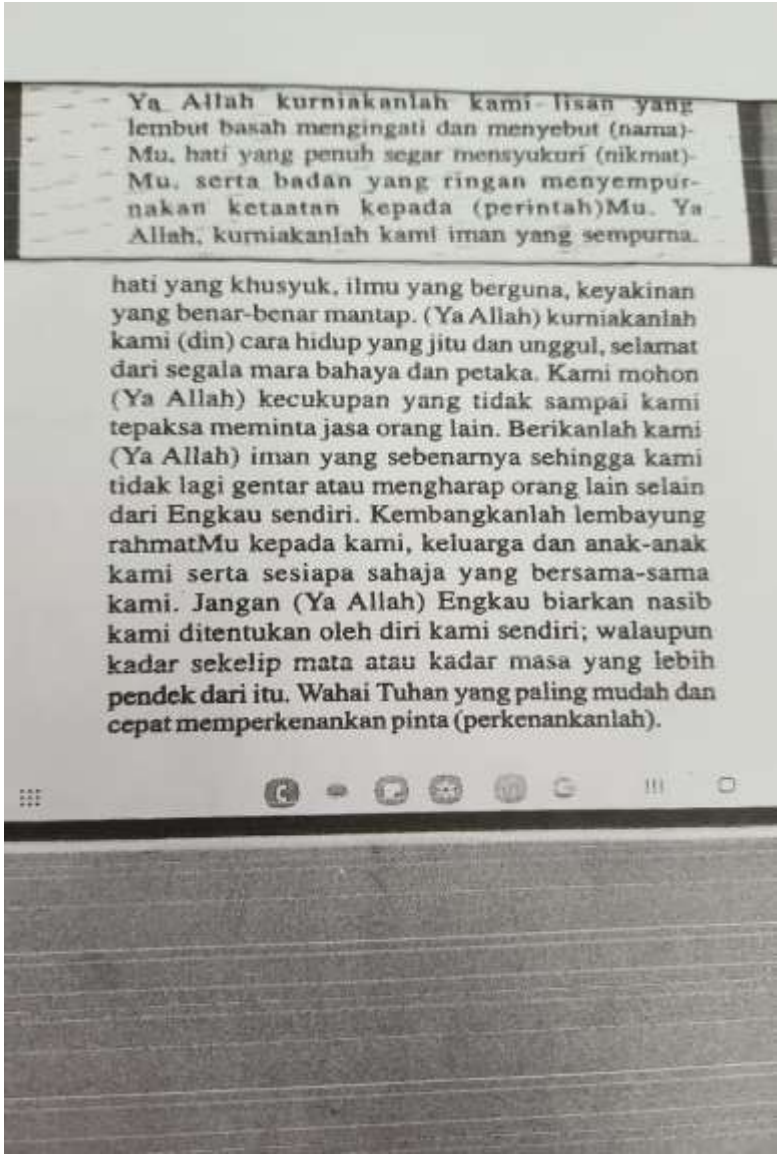
Grammars

* اَللّٰهُمَّ إِنَّا نَسْأَلُكَ لِسَانًا رَطِبًا بِذِكْرِكَ
وَقَلْبًا مَفْعَمًا بِشُكْرِكَ وَبَدَنًا هَيِّنًا لَيِّنًا
بِطَاعَتِكَ اَللّٰهُمَّ إِنَّا نَسْأَلُكَ إِيْمَانًا كَامِلاً

وَنَسْأَلُكَ قَلْبًا خَاشِعًا وَنَسْأَلُكَ عِلْمًا نَافِعًا
وَنَسْأَلُكَ يَقِيْنًا صَادِقًا وَنَسْأَلُكَ دِيْنًا قَيِّمًا
وَنَسْأَلُكَ الْعَافِيَةَ مِنْ كُلِّ بَلِيَّةٍ وَنَسْأَلُكَ
تَمَامَ الْغِنَى عَنِ النَّاسِ وَهَبْ لَنَا حَقِيْقَةَ
الْإِيْمَانِ بِكَ حَتَّى لاَ نَخَافَ وَلاَ نَرْجُوْ
غَيْرَكَ وَلاَ نَعْبُدَ شَيْئًا سِوَاكَ وَاجْعَلْ يَدَكَ
مَبْسُوْطَةً عَلَيْنَا وَعَلى أَهْلِيْنَا وَأَوْلاَدِنَا
وَمَنْ مَعَنَا بِرَحْمَتِكَ وَلاَ تَكِلْنَا إِلَى
أَنْفُسِنَا طَرْفَةَ عَيْنٍ وَلاَ أَقَلَّ مِنْ ذٰلِكَ يَا
نِعْمَ الْمُجِيْبُ.


Ya Allah kurniakanlah kami lisan yang lembut basah mengingati dan menyebut (nama)-Mu, hati yang penuh segar mensyukuri (nikmat)-Mu, serta badan yang ringan menyempurnakan ketaatan kepada (perintah)Mu. Ya Allah, kurniakanlah kami iman yang sempurna.
hati yang khusyuk, ilmu yang berguna, keyakinan yang benar-benar mantap. (Ya Allah) kurniakanlah kami (din) cara hidup yang jitu dan unggul, selamat dari segala mara bahaya dan petaka. Kami mohon (Ya Allah) kecukupan yang tidak sampai kami tepaksa meminta jasa orang lain. Berikanlah kami (Ya Allah) iman yang sebenarnya sehingga kami tidak lagi gentar atau mengharap orang lain selain dari Engkau sendiri. Kembangkanlah lembayung rahmatMu kepada kami, keluarga dan anak-anak kami serta sesiapa sahaja yang bersama-sama kami. Jangan (Ya Allah) Engkau biarkan nasib kami ditentukan oleh diri kami sendiri; walaupun kadar sekelip mata atau kadar masa yang lebih pendek dari itu. Wahai Tuhan yang paling mudah dan cepat memperkenankan pinta (perkenankanlah).

Grammars

- Grammars express languages
- Example: the English language

$$\langle \text{sentence} \rangle \rightarrow \langle \text{noun\_phrase} \rangle \langle \text{predicate} \rangle$$

$$\langle \text{noun\_phrase} \rangle \rightarrow \langle \text{article} \rangle \langle \text{noun} \rangle$$

$$\langle \text{predicate} \rangle \rightarrow \langle \text{verb} \rangle$$

Grammars

$\langle \text{article} \rangle \rightarrow \text{a}$

$\langle \text{article} \rangle \rightarrow \text{the}$

$\langle \text{noun} \rangle \rightarrow \text{boy}$

$\langle \text{noun} \rangle \rightarrow \text{dog}$

$\langle \text{verb} \rangle \rightarrow \text{runs}$

$\langle \text{verb} \rangle \rightarrow \text{walks}$

Grammars

- A **derivation** of “the boy walks”:

Grammars

- A **derivation** of "the boy walks":

$$\begin{aligned}
\langle \text{sentence} \rangle &\Rightarrow \langle \text{noun_phrase} \rangle \langle \text{predicate} \rangle \\
&\Rightarrow \langle \text{noun_phrase} \rangle \langle \text{verb} \rangle \\
&\Rightarrow \langle \text{article} \rangle \langle \text{noun} \rangle \langle \text{verb} \rangle \\
&\Rightarrow \text{the } \langle \text{noun} \rangle \langle \text{verb} \rangle \\
&\Rightarrow \text{the boy } \langle \text{verb} \rangle \\
&\Rightarrow \text{the boy walks}
\end{aligned}$$

Grammars

- A derivation of “a dog runs”:

Grammars

- A derivation of "a dog runs":

$$\begin{aligned}
\langle \text{sentence} \rangle &\Rightarrow \langle \text{noun_phrase} \rangle \langle \text{predicate} \rangle \\
&\Rightarrow \langle \text{noun_phrase} \rangle \langle \text{verb} \rangle \\
&\Rightarrow \langle \text{article} \rangle \langle \text{noun} \rangle \langle \text{verb} \rangle \\
&\Rightarrow \text{a} \langle \text{noun} \rangle \langle \text{verb} \rangle \\
&\Rightarrow \text{a dog} \langle \text{verb} \rangle \\
&\Rightarrow \text{a dog runs}
\end{aligned}$$

Grammars

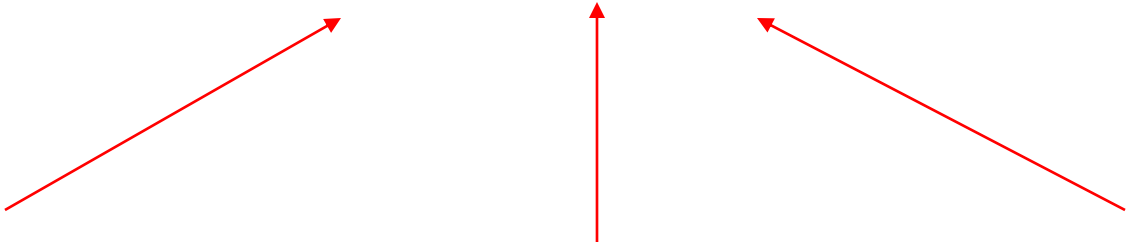
- Language of the grammar:

L = { “a boy runs”,
“a boy walks”,
“the boy runs”,
“the boy walks”,
“a dog runs”,
“a dog walks”,
“the dog runs”,
“the dog walks” }

Notations

$$\langle \text{noun} \rangle \rightarrow \text{boy}$$

$$\langle \text{noun} \rangle \rightarrow \text{dog}$$



Variable or Nonterminal

Production rule

Terminal

Example

- Grammar: $S \to aSb$

  $S \to \lambda$

- Derivation of sentence ab:

$$S \Rightarrow aSb \Rightarrow ab$$

$S \to aSb$ $\qquad$ $S \to \lambda$

Example

- Grammar: $S \to aSb$

  $S \to \lambda$

- Derivation of sentence $aabb$:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb$$

$S \to aSb$ $\quad$ $S \to \lambda$

Example

- Other derivations:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb$$
$$\Rightarrow aaaaSbbbb \Rightarrow aaaabbbb$$

Languages

- Language of the grammar:

$$S \rightarrow aSb$$

$$S \rightarrow \lambda$$

$$L = \{a^n b^n : n \geq 0\}$$

Notations

- Grammar $G = (V, T, S, P)$

  V: a set of **variables** (**nonterminal** symbols)

  T: a set of **terminal** symbols

  S: the **start variable** (the **axiom**)

  P: a set of **production rules**

Example

- Grammar G:

$$S \to aSb$$

$$S \to \lambda$$

$$G = (V, T, S, P)$$

$$V = \{S\} \qquad T = \{a, b\}$$

$$P = \{S \to aSb,\ S \to \lambda\}$$

Notations

- Sentential Form:

  A sentence that contains variables and terminals

- **Example**:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

Sentential Forms

Sentence

Notations

- We write: $S \Rightarrow^* aaabbb$

- Instead of:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

Example

- Grammar

$$S \rightarrow aSb$$

$$S \rightarrow \lambda$$

Derivations

$$S \Rightarrow^* \lambda$$

$$S \Rightarrow^* ab$$

$$S \Rightarrow^* aabb$$

$$S \Rightarrow^* aaabbb$$

Example

- Grammar G:

$$S \to Ab$$

$$A \to aAb$$

$$A \to \lambda$$

- Derivations:

$$S \Rightarrow Ab \Rightarrow b$$

$$S \Rightarrow Ab \Rightarrow aAbb \Rightarrow abb$$

$$S \Rightarrow Ab \Rightarrow aAbb \Rightarrow aaAbbb \Rightarrow aabbb$$

More Derivations

$$S \Rightarrow Ab \Rightarrow aAbb \Rightarrow aaAbbb \Rightarrow aaaAbbbb$$
$$\Rightarrow aaaaAbbbbb \Rightarrow aaaabbbbb$$

$$S \Rightarrow^* aaaabbbbb$$

$$S \Rightarrow^* aaaaaabbbbbbb$$

$$S \Rightarrow^* a^n b^n b$$

Language of a Grammar

- For a grammar G with start variable S:

$$L(G) = \{w : S \Rightarrow^* w\}$$

String of terminals

Example

- For grammar G:

$$S \rightarrow Ab$$

$$A \rightarrow aAb$$

$$A \rightarrow \lambda$$

$$L(G) = \{a^n b^n b : \ n \geq 0\}$$

- Since: $S \Rightarrow^* a^n b^n b$ for any $n \geq 0$

A Convenient Notation

$$A \rightarrow aAb$$
$$A \rightarrow \lambda$$

$$A \rightarrow aAb \mid \lambda$$

Linear Grammars

Linear Grammars

- Grammars with at most one variable at the right side of a production

- **Examples**:

$$S \rightarrow aSb$$
$$S \rightarrow \lambda$$

$$S \rightarrow Ab$$
$$A \rightarrow aAb$$
$$A \rightarrow \lambda$$

Another Linear Grammar

- Grammar G :

$$S \to A$$

$$A \to aB \mid \lambda$$

$$B \to Ab$$

$$L(G) = \{a^n b^n : n \geq 0\}$$

A Non-Linear Grammar

- Grammar G :

$$S \rightarrow SS$$

$$S \rightarrow \lambda$$

$$S \rightarrow aSb$$

$$S \rightarrow bSa$$

Right-Linear Grammars

- All productions have form:

$$A \rightarrow xB$$

or

$$A \rightarrow x$$

string of terminals

- **Example:**

$$S \rightarrow abS$$

$$S \rightarrow a$$

Left-Linear Grammars

- All productions have form:

$$A \to Bx$$

or

$$A \to x$$

- **Example:**

$$S \to Aab$$

$$A \to Aab \mid B$$

$$B \to a$$

string of terminals

Exercises

- For each of the following languages, find a grammar that generate it.

$$L_1 = \{a^n b^m : n \geq 1, m \geq 1\}$$

$$L_2 = \{a^n b^{2n} : n \geq 1\}$$

$$L_3 = \{a^{n+2} b^n : n \geq 1\}$$

$$L_4 = \{a^n b^{n-3} : n \geq 3\}$$

Regular Grammars

Observation

A **regular grammar** is any right-linear or left-linear grammar.

Examples:

G_1

$S \to abS$

$S \to a$

G_2

$S \to Aab$

$A \to Aab \mid B$

$B \to a$

Observation

A **regular grammar** is any right-linear or left-linear grammar.

Examples:

G_1

$S \to abS$

$S \to a$

G_2

$S \to Aab$

$A \to Aab \mid B$

$B \to a$

$L(G_1) = (ab)^* a$

$L(G_2) = aab(ab)^*$

Exercises

- Find grammars for $\Sigma = \{a, b\}$ that generate the sets (languages) of
  - all strings with exactly one a
  - all strings with at least one a
  - all strings with at least 3 a's
  - all strings with no more than 3 a's

Theorem

{ Languages Generated by Regular Grammars } = { Regular Languages }

The Case of Right-Linear Grammars

- Let G be a right-linear grammar
- We will prove: $L(G)$ is regular

Proof idea: We will construct NFA M with $L(M) = L(G)$

Example

- Grammar G is right-linear

- **Example:**

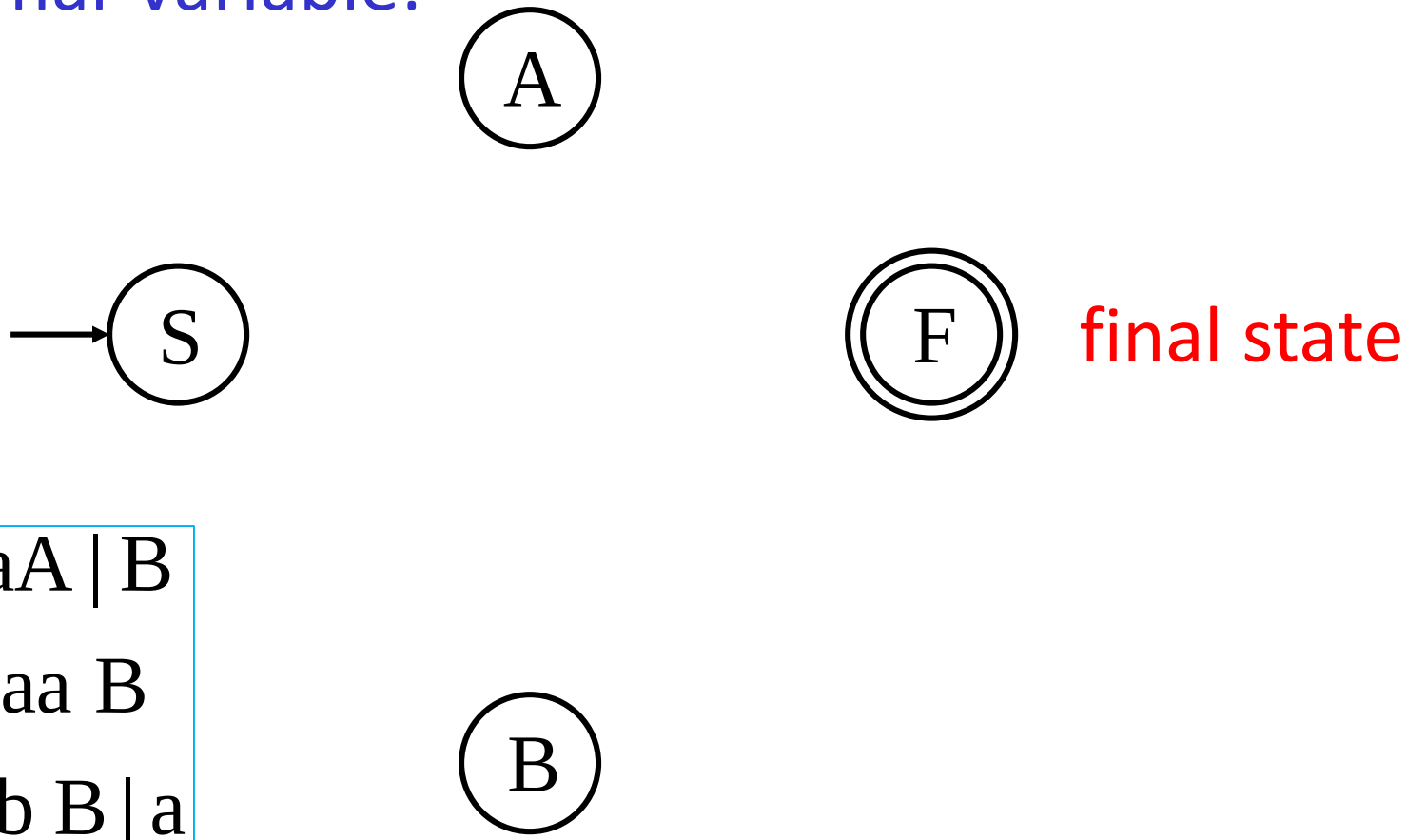
$$S \rightarrow aA \mid B$$

$$A \rightarrow aa\ B$$

$$B \rightarrow bB \mid a$$

Example

- Construct NFA M such that every state is a grammar variable:



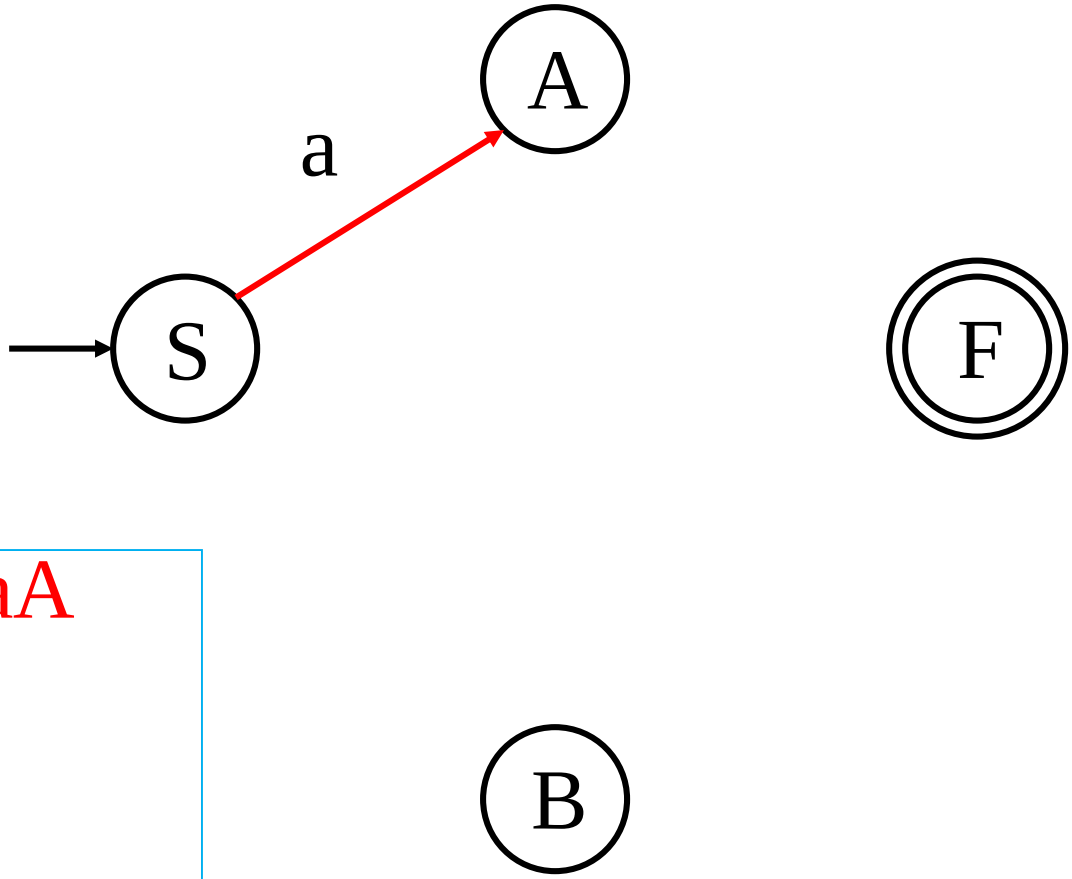
$$S \rightarrow aA \mid B$$

$$A \rightarrow aa\ B$$

$$B \rightarrow b\ B \mid a$$

Example

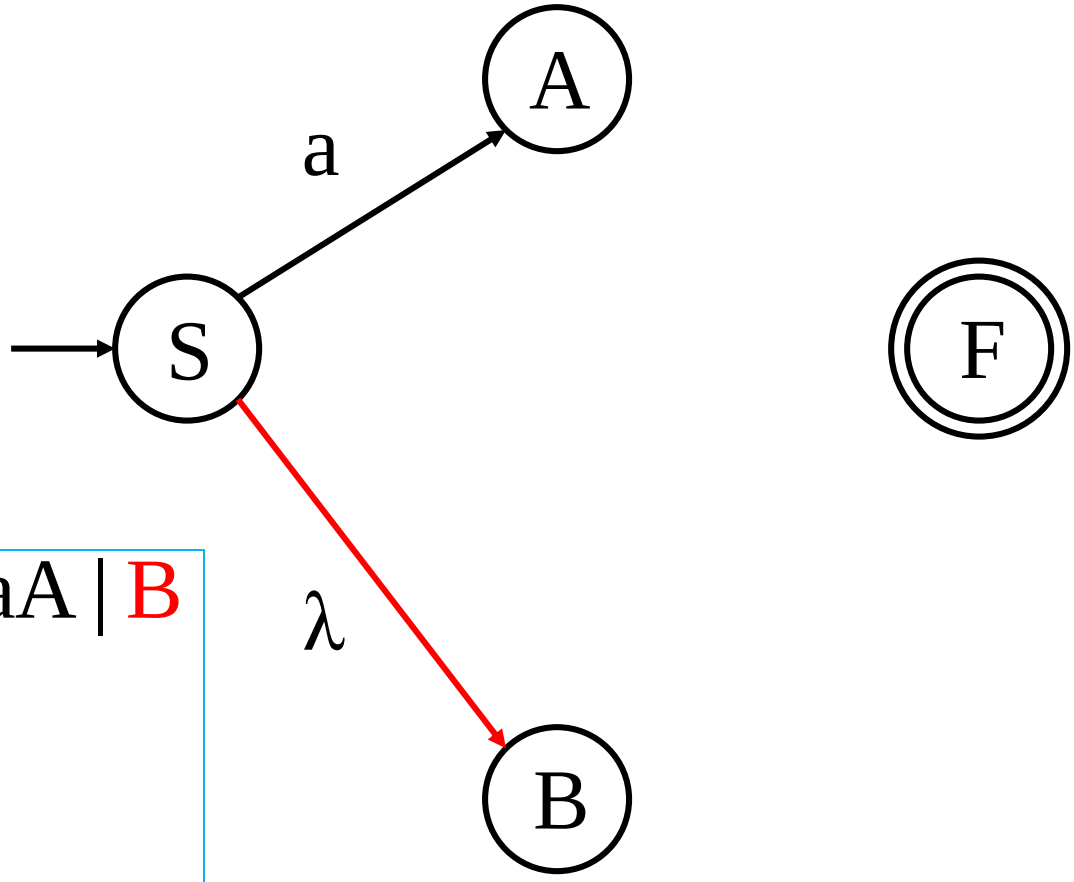
- Add edges for each production:



$S \rightarrow aA$

Example

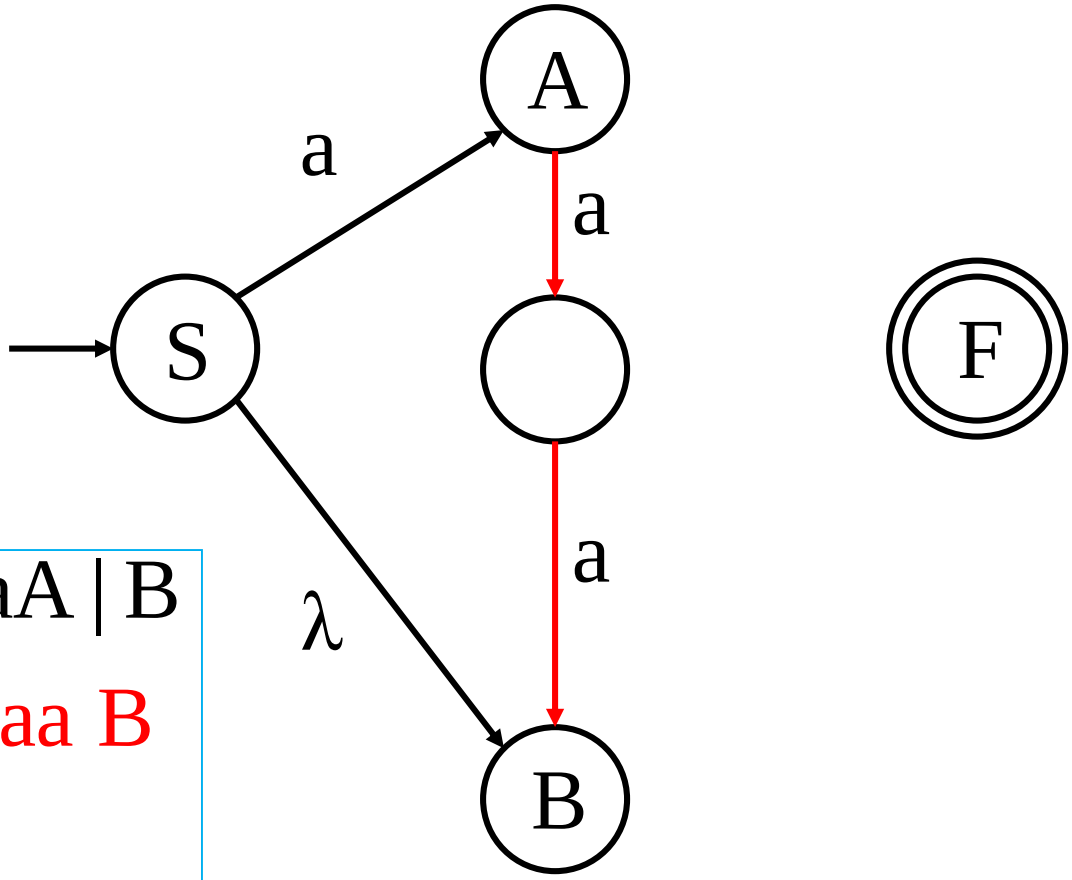
- Add edges for each production:



$S \rightarrow aA \mid B$

Example

- Add edges for each production:

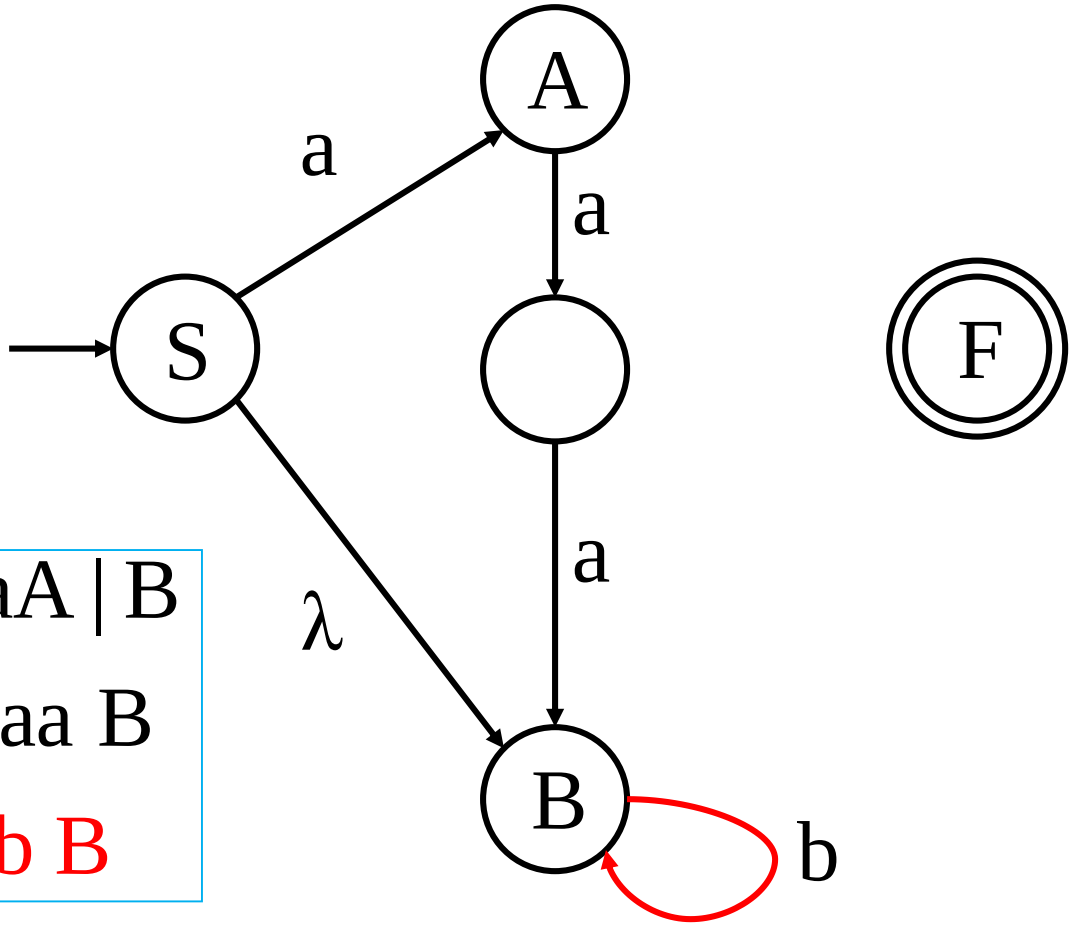


$S \rightarrow aA \mid B$

$A \rightarrow aa\ B$

Example

- Add edges for each production:



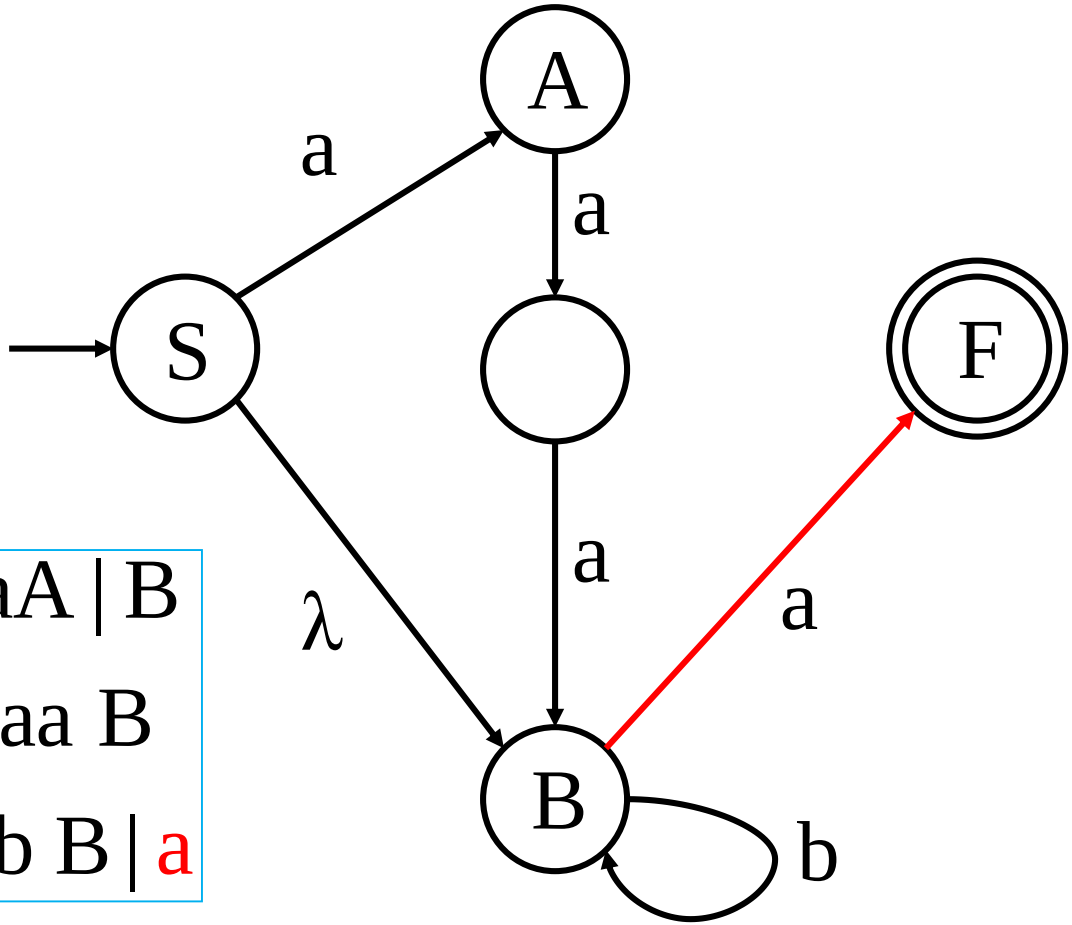
$S \rightarrow aA \mid B$

$A \rightarrow aa\ B$

$B \rightarrow b\ B$

Example

- Add edges for each production:



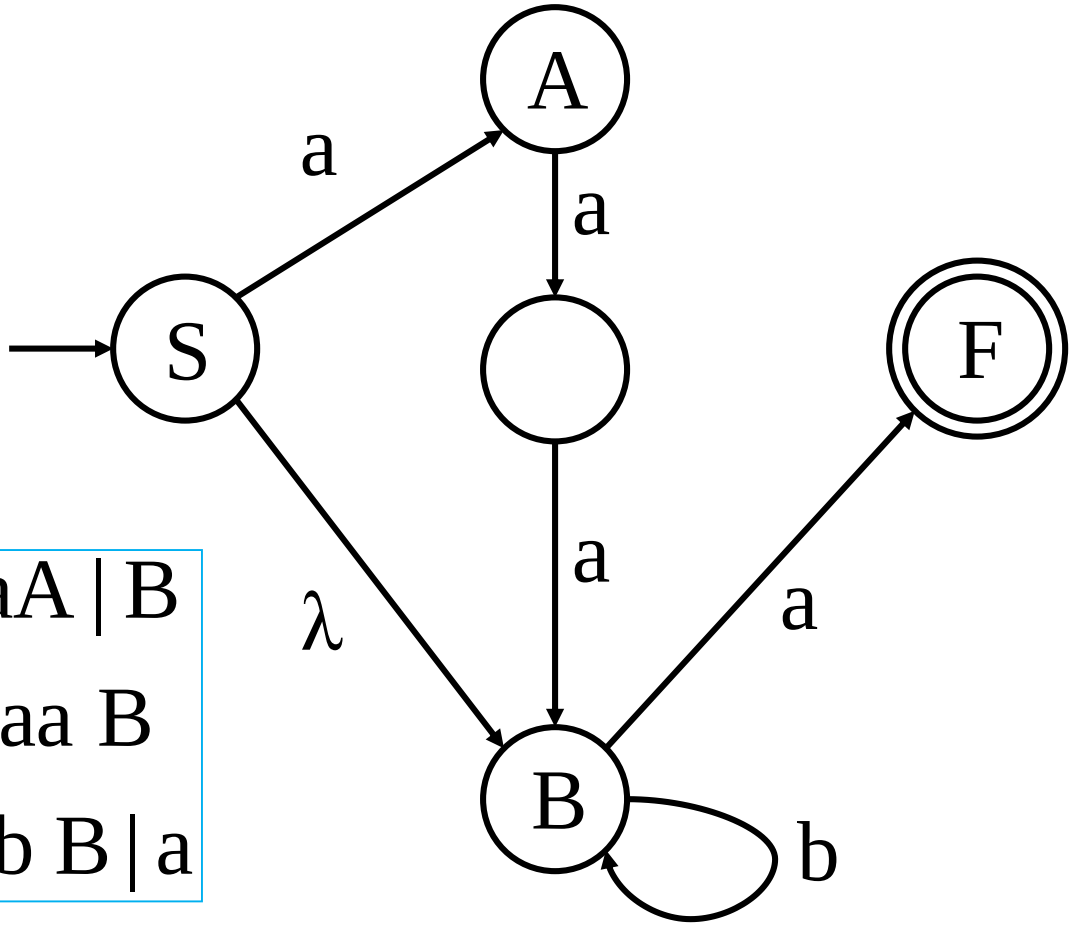
$S \rightarrow aA \mid B$

$A \rightarrow aa\ B$

$B \rightarrow b\ B \mid a$

Example

- NFA M:



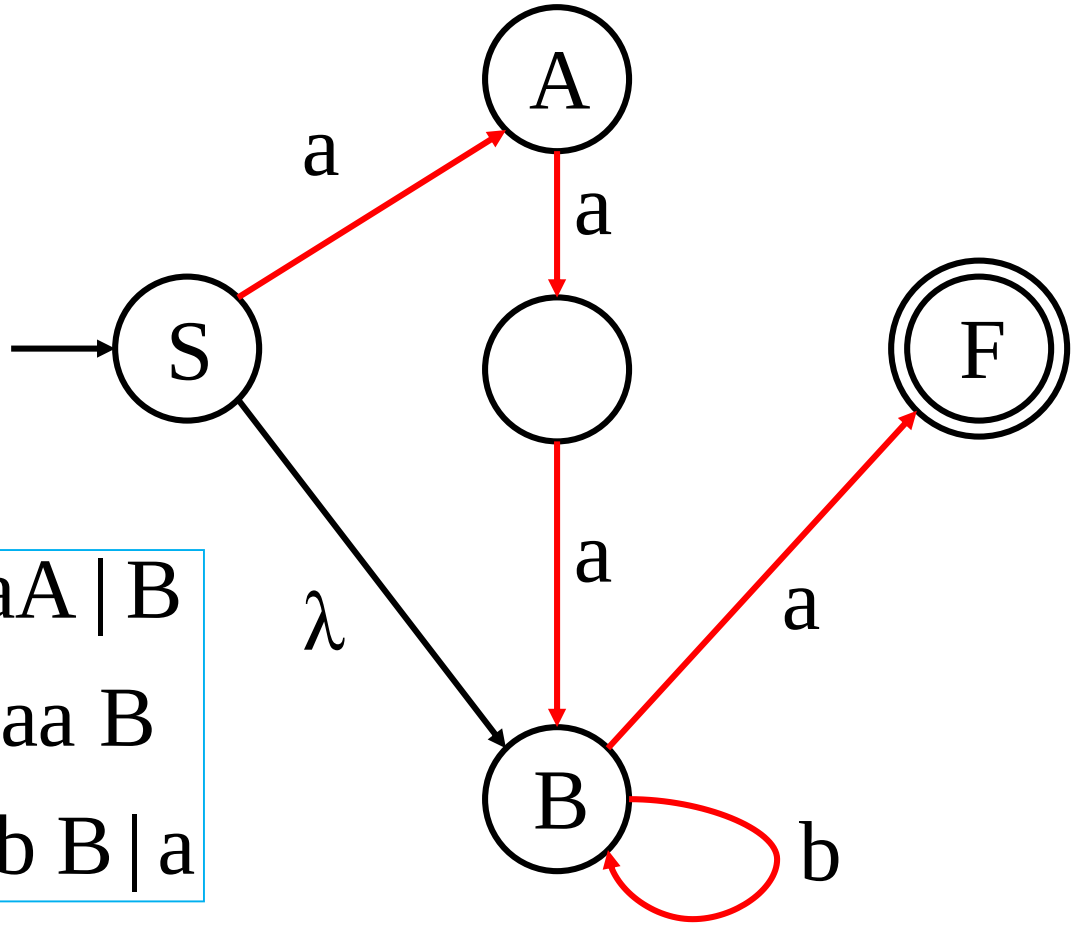
$S \rightarrow aA \mid B$

$A \rightarrow aa\ B$

$B \rightarrow b\ B \mid a$

Example

- NFA M: $S \Rightarrow aA \Rightarrow aaaB \Rightarrow aaabB \Rightarrow aaaba$



$S \rightarrow aA \mid B$

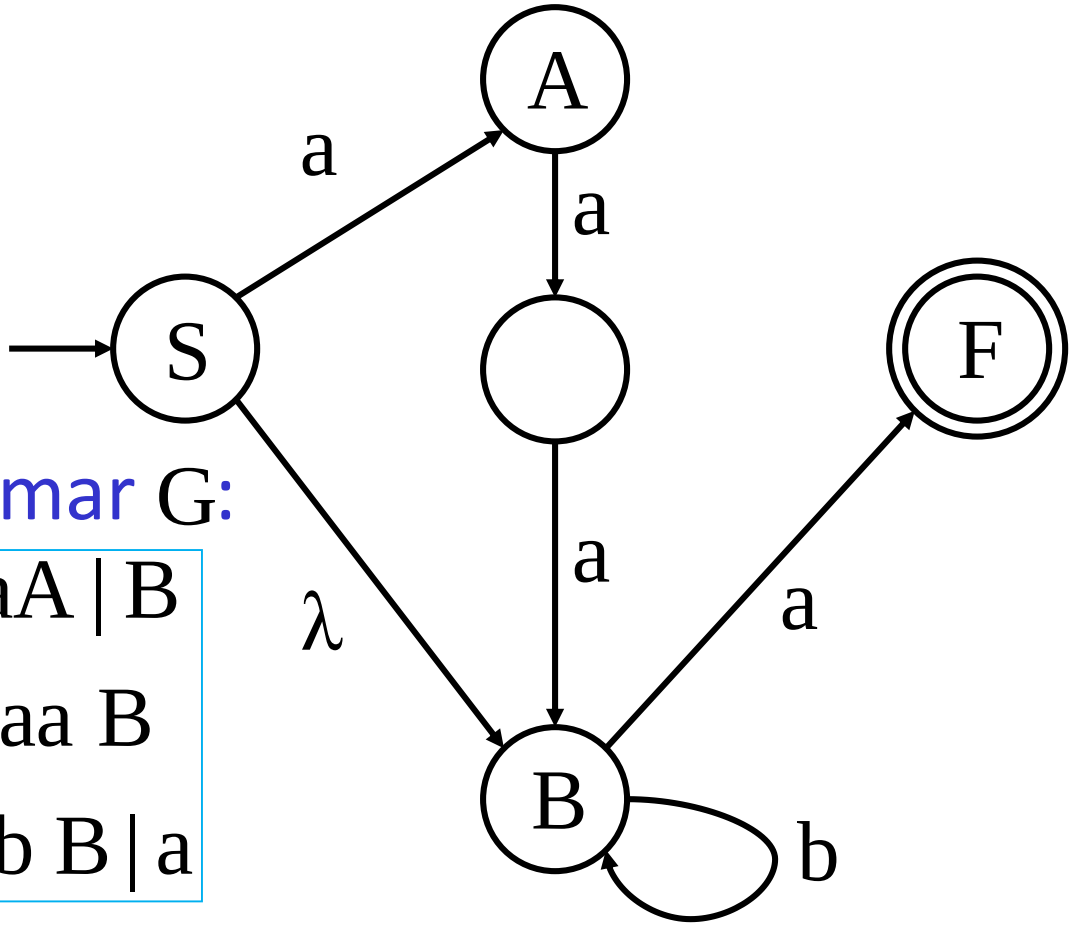
$A \rightarrow aa\ B$

$B \rightarrow b\ B \mid a$

Example

- NFA M:

$$L(G) = L(M) = aaab^*a + b^*a$$



- Grammar G:

$$S \rightarrow aA \mid B$$

$$A \rightarrow aa\ B$$

$$B \rightarrow b\ B \mid a$$

Proof

- Any regular language L is generated by some regular grammar G

Proof idea:

- Let M be the NFA with $L = L(M)$.

- Construct from M a regular grammar G such that

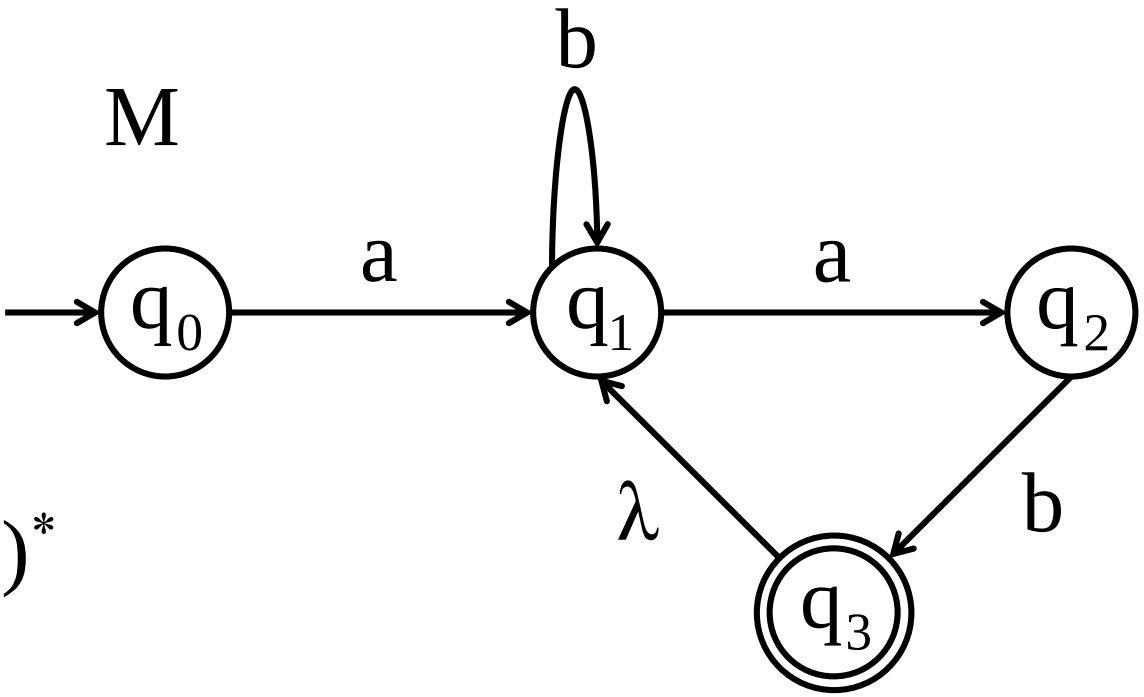
$$L(M) = L(G)$$

Proof

- Since L is regular there is an NFA M such that

$$L = L(M)$$

- **Example:**



$$L = ab^*ab(b^*ab)^*$$

$$L = L(M)$$

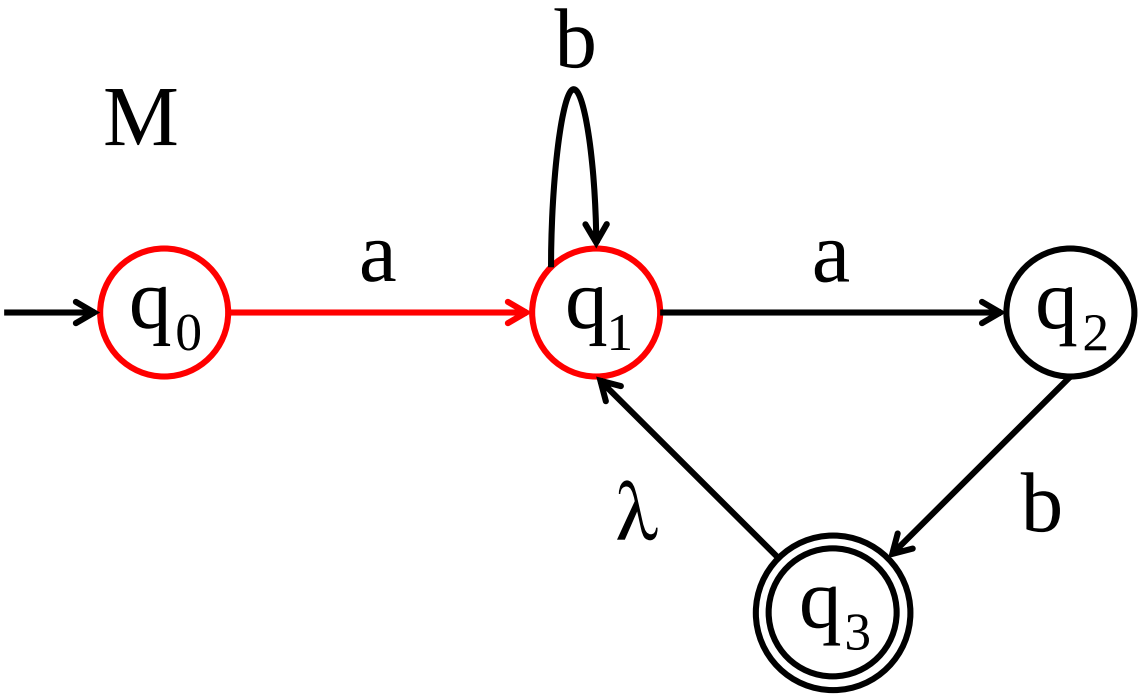
Proof

- Convert M to a right-linear grammar

G

$q_0 \rightarrow aq_1$

M



Proof

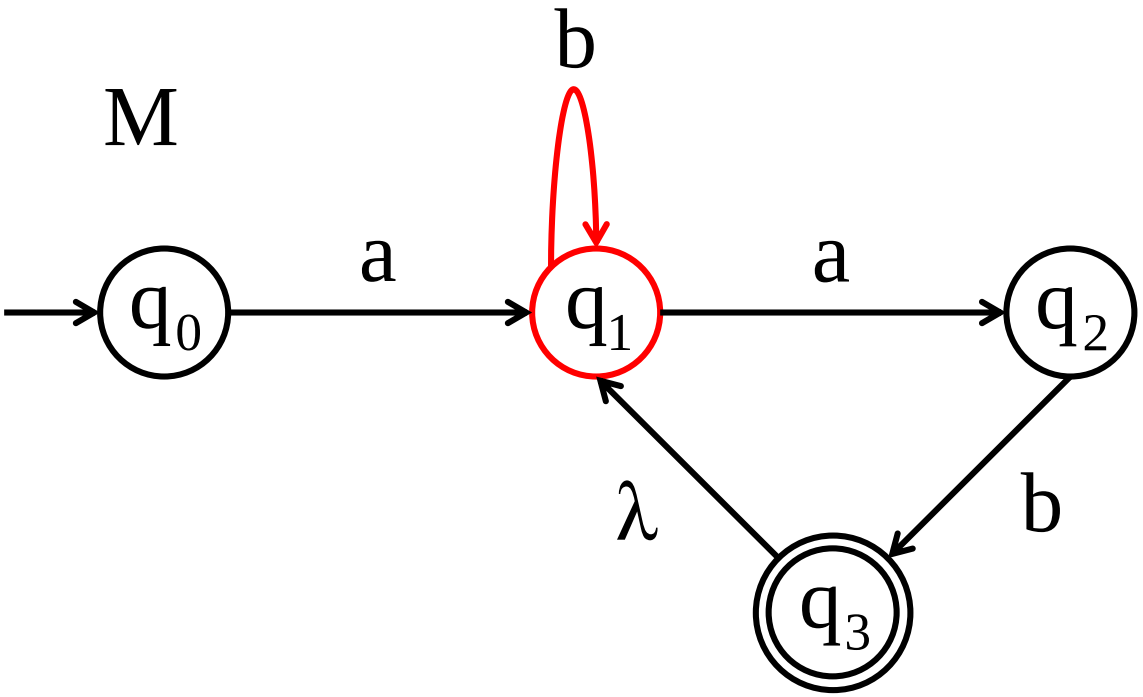
- Convert M to a right-linear grammar

G

$q_0 \rightarrow aq_1$

$q_1 \rightarrow bq_1$

M



Proof

- Convert M to a right-linear grammar

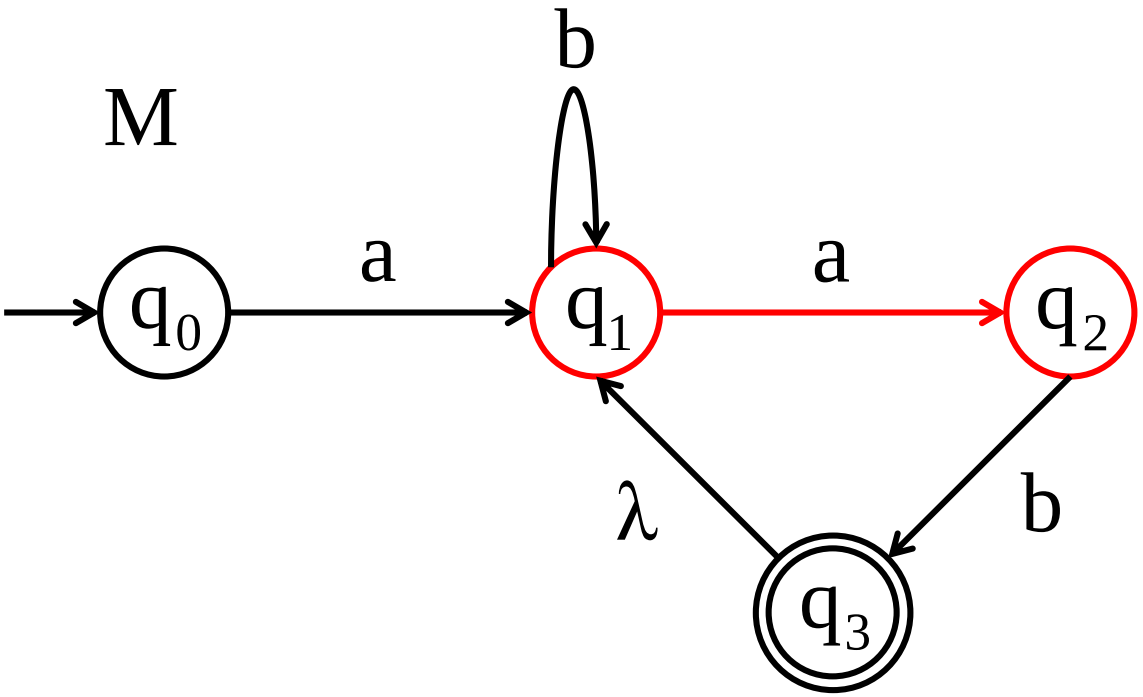
G

$q_0 \rightarrow aq_1$

$q_1 \rightarrow bq_1$

$q_1 \rightarrow aq_2$

M



Proof

- Convert M to a right-linear grammar

G

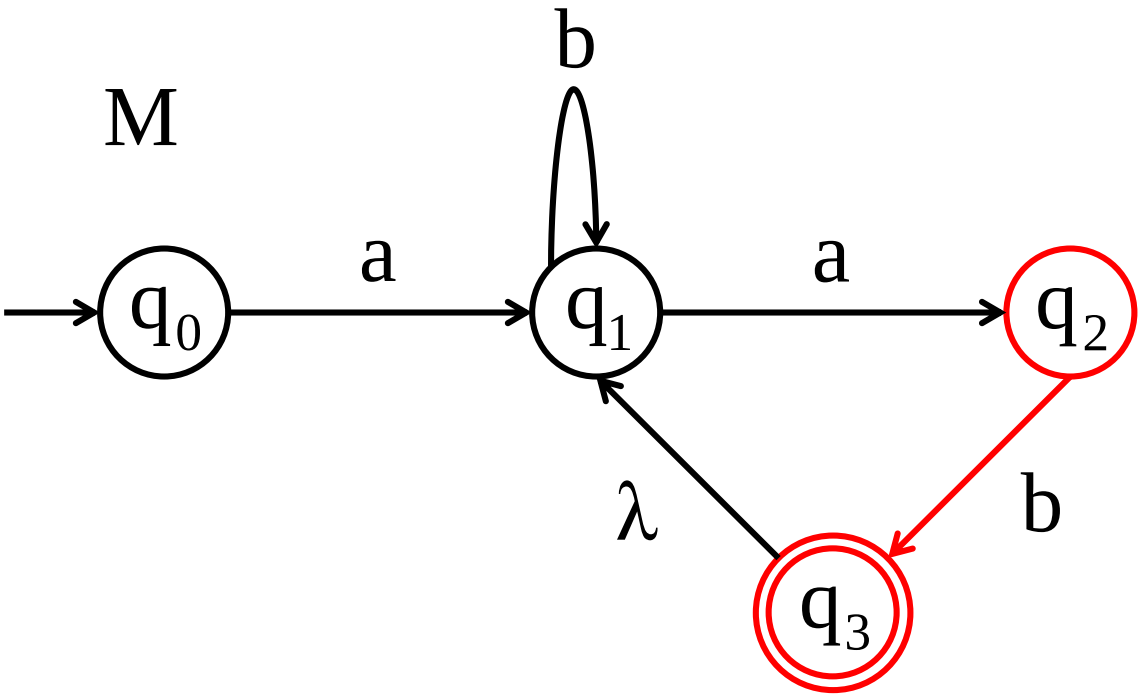
$q_0 \rightarrow aq_1$

$q_1 \rightarrow bq_1$

$q_1 \rightarrow aq_2$

$q_2 \rightarrow bq_3$

M



Proof

- Convert M to a right-linear grammar

G

$q_0 \rightarrow aq_1$

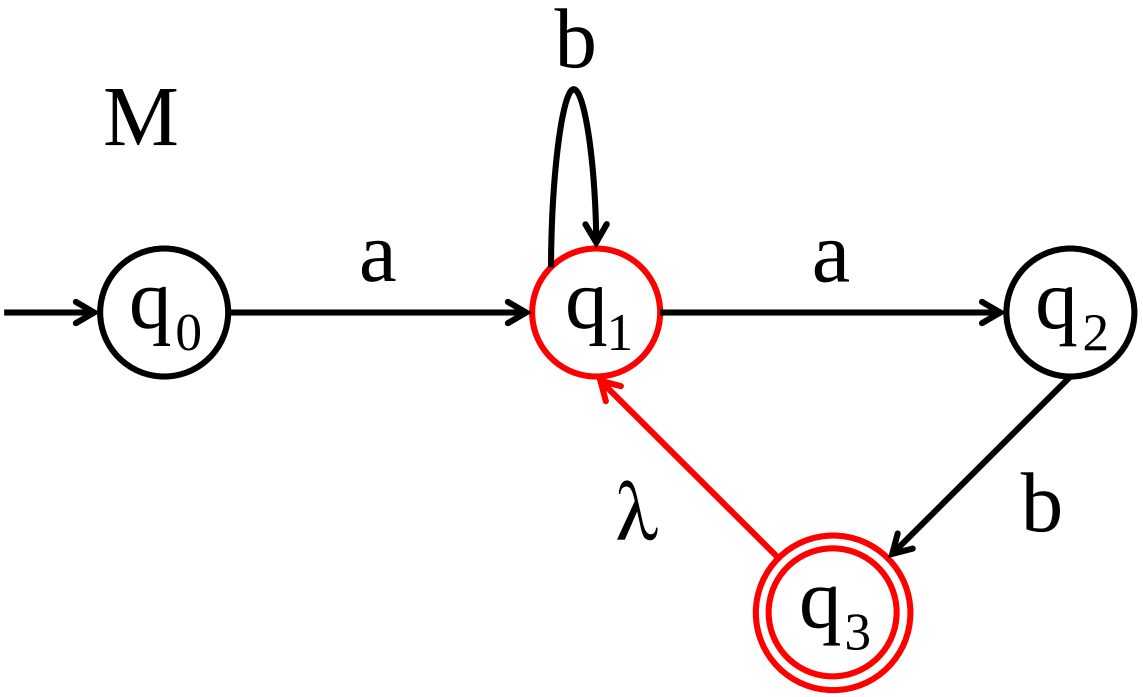
$q_1 \rightarrow bq_1$

$q_1 \rightarrow aq_2$

$q_2 \rightarrow bq_3$

$q_3 \rightarrow q_1$

M



Proof

- Convert M to a right-linear grammar

G

$q_0 \rightarrow aq_1$

$q_1 \rightarrow bq_1$

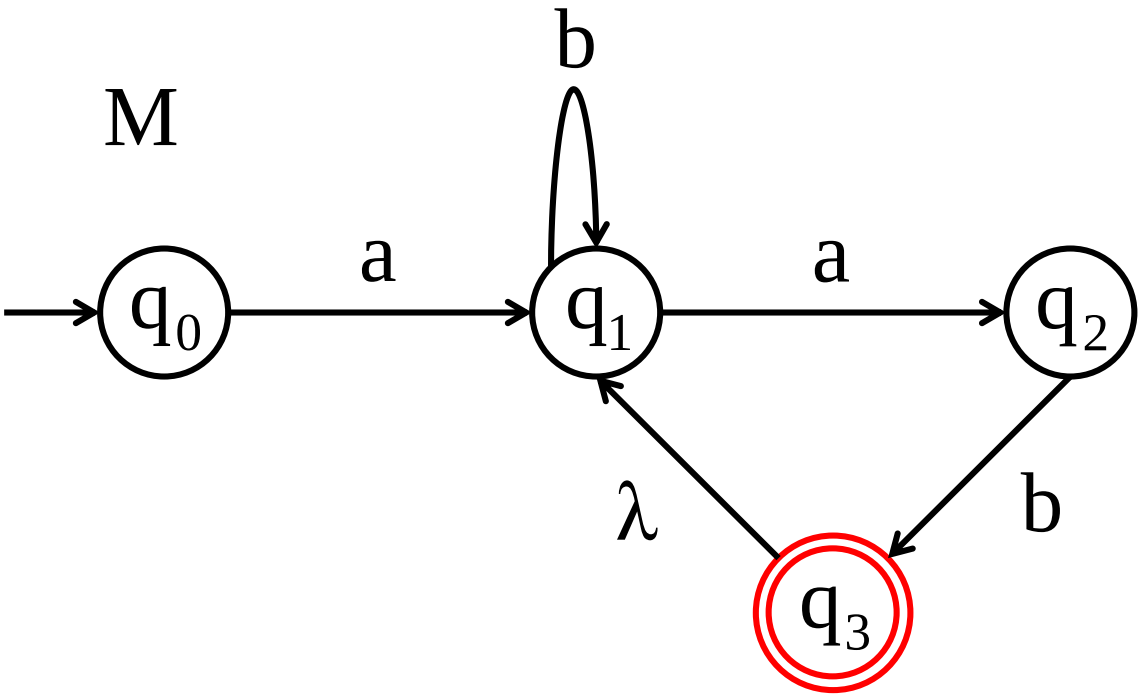
$q_1 \rightarrow aq_2$

$q_2 \rightarrow bq_3$

$q_3 \rightarrow q_1$

$q_3 \rightarrow \lambda$

M



Proof

G

$q_0 \rightarrow aq_1$

$q_1 \rightarrow bq_1$

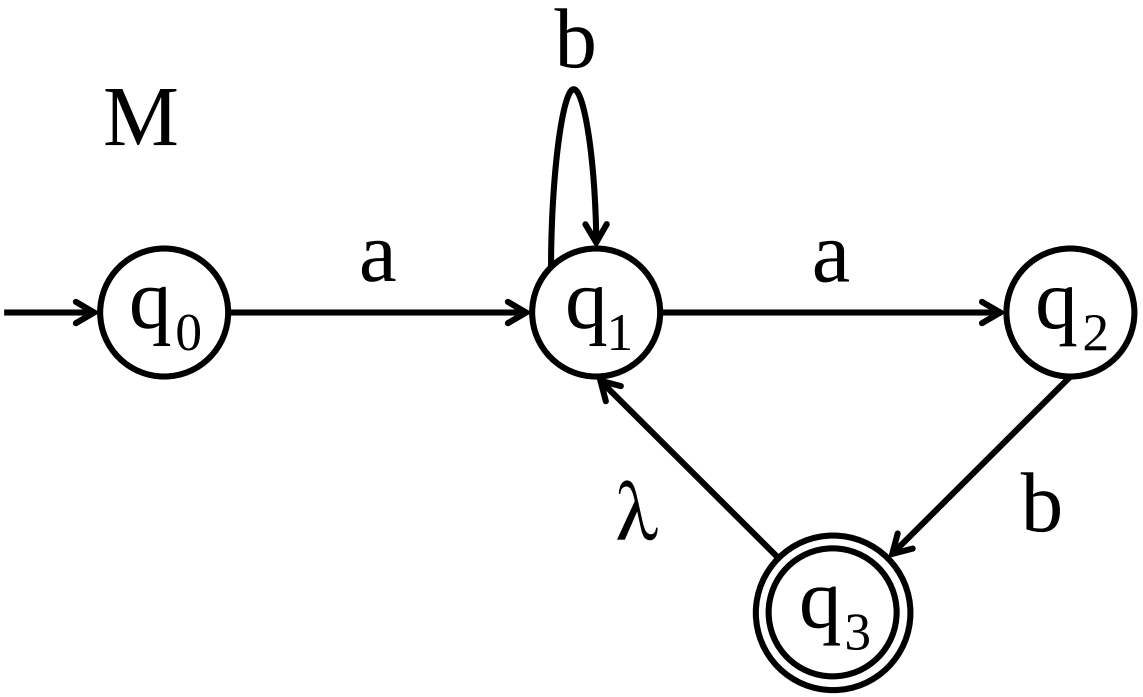
$q_1 \rightarrow aq_2$

$q_2 \rightarrow bq_3$

$q_3 \rightarrow q_1$

$q_3 \rightarrow \lambda$

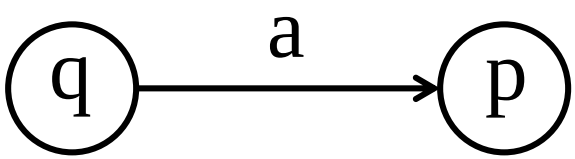
M



$$L(G) = L(M) = L$$

In General

- For any transition:



- Add production:

$$q \rightarrow ap$$

variable terminal variable

In General

- For any final state: 

- Add production: $q_f \rightarrow \lambda$

In General

- Since G is right-linear grammar

  G is also a regular grammar with

$$L(G) = L(M) = L$$

Exercises

- Find a regular grammar for the language

$$L = \{a^n b^m : n + m \text{ is even}\}$$

Exercises

- What languages do the grammars with the following productions generate?

1. $S \to aA$
   $A \to bS$
   $S \to \lambda$

2. $S \to Aa$
   $A \to B$
   $B \to Aa, \quad A \to a$

Construct an equivalent FA for each grammar.

Exercises

- Construct an FA that accepts the language generated by the grammar

$$S \rightarrow abA$$

$$A \rightarrow baB$$

$$B \rightarrow aA|bb$$

- Construct a left-linear grammar for the language generated by the grammar above.