GOPH 419 Lab 1 Report
Fazlie Latib 30067991

Abstract

The aim of this report is to examine the use of Taylor series and analysis of error in numerical approximations through planning and building an algorithm function in MATLAB to compute the allowable range of launch angles for a rocket system to reach a target maximum altitude within a specified tolerance. The launch angles mechanical formula is equivalent to $\sin^{-1} x$. $(\sin^{-1} x)^2$ can be approximated using a Maclaurin series (an example of Taylor series) which will be implemented in the algorithm through an iterative approach using a while loop. For the while loop to stop iterating, the absolute approximate error estimate had to fall below the specified stopping criterion. The stopping criterion ensured that the results of the launch angles approximation would be correct to the desired number of significant figures which in this report is 5. The launch_angle function created will accept three variable inputs (v_e/v_0, α and the tolerance for α) and give out two outputs (the minimum and maximum launch angles in radian). Using the function, two separate graphs were plotted to analyze the range of launch angles for a range of two variable inputs, v_e/v_0 and α. The error in $\sin \phi_0$ were found to be 0.12 radian and the condition number of equation (1) was calculated to be equal to 12.

Background / Theory

In mechanics, the formula to calculate the launch angles for a rocket system is shown below,

$$\sin \phi_0 = (1+\alpha)\sqrt{1-\frac{\alpha}{1+\alpha}\left(\frac{v_e}{v_0}\right)^2} \tag{1}$$

$$\phi_0 = \sin^{-1}\left((1+\alpha)\sqrt{1-\frac{\alpha}{1+\alpha}\left(\frac{v_e}{v_0}\right)^2}\right) \tag{2}$$

where ϕ_0 is the launch angle from vertical, α is the desired maximum altitude as a fraction of Earth's radius, v_e is the rocket's escape velocity (minimum velocity required to escape Earth's orbit) and v_0 is the terminal velocity of the rocket (the maximum velocity that the rocket reaches shortly after launch).

Since the embedded system for the automated delivery of supplies rocket system did not has a function that can compute inverse trigonometric functions, the following formula (Borwein and Chamberland, 2007) which is an example of Maclaurin series expansion will be implemented in the algorithm.

$$(\sin^{-1} x)^2 = 0.5\sum_{n=1}^{\infty}\left[\frac{(2x)^{2n}}{n^2\left(\frac{(2n)!}{(n!)^2}\right)}\right] \text{ where } x = \left((1+\alpha)\sqrt{1-\frac{\alpha}{1+\alpha}\left(\frac{v_e}{v_0}\right)^2}\right) \tag{3}$$

Methods / Algorithm

The function to compute the minimum angle and the maximum angle is called the launch_angle function which takes as input the ratio of escape velocity to terminal velocity (ve_v0), the desired maximum altitude as a fraction of Earth's radius (alpha) and the tolerance for maximum altitude (tol_alpha). The tolerance for the altitude was used to compute the maximum (4) and minimum (5) altitude which then were used to approximate the minimum and maximum angles respectively.

$$\text{Maximum amplitude} = (1 + tol\_alpha) \times alpha \tag{4}$$

$$\text{Minimum amplitude} = (1 - tol\_alpha) \times alpha \tag{5}$$

In the launch_angle function, while loops were implemented where the angles approximation will keep adding terms until the absolute value of the approximate error, ε_a falls below a specified stopping criterion, ε_s conforming to n significant figures. When the loops reached this condition where $\varepsilon_a < \varepsilon_s$, it can be assured that the launch angles approximation is correct to at least n significant figures. The function would compute the launch angles such that at least 5 significant figures of $(\sin^{-1} x)^2$ was correct where $\varepsilon_s = 0.5\times10^{-5}$. The initial ε_a can be set to any values larger than ε_s and in the function, it is set to be 1.

$$\varepsilon_a = \left|\frac{\text{current approximation} - \text{previous approximation}}{\text{current approximation}}\right| \tag{6}$$

$$\varepsilon_s = 0.5 \times 10^{-n} \tag{7}$$

Results and Discussion

The expected results from the launch_angle function when $v_e/v_0 = 2$, $\alpha = 0.25$ with a tolerance of ±2% was computed using verify_launch_angle function. The expected results, and the results from the launch_angle function was shown in Figure 1. Based on the figure, the results of the launch_angle function and the expected results are identical where the minimum angle = 0.5741 radian and the maximum angle = 0.6119 radian. This implied that they agree with each other.

```
Command Window
>> [min_angle,max_angle] = verify_launch_angle(2,0.25,0.02)

min_angle =

    0.5741


max_angle =

    0.6119

>> [min_angle,max_angle] = launch_angle(2,0.25,0.02)

min_angle =

    0.5741


max_angle =

    0.6119
```

Figure 1 The expected results using verify_launch_angle function, and the results using launch_angle function in Command Window.

With $v_e/v_0 = 2$ set to be constant, the maximum and minimum launch angles over a range of α with a tolerance of ±4% were computed and plotted using the launch_angle function (Figure 2a). The lower limit for both minimum and maximum angle is at at $\alpha = 0$. The upper limit for the minimum angle is at $\alpha = 0.32$, and the upper limit for the maximum angle is at $\alpha = 0.35$. Any values of α that is outside of the limits will give us complex number outputs. For a very small α, it can observe that the launch angle approaches to $\pi/2$ radian which basically means the rocket would have a horizontal launch. This observation makes sense because a horizontal launch would only let the rocket to stay on the ground level after launching. The maximum α for $v_e/v_0 = 2$ is near 0.35 which means the maximum height that the rocket can achieve is equal to 0.35 × Earth's radius. Earth's radius is approximately 6371 km, so the maximum height the rocket can achieve at $v_e/v_0 = 2$ is 0.35 × 6371 km ≈ 2230 km.

With $\alpha = 0.25$ set to be constant, the maximum and minimum launch angles over a range of v_e/v_0 with a tolerance of ±4% were computed and plotted using the launch_angle function (Figure 2b). The lower limit for both minimum and maximum angle is at $v_e/v_0 = 1.3$. The upper limit for the minimum angle is at $v_e/v_0 = 2.20$, and the upper limit for the maximum angle is at $v_e/v_0 = 2.28$. Any values of v_e/v_0 that is outside of the limits will give us complex number outputs. When the launch angle is practically vertical (launch angle = 0), the minimum velocity ratio to achieve this height is 2.20. At a very small launch angles, the maximum velocity ratio to achieve this height is 2.28.

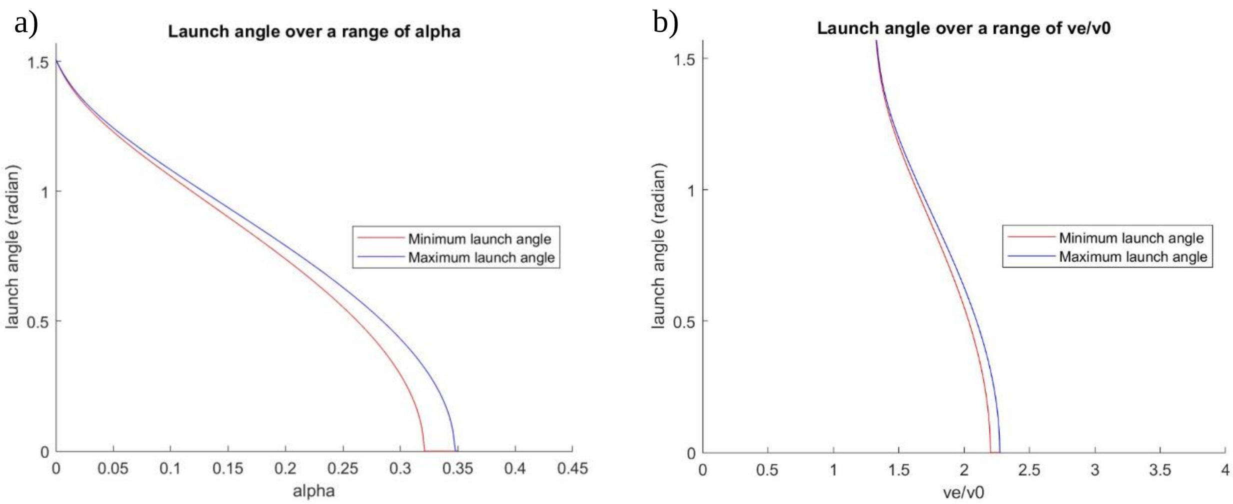

Figure 2 a) The plotted graph over a range of α using launch_angle function. b) The plotted graph over a range of v_e/v_0 using launch_angle function.

The error in $\sin\phi_0$, $\Delta(\sin\phi_0)$ can be expressed in terms of the error in v_e/v_0, $\Delta(v_e/v_0)$ and the error in α, $\Delta\alpha$ as shown in equation (8).

$$\Delta(\sin\phi_0) \approx \left|\frac{\delta(\sin\phi_0)}{\delta\left(\frac{v_e}{v_0}\right)}\right| \Delta\left(\frac{v_e}{v_0}\right) + \left|\frac{\delta(\sin\phi_0)}{\delta\alpha}\right| \Delta\alpha \tag{8}$$

Based on differentiation function on MATLAB, the partial differential of equation (1) with respect to v_e/v_0 and α is shown as equation (9) and (10) respectively.

$$\frac{\delta(\sin\phi_0)}{\delta\left(\frac{v_e}{v_0}\right)} = \sqrt{1-\frac{\alpha\left(\frac{v_e}{v_0}\right)^2}{\alpha}} - \frac{\left(\frac{\left(\frac{v_e}{v_0}\right)^2}{\alpha+1} - \frac{\alpha\left(\frac{v_e}{v_0}\right)^2}{(\alpha+1)^2}\right)(\alpha+1)}{2\sqrt{1-\frac{\alpha\left(\frac{v_e}{v_0}\right)^2}{\alpha+1}}} \tag{9}$$

$$\frac{\delta(\sin\phi_0)}{\delta\alpha} = -\frac{\left(\alpha\frac{v_e}{v_0}\right)(\alpha+1)}{\sqrt{1-\frac{\alpha\left(\frac{v_e}{v_0}\right)^2}{\alpha}}} \tag{10}$$

At $v_e/v_0 = 2$ and $\alpha = 0.25$, $\frac{\delta(\sin\phi_0)}{\delta\left(\frac{v_e}{v_0}\right)} = -\frac{7\sqrt{5}}{5}$ and $\frac{\delta(\sin\phi_0)}{\delta\alpha} = -\frac{\sqrt{5}}{2}$. When $\Delta(v_e/v_0) \approx 0.05$ and $\Delta\alpha \approx 0.02$, the error in $\sin\phi_0$ was calculated as shown below.

$$\Delta(\sin\phi_0) \approx \left|-\frac{7\sqrt{5}}{5}\right| 0.05 + \left|-\frac{\sqrt{5}}{2}\right| 0.02 = 0.12$$

To investigate the stability of equation (1) near a certain value of v_e/v_0 and α, the condition number of the equation was calculated. The condition number, CN for a multivariate function can be calculated using equation (11),

$$\mathrm{CN} = \frac{\|x\|\|J\|}{\|f\|} \tag{11}$$

where x is the vector of input parameters, J is the Jacobian matrix of partial derivatives of all outputs with respect to all inputs, f is the vector of output values, and $\|\cdot\|$ represents the Euclidean norm of a vector or matrix. The following values are the x and f vectors and the Jacobian matrix J near $v_e/v_0 = 2$ and $\alpha = 0.25$ and the corresponding $\|x\|$, $\|J\|$ and $\| f \|$ values.

$$x = \left[\frac{v_e}{v_0}, \alpha\right] = [0.25, 2] \qquad \|x\| = \sqrt{2^2 + 0.25^2} = \sqrt{\frac{65}{16}}$$

$$J = \left[\frac{\delta(\sin\phi_0)}{\delta\left(\frac{v_e}{v_0}\right)}, \frac{\delta(\sin\phi_0)}{\delta\alpha}\right] = \left[-\frac{7\sqrt{5}}{5}, -\frac{\sqrt{5}}{2}\right] \qquad \|J\| = \sqrt{\left(-\frac{7\sqrt{5}}{5}\right)^2 + \left(-\frac{\sqrt{5}}{2}\right)^2} = \frac{\sqrt{5}\sqrt{221}}{10}$$

$$f = [\sin\phi_0] = \frac{\sqrt{5}}{4} \qquad \|f\| = \frac{\sqrt{5}}{4}$$

Using the values above, the condition number can be calculated as shown below. The calculated condition number is greater than 1 indicates that equation (1) is unstable near $v_e/v_0 = 2$ and $\alpha = 0.25$ because the relative error is amplified near those points.

$$\mathrm{CN} = \frac{\sqrt{\frac{65}{16}} \cdot \frac{\sqrt{5}\sqrt{221}}{10}}{\frac{\sqrt{5}}{4}} = 12$$

Conclusion

The Taylor series, in particular the Maclaurin series for $(\sin^{-1} x)^2$ is a useful formula to approximate the launch angles of a rocket system in a MATLAB environment. The stopping criterion is a useful value so that a while loop can stop iterating when the absolute approximate error is less than the stopping criterion. It is also beneficial in estimating the approximation to a desired number of significant figures. The results of the function using the Maclaurin series matched with the expected results of using the inverse sine trigonometric function in MATLAB. Using the launch_angle function, it is found that at some values of v_e/v_0 and α, the outputs of the function would be complex numbers. To account for this problem, the function can be modified better in the future to account for this problem since the launch angles values cannot be imaginary. The error in $\sin \phi_0$ can be approximate using the error in v_e/v_0 and α, and the partial differential equation of $\sin \phi_0$ with respect to v_e/v_0 and α. The condition number of $\sin \phi_0$ is found to be greater than 1 which conclude that the equation is unstable near the specified value of v_e/v_0 and α.

References

Borwein, J. M. and Chamberland, M. (2007). Integer Powers of Arcsin, Int. J. Math. Math. Sci., Art. 19381, doi: 10.1155/2007/19381.

Chapra, S.C. and Canale, R.P. (2015). Numerical Methods for Engineers, 7th Ed. McGraw-Hill, New York, N