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A continuous time approach to intensive longitudinal data: What, Why and How?

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Abstract

The aim of this chapter is to: a) provide a broad didactical treatment of the first-order stochastic differential equation model - also known as the continuous-time (CT) first-order vector autoregressive (VAR(1)) model; and b) argue for and illustrate the potential of this model for the study of psychological processes using intensive longitudinal data. We begin by describing what the CT-VAR(1) model is, and how it relates to the more commonly used discrete-time VAR(1) models. Assuming no prior knowledge on the part of the reader, we introduce important concepts for the analysis of dynamic systems, such as stability and fixed points. In addition we examine why applied researchers should take a continuous-time approach to psychological phenomena, focusing on both the practical and conceptual benefits of this approach. Finally, we elucidate how researchers can interpret CT models, describing the direct interpretation of CT model parameters as well as tools such as impulse response functions, vector fields, and lagged-parameter plots. To illustrate this methodology we re-analyse a single-subject experience-sampling dataset with the R package ctsem; for didactical purposes, R code for this analysis is included, and the dataset itself is publically available.

A continuous time approach to intensive longitudinal data:

What, Why and How?

The increased availability of intensive longitudinal data – such as obtained with ambulatory assessments, experience sampling, ecological momentary assessments and electronic diaries – has opened up new opportunities for researchers to investigate the *dynamics* of psychological *processes*, that is, the way psychological variables evolve, vary and relate to one another over time (cf. Bolger & Laurenceau, 2013; Chow, Ferrer, & Hsieh, 2011; Hamaker, Dolan, & Molenaar, 2005). A useful concept in this respect is that of people being dynamic systems whose current state depends on their preceding states. For instance, we may be interested in the relationship between momentary stress and anxiety. We can think of stress and anxiety as each defining an axis in a two-dimensional space, and let the values of stress and anxiety at each moment in time define a position in this space. Over time, the point that represents a person's momentary stress and anxiety, moves through this two-dimensional space, and our goal is to understand the lawfulness that underlie these movements.

There are two frameworks that can be used to describe such movements: 1) the *discrete time* (DT) framework, in which the passage of time is treated in discrete steps; and 2) the *continuous time* (CT) framework, in which time is viewed as a continuous variable. Most psychological researchers are at least somewhat familiar with the DT approach, as it is the basis of the vast majority of longitudinal models used in the social sciences. In contrast, CT models have gained relatively little attention in fields such as psychology: This is despite the fact that many

psychological researchers have been advocating their use for a long time, claiming that the CT approach overcomes practical and conceptual problems associated with the DT approach (e.g., Boker, 2002; Chow et al., 2005; Oud & Delsing, 2010; Voelkle, Oud, Davidov, & Schmidt, 2012). We believe there are two major hurdles that hamper the adoption of the CT approach in psychological research. First, the estimation of CT models typically requires the use of specialized software (cf. Chow, Ferrer, & Nesselroade, 2007; Driver, Oud, & Voelkle, in press; Oravecz, Tuerlinckx, & Vandekerckhove, 2016) or unconventional use of more common software (cf. Boker, Deboeck, Edler, & Keel, 2010; Boker, Neale, & Rausch, 2004; Steele & Ferrer, 2011). Second, the results from CT models are not easily understood, and researchers may not know how to interpret and represent their findings.

Our goal in this chapter is twofold. First, we introduce readers to the perspective of psychological processes as CT processes; we focus on the *conceptual reasons* for which the CT perspective is extremely valuable in moving our understanding of processes in the right direction. Second, we provide a didactical description of how to interpret the results of a CT model, based on our analysis of an empirical dataset. We examine the direct interpretation of model parameters, examine different ways in which the dynamics described by the parameters can be understood and visualised, and explain how these are related to one another throughout. We will restrict our primary focus to the simplest DT and CT models, that is, first-order (vector) autoregressive models and first-order differential equations.

The organization of this chapter is as follows. First, we provide an overview of the DT and CT models under consideration. Second, we discuss the practical and

conceptual reasons researchers should adopt a CT modelling approach. Third, we illustrate the use and interpretation of the CT model using a bivariate model estimated from empirical data. Fourth, we conclude with a brief discussion of more complex models which may be of interest to substantive researchers.

Two frameworks

The relationship between the DT and CT frameworks has been discussed extensively by a variety of authors. Here, we briefly reiterate the main issues, as this is vital to the subsequent discussion. For a more thorough treatment of this topic, the reader is referred to Voelkle et al. (2012). We begin by presenting the first-order vector auto-regressive model in DT, followed by the presentation of the first-order differential equation in CT. Subsequently, we show how these models are connected, and discuss certain properties which can be inferred from the parameters of the model. For simplicity, and without loss of generalisation, we describe single-subject DT and CT models, in terms of observed variables. Extensions for multiple-subject data, and extensions for latent variables, in which the researchers can account for measurement error by additionally specifying a measurement model, are readily available (in the case of CT models, see for example Boker et al., 2004; Driver, Voelkle, & Oud, 2017; Oravecz & Tuerlinckx, 2011).

The discrete time framework

DT models are those models for longitudinal data in which the passage of time is accounted for only with regards to the order of observations. If the true data-generating model for a process is a DT model, then the process only takes on values at discrete moments in time (e.g., hours of sleep per day or monthly salary).

Such models are typically applied to data that consist of some set of variables measured repeatedly over time. These measurements typically show auto-correlation, that is, serial dependencies between the observed values of these variables at consecutive measurement occasions. We can model these serial dependencies using (discrete-time) auto-regressive equations, which describe the relationship between the values of variables observed at consecutive measurement occasions.

The specific type of DT model that we will focus on in this chapter is the first-order Vector Auto-Regressive (VAR(1)) model (cf. Hamilton, 1994). Given a set of V variables of interest measured at N different occasions, the VAR(1) describes the relationship between $\boldsymbol{y}_\tau$, a $V \times 1$ column vector of variables measured at occasion τ (for $\tau = 2, \ldots, N$) and the values those same variables took on at the preceding measurement occasion, the vector $\boldsymbol{y}_{\tau-1}$. This model can be expressed as

$$\boldsymbol{y}_\tau = \boldsymbol{c} + \boldsymbol{\Phi}\boldsymbol{y}_{\tau-1} + \boldsymbol{\epsilon}_\tau, \tag{1}$$

where $\boldsymbol{\Phi}$ represents a $V \times V$ matrix with autoregressive and cross-lagged coefficients that regresses $\boldsymbol{y}_\tau$ on $\boldsymbol{y}_{\tau-1}$. The $V \times 1$ column vector $\boldsymbol{\epsilon}_\tau$ represents the variable-specific random shocks or innovations at that occasion, which are normally distributed with mean zero and a $V \times V$ variance-covariance matrix $\boldsymbol{\Psi}$. Finally, $\boldsymbol{c}$ represents a $V \times 1$ column vector of intercepts.

In the case of a stationary process, the mean $\boldsymbol{\mu}$, and the variance-covariance matrix of the variables $\boldsymbol{y}_\tau$ (generally denoted $\boldsymbol{\Sigma}$), do not change over time.[1] Then, the vector $\boldsymbol{\mu}$ represents the long-run expected values of the random variables, $E(\boldsymbol{y}_\tau)$, and is a function of the vector of intercepts and the matrix with lagged regression coefficients, that is, $\boldsymbol{\mu} = (\boldsymbol{I} - \boldsymbol{\Phi})^{-1}\boldsymbol{c}$, where $\boldsymbol{I}$ is a $V \times V$ identity matrix

(cf. Hamilton, 1994). In terms of a V-dimensional dynamical system of interest, $\boldsymbol{\mu}$ represents the *equilibrium position* of the system. By definition, τ is limited to positive integers; that is, there is no .1th or 1.5th measurement occasion.

Both the single-subject and multilevel versions of the VAR(1) model have frequently been used to analyze intensive longitudinal data of psychological variables, including symptoms of psychopathology, such as mood- and affect-based measures (Bringmann, Lemmens, Huibers, Borsboom, & Tuerlinckx, 2015; Bringmann et al., 2016; Browne & Nesselroade, 2005; Moberly & Watkins, 2008; Rovine & Walls, 2006). In these cases, the auto-regressive parameters ϕ_{ii} are often interpreted as reflecting the *stability, inertia* or *carry-over* of a particular affect or behaviour (Koval, Kuppens, Allen, & Sheeber, 2012; Kuppens, Allen, & Sheeber, 2010; Kuppens et al., 2012). The cross-lagged effects (i.e., the off-diagonal elements ϕ_{ij} for $i \neq j$) quantify the lagged relationships, sometimes referred to as the *spill-over*, between different variables in the model. These parameters are often interpreted in substantive terms, either as predictive or Granger-causal relationships between different aspects of affect or behaviour (Bringmann et al., 2013; Gault-Sherman, 2012; Granger, 1969; Ichii, 1991; Watkins, Lei, & Canivez, 2007). For example, if the standardized cross-lagged effect of $y_{1,\tau-1}$ on $y_{2,\tau}$ is larger than the cross-lagged effect of $y_{2,\tau-1}$ on $y_{1,\tau}$, researchers may draw the conclusion that y_1 is the driving force or dominant variable of that pair (Schuurman, Ferrer, de Boer-Sonnenschein, & Hamaker, 2016). As such, substantive researchers are typically interested in the (relative) magnitudes and signs of these parameters.

The continuous time framework

In contrast to the DT framework, which treats values of processes indexed by observation τ, the CT framework treats processes as *functions* of the continuous variable time t: The processes being modeled are assumed to vary continuously with respect to time, meaning that these variables may take on values if observed at any imaginable moment. CT processes can be modeled using a broad class of differential equations, allowing for a wide degree of diversity in the types of dynamics that are being modeled. It is important to note that many DT models have a differential equation counterpart. For the VAR(1) model, the CT equivalent is the first-order Stochastic Differential Equation (SDE), where stochastic refers to the presence of random innovations or shocks.

The first-order SDE describes how the position of the V-dimensional system at a certain point in time, $\boldsymbol{y}(t)$, relative to the equilibrium position $\boldsymbol{\mu}$, is related to the *rate of change* of the process with respect to time (i.e., $\frac{d\boldsymbol{y}(t)}{dt}$) in that same instant. The latter can also be thought of as a vector of *velocities*, describing in what direction and with what magnitude the system will move an instant later in time (i.e., the ratio of the change in position over some time-interval, to the length of that time-interval, as the length of the time-interval approaches zero). The first-order SDE can be expressed as

$$\frac{d\boldsymbol{y}(t)}{dt} = \boldsymbol{A}(\boldsymbol{y}(t) - \boldsymbol{\mu}) + \boldsymbol{G}\frac{d\boldsymbol{W}(t)}{dt} \tag{2}$$

where $\boldsymbol{y}(t), \frac{d\boldsymbol{y}(t)}{dt}$ and $\boldsymbol{\mu}$ are $V \times 1$ column vectors described above, $\boldsymbol{y}(t) - \boldsymbol{\mu}$ represents the position as a deviation from the equilibrium, and the $V \times V$ matrix $\boldsymbol{A}$ represents the *drift* matrix relating $\frac{d\boldsymbol{y}(t)}{dt}$ to $(\boldsymbol{y}(t) - \boldsymbol{\mu})$. The diagonal elements of $\boldsymbol{A}$, relating the position in a certain dimension to the velocity in that same

dimension, are referred to as auto-effects while the off-diagonal elements are referred
to as cross-effects. The second part on the right-hand side of Equation (2)
represents the stochastic part of the model: $\boldsymbol{W}(t)$ denotes the so-called Wiener
process, broadly speaking a continuous-time analogue of a random walk. This
stochastic element has a variance-covariance matrix $\boldsymbol{GG'} = \boldsymbol{\Gamma}$, which is often
referred to as the *diffusion* matrix (for details see Voelkle et al., 2012).

The model representation in Equation (2) is referred to as the differential form
as it includes the derivative $\frac{d\boldsymbol{y}(t)}{dt}$. The same model can be represented in the
integral form, in which the derivatives are integrated out, sometimes referred to as
the solution of the derivative model. The integral form of this particular first-order
differential equation is known as the CT-VAR(1) or Ornstein-Uhlenbeck model
(Oravecz, Tuerlinckx, & Vandekerckhove, 2011). In this form, we can describe the
same system, but now in terms of the positions of the system (i.e., the values the
variables take on) at different points in time. For notational simplicity, we can
represent $\boldsymbol{y}(t) - \boldsymbol{\mu}$ as $\boldsymbol{y^c}(t)$, denoting the position of the process as a deviation from
its equilibrium.

The CT-VAR(1) model can be written as

$$\boldsymbol{y^c}(t) = \boldsymbol{e^{A\Delta t}}\boldsymbol{y^c}(t - \Delta t) + \boldsymbol{w}(\Delta t) \tag{3}$$

where $\boldsymbol{A}$ has the same meaning as above, the $V \times 1$ vector $\boldsymbol{y^c}(t - \Delta t)$ represents the
position as a deviation from equilibrium some time-interval Δt earlier, $\boldsymbol{e}$ represents
the matrix exponential function, and the $V \times 1$ column vector $\boldsymbol{w}(\Delta t)$ represents the
stochastic innovations, the integral form of the Wiener process in Equation (2).
These innovations are normally distributed with a variance-covariance matrix that
is a function of the time-interval between measurements Δt, the drift matrix $\boldsymbol{A}$, and

the diffusion matrix $\mathbf{\Gamma}$ (cf. Voelkle et al., 2012).[2] As the variables in the model have been centered around their equilibrium, we omit any intercept term. The relationship between lagged variables, that is, the relationships between the positions of the centered variables in the multivariate space, separated by some time-interval Δt, is an (exponential) function of the drift matrix $\boldsymbol{A}$ *and* the length of that time-interval.

Relating DT and CT models

It is clear from the integral form of the first-order SDE given in Equation (3) that the relationship between lagged values of variables is dependent on the length of the time-interval between these lagged values. As such, if the DT-VAR(1) model in Equation 1 is fitted to data generated by the CT model considered here, then the auto-regressive and cross lagged effects matrix $\mathbf{\Phi}$ will be a function of the time-interval Δt between the measurements. We denote this dependency by writing $\mathbf{\Phi}(\Delta t)$. This characteristic of the DT model has been referred to as the lag problem (Gollob & Reichardt, 1987; Reichardt, 2011).

The precise relationship between the CT-VAR(1) and DT-VAR(1) effects matrices is given by the well-known equality

$$\mathbf{\Phi}(\Delta t) = \boldsymbol{e}^{\boldsymbol{A}\Delta t}. \tag{4}$$

Despite this relatively simple relationship, it should be noted that taking the exponential of a matrix is not equivalent to taking the exponential of each of the elements of the matrix. That is, any lagged effect parameter $\phi_{ij}(\Delta t)$, relating variable i and variable j across time-points, is not only dependent on the corresponding CT cross-effect a_{ij}, but is a non-linear function of the interval and

every other element of the matrix $\boldsymbol{A}$. For example, in the bivariate case the DT cross-lagged effect of $y_1(t - \Delta t)$ on $y_2(t)$, denoted $\phi_{21}(\Delta t)$, is given by

$$\frac{a_{21}\left(e^{\frac{1}{2}\left(a_{11}+a_{22}+\sqrt{a_{11}^2+4a_{12}a_{21}-2a_{11}a_{22}+a_{22}^2}\right)\Delta t} - e^{\frac{1}{2}\left(a_{11}+a_{22}-\sqrt{a_{11}^2+4a_{12}a_{21}-2a_{11}a_{22}+a_{22}^2}\right)\Delta t}\right)}{\sqrt{a_{11}^2 + 4a_{12}a_{21} - 2a_{11}a_{22} + a_{22}^2}} \quad (5)$$

where e represents the scalar exponential. In higher dimensional models, these relationships quickly become intractable. For a derivation of Equation 5 we refer readers to Appendix A.

This complicated non-linear relationship between the elements of $\boldsymbol{\Phi}$ and the time-interval has major implications for applied researchers who wish to interpret the parameters of a DT-VAR(1) model in the substantive terms outlined above. In the general multivariate case, the size, sign, and relative strengths of both auto-regressive and cross-lagged effects may differ depending on the value of the time-interval used in data collection (Deboeck & Preacher, 2016; Dormann & Griffin, 2015; Oud, 2007; Reichardt, 2011). As such, conclusions that researchers draw regarding the stability of processes, and the nature of how different processes relate to one another may differ greatly depending on the time-interval used.

While the relationship in Equation 4 describes the DT-VAR(1) effects matrix we would find given data generated by a CT-VAR(1) model, the reader should note that not all DT-VAR(1) processes have a straightforward equivalent representation as a CT-VAR(1). For example, a univariate discrete-time AR(1) process with a negative auto-regressive parameter cannot be represented as a CT-AR(1) process; as the exponential function is always positive, there is no A that satisfies Equation 4 for $\Phi < 0$. As such, we can refer to DT-VAR(1) models with a CT-VAR(1) equivalent as those which exhibit 'positive autoregression'. We will focus throughout on the CT-VAR(1) as the data-generating model.[3]

Types of dynamics: Eigenvalues, stability and equilibrium

Both the DT-VAR(1) model and the CT-VAR(1) model can be used to describe a variety of different types of dynamic behaviour. As the dynamic behaviour of a system is always understood with regards to how the variables in the system move in relation to the equilibrium position, often dynamic behaviours are described by differentiating the type of equilibrium position or *fixed point* in the system (Strogatz, 2014). In the general multivariate case, we can understand these different types of dynamic behaviour or fixed points with respect to the *eigenvalues* of the effects matrices $\boldsymbol{A}$ or $\boldsymbol{\Phi}$ (see Appendix A for more information on for a more detailed explanation of the relationship between these two matrices and eigenvalues). In this chapter we will focus on *stable* processes, in which, given a perturbation, the system of interest will inevitably return to the equilibrium position. We limit our treatment to these types of processes, because we believe these are most common and most relevant for applied researchers. A brief description of other types of fixed points and how they relate to the eigenvalues of the effects matrix $\boldsymbol{A}$ is given in the discussion section - for a more complete taxonomy we refer readers to (Strogatz, 2014, p.136).

In DT settings, stable processes are those for which the absolute values of the eigenvalues of $\boldsymbol{\Phi}$ are smaller than one. In DT applications researchers also typically discuss the need for *stationarity*, that is, time-invariant mean and variance, as introduced above. Stability of a process ensures that stationarity in relation to the mean and variance hold. For CT-VAR(1) processes, stability is ensured if the real-parts of the eigenvalues of $\boldsymbol{A}$ are negative. It is interesting to note that the equilibrium position of stable processes can be related to our observed data in

various ways: In some applications $\boldsymbol{\mu}$ is constrained to be equal to the mean of the observed values (e.g., Hamaker et al., 2005; Hamaker & Grasman, 2015), while in others the equilibrium can be specified a-priori or estimated to be equal to some (asymptotic) value (e.g., Bisconti, Bergeman, & Boker, 2004).

We can further distinguish between dynamic processes that have real eigenvalues, complex eigenvalues, or in the case of systems with more than two variables, a mix of both. In the section "Making Sense of CT Models" we will focus on the interpretation of a CT-VAR(1) model with real, negative, non-equal eigenvalues. We can describe the equilibrium position of this system as a *stable node.* In the discussion section we examine another type of system which has been the focus of psychological research, sometimes described as a damped linear oscillator (e.g. Boker, Montpetit, Hunter, & Bergeman, 2010), in which the eigenvalues of $\boldsymbol{A}$ are complex, with a negative real part. The fixed point of such a system is described as a *stable spiral.* Further detail on the interpretation of these two types of systems is given in the corresponding sections.

Why researchers should adopt a CT process perspective

There are both practical and theoretical benefits to CT model estimation over DT model estimation. Here we will discus three of of these practical advantages which have received notable attention in the literature. We then discuss the fundamental conceptual benefits of treating psychological processes as continuous-time systems.

The first practical benefit to CT model estimation is that the CT model deals well with observations taken at unequal intervals, often the case in experience

sampling and ecological momentary assessment datasets (Oud & Jansen, 2000; Voelkle & Oud, 2013; Voelkle et al., 2012). Many studies use random intervals between measurements, for example to avoid participant-anticipation of measurement occassions, potentially resulting in unequal time-intervals both within and between participants. The DT model, however, is based on the assumption of equally spaced measurements, and as such estimating the DT model from unequally spaced data will result in an estimated $\mathbf{\Phi}$ matrix that is a blend of different $\mathbf{\Phi}(\Delta t)$ matrices for a range of values of Δt.

The second practical benefit of CT modeling over DT modeling is that, when measurements are equally spaced, the lagged effects estimated by the DT models are not generalizable beyond the time-interval used in data collection. Several different researchers have demonstrated that utilizing different time-intervals of measurement can lead researchers to reach very different conclusions regarding the values of parameters in $\mathbf{\Phi}$ (Oud & Jansen, 2000; Reichardt, 2011; Voelkle et al., 2012). The CT model has thus been promoted as facilitating better comparisons of results between studies, as the CT effects matrix $\boldsymbol{A}$ is independent of time-interval (assuming a sufficient frequency of measurement to capture the relevant dynamics).

Third, the application of CT models allows us to explore how cross-lagged effects are expected to change depending on the time-interval between measurements, using the relationship expressed in Equation (4). Some authors have used this relationship to identify the time-interval at which cross-lagged effects are expected to reach a maximum (Deboeck & Preacher, 2016; Dormann & Griffin, 2015). Such information could be used to decide upon the 'optimal' time-interval that should be used in gathering data in future research.

While these practical concerns regarding the use of DT models for CT processes are legitimate, there may be instances in which alternative practical solutions can be used, without necessitating the estimation of a CT model. For instance, the problem of unequally spaced measurements in DT modeling can be addressed by defining a time grid and adding missing data to your observations, to make the occasions approximately equally spaced in time. Some simulation studies indicate that this largely reduces the bias that results from using DT estimation of unequally spaced data (De Haan-Rietdijk, Voelkle, Keijsers, & Hamaker, 2017; Safarkhani, Vermunt, & Hamaker, in preperation).

Furthermore, the issue of comparability between studies that use different time-intervals can be solved, in certain circumstances, by a simple transformation of the estimated $\mathbf{\Phi}$ matrix, described in more detail by Kuiper and Ryan (under review). Given an estimate of $\mathbf{\Phi}(\Delta t)$ we can solve for the underlying $\mathbf{A}$ using Equation 4. This is known as the "indirect method" of CT model estimation (Oud, van Leeuwe, & Jansen, 1993). However this approach cannot be applied in all circumstances, as it involves using the matrix logarithm, the inverse of the matrix exponential function. As the matrix logarithm function in the general case does not give a unique solution, this method is only appropriate if both the estimated $\mathbf{\Phi}(\Delta t)$ and true underlying $\mathbf{A}$ matrices have real eigenvalues only (for further discussion of this issue see Hamerle, Nagl, & Singer, 1991).

However, the CT perspective has added value above and beyond the potential practical benefits discussed above. Multiple authors have argued that psychological phenomena, such as stress, affect and anxiety, do not vary in discrete steps over time, but likely vary and evolve in a continuous and smooth manner (Boker, 2002;

Gollob & Reichardt, 1987). Viewing psychological processes as CT dynamic systems has important implications for the way we conceptualise the influence of psychological variables on each other. Gollob and Reichardt (1987) give the example of a researcher who is interested in the effect of taking aspirin on headaches: This effect may be zero shortly after taking the painkiller, substantial an hour or so later, and near zero again after twenty-four hours. All of these results may be considered as accurately portraying the effect of painkillers on headaches *for a specific time-interval*, although each of these intervals considered separately represent only a snapshot of the process of interest.

It is only through examining the underlying dynamic trajectories, and exploring how the cross-lagged relationships evolve and vary as a function of the time-interval, that we can come to a more complete picture of the dynamic system of study. We believe that - while the practical benefits of CT modelling are substantial - the conceptual framework of viewing psychological variables as CT processes has the potential to transform longitudinal research in this field.

Making sense of CT models

In this section, we illustrate how researchers can evaluate psychological variables as dynamic CT processes by describing the interpretation of the drift matrix parameters $\boldsymbol{A}$. We describe multiple ways in which the dynamic behaviour of the model in general, as well as specific model parameters, can be understood. In order to aid researchers who are unfamiliar with this type of analysis, we take a broad approach in which we incorporate the different ways in which researchers interested in dynamical systems and similar models interpret their results. For

instance, Boker and colleagues (e.g., Boker, Montpetit, et al., 2010) typically interpret the differential form of the model directly; in the econometrics literature it is typical to plot specific trajectories using Impulse Response Functions (Johnston & DiNardo, 1972); in the physics tradition, the dynamics of the system are inspected using Vector Fields (e.g., Boker & McArdle, 1995); the work of Voelkle, Oud and others (e.g., Deboeck & Preacher, 2016; Voelkle et al., 2012) typically focuses on the integral form of the equation, and visually inspecting the time-interval dependency of lagged effects.

We will approach the interpretation of a single CT model from these four angles, and show how they each represent complimentary ways to understand the same system. For ease of interpretation we focus here on a bivariate system; the analysis of larger systems is addressed in the discussion section.

Substantive Example from Empirical Data

To illustrate the diverse ways in which the dynamics described by the CT-VAR(1) model can be understood, we make use of a substantive example. This example is based on our analysis of a publicly-available single-subject ESM dataset (Kossakowski, Groot, Haslbeck, Borsboom, & Wichers, 2017). The subject in question is a 57-year old male with a history of major depression. The data consists of momentary, daily and weekly items relating to affective states. The assessment period includes a double-blind phase in which the dosage of the participants anti-depression medication was reduced. We select only those measurements made in the initial phases of the study, before medication reduction; it is only during this period that we would expect the system of interest to be stable. The selected measurements consists of 286 momentary assessments over a period of 42

consecutive days. The modal time-interval between momentary assessments was 1.766 hours (inter-quartile range of 1.250 to 3.323).

For our analysis we selected two momentary assessment items, "I feel down" and "I am tired", which we will name Down (Do) and Tired (Ti), respectively. Feeling down is broadly related to assessments of negative affect (Meier & Robinson, 2004), and numerous cross-sectional analyses have suggested a relationship between negative affect and feelings of physical tiredness or fatigue (e.g., Denollet & De Vries, 2006). This dataset afforded us the opportunity to investigate the links between these two processes from a dynamic perspective. Each variable was standardized before the analysis to facilitate ease of interpretation of the parameter estimates. Positive values of Do indicate that the participant felt down more than average, negative values indicate below-average feelings of being down, and likewise for positive and negative values of Ti.

The analysis was conducted using the *ctsem* package in R (Driver et al., 2017). Full details of the analysis can be found in Appendix B. Parameter estimates and standard errors are given in Table 1, including estimates of the stochastic part of the CT model, represented by the diffusion matrix $\mathbf{\Gamma}$. The negative value of γ_{21} indicates that there is a negative co-variance between the stochastic input to the rates of change of Do and Ti; in terms of the CT-VAR(1) representation, there is a negative covariance between the residuals of Do and Ti in the same measurement occassion. Further interpretation of the diffusion matrix falls beyond the scope of the current chapter. As the analysis is meant as an illustrative example only, we will throughout interpret the estimated drift matrix parameter as though they are true population parameters.

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Insert Table 1 about here

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Interpreting the drift parameters

The drift matrix relating the processes Down ($Do(t)$) and Tired ($Ti(t)$) is given by

$$\boldsymbol{A} = \begin{bmatrix} -0.995 & 0.573 \\ 0.375 & -2.416 \end{bmatrix}. \tag{6}$$

As the variables are standardized, the equilibrium position is $\boldsymbol{\mu} = [0,0]$ (i.e., $E[Do(t)] = E[Ti(t)] = 0$). The drift matrix $\boldsymbol{A}$ describes how the position of the system at any particular time t (i.e., $Do(t)$ and $Ti(t)$) relates to the *velocity* or *rate of change* of the process, that is, how the position of the process is changing. The system of equations which describe the dynamic system made up of Down and Tired is given by

$$\begin{bmatrix} E[\frac{dDo(t)}{dt}] \\ E[\frac{dTi(t)}{dt}] \end{bmatrix} = \begin{bmatrix} -0.995 & 0.573 \\ 0.375 & -2.416 \end{bmatrix} \begin{bmatrix} Do(t) \\ Ti(t) \end{bmatrix} \tag{7}$$

such that

$$E[\frac{dDo(t)}{dt}] = -0.995 Do(t) + 0.573 Ti(t) \tag{8}$$

$$E[\frac{dTi(t)}{dt}] = 0.375 Do(t) - 2.416 Ti(t) \tag{9}$$

where the rates of change of Down and Tired at any point in time are both dependent on the positions of both Down and Tired at that time.

Before interpreting any particular parameter in the drift matrix, we can determine the type of dynamic process under consideration by inspecting the eigenvalues of $\boldsymbol{A}$. The eigenvalues of $\boldsymbol{A}$ are $\lambda_1 = -2.554$ and $\lambda_2 = -0.857$; since both eigenvalues are negative, the process under consideration is stable. This means that if the system takes on a position away from equilibrium (e.g., due to a random shock from the stochastic part of the model on either Down or Tired), the system will inevitably return to its equilibrium position over time. It is for this reason that the equilibrium position or fixed point in stable systems is also described as the *attractor point*, and stable systems are described as *equilibrium-reverting*. As the eigenvalues of the system are real-valued as well as negative, the system returns to equilibrium with an *exponential decay*; when the process is far away from the equilibrium, it takes on a greater velocity, that is, moves faster towards equilibrium. We can refer to the type of fixed point in this system as a *stable node* (Strogatz, 2014).

Typical of such an equilibrium-reverting process, we see negative CT auto-effects $a_{11} = -0.995$ and $a_{22} = -2.416$. This reflects that, if either variable in the system takes on a position away from the equilibrium, they will take on a velocity of opposite sign to this deviation, that is, a velocity which returns the process to equilibrium. For higher values of $Do(t)$, the rate of change of $Do(t)$ is of greater (negative) magnitude, that is, the velocity towards the equilibrium is higher. In addition, the auto-effect of $Ti(t)$ is more than twice as strong (in an absolute sense) as the auto-effect of $Do(t)$. If there were no cross-effects present, this would imply that $Ti(t)$ returns to equilibrium faster than $Do(t)$; however, as there are cross-effects present, such statements cannot be made in the general case from

inspecting the auto-effects alone.

In this case the cross-effects of $Do(t)$ and $Ti(t)$ on each others rates of change are positive rather than negative. Moreover, the cross-effect of $Ti(t)$ on the rate of change of $Do(t)$ ($a_{12} = 0.573$) is slightly stronger than the corresponding cross-effect of $Do(t)$ on the rate of change of $Ti(t)$ ($a_{21} = 0.375$). These cross-effects quantify the force that each component of the system exerts on the other. However, depending on what values each variable takes on at a particular point in time t, that is, the position of the system in each of the $Do(t)$ and $Ti(t)$ dimensions, this may translate to $Do(t)$ pushing $Ti(t)$ to return faster to its equilibrium or to deviate away from its equilibrium position, and vice versa. To better understand both the cross-effects and auto-effects described by $\boldsymbol{A}$, it is helpful to visualise the possible trajectories of our two-dimensional system.

Visualising Trajectories

We will now describe and apply two related tools which allow us to visualise the trajectories of the variables in our model over time: Impulse Response Functions and Vector Fields. These tools can help us to understand the dynamic system we are studying, by exploring the dynamic behaviour which results from the drift matrix parameters.

Impulse Response Functions. Impulse Response Functions (IRFs) are typically used in the econometrics literature to aid in making forecasts based on a DT-VAR model. The idea behind this is to allow us to explore how an impulse to one variable in the model at occasion τ will affect the values of both itself and the other variables in the model at occasions $\tau + 1$, $\tau + 2$, $\tau + 3$ and so on. In the stochastic

systems we focus on in this chapter, we can conceptualise these impulses as random perturbations or innovations, or alternatively as external interventions in the system.[4] IRFs thus represent the trajectories of the variables in the model over time, following a particular impulse, assuming no further stochastic innovations (see Johnston & DiNardo, 1972, Chapter 9).

To specify impulses in an IRF, we generally assign a value to a single variable in the system at some initial occasion, $y_{i,\tau}$. The corresponding values of the other variables at the initial occasion $y_{j,\tau}$, $j \neq i$ are usually calculated based on, for instance, the covariance in the stochastic innovations, $\boldsymbol{\Psi}$, or the stable covariance between the processes $\boldsymbol{\Sigma}$. Such an approach is beneficial in at least two ways: first, it allows researchers to specify impulses which are more likely to occur in an observed dataset; second, it aids researchers in making more accurate future predictions or forecasts. For a further discussion of this issue in relation to DT-VAR models, we refer the reader to Johnston and DiNardo (1972) pages 298-300. Below, we will take a simplified approach and specify bi-variate impulses at substantively interesting values.

The IRF can easily be extended for use with the CT-VAR(1) model. We can calculate the impulse response of our system by taking the integral form of the CT-VAR(1) model in Equation (3) and a) plugging in the $\boldsymbol{A}$ matrix for our system, b) choosing some substantively interesting set of impulses $\boldsymbol{y}(t = 0)$ and c) calculating $\boldsymbol{y}(t)$ for increasing values of $t > 0$. To illustrate this procedure, we will specify four substantively interesting sets of impulses. The four sets of impulses shown here include $\boldsymbol{y}(0) = [1, 0]$, reflecting what happens when $Do(0)$ takes on a positive value 1 standard deviation above the persons mean, while $Ti(0)$ is at

equilibrium; $\boldsymbol{y}(0) = [0, 1]$ reflecting when $Ti(0)$ takes on a positive value of

corresponding size while $Do(0)$ is at equilibrium; $\boldsymbol{y}(0) = [1, 1]$ reflecting what

happens when $Do(0)$ and $Ti(0)$ both take on values 1 standard deviation above the

mean; and $\boldsymbol{y}(0) = [1, -1]$ reflecting what happens when $Do(0)$ and $Ti(0)$ take on

values of equal magnitude but opposite valence (1SD more and 1SD less than the

mean respectively). Figures 1(a) to 1(d) contain the IRFs for both processes in each

of these four scenarios.

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Insert Figure 1 about here

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Examining the IRFs shows us the equilibrium-reverting behaviour of the

system: Given any set of starting values, the process eventually returns, in an

exponential fashion, to the bivariate equilibrium position where both processes take

on a value of zero.

In Figure 1(a), we can see that when $Ti(t)$ is at equilibrium and $Do(0)$ takes

on a value of plus one, then $Ti(t)$ is pushed away from equilibrium in the same (i.e.,

positive) direction. In substantive terms, when our participant is feeling down at a

particular moment, he begins to feel a little tired. Eventually, both $Do(t)$ and $Ti(t)$

return to equilibrium due to their negative auto-effects. The feelings of being down

and tired have returned to normal around $t = 4$, that is, four hours after the initial

impulse; stronger impulses ($|Do(0)| > 1$) will result in the system taking longer to

return to equilibrium, and weaker impulses ($|Do(0)| < 1$) would dissipate quicker.

Figure 1(b) shows the corresponding reaction of $Do(t)$ at equilibrium to a

positive value of $Ti(0)$. We can further see that the deviation of $Do(t)$ in

Figure 1(b) is greater than the deviation of $Ti(t)$ in Figure 1(a): a positive value of $Ti(t)$ exerts a greater push on $Do(t)$ than vice versa, because of the greater cross-effect of $Ti(t)$ on $Do(t)$. In this case this strong cross-effect, combined with the relatively weaker auto-effect of $Do(t)$, results in $Do(t)$ taking on a higher value than $Ti(t)$ at around $t = 1$, one hour after the initial impulse. Substantively, when our participant is feeling physically tired at a particular moment (Figure 1(b)), he begins to feel a down over the next couple of hours, before eventually these feelings return to normal (again in this case, around 4 hours later).

Figures 1(c) further demonstrates the role of the negative auto-effects and positive cross-effects in different scenarios. In Figure 1(c), both processes take on positive values at $t = 0$; the positive cross-effects result in both processes returning to equilibrium at a slower rate than in Figures 1(a) and 1(b). In substantive terms this means that, when the participant is feeling very down, and very tired, it takes longer for the participant to return to feeling normal. Here also the stronger auto-effect of $Ti(t)$ than $Do(t)$ is evident: although both processes start at the same value, an hour later $Ti(1)$ is much closer to zero than $Do(1)$, that is, $Ti(t)$ decays faster to equilibrium than $Do(t)$. In substantive terms, this tells us that when the participant is feeling down and physically tired, he recovers much quicker from the physical tiredness than he does from feeling down.

In Figure 1(d), we see that $Do(0)$ and $Ti(0)$ taking on values of opposite signs results in a speeding-up of the rate at which each variable decays to equilibrium. The auto-effect of $Do(t)$ is negative, which is added to by the positive cross-effect of $Ti(t)$ multiplied by the negative value of $Ti(0)$. This means that $Do(0)$ in Figure 1(d) takes on a stronger negative velocity, in comparison to Figures 1(a) or 1(c). A

positive value for $Do(0)$ has a corresponding effect of making $Ti(0)$ take on an even stronger positive velocity. Substantively, this means that when the participant feels down, but feels less tired (i.e. more energetic) than usual, both of these feelings wear off and return to normal quicker than in the other scenarios we examined. The stronger auto-effect of $Ti(t)$, in combination with the positive cross-effect of $Do(t)$ on $Ti(t)$, actually results in $Ti(t)$ shooting past the equilibrium position in the $Ti(t)$ dimension $(Ti(t) = 0)$ and taking on positive values around $t = 1.5$, before the system as a whole returns to equilibrium. Substantively, when the participant is feeling initially down but quite energetic, we expect that he feels a little bit more tired than usual about an hour and half later, before both feelings return to normal.

Vector Field. Vector fields are another technique which can be used to visualise the dynamic behaviour of the system by showing potential trajectories through a bivariate space. In our case the two axes of this space are $Do(t)$ and $Ti(t)$. The advantage of vector fields over IRFs in this context is that in one plot it shows how, for a range of possible starting positions, the process is expected to move in the (bivariate) space a moment later. For this reason, the vector field is particularly useful in bivariate models with complex dynamics, in which it may be difficult to obtain the full picture of the dynamic system from a few IRFs alone. Furthermore, by showing the dynamics for a grid of values, we can identify areas in which the movement of the process is similar or differs.

To create a vector field, $E[\frac{d\boldsymbol{y}(t)}{dt}]$ is calculated for a grid of possible values for $y_1(t)$ and $y_2(t)$ covering the full range of the values both variables can take on. The vector field for $Do(t)$ and $Ti(t)$ is shown in Figure 2. The base of each arrow represents a potential position of the process $\boldsymbol{y}(t)$. The head of the arrow represents

where the process will be if we take one small step in time forward, that is the value

of $\boldsymbol{y}(t + \Delta t)$ as Δt approaches zero. In other words, the arrows in this vector field

represent the information of two derivatives, $dDo(t)/dt$ and $dTi(t)/dt$. Specifically,

the direction the arrow is pointing is a function of the sign (positive or negative) of

the derivatives, while the length of the arrow represents the magnitude of this

movement, and is a function of the absolute values of the derivative(s).

Insert Figure 2 about here

If an arrow in the vector field is completely vertical, this means that, for that

position, taking one small step forward in time would result in a change in the

system's position along the $Ti(t)$ axis (i.e., a change in the value of Tired), but not

along the $Do(t)$ axis (that is, $dDo(t)/dt = 0$ and $dTi(t)/dt \neq 0$). The converse is

true for a horizontal arrow (that is, $dDo(t)/dt \neq 0$ and $dTi(t)/dt = 0$). The two

lines in Figure 2, blue and red, identify at which positions $dDo(t)/dt = 0$ and

$dTi(t)/dt = 0$, respectively; these are often referred to as *nullclines*. If the nullclines

are not perfectly perpendicular to one another, this is due to the presence of at least

one cross-effect. The point at which these nullclines cross represents the equilibrium

position in this two-dimensional space, here located at $Do(t) = 0, Ti(t) = 0$. The

crossing of these nullclines splits the vector field in four quadrants, each of which is

characterized by a different combination of negative and positive values for

$dDo(t)/dt$ and $dTi(t)/dt$. The top left and bottom right quadrants represent areas

in which the derivatives are of opposite sign, $dDo(t)/dt > 0$ & $dTi(t)/dt < 0$ and

$dDo(t)/dt < 0$ & $dTi(t)/dt > 0$, respectively. The top right and bottom left

quadrants represent areas where the derivatives are of the same sign, $dDo(t)/dt < 0$ & $dTi(t) < 0$ and $dDo(t)/dt > 0$ & $dTi(t) > 0$, respectively.

By tracing a path through the arrows, we can see the trajectory of the system of interest from any point in the possible space of values. In Figure 2, we include the same four bi-variate trajectories as we examined with the IRFs. Instead of the IRF representation of two variables whose values are changing, the vector field represents this as the movement of one process in a two-dimensional space. For instance, the trajectory starting at $Do(t) = 0$ and $Ti(t) = 1$ begins in the top-left quadrant, where $dDo(t)/dt$ is positive and $dTi(t)/dt$ is negative; this implies that the value of Down will increase, and the value of Tired will decrease (as can be seen in Figure 1(b)). Instead of moving directly to the equilibrium along the $Ti(t)$ dimension, the system moves away from equilibrium along the $Do(t)$ dimension, due to the cross-effect of $Ti(t)$ on $Do(t)$, until it moves into the top-right quadrant. In this quadrant, $dDo(t)/dt$ and $dTi(t)/dt$ are both negative; once in this quadrant the process moves towards equilibrium, tangent to the $dDo(t)/dt$ nullcline. The other trajectories in Figure 2 analogously describe the same trajectories as in Figure 1(b), 1(c) and 1(d).

In general, the trajectories in this vector field first decay quickest along the $Ti(t)$ dimension, and slowest along the $Do(t)$ dimension. This can be clearly seen in trajectories b), c), and d). Each of these trajectories first change steeply in the $Ti(t)$ dimension, before moving to equilibrium at a tangent to the red ($\frac{dDo(t)}{dt}$) nullcline. This general property of the bi-dimensional system is again related to the much stronger auto-effect of $Ti(t)$, and the relatively small cross-effects. In a technical sense we can say that that $Do(t)$ represents the 'slowest eigen-direction' (Strogatz, 2014, Chapter 5).

Inspecting the lagged parameters

Another way to gain insight into the processes of interest is by determining the relationships between lagged positions of the system, according to our drift matrix. To this end, we can use Equation (4) to determine $\mathbf{\Phi}(\Delta t)$ for some Δt. For instance, we can see that the auto-regressive and cross-lagged relationships between values of Competence and Exhaustion given $\Delta t = 1$ are

$$\mathbf{\Phi}(\Delta t = 1) = \begin{bmatrix} 0.396 & 0.117 \\ 0.077 & 0.106 \end{bmatrix}. \tag{10}$$

For this given time-interval, the cross-lagged effect of Down on Tired $(\phi_{21}(\Delta t = 1) = 0.077)$ is smaller than the cross-lagged effect of Down on Tired $(\phi_{12}(\Delta t = 1) = 0.117)$. However, as shown in Equation (5) the value of each of these lagged effects changes in a non-linear way depending on the time-interval chosen. To visualise this, we can calculate $\mathbf{\Phi}(\Delta t)$ for a range of Δt, and represent this information graphically in a lagged parameter plot, as in Figure 3.

Insert Figure 3 about here

From Figure 3, we can see that both cross-lagged effects reach their maximum (and have their maximum difference) at a time-interval of $\Delta t = 0.65$; furthermore, we can see that the greater cross-effect (a_{12}) results in a stronger cross-lagged effect $\phi_{12}(\Delta t)$ for a range of Δt. Moreover, we can visually inspect how the size of each of the effects of interest, as well as the difference between these effects, varies according to the time-interval. From a substantive viewpoint, we could say that the

effect of feeling physically tired has the strongest effect on feelings of being down around 40 minutes later.

While the shape of the lagged parameters may appear similar to the shapes of the trajectories plotted in the IRFs, lagged parameter plots and IRFs represent substantively different information. IRFs plot the positions of each variable in the system as they change over time, given some impulse ($\boldsymbol{y}(t)$ vs t given some $\boldsymbol{y}(0)$). In contrast, lagged parameter plots show how the lagged relationships change depending on the length of the time-interval between them, independent of impulse values ($e^{\boldsymbol{A}\Delta t}$ vs Δt). The lagged relationships can be thought of as the components which go into determining any specific trajectories.

Caution with interpreting estimated parameters

It is important to note that in the above interpretation of CT models, we have treated the matrix $\boldsymbol{A}$ as known. In practice of course researchers should take account of the uncertainty in parameter estimates. For example, the *ctsem* package also allows provides lagged parameter plots with credible intervals to account for this uncertainty.

Furthermore, researchers should be cautious about extrapolating beyond the data. For instance, when we consider a vector field, we should be careful about interpreting regions in which there is little or no observed data (cf. Boker & McArdle, 1995). The same logic applies for the interpretation of IRFs for impulses that do not match observed values. Moreover, we should also be aware that interpreting lagged parameter plots for time-intervals much shorter than those we observe data at is a form of extrapolation: It relies on strong model-based assumptions, such as ruling out the possibility of a high-frequency higher-order

process (Voelkle & Oud, 2013; Voelkle et al., 2012).

Discussion

In this chapter we have set out to clarify the connection between DT- and CT-VAR(1) models, and how we can interpret and represent the results from these models. So far we have focused on single-subject, two-dimensional, first-order systems with a stable node equilibrium. However, there are many ways in which these models can be extended, to match more complicated data and/or dynamic behaviour. Below we consider three such extensions: a) systems with more than two dimensions (i.e., the number of variables); b) different types of fixed points resulting from non-real eigenvalues of the drift matrix; and c) moving from single-subject to multi-level datasets.

Beyond two-dimensional systems

In the empirical illustration, we examined the interpretation of a drift matrix in the context of a bivariate CT-VAR(1) model. Notably, the current trend in applications of DT-VAR(1) models in psychology has been to focus more and more on the analysis of large systems of variables, as typified, for example, by the dynamic network approach of Bringmann et al. (2016, 2013). The complexity of these models grows rapidly as the number of variables is added: To estimate a full drift matrix for a system of three variables, we must estimate nine unique parameters, in contrast to four drift matrix parameters for a bivariate system. In addition, we must estimate a three-by-three covariance matrix for the residuals, rather than a two-by-two matrix.

The relationship between the elements of $\boldsymbol{A}$ and $\boldsymbol{\Phi}(\Delta t)$ becomes even less intuitive once the interest is in a system of three variables, because the lagged

parameter values are dependent on the drift matrix as a whole, as explained earlier. This means that both the relative sizes, as well as the signs of the cross-lagged effects may differ depending on the interval: The same lagged parameter may be negative for some time-intervals and positive for others, and zero-elements of $\boldsymbol{A}$ can result in corresponding non-zero elements of $\boldsymbol{\Phi}$ (cf. Aalen, Gran, Røysland, Stensrud, & Strohmaier, 2017; Aalen, Røysland, Gran, Kouyos, & Lange, 2016; Aalen, Røysland, Gran, & Ledergerber, 2012; Deboeck & Preacher, 2016). Therefore, although we saw in our bivariate example that, for instance, negative CT cross-effects resulted in negative DT cross-lagged effects, this does not necessarily hold in the general case (Kuiper & Ryan, under review).

Additionally, substantive interpretation of the lagged parameters in systems with more than two variables also becomes less straightforward. For example, Deboeck and Preacher (2016), Aalen et al. (2016, 2012) and Aalen et al. (2017) argue that the interpretation of $\boldsymbol{\Phi}(\Delta t)$ parameters in mediation models (with three variables and a triangular $\boldsymbol{A}$ matrix) as direct effects may be misleading: Deboeck and Preacher argue that instead they should be interpreted as *total* effects. This has major consequences for the practice of DT analyses and the interpretation of its results.

Complex and Positive Eigenvalues

The empirical illustration is characterized by a system with negative, real, non-equal eigenvalues, which implies that the fixed point in the system is a *stable node*. In theory, however, there is no reason that psychological processes must adhere to this type of dynamic behaviour. We can apply the tools we have defined already to understand the types of behaviour that might be described by other

types of drift matrices. Notably, some systems may have drift matrices with complex eigenvalues, that is, eigenvalues of the form $\alpha \pm \omega i$, where $i = \sqrt{-1}$ is the imaginary number, α is referred to as the real part, and ωi as the imaginary part of the eigenvalue. If the real component of these eigenvalues is negative ($\alpha < 0$), then the system is still stable, and given a deviation it will return eventually to a resting state at equilibrium. However, unlike the systems we have described before, these types of systems spiral or oscillate around the equilibrium point, before eventually coming to rest. Such systems have been described as *stable spirals*, or alternatively as *damped* (linear or harmonic) *oscillators* (Boker, Montpetit, et al., 2010; Voelkle & Oud, 2013).

A vector field for a process which exhibits this type of stable spiral behaviour is shown in Figure 4, with accompanying trajectories. The drift matrix corresponding to this vector field is

$$\boldsymbol{A} = \begin{bmatrix} -0.995 & 0.573 \\ -2.000 & -2.416 \end{bmatrix} \tag{11}$$

which is equivalent to our empirical example above, but with the value of a_{21} altered from 0.375 to -2.000. The eigenvalues of this matrix are $\lambda_1 = -1.706 + 0.800i$ and $\lambda_2 = -1.706 - 0.800i$. In contrast to our empirical example, we can see that the trajectories follow a spiral pattern; the trajectory which starts at $y_1(t) = 1, y_2(t) = 1$ actually overshoots the equilibrium in the $Ti(t)$ dimension before spiraling back once in the bottom quadrant. There are numerous examples of psychological systems that are modeled as damped linear oscillators using second-order differential equations, which include the first- and second-order derivatives (cf., Bisconti et al., 2004; Boker, Montpetit, et al., 2010; Boker & Nesselroade, 2002;

Horn, Strachan, & Turkheimer, 2015). However, as shown here, such behaviour may also result from a first-order model.

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Insert Figure 4 about here

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Stable nodes and spirals can be considered the two major types of stable fixed points, as they occur whenever the real part of the eigenvalues of $\boldsymbol{A}$ are negative, that is $\alpha < 0$. Many other types of stable fixed points can be considered as special cases: when we have real, negative eigenvalues that are exactly equal, the fixed point is called a *stable star node* (if the eigenvectors are distinct), or a *stable degenerate node* (if the eigenvalues are not distinct). In contrast, if the real-part of the eigenvalues of $\boldsymbol{A}$ are positive then the system is unstable (also referred to as non-stationary or a unit-root in the time series literature Hamilton, 1994). This implies that, given a deviation, the system will not return to equilibrium; in contrast to stable systems, in which trajectories are *attracted* to the fixed point, the trajectories of unstable systems are *repelled* by the fixed point. As such we can also encounter unstable nodes, spirals, star nodes and degenerate nodes. The estimation and interpretation of unstable systems in psychology may be fruitful ground for further research.

Two further types of fixed points may be of interest to researchers; in the special case where the eigenvalues of $\boldsymbol{A}$ have an imaginary part and no real part ($\alpha = 0$), the fixed point is called a *center*. In a system with a center fixed point, trajectories spiral around the fixed point without ever reaching it. Such systems exhibit oscillating behaviour, but without any damping of oscillations; certain

biological systems, such as the circadian rhythm, can be modelled as a dynamic system with a center fixed point. Such systems are on the borderline between stable and unstable systems, sometimes referred to as *neutrally stable*; trajectories are neither attracted to or repelled by the fixed point. Finally, a *saddle point* occurs when the eigenvalues of $\boldsymbol{A}$ are real but of opposite sign (one negative, one positive). Saddle points have one stable and one unstable component; only trajectories which start exactly on the stable axis return to equilibrium, and all others do not. Together spirals, nodes and saddle points cover the majority of the space of possible eigenvalues for $\boldsymbol{A}$. Strogatz (2014) describes the different dynamic behaviour generated by different combinations of eigenvalues of $\boldsymbol{A}$ in greater detail.

Multilevel extensions

The time series literature (such as from the field of econometrics) as well as the dynamic systems literature (such as from the field of physics) tends to be concerned with a single dynamic system, either because there is only one case ($N = 1$), or because all cases are exact replicates (e.g., molecules). In psychology however, we typically have data from more than one person, and we also know that people tend to be highly different. Hence, when we are interested in modeling their longitudinal data, we should take their differences into account somehow. The degree to which this can be done, depends on the number of time points we have per person. In traditional panel data, we typically have between two and six waves of data. In this case, we should allow for individual differences in means or intercepts, in order to separate the between-person, stable differences from the within-person dynamic process, while assuming the lagged relationships are the same across individuals (cf. Hamaker, Kuiper, & Grasman, 2015).

In contrast, experience sampling data and other forms of intensive longitudinal data consist of many repeated measurement per person, such that we can allow for individual differences in the lagged coefficients. This can be done by either analyzing the data of each person separately, or by using a dynamic multilevel model in which the individuals are allowed to have different parameters (cf. Boker, Staples, & Hu, 2016; Driver et al., 2017). Individual differences in dynamics has been the focus of many recent studies, which have shown that there are substantial individual differences in the dynamics, and that these differences can be meaningfully related to other person characteristics, such as personality traits, gender, age, and depressive symptomatology, but also to later health outcomes and psychological well-being (e.g., Bringmann et al., 2013; Kuppens et al., 2010, 2012).

While the current chapter has focused on elucidating the interpretation of a single-subject CT-VAR(1) model, the substantive interpretations and visualisation tools we describe here can be applied in a straightforward manner to, for example, the fixed effects estimated in a multilevel CT-VAR(1) model, or to individual-specific parameters estimated in a multilevel framework. The latter would however, lead to an overwhelming amount of visual information. The development of new ways of summarizing the individual differences in dynamics, based on the current tools, is a promising area.

Conclusion

There is no doubt that the development of dynamical systems modeling in the field of psychology has been hampered by the difficulty in obtaining suitable data to model such systems. However this is a barrier that recent advances in technology will shatter in the coming years. Along with this new source of psychological data,

new psychological theories are beginning to emerge, based on the notion of psychological processes as dynamic systems. Although the statistical models needed to investigate these theories may seem exotic or difficult to interpret at first, they reflect the simple intuitive and empirical notions we have about psychological processes: Human behaviour, emotion and cognition fluctuate continuously over time, and the models we use should reflect that. We hope that our treatment of CT-VAR(1) models and their interpretation will help researchers to overcome the knowledge-barrier to this approach, and can serve as a stepping stone towards a broader adaptation of the CT dynamical system approach to psychology.

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Appendix A

Matrix Exponential

Similarly to the scalar exponential, the matrix exponential can be defined as an infinite sum

$$e^{\boldsymbol{A}} = \sum_{k=0}^{\infty} \frac{1}{k!} \boldsymbol{A}^k$$

The exponential of a matrix is not equivalent to taking the scalar exponential of each element of the matrix, unless that matrix is diagonal. The exponential of a matrix can be found using an eigenvalue decomposition

$$\boldsymbol{A} = \boldsymbol{V}\boldsymbol{D}\boldsymbol{V}^{-1}$$

where $\boldsymbol{V}$ is a matrix of eigenvectors of $\boldsymbol{A}$ and $\boldsymbol{D}$ is a diagonal matrix of the eigenvalues of $\boldsymbol{A}$ (cf. Moler & Van Loan, 2003). The matrix exponential of $\boldsymbol{A}$ is given by

$$e^{\boldsymbol{A}} = \boldsymbol{V}e^{\boldsymbol{D}}\boldsymbol{V}^{-1}$$

where $e^{\boldsymbol{D}}$ is the diagonal matrix whose entries are the scalar exponential of the eigenvalues of $\boldsymbol{A}$. When we want to solve for the matrix exponential of a matrix multiplied by some constant Δt we get

$$e^{\boldsymbol{A}\Delta t} = \boldsymbol{V}e^{\boldsymbol{D}\Delta t}\boldsymbol{V}^{-1} \tag{12}$$

Take it that we have a 2×2 square matrix given by

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and we wish to solve for $e^{A\Delta t}$. The eigenvalues of A are given by

$$\lambda_1 = \frac{1}{2}(a + d - \sqrt{a^2 + 4bc - 2ad + d^2})$$

$$\lambda_2 = \frac{1}{2}(a + d + \sqrt{a^2 + 4bc - 2ad + d^2})$$

where we will from here on denote

$$R = \sqrt{a^2 + 4bc - 2ad + d^2}$$

for notational simplicity. The exponential of the diagonal matrix made up of eigenvalues, multiplied by the constant Δt is given by

$$e^{D\Delta t} = \begin{bmatrix} e^{\frac{1}{2}(a+d-R)\Delta t} & 0 \\ 0 & e^{\frac{1}{2}(a+d+R)\Delta t} \end{bmatrix}$$

The matrix of eigenvectors of A is given by

$$V = \begin{bmatrix} \frac{a-d-R}{2c} & \frac{a-d+R}{2c} \\ 1 & 1 \end{bmatrix}$$

assuming $c \neq 0$, with inverse

$$V^{-1} = \begin{bmatrix} \frac{-c}{R} & \frac{a-d+R}{2R} \\ \frac{c}{R} & \frac{-a+d+R}{2R} \end{bmatrix}.$$

Multiplying $V e^D V^{-1}$ gives us

$$e^{A\Delta t} = \begin{bmatrix} \frac{R-a+d}{2R}e^{\lambda_1\Delta t} + \frac{R+a-d}{2R}e^{\lambda_2\Delta t} & \frac{b(-e^{\lambda_1\Delta t}+e^{\lambda_2\Delta t})}{R} \\ \frac{c(-e^{\lambda_1\Delta t}+e^{\lambda_2\Delta t})}{R} & \frac{R+a-d}{2R}e^{\lambda_1\Delta t} + \frac{R-a+d}{2R}e^{\lambda_2\Delta t} \end{bmatrix} \tag{13}$$

Note that we present here only a worked out example for a 2×2 square matrix. For larger square matrices (representing models with more variables), the eigenvalue decomposition remains the same although the terms for the eigenvalues, eigenvectors and determinants become much less feasible to present.

Appendix B

Empirical Example Data Analysis

```
## Data Preperation

# Load the ctsem package

library(ctsem)


# Data is available form https://osf.io/c6xt4/ #

setwd("./ESMdata")


#Load Data#

rawdata<-read.csv("ESMdata.csv",header=TRUE, stringsAsFactors = FALSE)


#Select only measruements which take place in the control and initial (no medication

rawdata<-subset(rawdata,rawdata$phase==1|rawdata$phase==2)


#Select only the variables of interest

sel<-c("mood_down","phy_tired")

data<-rawdata[,(names(rawdata) %in% sel)]


# Standardise the selected variables

for(j in 1:dim(data)[2]){

  data[,j]<-(data[,j]-mean(data[,j]))/sd(data[,j])

}
```

```
# Create a time vector which represents hours since first measurement

# Required by ctsem function ctIntervalise

t1<-as.POSIXct(paste(rawdata$date,rawdata$resptime_s),

format="%d/%m/%y %H:%M:%S")

time<-as.numeric(difftime(t1,t1[1], units="hours"))


# Attach this time variable to the selected items

data$time=time


# Create an ID variable

data$id=rep(1,dim(data)[1])


# Rename pat_agitate = Y1, and event_import= Y2 for use with ctsem

colnames(data)=c("Y1","Y2","time","id")

# Get data in wide format for ctsem

datawide<-ctLongToWide(datalong=data,id="id",

time="time",manifestNames=c("Y1","Y2"))


# Create time-interval variable

datawide<-ctIntervalise(datawide=datawide,Tpoints=dim(data)[1],

                        n.manifest=2,manifestNames=c("Y1","Y2"))
```

Data analysis

```
# First specify the bivariate model, with 2 observed variables
model <- ctModel(n.manifest = 2, n.latent= 2, Tpoints = 286,

                LAMBDA = diag(nrow=2),

                MANIFESTMEANS = matrix(data=0, nrow=2, ncol=1),

                MANIFESTVAR = matrix(data=0, nrow=2, ncol=2),

                DRIFT = "auto",

                CINT = matrix(data=0, nrow=2, ncol=1),

                DIFFUSION = "auto",

                TRAITVAR = NULL,

                MANIFESTTRAITVAR = NULL,

                startValues = NULL)


# Fit the model to the data using carefulFit to get initial values
fit <- ctFit(data= datawide, ctmodelobj = model, objective = "Kalman",

             stationary = c("T0VAR", "T0MEANS"),

           iterationSummary = T, carefulFit = T, showInits = F, asymptotes = F,

            meanIntervals = F,

            plotOptimization = F, nofit = F, discreteTime = F,

            verbose = 0)


summary(fit)
```

Footnotes

[1]The variance-covariance matrix of the variables $\boldsymbol{\Sigma}$ is a function of both the lagged parameters and the variance-covariance matrix of the innovations, $vec(\boldsymbol{\Sigma}) = (\boldsymbol{I} - \boldsymbol{\Phi} \otimes \boldsymbol{\Phi})^{-1} vec(\boldsymbol{\Psi})$, where $vec(\boldsymbol{.})$ denotes the operation of putting the elements of an $N \times N$ matrix into an $NN \times 1$ column matrix (Kim & Nelson, 1999, p.27)

[2]Readers should note that there are multiple different possible ways to parameterize the CT stochastic process in integral form, and also multiple different notations used (e.g., Oravecz et al., 2011; Voelkle et al., 2012).

[3]In general, there is no straightforward CT-VAR(1) representation of DT-VAR(1) models with real, negative eigenvalues. However it may be possible to specify more complex continuous-time models which do not exhibit positive autoregression. Notably, Fisher (2001) demonstrates how a DT-AR(1) model with negative autoregressive parameter can be modeled with the use of two continuous-time (so-called) Ito processes.

[4]Similar functions can be used for deterministic systems (those without a random innovation part), however in these cases the term *initial value* is more typically used.

Table 1

Parameter estimates from substantive example

Parameter	Value	Std. Error
a_{11}	-0.995	0.250
a_{21}	0.375	0.441
a_{12}	0.573	0.595
a_{22}	-2.416	1.132
γ_{11}	1.734	0.612
γ_{21}	-0.016	0.650
γ_{22}	4.606	1.374

Figure Captions

Figure 1. Impulse response function for the model in Equation 9 for four different sets of impulses; red solid line $= Do(t)$ and blue dashed line $= Ti(t)$.

Figure (a). $Do(0) = 1, Ti(0) = 0$

Figure (b). $Do(0) = 0, Ti(0) = 1$

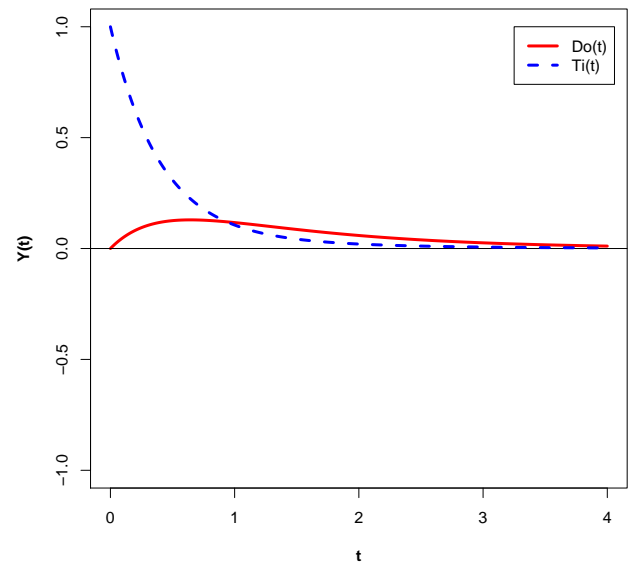*Figure (c).* $Do(0) = 1, Ti(0) = 1$

Figure (d). $Do(0) = 1, Ti(0) = -1$

Figure 2. Vector field for $Do(t)$ and $Ti(t)$, including blue and red nullclines.

Figure 3. The elements of $\boldsymbol{\phi}(\Delta t)$ for the bivariate example (i.e., $\phi_{11}(\Delta t)$, $\phi_{12}(\Delta t)$, $\phi_{21}(\Delta t)$, $\phi_{22}(\Delta t)$) plotted for a range of values for Δt.
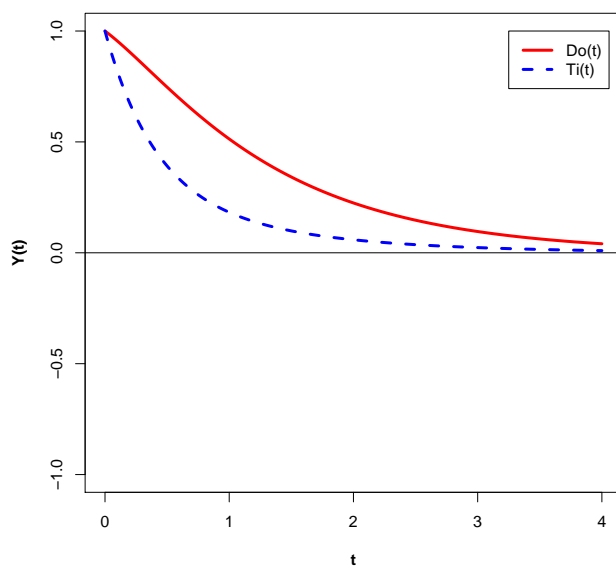
Figure 4. Vector field for a stable spiral corresponding to a drift matrix with negative real part complex eigenvalues.
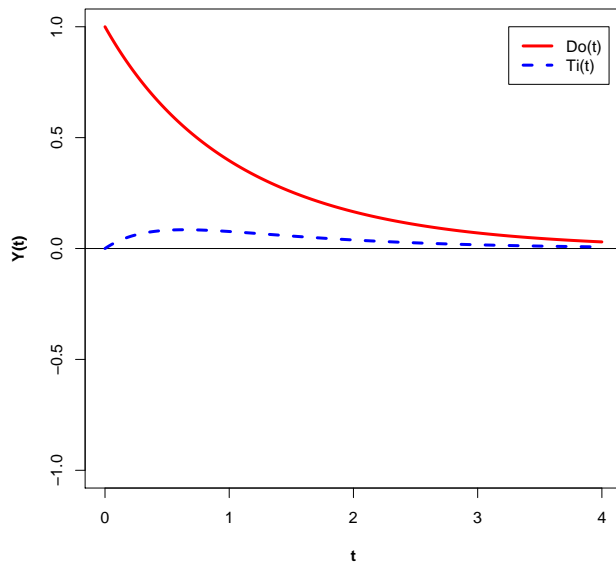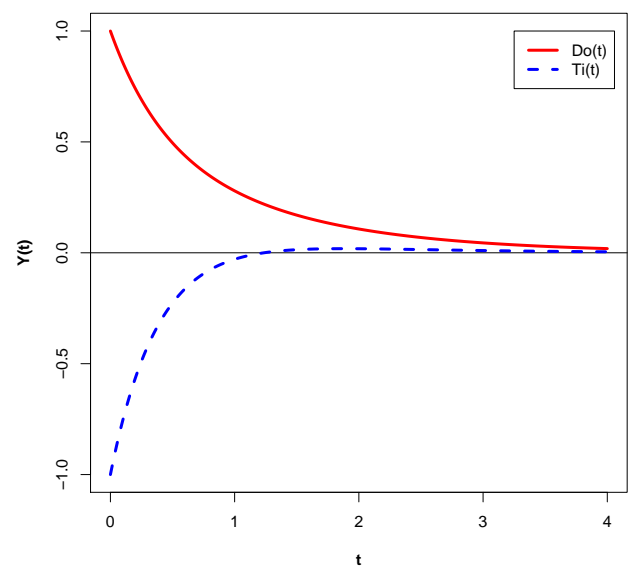
(a) $Do(0) = 1, Ti(0) = 0$

(b) $Do(0) = 0, Ti(0) = 1$

(c) $Do(0) = 1, Ti(0) = 1$

(d) $Do(0) = 1, Ti(0) = -1$