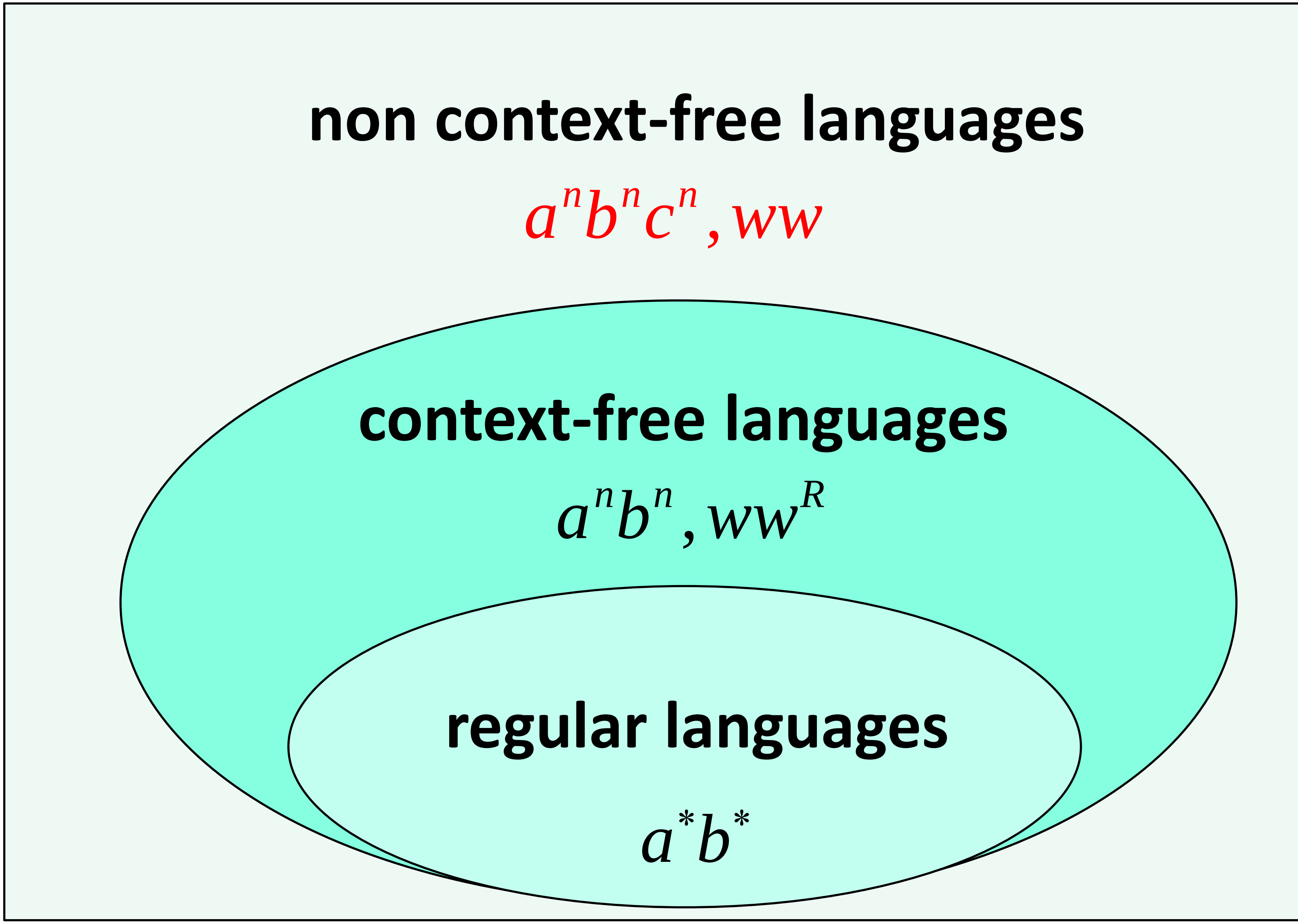
Deterministic Pushdown Automata – DPDAs

Non-Context-free Languages



Proof

- Any regular language L is generated by some regular grammar G
- (Linear Grammar: Right or Left)

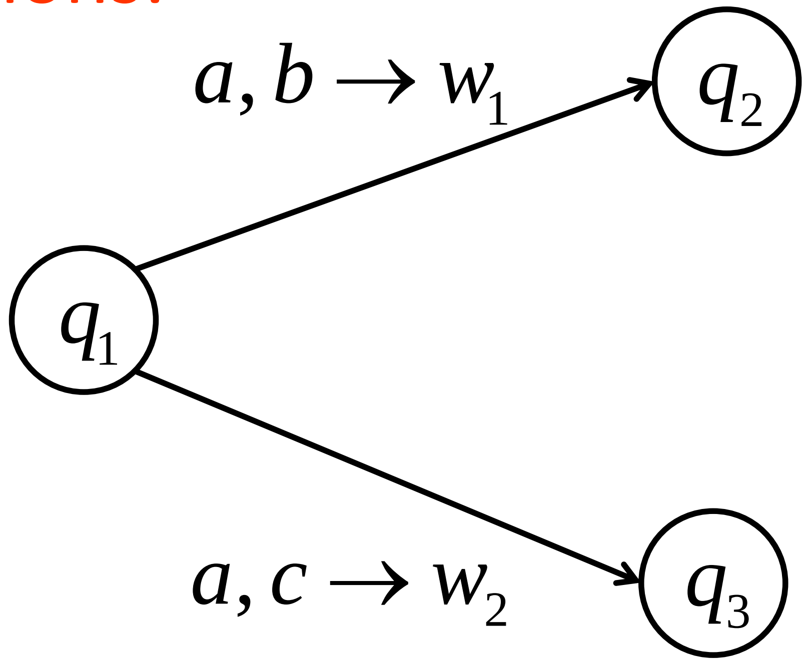
Proof idea:

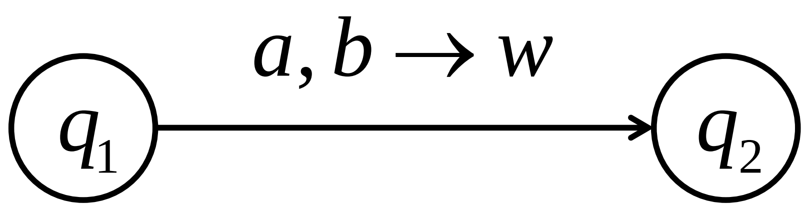
- Let M be the NFA with $L = L(M)$.
- Construct from M a regular grammar G such that

$$L(M) = L(G)$$

DPDAs

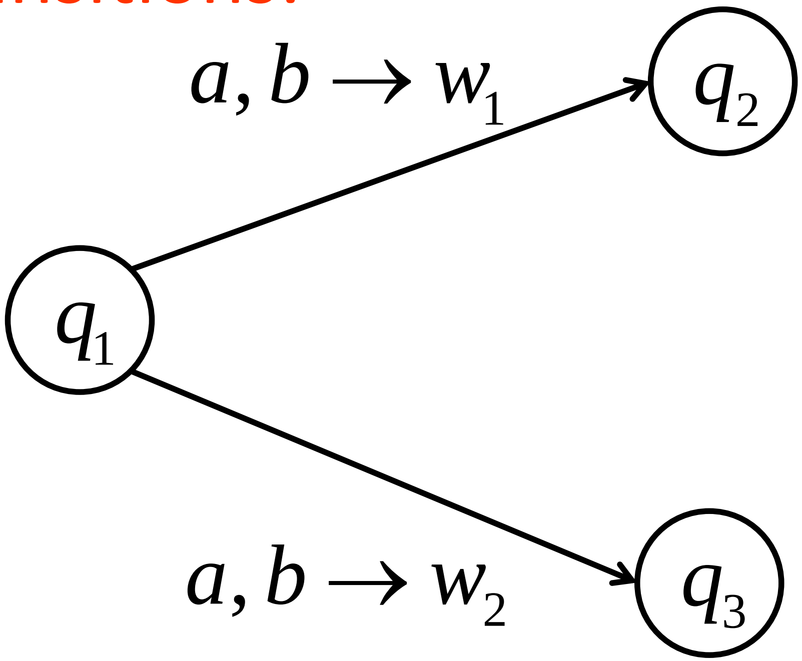
- Allowed transitions:





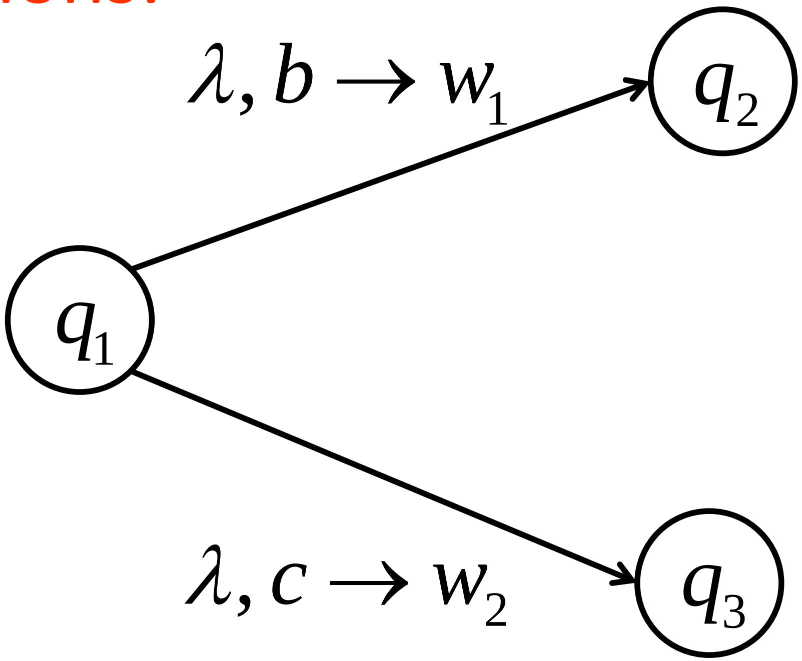
DPDAs

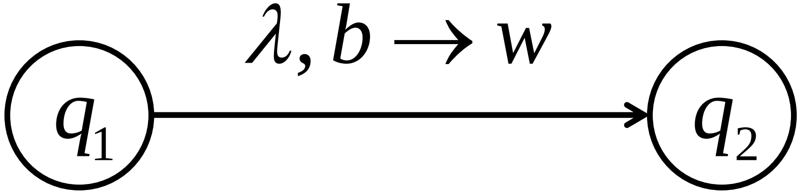
- Not allowed transitions:



DPDAs

- Allowed transitions:

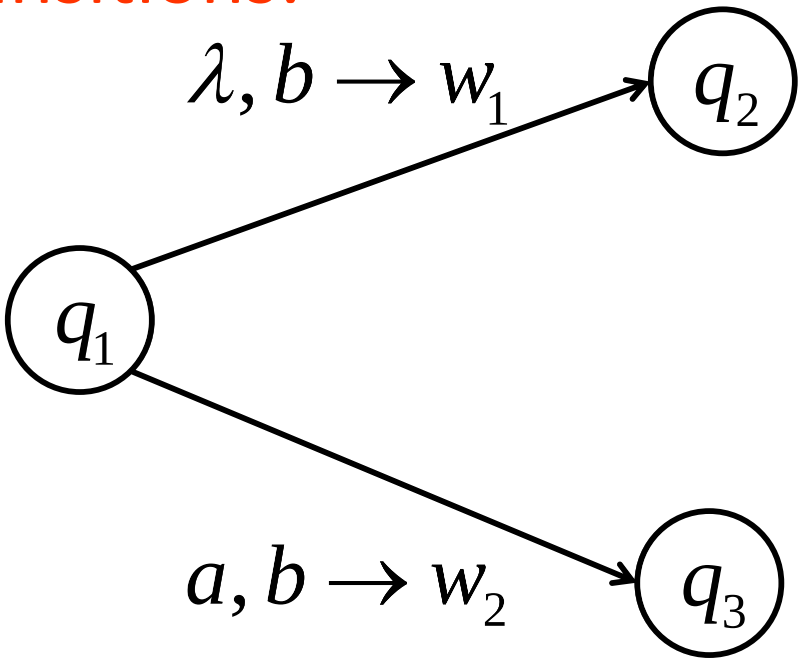




- Something must be matched from the stack

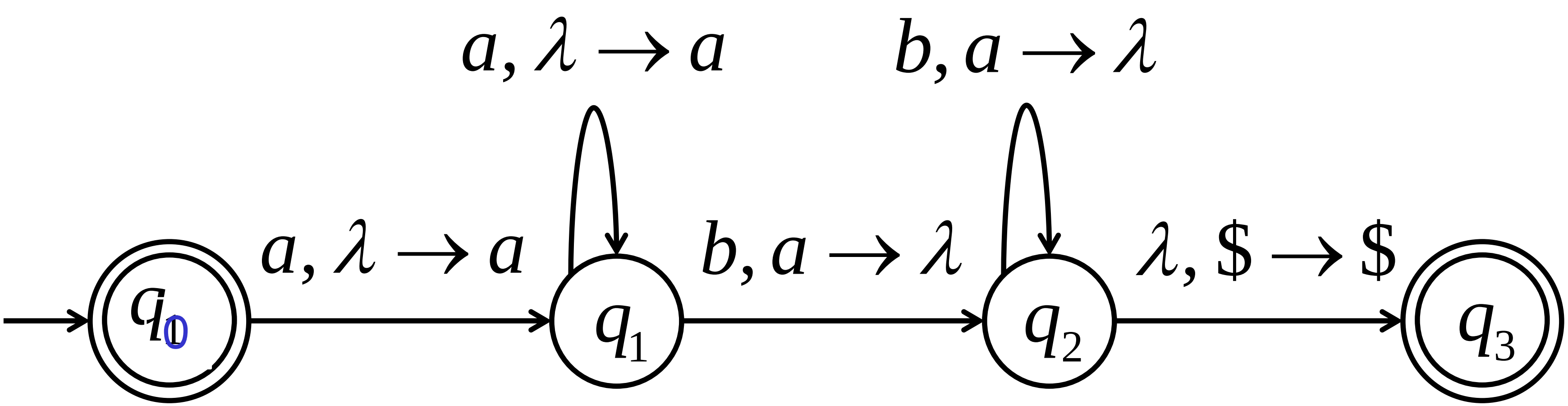
DPDAs

- Not allowed transitions:



DPDA example

$$L(M) = \{a^n b^n : n \geq 0\}$$



DPDAs

- The language

$$\boxed{L(M) = \{a^n b^n : n \geq 0\}}$$

is **deterministic context-free.**

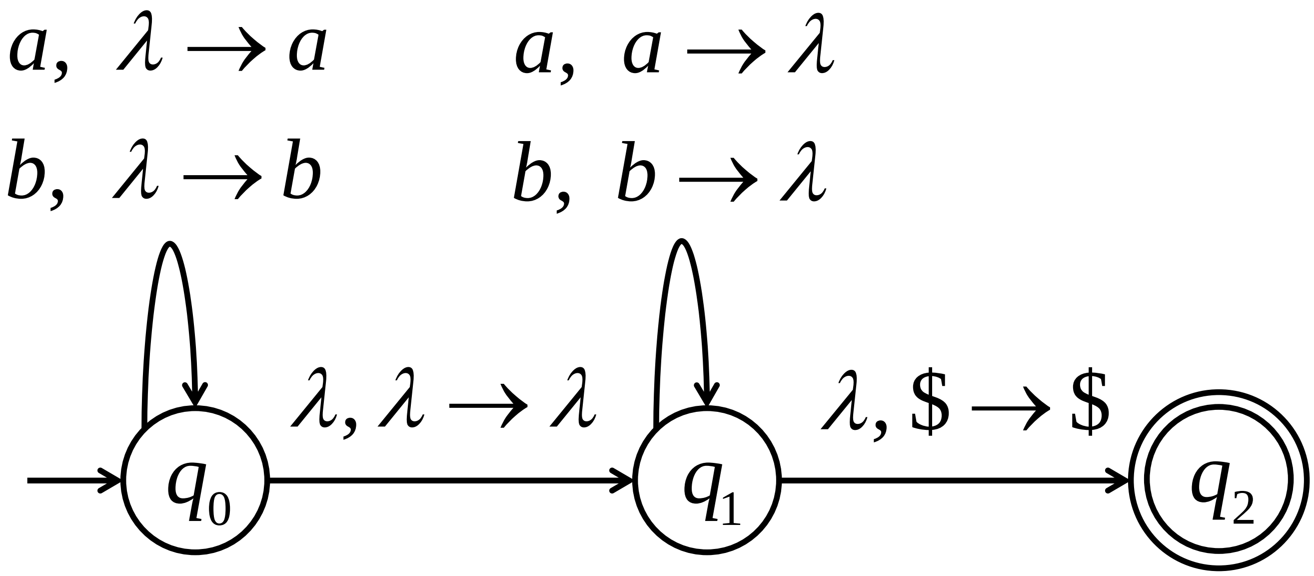
The Language of DPDA

Definition:

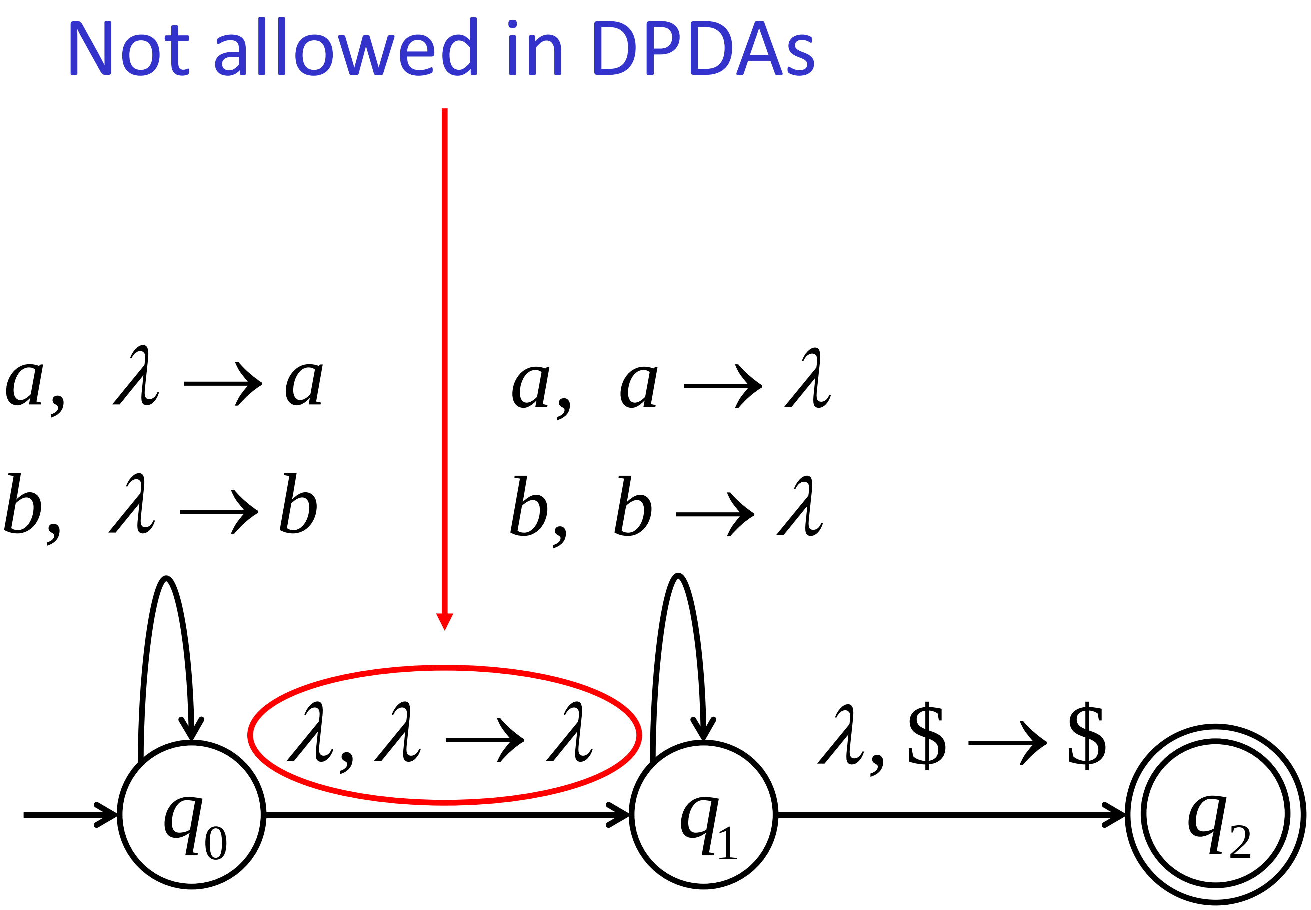
A language accepted by some deterministic pushdown automata (DPDA) is called a **context-free language**.

Non-DPDA example

$$\boxed{L(M) = \{ww^R\}}$$



Non-DPDA example



NPDAs Have More Power Than DPDAs

DPDAs vs. NDPAs

We will show:

there is a context-free language L
(accepted by a NPDA)
which is **not** deterministic context-free
(**not** accepted by a DPDA)

DPDAs vs. NDPAs

The language is:

$$\boxed{L = \{a^n b^n : n \geq 0\} \cup \{a^n b^{2n} : n \geq 0\}}$$

DPDAs vs. NDPAs

$$L = \{a^n b^n : n \geq 0\} \cup \{a^n b^{2n} : n \geq 0\}$$

- The language L is context-free since there is a context-free grammar G generating L;

$$S \to S_1 \mid S_2$$
$$S_1 \to aS_1b \mid \lambda$$
$$S_2 \to aS_2bb \mid \lambda$$

there is also an NPDA M that accepts L.

DPDAs vs. NDPAs

Theorem

The language

$$L = \{a^n b^n : n \geq 0\} \cup \{a^n b^{2n} : n \geq 0\}$$

is not deterministic context-free
(there is no DPDA M that accepts L).

Proof

Assume for contradiction that

$$L = \{a^n b^n : n \geq 0\} \cup \{a^n b^{2n} : n \geq 0\}$$

is deterministic context-free.

Therefore, there is a DPDA M that accepts L).

DPDAs vs. NDPAs

Proof

Assume for contradiction that

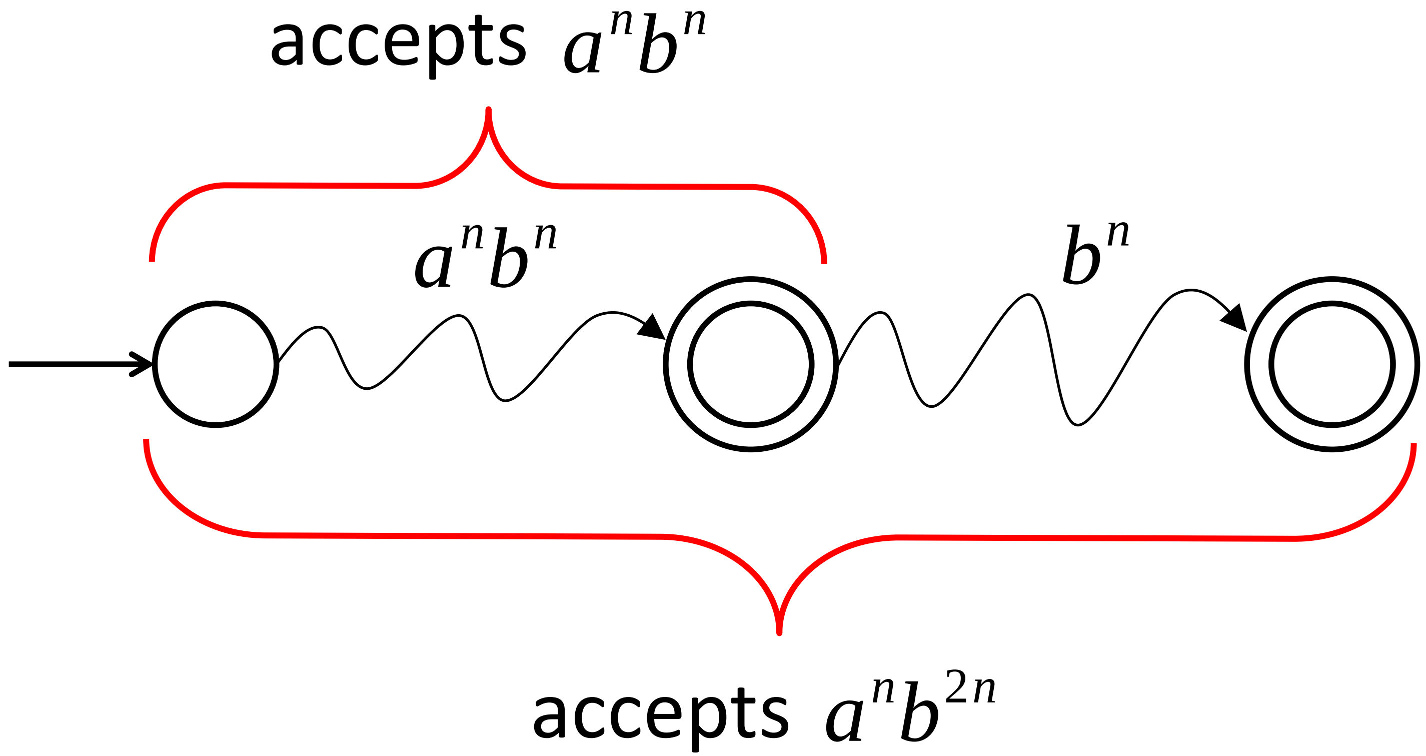
$$L = \{a^n b^n : n \geq 0\} \cup \{a^n b^{2n} : n \geq 0\}$$

is deterministic context-free.

Therefore, there is a DPDA M that accepts L).

DPDAs vs. NDPAs

- DPDA M with $L(M) = \{a^n b^n\} \cup \{a^n b^{2n}\}$



DPDAs vs. NDPAs

- **Fact 1:** The language $\{a^n b^n c^n\}$ is not context-free.

- **Fact2:** The language $L \cup \{a^n b^n c^n\}$ is not context-free, where

$$L = \{a^n b^n\} \cup \{a^n b^{2n}\}$$

(a consequence of Fact 1)

DPDAs vs. NDPAs

- We will construct a NPDA that accepts:

$$\boxed{L \cup \{a^n b^n c^n\}}$$

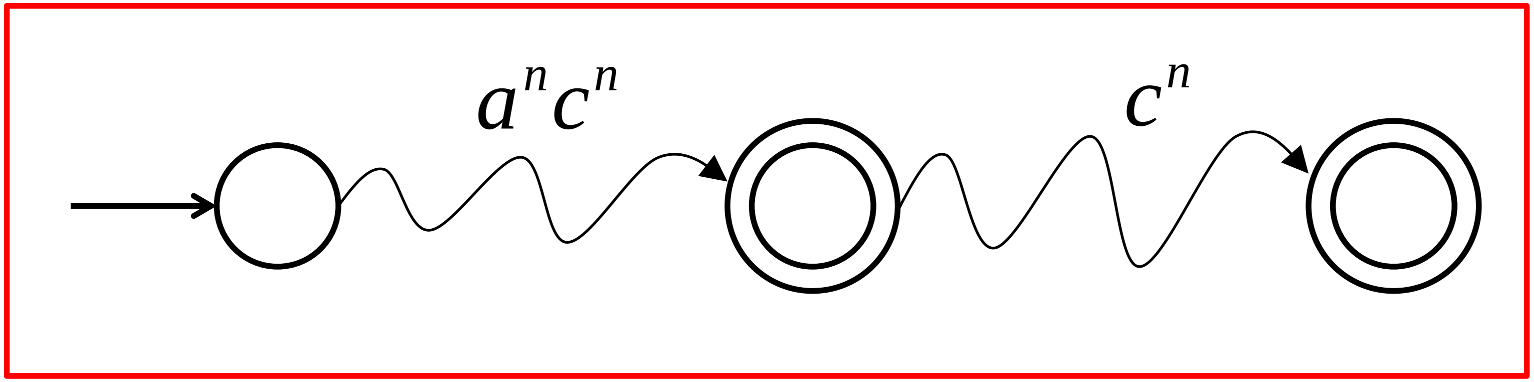
Contradiction!

DPDAs vs. NDPAs

- We modify the NPDA M $(L = \{a^n b^n\} \cup \{a^n b^{2n}\})$

- Replace b with c

Modified M $(L' = \{a^n c^n\} \cup \{a^n c^{2n}\})$

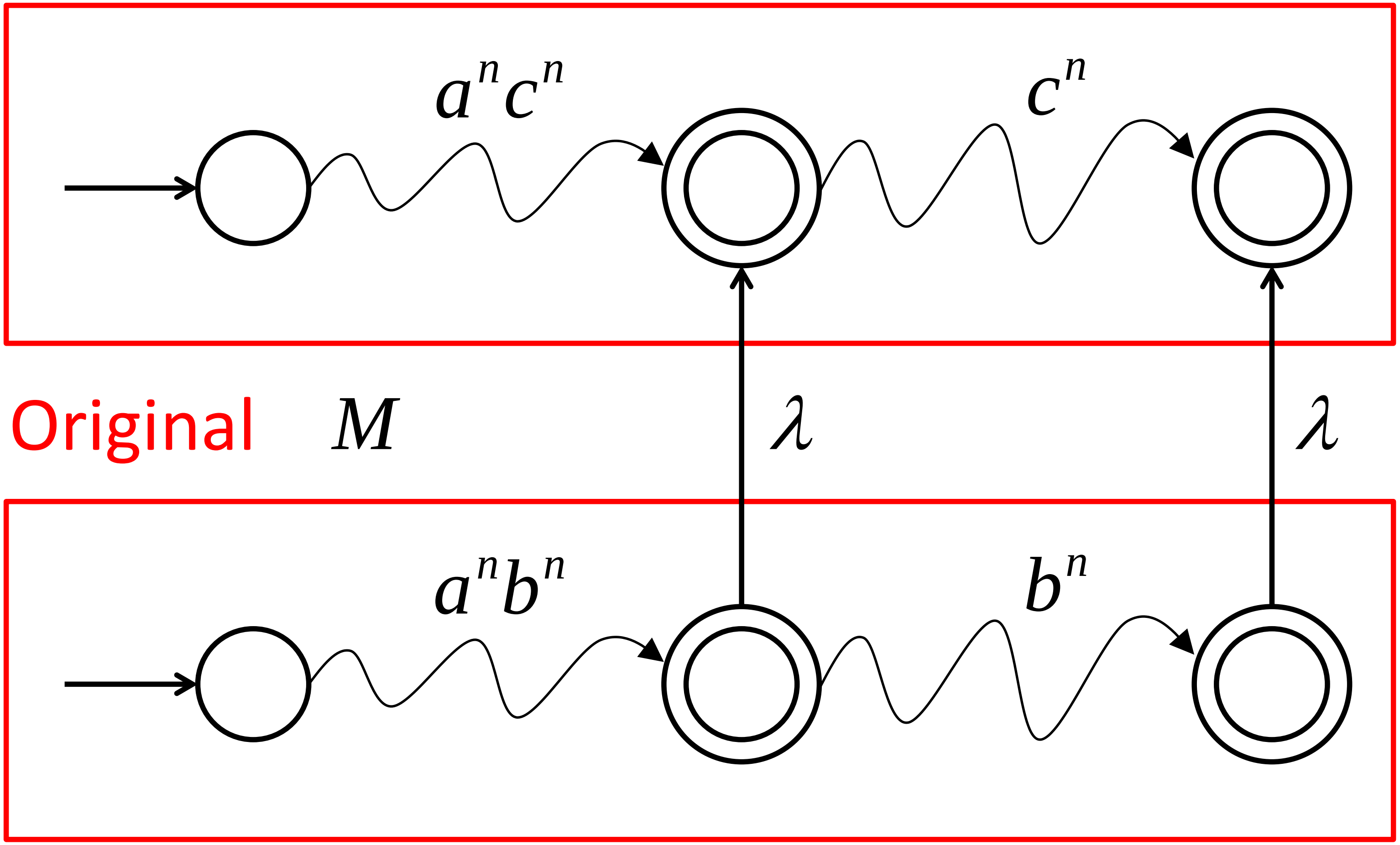


DPDAs vs. NDPAs

- The NPDA that accepts $L \cup \{a^n b^n c^n\}$

Modified M

Original M



DPDAs vs. NDPAs

- Since $L \cup \{a^n b^n c^n\}$ is accepted by a NDPA, it is context-free.

Contradiction!

DPDAs vs. NDPAs

Therefore:

There is **no** DPDA that accepts

$$\boxed{L = \{a^n b^n\} \cup \{a^n b^{2n}\}}$$

not deterministic context free

End of Proof

“Positive” Properties of Context-Free Languages

Union

- Context-free languages are closed under **union:**

  If L_1 and L_2 are context-free languages, then the **union** of the languages L_1 and L_2:

$$L_1 \cup L_2$$

  is also context-free.

Union: Example

$L_1 = \{a^n b^n : n \geq 0\}$ $\qquad$ $G_1 : S_1 \rightarrow aS_1b|\lambda$

$L_2 = \{ww^R : \{a, b\}^*\}$ $\qquad$ $G_2 : S_2 \rightarrow aS_2a|bS_2b|\lambda$

Union

$L = \{a^n b^n\} \cup \{ww^R\}$ $\qquad$ $G : S \rightarrow S_1|S_2$

Union: In General

- For context-free languages L_1 and L_2 with context-free grammars G_1 and G_2, and the start variables S_1 and S_2, respectively,

the grammar of the **union** $L_1 \cup L_2$ has the new start variable S and the additional production $S \rightarrow S_1 | S_2$.

Concatenation

- Context-free languages are closed under **concatenation:**

  If L_1 and L_2 are context-free languages, then the **concatenation** of the languages L_1 and L_2:

  $$L_1 \cdot L_2$$

  is also context-free.

Concatenation: Example

$L_1 = \{a^n b^n : n \geq 0\}$ $\qquad$ $G_1 : S_1 \rightarrow aS_1b|\lambda$

$L_2 = \{ww^R : \{a, b\}^*\}$ $\qquad$ $G_2 : S_2 \rightarrow aS_2a|bS_2b|\lambda$

Concatenation

$L = \{a^n b^n\} \cdot \{ww^R\}$ $\qquad$ $G : S \rightarrow S_1 \cdot S_2$

Concatenation: In General

- For context-free languages L_1 and L_2 with context-free grammars G_1 and G_2, and the start variables S_1 and S_2, respectively,

  the grammar of the **concatenation** $L_1 \cdot L_2$ has the new start variable S and the additional production $S \rightarrow S_1 \cdot S_2$.

Star-Operation

- Context-free languages are closed under **star-operation:**

  If L is context-free languages, then the star language:

  $$L^*$$

  is also context-free.

Star-Operation: Example

$L = \{a^n b^n : n \geq 0\}$ $G : S \to aSb|\lambda$

Star-Operation

$L = \{a^n b^n\}^*$ $G : S' \to S \cdot S'|\lambda$

Star-Operation: In General

- For context-free languages L with context-free grammars G and the start variable S,

  the grammar of the **star-operation** L^* has new start variable S' and the additional productions $S' \rightarrow SS'|\lambda$.

“Negative” Properties
of
Context-Free Languages

Intersection

- Context-free languages are **not** closed under **intersection:**

  If L_1 and L_2 are context-free languages, then the **intersection** of the languages L_1 and L_2:

  $$L_1 \cap L_2$$

  is not necessarily context-free.

Intersection: Example

$$L_1 = \{a^n b^n c^m\}$$

$$S_1 \to AC$$
$$A \to aAb|\lambda$$
$$C \to cC|\lambda$$

$$L_2 = \{a^n b^m c^m\}$$

$$S_2 \to AB$$
$$A \to aA|\lambda$$
$$B \to bBc|\lambda$$

Intersection

$$L_1 \cap L_2 = \{a^n b^n c^n\}$$

is not a context-free language.

Complement

- Context-free languages are **not** closed under **complement:**

  If L is context-free language, then the **complement** of the language L:

$$\overline{L}$$

  is not necessarily context-free.

Intersection: Example

$L_1 = \{a^n b^n c^m\}$

$S_1 \to AC$
$A \to aAb|\lambda$
$C \to cC|\lambda$

$L_2 = \{a^n b^m c^m\}$

$S_2 \to AB$
$A \to aA|\lambda$
$B \to bBc|\lambda$

Complement

$\overline{\overline{L_1} \cup \overline{L_2}} = L_1 \cap L_2 = \{a^n b^n c^n\}$

is not a context-free language.

Intersection
of
Context-free Languages
and
Regular Languages

Intersection with Regular Languages

- Context-free languages are closed under **intersection with regular languages:**

  If L_1 is a context-free language and L_2 is a regular language, then the **intersection** of the languages L_1 and L_2:

  $$L_1 \cap L_2$$

  is also context-free.

Intersection: Example

Machine M_1

NDPA for L_1
(context-free)

Machine M_2

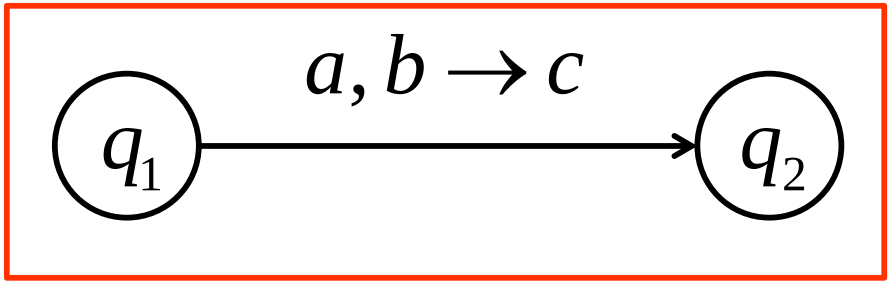
DFA for L_2
(regular language)

- Construct a new NPDA machine M that accepts $L_1 \cap L_2$.

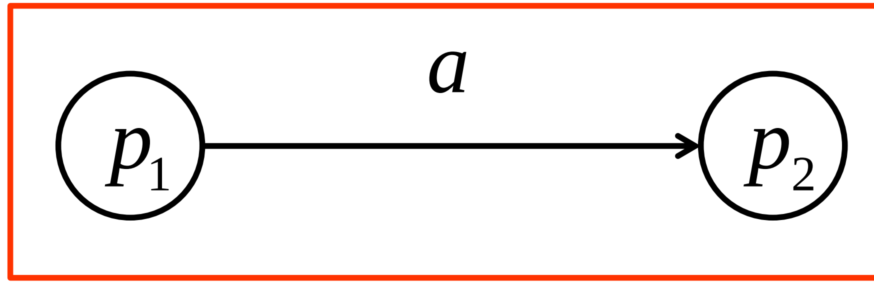
- M simulates in parallel M_1 and M_2.

Transition

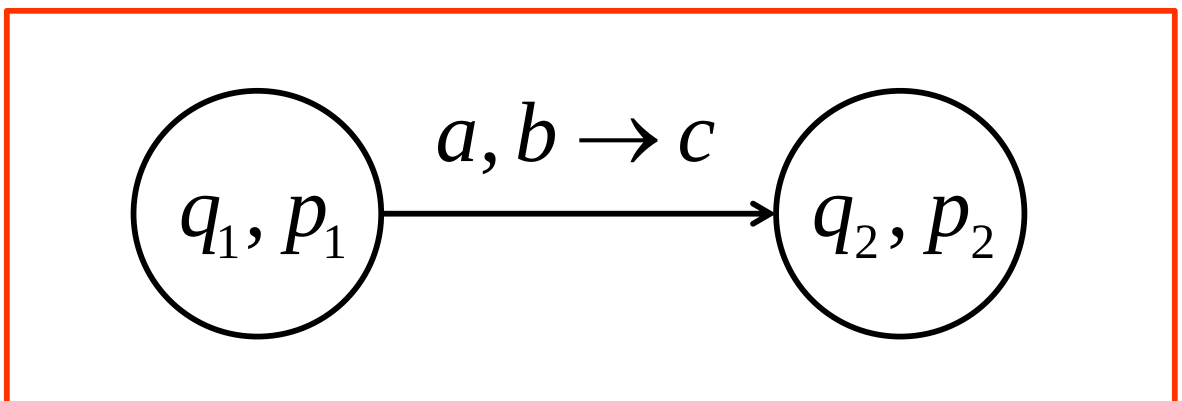
NDPA M_1



DFA M_2

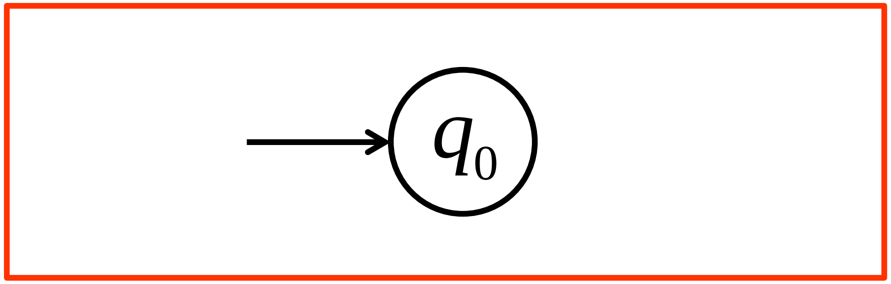


NDPA M_1

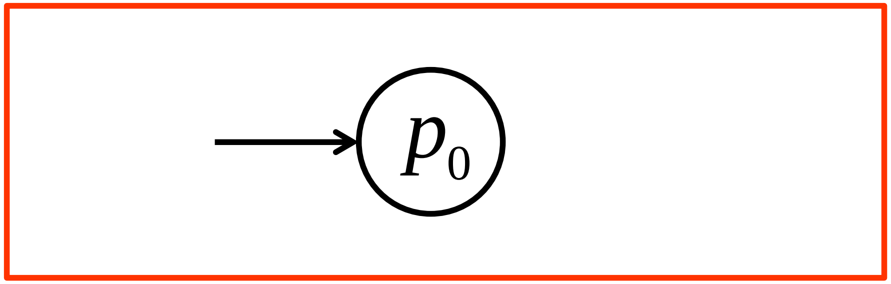


Initial State

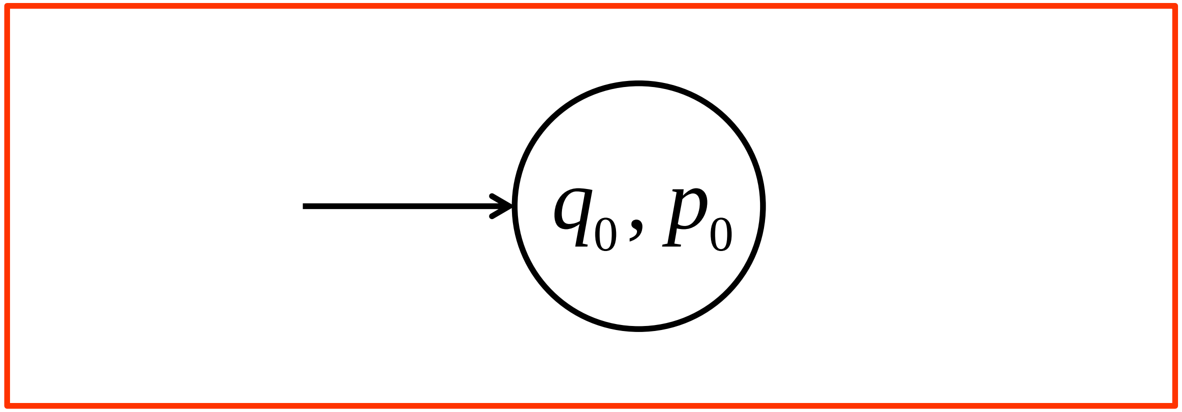
NDPA M_1



DFA M_2

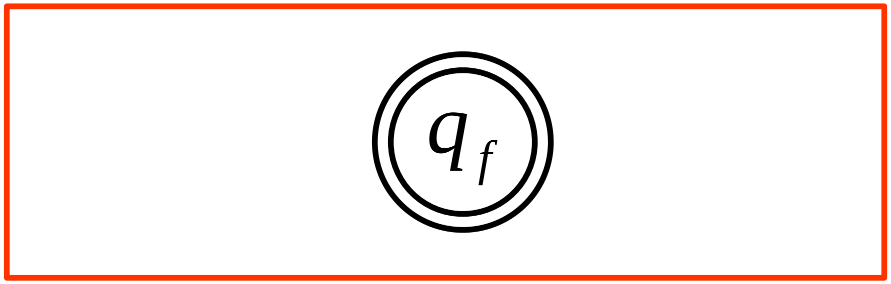


NDPA M_1

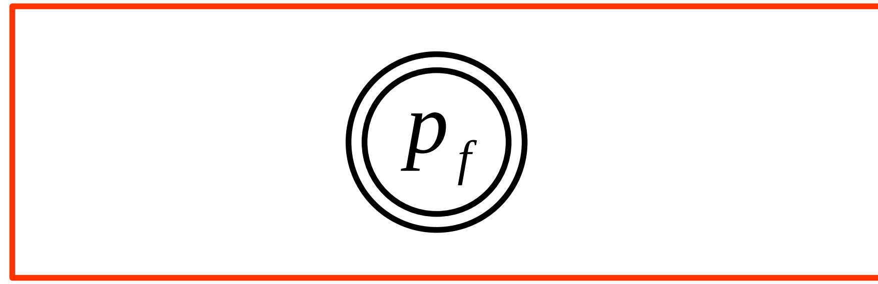


Final State

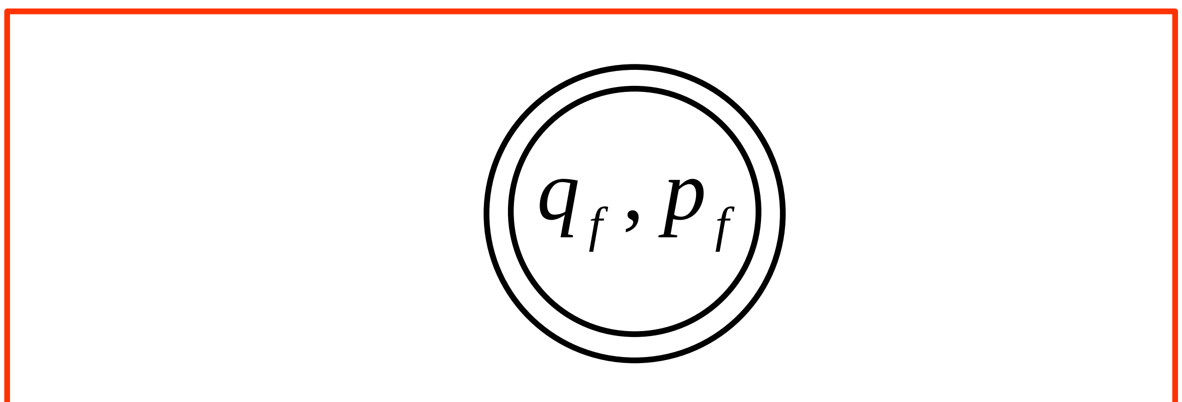
NDPA M_1



DFA M_2



NDPA M_1



Intersection with Regular Languages

- The NDPA M simulates in parallel the NDPA M_1 and the DFA M_2.

- The NDPA M accepts a string w if and only if
  - M_1 accepts the string w and
  - M_2 accepts the string w

- Therefore,

$$L_1 \cap L_2 = L(M) = L(M_1) \cap L(M_2)$$

is context-free.