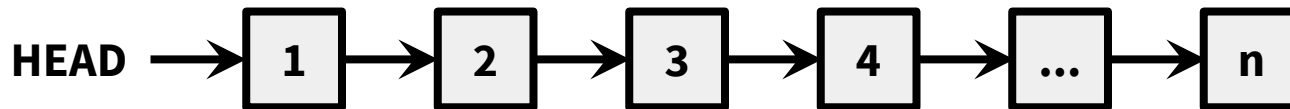# Binary Search Trees
Red-Black Trees

Binary Search Trees

Why BSTs?

Good data structures help us to design more efficient algorithms!

HEAD → `1` → `2` → `3` → `4` → `...` → `n`

Sorted linked lists: $O(n)$ search/select
$O(1)$ insert/delete

Assuming we already have a
pointer to the location of the
insert/delete

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | ... | n-1 | n |

Sorted arrays: $O(\log n)$ search
$O(1)$ select
$O(n)$ insert/delete

"Get the kth smallest element"

3

Why BSTs?

|	Sorted linked lists	Sorted arrays	Binary search trees
Search	O(n)	O(log n)	O(log n)
Insert/Delete	O(n) If we already find the place to insert or delete, then O(1)	O(n)	O(log n)

Tree Terminology

This is a "vertex" or "node"; it has key 3

The vertex without a parent is the "root"

5

3

6

2

4

7

1

The left child of 3 is 2

The right child of 3 is 4

Vertices without non-NIL children are "leaves"

Both children of 7 are NIL.

Tree Terminology

The left-descendants of 5 are 1, 2, 3, and 4.

5

3

6

2

4

7

1

The parent of 1 is 2; the ancestors of 1 are 2, 3, and 5.
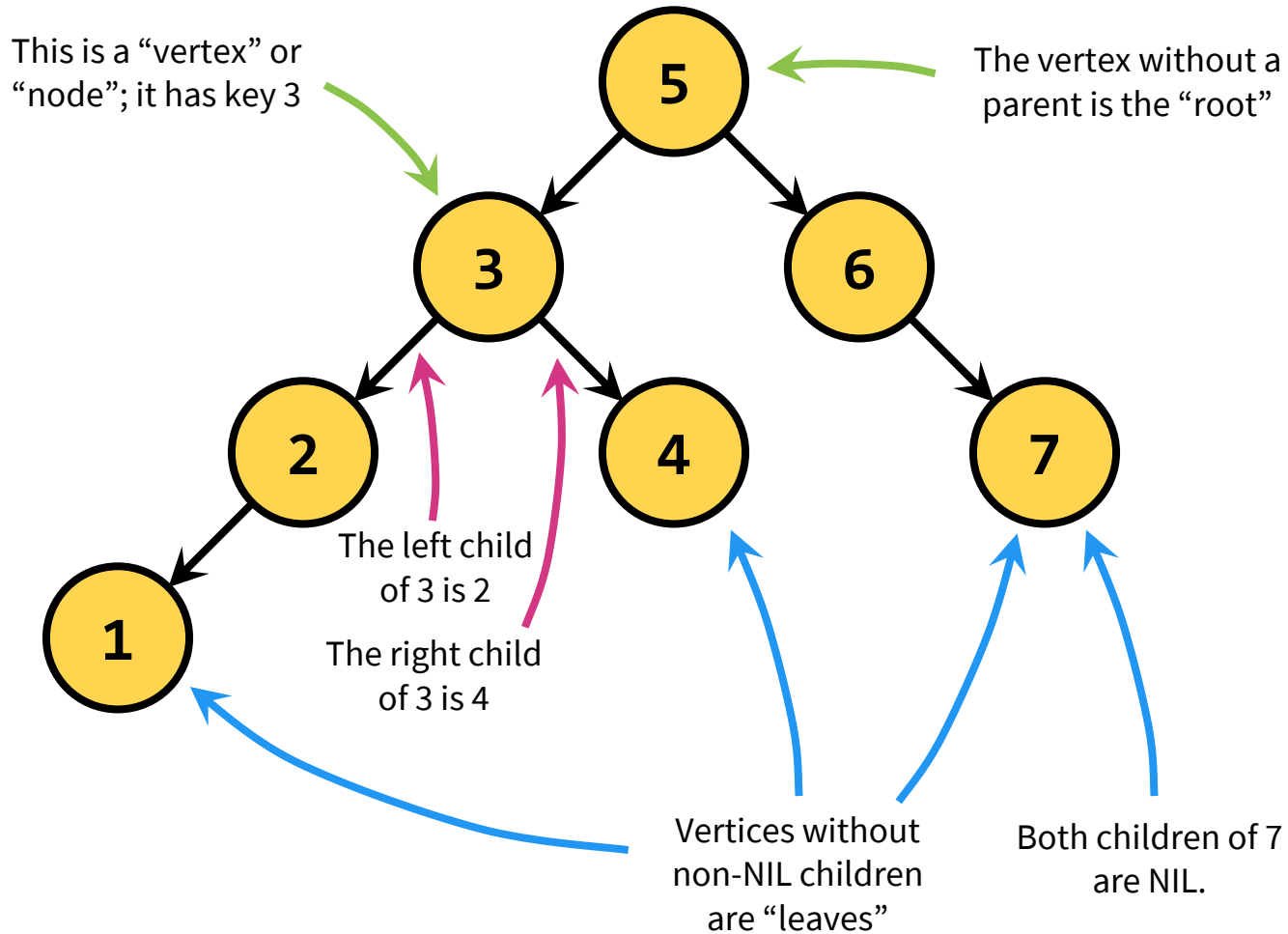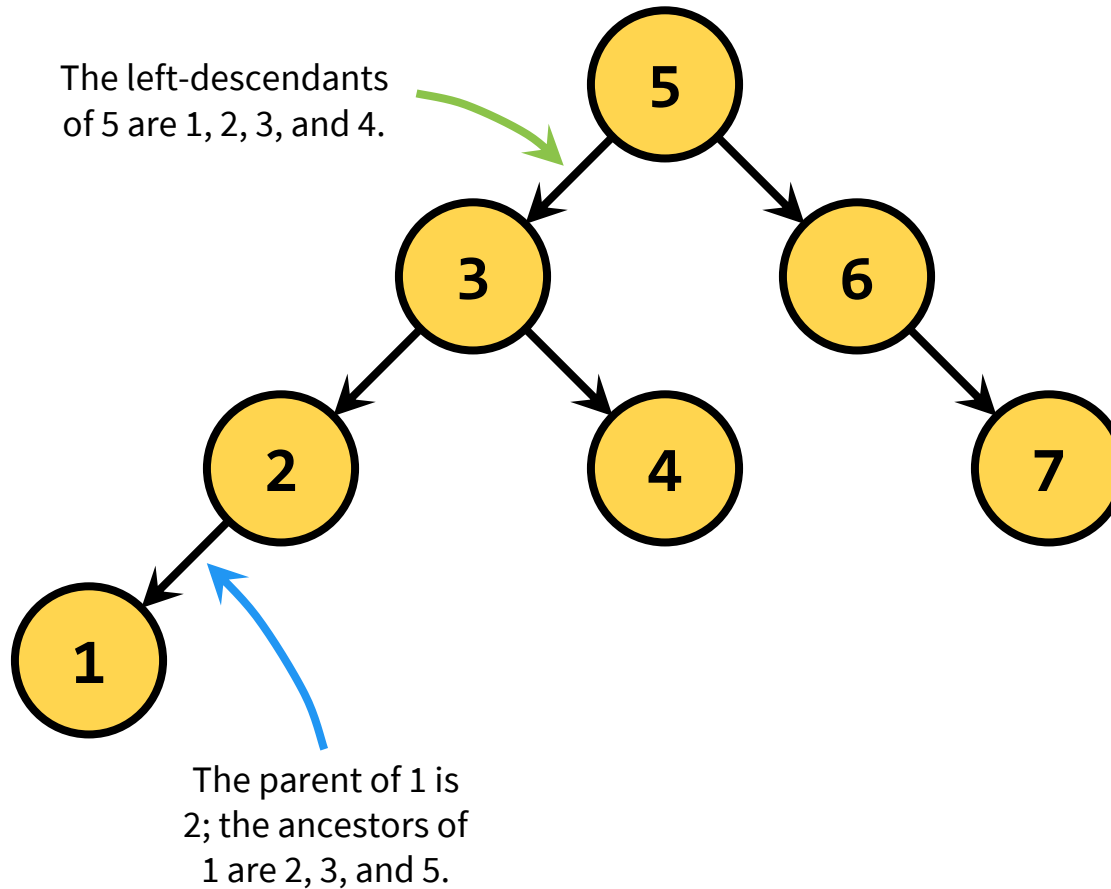
Tree Terminology

The **Predecessor** of a node is the right most element in its left subtree.
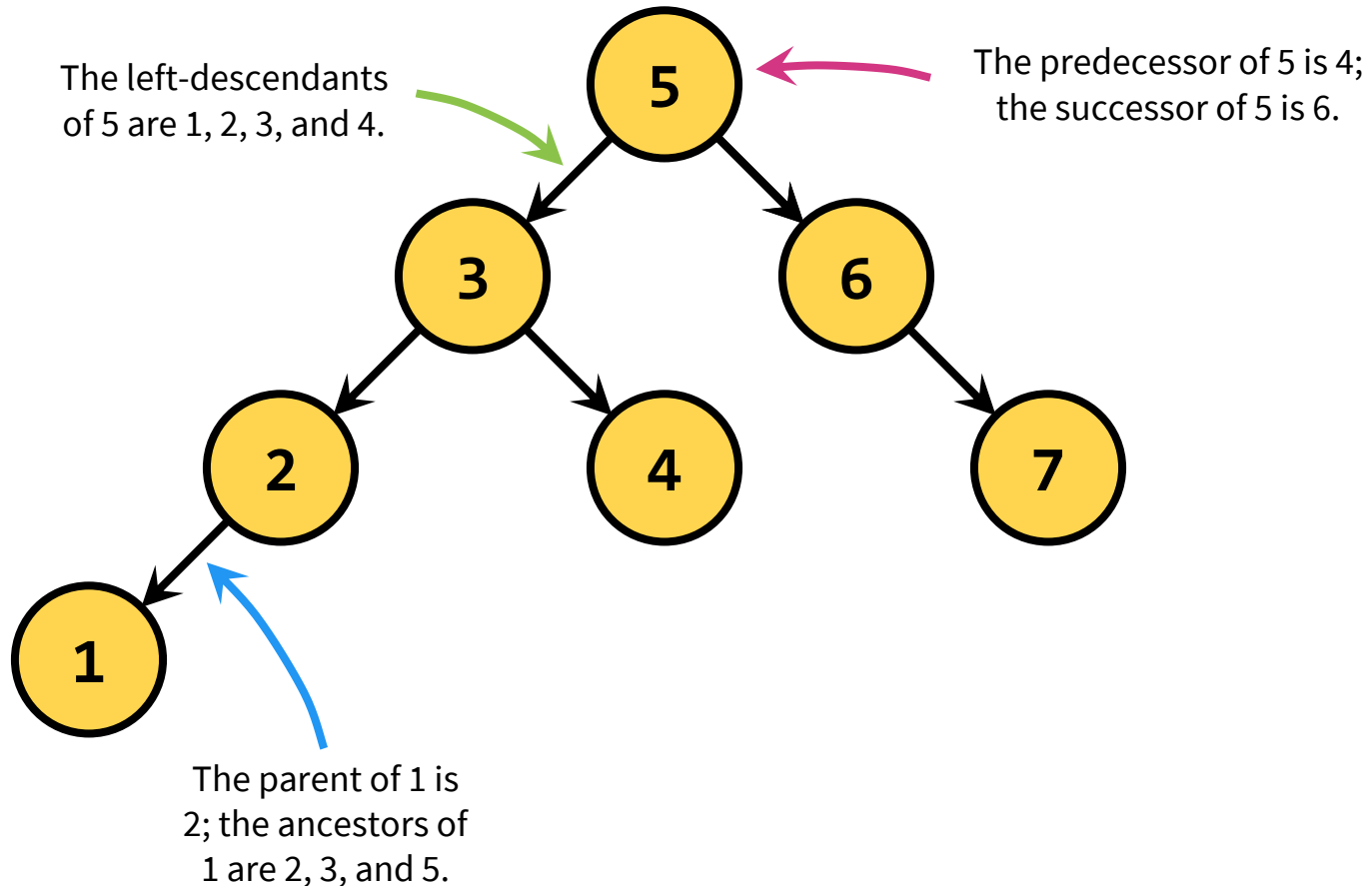The **Successor** of a node is the left most element in its right subtree.
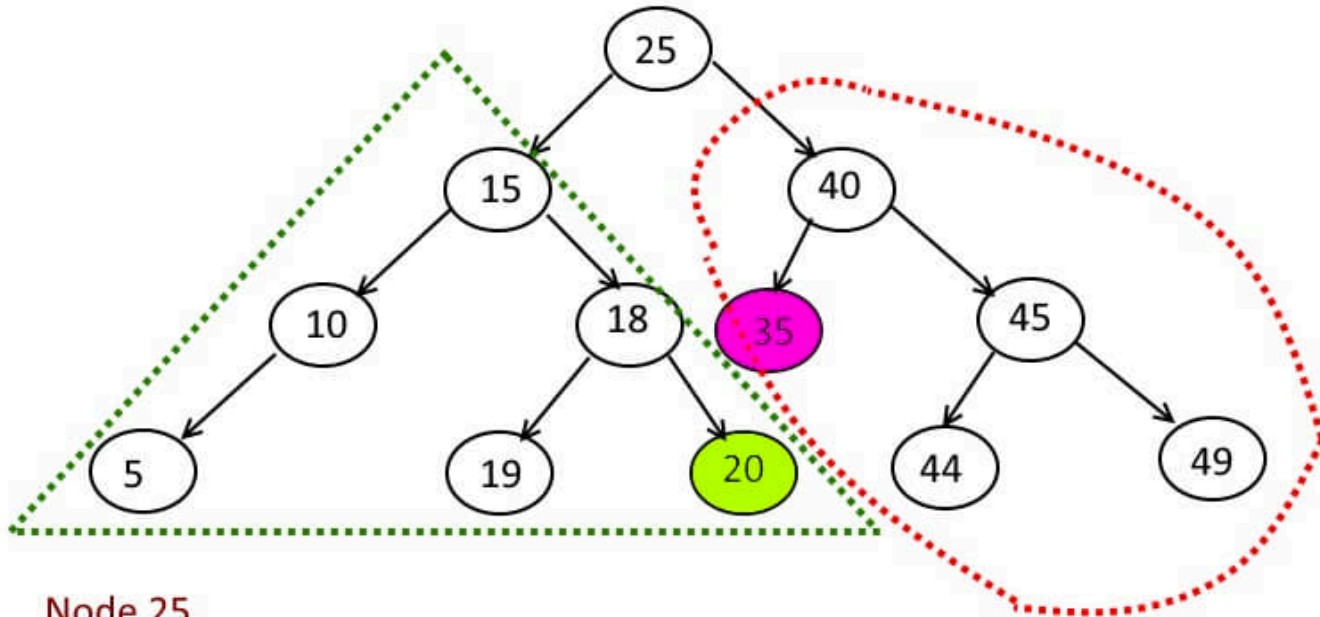
Node 25

The predecessor of node 25 is the right most node in its left subtree, which is 20

The successor of node 25 is the left most node in its right subtree, which is 35

Tree Terminology

The left-descendants of 5 are 1, 2, 3, and 4.

The predecessor of 5 is 4; the successor of 5 is 6.

The parent of 1 is 2; the ancestors of 1 are 2, 3, and 5.

8

Binary Search Trees

Binary Trees are trees such that each vertex has at most 2 children.

Binary Search Trees are Binary Trees such that:

Every left descendent of a vertex has key smaller than that vertex.

Every right descendent of a vertex has key greater than that vertex.

Take care: Not only the left and right child, but all the left and right descendants!
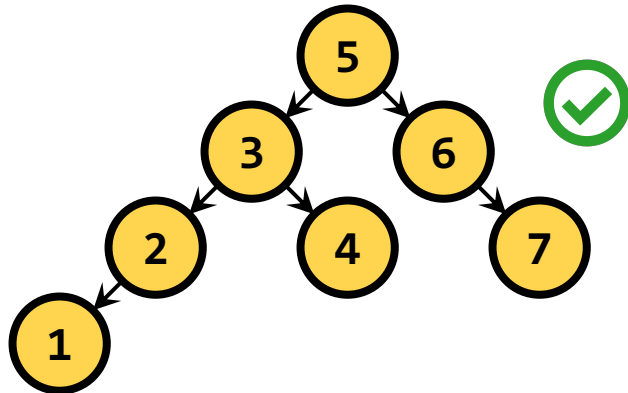
Binary Search Trees

Binary Search Trees are Binary Trees such that:

Every left descendent of a vertex has key smaller than that vertex.

Every right descendent of a vertex has key greater than that vertex.

Take care: Not only the left- and right-child, but all the left and right descendants!
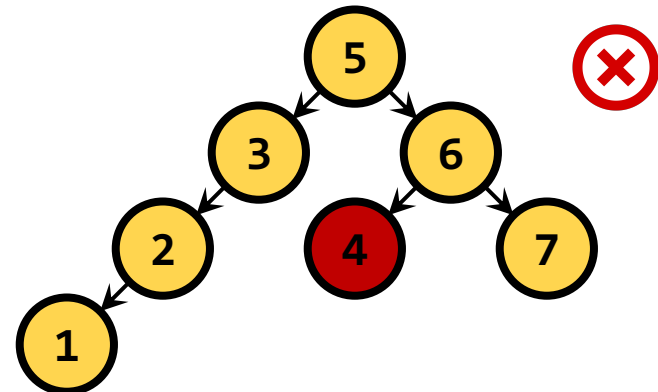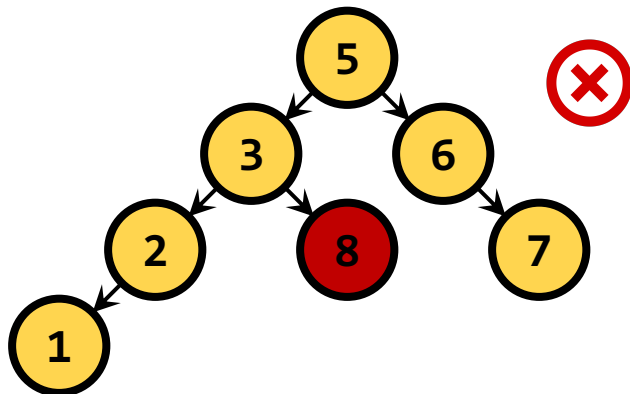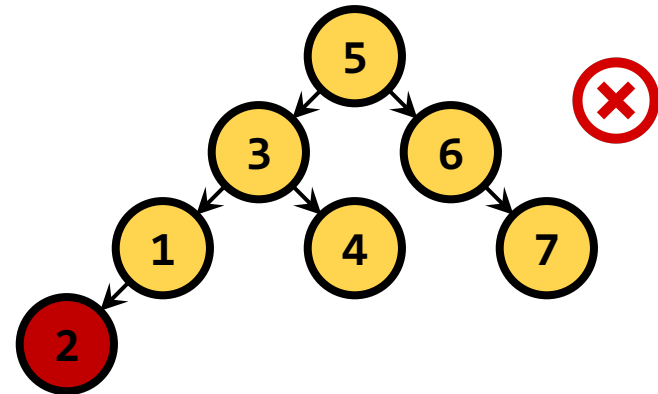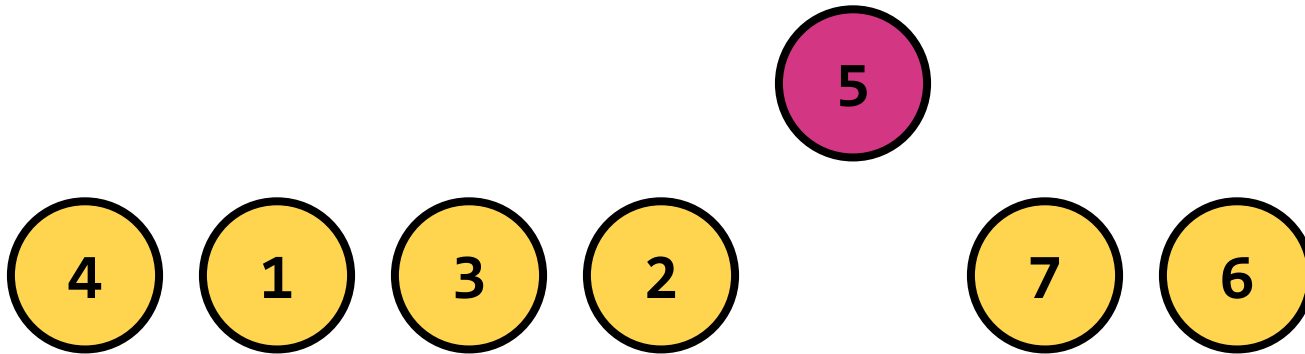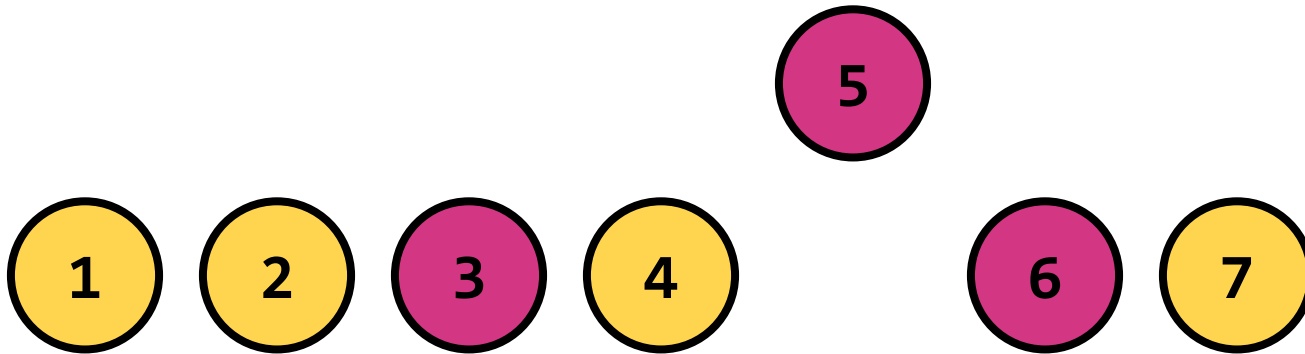
Building BSTs by Example

Building BSTs by Example

Let's partition about one of the vertices …

Building BSTs by Example

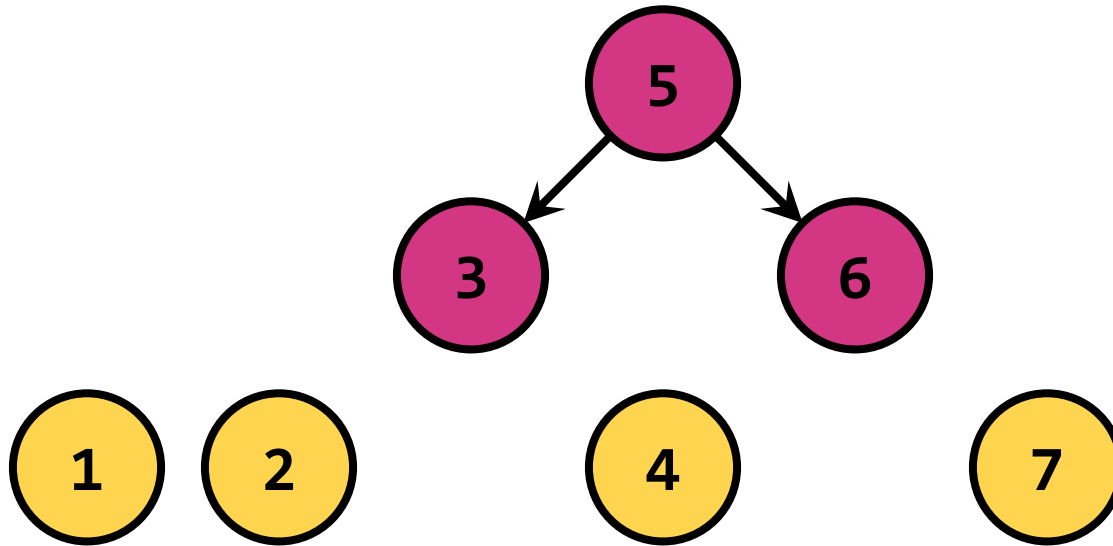… and build a tree with that vertex as the root.
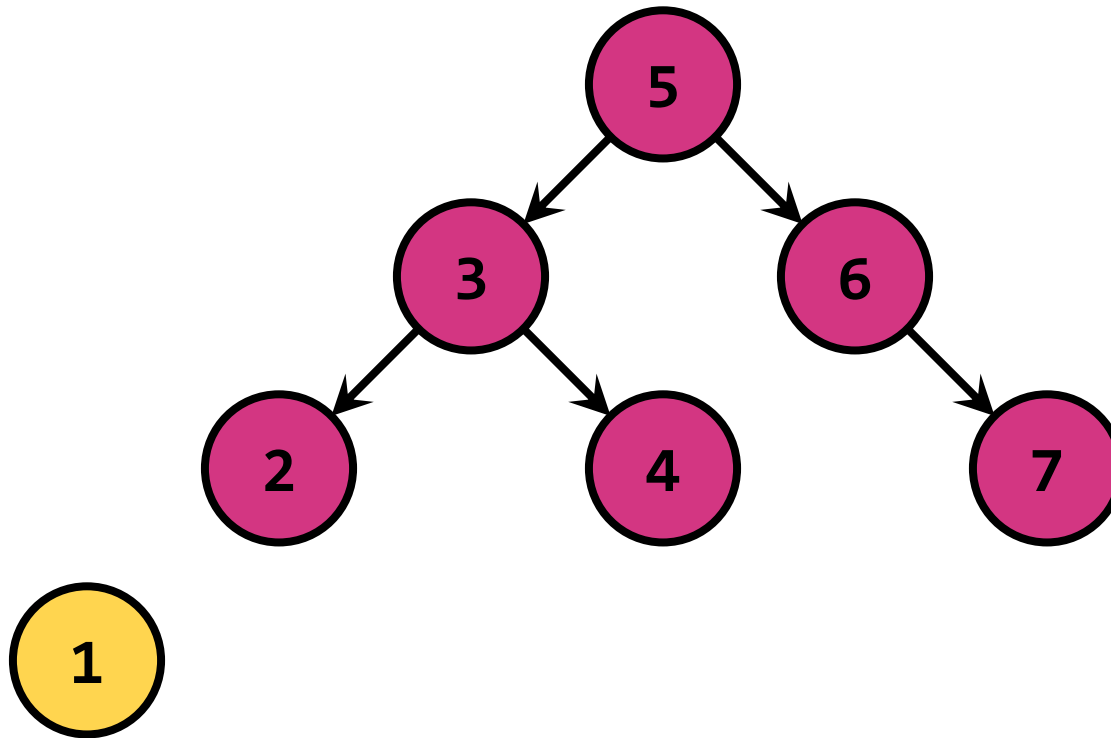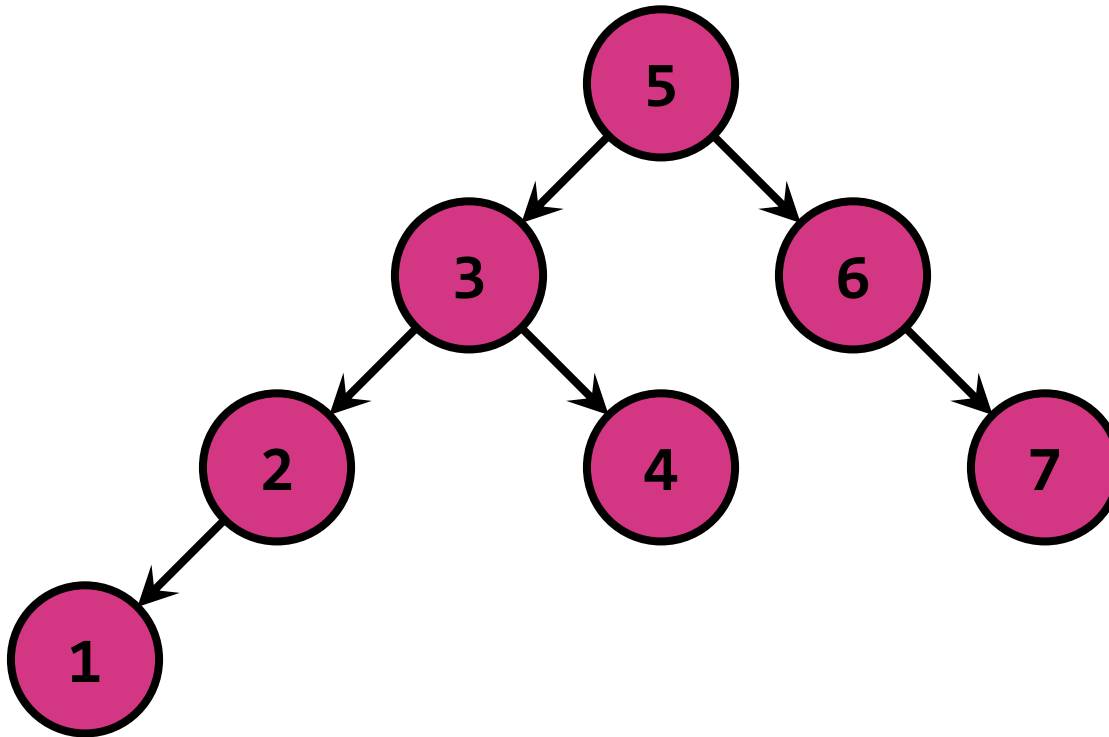
Building BSTs by Example

Then, recursively build trees out of its descendants.

Building BSTs by Example

Then, recursively build trees out of its descendants.

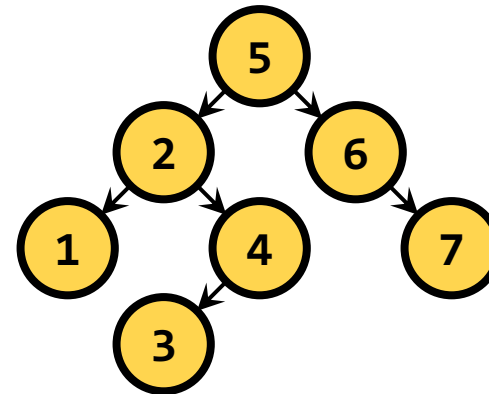# Building BSTs by Example

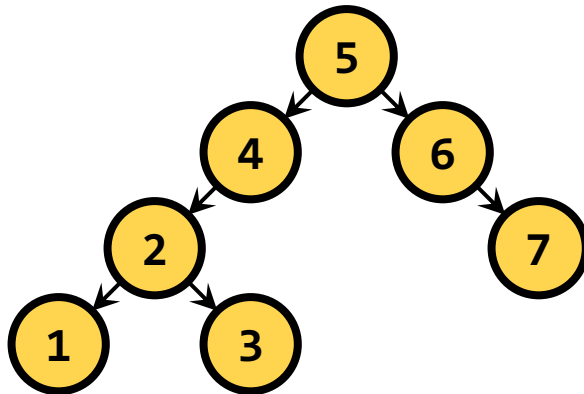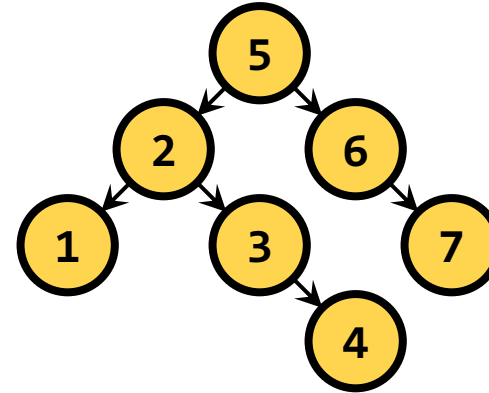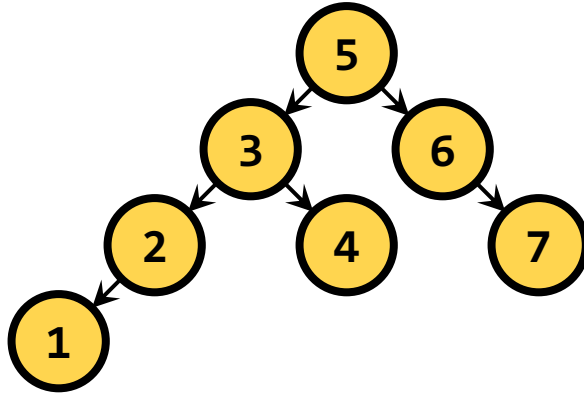Then, recursively build trees out of its descendants.

Building BSTs by Example

Then, recursively build trees out of its descendants.

There Exist Many Valid BSTs

Explanation on board: construction of another BST by selecting 4 as the root.
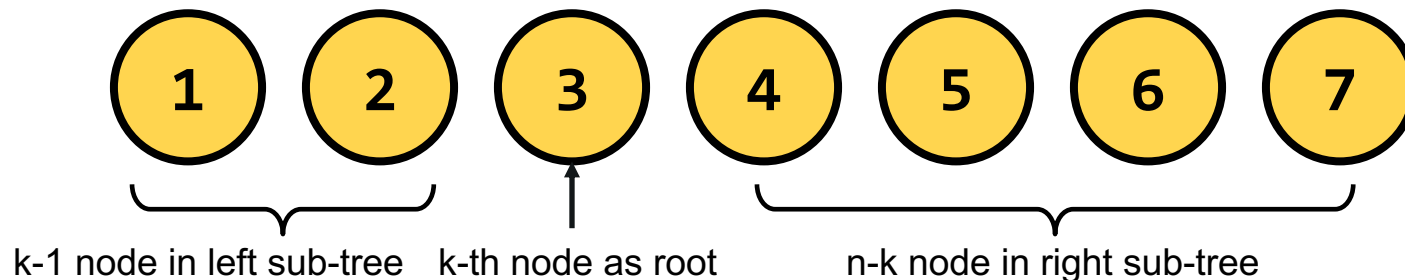
… and many more.

How many?

Catalan number:

$$C(n) = \frac{(2n)!}{n!\,(n+1)!}$$

There Exist Many Valid BSTs

Given *n* vertices, how many valid BSTs can we possibly build?

Let C(n) be the number of valid BSTs using n nodes.

k-1 node in left sub-tree   k-th node as root   n-k node in right sub-tree

C(0) = C(1) = 1
Rewrite C(n) by construction: $C(n) = \sum_{k=1}^{n} C(k-1)C(n-k)$

C(n) is a series of numbers defined by the following recurrence relation:
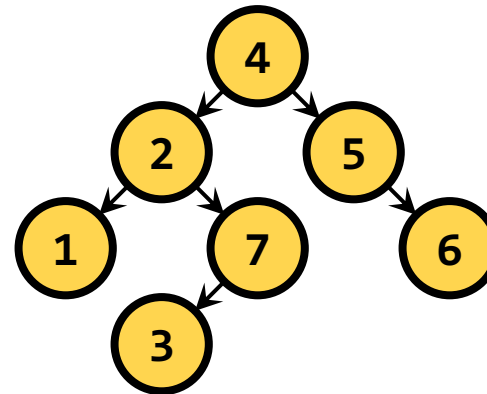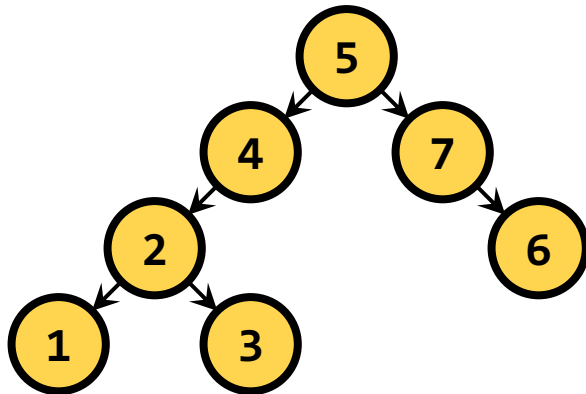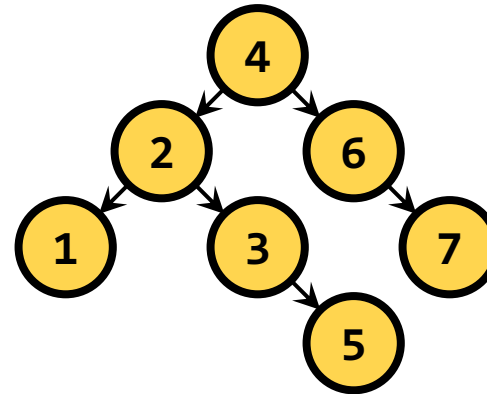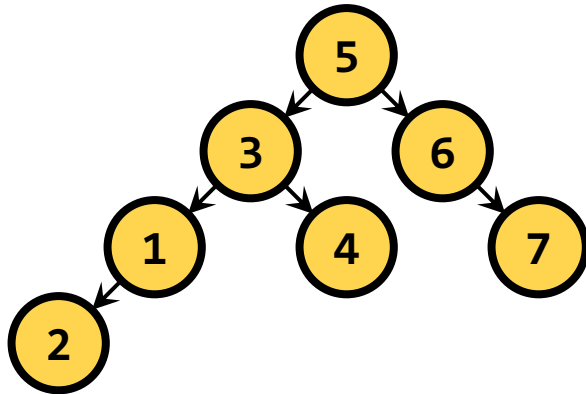$$C(0) = C(1) = 1$$
$$C(n) = C(0)C(n-1) + C(1)C(n-2) + \cdots + C(n-1)C(0)$$

This is the Catalan series: 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, …
Answer to this series is:

$$C(n) = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!\,(n+1)!}$$
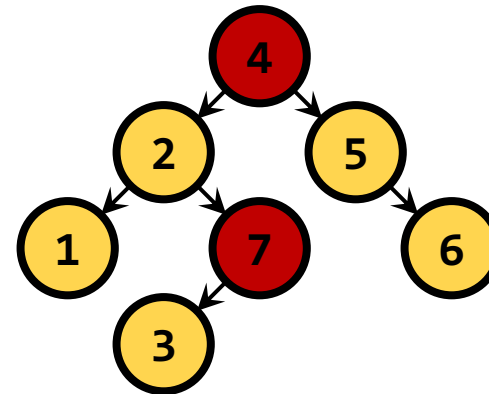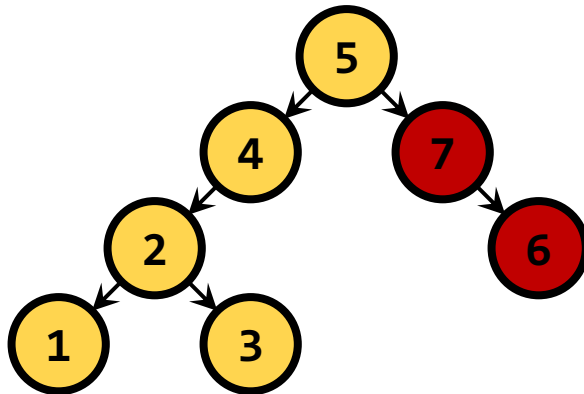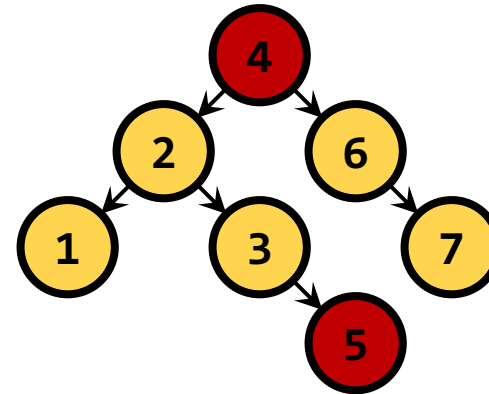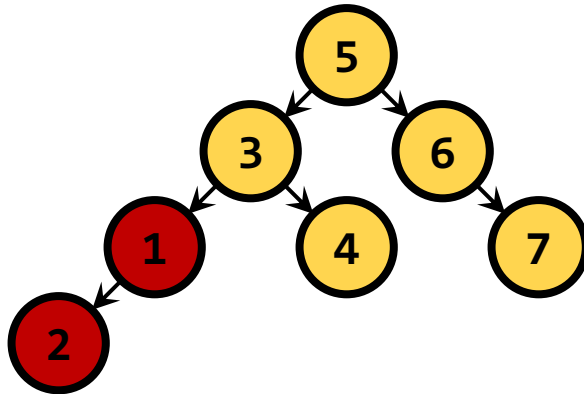
There Exist Many Invalid BSTs

… and many more.

How many?

There Exist Many Invalid BSTs

… and many more.

How many?

$$\frac{(2n)!}{(n+1)!} - \frac{(2n)!}{n!\,(n+1)!}$$

There Exist Many Invalid BSTs

Given *n* vertices, how many BTs can we possibly build?

Let T(n) be the number of binary trees (BTs) using n nodes, no matter being valid or invalid BST.

Any of the n nodes could be selected as root node

Any k nodes from the remaining nodes could be selected for the left sub-tree (0 ≤ k ≤ n-1)

The remaining n-k-1 nodes will be selected for the right sub-tree

T(0) = T(1) = 1

Rewrite T(n) by construction: $T(n) = \binom{n}{1} \sum_{k=0}^{n-1} \binom{n-1}{k} T(k) T(n-k-1)$

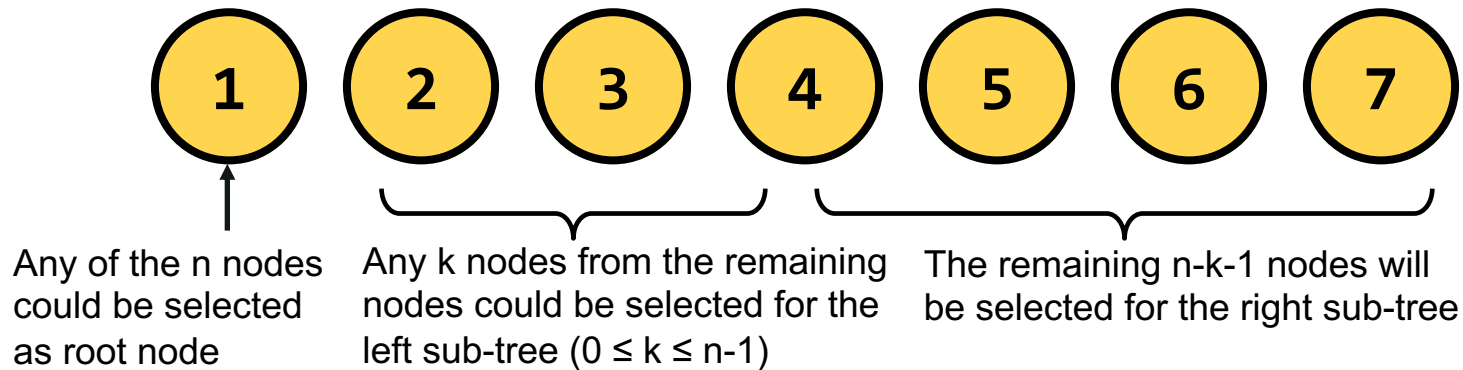Answer to this series is: $T(n) = \dfrac{(2n)!}{(n+1)!}$

There Exist Many Invalid BSTs

Given *n* vertices, how many BTs can we possibly build?

Let T(n) be the number of binary trees (BTs) using n nodes, no matter being valid or invalid BST.

T(0) = T(1) = 1

Rewrite T(n) by construction: $T(n) = \binom{n}{1} \sum_{k=0}^{n-1} \binom{n-1}{k} T(k) T(n-k-1)$

Answer to this series is: $T(n) = \dfrac{(2n)!}{(n+1)!}$

An easier way to get the answer:
Each unique BT structure corresponds to a unique valid BST, e.g., given the following structure, the root must be 5, because there are 4 nodes smaller than root and 4 greater than root. Similar for remaining nodes.
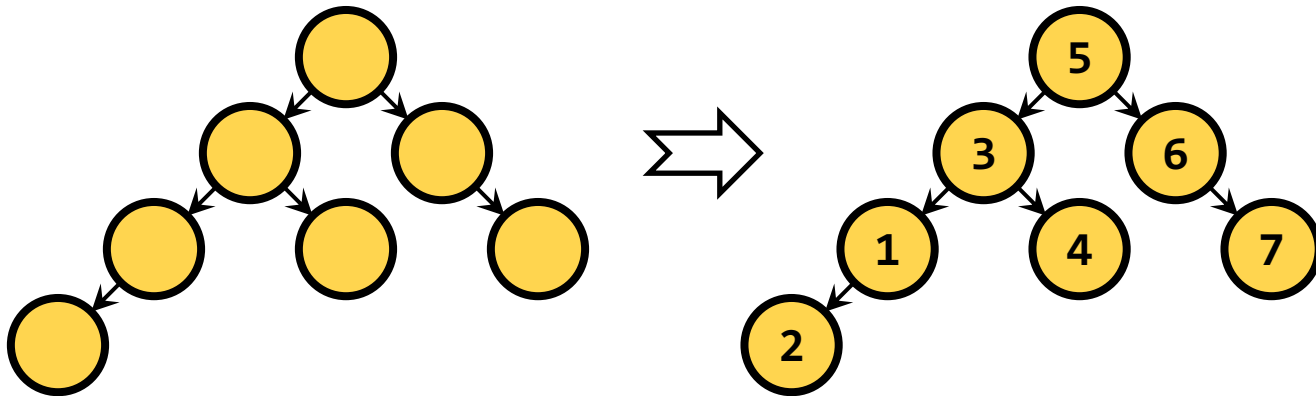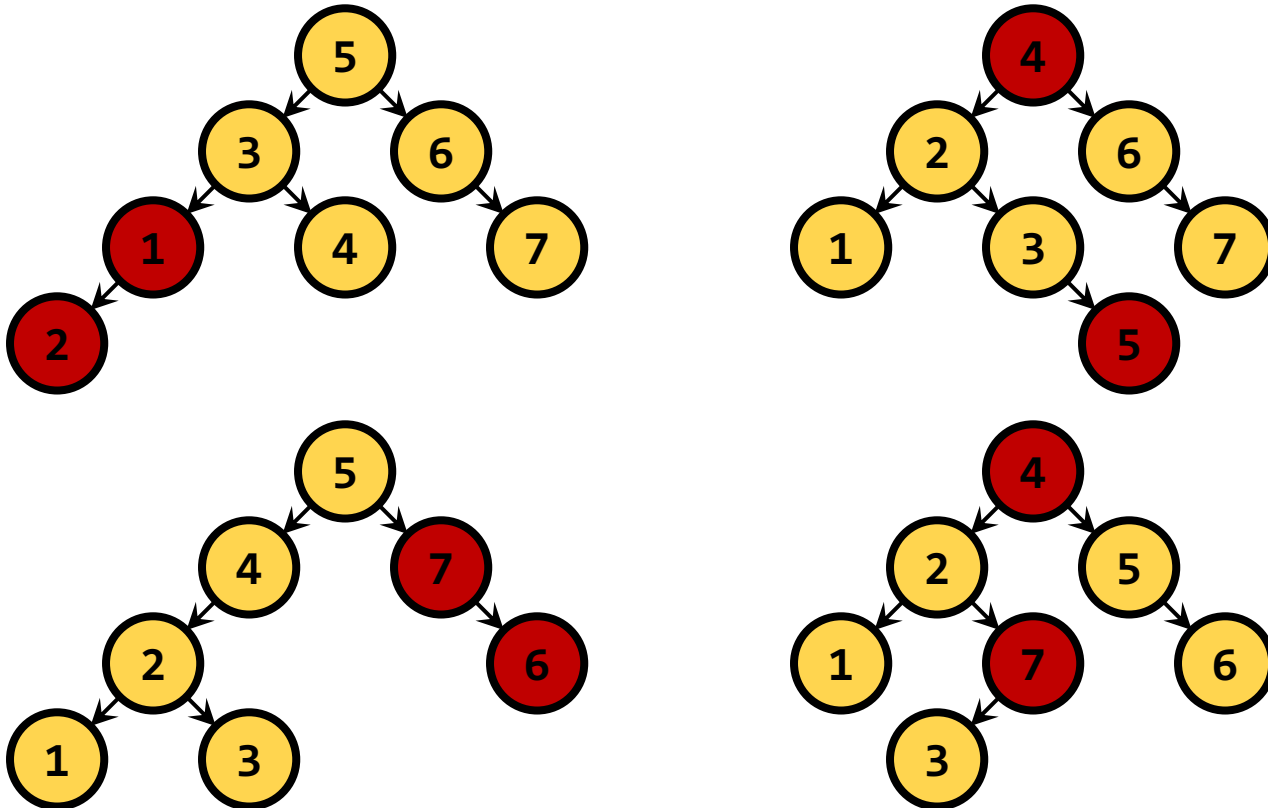

If we do not care about validity of BST, for each BST, we can permute n numbers arbitrarily to create BTs. n numbers will give n! permutations. As a result:

$$T(n) = C(n) \cdot n! = \frac{(2n)!}{n!\,(n+1)!} \cdot n! = \frac{(2n)!}{(n+1)!}$$

There Exist Many Invalid BSTs

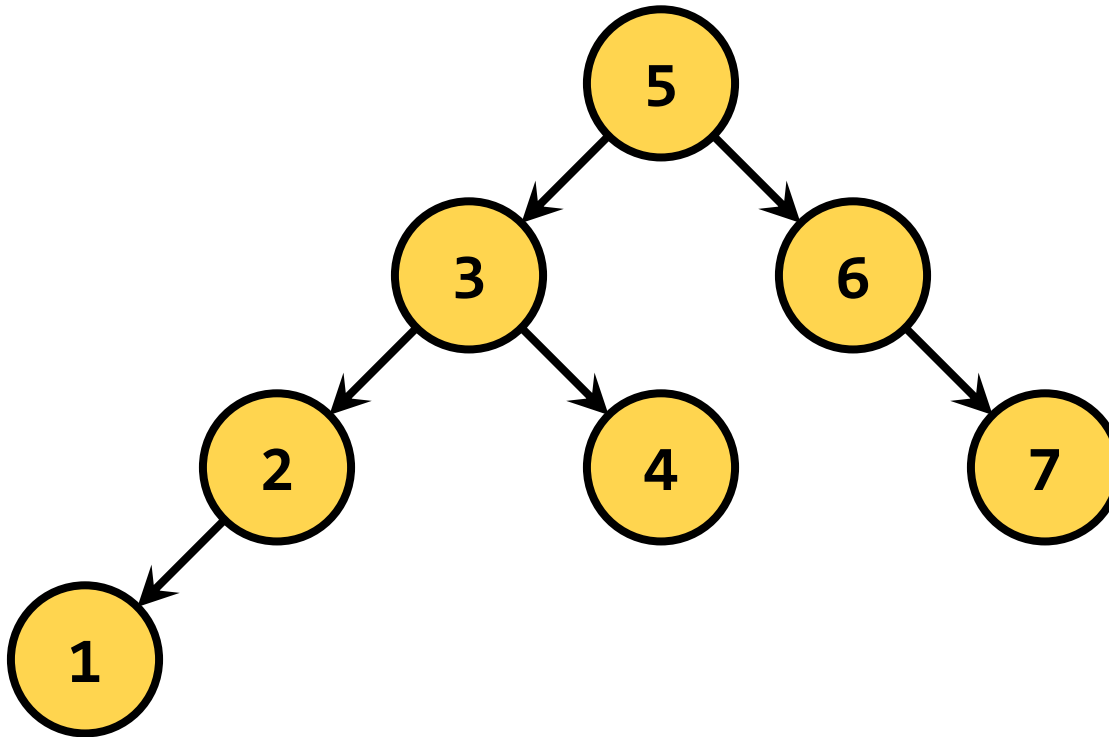Given n vertices, how many invalid BSTs can we possibly build?

Invalid BSTs = # BTs - # Valid BSTs $= T(n) - C(n) = \dfrac{(2n)!}{(n+1)!} - \dfrac{(2n)!}{n!\,(n+1)!}$
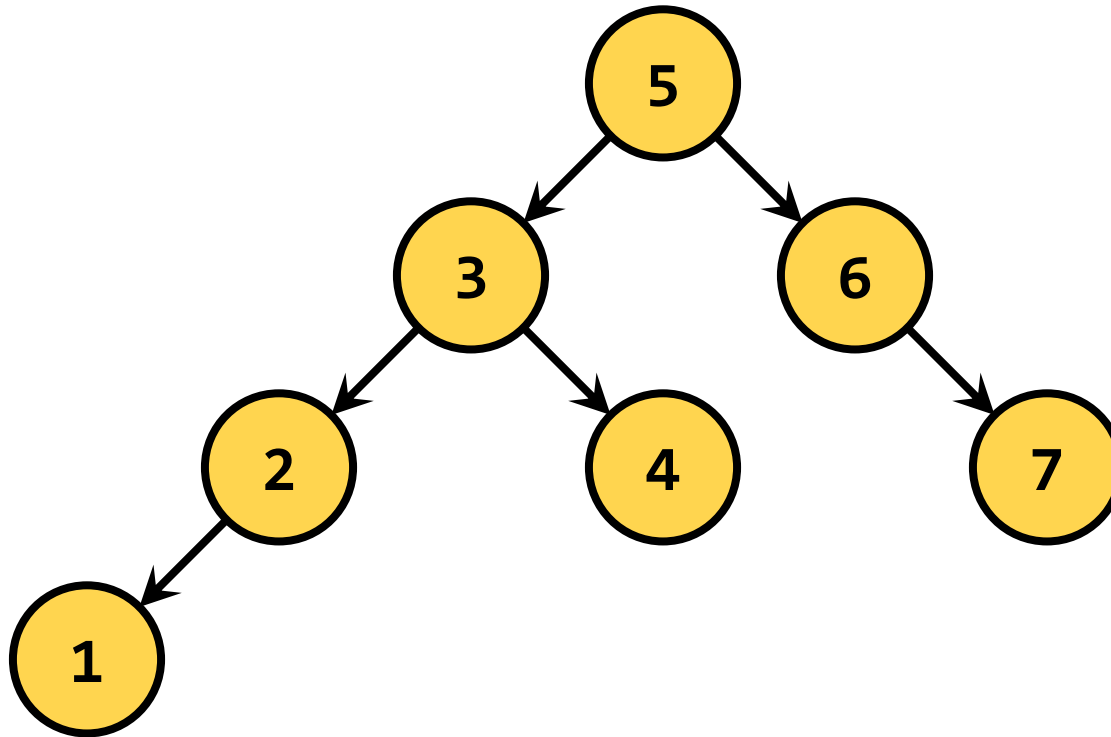
Operations on BSTs

search **in BSTs**

search compares the desired key to the current vertex, visiting left or right children as appropriate.
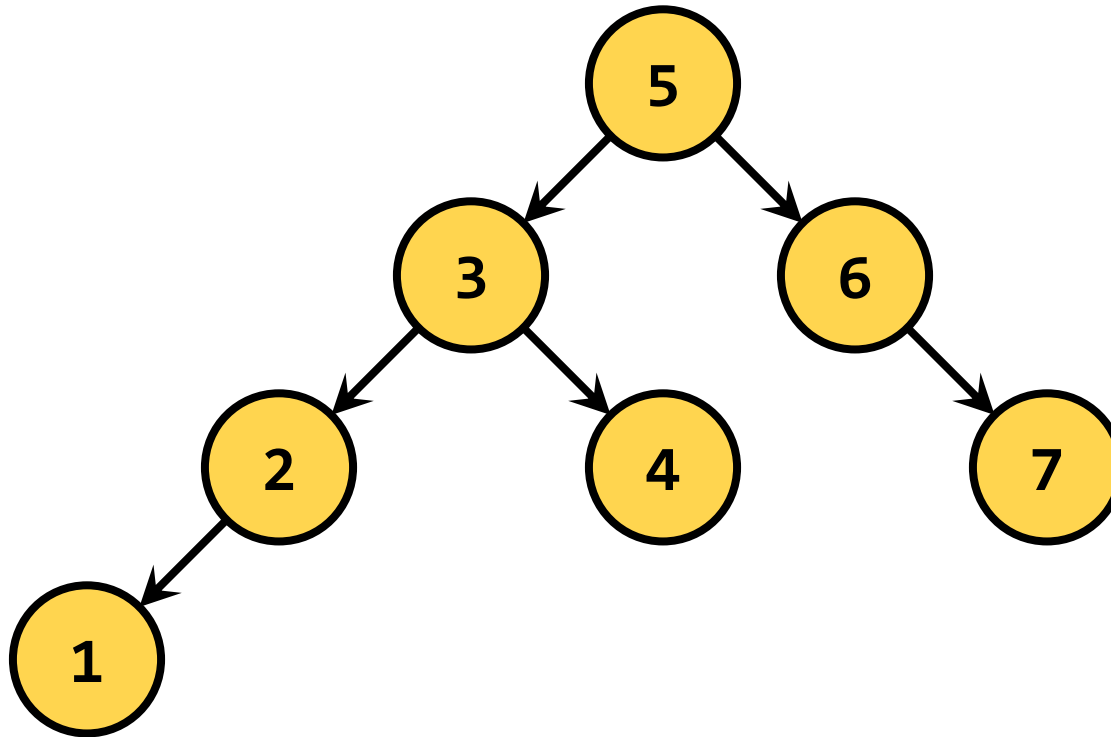
search **in BSTs**

For example, `search(4)` compares the 4 to the 5, then visits its left child of 3, then visits its right child of 4.

Write pseudocode to implement this algorithm!

search **in BSTs**

If we desire a non-existent key, such as `search(4.5)`, we can either return the last seen node (in this case, 4) or we can throw an exception. For now, let's do the former (helpful to other algorithms that may use search() function, e.g., insert.

insert **in BSTs**

```
algorithm insert(root, key_to_insert):
  x = search(root, key_to_insert)
  v = new vertex with key_to_insert
  if key_to_insert > x.key:
    x.right = v
  if key_to_insert < x.key:
    x.left = v
  if key_to_insert == x.key:
    return
```

Explain on board: insert(4.5), insert(6.5), insert(4)

Runtime: $O(\log n)$ if balanced, $O(n)$ otherwise
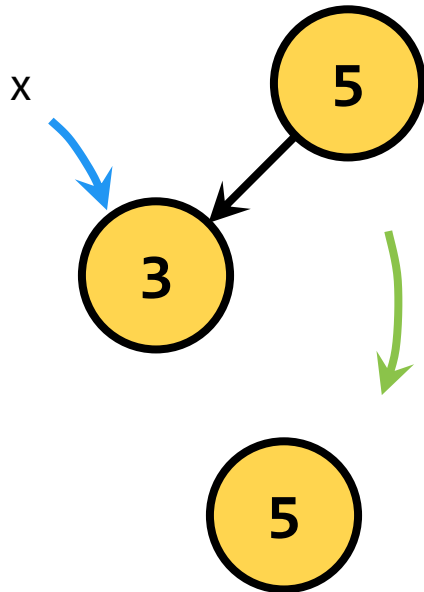Explain on board: an extremely unbalanced BST

delete **in BSTs**

```
algorithm delete(root, key_to_delete):
  x = search(root, key_to_delete)
  if key_to_delete == x.key:
    delete x
```

This is somewhat complicated …
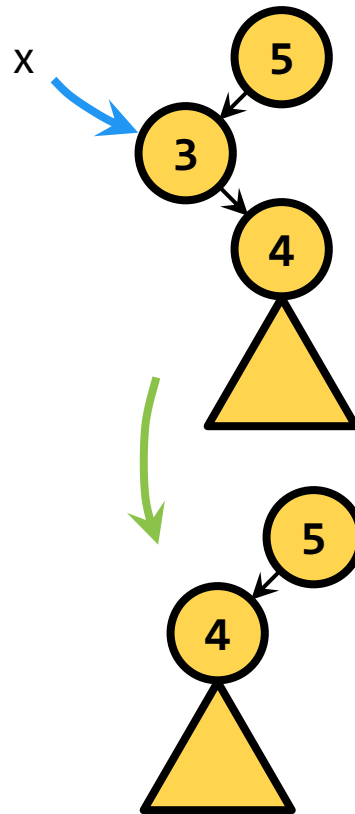
Runtime: **O(log n)** if balanced, **O(n)** otherwise

delete **in BSTs**
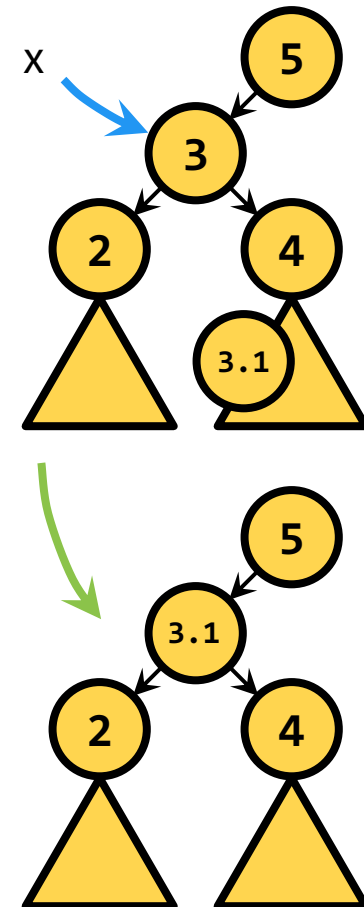
Case 1: x is a leaf
Just delete x

Case 2: x has 1 child
Move its child up
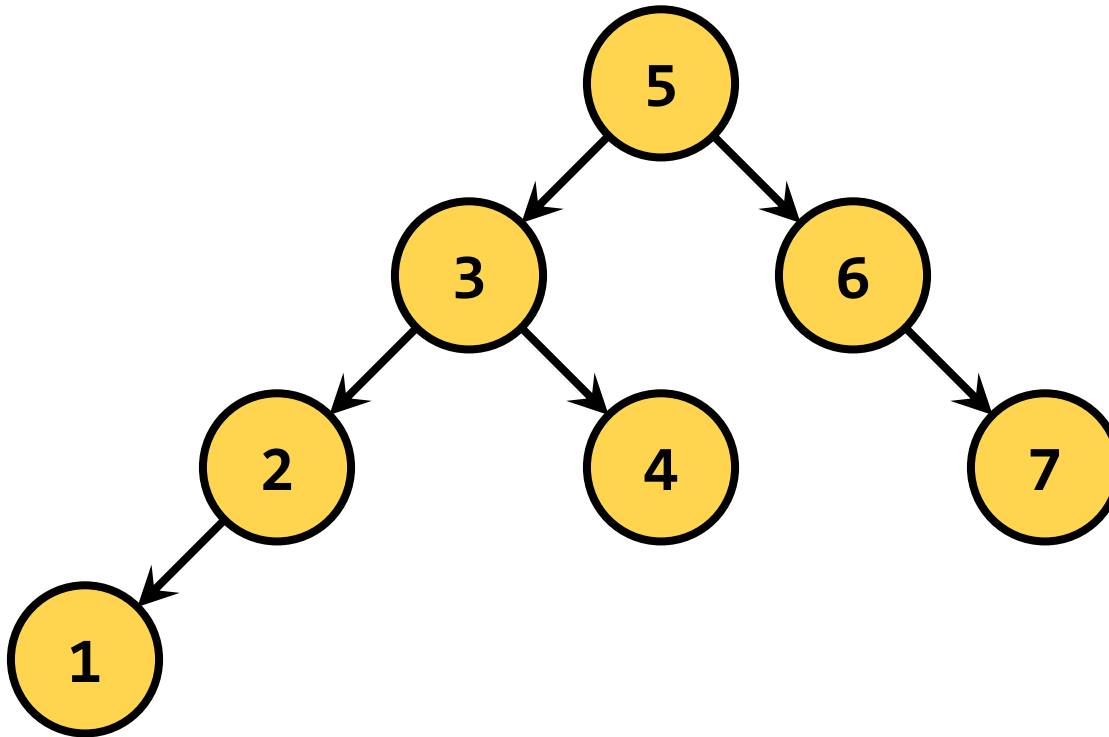
Case 3: x has 2 children
Replace x with its successor

Explain on board: why the successor is selected in case 3.
It should be greater than 2 and all its descendants, while being less than 4
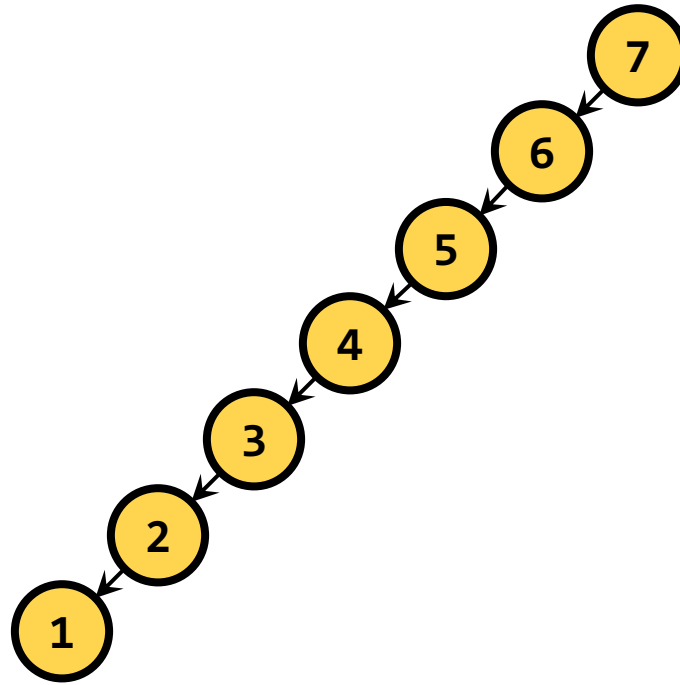and all its descendants. So we choose the smallest value in the right-descendants of 3, i.e., successor.

31

Runtime Analysis

Runtime of search (which insert and delete both call) is O(depth of tree).

Runtime Analysis

But this is a valid BST and the depth of the tree is n, resulting in a runtime of $O(n)$ for search.

In what order would we need to insert vertices to generate this tree?
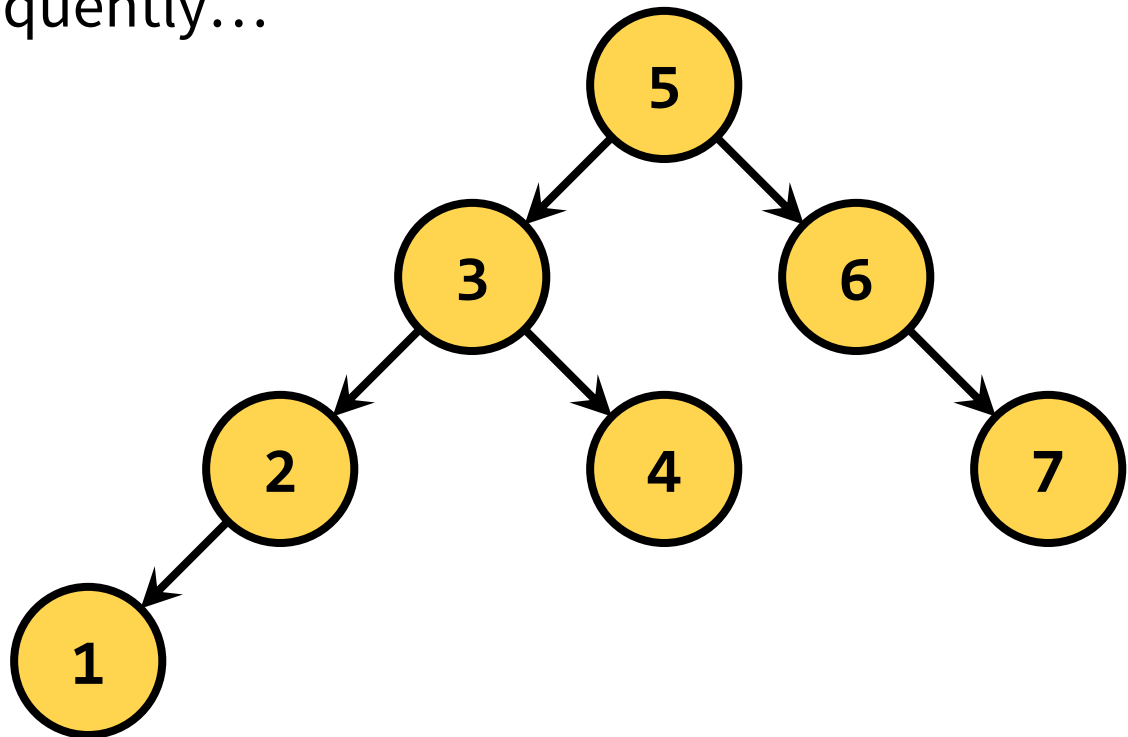
What To Do?

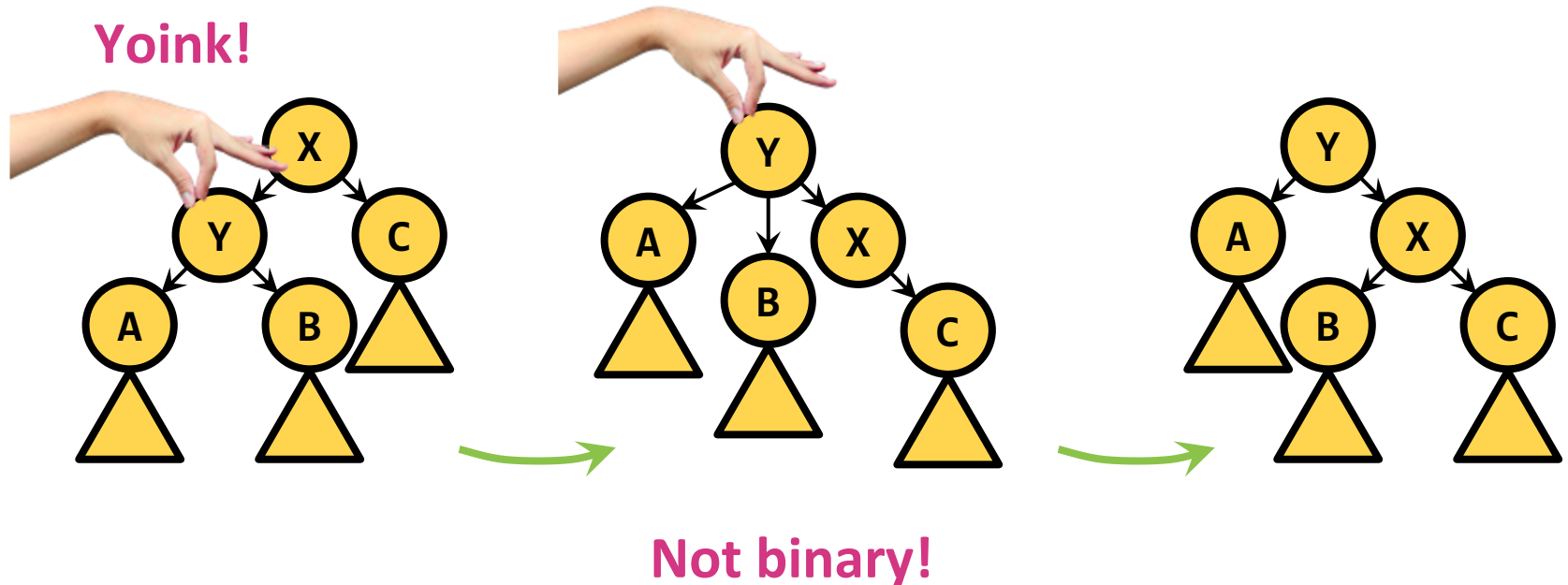We could keep track of the depth of the tree. If it gets too tall, re-do everything from scratch.

But this is time-consuming because we have to reconstruct the whole tree frequently…
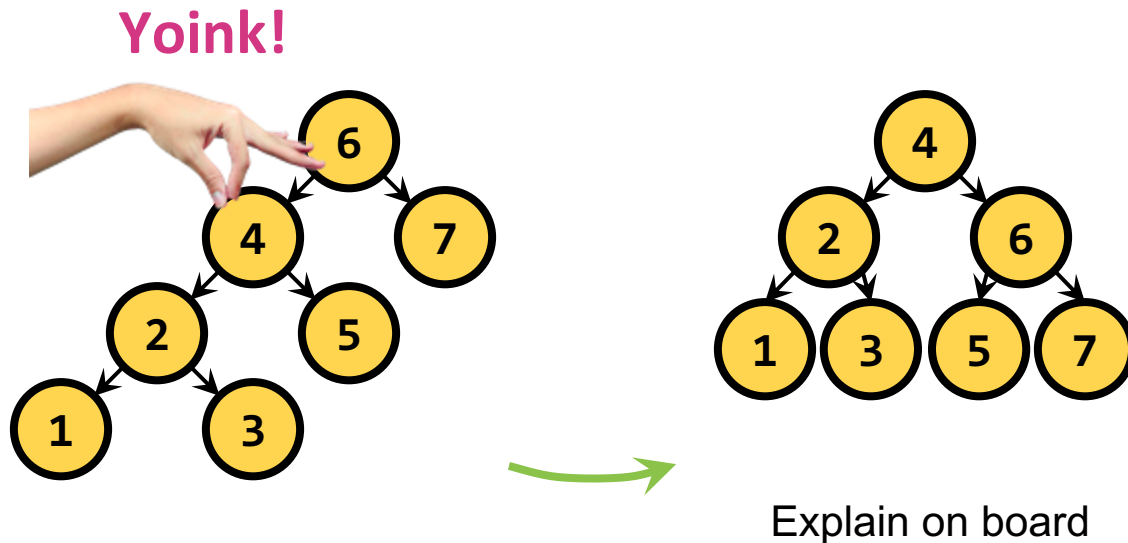
Any other ideas?

Idea 1: Rotations

Maintain the BST property, and move some of the vertices (but not all of them) around.

Check the BST properties on Y and X

Idea 1: Rotations

Maintain the BST property, and move some of the vertices (but not all of them) around.

Yoink!

Explain on board

Idea 2: Proxy for Balance

Maintaining **perfect balance** is too difficult.

> Checking for balance and which node to rotate is difficult

Instead, let's determine some proxy for balance.

i.e. If the tree satisfies some property, then it's "pretty balanced."

If during tree modification (construction, insert, delete, etc.) the property no long holds, we can maintain this property using rotations.

Red-Black Trees (RB-Tree)

Advanced Topic (Optional)

Red-Black Trees

There exist many ways to achieve this proxy for balance, but here we'll study the **red**-**black** **tree**.

1. Every vertex is colored **red** or **black**.

2. The root vertex is a **black** vertex.

3. A NIL child is a **black** vertex.

4. The child of a **red** vertex must be a **black** vertex.

5. For all vertices v, all paths from v to its NIL descendants have the same number of **black** vertices.

We can be sure that the tree is pretty balanced as long as these proxy properties hold.

Red-Black Trees by Example

1. Every vertex is colored **red** or **black**.

2. The root vertex is a **black** vertex.

3. A NIL child is a **black** vertex.

4. The child of a **red** vertex must be a **black** vertex.

5. For all vertices v, all paths from v to its NIL descendants have the same number of **black** vertices.
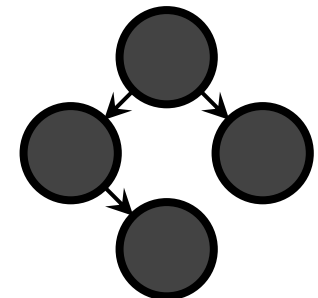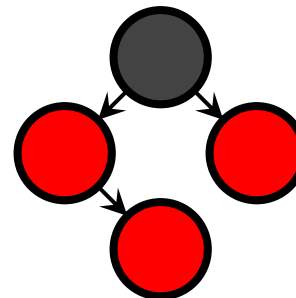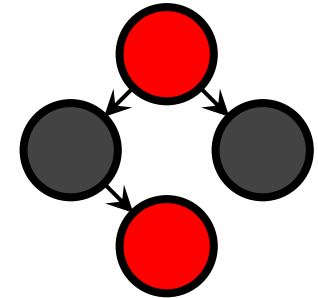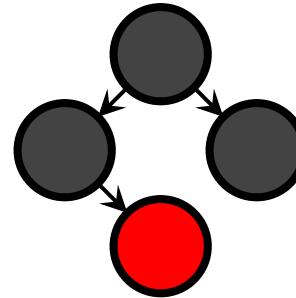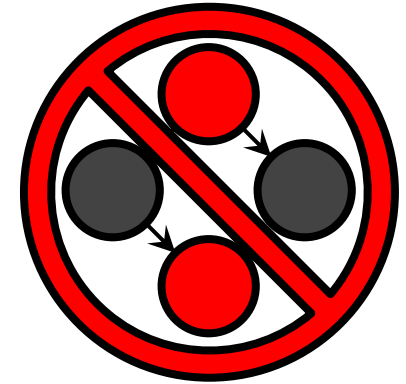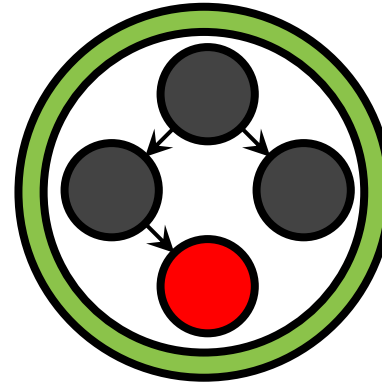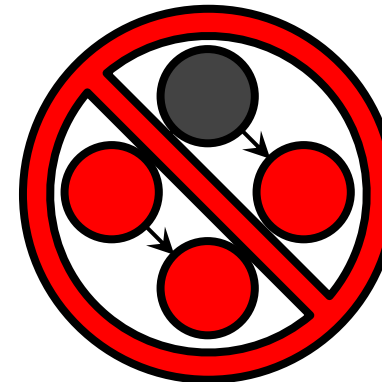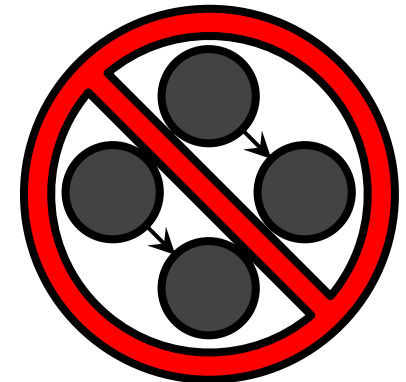
Red-Black Trees by Example

1. Every vertex is colored **red** or **black**.

2. The root vertex is a **black** vertex.

3. A NIL child is a **black** vertex.

4. The child of a **red** vertex must be a **black** vertex.

5. For all vertices v, all paths from v to its NIL descendants have the same number of **black** vertices.

Violates 2

Violates 4

Violates 5

Red-Black Trees

Maintaining these properties maintains a "pretty balanced" BST. Intuition:

The **black** vertices are balanced.
Rule #5: For all vertices v, all paths from v to its NIL descendants have the same number of **black** vertices

The **red** vertices are "spread out."
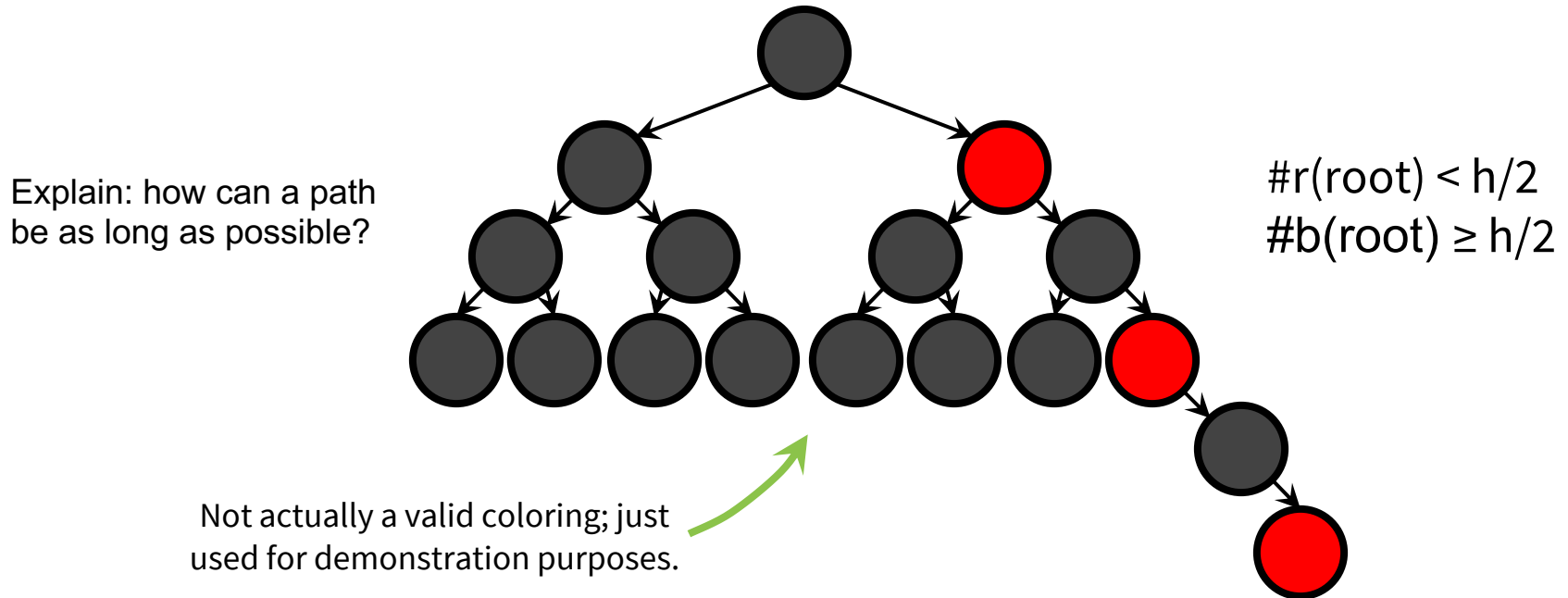Rule #4: The child of a **red** vertex must be a **black** vertex

We can maintain this property as we insert/delete vertices, using rotations.
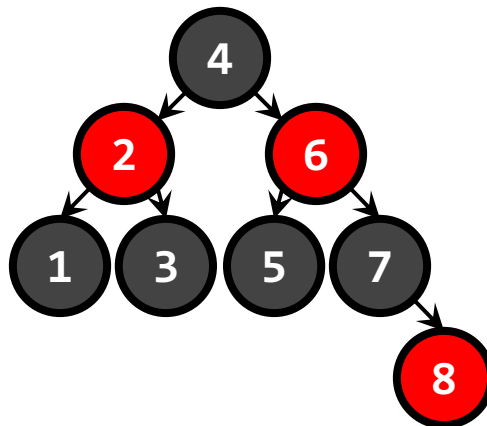
Red-Black Trees

To see why a red-black tree is "pretty balanced," consider that its height is at most O(log(n)).

Property: One path could be twice as long as the others if we pad it with red vertices, but at most twice as long as the others.

Explain: how can a path be as long as possible?

$\#r(\text{root}) < h/2$
$\#b(\text{root}) \geq h/2$

Not actually a valid coloring; just used for demonstration purposes.

43

Red-Black Trees

Lemma: The number of node in a subtree of x is at least $n(x) \geq 2^{b(x)} - 1$, b(x) is the number of black nodes from x to NIL descendants.

Proof:

To prove this statement, we proceed by induction.

For our base case, note that a NIL node has $b(x) = 0$ and at least $2^0 - 1 = 0$ non-NIL descendants; same for a single black or red node.

For our inductive step, let n(x) be the number of non-NIL descendants of x. Then:

$$n(x) = 1 + n(x.left) + n(x.right)$$
$$\geq 1 + (2^{b(x)-1} - 1) + (2^{b(x)-1} - 1)$$
$$= 2^{b(x)} - 1$$

Thus, the number of non-NIL descendants of x is at least $2^{b(x)} - 1$.

Explain on board

44

Red-Black Trees

Theorem: A Red-Black Tree has height $h \leq 2 \log_2(n+1) = O(\log n)$.

Proof:

By our lemma, the number of non-NIL descendants of x is at least $2^{b(x)} - 1$.

Notice that on any root to NIL path there are no two consecutive red vertices (otherwise the tree violates rule 4);

So the number of black vertices is at least the number of red vertices.
Thus, b(root) is at least half of the height, i.e., $b(root) \geq h/2$

Let n be the number of vertices in the tree
then $n \geq 2^{b(root)} - 1 \geq 2^{h/2} - 1$, and hence $h \leq 2 \log_2(n+1)$.

insert **in Red-Black Trees**

```
algorithm rb_insert(root, key_to_insert):
  x = search(root, key_to_insert)
  v = new red vertex with key_to_insert
  if key_to_insert > x.key:
    x.right = v
    fix things up, if necessary
  if key_to_insert < x.key:
    x.left = v
    fix things up, if necessary
  if key_to_insert == x.key:
    return
```
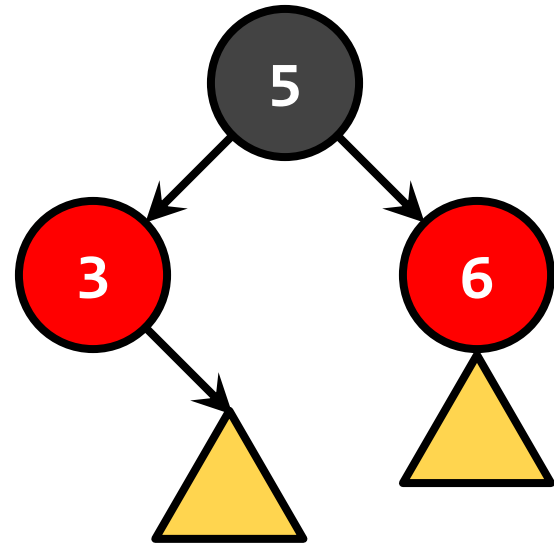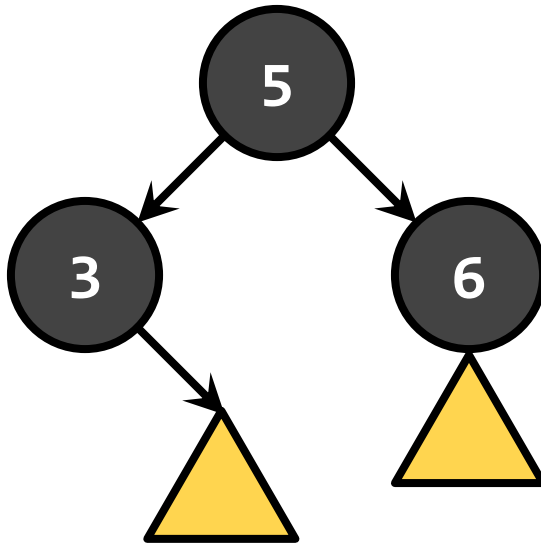
What does
that mean?

`insert` **in Red-Black Trees**

What does "if necessary" mean?

Suppose we want to `insert(1)`.

`insert` **in Red-Black Trees**

What does "if necessary" mean?

Suppose we want to `insert(1)`.

insert **in Red-Black Trees**
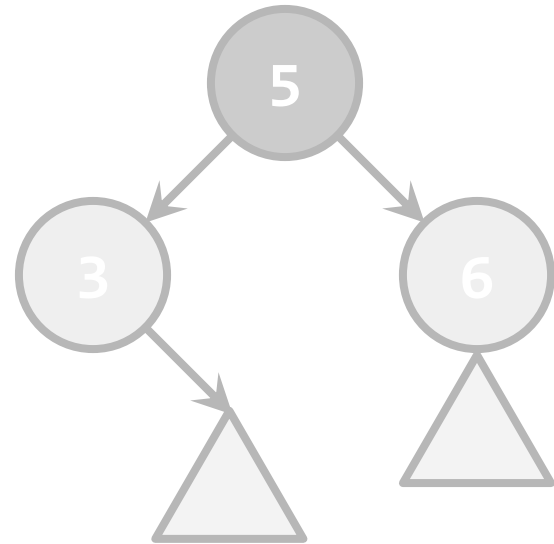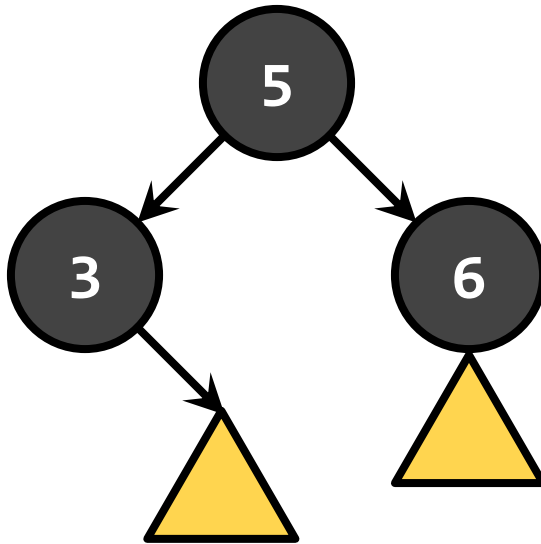
What does "if necessary" mean?
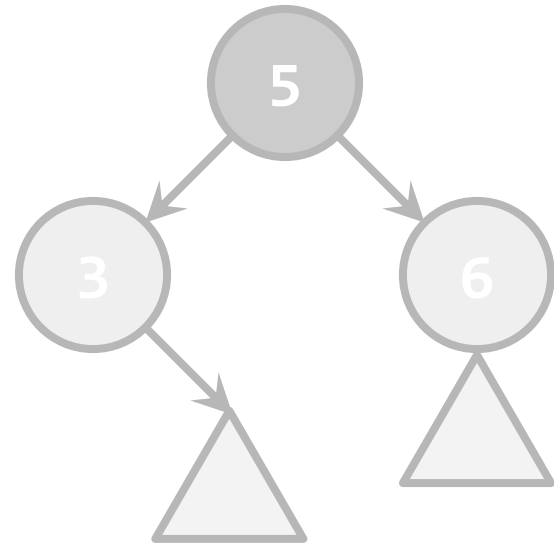
Suppose we want to insert(1).

insert in Red-Black Trees

What does "if necessary" mean?

Suppose we want to insert(1).

`insert` **in Red-Black Trees**

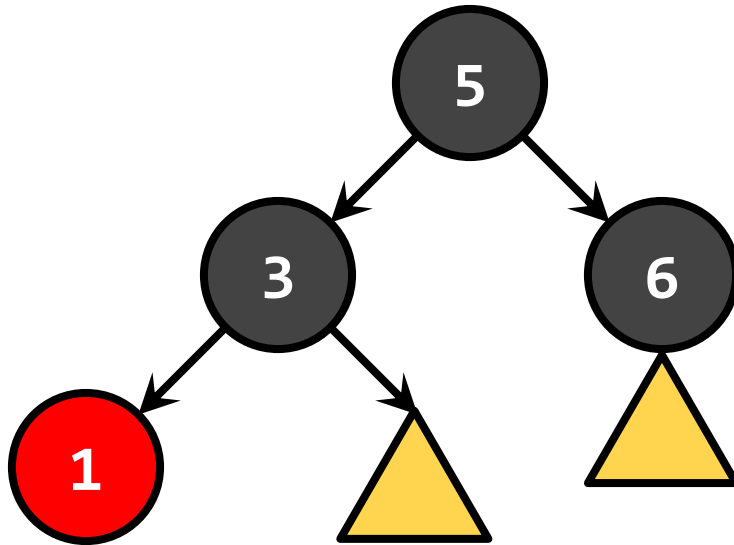What does "if necessary" mean?

Suppose we want to `insert(1)`.

insert **in Red-Black Trees**

What does "if necessary" mean?

Suppose we want to insert(1).

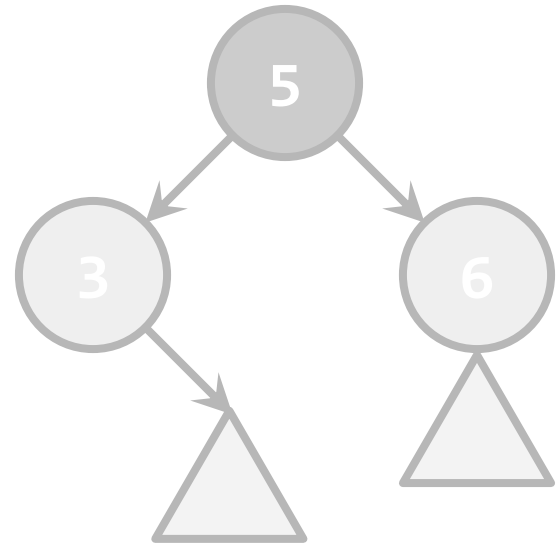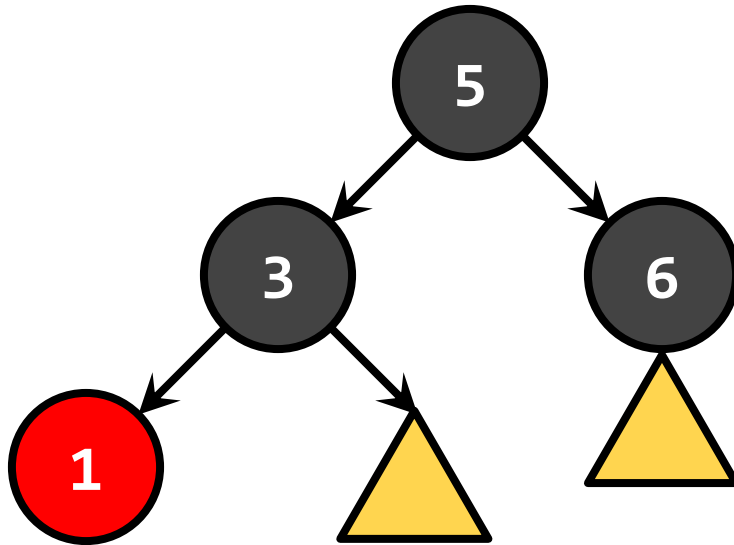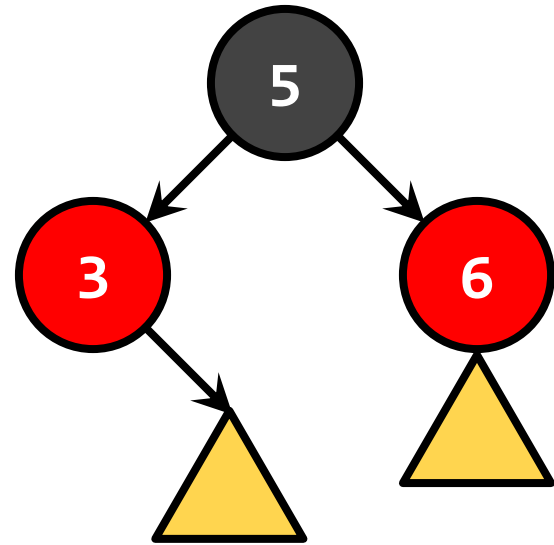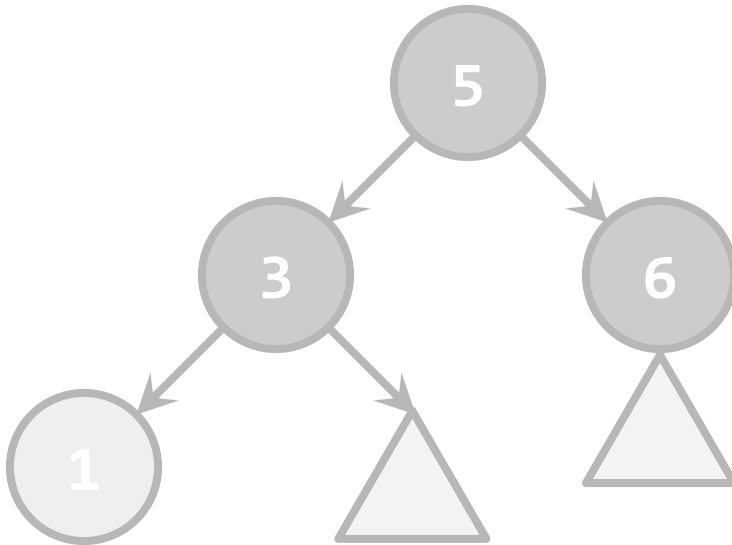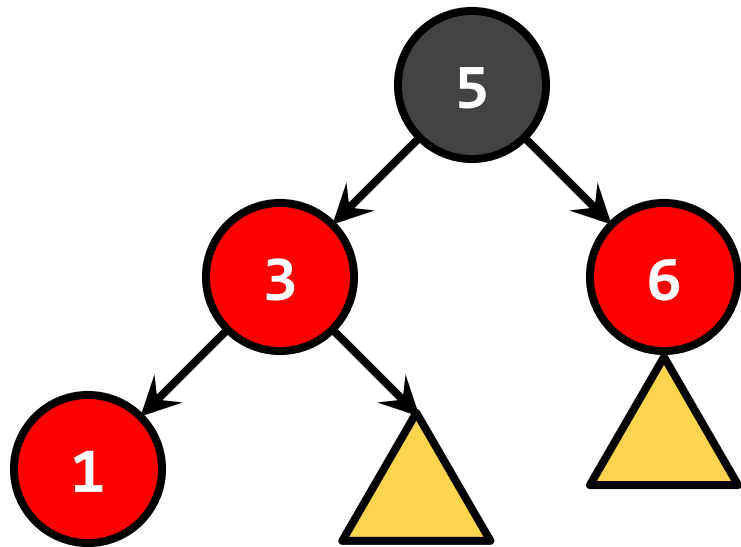# insert **in Red-Black Trees**

What does "if necessary" mean?

Suppose we want to insert(1).

Violates 4

insert **in Red-Black Trees**

What does "if necessary" mean?
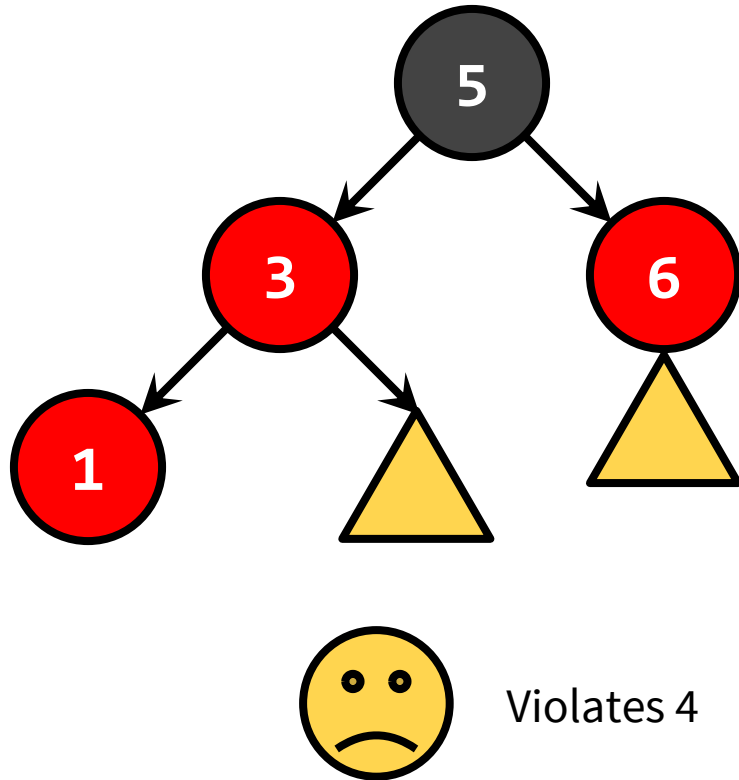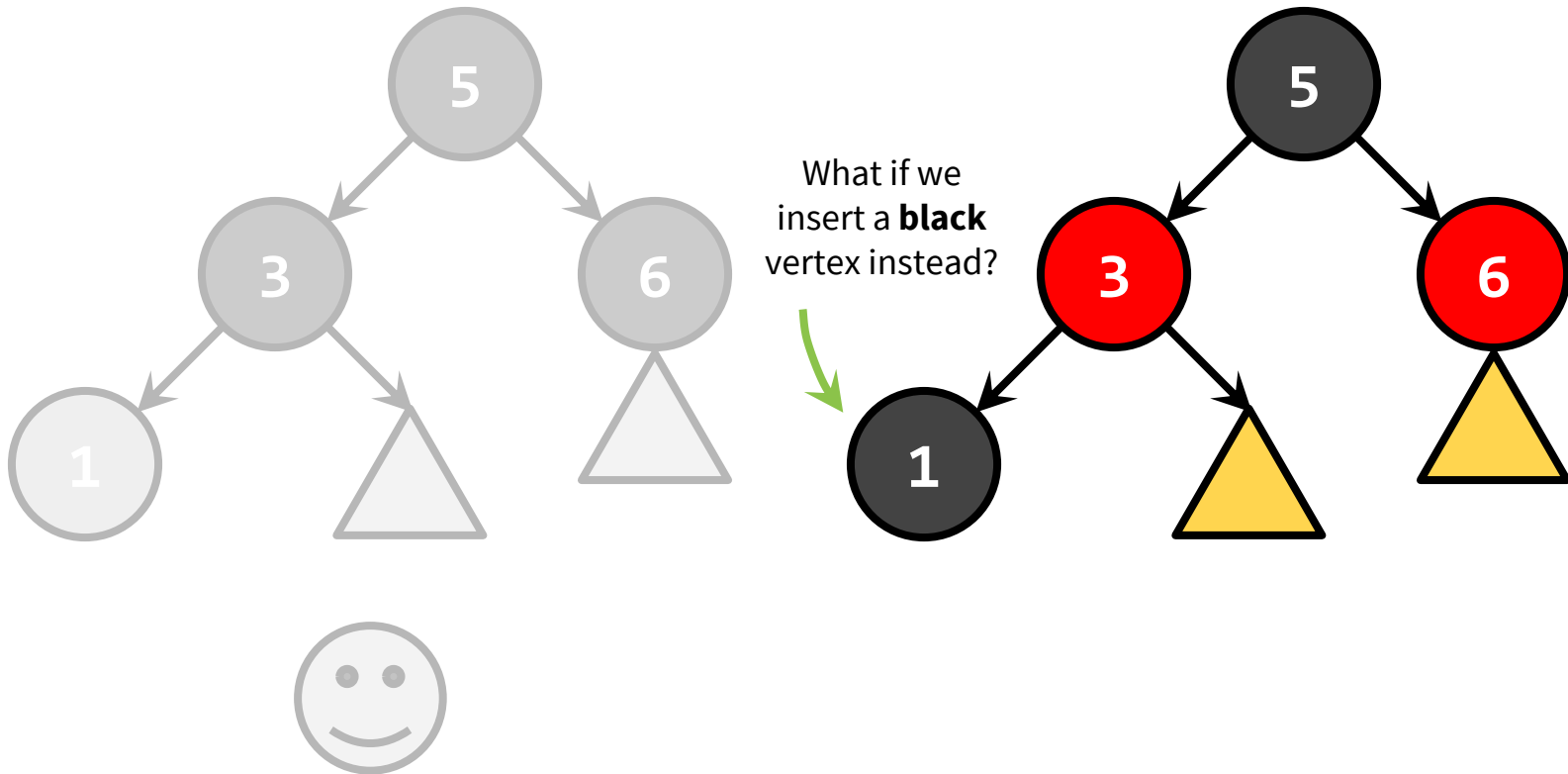
Suppose we want to insert(1).

What if we insert a **black** vertex instead?

insert **in Red-Black Trees**

What does "if necessary" mean?

Suppose we want to insert(1).

What if we insert a **black** vertex instead?

Violates 5

`insert` **in Red-Black Trees**

What does "if necessary" mean?

So it seems we're happy if the parent of the inserted vertex is **black**.

Check the rules:
1. Every vertex is colored **red** or **black**.

2. The root vertex is a **black** vertex.

3. A NIL child is a **black** vertex.

4. The child of a **red** vertex must be a **black** vertex.

5. For all vertices v, all paths from v to its NIL descendants have the same number of **black** vertices.

But there's an issue if the parent of the inserted vertex is **red**.

insert **in Red-Black Trees**

```
algorithm rb_insert(root, key_to_insert):
  x = search(root, key_to_insert)
  v = new red vertex with key_to_insert
  if key_to_insert > x.key:
    x.right = v
    recolor(v)
  if key_to_insert < x.key:
    x.left = v
    recolor(v)
  if key_to_insert == x.key:
    return
```

Runtime: `O(log n)`

insert **in Red-Black Trees**

```
algorithm recolor(v):
  if v == root:
    v.color = black
    return
  p = parent(v)
  if p.color == black:
    return
  grand_p = p.parent
  uncle = grand_p.right
  if uncle.color == red:
    p.color = black
    uncle.color = black
    grand_p.color = red # maintain number of black vertices
    recolor(grand_p) # fix up color recursively
  else:  # uncle.color == black
    p.color = black
    grand_p.color = red
    right_rotate(grand_p)  # yoink
```

Runtime: `O(log n)`

insert **in Red-Black Trees**

Grand Parent
grand_p

Parent p

Uncle u

New node v

Insert new node v as a red node

Fix the color of parent, uncle and grand_p

Fix the color of grand_p recursively

Red-Black Trees

Since we maintain the red-black property in **O(log n)**, then insert, delete, and search all require **O(log n)**-time.

	Search	Insertion	Deletion
Linked list	O(n)	O(n)	O(n)
Arrays	O(log n)	O(n)	O(n)
BST (unbalanced)	O(n)	O(n)	O(n)
BST (balanced)	O(log n)	O(log n)	O(log n)
RBT (always balanced)	O(log n)	O(log n)	O(log n)

Red-Black Trees

Since we maintain the red-black property in **O(log n)**, then insert, delete, and search all require **O(log n)**-time.

	Search	Insertion	Deletion
Linked list	O(n)	O(n)	O(n)
Arrays	O(log n)	O(n)	O(n)
BST (unbalanced)	O(n)	O(n)	O(n)
BST (balanced)	O(log n)	O(log n)	O(log n)
RBT (always balanced)	O(log n)	O(log n)	O(log n)