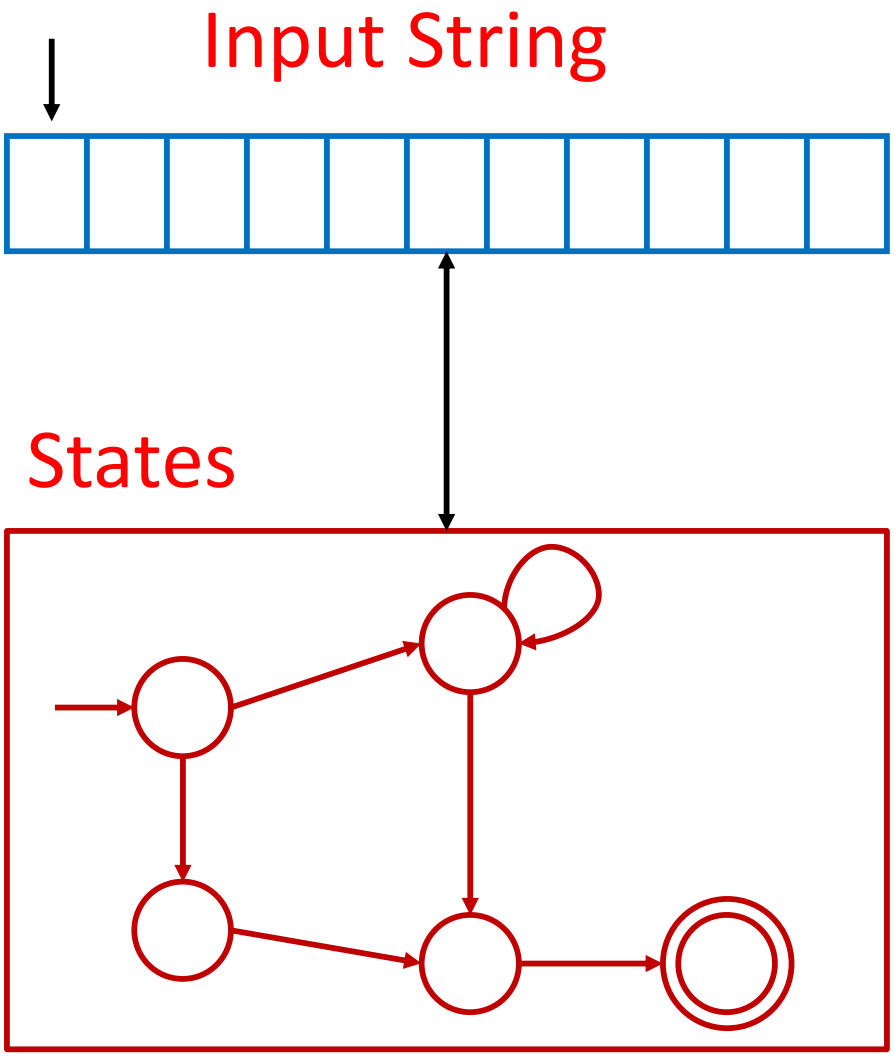
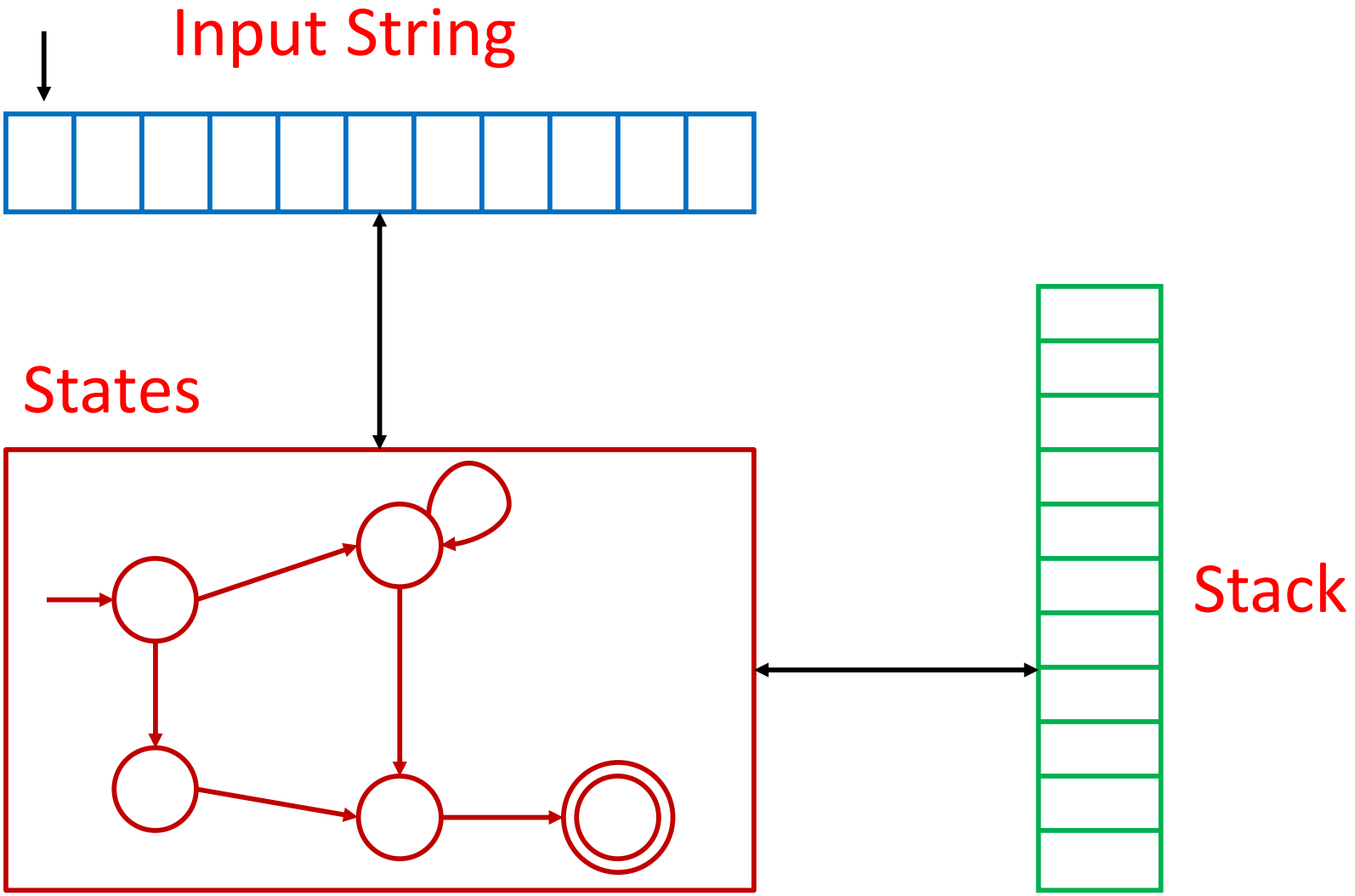
Pushdown Automata

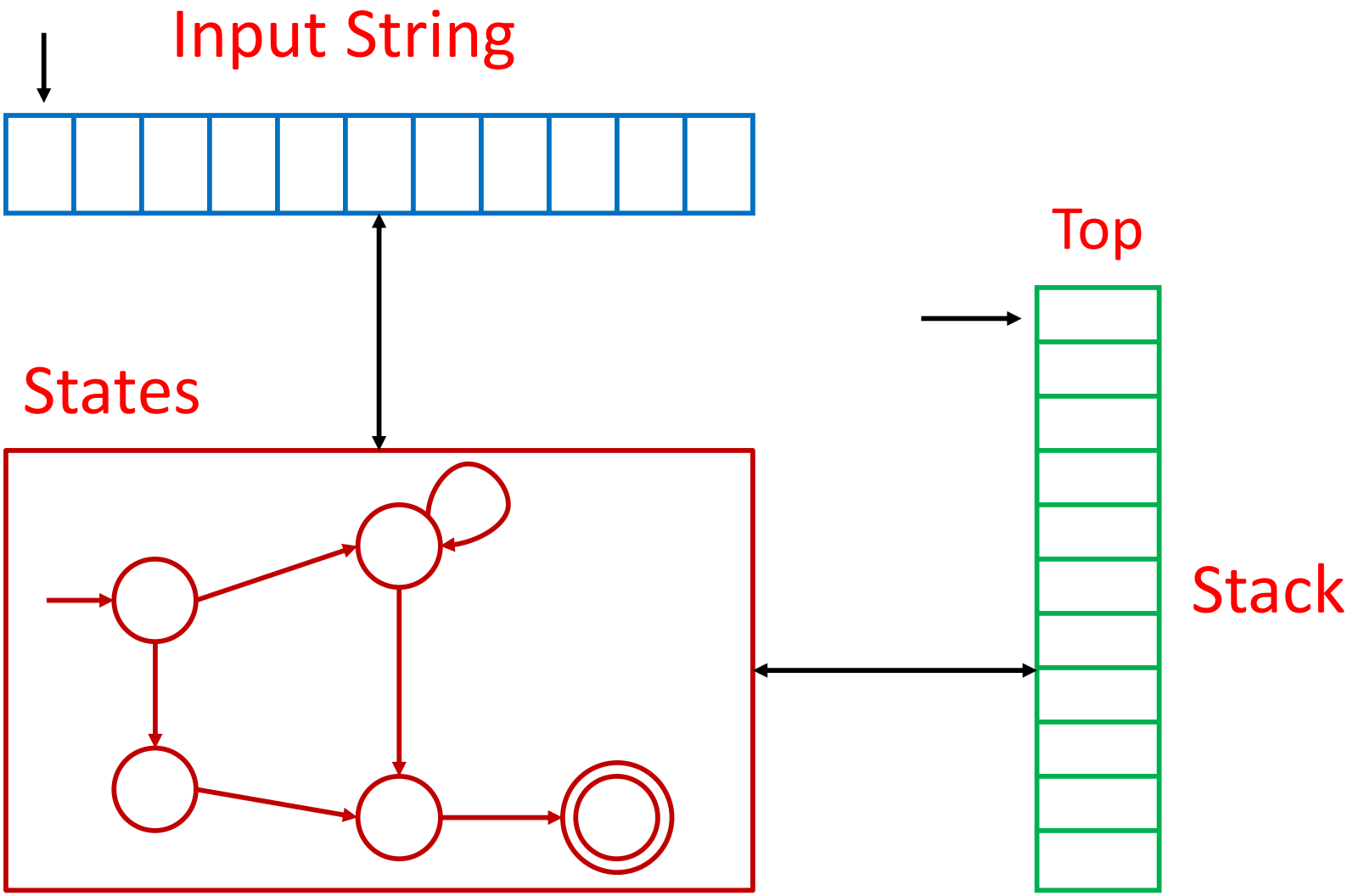
Finite Automaton



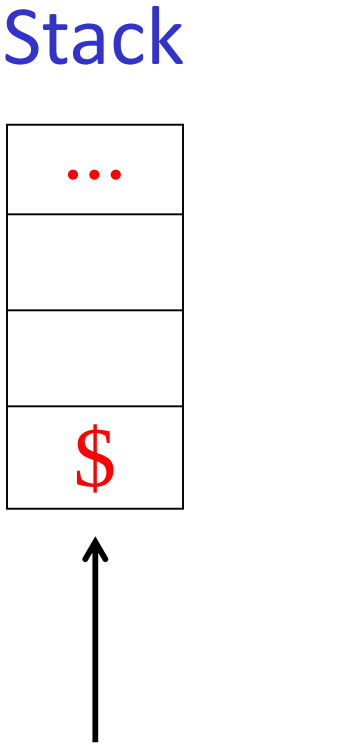
Pushdown Automaton - PDA



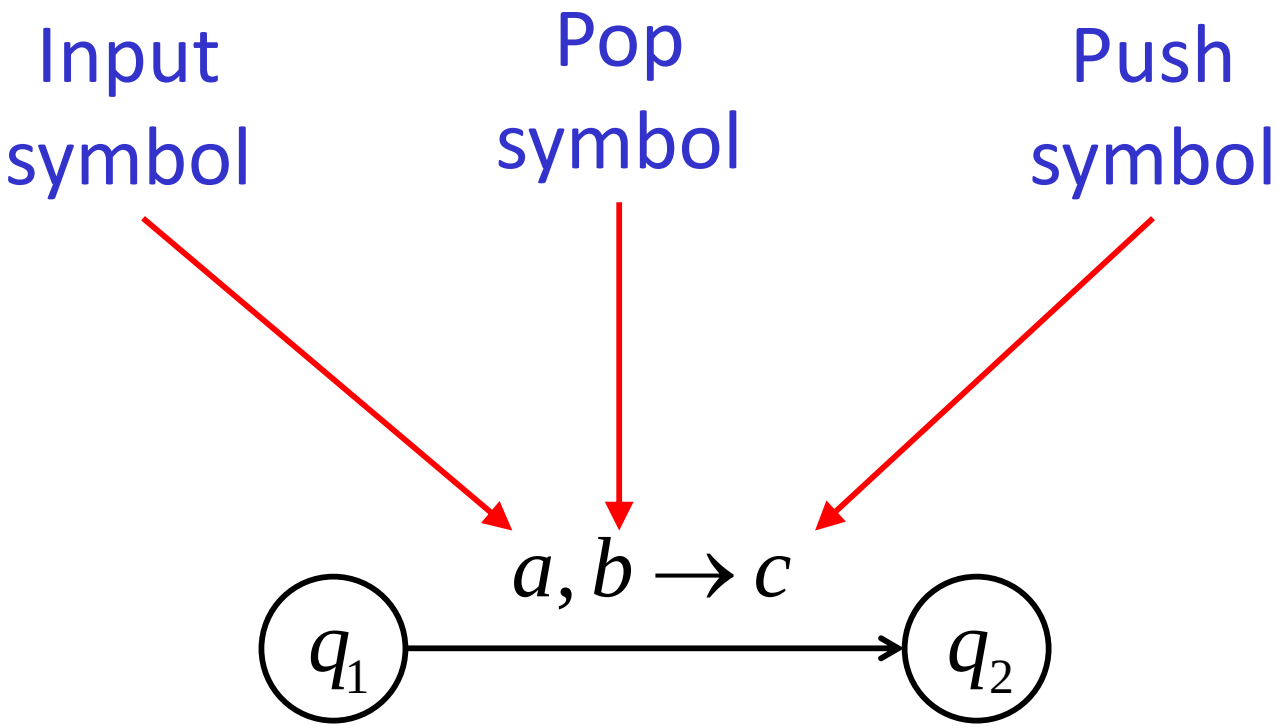
Pushdown Automaton - PDA



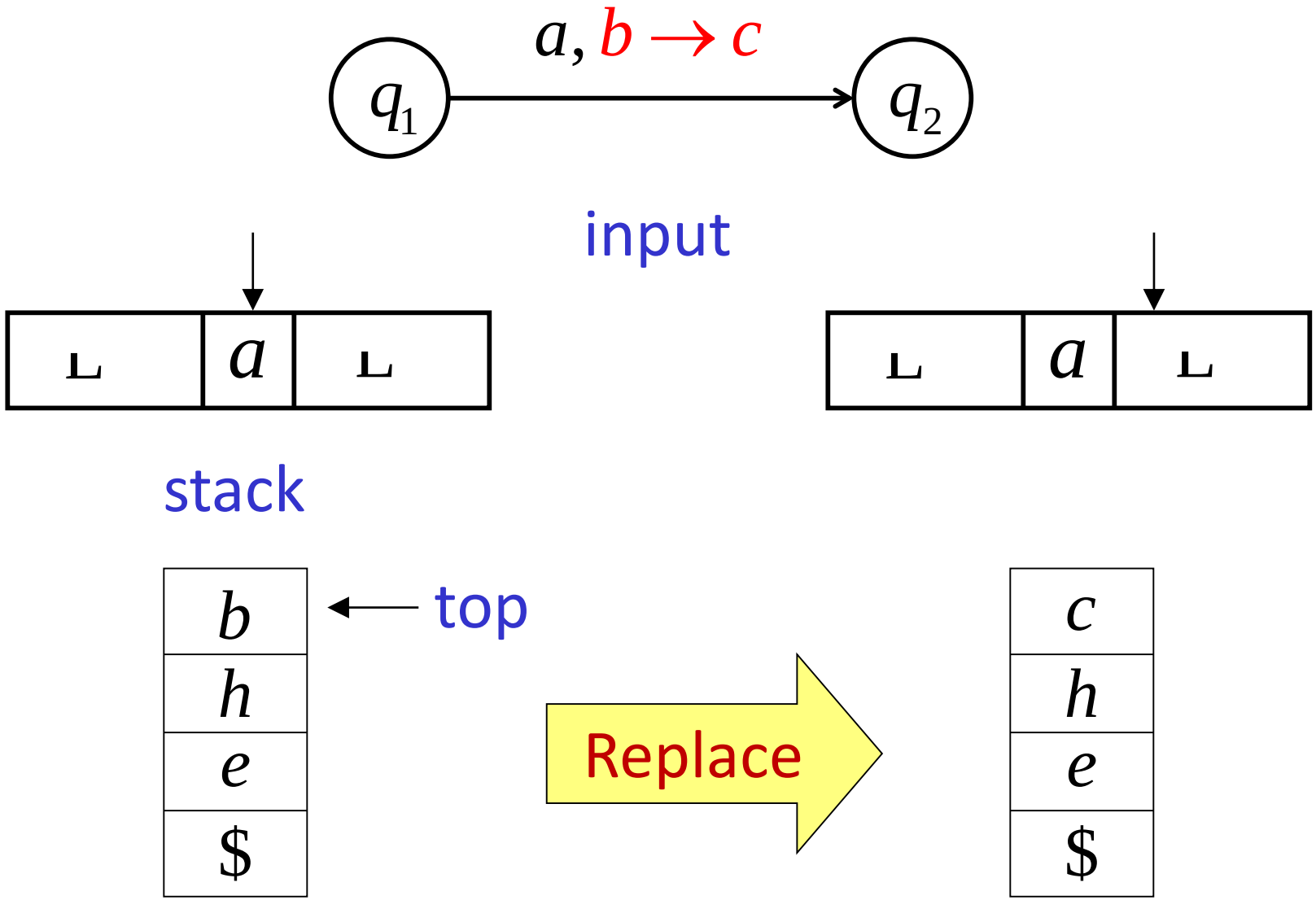
Initial Stack Symbol



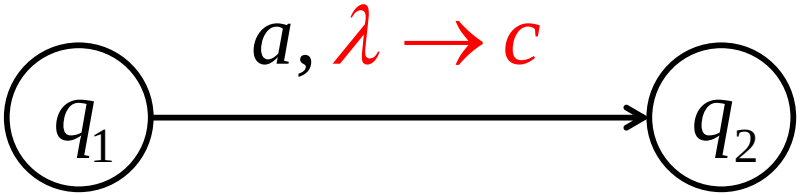
The States



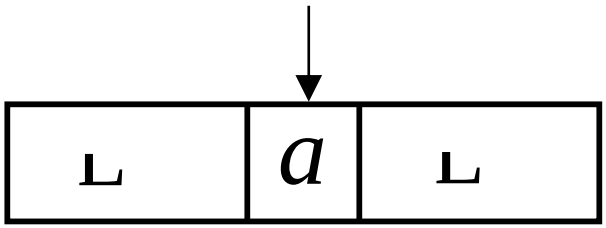
The Stack Operations

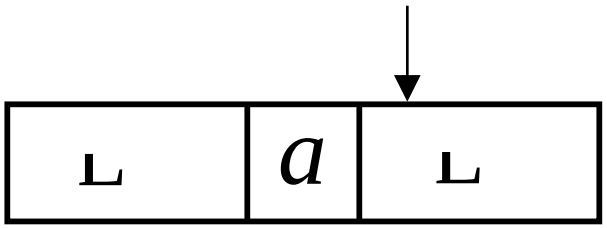


The Stack Operations

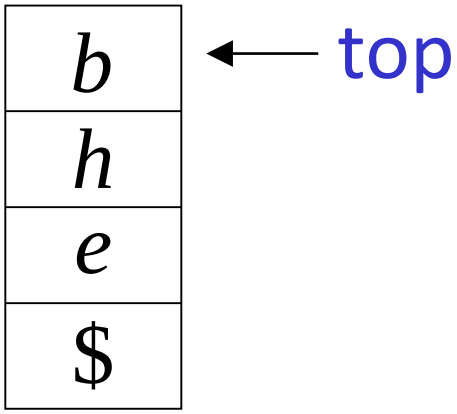


input

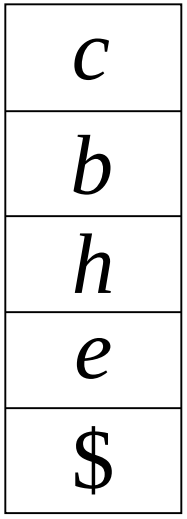




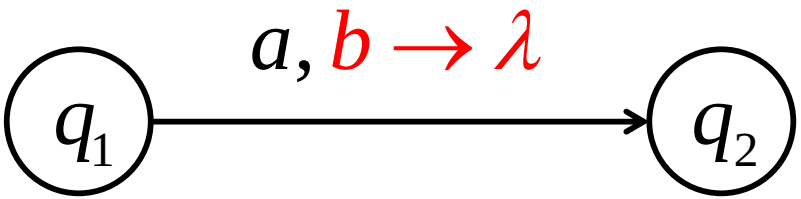
stack



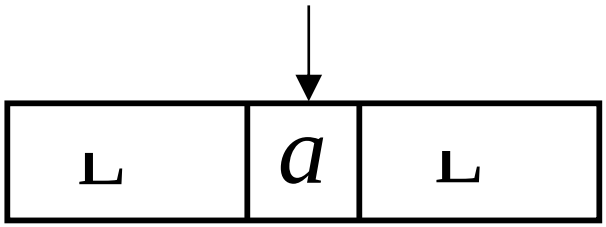
Push

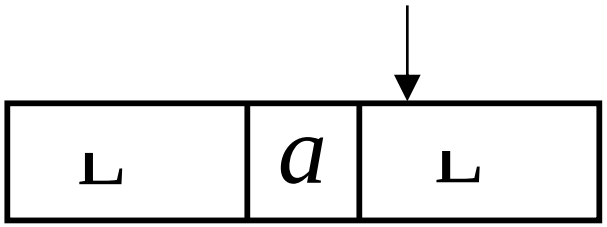


The Stack Operations

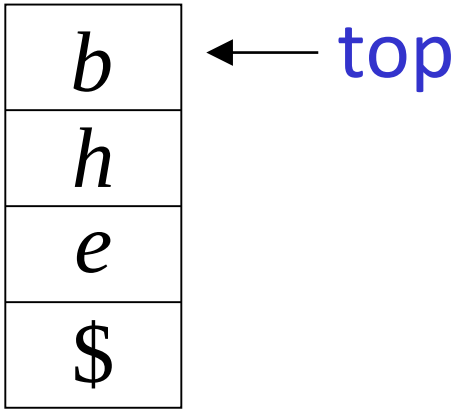


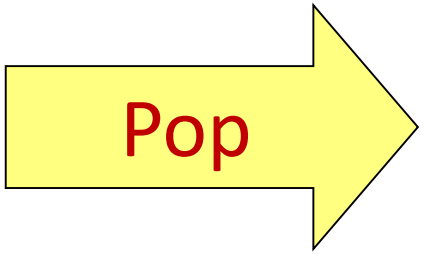
input

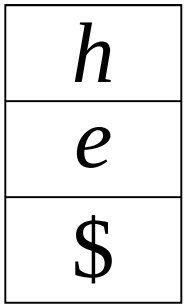




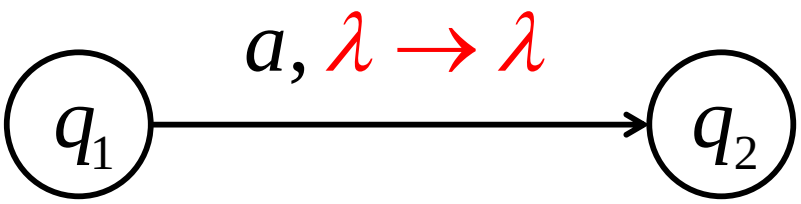
stack



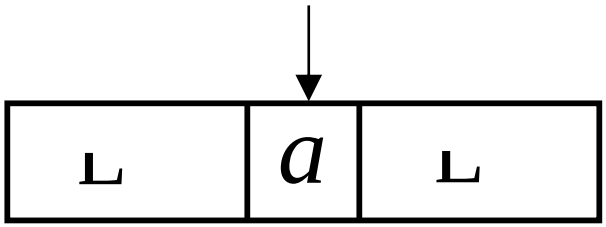


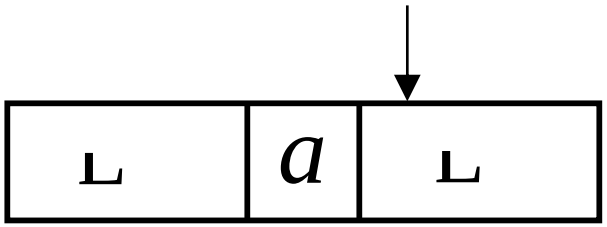


The Stack Operations

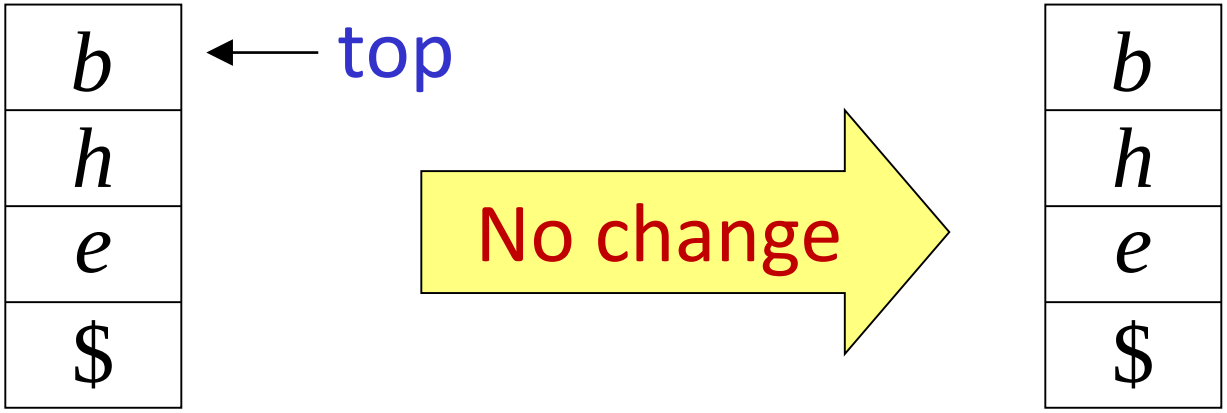


input

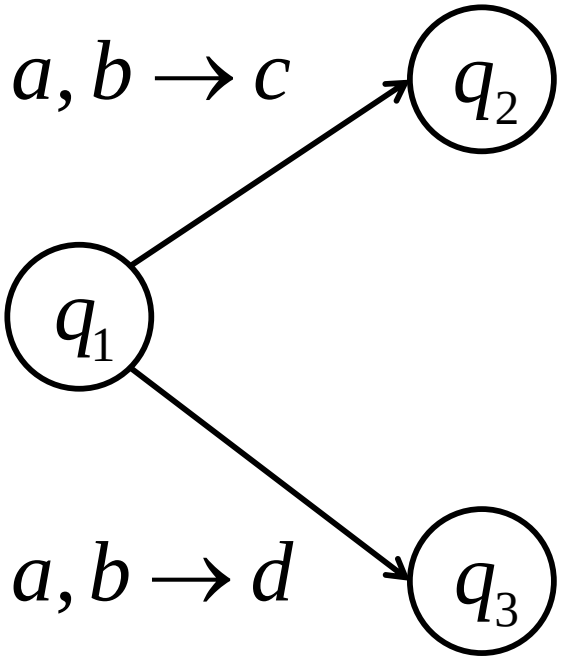


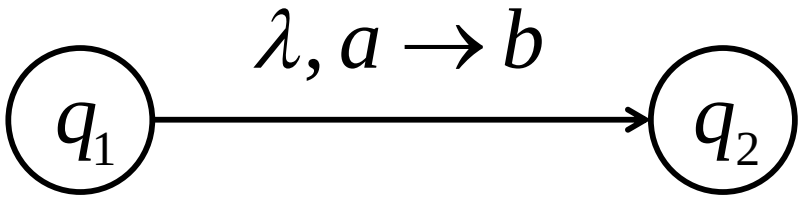


stack



Non-Determinism



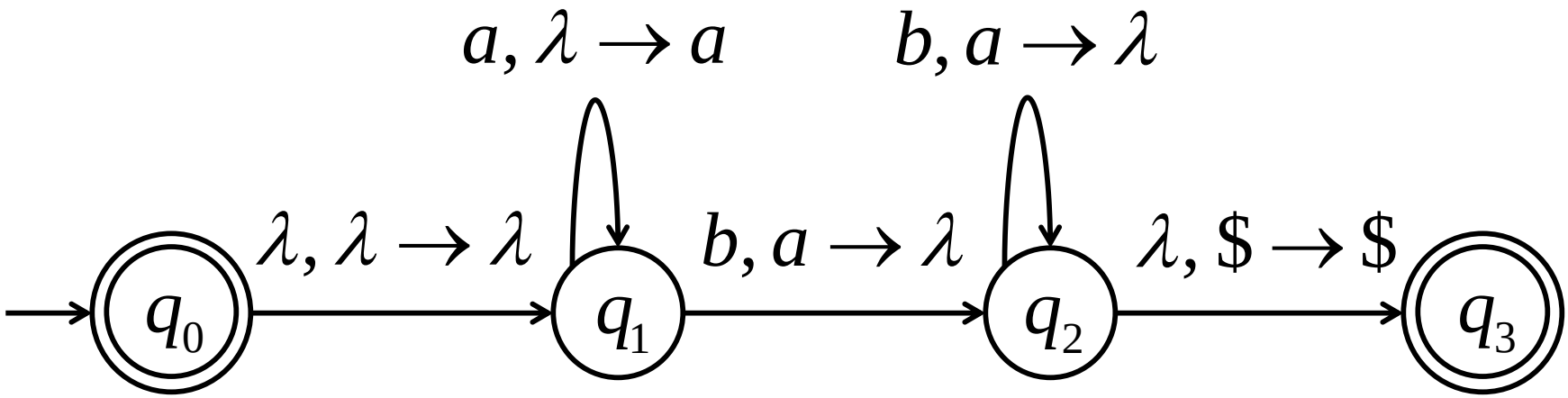


λ -transition

These are allowed transitions in a non-deterministic PDA (NPDA)

NPDA: Non-Deterministic PDA

- **Example:**

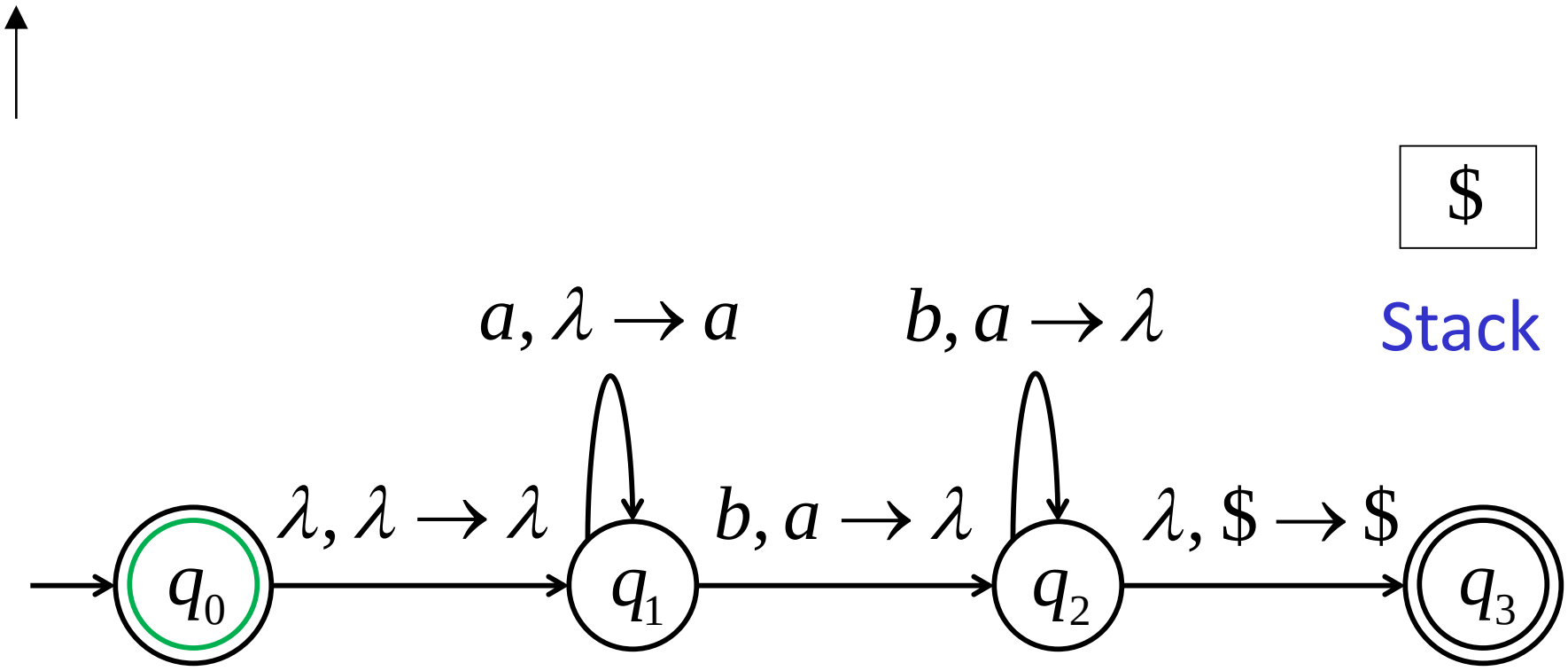


Execution Example

Time 0

Input

a	a	a	b	b	b



Execution Example

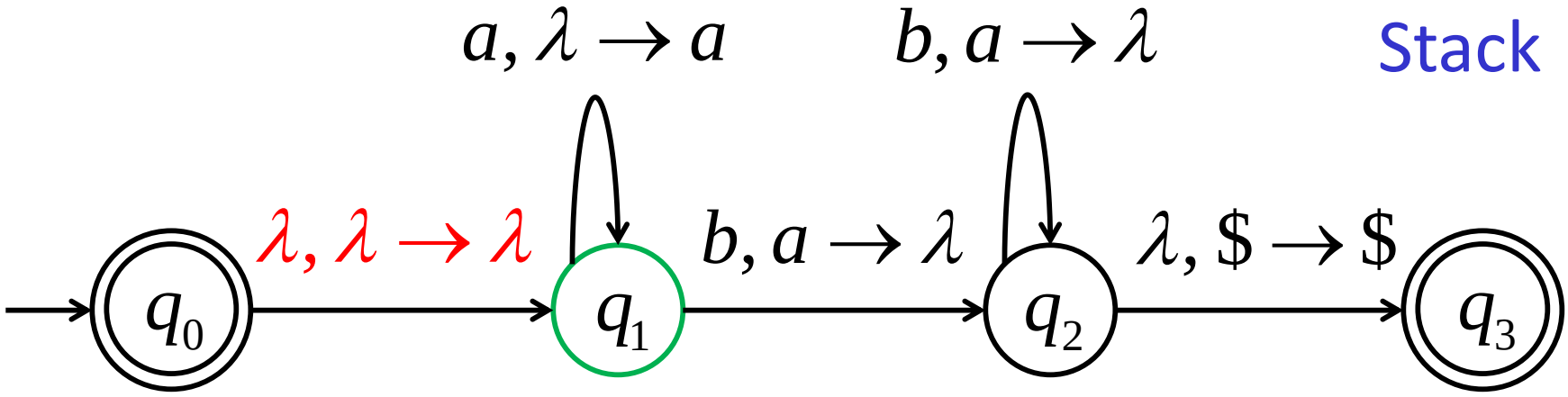
Input

Time 1

a	a	a	b	b	b

$

Stack



Execution Example

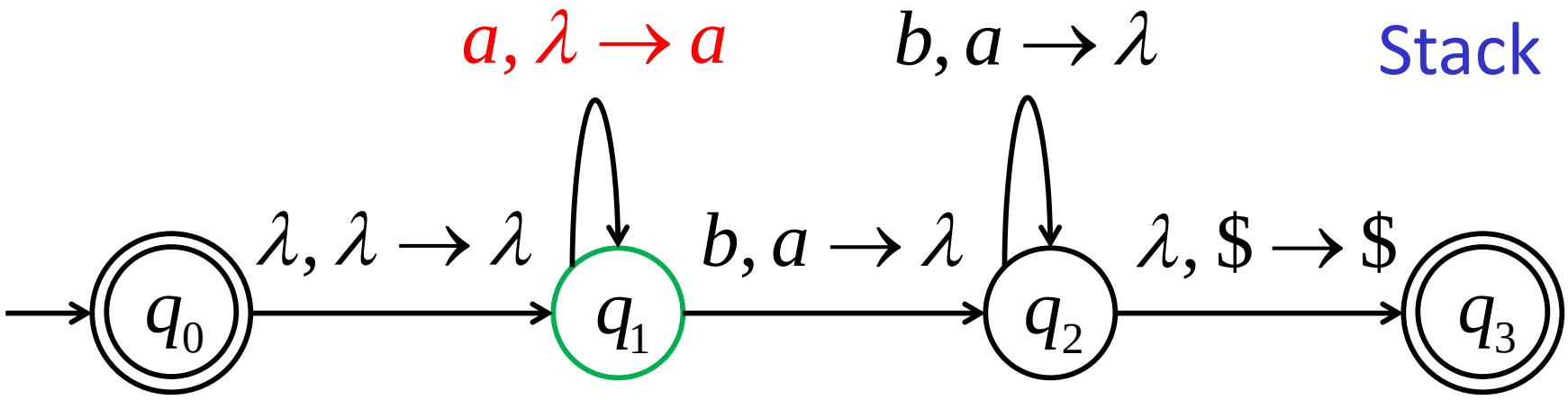
Input

Time 2

a	a	a	b	b	b

a
$\$$

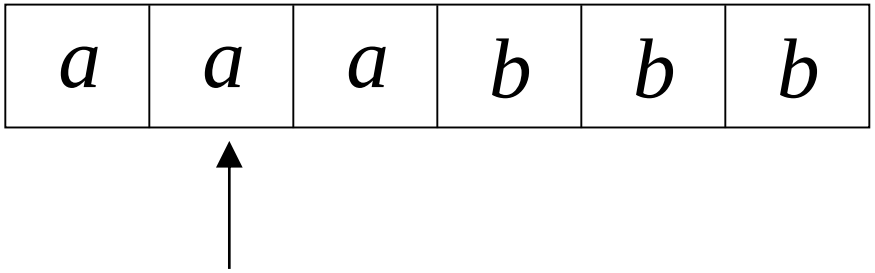
Stack

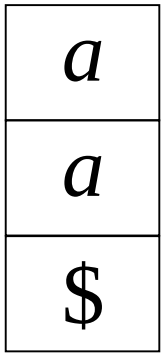


Execution Example

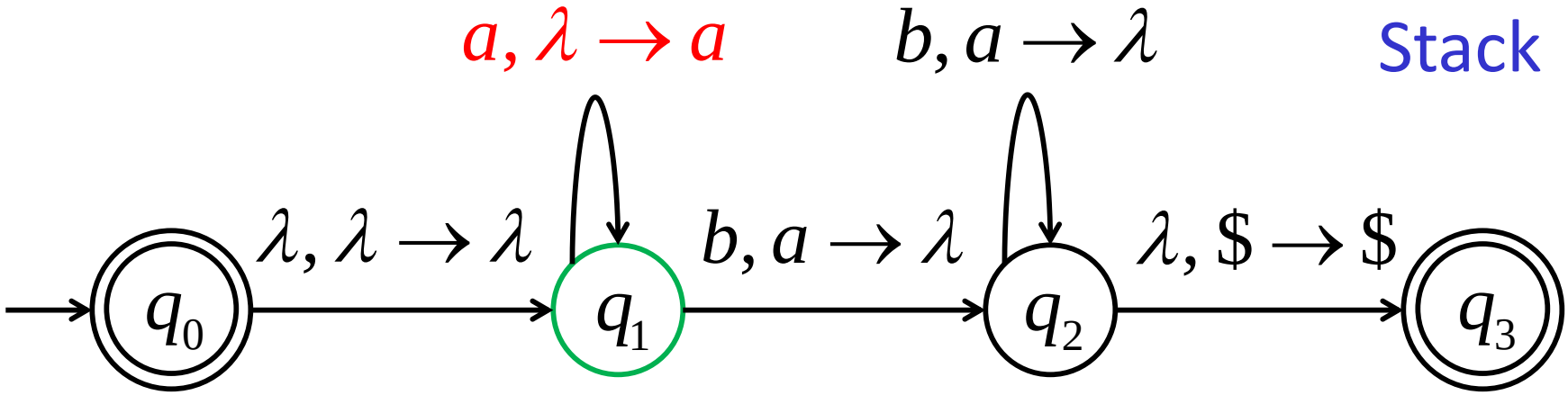
Input

Time 3





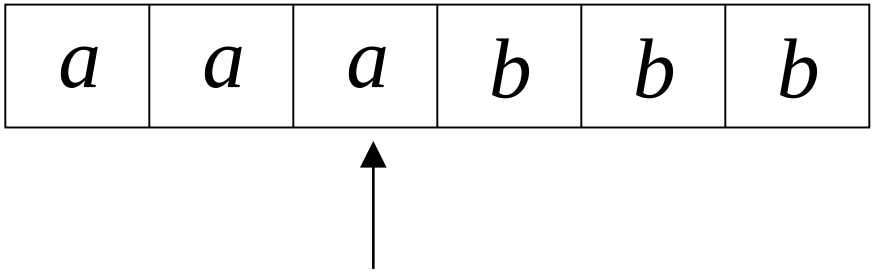
Stack

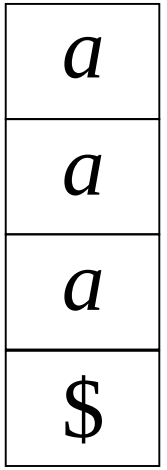


Execution Example

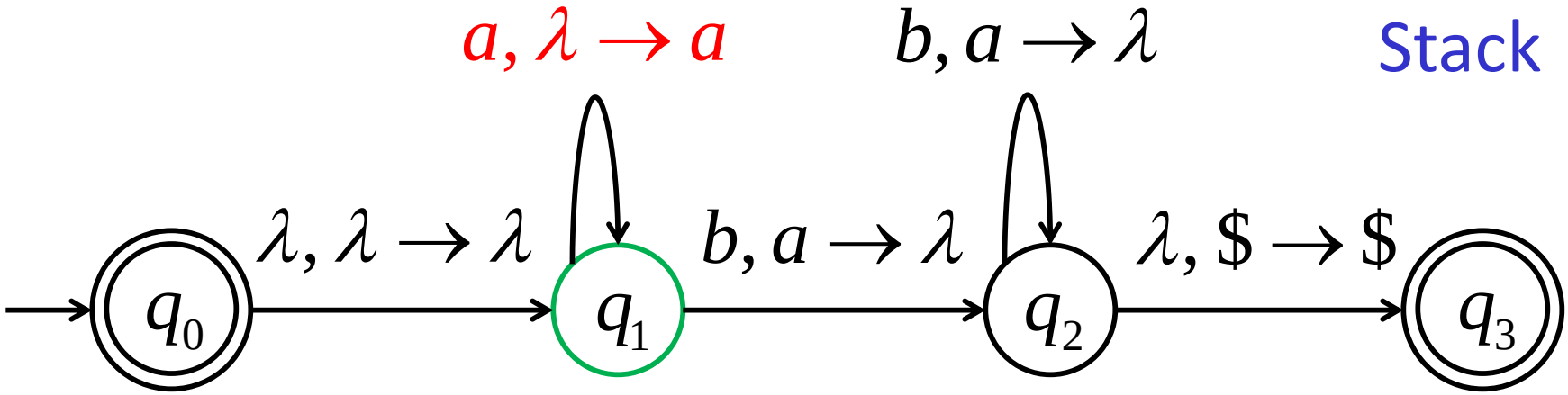
Input

Time 4





Stack



Execution Example

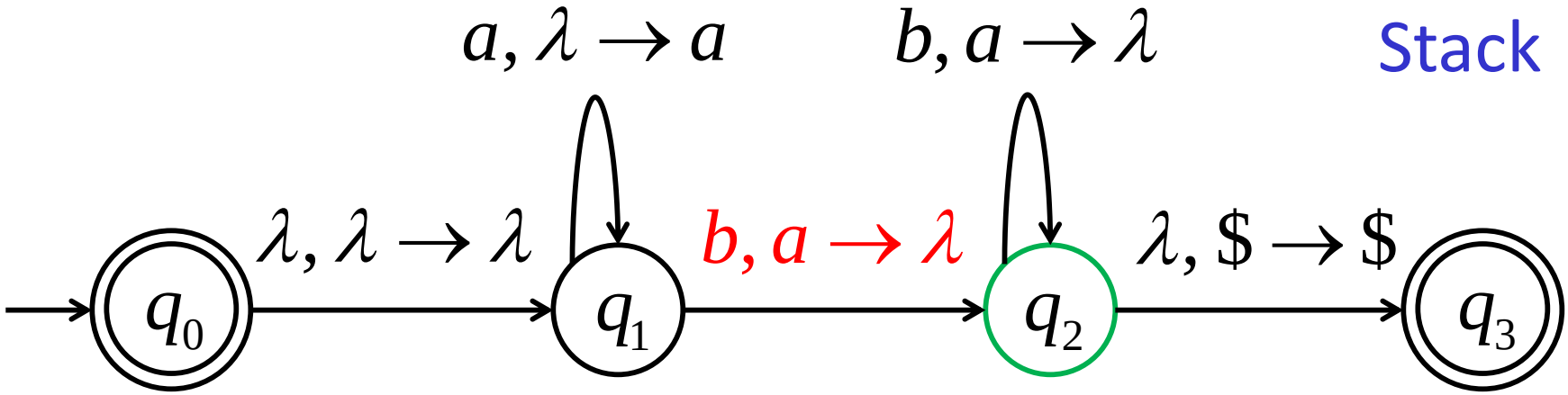
Input

a	a	a	b	b	b

Time 5

Stack

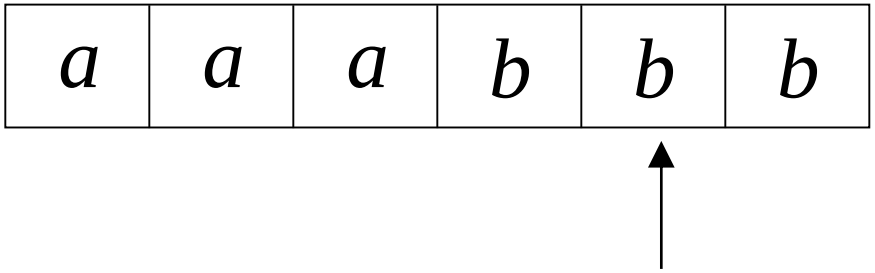
~~a~~
a
a
$\$$

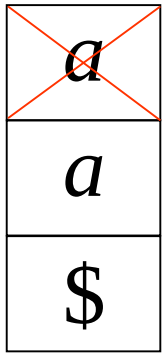


Execution Example

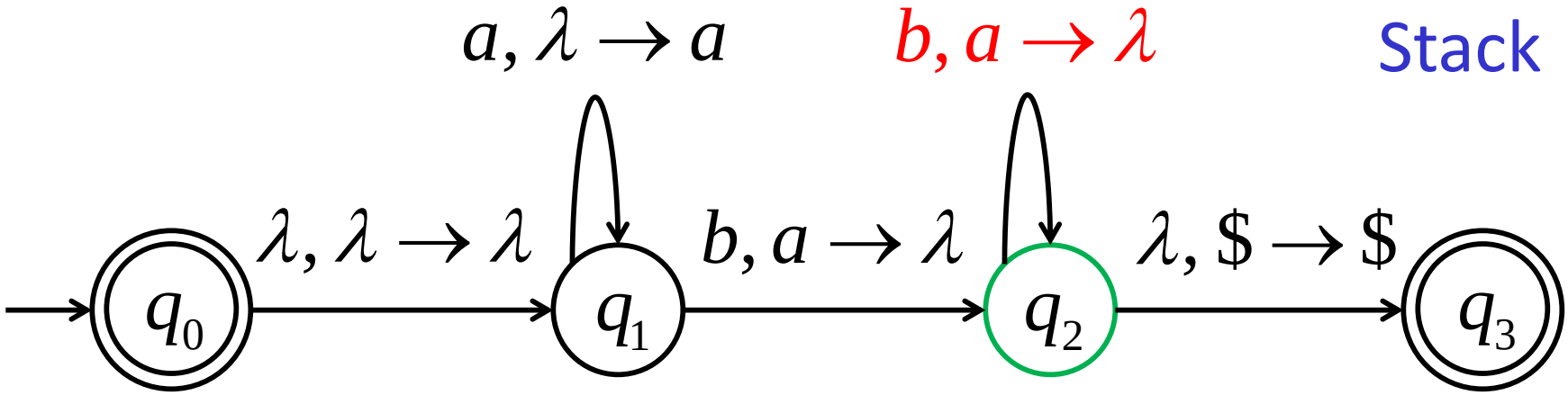
Input

Time 6





Stack

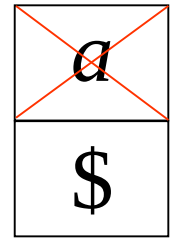


Execution Example

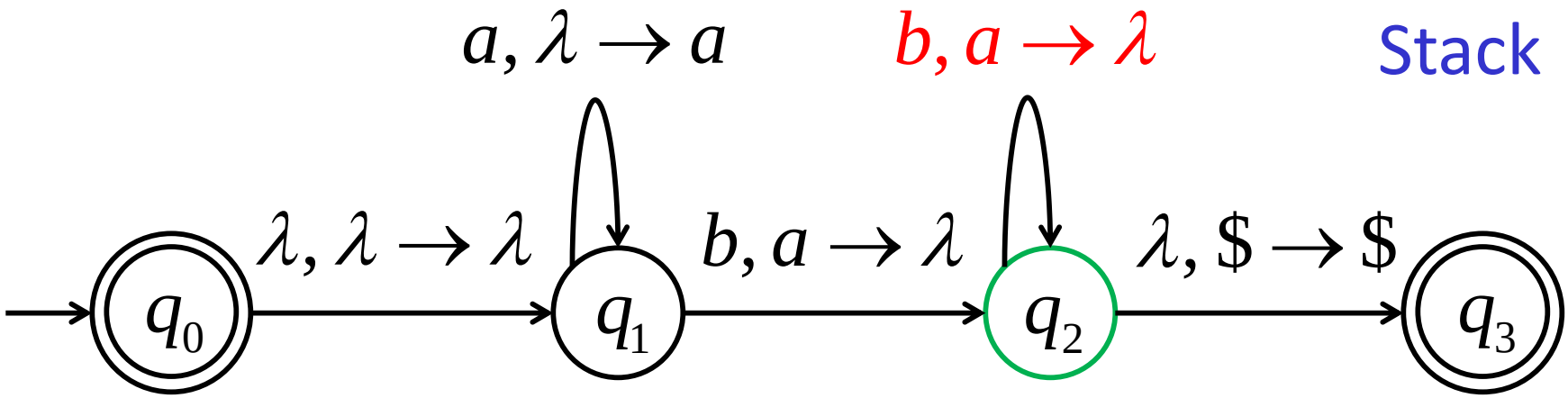
Input

Time 7

a	a	a	b	b	b



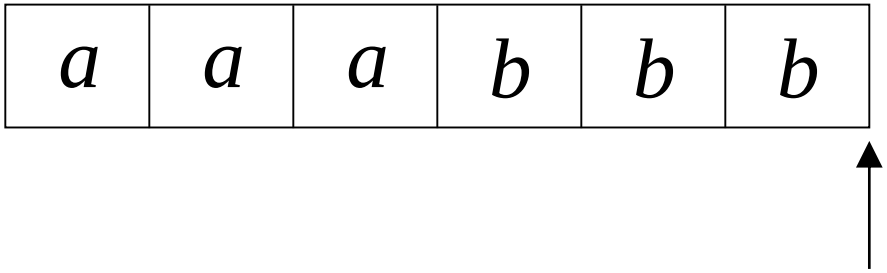
Stack

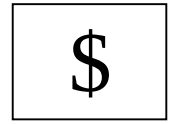


Execution Example

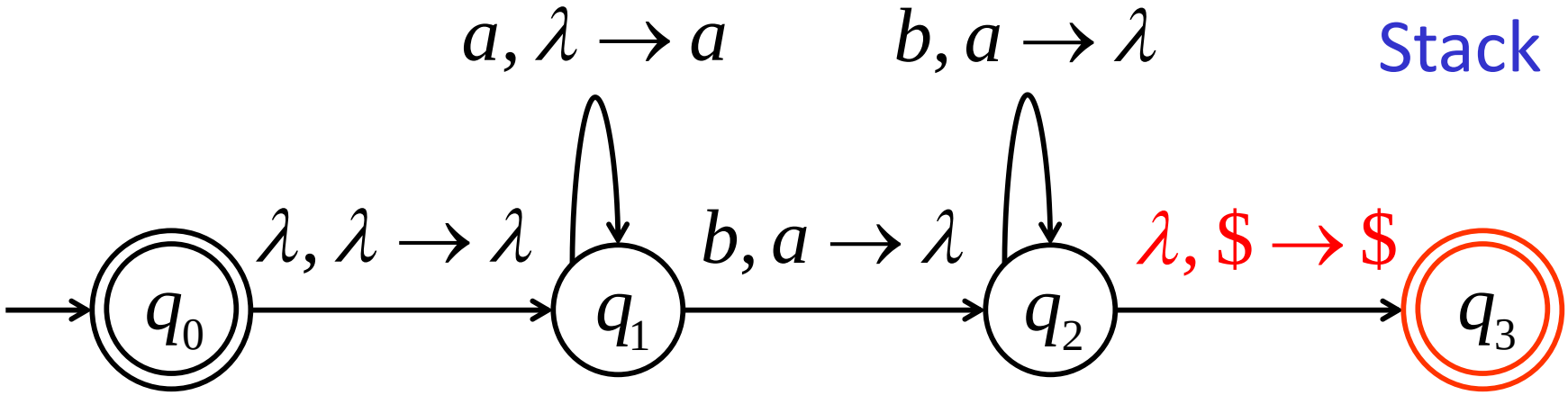
Input

Time 8





Stack



Pushdown Automaton - PDA

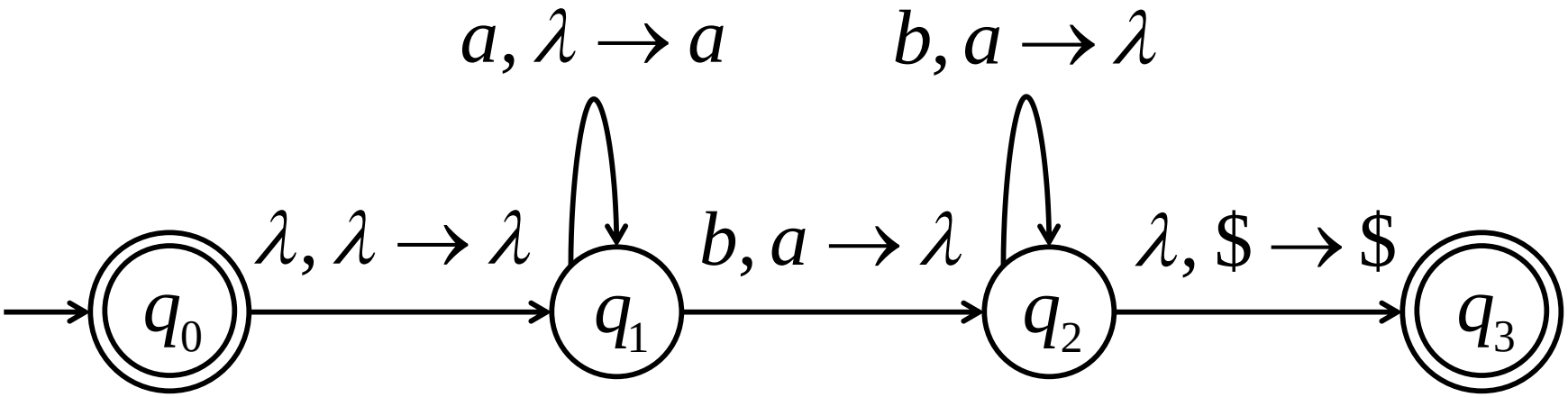
- A string is accepted if there is a computation such that:

  1. All the input is consumed
  2. The last state is a final state

  At the end of the computation, we do not care about the **stack contents**

Pushdown Automaton - PDA

- The input string *aaabbb* is accepted by the NPDA:

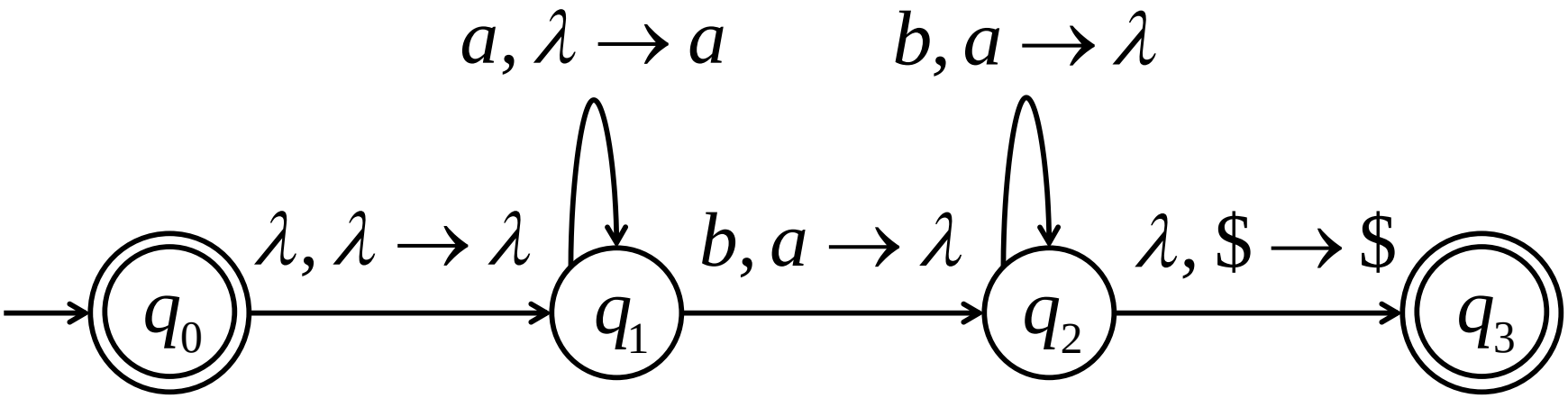


Pushdown Automaton - PDA

- In general,

$$L = \{a^n b^n : n \geq 0\}$$

is the language accepted by the NPDA:



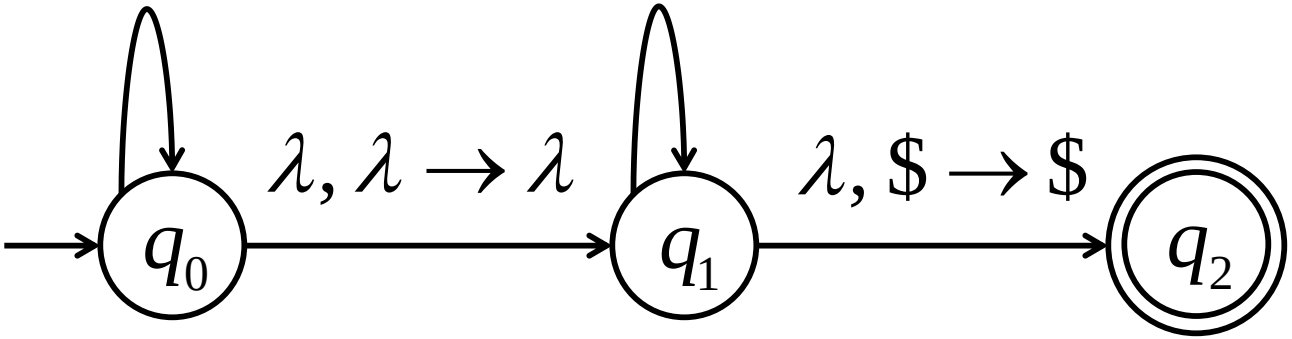
Example

- NDPA M

$$L(M) = \{ww^R\}$$

$a,\ \lambda \to a$ $\qquad$ $a,\ a \to \lambda$

$b,\ \lambda \to b$ $\qquad$ $b,\ b \to \lambda$



Execution Example

Input

Time 0

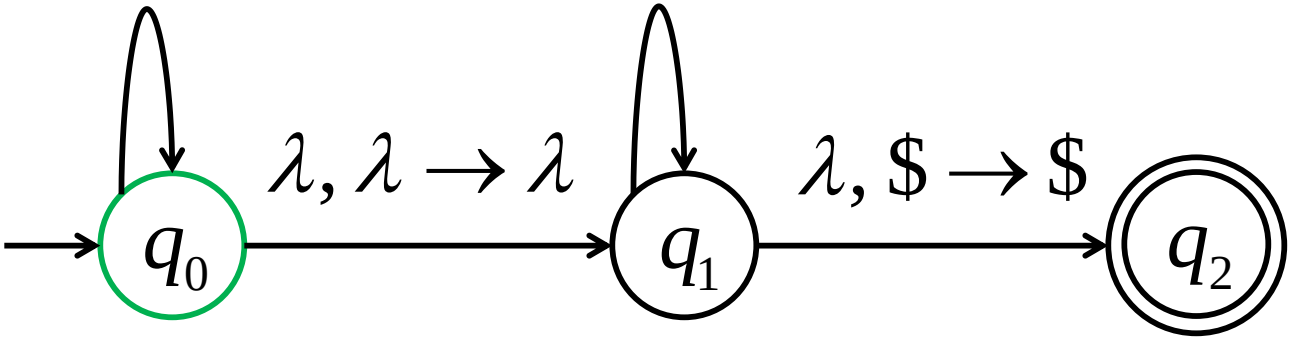
a	b	b	a

$a,\ \lambda \to a$ $\quad$ $a,\ a \to \lambda$

$b,\ \lambda \to b$ $\quad$ $b,\ b \to \lambda$

$

Stack



Execution Example

Input

a	b	b	a

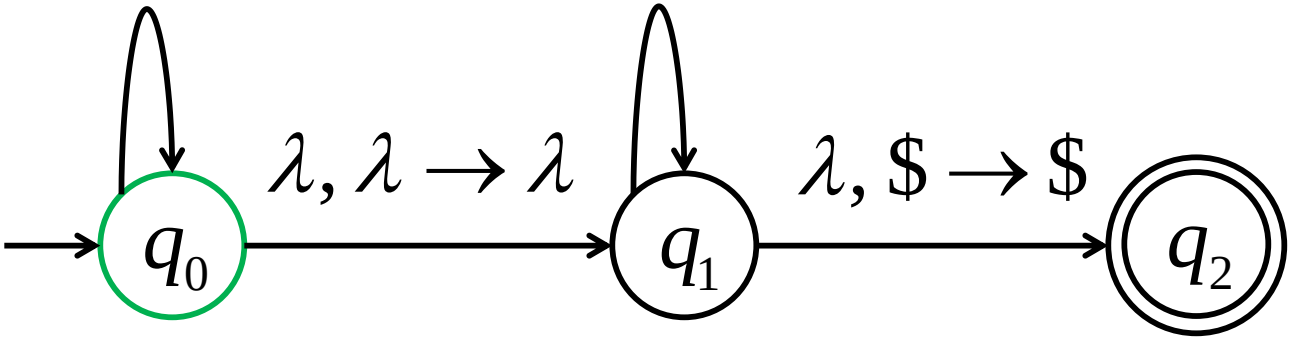
Time 1

a
$\$$

Stack

$a,\ \lambda \to a$ $\quad$ $a,\ a \to \lambda$

$b,\ \lambda \to b$ $\quad$ $b,\ b \to \lambda$



Execution Example

Input

a	b	b	a

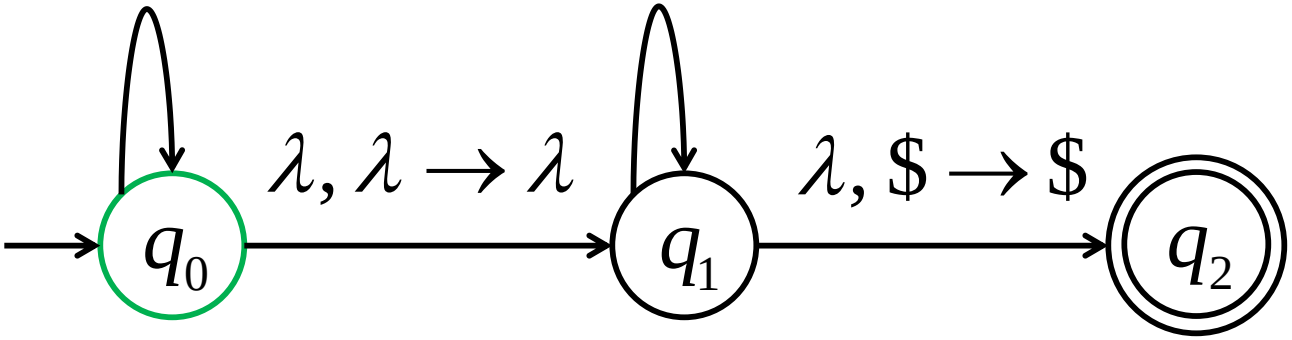
Time 2

b
a
\$

Stack

$a,\ \lambda \to a$ $\quad$ $a,\ a \to \lambda$

$b,\ \lambda \to b$ $\quad$ $b,\ b \to \lambda$



Execution Example

Input

a	b	b	a

Time 3

Guess the middle of string

b
a
\$

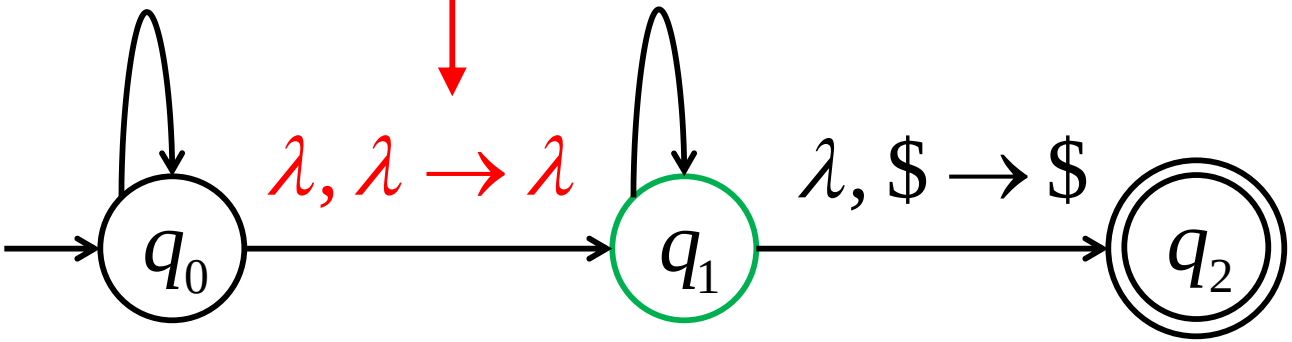
Stack

$a,\ \lambda \to a$

$b,\ \lambda \to b$

$a,\ a \to \lambda$

$b,\ b \to \lambda$



Execution Example

Input

a	b	b	a

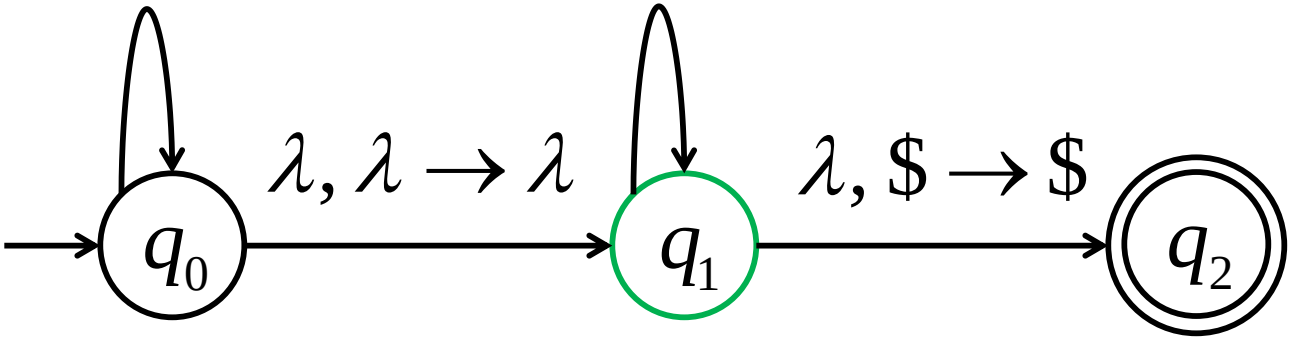
Time 4

b
a
$

Stack

$a,\ \lambda \to a$ $\quad$ $a,\ a \to \lambda$

$b,\ \lambda \to b$ $\quad$ $b,\ b \to \lambda$



Execution Example

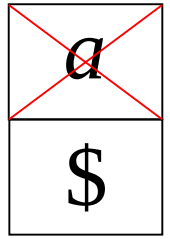
Input

a	b	b	a

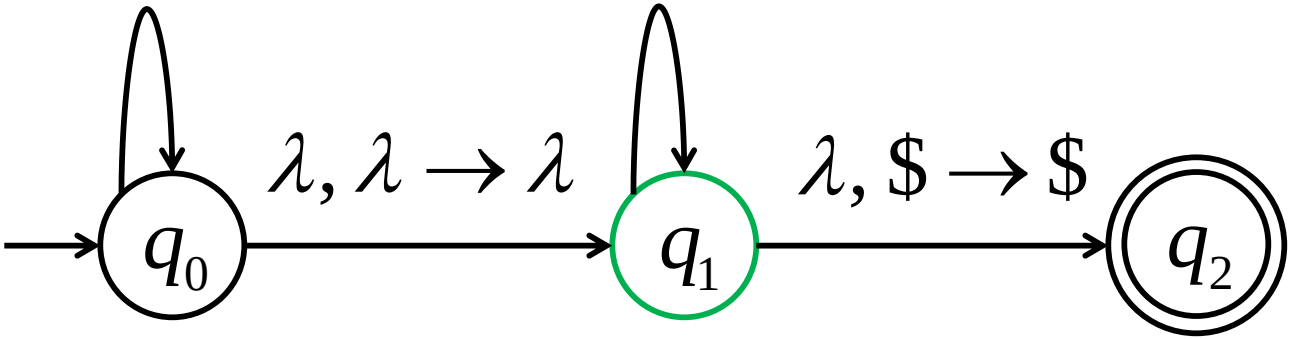
Time 5

$a,\ \lambda \to a$ $\quad$ $a,\ a \to \lambda$

$b,\ \lambda \to b$ $\quad$ $b,\ b \to \lambda$



Stack

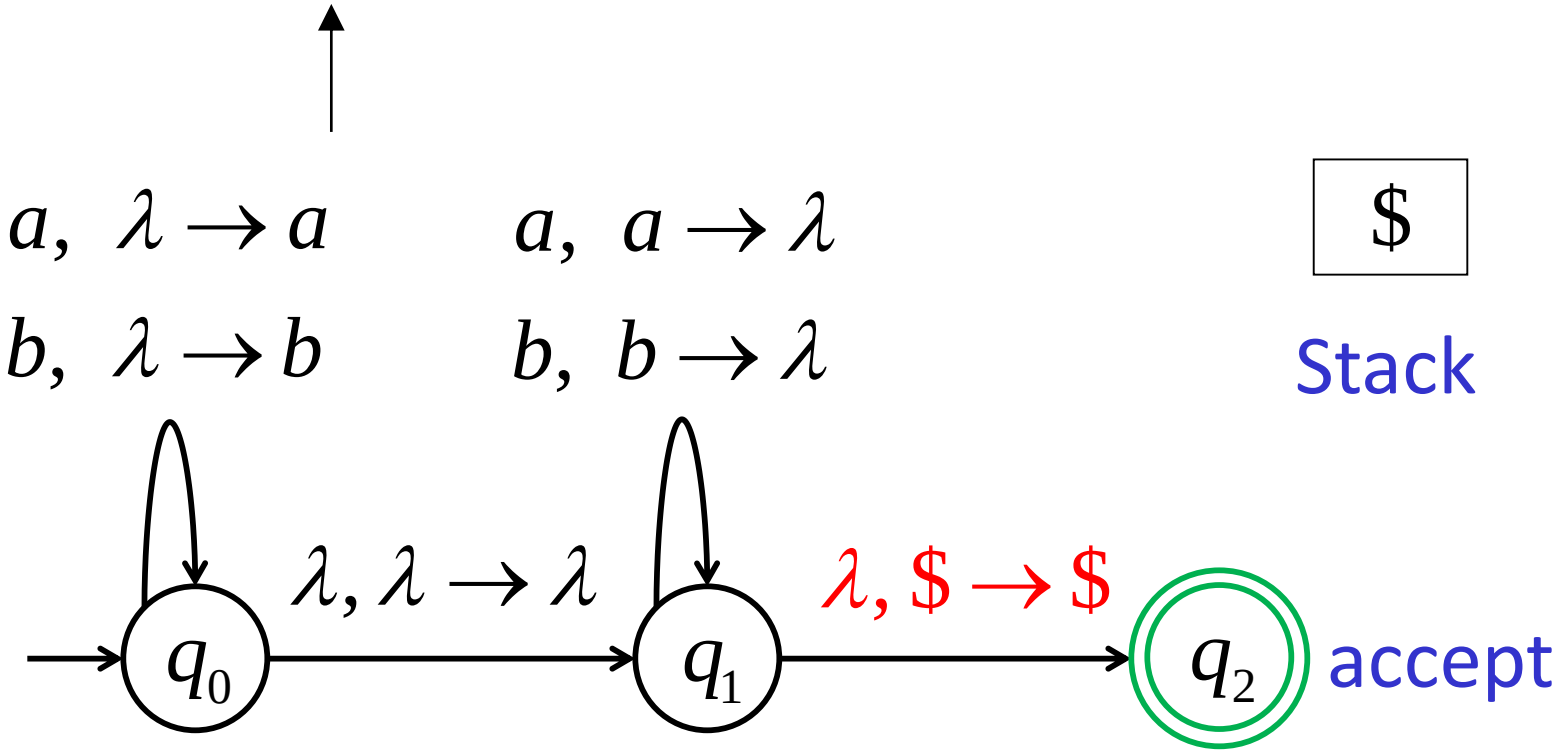


Execution Example

Input

a	b	b	a

Time 6



Rejection Example

Input

a	b	b	b

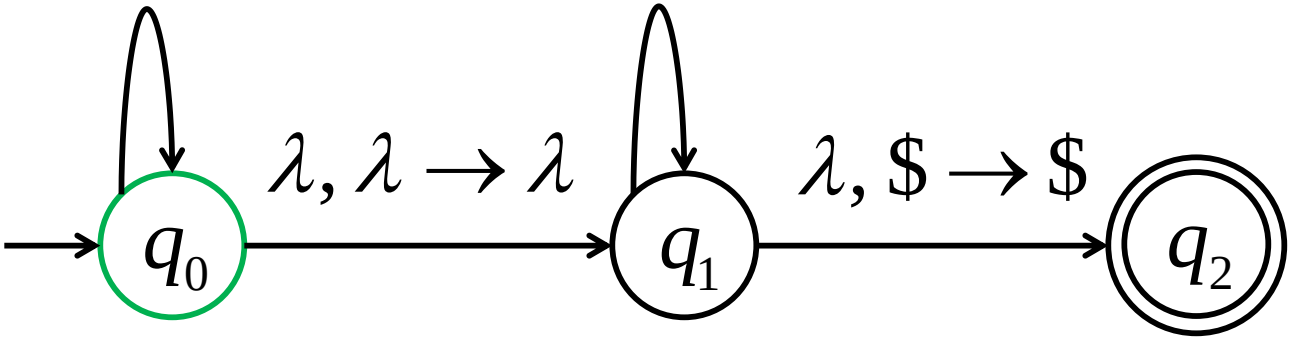
Time 0

$a,\ \lambda \to a$ $\quad$ $a,\ a \to \lambda$

$b,\ \lambda \to b$ $\quad$ $b,\ b \to \lambda$

$

Stack



Rejection Example

Input

a	b	b	b

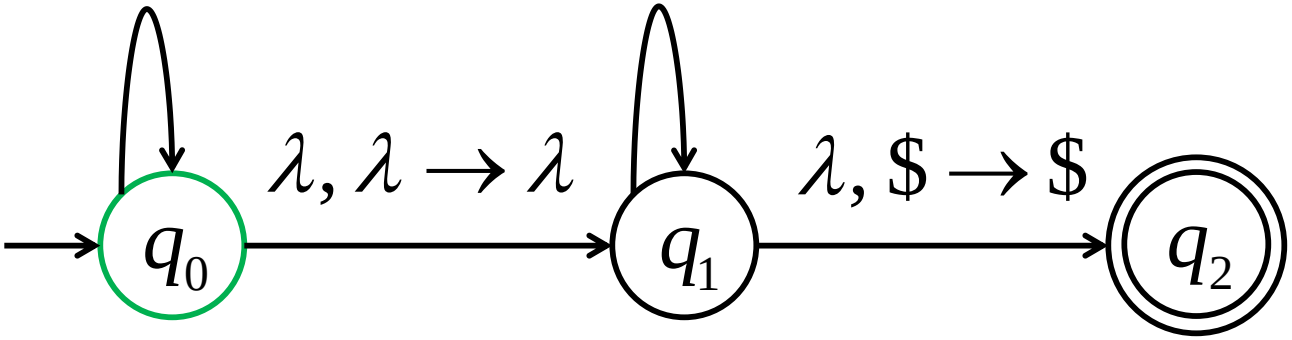
Time 1

a
$\$$

Stack

$a,\ \lambda \to a$ $\quad$ $a,\ a \to \lambda$

$b,\ \lambda \to b$ $\quad$ $b,\ b \to \lambda$



Rejection Example

Input

a	b	b	b

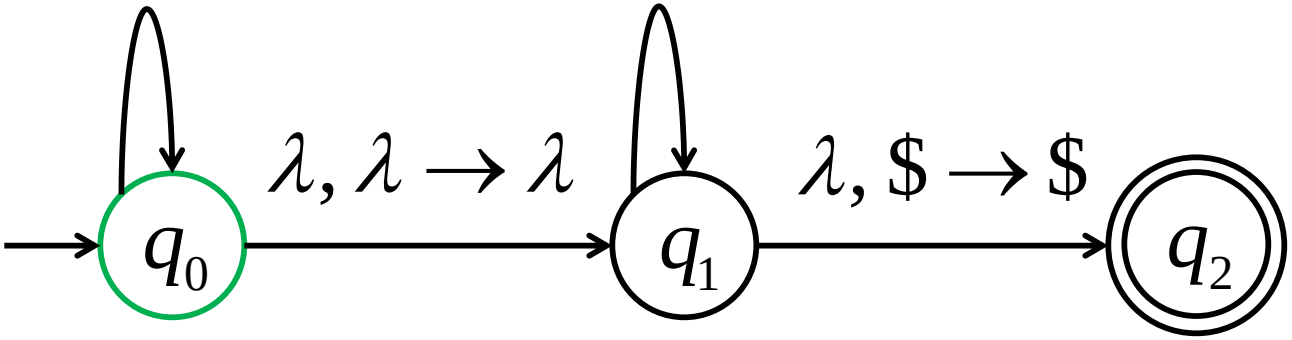
Time 2

b
a
\$

Stack

$a,\ \lambda \to a$ $\quad a,\ a \to \lambda$

$b,\ \lambda \to b$ $\quad b,\ b \to \lambda$



Rejection Example

Input

a	b	b	b

Time 3

Guess the middle of string

b
a
\$

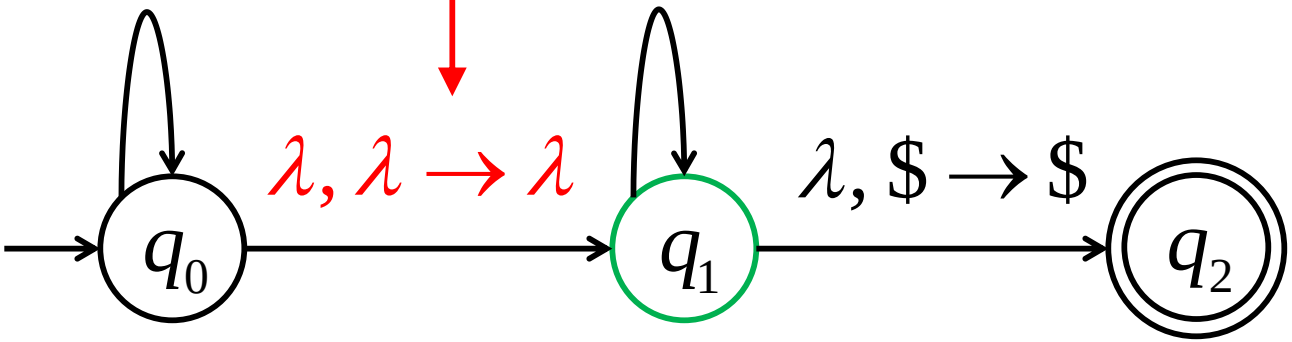
Stack

$a,\ \lambda \to a$

$b,\ \lambda \to b$

$a,\ a \to \lambda$

$b,\ b \to \lambda$



Rejection Example

Input

a	b	b	b

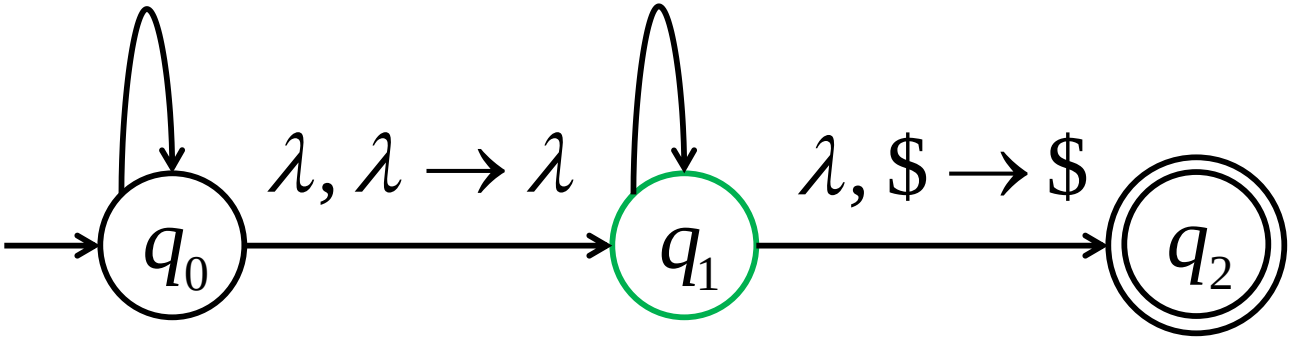
Time 4

b
a
$\$$

Stack

$a,\ \lambda \to a$ $\quad$ $a,\ a \to \lambda$

$b,\ \lambda \to b$ $\quad$ $b,\ b \to \lambda$



Rejection Example

Input

a	b	b	b

Time 5

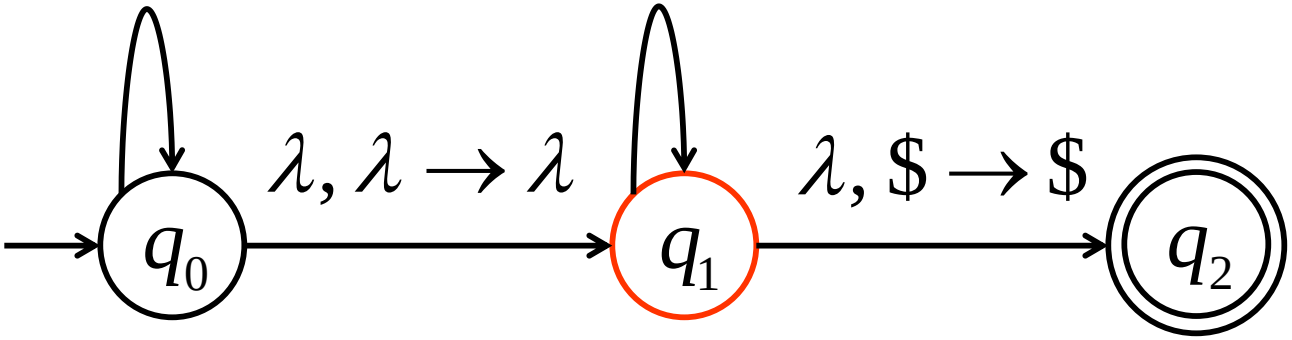
There is no possible transition;
input is not consumed

a
$

Stack

$a,\ \lambda \to a$ $\quad$ $a,\ a \to \lambda$

$b,\ \lambda \to b$ $\quad$ $b,\ b \to \lambda$



Rejection Example: Another Computation

Time 0

Input

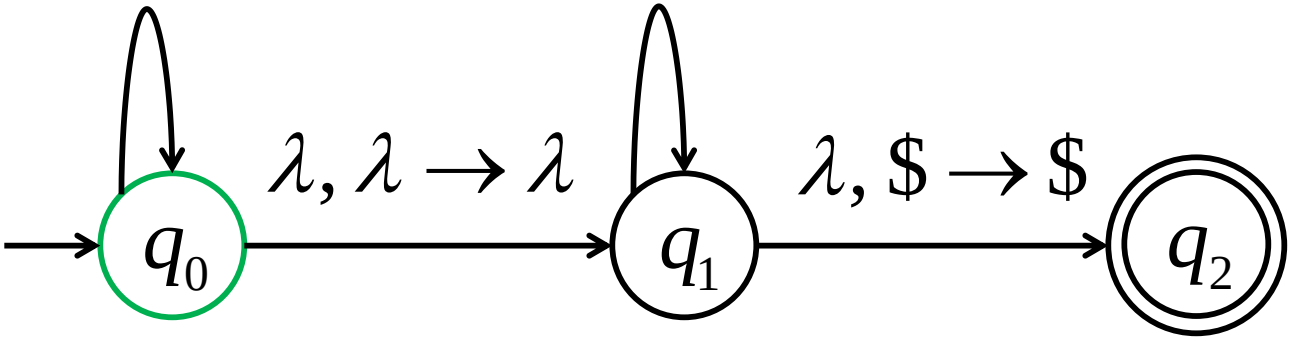
a	b	b	b

$a,\ \lambda \to a$ $\quad$ $a,\ a \to \lambda$

$b,\ \lambda \to b$ $\quad$ $b,\ b \to \lambda$

$

Stack



Rejection Example: Another Computation

Time 1

Input

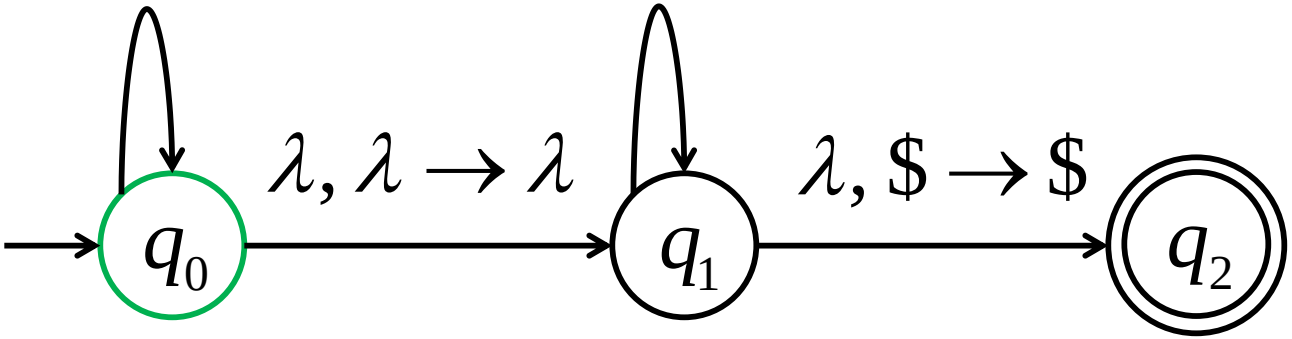
a	b	b	b

a
\$

Stack

$a,\ \lambda \rightarrow a$ $\quad$ $a,\ a \rightarrow \lambda$

$b,\ \lambda \rightarrow b$ $\quad$ $b,\ b \rightarrow \lambda$



Rejection Example: Another Computation

Time 2

Input

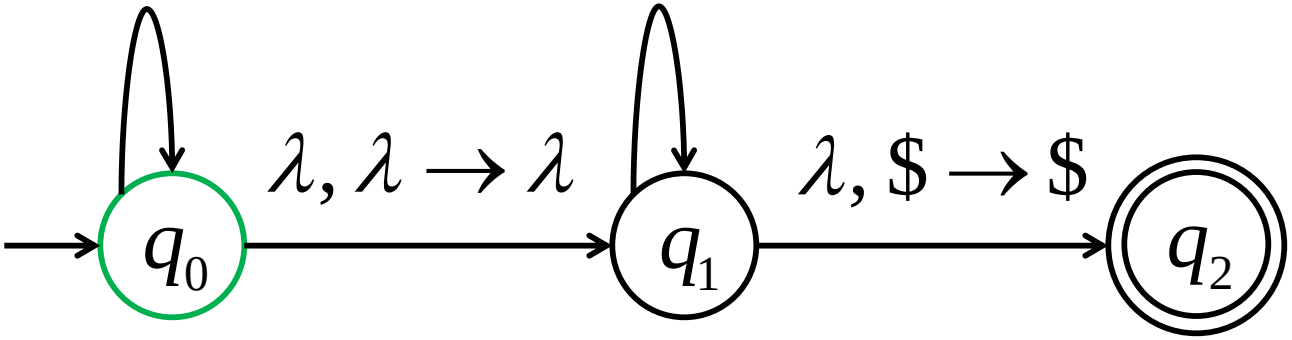
a	b	b	b

b
a
\$

Stack

$a,\ \lambda \to a$ $\quad a,\ a \to \lambda$

$b,\ \lambda \to b$ $\quad b,\ b \to \lambda$



Rejection Example: Another Computation

Time 3

Input

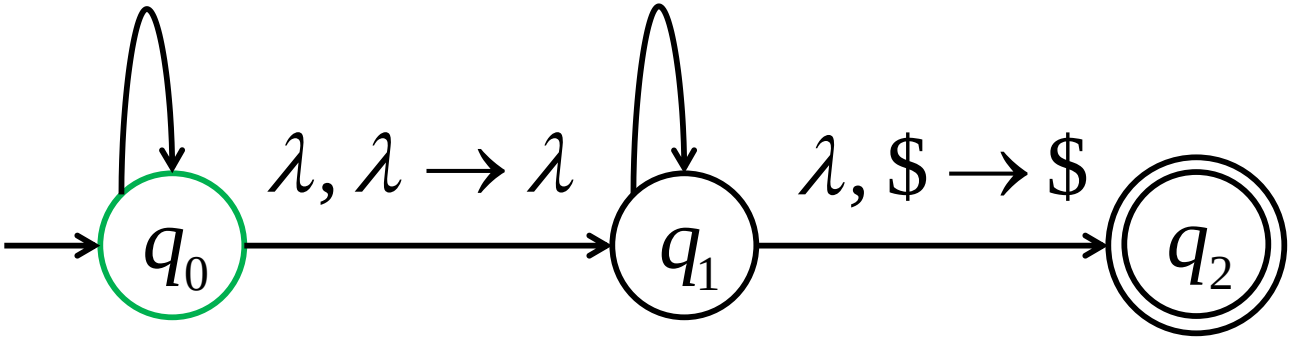
a	b	b	b

b
b
a
\$

Stack

$a,\ \lambda \to a$ $\quad$ $a,\ a \to \lambda$

$b,\ \lambda \to b$ $\quad$ $b,\ b \to \lambda$



Rejection Example: Another Computation

Input

a	b	b	b

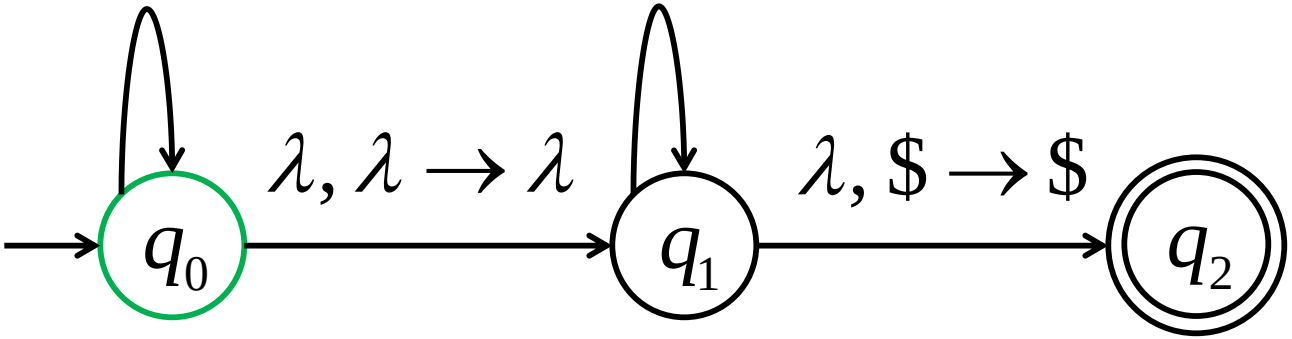
Time 4

b
b
b
a
\$

Stack

$a,\ \lambda \to a$ $\quad$ $a,\ a \to \lambda$

$b,\ \lambda \to b$ $\quad$ $b,\ b \to \lambda$



Rejection Example: Another Computation

Input

a	b	b	b

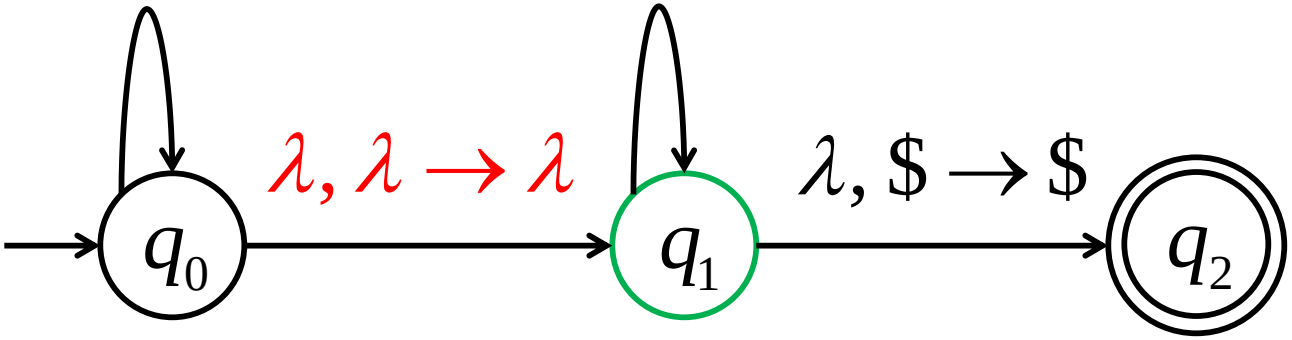
Time 5

b
b
b
a
\$

Stack

$a,\ \lambda \to a$ $\quad$ $a,\ a \to \lambda$

$b,\ \lambda \to b$ $\quad$ $b,\ b \to \lambda$



Rejection Example: Another Computation

Input

a	b	b	b

Time 6

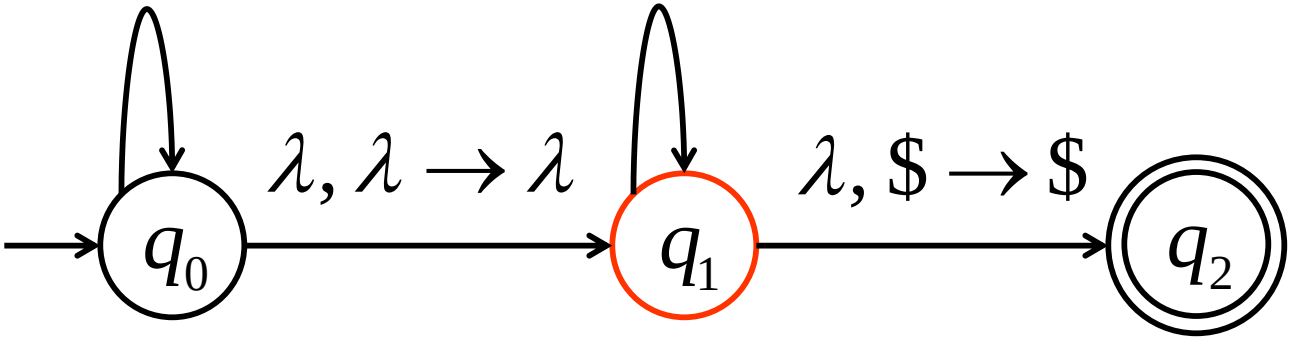
No final state is reached

b
b
b
a
\$

Stack

$a,\ \lambda \to a$ $\quad$ $a,\ a \to \lambda$

$b,\ \lambda \to b$ $\quad$ $b,\ b \to \lambda$



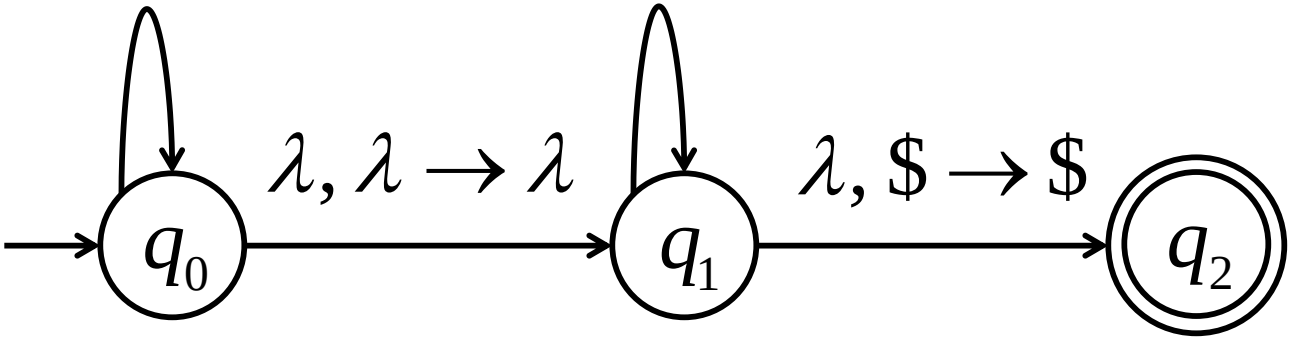
Rejection Example: Another Computation

There is no computation that accepts string *abbb*

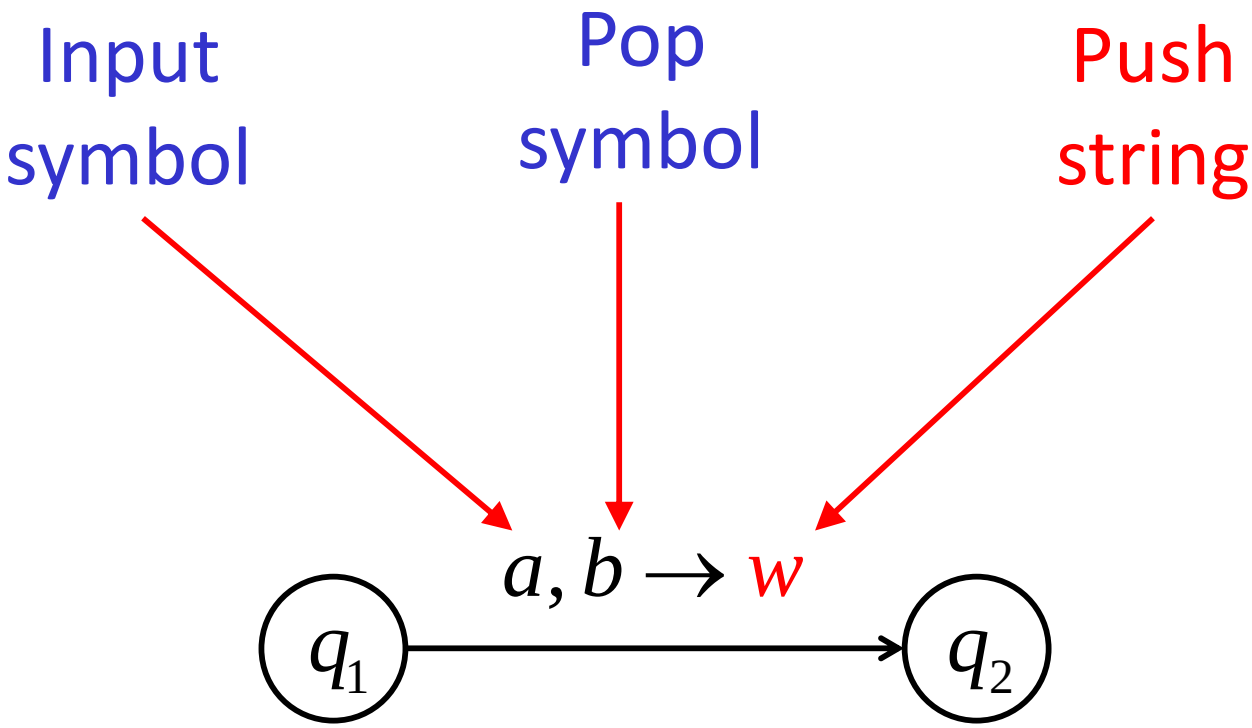
$$abbb \notin L(M)$$

$a,\ \lambda \to a$ $\quad$ $a,\ a \to \lambda$

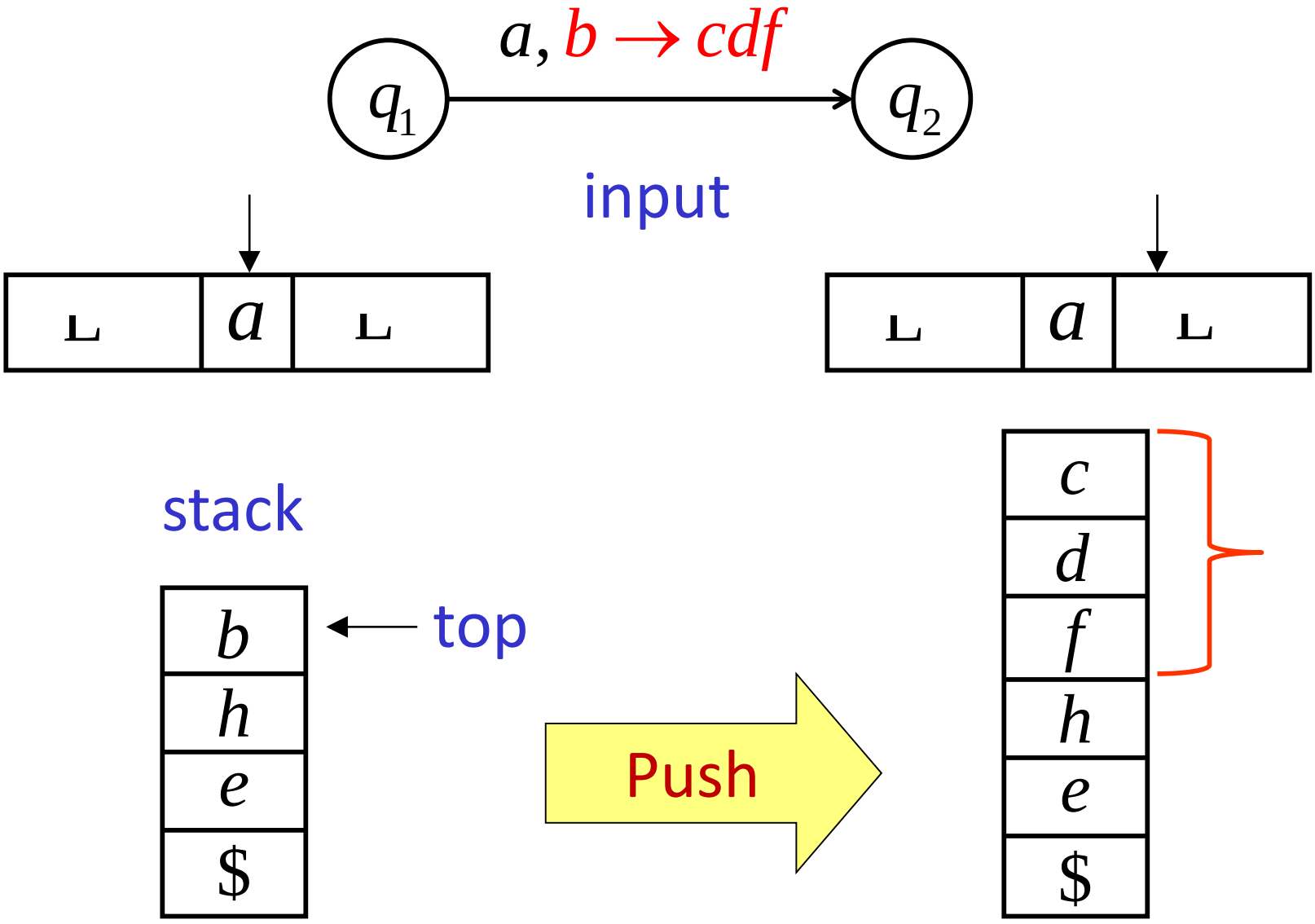
$b,\ \lambda \to b$ $\quad$ $b,\ b \to \lambda$



Pushing Strings



Example



Another NPDA example

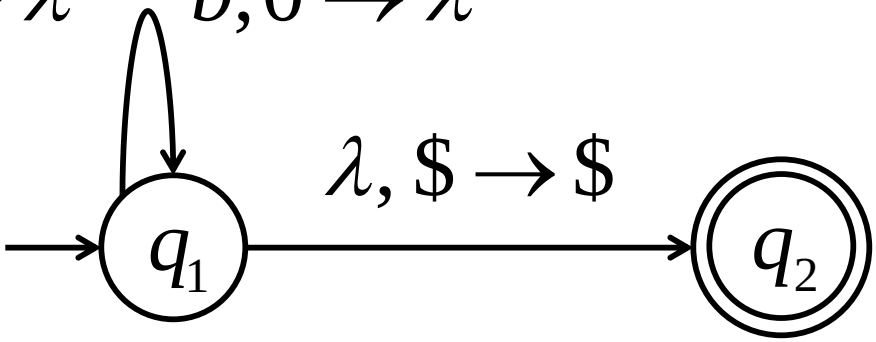
NPDA M

$$L(M) = \{w: \quad n_a(w) = n_b(w)\}$$

$a, \$ \to 0\$ \quad b, \$ \to 1\$$

$a, 0 \to 00 \quad b, 1 \to 11$

$a, 1 \to \lambda \quad b, 0 \to \lambda$



Execution Example

Input

Time 0

a	b	b	b	a	a

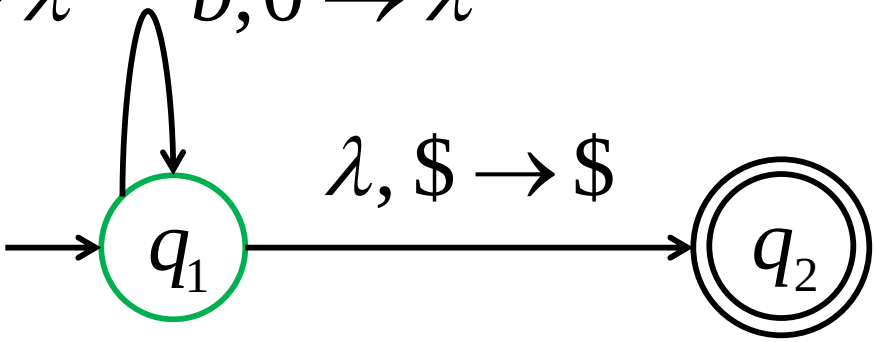
$a, \$ \rightarrow 0\$ \quad b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00 \quad b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda \quad b, 0 \rightarrow \lambda$

$

Stack



Execution Example

Time 1

Input

a	b	b	b	a	a

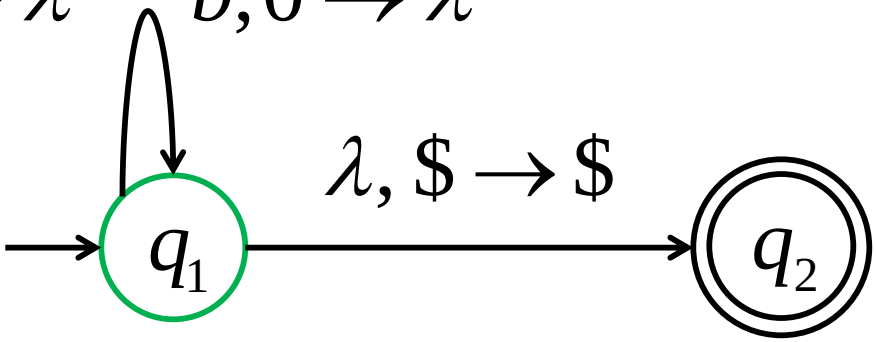
0
$

Stack

$a, \$ \to 0\$$ $b, \$ \to 1\$$

$a, 0 \to 00$ $b, 1 \to 11$

$a, 1 \to \lambda$ $b, 0 \to \lambda$



Execution Example

Time 2

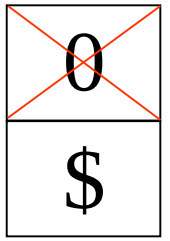
Input

a	b	b	b	a	a

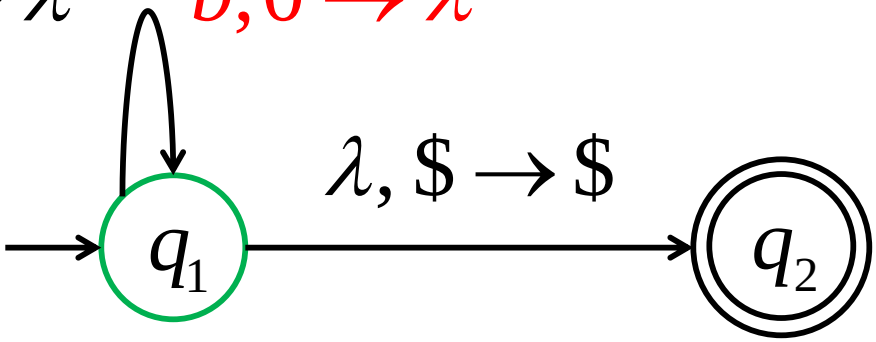
$a, \$ \rightarrow 0\$ \quad b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00 \quad b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda \quad b, 0 \rightarrow \lambda$



Stack



Execution Example

Time 3

Input

a	b	b	b	a	a

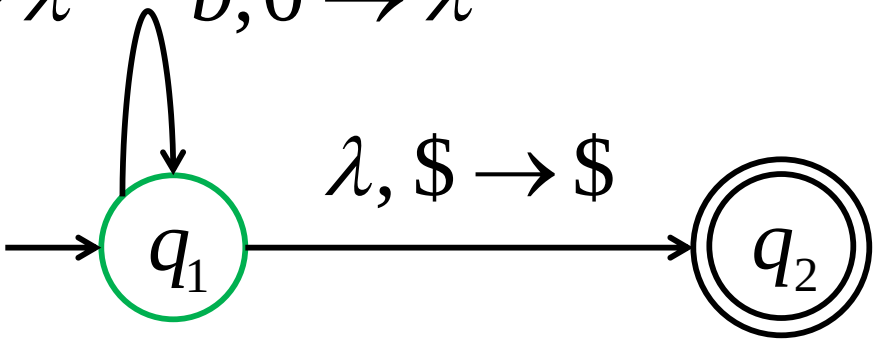
$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda$ $b, 0 \rightarrow \lambda$

1
$

Stack



Execution Example

Input

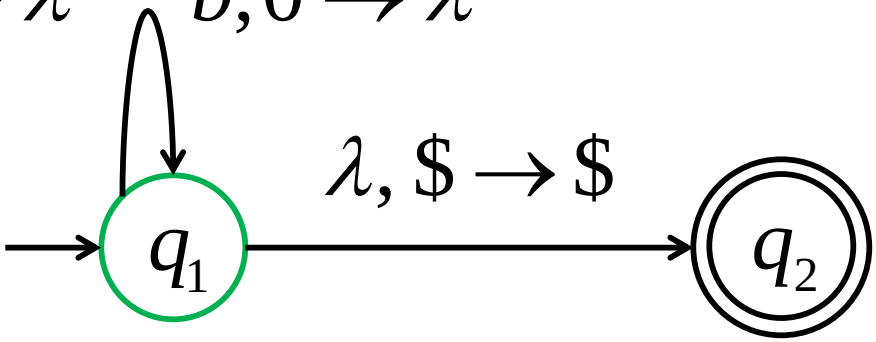
Time 4

a	b	b	b	a	a

$a, \$ \rightarrow 0\$ \quad b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00 \quad b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda \quad b, 0 \rightarrow \lambda$



1
1
$

Stack

Execution Example

Time 5

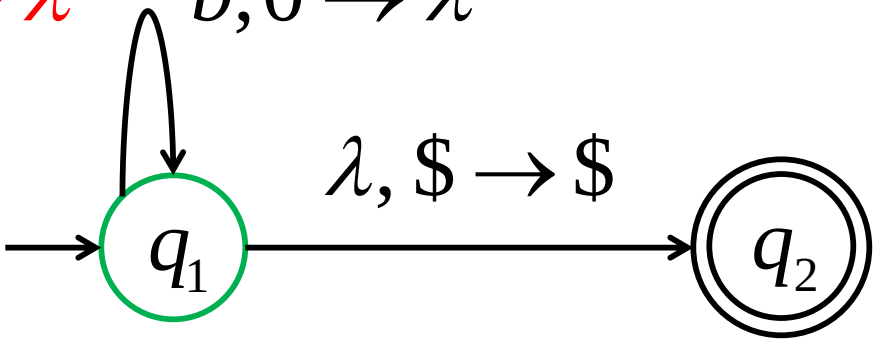
Input

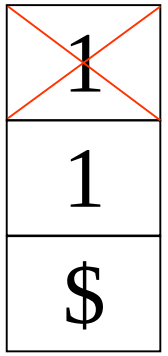
a	b	b	b	a	a

$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda$ $b, 0 \rightarrow \lambda$





Stack

Execution Example

Input

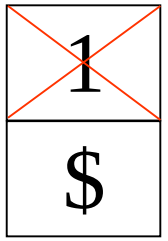
Time 6

a	b	b	b	a	a

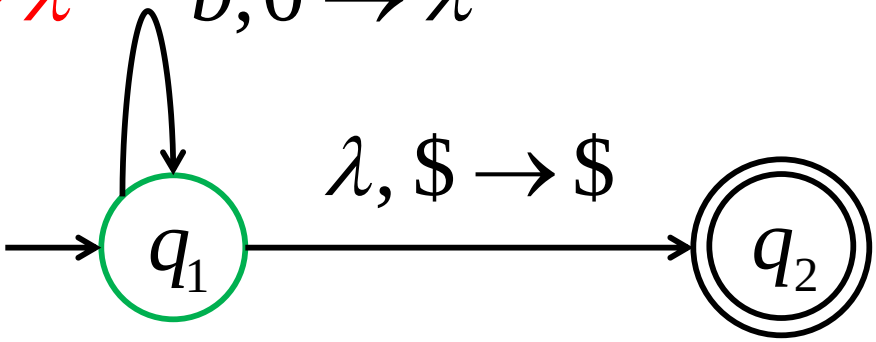
$a, \$ \rightarrow 0\$ \quad b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00 \quad b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda \quad b, 0 \rightarrow \lambda$



Stack



Execution Example

Time 7

Input

a	b	b	b	a	a

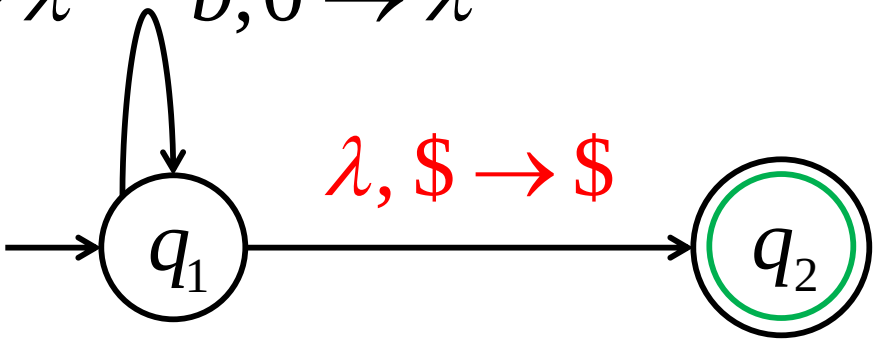
$a, \$ \to 0\$ \quad b, \$ \to 1\$$

$a, 0 \to 00 \quad b, 1 \to 11$

$a, 1 \to \lambda \quad b, 0 \to \lambda$

$

Stack



Execution Example

Input

Time 8

a	b	b	b	a	a

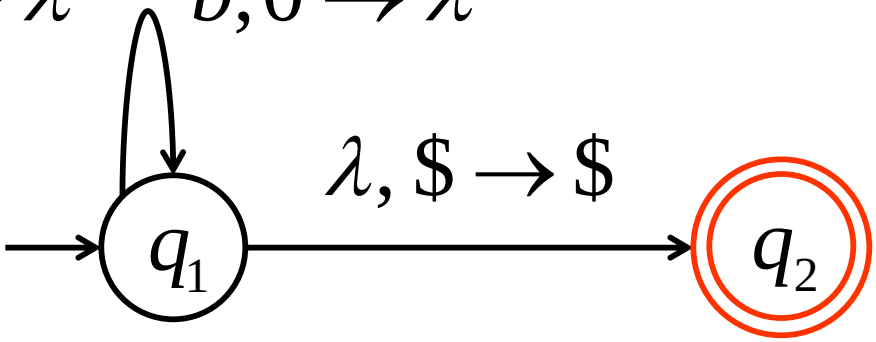
$a, \$ \rightarrow 0\$ \quad b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00 \quad b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda \quad b, 0 \rightarrow \lambda$

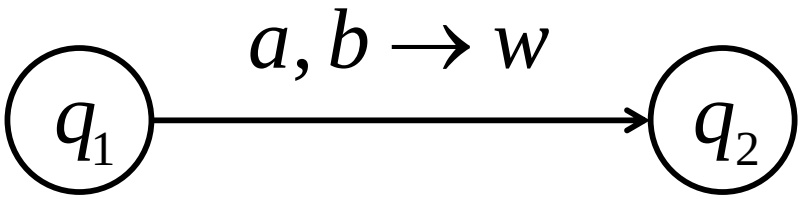
$\$$

Stack



Formalities for NPDAs

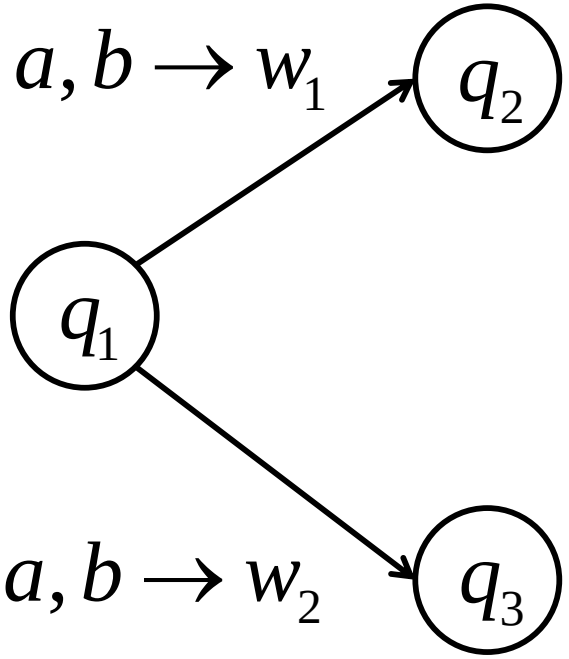
Formal Definition



- Transition function:

$$\delta(q_1, a, b) = \{(q_2, w)\}$$

Formal Definition



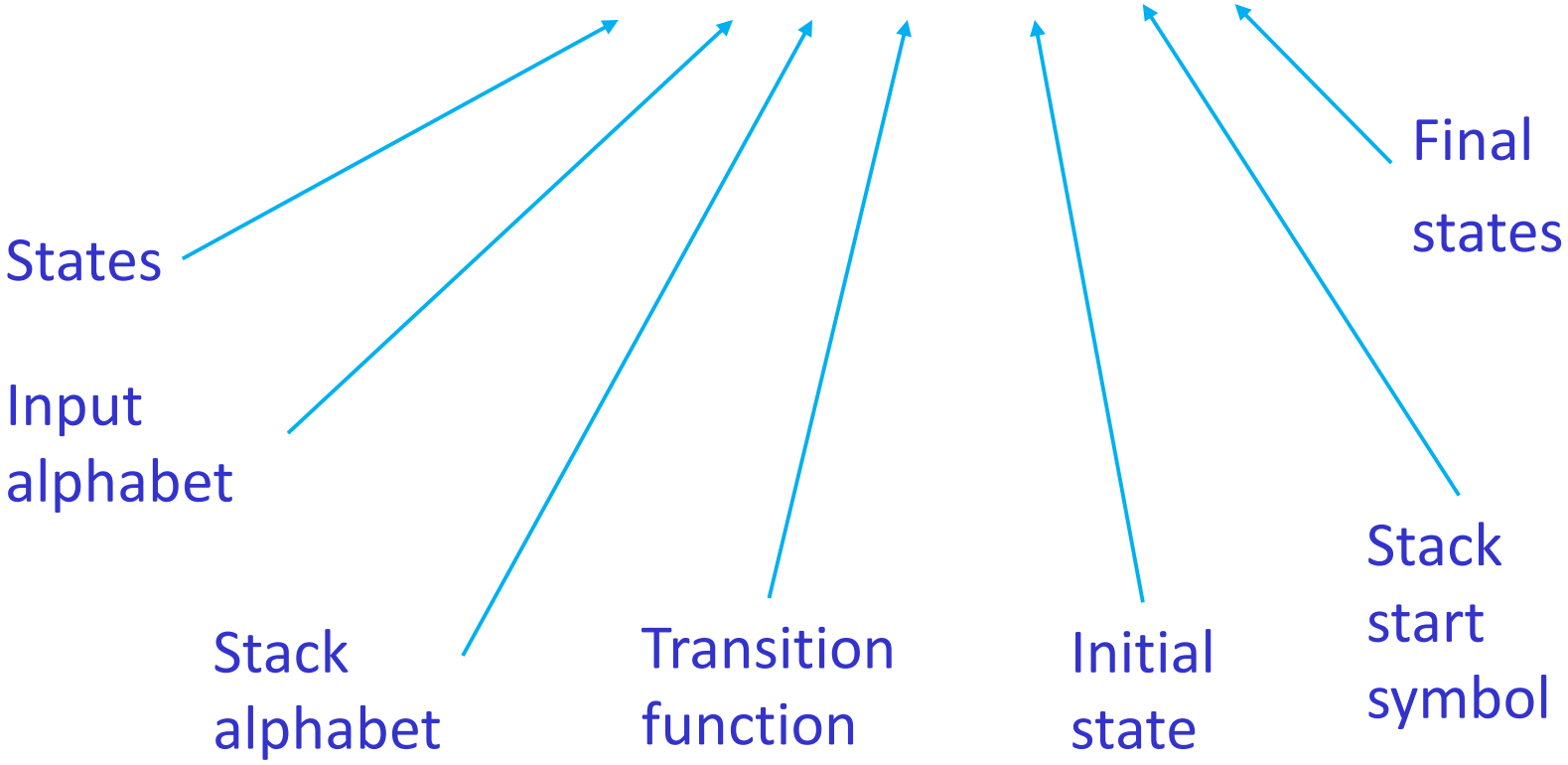
- Transition function:

$$\delta(q_1, a, b) = \{(q_2, w_1), (q_3, w_2)\}$$

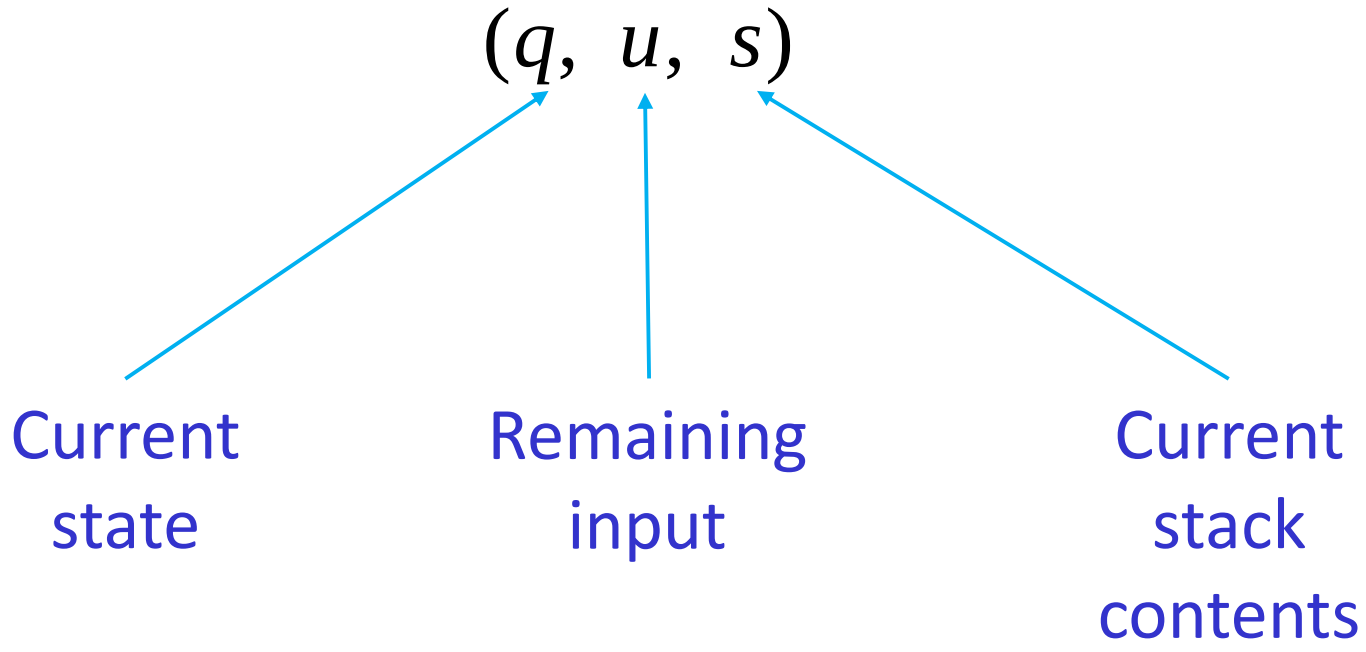
Formal Definition

Non-Deterministic Pushdown Automaton - NPDA

$$M = (Q, \Sigma, \Gamma, \delta, q_0, \$, F)$$



Instantaneous Description



Example: Instantaneous Description

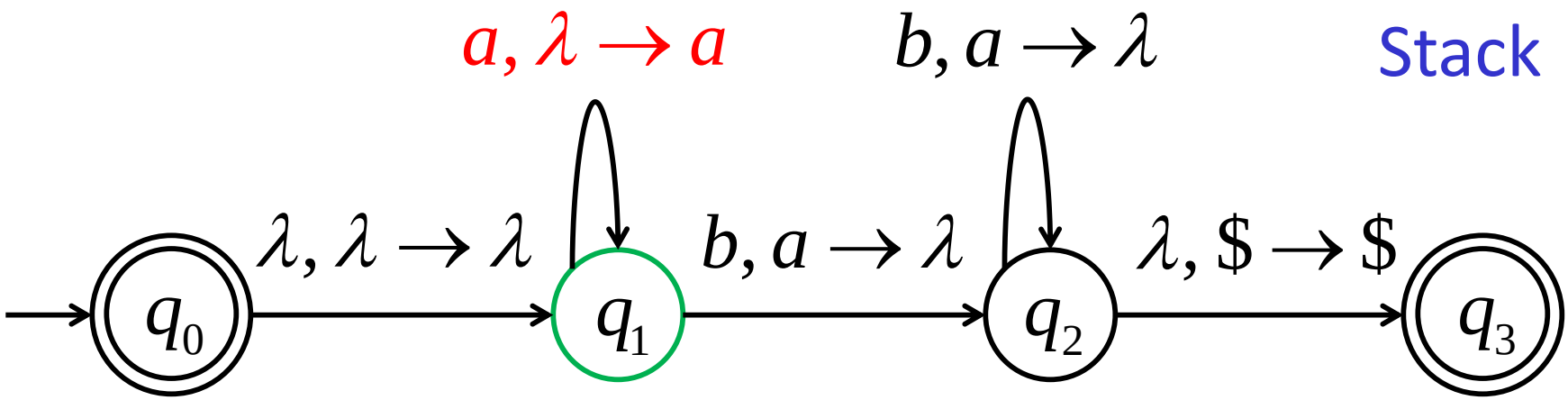
$(q_1, bbb, aaa\$)$

Input

a	a	a	b	b	b

a
a
a
$\$$

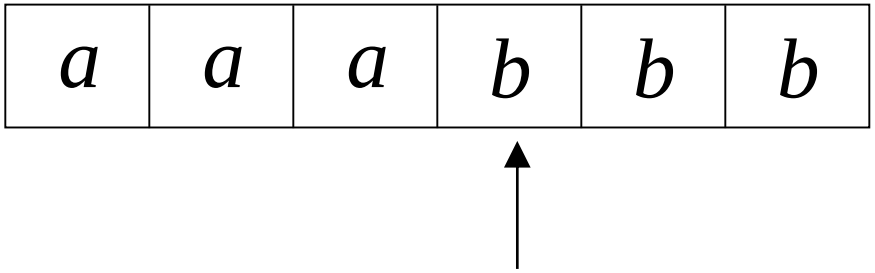
Stack

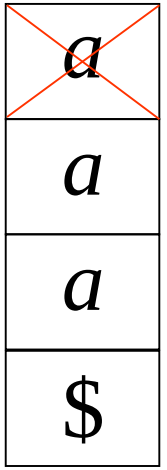


Example: Instantaneous Description

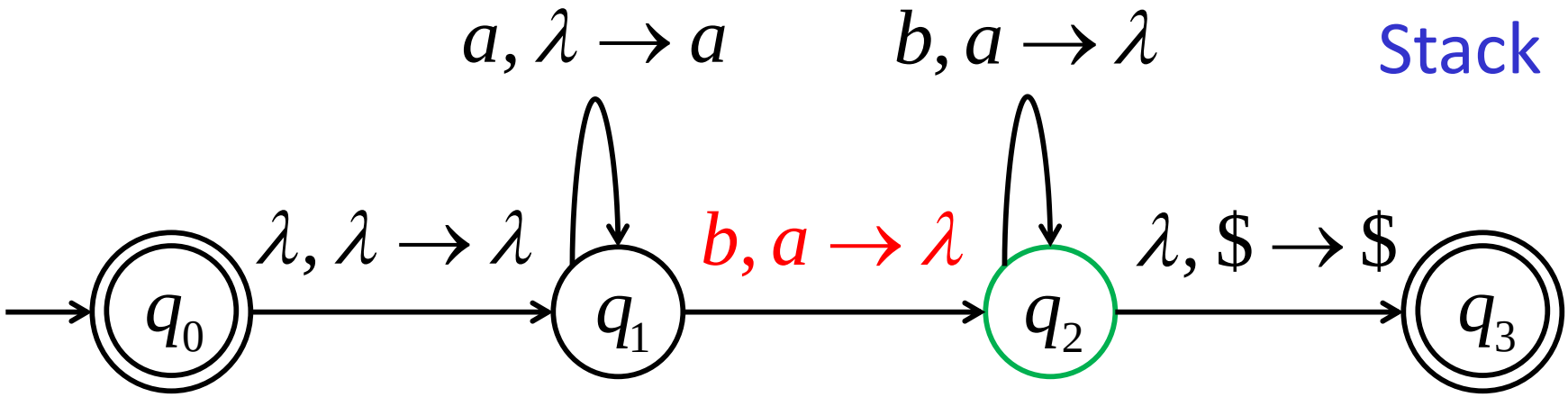
$(q_1, bb, aa\$)$

Input





Stack



Computation

We write:

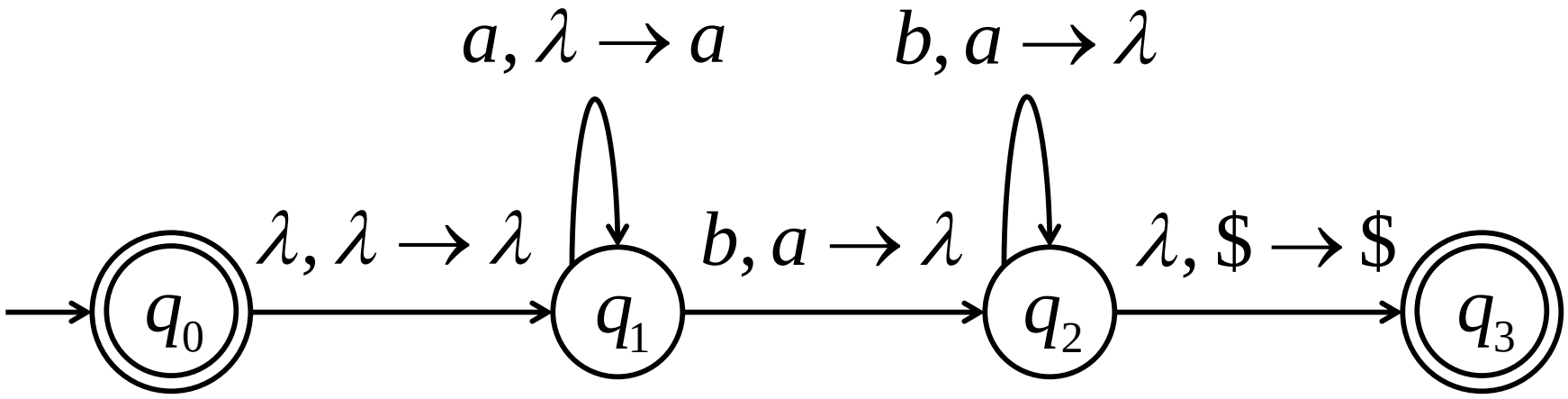
$$(q_1, bbb, aaa\$) \vdash (q_2, bb, aa\$)$$

Computation

$(q_0, aaabbb, \$) \vdash (q_1, aaabbb, \$) \vdash$

$(q_1, aabbb, a\$) \vdash (q_1, abbb, aa\$) \vdash (q_1, bbb, aaa\$) \vdash$

$(q_2, bb, aa\$) \vdash (q_2, b, a\$) \vdash (q_2, \lambda, \$) \vdash (q_3, \lambda, \$)$



Computation

$$(q_0, aaabbb, \$) \vdash (q_1, aaabbb, \$) \vdash$$

$$(q_1, aabbb, a\$) \vdash (q_1, abbb, aa\$) \vdash (q_1, bbb, aaa\$) \vdash$$

$$(q_2, bb, aa\$) \vdash (q_2, b, a\$) \vdash (q_2, \lambda, \$) \vdash (q_3, \lambda, \$)$$

For convenience we write:

$$(q_0, aaabbb, \$) \overset{*}{\vdash} (q_3, \lambda, \$)$$

Formal Definition

Language of NPDA M:

$$L(M) = \{w: \ (q_0, w, s) \ \overset{*}{\vdash} \ (q_f, \lambda, s')\}$$

Initial state

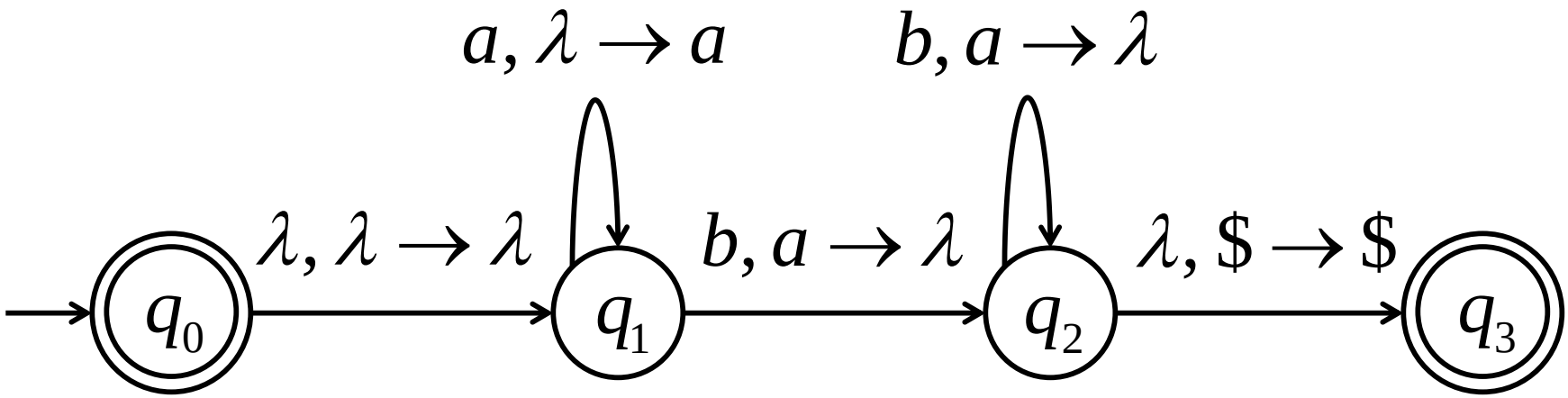
Final state

Example

$$(q_0, aaabbb, \$) \overset{*}{\vdash} (q_3, \lambda, \$)$$

$$aaabbb \in L(M)$$

NPDA M:

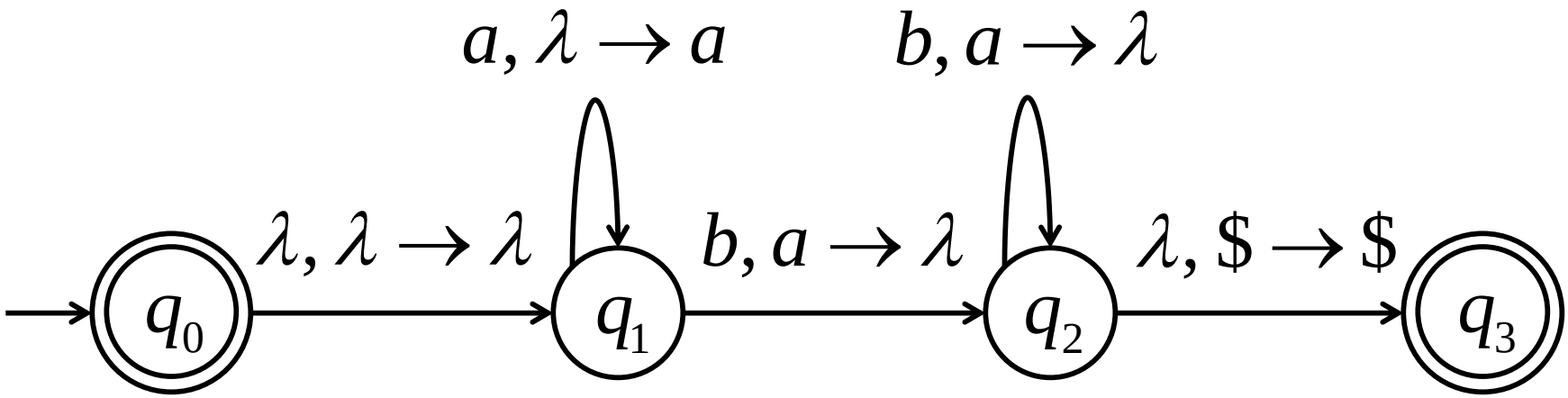


Example

$(q_0, a^n b^n, \$) \overset{*}{\vdash} (q_3, \lambda, \$)$

$a^n b^n \in L(M)$

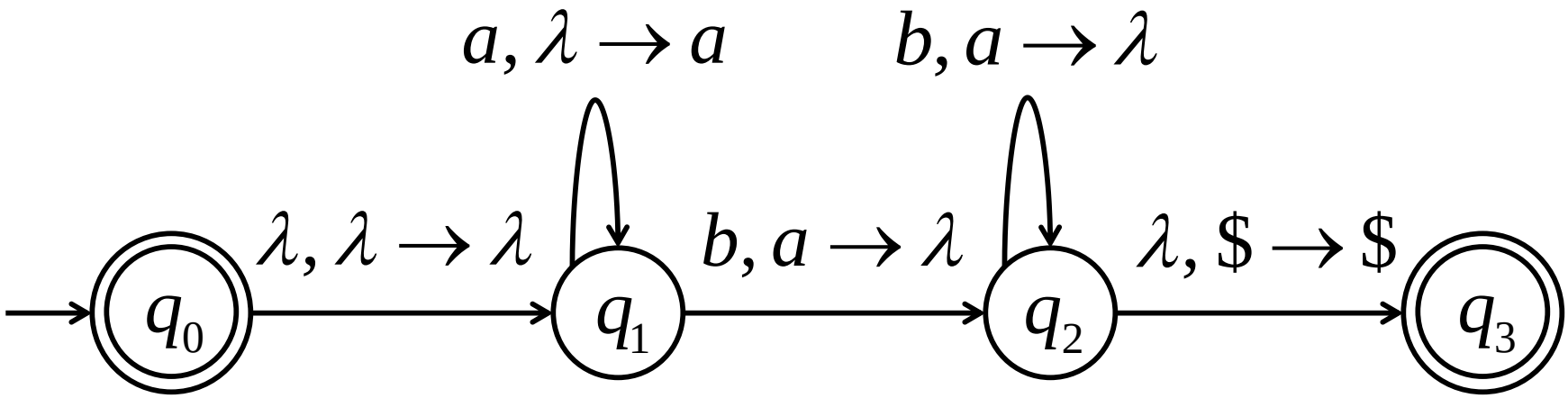
NPDA M:



Example

Therefore: $L(M) = \{a^n b^n : n \geq 0\}$

NPDA M:



Exercises

- Construct NPDAs that accept the following languages on $\Sigma = \{a, b, c\}$.

  a. $L = \{a^n b^{2n} : n \geq 0\}$

  b. $L = \{wcw^R : w \in \{a, b\}^*\}$

  c. $L = \{a^n b^{n+m} c^m : n, m \geq 0\}$

  d. $L = \{a^n b^m c^{n+m} : n, m \geq 0\}$

  e. $L = \{a^3 b^n c^n : n \geq 0\}$

Exercises

- Construct an NPDA that accepts the language

$$L = \{w : n_a(w) = n_b(w) + 1\}$$

- Construct an NPDA that accepts the language

$$L = \{w : n_a(w) = 2n_b(w)\}$$

- Construct an NPDA that accepts the language

$$L = \{w : \ n_a(w) + n_b(w) = n_c(w)\}$$