1.

Theorem: $log_{72}48$ is irrational

Proof: We can suppose for the sake of a proof by contradiction that $log_{72}48$ is rational. We can express this as,

$$log_{72}48 = \frac{q}{p} \text{ for integers } q \text{ and } p.$$

Multiplying by p,

$$p * log_{72}48 = q.$$

Rearranging using logarithm laws,

$$log_{72}48^{p} = q.$$

Simplifying further,

$$48^{p} = 72^{q}.$$

Reducing to prime factors,

$$(2^{4} * 3)^{p} = (2^{3} * 3^{2})^{q}.$$

Simplifying powers,

$$2^{4p} * 3^{p} = 2^{3q} * 3^{2q}.$$

Rearranging to have one integer on one side, we get

$$3^{p-2q} = 2^{3q-4p}.$$

However, for any integers p and q,

$$3^{p-2q} \neq 2^{3q-4p}.$$

Hence, by contradiction, we have proven that $log_{72}48$ must be irrational. QED

2.

a)

Theorem: A relation $\star$ defined on the set $\mathbb{N}$ by $x \star y$ if and only if there exists a natural number k such that $x = y + 3k$ is reflexive if every element x in $\mathbb{N}$ satisfies $x \star x$.

Proof: Let x be a natural number. We need to find an integer k such that

$$x = x + 3k.$$

By inspection,

$$3k = 0.$$

And thus, we get

$$k = 0.$$

However, $k \neq 0$ as k is defined to be a natural number. Therefore, the relation is not reflexive. QED

b)
Theorem: A relation $\star$ defined on the set $\mathbb{N}$ by $x \star$ y if and only if there exists a natural number k such that $x = y + 3k$ is symmetric if for every element x, y in $\mathbb{N}$, $x \star y$ implies $y \star x$.

Proof: Let x, y be natural numbers. To prove that there is symmetry, we need to also check that there exists a natural number l such that,

$$y = x + 3l.$$

Using the assumption that $x \star y$, we can rewrite it as,

$$x = y + 3k, \text{ for a natural number } k.$$

Rearranging to make y the subject,

$$y = x - 3k,$$

Which is equivalent to,

$$y = x + 3(-k).$$

Simplifying,

$$y = x + 3c, \text{where } c = -k$$

However, as k is defined to be a natural number, this means that c cannot be a natural number. Thus, there doesn't exist a natural number l such that $y = x + 3l$, and therefore, the relation is not symmetric. QED

c)
Theorem: A relation $\star$ defined on the set $\mathbb{N}$ by $x \star$ y if and only if there exists a natural number k such that $x = y + 3k$ is anti-symmetric if and only if for every element x, y in $\mathbb{N}$ whenever $x \star$ y and $x \star$ y then $x =$ y.

Proof: Let x, y be natural numbers. We need to find natural numbers k and l such that,

$$x = y + 3k,$$
$$y = x + 3l.$$

Substituting the expression for x into y,

$$y = y + 3k + 3l.$$

Simplifying,

$$k + l = 0.$$

However, $k + l \neq 0$ as k and l are defined to be natural numbers. Thus, the only solution can be when x = y. Hence, whenever $x \star$ y and $x \star$ y then $x =$ y and therefore, the relation is anti-symmetric. QED

d)

Theorem: A relation $\star$ defined on the set $\mathbb{N}$ by $x \star$ y if and only if there exists a natural number k such that $x = y + 3k$ is transitive if for every element x, y and z in $\mathbb{N}$ when $x \star y$ and $y \star z$, it follows that $x \star$ z.

Proof: Let x, y and z be natural numbers. Using the assumption that $x \star y$ and $y \star z$, we can express this as

$$x = y + 3k \text{ for a natural number } k,$$
$$y = \mathrm{z} + 3l \text{ for a natural number } l.$$

Substituting the expression for y into x,

$$x = z + 3l + 3k.$$

And thus, we get

$$x = z + 3(l + k)$$

As k + l is a natural number, it follows that there exists a natural number c where c = k + l such that,

$$x = z + 3\mathrm{c}.$$

Therefore, $x \star$ z . Since we have proven that whenever $x \star y$ and $y \star z$, it follows that $x \star$ z, then we have shown that the relation is transitive. QED

e) To determine if a relation $\star$ defined on the set $\mathbb{N}$ by $x \star$ y if and only if there exists a natural number k such that $x = y + 3k$ is an Equivalence Relation, Partial Order, both or neither, we need to consider whether the relation $\star$ is reflexive, symmetric, antisymmetric and/or transitive. An equivalence relation is a relation which is reflexive, symmetric and transitive while a partial order is a relation which is reflexive, antisymmetric and transitive.

As proven in part a), the relation is not reflexive. Therefore, we can conclude that it cannot be an Equivalence Relation or a Partial Order as they both require reflexivity. Therefore, it must be neither.

3.

a)

Theorem: For all integers n, let d be an integer such that if $d \mid n + 3$ and $d \mid n^2 + 5$ then $d \mid 14$.

Proof: Let n be an integer. We shall then consider the cases when n is either even or odd. First, suppose that n is even.

We can express that as,

$$n = 2k \text{ for an integer } k.$$

We can then express the initial assumptions in terms of the new expression for n,

$$d \mid 2k + 3 \text{ and } d \mid (2k)^2 + 5$$

which is equivalent to,

$$2k + 3 = da \text{ and } (2k)^2 + 5 = db \text{ for integers } a, b.$$

Rearranging to obtain an expression for k,

$$k = (da - 3)/2.$$

Substituting into the second expression,

$$(2(da - 3)/2))^2 + 5 = db$$

Which simplifies to,

$$(da - 3)^2 + 5 = db$$

Expanding the square,

$$(da)^2 - 6da + 9 + 5 = db.$$

Moving constant terms to RHS and variables to LHS,

$$(da)^2 - 6ad - db = -14.$$

Multiplying by -1,

$$6ad + db - (da)^2 = 14.$$

Factorising d,

$$d(6a + b - (da)^{\square}) = 14.$$

Thus, we get that

$$14 = d(6a + b - da) = dc, \text{ where c = } 6a + b - (da), \text{ for integers } a, b \text{ and } c.$$

Hence, for even integers $n,\ d \mid 14$.

Similarly, we can suppose that n is odd.
We can express that as,

$$n = 2k + 1 \text{ for an integer } k.$$

We can then express the initial assumptions in terms of the new expression for n,

$$d \mid 2k + 1 + 3 \text{ and } d \mid (2k + 1)^2 + 5$$

which is equivalent to,

$$2k + 4 = da \text{ and } (2k + 1)^2 + 5 = db \text{ for integers } a, b.$$

Rearranging to obtain an expression for k,

$$k = (da - 4)/2.$$

Substituting into the second expression,

$$\left(\frac{2(da-4)}{2}\right) + 1)^2 + 5 = db$$

Which simplifies to,

$$(da - 4 + 1)^2 + 5 = db$$

Expanding the square,

$$(da)^2 - 6da + 9 + 5 = db.$$

Moving constant terms to RHS and variables to LHS,

$$(da)^2 - 6ad - db = -14.$$

Multiplying by -1,

$$6ad + db - (da)^2 = 14.$$

Factorising d,

$$d(6a + b - (da)^{\square}) = 14.$$

Thus, we get that

$14 = d(6a + b - da) = dc,$ where c = $6a + b - (da)$, for integers a, b and c.

Hence, for odd integers n, $d \mid 14$. Thus, for all integers n, if d is an integer such that $d \mid n + 3$ and $d \mid n^2 + 5$ then $d \mid 14$. QED

b)

Theorem: For all integers n, if n is a multiple of 14 then $n^2 + 5$ and $n + 3$ are coprime.

Proof: Let n be an integer. As n is a multiple of 14, we can express this as,

$n = 14k$ for an integer k.

Expressing the other equations in terms of k,

$$\mathrm{a} = n + 3 = 14k + 3\ ,$$
$$\mathrm{b} = n^2 + 5 = (14\mathrm{k})^2 + 5 = 196\mathrm{k}^2 + 5\ .$$

We can assume that there is a common divisor d such that $d \mid 14k + 3$ and $\mathrm{d} \mid 196\mathrm{k}^2 + 5$.

If d divides $14k + 3$, we can express this using modular arithmetic,

$$14k + 3 \equiv 0 \ (\text{mod } d) \ ,$$
$$14k \equiv -3 \ (\text{mod } d) \ .$$

Similarly, if d divides $196k^2 + 5$, we can expressing this using modular arithmetic,

$$196k^2 + 5 \ \equiv 0 \ (\text{mod } d) \ ,$$
$$196k^2 \equiv -5 \ (\text{mod } d) \ .$$

Squaring both sides of the first modular arithmetic equation,

$$196k^2 \equiv 9 \ (\text{mod } d) \ .$$

Equating both equations for $196k^2$,

$$-5 \equiv 9 \ (\text{mod } d) \ ,$$

Which is equivalent to,

$$14 \equiv 0 \ (\text{mod } d) \ .$$

Hence, d divides 14. Therefore, the only possible values of d are 0,1,14,2 and 7.
When d = 2,

$$14k + 3 \equiv 1 \ (\text{mod } 2) \ ,$$

$$196k^2 + 5 \equiv 1 \ (\text{mod } 2) \ .$$

Hence, 2 doesn't divide both.