

Dynamic Systems Modelling

Coursework Simulink-based modelling

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CONTENTS

1. Introduction	3
2. Model Construction and Verification	4
1. Body Model	4
2. Suspension Model	7
3. Wheel Model	8
4. Complete half-car Model	10
3. Investigation of Car Performance	15
1. Sinusoidal Road Input	15
2. Speed Bump	17
A Appendix: Source Code	20
References	24

LIST OF FIGURES

1	Speed bump integrated with pedestrian crossing(Brancquart, 2017)	3
2	Representation of a half-car with two suspensions (Cookson, 2023)	4
3	Half-Car body icon block	5
4	Free fall test body movement	6
5	Force equilibrium test body pitch	6
6	Lifting force test body movement	7
7	Lifting force test body pitch	7
8	Half-Car suspension icon block	7
9	Half-Car wheel icon block	8
10	Free fall test wheel movement	9
11	Force equilibrium test wheel	10
12	Road step test wheel movement	10
13	Half-Car model assembly	11
14	Stationary car test body and wheel movement	12
15	Road step test body movement with a rigid wheel	12
16	Road step test body and wheel movement with a rigid suspension	13
17	Suspension spring non-linear coefficient	13
18	Non-linear comparison on a stationary car test showing body and wheel movement	14
19	Road step test body movement with non-linear parameters	14
20	Sinusoidal road test body movement with amplitude $A = 0.01m$	16
21	Sinusoidal road test body movement with amplitude $v = 10ms^{-1}$	17
22	Watts and Seminole profile speed bumps (Shwaly, Zakaria and Al-Ayaat, 2018)	17
23	Car movement while driving at $8m/s$ over a single Watts profile speed bump around 5.5s	18
24	Car movement while driving at $8m/s$ over a single Seminole profile speed bump around 4.3s	19
25	Simulink body subsystem	20
26	Simulink suspension subsystem	21
27	Simulink wheel subsystem	21
28	Simulink tests for each subsystem of the car model	22
29	Simulink tests for the car model	22
30	Simulink tests for sinusoidal road effect	23
31	Simulink tests for different speed bumps effects	23

LIST OF TABLES

1	Car model parameters (Cookson, 2023)	4
2	List of tests used to assert the body subsystem	5
3	Input parameters ($[\dot{s}_B, s_B, \dot{s}_W, s_W]$) combinations used to assert the suspension subsystem	8
4	List of tests used to assert the tyre subsystem	9
5	List of tests used to assert the complete model	11
6	Sinusoidal road testing parameters	15
7	Speed bump testing parameters	18

LISTINGS

1	Model parameters used in testing and simulations	20
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1. INTRODUCTION

Road vehicles such as cars use a suspension system to improve performance and comfort. A suspension effectively separates the wheels vertical movement from the body. Most cars implement independent suspensions for each wheel, for ease of computation only one side, or two wheels, are considered hereafter.

Section 2. of this report goes through the steps to implement and test the half-car model shown in Figure 2. The model is constructed in MATLAB's Simulink using a series of subsystem blocks to resolve the body movement given by Equation 1 and Equation 2. All diagrams created can be found in Appendix A.

After the diagrams are proven to work correctly, in Section 3. the model is applied to a real world scenario: driving over a speed bump. The main purpose of installation of speed humps is to reduce speed and volume of traffic to acceptable limits in order to enhance safety for pedestrians and motorists (Shwaly, Zakaria and Al-Ayaat, 2018). Investigation of car behaviour can be important in suspension design as speed bump have different effects for each vehicle using the road.

Different speed bump shapes, such as the one in Figure 1, can be tested and the optimal suspension for passenger comfort can be found.



Figure 1. Speed bump integrated with pedestrian crossing(Brancquart, 2017).

This problem takes into account the vertical motion and the pitch of the car chassis; while the roll is ignored due to simplified nature of the model.

$$m_B \ddot{s}_B = F_{front} + F_{rear} - m_B g \quad (1)$$

$$I \ddot{\theta} = aF_{front} - bF_{rear} \quad (2)$$

2. MODEL CONSTRUCTION AND VERIFICATION

The half-car model used is shown in Figure 2, this includes body, suspension and wheel for the front and back of the vehicle. For the purposes of testing and simulation some parameters for this vehicle are chosen and listed in Table 1.

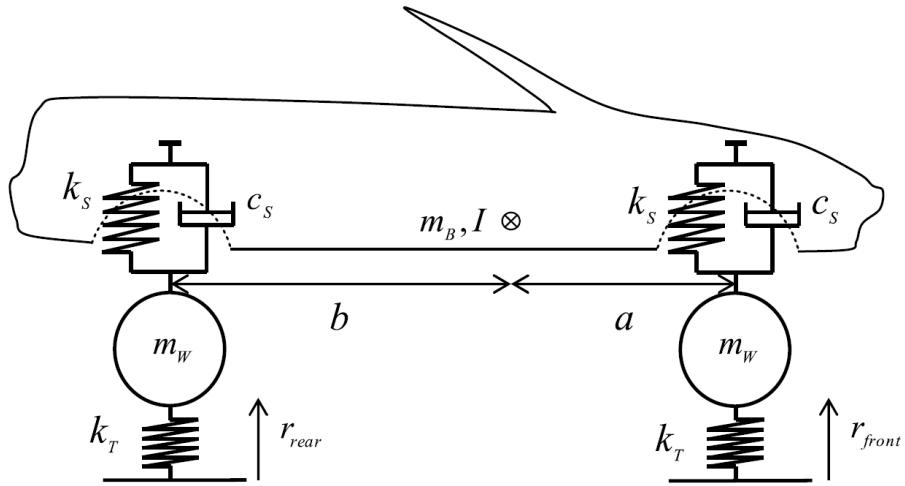


Figure 2. Representation of a half-car with two suspensions (Cookson, 2023).

Table 1. Car model parameters (Cookson, 2023)

Parameter	Description	Value
m_W	Wheel mass	20kg
k_T	Wheel stiffness	$14 \times 10^4 Nm^{-1}$
k_S	Spring stiffness	$2 \times 10^4 Nm^{-1}$
c_S	Bumping rate contraction and extension	$[600, 1200] Nsm^{-1}$
k_{Sstiff}	Spring hardening	$20 * k_S$
m_B	Body mass	700kg
I	Body inertia	$650kgm^{-2}$
a	Front wheel distance from CoG	1.35m
b	Rear wheel distance from CoG	1.5m

A free body diagram is created for the model and by applying Newton's Second Law the forces and moments are resolved resulting in the above Equations 1 and 2.

Where \ddot{s}_B and $\dot{\theta}$ are respectively the vertical and angular acceleration of the body.

The equations are further developed to solve the kinematics at each suspension resulting for displacement in:

$$s_{Bf} = s_B + a\theta \quad s_{Br} = s_B - b\theta \quad (3)$$

The equations above are implemented through simulink block models arranged in subsystems for the body, suspension and wheel. These can be found in the library file *CarModelLibrary.slx*.

1. Body Model

The half-car body model, shown in the icon block in Figure 3, was implemented in Simulink using the subsystem illustrated in Figure 25. This subsystem produces outputs for linear and angular kinematics, along with the single suspension points.

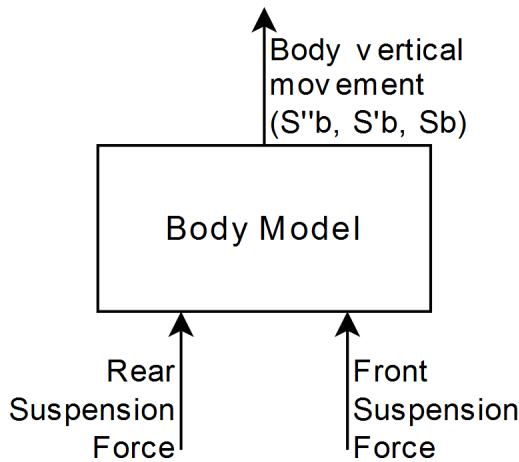


Figure 3. Half-Car body icon block.

A series of tests were performed (Figure 28) to verify the correctness of the subsystem, these are listed in Table 2

Table 2. List of tests used to assert the body subsystem.

Type	Description	Expected Result
Free Fall	No force applied to the body ($F_f \& F_r = 0$)	Free fall due to gravity, with acceleration equal to g , velocity linearly decreasing and displacement negative parabolic trajectory
Force Equilibrium	Total force applied to body is equal to body weight ($F_f + F_r = m_B g$)	According to Newton's First Law when the forces cancel out there is no movement in the body
Lifting Force	Force equal to body weight applied to each suspension ($F_f \& F_r = m_B g$)	The resultant force on the body is equal to the weight but points upwards. Similar to the free fall test but this time moving upwards

The free fall test validates the accurate implementation of Equation 1 through the examination of body displacement. With null inputs, the test ensures the accurate calculation of internal forces, such as weight, without external force interference. The result in Figure 4 showcases a parabolic displacement with linear velocity and a constant acceleration equal to ' g '.

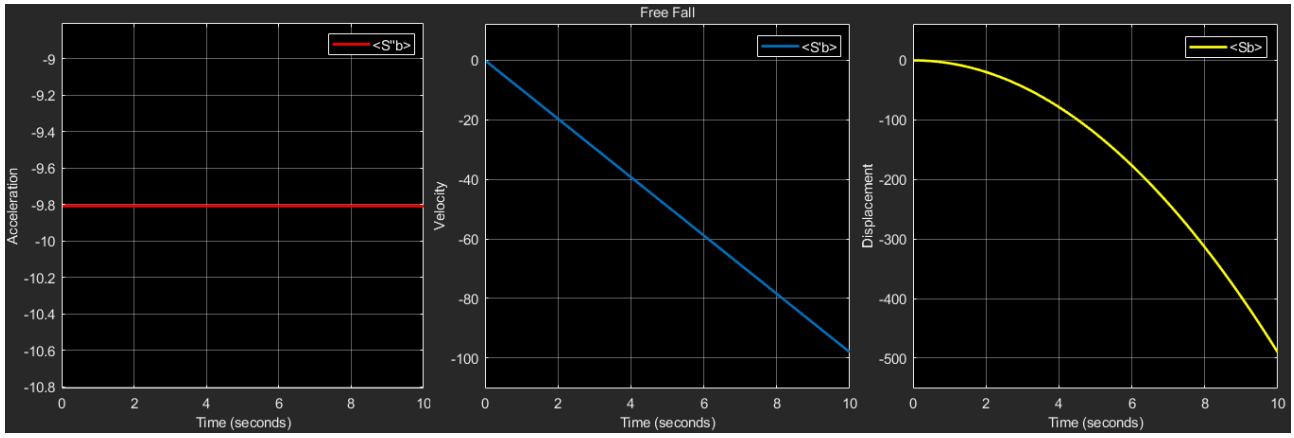


Figure 4. Free fall test body movement

The force equilibrium test validates the accurate implementation of Equation 2 by examining the body pitch. The upward force applied to the car is distributed proportionally across the two suspensions to prevent the introduction of any moments around the centre of gravity (CoG). The outcome shown in Figure 5, reveals a pitch of 0, indicating the cancellation of moments through a correct implementation of the free body diagram. It's worth noting that the y-axis was manually scaled to visualise these results, as a pitch angle of approximately 10^{-14} radians was observed. This minute angle is likely attributable to rounding errors, deemed negligible due to its small magnitude, and consequently, the test is considered successful.

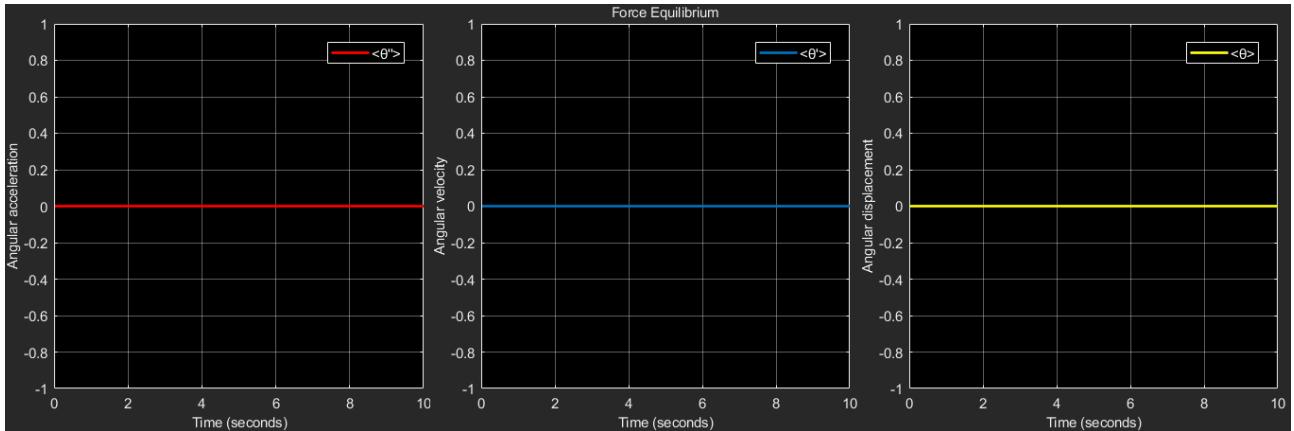


Figure 5. Force equilibrium test body pitch

The final test, the lifting force, validates both resultant forces and moments acting together on the body. Solving Equation 1 the output force is equal to the weight but with opposite sign resulting in a displacement similar to free fall but in opposite direction (Figure 6). As the same force is applied to each suspension a moment is induced in the body due to its asymmetry. This results in a downward pitch of the car as the back wheel is further from the CoG (Figure 7).

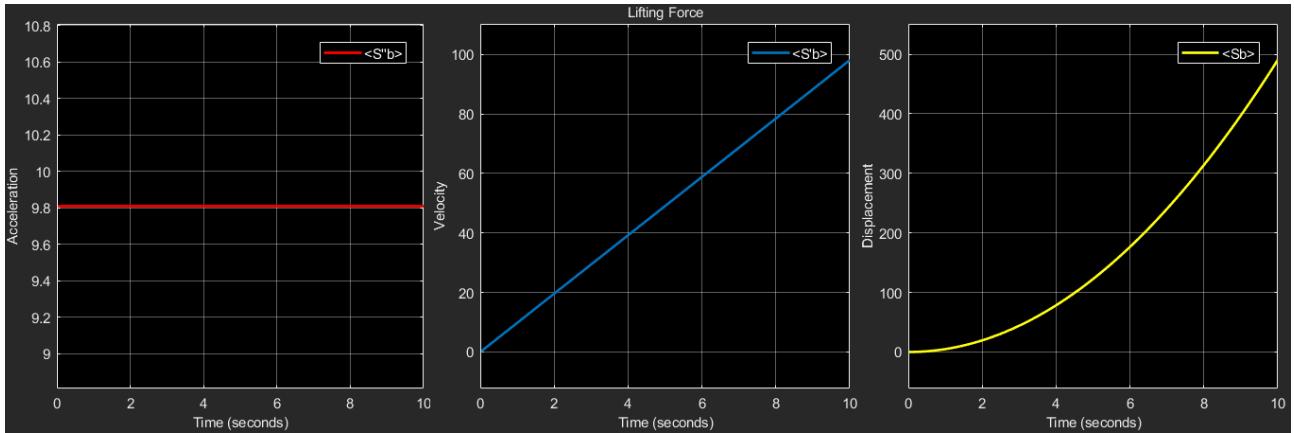


Figure 6. Lifting force test body movement

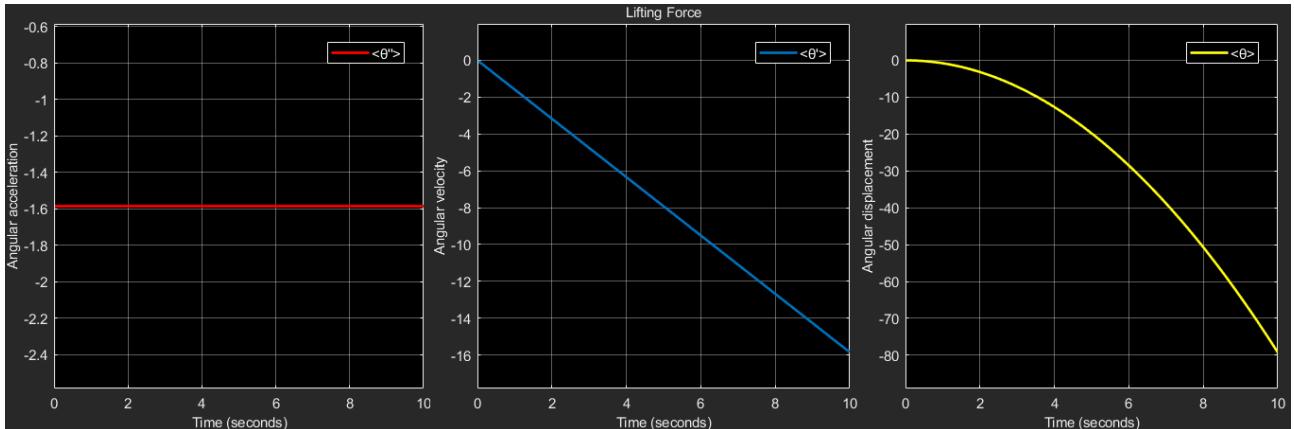


Figure 7. Lifting force test body pitch

2. Suspension Model

The half-car suspension model, shown in the icon block in Figure 8, was implemented in Simulink using the subsystem illustrated in Figure 26. This subsystem outputs the single suspension force which will then apply to the body.

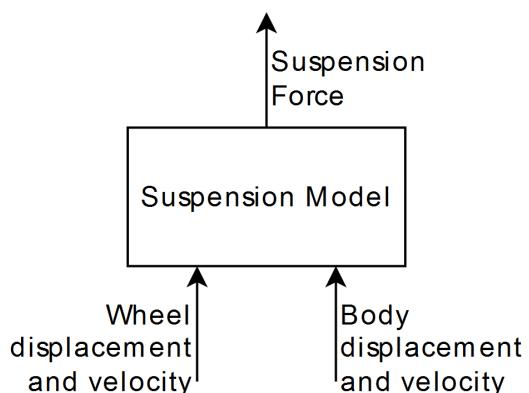


Figure 8. Half-Car suspension icon block.

A car suspension system is made of a spring and a dampener working together to stabilise the car and reduce the impacts (Figure 2). The relationship between the input displacement and velocity and the output force is given by Equation 4.

$$F = k_s(s_B - s_W) + c_S(\dot{s}_B - \dot{s}_W) \quad (4)$$

A direct comparison with the analytical solution was chosen as a suitable test for the suspension subsystem modelled by Equation 4. Six combinations of the input parameters $[\dot{s}_B, s_B, \dot{s}_W, s_W]$ are listed in Table 3 and the solution was compared using the values in Table 1 (Figure 28).

Table 3. Input parameters ($[\dot{s}_B, s_B, \dot{s}_W, s_W]$) combinations used to assert the suspension subsystem.

Inputs	Expected Solution
[0, 0, 0, 0]	0
[0, 0, 0, 1]	20000
[0, 0, 1, 1]	20900
[0, 1, 0, 0]	-20000
[1, 1, 0, 0]	-20900
[1, 1, 1, 1]	0

3. Wheel Model

The half-car wheel model, shown in the icon block in Figure 9, was implemented in Simulink using the subsystem illustrated in Figure 27. This subsystem produces outputs for linear displacement and velocity of the wheel.

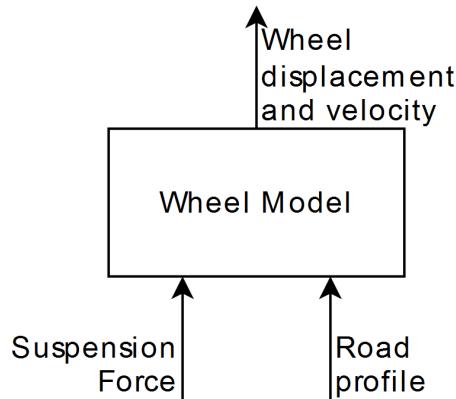


Figure 9. Half-Car wheel icon block.

From the free body diagram of the wheel the sum of all the forces is given by Equation 5. The displacement and velocity can be obtained by integrating the acceleration.

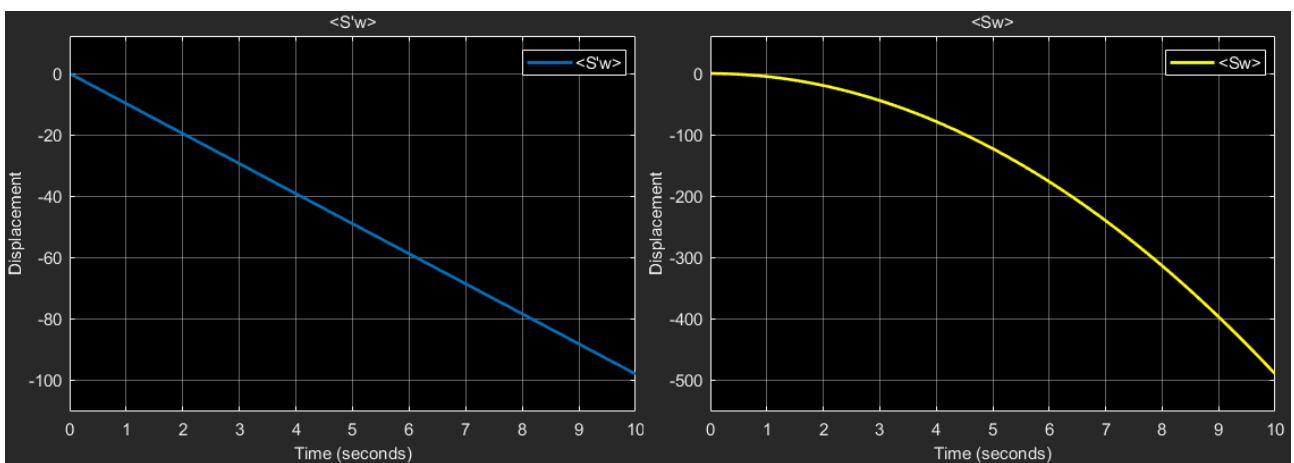
$$m_W \ddot{s}_w = -F + k_T(r - s_W) - m_W g \quad (5)$$

A series of tests were performed (Figure 28) to verify the correctness of the subsystem, these are listed in Table 4

Table 4. List of tests used to assert the tyre subsystem.

Type	Description	Expected Result
Free Fall	No suspension force applied to the wheel and no tyre stiffness ($F = 0, k_T = 0$)	Free fall due to gravity, with velocity linearly decreasing and displacement negative parabolic trajectory
Force Equilibrium	Total force applied to tyre is equal to tyre weight and no tyre stiffness ($F = m_W g, k_T = 0$)	According to Newton's First Law when the forces cancel out there is no movement in the body
Road Step	Step input as road surface with no suspension for nor gravity ($F = 0, g = 0$)	Constant oscillation of displacement and velocity with frequency= $\sqrt{k_T/m_W}$

The free fall and force equilibrium test validates the accurate implementation of Equation 5 through the examination of body displacement. With null inputs, the test ensures the accurate calculation of internal forces, such as weight, without external force interference. With the resultant force equal to zero results in a tyre equilibrium with no vertical displacement nor velocity. The results in Figures 10 and 11 showcases a constant acceleration equal to 'g' in the former and no movement in the latter.

**Figure 10.** Free fall test wheel movement

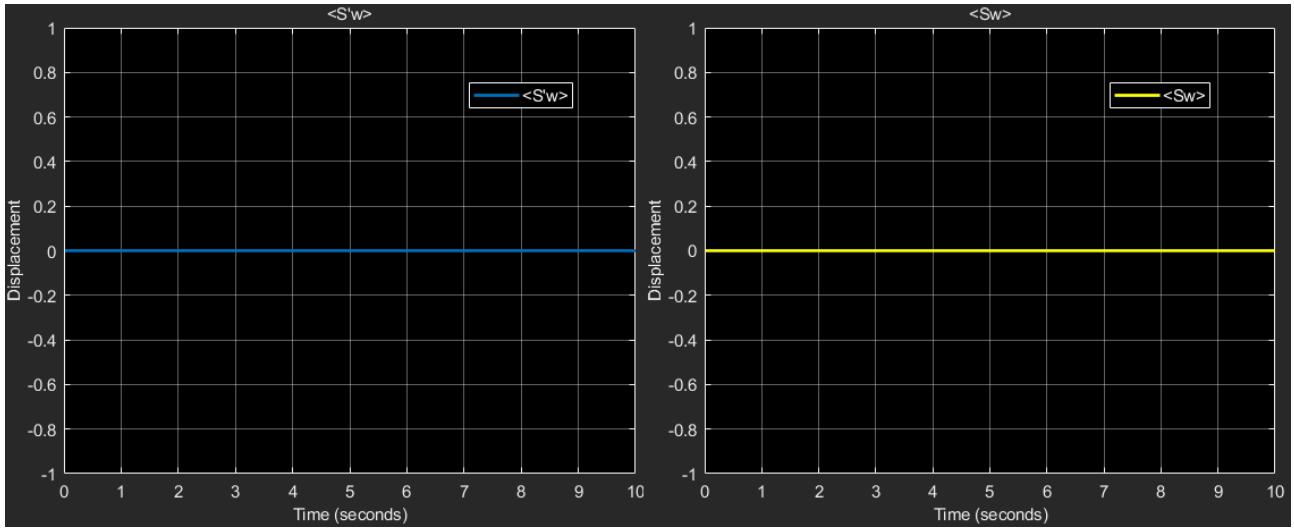


Figure 11. Force equilibrium test wheel

The final test, the road step, validates the third component of the equation: the spring like tyre. With the remaining terms as 0 the output movement is only due to the spring response. When a road step is applied the spring starts oscillating at its natural frequency and due to the lack of other forces the oscillation continues indefinitely (Figure 12). The natural frequency in Hertz of the spring is given by $f = \frac{\sqrt{k_T/m_W}}{2\pi}$, using the values in Table 1 this gives a value of 13.3159Hz . Using cursors in Figure 12 the period and therefore the frequency was measured at 13.3321Hz proving the correct implementation of the wheel subsystem

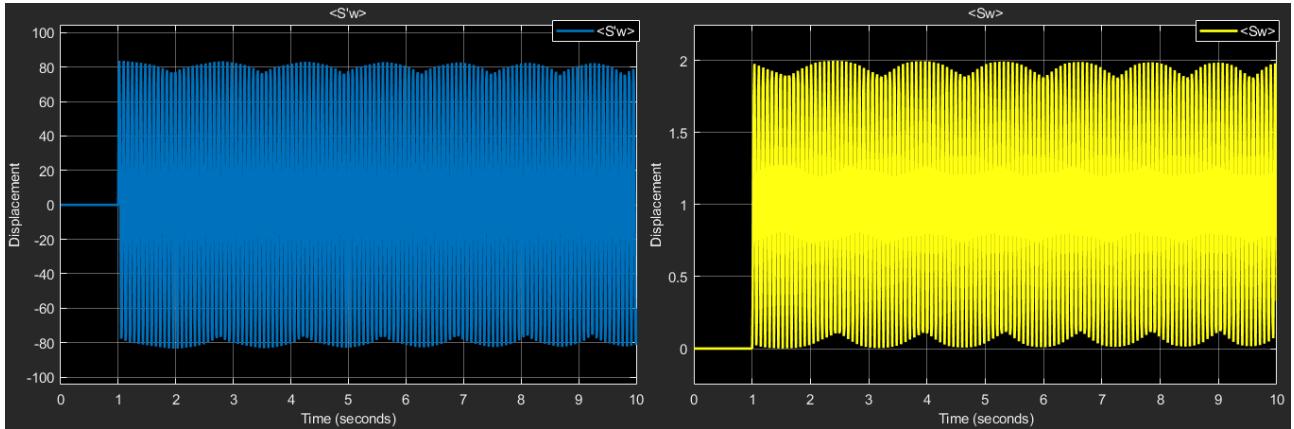
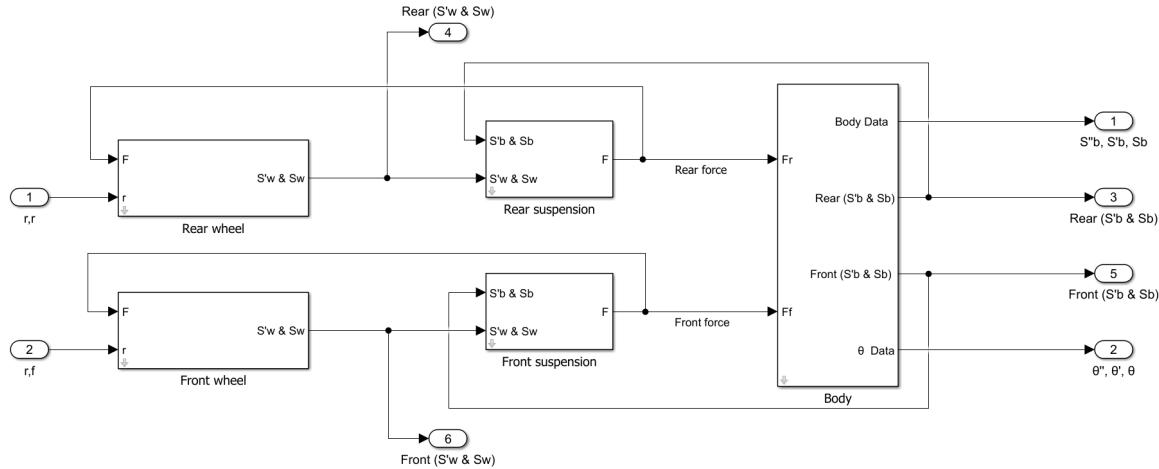


Figure 12. Road step test wheel movement

4. Complete half-car Model

Finally, the complete half-car model can be implemented in Simulink by assembling the single subsystems and connecting the relative inputs and outputs. The half-car model, shown in Figure 13, is a combination of the subsystems for both the front and rear wheel and suspensions. The model takes the road surface as input and outputs a series of data relevant to the car kinematics, including vertical and pitch movements.

**Figure 13.** Half-Car model assembly.

A series of tests were performed (Figure 29) to verify the correctness of the complete model, these are listed in Table 5

Table 5. List of tests used to assert the complete model.

Type	Description	Expected Result
Stationary	No road input ($r = 0$)	The system is only subject to gravity, the suspension and wheel are compressed under the car weight and should soon settle
Suspension Test	Road step input and rigid wheel with k_T very high and m_W very low allows to remove the wheel	The system starts oscillating due to the road and settle soon after.
Wheel Test	Road step input and rigid suspension with k_S and c_S very high allows to remove the suspension	The system starts oscillating due to the road and keeps the osculation due to a lack of external forces.
Non-Linear comparison	Introduce non-linear parameters in springs and dampener and compare it to the linear outputs	Differences in periods and amplitudes of the oscillations.
Non-Linear step input	Observe body movement under a step road with non-linear parameters	Wheel bouncing on the ground proving correct behaviour

The stationary test investigates the body movement on a flat road. When the simulation is started the suspension is unloaded therefore the weight of the car is the first force acting on the system. This causes both the wheel and the suspension to compress and oscillate. As the road surface is constantly 0 the system soon stabilises and the body reaches an equilibrium roughly 20cm below the initial position (Figure 14).

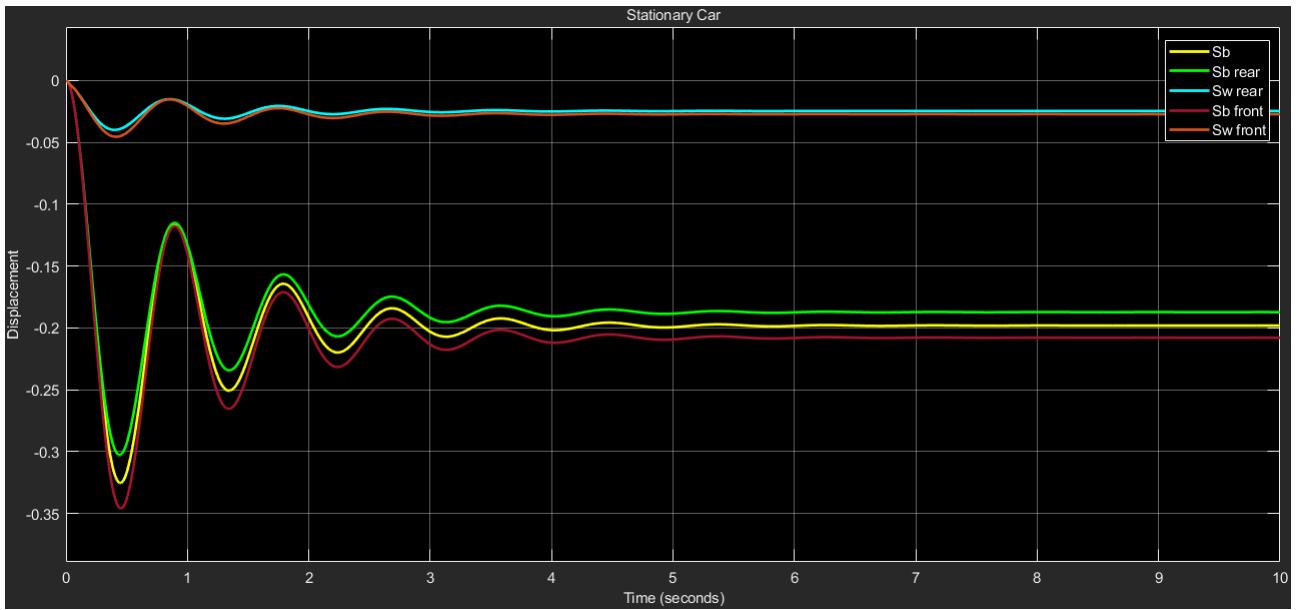


Figure 14. Stationary car test body and wheel movement

The suspension and wheel tests investigate the body movement under a step input road respectively when the wheel or the suspension are disabled. To make the wheel rigid the stiffness is increased by a factor of 10^4 while the mass is decreased by the same factor. The result in Figure 15 illustrates the oscillation given by the suspension spring and the stabilisation resulting from the suspension dampener. To make the suspension rigid the stiffness and dampening coefficients are increased by a factor of 10^6 . The result in Figure 16 illustrates the oscillation given by the wheel springiness, due to a lack of dampening this oscillation continues indefinitely. In addition it can be observed the negligible suspension movement as the body and the wheel masses act as a single rigid body.

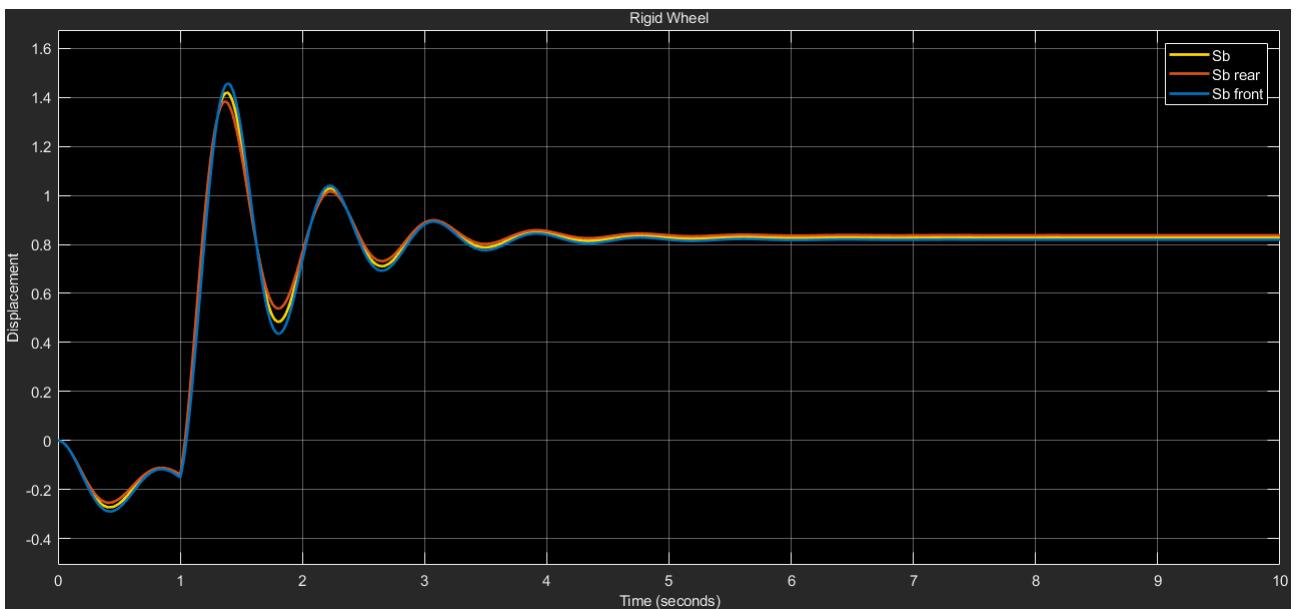


Figure 15. Road step test body movement with a rigid wheel

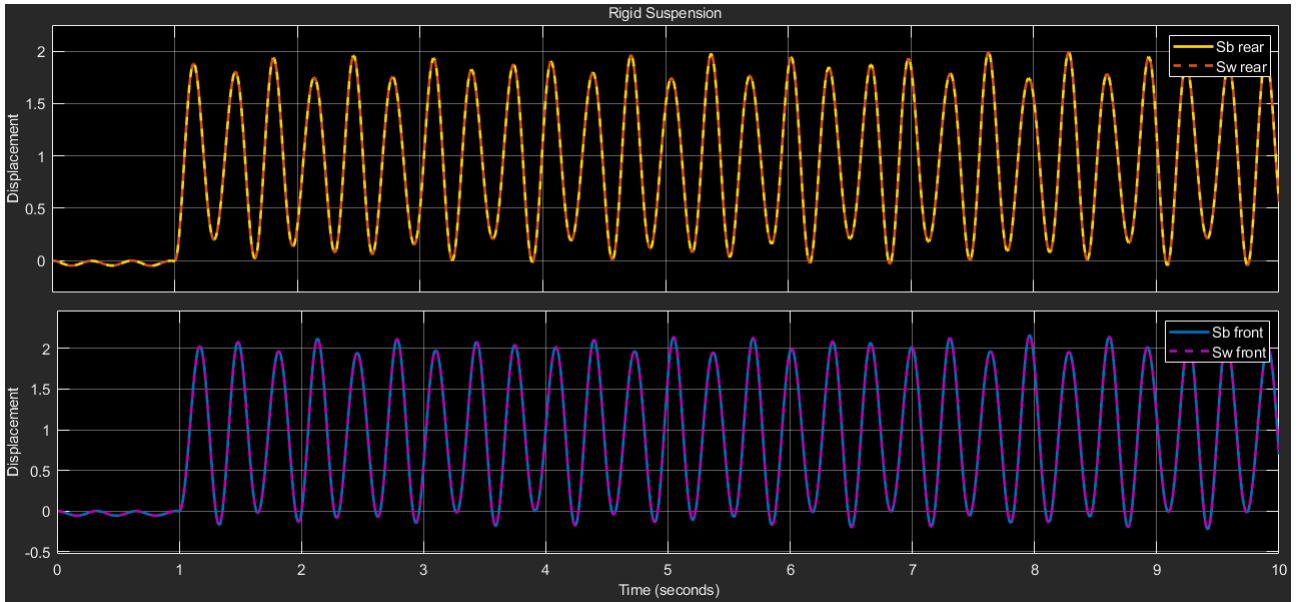


Figure 16. Road step test body and wheel movement with a rigid suspension

So far all tests were conducted with linear parameters for the springs and the dampener. This meant that the springs behaved linearly no matter the elongation while the dampener ignore the compression or extension phase. To better represent real world behaviour the spring hardening and the different damping rates are introduced. The spring hardening curve is given by Figure 17; the damping rates, which were previously averaged, are given in Table 1.

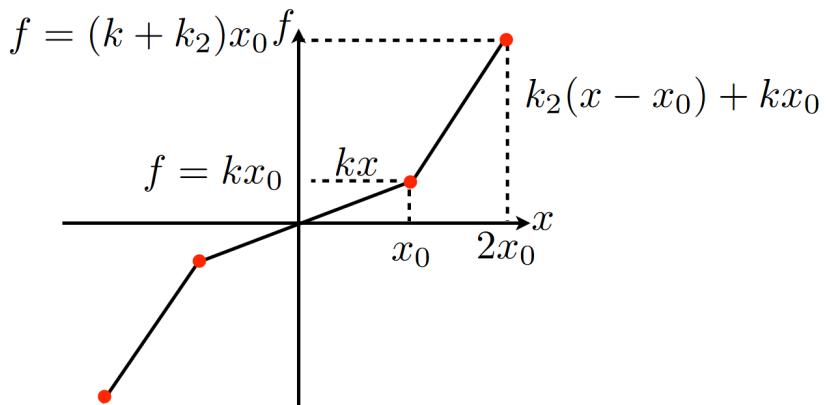


Figure 17. Suspension spring non-linear coefficient

The first non-linear test is a direct comparison with the linear one. The same stationary conditions of the previous test were applied and the results compared in Figure 18. The result shows the correct implementation of the non-linear parameters through look-up tables as the same equilibrium is reached.

To major effects, noticed with the implantation of non-linearities, is a change in amplitude and in period. The amplitude is observed to decrease in one direction while increasing in the other, this is because the new damping coefficients are larger for extension and smaller for compression. The increase in frequency is due to the larger spring coefficient. The car mostly oscillates in a range above the hardening limit (x_0) increasing k_S . A higher coefficient gives a higher frequency due to the relationship $f = \frac{\sqrt{2k_S/m_B}}{2\pi}$.

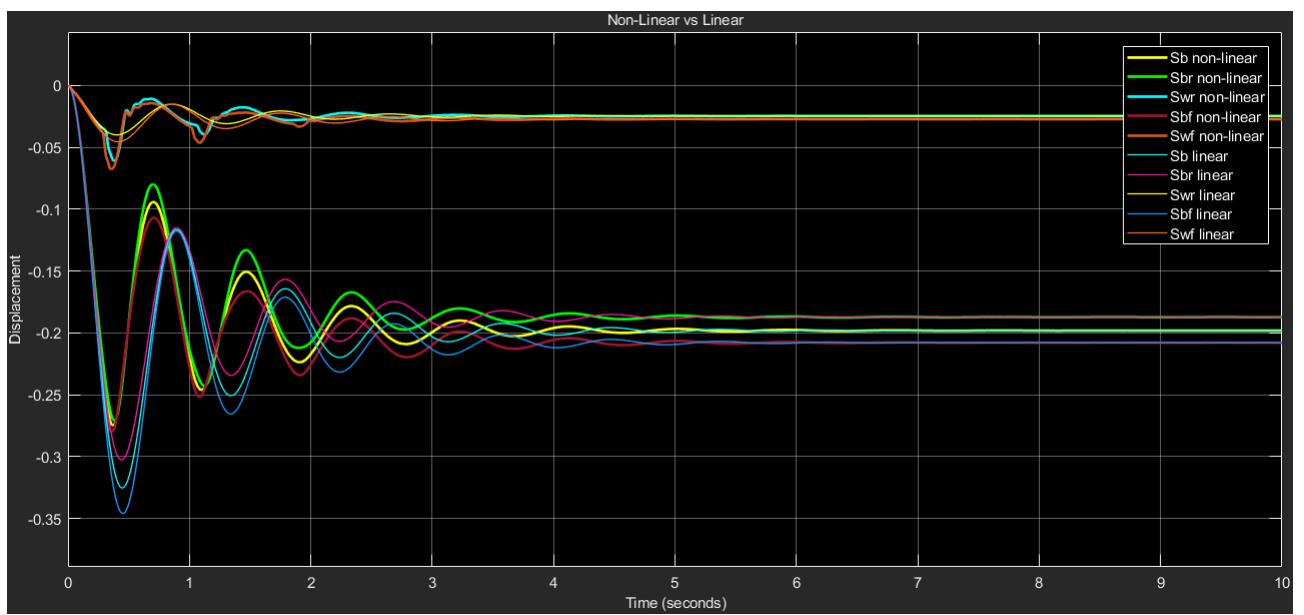


Figure 18. Non-linear comparison on a stationary car test showing body and wheel movement

The last test is the application of a road step input To the non-linear model. By introducing a non linear tyre stiffness the idealised constant contact with ground is removed and the wheel is allowed to lift off. When plotting the body displacement in Figure 19 a series of large bounces are observed as the result of the 1m input. The wheel correctly detaches from the ground and falls back due to gravity until it stabilises due to the suspension dampening.

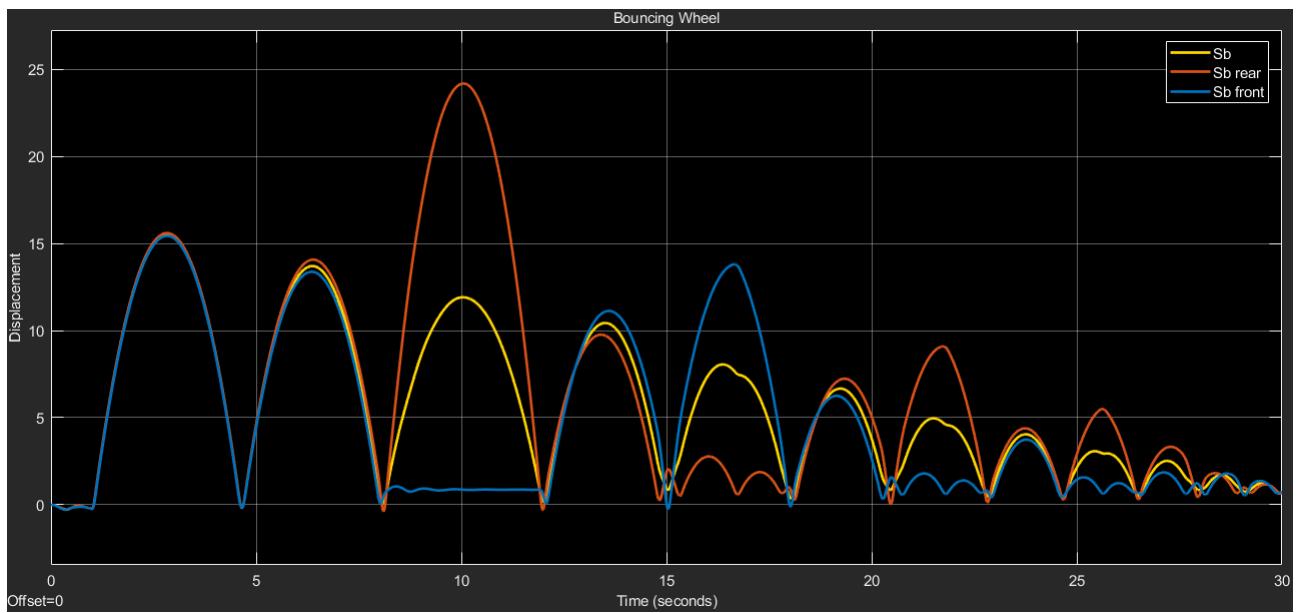


Figure 19. Road step test body movement with non-linear parameters

3. INVESTIGATION OF CAR PERFORMANCE

Once the model is constructed and its performance is verified it can then be applied to a real road scenario. The constructed car is driven over two road conditions to analyse its performance and optimise the suspension characteristics to improve passenger safety and comfort.

1. Sinusoidal Road Input

The first road condition is a perfect sinusoidal road. Despite this is rarely the case in a real world scenario its a first step in the simulation process. The road input r is consequently given by:

$$r = A \sin 2\pi f t \quad (6)$$

where, A is the amplitude in metres, f is the sinusoidal frequency (v/λ) and t the time in seconds. As the car is assumed to drive forward a phase lag is introduced in the back wheel equal to $\phi = 2\pi \frac{(a+b)}{\lambda}$. A range of car velocities and road amplitudes were tested (Figure 30) and reported in Table 6, while the wavelength was fixed at 1m.

Table 6. Sinusoidal road testing parameters

Parameter	Description	Value
v	Car velocity	$[1, 2, 3, 5, 8, 10] ms^{-1}$
A	Road amplitude	$[0.01, 0.05, 0.1] m$
λ	Road wavelength	1m

The initial test consists in analysing the effect of vehicle speed on the system. The road amplitude was fixed to $A = 0.01m$ while the range of velocity was used. The results in Figure 20 illustrates the initial oscillation due to the car weight applied to the unloaded suspensions and tyres and subsequently the oscillation induced by the road. All velocities roughly converge to the same value despite the variation in amplitude and frequency. These variations are direct consequence of the speed as the faster the car travels the more bumps it encounters per unit time. The highest velocity ($10ms^{-1}$) while producing the smallest displacement it also has the highest body acceleration which may negatively impact passenger comfort.

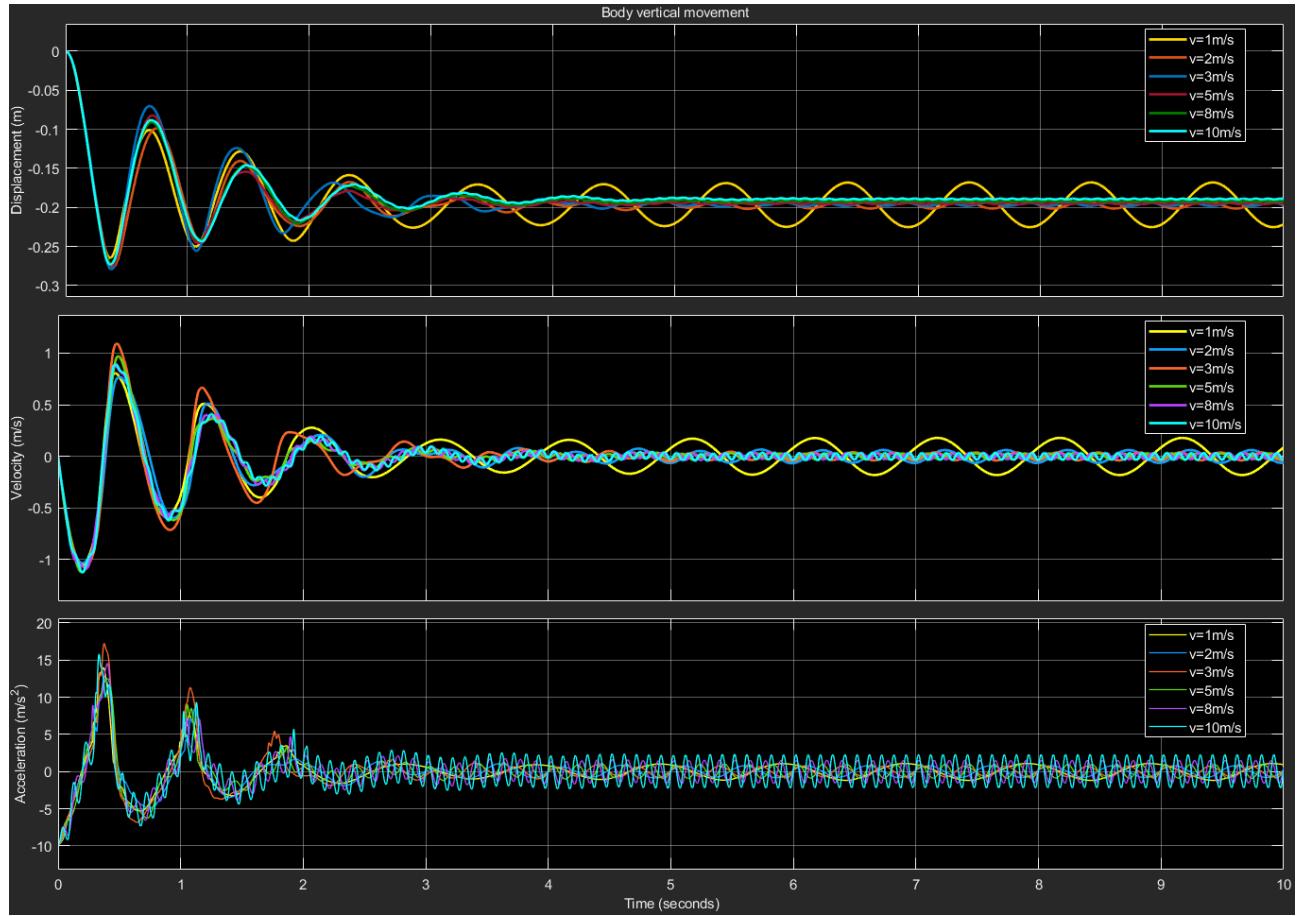


Figure 20. Sinusoidal road test body movement with amplitude $A = 0.01m$

The second test consists in analysing the effect of road amplitude on the system. The vehicle speed was fixed to $v = 10\text{ms}^{-1}$ while the range of amplitudes was used. The results in Figure 21 illustrates a similar behaviour between the first two amplitudes only showing a slight decrease in displacement and frequency of oscillation. On the other hand the 10cm amplitude shows major effects on the car body which does not stabilise within reasonable time. The body displacement is high and the wheels are detaching from the ground causing an infinite oscillation. The large displacements also cause rapid and large changes in acceleration.

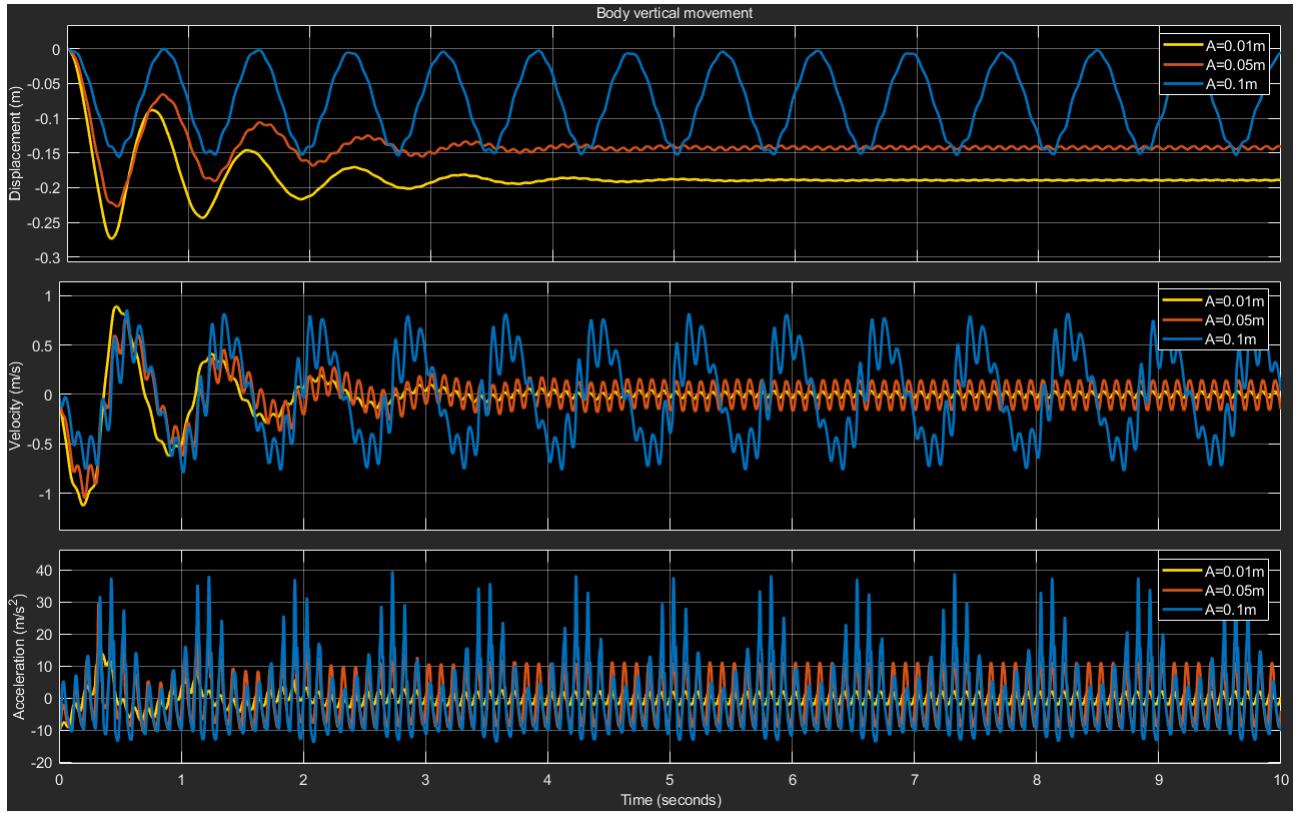


Figure 21. Sinusoidal road test body movement with amplitude $v = 10 m s^{-1}$

2. Speed Bump

The final road condition is a speed bump. These are traffic calming devices installed on roads to reduce vehicle speeds. Speed bumps work by acting on the wheels, changing the vehicle vertical movements resulting in increased passenger discomfort. As the discomfort increases with vehicle speed the driver tends to slow down in proximity of speed bumps (Antić et al., 2013).

Two speed bumps are defined: a Watts profile and a Seminole profile shown in Figure 22

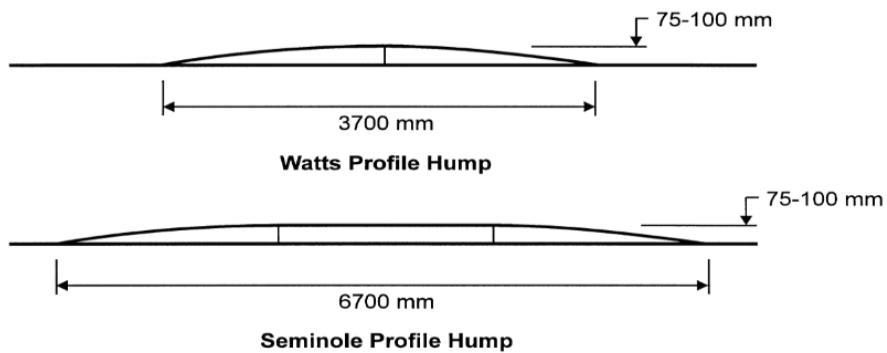


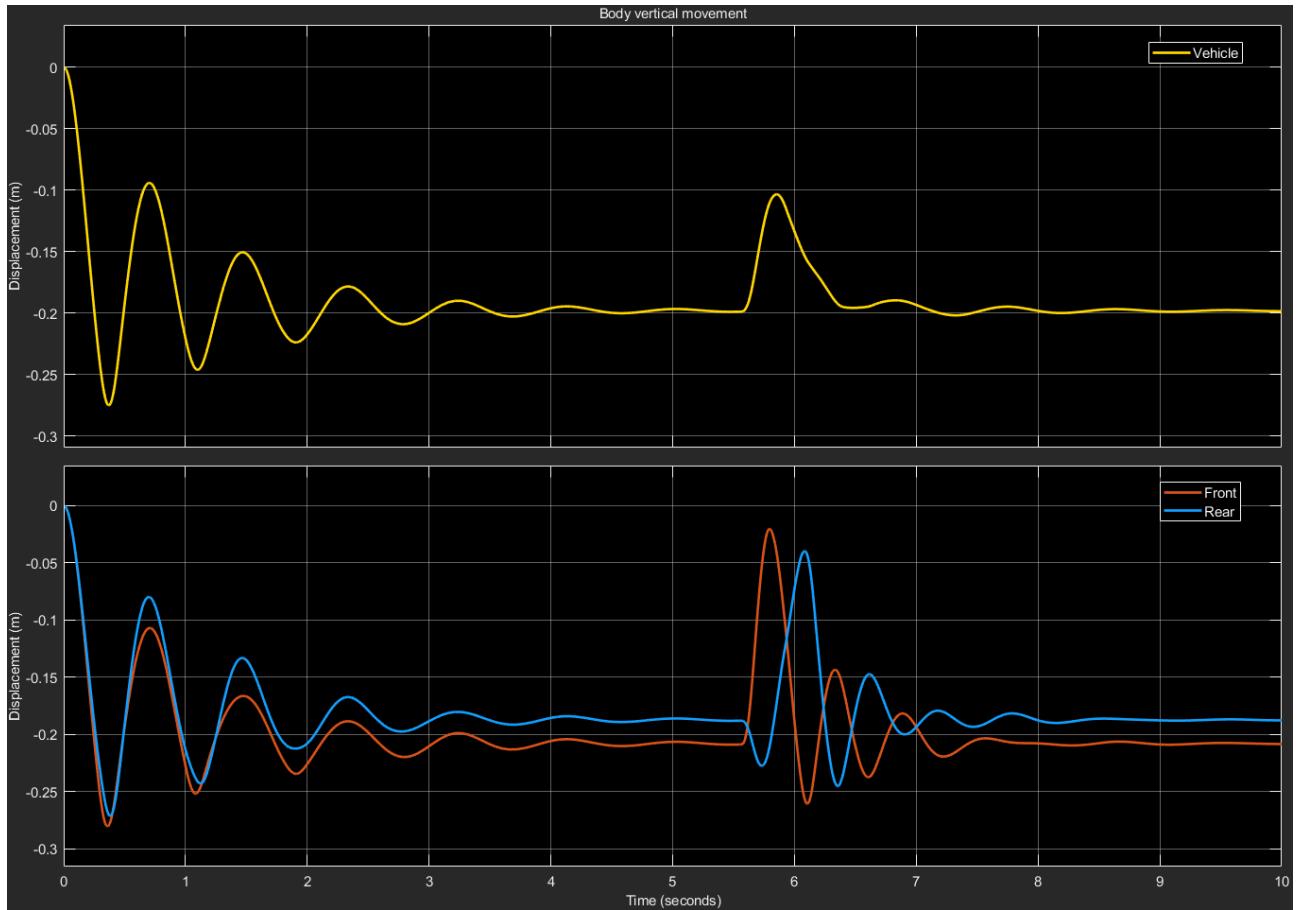
Figure 22. Watts and Seminole profile speed bumps (Shwaly, Zakaria and Al-Ayaat, 2018)

The Watts profile was modelled with a simple window around a sinusoidal road to only extract one bump. The Seminole profile instead starts with the Watts profile and adds a flat constant section at the peak. The two bumps are tested in Figure 31 with the parameters given in Table 7.

Table 7. Speed bump testing parameters

Parameter	Description	Value
h	Bump height	0.1m
L	Watts bump length	3.7m
d	Seminole top surface length	3m
v	Car velocity	$8ms^{-1}$

The result of driving over a Watts bump is shown in Figure 23. After the usual initial oscillation due the suspensions compression the car stabilises and around 5.5 seconds into the simulation hits the bump. The front suspension is affected first while the rear has a lag due to the inter-axis distance. While the body vertical movement is uniform a moment in the car is generated resulting in an oscillation of the pitch. It can be noticed how the single wheels move by the bump height while the car is only displaced by less than half. An ideal suspension would only move the wheel while the car body is stationary.

**Figure 23.** Car movement while driving at $8m/s$ over a single Watts profile speed bump around 5.5s

The Seminole profile bump has a very similar effect shown in Figure 24. This time though the top flat surface is long enough for the car to stabilise before descending the bump.

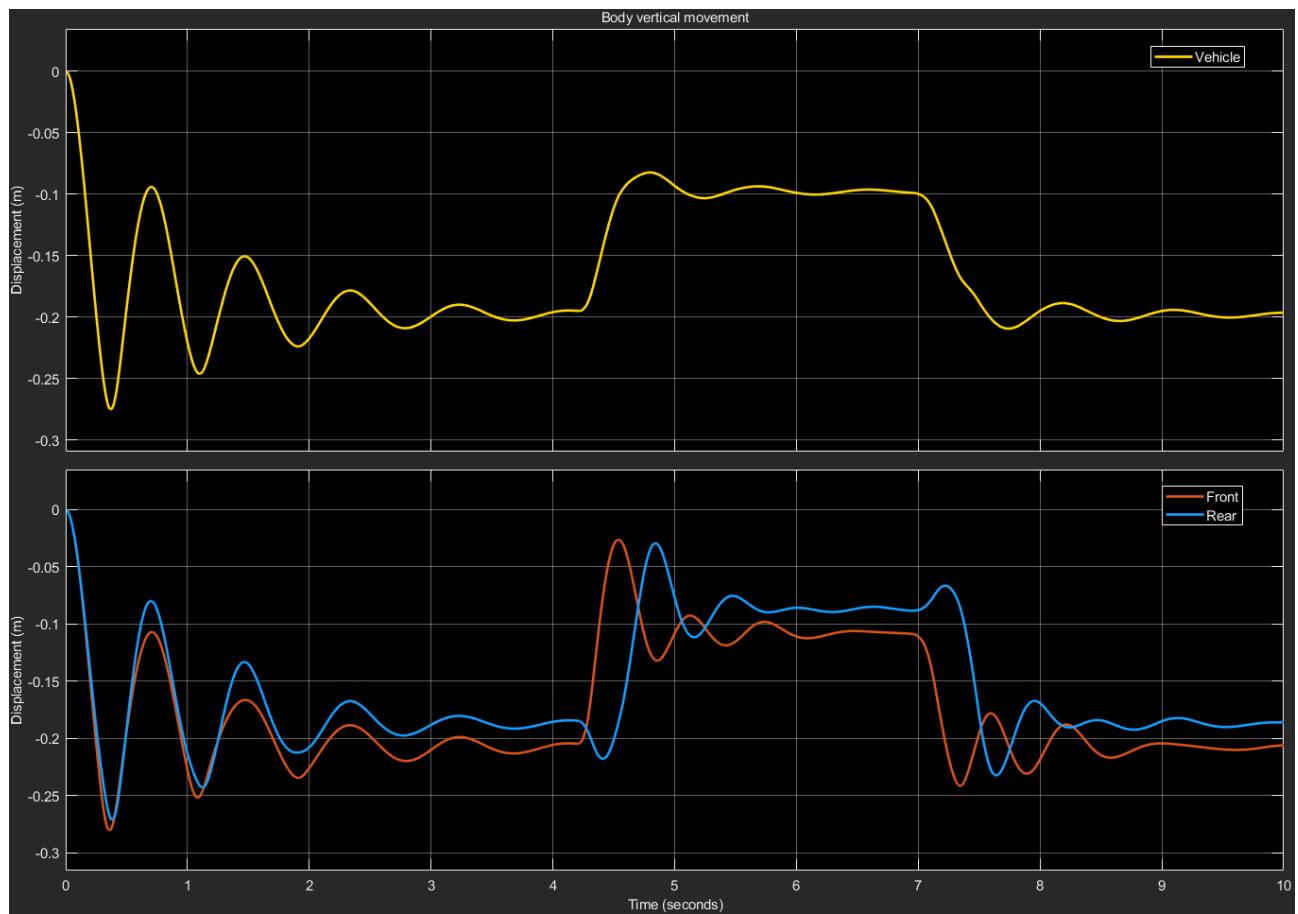


Figure 24. Car movement while driving at $8m/s$ over a single Seminole profile speed bump around 4.3s

A APPENDIX: SOURCE CODE

Listing 1. Model parameters used in testing and simulations

```

1 %% Code initialisation parameters
2
3 % Wheel parameters
4 Mw = 20;           % Wheel mass (kg)
5 Kt = 14e4;         % Wheel stiffness (N/m)
6
7 % Suspension parameters
8 Ks = 2e4;           % Spring stiffness (N/m)
9 Csc = 600;          % Bumping rate contraction (Ns/m)
10 Cse = 1200;         % Bumping rate extension (Ns/m)
11 Kstiff = 20;        % Spring hardening (*Ks)
12 x0 = 0.2;           % Spring hardening point (m)
13
14 % Body parameters
15 a = 1.35;          % Front wheel distance from CoG (m)
16 b = 1.5;            % Rear wheel distance from CoG (m)
17 Mb = 700;           % Body mass (kg)
18 I = 650;            % Body inertia (kg/m^2)
19
20
21 g = 9.81;          % Gravitational acceleration (m/s^2)
22
23 % Test parameters
24 lambda = 1;          % Sinusoidal road wavelength (m)
25 A = [0.01, 0.05, 0.1];    % Sinusoidal road amplitudes (m)
26 v = [1, 2, 3, 5, 8, 10];   % Vehicle speed (m/s)
27
28 h = 0.1;             % Bump height (m)
29 L = 3.7;              % Bump length (m)
30 d = 3;                % Seminole bump top length (m)

```

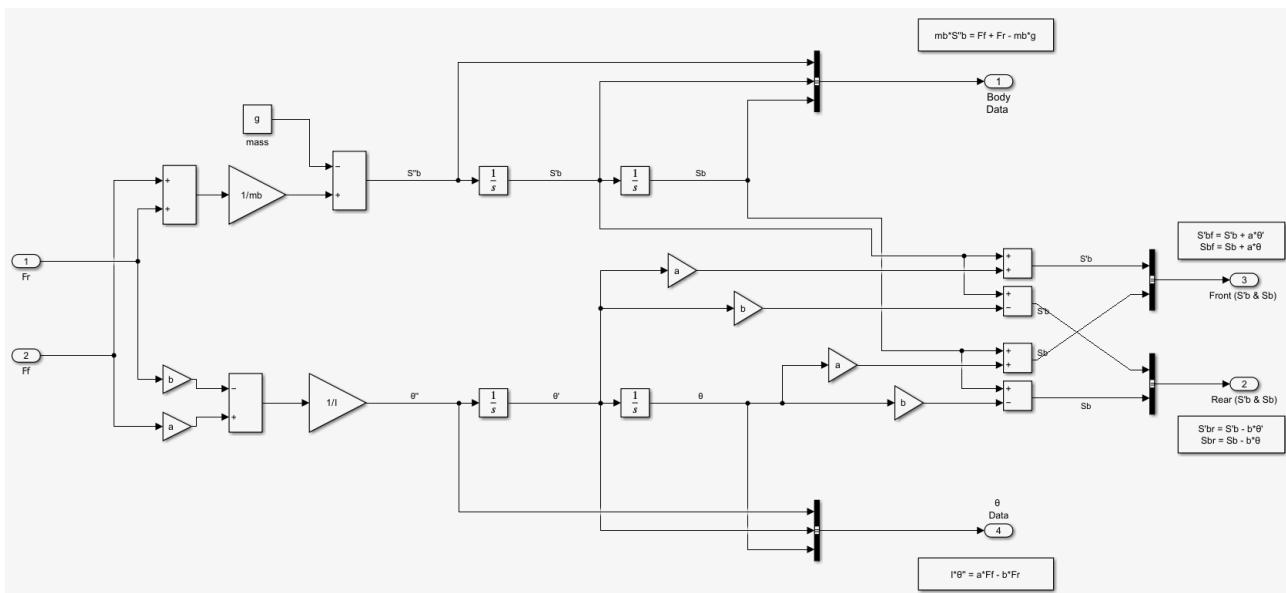


Figure 25. Simulink body subsystem

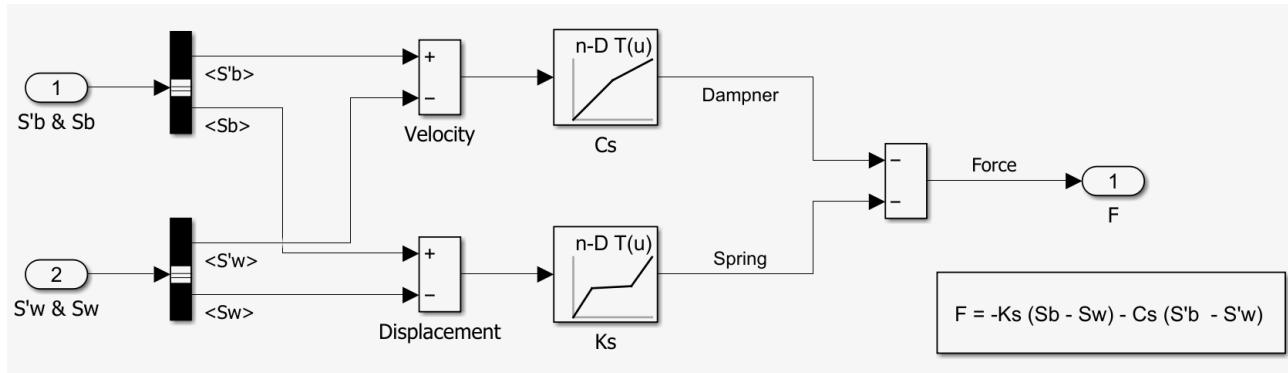


Figure 26. Simulink suspension subsystem

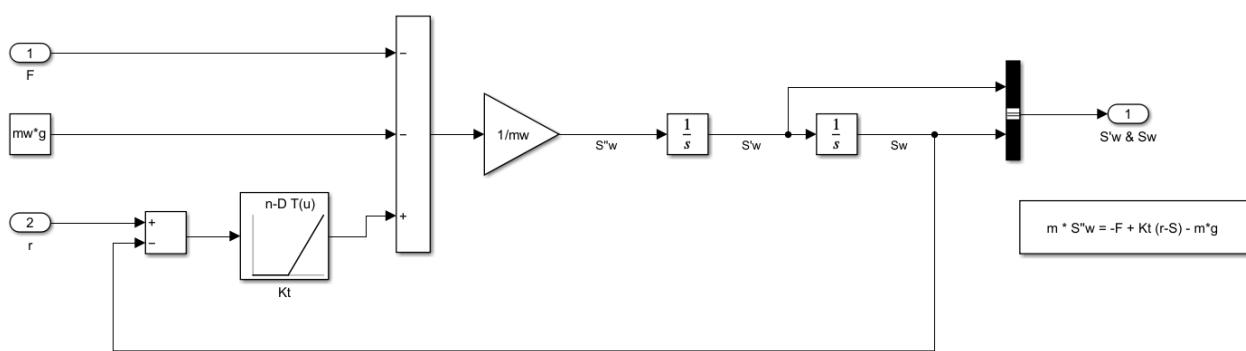
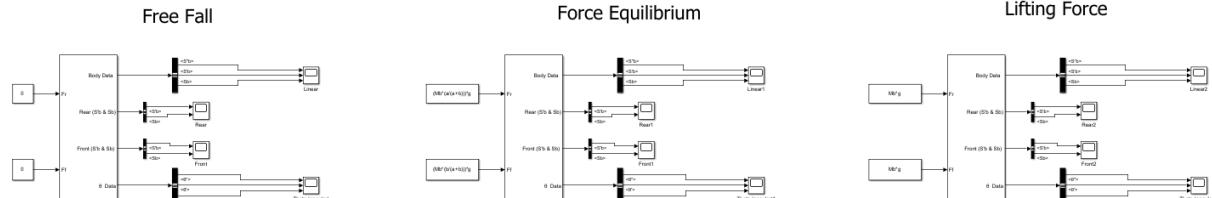
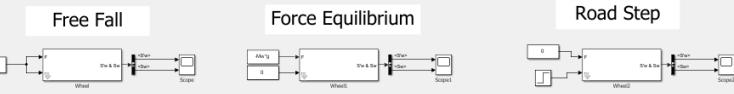


Figure 27. Simulink wheel subsystem

Body Model Tests



Wheel Model Tests

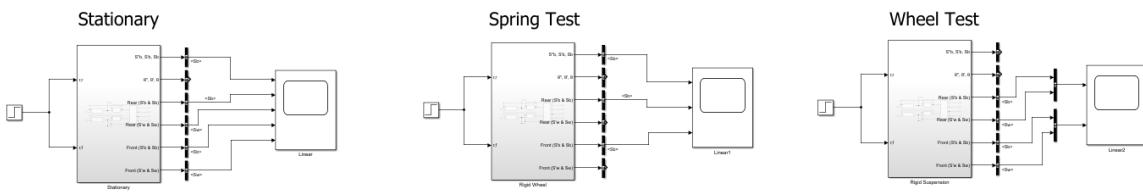


Suspension Model Test



Figure 28. Simulink tests for each subsystem of the car model

Complete Model Tests



Non-Linear Complete Model Tests

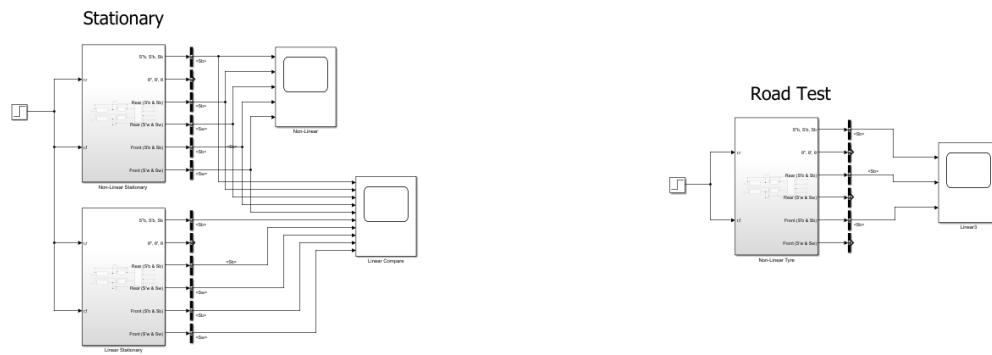


Figure 29. Simulink tests for the car model

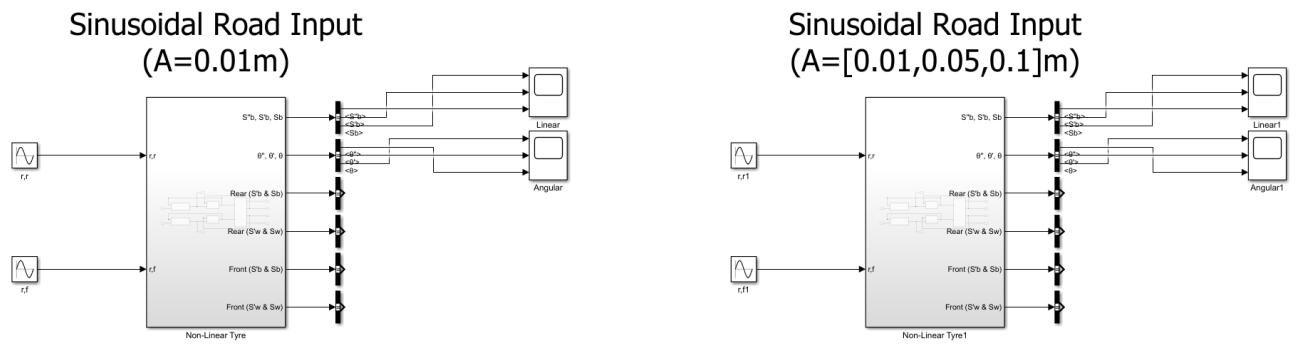


Figure 30. Simulink tests for sinusoidal road effect

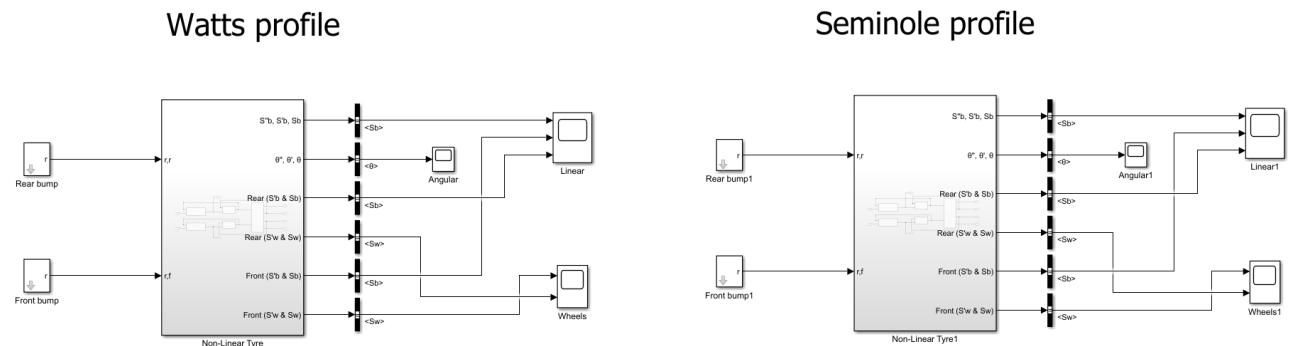


Figure 31. Simulink tests for different speed bumps effects

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