CS 211: Computer Architecture
Digital Logic

Topics:

- Converting truth tables to expressions
- Karnaugh maps

Converting Truth Table to Boolean Expression

sensor inputs

A	B	C	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Given a circuit, isolate the rows in which the output of the circuit should be true

Converting Truth Table to Boolean Expression

sensor inputs

A	B	C	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

sensor inputs

A	B	C	Output	
0	0	0	0	
0	0	1	0	
0	1	0	0	
0	1	1	1	$\overline{A}BC = 1$
1	0	0	0	
1	0	1	1	$A\overline{B}C = 1$
1	1	0	1	$AB\overline{C} = 1$
1	1	1	1	$ABC = 1$

Given a circuit, isolate that rows in which the output of the circuit should be true

A product term that contains exactly one instance of every variable is called a minterm

Converting Truth Table to Boolean Expression

sensor inputs

A	B	C	Output	
0	0	0	0	
0	0	1	0	
0	1	0	0	
0	1	1	1	$\overline{A}BC = 1$
1	0	0	0	
1	0	1	1	$A\overline{B}C = 1$
1	1	0	1	$AB\overline{C} = 1$
1	1	1	1	$ABC = 1$

$$\text{Output} = \overline{A}BC + A\overline{B}C + AB\overline{C} + ABC$$

Given the expressions for each row, build a larger Boolean expression for the entire table.

- This is a sum-of-products (SOP) form.

Converting Truth Table to Boolean Expression

sensor inputs

A	B	C	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

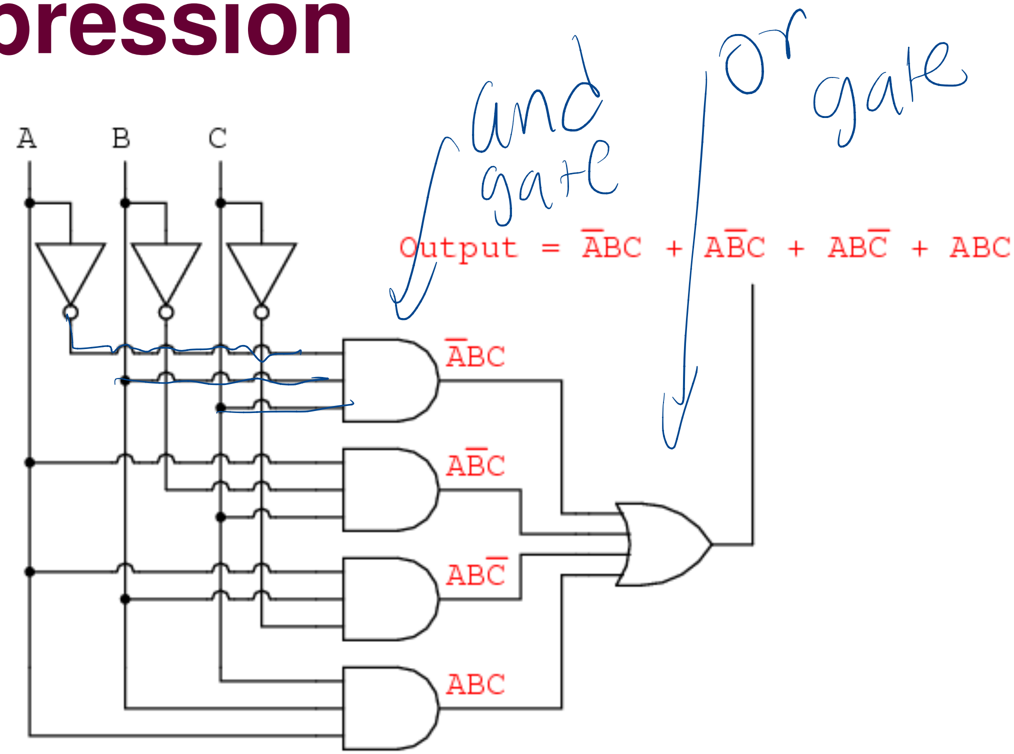
$\overline{A}BC = 1$

$A\overline{B}C = 1$

$AB\overline{C} = 1$

$ABC = 1$

$\text{Output} = \overline{A}BC + A\overline{B}C + AB\overline{C} + ABC$



Finally build the circuit.

- Problem: SOP forms are often not minimal.
- Solution: Make it minimal. We'll go over two ways.

First approach: algebraic

$BC(\bar{A}+A)+A\bar{B}C+AB\bar{C}$

$BC+A\bar{B}C+AB\bar{C}$

$BC+A\bar{B}C+AB\bar{C}$

$B(C+A\bar{C})+A\bar{B}C$

$B(C+A)+A\bar{B}C$

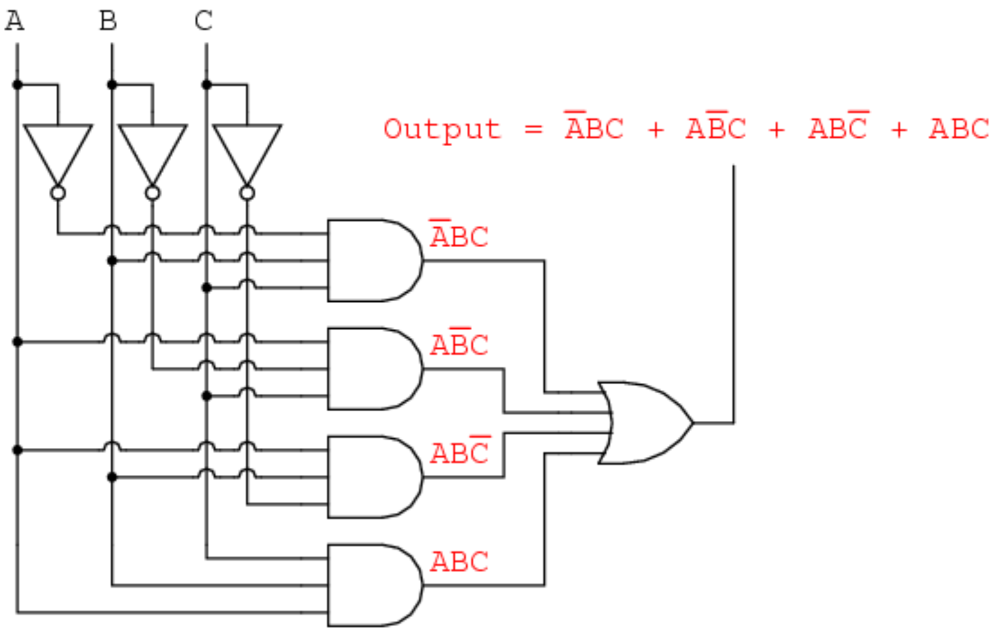
$BC+AB+A\bar{B}C$

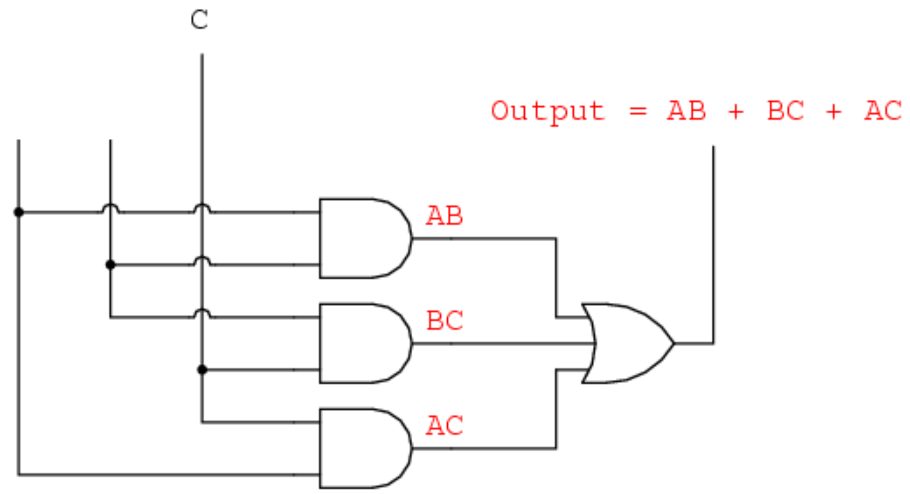
$BC+A(B+\bar{B}C)$

$BC+AB+AC$

$$\bar{A}BC + A\bar{B}C + AB\bar{C} + ABC$$
$$\Downarrow$$
$$BC(\bar{A}+A) + A\bar{B}C + AB\bar{C}$$
$$\Downarrow$$
$$BC(1) + A\bar{B}C + AB\bar{C}$$
$$\Downarrow$$
$$BC + A\bar{B}C + AB\bar{C}$$
$$\Downarrow$$
$$B(C+A\bar{C}) + A\bar{B}C$$
$$\Downarrow$$
$$B(C+A) + A\bar{B}C$$
$$\Downarrow$$
$$BC + AB + A\bar{B}C$$
$$\Downarrow$$
$$BC + A(B+\bar{B}C)$$
$$\Downarrow$$
$$BC + A(B+C)$$
$$\Downarrow$$
$$BC + AB + AC$$

The Result


A
B
C
Output = $\overline{A}BC + A\overline{B}C + AB\overline{C} + ABC$
$\overline{A}BC$
$A\overline{B}C$
$AB\overline{C}$
ABC

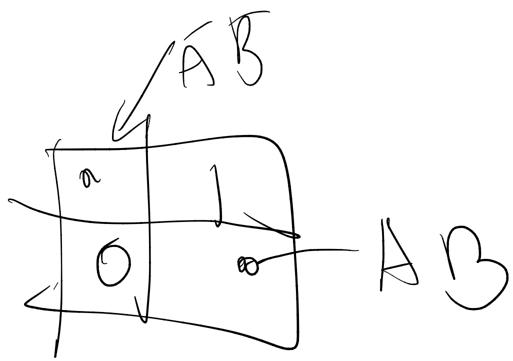

C
Output = AB + BC + AC
AB
BC
AC

Karnaugh Maps or K-Maps

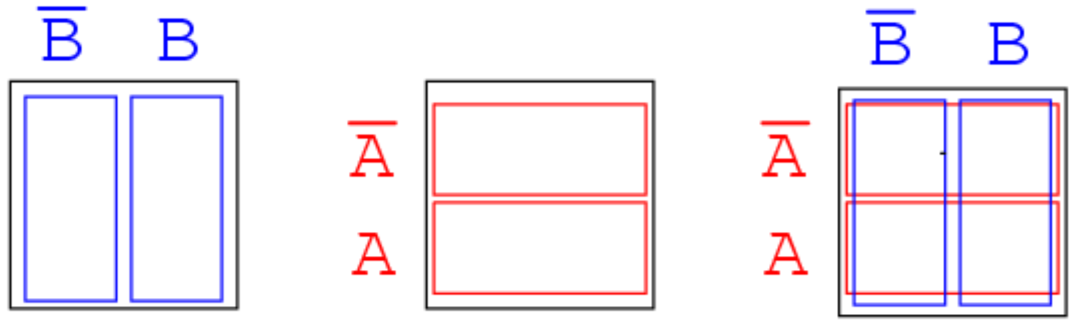
K-maps are a graphical technique to view minterms and how they relate.

The “map” is a diagram made up of squares, with each square representing a single minterm.

Minterms resulting in a “1” are marked as “1”, all others are marked “0”



2 Variable K-Map



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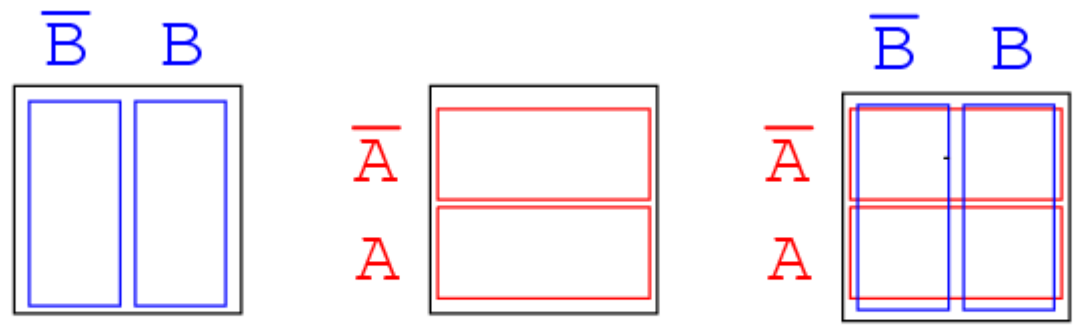
2 Variable K-Map



A	B	Output
0	0	0
0	1	1
1	0	0
1	1	1

A \ B	0	1
0	0	1
1		

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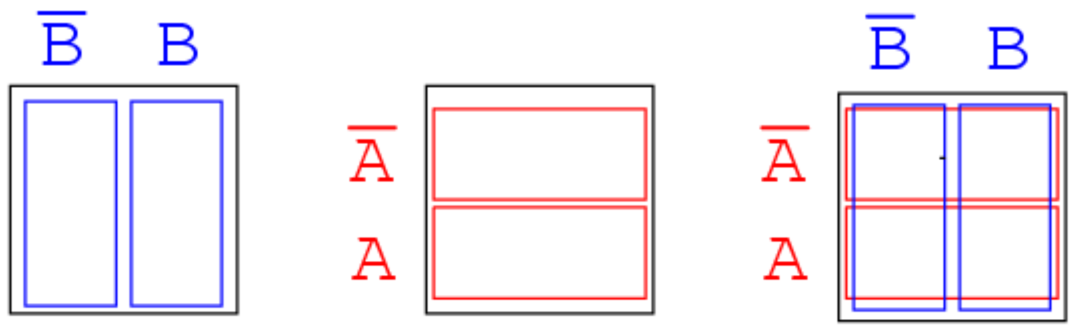
2 Variable K-Map



A	B	Output
0	0	0
0	1	1
1	0	0
1	1	1

A \ B	0	1
0	0	1
1	0	1

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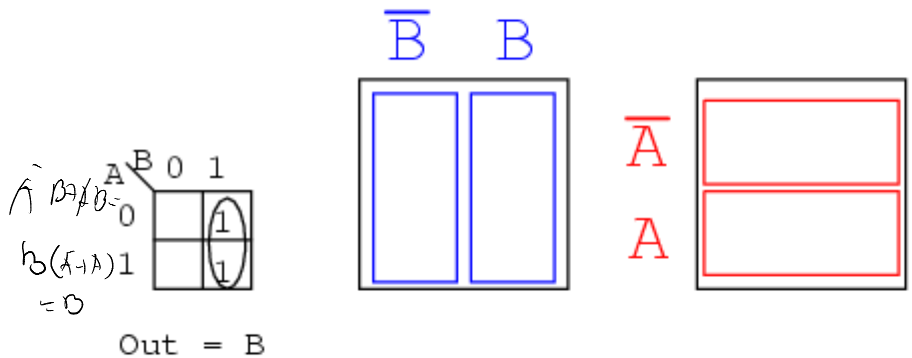
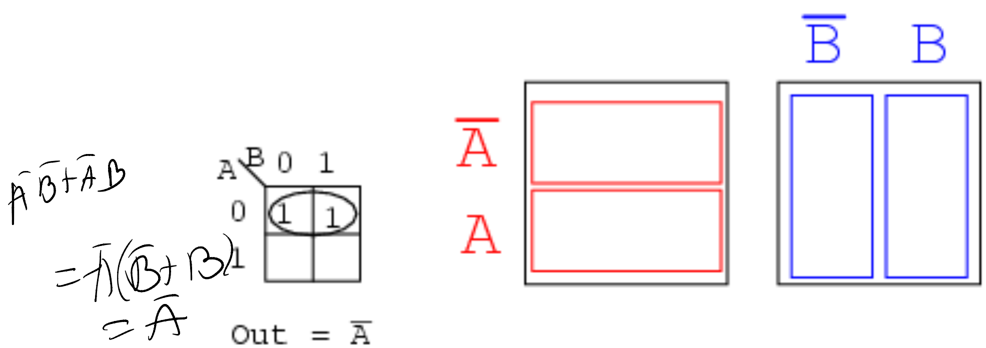
Finding Commonality





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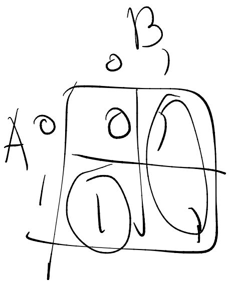
Finding the “best” solution

$(A+B)(\overline{B}+B)$

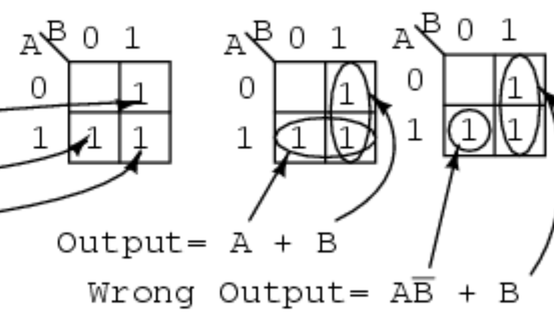
$= A+B$



A	B	Output
0	0	0
0	1	1
1	0	1
1	1	1

Og: $A\overline{B}+B$

$(A+B)(\overline{B}+B)$



$A\overline{B}+AB \quad A(\overline{B}+B)$

Grouping become simplified products.

Both are “correct”. “A+B” is preferred.

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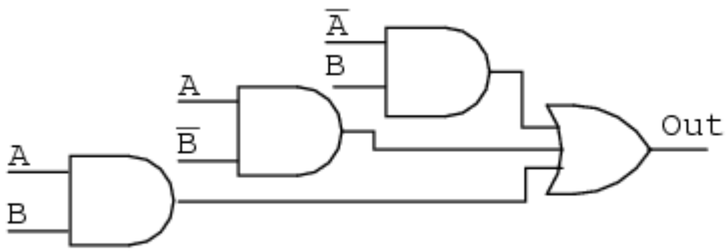
Simplify Example



A \ B	0	1
0	0	1
1	1	0

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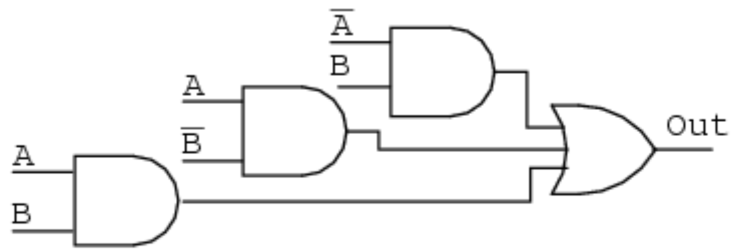
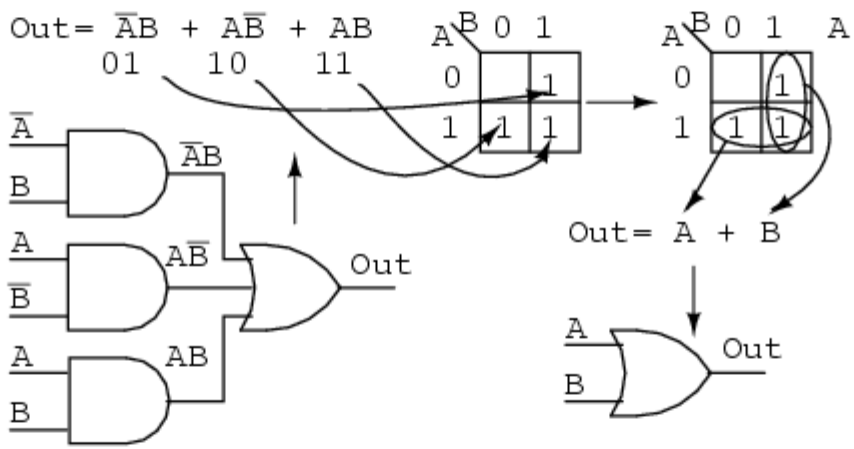
Simplify Example





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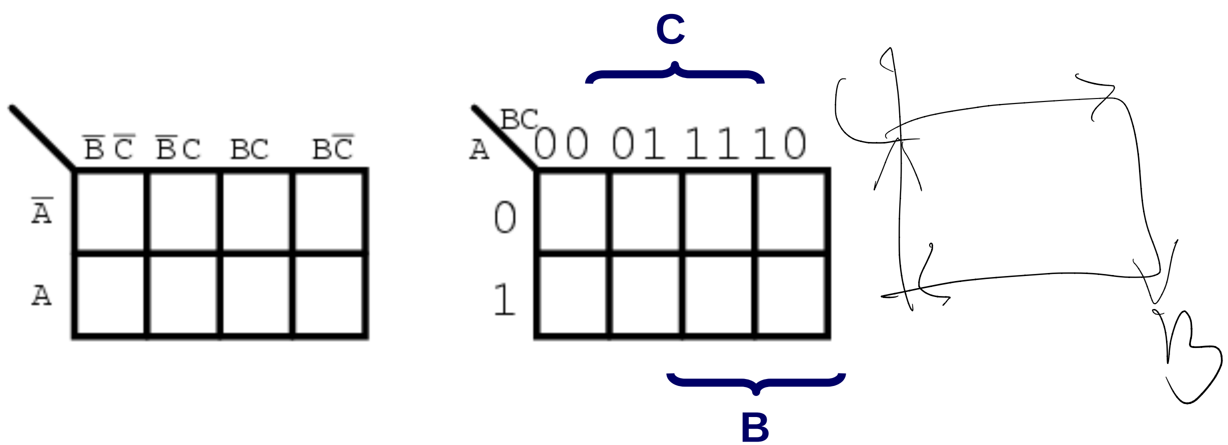
3 Variable K-Maps



- Note in higher maps, several variables occupy a given axis
- The sequence of 1s and 0s follow a Gray Code Sequence.

3 Variable K-Maps

	$\bar{B}\bar{C}$	$\bar{B}C$	BC	$B\bar{C}$
$\bar{A}$				
A				

A \ BC	00	01	11	10
0	1	1	0	0
1	0	0	0	0

$\bar{C}$

$\bar{A} = 0$

$$\text{Out} = \bar{A}\bar{B}\bar{C} + \bar{A}\bar{B}C = \bar{A}\bar{B}(\bar{C} + C)$$

$$= \bar{A}\bar{B}$$

A \ BC	00	01	11	10
0	1	1		
1				

$\text{Out} = \bar{A}\bar{B}$

$\bar{A}$ $\bar{C}$

$\bar{C}$

3 Variable K-Maps

	$\bar{B}\bar{C}$	$\bar{B}C$	BC	$B\bar{C}$
$\bar{A}$				
A				

C

A \ BC	00	01	11	10
0				
1				

B

$\bar{A}\bar{B} + \bar{A}B$

$\bar{A}(\bar{B}+B)$

$$Out = \bar{A}\bar{B}\bar{C} + \bar{A}\bar{B}C + \bar{A}BC + \bar{A}B\bar{C}$$

$\bar{A}$

A \ BC	00	01	11	10
0	1	1	1	1
1				

$\bar{A}\bar{B} + \bar{A}B$

$\bar{A}(B+\bar{B})$

$$Out = \bar{A}$$

3 Variable K-Maps

$\bar{A}C(\bar{B}+B)$

	$\bar{B}\bar{C}$	$\bar{B}C$	BC	$B\bar{C}$
$\bar{A}$				
A				

A \ BC	00	01	11	10
0				
1				

$\bar{A}C + AC$

$C(\bar{A}+A)$

C

$$Out = \bar{A}\bar{B}C + \bar{A}BC + A\bar{B}C + ABC$$

A \ BC	00	01	11	10
0		1	1	
1		1	1	

$$Out = C$$

3 Variable K-Maps

	$\bar{B}\bar{C}$	$\bar{B}C$	BC	$B\bar{C}$
$\bar{A}$				
A				

A \ BC	00	01	11	10
0				
1				

$A + \bar{A}B$

$(\bar{A} + B)\cancel{(A + \bar{A})}$

$$Out = \bar{A}\bar{B}\bar{C} + \bar{A}\bar{B}C + \bar{A}BC + \bar{A}B\bar{C} + ABC + AB\bar{C}$$

A \ BC	00	01	11	10
0	1	1	1	1
1			1	1

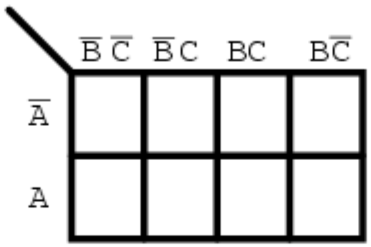
$\bar{A}\bar{B} + \bar{A}B + AB$

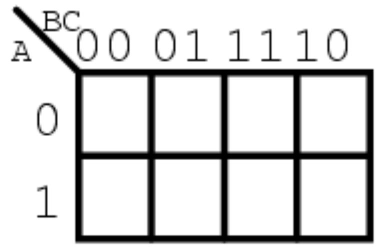
$\bar{A}\bar{B} + AB + \bar{A}B$

$A(\bar{B} + B) + \bar{A}B$

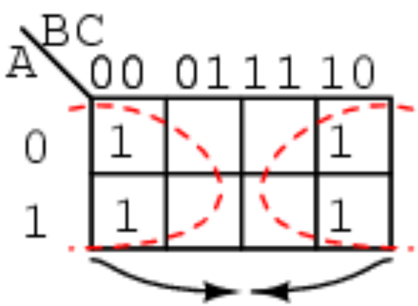
$$Out = \bar{A} + B$$

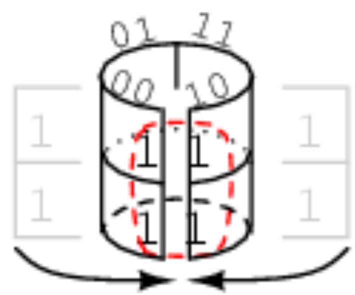
3 Variable K-Maps





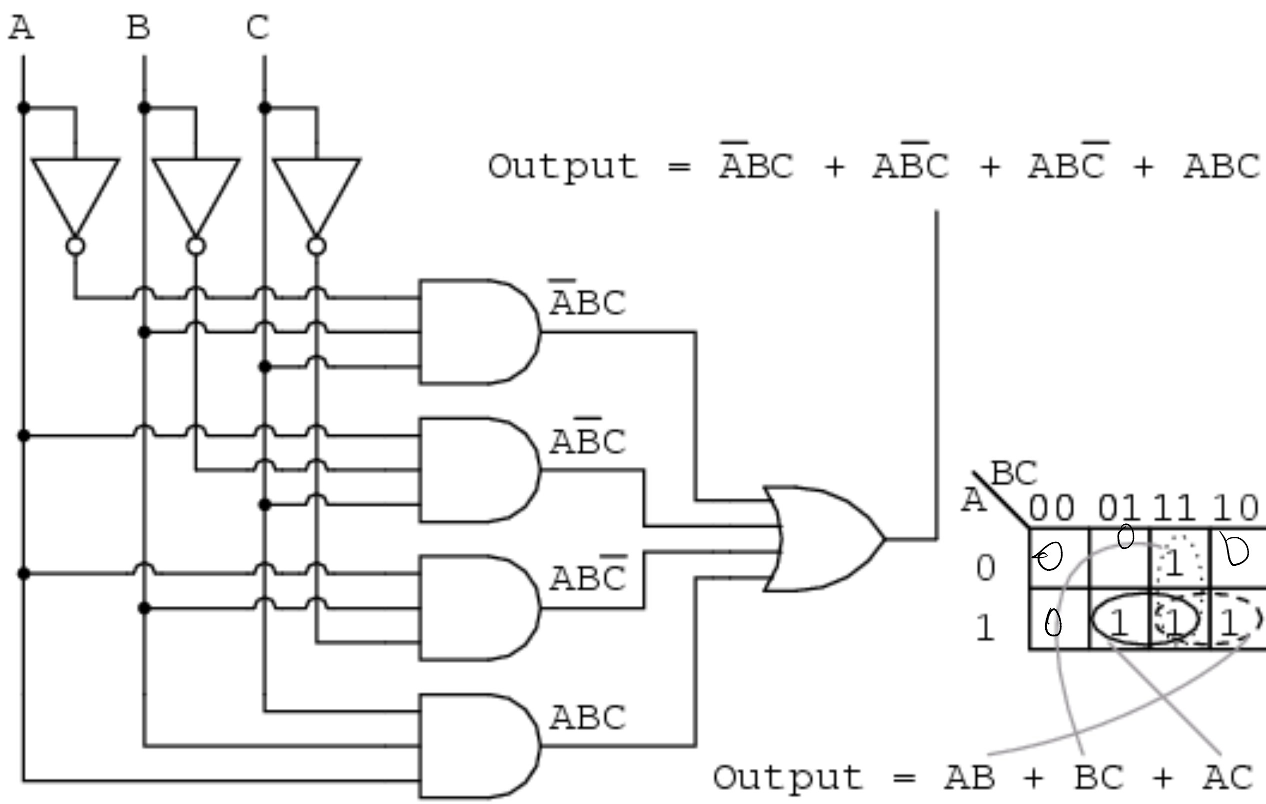
$$Out = \bar{A}\bar{B}\bar{C} + A\bar{B}\bar{C} + \bar{A}B\bar{C} + AB\bar{C}$$

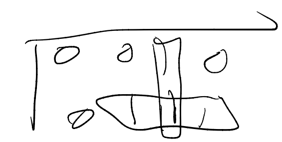




$$Out = \bar{C}$$

Back to our earlier example…..





The K-map and the algebraic produce the same result.

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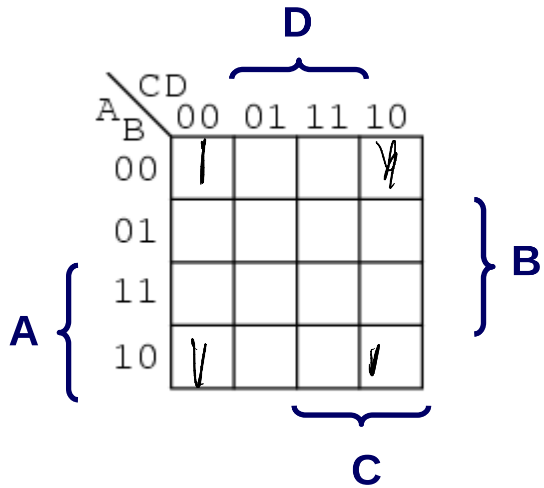
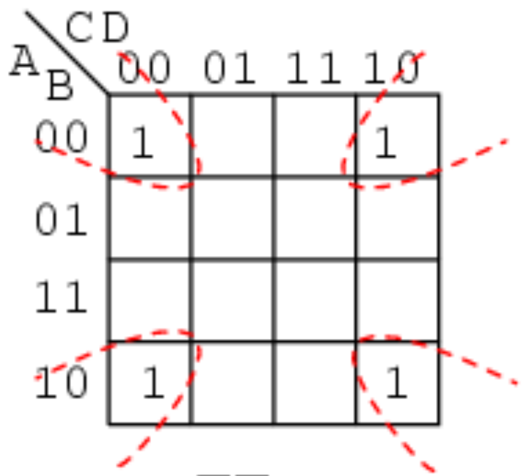
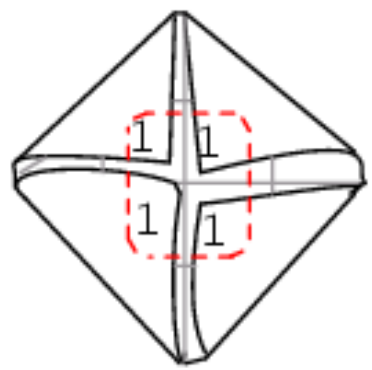
Up… up… and let’s keep going



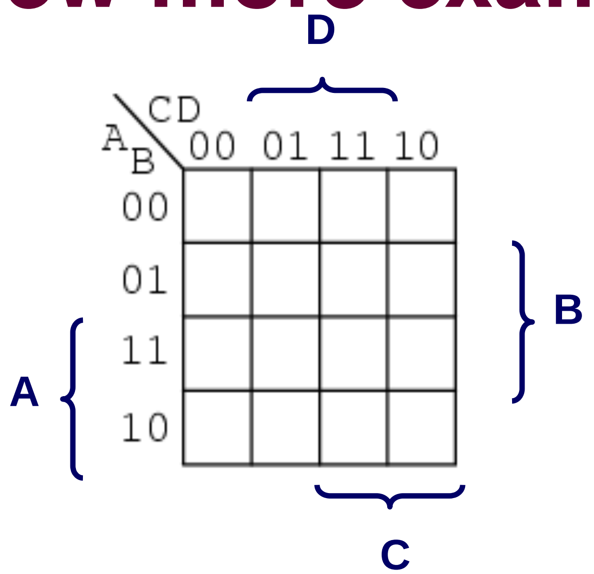
$$Out= \overline{A}\,\overline{B}\,\overline{C}\,\overline{D} + \overline{A}\,\overline{B}\,C\,\overline{D} + A\,\overline{B}\,\overline{C}\,\overline{D} + A\,\overline{B}\,C\,\overline{D}$$



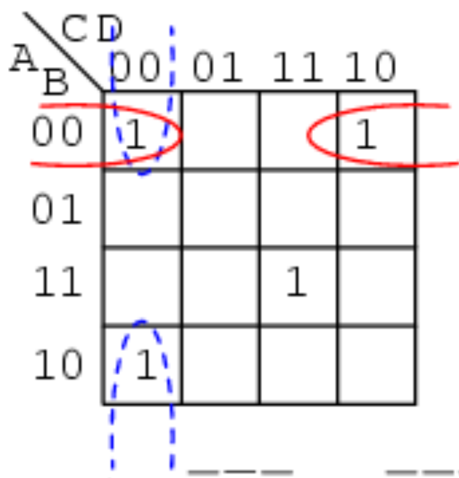


$$Out= \overline{B}\,\overline{D}$$

Few more examples

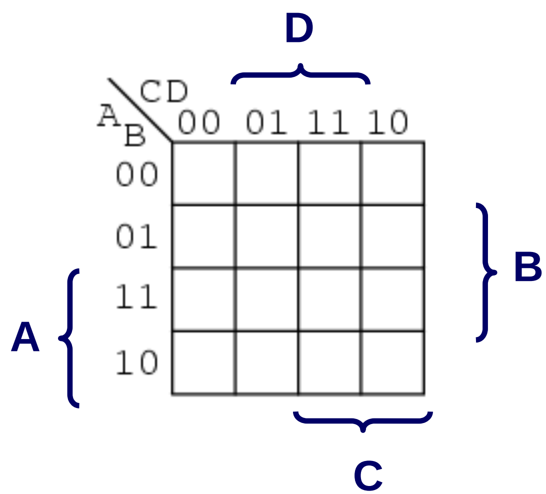


$$Out = \overline{A}\,\overline{B}\,\overline{C}\,\overline{D} + \overline{A}\,\overline{B}\,C\,\overline{D} + A\,\overline{B}\,\overline{C}\,\overline{D} + A\,B\,C\,D$$

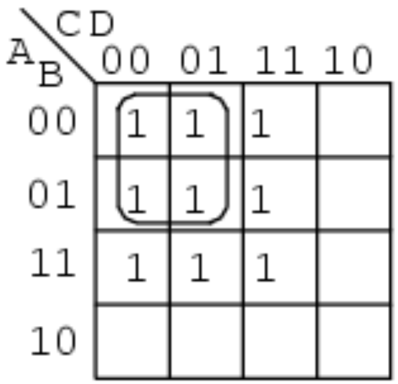


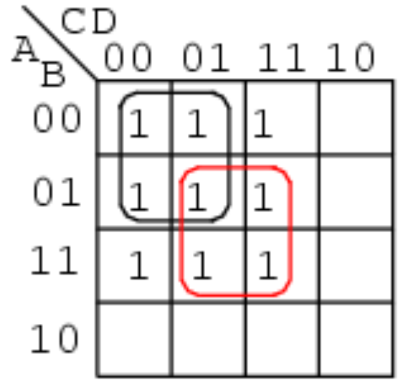
$$Out = \overline{B}\,\overline{C}\,\overline{D} + \overline{A}\,\overline{B}\,\overline{D} + A\,B\,C\,D$$

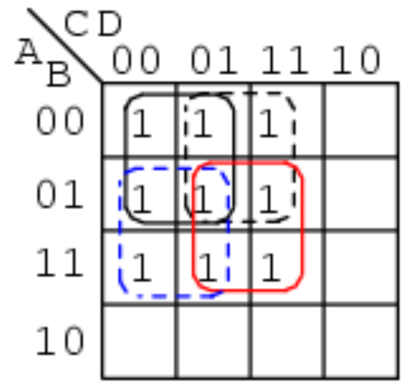
Few more examples



$$Out = \bar{A}\bar{B}\bar{C}\bar{D} + \bar{A}\bar{B}\bar{C}D + \bar{A}\bar{B}CD + \bar{A}B\bar{C}\bar{D} + \bar{A}B\bar{C}D + \bar{A}BCD + AB\bar{C}\bar{D} + AB\bar{C}D + ABCD$$







$$Out = \bar{A}\bar{C} + \bar{A}D + B\bar{C} + BD$$

Don't Care Conditions

- Let $F = AB + \overline{A}\overline{B}$
- Suppose we know that a disallowed input combo is A=1, B=0
- Can we replace F with a simpler function G whose output matches for all inputs we do care about?
- Let H be the function with Don't-care conditions for obsolete inputs

Inputs will not occur

A	B	F	H	G
0	0	1	1	1
0	1	0	0	0
1	0	0	X	1
1	1	1	1	1

$$G = AB + \overline{B}$$

- Both F & G are appropriate functions for H
- G can substitute for F for valid input combinations

Don’t Cares can Greatly Simplify Circuits

Sometimes “don’t cares” greatly simplify circuitry

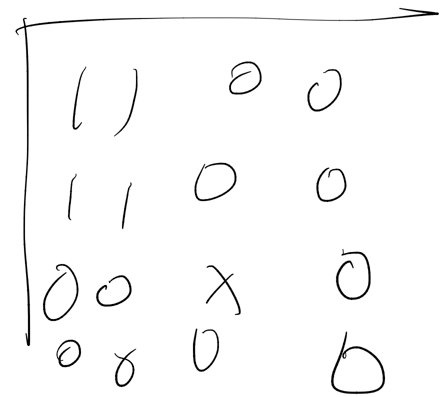
D

1	X	X	X
X	1	X	X
0	0	1	X
0	0	X	1

B

A

C



$$\bar{A}\bar{B}\bar{C}\bar{D} + \bar{A}B\bar{C}D + ABCD + A\bar{B}C\bar{D} \quad \text{vs.} \quad \bar{A} + C$$

Formal Definition of Minterms

e.g., Minterms for 3 variables A,B,C

A	B	C	minterm
0	0	0	m0 $\bar{A}\bar{B}\bar{C}$
0	0	1	m1 $\bar{A}\bar{B}C$
0	1	0	m2 $\bar{A}B\bar{C}$
0	1	1	m3 $\bar{A}BC$
1	0	0	m4 $A\bar{B}\bar{C}$
1	0	1	m5 $A\bar{B}C$
1	1	0	m6 $AB\bar{C}$
1	1	1	m7 ABC

- A product term in which all variables appear once, either complemented or uncomplemented (i.e., an entry in the truth table).
- Each minterm evaluates to 1 for exactly one variable assignment, 0 for all others.
- Denoted by mX where X corresponds to the variable assignment for which mX = 1.

Minterm Example

(variables appear once in each minterm)

A	B	C	F	$\overline{F}$	minterm	
0	0	0	1	0	m0	$\overline{A}\,\overline{B}\,\overline{C}$
0	0	1	1	0	m1	$\overline{A}\,\overline{B}C$
0	1	0	1	0	m2	$\overline{A}B\overline{C}$
0	1	1	0	1	m3	$\overline{A}BC$
1	0	0	1	0	m4	$A\overline{B}\,\overline{C}$
1	0	1	1	0	m5	$A\overline{B}C$
1	1	0	0	1	m6	$AB\overline{C}$
1	1	1	0	1	m7	ABC

$$F = \overline{A}\,\overline{B}\,\overline{C} + \overline{A}\,\overline{B}C + \overline{A}B\overline{C} + A\overline{B}\,\overline{C} + A\overline{B}C$$
$$= m0 + m1 + m2 + m4 + m5$$
$$= \sum m(0,1,2,4,5)$$

$$\overline{F} = \overline{A}BC + AB\overline{C} + ABC$$
$$= m3 + m6 + m7$$
$$= \sum m(3,6,7)$$

Minterm Example

Circuits may be specified in the form $\sum \mathrm{m}(0,2,4,5,6,7,13,15)$, where the indices correspond to positions in a Karnaugh map:

CD

AB	00	01	11	10
00	0	1	3	2
01	4	5	7	6
11	12	13	15	14
10	8	9	11	10

Minterm Example

Circuits may be specified in the form $\sum \mathrm{m}(0, 2, 4, 5, 6, 7, 13, 15)$, where the indices correspond to positions in a Karnaugh map:

AB \ CD	00	01	11	10
00	1	0	0	1
01	1	1	1	1
11	0	1	1	0
10	0	0	0	0

Formal Definition of Maxterms

A	B	C	maxterm
0	0	0	M0 $A+B+C$
0	0	1	M1 $A+B+\overline{C}$
0	1	0	M2 $A+\overline{B}+C$
0	1	1	M3 $A+\overline{B}+\overline{C}$
1	0	0	M4 $\overline{A}+B+C$
1	0	1	M5 $\overline{A}+B+\overline{C}$
1	1	0	M6 $\overline{A}+\overline{B}+C$
1	1	1	M7 $\overline{A}+\overline{B}+\overline{C}$

- A sum term in which all variables appear once, either complemented or uncomplemented.
- Each maxterm evaluates to 0 for exactly one variable assignment, 1 for all others.
- Denoted by MX where X corresponds to the variable assignment for which MX = 0.

Maxterm Example

A	B	C	maxterm	F
0	0	0	M0 $A+B+C$	1
0	0	1	M1 $A+B+\overline{C}$	1
0	1	0	M2 $A+\overline{B}+C$	1
0	1	1	M3 $A+\overline{B}+\overline{C}$	0
1	0	0	M4 $\overline{A}+B+C$	1
1	0	1	M5 $\overline{A}+B+\overline{C}$	1
1	1	0	M6 $\overline{A}+\overline{B}+C$	0
1	1	1	M7 $\overline{A}+\overline{B}+\overline{C}$	0

$$F = (A+\overline{B}+\overline{C})\,(\overline{A}+\overline{B}+C)\,(\overline{A}+\overline{B}+\overline{C})$$

$$= (M3)\,(M6)\,(M7)$$

$$= \prod M(3,6,7)$$

Maxterm Example

Then we can find the usual sum of products:

AB \ CD	00	01	11	10
00	1	0	0	1
01	1	1	1	1
11	0	1	1	0
10	0	0	0	0

$$F = BD + \bar{A}\bar{D}$$

$\bar{A}\bar{C}\bar{D} + B + BD$

$+ \bar{A}CD$

$\bar{A}(\bar{C}\bar{D} + CD)$

$\bar{A}\bar{D} + B\ D$

Maxterm Example

Or the product of sums:

AB \ CD	00	01	11	10
00	1	0	0	1
01	1	1	1	1
11	0	1	1	0
10	0	0	0	0

$$F = (\bar{A} + D)(B + \bar{D})$$

$\bar{A} + D$

Converting Between Canonical Forms

	F	$\overline{F}$
Minterms (SOP)	$\sum m(F = 1)$	$\sum m(F = 0)$
Maxterms (POS)	$\prod M(F = 0)$	$\prod M(F = 1)$

DeMorgans: same terms

Product of Sums Example

WX \ YZ	00	01	11	10
00	1	1	0	1
01	1	0	0	0
11	1	0	0	0
10	1	1	0	1

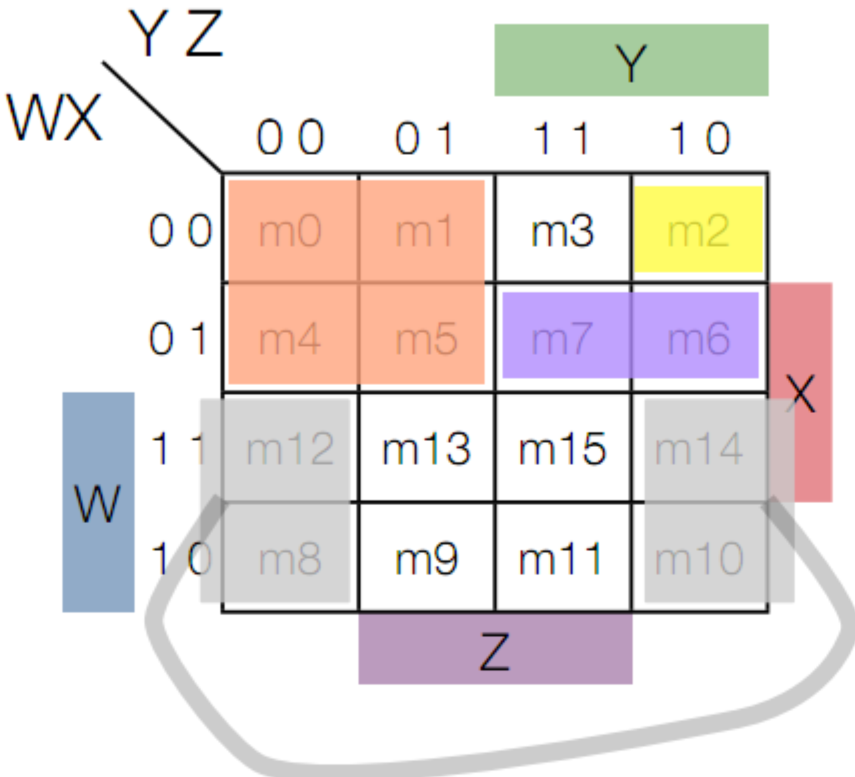
$$\overline{F} = YZ + XZ + YX$$

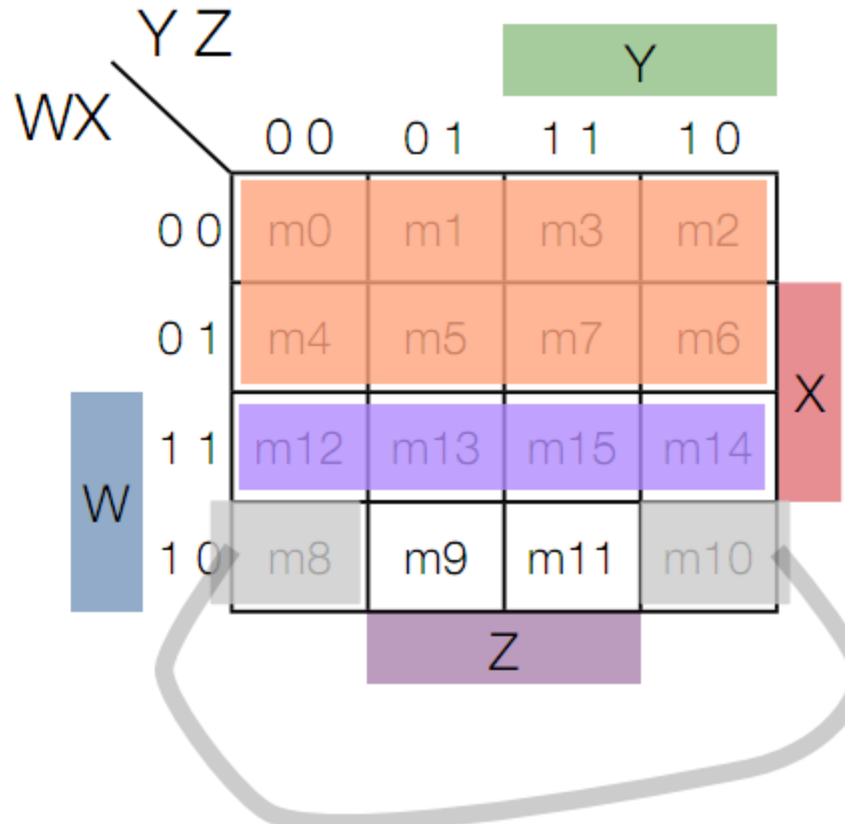
DeMorgan's

$$F = (\overline{Y}+\overline{Z})(\overline{Z}+\overline{X})(\overline{Y}+\overline{X})$$

K-maps and Implicants

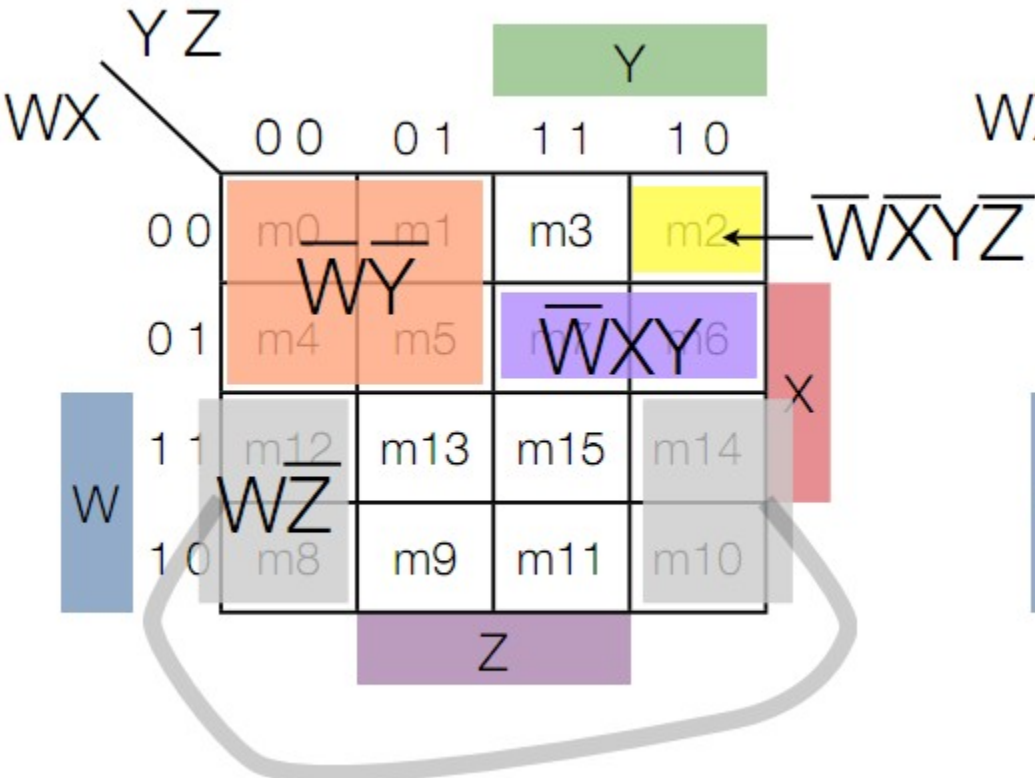
- **Implicant**: a product term, which, viewed in a K-Map is a $2^i \times 2^j$ size "rectangle" (possibly wrapping around) where i=0,1,2, j=0,1,2

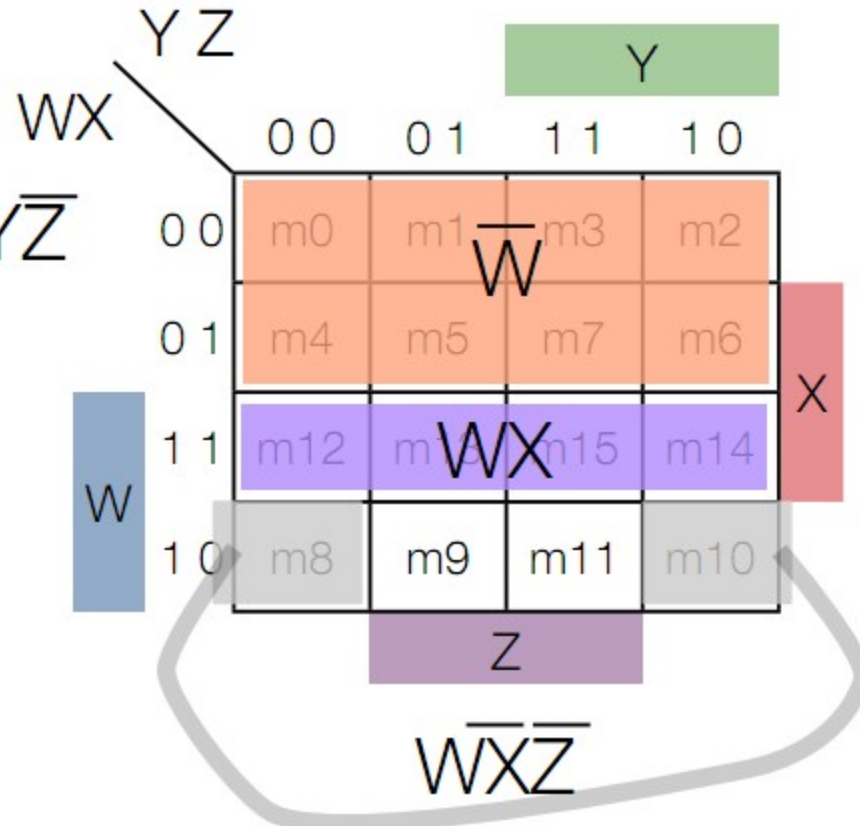




Implicants

- **Implicant**: a product term, which, viewed in a K-Map is a $2^i \times 2^j$ size "rectangle" (possibly wrapping around) where i=0,1,2, j=0,1,2





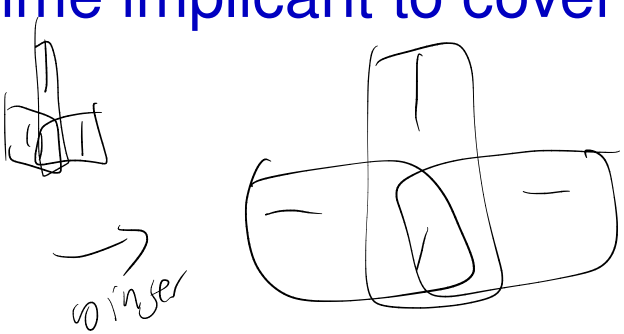
Note: bigger rectangles = fewer literals

More Implicant Terminology

Implicant: product term, which when viewed in a K-map, is a rectangle of 1s

Prime implicant: an implicant not contained in another implicant

Essential prime implicant: a prime implicant that is the only prime implicant to cover some minterm



Example

- List all of the prime implicants for this function
- Is any of them an essential prime implicant?
- What is a simplified expression for this function?

WX \ YZ	00	01	11	10
00	0	0	1	0
01	1	1	1	0
11	0	1	1	1
10	0	1	0	0

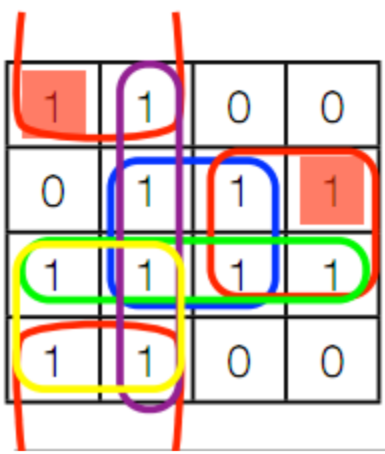
Y spans columns 11 and 10; Z spans columns 01 and 11; X spans rows 01 and 11; W spans rows 11 and 10.

$\bar{W} Y Z + W X Y + W \bar{Y} Z + \bar{W} X \bar{Y}$

Example

- Step 1: Identify all PIs and essential PIs
- Step 2: Include all Essential PIs in the circuit (Why?)
- Step 3: If any 1-valued minterms are uncovered by EPIs, choose PIs that are "big" and do a good job covering
- Selection Rule: a heuristic for usually choosing "good" PIs: choose the PIs that minimize overlap with one another and with EPIs

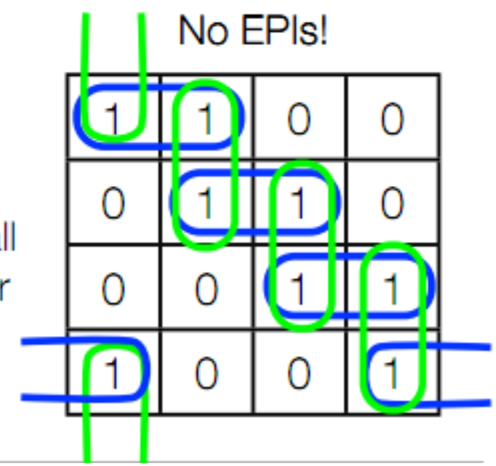
Red bounds are EPIs (solo-covered minterm shown in red)



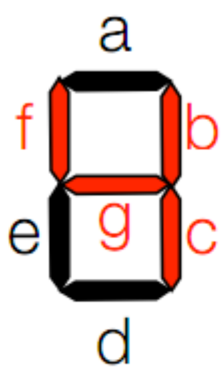
Also need (purple or blue) and (yellow or green)

All blue PIs or all green PIs cover

No EPIs!



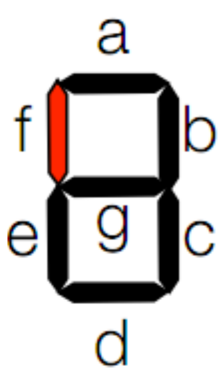
Design Example



Input					Output						
Va	W	X	Y	Z	a	b	c	d	e	f	g
0	0	0	0	0	1	1	1	1	1	1	0
1	0	0	0	1	0	1	1	0	0	0	0
2	0	0	1	0	1	1	0	1	1	0	1
3	0	0	1	1	1	1	1	1	0	0	1
4	0	1	0	0	0	1	1	0	0	1	1
5	0	1	0	1	1	0	1	1	0	1	1
6	0	1	1	0	1	0	1	1	1	1	1
7	0	1	1	1	1	1	1	0	0	0	0
8	1	0	0	0	1	1	1	1	1	1	1
9	1	0	0	1	1	1	1	0	0	1	1
X	1	0	1	0	X	X	X	X	X	X	X
X	1	0	1	1	X	X	X	X	X	X	X
X	1	1	0	0	X	X	X	X	X	X	X
X	1	1	0	1	X	X	X	X	X	X	X
X	1	1	1	0	X	X	X	X	X	X	X
X	1	1	1	1	X	X	X	X	X	X	X

e.g., what outputs "lights up" when input V=4?

Design Example



For what values does output f "light up" for?

Y=1

	00	01	11	10
00	1	0	0	0
01	1	1	0	1
11	X	X	X	X
10	1	1	X	X

X=1

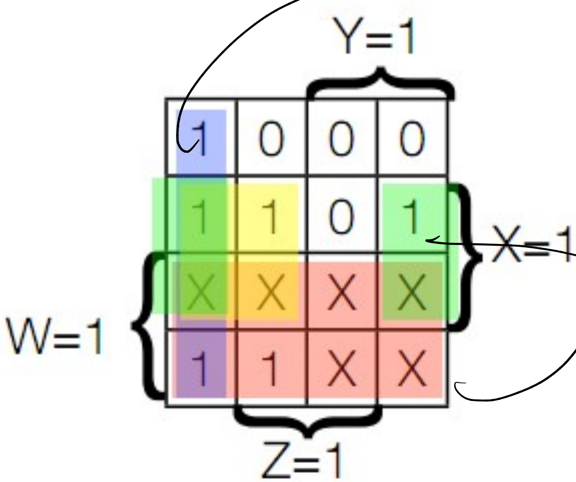
W=1

Z=1

Input / Output

Va	W	X	Y	Z	a	b	c	d	e	f	g
0	0	0	0	0	1	1	1	1	1	1	0
1	0	0	0	1	0	1	1	0	0	0	0
2	0	0	1	0	1	1	0	1	1	0	1
3	0	0	1	1	1	1	1	1	0	0	1
4	0	1	0	0	0	1	1	0	0	1	1
5	0	1	0	1	1	0	1	1	0	1	1
6	0	1	1	0	1	0	1	1	1	1	1
7	0	1	1	1	1	1	1	0	0	0	0
8	1	0	0	0	1	1	1	1	1	1	1
9	1	0	0	1	1	1	1	0	0	1	1
X	1	0	1	0	X	X	X	X	X	X	X
X	1	0	1	1	X	X	X	X	X	X	X
X	1	1	0	0	X	X	X	X	X	X	X
X	1	1	0	1	X	X	X	X	X	X	X
X	1	1	1	0	X	X	X	X	X	X	X
X	1	1	1	1	X	X	X	X	X	X	X

Design Example



$$f = W + \overline{Y}\,\overline{Z} + X\overline{Z} + \overline{X}\,\overline{Y} = W + (X+\overline{Y})\overline{Z} + \overline{X}\,\overline{Y}$$