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Q1

Assume that 1 is not the smallest positive integer. From Well-Ordering Property, since 1 is positive there exists a smallest positive integer that is smaller than 1.
Let this smallest integer to be a, from the above statement $0 < a < 1$
Multiplying the inequality with a, we get,
$0 < a^2 < a$
Since the square of a positive integer is a positive integer that is bigger than itself, $0 < a^2 < a$ inequality does not hold. Hence, we reached a contradiction.
Therefore, a is not the smallest positive integer. Therefore our assumption is not correct. Hence, 1 is the smallest positive integer.

Q2

Let's first prove $S(m, 1)$ and $S(1, n)$ using induction.
1) Starting with S(m,1);
Basis: S(2,1): $\{(0, 1), (1, 0)\}$ So, there are 2 ordered solutions. $f(2, 1) = \frac{(2+1-1)!}{1!(2-1)!} = C(2, 1) = 2$. Hence, base step is completed.
Inductive Step: Assume that $S(k, 1)$ is true.
S(k+1, 1):
2 cases: a) x_{k+1} is 0. Same solutions but will place the 0 in them. It makes $C(k, 1)$ from the inductive hypothesis.
b)x_{k+1} is 1. Only one solution will be added to the solution space.
Since the cases are mutually exclusive, we can add them.
$f(k+1, 1) = C(k, 1) + 1 = C(k, 1) + C(k, 0) = C(k+1, 1)$ (By Pascal's Identity)$= f(k+1, 1)$
Hence, the inductive step is completed. Hence, we proved by induction that $S(k+1, 1)$ is true under our assumption.

2) Now S(1, n);
Basis: S(1, 2) $= \{(2)\} = 1 =$ C(2, 2) $= 1$
Hence, base step is completed.
Inductive Step: Assume S(1, k) is true.
S(1, k+1):
$x_a + 1 = k + 1$ from inductive step.
Let $x_{k+1} = x_a + 1$, Hence;
$x_{k+1} = k + 1 =$ f(1, k+1) $= 1$
Hence, the inductive step is completed. Hence, we proved by induction that $S(1, k+1)$ is true under our assumption.

I could not do the rest of the proof.

Q3

a. 91
We can draw 4 triangles with the same size from 1x1 squares. The number of squares that are 1x1 in the figure is 21.
The other triangles will be counted from the right most triangles. There are 7 such triangles.
Hence, the number of triangles can be constructed is $4 * 21 + 7$, which is equal to 91.

b. 1560
We can first find the number of all the functions and then subtract the number of ones that are not onto. However, when we subtract the not onto functions (with excluding elements from the range) we should not subtract the same not onto function twice. So, we need to use the inclusion-exclusion principle.
$4^6 - C(4,1)3^6 + C(4,2)2^6 - C(4,3)1^6 = 4096 - 2916 + 384 - 4 = 1560$

Q4

a. There are two disjoint cases of an n-1 length string.
Case 1) The string contains two consecutive symbols that are the same.
Case 2) The string does not contain two consecutive symbols that are the same

For n-1 length such strings, a_{n-1} would show the number of strings that contain two consecutive symbols that are the same. And since the cases are disjoint, $3^{n-1} - a_{n-1}$ would show the number of strings that do not contain two consecutive symbols that are the same.

If the string that has n-1 length is from case 1, we can add all 3 symbols to the end of the string.
If the string that has n-1 length is from case 2, for a_n to contain two consecutive symbols that are the same, we can only use the last symbol of the n-1 length string.
Hence, the recurrence relation becomes;
$a_n = 3a_{n-1} + (3^{n-1} - a_{n-1})$

After algebraic operations, for $n > 1$;
$a_n = 3^{n-1} + 2a_{n-1}$

b. We only need 1 initial condition.
If the string has length 1, the number of strings that contain two consecutive symbols that are the same is 0.
$a_1 = 0$

c. $a_n - 2a_{n-1} = 3^{n-1}$

First, homogeneous solution;
We need to find the roots of characteristic equation.
Characteristic equation is $\alpha - 2 = 0$
Hence, there is only one root and it is $\alpha_1 = 2$
So, $a_n^{(h)} = A2^n$

Now the particular solution;
Since 3 is not a root of the characteristic equation, particular solution is;
$a_n^{(p)} = B3^n$
Now substituting the $a_n^{(p)}$ to the recurrence equation;
$B3^n - 2B3^{n-1} = 3^{n-1}$
$3B3^{n-1} - 2B3^{n-1} = 3^{n-1}$
$B3^{n-1} = 3^{n-1}$
$B = 1$

$a_n = a_n^{(h)} + a_n^{(p)} = A2^n + 3^n$
From the initial condition, we get;
$2A + 3 = 0$
$A = \frac{-3}{2}$

Hence, the solution of the recurrence relation is;
$a_n = \frac{-3}{2}2^n + 3^n$