Question 1:

A) Prove that when working modulo 9, it is not possible for a perfect square to be congruent to 2, 3, 5, 6 or 8.

This question can be interpreted that we must prove that it is impossible for us to get a remainder of 2, 3, 5, 6 or 8 when dividing a perfect square by 9.

We know that in general working with modulo n, we can only get numbers between 0 to n – 1 inclusive. Based on this we could get the possible values of perfect squares ranging from 0 to 8 inclusive. Doing so we get

$$0^2 \equiv 0(mod(9)),$$

$$1^2 \equiv 1(mod(9)),$$

$$2^2 \equiv 4(mod(9)),$$

$$3^2 \equiv 0(mod(9)),$$

$$4^2 \equiv 7(mod(9)),$$

$$5^2 \equiv 7(mod(9)),$$

$$6^2 \equiv 0(mod(9)),$$

$$7^2 \equiv 4(mod(9)),$$

$$8^2 \equiv 1(mod(9)).$$

As can be seen, the only possible values for which a perfect square could be congruent to when working with modulo 9 are 0, 1, 4 or 7. Therefore perfect squares can not be congruent to 2, 3, 5, 6 or 8.

B) Hence (and not otherwise) prove that there do not exist three consecutive integer values of n for which 47n+16 is a perfect square.

Using the fact from the previous part, we now need to prove that there does not exist three consecutive integers of n where 47n + 16 is a perfect square.

Using proof by contradiction, we assume that there exists three consecutive integers (n - 1), n and (n + 1) which would give us perfect squares $a, b\ and\ c$ in the form of

$$a = 47(\mathrm{n} - 1) + 16,$$

$$b = 47\mathrm{n} + 16,$$

$$c = 47(\mathrm{n} + 1) + 16.$$

We first consider b. If b is a perfect square, then we should get a remainder of 0, 1, 4 or 7 when we modulo with 9. So, we get $47n + 16 \equiv k^2(mod(9))$ where k is an integer. Since $47 \equiv 2(mod(9))$ we can convert the previous expression into $47n + 16 \equiv 2n + 16\ (mod(9))$. We then manually check the expression for when n is 0, 1, 4 or 7 to get possible values that make 47n + 16 a perfect square. We get

$$2n + 16 \equiv 0\big(mod(9)\big) \rightarrow n \equiv 1\big(mod(9)\big),$$

$$2n + 16 \equiv 1\big(mod(9)\big) \rightarrow n \equiv 6\big(mod(9)\big),$$

$$2n + 16 \equiv 4\big(mod(9)\big) \rightarrow n \equiv 3\big(mod(9)\big),$$

$$2n + 16 \equiv 7\big(mod(9)\big) \rightarrow n \equiv 0\big(mod(9)\big).$$

Therefore, the possible values of n which let 47n + 16 be a perfect square are given by $n \equiv 0, 1, 3, or\ 6\big(mod(9)\big)$. However, as we need three consecutive integers

n – 1, n and n + 1 such that a, b and c are perfect squares, each of them must also be congruent to $0, 1, 4, or\ 7\big(mod(9)\big)$. But because $n\big(mod(9)\big)$ only gives us four valid results which are not consecutive, then there are no three consecutive values which satisfy the congruence conditions.

Hence there does not exist three consecutive integers of n for which 47n + 16 is a perfect square.

Question 2:

A relation $\star$ defined on the set $\mathbb{Z}^2$ by $(x_1, x_2) \star (y_1, y_2)$ if and only if there exists a real number $0 < k \leq 1$ such that $y_2 = kx_1\ and\ x_2 = ky_1$.

A) Is $\star$ reflexive?

$\star$ can only be reflexive if $(x_1, x_2) \star (x_1, x_2)$ for all pairs of elements that follow the mentioned restrictions. Testing $(1,4) \star (1,4)$ we get

$$4 = k(1) \; and \; 4 = k(1).$$

Because solving for k we get $k = 4$ which does not fit the parameter of

$$0 < k \leq 1.$$

Therefore $\star$ is not reflexive.

B) Is $\star$ symmetric?

$\star$ can only be symmetric if $(x_1, x_2) \star (y_1, y_2)$ implies $(y_1, y_2) \star (x_1, x_2)$.

By applying both operations we get

$$y_2 = k(x_1)$$

and

$$x_2 = k(y_1)$$

along with

$$x_2 = n(y_1)$$

and

$$y_2 = n(x_1).$$

Replacing y_2 and x_2 into the first two expressions we get

$$k(x_1) = n(x_1)$$

and

$$k(y_1) = n(y_1)$$

Knowing that neither $x_1 = 0$ or $y_1 = 0$. We can simplify the expressions to get $k = n$. Now assuming that $(x_1, x_2) \star (y_1, y_2)$ does follow the criteria of $\star$, then k must be $0 < k \leq 1$. Based on this and that $n = k$, then n must be $0 < n \leq 1$. Hence, n will always meet the criteria of being $0 < n \leq 1$.

Therefore $\star$ is symmetric.

C) Is $\star$ anti-symmetric?

$\star$ can only be anti-symmetric if $(x_1, x_2) \star (y_1, y_2)$ and $(y_1, y_2) \star (x_1, x_2)$ are true implies $(x_1, x_2) = (y_1, y_2)$.

By applying both operations we get

$$y_2 = k(x_1)$$

and

$$x_2 = k(y_1)$$

along with

$$x_2 = n(y_1)$$

and

$$y_2 = n(x_1).$$

Replacing y_2 and x_2 into the first two expressions we get

$$k(x_1) = n(x_1)$$

and

$$k(y_1) = n(y_1)$$

If $x_1 = 0$ or $y_1 = 0$ then that would mean that x_2 and $y_2$would be 0 based on their definitions, otherwise if $x_1 \neq 0$ and $y_1 \neq 0$. Then that would mean that $n = k$ which would then mean that $x_1 = y_1$ and $x_2 = y_2$. In both cases $(x_1, x_2) = (y_1, y_2)$.

Therefore $\star$ is anti-symmetric.

D) Is $\star$ transitive?

Assume $(\boldsymbol{x_1}, \boldsymbol{x_2}) \star (y_1, y_2)$ is valid, then $y_2 = k(x_1)$ and $x_2 = k(y_1)$.

Also Assume $(y_1, y_2) \star (z_1, z_2)$ is valid, then $z_2 = m(y_1)$ and $y_2 = m(z_1)$.

We need to prove $(x_1, x_2) \star (z_1, z_2)$ is valid which gives $z_2 = n(x_1)$ and $x_2 = n(z_1)$. For n which is $0 < n \leq 1$.

Since y_2 is shared between the first two expressions, we can get the expression $kx_1 = mz_1$. We can now find n via the expression $z_2 = nx_1$, which gives us $z_2 = \frac{mx_2}{k}$. Which after all substitution gives us $n = \frac{kz_2}{mz_1}$ hence such an n exists.

Therefore $\star$ is transitive.

E) Is $\star$ an equivalence relation, a partial order, both, or neither?

For $\star$ to be an equivalence relation it must meet the following:

- Be reflexive

- Be symmetric
- Be Transitive

Since $\star$ is not reflexive, we can determine that $\star$ is not an equivalence relation.

For $\star$ to be a partial order it must meet the following:

- Be reflexive
- Be Anti-symmetric
- Be Transitive

Since $\star$ is not reflexive, we can determine that $\star$ is not a partial order.

Therefore $\star$ is neither equivalence relation nor partial order.

Question 3:

Consider two functions $\boldsymbol{g}: \boldsymbol{X} \rightarrow \boldsymbol{Y}$ and $\boldsymbol{h}: \boldsymbol{Y} \rightarrow \boldsymbol{Z}$ for non-empty sets *X, Y, Z.*

A) If $\boldsymbol{h} \circ \boldsymbol{g}$ is surjective, then h is surjective.

Assume $h \circ g$ is surjective. Then for all $z \in Z$ there exists an $x \in X$ such that $h\big(g(x)\big) = z$.

The function h is surjective if for all $z \in Z$ there exists $y \in Y$ such that $h(y) = z$.

Since $h\big(g(x)\big) = z$ for some $x \in X$, we can get $y = g(x)$. Hence, $h(y) = h\big(g(x)\big) = z$.

Therefore, h is surjective.

B) If $\boldsymbol{h} \circ \boldsymbol{g}$ is surjective, then g is surjective.

Let sets $X = \{1\}, Y = \{1, 4\}$ and $Z = \{1\}$. Based on the previous definitions of functions g and h, let $g(1) = 1$, $h(1) = 1$ and $h(4) = 1$.

We can see that $h\big(g(1)\big) = h(1) = 1$. Hence $h \circ g$ is surjective as every element of Z is mapped to by an element of X.

However, g(x) is not surjective as $4 \in Y$ is not mapped to by any element in X.

Therefore, g is not surjective.

C) If $\boldsymbol{h \circ g}$ is surjective and h is injective, then g is surjective.

Assume $h \circ g$ is surjective. Then for all $z \in Z$ there exists an $x \in X$ such that $h\big(g(x)\big) = z$.

Assume h is injective. That means if $h(y_1) = h(y_2)$ then $y_1 = y_2$.

Suppose there is a $y \in Y$ that is not an image of g. Then h(y) would not be in the image of $h(g(x))$ which contradicts The first assumption. Hence, such a y does not exist and that for every $y \in Y$ there must exist some $x \in X$ so that $g(x) = y$.

Therefore, g is surjective.

Generative AI was used to help generate initial ways of solving a question.