Pumping Lemma

Non-regular languages

* اَللّٰـهُمَّ إِنَّـا نَسْأَلُكَ لِسَانًا رَطِبًا بِذِكْرِكَ وَقَلْبًـا مُفْعَمًـا بِشُكْـرِكَ وَبَدَنًا هَيِّنًا لَيِّنًا بِطَاعَتِكَ اَللّٰهُمَّ إِنَّا نَسْأَلُكَ إِيْمَانًا كَامِلاً

وَنَسْأَلُكَ قَلْبًا خَاشِعًا وَنَسْأَلُكَ عِلْمًا نَافِعًا وَنَسْأَلُكَ يَقِيْنًا صَادِقًا وَنَسْأَلُكَ دِيْنًا قَيِّمًا وَنَسْأَلُكَ الْعَافِيَةَ مِنْ كُلِّ بَلِيَّةٍ وَنَسْأَلُكَ تَمَامَ الْغِنَى عَنِ النَّاسِ وَهَبْ لَنَا حَقِيْقَةَ الْإِيْمَـانِ بِكَ حَتَّى لاَ نَخَافَ وَلاَ نَرْجُوْ غَيْرَكَ وَلاَ نَعْبُدَ شَيْئًا سِوَاكَ وَاجْعَلْ يَدَكَ مَبْسُـوْطَةً عَلَيْنَا وَعَلى أَهْلِيْنَا وَأَوْلاَدِنَا وَمَنْ مَعَنَـا بِرَحْمَتِـكَ وَلاَ تَكِـلْنَـا إِلَى أَنْفُسِنَا طَرْفَةَ عَيْنٍ وَلاَ أَقَلَّ مِنْ ذٰلِكَ يَا نِعْمَ الْمُجِيْبُ.

Ya Allah kurniakanlah kami lisan yang lembut basah mengingati dan menyebut (nama)-Mu, hati yang penuh segar mensyukuri (nikmat)-Mu, serta badan yang ringan menyempurnakan ketaatan kepada (perintah)Mu. Ya Allah, kurniakanlah kami iman yang sempurna,

hati yang khusyuk, ilmu yang berguna, keyakinan yang benar-benar mantap. (Ya Allah) kurniakanlah kami (din) cara hidup yang jitu dan unggul, selamat dari segala mara bahaya dan petaka. Kami mohon (Ya Allah) kecukupan yang tidak sampai kami tepaksa meminta jasa orang lain. Berikanlah kami (Ya Allah) iman yang sebenarnya sehingga kami tidak lagi gentar atau mengharap orang lain selain dari Engkau sendiri. Kembangkanlah lembayung rahmatMu kepada kami, keluarga dan anak-anak kami serta sesiapa sahaja yang bersama-sama kami. Jangan (Ya Allah) Engkau biarkan nasib kami ditentukan oleh diri kami sendiri; walaupun kadar sekelip mata atau kadar masa yang lebih pendek dari itu. Wahai Tuhan yang paling mudah dan cepat memperkenankan pinta (perkenankanlah).

Non-regularity

Non-regular languages

$$\{a^n b^n : n \geq 0\}$$

$$\{ww^R : w \in \{a,b\}^*\}$$

Regular languages

$$a^* b \qquad b^* c + a$$

$$b + c(a+b)^*$$

etc...

Non-regularity

How can we prove that a language L is not regular?

Prove that there is no DFA that accepts L

Problem: this is not easy to prove

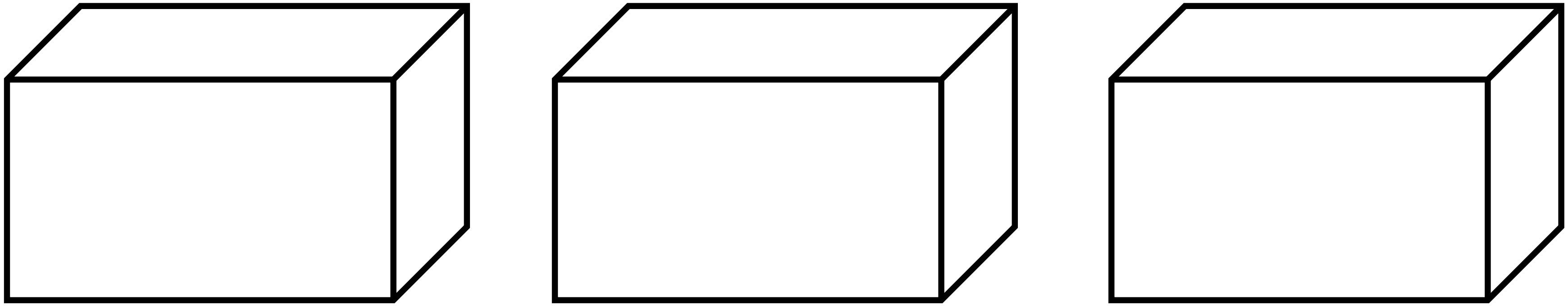
Solution: the Pumping Lemma !!!

The Pigeonhole Principle

Pigeonhole Principle

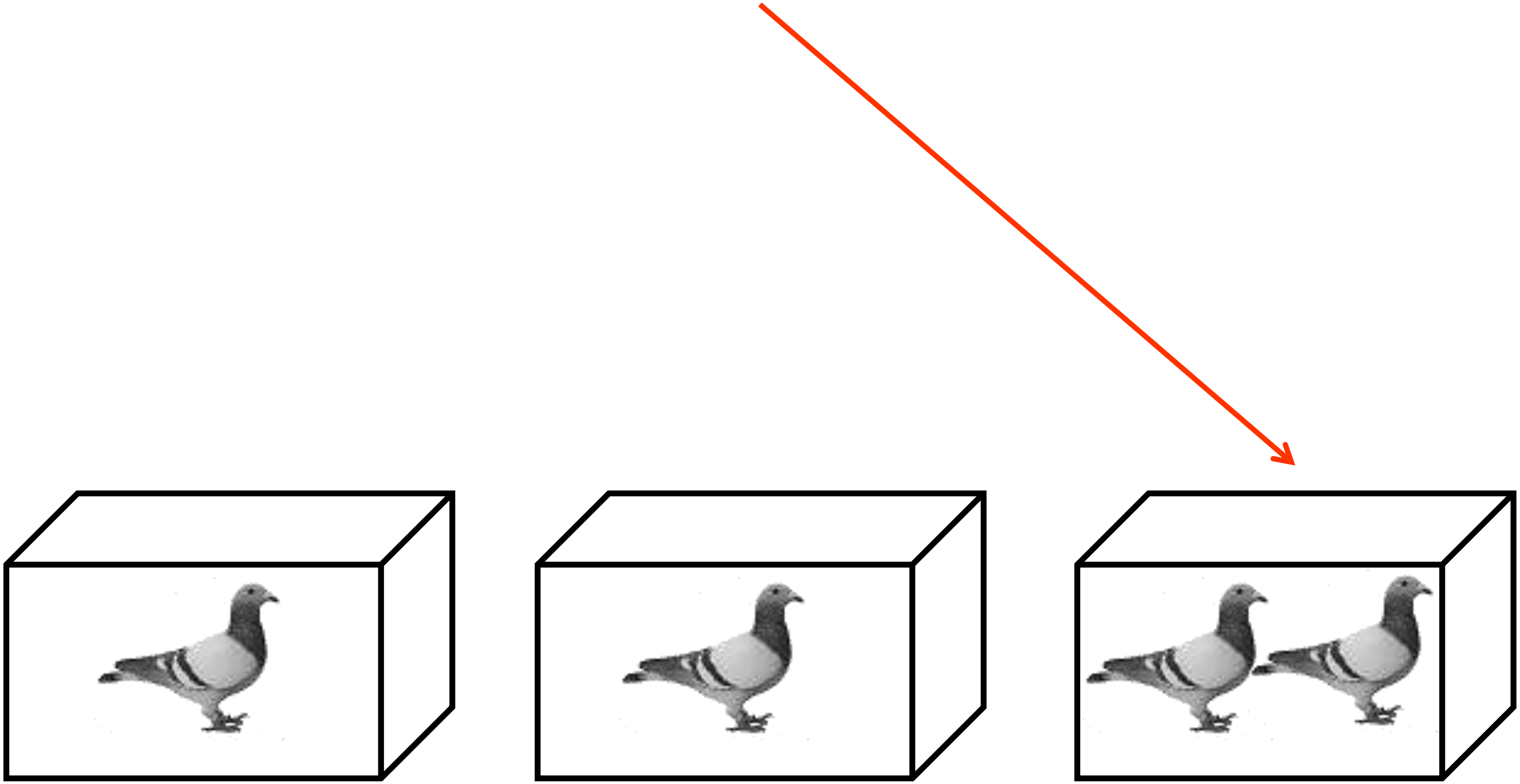
4 pigeons

3 pigeonholes



Pigeonhole Principle

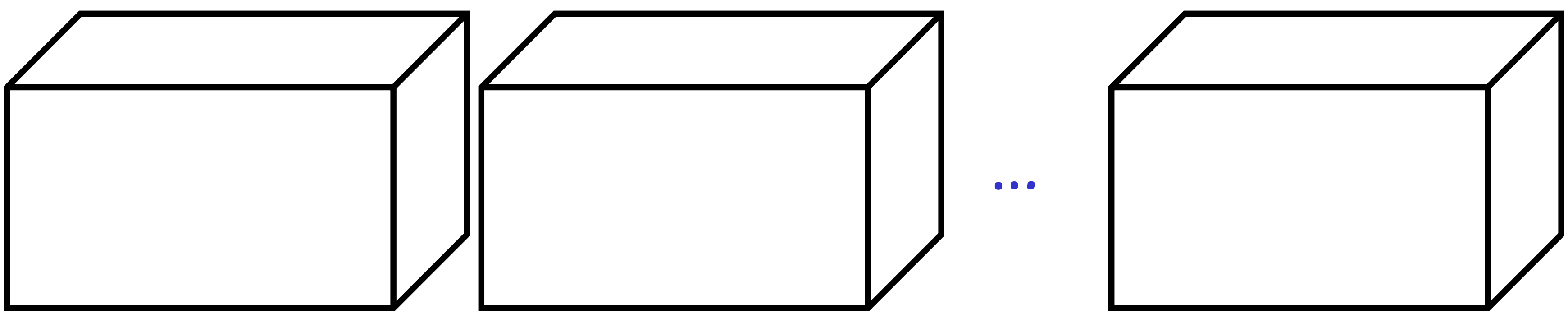
- A pigeonhole must contain at least two pigeons



Pigeonhole Principle

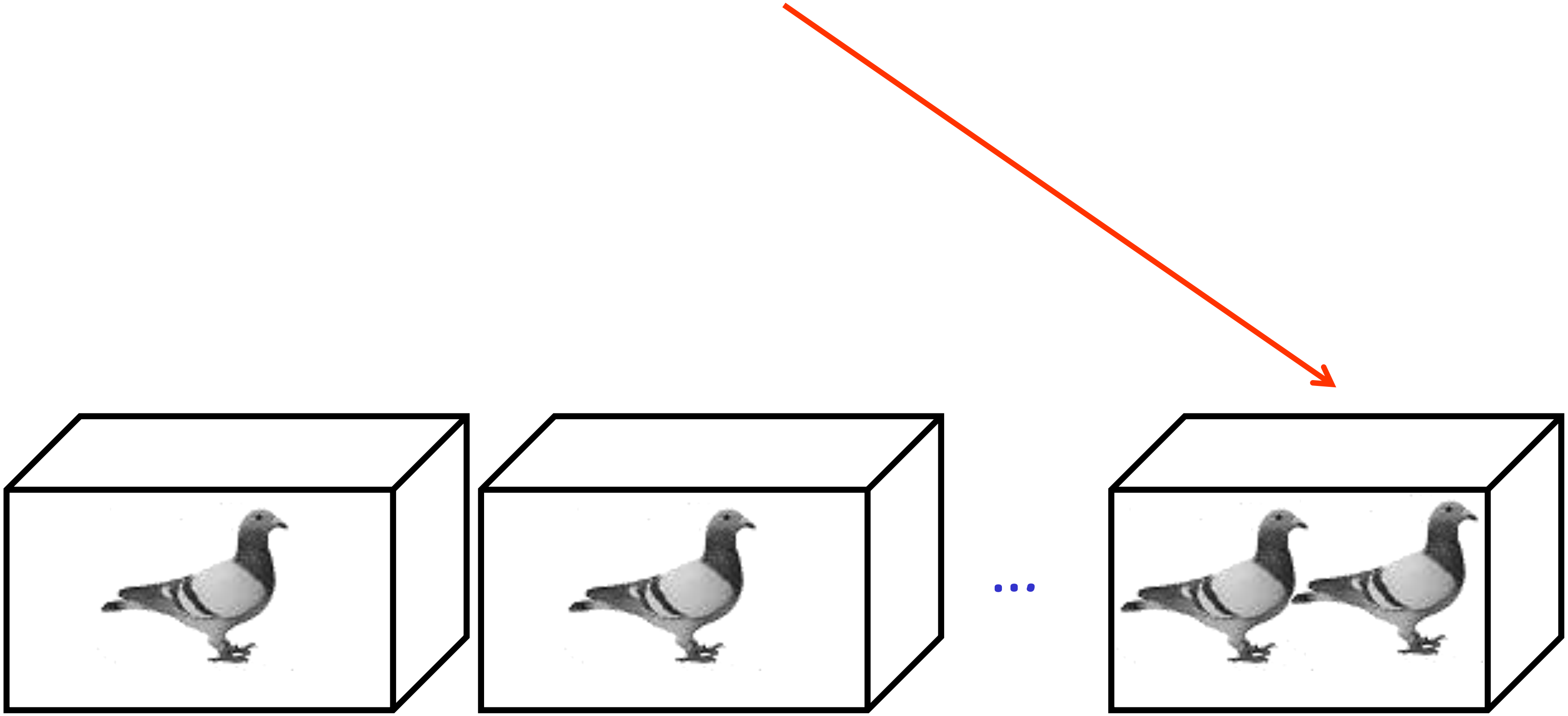
n pigeons

m pigeonholes and $n > m$



Pigeonhole Principle

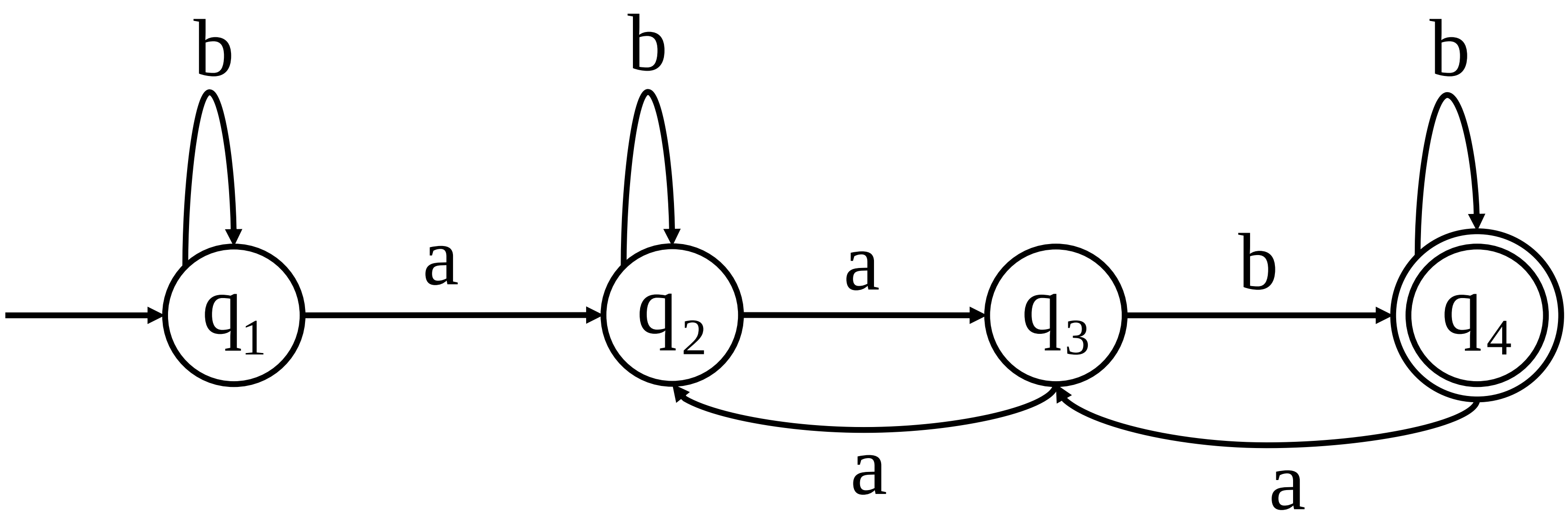
- Since $n > m$, there is a pigeonhole with at least two pigeons



The Pigeonhole Principle
and
DFAs

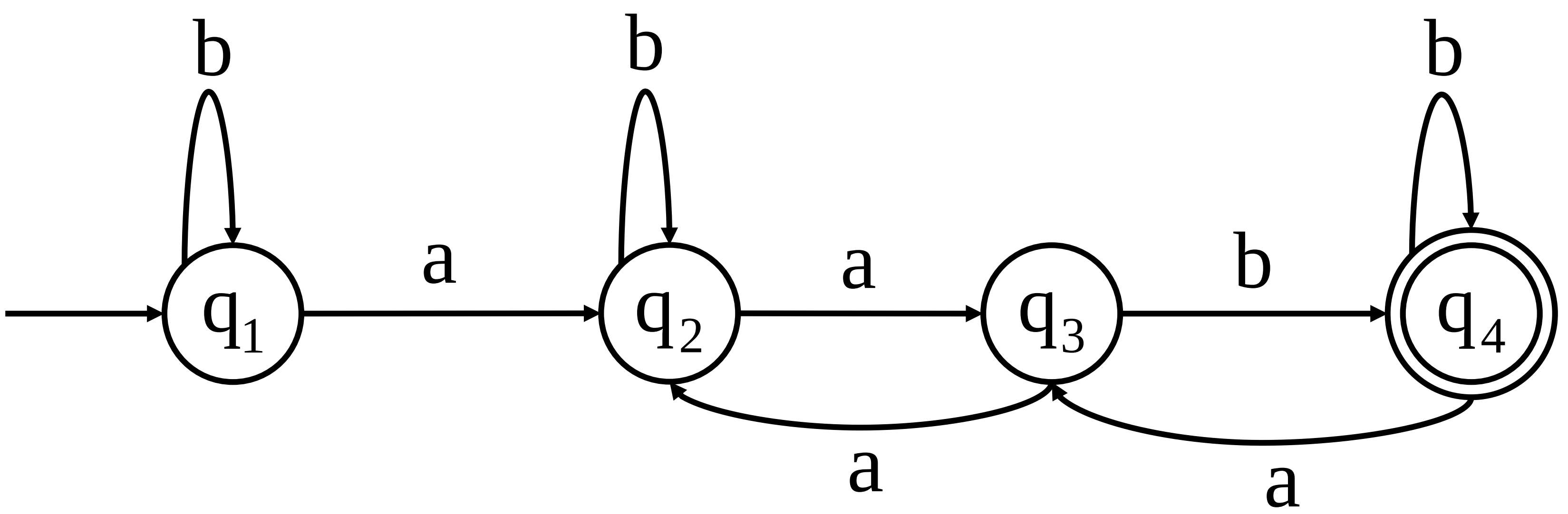
The Pigeonhole Principle and DFAs

DFA with 4 states



The Pigeonhole Principle and DFAs

In walks of strings a, aa, aab no state is repeated



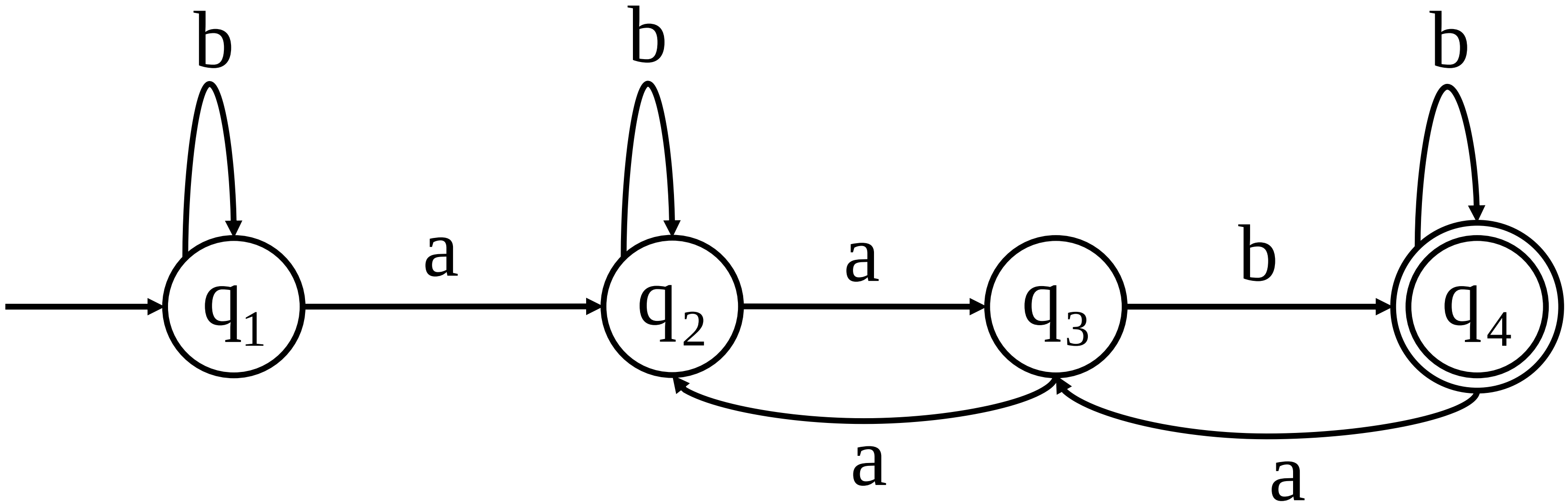
The Pigeonhole Principle and DFAs

In walks of strings aabb a state is repeated

bbaa

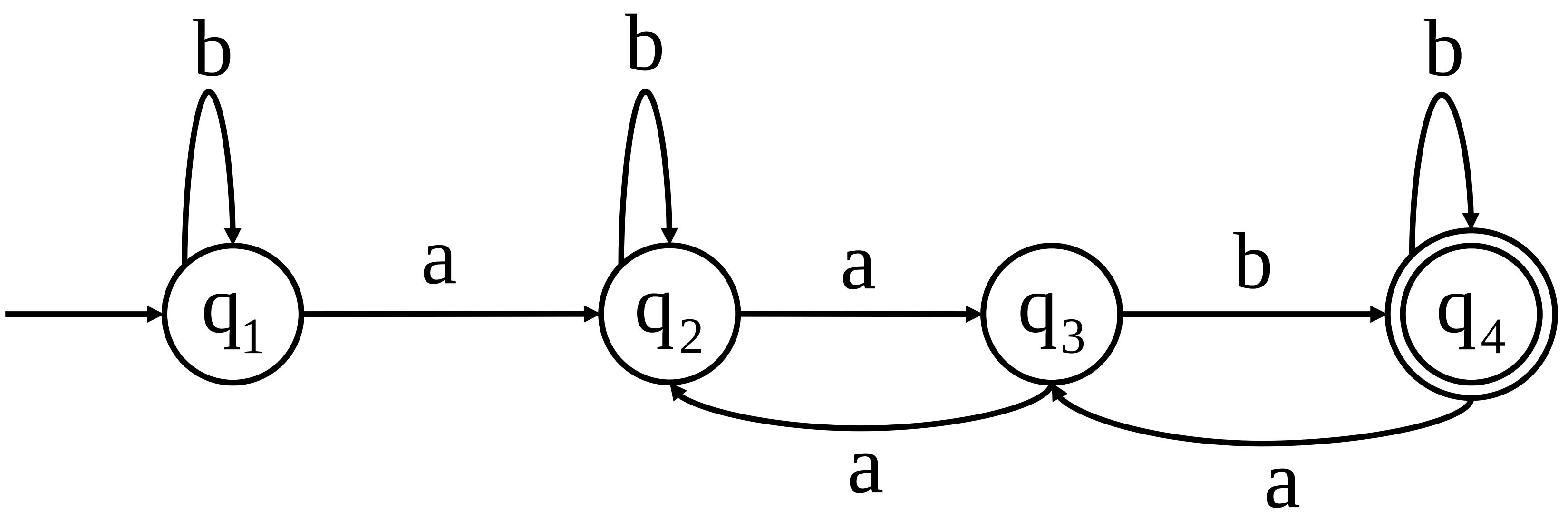
abbabb

abbbabbabb...



The Pigeonhole Principle and DFAs

- If string w has length $|w| \geq 4$, then the transitions of string w are more than the states of the DFA.
- Thus, a state must be repeated.

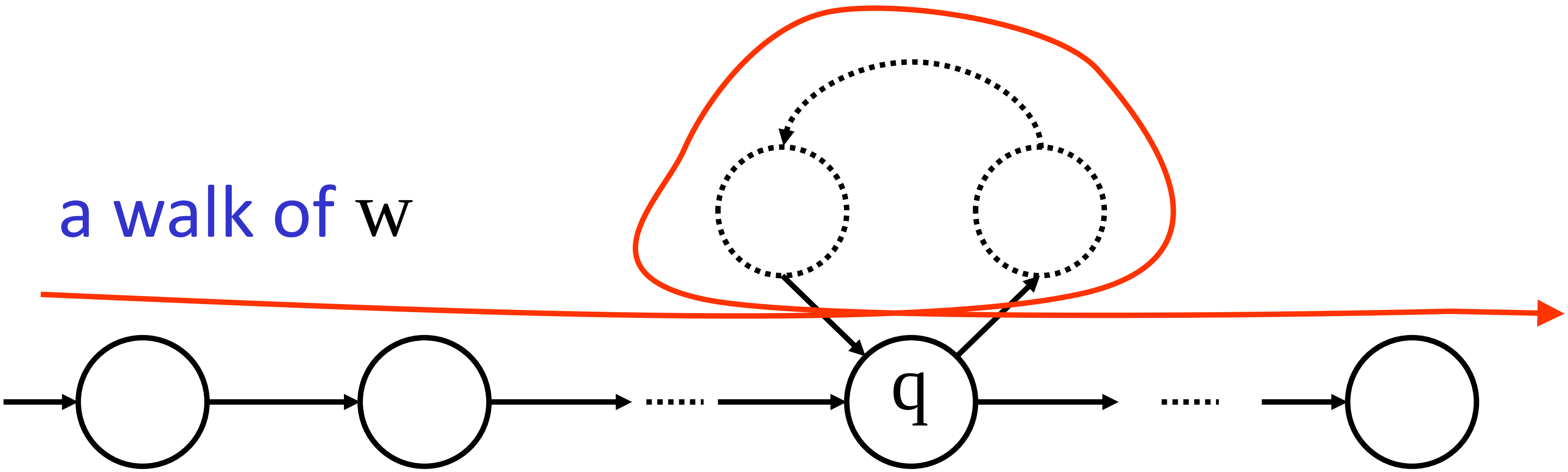


The Pigeonhole Principle and DFAs

- In general, for any DFA:

  String w has length $|w| \geq$ number of states,
  then some state q in the walk of w.

a walk of w

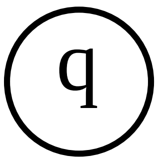


repeated state

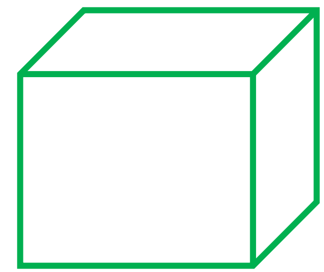
The Pigeonhole Principle and DFAs

- In other words for a string w:

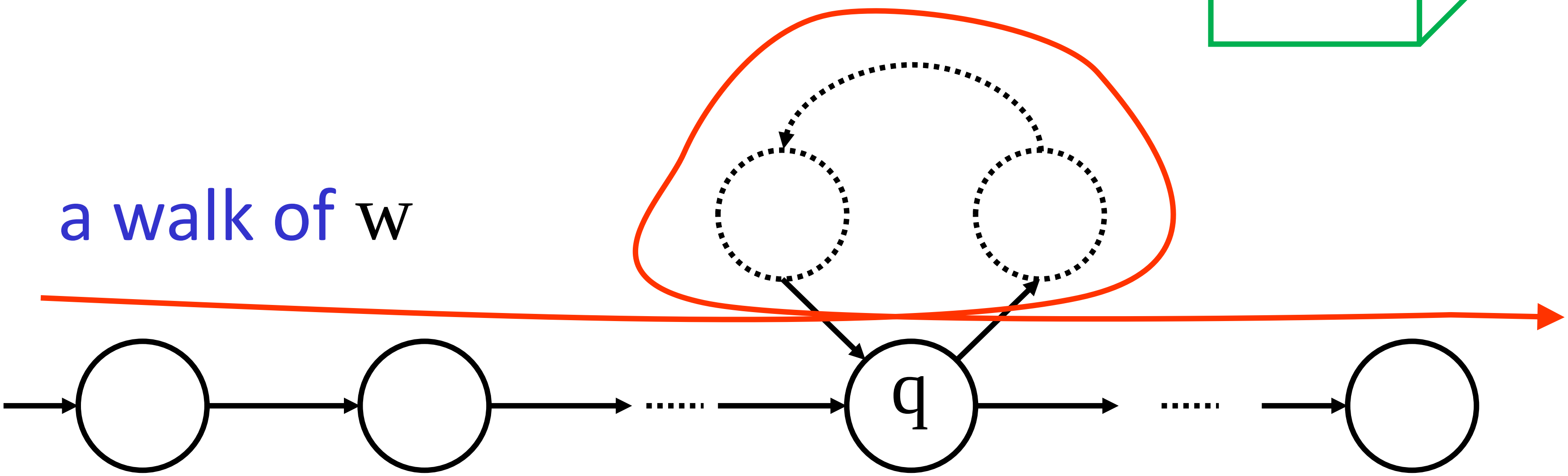
transitions are pigeons



states are pigeonholes



a walk of w



repeated state

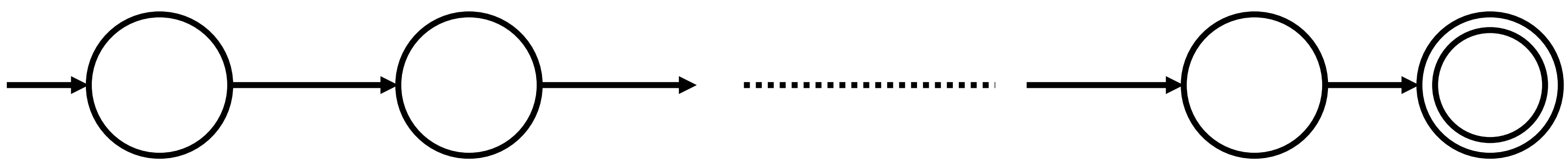
The Pumping Lemma

The Pigeonhole Principle and DFAs

- Let L be infinite regular language. Then there is a DFA M with m states that accepts L.

- Let $w \in L$. Then there is a walk in the DFA M with the label w

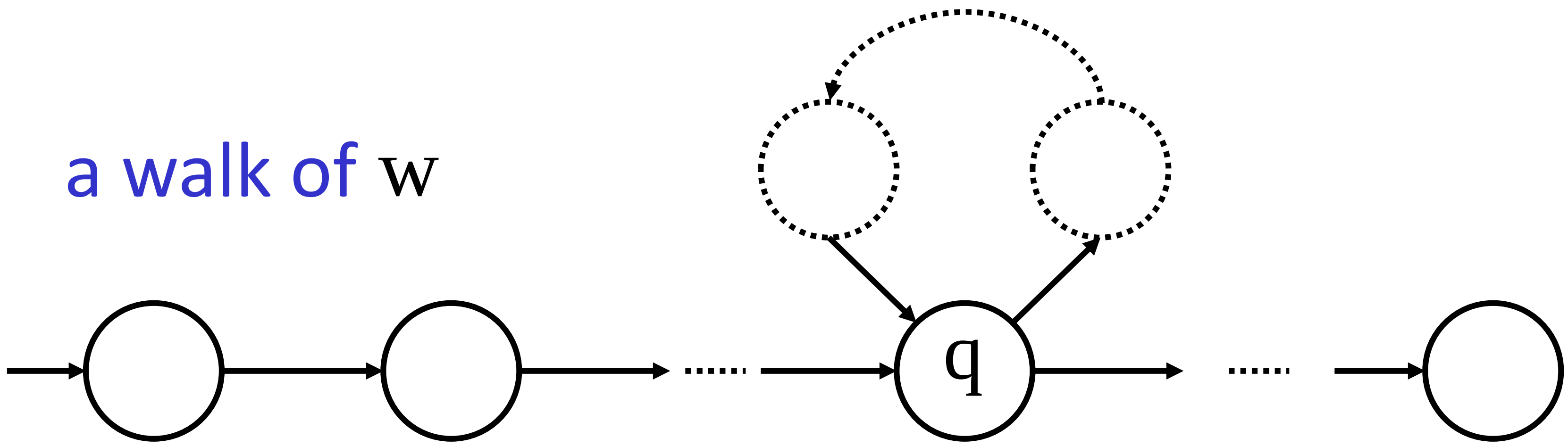
a walk of w



The Pigeonhole Principle and DFAs

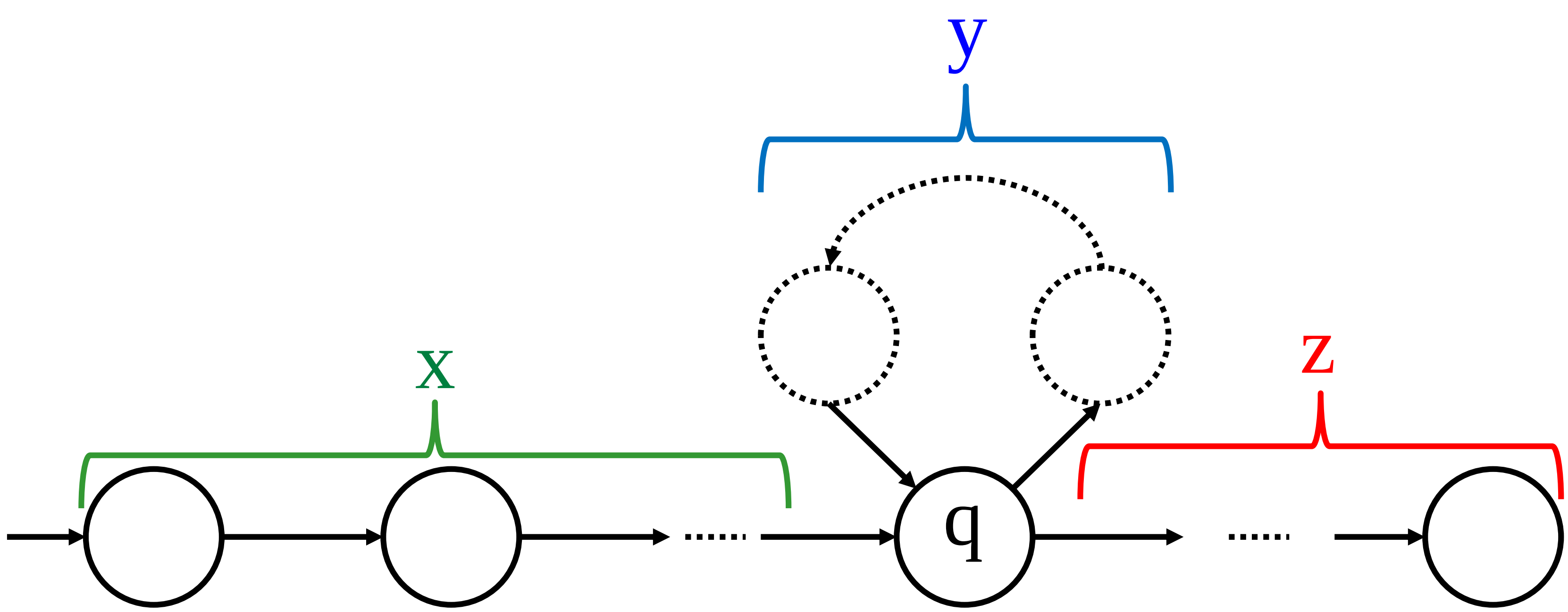
- If the length of the string w greater than or equal to m, i.e., $|w| \geq m$ then some state q is repeated in the walk.
- We assume that q the first state repeated in the walk.

a walk of w



The Pigeonhole Principle and DFAs

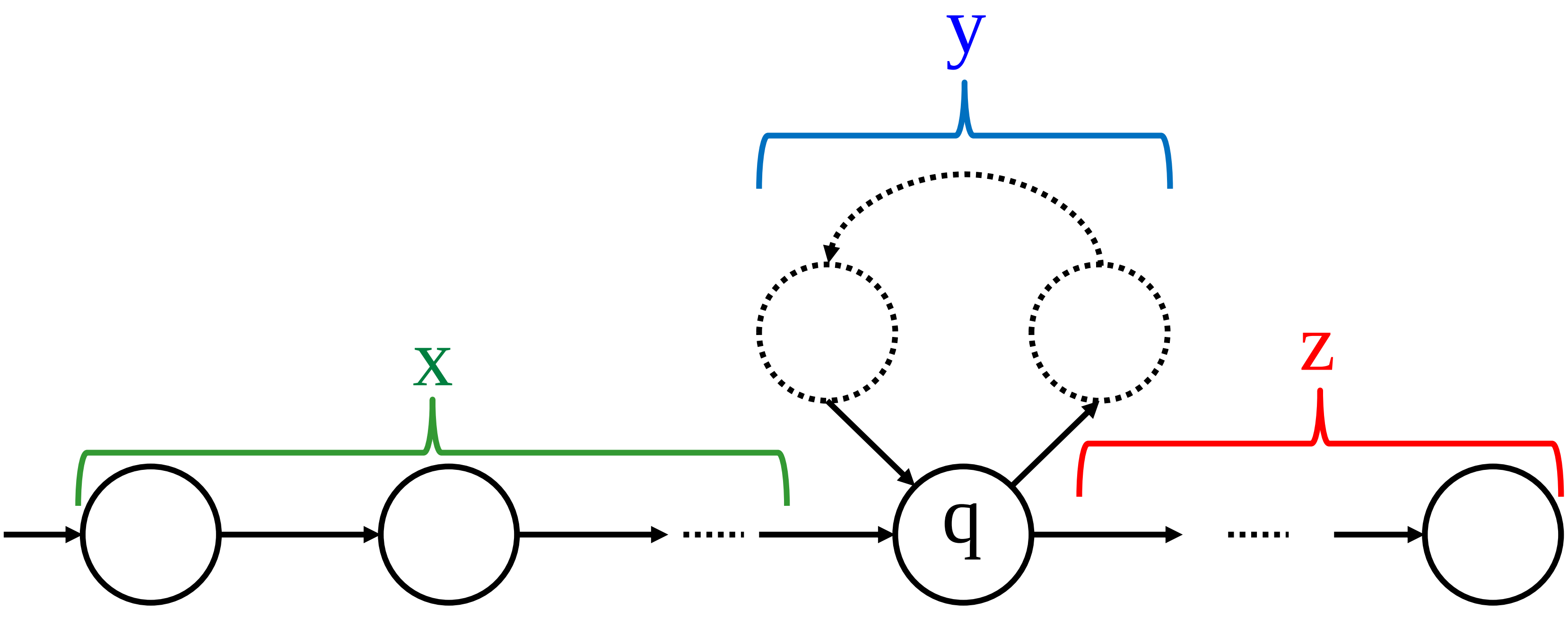
- Then we can divide w into three substrings $\mathrm{w} = \mathrm{x} \cdot \mathrm{y} \cdot \mathrm{z}$ such that



The Pigeonhole Principle and DFAs

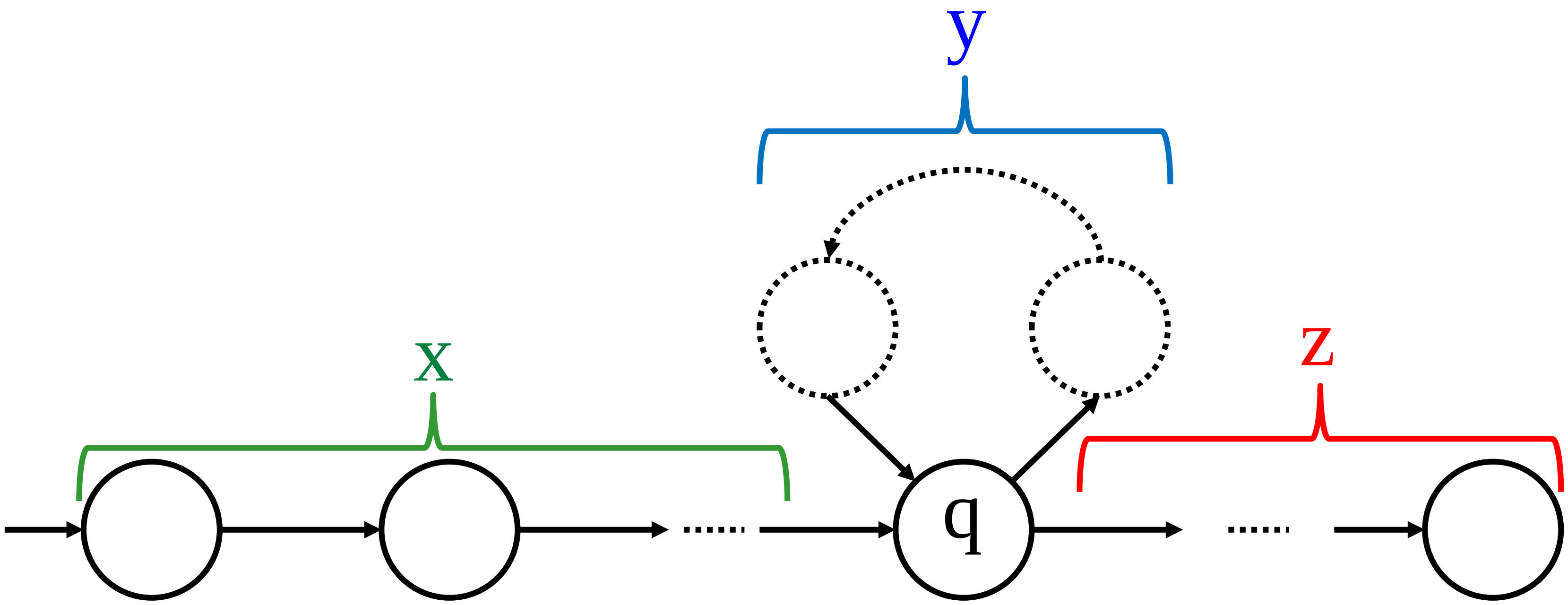
- Observation: the length of $x \cdot y$: $|x \cdot y| \leq m$

  and the length of y: $|y| \geq 1$



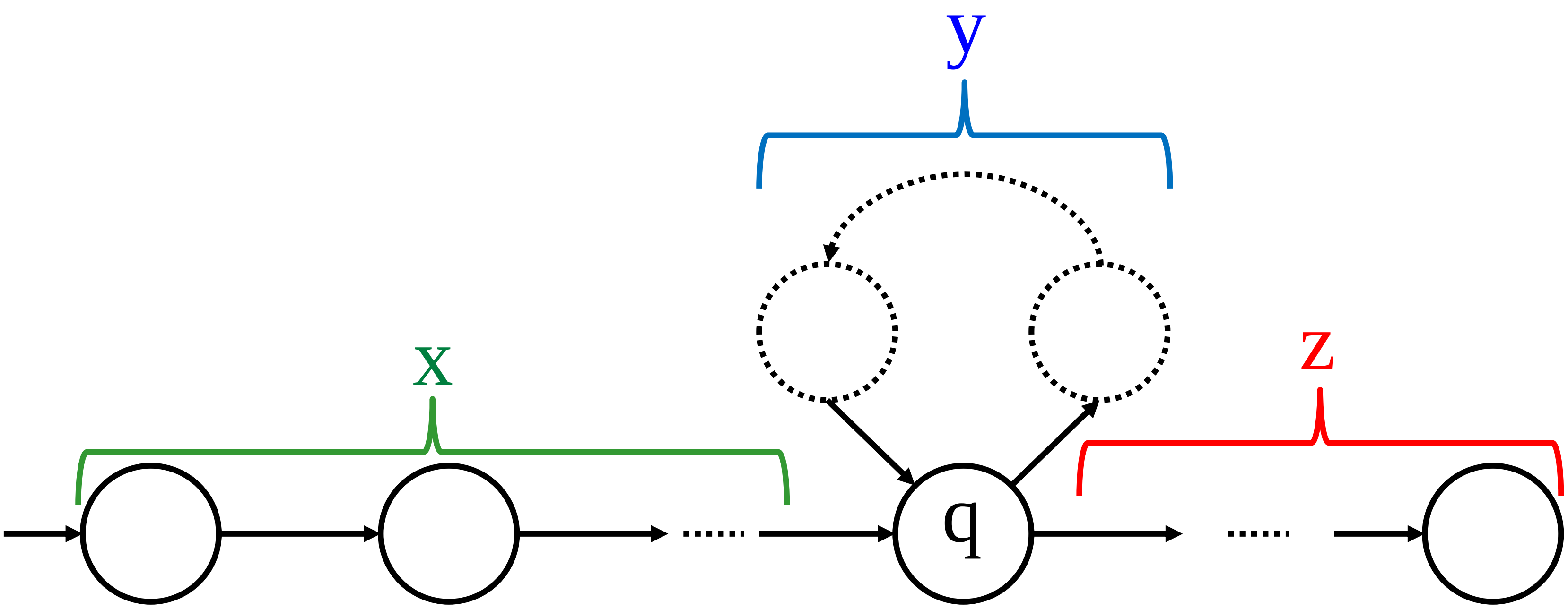
The Pigeonhole Principle and DFAs

- Observation: the string $x \cdot z$ is accepted



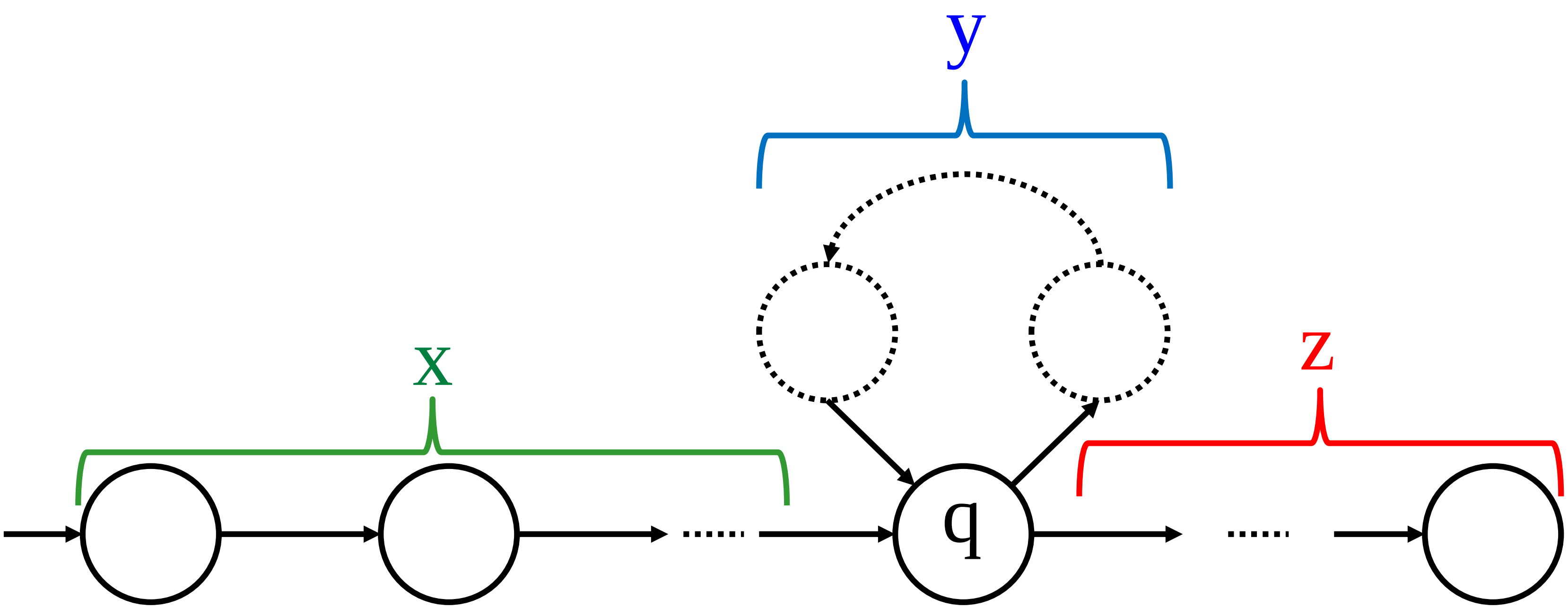
The Pigeonhole Principle and DFAs

- Observation: the string $x \cdot y \cdot y \cdot z$ is accepted



The Pigeonhole Principle and DFAs

- In general : the string $x \cdot y^i \cdot z$ is accepted, where $i = 0, 1, 2, \ldots$



The Pumping Lemma

- Let L be an infinite regular language. Then there exists some positive integer m such that any $w \in L$ with $|w| \geq m$ can be decomposed as

$$w = x \cdot y \cdot z$$

with

$$|x \cdot y| \leq m$$

and

$$|y| \geq 1$$

such that $w_i = x \cdot y^i \cdot z \in L$ for all $i = 0, 1, 2, \ldots$

Applications
of
the Pumping Lemma

Theorem

- The language $L = \{a^n b^n : n \geq 0\}$ is **not** regular.

- **Proof**. By contradiction and using the Pumping Lemma.

  Assume that L is a regular language. Since L infinite language, we can apply the Pumping Lemma.

Proof

- Let the language L is accepted by a DFA M with m states.
- Let $w = a^m b^m \in L$.
- Obviously, $|w| \geq m$.
- Since, $|x \cdot y| \leq m$, y consists of only a's.
- So,

$$x \cdot y \cdot z = a^m b^m = \overbrace{\underbrace{a \cdots a}_{x}\,\underbrace{a \cdots a}_{y}\,\underbrace{a \cdots a\,}_{}}^{m}\!\!\!\!\!\underbrace{\overbrace{b \cdots b}^{m}}_{z}$$

where $y = a^k, k \geq 1$.

Proof

- By the Pumping Lemma, $x \cdot y^i \cdot z$ is also in L, for all $i = 0,1,2, \ldots$.

- Thus, $x \cdot y^2 \cdot z \in L$.

$$x \cdot y^2 \cdot z = \overbrace{\underbrace{a \cdots a}_{x}\,\underbrace{a \cdots a}_{y}\,\underbrace{a \cdots a}_{y}\,a \cdots a}^{m+k}\overbrace{b \cdots b}^{m}$$

(where $z = a \cdots a\, b \cdots b$)

- Hence, $a^{m+k} b^m \in L, k \geq 1$.

- But this is a contradiction: $a^{m+k} b^m \notin L$.

Proof

- Therefore, our assumption that L is a regular language is not true

- Conclusion: L is not a regular language!

Exercises

- Prove that the following languages are not regular:

a. $L = \{ww^R : w \in \{a, b\}^*\}$

b. $L = \{ww : w \in \{a, b\}^*\}$

c. $L = \{a^n b^n c^n : n \geq 0\}$