Greedy Algorithms I

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Mid-Course Review

We've covered a lot so far!

- Techniques for algorithmic analysis
  - Asymptotics, lower-bounding functions, proofs of correctness, runtime analysis of algorithms
- 2 algorithmic paradigms: divide and conquer, randomized.
  - Randomized/graph: `karger`, for finding minimum cut
  - Divide and conquer/randomized: `quicksort`, `quickselect`
- Several problems: sorting, single-source shortest path (Dijkstra's), global minimum cut (karger), hashing, SCC finding (Kosaraju's), topological sorting (by DFS), bipartite finding (by BFS).

A lot of cool stuff ahead!

- 2 more algorithmic paradigms: greedy algorithm (today's topic) and dynamic programming.
- Approximation algorithms, amortized analysis, intractability.

Outline for Today

- Greedy algorithms
  - Frog Hopping
  - Greedy graph algorithms
    - Minimum Spanning Trees
    - Prim's Algorithm
    - Kruskal's Algorithm

Frog Hopping

A warmup example

Greedy Algorithms

Greedy algorithms construct solutions one step at a time, at each step choosing the locally best option.

Advantages: simple to design, often efficient

Disadvantages: difficult to verify correctness or optimality

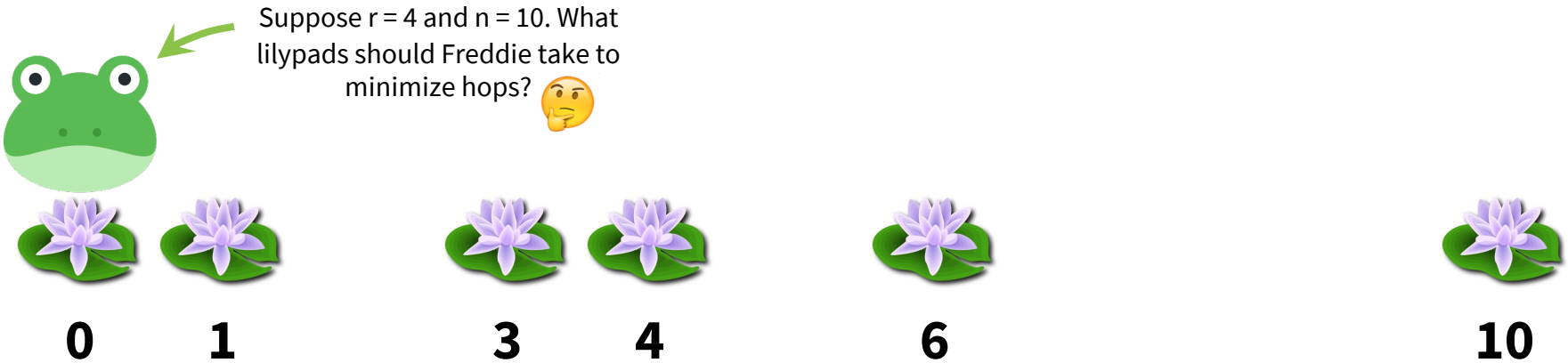
Freddie the Frog

Freddie the Frog starts at position 0 along a river. His goal is to reach position n.

There are lilypads at various positions, including at position 0 and position n.

Freddie can hop at most r units at a time.

Task: Find the path Freddie should take to minimize hops, assuming such a path exists.



Frog Hopping

```
algorithm frog_hopping(lilys, r, n):
  # lilys = [0, 1, 3, 4, 6, 10] in the previous example
  H = [0]  # contains hops
  cur_lily = {"index": 0, "position": 0}
  while cur_lily["position"] < n:
    next_lily = furthest_reachable_lily(
      cur_lily, lilys, r
    )  # finds the furthest lilypad still reachable
       # from cur_lily
    H.append(next_lily["position"])
    cur_lily = next_lily
  return H
```

You should be able to implement this function yourself.

Runtime: `O(n)`

Frog Hopping

We need to prove two properties about the algorithm to guarantee correctness.

(1) **Feasibility.** The algorithm finds a feasible (aka legal) series of hops (i.e. it doesn't "get stuck" or break any rules).

(2) **Optimality.** The algorithm finds an optimal series of hops (i.e. there isn't a better path available).

Frog Hopping

Lemma 1: `frog_hopping` always finds a feasible series of hops for Freddie.

Proof: We proceed by contradiction.

Suppose it did not. A path might be infeasible for one of three reasons:
(1) H.first ≠ 0, (2) H[k] + r < H[k+1] for some k, or (3) H.last ≠ n.

Notation for the first element in list H.

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Since the algorithm initializes H to [0], (1) is impossible.

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By construction of the algorithm, `next_lily` will always be reachable from the `cur_lily`; therefore (2) is impossible.

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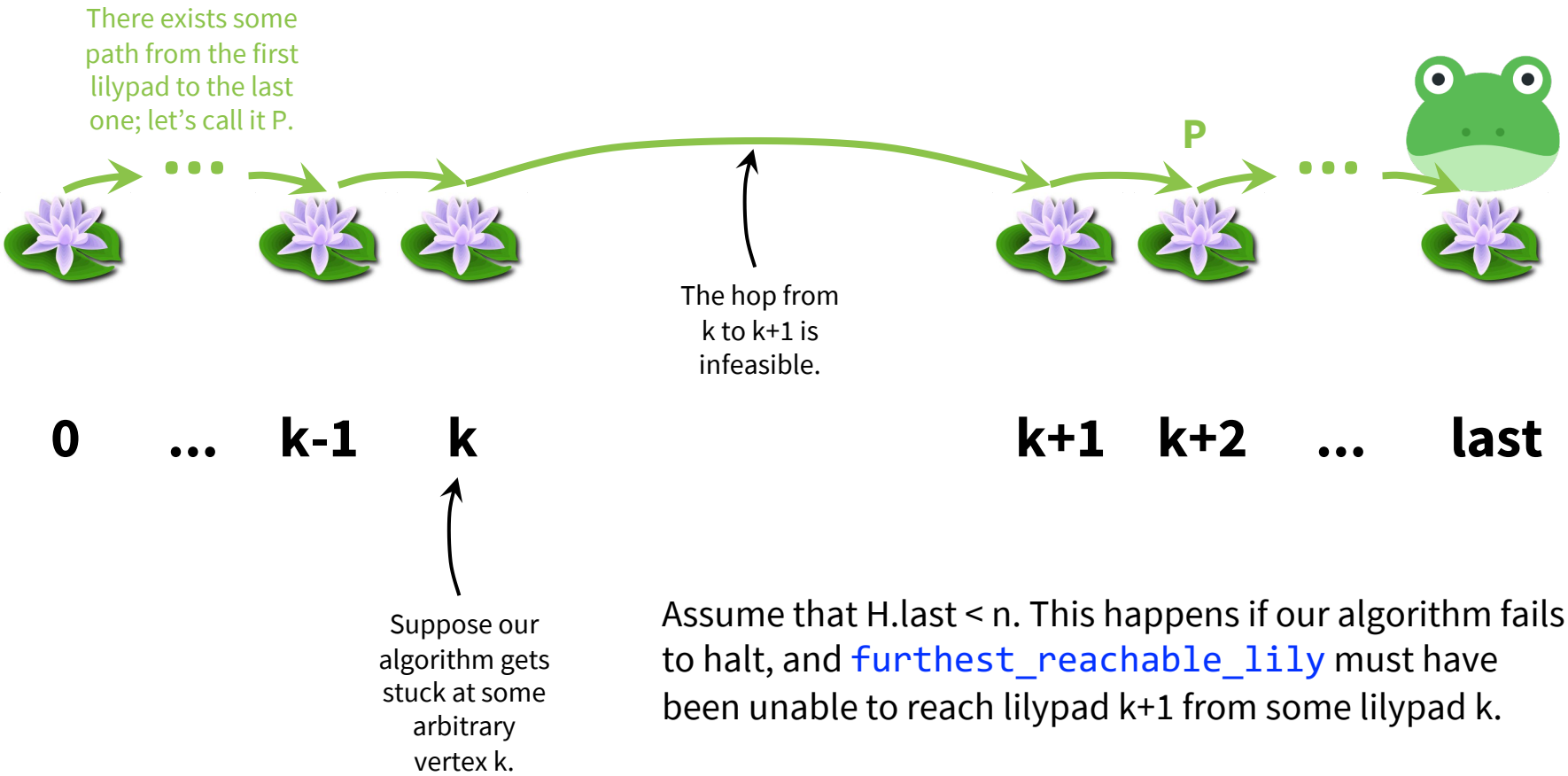
Since the algorithm initializes H to [0], (1) is impossible.

By construction of the algorithm, `next_lily` will always be reachable from the `cur_lily`; therefore (2) is impossible.

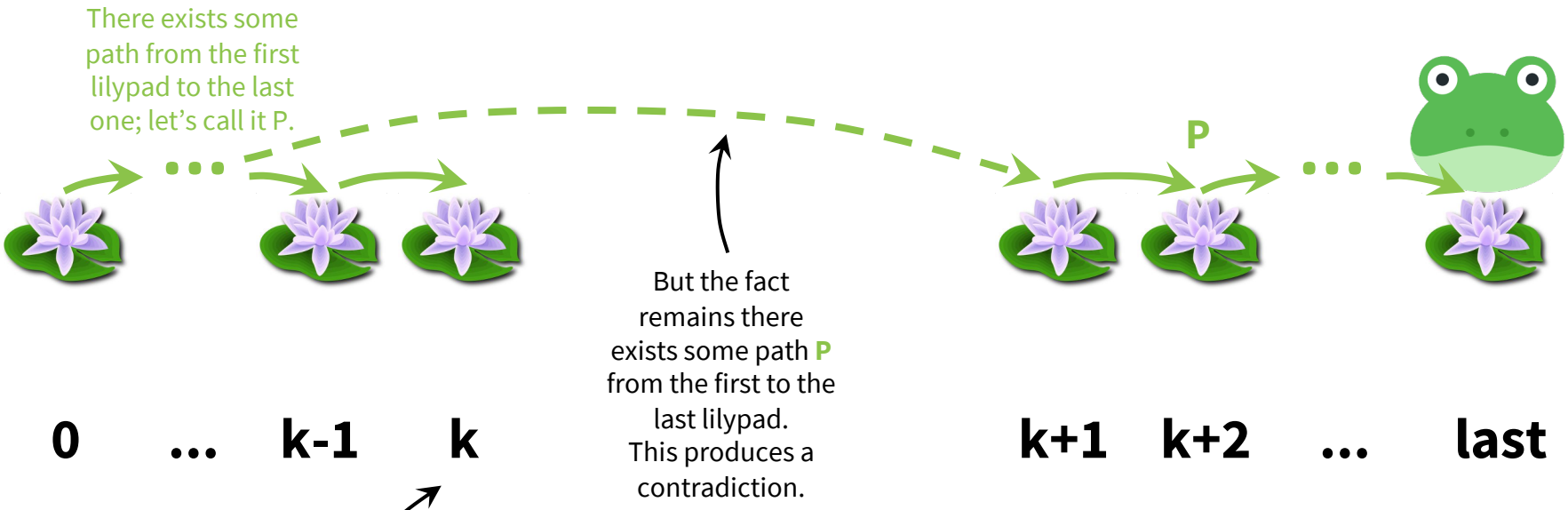
To prove that H.last ≠ n is impossible, we proceed by contradiction.

Frog Hopping



Assume that H.last < n. This happens if our algorithm fails to halt, and `furthest_reachable_lily` must have been unable to reach lilypad k+1 from some lilypad k.

Frog Hopping



But then for some lily s < k, we will have lily[s] + r < lily[k] + r < lily[k+1]. This would imply that any Lilypad[0:k] is unreachable to Lilypad[k+1], this contradicts with our knowledge that there exists a path from the first to the last Lilypad.
We have reached a contradiction, so our assumption must have been incorrect; therefore, (3) is impossible.

Frog Hopping

We need to prove two properties about the algorithm to guarantee correctness.

(1) **Feasibility.** The algorithm finds a feasible (aka legal) series of hops (i.e. it doesn't "get stuck" or break any rules). 👌

(2) **Optimality.** The algorithm finds an optimal series of hops (i.e. there isn't a better path available). 🤔

5-Minute Break

Proof of Optimality

Frog Hopping

Now for the difficult part: How might we prove that `frog_hopping` always finds an optimal series of hops?

- Let's introduce notation to talk about the algorithm with greater precision …
- Let H be the series of hops produced by our algorithm and H* be **an arbitrary** (not necessarily **the only**) optimal series of hops. Then |H| and |H*| denote the number of hops in H and H*, respectively.
- Note that |H| ≥ |H*|. Why? 🤔

Frog Hopping

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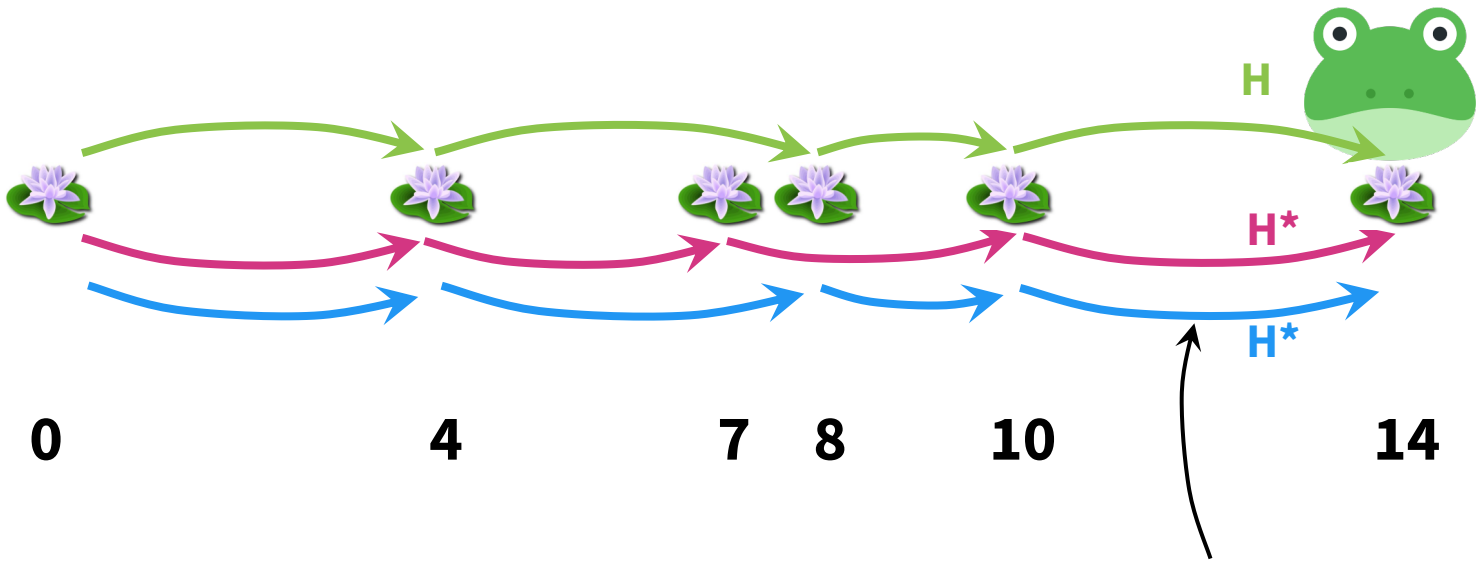
- Note that |H| ≥ |H*|. Why? 🤔 Otherwise, H* wouldn't be optimal.
- We want to prove that |H| = |H*|. How?

Intuition: Consider an arbitrary optimal series of hops H*, then show that our greedy algorithm produces a series of hops H no worse than H*, i.e., |H| ≤ |H*|

What Does Arbitrary H* Mean?

Suppose Freddie's longest hop at each time is r=4.



There could be many optimal H* (this series of lilypads has 2 optimal H*); this proof relies on an arbitrary choice from among these H*.

Suppose we choose **H***.

Frog Hopping

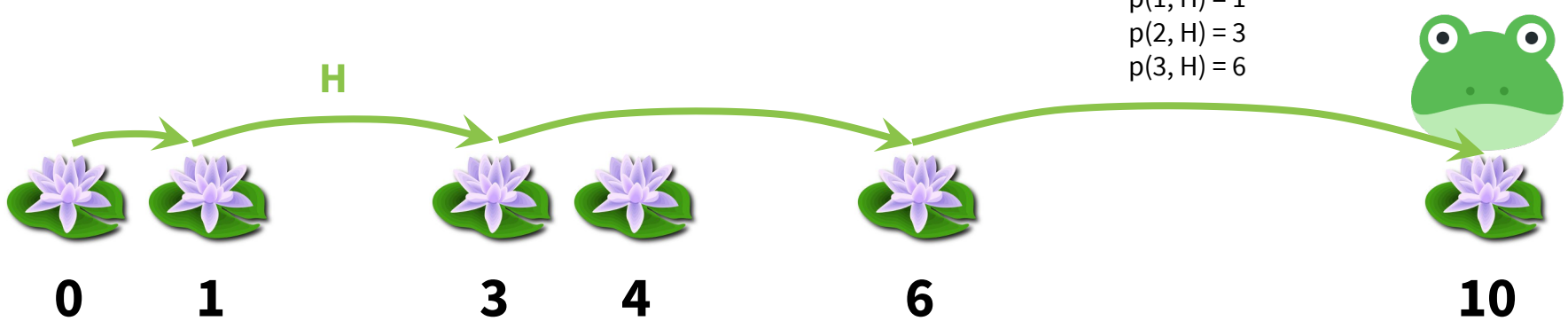
Let p(i, H) denote Freddie's position after taking the first i hops from series H.

Lemma: For any i in $0 \le i \le |H^*|$, we have $p(i, H) \ge p(i, H^*)$, constructing H from `frog_hopping`.

i.e. After taking i hops according to our greedy algorithm, Freddie will be at least as far forward as if it took i jumps according to an optimal solution.

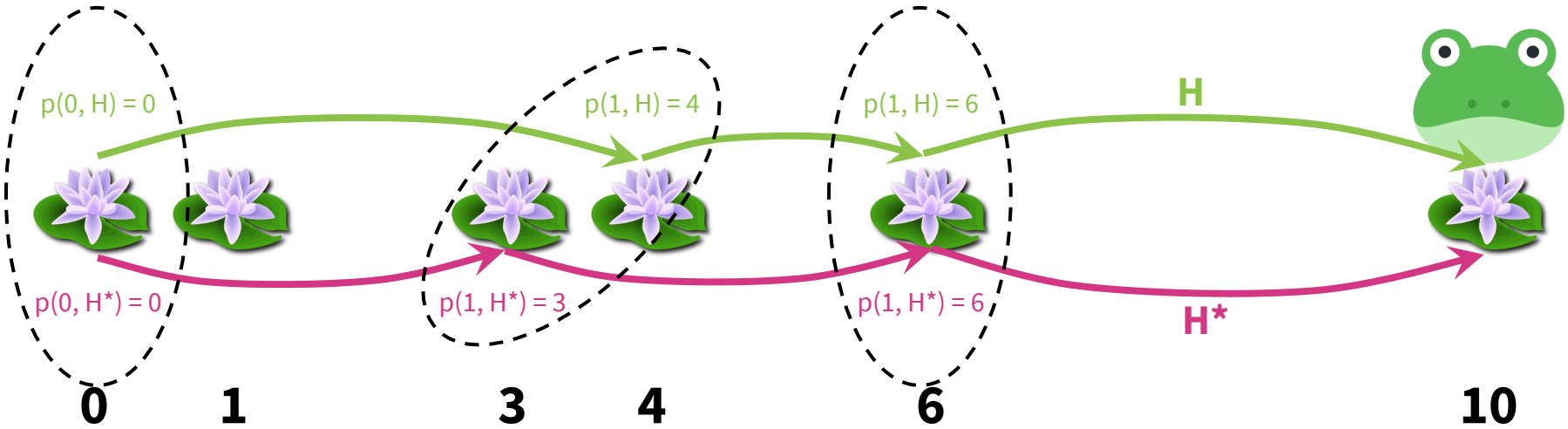
Let's formalize this using induction.

For this arbitrary **H**:
(unrelated to our alg)
p(0, H) = 0
p(1, H) = 1
p(2, H) = 3
p(3, H) = 6



Frog Hopping

Lemma: For any i in $0 \le i \le |H^*|$, we have $p(i, H) \ge p(i, H^*)$, constructing **H** from `frog_hopping`.



Frog Hopping

Lemma 2: For all $0 \le i \le |H^*|$, we have $p(i, H) \ge p(i, H^*)$, constructing H from `frog_hopping`.

Proof: We proceed by induction.

As a base case, when $i = 0$, then $p(0, H) = 0 \ge 0 = p(0, H^*)$ since Freddie hasn't moved.

For the inductive step, assume that the claim holds for some $0 \le i < |H^*|$. We'll prove the claim holds for $i + 1$ by considering two cases:

Case 1: $p(i, H) \ge p(i+1, H^*)$. Since each hop moves forward, we have $p(i+1, H) \ge p(i, H)$, so we have $p(i+1, H) \ge p(i+1, H^*)$.

Case 2: $p(i, H) < p(i+1, H^*)$. Each hop is of size at most r, so $p(i+1, H^*) \le p(i, H^*) + r$. By our inductive hypothesis, we know $p(i, H) \ge p(i, H^*)$, so $p(i+1, H^*) \le p(i, H) + r$; i.e. position $p(i+1, H^*)$ is reachable from position $p(i, H)$. Since the greedy algorithm hops to the furthest lilypad still reachable from position $p(i, H)$, it hops to at least position $p(i+1, H^*)$. Therefore, $p(i+1, H) \ge p(i+1, H^*)$.

So $p(i+1, H) \ge p(i+1, H^*)$, completing the induction. ■

Frog Hopping

Now for the theorem: `frog_hopping` produces an optimal solution for Freddie.

Frog Hopping

Theorem: `frog_hopping` produces an optimal solution for Freddie.

Proof:

Since H^* is an optimal solution, we know that $|H^*| \leq |H|$. We will prove $|H^*| = |H|$.

Let $k = |H^*|$. By **Lemma 2**, we have $p(k, H) \geq p(k, H^*)$. Since Freddie arrives at position n after k hops along series H^*, we know that $p(k, H) \geq p(k, H^*) = n$. Because the greedy algorithm never hops past position n, we know $p(k, H) \leq n$. Since $n \leq p(k, H) \leq n$, then $p(k, H) = n$.

The greedy algorithm arrives at position n after k hops, so $|H| = k$. Importantly, it's impossible to reach position n in fewer than k hops since doing so would contradict the optimality of H^*. Thus, $|H| = k = |H^*|$, so `frog_hopping` produces an optimal solution. ■

Frog Hopping

We need to prove two properties about the algorithm to guarantee correctness.

(1) **Feasibility.** The algorithm finds a feasible (aka legal) series of hops (i.e. it doesn't "get stuck" or break any rules). 👌

(2) **Optimality.** The algorithm finds an optimal series of hops (i.e. there isn't a better path available). 👌

Greedy Stays Ahead

The style of proof we just wrote is an example of a **greedy stays ahead** proof.

(1) Find intermediate values that evaluate the solution produced by any algorithm, including the greedy one.

What's our values in `frog_hopping`? 🤔 The position after i hops: p(i, H).

(2) Show the greedy algorithm produces values at least as good as any solution's (using induction).

(3) Prove that since the greedy algorithm produces values at least as good as any solution's, it must be optimal (using direct proof or proof by contradiction).

Greedy Exchange Argument

There's another style of proof that uses **greedy exchange argument**.

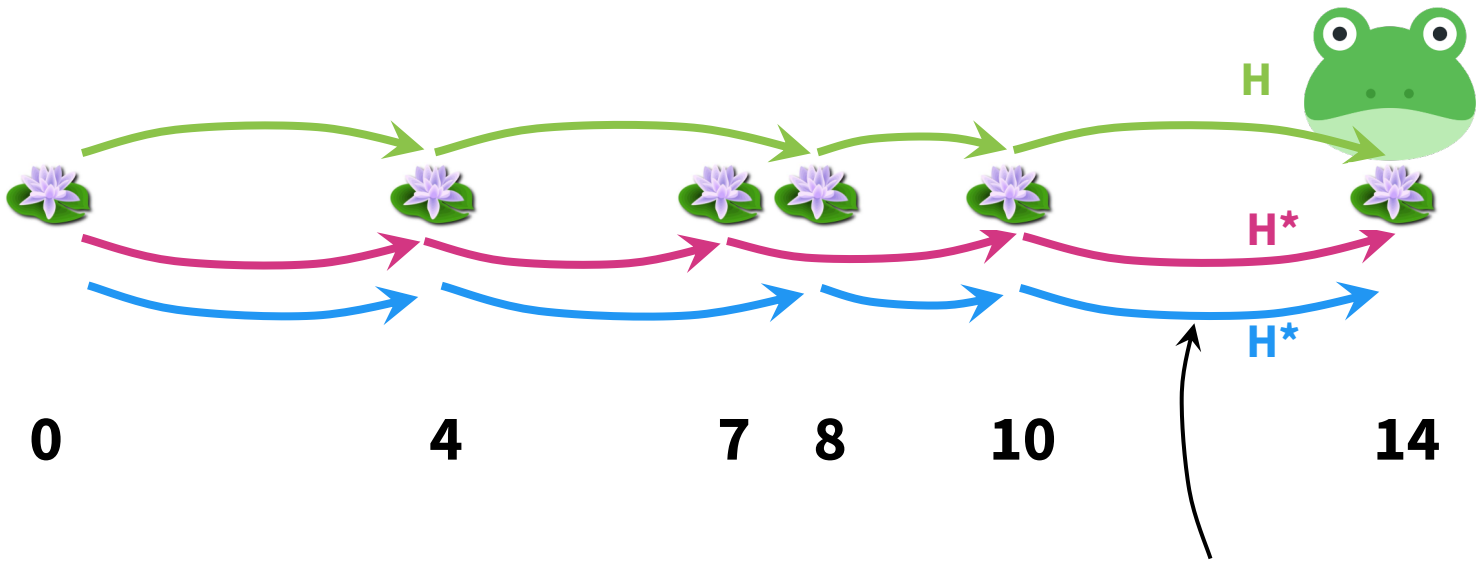
- If we swap an optimal solution out for the greedy solution, argue that we're still optimal.

5-Minute Break

Greedy Exchange Argument Proof

Greedy Exchange Argument

Again, this proof will rely on an arbitrary choice of H*.



There could be many optimal H* (this series of lilypads has 2 optimal H*); this proof relies on an arbitrary choice from among these H*.

Suppose we choose **H***.

Greedy Exchange Argument

Theorem: `frog_hopping` produces an optimal solution.

Proof: We proceed by induction.

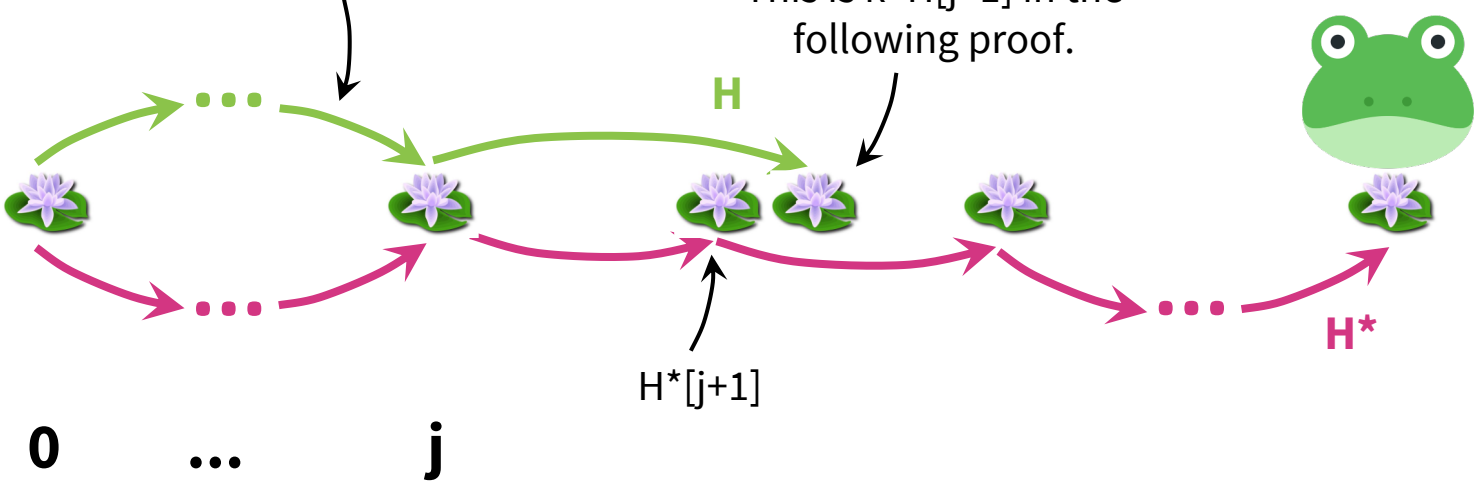
As a base case, we initialize H to `[0]` and all feasible hops H* must have H*[0] = 0, as a result, H*[0] = H[0].

For the inductive step, assume that after hop j has been added to H, there exists an optimal feasible series of hops H* such that H*[0..j] = H[0..j]. We'll prove that after hop j+1 has been added to H, there still exists an optimal series of hops H_{new}* such that H_{new}*[0..j+1] = H[0..j+1].

Greedy Exchange Argument

By the inductive hypothesis, **H** and **H*** are the same from 0 to j.
i.e., for our **H**, there exists an **H*** that is the same as **H** from 0 to j.

This is k=H[j+1] in the following proof.



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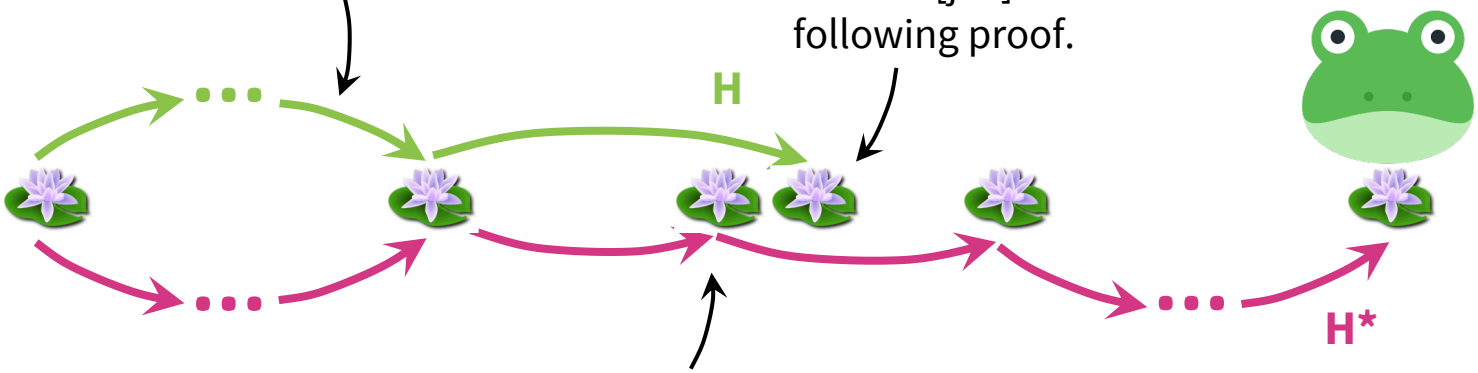

0 **...** **j**

The "exchange argument" contends that swapping H*[j+1] out for k = H[j+1] is still feasible and optimal.

Greedy Exchange Argument

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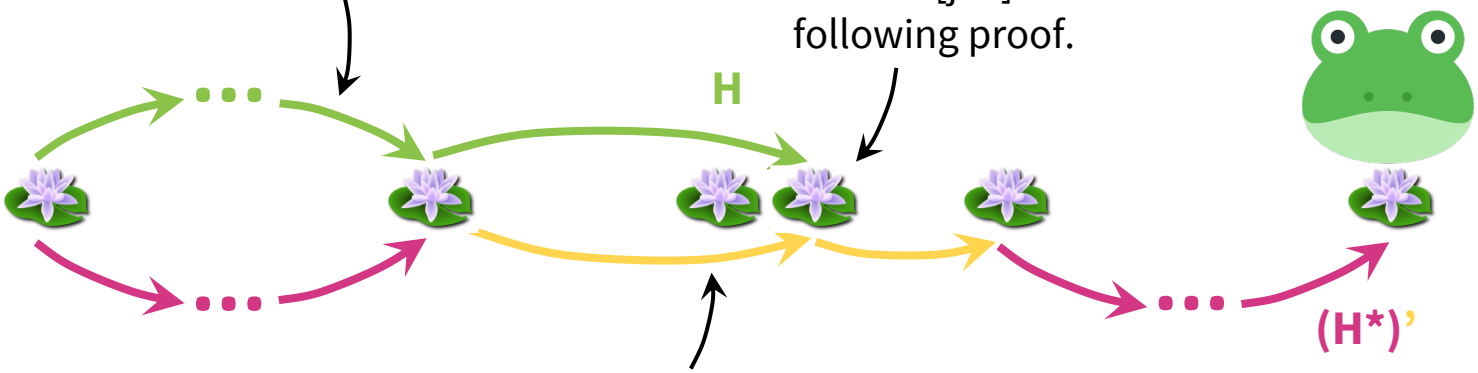

The "exchange argument" contends that swapping H*[j+1] out for k = H[j+1] is still feasible and optimal, giving a new path (H*)'.

Greedy Exchange Argument

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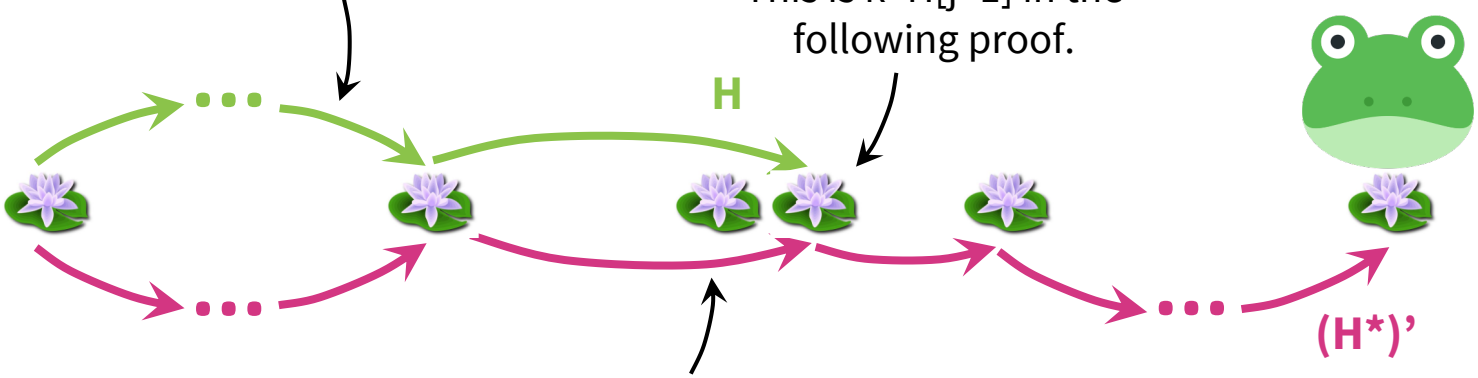

0 **…** **j**

So if for H[0…j] there exists an optimal solution H* such that H[0…j] = H*[0…j], then for H[0…j+1] there also exists an optimal solution (H*)' such that H[0…j+1] = (H*)'[0…j+1].

So when the greedy algorithm concludes and the algorithm returns H[0…K], there exists an optimal solution H* such as H[0…K] = H*[0…K]. Thus H is an optimal solution.

Greedy Exchange Argument

Theorem: `frog_hopping` produces an optimal solution.

Proof: We proceed by induction.

As a base case, we initialize H to `[0]` and all feasible hops H* must have H*[0] = 0, as a result, H*[0] = H[0].

For the inductive step, assume that after hop j has been added to H, there exists an optimal feasible series of hops H* such that H*[0..j] = H[0..j]. We'll prove that after hop j+1 has been added to H, there still exists an optimal series of hops H_{new}* such that H_{new}*[0..j+1] = H[0..j+1].

Let H* be an optimal series of hops such that H*[0..j] = H[0..j]. Suppose we add k as H[j+1]. Then k ≥ H*[j+1], because (1:) H*[j+1] ≤ r + H*[j] = r + H[j] and, (2:) by construction, k is the furthest lilypad such that k ≤ r + H[j].

Consider (H*)' obtained from H* by setting H*[j+1] = k.

This is still feasible because (1:) (H*)'[j+1] = k ≤ r + (H*)'[j] and (2:) (H*)'[j+2] = H*[j+2] ≤ r + H*[j+1] ≤ r + k = r + (H*)'[j+1].

Since H* and (H*)' are the same except at position j+1.

This is still optimal since (H*)' has the same number of hops as H*.

Activity Selection

Optional topic

Planning Your Life

You have a list of activities $(s_1, e_1), (s_2, e_2), \ldots, (s_n, e_n)$ denoted by their start and end times.

All activates are equally attractive to you, and you want to maximize the number of activities you do.

Task: Choose the largest number of non-overlapping activities possible.

Greedy Stays Ahead

What are a few ways of picking activities greedily? 🤔

Be impulsive: choose activities in ascending order of start times.

Avoid commitment: choose activities in ascending order of length.

Finish fast: Choose activities in ascending order of end times.

Only the third one seems to work.

Activity Selection

```
algorithm activity_selection(activities):
  sort activities into ascending order by end time
  U = set of all activities
  S = an empty set
  while U not empty:
    choose any activity with the earliest finishing time;
    add that activity to S;
    remove other activities that overlap with it from U;
  return S
```

Runtime: $O(n^2)$

Activity Selection

We need to prove two properties about the algorithm to guarantee correctness.

(1) **Legality.** The algorithm finds a legal schedule of activities (i.e. it doesn't "schedule conflicting activities").

(2) **Optimality.** The algorithm finds an optimal schedule of activities (i.e. there isn't a better schedule available).

Activity Selection

Lemma: The schedule produced by `activity_selection` is a legal schedule.

Intuition: Use induction to show that at each step, the set U only contains activities that do not conflict with the selected activities in S.

Activity Selection

We need to prove two properties about the algorithm to guarantee correctness.

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(2) **Optimality.** The algorithm finds an optimal schedule of activities (i.e. there isn't a better schedule available). 🤔

Activity Selection

To prove that the schedule S produced by the algorithm is optimal, we will use another "greedy stays ahead" argument.

(1) Find intermediate values that evaluate the solution produced by any algorithm, including the greedy one. **Here, the end_time of the k-th activity chosen.**

(2) Show the greedy algorithm produces values at least as good as any solution's (using induction).

(3) Prove that since the greedy algorithm produces values at least as good as any solution's, it must be optimal (using direct proof or proof by contradiction).

Activity Selection

How might we prove that `activity_selection` finds an optimal schedule of activities?

- Let's introduce notation to talk with greater precision about the algorithm …
- Let S be the schedule produced by our algorithm and S* be **an arbitrary** (not necessarily **the only**) optimal schedule. Then |S| and |S*| denote the number of activities in S and S*, respectively.
- Note that |S| ≤ |S*|. Why? 🤔 Because otherwise S* would not be optimal
- We want to prove that |S| = |S*|. How?

Intuition: Consider an arbitrary optimal schedule S*, then show that our greedy algorithm produces a schedule S no worse than S*.

Activity Selection

Let $f(i, S)$ denote the time that the i-th activity finishes in schedule S.

Lemma: For any $1 \leq i \leq |S|$, we have $f(i, S) \leq f(i, S^*)$.

i.e. After scheduling i activities according to the greedy algorithm, you will be at most as late as if you scheduled i activities according to an optimal solution.

Let's formalize this using induction!

Proving Optimality

Lemma: For all $1 \le i \le |S|$, we have $f(i, S) \le f(i, S^*)$.

Proof: We proceed by induction.

As a base case, the first activity the greedy algorithm selects must be an activity that ends no later than any other activity, so $f(1, S) \le f(1, S^*)$.

For the inductive step, assume that the claim holds for some $1 \le i < |S|$. We will prove the claims holds for $i + 1$. According to the induction assumption, we have $f(i, S) \le f(i, S^*)$, which means the i-th activity in S finishes before the i-th activity in S* finishes. Since the (i+1)-th activity in S* must start after the i-th activity in S* ends (otherwise the two activities in S* will conflict), we have $f(i, S^*) \le b(i+1, S^*)$, (b(*) means the beginning time of an activity). Combining the two inequalities, we have $f(i, S) \le b(i+1, S^*)$, meaning that the (i+1)-th activity in S* must start after the i-th activity in S ends.

Therefore, the (i+1)-th activity in S* must remain in U after you select the i-th activity for S. So when you use the greedy algorithm to select the (i+1)-th activity for S, you will select the activity in U with the lowest end time, so we must have $f(i+1, S) \le f(i+1, S^*)$, completing the induction.

Proving Optimality

Theorem: `activity_selection` produces an optimal solution.

Proof: Since S^* is optimal, we have $|S| \leq |S^*|$. We will prove $|S| = |S^*|$.

We proceed by contradiction. Suppose that $|S| \neq |S^*|$, we must have $|S| < |S^*|$.

Let $k = |S|$. By our lemma, we know $f(k, S) \leq f(k, S^*)$, so the k-th activity in S finishes no later than the k-th activity in S^*. Since $|S| < |S^*|$, there is a (k+1)-th activity in S^*, and its start time must be after $f(k, S^*)$ and therefore after $f(k, S)$.

Thus after the greedy algorithm added its k-th activity to S, the (k+1)-th activity from S^* would still belong to U, because it does not conflict with $f(k, S)$.

But the greedy algorithm ended after k activities, so U must have been empty.

We have reached a contradiction, so our assumption was wrong and $|S^*| = |S|$, so the greedy algorithm produces an optimal solution. ■

Activity Selection

We need to prove two properties about the algorithm to guarantee correctness.

(1) **Legality.** The algorithm finds a legal schedule of activities (i.e. it doesn't "schedule conflicting activities"). 👌

(2) **Optimality.** The algorithm finds an optimal schedule of activities (i.e. there isn't a better schedule available). 👌

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Acknowledgement: Part of the materials are adapted from Virginia Williams and David Eng's lectures on algorithms. We appreciate their contributions.