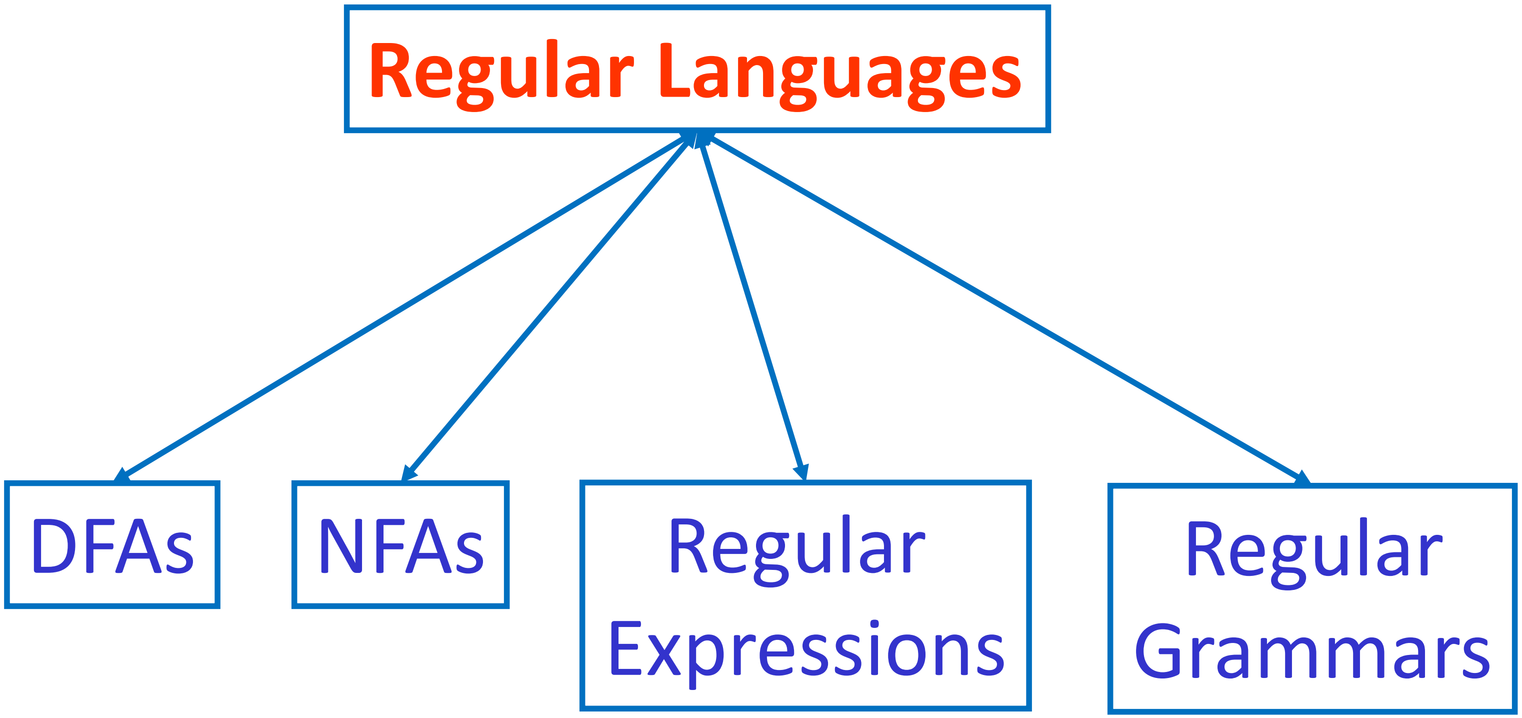
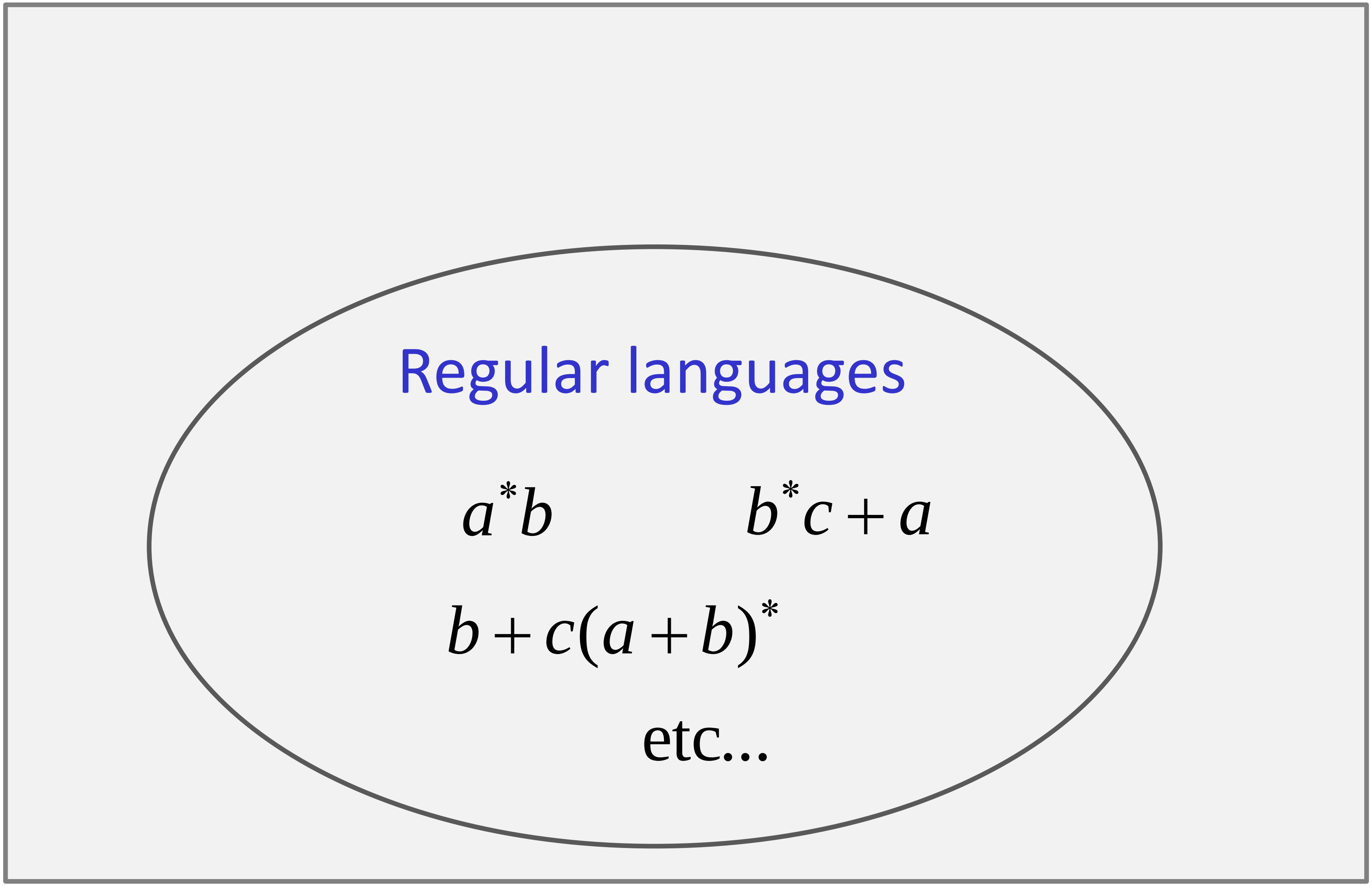
Standard Representations



Context-Free Languages

Non-regular Languages



Non-regular Languages

$$\{a^n b^n : n \geq 0\}$$

Non-regular languages

$$\{ww^R : w \in \{a,b\}^*\}$$

Regular languages

$$a^* b \qquad b^* c + a$$

$$b + c(a+b)^*$$

etc...

Non-regular Languages

$$\{a^n b^n : \ n \geq 0\}$$

Context-free languages

$$\{ww^R : \ w \in \{a,b\}^*\}$$

Regular languages

$$a^* b \qquad b^* c + a$$

$$b + c(a+b)^*$$

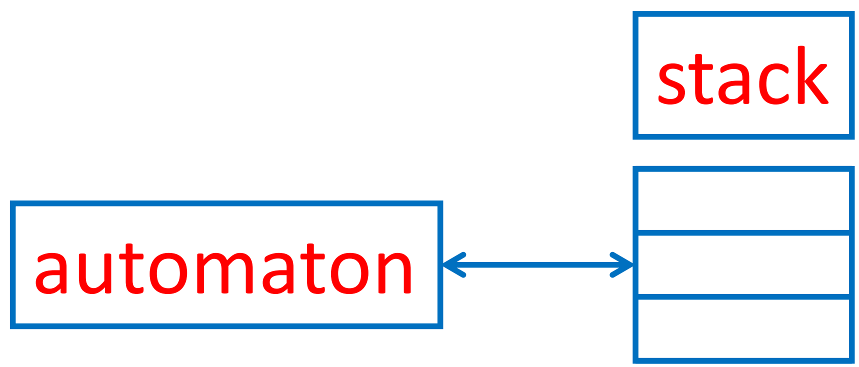
etc...

Context-free Languages

Context-Free Languages

Context-Free Grammars

Pushdown Automata



Context-free Grammars

- A context-free grammar:

$$G:$$

$$S \to aSb$$

$$S \to \lambda$$

- A derivation:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb = a^3b^3$$

Context-free Grammars

- A context-free grammar:

$$G:$$

$$S \to aSb$$

$$S \to \lambda$$

- Another derivation:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb$$

$$\Rightarrow aaaaSbbbb \Rightarrow aaaabbbb = a^4b^4$$

Context-free Grammars

- A context-free grammar:

$$G:$$

$$S \rightarrow aSb$$

$$S \rightarrow \lambda$$

- The language:

$$L(G) = \{a^n b^n : n \geq 0\}$$

Context-free Grammars

- Another context-free grammar:

$$G:$$

$$S \to aSa$$

$$S \to bSb$$

$$S \to \lambda$$

- A derivation:

$$S \Rightarrow aSa \Rightarrow abSba \Rightarrow abba$$

Context-free Grammars

- Another context-free grammar:

$$G:$$
$$S \to aSa$$
$$S \to bSb$$
$$S \to \lambda$$

- Another derivation:

$$S \Rightarrow aSa \Rightarrow abSba \Rightarrow abaSaba \Rightarrow abaaba$$

Context-free Grammars

- Another context-free grammar:

$$G:$$

$$S \to aSa$$

$$S \to bSb$$

$$S \to \lambda$$

- The language generated by G:

$$L(G) = \{ww^R : w \in \{a,b\}^*\}$$

Context-free Grammars

- A context-free grammar (not linear!):

$$G:$$

$$S \to SS$$

$$S \to aSb$$

$$S \to \lambda$$

- A derivation:

$$S \Rightarrow SS \Rightarrow aSbS \Rightarrow abS \Rightarrow ab$$

Context-free Grammars

- A context-free grammar:

$$G:$$

$$S \to SS$$

$$S \to aSb$$

$$S \to \lambda$$

- A derivation:

$$S \Rightarrow SS \Rightarrow aSbS \Rightarrow abS \Rightarrow abaSb \Rightarrow abab$$

Context-free Grammars

- A context-free grammar:

$$G:$$

$$S \rightarrow SS$$

$$S \rightarrow aSb$$

$$S \rightarrow \lambda$$

- The language generated by G:

$$L(G) = \{w : n_a(w) = n_b(w) \text{ and } n_a(v) \geq n_b(v) \text{ in any prefix } v\}$$

Context-free Grammars

- A context-free grammar:

$$G = (V, T, S, P)$$

- V: a set of variables (nonterminals)
- T: a set of terminals
- S: the start symbol (the axiom)
- P: a set of production rules of the form

$$A \to x$$

- x is a string of variables and terminals

Context-free Languages

- A language L is context-free if and only if there is a context-free grammar G with

$$\boxed{L = L(G)}$$

Derivation Order

$$1.\ S \to AB \qquad 2.\ A \to aaA \qquad 4.\ B \to Bb$$

$$3.\ A \to \lambda \qquad 5.\ B \to \lambda$$

- Leftmost derivation

$$S \overset{1}{\Rightarrow} AB \overset{2}{\Rightarrow} aaAB \overset{3}{\Rightarrow} aaB \overset{4}{\Rightarrow} aaBb \overset{5}{\Rightarrow} aab$$

- Rightmost derivation

$$S \overset{1}{\Rightarrow} AB \overset{4}{\Rightarrow} ABb \overset{5}{\Rightarrow} Ab \overset{2}{\Rightarrow} aaAb \overset{3}{\Rightarrow} aab$$

Derivation Order

$$S \to aAB, \quad A \to bBb, \quad B \to A \mid \lambda$$

- Leftmost derivation

$$S \Rightarrow aAB \Rightarrow abBbB \Rightarrow abAbB \Rightarrow abbBbbB$$
$$\Rightarrow abbbbB \Rightarrow abbbb$$

- Rightmost derivation

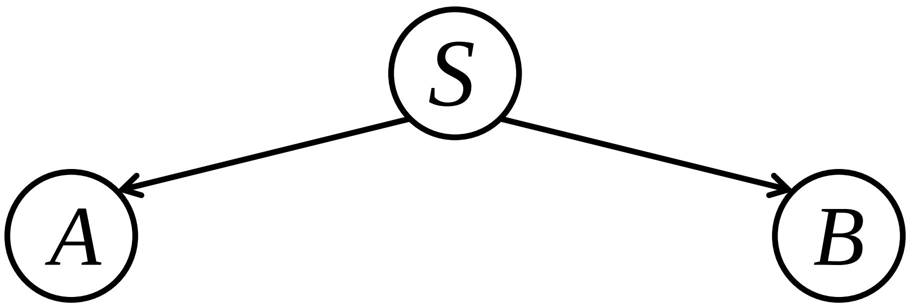
$$S \Rightarrow aAB \Rightarrow aA \Rightarrow abBb \Rightarrow abAb$$
$$\Rightarrow abbBbb \Rightarrow abbbb$$

Derivation Trees

Derivation Trees

$$S \to AB \qquad A \to aaA \mid \lambda \qquad B \to Bb \mid \lambda$$

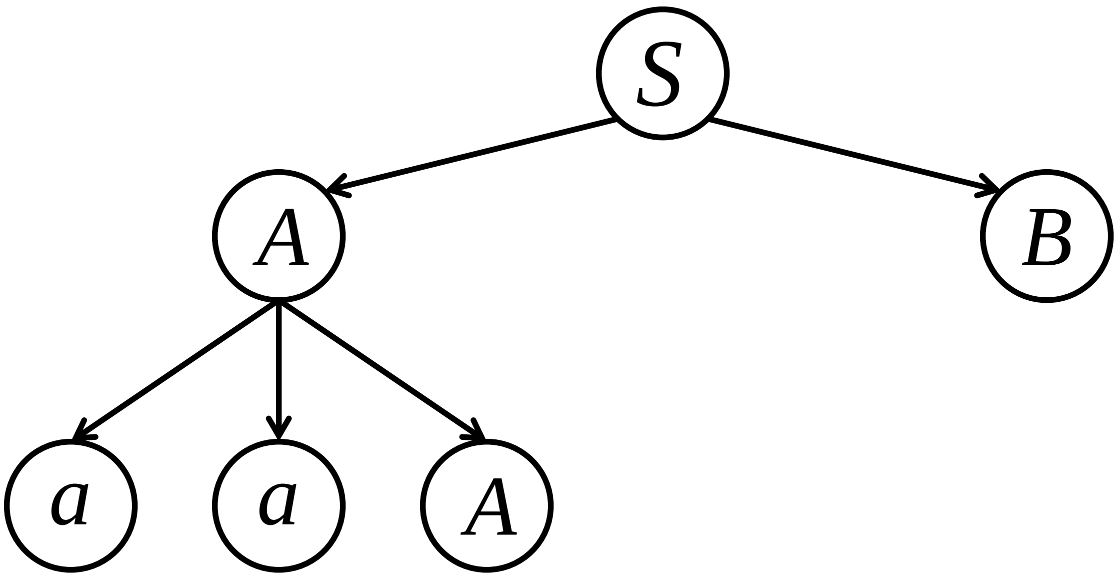
$$S \Rightarrow AB$$



Derivation Trees

$S \to AB$ $\quad A \to aaA \mid \lambda$ $\quad B \to Bb \mid \lambda$

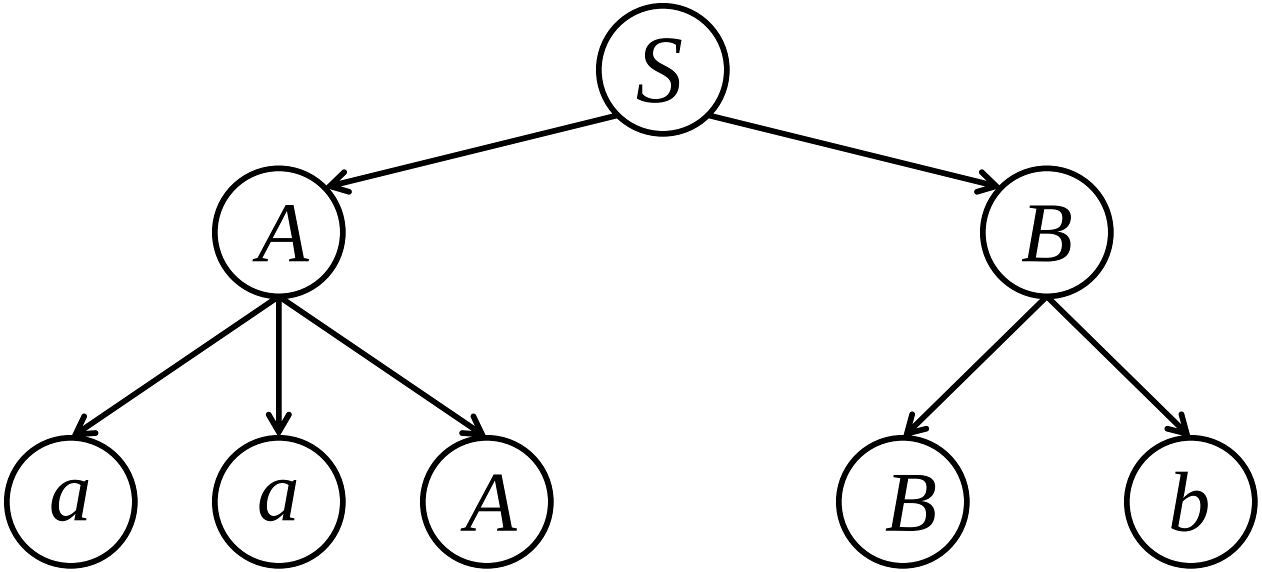
$$S \Rightarrow AB \Rightarrow aaAB$$



Derivation Trees

$$S \rightarrow AB \qquad A \rightarrow aaA \mid \lambda \qquad B \rightarrow Bb \mid \lambda$$

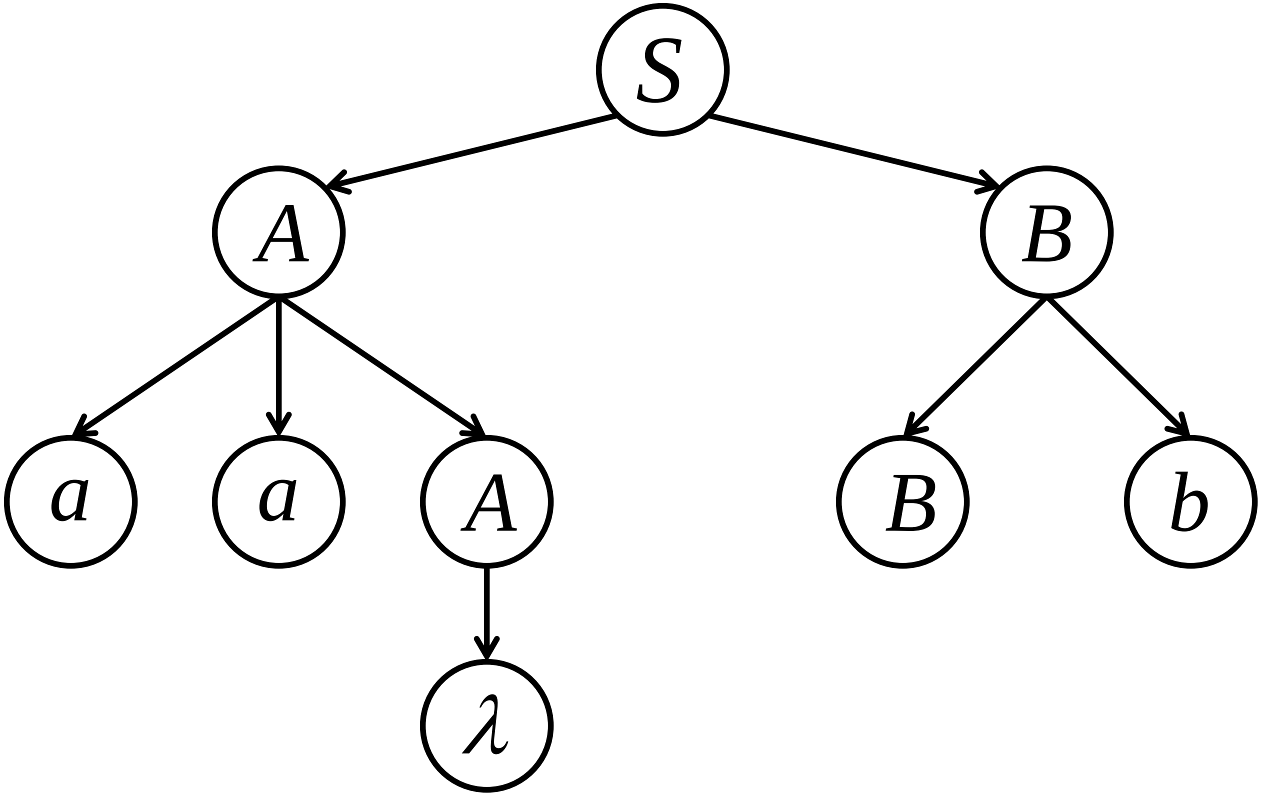
$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb$$



Derivation Trees

$$S \rightarrow AB \qquad A \rightarrow aaA \mid \lambda \qquad B \rightarrow Bb \mid \lambda$$

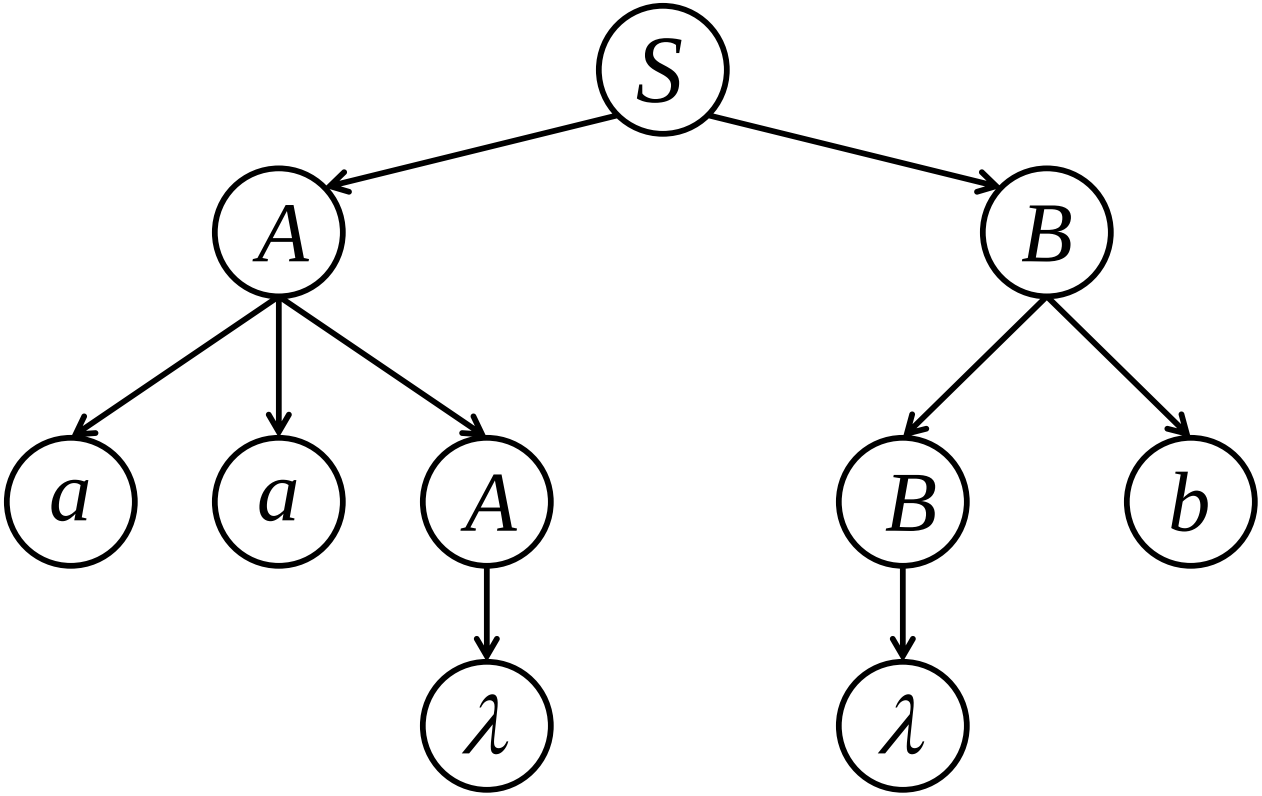
$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb \Rightarrow aaBb$$



Derivation Trees

$$S \to AB \qquad A \to aaA \mid \lambda \qquad B \to Bb \mid \lambda$$

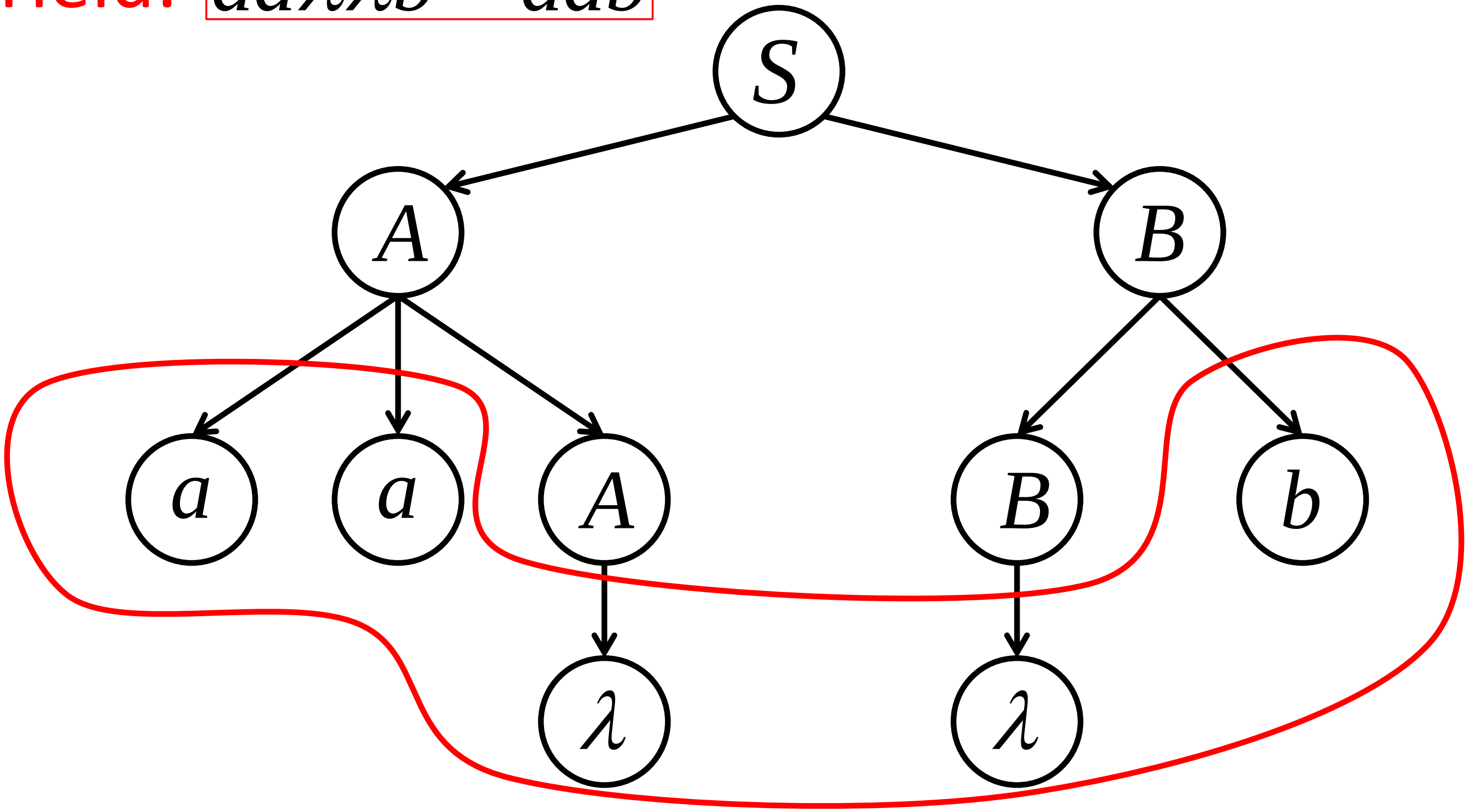
$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb \Rightarrow aaBb \Rightarrow aab$$



Derivation Trees

$S \to AB$ $\qquad A \to aaA \mid \lambda$ $\qquad B \to Bb \mid \lambda$

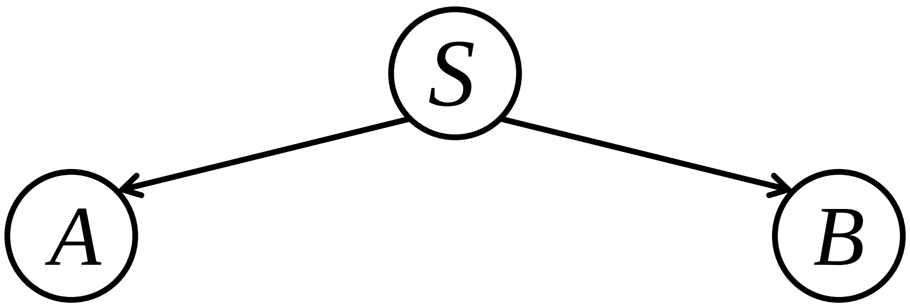
- Yield: $\boxed{aa\lambda\lambda b = aab}$



Partial Derivation Trees

$$S \to AB \qquad A \to aaA \mid \lambda \qquad B \to Bb \mid \lambda$$

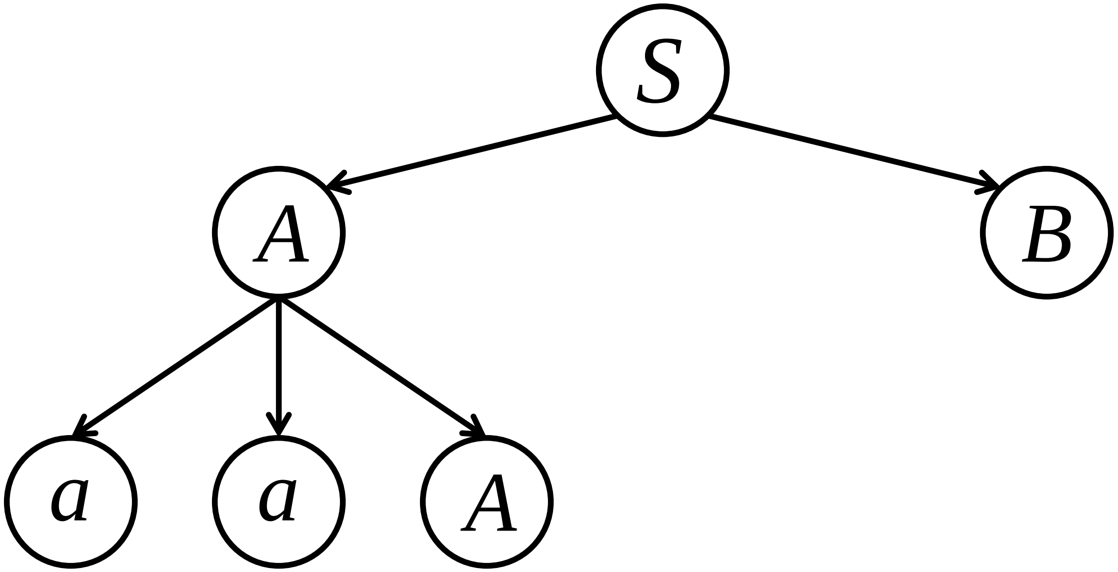
$$S \Rightarrow AB$$



Partial Derivation Trees

$$S \to AB \qquad A \to aaA \mid \lambda \qquad B \to Bb \mid \lambda$$

$$S \Rightarrow AB \Rightarrow aaAB$$

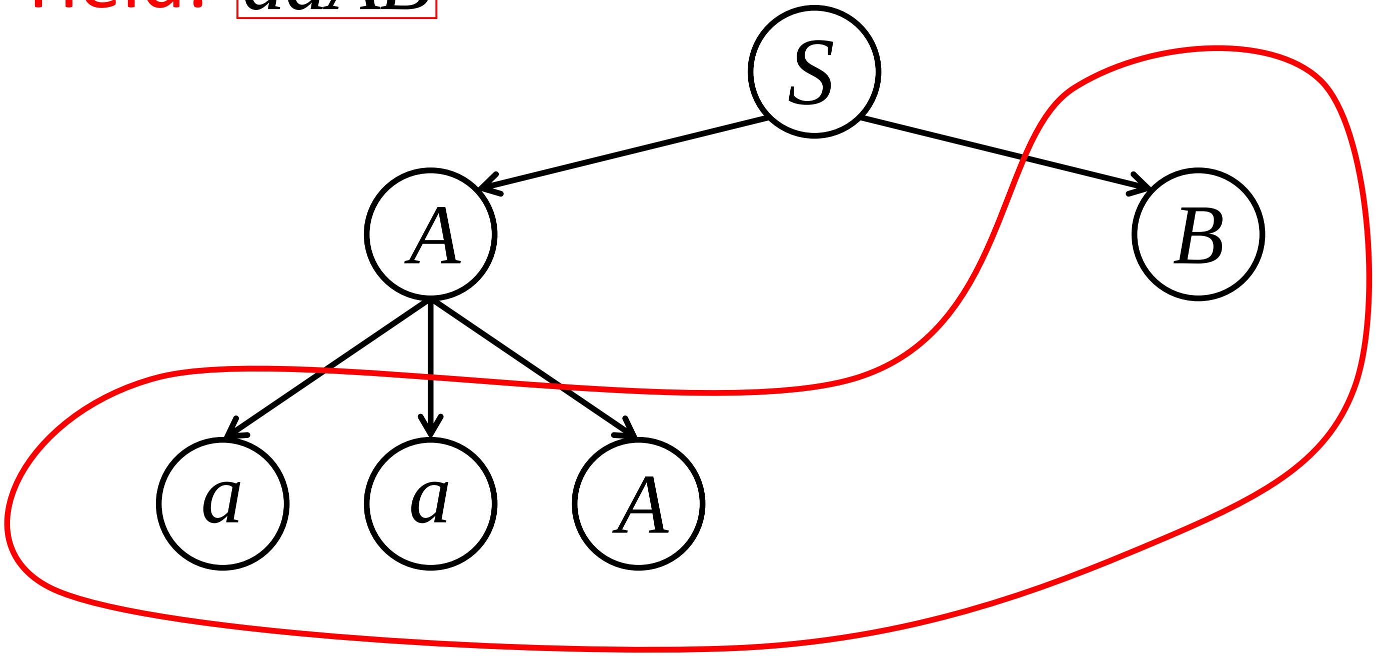


Partial Derivation Trees

$$S \to AB \qquad A \to aaA \mid \lambda \qquad B \to Bb \mid \lambda$$

$$S \Rightarrow AB \Rightarrow aaAB$$: sentential form

- Yield: $\boxed{aaAB}$



Exercises

1. Show a derivation tree for the string $aabbbb$ with the grammar:

$$S \to AB \mid \lambda$$
$$A \to aB$$
$$B \to Sb$$

Exercises

2. Find context-free grammars for the following languages $(n, m \geq 0)$:

- $L = \{a^n b^m : n \neq m - 1\}$
- $L = \{a^n b^m : 2n \leq m \leq 3n\}$

Exercises

3. Find context-free grammars for the following languages ($n, m, k \geq 0$)

- $L = \{a^n b^m c^k : n = m \text{ or } m \leq k\}$
- $L = \{a^n b^m c^k : n = m \text{ or } m \neq k\}$
- $L = \{a^n b^m c^k : k = n + m\}$

Simplifications
of
Context-Free Grammars

* اَللّٰهُمَّ إِنَّـا نَسْأَلُكَ لِسَانًا رَطِبًا بِذِكْرِكَ وَقَلْبًـا مُفْعَمًـا بِشُكْـرِكَ وَبَدَنًا هَيِّنًا لَيِّنًا بِطَاعَتِكَ اَللّٰهُمَّ إِنَّا نَسْأَلُكَ إِيْمَانًا كَامِلًا

وَنَسْأَلُكَ قَلْبًا خَاشِعًا وَنَسْأَلُكَ عِلْمًا نَافِعًا وَنَسْأَلُكَ يَقِيْنًا صَادِقًا وَنَسْأَلُكَ دِيْنًا قَيِّمًا وَنَسْأَلُكَ الْعَافِيَةَ مِنْ كُلِّ بَلِيَّةٍ وَنَسْأَلُكَ تَمَامَ الْغِنَى عَنِ النَّاسِ وَهَبْ لَنَا حَقِيْقَةَ الْإِيْمَـانِ بِكَ حَتَّى لَا نَخَافَ وَلَا نَرْجُوْ غَيْرَكَ وَلَا نَعْبُدَ شَيْئًا سِوَاكَ وَاجْعَلْ يَدَكَ مَبْسُـوْطَةً عَلَيْنَا وَعَلَى أَهْلِيْنَا وَأَوْلَادِنَا وَمَنْ مَعَنَـا بِرَحْمَتِـكَ وَلَا تَكِـلْنَـا إِلَى أَنْفُسِنَا طَرْفَةَ عَيْنٍ وَلَا أَقَلَّ مِنْ ذٰلِكَ يَا نِعْمَ الْمُجِيْبُ.

Useless Productions

$$S \rightarrow aSb$$

$$S \rightarrow \lambda$$

$$S \rightarrow A$$

$$A \rightarrow aA$$

Useless Production

Some derivations never terminate...

$$S \Rightarrow A \Rightarrow aA \Rightarrow aaA \Rightarrow \ldots \Rightarrow aa\ldots aA \Rightarrow \ldots$$

Useless Productions

$$S \to A$$

$$A \to aA$$

$$A \to \lambda$$

$$B \to bA$$ Useless Production

Not reachable from S

Useless Productions

- In general

- If

$$S \Rightarrow \cdots \Rightarrow xAy \Rightarrow \cdots \Rightarrow w \in L(G)$$

- Then variable A is useful, otherwise it is useless

Useless Productions

- A production $A \to x$ is useful if all its variables are useful

Removing Useless Productions

- Example Grammar:

$$S \rightarrow aS \mid A \mid C$$
$$A \rightarrow a$$
$$B \rightarrow aa$$
$$C \rightarrow aCb$$

Removing Useless Productions

- **First:** find all variables that produce strings with only terminals

- Round 1: $\{A, B\}$

- Round 2: $\{A, B, S\}$

$$\boxed{\begin{aligned} &S \to aS \mid A \mid C \\ &A \to a \\ &B \to aa \\ &C \to aCb \end{aligned}}$$

Removing Useless Productions

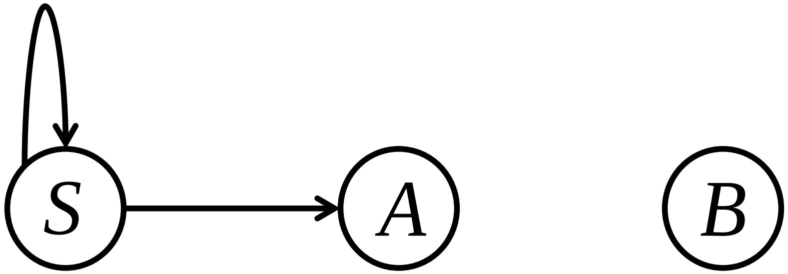
- Keep only the variables that produce terminal symbols: $\{A, B, S\}$

$$\boxed{\begin{aligned} &S \to aS \mid A \mid C \\ &A \to a \\ &B \to aa \\ &C \to aCb \end{aligned}} \quad \Longrightarrow \quad \boxed{\begin{aligned} &S \to aS \mid A \\ &A \to a \\ &B \to aa \end{aligned}}$$

Removing Useless Productions

- **Second:** find all variables reachable from S :

- Dependency Graph



$$S \to aS \mid A$$
$$A \to a$$
$$B \to aa$$

not reachable

Removing Useless Productions

- Keep only the variables reachable from S :

$$
\begin{aligned}
S &\to aS \mid A \\
A &\to a \\
B &\to aa
\end{aligned}
$$

$\Longrightarrow$

Final Grammar

$$
\begin{aligned}
S &\to aS \mid A \\
A &\to a
\end{aligned}
$$

A Substitution Rule: Example

$$A \to a$$
$$A \to aaA$$
$$A \to abBc$$
$$B \to abbA$$
$$B \to b$$

Substitute B

Equivalent grammar

$$A \to a$$
$$A \to aaA$$
$$A \to ababbAc$$
$$A \to abbc$$

A Substitution Rule: In General

$$A \to xBz$$

$$B \to y_1 \mid y_2 \mid \cdots \mid y_n$$

Equivalent grammar

$$A \to xy_1z \mid xy_2z \mid \cdots \mid xy_nz$$

Nullable Variables

$\lambda-\text{production:}$ $A \to \lambda$

Nullable Variable: $A \Rightarrow \ldots \Rightarrow \lambda$

Removing Nullable Variables

- Example Grammar:

$$S \rightarrow aMb$$
$$M \rightarrow aMb$$
$$M \rightarrow \lambda$$

- Nullable variable

Removing Nullable Variables

- Example Grammar:

$$\boxed{\begin{aligned} S &\to aMb \\ M &\to aMb \\ M &\to \lambda \end{aligned}}$$

Substitute $M \to \lambda$

- Final grammar:

$$\boxed{\begin{aligned} S &\to aMb \\ S &\to ab \\ M &\to aMb \\ M &\to ab \end{aligned}}$$

Unit Production

- Unit production:

$$A \rightarrow B$$

Removing Unit Productions

- Observation:

$$A \to A$$

is removed immediately

Removing Unit Productions

- Grammar:

$$
\begin{aligned}
S &\rightarrow aA \\
A &\rightarrow a \\
A &\rightarrow B \\
B &\rightarrow A \\
B &\rightarrow bb
\end{aligned}
$$

Removing Unit Productions

- Grammar:

$$S \to aA$$
$$A \to a$$
$$A \to B$$
$$B \to A$$
$$B \to bb$$

substitute

$$A \to B$$

$$S \to aA | aB$$
$$A \to a$$
$$B \to A$$
$$B \to B$$
$$B \to bb$$

Removing Unit Productions

- Grammar:

$$S \to aA|aB$$
$$A \to a$$
$$B \to A$$
$$B \to B$$
$$B \to bb$$

remove

$$B \to B$$

$$S \to aA \mid aB$$
$$A \to a$$
$$B \to A$$
$$B \to bb$$

Removing Unit Productions

remove repeated productions

$$S \to aA \mid aB$$
$$A \to a$$
$$B \to A$$
$$B \to bb$$

$$S \to aA \mid aB$$
$$A \to a$$
$$B \to bb$$

Final grammar

Simplification Rules

- **Step 1:** Remove Nullable Variables

- **Step 2:** Remove Unit-Productions

- **Step 3:** Remove Useless Variables