BRAUER-MANIN OBSTRUCTION AND FAMILIES OF GENERALISED CHÂTELET SURFACES OVER NUMBER FIELDS

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Abstract. Over an infinite class of number fields k (including all finite Galois extensions $k/\mathbf{Q}$ of odd degree unramified at 2), we construct infinite families of generalised Châtelet surfaces X over k associated to the normic equation $N_{k(\sqrt{-1})/k}(\vec{z}) = h(x)$ where $\deg(h) \geq 4$ is even and arbitrary large, and with the property that $X(\mathbf{A}_k)^{\mathrm{Br}} = \emptyset$ but $X(\mathbf{A}_k) \neq \emptyset$. We also construct infinite families of generalised Châtelet surfaces X over k associated to the normic equation $N_{k(\sqrt{-1})/k}(\vec{z}) = h(x)$ where $\deg(h) \geq 4$ is even and arbitrarily large, and with the property that $X(\mathbf{A}_k)^{\mathrm{Br}} = X(\mathbf{A}_k) \neq \emptyset$ and $\mathrm{Br}\, X/\mathrm{Br}\, k \neq 0$. Finally, as an application of the main theorems of this paper, we prove that, for a certain family of generalised Châtelet surfaces over $\mathbf{Q}$, a positive proportion (but not 100%) of its members exhibit a violation of the Hasse principle explained by the Brauer-Manin obstruction.

1. Introduction

Let k be a number field, Ω_k its set of places, and $\mathbf{A}_k$ its adelic ring. We start by recalling some standard definitions. For $\{X_\omega\}_\omega$ a family of smooth, projective, geometrically integral varieties over k, we say that $\{X_\omega\}_\omega$ satisfies the Hasse principle if $X_\omega(\mathbf{A}_k) \neq \emptyset$ implies that $X_\omega(k) \neq \emptyset$, for all ω. By the Lang-Nishimura theorem (cf. [Lan54],[Nis55]), the Hasse principle is a birational invariant of smooth, projective, geometrically integral varieties. Now let X be a smooth, quasi-projective, geometrically integral variety over k and let $\mathrm{Br}\, X := H^2_{\text{ét}}(X, \mathbf{G}_m)$ be the (cohomological) Brauer group of X. As a consequence of the fact that $\mathrm{Br}\, X \hookrightarrow \mathrm{Br}\, k(X)$ and of a result by Gabber (see [dJ]), for such an X we have that $\mathrm{Br}\, X$ is the same as the Brauer group defined in terms of Azumaya algebras. The Brauer-Manin set of X (first introduced by Manin in [Man71]) is the set

$$X(\mathbf{A}_k)^{\mathrm{Br}} := \left\{ (x_v) \in X(\mathbf{A}_k) : \sum_{v \in \Omega_k} \mathrm{inv}_v\, \alpha(x_v) = 0 \text{ for all } \alpha \in \mathrm{Br}(X) \right\},$$

where $\mathrm{inv}_v : k_v \to \mathbf{Q}/\mathbf{Z}$ are the invariant maps from local class field theory. One can check that $X(k) \subset X(\mathbf{A}_k)^{\mathrm{Br}}$. If $X(\mathbf{A}_k) \neq \emptyset$ but $X(\mathbf{A}_k)^{\mathrm{Br}} = \emptyset$, we say that X is a counterexample to the Hasse principle explained by the Brauer-Manin obstruction.

The aim of this paper is to provide large classes of examples supporting the following conjecture (stated here for the Hasse principle only; the full conjecture is for weak approximation).

Conjecture 1.1 (Colliot-Thélène and Sansuc)**.** Let X be a smooth, projective, geometrically rational surface over a number field k. Then $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ implies that $X(k) \neq \emptyset$.

By a theorem by Iskovskikh (cf. [Isk79, Thm 1]), any smooth, proper, geometrically rational surface over k is k-birational to either a del Pezzo surface or a conic bundle surface (or both). Since the property "$X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ implies $X(k) \neq \emptyset$" is birationally invariant for smooth, projective, and geometrically integral varieties (see e.g. [CTPS16, §6]), it follows that Conjecture 1.1 needs only be verified for del Pezzo surfaces and conic bundle surfaces. In this paper, we are concerned with the following types of (k-birational classes of) conic bundle surfaces. Let K/k be a quadratic extension

MSC2010: 14G05, 14J26. *Keywords:* Brauer-Manin obstruction, Hasse principle, Châtelet surfaces, geometrically rational surfaces, rational points.

of number field with $\{\omega_1, \omega_2\}$ a k-basis for K. Let $h(x) \in k[x]$ be a separable polynomial of degree $\deg(h) \geq 2$. Let $X_0 \subset \mathbf{A}_k^3$ be given by the equation

$$N_{K/k}(\vec{z}) = h(x),$$

where $\vec{z} := \omega_1 z_1 + \omega_2 z_2$. By making a change of variables if necessary, we can assume without loss of generality that h is separable, $\deg(h) = 2n$ for some $n \geq 1$, and $\sqrt{-1}$ is not in the splitting field of h over k. A smooth proper model X of X_0 that extends the map $X_0 \to \mathbf{A}_k^3$ given by $(\vec{z}, x) \mapsto x$ to a map $X \to \mathbf{P}_k^3$ can be constructed as follows. Let $\mathcal{E} := (\oplus_{i=1}^2 \mathcal{O}_{\mathbf{P}_k^1}) \oplus \mathcal{O}_{\mathbf{P}_k^1}(n)$ be a vector sheaf on $\mathbf{P}_k^1$ of rank 3. Let s_2 be the homogeneisation $\tilde{h}(x,t) := t^{2n} h(x/t)$ in $\Gamma(\mathbf{P}_k^1, \mathcal{O}_{\mathbf{P}_k^1}(n)^{\otimes 2})$ and let $s_1 := N_{K/k}(\vec{z}) \in \Gamma(\mathbf{P}_k^1, \mathrm{Sym}^2(\oplus_{i=1}^2 \mathcal{O}_{\mathbf{P}_k^1}))$. Then $X := \mathbb{V}(s_1 - s_2) \in \mathbf{P}\mathcal{E}$ is a compactification of X_0. Moreover, one can check that X is smooth over k (using the fact that $h(x)$ is separable), and that X becomes rational over $k(\sqrt{-1})$.

Definition 1.2. A generalised Châtelet surface over k associated to X_0 is the smooth compactification X of X_0 as above.

Assuming Schinzel's hypothesis (cf. [SS58]; see e.g. [VAV12] for the statement for number fields), Conjecture 1.1 holds for generalised Châtelet surfaces over any number field k (cf. [CTSD94]). Unconditionally, we mention the following results in the literature. When $\deg(h) = 4$, generalised Châtelet surfaces are usually called Châtelet surfaces and their arithmetic has been studied extensively (see e.g. [CTSSD87a] and [CTSSD87b]); in particular, the full Conjecture 1.1 (for weak approximation) has been verified for Châtelet surfaces over any number field. When $\deg(h) = 6$ and $h(x) = f(x)g(x)$ is the product of two irreducible polynomials over k with $\deg(f) = 2$ and $\deg(g) = 4$, Conjecture 1.1 has been verified by Swinnerton-Dyer in [SD99]. A detailed account of these results can be found in [Sko01, §7]. For higher degrees of $h(x)$, we also mention the work [BMS14], which verifies the full Conjecture 1.1 (for weak approximation) when $h(x)$ completely splits over $\mathbf{Q}$.

We focus here on the case when $K = k(\sqrt{-1})$ and $\deg(h) \geq 4$ is even. More precisely, our aim is to give large classes of examples of generalised Châtelet surfaces for which the failure of the Hasse principle is explained by the Brauer-Manin obstruction, and for which the Brauer-Manin obstruction is empty, the Brauer group (modulo constants) is non-trivial, and the set of adelic points is non-empty (in particular, there is no Brauer-Manin obstruction to weak approximation for such examples). The first class of examples provides direct evidence towards Conjecture 1.1. The surfaces in the second class of examples conjecturally have a k-rational point and can be used, with the help of a computer algebra system, as a testing ground for Conjecture 1.1. In general, to the best of our knowledge, such general examples for large $\deg(h)$ have not yet appeared in the literature. We now state our main results. Let $\Omega_k^{\mathrm{even}\#}$ be the subset of even places v of k with $[k_v : \mathbf{Q}_2]$ odd. Let

$$\mathscr{K} := \{k \text{ number field} : \sqrt{-1} \notin k,\ \Omega_k^{\mathrm{even}} = \Omega_k^{\mathrm{even}\#}, \text{ and } k_v/\mathbf{Q}_2 \text{ is unramified for all } v \in \Omega_k^{\mathrm{even}}\}.$$

Remark 1.3. Examples of number fields $k \in \mathscr{K}$ are Galois extensions $k/\mathbf{Q}$ of odd degree unramified at 2.

For any $k \in \mathscr{K}$ and $N \geq 4$ even, we consider the family $\mathscr{F}_{k,N}$ of generalised Châtelet surfaces X over k associated to affine varieties of the form

$$X_0 : N_{k(\sqrt{-1})/k}(\vec{z}) = f(x)(\lambda + \mu(f(x))^\nu),$$

where $f(x) \in \mathcal{O}_k[x]$ has even degree, $\lambda, \mu \in \mathcal{O}_k^\times$, $\nu \in \mathbf{Z}_{\geq 1}$, and $\deg(f(\lambda + \mu f^\nu)) = N$.

Theorem 1.4 (Simplified version of Theorem 4.12)**.** *Let $k \in \mathscr{K}$ be such that $|\Omega_k^{\text{even}}|$ is odd. Then, for any $M \in \mathbf{Z}_{\geq 1}$, there exists an integer $N \geq M$ and a "flexible" way to construct infinitely many generalised Châtelet surfaces $X \in \mathscr{F}_{k,N}$ such that $X(\mathbf{A}_k) \neq \emptyset$ and $X(\mathbf{A}_k)^{\mathrm{Br}} = \emptyset$.*

Theorem 1.5 (Simplified version of Theorem 4.14)**.** *Let $k \in \mathscr{K}$. Then, for any $M \in \mathbf{Z}_{\geq 1}$, there exists an integer $N \geq M$ and a "flexible" way to construct infinitely many generalised Châtelet surfaces $X \in \mathscr{F}_{k,N}$ such that $\operatorname{Br} X / \operatorname{Br}_0 X \neq 0$ and $X(\mathbf{A}_k)^{\mathrm{Br}} = X(\mathbf{A}_k) \neq \emptyset$.*

Remark 1.6. The point of Theorem 1.5 is that the generalised Châtelet surfaces constructed there don't have any "obvious" k-rational points, generally.

Specialising to $k = \mathbf{Q}$, we obtain more refined results.

Theorem 1.7. *Let $N \geq 4$ be any even number such that $N/2$ is not an odd prime ≥ 5. Then there exist a "flexible" way to construct infinitely many generalised Châtelet surfaces $X \in \mathscr{F}_{\mathbf{Q},N}$ such that $X(\mathbf{A}_{\mathbf{Q}}) \neq \emptyset$ and $X(\mathbf{A}_{\mathbf{Q}})^{\mathrm{Br}} = \emptyset$.*

Theorem 1.8. *Let $N \geq 4$ be any even number. Then there exist a "flexible" way to construct infinitely many generalised Châtelet surfaces $X \in \mathscr{F}_{\mathbf{Q},N}$ such that $X(\mathbf{A}_k) = X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$.*

As an application of Theorem 4.12, one can prove positive density results of the following kind: for a certain family of generalised Châtelet surfaces over $\mathbf{Q}$, a positive proportion of its members exhibit a violation of the Hasse principle explained by the Brauer-Manin obstruction (see Theorem 7.1 for more details).

Structure of the paper. In §§2, 3, we recall some results useful for computing the Brauer group of generalised Châtelet surfaces and the Hilbert symbol. In §4, we prove the main theorems of this paper, namely Theorems 4.12 and 4.14; their proof is not difficult, but is rather computational. In §5, we specialise to $k = \mathbf{Q}$ and prove Theorems 1.7 and 1.8. In §6, we give some examples of many-parameters families of generalised Châtelet surfaces over a number field $k \neq \mathbf{Q}$ satisfying the conditions of Theorems 4.12 and 4.14. We conclude in §7 with an application (Theorem 7.1) of Theorem 4.12.

General notation. We fix once and for all an algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$, and we take any algebraic extension of $\mathbf{Q}$ to be inside $\overline{\mathbf{Q}}$. Let k be a number field. We denote by Ω_k the set of places of k, and by k_v the completion of k at the place $v \in \Omega_k$. We denote by $\Omega_k^{\mathbf{R}}$ and $\Omega_k^{\mathbf{C}}$ the real and complex places of k, respectively, and we denote by Ω_k^{even} and Ω_k^{odd} the finite places of k above the rational prime 2 and above odd rational primes, respectively. We further denote by $\Omega_k^{\text{odd}_{\text{QR}}}$ and $\Omega_k^{\text{odd}_{\text{QNR}}}$ the subsets of places $v \in \Omega_k^{\text{odd}}$ such that $\operatorname{red}_v(-1)$, respectively, is and is not a square in $\mathbf{F}_v$, and by $\Omega_k^{\text{even}\#}$ the subset of places $v \in \Omega_k^{\text{even}}$ with $[k_v : \mathbf{Q}_2]$ odd. If $v \in \Omega_k$ is a finite place, we write $\mathbf{F}_v := \mathcal{O}_{k_v}/\mathfrak{m}_v$ for the residue field at v, where $\mathfrak{m}_v$ is the maximal ideal of $\mathcal{O}_{k_v}$; we write $\operatorname{red}_v : \mathcal{O}_{k_v} \to \mathbf{F}_v$ for the reduction map. If $f \in k[x]$, we denote by $\operatorname{Split}_k(f)$ the splitting field of f over k. If X is a variety over a number field k, we denote by $\overline{X}$ the base-change of X to $\overline{k}$.

Acknowledgements. The author would like to thank Efthymios Sofos for some helpful comments. The author is financially supported by a Postdoctoral Fellowship from the Max-Planck-Institut für Mathematik in Bonn.

2. The Hilbert symbol

We recall, for convenience, some of the explicit formulas for computing the Hilbert symbol. For $v \in \Omega_k^{\mathrm{odd}}$ and $a, b \in k_v^\times$, we have (see e.g. [Ser79, Chap. XIV,§3])

$$(a,b)_{k_v} = \left(\mathrm{red}_v\left((-1)^{v(a)v(b)}\frac{a^{v(b)}}{b^{v(a)}}\right)\right)^{\frac{\#\mathbf{F}_v-1}{2}}.$$

Remark 2.1. If $c \in \mathbf{F}_v^\times$, then $c^{\frac{\#\mathbf{F}_v-1}{2}}$ is the Legendre symbol, which is equal to 1 if and only if c is a square in $\mathbf{F}_v$.

In particular, letting $a = -1$ yields the following.

Lemma 2.2. *Let $v \in \Omega_k^{\mathrm{odd}}$. For $b \in b \in k_v^\times$, we have $(-1,b)_{k_v} = -1$ if and only if $v(b)$ is odd and* $\mathrm{red}_v(-1)$ *is not a square in $\mathbf{F}_v$.*

Now let $v \in \Omega_k^{\mathrm{even}}$. Let $a \in k_v^\times$ and $b \in \mathbf{Q}_2^\times$, and write $N_{k_v/\mathbf{Q}_2}(a) = 2^\alpha u_a$ and $b = 2^\beta u_b$, where $u_a, u_b \in \mathcal{O}_{\mathbf{Q}_2}^\times$. By [Ben73, Theorem 1], the Hilbert symbol for k_v descends to the Hilbert symbol for $\mathbf{Q}_2$ as

$$(a,b)_{k_v} = (N_{k_v/\mathbf{Q}_2}(a), b)_{\mathbf{Q}_2} = (-1)^{\epsilon(u_a)\epsilon(u_b)+\alpha\omega(u_b)+\beta\omega(u_a)}, \tag{2.1}$$

where $\epsilon(x) := \frac{x-1}{2} \pmod 2$, $\omega(x) := \frac{x^2-1}{8} \pmod 2$, and where the right-most equality of (2.1) follows from the well-known formula of the Hilbert symbol for $\mathbf{Q}_2$. In particular, letting $b = -1$ yields the following.

Lemma 2.3. *Let $v \in \Omega_k^{\mathrm{even}}$. For $a \in k_v^\times$, we have $(a,-1)_{k_v} = (N_{k_v/\mathbf{Q}_2}(a), -1)_{\mathbf{Q}_2} = (-1)^{\epsilon(u_a)}$.*

Let $v \in \Omega_k$ be a finite place. We briefly recall some results about the structure of the group of units of k_v. For any integer $m \geq 1$, we define the group of m-principal units of k_v to be

$$U_{k_v}^m := \{u \in \mathcal{O}_{k_v}^\times : u = 1 + \mathfrak{m}_v^m\},$$

and we define the set

$$\overline{U}_{k_v}^m := \{u \in \mathcal{O}_{k_v}^\times : u = -1 + \mathfrak{m}_v^m\}.$$

For any integer $r \geq 1$, we denote by $\mu_r(k_v)$ the subgroup of $\mathcal{O}_{k_v}^\times$ consisting of all r-th roots of unity in k_v. From these definitions, we immediately have the following.

Lemma 2.4. *$\mathcal{O}_{k_v}^\times / U_{k_v}^1 = \mathbf{F}_v^\times$. Consequently, we can write any unit $u \in \mathcal{O}_{k_v}^\times$ as $u = \epsilon u_1$, where $\epsilon \in \mu_{\#\mathbf{F}_v-1}(k_v)$ and $u_1 \in U_{k_v}^1$.*

Remark 2.5. $\mathcal{O}_{\mathbf{Q}_2}^\times = U_{\mathbf{Q}_2}^1$.

Lemma 2.6. *Suppose that $k_v/\mathbf{Q}_2$ is an unramified (Galois) extension and let $a = 2^{v(a)}u \in k_v^\times$, where $u \in \mathcal{O}_{k_v}^\times$. Write $u = \epsilon u_1 t^2$, where $\epsilon \in \mu_{2^{[k_v:\mathbf{Q}_2]}-1}(k_v)$ and $u_1 \in U_{k_v}^1$ are square-free, and $t \in \mathcal{O}_{k_v}^\times$. If $[k_v : \mathbf{Q}_2]$ is odd and $u_1 \in \overline{U}_{k_v}^2$, then $(-1,a)_{k_v} = -1$.*

Proof. First, we note that $N_{k_v/\mathbf{Q}_2}(a) = N_{k_v/\mathbf{Q}_2}(2^{v(a)}u) = 2^{[k_v:\mathbf{Q}_2]v(a)}N_{k_v/\mathbf{Q}_2}(u)$. By Lemma 2.3, it follows that

$$(-1,a)_{k_v} = (-1, N_{k_v/\mathbf{Q}_2}(a))_{\mathbf{Q}_2} = (-1, N_{k_v/\mathbf{Q}_2}(u))_{\mathbf{Q}_2}.$$

Let $f_v := [\mathbf{F}_v : \mathbf{F}_2] = [k_v : \mathbf{Q}_2]$ be the residue degree. Since $u \in \mathcal{O}_{k_v}^\times$, we can write $u = \epsilon u_1 t^2$, where $\epsilon \in \mu_{2^{f_v}-1}(k_v)$ and $u_1 \in U_{k_v}^1$ are square-free, and $t \in \mathcal{O}_{k_v}^\times$. Since $N_{k_v/\mathbf{Q}_2}(t^2)$ is a square, by the properties of the Hilbert symbol we have

$$(-1, N_{k_v/\mathbf{Q}_2}(u))_{\mathbf{Q}_2} = (-1, N_{k_v/\mathbf{Q}_2}(\epsilon u_1))_{\mathbf{Q}_2}.$$

Since $2^{f_v}-1$ is odd and $\epsilon^{2^{f_v}-1}=1$, it follows that $(-1,N_{k_v/\mathbf{Q}_2}(\epsilon))_{\mathbf{Q}_2}=((-1,N_{k_v/\mathbf{Q}_2}(\epsilon))_{\mathbf{Q}_2})^{2^{f_v}-1}=1$. Hence, by the multiplicativity property of the Hilbert symbol we have

$$(-1,N_{k_v/\mathbf{Q}_2}(\epsilon u_1))_{\mathbf{Q}_2}=(-1,N_{k_v/\mathbf{Q}_2}(u_1))_{\mathbf{Q}_2}.$$

It remains to compute $(-1,N_{k_v/\mathbf{Q}_2}(u_1))_{\mathbf{Q}_2}$. If $u_1\in\overline{U}^2_{k_\mathfrak{q}}$, then we can write $u_1=-1+4h$ for some $h\in\mathcal{O}_{k_v}$. Then

$$\begin{aligned}N_{k_v/\mathbf{Q}_2}(u_1)&=\textstyle\prod_{\sigma\in\mathrm{Gal}(k_v/\mathbf{Q}_2)}\sigma(u_1)\\&=\textstyle\prod_{\sigma\in\mathrm{Gal}(k_v/\mathbf{Q}_2)}\sigma(-1+4h)\\&=\textstyle\prod_{\sigma\in\mathrm{Gal}(k_v/\mathbf{Q}_2)}(-1+4\sigma(h))\\&=(-1)^{[k_v:\mathbf{Q}_2]}+4h',\end{aligned}$$

for some $h'\in\mathcal{O}_{\mathbf{Q}_2}$. If $[k_v:\mathbf{Q}_2]$ is odd, it follows that $N_{k_v/\mathbf{Q}_2}(u_1)\in\overline{U}^2_{k_v}$. Hence, by Lemma 2.3 we obtain $(-1,a)_{k_v}=(-1,N_{k_v/\mathbf{Q}_2}(a))_{\mathbf{Q}_2}=(-1)^{\epsilon(N_{k_v/\mathbf{Q}_2}(u_1))}=-1$, as required. ☐

Remark 2.7. If $u_1\in U^2_{k_v}$, then a similar proof as the above yields $(-1,a)_{k_v}=1$ for any $[k_v:\mathbf{Q}_2]$.

3. The Brauer group

Let X be the generalised Châtelet surface over a number field k associated to

$$N_{k(\sqrt{-1})/k}(\vec{z})=f(x)g(x),$$

where f and g are polynomials over k of even degrees $\deg f,\deg g\geq 2$ and $f(x)g(x)$ is separable. Let $\mathscr{A}\in\operatorname{Br}k(X)$ be the class of the quaternion algebra $(-1,f)_{k(X)}$. Since the class of $(-1,f)_{k(X)}$ is unaffected if we multiply f by a square or by a norm of $k(X)(\sqrt{-1})/k(X)$, it follows that $\mathscr{A}=[(-1,g)]_{k(X)}$.

Lemma 3.1. *For X, k, and $\mathscr{A}$ as above, we have $\mathscr{A}+\operatorname{Br}_0X\in\operatorname{Br}X/\operatorname{Br}_0X$.*

Proof. This follows from the purity theorem (cf. [Fuj02]) and by the fact that $\deg f$ is even. ☐

Lemma 3.2. *Let X, k, and $\mathscr{A}$ as above. If moreover f and g are irreducible over k and neither $\operatorname{Split}_k(f)$ nor $\operatorname{Split}_k(g)$ contain $\sqrt{-1}$, then $\operatorname{Br}X/\operatorname{Br}_0X=\langle\mathscr{A}+\operatorname{Br}_0X\rangle\cong\mathbf{Z}/2\mathbf{Z}$.*

Proof. Since $\operatorname{Br}\overline{X}=0$ and k is a number field, by the low-degree terms sequence of the Hochschild-Serre spectral sequence $H^p(\mathrm{Gal}(\overline{k}/k),H^q_{\text{ét}}(\overline{X},\mathbf{G}_m))\Rightarrow H^{p+q}_{\text{ét}}(X,\mathbf{G}_m)$ we deduce that $\operatorname{Br}X/\operatorname{Br}_0X=H^1(k,\operatorname{Pic}\overline{X})$. By [Sko01, Prop.7.1.1] (see also [VAV12]), we have an isomorphism

$$\frac{\{(n_1,n_2)\in(\mathbf{Z}/2\mathbf{Z})^2:n_1\deg f+n_2\deg g\equiv 0\pmod 2\}}{(1,1)}\xrightarrow{\sim}\frac{\operatorname{Br}X}{\operatorname{Br}_0X}$$

given by $[(n_1,n_2)]\mapsto[(-1,f(x)^{n_1}g(x)^{n_2})_{k(X)}]+\operatorname{Br}_0X$. Since $\deg f$ and $\deg g$ are both even, it follows that $n_1\deg f+n_2\deg g\equiv 0\pmod 2$ for all $(n_1,n_2)\in(\mathbf{Z}/2\mathbf{Z})^2$. Hence, $\operatorname{Br}X/\operatorname{Br}_0X=\langle\mathscr{A}+\operatorname{Br}_0X\rangle\cong\mathbf{Z}/2\mathbf{Z}$. ☐

4. Constructing the generalised Châtelet surfaces

Let $k\in\mathscr{K}$. For any $f(x):=\sum_{i=0}^n f_ix^i\in\mathcal{O}_k[x]$ and any tuple $(\lambda,\mu,\nu)\in\mathcal{O}_k\times\mathcal{O}_k\times\mathbf{Z}_{\geq 1}$, we define the following set of conditions, which we collectively call Conditions (ELS).

Condition 4.1 (ELS-f). We have $n\geq 2$ even, $f(\lambda+\mu f^\nu)$ separable, and $f_n,f_0,\lambda+\mu f_0^\nu\neq 0$.

Condition 4.2 (ELS-odd$_{\mathbf{QR}}$). For any $v\in\Omega_k^{\mathrm{odd}_{\mathrm{QR}}}$, there is $x_v\in k_v$ such that $v(f(x_v)(\lambda+\mu f(x_v)^\nu))$ is even.

Remark 4.3. If e.g. $f_n = \pm 1$, $v(\mu) = 0$ for all $v \in \Omega_k^{\mathrm{odd}_{\mathrm{QR}}}$, and $n \geq 2$ is even, then for any $v \in \Omega_k^{\mathrm{odd}_{\mathrm{QR}}}$ we can take $x_v = u_v/\pi$, where π is a uniformiser and $u_v \in \mathcal{O}_{k_v}^{\times}$.

Condition 4.4 (**ELS-odd$_{\mathbf{QNR}}$**)**.** For $v \in \Omega_k^{\mathrm{odd}_{\mathrm{QNR}}}$, we have $v(f_n) = v(\lambda) = v(\mu) = 0$.

Condition 4.5 (**ELS-even**)**.** For $v \in \Omega_k^{\mathrm{even}}$, we have $(-1, f_0)_{k_v} = (-1, \lambda + \mu f_0^{\nu})_{k_v}$.

Condition 4.6 (**ELS-R**)**.** If $\Omega_k^{\mathbf{R}} \neq \emptyset$, then, for $v \in \Omega_k^{\mathbf{R}}$, there exists some $x_v \in k_v$ such that $f(x_v)(\lambda + \mu f(x_v)^{\nu}) > 0$ in k_v.

We also define the following set of conditions, which we collectively call Conditions (Br).

Condition 4.7 (**Br-k-f**)**.** Let $\chi_k := \mathrm{lcm}_{v \in \Omega_k^{\mathrm{even}}} \{2^{[k_v:\mathbf{Q}_2]} - 1\}$.

(1) If $f_i \notin 4\mathcal{O}_k$, then $\chi_k | i$;
(2) both f and $\lambda + \mu f^{\nu}$ have no roots over k.

Condition 4.8 (**Br-even**)**.** For $v \in \Omega_k^{\mathrm{even}}$, writing $\lambda = 2^{v(\lambda)} u_\lambda$, $\mu = 2^{v(\mu)} u_\mu$, and $\lambda + \mu f_0^{\nu} = 2^{v(\lambda + \mu f_0^{\nu})} u_{\lambda + \mu f_0^{\nu}}$, we have:

(1) if j is odd, then either $f_j = 0$ or $v(f_j) \geq 1$;
(2) $f_n, f_0, u_{\lambda + \mu f_0^{\nu}} \in \overline{U}_{k_v}^2$;
(3) $u_\lambda \in U_{k_v}^2$;
(4) if ν is odd, then $u_\mu \in \overline{U}_{k_v}^2$, while if ν is even, then $u_\mu \in U_{k_v}^2$;
(5) either $\sum_{i=1}^{n-1} f_i \in U_{k_v}^2$, or $\sum_{i=1}^{n-1} f_i \in \overline{U}_{k_v}^2$, or $\sum_{i=1}^{n-1} f_i \in 4\mathcal{O}_{k_v}$. Moreover,
   (a) if $\sum_{i=1}^{n-1} f_i \in \overline{U}_{k_v}^2$, then
      (i) if ν is odd, then $v(\mu) = v(\lambda) + 1$;
      (ii) if ν is even, then either $v(\mu) = v(\lambda) + 1$ or $v(\lambda) = v(\mu) + 1$;
   (b) if $\sum_{i=1}^{n-1} f_i \in 4\mathcal{O}_{k_v}$, then $v(\mu) + \nu = v(\lambda) + 1$ and $\nu \leq 2$.

Condition 4.9 (**Br-R**)**.** If $\Omega_k^{\mathbf{R}} \neq \emptyset$, then, for $v \in \Omega_k^{\mathbf{R}}$, we have:

(1) if ν is odd, then $\lambda > 0$ and $\mu < 0$ in k_v;
(2) if ν is even, then $\lambda, \mu > 0$ in k_v.

Finally, we let Conditions (HP) be Conditions (**Br-k-f**) and (**Br-R**) together with the following two conditions.

Condition 4.10 (**HP-k-f**)**.** Both f and $\lambda + \mu f^{\nu}$ are irreducible over k, and neither $\mathrm{Split}_k(f)$ nor $\mathrm{Split}_k(\lambda + \mu f^{\nu})$ contains $\sqrt{-1}$.

Condition 4.11 (**HP-even**)**.** For $v \in \Omega_k^{\mathrm{even}}$, writing $\lambda = 2^{v(\lambda)} u_\lambda$, $\mu = 2^{v(\mu)} u_\mu$, and $\lambda + \mu f_0^{\nu} = 2^{v(\lambda + \mu f_0^{\nu})} u_{\lambda + \mu f_0^{\nu}}$, we have:

(1) if j is odd, then either $f_j = 0$ or $v(f_j) \geq 1$;
(2) $f_n, f_0, u_{\lambda + \mu f_0^{\nu}} \in U_{k_v}^2$;
(3) $u_\lambda \in U_{k_v}^2$;
(4) if ν is odd, then $u_\mu \in \overline{U}_{k_v}^2$, while if ν is even, then $u_\mu \in U_{k_v}^2$;
(5) either $\sum_{i=1}^{n-1} f_i \in U_{k_v}^2$, or $\sum_{i=1}^{n-1} f_i \in \overline{U}_{k_v}^2$, or $\sum_{i=1}^{n-1} f_i \in 4\mathcal{O}_{k_v}$, or $\sum_{i=1}^{n-1} f_i \in 2 + 4\mathcal{O}_{k_v}$. Moreover,
   (a) if $\sum_{i=1}^{n-1} f_i \in U_{k_v}^2$, then
      (i) if ν is odd, then $v(\mu) \geq v(\lambda) + 2$;
      (ii) if ν is even, then either $v(\mu) \geq v(\lambda) + 2$ or $v(\lambda) \geq v(\mu) + 2$;
   (b) if $\sum_{i=1}^{n-1} f_i \in 4\mathcal{O}_{k_v}$, then

(i) if ν is odd, then either $v(\mu) \geq v(\lambda)+2$, or $v(\lambda) = v(\mu)+1$ and $\nu \geq 3$;
(ii) if ν is even, then either $v(\mu) \geq v(\lambda)+2$, or $v(\mu)+2 \leq v(\lambda) \leq v(\mu)+\nu-2$;

(c) if $\sum_{i=1}^{n-1} f_i \in 2+4\mathcal{O}_{k_v}$, then
(i) if ν is odd, then either $v(\mu) \geq v(\lambda)+2$, or $v(\lambda) = v(\mu)+1$ and $\nu \geq 3$;
(ii) if ν is even, then either $v(\mu) \geq v(\lambda)+2$, or $v(\mu)+2 \leq v(\lambda) \leq v(\mu)+2\nu-2$.

The main theorems of this section are the following.

Theorem 4.12. *Let $k \in \mathscr{K}$ be such that $|\Omega_k^{\text{even}}|$ is odd. Let $f(x) := \sum_{i=0}^{n} f_i x^i \in \mathcal{O}_k[x]$ and $(\lambda, \mu, \nu) \in \mathcal{O}_k \times \mathcal{O}_k \times \mathbf{Z}_{\geq 1}$ satisfy Conditions* (**ELS**) *and* (**Br**). *Let X be the generalised Châtelet surface over k associated to $X_0 : N_{k(\sqrt{-1})/k}(\vec{z}) = f(x)(\lambda + \mu(f(x))^\nu)$. Then $X(\mathbf{A}_k) \neq \emptyset$ and $X(k) \subset X(\mathbf{A}_k)^{\mathrm{Br}} = \emptyset$.*

Remark 4.13. For $k = \mathbf{Q}$, $f(x) = -x^2+3$, and $(\lambda, \mu, \nu) = (1, -1, 1)$, we retrieve Iskovskikh's famous counterexample to the Hasse principle (see [Isk71]).

Theorem 4.14. *Let $k \in \mathscr{K}$, $f(x) := \sum_{i=0}^{n} f_i x^i \in \mathcal{O}_k[x]$, and $(\lambda, \mu, \nu) \in \mathcal{O}_k \times \mathcal{O}_k \times \mathbf{Z}_{\geq 1}$ satisfy Conditions* (**ELS**) *and* (**HP**). *Let X be the generalised Châtelet surface over k associated to $X_0 : N_{k(\sqrt{-1})/k}(\vec{z}) = f(x)(\lambda + \mu(f(x))^\nu)$. Then $\operatorname{Br} X / \operatorname{Br}_0 X \neq 0$ and $X(\mathbf{A}_k)^{\mathrm{Br}} = X(\mathbf{A}_k) \neq \emptyset$.*

We first prove some preliminary results.

Lemma 4.15. *Let k be a number field with $\sqrt{-1} \notin k$. Let $v \in \Omega_k$ be such that there exists $\gamma \in k_v^\times$ with $(-1, \gamma)_{k_v} = -1$. Then, for any $\alpha, \beta \in k_v^\times$, there is a k_v-solution $\vec{z_v}$ to $N_{k(\sqrt{-1})/k}(\vec{z}) = \alpha\beta$ if and only if $(-1, \alpha)_{k_v} = (-1, \beta)_{k_v}$.*

Proof. The "only if" direction is clear, by the definition and the multiplicativity property of the Hilbert symbol. For the other direction, let us assume that $(-1, \alpha)_{k_v} = (-1, \beta)_{k_v}$. By multiplicativity, this is equivalent to $(-1, \alpha\beta)_{k_v} = 1$. By the definition of the Hilbert symbol, this implies that there is a non-trivial k_v-solution $(\vec{z_v}, t_v)$ to the equation $N_{k(\sqrt{-1})/k}(\vec{z}) = \alpha\beta t^2$. If we can show that $t_v \neq 0$, then clearly $\vec{z_v}/t_v$ is a k_v-solution to $N_{k(\sqrt{-1})/k}(\vec{z}) = \alpha\beta$. By hypothesis, there exists some $\gamma \in k_v^\times$ with $(-1, \gamma)_{k_v} = -1$. Hence, there is no non-trivial k_v-solution to $N_{k(\sqrt{-1})/k}(\vec{z}) = \gamma t^2$. In particular, there is no non-trivial k_v-solution to $N_{k(\sqrt{-1})/k}(\vec{z}) = 0$. This is sufficient to rule out the case $t_v = 0$ in our solution $(\vec{z_v}, t_v)$ to $N_{k(\sqrt{-1})/k}(\vec{z}) = \alpha\beta t^2$. ☐

Remark 4.16. If $k \in \mathscr{K}$ holds, then Lemma 4.15 applies to all $v \in \Omega_k$ with the exception of $v \in \Omega_k^{\mathbf{C}}$ and $v \in \Omega_k^{\text{odd}}$ with $\operatorname{red}_v(-1) \in \mathbf{F}_v^2$.

The following follows from the theory of local fields.

Lemma 4.17. *Let k be a number field and let $v \in \Omega_k^{\text{odd}}$. Then $k_v(\sqrt{-1})/k_v$ is unramified. Consequently, $N_{k_v(\sqrt{-1})/k_v} : \mathcal{O}_{k_v(\sqrt{-1})}^\times \to \mathcal{O}_{k_v}^\times$ is a surjective homomorphism.*

Proposition 4.18. *Let $k \in \mathscr{K}$, $f(x) := \sum_{i=0}^{n} f_i x^i \in \mathcal{O}_k[x]$, and $(\lambda, \mu, \nu) \in \mathcal{O}_k \times \mathcal{O}_k \times \mathbf{Z}_{\geq 1}$ satisfy Conditions* (**ELS**). *Let X be the generalised Châtelet surface over k associated to $X_0 : N_{k(\sqrt{-1})/k}(\vec{z}) = f(x)(\lambda + \mu(f(x))^\nu)$. Then $X(\mathbf{A}_k) \neq \emptyset$.*

Proof. It suffices to show that $X_0(k_v) \neq \emptyset$ for all $v \in \Omega_k$, since then $\emptyset \neq \prod_{v \in \Omega_k} X_0(k_v) \subset \prod_{v \in \Omega_k} X(k_v) = X(\mathbf{A}_k)$. For $v \in \Omega_k^{\mathbf{C}}$, it is clear that $X_0(k_v) \neq \emptyset$. For $v \in \Omega_k^{\mathbf{R}} \cup \Omega_k^{\text{odd}_{QNR}} \cup \Omega_k^{\text{even}}$ we use Lemma 4.15. More precisely, if $\Omega_k^{\mathbf{R}} \neq \emptyset$, then for $v \in \Omega_k^{\mathbf{R}}$ there exists, by assumption, some $x_v \in k_v$ such that $f(x_v), \lambda + \mu f(x_v)^\nu \in k_v^\times$ and $(-1, f(x_v)(\lambda + \mu f(x_v)^\nu))_{k_v} = 1$. For $v \in \Omega_k^{\text{odd}_{QNR}}$, we can choose some $x_v \in k_v$ with $v(x_v) < 0$ and $f(x_v), \lambda + \mu f(x_v)^\nu \in k_v^\times$. Then, using that $f, \lambda + \mu f^\nu \in \mathcal{O}_k[x]$, that $v(f_n) = v(\mu f_n^\nu) = 0$, and that n is even, we deduce that both $v(f(x_v))$ and $v(\lambda + \mu f(x_v)^\nu)$ are even. By Lemma 2.2, it follows that $(-1, f(x_v))_{k_v} = (-1, \lambda + \mu f(x_v)^\nu)_{k_v}$.

For $v \in \Omega_k^{\text{even}}$, we take $x_v = 0$. Then, by assumption, we have $(-1, f_0)_{k_v} = (-1, \lambda + \mu f_0^\nu)_{k_v}$. In order to prove that $X_0(k_v)$ for $v \in \Omega_k^{\text{odd}_{QR}}$, we take the $x_v \in k_v$ that, by assumption, has $v(f(x_v)(\lambda + \mu f(x_v)^\nu))$ even and use Lemma 4.17. □

Proposition 4.19. *Let $k \in \mathscr{K}$, $f(x) := \sum_{i=0}^n f_i x^i \in \mathcal{O}_k[x]$, and $(\lambda, \mu, \nu) \in \mathcal{O}_k \times \mathcal{O}_k \times \mathbf{Z}_{\geq 1}$ satisfy Conditions* (**ELS-**f)*,* (**ELS-odd$_{\mathbf{QNR}}$**)*,* (**Br-**k**-**f)(2),(3)*, and* (**Br-R**)*. Let X be the generalised Châtelet surface over k associated to $X_0 : N_{k(\sqrt{-1})/k}(\vec{z}) = f(x)(\lambda + \mu(f(x))^\nu)$. Let $\mathscr{A} := [(-1, f)_{k(X)}] = [(-1, \lambda + \mu f^\nu)_{k(X)}] \in \operatorname{Br} X$. Then for any $\boldsymbol{x}_v \in X_0(k_v)$ and for any $v \notin \Omega_k^{\text{even}}$, we have $\operatorname{inv}_v \mathscr{A}(\boldsymbol{x}_v) = 0$.*

Proof. Since $\operatorname{inv}_v \mathscr{A} : X(k_v) \to \mathbf{Q}/\mathbf{Z}$ is continuous for the local topology for any $v \in \Omega_k$, by deforming locally if necessary we may assume without loss of generality that $f(x_v), \lambda + \mu f(x_v)^\nu \neq 0$ for all $v \notin \Omega_k^{\text{even}}$. Since $\mathscr{A}(\mathbf{x}_v) = [(f(x_v), -1)_{k_v}] = [(\lambda + \mu f(x_v)^\nu, -1)_{k_v}]$, it suffices to compute the Hilbert symbols $(f(x_v), -1)_{k_v} = (\lambda + \mu f(x_v)^\nu, -1)_{k_v}$ for each $v \in \Omega_k$.

If $v \in \Omega_k^{\mathbf{C}}$, then $(f(x_v), -1)_{k_v} = 1$, as $-1 \in k_v^2$. Hence, $\operatorname{inv}_v \mathscr{A}(\mathbf{x}_v) = 0$. If $\Omega_k^{\mathbf{R}} \neq \emptyset$ and $v \in \Omega_k^{\mathbf{R}}$, then $f(x_v) > 0$ in k_v: if $f(x_v) < 0$, then by Condition (**Br-R**) we would have $\lambda + \mu f(x_v)^\nu > 0$, and thus that $(f(x_v), -1)_{k_v} \neq (\lambda + \mu f(x_v)^\nu, -1)_{k_v}$, which in turn, by the correspondence between Hilbert symbols and quaternion algebras, would give $\operatorname{inv}_v \mathscr{A}(\mathbf{x}_v) = 1/2$ and $\operatorname{inv}_v \mathscr{A}(\mathbf{x}_v) = 0$, a contradiction. Hence, $f(x_v) > 0$ and $\operatorname{inv}_v \mathscr{A}(\mathbf{x}_v) = 0$.

If $v \in \Omega_k^{\text{odd}_{\text{QR}}}$, then by Lemma 2.2 we have $(f(x_v), -1)_{k_v} = 1$ and thus $\operatorname{inv}_v \mathscr{A}(\mathbf{x}_v) = 0$. If $v \in \Omega_k^{\text{odd}_{\text{QNR}}}$, we need to distinguish some cases. We write $x_v = \pi^\alpha u$, where π is a uniformiser, $\alpha \in \mathbf{Z}$, and $u \in \mathcal{O}_{k_v}^\times$. If $\alpha < 0$, we have already seen in the proof of Proposition 4.18 that $(f(x_v), -1)_{k_v} = 1$ and thus $\operatorname{inv}_v \mathscr{A}(\mathbf{x}_v) = 0$. If $\alpha > 0$, then it easy to see that $v(f(x_v)) > 0$ if and only if $v(f_0) > 0$, and similarly for $\lambda + \mu f(x_v)$. If $v(f_0) > 0$, then $v(\lambda + \mu f_0^\nu) = 0$ since $v(\lambda) = 0$. Hence, either $v(f_0) = 0$ or $v(\lambda + \mu f_0^\nu) = 0$, meaning that either $v(f(x_v)) = 0$ or $v(\lambda + \mu f(x_v)) = 0$. Using $(f(x_v), -1)_{k_v} = (\lambda + \mu f(x_v), -1)_{k_v}$ and Lemma 2.2, we deduce that $\operatorname{inv}_v \mathscr{A}(\mathbf{x}_v) = 0$. Finally, if $\alpha = 0$, then $v(f(x_v))$ is even: if $v(f(x_v)) > 0$ were odd, then $v(\lambda + \mu f(x_v)^\nu) = 0$ as $v(\lambda) = 0$ and we would have by Lemma 2.2 that $(f(x_v), -1)_{k_v} \neq (\lambda + \mu f(x_v)^\nu, -1)_{k_v}$, which in turn would give at the same time $\operatorname{inv}_v \mathscr{A}(\mathbf{x}_v) = 1/2$ and $\operatorname{inv}_v \mathscr{A}(\mathbf{x}_v) = 0$, a contradiction. Hence, by Lemma 2.2, $\operatorname{inv}_v \mathscr{A}(\mathbf{x}_v) = 0$. □

Proposition 4.20. *Let $k \in \mathscr{K}$, $f(x) := \sum_{i=0}^n f_i x^i \in \mathcal{O}_k[x]$, and $(\lambda, \mu, \nu) \in \mathcal{O}_k \times \mathcal{O}_k \times \mathbf{Z}_{\geq 1}$ satisfy Conditions* (**ELS-**f)*,* (**Br-**k**-**f)*, and either* (**Br-even**) *or* (**HP-even**)*. Let X be the generalised Châtelet surface over k associated to $X_0 : N_{k(\sqrt{-1})/k}(\vec{z}) = f(x)(\lambda + \mu(f(x))^\nu)$. Let $\mathscr{A} := [(-1, f)_{k(X)}] = [(-1, \lambda + \mu f^\nu)_{k(X)}] \in \operatorname{Br} X$. Then for any $\boldsymbol{x}_v \in X_0(k_v)$ and for any $v \in \Omega_k^{\text{even}}$, we have*

$$\operatorname{inv}_v \mathscr{A}(\boldsymbol{x}_v) = \begin{cases} 0 & \text{if } (\textbf{HP-even}) \text{ holds,} \\ \frac{1}{2} & \text{if } (\textbf{Br-even}) \text{ holds.} \end{cases}$$

Proof. Since $\operatorname{inv}_v \mathscr{A} : X(k_v) \to \mathbf{Q}/\mathbf{Z}$ is continuous for the local topology, we may assume without loss of generality that $f(x_v), \lambda + \mu f(x_v)^\nu \neq 0$ for all $v \in \Omega_k^{\text{even}}$. Since $\mathscr{A}(\mathbf{x}_v) = [(f(x_v), -1)_{k_v}] = [(\lambda + \mu f(x_v)^\nu, -1)_{k_v}]$, it suffices to compute the Hilbert symbols $(f(x_v), -1)_{k_v} = (\lambda + \mu f(x_v)^\nu, -1)_{k_v}$ for each $v \in \Omega_k^{\text{even}}$.

We give the proof for when (**Br-even**) holds; the case for (**HP-even**) is similar. We write $x_v = 2^\alpha u$, for some $\alpha \in \mathbf{Z}$ and $u \in \mathcal{O}_{k_v}^\times$. If $\alpha > 0$, then by using (**Br-even**)(1),(2) we have $f(x_v) - f_0 \in 4\mathcal{O}_{k_v}$ and thus that $f(x_v) \in \overline{U}_{k_v}^2$. Hence, by Lemma 2.6 we have $\operatorname{inv}_v \mathscr{A}(\mathbf{x}_v) = 1/2$. If $\alpha < 0$, then by by using (**Br-even**)(1),(2) we have $2^{-n\alpha} u^{-n} f(x_v) - f_n \in 4\mathcal{O}_{k_v}$ and thus that $2^{-n\alpha} u^{-n} f(x_v) \in \overline{U}_{k_v}^2$. Hence, by Lemma 2.6 we have $\operatorname{inv}_v \mathscr{A}(\mathbf{x}_v) = 1/2$. Finally, if $\alpha = 0$, then we write $x_v = \epsilon u_1$ for some $\epsilon \in \mu_{2^{[k_v:\mathbf{Q}_2]}-1}(k_v)$ and some $u_1 \in U_{k_v}^1$. In this case, by (**Br-**k**-**f)

we have that $f(x_v) - \sum_{i=0}^n f_i \in 4\mathcal{O}_{k_v}$, since $f_i \in 4\mathcal{O}_k$ whenever $\chi_k \not| i$ (and for those indices i such that $\chi_k | i$, we have $f_i x_v^i = f_i(\epsilon u_1)^i = f_i u_1^i$). Depending on the value of $\sum_{i=0}^n f_i \bmod 4\mathcal{O}_{k_v}$, using the relevant assumptions in (**Br-even**) it is easy to check that either $f(x_v) = 2^{v(f(x_v))}u'$ with $u' \in \overline{U}_{k_v}^2$ or $\lambda + \mu f(x_v)^\nu = 2^{v(\lambda+\mu f(x_v)^\nu)}u''$ with $u'' \in \overline{U}_{k_v}^2$. In any case, by Lemma 2.6 we have $\mathrm{inv}_v \mathscr{A}(\mathbf{x}_v) = 1/2$. $\square$

Proof of Theorem 4.12. By Proposition 4.18, $X(\mathbf{A}_k) \neq \emptyset$. Now let $(\mathbf{x}_v) \in X(\mathbf{A}_k)$. We want to show that $(\mathbf{x}_v) \notin X(\mathbf{A}_k)^{\mathscr{A}}$, where $\mathscr{A} := [(-1, f)_{k(X)}] \in \operatorname{Br} X$ (cf. §3). Since X is smooth, by the Implicit Function Theorem we have that $X_0(k_v)$ is dense in $X(k_v)$ for the local topology, for any $v \in \Omega_k$. Since moreover $\mathrm{inv}_v \mathscr{A} : X(k_v) \to \mathbf{Q}/\mathbf{Z}$ is continuous for the local topology for any $v \in \Omega_k$, by deforming locally if necessary we may assume without loss of generality that $\mathbf{x}_v \in X_0(k_v)$ for all $v \in \Omega_k$. By Propositions 4.19 and 4.20, $\mathrm{inv}_v(\mathbf{x}_v) = 0$ if $v \notin \Omega_k^{\mathrm{even}}$ and $\mathrm{inv}_v(\mathbf{x}_v) = 1/2$ if $v \in \Omega_k^{\mathrm{even}}$. Since by assumption $|\Omega_k^{\mathrm{even}}|$ is odd, it follows that

$$\sum_{v\in\Omega_k} \mathrm{inv}_v \mathscr{A}(\mathbf{x}_v) = \frac{1}{2},$$

implying that $(\mathbf{x}_v) \notin X(\mathbf{A}_k)^{\mathscr{A}}$. Hence, $X(\mathbf{A}_k)^{\mathscr{A}} = \emptyset$, which implies that $X(\mathbf{A}_k)^{\mathrm{Br}} = \emptyset$. $\square$

Proof of Theorem4.14. By Proposition 4.18, $X(\mathbf{A}_k) \neq \emptyset$. Now let $(\mathbf{x}_v) \in X(\mathbf{A}_k)$. We want to show that $(\mathbf{x}_v) \in X(\mathbf{A}_k)^{\mathrm{Br}} = X(\mathbf{A}_k)^{\mathscr{A}}$, where $\mathscr{A} := [(-1, f)_{k(X)}] \in \operatorname{Br} X$ (cf.§3). Since X is smooth, by the Implicit Function Theorem we have that $X_0(k_v)$ is dense in $X(k_v)$ for the local topology, for any $v \in \Omega_k$. Since moreover $\mathrm{inv}_v \mathscr{A} : X(k_v) \to \mathbf{Q}/\mathbf{Z}$ is continuous for the local topology for any $v \in \Omega_k$, by deforming locally if necessary we may assume without loss of generality that $\mathbf{x}_v \in X_0(k_v)$ for all $v \in \Omega_k$. By Propositions 4.19 and 4.20, $\mathrm{inv}_v(\mathbf{x}_v) = 0$ for all $v \in \Omega_k$. Hence,

$$\sum_{v\in\Omega_k} \mathrm{inv}_v \mathscr{A}(\mathbf{x}_v) = 0,$$

implying that $(\mathbf{x}_v) \in X(\mathbf{A}_k)^{\mathscr{A}}$. Since $(\mathbf{x}_v) \in X(\mathbf{A}_k)$ was arbitrary, it follows that $X(\mathbf{A}_k) = X(\mathbf{A}_k)^{\mathrm{Br}}$. Finally, we remark that $\operatorname{Br} X / \operatorname{Br}_0 X \neq 0$, as it is generated by the non-constant element $\mathscr{A}$ (modulo constants). $\square$

5. Proofs of Theorem 1.7 and Theorem 1.8

As corollaries of Theorems 4.12 and 4.14, we can now prove Theorems 1.7 and 1.8.

Proof of Theorem 1.7. If $N/2$ is even and $\neq 2$, let $(\lambda, \mu, \nu) := (1, -1, 1)$ and let $f(x) := -x^{N/2} + 4x\sum_{i=0}^{N/4-1} f_{2i+1}x^{2i} + \sum_{i=1}^{N/4-1} f_{2i}x^{2i} + 3 \in \mathbf{Z}[x]$, where $\sum_{i=1}^{N/4-1} f_{2i} \equiv 1 \pmod{4\mathbf{Z}}$.

If $N/2$ is odd and not equal to 3, then we let p be any prime dividing $N/2$. Since, by assumption, $N/2$ is not a prime, it follows that $N/(2p) \neq 1$. In this case, we let $(\lambda, \mu, \nu) := (2, 1, N/(2p) - 1)$ and let $f(x) := -x^{2p} + 4x\sum_{i=0}^{p-1} f_{2i+1}x^{2i} + \sum_{i=1}^{p-1} f_{2i}x^{2i} + 3 \in \mathbf{Z}[x]$, where $\sum_{i=1}^{p-1} f_{2i} \equiv 1 \pmod{4\mathbf{Z}}$.

If $N = 4$, then we let $(\lambda, \mu, \nu) := (1, -1, 1)$ and let $f(x) := -x^2 + 4f_1x + 3 \in \mathbf{Z}[x]$.

If $N = 6$, then we let $(\lambda, \mu, \nu) := (2, 1, 2)$ and let $f(x) := -x^2 + 4f_1x + 3 \in \mathbf{Z}[x]$.

In any case, we let the coefficients f_i be such that both f and $\lambda + \mu f^\nu$ have no roots over $\mathbf{Q}$. Let X be the generalised Châtelet surface over $\mathbf{Q}$ with affine equation given by

$$N_{k(\sqrt{-1})/k}(\vec{z}) = f(x)(\lambda + \mu(f(x))^\nu).$$

By construction, $\deg(f(\lambda + \mu f^\nu)) = N$. It is clear that the conditions on $f(x)$ and $\lambda + \mu(f(x))^\nu$ in the statement of Theorem 4.12 are satisfied. For the real place $v = \infty$, we just note that $(-1) \cdot 3 < 0$ in $\mathbf{R}$, which implies that f has a root in $\mathbf{R}$. Near such a root, we can find an $\tilde{x} \in \mathbf{R}$ such that $f(\tilde{x})(\lambda + \mu(f(x))^\nu) > 0$. Hence, we can apply Theorem 4.12 to deduce that $X(\mathbf{A}_\mathbf{Q}) \neq \emptyset$ and $X(\mathbf{A}_\mathbf{Q})^{\mathrm{Br}} = \emptyset$. $\square$

Proof of Theorem 1.8. If $N \equiv 0 \pmod 4$, let $(\lambda,\mu,\nu) := (1,-8,1)$ and let $f(x) := x^{N/2} + 4\sum_{i=0}^{N/2-1} f_i x^i - 3 \in \mathbf{Z}[x]$.

If $N \equiv 2 \pmod 4$, let $(\lambda,\mu,\nu) := (1,4,(N-2)/2)$ and let $f(x) := x^2 + 4f_1x - 3 \in \mathbf{Z}[x]$.

In any case, we let the coefficients f_i be such that $f(\lambda+\mu f^\nu)$ is separable, f and $\lambda+\mu f^\nu$ are both irreducible over $\mathbf{Q}$, and $\mathbf{Q}(\sqrt{-1})/\mathbf{Q}$ is not a subfield of the splitting fields of f and $\lambda+\mu f^\nu$ over $\mathbf{Q}$. Let X be the generalised Châtelet surface over $\mathbf{Q}$ associated to

$$N_{k(\sqrt{-1})/k}(\vec{z}) = f(x)(\lambda+\mu(f(x))^\nu).$$

By construction, $\deg(f(\lambda+\mu f^\nu)) = N$. It is clear that the conditions on $f(x)$ and $\lambda+\mu(f(x))^\nu$ in the statement of Theorem 4.12 are satisfied. For the real place $v=\infty$, we just note that $1\cdot(-3) < 0$ in $\mathbf{R}$, which implies that f has a root in $\mathbf{R}$. Near such a root, we can find an $\tilde{x} \in \mathbf{R}$ such that $f(\tilde{x})(\lambda+\mu(f(x))^\nu) > 0$. Hence, we can apply Theorem 4.12 to deduce that $X(\mathbf{A_Q}) = X(\mathbf{A_Q})^{\mathrm{Br}} \neq \emptyset$ and $\operatorname{Br} X/\operatorname{Br}_0 X \neq 0$. □

6. Many-parameters examples over a field $k \neq \mathbf{Q}$

Example 6.1. Let $k := \mathbf{Q}(\alpha)$ be the totally real cubic Galois extension where α satisfies $\alpha^3 - \alpha^2 - 2\alpha + 1 = 0$. Then $k \in \mathscr{K}$ and k has class number equal to 1 (so Eisenstein's criterion works over k). For any $v \in \Omega_k^{\text{even}}$, we have $f := f_v := [\mathbf{F}_v : \mathbf{F}_2] = [k:\mathbf{Q}] = 3$. Consequently, the number χ_k appearing in Condition (**Br**-k-f)(1) is $\chi_k = 2^3 - 1 = 7$. For any $n \equiv 0 \pmod{28}$, we let $f_{n,\mathbf{f}}(x) := -x^{n/2} + 12\sum_{i=1}^{n/14-1} f_{7i}x^{7i} + 3 \in \mathbf{Z}[x]$ and let $X_{n,\mathbf{f}}$ be the generalised Châtelet surface associated to

$$N_{k(\sqrt{-1})/k}(\vec{z}) = f_{n,\mathbf{f}}(x)(1 - f_{n,\mathbf{f}}(x)).$$

We note that the rational primes 2 and 3 are inert in k. By Eisenstein's criterion for 3 we deduce that $f_{n,\mathbf{f}}(x)$ is irreducible over k. Similarly, by Eisenstein's criterion for 2 we deduce that $1 - f_{n,\mathbf{f}}(x)$ is irreducible over k. Apart from those $\mathbf{f} \in \mathbf{Z}^{n/2-1}$ for which $f_{n,\mathbf{f}}(1-f_{n,\mathbf{f}})$ is not separable, it is easy to check that all the other hypotheses of Theorem 4.12 are satisfied. Hence, the family $\{X_{n,\mathbf{f}}\}_{\mathbf{f}\in\mathbf{Z}^{n/2-1}}$ of generalised Châtelet surfaces over k is such that $X_{n,\mathbf{f}}(\mathbf{A}_k) \neq \emptyset$ and $X_{n,\mathbf{f}}(k) \subseteq X_{n,\mathbf{f}}(\mathbf{A}_k)^{\mathrm{Br}} = \emptyset$, for all $\mathbf{f} \in \mathbf{Z}^{n/2-1}$ such that $f_{n,\mathbf{f}}(1-f_{n,\mathbf{f}})$ is separable.

Example 6.2. Again, we let $k := \mathbf{Q}(\alpha) \in \mathscr{K}$ and compute $\chi_k = 2^3 - 1 = 7$. For any $n \equiv 0 \pmod{28}$, we let $f_{n,\mathbf{f}}(x) := x^{n/2} + 12\cdot 13\sum_{i=1}^{n/14-1} f_{7i}x^{7i} - 3 \in \mathbf{Z}[x]$ and let $X_{n,\mathbf{f}}$ be the generalised Châtelet surface associated to

$$N_{k(\sqrt{-1})/k}(\vec{z}) = f_{n,\mathbf{f}}(x)(1 - 4f_{n,\mathbf{f}}(x)).$$

We note that the rational prime 13 is totally split in k; we fix some prime $\mathfrak{p}_{13}$ above 13. By Eisenstein's criterion for 3 and $\mathfrak{p}_{13}$, we deduce that both $f_{n,\mathbf{f}}(x)$ and $1 - 4f_{n,\mathbf{f}}(x)$ are irreducible over k. Apart from those $\mathbf{f} \in \mathbf{Z}^{n/2-1}$ for which $f_{n,\mathbf{f}}(1-4f_{n,\mathbf{f}})$ is not separable or for which $\sqrt{-1}$ is contained in $\operatorname{Split}_k(f_{n,\mathbf{f}}(1-4f_{n,\mathbf{f}}))$, it is easy to check that all the other hypotheses of Theorem 4.14 are satisfied. Hence, the family $\{X_{n,\mathbf{f}}\}_{\mathbf{f}\in\mathbf{Z}^{n/2-1}}$ of generalised Châtelet surfaces over k is such that $X_{n,\mathbf{f}}(\mathbf{A}_k)^{\mathrm{Br}} = X_{n,\mathbf{f}}(\mathbf{A}_k) \neq \emptyset$, for all $\mathbf{f} \in \mathbf{Z}^{n/2-1}$ such that $f_{n,\mathbf{f}}(1-4f_{n,\mathbf{f}})$ is separable and $\sqrt{-1}$ is not contained in $\operatorname{Split}_k(f_{n,\mathbf{f}}(1-4f_{n,\mathbf{f}}))$.

7. An application: some density considerations for a family of generalised Châtelet surfaces over $\mathbf{Q}$

Consider the family of generalised Châtelet surfaces over $\mathbf{Q}$

$$\mathscr{G}_n := \{X_{\mathbf{f}} : \text{ gen. Châtelet surf. associated to } N_{\mathbf{Q}(\sqrt{-1})/\mathbf{Q}}(\vec{z}) = f(x)(1-f(x))\}_{\mathbf{f}:=(f_i)_{i=0}^{n-1}\in\mathbf{Z}^n},$$

where $f(x) := -x^n + \sum_{i=1}^{n-1} f_i x^i + f_0$. We define the counting function

$$\mathscr{N}_n^{\text{Br-HP}}(B) \ := \frac{\#\{X_{\mathbf{f}} \in \mathscr{G}_n : \max_{0\le i\le n-1}|f_i| \le B \text{ and } X_{\mathbf{f}}(\mathbf{A_Q}) \neq \emptyset, X_{\mathbf{f}}(\mathbf{A_Q})^{\text{Br}} = \emptyset\}}{\#\{X_{\mathbf{f}} \in \mathscr{G}_n : \max_{0\le i\le n-1}|f_i| \le B\}}.$$

Theorem 7.1. $\liminf_{B\to+\infty} \mathscr{N}_n^{\text{Br-HP}}(B) \geq \delta_n$*, where* $\delta_2 := 2^{-6} > 0$ *and* $\delta_n := 2^{-(n/2+6)} > 0$ *for* $n \geq 4$.

Proof. Let $X_{\mathbf{f}} \in \mathscr{G}_n$ with $\max_{0\le i\le n-1}|f_i| \le B$. Let $f_0 \in [-B, B] \cap \mathbf{Z}$ be such that $f_0 \equiv 3 \pmod 8$. Then $1 - f_0 = 2u_{1-f_0}$ where $u_{1-f_0} \in \overline{U}^2_{\mathbf{Q_2}}$. We have

$$\#\{f_0 \in [-B, B] \cap \mathbf{Z} : f_0 > 0 \text{ and } f_0 \equiv 3 \pmod 8\} = \frac{1}{8}B + O(1).$$

If $n = 2$, then we want to count e.g. the number of $f_1 \in [-B, B] \cap \mathbf{Z}$ such that $f_1 \in 4\mathbf{Z}$, which is $B/2 + O(1)$. If $n \geq 4$, then we want to count e.g. the number of $(f_i)_{i=1}^{n-1} \in ([-B, B] \cap \mathbf{Z})^{n-1}$ such that $\sum_{i=1}^{n-1} f_i \in 4\mathbf{Z}$ and f_i is even whenever i is odd. This number has a very crude lower bound of $2^{n/2-3}B^{n-1} + O(B^{n-2})$: indeed, if write $f_{2j+1} = 2g_{2j+1}$ for $g_{2j+1} \in [-B/2, B/2] \cap \mathbf{Z}$ and $j = 0, ..., n/2 - 1$, then $2\sum_{j=0}^{n/2-1} g_{2j+1} + \sum_{j=1}^{n/2-1} f_{2j} \equiv 0 \pmod 4$ can be viewed as an equation in f_2 (where all the other parameters f_j with $j \neq 2$ are free), and the value of f_2 is determined mod 4; hence, we get a lower bound of $(2B)^{n/2-2}B^{n/2}(2B/4) + O(B^{n-2})$.

The number of $(f_i)_{i=0}^{n-1} \in ([-B, B] \cap \mathbf{Z})^n$ such that $f(x)$ and $1 - f(x)$ are either not separable or have $\mathbf{Q}$-roots, or such that $f_0 = 0$ or $1 - f_0 = 0$, is negligible as $B \to +\infty$.

Hence, putting everything together, we obtain that the number of $(f_i)_{i=0}^{n-1} \in ([-B, B] \cap \mathbf{Z})^n$ such that $X_{\mathbf{f}}$ satisfies Conditions (**ELS**) and (**Br**) is at least

$$\begin{cases} 2^{-4}B^2 + O(B) & \text{if } n = 2, \\ 2^{n/2-6}B^n + O(B^{n-1}) & \text{if } n \geq 4. \end{cases}$$

By Theorem 4.12, it follows that any such $X_{\mathbf{f}}$ satisfies $X_{\mathbf{f}}(\mathbf{A_Q}) \neq \emptyset$ and $X_{\mathbf{f}}(\mathbf{A_Q})^{\text{Br}} = \emptyset$. Hence,

$$\liminf_{B\to+\infty} \mathscr{N}_n^{\text{Br-HP}}(B) \geq \begin{cases} \lim_{B\to+\infty} 2^{-4}B^2/(2^2B^2) = 2^{-6} & \text{if } n = 2, \\ \lim_{B\to+\infty} 2^{n/2-6}B^n/(2^nB^n) = 2^{-(n/2+6)} & \text{if } n \geq 4, \end{cases}$$

so we can take $\delta_2 := 2^{-6}$ and $\delta_n := 2^{-(n/2+6)}$ for $n \geq 4$. ☐

Remark 7.2. In a similar way, one can show that $\limsup_{B\to+\infty} \mathscr{N}_n^{\text{Br-HP}}(B) \leq \Delta_n$ for $\Delta_n < 1$ by considering the members of $\mathscr{G}_n$ satisfying the conditions in Theorem 4.14.

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