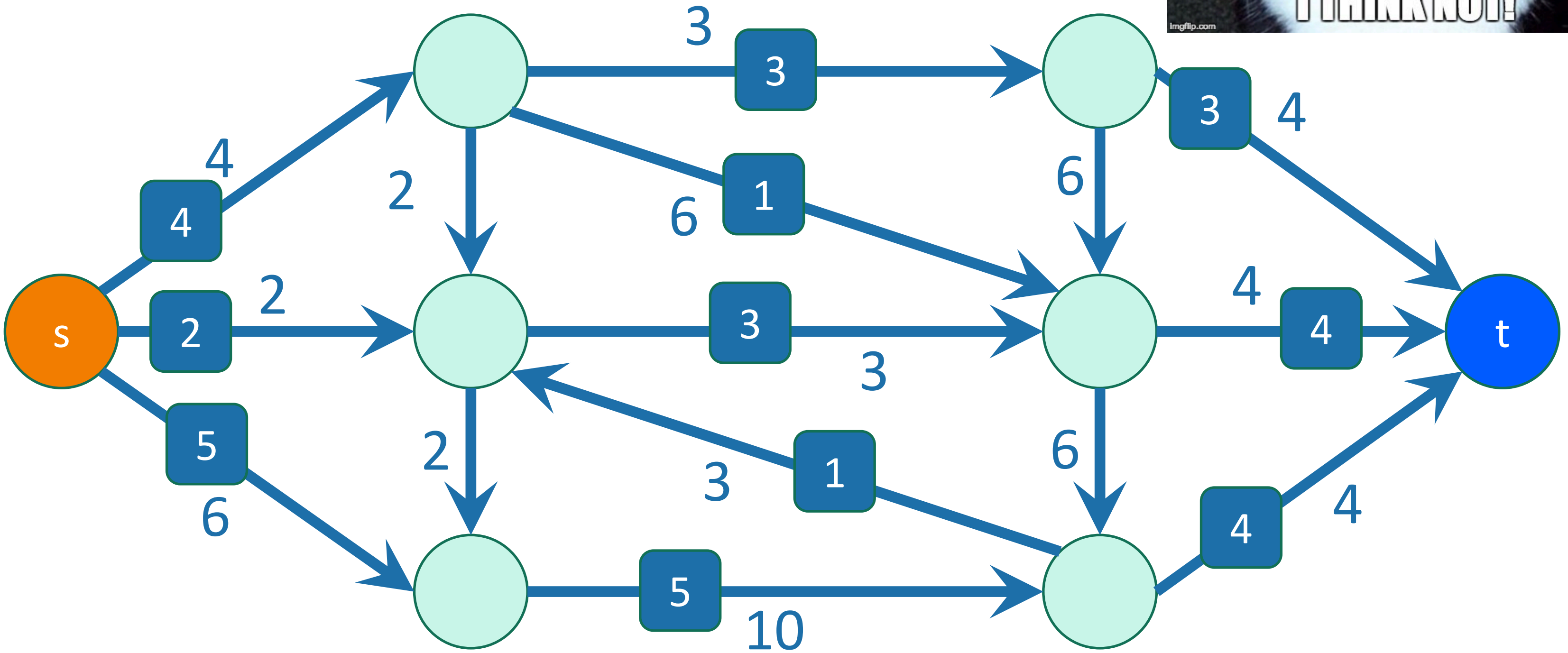
A maximum flow

is a flow of maximum value.

- This one is maximal; it has value 11.

That's the same as the minimum cut in this graph!



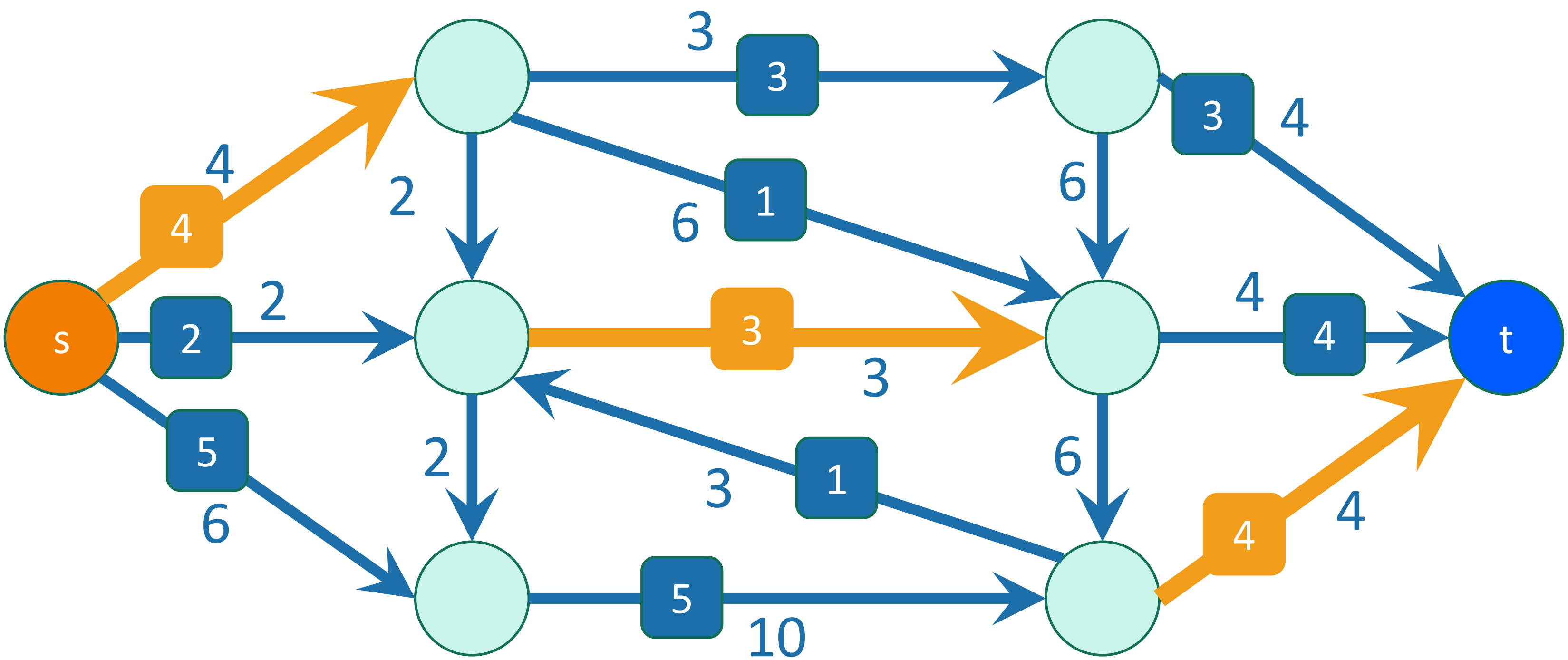


Theorem

Max-flow min-cut theorem

The value of a max flow from s to t
is equal to
the cost of a min s-t cut.

Intuition: in a max flow, the min cut better fill up, and this is the bottleneck.



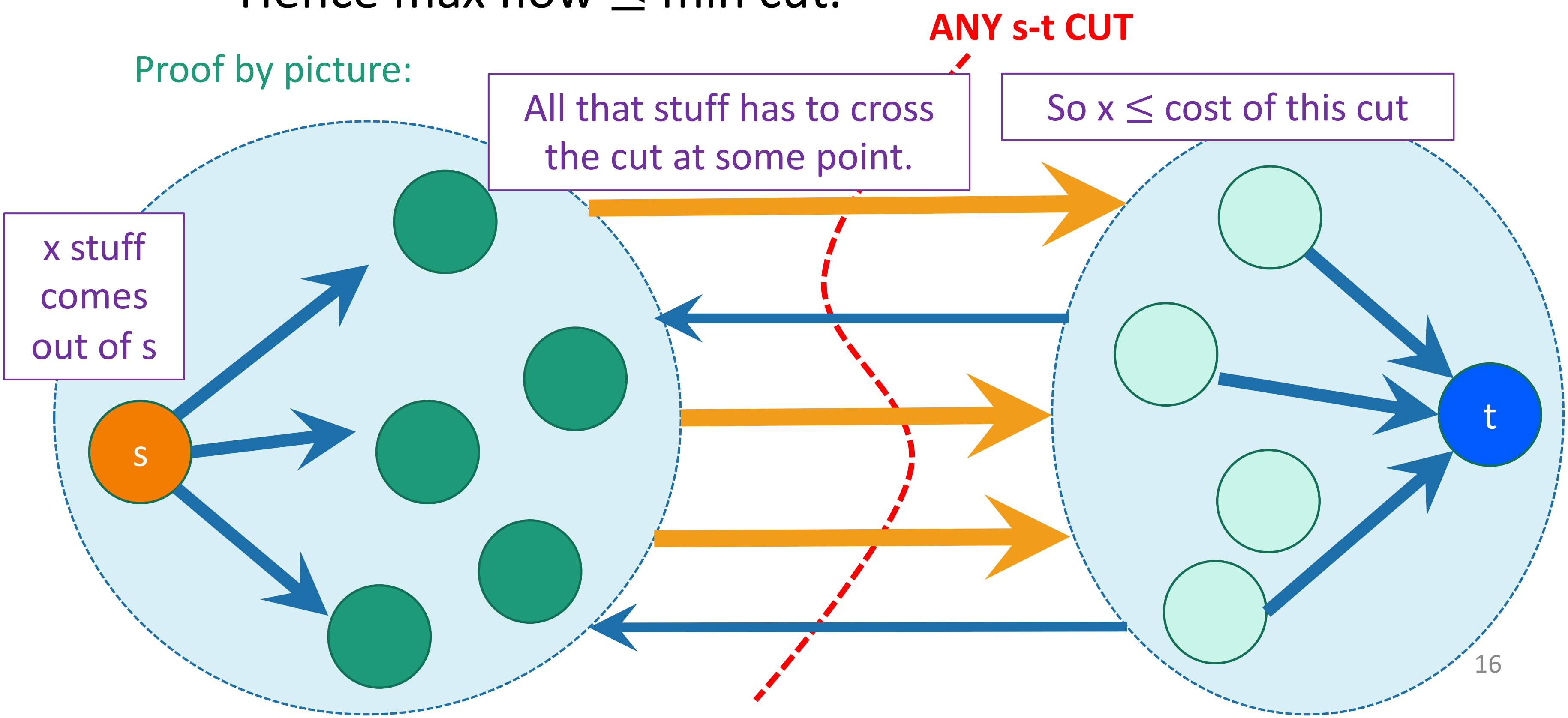
Proof outline

- Lemma 1: max flow $\leq$ min cut.
  - Proof-by-picture
- Lemma 2: max flow $\geq$ min cut.
  - Proof-by-algorithm, using a "Residual graph" G_f
  - Sub-Lemma: t is not reachable from s in G_f $\Leftrightarrow$ f is a max flow.
    - $\Leftarrow$ first we do this direction:
    - Claim: If there is a path from s to t in G_f, then we can increase the flow in *G.*
    - Hence we couldn't have started with a max flow.
    - $\Rightarrow$ for this direction, proof-by-picture again.

This claim actually gives us an algorithm: Find paths from s to t in G_f and keep increasing the flow until you can't anymore.

Proof of Min-Cut Max-Flow Theorem

- **Lemma 1:**
  - For ANY s-t flow and ANY s-t cut, the value of the flow is at most the cost of the cut.
  - Hence max flow ≤ min cut.



Proof of Min-Cut Max-Flow Theorem

- **Lemma 1:**
  - For ANY s-t flow and ANY s-t cut, the value of the flow is at most the cost of the cut.
  - Hence max flow $\leq$ min cut.

- That was proof-by-picture.
- See the notes for proof-by-proof.
  - You are **not** responsible for proof-by-proof on the final.

Proof of Min-Cut Max-Flow Theorem

- **Lemma 1:**
  - For ANY s-t flow and ANY s-t cut, the value of the flow is at most the cost of the cut.
  - Hence max flow $\leq$ min cut.
- The theorem is stronger:
  - max flow $=$ min cut
  - Need to show max flow $\geq$ min cut.
  - Next: Proof by algorithm!

5-min Break

Proof of Max-Flow Min-Cut Theorem I

Ford-Fulkerson algorithm

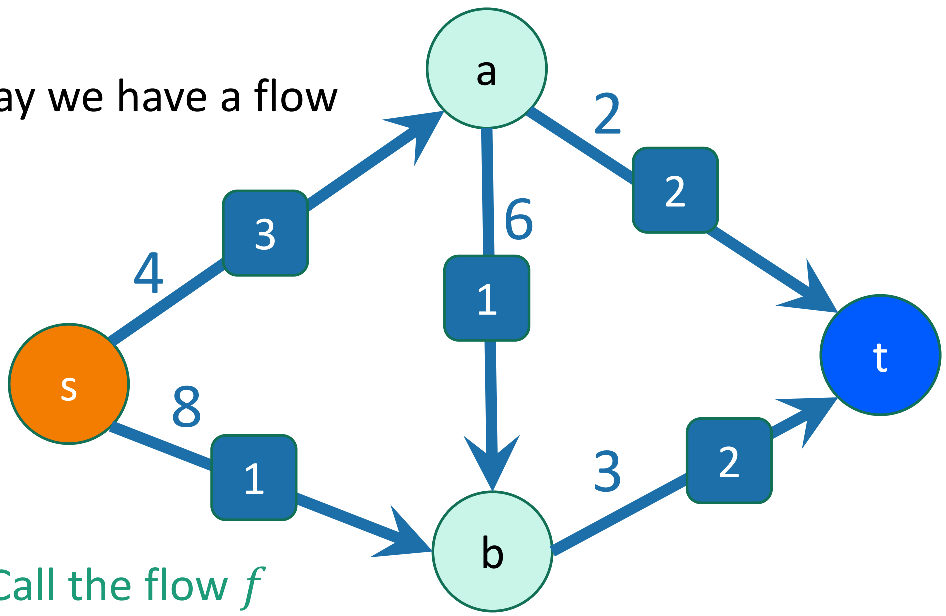
- Usually we state the algorithm first and then prove that it works.
- Today we're going to just start with the proof, and this will inspire the algorithm.

Outline of algorithm:

- Start with zero flow
- We will maintain a **"residual graph"** G_f
- A path from s to t in G_f will give us a way to improve our flow.
- We will continue until there are no s-t paths left.

Tool: Residual networks

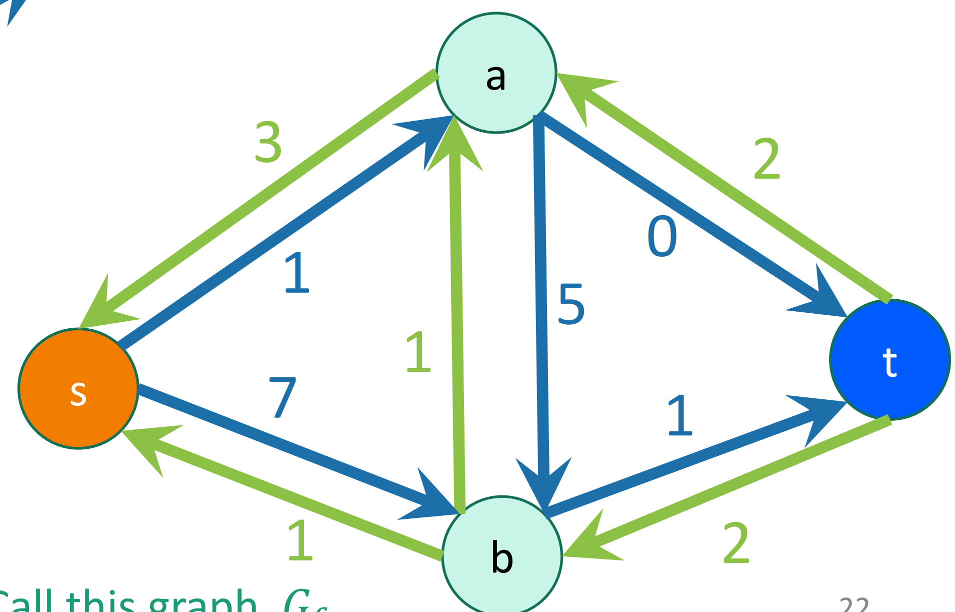
Say we have a flow



Call the flow f
Call the graph G

Create a new **residual network** from this flow:

Forward edges are the amount that's left.
Backwards edges are the amount that's been used.



Call this graph G_f

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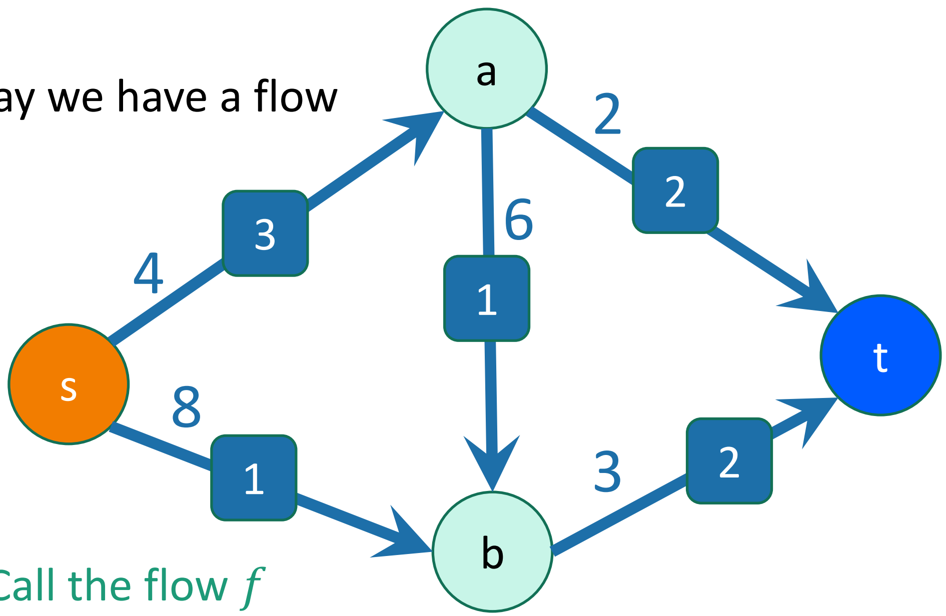

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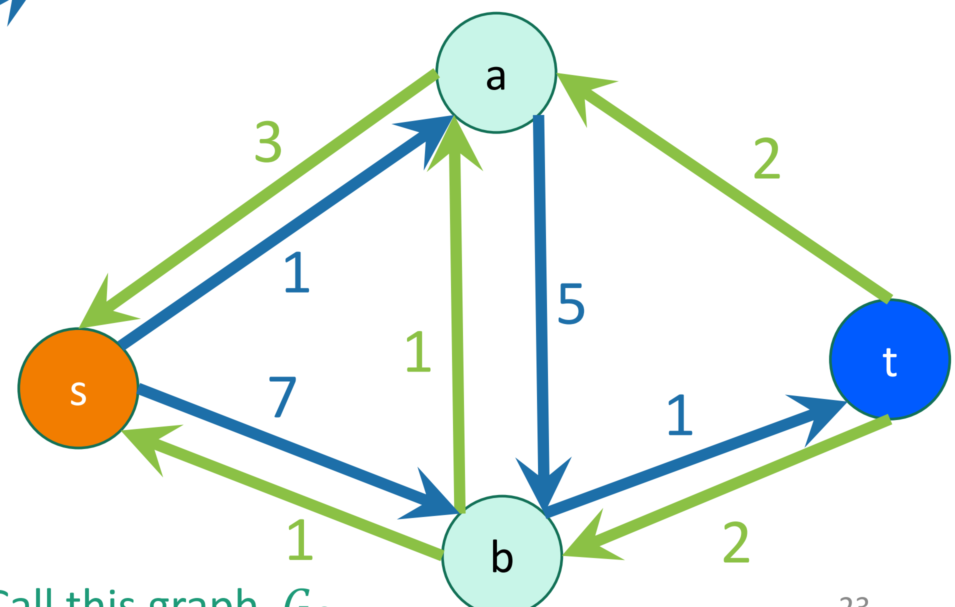


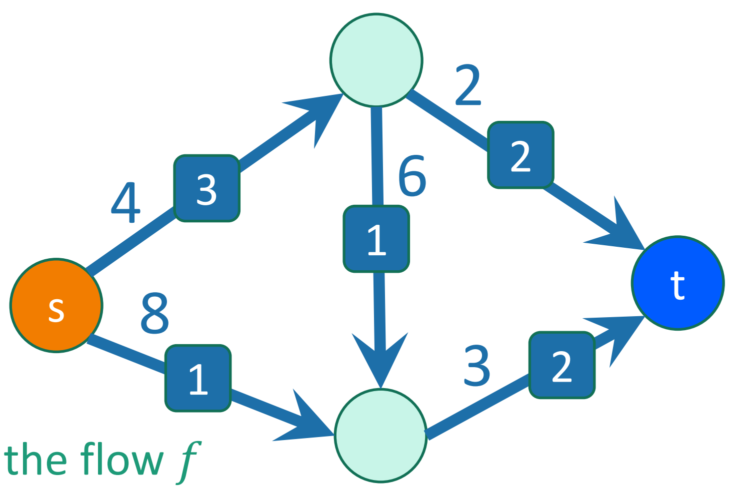
Call this graph G_f

Why look at residual networks?

Lemma:

- t is not reachable from s in G_f $\Leftrightarrow$ f is a max flow.

Example: **t is reachable from s in this example, so not a max flow.**



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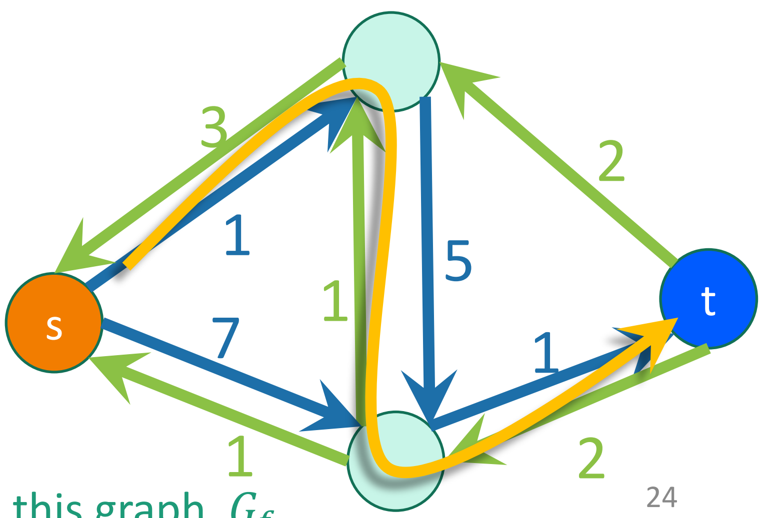
Call the graph G



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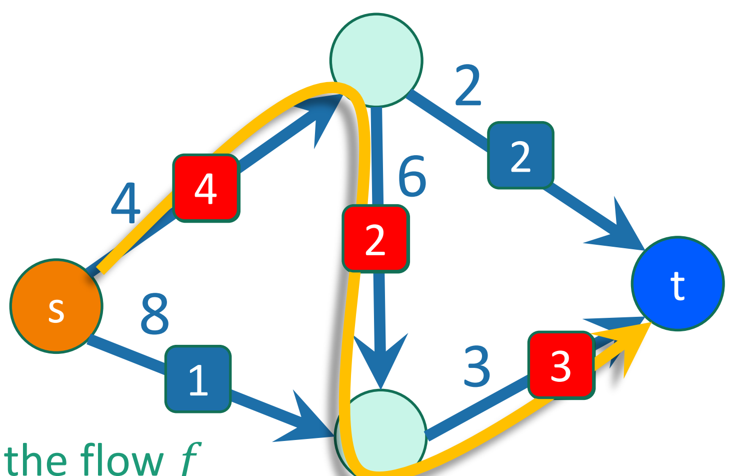
To see that this flow is not maximal, notice that we can improve it by sending one more unit more stuff along this path:



Call the flow f

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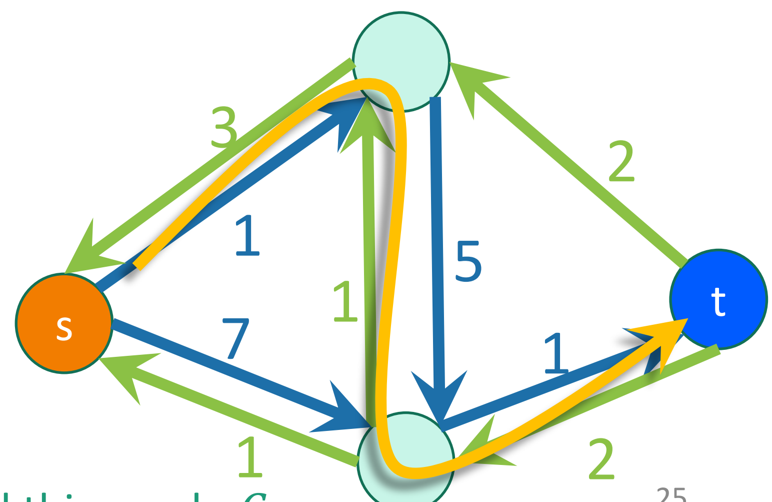
Example: **t is reachable from s in this example, so not a max flow.**

Now update the residual graph…



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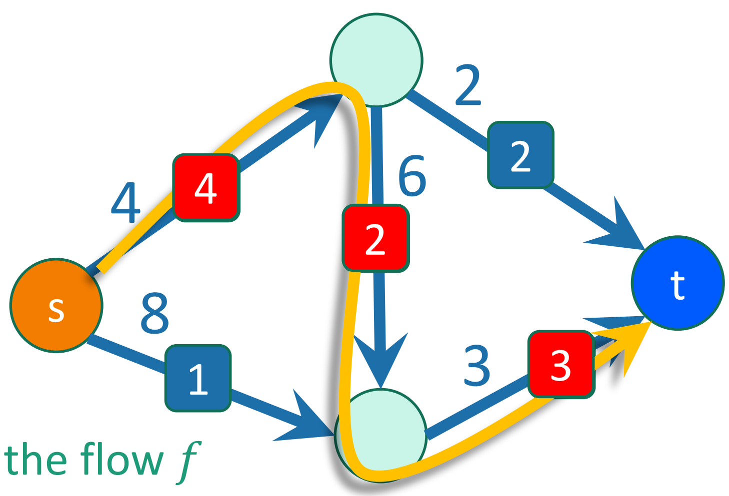
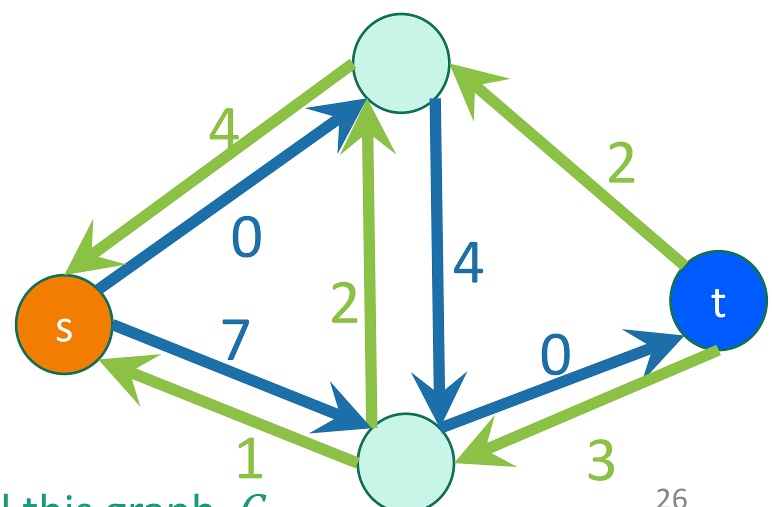


Call the flow f

Call the graph G

Example:

Now we get this residual graph:



Call this graph G_f

Why look at residual networks?

Lemma:

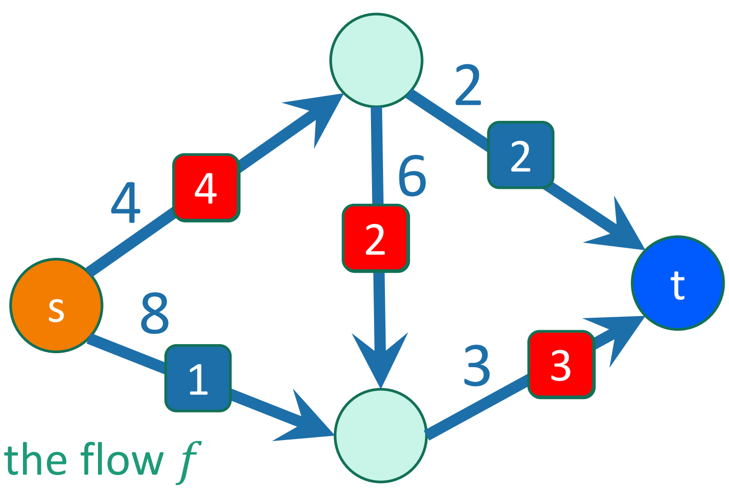
- t is not reachable from s in G_f $\Leftrightarrow$ f is a max flow.

Example:

Now we get this residual graph:

Now we can't reach t from s.

So the lemma says that f is a max flow.



Call the flow f

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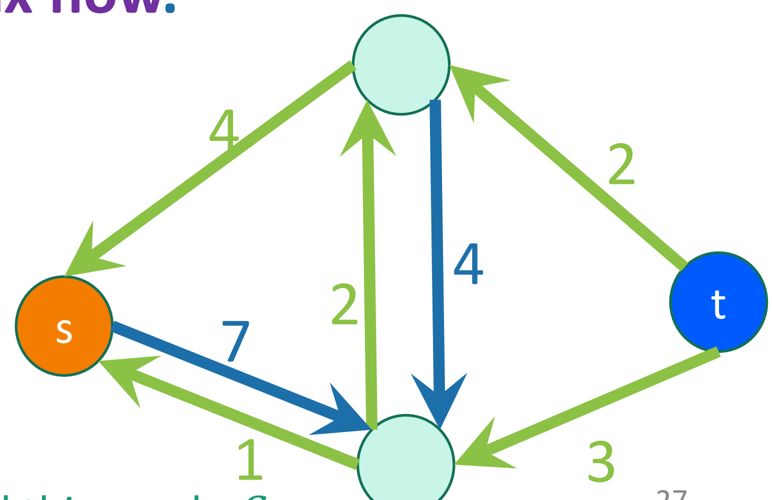


Call this graph G_f

Let’s prove the Lemma

- t is not reachable from s in G_f $\Leftrightarrow$ f is a max flow.

Lemma:

$\Leftarrow$ first this direction $\Leftarrow$

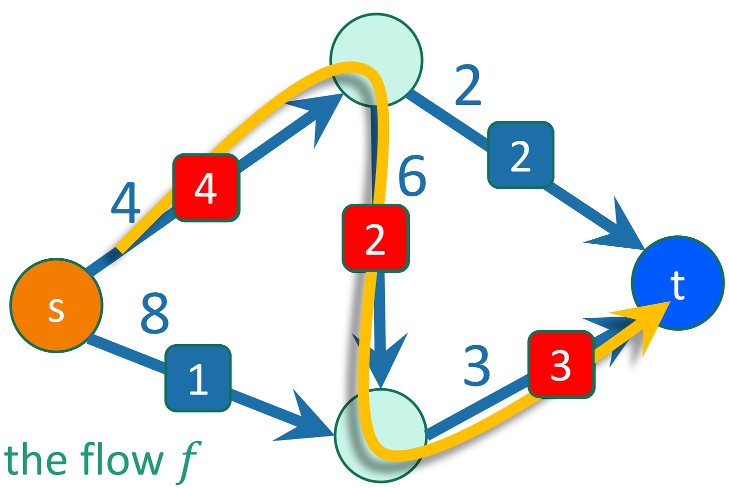
We will prove the contrapositive

t is not reachable from s in G_f $\Leftrightarrow$ f is a max flow.

- Suppose there is a path from s to t in G_f.
  - This is called an augmenting path.
- **Claim:** if there is an augmenting path, we can increase the flow along that path.

we will come back to this in a second.

- This results in a bigger flow
  - so we can't have started with a max flow.



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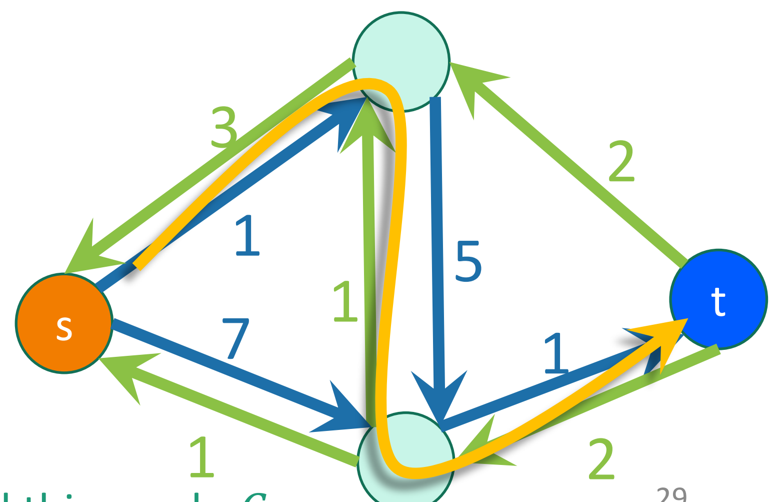
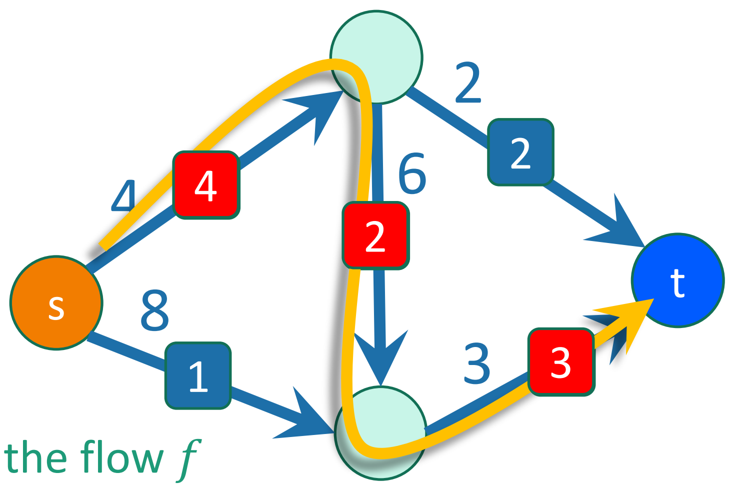


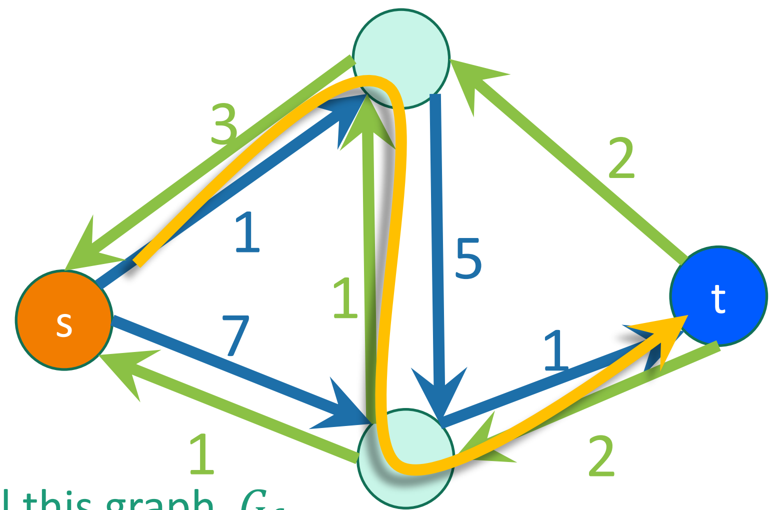
Call this graph G_f

claim:

if there is an augmenting path, we can increase the flow along that path.

- In the situation we just saw, this is pretty obvious.





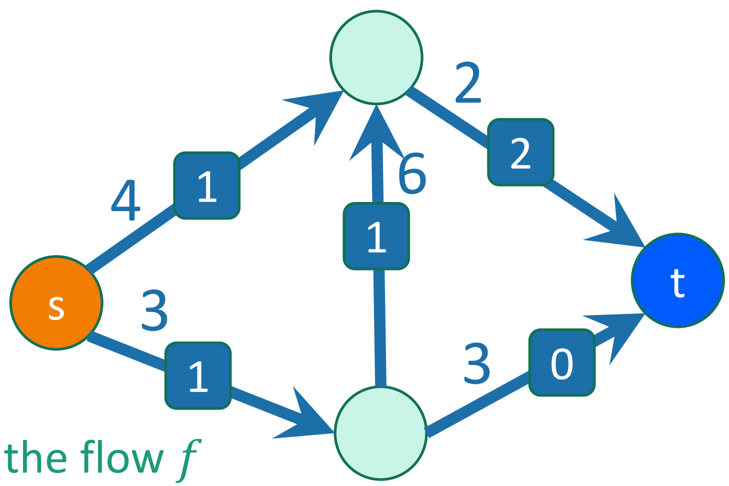
- Every edge on the path in G_f was a **forward edge**, so increase the flow on all the edges.

aka, an edge indicating how much stuff can still go through

claim:

if there is an augmenting path, we can increase the flow along that path.

- But maybe there are **backward edges** in the path.
  - Here's a slightly different example of a flow:



Call the flow f
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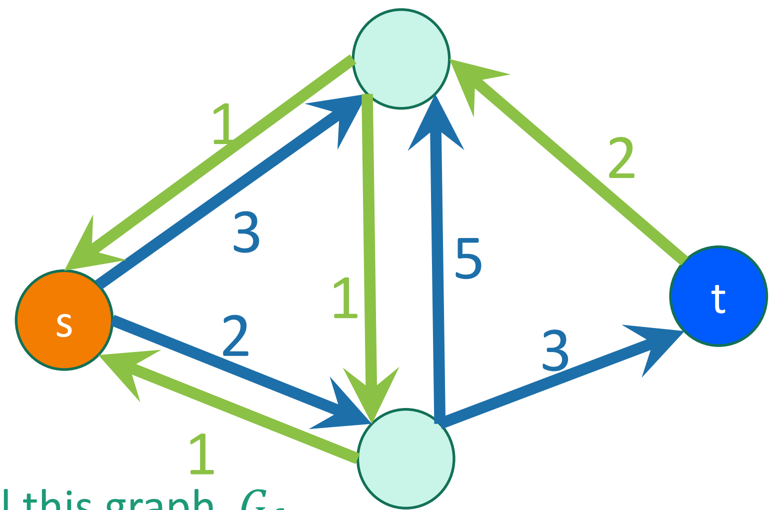

Call this graph G_f

I changed some of the weights and edge directions.

claim:

if there is an augmenting path, we can increase the flow along that path.

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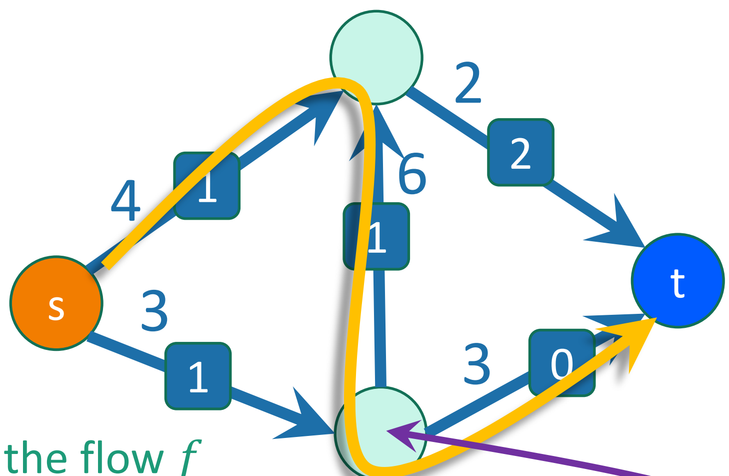

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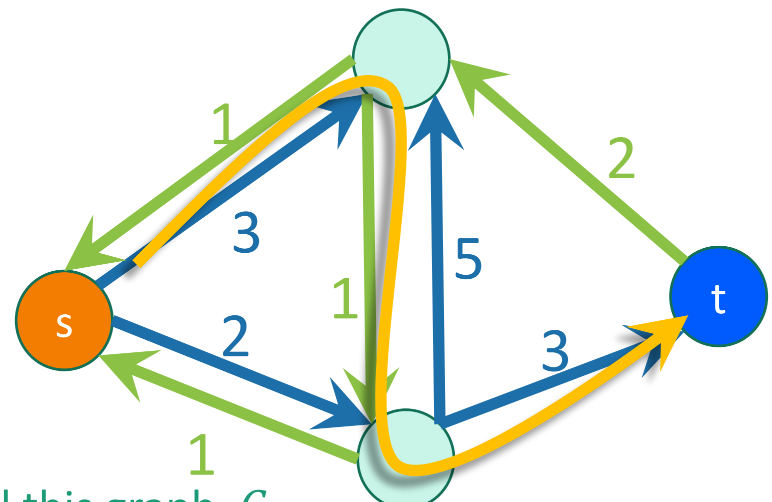


Call this graph G_f

Now we should NOT increase the flow at all the edges along the path!

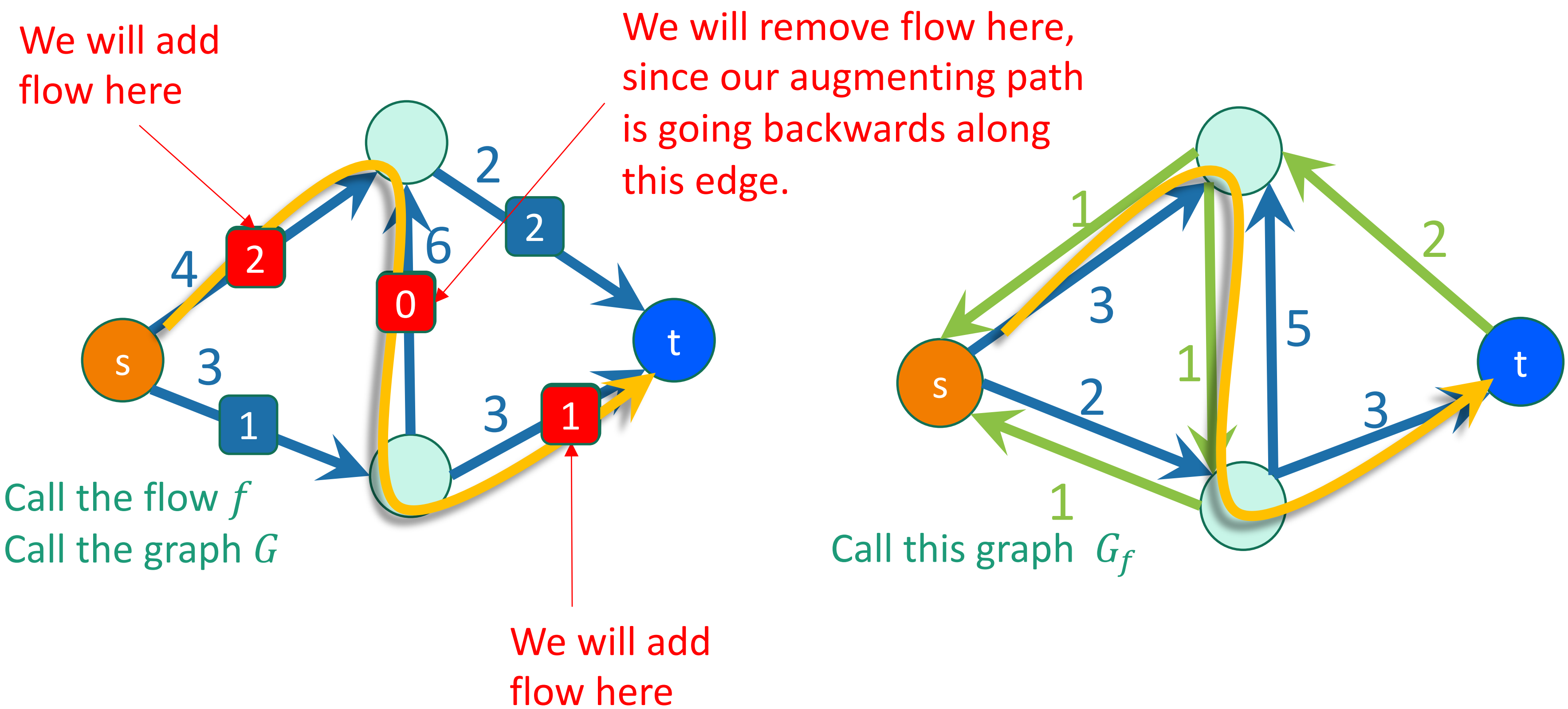
- For example, that will mess up the conservation of stuff at this vertex.

I changed some of the weights and edge directions.

claim:

if there is an augmenting path, we can increase the flow along that path.

- In this case we do something a bit different:



claim:

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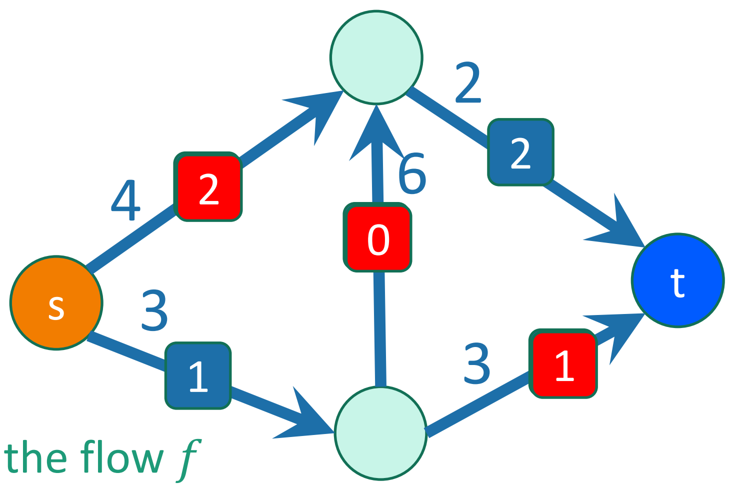
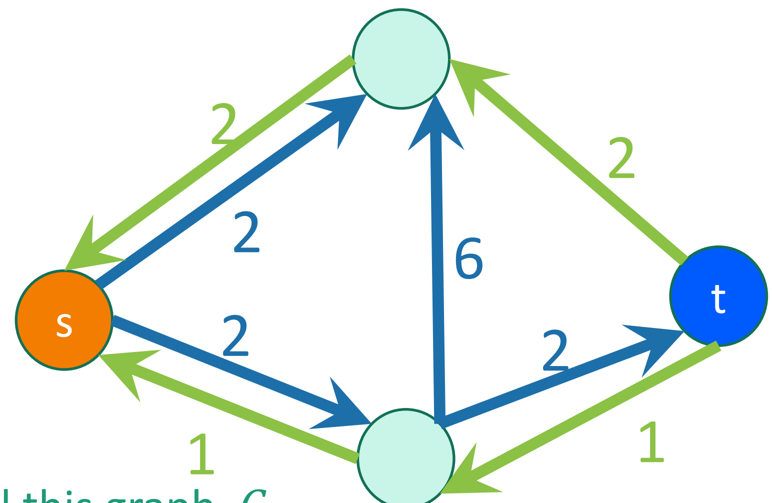
- In this case we do something a bit different:

Then we'll update the residual graph:



Call the flow f
Call the graph G

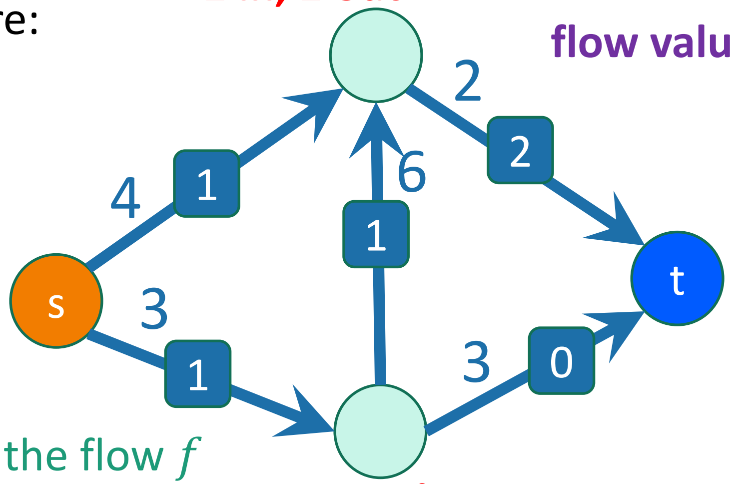


Call this graph G_f

Before:

2 in, 2 out

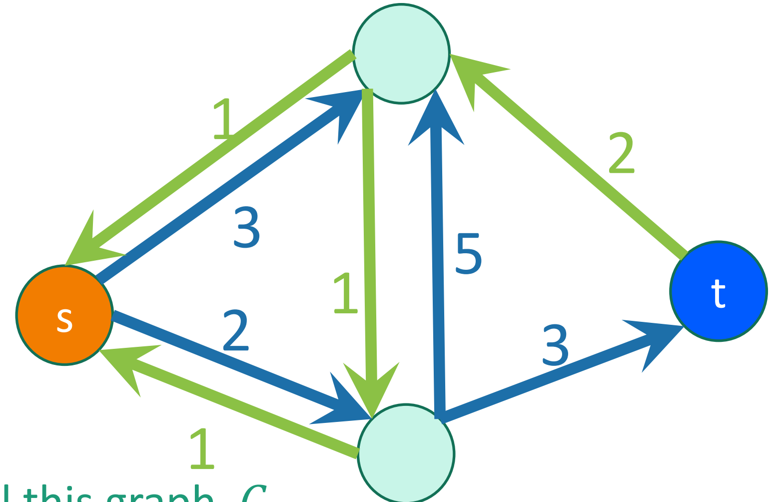
flow value is 2



1 in, 1 out

Call the flow f

Call the graph G

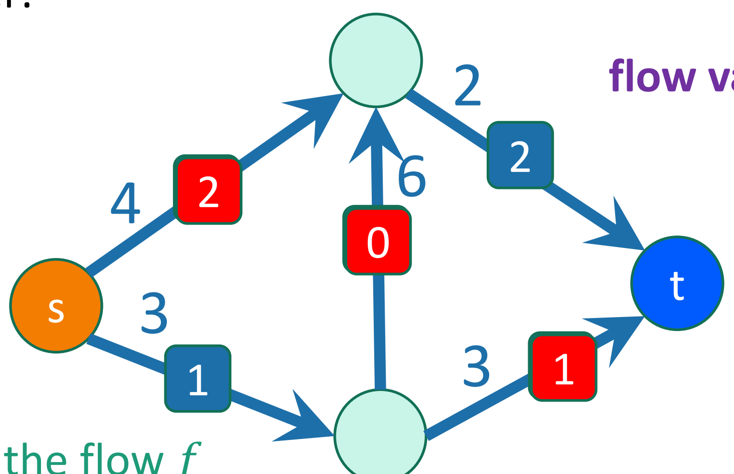


Call this graph G_f

After:

2 in, 2 out

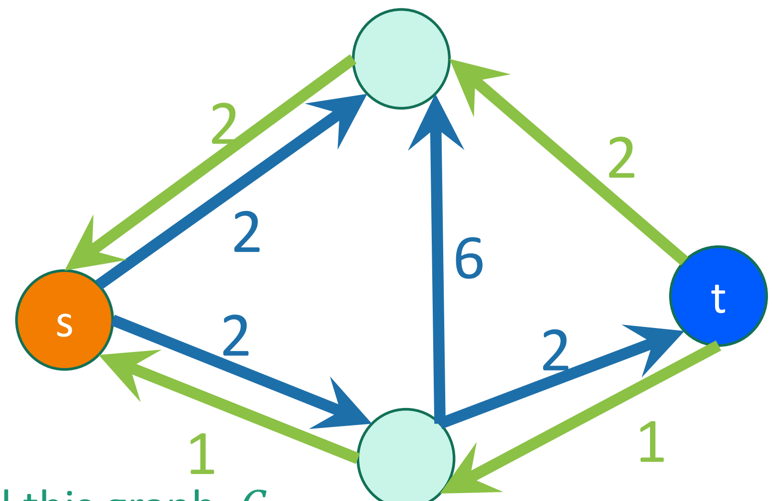
flow value is 3



1 in, 1 out

Call the flow f

Call the graph G



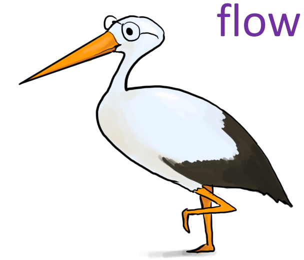
Call this graph G_f

Still a legit flow, but with a bigger value!

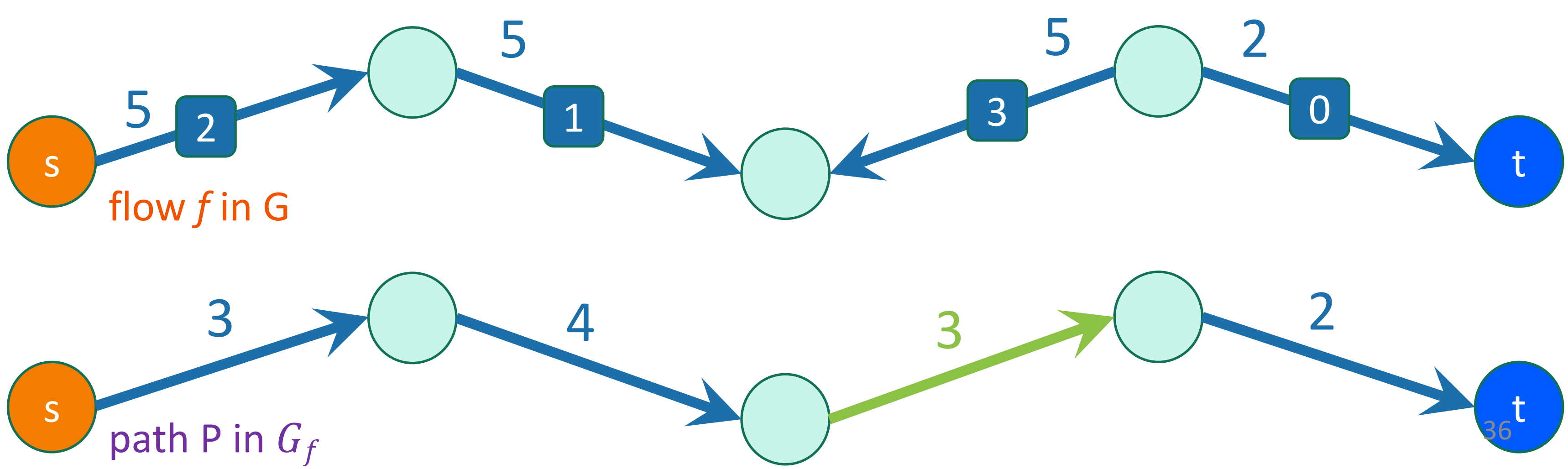
claim:

if there is an augmenting path, we can increase the flow along that path.

Check that this always makes a bigger (and legit) flow!



- increaseFlow(path P in G_f , flow f):
  - x = min weight on any edge in P
  - **for** (u,v) in P:
    - **if** (u,v) in E, $f'(u,v) \leftarrow f(u,v) + x$.
    - **if** (v,u) in E, $f'(v,u) \leftarrow f(v,u) - x$
  - **return** f'



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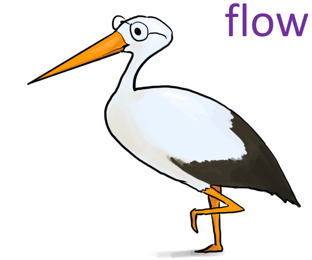

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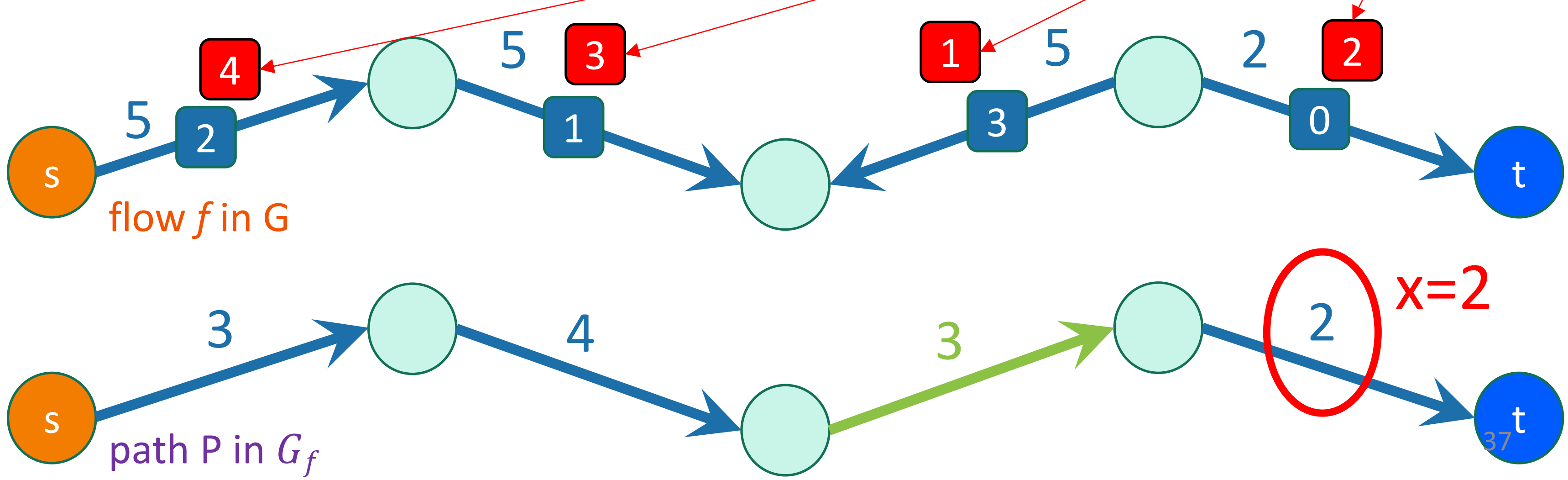
  - **return** f'

This is f'



That proves the **claim**

t ***is*** reachable from s in $G_f \Rightarrow f$ ***is not*** a max flow.

t ***is not*** reachable from s in $G_f \Leftarrow f$ ***is*** a max flow.

Converse-negative propositions are equivalent

If there is an augmenting path, we can increase the flow along that path

Question: When do we stop the process?

i.e., if there is no longer an augmenting path to increase the flow, does it mean that we have reached the maximum flow?

5-min Break

Proof of Max-Flow Min-Cut Theorem II

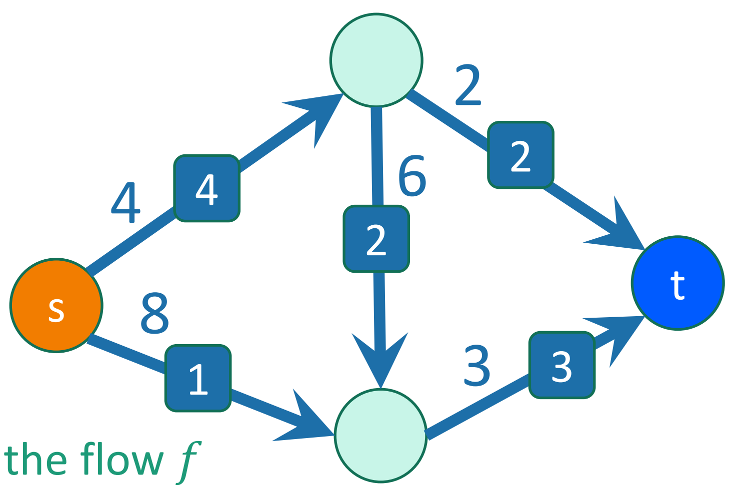
Lemma:

⇒ now this direction ⇒

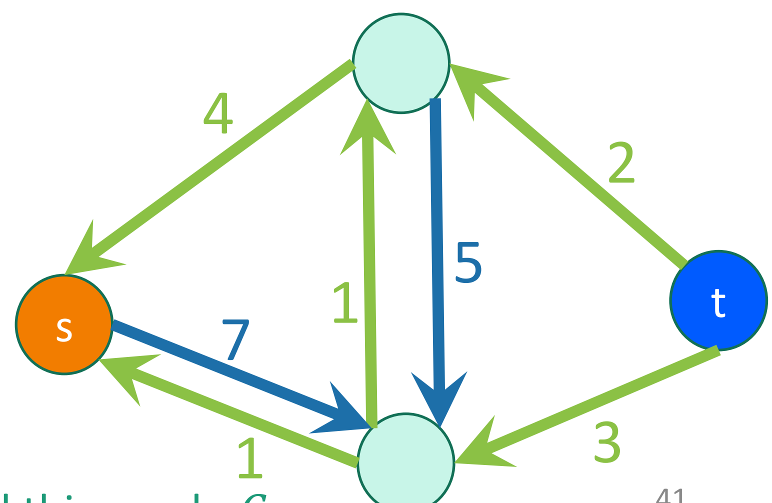
t is not reachable from s in G_f $\Leftrightarrow$ f is a max flow.

- Suppose there is not a path from s to t in G_f.
- Consider the cut given by:

{things reachable from s} , {things not reachable from s}



Call the flow f
Call the graph G



Call this graph G_f

Lemma:

⇒ now this direction ⇒

t is not reachable from s in G_f ⇔ f is a max flow.

- Suppose there is not a path from s to t in G_f.
- Consider the cut given by:

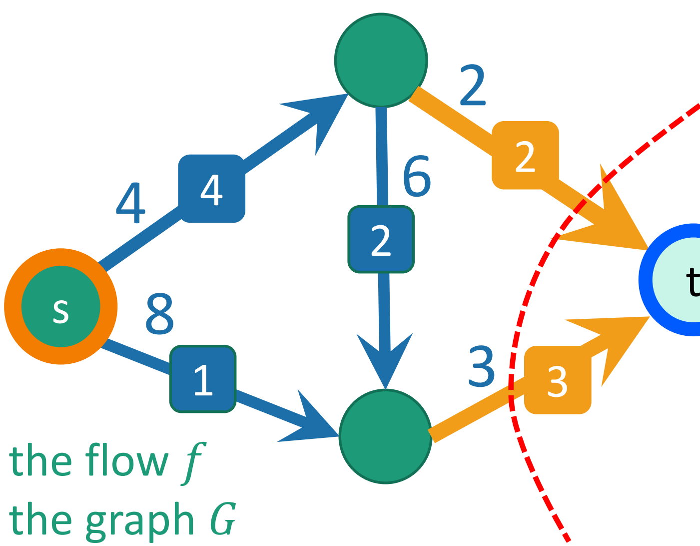
t lives here

{things reachable from s} , {things not reachable from s}

- The flow from s to t is **equal** to the cost of this cut.
  - Similar to proof-by-picture we saw before:
    - All of the stuff has to **cross the cut**.

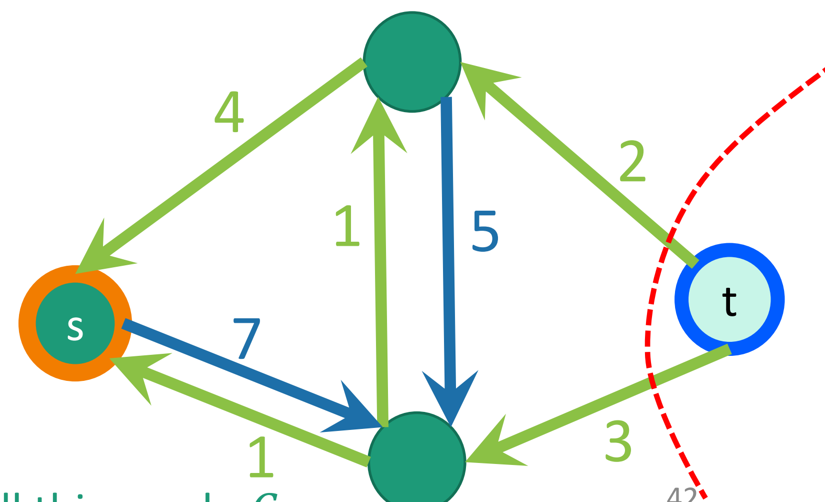
Lemma 1

- **thus:** $\text{this flow value} = \text{cost of this cut} \geq \text{cost of min cut} \geq \text{max flow}$



Call the flow f

Call the graph G



Call this graph G_f

Lemma:

$\Rightarrow$ now this direction $\Rightarrow$

t is not reachable from s in G_f $\Leftrightarrow$ f is a max flow.

- Suppose there is not a path from s to t in G_f.
- Consider the cut given by:

t lives here

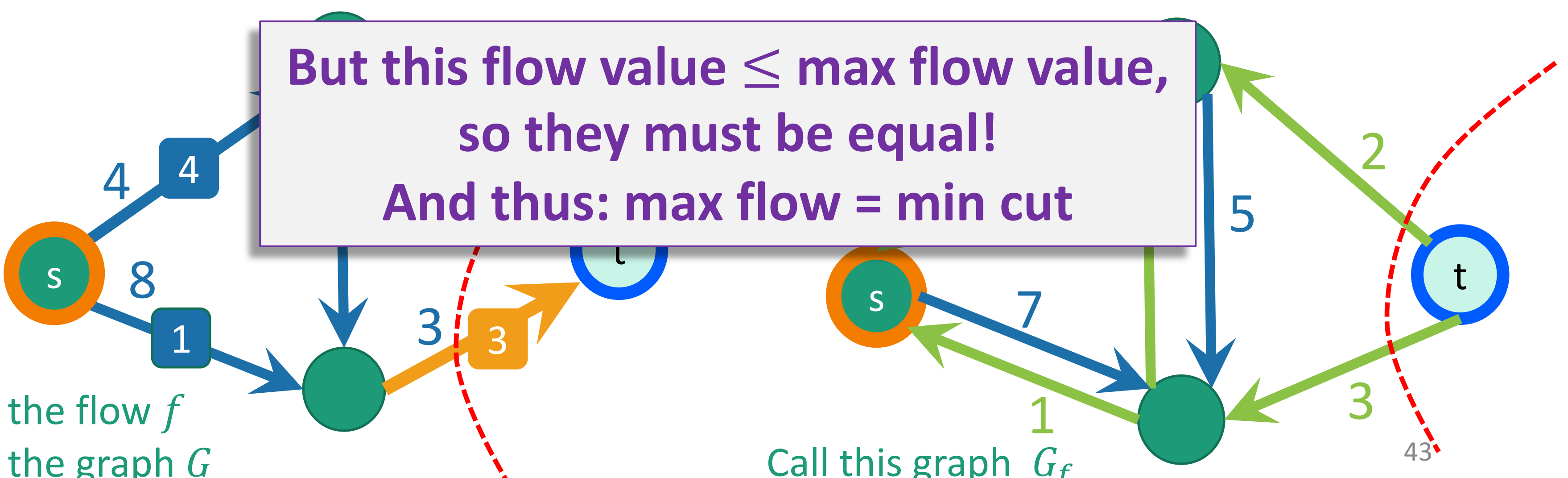
{things reachable from s} , {things not reachable from s}

- The flow from s to t is **equal** to the cost of this cut.
  - Similar to proof-by-picture we saw before:
    - All of the stuff has to **cross the cut**.

Lemma 1

- **thus:** this flow value $=$ cost of this cut $\geq$ cost of min cut $\geq$ max flow

But this flow value $\leq$ max flow value, so they must be equal! And thus: max flow = min cut



Call the flow f
Call the graph G

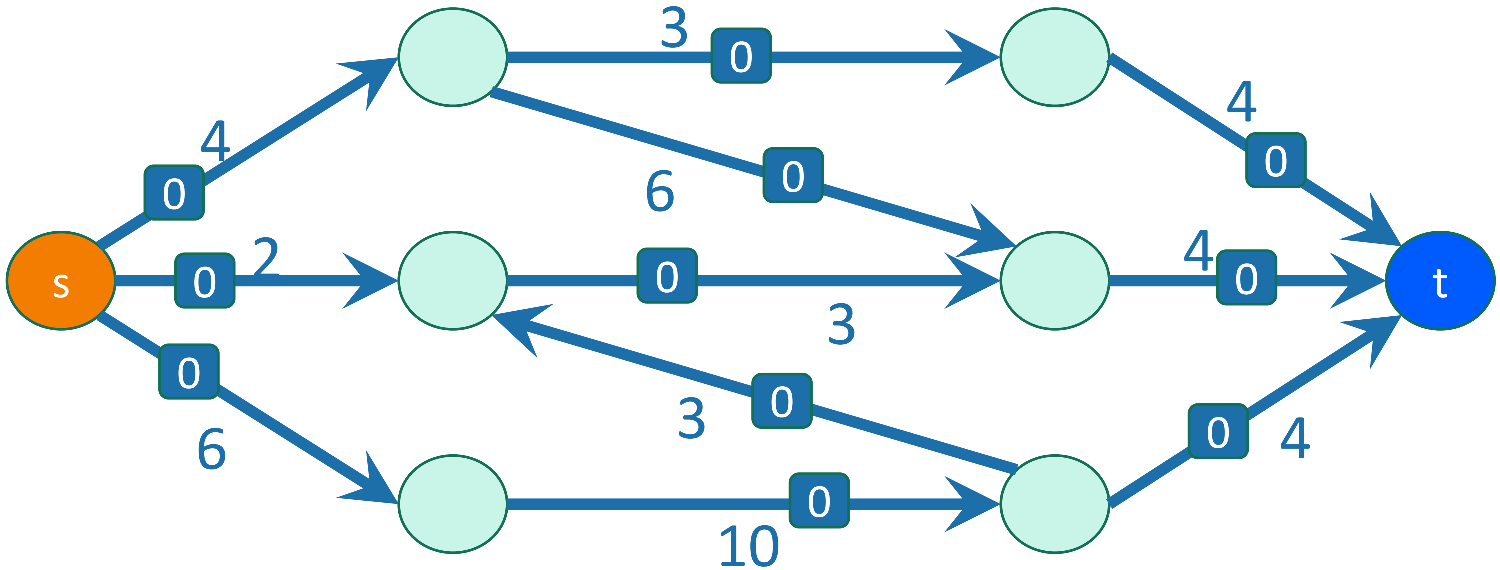
Call this graph G_f

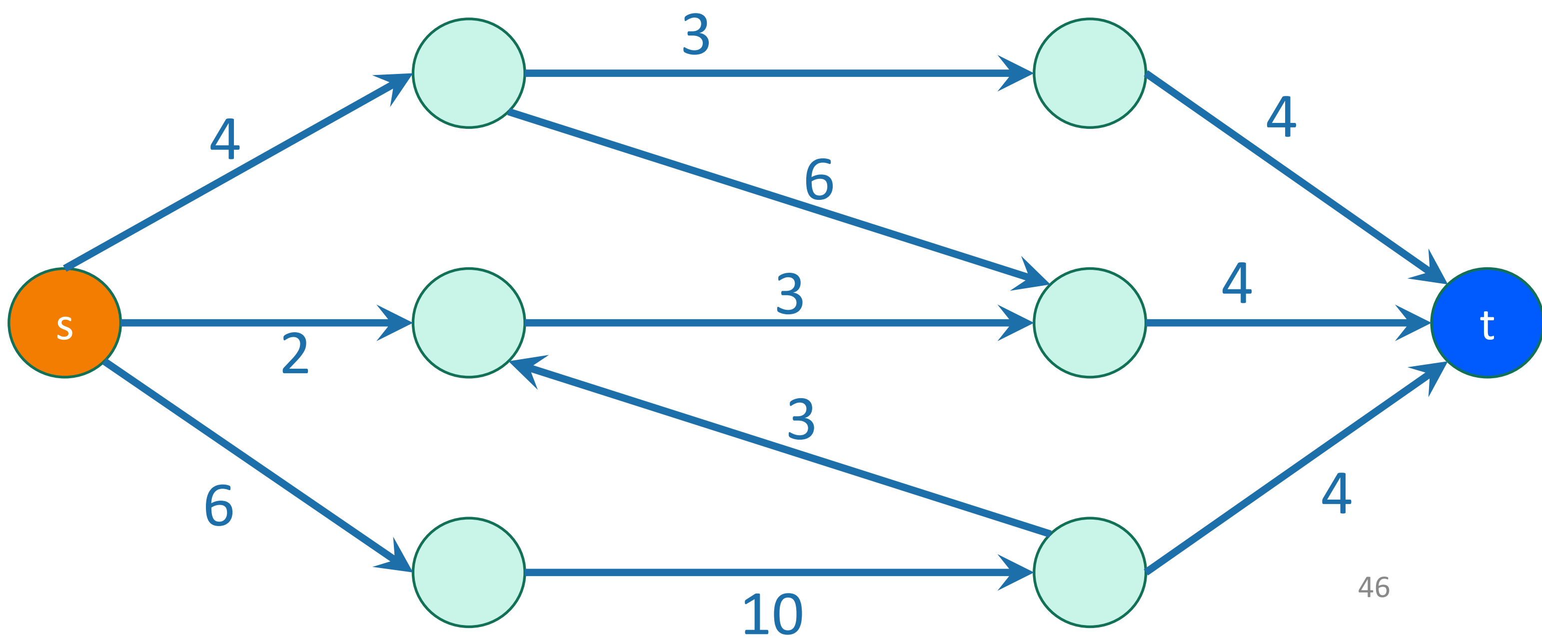
We've proved:

- t is not reachable from s in G_f $\Leftrightarrow$ f is a max flow

- This is inspires an **algorithm**:
- **Ford-Fulkerson**(G):
  - $f \leftarrow$ all zero flow.
  - $G_f \leftarrow G$
  - **while** t is reachable from s in G_f
    - Find a path P from s to t in G_f // eg, use BFS
    - $f \leftarrow$ **increaseFlow**(P, f)
    - update G_f
  - **return** f

Example of Ford-Fulkerson





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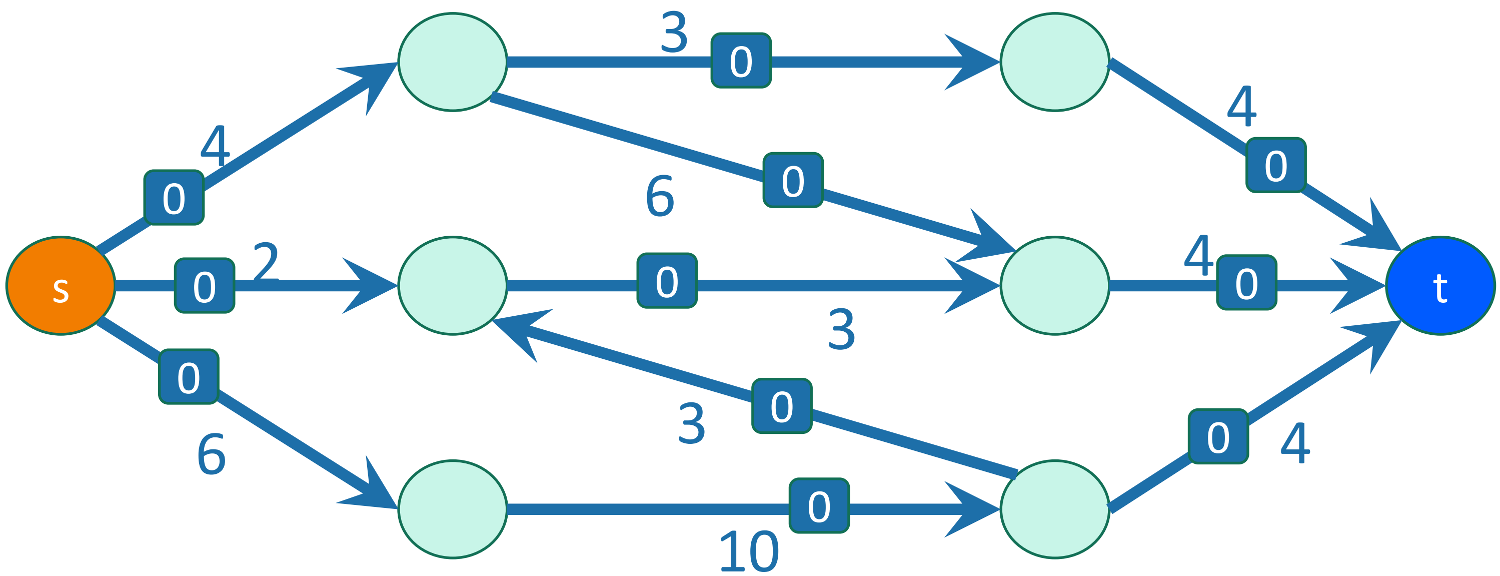

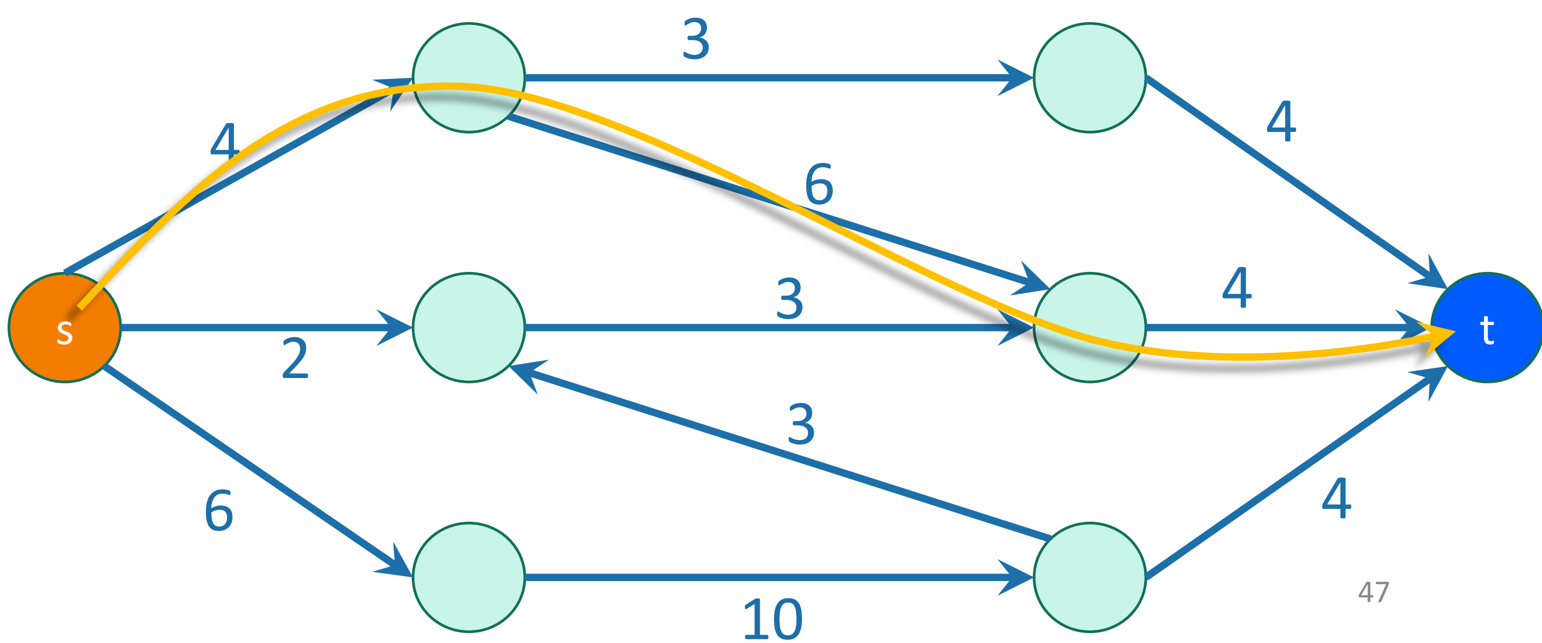

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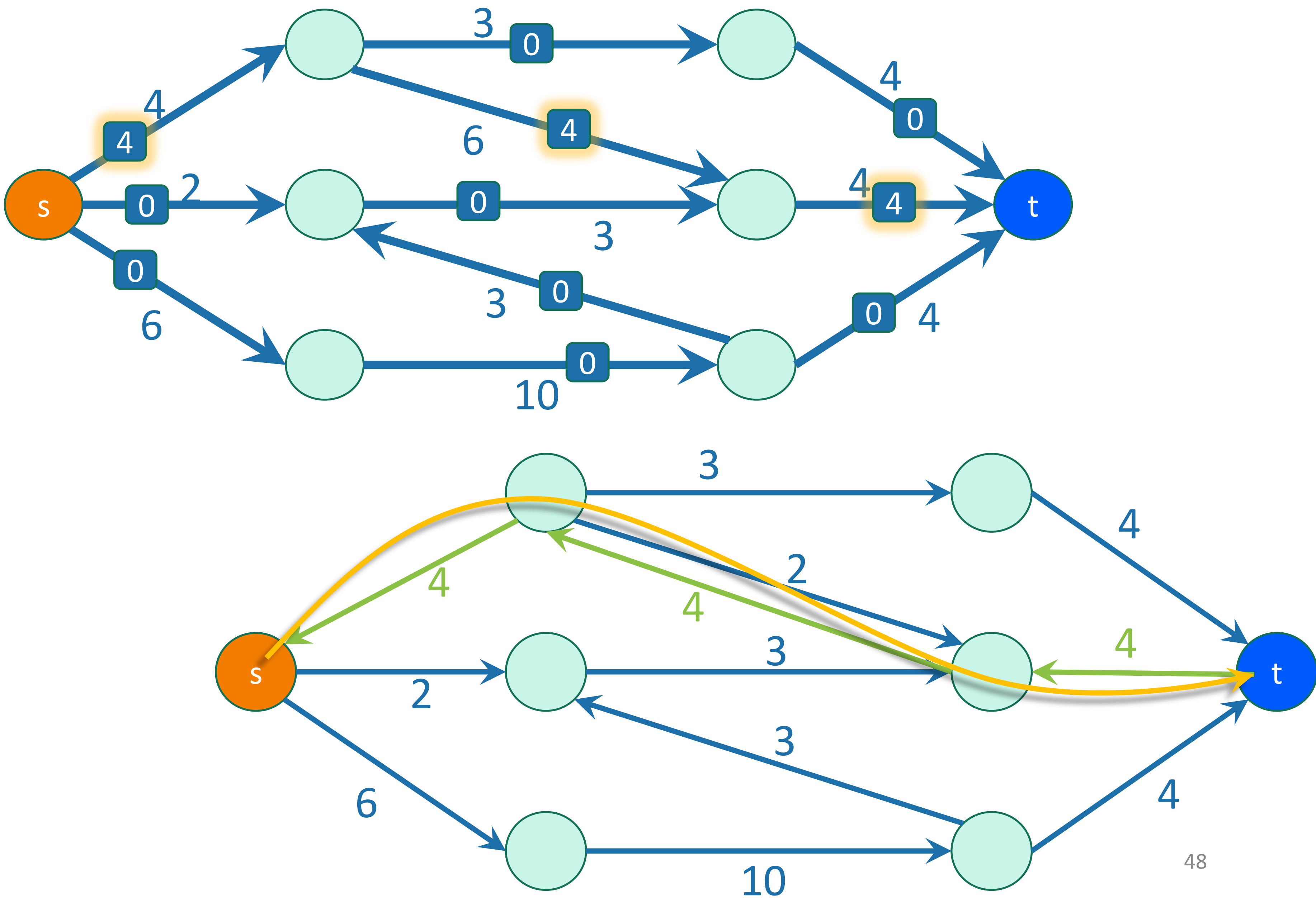

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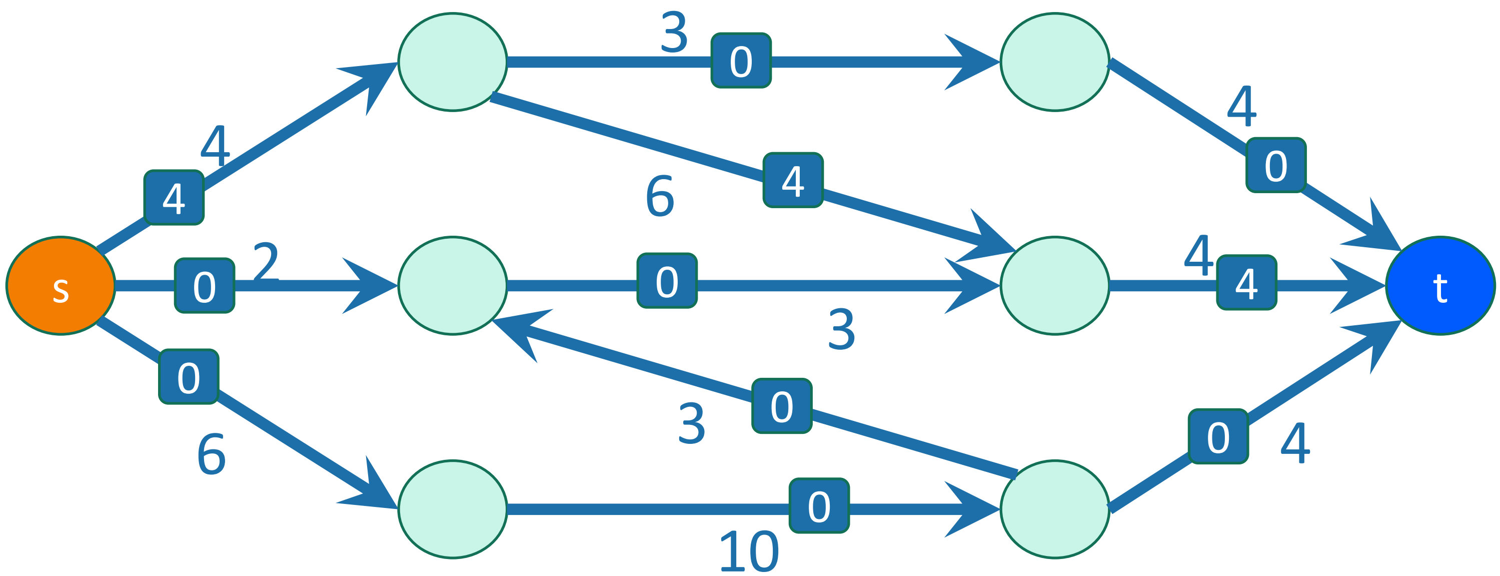

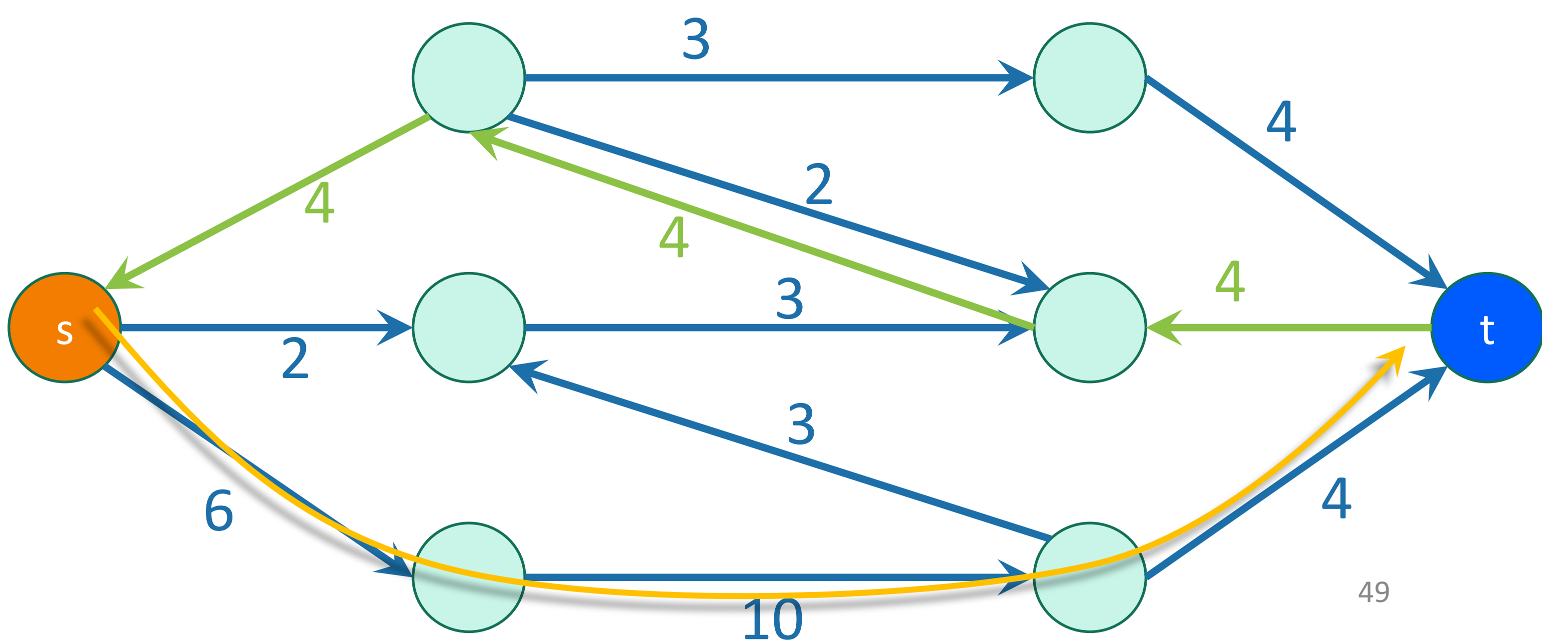

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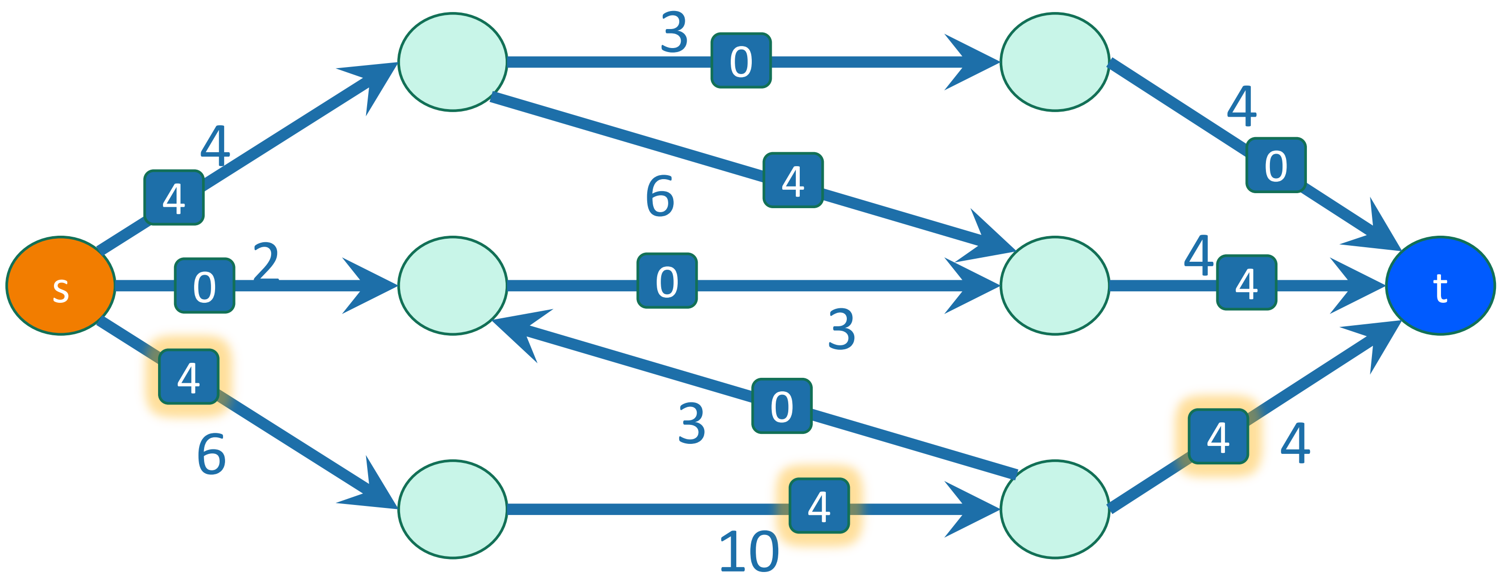
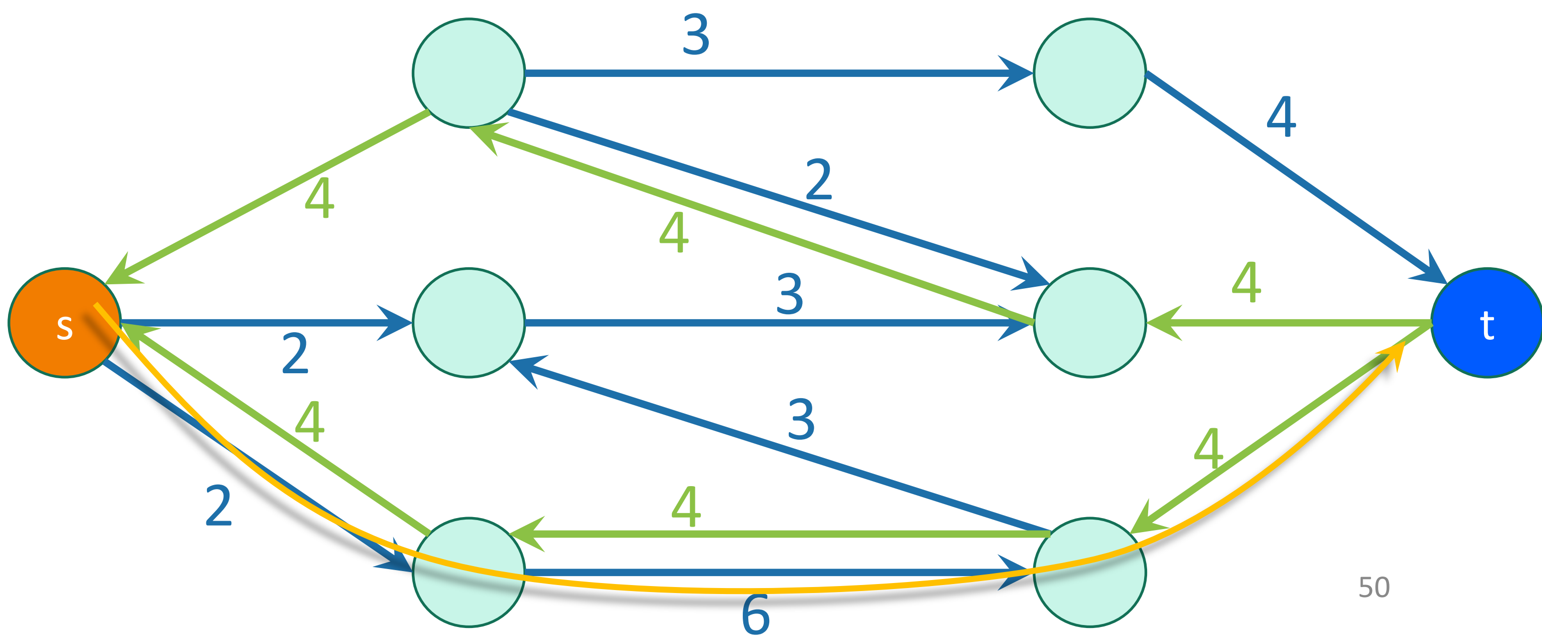
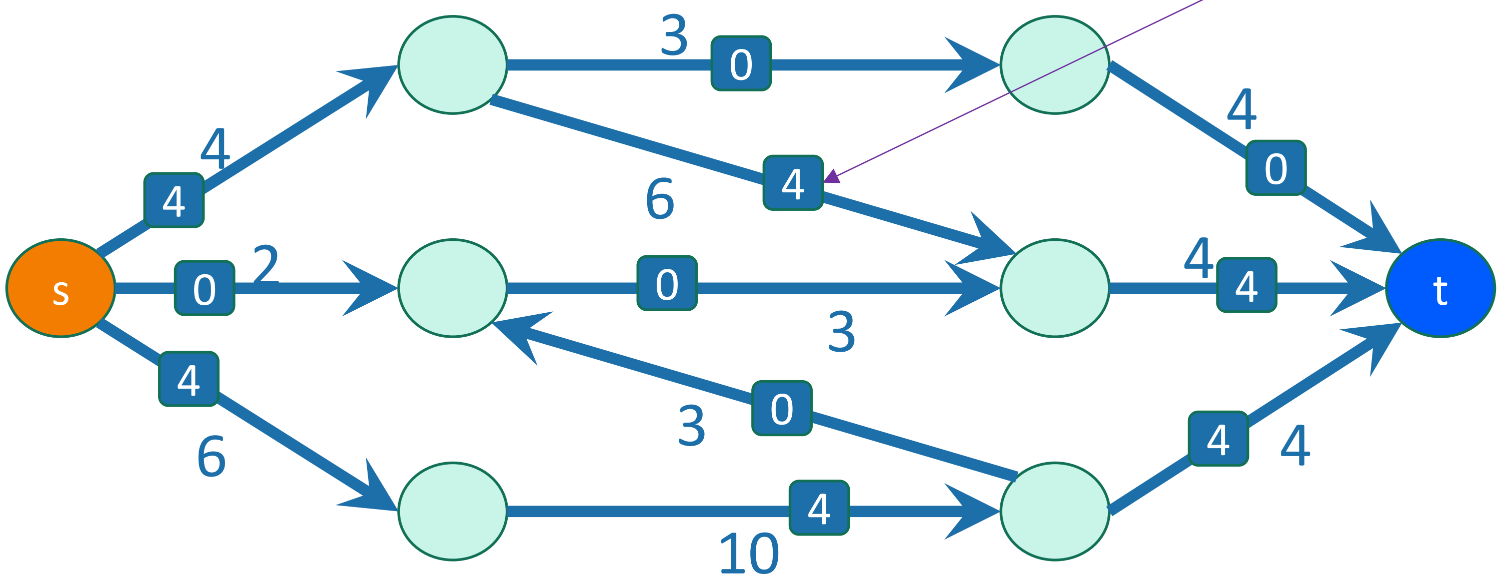


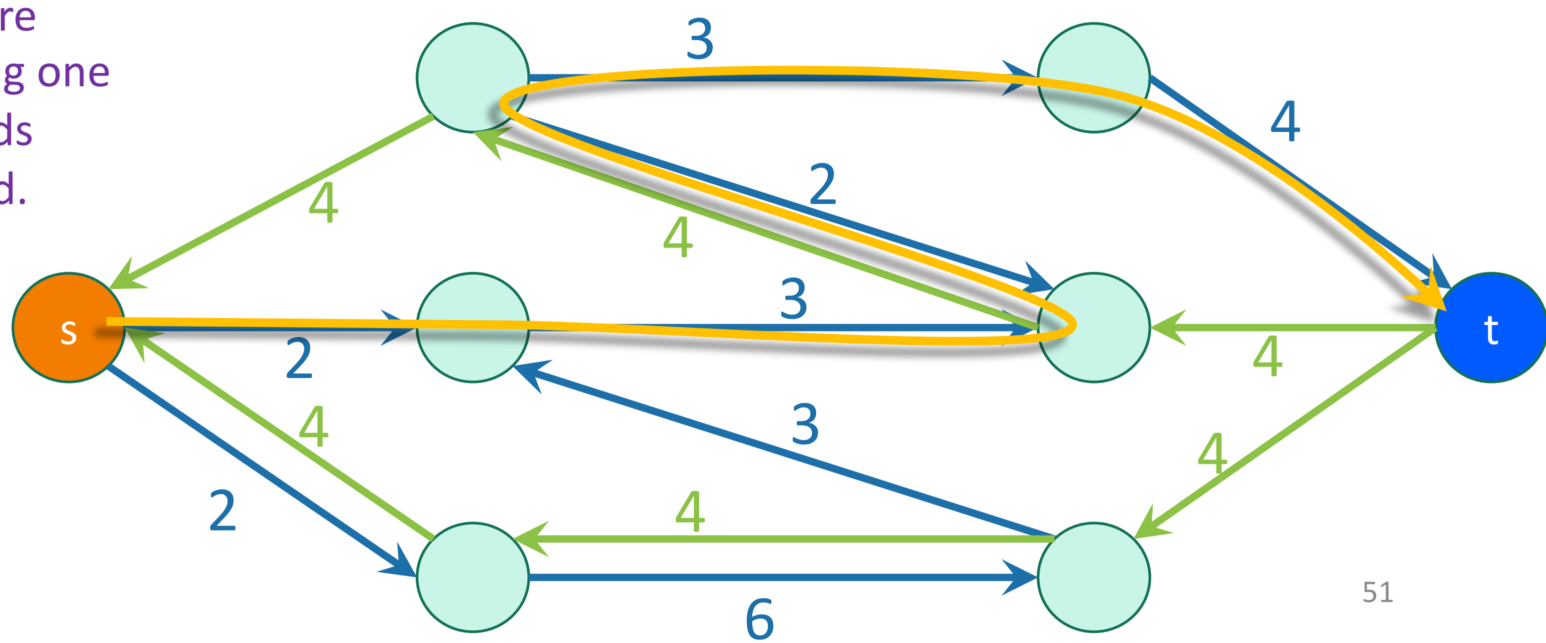


Example of Ford-Fulkerson

We will **remove** flow from this edge.



Notice that we're going back along one of the backwards edges we added.



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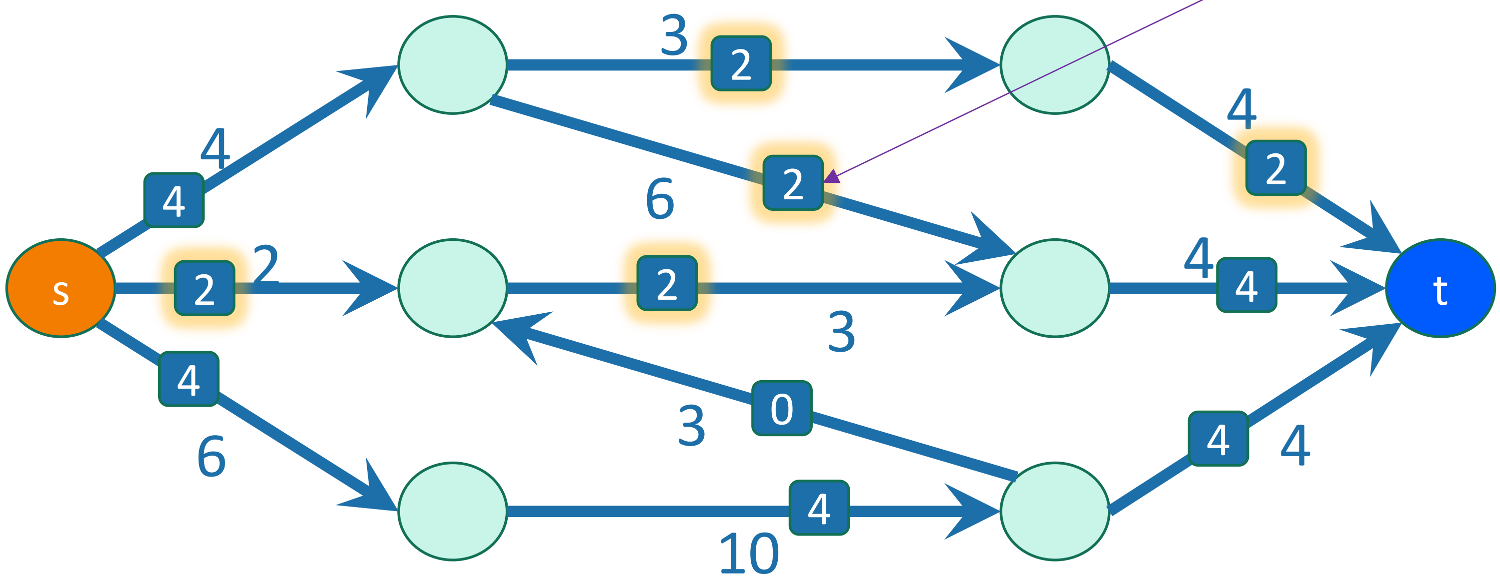

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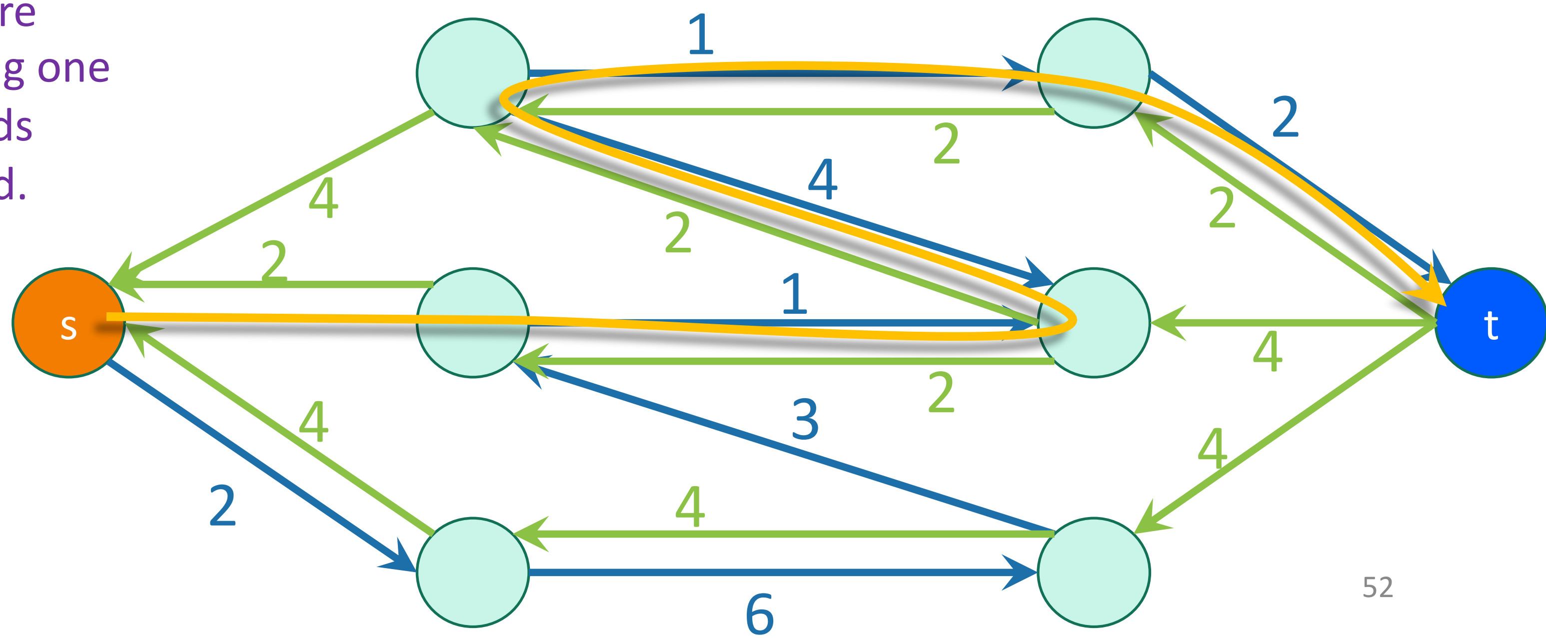
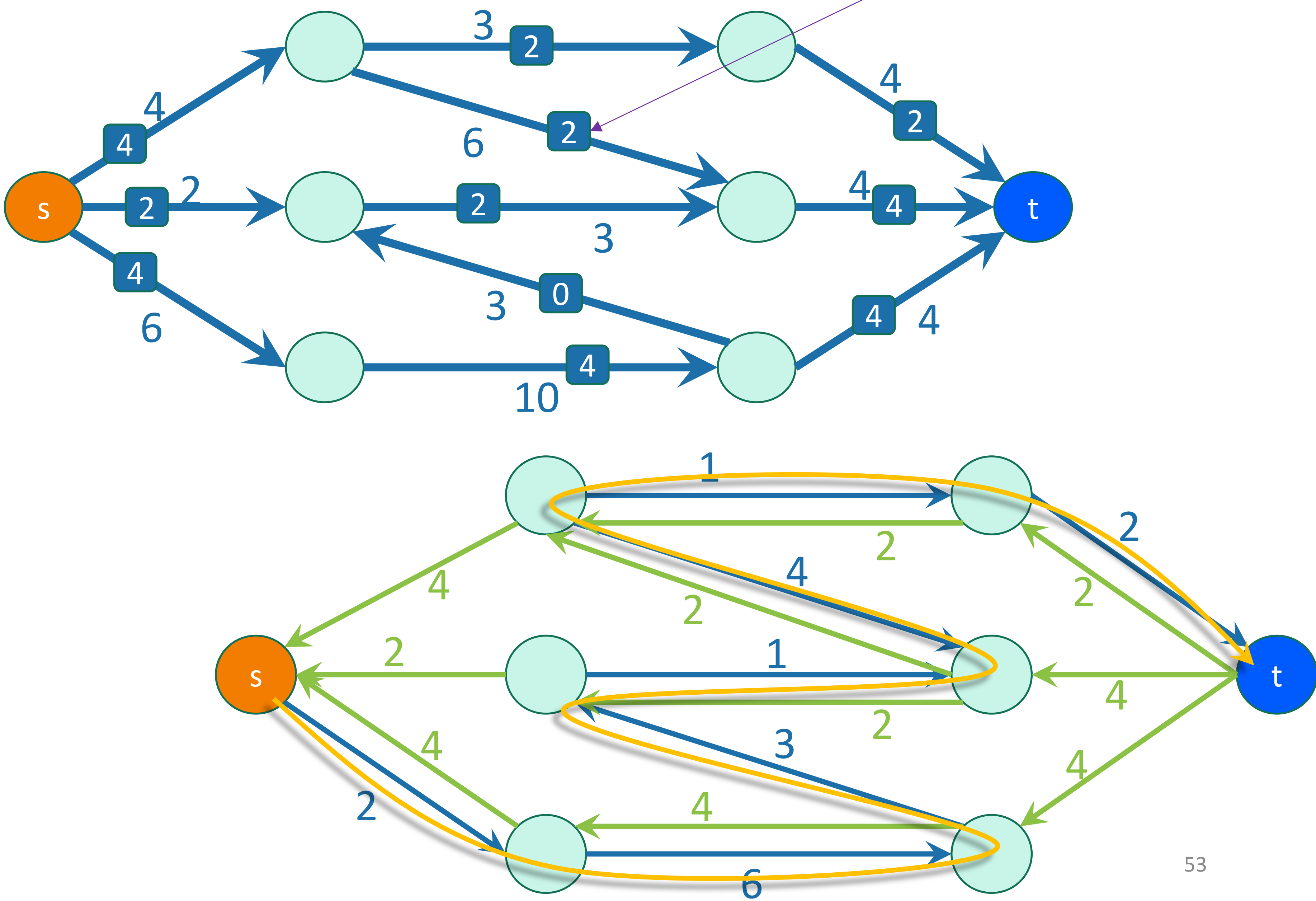


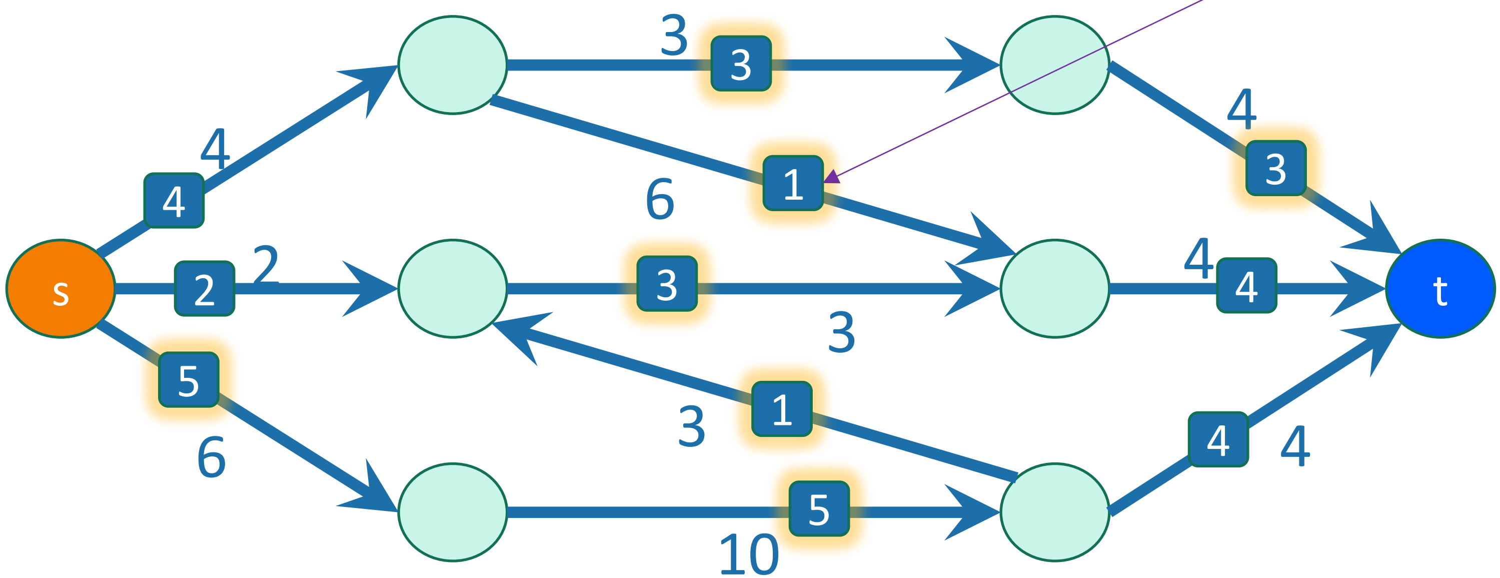
Example of Ford-Fulkerson

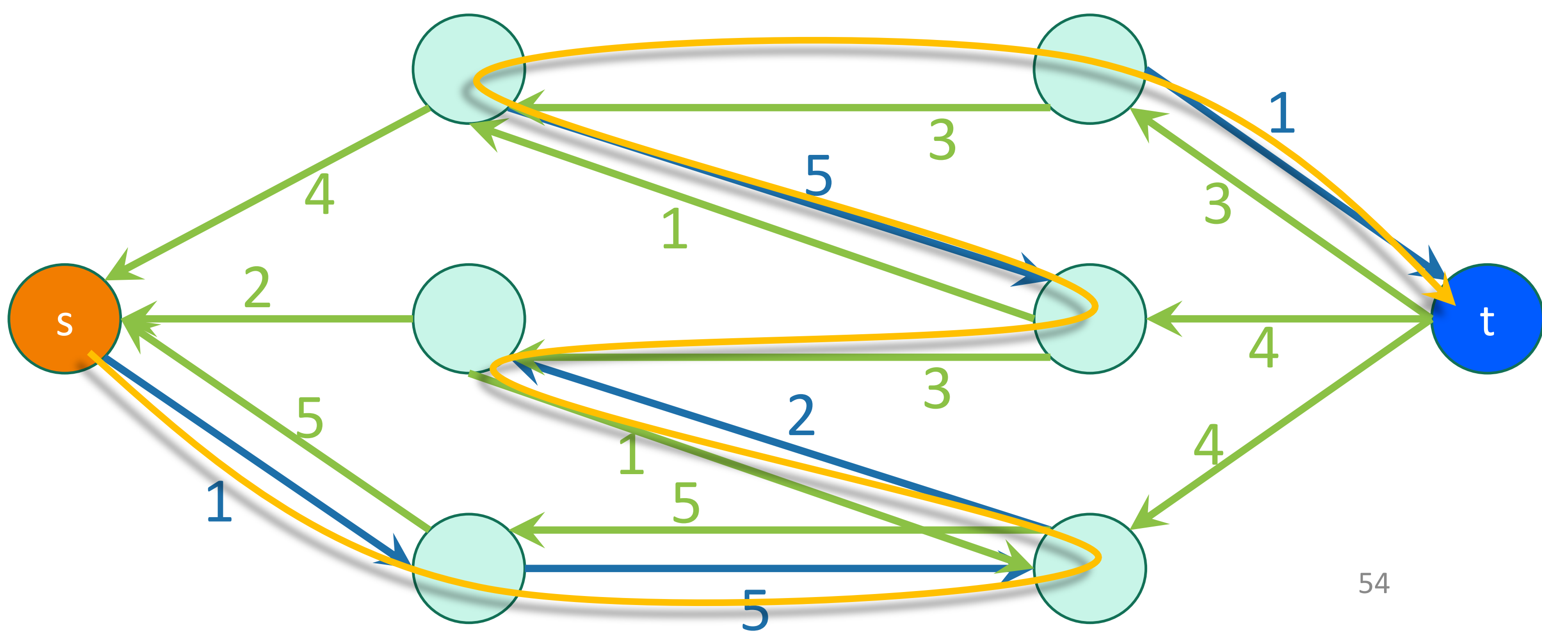
We will remove flow from this edge AGAIN.



Example of Ford-Fulkerson

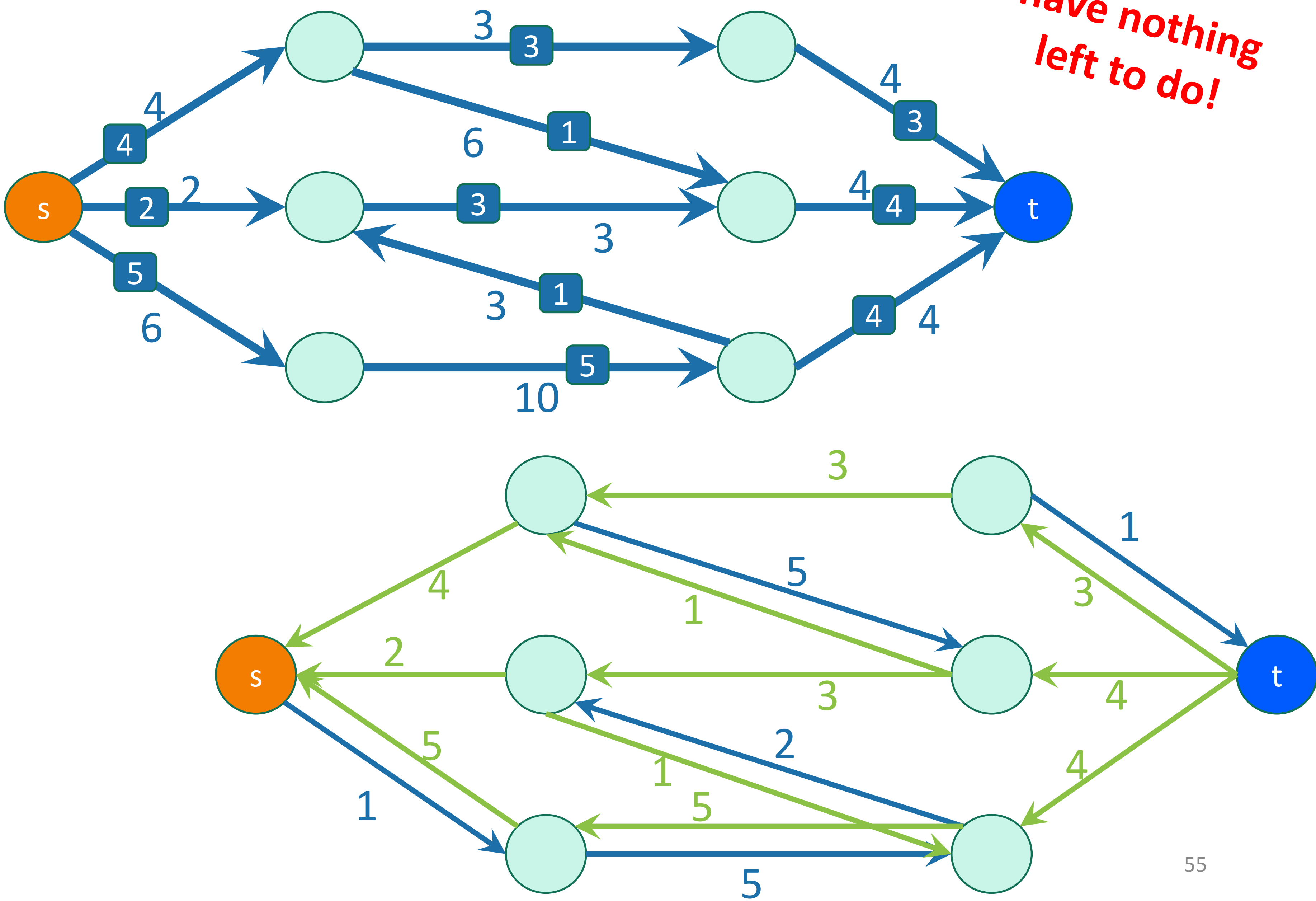
We will remove flow from this edge AGAIN.



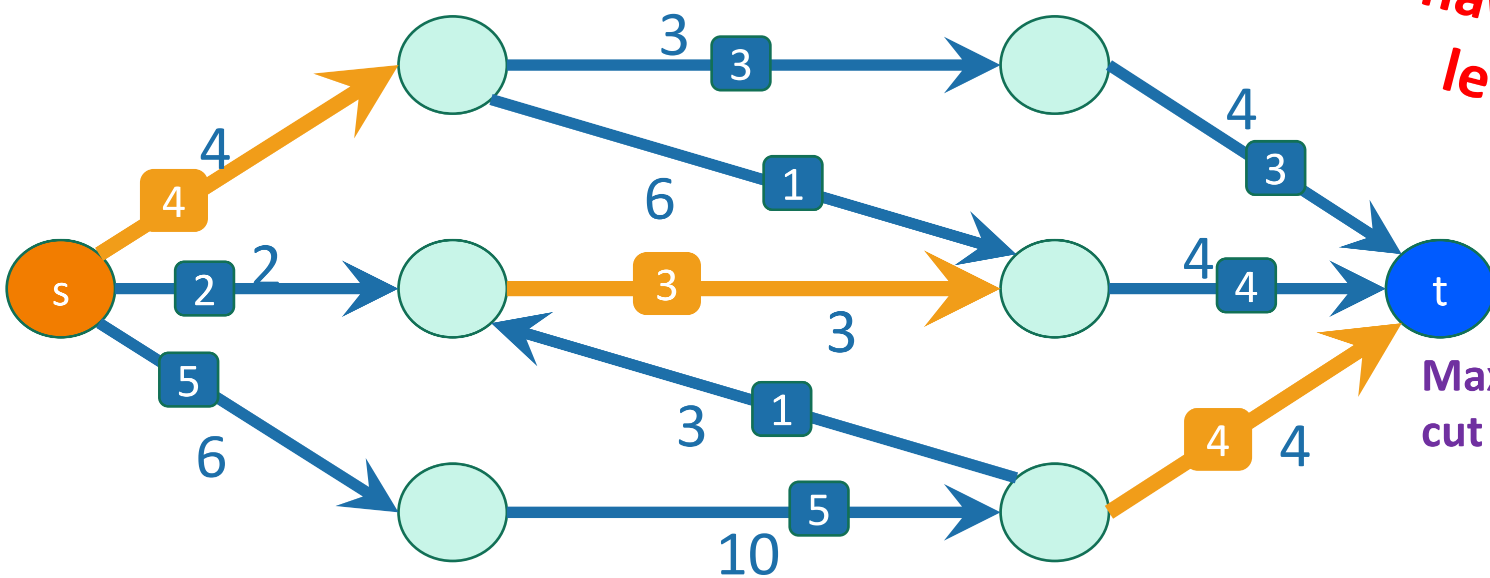


Example of Ford-Fulkerson

Now we have nothing left to do!

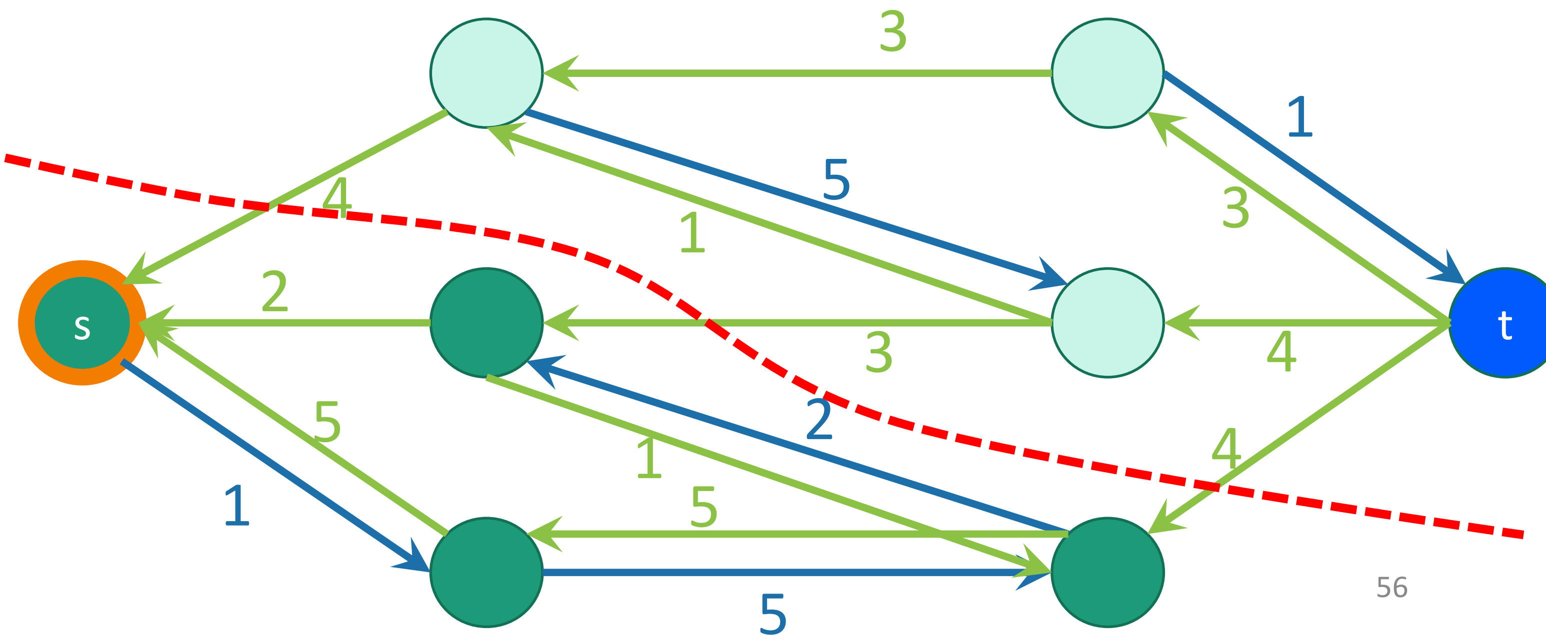


Example of Ford-Fulkerson



Now we have nothing left to do!

Max flow and min cut are both 11.



There's no path from s to t, and here's the cut to prove it.

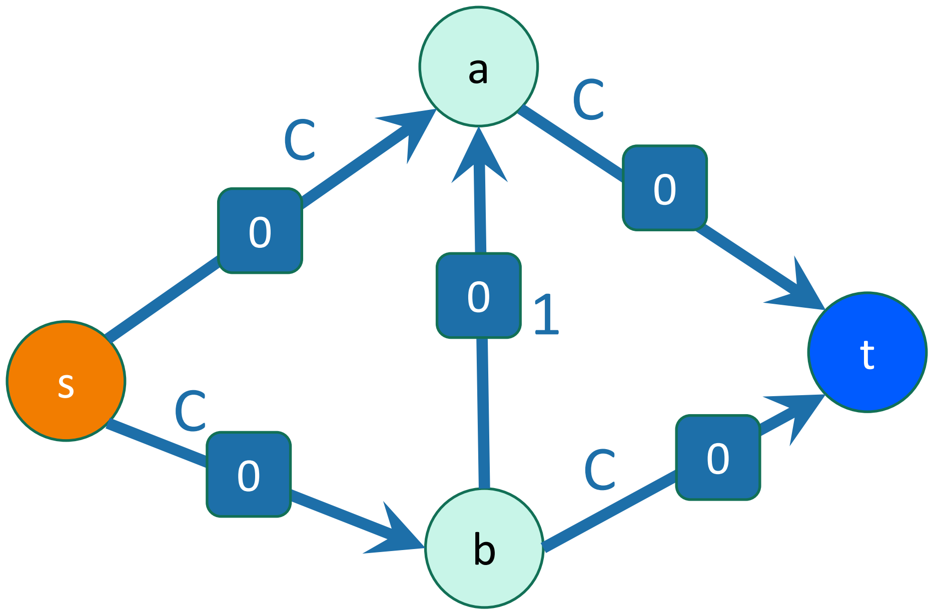
What have we learned?

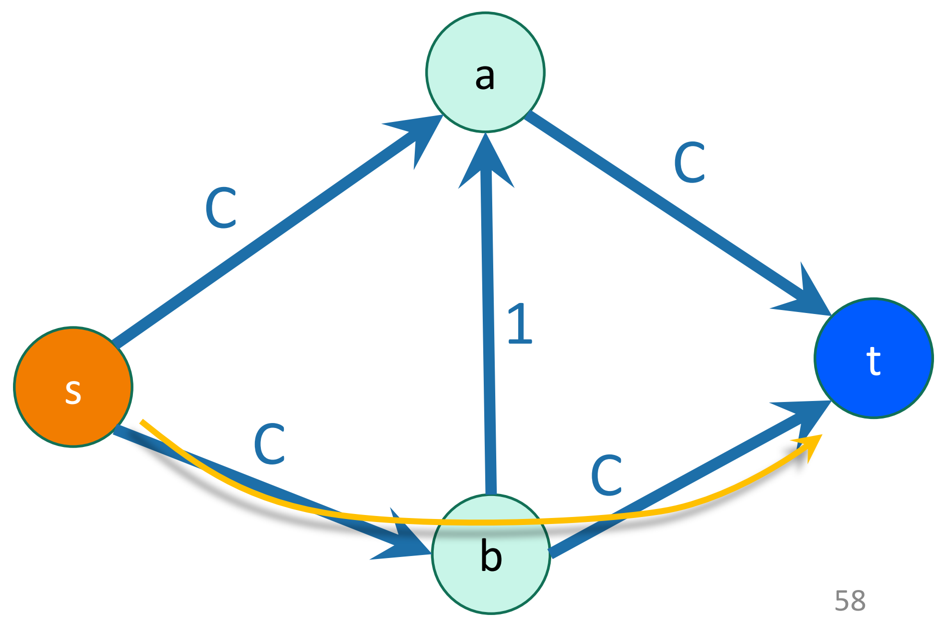
- Max s-t flow is equal to min s-t cut!
- The Ford-Fulkerson algorithm can find the max-flow/min-cut.
  - Repeatedly improve your flow along an augmenting path.
- **How long does this take???**

Why should we be concerned?

Suppose we picked paths **smartly**.

Choose a really big number C.





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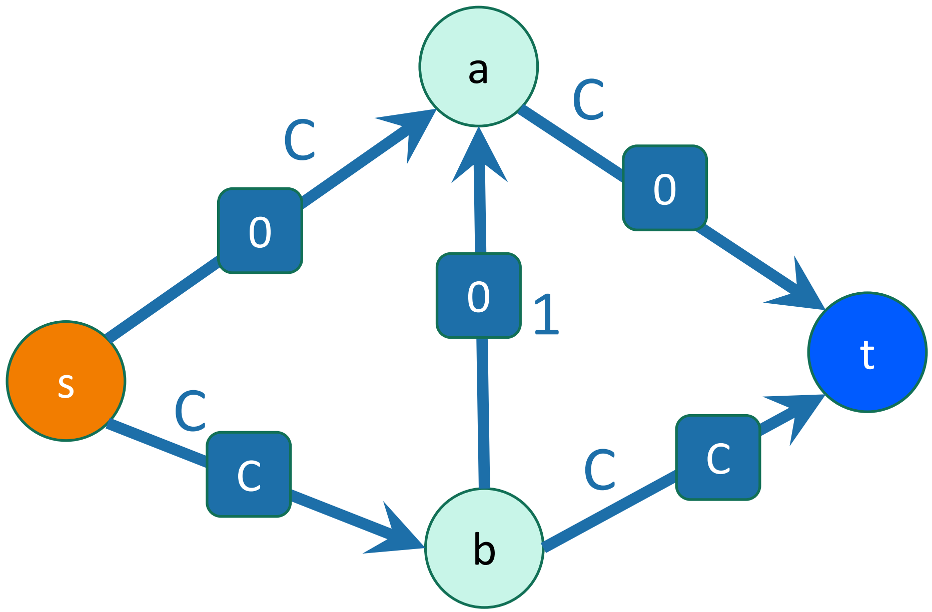

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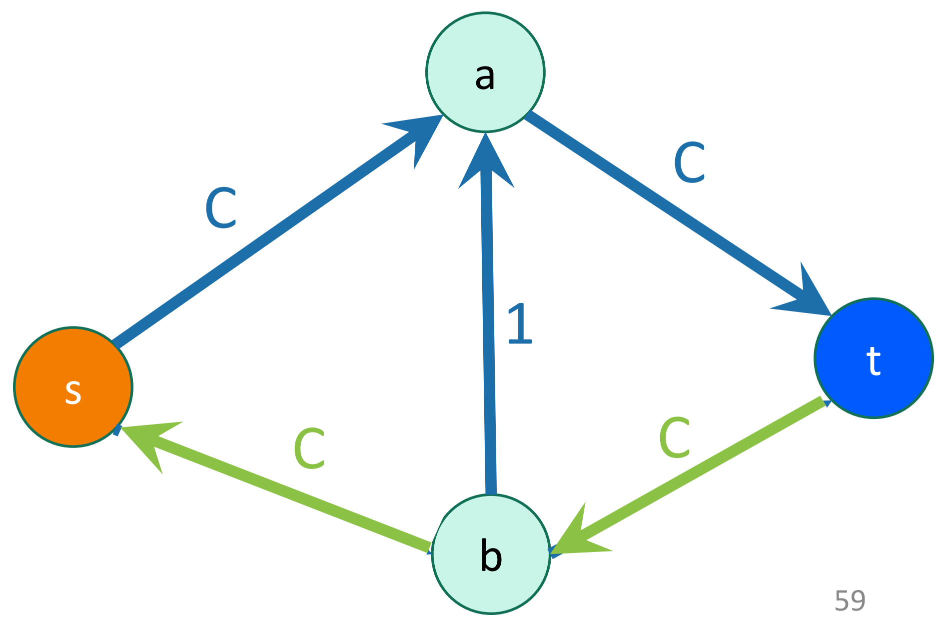

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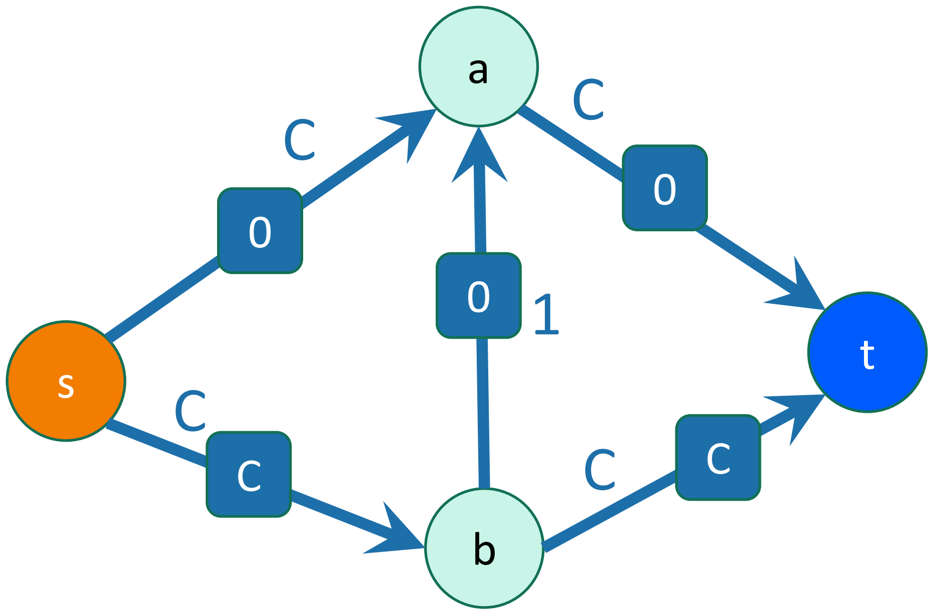

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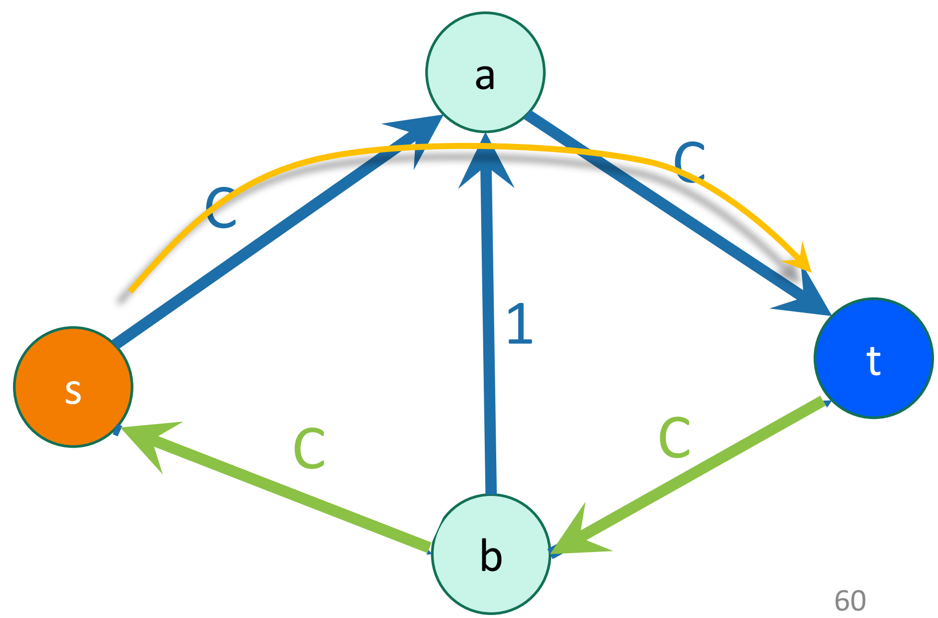

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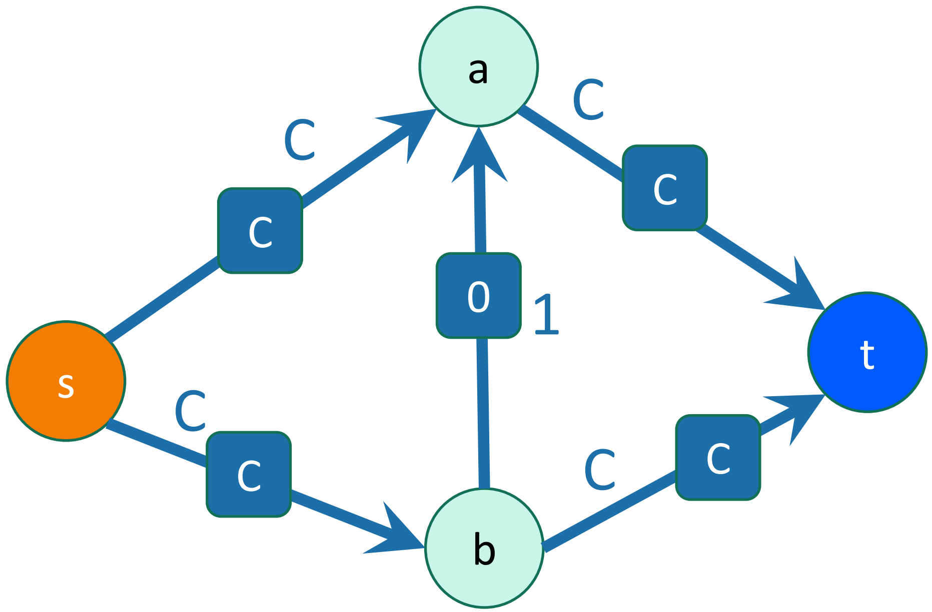

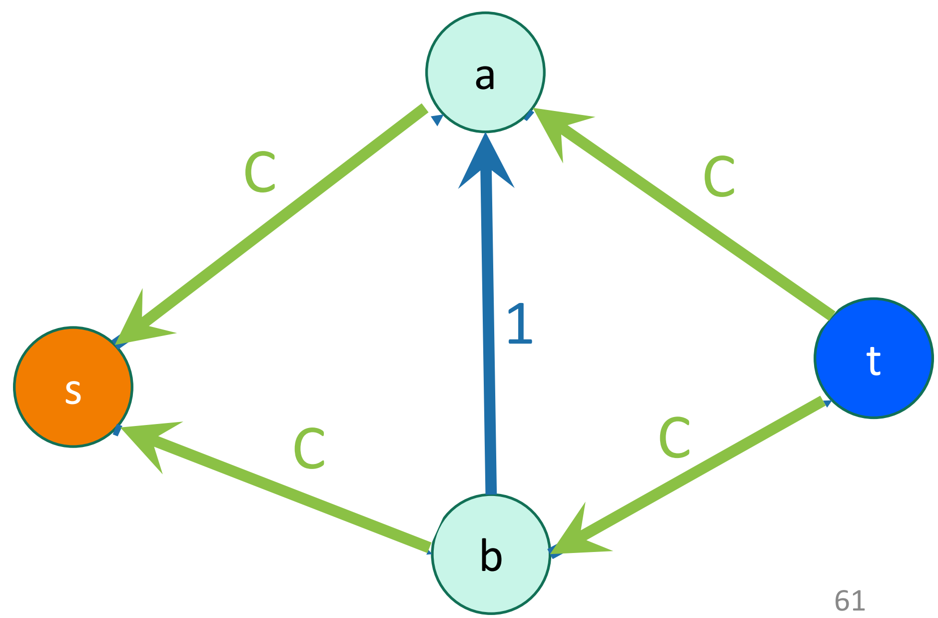

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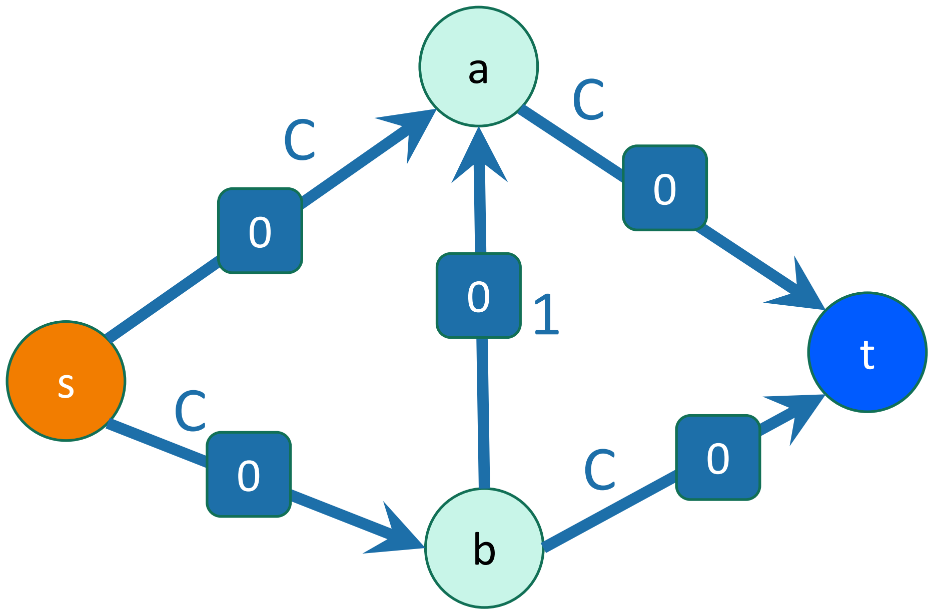

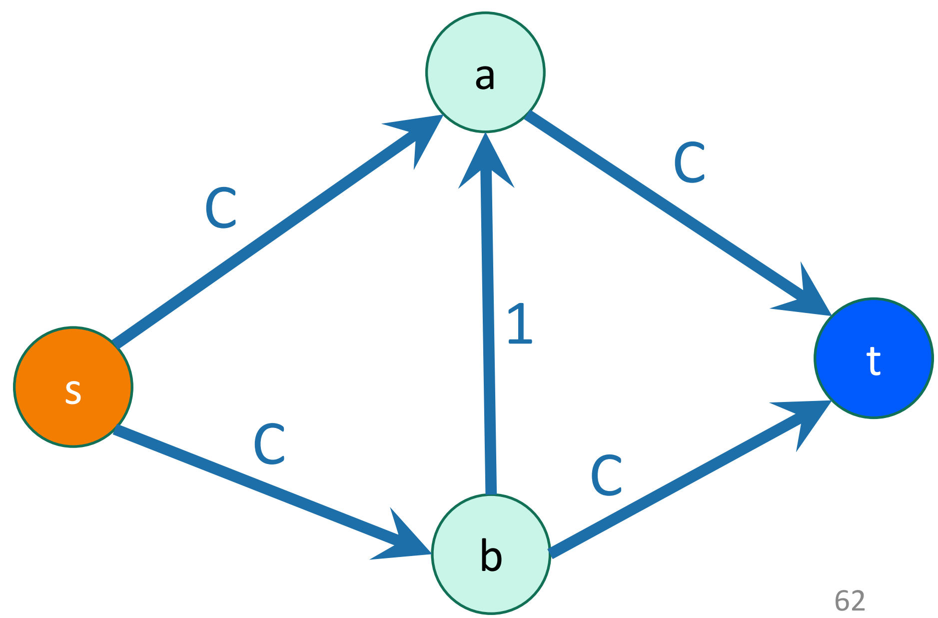

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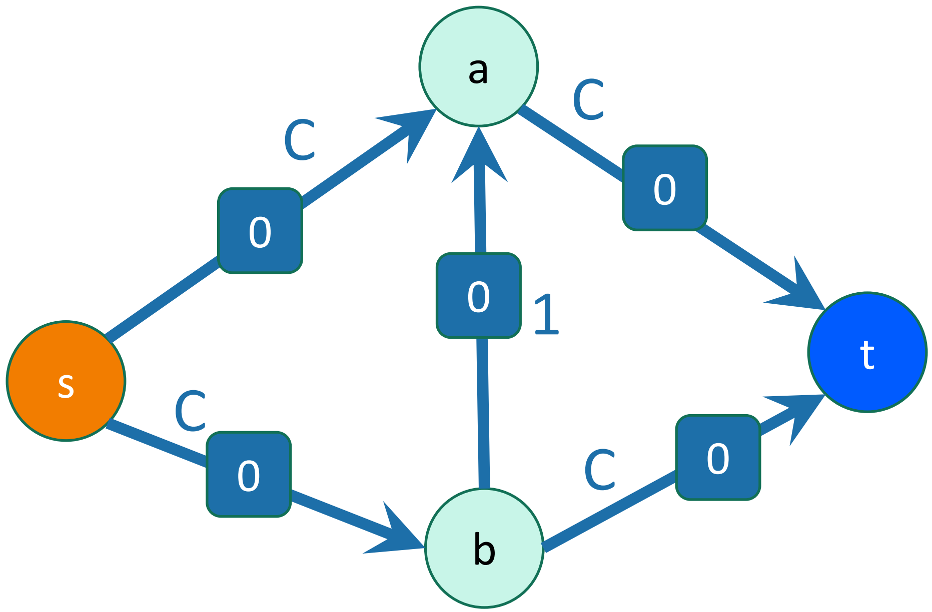

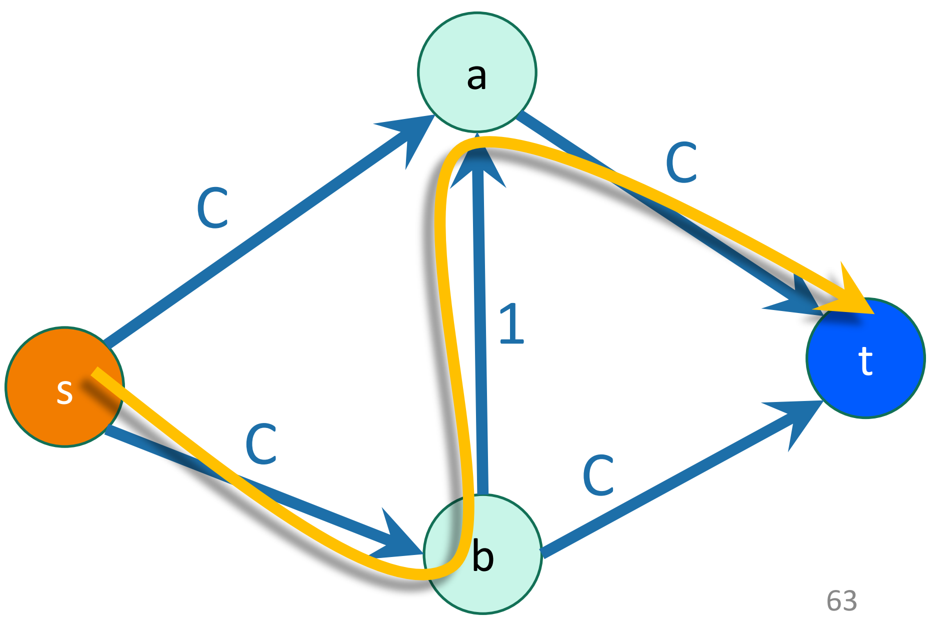

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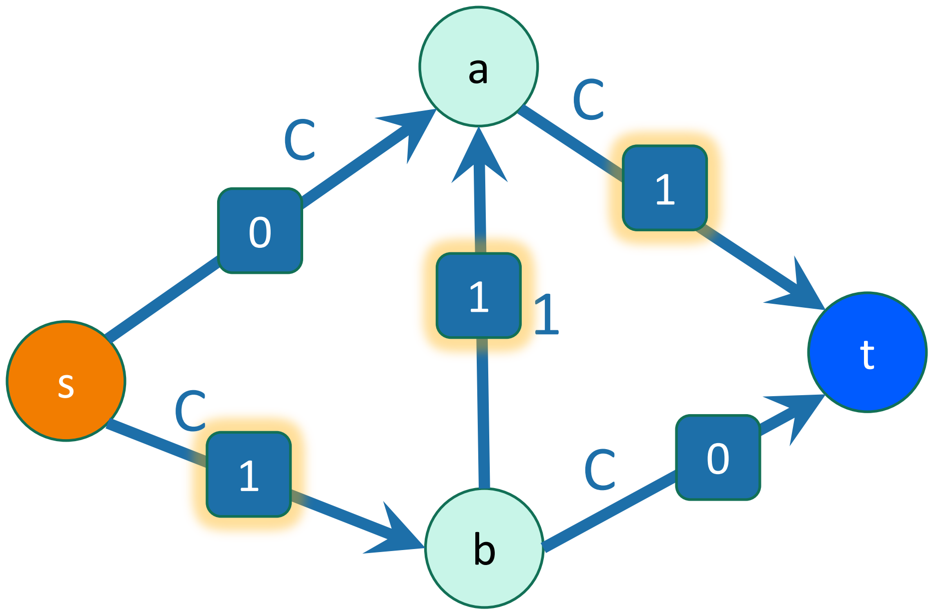

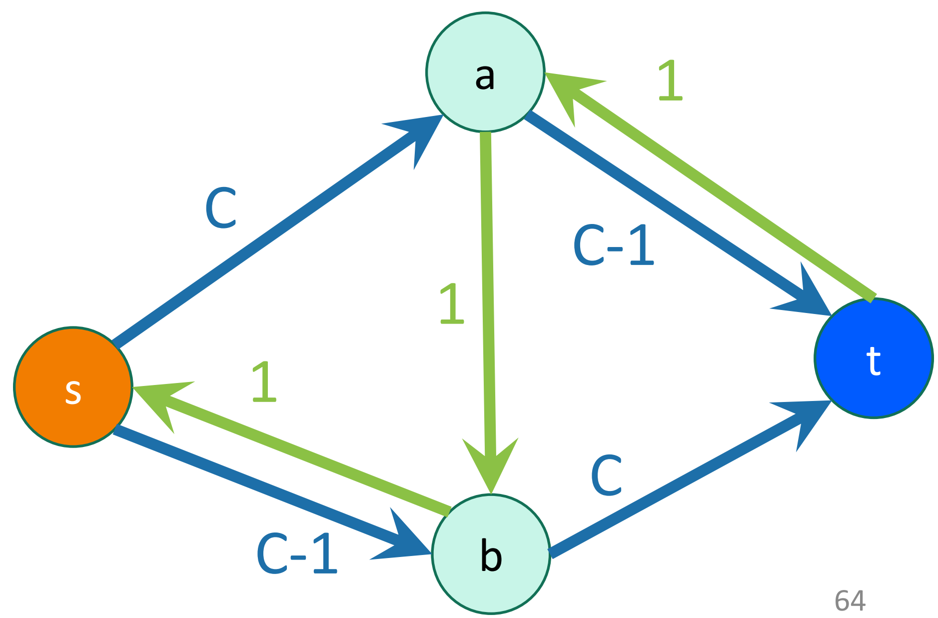

The edge (b,a) disappeared from the residual graph!

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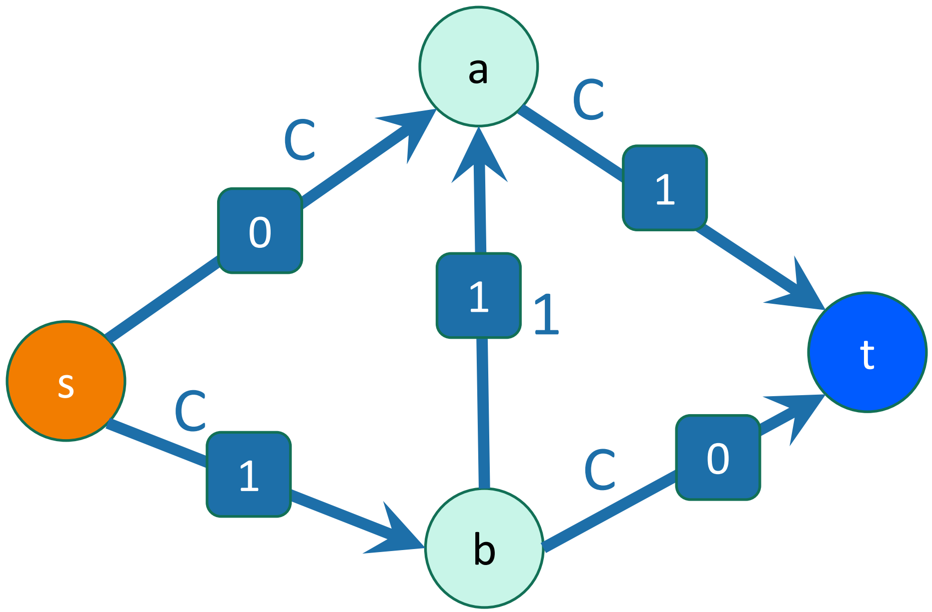

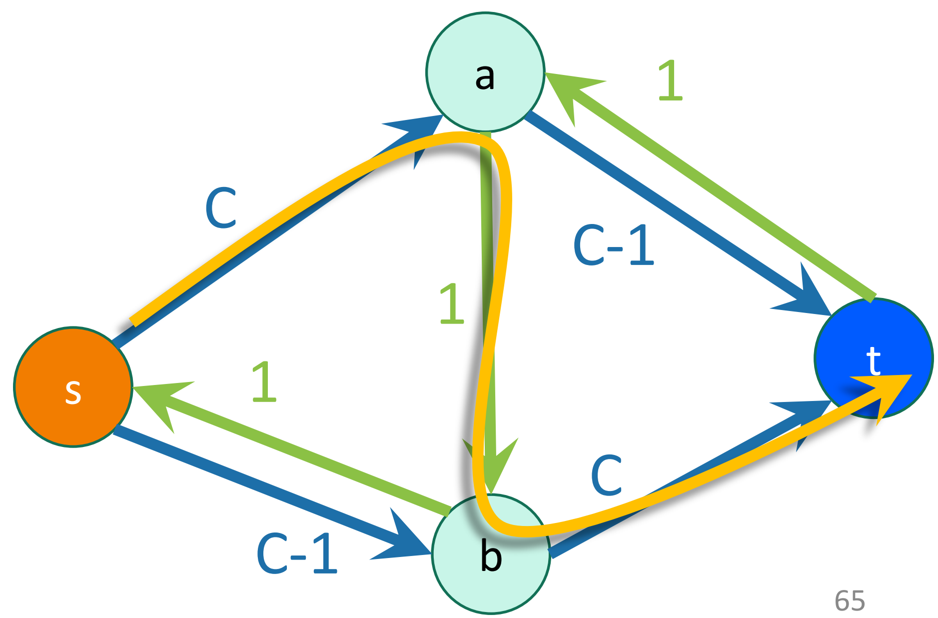

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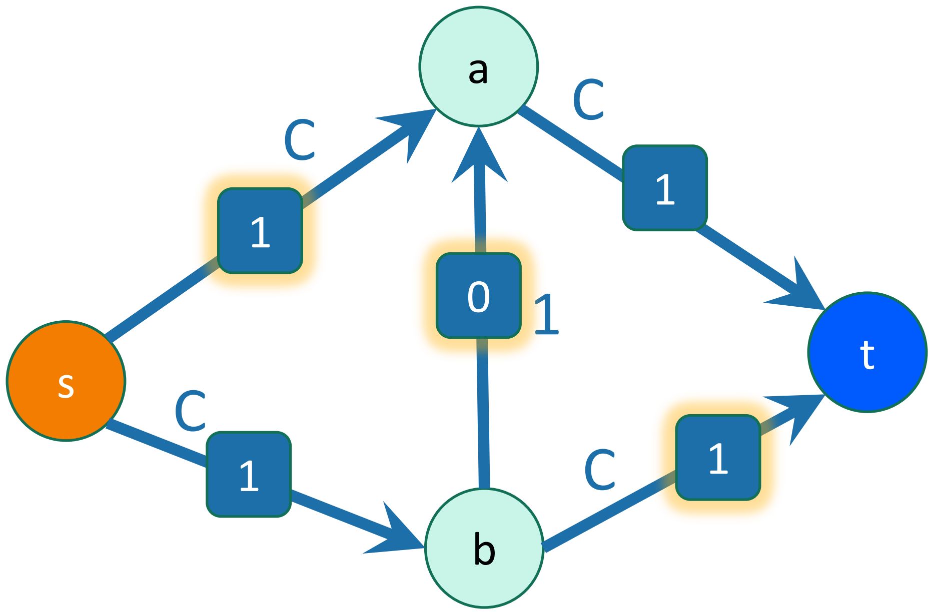

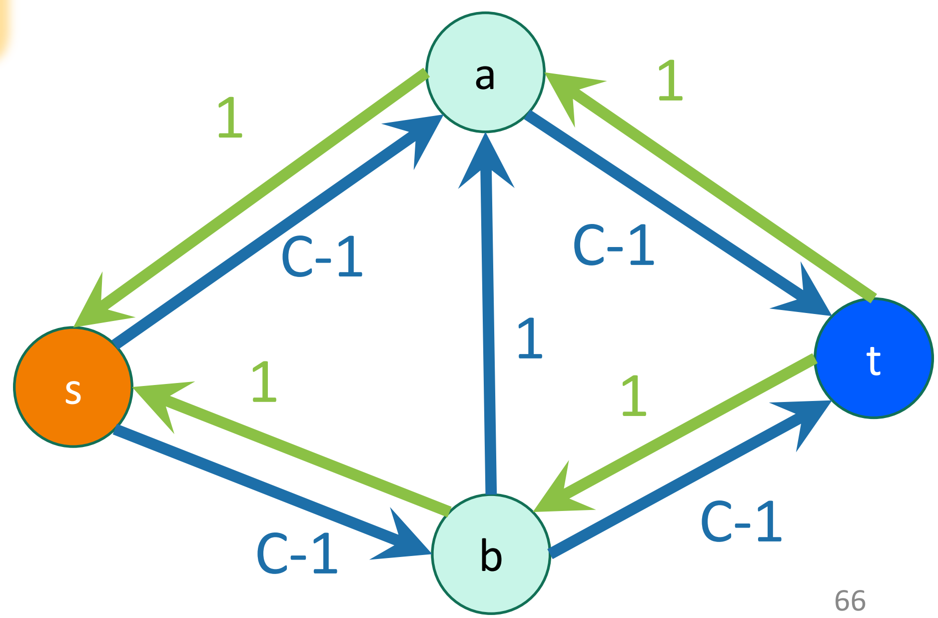

The edge (b,a) re-appeared in the residual graph!

Why should we be concerned?

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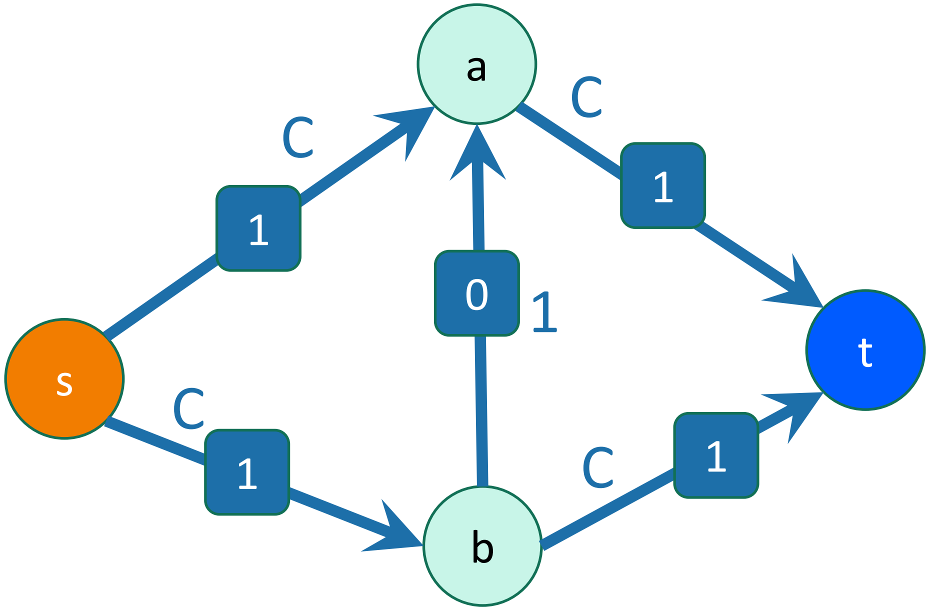

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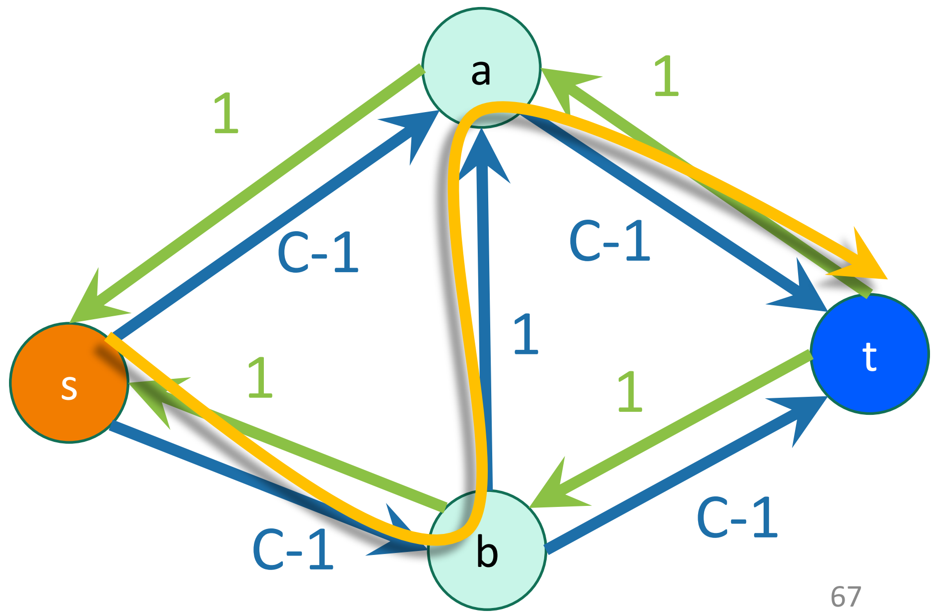

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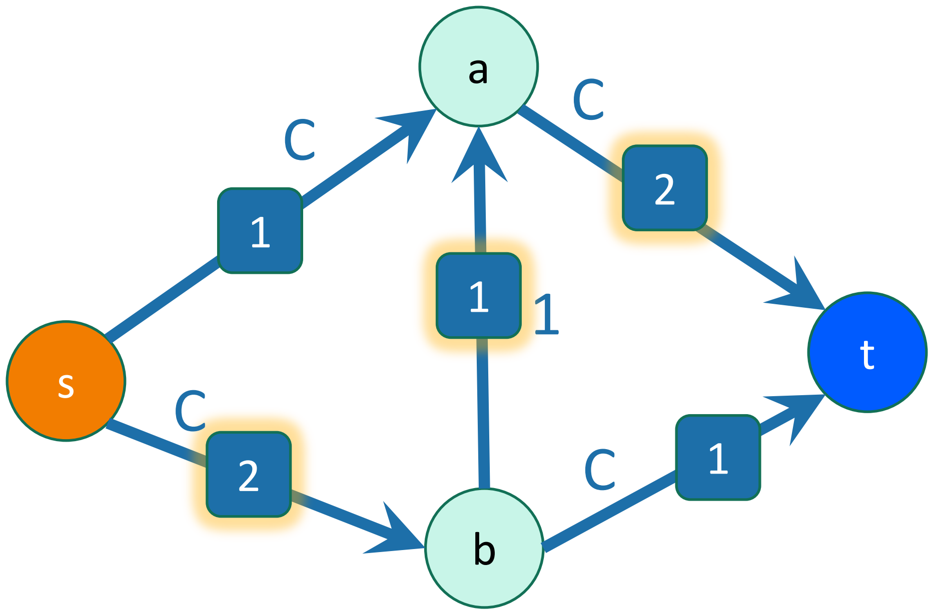
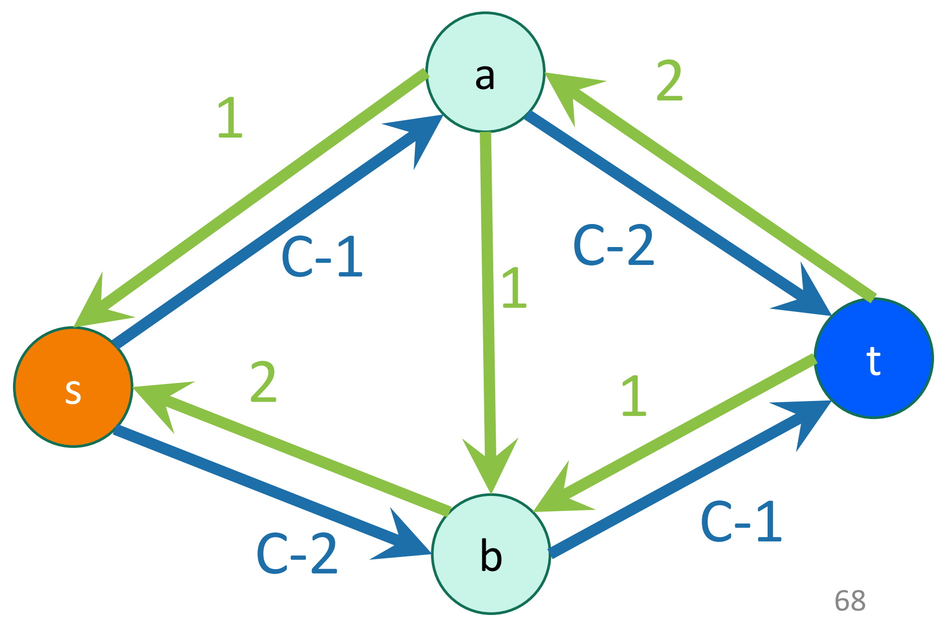


Choose a really big number C.

The edge (b,a) disappeared from the residual graph!



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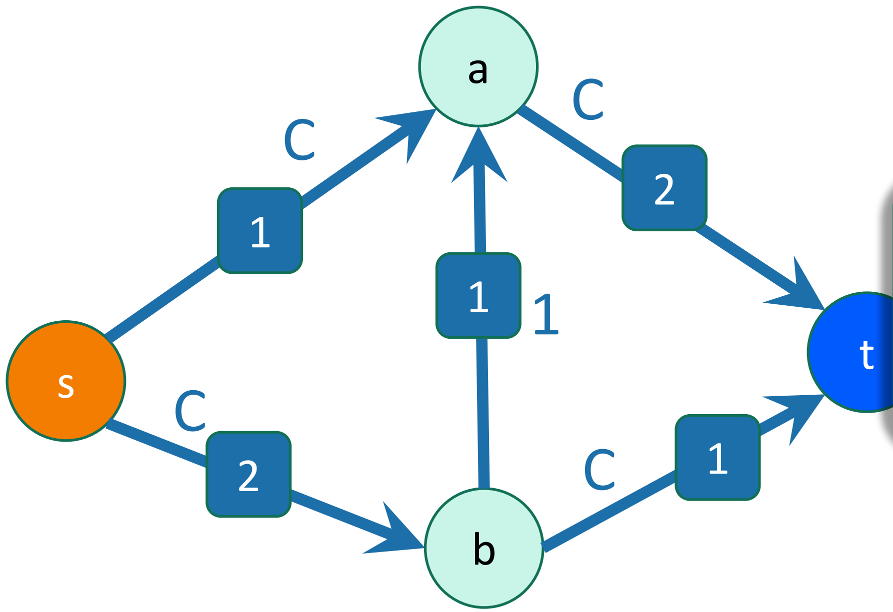
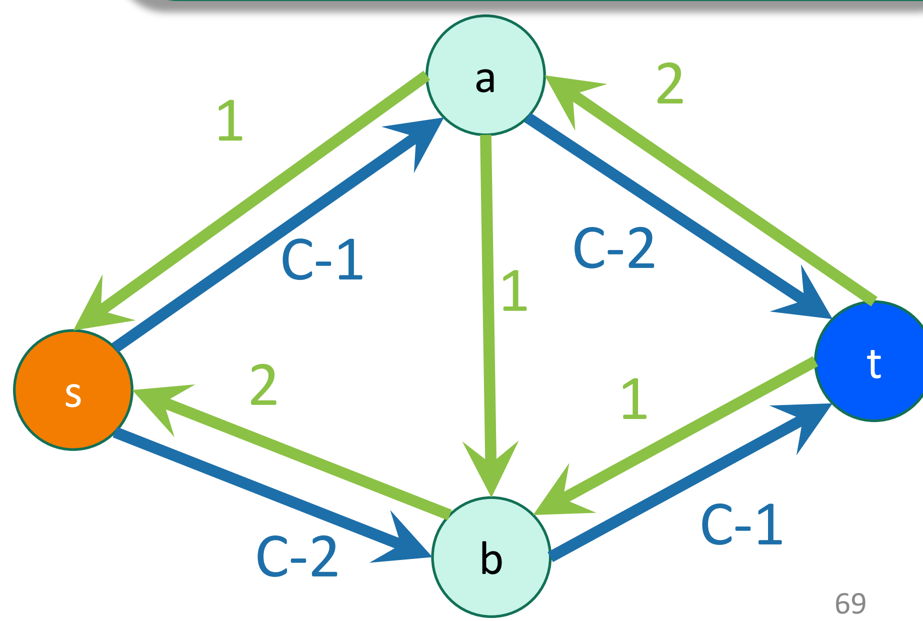


This will go on for C steps, adding flow along (b,a) and then subtracting it again.



The edge (b,a) disappeared from the residual graph!

Doing Ford-Fulkerson with BFS is called the **Edmonds-Karp algorithm**.

Theorem

- If you use BFS, the Ford-Fulkerson algorithm runs in time **$O(nm^2)$.**

Doesn't have anything to do with the edge weights!

- We will skip the proof in class.
  - You can check it out in the notes if you are interested.
  - It will **not** be on the exam.

- Basic idea:
  - The number of times you remove an edge from the residual graph is O(n).
    - This is the hard part
  - There are at most m edges.
  - Each time we remove an edge we run BFS, which takes time O(n+m).
    - Actually, O(m), since we don't need to explore the whole graph, just the stuff reachable from s.

Recap

- Today we talked about s-t cuts and s-t flows.
- The **Min-Cut Max-Flow Theorem** says that minimizing the cost of cuts is the same as maximizing the value of flows.
- The Ford-Fulkerson algorithm does this!
  - Find an augmenting path
  - Increase the flow along that path
  - Repeat until you can't find any more paths and then you're done!

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Acknowledgement: Part of the materials are adapted from Virginia Williams and David Eng's lectures on algorithms. We appreciate their contributions.