SOME REMARKS ON STRONG APPROXIMATION AND APPLICATIONS TO HOMOGENEOUS SPACES OF LINEAR ALGEBRAIC GROUPS

F. BALESTRIERI

Abstract. Let k be a number field and X a smooth, geometrically integral variety over k. For any linear algebraic group G over k and any G-torsor $g : Z \to X$, we observe that if the étale-Brauer obstruction is the only one for strong approximation off a finite set of places S for all twists of Z by elements in $H^1_{\text{ét}}(k, G)$, then the étale-Brauer obstruction is the only one for strong approximation off a finite set of places S for X. As an application, we show that any homogeneous space of the form G/H with G a connected linear algebraic group over k satisfies strong approximation off the infinite places with étale-Brauer obstruction. We also prove more refined strong approximation results for homogeneous spaces of the form G/H with G semisimple simply connected and H finite, using the theory of torsors and descent.

1. Introduction and main results

Motivation. The aim of this short note is to collect together various observations on strong approximation for points on varieties over number fields, with a view towards applications to homogeneous spaces of connected linear algebraic groups. In particular, in Theorem 1.5 we will show that, if k is a number field, G is a connected linear algebraic group over k, $H \subset G$ is any closed k-subgroup (not necessarily normal nor connected), and S is any finite set of places containing all the infinite places such that some non-compactness conditions for the adelic points at S of the semisimple part of G hold, then the homogeneous space G/H satisfies strong approximation with étale-Brauer obstruction off S. (A similar result has been recently found independently by Demeio in [Dem20]; see also Remark 1.6.) In the special case when G is semisimple simply connected (e.g. $\mathrm{SL}_{n,k}$) and H is finite, we will give even more precise statements for strong approximation, using the theory of torsors and descent (see Theorem 1.10 below). This approach using torsors complements the one trying to obtain information about strong approximation using the Brauer-Manin obstruction (e.g. [BD13], [Dem17], [CX18a],[CX18b], [Cao18], [CDX19]). Homogeneous spaces of the form $\mathrm{SL}_{n,k}/H$ with H finite and constant are of particular interest for their connection to the inverse Galois problem: if $\mathrm{SL}_{n,k}/H$ satisfies strong approximation off some finite set S, then it satisfies weak weak approximation (that is, weak approximation away from some finite set of places), and thus, by [Ser08, §3] and [Eke90], the inverse Galois problem for H has a positive answer.

Obstruction sets and strong approximation. Let us now give more specific details strong approximation and the types of obstruction sets that we are going to use. Let k be a number field with a fixed algebraic closure $\overline{k}$ and let $\mathbf{A}_k$ the adelic ring of k. Let X a smooth, geometrically integral variety over k (where "variety over k" means "separated scheme of finite type over k"). We write $\overline{X} := X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$. We now define some useful subsets of the set of adelic points $X(\mathbf{A}_k)$, all containing the adelic closure of $X(k)$.

Definition 1.1. Let $\operatorname{Br} X := H^2_{\text{ét}}(X, \mathbf{G}_m)$ be the Brauer group of X. For any subset $B \subset \operatorname{Br} X$, we define the set

$$X(\mathbf{A}_k)^B := \bigcap_{\alpha \in B} \left\{ (x_v) \in X(\mathbf{A}_k) : \sum_{v \in \Omega_k} \mathrm{inv}_v(\alpha(x_v)) = 0 \right\},$$

Date: June 5, 2020.

where the maps $\mathrm{inv}_v : \mathrm{Br}\, k_v \to \mathbf{Q}/\mathbf{Z}$ are the invariant maps coming from local class field theory. When $B = \mathrm{Br}\, X$ (respectively, $B = \mathrm{Br}_1 X := \ker(\mathrm{Br}\, X \to \mathrm{Br}\, \overline{X})$), we call $X(\mathbf{A}_k)^B$ the Brauer-Manin set (respectively, the algebraic Brauer-Manin set) of X.

Recall that, for any linear algebraic group G over k, the cohomology set $H^1_{\text{ét}}(X, G)$ classify (k-isomorphism) classes of G-torsors $g : Z \to X$ and that, for any element $\sigma \in H^1_{\text{ét}}(k, G)$, we can twist the torsor $g : Z \to X$ by σ and obtain a G^σ-torsor $g^\sigma : Z^\sigma \to X$ (see e.g. [Sko01] for more details). Let $\mathcal{F}_k$ (respectively, $\mathcal{F}_k^{\text{ab}}$, $\mathcal{F}_k^{\text{nilp}}$, $\mathcal{F}_k^{\text{sol}}$) be the set of all k-isomorphism classes of finite (respectively, finite abelian, finite nilpotent, finite solvable) linear algebraic groups over k.

Definition 1.2. For $\star \in \{\mathcal{F}_k, \mathcal{F}_k^{\text{ab}}, \mathcal{F}_k^{\text{nilp}}, \mathcal{F}_k^{\text{sol}}\}$ and $\square \in \{\ , \mathrm{Br}\}$, consider the set

$$\bigcap_{F \in \star} \ \bigcap_{[f : Y \to X] \in H^1_{\text{ét}}(X, F)} \ \bigcup_{\sigma \in H^1_{\text{ét}}(k, F)} f^\sigma(Y^\sigma(\mathbf{A}_k)^\square).$$

We call such a set the

- finite descent set of X, denoted by $X(\mathbf{A}_k)^{\text{ét}}$, if $(\star, \square) = (\mathcal{F}_k, \ )$;
- finite abelian descent set of X, denoted by $X(\mathbf{A}_k)^{\text{f-ab}}$, if $(\star, \square) = (\mathcal{F}_k^{\text{ab}}, \ )$;
- finite nilpotent descent set of X, denoted by $X(\mathbf{A}_k)^{\text{f-nilp}}$, if $(\star, \square) = (\mathcal{F}_k^{\text{nilp}}, \ )$;
- finite solvable descent set of X, denoted by $X(\mathbf{A}_k)^{\text{f-sol}}$, if $(\star, \square) = (\mathcal{F}_k^{\text{sol}}, \ )$;
- étale-Brauer set of X, denoted by $X(\mathbf{A}_k)^{\text{ét}\,\mathrm{Br}}$, if $(\star, \square) = (\mathcal{F}_k, \mathrm{Br})$.

Let Ω_k be the set of non-trivial places of k and $\infty_k \subset \Omega_k$ be the subset of infinite places. If $S \subset \Omega_k$ is finite, we let

$$\mathrm{pr}^S : X(\mathbf{A}_k) \to X(\mathbf{A}_k^S) := \prod_{v \notin S}{}' (X(k_v), \mathcal{X}(\mathcal{O}_v))$$

be the natural projection, where $\prod'$ denotes the restricted product (as convention, when $v \in \infty_k$ we take $X(k_v)$ in the restricted product). The map pr^S is surjective if $\prod_{v \in S} X(k_v) \neq \emptyset$. If ω is some "obstruction", we let $X(\mathbf{A}_k^S)^\omega := \mathrm{pr}^S(X(\mathbf{A}_k)^\omega)$. We denote by $\delta : X(k) \to X(\mathbf{A}_k)$ the natural diagonal map. To ease notation, we denote the closure of $\mathrm{pr}_S(\delta(X(k))$ in $X(\mathbf{A}_k^S)$ by $\overline{X(k)}^S$.

Definition 1.3. Let X be a geometrically integral variety over a number field k. Let $S \subset \Omega_k$ be a finite (possibly empty) subset of places of k. Let ω be some "obstruction" and suppose that $X(\mathbf{A}_k)^\omega \neq \emptyset$. We say that X satisfies strong approximation off S with obstruction ω if $\mathrm{pr}^S(\delta(X(k)))$ is dense in $(X(\mathbf{A}_k^S)^\omega) \subset X(\mathbf{A}_k^S)$. In other words, given any non-empty open set $U \subset X(\mathbf{A}_k^S)$, if $(U \times \prod_{v \in S} X(k_v)) \cap X(\mathbf{A}_k)^\omega \neq \emptyset$ then $(U \times \prod_{v \in S} X(k_v)) \cap \delta(X(k)) \neq \emptyset$.

Main results. We can now state the main results of this short note. We start with the following observation, which is a corollary of [Cao17, Thm 1.1].

Theorem 1.4. *Let X be a smooth and geometrically integral variety over a number field k with $X(\mathbf{A}_k)^{\text{ét}\,\mathrm{Br}} \neq \emptyset$. Let G be a linear algebraic group over k and let $g : Z \to X$ be a G-torsor. Let $S \subset \Omega_k$ be a finite (possibly empty) subset of places of k. If $\overline{Z^\sigma(k)}^S \cap Z^\sigma(\mathbf{A}_k^S)^{\text{ét}\,\mathrm{Br}} = Z^\sigma(\mathbf{A}_k^S)^{\text{ét}\,\mathrm{Br}}$ for all $\sigma \in H^1_{\text{ét}}(k, G)$ such that $Z^\sigma(\mathbf{A}_k)^{\text{ét}\,\mathrm{Br}} \neq \emptyset$, then $\overline{X(k)}^S \cap X(\mathbf{A}_k^S)^{\text{ét}\,\mathrm{Br}} = X(\mathbf{A}_k^S)^{\text{ét}\,\mathrm{Br}}$.*

Theorem 1.4, together with some recent results on strong approximation for connected linear algebraic groups, yields the following quite general application to homogeneous spaces of connected linear algebraic groups.

Theorem 1.5. *Let G be a connected linear algebraic group over a number field k and let $H \subset G$ be a closed (not necessarily normal nor connected) k-subsgroup of G. Let $S \subset \Omega_k$ be a finite non-empty subset of places of k with $\infty_k \subset S$ and such that, for any $\sigma \in H^1_{\text{ét}}(k, H)$ with $G^\sigma(\mathbf{A}_k) \neq \emptyset$, we have that $\prod_{v \in S} G'_\sigma(k_v)$ is non-compact for each non-trivial k-almost simple factor G'_σ of the semisimple part of G^σ. Then G/H satisfies strong approximation off S with étale-Brauer obstruction.*

Remark 1.6. A similar result as Theorem 1.5 has been recently proved, independently and with different techniques, by Julian Demeio (see [Dem20, Theorem 5.0.1]). In his work, Demeio then obtains further nice results relating $X(\mathbf{A}_k^S)^\omega = \mathrm{pr}^S(X(\mathbf{A}_k)^\omega)$ and the set $X(\mathbf{A}_k^S)^{\omega_S}$ (i.e. the set of points in $X(\mathbf{A}_k^S)$ which survive the obstruction ω_S given by, roughly speaking, the "obstruction ω with the places S removed"). In this paper, we go on a different direction, looking instead at descent obstructions for homogeneous spaces of the form G/H, where G is semisimple simply connected and H is finite.

Remark 1.7. The non-compactness conditions in Theorem 1.5 are vacuous unless k is totally real.

Remark 1.8. If $\overline{k}[G]^\times = \overline{k}^\times$, then by the main theorem in [Cao18] we can take $S \subset \Omega_k$ to be any non-empty finite subset (not necessarily containing ∞_k) in Theorem 1.5.

Remark 1.9. There are examples of homogeneous spaces of connected linear algebraic groups for which strong approximation with Brauer-Manin obstruction fails (see e.g. [Dem17] for an example of such a homogeneous space G/H with G semisimple simply connected and H finite nilpotent).

If we specialise to the case when G is a semisimple simply connected linear algebraic group over k (e.g. $G = \mathrm{SL}_{n,k}$ for $n \geq 2$) and $H \subset G$ is a finite closed k-subgroup, we can obtain more precise results on strong approximation, using the theory of torsors. Recall that, if $\mathscr{F}$ is a subset of (k-isomorphism classes of) finite linear algebraic groups over k and if X is a variety over k, then

$$X(\mathbf{A}_k)^{\mathscr{F}} := \bigcap_{F \in \mathscr{F}} \bigcap_{[f:Y \to X] \in H^1_{\text{ét}}(X,F)} \bigcup_{\sigma \in H^1_{\text{ét}}(k,F)} f^\sigma(Y^\sigma(\mathbf{A}_k)).$$

Theorem 1.10. *Let k be a number field and let $\mathscr{F} \subset \mathcal{F}_k$ be a set of (k-isomorphism classes of) finite linear algebraic groups over k which is geometrically closed under taking quotients in the sense that if $F \in \mathcal{F}_k$ is such that there exist $F_1 \in \mathscr{F}$ and $F_2 \in \mathcal{F}_{\overline{k}}$ satisfying $F(\overline{k}) = F_1(\overline{k})/F_2(\overline{k})$, then $F \in \mathscr{F}$. Let G be a semisimple simply connected linear algebraic group over k and let $H \subset G$ be any finite closed k-subgroup with $H \in \mathscr{F}$. Let $S \subset \Omega_k$ be a finite non-empty subset of places of k such that $\prod_{v \in S} G'_\sigma(k_v)$ is not compact for each non-trivial k-almost simple factor G'_σ of G^σ, for any $\sigma \in H^1_{\text{ét}}(k,H)$ such that $G'_\sigma(\mathbf{A}_k) \neq \emptyset$. Then (G/H) satisfies strong approximation off S with the descent obstruction associated to $\mathscr{F}$, that is, $\overline{(G/H)(k)}^S \cap (G/H)(\mathbf{A}_k^S)^{\mathscr{F}} = (G/H)(\mathbf{A}_k^S)^{\mathscr{F}}$.*

Example 1.11. Feasible (in the sense of Theorem 1.10) choices for $\mathscr{F}$ include: all {finite, finite abelian, finite solvable, finite supersolvable, finite nilpotent, ...} linear algebraic groups over k. When $\mathscr{F} = \mathcal{F}_k^{\text{ab}}$, then $\overline{(G/H)(k)}^S \cap (G/H)(\mathbf{A}_k^S)^{\text{f-ab}} = (G/H)(\mathbf{A}_k^S)^{\text{f-ab}}$. Since $\overline{k}[G/H]^\times = \overline{k}^\times$, by [Har02, Thm 2] we have $(G/H)(\mathbf{A}_k)^{\mathrm{Br}_1} \subset (G/H)(\mathbf{A}_k)^{\text{f-ab}}$, so that $\overline{(G/H)(k)}^S \cap (G/H)(\mathbf{A}_k^S)^{\mathrm{Br}_1} = (G/H)(\mathbf{A}_k^S)^{\mathrm{Br}_1}$ as well; we compare this with [Bor96] for a similar result concerning *weak* approximation with algebraic Brauer-Manin obstruction. When $\mathscr{F} = \mathcal{F}_k^{\text{sol}}$, we refer to [HW18] for results concerning *weak* approximation with Brauer-Manin obstruction in the case when H is supersolvable (in the sense of [HW18, Def. 6.4]), and hence solvable. It is unclear to the author what the relation between $(G/H)(\mathbf{A}_k^S)^{\mathrm{Br}}$ and $(G/H)(\mathbf{A}_k^S)^{\text{f-sol}}$ is. When $\mathscr{F} = \mathcal{F}_k^{\text{nilp}}$, then we know by [Dem17] that there are examples of number fields k, finite subsets $S \subset \Omega_k$ of places of k, and homogeneous spaces G/H with G semisimple simply connected and H finite nilpotent such that strong approximation off S with Brauer-Manin obstrution *fails*. Explicitly, one could take, for example, $k = \mathbf{Q}(\sqrt{-21})$, $S = \infty_k \cup \{v|2\}$, $G = \mathrm{SL}_{4,k}$, and $H = Q_8$ (the quaternion group of order 8). By our Theorem 1.10, these counterexamples by Demarche are all explained by the finite nilpotent descent obstruction.

Remark 1.12. In the light of Theorem 1.10, it would be interesting to study further the relationships (if any) between descent sets under classes of finite linear algebraic groups and Brauer-Manin sets.

2. Proofs

In this section we give proofs of the results presented in the previous section.

Proof of Theorem 1.4. Since by [Cao17] we have $X(\mathbf{A}_k)^{\text{ét Br}} = \bigcup_\sigma g^\sigma(Z^\sigma(\mathbf{A}_k)^{\text{ét Br}})$, it follows that $X(\mathbf{A}_k^S)^{\text{ét Br}} = \mathrm{pr}^S(\bigcup_\sigma g^\sigma(Z^\sigma(\mathbf{A}_k)^{\text{ét Br}})) = \bigcup_\sigma g^\sigma(Z^\sigma(\mathbf{A}_k^S)^{\text{ét Br}})$. By our assumptions, we deduce that

$$\begin{aligned} X(\mathbf{A}_k^S)^{\text{ét Br}} &= \textstyle\bigcup_\sigma g^\sigma(Z^\sigma(\mathbf{A}_k^S)^{\text{ét Br}}) \\ &= \textstyle\bigcup_\sigma g^\sigma(\overline{Z^\sigma(k)}^S \cap Z^\sigma(\mathbf{A}_k^S)^{\text{ét Br}}) \\ &\subset \textstyle\bigcup_\sigma \left(g^\sigma(\overline{Z^\sigma(k)}^S) \cap g^\sigma(Z^\sigma(\mathbf{A}_k^S)^{\text{ét Br}})\right) \\ &\subset \left(\textstyle\bigcup_\sigma g^\sigma(\overline{Z^\sigma(k)}^S)\right) \cap \left(\textstyle\bigcup_\sigma g^\sigma(Z^\sigma(\mathbf{A}_k^S)^{\text{ét Br}})\right) \\ &\subset \overline{X(k)}^S \cap X(\mathbf{A}_k^S)^{\text{ét Br}}. \end{aligned}$$

Since the opposite inclusion is clear, we obtain the required result. □

Proof of Theorem 1.5. We have a natural H-torsor structure on $G \to G/H$. Let $\sigma \in H^1_{\text{ét}}(k, H)$ be such that $G^\sigma(\mathbf{A}_k) \neq \emptyset$. Since G^σ is a connected linear algebraic group over k, it follows by our hypotheses and by [Cao18, Thm 1.4] that G^σ satisfies strong approximation off S with Brauer-Manin obstruction (and thus with étale-Brauer obstruction). Hence, Theorem 1.4 yields that G/H satisfies strong approximation off S with étale-Brauer obstruction. □

Proof of Theorem 1.10. It is well-known that $\overline{k}[G']^\times = \overline{k}^\times$ and $\mathrm{Br}(G') = \mathrm{Br}\, k$ for any G' semisimple and simply connected (see, for example, [Vos11, Chap 2, §4.3]). By [Cao18], G^σ satisfies strong approximation off S with Brauer-Manin obstruction for all $\sigma \in H^1_{\text{ét}}(k, H)$ such that $G^\sigma(\mathbf{A}_k^S)^{\mathrm{Br}} \neq \emptyset$. Since $\mathrm{Br}(G^\sigma) = \mathrm{Br}\, k$, it follows that G^σ satisfies strong approximation off S for any such σ. By an easy modification of [Sto07, Proposition 5.17] using [CDX19, §6], a similar proof of Theorem 1.5 applied to the H-torsor $G \to G/H$ yields

$$\overline{(G/H)(k)}^S \cap (G/H)(\mathbf{A}_k^S)^{\text{ét}} = (G/H)(\mathbf{A}_k^S)^{\text{ét}}. \tag{2.1}$$

The fibration obtained by applying the analytification functor $(-)^{\mathrm{an}}$ to the $H_{\mathbf{C}}$-torsor $G_{\mathbf{C}} \to (G/H)_{\mathbf{C}}$ induces the homotopy (exact) sequence

$$\pi_1^{\mathrm{top}}((G_{\mathbf{C}})^{\mathrm{an}}) \to \pi_1^{\mathrm{top}}(((G/H)_{\mathbf{C}})^{\mathrm{an}}) \to \pi_0^{\mathrm{top}}((H_{\mathbf{C}})^{\mathrm{an}}) \to \pi_0^{\mathrm{top}}((G_{\mathbf{C}})^{\mathrm{an}}),$$

where $\pi_0^{\mathrm{top}}((G_{\mathbf{C}})^{\mathrm{an}}) = 0$ since $(G_{\mathbf{C}})^{\mathrm{an}}$ is (topologically) connected as $G_{\mathbf{C}}$ is (see [Gro71, XII, Prop. 2.4]), and where $\pi_1^{\mathrm{top}}((G_{\mathbf{C}})^{\mathrm{an}}) = 0$ since $(G_{\mathbf{C}})^{\mathrm{an}}$ is (topologically) simply connected. In order to see this last claim, note that by [Gro71, XII, Cor. 5.2] we have (omitting base-points) that $\widehat{\pi_1^{\mathrm{top}}((G_{\mathbf{C}})^{\mathrm{an}})} \cong \pi_1^{\text{ét}}(G_{\mathbf{C}})$. Moreover, since G is simply connected, we have $\pi_1^{\text{ét}}(G) = 0$. By the exact sequence of étale fundamental groups (with base-points omitted)

$$1 \to \pi_1^{\text{ét}}(G_{\overline{k}}) \to \pi_1^{\text{ét}}(G) \to \mathrm{Gal}(\overline{k}/k) \to 1$$

it follows that $\pi_1^{\text{ét}}(G_{\overline{k}}) = 0$. By the Lefschetz principle for étale fundamental groups (see [Sza09, Second proof of Cor. 5.7.6 and Rmk 5.7.8] together with [Gro71, XII] and [Org03]), this is equivalent to $\pi_1^{\text{ét}}(G_{\mathbf{C}}) = 0$. But since $\pi_1^{\mathrm{top}}((G_{\mathbf{C}})^{\mathrm{an}})$ is finitely generated and abelian, it follows that $\widehat{\pi_1^{\mathrm{top}}((G_{\mathbf{C}})^{\mathrm{an}})} = 0$ implies that $\pi_1^{\mathrm{top}}((G_{\mathbf{C}})^{\mathrm{an}}) = 0$, which proves the claim.

Since H is finite, since $((G/H)_{\mathbf{C}})^{\mathrm{an}}$ is (topologically) connected as $(G/H)_{\mathbf{C}}$ is, and by the Lefschetz principle for étale fundamental groups, it follows that taking the profinite completion of $\pi_1^{\mathrm{top}}(((G/H)_{\mathbf{C}})^{\mathrm{an}}) = \pi_0^{\mathrm{top}}((H_{\mathbf{C}})^{\mathrm{an}})$ yields that

$$\pi_1^{\text{ét}}((G/H)_{\overline{k}}) = \widehat{\pi_1^{\mathrm{top}}((H_{\mathbf{C}})^{\mathrm{an}})}$$

is finite of size $\#H(\overline{k})$. Hence, using the notion of cofinal family of coverings as in [Sto07], it is easy to check that

$$(G/H)(\mathbf{A}_k)^{\text{ét}} = \bigcap_{i\in I} \bigcup_{\sigma\in H^1_{\text{ét}}(k,F_i)} f_i^\sigma(Y_i^\sigma(\mathbf{A}_k)),$$

where each $f_i : Y_i \to G/H$ is some F_i-torsor over G/H under a finite linear algebraic group F_i over k where $F_i(\overline{k})$ is a quotient of $H(\overline{k})$ (in particular, $\#F_i(\overline{k})|\#H(\overline{k})$) and where Y_i is geometrically connected. By hypothesis, H belongs to a (k-isomorphism) class $\mathscr{F} \subset \mathcal{F}_k$ satisfying the following property: for all $F \in \mathcal{F}_k$, if $F(\overline{k})$ is the quotient of some $F'(\overline{k})$ by some $F''(\overline{k})$ with $F' \in \mathscr{F}$ and $F'' \in \mathcal{F}_{\overline{k}}$, then $F \in \mathscr{F}$. It is clear then that $F_i \in \mathscr{F}$ for all $i \in I$. It follows that

$$(G/H)(\mathbf{A}_k)^{\text{ét}} = \bigcap_{F\in\mathscr{F}} \bigcap_{[f:Y\to(G/H)]} \bigcup_{\sigma} f^\sigma(Y^\sigma(\mathbf{A}_k)) =: (G/H)(\mathbf{A}_k)^{\mathscr{F}}.$$

Hence, by (2.1) the result follows. □

Acknowledgements

The author would like to thank Yang Cao for suggesting the application to homogeneous spaces and Julian Demeio for sharing his preprint [Dem20] with her. This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement 840684.

References

[BD13] M. Borovoi and C. Demarche, *Manin obstruction to strong approximation for homogeneous spaces*, Comment. Math. Helv. **88** (2013), no. 1, 1–54.

[Bor96] M. Borovoi, *The brauer-manin obstructions for homogeneous spaces with connected or abelian stabilizer*, J. Reine Angew. Math. **473** (1996), 181–194.

[Cao17] Y. Cao, *Sous-groupe de brauer invariant et obstruction de descente itérée*, 2017. Preprint,`ArXiv: 1704.05425`.

[Cao18] ———, *Approximation forte pour les variétés avec une action d'un groupe linéaire*, Compositio Mathematica **154** (2018), no. 4, 773–819.

[CDX19] Y. Cao, C. Demarche, and F. Xu, *Comparing descent obstruction and Brauer-Manin obstruction for open varieties*, Trans. Amer. Math. Soc. **371** (2019), 8625–8650.

[CX18a] Y. Cao and F. Xu, *Strong approximation with Brauer-Manin obstruction for groupic varieties*, Proc. London Math. Soc. **117** (2018), no. 3, 727–750.

[CX18b] ———, *Strong approximation with Brauer-Manin obstruction for toric varieties*, Annales de l'Institut Fourier (Grenoble) **68** (2018), no. 5, 1879–1908.

[Dem17] C. Demarche, *Obstructions de brauer-manin entières sur les espaces homogènes à stabilisateurs finis nilpotents*, Bulletin de la SMF **145** (2017), no. 2, 225–236.

[Dem20] J. L. Demeio, *The étale Brauer-Manin obstruction to strong approximation on homogeneous spaces* (2020). Preprint.

[Eke90] T. Ekedahl, *An effective version of Hilbert's irreducibility theorem*, Séminaire de Théorie des Nombres, Paris 1988–1989, Progr. Math., vol. 91, Birkhäuser Boston, 1990.

[Gro71] A. Grothendieck, *Revêtements étales et groupe fondamental (SGA 1)*, Lecture notes in mathematics, vol. 224, Springer-Verlag, 1971.

[Har02] D. Harari, *Groupes algébriques et points rationnels*, Math. Ann. **322** (2002), no. 4, 811–826.

[HW18] Y. Harpaz and O. Wittenberg, *Zéro-cycles sur les espaces homogènes et problème de galois inverse*, 2018. arXiv:1802.09605.

[Org03] F. Orgogozo, *Altérations et groupe fondamental premier à p*, Bulletin de la société mathématique de France **131** (2003), no. 1, 123–147.

[Ser08] J.-P. Serre, *Topics in galois theory (2nd ed.)*, Research Notes in Mathematics, vol. 1, A K Peters, 2008.

[Sko01] A. N. Skorobogatov, *Torsors and rational points*, Cambridge Tracts in Mathematics, vol. 144, Cambridge University Press, 2001.

[Sto07] M. Stoll, *Finite descent obstructions and rational points on curves*, Algebra and Number Theory **1** (2007), no. 4, 349–391.

[Sza09] T. Szamuely, *Galois groups and fundamental groups*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2009.

[Vos11] V. E. Voskresenskii, *Algebraic Groups and Their Birational Invariants*, Translations of Mathematical Monographs, American Mathematical Society, 2011.

IST Austria, Am Campus 1, 3400 Klosterneuburg, Austria
Email address: fbalestr@ist.ac.at
URL: https://fbalestrieri.github.io