# Introduction to Complex Numbers

## 1 Introduction

When solving a quadratic equation $ax^2+bx+c=0$, we can find the roots by using the quadratic formula, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

In the event when $b^2-4ac<0$, then the equation $ax^2+bx+c=0$ **has no real roots**.

Taking the example $z^2-4z+13=0$, solving this quadratic equation will result in the following:

$$
\begin{aligned}
z &= \frac{-(-4)\pm\sqrt{(-4)^2-4(1)(13)}}{2(1)} \\
&= \frac{4\pm\sqrt{-36}}{2}
\end{aligned}
$$

Hence, by introducing the **imaginary number i** which represents $\sqrt{-1}$, we have :

$$
\begin{aligned}
x &= \frac{4\pm\sqrt{-36}}{2} \\
&= \frac{4\pm\sqrt{36}\sqrt{-1}}{2} \\
&= \frac{4\pm 6\mathrm{i}}{2} \\
&= 2\pm 3\mathrm{i}
\end{aligned}
$$

Therefore, the equation $z^2-4z+13=0$ has two roots, $2+3\mathrm{i}$ and $2-3\mathrm{i}$.

## 2. Complex Numbers in Cartesian (or Rectangular) Form

A ***complex number*** z is represented in the form $x+y\mathrm{i}$,

$x$ is called the ***real part*** of $z$, denoted by Re($z$).
$y$ is called the ***imaginary part*** of $z$, denoted by Im($z$).

Note that both $x$ and $y$ are real numbers..

| | |
|---|---|
| If | $z=2+3\mathrm{i}$, |
| then | Re($z$) = 2 ; Im($z$) = 3. |

## 3. Mathematical Operations :

**Addition, Subtraction, Multiplication and Division of Complex Numbers**

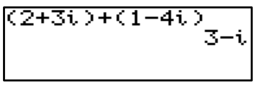
Let $z_1 = x_1 + y_1\text{i}$ and $z_2 = x_2 + y_2\text{i}$ be two complex numbers, where $x_1$, $y_1$, $x_2$ and $y_2$ are all real numbers.

| Operations | Algebraic Method | Examples |
|---|---|---|
| Addition | $z_1 + z_2 = (x_1 + y_1\text{i}) + (x_2 + y_2\text{i})$<br>$=(\quad\quad) + (\quad\quad)\,\text{i}$ | $(2+3\text{i})+(1-4\text{i})$<br>$=2+3\text{i}+1-4\text{i}$<br>$=2+1+3\text{i}-4\text{i}$<br>$=3-\text{i}$ |
| Subtraction | $z_1 - z_2 = (x_1 + y_1\text{i}) - (x_2 + y_2\text{i})$<br>$=(\quad\quad) + (\quad\quad)\,\text{i}$ | $(2+3\text{i})-(1-4\text{i})$<br>$=2+3\text{i}-1+4\text{i}$<br>$=2-1+3\text{i}+4\text{i}$<br>$=1+7\text{i}$ |
| Multiplication | $z_1 z_2 = (x_1 + y_1\text{i})(x_2 + y_2\text{i})$<br>$= \dots$<br>$= \dots$<br>$=(\quad\quad) + (\quad\quad)\,\text{i}$ | $(2+3\text{i})(1-4\text{i})$<br>$=2(1)+3\text{i}(1)+2(-4\text{i})+(3\text{i})(-4\text{i})$<br>$=2+3\text{i}-8\text{i}-4\text{i}^2$<br>$=2+3\text{i}-8\text{i}-4(-1)$<br>$=2+4+3\text{i}-8\text{i}$<br>$=6-5\text{i}$ |
| Division | $\frac{z_1}{z_2} = \frac{x_1 + y_1\text{i}}{x_2 + y_2\text{i}}$<br>$=\frac{(x_1 + y_1\text{i})}{(x_2 + y_2\text{i})} \cdot \frac{(x_1 - y_1\text{i})}{(x_2 - y_2\text{i})}$<br>$= \dots$<br>$= \dots$<br>$=(\quad\quad) + (\quad\quad)\,\text{i}$ | $\frac{2+3\text{i}}{1-4\text{i}} = \frac{2+3\text{i}}{1-4\text{i}} \times \frac{1+4\text{i}}{1+4\text{i}}$<br>$=\frac{(2+3\text{i})(1+4\text{i})}{1^2+4^2}$<br>$=\frac{-10+11\text{i}}{1+16}$<br>$=-\frac{10}{17}+\frac{11}{17}\text{i}$<br>$=-0.59+0.65\text{i}$ |

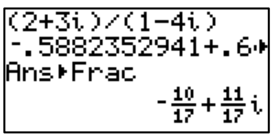
**For the purpose of this task, we will express all complex numbers as floats, rounding off to 2 decimal places.**

Using GC, ‘i’ can be called out by pressing $\boxed{\text{2nd}}\boxed{.}$

**Addition**

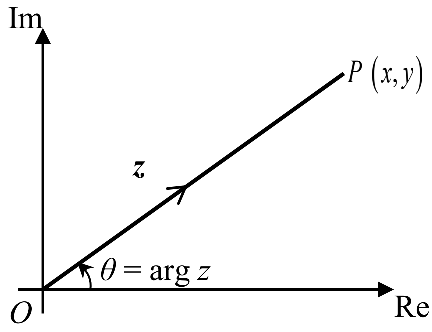



**Division**



## 4. Modulus and Argument of a Complex Number

Let the complex number $z = x + y\text{i}$ be represented by the point $P$ in an Argand diagram, where $x, y \in \mathbb{R}$.




- ***Modulus* of $z$** = $|z| = |\overrightarrow{OP}|$.

$\Rightarrow \quad |z| = \sqrt{x^2 + y^2}$

- ***Argument* of $z$** = $\arg(z)$

= the **directed** angle $\overrightarrow{OP}$ makes with the **positive real axis**

= $\theta$ in the Argand diagram

$\Rightarrow \quad \theta = \arg(z)$, where $\boxed{-\pi < \arg(z) \le \pi}$.

**Note:**

- The argument in the interval $(-\pi, \pi]$ is known as the principal argument.
- $\arg z \ge 0$ if the angle is measured in the anti-clockwise direction from the positive real axis.
- $\arg z < 0$ if the angle is measured in the clockwise direction from the positive real axis.
- $\arg 0 = \arg(0 + 0\text{i}) = 0$

**Steps for finding argument of $z = x + y\text{i}$:**

**Step 1:** Find the basic angle, $\alpha$, of the argument.

Basic angle = $\tan^{-1}\left|\frac{y}{x}\right|$.

**Step 2:** Determine the quadrant in which $z$ lies.

**Step 3:** Deduce the value of arg($z$) in the range $(-\pi, \pi]$.

Basic $\angle = \alpha$

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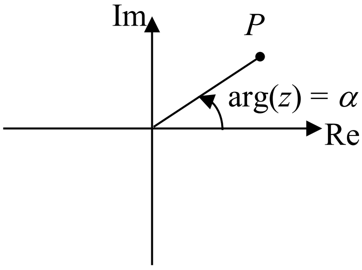
If $z$ falls in the $\mathbf{1^{st}}$ **quadrant,**

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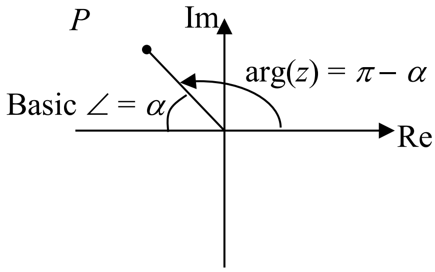
If $z$ falls in the $\mathbf{2^{nd}}$ **quadrant,**

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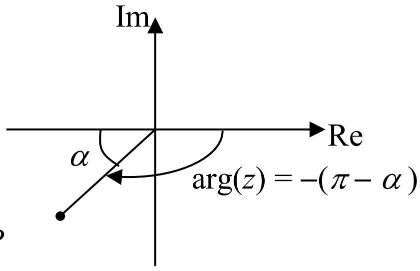
If $z$ falls in the $\mathbf{3^{rd}}$ **quadrant,**

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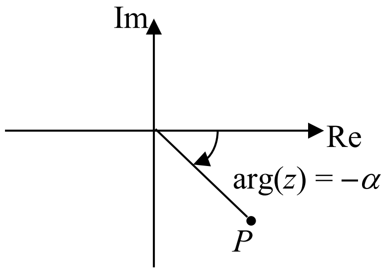
If $z$ falls in the $\mathbf{4^{th}}$ **quadrant,**

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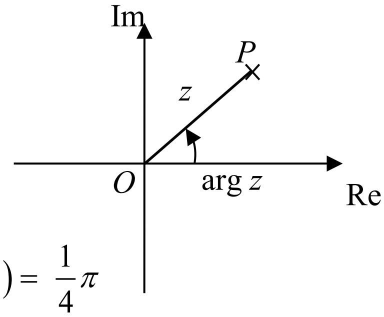
Find the exact value of the modulus and argument of the complex number $1+\mathrm{i}$.

Let $z=1+\mathrm{i}$

$|z|=\sqrt{1^2+1^2}=\sqrt{2}$

Basic angle $=\frac{1}{4}\pi$

Since $z=1+\mathrm{i}$ lies in the 1st quadrant, $\arg(z)=\frac{1}{4}\pi$

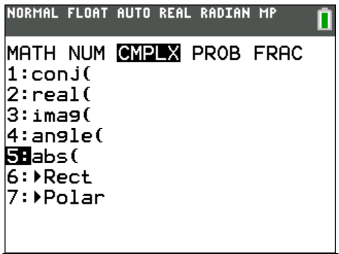


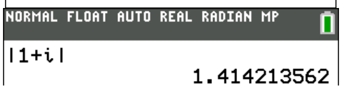

Using the GC to find the Modulus and Argument of a Complex Number

**Finding Modulus of a Complex Number**

To find the modulus of $1+\mathrm{i}$:
Press $\boxed{\text{MATH}}$ and go to `CMPLX`.
Select `5:abs(`
Key in the complex number and press Enter.

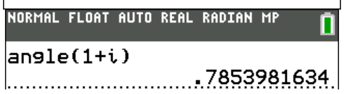
**Note:** The G.C. is not able to give the exact answer in the case where the modulus of a complex number is a surd number.



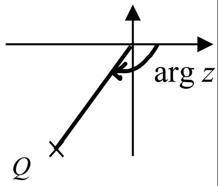
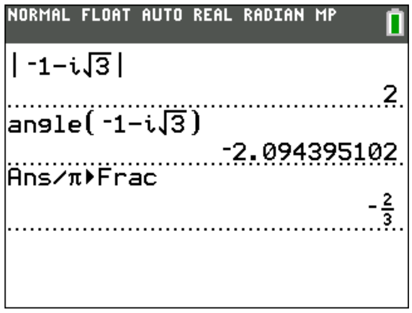
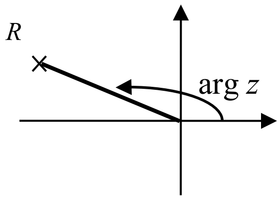
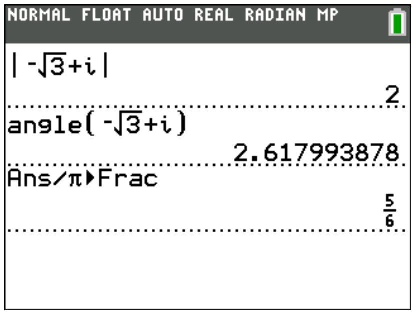
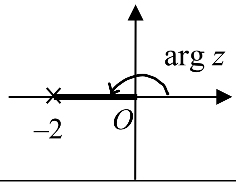
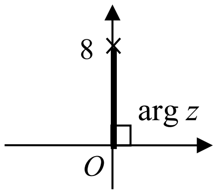




**Finding Argument of a Complex Number**

To find the argument of $1+\mathrm{i}$:
Press $\boxed{\text{MATH}}$ and go to `CMPLX`.
Select `4:angle(`
Key in the complex number and press Enter.
Ans: $0.785$ (3 s.f.)

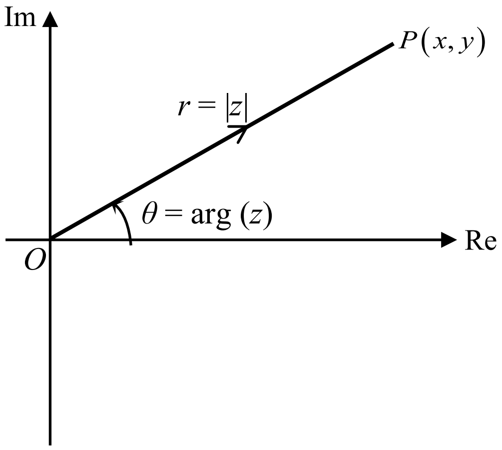



## More Examples :

| Algebraically | Using GC |
|---|---|
| (b) Let $z = -1 - \mathrm{i}\sqrt{3}$<br>$\lvert z \rvert = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$<br>Basic angle $= \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$<br><br>Since $-1 - \mathrm{i}\sqrt{3}$ lies in the 3rd quadrant,<br>arg $(z) = -\left(\pi - \frac{\pi}{3}\right) = -\frac{2\pi}{3}$ |  |
| (c)<br>Let $z = -\sqrt{3} + \mathrm{i}$<br>$\lvert z \rvert = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$<br>Basic angle $= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$<br><br>Since $-\sqrt{3} + \mathrm{i}$ lies in the 2nd quadrant,<br>$\arg(z) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ |  |
| (d)<br>Let $z = -2 + 0\mathrm{i}$<br>$\lvert z \rvert = 2$<br>From the diagram, $\arg(-2) = \pi$<br> | |
| (e)<br>Let $z = 0 + 8\mathrm{i}$<br>$\lvert z \rvert = 8$<br>From the diagram, $\arg(8\mathrm{i}) = \frac{\pi}{2}$<br> | |

## 5. Complex Numbers in Polar Form

Let the complex number $z = x + y\mathrm{i}$ be represented by the point $P$ on an Argand diagram, where $x, y \in \mathbb{R}$, such that $|z| = r$ and $\arg(z) = \theta$.




Then,

$$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$$

Since $z = x + y\mathrm{i}$, we have $z = (r\cos\theta) + (r\sin\theta)\mathrm{i}$.

$$\boxed{z = r(\cos\theta + \mathrm{i}\sin\theta)}$$

This is called the **polar form** of $z$, (or trigonometric form or modulus-argument form of the complex number $z$).

## 6. De Moivre's Theorem

For any complex number $z$ and integer $n$,

$$\begin{aligned} z^n &= r^n\left(\cos\theta + \mathrm{i}\sin\theta\right)^n \\ &= r^n\left(\cos n\theta + \mathrm{i}\sin n\theta\right) \\ &= r^n\cos n\theta + r^n\sin n\theta\,\mathrm{i} \end{aligned}$$

**Example**

Express the complex number $-\sqrt{3}+\mathrm{i}$ in the polar (or trigonometric) form, $r\left(\cos\theta + \mathrm{i}\sin\theta\right)$, where $r > 0, -\pi < \theta \le \pi$. Hence calculate $\left(-\sqrt{3}+\mathrm{i}\right)^6$.

Solution

Let $z = -\sqrt{3}+\mathrm{i}$,

$$|z| = \left|-\sqrt{3}+\mathrm{i}\right| = 2$$

Basic Angle of $\arg(z) = \tan^{-1}\left(\left|\frac{1}{-\sqrt{3}}\right|\right) = \frac{\pi}{6}$

Since $z = -\sqrt{3}+\mathrm{i}$ is in the 2nd quadrant,

$\therefore\quad \arg(z) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

Hence, $z = 2\left(\cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}\right)$

$$\begin{aligned} z^6 &= \left(-\sqrt{3}+\mathrm{i}\right)^6 \\ &= \left(2\left(\cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}\right)\right)^6 \\ &= (2)^6\left(\cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}\right)^6 \\ &= (2)^6\left(\cos(6)\frac{5\pi}{6} + \mathrm{i}\sin(6)\frac{5\pi}{6}\right) \\ &= 64\left(\cos 5\pi + \mathrm{i}\sin 5\pi\right) \\ &= 64\left(-1 + 0\mathrm{i}\right) \\ &= -64 \end{aligned}$$

**7. Complex Numbers in Exponential Form**

Applying the series expansions of $\mathrm{e}^x$, cos $x$ and sin $x$, from MF26, we may write

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}\theta} &= 1+\mathrm{i}\theta+\frac{(\mathrm{i}\theta)^2}{2!}+\frac{(\mathrm{i}\theta)^3}{3!}+\frac{(\mathrm{i}\theta)^4}{4!}+\frac{(\mathrm{i}\theta)^5}{5!}+\dots \\
&= 1+\mathrm{i}\theta-\frac{\theta^2}{2!}-\frac{\mathrm{i}\theta^3}{3!}+\frac{\theta^4}{4!}+\frac{\mathrm{i}\theta^5}{5!}+\dots \\
&= \left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}+\dots\right)+\mathrm{i}\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}+\dots\right) \\
&= \cos\theta + \mathrm{i}\sin\theta
\end{aligned}$$

In short, $\boxed{\mathrm{e}^{\mathrm{i}\theta}=\cos\theta+\mathrm{i}\sin\theta}$ , where $-\pi<\theta\le\pi$.

If $z=r(\cos\theta+\mathrm{i}\sin\theta)$ , then $\boxed{z=r\,\mathrm{e}^{\mathrm{i}\theta}}$ , where $r>0, -\pi<\theta\le\pi$.

This is the ***exponential form*** of the complex number $z$, where $r=|z|$ and $\theta=\arg(z)$.