DEGREES OF CLOSED POINTS ON DIAGONAL-FULL HYPERSURFACES

F. BALESTRIERI

Abstract. Let k be any field. Let $X \subset \mathbb{P}_k^N$ be a diagonal-full degree d hypersurface, where d is an odd prime. We prove that if $X(K) \neq \emptyset$ for some extension K/k with $n := [K : k]$ prime and $\gcd(n, d) = 1$, then $X(L) \neq \emptyset$ for some extension L/k with $\gcd([L : k], nd) = 1$ and $[L : k] \leq nd - n - d$. Moreover, if a K-solution is known explicitly, then we can compute L/k explicitly as well. When n or d is not prime, we can still say something about the possible values of $[L : k]$. As an example, we improve upon a theorem by Coray on smooth cubic surfaces $X \subset \mathbb{P}_k^3$, in the case when X is diagonal-full, by showing that if $X(K) \neq \emptyset$ for some extension K/k with $\gcd([K : k], 3) = 1$, then $X(L) \neq \emptyset$ for some L/k with $[L : k] \in \{1, 10\}$.

1. Introduction

Springer's theorem for quadratic forms famously states that, if a quadratic form φ on a finite-dimensional vector space over a field k is isotropic over some extension L/k of odd degree, then it is already isotropic over k (see [Spr52] for the case when the characteristic is not 2 and [EKN08, Corollary 18.5] for any characteristic). Equivalently, in more geometric terms, if $X \subset \mathbb{P}_k^N$ is a degree 2 hypersurface, then $X(L) \neq \emptyset$ for some extension L/k of odd degree implies that $X(k) \neq \emptyset$. A natural question to ask is whether Springer's theorem generalises to higher degree forms.

Question 1.1. Given a degree $d \geq 3$ hypersurface $X \subset \mathbb{P}_k^N$ over a field k, is it true that if $X(L) \neq \emptyset$ for some extension L/k with $\gcd([L : k], d) = 1$, then $X(k) \neq \emptyset$?

When $d \geq 4$, the general answer to Question 1.1 seems to be *no*, while, when $d = 3$, Cassels and Swinnerton-Dyer have conjectured that the answer to Question 1.1 should be *yes*. Some progress towards the conjecture by Cassels and Swinnerton-Dyer has been obtained by Coray (see [Cor76]), who proved, for any smooth cubic surface $X \subset \mathbb{P}_k^3$ over a perfect field k, that if $X(K) \neq \emptyset$ for some extension K/k with $\gcd([K : k], 3) = 1$, then $X(L) \neq \emptyset$ for some extension L/k with $[L : k] \in \{1, 4, 10\}$. In recent work, Ma has been able to remove the condition on the field being perfect, proving Coray's result for any field (see [Ma21]). Moreover, when k is a global field, Rivera and Viray have shown that, if the Brauer-Manin obstruction is the only one for the Hasse principle for rational points on smooth cubic surfaces in $\mathbb{P}^3$ over k (and, by a conjecture by Colliot-Thélène and Sansuc, this should always be the case), then the conjecture by Cassels and Swinnerton-Dyer holds for such surfaces (see [RV21]).

In this paper, we are concerned with the following much weaker version of Question 1.1.

Question 1.2. Let $X \subset \mathbb{P}_k^N$ be a degree $d \geq 3$ hypersurface over a field k. If $X(K) \neq \emptyset$ for some finite extension K/k with $\gcd([K : k], d) = 1$, can we find some (somewhat explicit) finite extension L/k with $\gcd([L : k], d) = 1$, $[K : k] \nmid [L : k]$, and $X(L) \neq \emptyset$?

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Our main theorem gives a positive answer to Question 1.2 for the class of *diagonal-full* (see Definition 2.3) hypersurfaces of degree d, under some assumptions on d and $[K:k]$.

Theorem (Theorem 3.1)**.** *Let k be any field. Let $X \subset \mathbb{P}_k^N$ be a diagonal-full degree d hypersurface over k, where d is an odd prime. If $X(K) \neq \emptyset$ for some extension K/k with $n := [K:k]$ prime and $\gcd(n,d)=1$, then $X(L) \neq \emptyset$ for some extension L/k with $\gcd([L:k], nd) = 1$ and $[L:k] \leq nd - n - d$. Moreover, if a K-solution is known explicitly, then L/k can be computed explicitly as well.*

When n or d is not prime, the proof of Theorem 3.1 can still say something about the possible values of $[L:k]$. As an example, we prove the following result, which implies an improvement upon Coray's and Ma's theorems when considering diagonal-full forms.

Theorem (Theorem 3.10)**.** *Let k be a field and let $X \subset \mathbb{P}_k^N$ be a cubic diagonal-full hypersurface over k. If $X(K) \neq \emptyset$ for some simple field extension K/k with $[K:k] = 4$, then $X(L) \neq \emptyset$ for some extension L/k with $[L:k] \in \{1,5\}$.*

Corollary 1.3. *Let k be a field and let $X \subset \mathbb{P}_k^3$ be a smooth diagonal-full cubic surface over k. If $X(K) \neq \emptyset$ for some extension K/k with $\gcd([K:k],3) = 1$, then $X(L) \neq \emptyset$ for some L/k with $[L:k] \in \{1,10\}$.*

Proof. By Coray's and Ma's results, we know, under the hypothesis of the corollary, that there exists some L/k with $[L:k] \in \{1,4,10\}$ and $X(L) \neq \emptyset$. If $[L:k]=4$, then either L/k is simple, in which case, by Theorem 3.10, there is some other L'/k with $[L':k] \in \{1,5\}$ and $X(L') \neq \emptyset$, or L/k is not simple. If L/k is not simple, then, since it is finite, it must be a tower of simple extensions $L/k(\alpha)/k$ with $[L:k(\alpha)] = [k(\alpha):k] = 2$. Then $X_{k(\alpha)}$ is a smooth diagonal-full cubic surface as well, and $X_{k(\alpha)}(L) \neq \emptyset$, where $[L:k(\alpha)] = 2$; this implies that $X(k(\alpha)) \neq \emptyset$. Repeating the same argument with $k(\alpha)$ and k, we get that $X(k) \neq \emptyset$ and we can let $L' = k$. In any case, we have found some L'/k with $[L':k] \in \{1,5\}$ and $X(L') \neq \emptyset$. If $L' = k$ we are done, and if $[L':k] = 5$, then any quadratic extension L''/L' (thus with $[L'':k] = 10$) satisfies $X(L'') \neq \emptyset$. □

2. Preliminaries on degree d forms

Hypersurfaces $X \subset \mathbb{P}_k^N$ of degree d over a field k are equivalent to degree d (homogeneous) forms in $N+1$ variables over k. Since we are going to prove our main theorems in the language of forms, we start by recalling some basic definitions.

Definition 2.1. Let φ be a form of degree d on a finite-dimensional vector space V over a field k. We say that φ is isotropic if there exists some non-zero $v \in V$ with $\varphi(v) = 0$. Otherwise, we say that φ is anisotropic.

Remark 2.2. If $X \subset \mathbb{P}_k^N$ is a degree d hypersurface over a field k corresponding to the degree d form φ on k^{N+1}, then, for any extension L/k, we have that $X(L) \neq \emptyset$ if and only if φ_L is isotropic.

If $(i_0,...,i_N) \in \mathbb{Z}_{\geq 0}^{N+1}$ and $x := (x_0,...,x_N)$, we denote by $\underline{x}^{(i_0,...,i_N)}$ the monomial in which x_j appears with exponent i_j if $i_j > 0$ and does not appear at all if $i_j = 0$.

Definition 2.3. Let φ be a form of degree d on a finite-dimensional vector space $V \cong k^{N+1}$ over a field k, say

$$\varphi(x_0,...,x_N) = \sum_{\substack{(i_0,...,i_N) \in \mathbb{Z}_{\geq 0}^{N+1}: \\ i_0+...+i_N=d}} a_{(i_0,...,i_N)} \underline{x}^{(i_0,...,i_N)},$$

with $a_{(i_0,\dots,i_N)} \in k$. We say that φ is diagonal-full if $a_{(d,0,\dots,0)}, a_{(0,d,0,\dots,0)}, \dots, a_{(0,\dots,0,d)} \neq 0$.
(In more geometric terms, a degree d hypersurface $X \subset \mathbb{P}^N_k$ is diagonal-full if X is given by an equation

$$\sum_{\substack{(i_0,\dots,i_N)\in\mathbb{Z}^{N+1}_{\geq 0}:\\ i_0+\dots+i_N=d}} \underline{x}^{(i_0,\dots,i_N)} = 0$$

with $a_{(i_0,\dots,i_N)} \in k$ and $a_{(d,0,\dots,0)}, a_{(0,d,0,\dots,0)}, \dots, a_{(0,\dots,0,d)} \neq 0$.)

Example 2.4. Any non-degenerate diagonal form is diagonal-full.

Definition 2.5. We let $D(\varphi_V) := \{\varphi(v) \neq 0 : v \in V\}$.

The following is a straightforward modification of [EKN08, Theorem 18.3, proof of (2) $\Rightarrow$ (3)].

Lemma 2.6. *Let φ be a form of degree d on a finite-dimensional vector space V over k and let $f \in k[t]$ be a non-constant polynomial. If there exists some $a \in k^\times$ such that $af \in \langle D(\varphi_{k(t)})\rangle$, then $\varphi_{k(g)}$ is isotropic for each irreducible polynomial g occurring to a power coprime to d in the factorisation of f, where $k(g) := k[t]/(g(t))$.*

Proof. Since $af \in \langle D(\varphi_{k(t)})\rangle$, there exist some $0 \neq h \in k[t]$ and $v_1, \dots, v_m \in V[t]$ such that

$$afh^d = \prod_{i=1}^m \varphi(v_i).$$

If it exists, let $p \in k[t]$ be a non-constant monic irreducible factor of f appearing with exponent λ coprime to d in the factorisation of f into irreducible polynomials, i.e. say $f = p^\lambda f'$ with p monic irreducible, $\deg(p) \geq 1$, $p \nmid f'$, and $\gcd(\lambda, d) = 1$. Write $v_i = p^{k_i} v_i'$, where $k_i \geq 0$ and $p \nmid v_i'$, for each $i = 1, \dots, m$. Then

$$ap^\lambda f' h^d = afh^d = \prod_{i=1}^m \varphi(v_i) = \prod_{i=1}^m p^{dk_i}\varphi(v_i') = p^{d\sum_{i=1}^m k_i} \prod_{i=1}^m \varphi(v_i').$$

Since

$$\lambda + d\nu_p(h) = \nu_p(ap^\lambda f' h^d) = \nu_p\left(\prod_{i=1}^m p^{dk_i}\varphi(v_i')\right) = d\sum_{i=1}^m k_i + \sum_{i=1}^m \nu_p(\varphi(v_i')),$$

where $\nu_p(-)$ denotes the valuation at p, and since $\gcd(\lambda, d) = 1$, it follows that $\nu_p(\varphi(v_j')) \geq 1$ for some $j \in \{1, \dots, m\}$. This means that $\varphi(v_j') \equiv 0 \bmod p$. Since by construction $p \nmid v_j'$, we also have that $v_j' \not\equiv 0 \bmod p$. Hence, $\varphi_{k(p)}$ is isotropic, as required. $\square$

Lemma 2.7. *Let d be a positive integer. Let k be a field and let φ be a diagonal-full form of degree d on a finite-dimensional vector space $V \cong k^{N+1}$ over k. Suppose that φ is anisotropic. Let $0 \neq r \in V[t]$. Then $\deg(\varphi(r)) = d\deg(r)$, where $deg(r) := \max_{i=0,\dots,N}(\deg(r_i))$.*

Proof. Since $r \in V[t]$ and since $V \cong k^{N+1}$, we can write $r = (r_0, \dots, r_N)$ with $r_i \in k[t]$ for each $i = 0, \dots, N$. Let $\deg(r) := \max_{i=0,\dots,N}(\deg(r_i))$, and let

$$I_{\deg(r)} := \{i \in \{0, \dots, N\} : \deg(r_i) = \deg(r)\}.$$

Since φ is diagonal-full, we can write it as

$$\varphi(x_0, \dots, x_N) = \sum_{\substack{(i_0,\dots,i_N)\in\mathbb{Z}^{N+1}_{\geq 0}:\\ i_0+\dots+i_N=d}} a_{(i_0,\dots,i_N)}\underline{x}^{(i_0,\dots,i_N)},$$

with $a_{(i_0,\dots,i_N)} \in k$ and $a_{(d,0,\dots,0)}, a_{(0,d,0,\dots,0)}, \dots, a_{(0,\dots,0,d)} \neq 0$. If $\deg(\varphi(r)) \neq d\deg(r)$, then some cancellation must have occured among the leading coefficients (not all 0, since φ is diagonal-full) of those polynomials $a_{(i_0,\dots,i_N)}\underline{r(t)}^{(i_0,\dots,i_N)}$ of degree $d\deg(r)$. (We note that, since $d\deg(r)$ is the maximal degree that can possibly be attained, the polynomial $\underline{r(t)}^{(i_0,\dots,i_N)}$ has degree $d\deg(r)$ if and only if $i_j = 0$ for all $j \notin I_{\deg(r)}$.) In particular, if we let $0 \neq \tilde{r} \in k^{N+1} \cong V$ be defined by

$$\tilde{r}_i = \begin{cases} r_i^* & \text{if } i \in I_{\deg(r)}, \\ 0 & \text{if } i \notin I_{\deg(r)}, \end{cases}$$

where $r_i^* \in k$ denotes the leading coefficient of $r_i(t)$, then $\tilde{r}$ must satisfy

$$\varphi(\tilde{r}) = \sum_{\substack{(i_0,\dots,i_N)\in\mathbb{Z}_{\geq 0}^{N+1}:\\ i_0+\dots+i_N=d}} a_{(i_0,\dots,i_N)}\underline{\tilde{r}}^{(i_0,\dots,i_N)} = 0,$$

which would imply that φ is isotropic, a contradiction. Hence, $\deg(\varphi(r)) = d\deg(r)$, as required. $\square$

3. Proof of the main theorems

In this section, using fairly simple arguments, we prove (in the language of forms) the two main theorems of the paper.

Theorem 3.1. *Let k be any field. Let $X \subset \mathbb{P}_k^N$ be a diagonal-full degree d hypersurface over k, where d is an odd prime. If $X(K) \neq \emptyset$ for some extension K/k with $n := [K : k]$ prime and $\gcd(n, d) = 1$, then $X(L) \neq \emptyset$ for some extension L/k with $\gcd([L : k], nd) = 1$ and $[L : k] \leq nd - n - d$. Moreover, if a K-solution is known explicitly, then L/k can be computed explicitly as well.*

Proof. If φ is isotropic over k, we can take $L = k$ and $[L : k] = 1$ is coprime to nd. So, from now on, we assume that φ is anistropic over k.

Since $[K : k]$ is prime, K/k is a simple extension. Let $K = k(\alpha)$ and let $f \in k[t]$ be the minimal (irreducible) polynomial of α over k. Since, by assumption, $\varphi_{k(f)}$ is isotropic, it follows that there exists some $v \in V[t]$ such that $\varphi(v) \equiv 0 \bmod f$ but $v \not\equiv 0 \bmod f$. By the division algorithm, there exist some $0 \neq h \in k[t]$ and $w, r \in V[t]$ such that

$$hv = fw + r$$

and with $\deg(h) < \deg(f) = n$ and $\deg(r) < \deg(f) = n$. Since

$$h^d\varphi(v) = \varphi(hv) = \varphi(fw + r) = f^d\varphi(w) + f(\text{other stuff}) + \varphi(r)$$

and since $f \mid \varphi(v)$, it follows that $f \mid \varphi(r)$.

If $r = 0$, then $f \mid hv$. But since f is irreducible and $f \nmid v$, it follows that $f \mid h$, which is a contradiction since $\deg(h) < \deg(f)$. Hence, $r \neq 0$. Let $\varphi(r) = fg$ for some $g \in k[t]$. Since $r \neq 0$ and since, by assumption, φ is anisotropic, it follows that $\varphi(r) \neq 0$: indeed, since $r(t) \neq 0$, there is some $\tilde{t} \in k$ such that the specialisation $r(\tilde{t}) \in V$ is also not 0; if, however, $\varphi(r) = 0$, then we would have in particular that $\varphi(r(\tilde{t})) = 0$, which would imply that φ is isotropic over k, a contradiction. Since $\varphi(r) \neq 0$, it follows that $g \neq 0$. Hence, we have that $fg \in \langle D(\varphi_{k(t)})\rangle$. Since $\varphi(r) = fg$ and $\deg(r) < \deg(f)$, it follows that

$$\deg(g) + \deg(f) = \deg(\varphi(r)) < d\deg(f) = dn,$$

that is, $\deg(g) < n(d-1)$. Notice also that $\deg(g) \geq 1$, since otherwise we would get, by Lemma 2.7, that $d\deg(r) = \deg(\varphi(r)) = \deg(f) = n$, which is a contradiction to the fact that $\gcd(d, n) = 1$.

In the remainder of the proof, we aim to show that there exists an irreducible factor p of fg of exponent λ coprime to d and with $\gcd(\deg(p), dn) = 1$ and $\deg(p) > 1$ (with the goal of then applying Lemma 2.6 to it). Let the factorisation of g into irreducible factors be

$$g = g^* \prod_{i=1}^{r} p_i^{\lambda_i}$$

where $g^* \in k^\times$ and, for each $i = 1, ..., r$, the distinct polynomials $p_i \in k[t]$ are monic and irreducible, with $\deg(p_i) =: u_i$ and $\lambda_i \geq 1$. Then

$$\deg(g) = \sum_{i=1}^{r} u_i \lambda_i < n(d-1).$$

We now introduce some terminology and notation.

Definition 3.2. Let $n^* \in \{1, ..., d-1\}$ be the unique integer such that $n^* \equiv -n \bmod d$. We define the set

$$S_{d,n} := \{n^* + jd : j \in \mathbb{Z}_{\geq 0} \text{ and } n^* + jd < n(d-1)\}.$$

Definition 3.3. Let $u \in S_{d,n}$. We call any partition $u = \lambda_1 u_1 + ... + \lambda_r u_r$ in which there exists some i with $u_i = 1$ and $\gcd(\lambda_i, d) = 1$ an inadmissible partition. We call all the other partitions of u admissible.

Claim 3.4. *Let $\lambda_1 u_1 + ... + \lambda_r u_r$ be an admissible partition of $u \in S_{d,n}$. Then there exists at least one $i \in \{1, ..., r\}$ with λ_i coprime to d and $u_i > 1$ coprime to both n and d.*

Proof. Indeed, suppose, for a contradiction, that this is not the case. Then, for any $i \in \{1, ..., r\}$, either λ_i is not coprime to d or u_i is not comprime to both n and d. (We note that if $\gcd(\lambda_i, d) = 1$, then $u_i > 1$ since the partition is admissible.) Since $u \in S_{d,n}$ and since d is prime and $\gcd(d, n) = 1$, it follows that u is coprime to d. Hence, since $u = \sum_{i=1}^{r} u_i \lambda_i$, there exists at least one i with λ_i coprime to d. Let

$$I_d := \{i \in \{1, ..., r\} : \lambda_i \text{ is coprime to } d\},$$

which is non-empty, as noted above. Since the partition is admissible, we must have that $u_i > 1$ for all $i \in I_d$. Moreover, by assumption, we must have that u_i is not coprime to both n and d for all $i \in I_d$. Consider the subset of I_d defined by

$$J_d := \{j \in I_d : u_j \text{ is coprime to } d\}.$$

Since we are assuming that, for any i, either λ_i is not coprime to d or u_i is not coprime to both n and d, using the fact that n is prime we must have that u_j is divisible by n for all $j \in J_d$, and thus that $\sum_{j \in J_d} \lambda_j u_j \in n\mathbb{Z}$. Moreover, again using our assumptions, it follows by definition that if $i \in \{1, ..., r\} - J_d$, then $\lambda_i u_i \in d\mathbb{Z}$. Hence, we can write u as

$$u = \underbrace{\sum_{i \notin I_d} \lambda_i u_i}_{\in d\mathbb{Z}} + \underbrace{\sum_{j \in (I_d - J_d)} \lambda_j u_j}_{\in d\mathbb{Z}} + \underbrace{\sum_{j \in J_d} \lambda_j u_j}_{\in n\mathbb{Z}}. \tag{3.1}$$

Using the fact that $\gcd(u, d) = 1$, it follows that $J_d \neq \emptyset$. Moreover, we must also have that $\sum_{j \in J_d \neq \emptyset} \lambda_j u_j \notin d\mathbb{Z}$.

Write $\sum_{j\in J_d}\lambda_j u_j = nc$, for some $c\in\mathbb{Z}_{>0}$. Recall that $u = n^* + md$, for some $m\in\mathbb{Z}_{\geq 0}$ such that $u < (d-1)n$. It follows that $c < d-1$ (note that $d \geq 3$). Moreover, reducing (3.1) modulo d, we get

$$\begin{array}{lll} & n^* & \equiv nc \pmod d \\ \therefore & -n & \equiv nc \pmod d \\ \therefore & n(c+1) & \equiv 0 \pmod d. \end{array}$$

Since n is coprime to d, it follows that $c+1\equiv 0 \pmod d$. But $c\in\{1,...,d-2\}$, meaning that $c+1\in\{2,...,d-1\}$ is coprime to d, a contradiction. Hence, for each admissible partition $u=\lambda_1u_1+...+\lambda_ru_r$, there exists at least one $i\in\{1,...,r\}$ with λ_i coprime to d and $u_i>1$ coprime to both n and d. □

We now resume the proof of the main theorem. We make two claims.

Claim 3.5. *In the notation and assumptions as above,*

(1) $\deg(g)\in S_{d,n}$*;*
(2) $\deg(g)=\sum_{i=1}^r\lambda_iu_i$ *is an admissible partition.*

Proof. (1) By Lemma 2.7, we have that $\deg(\varphi(r)) = d\deg(r)$. Since $\varphi(r)=fg$, it follows that $\deg(g) = -n + d\deg(r)$. Since, moreover, $1\leq \deg(g) < n(d-1)$, it follows that $\deg(g)\in S_{d,n}$, as claimed.

(2) Suppose, for a contradiction, that

$$\deg(g)=\sum_{i=1}^r\lambda_iu_i$$

is an inadmissible partition. This means that there exists some i with $u_i=1$ and $\gcd(\lambda_i,d)=1$. Since $u_i=\deg(p_i)$, this means that g has a monic linear factor $p_i\in k[t]$ with exponent coprime to d. We note that $p_i\nmid f$, since f is irreducible and $\deg(p_i)=1<\deg(f)=n$. Hence, p_i is a monic linear factor of fg appearing with exponent coprime to d in the factorisation of fg. By Lemma 2.6, this means that $\varphi_{k(p_i)}$ is isotropic. But $k(p_i)\cong k$, which implies that φ is isotropic, a contradiction to the assumption that φ is anisotropic. Hence, $\deg(g)=\sum_{i=1}^r\lambda_iu_i$ is an admissible partition, as claimed. □

By Claims 3.5 and 3.4, there exists some $i\in\{1,...,r\}$ with $\gcd(\lambda_i,d)=1$ and $u_i>1$ with $\gcd(u_i,nd)=1$. This corresponds to an irreducible factor p_i of degree u_i of g with exponent λ_i coprime to d. We notice that $p_i\nmid f$, since both f and p_i are irreducible and $\deg(p_i)=u_i\neq n=\deg(f)$. Hence, p_i is a monic irreducible factor of fg of exponent λ_i coprime to d. By Lemma 2.6, this implies that $\varphi_{k(p_i)}$ is isotropic. By letting $L:=k(p_i)=k[t]/(p_i(t))$, we see that $[L:k]=u_i$ satisfies $\gcd([L:k],nd)=1$, as required.

In order to show that any L/k found by using the above method satisfies $[L:k]\leq nd-n-d$, it suffices to show that $u_{\max}:=\max S_{d,n}=nd-n-d$, because then we can just notice that, for any $u\in S_{d,n}$ with $u\neq u_{\max}$, if $(a_1,...,a_r)$ is an admissible (in the sense that $a_i>1$ for all $i=1,...,r$) partition into positive integers for u, then $(a_1,...,a_r,jd)$ is an admissible partition for $u_{\max}$, for some $j\in\mathbb{Z}_{\geq 1}$, and so $[L:k]$ will necessarily come from some admissible partition of $u_{\max}$.

Claim 3.6. *For any positive integers* $n,d\geq 2$ *with* $\gcd(d,n)=1$ *we have*

$$\max S_{d,n}=nd-n-d.$$

Proof. We assume first that $n < d$. If $n^* \in \{1, 2, ..., d-1\}$ is such that $n^* \equiv -n \bmod d$, then, since $n < d$, we have $n^* = d - n$. Hence,

$$\begin{aligned} S_{d,n} &= \{d - n + jd : j \in \mathbb{Z}_{\geq 0} \text{ and } d - n + jd < (d-1)n\} \\ &= \{d - n + jd : j \in \{0, 1, ..., n-2\}\} \end{aligned}$$

and so $\max S_{d,n} = d - n + (n-2)d = dn - n - d$.

Assume now that $d < n$. If $n^* \in \{1, 2, ..., d-1\}$ is such that $n^* \equiv -n \bmod d$, then, since $d < n$, we can write $n^* = \alpha d - n$ where α is the unique positive integer strictly between $\frac{n}{d}$ and $\frac{d+n}{d}$. Hence,

$$\begin{aligned} S_{n,d} &= \{\alpha d - n + jd : j \in \mathbb{Z}_{\geq 0} \text{ and } \alpha d - n + jd < (d-1)n\} \\ &= \{\alpha d - n + jd : j \in \{0, 1, ..., n - \alpha - 1\}\} \end{aligned}$$

and so $\max S_{n,d} = \alpha d - n + (n - \alpha - 1)d = dn - n - d$.

So, in any case, $\max S_{n,d} = dn - n - d$, as required. □

Finally, if we have an explicit non-trivial solution over K, then, in the above proof, we also have an explicit $v \in V[t]$, which implies that h, w, r are also explicit, and thus that g is explicit as well. Then the factorisation $g = g^* \prod_{i=1}^{r} p_i^{\lambda_i}$ into its irreducible factors is also explicit, and we get all its irreducible factors p_i with $\gcd(\deg(p_i), nd) = 1$ and $\gcd(\lambda_i, d) = 1$; for each such factor, $L = k[t]/(p_i(t))$ is explicitly computed. □

Remark 3.7. We remark that the statement of Theorem 3.1 is completely symmetric in n and d (taking also into account Springer's theorem in the case when either d or n is equal to 2).

Example 3.8. Let φ be a diagonal-full cubic form on a finite-dimensional vector space V over a field k with φ_K isotropic for some simple extension K/k of degree $n := [K : k] = 2$. Since $(d, n) = (3, 2)$, we have $S_{d,n} = \{1\}$. By following the proof of Theorem 3.1, this implies that φ is already isotropic over k.

Example 3.9. Let φ be a diagonal-full cubic form on a finite-dimensional vector space V over a field k with φ_K isotropic for some simple extension K/k of degree $n := [K : k] = 5$. Since $(d, n) = (3, 5)$, we have $S_{d,n} = \{1, 4, 7\}$. By considering the partitions into positive integers of each $u \in S_{d,n}$, we can find possible values for $[L : k]$. We note that all the partitions into positive integers of $u \in \{1, 4\}$ appear as subpartitions of $u = 7$, so we just need to consider $u = 7$.

- $u = 7$. If a partition of 7 into positive integers involves 1 or 2, then, following the notation in the proof of Theorem 3.1, we know that there exists some $i \in \{1, ..., r\}$ with $u_i \in \{1, 2\}$ and $\gcd(\lambda_i, 3) = 1$, which implies that there exists some L/k with $[L : k] \in \{1, 2\}$ and φ_L isotropic; if $[L : k] = 2$, then we can use Example 3.8 to conclude that φ is isotropic over k. Hence, it suffices to consider those partitions of 7 not involving 1 or 2. The only partitions of 7 into positive integers not involving 1 or 2 are (7) and (4, 3). Hence, in this case, there always exists some $i \in \{1, ..., r\}$ with $\gcd(\lambda_i, 3) = 1$ and $u_i \in \{1, 2(\leftrightarrow 1), 4, 7\}$.

Hence, we conclude that there is always an extension L/k with $[L : k] \in \{1, 4, 7\}$ and φ_L isotropic.

3.1. **The case when n or d is not prime.** When $n := [K : k]$ or d is not prime, the proof of Claim 3.4 might fail. However, even in this case, we can use the proof of Theorem 3.1, with some care, to determine the possible degrees of L/k with φ_L isotropic. We illustrate this procedure by specialising to the case when $d = 3$ and $n = 4$.

Theorem 3.10. *Let k be a field and let φ be a cubic diagonal-full form on a finite-dimensional vector space V over k. If there exists a simple extension K/k with $[K : k] = 4$ such that φ_K is isotropic, then there exists a finite extension L/k with $[L : k] \in \{1, 5\}$ such that φ_L is isotropic.*

Proof. The first part of the proof of Theorem 3.10 is identical to that of Theorem 3.1, so we just sketch it. Let $K = k(\alpha)$ and let $f \in k[t]$ be the minimal (irreducible) polynomial of α over k. Since, by assumption, $\varphi_{k(f)}$ is isotropic, it follows that there exists some $v \in V[t]$ such that $\varphi(v) \equiv 0 \bmod f$ but $v \not\equiv 0 \bmod f$. By the division algorithm, there exist some $0 \neq h \in k[t]$ and $w, r \in V[t]$ such that

$$hv = fw + r,$$

with $\deg(r) < \deg(f) = 4$, with $f \mid \varphi(r)$, and with $\varphi(r) \neq 0$. We write $\varphi(r) = fg$ for some $g \in k[t]$, which we can show satisfies $0 < \deg(g) < n(d-1) = 8$.

Let the factorisation of g into irreducible factors be

$$g = g^* \prod_{i=1}^{r} p_i^{\lambda_i}$$

where $g^* \in k^\times$ and, for each $i = 1, ..., r$, the distinct polynomials $p_i \in k[t]$ are monic and irreducible, with $\deg(p_i) =: u_i$ and $\lambda_i \geq 1$. Then

$$\deg(g) = \sum_{i=1}^{r} u_i \lambda_i < 8.$$

Let us compute $S_{d,n}$ for $(d, n) = (3, 4)$. We have $n^* = 2$. Hence,

$$S_{3,4} = \{2 + 3j : j \in \mathbb{Z}_{\geq 0} \text{ and } 2 + 3j < 8\} = \{2, 5\}.$$

We notice that $\deg(g) \in S_{3,4} = \{2, 5\}$, since, by Lemma 2.7, we have that $\deg(\varphi(r)) = 3\deg(r)$ and since $\varphi(r) = fg$, implying that $0 < \deg(g) = -4 + 3\deg(r) < 8$.

We remark that if $\deg(g) = \sum_{j=1}^{r} u_j \lambda_j$ is such that $u_i \in \{1, 2\}$ and $\gcd(\lambda_i, 3) = 1$ for some $i \in \{1, ..., r\}$, then we know that g has an irreducible factor p_i of degree either 1 or 2 appearing in the factorisation of g with exponent λ_i. Moreover, such a factor p_i cannot divide f, since f is irreducible and $\deg(f) = 4$, while $\deg(p_i) < \deg(f) = 4$. Hence, p_i is an irreducible factor of fg appearing with exponent λ_i, and thus Lemma 2.6 yields that $L := k[t]/(p_i(t))$ is a field of degree 1 or 2 with φ_L isotropic. But if $[L : k] = 2$, it is easy to check that $S_{3,2} = \{1\}$ and thus φ is already isotropic over k. Hence, since if $\deg(g) = \sum_{j=1}^{r} u_j \lambda_j$ satisfies $u_i \in \{1, 2\}$ and $\gcd(\lambda_i, 3) = 1$ for some $i \in \{1, ..., r\}$ then φ is already isotropic over k, in the considerations below we will omit considering any partitions of 2 or 5 into positive integers having a 1 or a 2 in them, since any such partition would imply that $u_i \lambda_i \in \{1, 2\}$ for some i, meaning that $u_i \in \{1, 2\}$.

We distinguish two cases, depending on whether $\deg(g)$ is 2 or 5.

- **Case** $\deg(g) = 2$**.** Since any partition of 2 into positive integers involves a 1 or a 2, this implies that, in $2 = \sum_{j=1}^{r} u_j \lambda_j$, there is always some $u_i \in \{1, 2\}$ with $\gcd(\lambda_i, 3) = 1$. Hence, by the discussion above, φ is already isotropic over k.

- **Case** $\deg(g) = 5$**.** Since the only partition of 5 into positive integers that does not involve a 1 or a 2 is (5), we have, for this partition, that $r = 1$ and $u_1\lambda_1 = 5$, implying that $u_1 \in \{1, 5\}$ and $\gcd(\lambda_1, 3) = 1$. Notice that $p_1 \nmid f$ since f is irreducible of degree 4 and $\deg(p_1) = u_1 \in \{1, 5\}$; hence, p_1 is an irreducible factor in fg of exponent λ_i coprime to 3 and by Lemma 2.6, there is some L/k with $[L : k] \in \{1, 5\}$ and φ_L isotropic. It follows that, by considering all the partitions of 5 into positive integers, if $\deg(g) = 5 = \sum_{j=1}^{r} u_j\lambda_j$, then we can always find some L/k with $[L : k] \in \{1, 5\}$ and φ_L isotropic.

Hence, putting together all the possibilities from the two cases above, we conclude that there is always an extension L/k with $[L : k] \in \{1, 5\}$ and φ_L isotropic, as required. □

A similar proof as the one of Theorem 3.10 yields a procedure that can also give information about the possible degrees $[L : k]$ for other values of n (and d).

Example 3.11. Let φ be a diagonal-full cubic form on a finite-dimensional vector space V over a field k with φ_K isotropic for some simple extension K/k of degree $n := [K : k] = 10$. Since $(d, n) = (3, 10)$, we have $S_{d,n} = \{2, 5, 8, 11, 14, 17\}$. By considering the partitions into positive integers of each $u \in S_{d,n}$, and by using the knowledge that we have about the cases $(d, n) \in \{(3, 2), (3, 4)\}$, we can find possible values for $[L : k]$. We note that all the partitions into positive integers of $u \in \{2, 5, 8, 11, 14\}$ appear as subpartitions of $u = 17$, so we just need to consider $u = 17$.

- $u = 17$. If a partition of 17 into positive integers involves 1,2, or 4, then we know that there exists some L/k with $[L : k] \in \{1, 5\}$ and φ_L isotropic. The only partitions of 17 into positive integers not involving 1, 2, or 4 are (17), (14, 3), (12, 5), (11, 6), (11, 3, 3), (10, 7), (9, 8), (9, 5, 3), (8, 6, 3), (8, 3, 3, 3), (7, 7, 3), (7, 5, 5), (6, 6, 5), (6, 5, 3, 3), and (5, 3, 3, 3, 3). Hence, in this case, there always exists some $i \in \{1, ..., r\}$ with $\gcd(\lambda_i, 3) = 1$ and $u_i \in \{1, 2(\leftrightarrow 1), 4(\leftrightarrow 1 \text{ or } 5), 5, 7, 8, 11, 14, 17\}$.

Hence, we conclude that there is always an extension L/k with $[L : k] \in \{1, 5, 7, 8, 11, 14, 17\}$ and φ_L isotropic.

Summary of the procedure. To summarise, the general procedure for any positive integers $d, n \geq 2$ with $\gcd(d, n) = 1$ is the following.

(1) If φ is isotropic, we are done. Assume that φ is anisotropic over k and that φ_K is isotropic for some simple extension K/k of degree $n := [K : k] \geq 2$ coprime to d.
(2) Let f be the (irreducible) minimal polynomial of K/k, so that $\deg(f) = n$.
(3) There is some $0 \neq r \in V[t]$ with $\deg(r) < n$ and $0 \neq \varphi(r) = fg$, for some $g \in k[t]$ with $0 < \deg(g) < n(d - 1)$ and $\deg(g) \in S_{d,n}$, since $\deg(\varphi(r)) = d \deg(r)$.
(4) Let $g = g^* \prod_{i=1}^{r} p_i^{\lambda_i}$ be the factorisation of g into irreducible polynomials over k, where g^* is the leading coefficient of g and the p_i's are distinct monic irreducible polynomials of degree $u_i := \deg(p_i)$. Then $\deg(g) = \sum_{i=1}^{r} u_i\lambda_i \in S_{d,n}$.
(5) Let $u_{\max} \in S_{d,n}$ be the largest element; we have seen that $u_{\max} = nd - n - d$ (see the proof of Claim 3.6). For any $u \in S_{d,n}$ with $u \neq u_{\max}$, any partition $(a_1, ..., a_t)$ of u into positive integers is a subpartition of the partition $(a_1, ..., a_t, jd)$ of $u_{\max}$, for some $j \geq 1$.
(6) Let $(a_1, ..., a_r)$ be a partition of $u_{\max}$ into positive integers. For each a_i with $\gcd(a_i, d) = 1$, writing $a_i = u_i\lambda_i$ yields that u_i can be any positive divisor of a_i; if $u_i \mid a_i$ and $n \nmid u_i$, then we have found a $p_i \nmid f$ (since f is irreducible and $\deg(p_i) = u_i \neq n = \deg(f)$)

appearing in the factorisation of $fg = \varphi(r)$ with exponent λ_i coprime to d. By Lemma 2.6, any such p_i yields a field $L := k[t]/(p_i(t))$ with $\gcd([L:k], d) = 1$, $n \nmid [L:k]$ and φ_L isotropic.

Remark 3.12. If there exists a partition $(a_1, ..., a_r)$ of $u_{\max}$ into positive integers with $\gcd(a_i, d) > 1$ for all $i = 1, ..., r$, then unfortunately we cannot get any new information from the procedure. Moreover, if there exists a partition $(a_1, ..., a_r)$ of $u_{\max}$ into positive integers such that, for any a_i with $\gcd(a_i, d) = 1$, we have that $n \mid a_i$, then unfortunately we cannot get any new information in this case as well, because the existence of such a partition implies that n could possibly divide $[L:k]$ and that possibly $K \subset L$.

(7) Hence, by considering all the possible partitions of $u_{\max}$ into positive integers, if the situations described in Remark 3.12 do not occur, then we know that there exists some L/k with $\gcd([L:k], d) = 1$, $n \nmid [L:k]$, and φ_L isotropic, where $[L:k]$ is in the set of all possible degrees found by considering all the partitions of $u_{\max}$.

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The American University of Paris, 5 Boulevard de La Tour-Maubourg, 75007 Paris, France

Email address: fbalestrieri@aup.edu