BRAUER-MANIN OBSTRUCTION AND FAMILIES OF GENERALISED CHÂTELET SURFACES OVER NUMBER FIELDS

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Abstract. Over infinitely many number fields k (including all finite Galois extensions $k/\mathbf{Q}$ of odd degree unramified at 2), we give general sufficient conditions in order for the generalised Châtelet surfaces X over k associated to the normic equation $N_{k(\sqrt{-1})/k}(\vec{z}) = h(x)$, where $\deg(h) \geq 4$ is even and arbitrarily large, to have the property that $X(\mathbf{A}_k)^{\mathrm{Br}} = \emptyset$ but $X(\mathbf{A}_k) \neq \emptyset$. We also give general sufficient conditions in order for the generalised Châtelet surfaces X over k of the same form as above to have the property that $X(\mathbf{A}_k)^{\mathrm{Br}} = X(\mathbf{A}_k) \neq \emptyset$ and $\mathrm{Br}\, X/\mathrm{Br}\, k \neq 0$. As an application, we prove that, for a certain family of generalised Châtelet surfaces over $\mathbf{Q}$, a positive proportion (but not 100%) of its members exhibit a violation of the Hasse principle explained by the Brauer-Manin obstruction.

1. Introduction

Let k be a number field, Ω_k its set of places, and $\mathbf{A}_k$ its adelic ring. We start by recalling some standard definitions. For $\{X_\omega\}_\omega$ a family of smooth, projective, geometrically integral varieties over k, we say that $\{X_\omega\}_\omega$ satisfies the Hasse principle if $X_\omega(\mathbf{A}_k) \neq \emptyset$ implies that $X_\omega(k) \neq \emptyset$, for all ω. By the Lang-Nishimura theorem (cf. [Lan54],[Nis55]), the Hasse principle is a birational invariant of smooth, projective, geometrically integral varieties. Now let X be a smooth, quasi-projective, geometrically integral variety over k and let $\mathrm{Br}\, X := H^2_{\text{ét}}(X, \mathbf{G}_m)$ be the (cohomological) Brauer group of X. The Brauer-Manin set of X (first introduced by Manin in [Man71]) is the set

$$X(\mathbf{A}_k)^{\mathrm{Br}} := \left\{ (x_v) \in X(\mathbf{A}_k) : \sum_{v \in \Omega_k} \mathrm{inv}_v\, \alpha(x_v) = 0 \text{ for all } \alpha \in \mathrm{Br}(X) \right\},$$

where $\mathrm{inv}_v : \mathrm{Br}(k_v) \to \mathbf{Q}/\mathbf{Z}$ are the Hasse invariant maps from local class field theory. One can check that $X(k) \subset X(\mathbf{A}_k)^{\mathrm{Br}}$. If $X(\mathbf{A}_k) \neq \emptyset$ but $X(\mathbf{A}_k)^{\mathrm{Br}} = \emptyset$, we say that X is a counterexample to the Hasse principle explained by the Brauer-Manin obstruction.

The aim of this paper is to provide large classes of examples supporting the following conjecture for the Hasse principle, which is a special case of the conjecture in [CTS80] for weak approximation.

Conjecture 1.1 (Colliot-Thélène and Sansuc)**.** Let X be a smooth, projective, geometrically rational surface over a number field k. Then $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ implies that $X(k) \neq \emptyset$.

By a theorem of Iskovskikh (cf. [Isk79, Thm 1]), any smooth, proper, geometrically rational surface over k is k-birationally equivalent to either a del Pezzo surface or a smooth conic bundle surface (or both). Since the property "$X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ implies $X(k) \neq \emptyset$" is birationally invariant for smooth, projective, and geometrically integral varieties (see e.g. [CTPS16, §6]), it follows that Conjecture 1.1 needs only be verified for del Pezzo surfaces and conic bundle surfaces. In this paper, we are concerned with the following types of (k-birational classes of) conic bundle surfaces. Let $K := k(\sqrt{d})$ be a quadratic field extension of k, and let $\{\omega_1, \omega_2\}$ be a k-basis for K. Let

MSC2010: 14G05, 14J26. *Keywords:* Brauer-Manin obstruction, Hasse principle, Châtelet surfaces, geometrically rational surfaces, rational points.

$h(x) \in k[x]$ be a separable polynomial of degree $\deg(h) \geq 2$. Let $X_0 \subset \mathbf{A}_k^3$ be given by the equation

$$N_{K/k}(\vec{z}) = h(x),$$

where $\vec{z} := \omega_1 z_1 + \omega_2 z_2$. As explained in e.g. [Sko01, §7.1], by making a change of variables in $\mathbf{A}_k^3$ if necessary we can assume without loss of generality that $\deg(h) = 2n$ for some $n \geq 1$ and $\sqrt{d}$ is not in the splitting field of h over k. A smooth proper model X of X_0 that extends the map $X_0 \to \mathbf{A}_k^1$ given by $(\vec{z}, x) \mapsto x$ to a map $X \to \mathbf{P}_k^1$ can be constructed as follows (see e.g. [VAV12, §2]). Let $\mathcal{E} := (\oplus_{i=1}^2 \mathcal{O}_{\mathbf{P}_k^1}) \oplus \mathcal{O}_{\mathbf{P}_k^1}(n)$ be a vector sheaf on $\mathbf{P}_k^1$ of rank 3. Let s_2 be the homogeneisation $\tilde{h}(x,t) := t^{2n} h(x/t)$ in $\Gamma(\mathbf{P}_k^1, \mathcal{O}_{\mathbf{P}_k^1}(n)^{\otimes 2})$ and let $s_1 := N_{K/k}(\vec{z}) \in \Gamma(\mathbf{P}_k^1, \mathrm{Sym}^2(\oplus_{i=1}^2 \mathcal{O}_{\mathbf{P}_k^1}))$. Then $X := \mathbb{V}(s_1 - s_2) \subset \mathbf{P}\mathcal{E}$ is a compactification of X_0. Moreover, one can check that X is smooth over k (using the fact that $h(x)$ is separable), and that X becomes rational over $k(\sqrt{d})$.

Definition 1.2. A generalised Châtelet surface over k associated to X_0 is the smooth compactification X of X_0 as above.

Assuming Schinzel's hypothesis (cf. [SS58]; see e.g. [VAV12] for the statement for number fields), Conjecture 1.1 holds for generalised Châtelet surfaces over any number field k (cf. [CTSD94]). We mention the following unconditional results in the literature. When $\deg(h) = 4$, generalised Châtelet surfaces are usually called Châtelet surfaces and their arithmetic has been studied extensively (see e.g. [CTSSD87a] and [CTSSD87b]); in particular, the full Conjecture 1.1 (for weak approximation) has been verified for Châtelet surfaces over any number field. When $\deg(h) = 6$ and $h(x) = f(x)g(x)$ is the product of two irreducible polynomials over k with $\deg(f) = 2$ and $\deg(g) = 4$, Conjecture 1.1 has been verified by Swinnerton-Dyer in [SD99]. A detailed account of these results can be found in [Sko01, §7]. For higher degrees of $h(x)$, we also mention the work [BMS14], which verifies the full Conjecture 1.1 (for weak approximation) when $h(x)$ completely splits over $\mathbf{Q}$.

We focus here on the case when $K := k(\sqrt{-1})$ and $\deg(h) \geq 4$ is even. Our aim is to give large classes of examples of generalised Châtelet surfaces for which the failure of the Hasse principle is explained by the Brauer-Manin obstruction, and for which the Brauer-Manin obstruction is empty, the Brauer group (modulo constants) is non-trivial, and the set of adelic points is non-empty (in particular, there is no Brauer-Manin obstruction to weak approximation for such examples). The first class of examples provides direct evidence towards Conjecture 1.1. The surfaces in the second class of examples conjecturally have a k-rational point and can be used, with the help of a computer algebra system, as a testing ground for Conjecture 1.1. In general, to the best of our knowledge, such general examples for large $\deg(h)$ have not yet appeared in the literature.

We now state our main results. Let Ω_k^{even} be the set of even places of k and let $\Omega_k^{\text{even}\#}$ be the subset of places v in Ω_k^{even} with $[k_v : \mathbf{Q}_2]$ odd. Let

$$\mathscr{K} := \{k \text{ number field} : \sqrt{-1} \notin k,\ \Omega_k^{\text{even}} = \Omega_k^{\text{even}\#}, \text{ and } k_v/\mathbf{Q}_2 \text{ is unramified for all } v \in \Omega_k^{\text{even}}\}.$$

Remark 1.3. Examples of number fields $k \in \mathscr{K}$ are Galois extensions $k/\mathbf{Q}$ of odd degree unramified at 2.

Our first main result is the following.

Theorem 1.4. *Let $k \in \mathscr{K}$ be such that $|\Omega_k^{\text{even}}|$ is odd. Let $f(x) := \sum_{i=0}^n f_i x^i \in \mathcal{O}_k[x]$ and $(\lambda, \mu, \nu) \in \mathcal{O}_k \times \mathcal{O}_k \times \mathbf{Z}_{\geq 1}$ satisfy Conditions* (**ELS**) *and* (**Br**) *in Section §4. Let X be the generalised Châtelet surface over k associated to $X_0 : N_{k(\sqrt{-1})/k}(\vec{z}) = f(x)(\lambda + \mu f(x)^\nu)$. Then $X(\mathbf{A}_k) \neq \emptyset$ and $X(k) \subset X(\mathbf{A}_k)^{\mathrm{Br}} = \emptyset$.*

Remark 1.5. For $k = \mathbf{Q}$, $f(x) = -x^2 + 3$, and $(\lambda, \mu, \nu) = (1, -1, 1)$, we retrieve Iskovskikh's famous counterexample to the Hasse principle (see [Isk71]).

For any $k \in \mathscr{K}$ and any even integer $N \geq 4$, we define $\mathscr{F}_{k,N}$ to be the family of generalised Châtelet surfaces X over k associated to affine varieties of the form

$$X_0 : N_{k(\sqrt{-1})/k}(\vec{z}) = f(x)(\lambda + \mu f(x)^\nu),$$

where $f(x) \in \mathcal{O}_k[x]$ has even degree, $\lambda, \mu \in \mathcal{O}_k^\times$, $\nu \in \mathbf{Z}_{\geq 1}$, and $\deg(f \cdot (\lambda + \mu f^\nu)) = N$. Specialising to $k = \mathbf{Q}$, we obtain the following corollary to Theorem 1.4.

Corollary 1.6. *Let $N \geq 4$ be any even number such that $N/2$ is not an odd prime ≥ 5. Then there exist infinitely many generalised Châtelet surfaces $X \in \mathscr{F}_{\mathbf{Q},N}$ such that $X(\mathbf{A_Q}) \neq \emptyset$ and $X(\mathbf{A_Q})^{\mathrm{Br}} = \emptyset$.*

For any variety X over k, we let $\mathrm{Br}_0\, X := \mathrm{im}(\mathrm{Br}\, k \to \mathrm{Br}\, X)$, where $\mathrm{Br}\, k \to \mathrm{Br}\, X$ is the natural morphism induced by the structure morphism $X \to \mathrm{Spec}\, k$. Our second main theorem is the following.

Theorem 1.7. *Let $k \in \mathscr{K}$, $f(x) := \sum_{i=0}^n f_i x^i \in \mathcal{O}_k[x]$, and $(\lambda, \mu, \nu) \in \mathcal{O}_k \times \mathcal{O}_k \times \mathbf{Z}_{\geq 1}$ satisfy Conditions* (**ELS**) *and* (**HP**) *in Section §4. Let X be the generalised Châtelet surface over k associated to $X_0 : N_{k(\sqrt{-1})/k}(\vec{z}) = f(x)(\lambda + \mu f(x)^\nu)$. Then $\mathrm{Br}\, X/\mathrm{Br}_0\, X \neq 0$ and $X(\mathbf{A}_k)^{\mathrm{Br}} = X(\mathbf{A}_k) \neq \emptyset$.*

Specialising to $k = \mathbf{Q}$, we obtain the following corollary.

Corollary 1.8. *Let $N \geq 4$ be any even number. Then there exist infinitely many generalised Châtelet surfaces $X \in \mathscr{F}_{\mathbf{Q},N}$ such that $X(\mathbf{A_Q}) = X(\mathbf{A_Q})^{\mathrm{Br}} \neq \emptyset$ and $\mathrm{Br}\, X/\mathrm{Br}_0\, X \neq 0$.*

Remark 1.9. The generalised Châtelet surfaces constructed in Corollary 1.8 don't usually have any "obvious" rational points.

As an application of Theorems 1.4 and 1.7, one can prove positive density results of the following kind: for a certain family of generalised Châtelet surfaces over $\mathbf{Q}$, a positive proportion (but not 100%) of its members exhibit a violation of the Hasse principle explained by the Brauer-Manin obstruction (see Theorem 7.1 and Remark 7.2 for more details).

Structure of the paper. In §§2, 3, we recall some results useful for computing the Brauer group of generalised Châtelet surfaces and the Hilbert symbol. In §4, we prove the main theorems of this paper, namely Theorems 1.4 and 1.7; their proof is not difficult, but rather computational. In §5, we specialise to $k = \mathbf{Q}$ and prove Corollaries 1.6 and 1.8. In §6, we give some examples of many-parameters families of generalised Châtelet surfaces over a number field $k \neq \mathbf{Q}$ satisfying the conditions of Theorems 1.4 and 1.7. We conclude in §7 with an application (Theorem 7.1 and Remark 7.2) of Theorems 1.4 and 1.7.

General notation. We fix once and for all an algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$, and we take any algebraic extension of $\mathbf{Q}$ to be inside $\overline{\mathbf{Q}}$. Let k be a number field. We denote by Ω_k the set of places of k, and by k_v the completion of k at the place $v \in \Omega_k$. We denote by $\Omega_k^{\mathbf{R}}$ and $\Omega_k^{\mathbf{C}}$ the real and complex places of k, respectively, and we denote by Ω_k^{even} and Ω_k^{odd} the finite places of k above the rational prime 2 and above odd rational primes, respectively. If $v \in \Omega_k$ is a finite place, we write $\mathbf{F}_v := \mathcal{O}_{k_v}/\mathfrak{m}_v$ for the residue field at v, where $\mathfrak{m}_v$ is the maximal ideal of $\mathcal{O}_{k_v}$; we write $\mathrm{red}_v : \mathcal{O}_{k_v} \to \mathbf{F}_v$ for the reduction map. We further denote by $\Omega_k^{\mathrm{odd_{QR}}}$ and $\Omega_k^{\mathrm{odd_{QNR}}}$ the subsets of places $v \in \Omega_k^{\mathrm{odd}}$ such that $\mathrm{red}_v(-1)$ is and is not, respectively, a square in $\mathbf{F}_v$, and by $\Omega_k^{\mathrm{even}\#}$ the subset of places $v \in \Omega_k^{\mathrm{even}}$ with $[k_v : \mathbf{Q}_2]$ odd. If $f \in k[x]$, we denote by $\mathrm{Split}_k(f)$ the splitting field of f over k. If X is a variety over a number field k, we denote by $\overline{X}$ the base-change of X to $\overline{Q}$.

Acknowledgements. The author would like to thank Efthymios Sofos for some helpful comments and the anonymous referee for the very useful suggestions that greatly improved the paper. The author is financially supported by a Postdoctoral Fellowship from the Max-Planck-Institut für Mathematik in Bonn.

2. The Hilbert symbol

We recall, for convenience, some of the explicit formulas for computing the Hilbert symbol. For $v \in \Omega_k^{\mathrm{odd}}$ and $a, b \in k_v^\times$, we have (see e.g. [Ser79, Chap. XIV,§3])

$$(a,b)_{k_v} = \left(\mathrm{red}_v\left((-1)^{v(a)v(b)}\frac{a^{v(b)}}{b^{v(a)}}\right)\right)^{\frac{\#\mathbf{F}_v - 1}{2}}.$$

Remark 2.1. If $c \in \mathbf{F}_v^\times$, then $c^{\frac{\#\mathbf{F}_v-1}{2}}$ is the Legendre symbol, which is equal to 1 if and only if c is a square in $\mathbf{F}_v$.

In particular, letting $a = -1$ yields the following.

Lemma 2.2. *Let $v \in \Omega_k^{\mathrm{odd}}$. For $b \in k_v^\times$, we have $(-1,b)_{k_v} = -1$ if and only if $v(b)$ is odd and* $\mathrm{red}_v(-1)$ *is not a square in $\mathbf{F}_v$.*

Now let $v \in \Omega_k^{\mathrm{even}}$. Let $a \in k_v^\times$ and $b \in \mathbf{Q}_2^\times$, and write $N_{k_v/\mathbf{Q}_2}(a) = 2^\alpha u_a$ and $b = 2^\beta u_b$, where $u_a, u_b \in \mathcal{O}_{\mathbf{Q}_2}^\times$. By [Ben73, Theorem 1], the Hilbert symbol for k_v descends to the Hilbert symbol for $\mathbf{Q}_2$ as

$$(a,b)_{k_v} = (N_{k_v/\mathbf{Q}_2}(a), b)_{\mathbf{Q}_2} = (-1)^{\epsilon(u_a)\epsilon(u_b)+\alpha\omega(u_b)+\beta\omega(u_a)}, \tag{2.1}$$

where $\epsilon(x) := \frac{x-1}{2} \pmod 2$, $\omega(x) := \frac{x^2-1}{8} \pmod 2$, and where the right-most equality in (2.1) follows from the well-known formula of the Hilbert symbol for $\mathbf{Q}_2$. In particular, letting $b = -1$ yields the following.

Lemma 2.3. *Let $v \in \Omega_k^{\mathrm{even}}$. For $a \in k_v^\times$, we have $(a,-1)_{k_v} = (N_{k_v/\mathbf{Q}_2}(a), -1)_{\mathbf{Q}_2} = (-1)^{\epsilon(u_a)}$.*

Let $v \in \Omega_k$ be a finite place. We briefly recall some results about the structure of the group of units of k_v. For any integer $m \geq 1$, we define the group of m-principal units of k_v to be

$$U_{k_v}^m := 1 + \mathfrak{m}_v^m,$$

and we define the set

$$\overline{U}_{k_v}^m := -1 + \mathfrak{m}_v^m.$$

For any integer $r \geq 1$, we denote by $\mu_r(k_v)$ the subgroup of $\mathcal{O}_{k_v}^\times$ consisting of all r-th roots of unity in k_v. From these definitions, we immediately have the following.

Lemma 2.4. $\mathcal{O}_{k_v}^\times / U_{k_v}^1 = \mathbf{F}_v^\times$. *Consequently, we can write any unit $u \in \mathcal{O}_{k_v}^\times$ as $u = \epsilon u_1$, where $\epsilon \in \mu_{\#\mathbf{F}_v - 1}(k_v)$ and $u_1 \in U_{k_v}^1$.*

Remark 2.5. $\mathcal{O}_{\mathbf{Q}_2}^\times = U_{\mathbf{Q}_2}^1$.

Lemma 2.6. *Suppose that $k_v/\mathbf{Q}_2$ is an unramified (Galois) extension and let $a = 2^{v(a)}u \in k_v^\times$, where $u \in \mathcal{O}_{k_v}^\times$. Write $u = \epsilon u_1 t^2$, where $\epsilon \in \mu_{2^{[k_v:\mathbf{Q}_2]}-1}(k_v)$ and $u_1 \in U_{k_v}^1$ are square-free, and $t \in \mathcal{O}_{k_v}^\times$. If $[k_v : \mathbf{Q}_2]$ is odd and $u_1 \in \overline{U}_{k_v}^2$, then $(-1,a)_{k_v} = -1$.*

Proof. First, we note that $N_{k_v/\mathbf{Q}_2}(a) = N_{k_v/\mathbf{Q}_2}(2^{v(a)}u) = 2^{[k_v:\mathbf{Q}_2]v(a)}N_{k_v/\mathbf{Q}_2}(u)$. By Lemma 2.3, it follows that

$$(-1,a)_{k_v} = (-1, N_{k_v/\mathbf{Q}_2}(a))_{\mathbf{Q}_2} = (-1, N_{k_v/\mathbf{Q}_2}(u))_{\mathbf{Q}_2}.$$

Let $f_v := [\mathbf{F}_v : \mathbf{F}_2] = [k_v : \mathbf{Q}_2]$ be the residue degree. Since $N_{k_v/\mathbf{Q}_2}(t^2)$ is a square, by the properties of the Hilbert symbol we have

$$(-1, N_{k_v/\mathbf{Q}_2}(u))_{\mathbf{Q}_2} = (-1, N_{k_v/\mathbf{Q}_2}(\epsilon u_1))_{\mathbf{Q}_2}.$$

Since $2^{f_v}-1$ is odd and $\epsilon^{2^{f_v}-1} = 1$, by the multiplicativity property of the Hilbert symbol it follows that $(-1, N_{k_v/\mathbf{Q}_2}(\epsilon))_{\mathbf{Q}_2} = ((-1, N_{k_v/\mathbf{Q}_2}(\epsilon))_{\mathbf{Q}_2})^{2^{f_v}-1} = 1$. Hence, by the multiplicativity property of the Hilbert symbol we have

$$(-1, N_{k_v/\mathbf{Q}_2}(\epsilon u_1))_{\mathbf{Q}_2} = (-1, N_{k_v/\mathbf{Q}_2}(u_1))_{\mathbf{Q}_2}.$$

It remains to compute $(-1, N_{k_v/\mathbf{Q}_2}(u_1))_{\mathbf{Q}_2}$. If $u_1 \in \overline{U}^2_{k_v}$, then we can write $u_1 = -1 + 4h$ for some $h \in \mathcal{O}_{k_v}$. Then

$$\begin{aligned} N_{k_v/\mathbf{Q}_2}(u_1) &= \textstyle\prod_{\sigma\in\mathrm{Gal}(k_v/\mathbf{Q}_2)} \sigma(u_1) \\ &= \textstyle\prod_{\sigma\in\mathrm{Gal}(k_v/\mathbf{Q}_2)} \sigma(-1+4h) \\ &= \textstyle\prod_{\sigma\in\mathrm{Gal}(k_v/\mathbf{Q}_2)} (-1+4\sigma(h)) \\ &= (-1)^{[k_v:\mathbf{Q}_2]} + 4h', \end{aligned}$$

for some $h' \in \mathcal{O}_{\mathbf{Q}_2}$. If $[k_v : \mathbf{Q}_2]$ is odd, it follows that $N_{k_v/\mathbf{Q}_2}(u_1) \in \overline{U}^2_{k_v}$. Hence, by Lemma 2.3 we obtain $(-1, a)_{k_v} = (-1)^{\epsilon(N_{k_v/\mathbf{Q}_2}(u_1))} = -1$, as required. □

Remark 2.7. If $u_1 \in U^2_{k_v}$, then a similar proof as the above yields $(-1, a)_{k_v} = 1$ for any degree $[k_v : \mathbf{Q}_2]$.

3. The Brauer group

Let X be a smooth, quasi-projective, geometrically integral variety over a number field k. As a consequence of the fact that $\operatorname{Br} X$ injects into $\operatorname{Br} k(X)$ and of a result by Gabber (see [dJ]), for such an X we have that $\operatorname{Br} X$ is the same as the Brauer group defined in terms of Azumaya algebras. We briefly recall the relations between Hilbert symbols, quaternion algebras, and evaluation of the Hasse invariant maps, as we will use them in the subsequent sections. For more details, we refer the reader to e.g. [Poo17, Chap. 1], [GS06, Chapters 1, 2, 4, 8], [KKS11, Chap. 8]. Let K be a field of characteristic different from 2, let $a, b \in K^\times$, and let $Q(a, b; K)$ be the corresponding quaternion (Azumaya) algebra. By definition of the Hilbert symbol, we have $(a, b)_K = 1$ if and only if $Q(a, b; K)$ splits. Now let $K := k_v$ be the completion of a number field k at a place v of k. Let $\mathscr{Q} := [Q(a, b; k_v)]$ be the class of $Q(a, b; k_v)$ in $\mathrm{Br}(k_v)$. Then $\mathscr{Q}$ is a 2-torsion element in $\mathrm{Br}(k_v)$ and the Hasse invariant map $\mathrm{inv}_v : \mathrm{Br}(k_v)[2] \to \frac{1}{2}\mathbf{Z}/\mathbf{Z}$ sends $\mathscr{Q}$ to 0 if and only if $Q(a, b; k_v)$ splits, and hence if and only if $(a, b)_K = 1$. In what follows, for any field K with $\operatorname{char} K \neq 2$ and for any $a, b \in K^\times$, we will denote the quaternion algebra $Q(a, b; K)$ by $(a, b)_K$; hopefully this will not cause any confusion.

In the subsequent sections, we will also need some results on the Brauer group of our generalised Châtelet surfaces. Let X be the generalised Châtelet surface over a number field k associated to

$$N_{k(\sqrt{-1})/k}(\vec{z}) = f(x)g(x),$$

where f and g are non-zero polynomials over k of even degrees $\deg f, \deg g \geq 2$ and $f(x)g(x)$ is separable. Let $\mathscr{A} \in \operatorname{Br} k(X)$ be the class of the quaternion algebra $(-1, f)_{k(X)}$. Since the class of $(-1, f)_{k(X)}$ is unaffected if we multiply f by a square or by a norm of $k(X)(\sqrt{-1})/k(X)$, it follows that $\mathscr{A} = [(-1, g)_{k(X)}]$.

Lemma 3.1. *Let X, k, and $\mathscr{A}$ as above. If moreover f and g are irreducible over k and neither* $\mathrm{Split}_k(f)$ *nor* $\mathrm{Split}_k(g)$ *contain $\sqrt{-1}$, then* $\operatorname{Br} X/\operatorname{Br}_0 X = \langle \mathscr{A} + \operatorname{Br}_0 X\rangle \cong \mathbf{Z}/2\mathbf{Z}$.

Proof. By [VAV12, Theorem 3.2], we have an isomorphism

$$\frac{\{(n_1,n_2)\in(\mathbf{Z}/2\mathbf{Z})^2 : n_1\deg f+n_2\deg g\equiv 0 \pmod 2\}}{(1,1)} \xrightarrow{\sim} \frac{\operatorname{Br} X}{\operatorname{Br}_0 X}$$

given by $[(n_1,n_2)] \mapsto [(-1, f(x)^{n_1}g(x)^{n_2})_{k(X)}] + \operatorname{Br}_0 X$. Since $\deg f$ and $\deg g$ are both even, it follows that $n_1 \deg f + n_2 \deg g \equiv 0 \pmod 2$ for all $(n_1,n_2)\in(\mathbf{Z}/2\mathbf{Z})^2$. Hence, $\operatorname{Br} X/\operatorname{Br}_0 X = \langle \mathscr{A} + \operatorname{Br}_0 X\rangle \cong \mathbf{Z}/2\mathbf{Z}$. $\square$

4. Proofs of the main theorems

In this section we prove Theorems 1.4 and 1.7.

Let $k\in\mathscr{K}$. For any $f(x) := \sum_{i=0}^n f_i x^i \in \mathcal{O}_k[x]$ and any tuple $(\lambda,\mu,\nu)\in\mathcal{O}_k\times\mathcal{O}_k\times\mathbf{Z}_{\geq 1}$, we define the following set of conditions, which we collectively call Conditions (ELS).

Condition 4.1 (**ELS-f**)**.** We have $n\geq 2$ even, $f\cdot(\lambda+\mu f^\nu)$ separable, and $f_n, f_0, \lambda+\mu f_0^\nu \neq 0$.

Condition 4.2 (**ELS-odd$_{\mathbf{QR}}$**)**.** For any $v\in\Omega_k^{\mathrm{odd}_{\mathrm{QR}}}$, there is $x_v\in k_v$ such that $v(f(x_v)(\lambda+\mu f(x_v)^\nu))$ is even.

Remark 4.3. If $f_n=\pm 1$, $v(\mu)=0$ for all $v\in\Omega_k^{\mathrm{odd}_{\mathrm{QR}}}$, and $n\geq 2$ is even, then for any $v\in\Omega_k^{\mathrm{odd}_{\mathrm{QR}}}$ we can take $x_v = u_v/\pi$, where π is a uniformiser of $\mathcal{O}_{k_v}$ and $u_v\in\mathcal{O}_{k_v}^\times$.

Condition 4.4 (**ELS-odd$_{\mathbf{QNR}}$**)**.** For $v\in\Omega_k^{\mathrm{odd}_{\mathrm{QNR}}}$, we have $v(f_n)=v(\lambda)=v(\mu)=0$.

Condition 4.5 (**ELS-even**)**.** For $v\in\Omega_k^{\mathrm{even}}$, we have $(-1,f_0)_{k_v} = (-1,\lambda+\mu f_0^\nu)_{k_v}$.

Condition 4.6 (**ELS-R**)**.** If $\Omega_k^{\mathbf{R}}\neq\emptyset$, then, for $v\in\Omega_k^{\mathbf{R}}$, there exists some $x_v\in k_v$ such that $f(x_v)(\lambda+\mu f(x_v)^\nu)>0$ in k_v.

We also define the following set of conditions, which we collectively call Conditions (Br).

Condition 4.7 (**Br-k-f**)**.** Let $\chi_k := \operatorname{lcm}_{v\in\Omega_k^{\mathrm{even}}}\{2^{[k_v:\mathbf{Q}_2]}-1\}$.

(1) If $f_i\notin 4\mathcal{O}_k$, then $\chi_k | i$;
(2) both f and $\lambda+\mu f^\nu$ have no zeros in k.

Condition 4.8 (**Br-even**)**.** For $v\in\Omega_k^{\mathrm{even}}$, writing $\lambda = 2^{v(\lambda)}u_\lambda$, $\mu=2^{v(\mu)}u_\mu$, and $\lambda+\mu f_0^\nu = 2^{v(\lambda+\mu f_0^\nu)}u_{\lambda+\mu f_0^\nu}$, we have:

(1) if j is odd, then either $f_j=0$ or $v(f_j)\geq 1$;
(2) $f_n, f_0, u_{\lambda+\mu f_0^\nu}\in\overline{U}_{k_v}^2$;
(3) $u_\lambda\in U_{k_v}^2$;
(4) if ν is odd, then $u_\mu\in\overline{U}_{k_v}^2$, while if ν is even, then $u_\mu\in U_{k_v}^2$;
(5) either $\sum_{i=1}^{n-1} f_i\in U_{k_v}^2$, or $\sum_{i=1}^{n-1} f_i\in\overline{U}_{k_v}^2$, or $\sum_{i=1}^{n-1} f_i\in 4\mathcal{O}_{k_v}$. Moreover,
  (a) if $\sum_{i=1}^{n-1} f_i\in\overline{U}_{k_v}^2$, then
    (i) if ν is odd, then $v(\mu)=v(\lambda)+1$;
    (ii) if ν is even, then either $v(\mu)=v(\lambda)+1$ or $v(\lambda)=v(\mu)+1$;
  (b) if $\sum_{i=1}^{n-1} f_i\in 4\mathcal{O}_{k_v}$, then $v(\mu)+\nu = v(\lambda)+1$ and $\nu\leq 2$.

Condition 4.9 (**Br-R**)**.** If $\Omega_k^{\mathbf{R}}\neq\emptyset$, then, for $v\in\Omega_k^{\mathbf{R}}$, we have:

(1) if ν is odd, then $\lambda>0$ and $\mu<0$ in k_v;
(2) if ν is even, then $\lambda,\mu>0$ in k_v.

Finally, we let Conditions (HP) be Conditions (**Br-k-f**) and (**Br-R**) together with the following two conditions.

Condition 4.10 (**HP-k-f**)**.** Both f and $\lambda+\mu f^\nu$ are irreducible over k, and neither $\mathrm{Split}_k(f)$ nor $\mathrm{Split}_k(\lambda+\mu f^\nu)$ contains $\sqrt{-1}$.

Condition 4.11 (**HP-even**)**.** For $v \in \Omega_k^{\mathrm{even}}$, writing $\lambda = 2^{v(\lambda)}u_\lambda$, $\mu = 2^{v(\mu)}u_\mu$, and $\lambda+\mu f_0^\nu = 2^{v(\lambda+\mu f_0^\nu)}u_{\lambda+\mu f_0^\nu}$, we have:

(1) if j is odd, then either $f_j = 0$ or $v(f_j) \geq 1$;
(2) $f_n, f_0, u_{\lambda+\mu f_0^\nu} \in U_{k_v}^2$;
(3) $u_\lambda \in U_{k_v}^2$;
(4) if ν is odd, then $u_\mu \in \overline{U}_{k_v}^2$, while if ν is even, then $u_\mu \in U_{k_v}^2$;
(5) either $\sum_{i=1}^{n-1} f_i \in U_{k_v}^2$, or $\sum_{i=1}^{n-1} f_i \in \overline{U}_{k_v}^2$, or $\sum_{i=1}^{n-1} f_i \in 4\mathcal{O}_{k_v}$, or $\sum_{i=1}^{n-1} f_i \in 2+4\mathcal{O}_{k_v}$. Moreover,
  (a) if $\sum_{i=1}^{n-1} f_i \in U_{k_v}^2$, then
    (i) if ν is odd, then $v(\mu) \geq v(\lambda)+2$;
    (ii) if ν is even, then either $v(\mu) \geq v(\lambda)+2$ or $v(\lambda) \geq v(\mu)+2$;
  (b) if $\sum_{i=1}^{n-1} f_i \in 4\mathcal{O}_{k_v}$, then
    (i) if ν is odd, then either $v(\mu) \geq v(\lambda)+2$, or $v(\lambda) = v(\mu)+1$ and $\nu \geq 3$;
    (ii) if ν is even, then either $v(\mu) \geq v(\lambda)+2$, or $v(\mu)+2 \leq v(\lambda) \leq v(\mu)+\nu-2$;
  (c) if $\sum_{i=1}^{n-1} f_i \in 2+4\mathcal{O}_{k_v}$, then
    (i) if ν is odd, then either $v(\mu) \geq v(\lambda)+2$, or $v(\lambda) = v(\mu)+1$ and $\nu \geq 3$;
    (ii) if ν is even, then either $v(\mu) \geq v(\lambda)+2$, or $v(\mu)+2 \leq v(\lambda) \leq v(\mu)+2\nu-2$.

Before proving Theorems 1.4 and 1.7 we need some preliminary results.

Lemma 4.12. *Let k be a number field with $\sqrt{-1} \notin k$. Let $v \in \Omega_k$ be such that there exists $\gamma \in k_v^\times$ with $(-1,\gamma)_{k_v} = -1$. Then, for any $\alpha, \beta \in k_v^\times$, there is a k_v-solution $\vec{z_v}$ to $N_{k(\sqrt{-1})/k}(\vec{z}) = \alpha\beta$ if and only if $(-1,\alpha)_{k_v} = (-1,\beta)_{k_v}$.*

Proof. The "only if" direction is clear, by the definition and the multiplicativity property of the Hilbert symbol. For the other direction, let us assume that $(-1,\alpha)_{k_v} = (-1,\beta)_{k_v}$. By multiplicativity, this is equivalent to $(-1,\alpha\beta)_{k_v} = 1$. By the definition of the Hilbert symbol, this implies that there is a non-trivial k_v-solution $(\vec{z_v}, t_v)$ to the equation $N_{k(\sqrt{-1})/k}(\vec{z}) = \alpha\beta t^2$. If we can show that $t_v \neq 0$, then clearly $\vec{z_v}/t_v$ is a k_v-solution to $N_{k(\sqrt{-1})/k}(\vec{z}) = \alpha\beta$. By hypothesis, there exists some $\gamma \in k_v^\times$ with $(-1,\gamma)_{k_v} = -1$. Hence, there is no non-trivial k_v-solution to $N_{k(\sqrt{-1})/k}(\vec{z}) = \gamma t^2$. In particular, there is no non-trivial k_v-solution to $N_{k(\sqrt{-1})/k}(\vec{z}) = 0$. This is sufficient to rule out the case $t_v = 0$ in our solution $(\vec{z_v}, t_v)$ to $N_{k(\sqrt{-1})/k}(\vec{z}) = \alpha\beta t^2$. ☐

Remark 4.13. If $k \in \mathscr{K}$ holds, then Lemma 4.12 applies to all $v \in \Omega_k$ with the exception of $v \in \Omega_k^{\mathbf{C}}$ and $v \in \Omega_k^{\mathrm{odd}}$ with $\mathrm{red}_v(-1) \in \mathbf{F}_v^2$ (cf. Lemma 2.2 and Lemma 2.6).

The following lemma follows from deduced from e.g. [Neu99, Chap. V, §1, Cor 1.2] and by using the fact that adjoining to a local field K a root of unity of order coprime to the residue characteristic of K yields an unramified extension of K.

Lemma 4.14. *Let k be a number field and let $v \in \Omega_k^{\mathrm{odd}}$. Then $k_v(\sqrt{-1})/k_v$ is unramified. Consequently, $N_{k_v(\sqrt{-1})/k_v} : \mathcal{O}_{k_v(\sqrt{-1})}^\times \to \mathcal{O}_{k_v}^\times$ is a surjective homomorphism.*

Proposition 4.15. *Let $k \in \mathscr{K}$, $f(x) := \sum_{i=0}^n f_i x^i \in \mathcal{O}_k[x]$, and $(\lambda,\mu,\nu) \in \mathcal{O}_k \times \mathcal{O}_k \times \mathbf{Z}_{\geq 1}$ satisfy Conditions* (**ELS**)*. Let X be the generalised Châtelet surface over k associated to $X_0 : N_{k(\sqrt{-1})/k}(\vec{z}) = f(x)(\lambda+\mu f(x)^\nu)$. Then $X(\mathbf{A}_k) \neq \emptyset$.*

Proof. It suffices to show that $X_0(k_v) \neq \emptyset$ for all $v \in \Omega_k$, since then $\emptyset \neq \prod_{v\in\Omega_k} X_0(k_v) \subset \prod_{v\in\Omega_k} X(k_v) = X(\mathbf{A}_k)$. For $v \in \Omega_k^{\mathbf{C}}$, it is clear that $X_0(k_v) \neq \emptyset$. For $v \in \Omega_k^{\mathbf{R}} \cup \Omega_k^{\mathrm{odd}_{QNR}} \cup \Omega_k^{\mathrm{even}}$ we use Lemma 4.12. More precisely, if $\Omega_k^{\mathbf{R}} \neq \emptyset$, then for $v \in \Omega_k^{\mathbf{R}}$ there exists, by assumption, some $x_v \in k_v$ such that $f(x_v), \lambda + \mu f(x_v)^\nu \in k_v^\times$ and $(-1, f(x_v)(\lambda + \mu f(x_v)^\nu))_{k_v} = 1$. For $v \in \Omega_k^{\mathrm{odd}_{QNR}}$, we can choose some $x_v \in k_v$ with $v(x_v) < 0$ and $f(x_v), \lambda + \mu f(x_v)^\nu \in k_v^\times$. Then, using that $f, \lambda + \mu f^\nu \in \mathcal{O}_k[x]$, that $v(f_n) = v(\mu f_n^\nu) = 0$, and that n is even, we deduce that both $v(f(x_v))$ and $v(\lambda + \mu f(x_v)^\nu)$ are even. By Lemma 2.2, it follows that $(-1, f(x_v))_{k_v} = (-1, \lambda + \mu f(x_v)^\nu)_{k_v}$. For $v \in \Omega_k^{\mathrm{even}}$, we take $x_v = 0$. Then, by assumption, we have $(-1, f_0)_{k_v} = (-1, \lambda + \mu f_0^\nu)_{k_v}$. In order to prove that $X_0(k_v) \neq \emptyset$ for $v \in \Omega_k^{\mathrm{odd}_{QR}}$, we take the $x_v \in k_v$ that, by assumption, has $v(f(x_v)(\lambda + \mu f(x_v)^\nu))$ even and use Lemma 4.14. $\square$

Proposition 4.16. *Let $k \in \mathscr{K}$, $f(x) := \sum_{i=0}^n f_i x^i \in \mathcal{O}_k[x]$, and $(\lambda, \mu, \nu) \in \mathcal{O}_k \times \mathcal{O}_k \times \mathbf{Z}_{\geq 1}$ satisfy Conditions* (**ELS-**f), (**ELS-odd$_{\mathbf{QNR}}$**), *and* (**Br-R**). *Let X be the generalised Châtelet surface over k associated to $X_0 : N_{k(\sqrt{-1})/k}(\vec{z}) = f(x)(\lambda + \mu f(x)^\nu)$. Let $\mathscr{A} := [(-1, f)_{k(X)}] = [(-1, \lambda + \mu f^\nu)_{k(X)}] \in$* Br X. *Then for any $\boldsymbol{x}_v \in X_0(k_v)$ and for any $v \notin \Omega_k^{\mathrm{even}}$, we have* $\mathrm{inv}_v \mathscr{A}(\boldsymbol{x}_v) = 0$.

Proof. Since $\mathrm{inv}_v \mathscr{A} : X(k_v) \to \mathbf{Q}/\mathbf{Z}$ is continuous for the local topology for any $v \in \Omega_k$, by deforming locally if necessary we may assume without loss of generality that $f(x_v), \lambda + \mu f(x_v)^\nu \neq 0$ for all $v \notin \Omega_k^{\mathrm{even}}$. Since $\mathscr{A}(\mathbf{x}_v) = [(f(x_v), -1)_{k_v}] = [(\lambda + \mu f(x_v)^\nu, -1)_{k_v}]$, it suffices to compute the Hilbert symbols $(f(x_v), -1)_{k_v} = (\lambda + \mu f(x_v)^\nu, -1)_{k_v}$ for each $v \in \Omega_k$.

If $v \in \Omega_k^{\mathbf{C}}$, then $(f(x_v), -1)_{k_v} = 1$, as $-1 \in k_v^2$. Hence, $\mathrm{inv}_v \mathscr{A}(\mathbf{x}_v) = 0$. If $\Omega_k^{\mathbf{R}} \neq \emptyset$ and $v \in \Omega_k^{\mathbf{R}}$, then $f(x_v) > 0$ in k_v: if $f(x_v) < 0$, then by Condition (**Br-R**) we would have $\lambda + \mu f(x_v)^\nu > 0$, and thus that $(f(x_v), -1)_{k_v} \neq (\lambda + \mu f(x_v)^\nu, -1)_{k_v}$, which in turn, by the correspondence between Hilbert symbols and quaternion algebras, would give $\mathrm{inv}_v \mathscr{A}(\mathbf{x}_v) = 1/2$ and $\mathrm{inv}_v \mathscr{A}(\mathbf{x}_v) = 0$, a contradiction. Hence, $f(x_v) > 0$ and $\mathrm{inv}_v \mathscr{A}(\mathbf{x}_v) = 0$.

If $v \in \Omega_k^{\mathrm{odd}_{\mathrm{QR}}}$, then by Lemma 2.2 we have $(f(x_v), -1)_{k_v} = 1$ and thus $\mathrm{inv}_v \mathscr{A}(\mathbf{x}_v) = 0$. If $v \in \Omega_k^{\mathrm{odd}_{\mathrm{QNR}}}$, we need to distinguish some cases. We write $x_v = \pi^\alpha u$, where π is a uniformiser of $\mathcal{O}_{k_v}$, $\alpha \in \mathbf{Z}$, and $u \in \mathcal{O}_{k_v}^\times$. If $\alpha < 0$, we have already seen in the proof of Proposition 4.15 that $(f(x_v), -1)_{k_v} = 1$ and thus $\mathrm{inv}_v \mathscr{A}(\mathbf{x}_v) = 0$. If $\alpha > 0$, then it easy to see that $v(f(x_v)) > 0$ if and only if $v(f_0) > 0$, and, similarly, that $v(\lambda + \mu f(x_v)^\nu) > 0$ if and only if $v(\lambda + \mu f_0^\nu) > 0$. If $v(f_0) > 0$, then $v(\lambda + \mu f_0^\nu) = 0$ as $v(\lambda) = 0$. Hence, either $v(f_0) = 0$ or $v(\lambda + \mu f_0^\nu) = 0$, meaning that either $v(f(x_v)) = 0$ or $v(\lambda + \mu f(x_v)) = 0$. Using $(f(x_v), -1)_{k_v} = (\lambda + \mu f(x_v), -1)_{k_v}$ and Lemma 2.2, we deduce that $\mathrm{inv}_v \mathscr{A}(\mathbf{x}_v) = 0$. Finally, if $\alpha = 0$, then $v(f(x_v))$ is even. Indeed, if $v(f(x_v)) > 0$ were odd, then $v(\lambda + \mu f(x_v)^\nu) = 0$ as $v(\lambda) = 0$ and we would have by Lemma 2.2 that $(f(x_v), -1)_{k_v} \neq (\lambda + \mu f(x_v)^\nu, -1)_{k_v}$, which in turn would give at the same time $\mathrm{inv}_v \mathscr{A}(\mathbf{x}_v) = 1/2$ and $\mathrm{inv}_v \mathscr{A}(\mathbf{x}_v) = 0$, a contradiction. Hence, by Lemma 2.2, $\mathrm{inv}_v \mathscr{A}(\mathbf{x}_v) = 0$. $\square$

Proposition 4.17. *Let $k \in \mathscr{K}$, $f(x) := \sum_{i=0}^n f_i x^i \in \mathcal{O}_k[x]$, and $(\lambda, \mu, \nu) \in \mathcal{O}_k \times \mathcal{O}_k \times \mathbf{Z}_{\geq 1}$ satisfy Conditions* (**ELS-**f) *and* (**Br-**k**-**f). *Let X be the generalised Châtelet surface over k associated to $X_0 : N_{k(\sqrt{-1})/k}(\vec{z}) = f(x)(\lambda + \mu f(x)^\nu)$. Let $\mathscr{A} := [(-1, f)_{k(X)}] = [(-1, \lambda + \mu f^\nu)_{k(X)}] \in$* Br X. *Then for any $\boldsymbol{x}_v \in X_0(k_v)$ and for any $v \in \Omega_k^{\mathrm{even}}$, we have*

$$\mathrm{inv}_v \mathscr{A}(\boldsymbol{x}_v) = \begin{cases} 0 & \text{if } (\textbf{HP-even}) \text{ holds,} \\ \frac{1}{2} & \text{if } (\textbf{Br-even}) \text{ holds.} \end{cases}$$

Proof. Since $\mathrm{inv}_v \mathscr{A} : X(k_v) \to \mathbf{Q}/\mathbf{Z}$ is continuous for the local topology, we may assume without loss of generality that $f(x_v), \lambda + \mu f(x_v)^\nu \neq 0$ for all $v \in \Omega_k^{\mathrm{even}}$. Since $\mathscr{A}(\mathbf{x}_v) = [(f(x_v), -1)_{k_v}] = [(\lambda + \mu f(x_v)^\nu, -1)_{k_v}]$, it suffices to compute the value of either $(f(x_v), -1)_{k_v}$ or $(\lambda + \mu f(x_v)^\nu, -1)_{k_v}$ for each $v \in \Omega_k^{\mathrm{even}}$. We remark that $(f(x_v), -1)_{k_v} = (\lambda + \mu f(x_v)^\nu, -1)_{k_v}$ as these two Hilbert symbols represent the same element $\mathscr{A}(\mathbf{x}_v) \in \mathrm{Br}(k_v)$. We give the proof for when (**HP-even**) holds; the proof for (**Br-even**) is similar. We will show that either $(-1, f(x_v))_{k_v} = 1$ or $(-1, \lambda + \mu f(x_v)^\nu)_{k_v} = 1$ for each $v \in \Omega_k^{\mathrm{even}}$, thus implying that $\mathrm{inv}_v \mathscr{A}(\mathbf{x}_v) = 0$ for each $v \in \Omega_k^{\mathrm{even}}$.

If $x_v = 0$, then by (**HP-even**)(2) we have $(-1, f(x_v))_{k_v} = (-1, f_0)_{k_v} = 1$ and $(-1, \lambda + \mu f(x_v)^\nu)_{k_v} = (-1, u_{\lambda+\mu f_0^\nu})_{k_v} = 1$. So let us assume that $x_v \neq 0$. Write $x_v = 2^\alpha u$, for some $\alpha \in \mathbf{Z}$ and $u \in \mathcal{O}_{k_v}^\times$. We distinguish some cases depending on α.

If $\alpha > 0$, then by using (**HP-even**)(1),(2) we have $f(x_v) - f_0 \in 4\mathcal{O}_{k_v}$ and thus that $f(x_v) \in U_{k_v}^2$. Hence, by Remark 2.7 we have $(-1, f(x_v))_{k_v} = 1$.

If $\alpha < 0$, then by by using (**HP-even**)(1),(2) we have $2^{-n\alpha}u^{-n}f(x_v) - f_n \in 4\mathcal{O}_{k_v}$ and thus that $2^{-n\alpha}u^{-n}f(x_v) \in U_{k_v}^2$. Hence, since n is even and by Remark 2.7, we have $\mathrm{inv}_v \mathscr{A}(\mathbf{x}_v) = 0$.

Finally, if $\alpha = 0$, then we write $x_v = \epsilon u_1$ for some $\epsilon \in \mu_{2^{[k_v:\mathbf{Q}_2]}-1}(k_v)$ and some $u_1 \in U_{k_v}^1$. In this case, by (**Br**-k-f) we have that $f(x_v) - \sum_{i=0}^n f_i \in 4\mathcal{O}_{k_v}$, since $f_i \in 4\mathcal{O}_k$ whenever $\chi_k \not| i$ (and for those indices i such that $\chi_k | i$, we have $f_i x_v^i = f_i(\epsilon u_1)^i = f_i u_1^i$). We distinguish some subcases depending on the value of $\sum_{i=0}^n f_i \bmod 4\mathcal{O}_{k_v}$.

(1) Let us begin with the subcase $\sum_{i=1}^{n-1} f_i \in \overline{U}_{k_v}^2$, i.e. $\sum_{i=0}^n f_i \in U_{k_v}^2$. Since $f(x_v) - \sum_{i=0}^n f_i \in 4\mathcal{O}_{k_v}$, we have $f(x_v) \in U_{k_v}^2$ and thus, by Remark 2.7, that $(-1, f(x_v))_{k_v} = 1$.

(2) Let us now deal with the subcase $\sum_{i=1}^{n-1} f_i \in U_{k_v}^2$, i.e. $\sum_{i=0}^n f_i \in \overline{U}_{k_v}^2$. We show that $(-1, \lambda + \mu f(x_v)^\nu)_{k_v} = 1$. Write
$$\lambda + \mu f(x_v)^\nu = 2^{v(\lambda)}u_\lambda + 2^{v(\mu)}u_\mu f(x_v)^\nu.$$

(a) If ν is odd, then by (**HP-even**)(5)(a)(i) we have that $v(\mu) \geq v(\lambda) + 2$. Hence,
$$\lambda + \mu f(x_v)^\nu = 2^{v(\lambda)}(u_\lambda + 2^{v(\mu)-v(\lambda)}u_\mu f(x_v)^\nu).$$
Moreover, by (**HP-even**)(3),(4) we have $u_\lambda \in U_{k_v}^2$. Since $v(\mu) - v(\lambda) \geq 2$, we can deduce that $u_\lambda + 2^{v(\mu)-v(\lambda)}u_\mu f(x_v)^\nu \in U_{k_v}^2$. By Remark 2.7, this implies that $(-1, \lambda + \mu f(x_v)^\nu)_{k_v} = 1$.

(b) If ν is even, then by (**HP-even**)(5)(a)(ii) we have that either $v(\lambda) \geq v(\mu) + 2$ or $v(\mu) \geq v(\lambda) + 2$. We will do the first subsubcase, the second subsubcase being very similar. Assume that $v(\lambda) \geq v(\mu) + 2$. Then
$$\lambda + \mu f(x_v)^\nu = 2^{v(\mu)}(2^{v(\lambda)-v(\mu)}u_\lambda + u_\mu f(x_v)^\nu).$$
Moreover, by (**HP-even**)(3),(4) we have $u_\lambda, u_\mu \in U_{k_v}^2$. Since $f(x_v) \in \overline{U}_{k_v}^2$, we deduce that $2^{v(\lambda)-v(\mu)}u_\lambda + u_\mu f(x_v)^\nu \in U_{k_v}^2$. By Remark 2.7, this implies that $(-1, \lambda + \mu f(x_v)^\nu)_{k_v} = 1$.

As a side note, we remark that this subcase never occurs, that is, if $\sum_{i=1}^{n-1} f_i \in U_{k_v}^2$ then $\alpha \neq 0$. Indeed, if $\alpha = 0$, then we have just shown that $(-1, \lambda + \mu f(x_v)^\nu)_{k_v} = 1$. But since $f(x_v) - \sum_{i=0}^n f_i \in 4\mathcal{O}_{k_v}$, we have $f(x_v) \in \overline{U}_{k_v}^2$ and thus, by Lemma 2.6, $(-1, f(x_v))_{k_v} = -1$. Since $(-1, f(x_v))_{k_v} = (-1, \lambda + \mu f(x_v)^\nu)_{k_v}$, we obtain a contradiction. Hence, if $\sum_{i=1}^{n-1} f_i \in U_{k_v}^2$ then $\alpha \neq 0$.

(3) Next, we deal with the subcase $\sum_{i=1}^{n-1} f_i \in 4\mathcal{O}_{k_v}$, i.e. $\sum_{i=0}^n f_i \in 2 + 4\mathcal{O}_{k_v}$. Since $f(x_v) - \sum_{i=0}^n f_i \in 4\mathcal{O}_{k_v}$, we have that $f(x_v) = 2 + 4t = 2(1 + 2t)$ for some $t \in \mathcal{O}_{k_v}$. We will show that $(-1, \lambda + \mu f(x_v)^\nu)_{k_v} = 1$. Write
$$\lambda + \mu f(x_v)^\nu = 2^{v(\lambda)}u_\lambda + 2^{v(\mu)}u_\mu f(x_v)^\nu.$$

(a) If ν is odd, then by (**HP-even**)(5)(b)(i) we have that either $v(\mu) \geq v(\lambda) + 2$ or $v(\lambda) = v(\mu) + 1$ and $\nu \geq 3$. The first subsubcase is similar to the subsubcase in (2)(a) above. For the second subsubcase, we have
$$\lambda + \mu f(x_v)^\nu = 2^{v(\mu)+1}(u_\lambda + 2^{\nu-1}u_\mu(1 + 2t)^\nu).$$
Since $\nu - 1 \geq 2$ and $u_\lambda \in U_{k_v}^2$, we can deduce that $u_\lambda + 2^{\nu-1}u_\mu(1 + 2t)^\nu \in U_{k_v}^2$. By Remark 2.7, this implies that $(-1, \lambda + \mu f(x_v)^\nu)_{k_v} = 1$.

We remark that the condition $v(\lambda) = v(\mu) + 1$ is used, together with $u_\mu \in \overline{U}^2_{k_v}$ and $f_0 \in U^2_{k_v}$, to ensure that $u_{\lambda+\mu f_0^\nu} \in U^2_{k_v}$ in (**HP-even**)(2) is satisfied.

(b) If ν is even, then by (**HP-even**)(5)(b)(ii) we have that either $v(\mu) \geq v(\lambda) + 2$ or $v(\mu) + 2 \geq v(\lambda) \geq v(\mu) + \nu - 2$. The first subsubcase is similar to the subsubcase in (2)(b) above. For the second subsubcase, since $v(\lambda) \geq v(\mu) + \nu - 2$ we have

$$\lambda + \mu f(x_v)^\nu = 2^{v(\lambda)}(u_\lambda + 2^{\nu+v(\mu)-v(\lambda)}u_\mu(1 + 2t)^\nu).$$

Moreover, by (**HP-even**)(3),(4) we have $u_\lambda \in U^2_{k_v}$. Since $\nu + v(\mu) - v(\lambda)) \geq 2$, we can deduce that $u_\lambda + 2^{\nu+v(\mu)-v(\lambda)}u_\mu(1 + 2t)^\nu \in U^2_{k_v}$. By Remark 2.7, this implies that $(-1, \lambda + \mu f(x_v)^\nu)_{k_v} = 1$. We remark that the condition $v(\mu) + 2 \geq v(\lambda)$ is used to ensure that $u_{\lambda+\mu f_0^\nu} \in U^2_{k_v}$ in (**HP-even**)(2) is satisfied.

(4) Finally, let us deal with the case $\sum_{i=1}^{n-1} f_i \in 2 + 4\mathcal{O}_{k_v}$, i.e. $\sum_{i=0}^{n} f_i \in 4\mathcal{O}_{k_v}$. Since $f(x_v) - \sum_{i=0}^{n} f_i \in 4\mathcal{O}_{k_v}$, we have that $f(x_v) = 4t$ for some $t \in \mathcal{O}_{k_v}$. We will show that $(-1, \lambda + \mu f(x_v)^\nu)_{k_v} = 1$. Write

$$\lambda + \mu f(x_v)^\nu = 2^{v(\lambda)}u_\lambda + 2^{v(\mu)}u_\mu f(x_v)^\nu.$$

(a) If ν is odd, then by (**HP-even**)(5)(c)(i) we have that either $v(\mu) \geq v(\lambda) + 2$ or $v(\lambda) = v(\mu) + 1$ and $\nu \geq 3$. These subsubcases are similar to those in (3)(a) above.

(b) If ν is even, then by (**HP-even**)(5)(c)(ii) we have that either $v(\mu) \geq v(\lambda) + 2$ or $v(\mu)+2 \geq v(\lambda) \geq v(\mu)+2\nu-2$. These subsubcases are similar to those in (3)(b) above. We remark that the condition $v(\mu) + 2 \geq v(\lambda)$ is used to ensure that $u_{\lambda+\mu f_0^\nu} \in U^2_{k_v}$ in (**HP-even**)(2) is satisfied. □

Proof of Theorem 1.4. By Proposition 4.15, $X(\mathbf{A}_k) \neq \emptyset$. Now let $(\mathbf{x}_v) \in X(\mathbf{A}_k)$. We want to show that $(\mathbf{x}_v) \notin X(\mathbf{A}_k)^{\mathscr{A}}$, where $\mathscr{A} := [(-1, f)_{k(X)}] \in \operatorname{Br} X$ (cf. §3). Since X is smooth, by the Implicit Function Theorem we have that $X_0(k_v)$ is dense in $X(k_v)$ for the local topology, for any $v \in \Omega_k$. Since moreover $\operatorname{inv}_v \mathscr{A} : X(k_v) \to \mathbf{Q}/\mathbf{Z}$ is continuous for the local topology for any $v \in \Omega_k$, by deforming locally if necessary we may assume without loss of generality that $\mathbf{x}_v \in X_0(k_v)$ for all $v \in \Omega_k$. By Propositions 4.16 and 4.17, $\operatorname{inv}_v(\mathscr{A}(\mathbf{x}_v)) = 0$ if $v \notin \Omega_k^{\text{even}}$ and $\operatorname{inv}_v(\mathscr{A}(\mathbf{x}_v)) = 1/2$ if $v \in \Omega_k^{\text{even}}$. Since by assumption $|\Omega_k^{\text{even}}|$ is odd, it follows that

$$\sum_{v\in\Omega_k} \operatorname{inv}_v \mathscr{A}(\mathbf{x}_v) = \frac{1}{2},$$

implying that $(\mathbf{x}_v) \notin X(\mathbf{A}_k)^{\mathscr{A}}$. Hence, $X(\mathbf{A}_k)^{\mathscr{A}} = \emptyset$, which implies that $X(\mathbf{A}_k)^{\text{Br}} = \emptyset$. □

Proof of Theorem 1.7. By Proposition 4.15, $X(\mathbf{A}_k) \neq \emptyset$. Now let $(\mathbf{x}_v) \in X(\mathbf{A}_k)$. We want to show that $(\mathbf{x}_v) \in X(\mathbf{A}_k)^{\text{Br}} = X(\mathbf{A}_k)^{\mathscr{A}}$, where $\mathscr{A} := [(-1, f)_{k(X)}] \in \operatorname{Br} X$ (cf. §3). Since X is smooth, by the Implicit Function Theorem we have that $X_0(k_v)$ is dense in $X(k_v)$ for the local topology, for any $v \in \Omega_k$. Since moreover $\operatorname{inv}_v \mathscr{A} : X(k_v) \to \mathbf{Q}/\mathbf{Z}$ is continuous for the local topology for any $v \in \Omega_k$, by deforming locally if necessary we may assume without loss of generality that $\mathbf{x}_v \in X_0(k_v)$ for all $v \in \Omega_k$. By Propositions 4.16 and 4.17, $\operatorname{inv}_v(\mathscr{A}(\mathbf{x}_v)) = 0$ for all $v \in \Omega_k$. Hence,

$$\sum_{v\in\Omega_k} \operatorname{inv}_v \mathscr{A}(\mathbf{x}_v) = 0,$$

implying that $(\mathbf{x}_v) \in X(\mathbf{A}_k)^{\mathscr{A}}$. Since $(\mathbf{x}_v) \in X(\mathbf{A}_k)$ was arbitrary, it follows that $X(\mathbf{A}_k) = X(\mathbf{A}_k)^{\text{Br}}$. Finally, we recall that $\operatorname{Br} X/\operatorname{Br}_0 X \neq 0$ by Lemma 3.1. □

5. Proofs of Corollaries 1.6 and 1.8

As corollaries of Theorems 1.4 and 1.7, we can now prove Corollaries 1.6 and 1.8.

Proof of Corollary 1.6. If $N/2$ is even and not equal to 2, we can let, for example, $(\lambda, \mu, \nu) := (1, -1, 1)$ and $f(x) := -x^{N/2} + 4x \sum_{i=0}^{N/4-1} f_{2i+1}x^{2i} + \sum_{i=1}^{N/4-1} f_{2i}x^{2i} + 3 \in \mathbf{Z}[x]$, where $f_j \in \mathbf{Z}$ for all $j \in \{1, ..., N/2\}$ and $\sum_{i=1}^{N/4-1} f_{2i} \equiv 1 \pmod{4\mathbf{Z}}$, or we can let $(\lambda, \mu, \nu) := (1, -2, 1)$ and $f(x) := -x^{N/2} + 4x \sum_{i=0}^{N/4-1} f_{2i+1}x^{2i} + \sum_{i=1}^{N/4-1} f_{2i}x^{2i} + 3 \in \mathbf{Z}[x]$, where $f_j \in \mathbf{Z}$ for all $j \in \{1, ..., N/2\}$ and $\sum_{i=1}^{N/4-1} f_{2i} \equiv -1 \pmod{4\mathbf{Z}}$.

If $N/2$ is odd and not equal to 3, then we let p be any prime dividing $N/2$. Since, by assumption, $N/2$ is not a prime, it follows that $N/(2p) \neq 1$. In this case, we can let, for example, $(\lambda, \mu, \nu) := (2, 1, N/(2p) - 1)$ and let $f(x) := -x^{2p} + 4x \sum_{i=0}^{p-1} f_{2i+1}x^{2i} + \sum_{i=1}^{p-1} f_{2i}x^{2i} + 3 \in \mathbf{Z}[x]$, where $f_j \in \mathbf{Z}$ for all $j \in \{1, ..., 2p-1\}$ and $\sum_{i=1}^{p-1} f_{2i} \equiv 1 \pmod{4\mathbf{Z}}$, or we can let $(\lambda, \mu, \nu) := (1, 2, N/(2p) - 1)$ and $f(x) := -x^{2p} + 4x \sum_{i=0}^{p-1} f_{2i+1}x^{2i} + \sum_{i=1}^{p-1} f_{2i}x^{2i} + 3 \in \mathbf{Z}[x]$, where $f_j \in \mathbf{Z}$ for all $j \in \{1, ..., 2p-1\}$ and $\sum_{i=1}^{p-1} f_{2i} \equiv -1 \pmod{4\mathbf{Z}}$.

If $N = 4$, then we let $(\lambda, \mu, \nu) := (1, -1, 1)$ and let $f(x) := -x^2 + 4f_1x + 3 \in \mathbf{Z}[x]$, where $f_1 \in \mathbf{Z}$.

If $N = 6$, then we let $(\lambda, \mu, \nu) := (2, 1, 2)$ and let $f(x) := -x^2 + 4f_1x + 3 \in \mathbf{Z}[x]$, where $f_1 \in \mathbf{Z}$.

In any case, we let the coefficients f_i be such that both f and $\lambda + \mu f^\nu$ have no roots in $\mathbf{Q}$, and such that $f \cdot (\lambda + \mu f^\nu)$ is separable. Let X be the generalised Châtelet surface over $\mathbf{Q}$ with affine equation given by

$$N_{k(\sqrt{-1})/k}(\vec{z}) = f(x)(\lambda + \mu f(x)^\nu).$$

By construction, $\deg(f \cdot (\lambda + \mu f^\nu)) = N$. It is clear that the conditions on $f(x)$ and $\lambda + \mu f(x)^\nu$ in the statement of Theorem 1.4 are satisfied. For example, for the real place $v = \infty$, we just note that multiplying the leading and the constant coefficients of f gives $(-1)\cdot 3 < 0$ in $\mathbf{R}$, which implies that f has a root in $\mathbf{R}$. Near such a root, we can find an $\tilde{x} \in \mathbf{R}$ such that $f(\tilde{x})(\lambda + \mu f(\tilde{x})^\nu) > 0$. Hence, we can apply Theorem 1.4 to deduce that $X(\mathbf{A_Q}) \neq \emptyset$ and $X(\mathbf{A_Q})^{\mathrm{Br}} = \emptyset$. ☐

Proof of Corollary 1.8. If $N \equiv 0 \pmod 4$, let can let $(\lambda, \mu, \nu) := (1, -4, 1)$ and let $f(x) := x^{N/2} + 4\sum_{i=0}^{N/2-1} f_i x^i - 3 \in \mathbf{Z}[x]$, where $f_j \in \mathbf{Z}$ for all $j \in \{1, ..., N/2\}$. Alternatively, if $N \equiv 0 \pmod 4$ with $N > 4$, we could also let $(\lambda, \mu, \nu) := (1, -4, 1)$ and $f(x) := x^{N/2} + 4x \sum_{i=0}^{N/4-1} f_{2i+1}x^{2i} + \sum_{i=1}^{N/4-1} f_{2i}x^{2i} - 3 \in \mathbf{Z}[x]$, where $f_j \in \mathbf{Z}$ for all $j \in \{1, ..., N/2\}$ and $\sum_{i=1}^{N/4-1} f_{2i} \equiv 1 \pmod{4\mathbf{Z}}$. If $N = 4$, we could let $(\lambda, \mu, \nu) := (1, -4, 1)$ and $f(x) := x^2 + 2f_1 - 3 \in \mathbf{Z}[x]$, where $f_1 \in \mathbf{Z}$ and $f_1 \not\equiv 0 \pmod{2\mathbf{Z}}$.

If $N \equiv 2 \pmod 4$, let $(\lambda, \mu, \nu) := (1, 4, (N-2)/2)$ and let $f(x) := x^2 + 4f_1x - 3 \in \mathbf{Z}[x]$, where $f_1 \in \mathbf{Z}$.

In any case, we let the coefficients f_i be such that $f \cdot (\lambda + \mu f^\nu)$ is separable, f and $\lambda + \mu f^\nu$ are both irreducible over $\mathbf{Q}$, and $\mathbf{Q}(\sqrt{-1})/\mathbf{Q}$ is not a subfield of the splitting fields of f and $\lambda + \mu f^\nu$ over $\mathbf{Q}$. Let X be the generalised Châtelet surface over $\mathbf{Q}$ associated to

$$N_{k(\sqrt{-1})/k}(\vec{z}) = f(x)(\lambda + \mu f(x)^\nu).$$

By construction, $\deg(f \cdot (\lambda + \mu f^\nu)) = N$. It is clear that the conditions on $f(x)$ and $\lambda + \mu f(x)^\nu$ in the statement of Theorem 1.7 are satisfied. For example, for the real place $v = \infty$, we just note that multiplying the leading and the constant coefficients of f gives $1\cdot(-3) < 0$ in $\mathbf{R}$, which implies that f has a root in $\mathbf{R}$. Near such a root, we can find an $\tilde{x} \in \mathbf{R}$ such that $f(\tilde{x})(\lambda + \mu f(\tilde{x})^\nu) > 0$. Hence, we can apply Theorem 1.7 to deduce that $X(\mathbf{A_Q}) = X(\mathbf{A_Q})^{\mathrm{Br}} \neq \emptyset$ and $\mathrm{Br}\, X / \mathrm{Br}_0\, X \neq 0$. ☐

6. Many-parameters examples over a field $k \neq \mathbf{Q}$

In this section, we give examples of infinite families of generalised Châtelet surfaces over number fields $k \neq \mathbf{Q}$ satisfying the conditions in Theorems 1.4 and 1.7. (All the properties of the number fields used in the examples have been check with a computer algebra system.)

Example 6.1. Let $k := \mathbf{Q}(\alpha)$ be the totally real cubic Galois extension where α satisfies $\alpha^3 - \alpha^2 - 2\alpha + 1 = 0$. Then $k \in \mathscr{K}$ and k has class number equal to 1 (so Eisenstein's criterion works over k). For any $v \in \Omega_k^{\text{even}}$, we have $f := f_v := [\mathbf{F}_v : \mathbf{F}_2] = [k : \mathbf{Q}] = 3$. Consequently, the number χ_k appearing in Condition (**Br**-k-f)(1) is $\chi_k = 2^3 - 1 = 7$. For any $n \equiv 0 \pmod{28}$, we let $\mathbf{f} := (f_i)_{i=1}^{n/14-1} \in \mathbf{Z}^{n/14-1}$ and we let $f_{n,\mathbf{f}}(x) := -x^{n/2} + 12\sum_{i=1}^{n/14-1} f_i x^{7i} + 3 \in \mathbf{Z}[x]$. Let $X_{n,\mathbf{f}}$ be the generalised Châtelet surface associated to

$$N_{k(\sqrt{-1})/k}(\vec{z}) = f_{n,\mathbf{f}}(x)(1 - f_{n,\mathbf{f}}(x)).$$

We note that the rational primes 2 and 3 are inert in k. By Eisenstein's criterion for 3 we deduce that $f_{n,\mathbf{f}}(x)$ is irreducible over k. Similarly, by Eisenstein's criterion for 2 we deduce that $1 - f_{n,\mathbf{f}}(x)$ is irreducible over k. Apart from those $\mathbf{f} \in \mathbf{Z}^{n/14-1}$ for which $f_{n,\mathbf{f}} \cdot (1 - f_{n,\mathbf{f}})$ is not separable, it is easy to check that all the other hypotheses of Theorem 1.4 are satisfied. Hence, the family $\{X_{n,\mathbf{f}}\}_{\mathbf{f} \in \mathbf{Z}^{n/2-1}}$ of generalised Châtelet surfaces over k is such that $X_{n,\mathbf{f}}(\mathbf{A}_k) \neq \emptyset$ and $X_{n,\mathbf{f}}(k) \subseteq X_{n,\mathbf{f}}(\mathbf{A}_k)^{\mathrm{Br}} = \emptyset$, for all $\mathbf{f} \in \mathbf{Z}^{n/2-1}$ such that $f_{n,\mathbf{f}} \cdot (1 - f_{n,\mathbf{f}})$ is separable.

Example 6.2. We let $k := \mathbf{Q}(\alpha) \in \mathscr{K}$ be as in Example 6.1. For any $n \equiv 0 \pmod{28}$, we let $\mathbf{f} := (f_i)_{i=1}^{n/14-1} \in \mathbf{Z}^{n/14-1}$ and we let $f_{n,\mathbf{f}}(x) := x^{n/2} + 12 \cdot 13 \sum_{i=1}^{n/14-1} f_{7i} x^{7i} - 3 \in \mathbf{Z}[x]$. Let $X_{n,\mathbf{f}}$ be the generalised Châtelet surface associated to

$$N_{k(\sqrt{-1})/k}(\vec{z}) = f_{n,\mathbf{f}}(x)(1 - 4f_{n,\mathbf{f}}(x)).$$

We note that the rational prime 13 is totally split in k; we fix some prime $\mathfrak{p}_{13}$ above 13. By Eisenstein's criterion for 3 and $\mathfrak{p}_{13}$, we deduce that both $f_{n,\mathbf{f}}(x)$ and $1 - 4f_{n,\mathbf{f}}(x)$ are irreducible over k. Apart from those $\mathbf{f} \in \mathbf{Z}^{n/14-1}$ for which $f_{n,\mathbf{f}} \cdot (1 - 4f_{n,\mathbf{f}})$ is not separable or for which $\sqrt{-1}$ is contained in $\mathrm{Split}_k(f_{n,\mathbf{f}} \cdot (1 - 4f_{n,\mathbf{f}}))$, it is easy to check that all the other hypotheses of Theorem 1.7 are satisfied. Hence, the family $\{X_{n,\mathbf{f}}\}_{\mathbf{f} \in \mathbf{Z}^{n/2-1}}$ of generalised Châtelet surfaces over k is such that $X_{n,\mathbf{f}}(\mathbf{A}_k)^{\mathrm{Br}} = X_{n,\mathbf{f}}(\mathbf{A}_k) \neq \emptyset$ and $\mathrm{Br}\, X / \mathrm{Br}_0\, X \neq 0$, for all $\mathbf{f} \in \mathbf{Z}^{n/2-1}$ such that $f_{n,\mathbf{f}} \cdot (1 - 4f_{n,\mathbf{f}})$ is separable and $\sqrt{-1}$ is not contained in $\mathrm{Split}_k(f_{n,\mathbf{f}} \cdot (1 - 4f_{n,\mathbf{f}}))$.

7. An application: some density considerations for a family of generalised Châtelet surfaces over $\mathbf{Q}$

For any even integer $n \geq 2$, consider the family of generalised Châtelet surfaces over $\mathbf{Q}$

$$\mathscr{G}_n := \{X_{\epsilon,\mathbf{f}} : \text{ gen. Châtelet surf. associated to } N_{\mathbf{Q}(\sqrt{-1})/\mathbf{Q}}(\vec{z}) = f_{n,\epsilon,\mathbf{f}}(x)(1 - f_{n,\epsilon,\mathbf{f}}(x))\}_{\mathbf{f} := (f_i)_{i=0}^{n-1} \in \mathbf{Z}^n, \epsilon \in \{0,1\}},$$

where $f_{n,\epsilon,\mathbf{f}}(x) := (-1)^\epsilon x^n + \sum_{i=1}^{n-1} f_i x^i + f_0$. We define the counting function

$$\mathscr{N}_n^{\text{Br-HP}}(B) := \frac{\#\{X_{\epsilon,\mathbf{f}} \in \mathscr{G}_n : \max_{0 \leq i \leq n-1} |f_i| \leq B \text{ and } X_{\epsilon,\mathbf{f}}(\mathbf{A}_\mathbf{Q}) \neq \emptyset, X_{\epsilon,\mathbf{f}}(\mathbf{A}_\mathbf{Q})^{\mathrm{Br}} = \emptyset\}}{\#\{X_{\epsilon,\mathbf{f}} \in \mathscr{G}_n : \max_{0 \leq i \leq n-1} |f_i| \leq B\}}.$$

Theorem 7.1. $\liminf_{B \to +\infty} \mathscr{N}_n^{\text{Br-HP}}(B) \geq \delta_n$, *where* $\delta_2 := 2^{-7} > 0$ *and* $\delta_n := 2^{-(n/2+7)} > 0$ *for* $n \geq 4$.

Proof. First of all, we note that, for any $B > 1$,

$$\mathscr{N}_n^{\text{Br-HP}}(B) \geq \mathscr{N}_{n,\epsilon=1}^{\text{Br-HP}}(B) := \frac{\#\{X_{1,\mathbf{f}} \in \mathscr{G}_n : \max_{0 \leq i \leq n-1} |f_i| \leq B \text{ and } X_{1,\mathbf{f}}(\mathbf{A}_\mathbf{Q}) \neq \emptyset, X_{1,\mathbf{f}}(\mathbf{A}_\mathbf{Q})^{\mathrm{Br}} = \emptyset\}}{\#\{X_{1,\mathbf{f}} \in \mathscr{G}_n : \max_{0 \leq i \leq n-1} |f_i| \leq B\}}$$

and thus that $\liminf_{B \to +\infty} \mathscr{N}_n^{\text{Br-HP}}(B) \geq \liminf_{B \to +\infty} \mathscr{N}_{n,\epsilon=1}^{\text{Br-HP}}(B)$. So we just focus on those $X_{\epsilon,\mathbf{f}} \in \mathscr{G}_n$ with $\epsilon = 1$ and look for a lower bound to $\liminf_{B \to +\infty} \mathscr{N}_{n,\epsilon=1}^{\text{Br-HP}}(B)$. Let $X_{1,\mathbf{f}} \in \mathscr{G}_n$ with

$\max_{0\le i\le n-1}|f_i|\le B$. Let $f_0\in[-B,B]\cap\mathbf{Z}$ be such that $f_0\equiv 3 \pmod 8$. Then $1-f_0=2u_{1-f_0}$ where $u_{1-f_0}\in\overline{U}^2_{\mathbf{Q}_2}$. We have

$$\#\{f_0\in[-B,B]\cap\mathbf{Z}: f_0>0 \text{ and } f_0\equiv 3 \pmod 8\}=\frac{1}{8}B+O(1).$$

If $n=2$, then to get a lower bound it suffices to count the number of $f_1\in[-B,B]\cap\mathbf{Z}$ such that $f_1\in 4\mathbf{Z}$, which is $B/2+O(1)$. If $n\ge 4$, then to get a lower bound it suffices to count the number of $(f_i)_{i=1}^{n-1}\in([-B,B]\cap\mathbf{Z})^{n-1}$ such that $\sum_{i=1}^{n-1}f_i\in 4\mathbf{Z}$ and f_i is even whenever i is odd. This number has a very crude lower bound of $2^{n/2-3}B^{n-1}+O(B^{n-2})$: indeed, if write $f_{2j+1}=2g_{2j+1}$ for $g_{2j+1}\in[-B/2,B/2]\cap\mathbf{Z}$ and $j=0,...,n/2-1$, then $2\sum_{j=0}^{n/2-1}g_{2j+1}+\sum_{j=1}^{n/2-1}f_{2j}\equiv 0 \pmod 4$ can be viewed as an equation in f_2 (where all the other parameters f_j with $j\ne 2$ are free), and the value of f_2 is determined mod 4; hence, we get a lower bound of $(2B)^{n/2-2}B^{n/2}(2B/4)+O(B^{n-2})$.

We note that the number of $(f_i)_{i=0}^{n-1}\in([-B,B]\cap\mathbf{Z})^n$ such that $f_{n,1,\mathbf{f}}(x)$ and $1-f_{n,1,\mathbf{f}}(x)$ are either not separable or have $\mathbf{Q}$-roots, is negligible as $B\to+\infty$. Indeed, by [Kub09] the right order of magnitude of monic polynomials of degree n with integer coefficients bounded in absolute value by B which are reducible is $O(B^{n-1})$ if $n\ge 3$ and $O(B\log B)$ if $n=2$. Hence, by an inclusion-exclusion argument, the number of $(f_i)_{i=0}^{n-1}\in([-B,B]\cap\mathbf{Z})^n$ such that $f_{n,\mathbf{f}}$ and $1-f_{n,1,\mathbf{f}}$ are both irreducible over $\mathbf{Z}$ is $(2B)^n-O(B^{n-1})$ if $n\ge 3$ and $(2B)^2-O(B\log B)$ if $n=2$. By Gauss' lemma, since $f_{n,1,\mathbf{f}}$ and $1-f_{n,1,\mathbf{f}}$ are both primitive, we have that their irreducibility over $\mathbf{Z}$ is the same as their irreducibility over $\mathbf{Q}$. It is clear that if $f_{n,1,\mathbf{f}}$ and $1-f_{n,1,\mathbf{f}}$ are both irreducible over $\mathbf{Q}$, then they have no $\mathbf{Q}$-roots (as $n\ge 2$) and they are both separable, implying, since $f_{n,1,\mathbf{f}}$ and $1-f_{n,1,\mathbf{f}}$ are coprime, that $f_{n,1,\mathbf{f}}\cdot(1-f_{n,1,\mathbf{f}})$ is also separable. It follows that

$$\lim_{B\to+\infty}\frac{\#\{\mathbf{f}\in([-B,B]\cap\mathbf{Z})^n:\ f_{n,1,\mathbf{f}}\cdot(1-f_{n,1,\mathbf{f}}) \text{ separable and } f_{n,1,\mathbf{f}} \text{ and } 1-f_{n,1,\mathbf{f}} \text{ have no } \mathbf{Q}\text{-roots}\}}{(2B+1)^n}=1.$$

Hence, putting everything together, we obtain that the number of $(f_i)_{i=0}^{n-1}\in([-B,B]\cap\mathbf{Z})^n$ such that $X_{1,\mathbf{f}}$ satisfies Conditions (**ELS**) and (**Br**) is at least

$$\begin{cases}2^{-4}B^2+O(B\,\log B) & \text{if } n=2,\\ 2^{n/2-6}B^n+O(B^{n-1}) & \text{if } n\ge 4.\end{cases}$$

By Theorem 1.4, it follows that any such $X_{1,\mathbf{f}}$ satisfies $X_{1,\mathbf{f}}(\mathbf{A}_\mathbf{Q})\ne\emptyset$ and $X_{1,\mathbf{f}}(\mathbf{A}_\mathbf{Q})^{\mathrm{Br}}=\emptyset$. Hence, since

$$\liminf_{B\to+\infty}\mathscr{N}^{\text{Br-HP}}_{n,\epsilon=1}(B)\ge\begin{cases}\lim_{B\to+\infty}(2^{-4}B^2+O(B\,\log B))/(2\cdot(2B+1)^2)=2^{-7} & \text{if } n=2,\\ \lim_{B\to+\infty}(2^{n/2-6}B^n+O(B^{n-1}))/(2\cdot(2B+1)^n)=2^{-(n/2+7)} & \text{if } n\ge 4,\end{cases}$$

we deduce that $\liminf_{B\to+\infty}\mathscr{N}^{\text{Br-HP}}_n(B)\ge\delta_n$, where $\delta_2:=2^{-7}$ and $\delta_n:=2^{-(n/2+7)}$ for $n\ge 4$. □

Remark 7.2. In a similar way, one can show that $\limsup_{B\to+\infty}\mathscr{N}^{\text{Br-HP}}_n(B)\le\Delta_n$ for some $\Delta_n<1$ by considering the members $X_{\epsilon,\mathbf{f}}\in\mathscr{G}_n$ with $\epsilon=0$ satisfying the conditions in Theorem 1.7. Indeed, analogous arguments as those in the proof of Theorem 7.1 show that a positive proportion of members in $\mathscr{G}_n$ have non-empty Brauer-Manin set. In particular, if we assume Schinzel's hypothesis, we deduce that a positive proportion of members in $\mathscr{G}_n$ satisfy the Hasse principle.

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