

DESCENT AND ÉTALE-BRAUER OBSTRUCTIONS FOR 0-CYCLES

FRANCESCA BALESTRIERI AND JENNIFER BERG

ABSTRACT. We fill a gap in the theory of obstructions for 0-cycles by defining, in this context, an analogue of the classical descent set for rational points on varieties over number fields. This leads to, among other things, a definition of the étale-Brauer obstruction set for 0-cycles, which we show is contained in the Brauer-Manin set. We then transfer some tools and techniques used to study the arithmetic of rational points into the setting of 0-cycles. For example, we extend the strategy developed by Y. Liang, relating the arithmetic of rational points over finite extensions of the base field to that of 0-cycles, to torsors. We give applications of our results to study the arithmetic behaviour of 0-cycles for Enriques surfaces, torsors given by (twisted) Kummer varieties, universal torsors, and torsors under tori.

1. INTRODUCTION

1.1. Obstruction sets for rational points. Let k be a number field and X be a smooth, proper, geometrically integral variety over k . In order to investigate questions concerning the qualitative arithmetic behaviour of the set $X(k)$ of rational points on X , a well-known strategy is that of defining so-called *obstruction sets* – namely, certain subsets of the set $X(\mathbb{A}_k)$ of adelic points on X that still contain $X(k)$ – and trying to exploit the more tractable local nature of these sets to study the specific question at hand. Such sets can be used, for example, to determine whether $X(k) = \emptyset$ by refining local-to-global principles, or to classify varieties according to their arithmetic behaviour. The theory of obstruction sets in the context of rational points has been developed quite extensively over the last several decades. In [Man71], Manin first constructed an obstruction set by using the Brauer group $\mathrm{Br}(X) := H_{\mathrm{et}}^2(X, \mathbb{G}_m)$ and class field theory to define what is now known as the *Brauer-Manin set*, namely

$$X(\mathbb{A}_k)^{\mathrm{Br}} := \bigcap_{\alpha \in \mathrm{Br} X} \left\{ (x_v)_{v \in \Omega_k} \in X(\mathbb{A}_k) : \sum_{v \in \Omega_k} \mathrm{inv}_v \alpha(x_v) = 0 \right\}.$$

Later, Colliot-Thélène and Sansuc [CTS87] defined a new type of obstruction sets known as *descent sets*, based not on the Brauer group, but rather on the notion of torsors under algebraic groups. That is, if G is a linear algebraic group over k and if $[g : Y \rightarrow X] \in H_{\mathrm{et}}^1(X, G)$ is (the k -class of) a G -torsor over X , then we can define the *descent set associated to g* as

$$X(\mathbb{A}_k)^g := \bigcup_{[\tau] \in H^1(k, G)} g^\tau(Y^\tau(\mathbb{A}_k)),$$

where the G^τ -torsor $g^\tau : Y^\tau \rightarrow X$ is the twist of the torsor $g : Y \rightarrow X$ by τ (see [Sko01, Ch. 2]). We can also combine these two main types of obstructions together to yield potentially finer obstruction sets. For example, the *étale-Brauer set*,

$$X(\mathbb{A}_k)^{\mathrm{et}, \mathrm{Br}} := \bigcap_{F \text{ finite lin alg } k\text{-group}} \bigcap_{[f : Y \rightarrow X] \in H_{\mathrm{et}}^1(X, F)} \bigcup_{\tau \in H_{\mathrm{et}}^1(k, F)} f^\tau(Y^\tau(\mathbb{A}_k)^{\mathrm{Br}}),$$

can be obtained by considering the Brauer-Manin sets of all finite étale covers of X . Although the obstruction sets defined using the Brauer group and those defined by using torsors have quite different natures, there is sometimes a close interrelation between them – see, for example [Sko01], [Har02], [Sto07], [Dem09], [Sko09a], [Bal16], [Cao20]. Unfortunately, despite the richness of the different obstructions available, the

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arithmetic behaviour of rational points on varieties is still not completely understood in general: for example, in [Poo10] Poonen constructed a variety X over a number field k such that $X(\mathbb{A}_k)^{\text{et Br}} \neq \emptyset$ but $X(k) = \emptyset$, showing that even the finest obstruction set currently at our disposal, i.e., the étale-Brauer set, is not quite refined enough to capture the lack of rational points.

1.2. Obstruction sets for 0-cycles. Let us now consider the theory of obstruction sets in a different context, namely, that of 0-cycles on X , which can be viewed as generalisations of rational points. Given a fixed integer d , the main object of interest is the set $Z_0^d(X)$ of 0-cycles of degree d on X , that is, the set of all formal \mathbb{Z} -sums $z := \sum_x n_x x$ of closed points x on X such that $\deg(z) := \sum_{x \in X} n_x [k(x) : k] = d$, where $k(x)$ is the residue field of x . Much in the same way as we did for rational points, we can ask qualitative arithmetic questions about $Z_0^d(X)$ – for example, whether $Z_0^d(X) = \emptyset$. The basic strategy to tackle such questions remains the same, that is, we consider subsets of the set $Z_0^d(X_{\mathbb{A}_k})$ of adelic degree d 0-cycles on X and try to exploit the local nature of these subsets to draw conclusions about $Z_0^d(X)$.

There are two striking differences between the theory of obstructions for rational points and that for 0-cycles, however. First, while there is a wide range of obstruction sets and tools currently available for rational points, the same cannot be said for 0-cycles. One key complication is that, in the definition of 0-cycles, several different field extensions need to be considered at once, thereby creating challenges in generalising obstruction sets from rational points to this new context. Currently, the only obstruction set that has been generalised is the Brauer-Manin obstruction [CT95], defined in a similar way as in the context of rational points, but with the extra use of corestriction maps to deal with the different residue fields of the support of the 0-cycles, namely

$$Z_0^d(X_{\mathbb{A}_k})^{\text{Br}} := \bigcap_{\alpha \in \text{Br } X} \left\{ \left(\sum_{x_v \in X_{k_v}} n_{x_v} x_v \right)_{v \in \Omega_k} \in Z_0^d(X_{\mathbb{A}_k}) : \sum_{v \in \Omega_k} \sum_{x_v \in X_{k_v}} n_{x_v} \text{inv}_v (\text{cores}_{k_v(x_v)/k_v} \alpha(x_v)) = 0 \right\}.$$

The second difference concerns the arithmetic behaviour of 0-cycles. While, as previously mentioned, even the finest known obstruction cannot explain all failures of the Hasse principle for rational points, the qualitative arithmetic of 0-cycles is, conjecturally, completely captured by the Brauer-Manin obstruction – and thus much more well behaved. Indeed, Colliot-Thélène [CT95] conjectured that the Brauer-Manin obstruction is the only one to weak approximation for 0-cycles of degree 1 on any smooth, proper, geometrically integral variety X over k (see also [KS86]). Colliot-Thélène’s conjecture is encompassed by the following main guiding conjecture in the context of 0-cycles, which is known for a few cases (see [Sal88], [Sai89], [CT99], [ES08], [Wit12], [Lia13], and [CTS21, p. 383-4] for a detailed list of references) but remains open in general.

Conjecture 1.1 (Conjecture (E)). *For any smooth, projective variety X over a number field k , the following complex is exact,*

$$\widehat{\text{CH}_0(X)} \longrightarrow \prod_{v \in \Omega_k} \widehat{\text{CH}'_0(X_{k_v})} \longrightarrow \text{Hom}(\text{Br } X, \mathbb{Q}/\mathbb{Z}),$$

where $\widehat{}$ denotes the profinite completion, and where $\text{CH}'_0(X_{k_v})$ coincides with $\text{CH}_0(X_{k_v})$ at finite places and is a modification of the Chow group at infinite places (see [Wit12, §1.1] or [CT95, §1] for further details).

1.3. Main results. The main aim of this paper is to fill in some significant gaps in the theory of obstructions for 0-cycles by defining an analogue of the descent set associated to a torsor for 0-cycles. Given a fixed $d \in \mathbb{Z}$, a fixed torsor $g : Y \rightarrow X$ under a linear algebraic group G over k , and a finite non-empty set S of field extensions of k of finite degree, we define the *degree d descent set associated to g* (see Definition

3.3) to be

$$Z_0^d(X_{\mathbb{A}_k})^g := \bigcup_{\substack{S \text{ finite non-empty} \\ \text{set of finite field} \\ \text{extensions of } k}} \mathbf{g}_{*,\text{ad}} \left(\prod_{\substack{(T_K \subset H^1(K,G))_{K \in S}: \\ T_K \neq \emptyset \text{ finite for all } K \in S}} \prod_{\substack{((\Delta_\tau)_{\tau \in T_K})_{K \in S}: \\ \Delta_\tau \in \mathbb{Z} \text{ for all } \tau, \\ \sum_{K \in S} [K:k] \sum_{\tau \in T_K} \Delta_\tau = d}} \prod_{K \in S} \prod_{\tau \in T_K} Z_0^{\Delta_\tau}(Y_{\mathbb{A}_K}^\tau) \right),$$

where

$$\left(\left(\left(\left(\sum_{y \in Y_{K_w}^\tau} n_y y \right)_{w \in \Omega_K: w|v} \right)_{\tau \in T_K} \right)_{K \in S} \right)_{v \in \Omega_k} \xrightarrow{\mathbf{g}_{*,\text{ad}}} \left(\sum_{K \in S} \sum_{\tau \in T_K} \sum_{w \in \Omega_K: w|v} \sum_{y \in Y_{K_w}^\tau} n_y (s_{K_w} \circ g_{K_w}^\tau)_*(y) \right)_{v \in \Omega_k}$$

and where $g_{K_w}^\tau : Y_{K_w}^\tau \rightarrow X_{K_w}$ and $s_{K_w} : X_{K_w} \rightarrow X_{k_v}$ are the natural maps. This set $Z_0^d(X_{\mathbb{A}_k})^g$ contains $Z_0^d(X)$ and, moreover, generalises the descent set $X(\mathbb{A}_k)^g$ when $d = 1$.

The definition of the descent set leads to, among other results, a definition of the étale-Brauer obstruction for 0-cycles (Definition 3.5), namely

$$Z_0^d(X_{\mathbb{A}_k})^{\text{et,Br}} := \bigcup_{\substack{S \neq \emptyset \text{ finite set} \\ \text{of finite field} \\ \text{extns of } k}} \bigcap_{\substack{G \text{ finite lin.} \\ \text{alg. } k\text{-group}}} \bigcap_{\substack{[f:Y \rightarrow X] \\ \in H^1(X,G)}} \mathbf{f}_{*,\text{ad}} \left(\prod_{\substack{(T_K \subset H^1(K,G))_{K \in S} \\ T_K \neq \emptyset \text{ finite} \\ \text{for all } K \in S}} \prod_{\substack{((\Delta_\tau)_{\tau \in T_K})_{K \in S}: \\ \Delta_\tau \in \mathbb{Z} \text{ for all } \tau, \\ \sum_{K \in S} [K:k] \sum_{\tau \in T_K} \Delta_\tau = d}} \prod_{K \in S} \prod_{\tau \in T_K} Z_0^{\Delta_\tau}(Y_{\mathbb{A}_K}^\tau)^{\text{Br}} \right).$$

More generally, it affords the possibility of a more systematic theory of obstructions for 0-cycles, analogous to that of rational points. It also provides new potential avenues for investigating Conjecture 1.1: if Conjecture 1.1 holds, it implies that the Brauer-Manin obstruction is the only one for weak approximation for 0-cycles (see [Lia13, Theorem A]); this, on the other hand, should imply that the étale-Brauer set and the Brauer-Manin set for 0-cycles are equal in the Chow group.

Equipped with our new definitions, in §4 we begin to transfer some of the tools and techniques used to study descent obstructions to rational points into the 0-cycles setting. We do so in the spirit of some of the ideas developed by Liang in [Lia13]. His strategy consists of trying to show that, under certain conditions, if some arithmetic property (such as, for example, the Brauer-Manin obstruction being the only one for weak approximation) holds for rational points for enough field extensions of finite degree of the base field, then an analogous arithmetic property holds for 0-cycles as well. In §4.2, we extend Liang's strategy to torsors.

Theorem (Theorem 4.2). *Let X be a smooth, proper, geometrically integral variety over a number field k . Let $f : Y \rightarrow X$ be (respectively, a proper map which is) an F -torsor for some linear algebraic group F over k and with Y geometrically integral. Let d be any integer. Assume that*

- (i) *for any finite extension K/k and any $\tau \in H^1(K, F)$, the quotient $\text{Br}_{nr}(Y_K^\tau) / \text{Br}_0(Y_K^\tau)$ is finite, and there exists a finite extension K'_τ of K so that for all finite extensions L of K linearly disjoint from K'_τ over K , the homomorphism*

$$\text{res}_{L/K} : \text{Br}_{nr}(Y_K^\tau) / \text{Br}_0(Y_K^\tau) \rightarrow \text{Br}_{nr}(Y_L^\tau) / \text{Br}_0(Y_L^\tau)$$

is surjective;

- (ii) *for any finite extension L/k , we have that $X_L(\mathbb{A}_L)^{f_L, \text{Br}_{nr}} \neq \emptyset$ if and only if $X(L) \neq \emptyset$ (respectively, if $X_L(\mathbb{A}_L)^{f_L, \text{Br}_{nr}} \neq \emptyset$, then weak approximation holds for X_L).*

Then f -descent with unramified Brauer obstruction is the only obstruction to the Hasse principle (respectively, weak approximation) for 0-cycles of degree d on X .

In §5 we consider several applications of our tools, often in contexts in which there is deeper knowledge of the obstructions that govern the existence and density of rational points. Firstly, building on work by Ieronymou [Ier21], we get an application of the étale-Brauer set for Enriques surfaces – at least, conditionally on a conjecture by Skorobogatov.

Theorem (Theorem 5.3). *Let X be an Enriques surface over a number field k and let $f : Y \rightarrow X$ be a K3 covering of X , i.e. a $\mathbb{Z}/2\mathbb{Z}$ -torsor over X with Y a K3 surface. Let $d \in \mathbb{Z}$. Assume that Conjecture 5.1 holds. Then, for any positive integer n , if $(z_v)_v \in Z_0^d(X_{\mathbb{A}_k})^{f, \text{Br}}$ then there exists a global 0-cycle $z_n \in Z_0^d(X)$ such that z_n and $(z_v)_v$ have the same image in $\text{CH}_0(X_{k_v})/n$ for all $v \in \Omega_k$.*

As a further application, we study the arithmetic of 0-cycles in the context of what can be considered as higher-dimensional generalisations of Enriques surfaces, where we look at the case where our torsors are (twisted) Kummer varieties.

Theorem (Theorem 5.6). *Let X be a smooth, proper, geometrically integral variety over a number field k . Let $f : Y \rightarrow X$ be a torsor under some linear algebraic group F over k , where Y is a (twisted) Kummer variety over k . Let $d \in \mathbb{Z}$ be odd. Assume that Question 5.5 has a positive answer. Then $Z_0^d(X_{\mathbb{A}_k})^{f, \text{Br}\{2\}} \neq \emptyset$ implies $Z_0^d(X) \neq \emptyset$.*

We remark that the recent preprint [Ier22] by Ieronymou should remove both the condition that d be odd and the need to restrict to the 2-primary part of the Brauer group. We nonetheless give the theorem in this form, as its proof can potentially be applied to other situations.

Finally, we consider universal torsors and torsors under tori. In the rational points setting, it is well-known that, if a universal torsor $g : W \rightarrow X$ exists, then $X(\mathbb{A}_k)^g = X(\mathbb{A}_k)^{\text{Br}_1}$ [Sko01, Lemma 2.3.1]. In Theorem 5.10, we recover a slightly weaker analogous result for 0-cycles.

Theorem (Theorem 5.10). *Let X be a smooth, proper, geometrically integral variety over k with $\text{Pic } \overline{X}$ finitely generated as a \mathbb{Z} -module. Suppose that a universal torsor $g : W \rightarrow X$ under some group G of multiplicative type over k exists. Then, for any integer $d \in \mathbb{Z}$, for any positive integer n , and for any finite subset $S' \subset \Omega_k$ of places of k , we have that*

- (1) $\mathbb{Z}_0^d(X_{\mathbb{A}_k})^g \neq \emptyset$ implies $\mathbb{Z}_0^d(X_{\mathbb{A}_k})^{\text{Br}_1} \neq \emptyset$;
- (2) if, moreover, $\text{Br}_1(X)/\text{Br}_0(X)$ is finite, then $(z_v)_v \in \mathbb{Z}_0^d(X_{\mathbb{A}_k})^{\text{Br}_1}$ implies that there exists some $(u_v)_v \in \mathbb{Z}_0^d(X_{\mathbb{A}_k})^g$ such that z_v and u_v have the same image in $\text{CH}_0(X_{k_v})/n$ for all $v \in S'$.

Similarly, for torsors under tori we obtain the following result, in the spirit of a result by Harpaz and Wittenberg (see [HW20, Théorème 2.1]).

Theorem (Theorem 5.13). *Let X be a smooth, proper, geometrically integral variety over k . Let $f : Y \rightarrow X$ be a torsor under a k -torus T . Assume that $\text{Br } X/\text{Br}_0 X$ is finite and that there is some finite extension F/k such that $\text{res}_{l/k} : \text{Br } X/\text{Br}_0 X \rightarrow \text{Br}(X_l)/\text{Br}_0(X_l)$ is surjective for all finite extensions l/k linearly disjoint from F over k . Then, for any integer $d \in \mathbb{Z}$, for any positive integer n , and for any finite subset $S' \subset \Omega_k$ of places of k , we have that $(z_v)_v \in \mathbb{Z}_0^d(X_{\mathbb{A}_k})^{\text{Br}}$ implies that there exists some $(u_v)_v \in \mathbb{Z}_0^d(X_{\mathbb{A}_k})^{f, \text{Br}_{nr}}$ such that z_v and u_v have the same image in $\text{CH}_0(X_{k_v})/n$ for all $v \in S'$.*

1.4. Notation and terminology. Let k be a number field and let \overline{k} denote a fixed algebraic closure of k . Let Ω_k denote the set of non-trivial places of k and \mathbb{A}_k the ring of adèles of k . For a k -scheme X , write $X_K := X \times_{\text{Spec } k} \text{Spec } K$ for the base change of X to the extension K/k , and $\overline{X} := X_{\overline{k}}$. A variety over k is defined as a separated scheme of finite type over k . For a closed point x of a k -scheme X , let $k(x)$ denote the residue field (over k). The Brauer group $\text{Br}(X) := H_{\text{et}}^2(X, \mathbb{G}_m)$ of a k -variety X is equipped with a natural filtration, $\text{Br}_0(X) \subset \text{Br}_1(X) \subset \text{Br}(X)$, where $\text{Br}_0(X) := \text{im}(\text{Br}(k) \rightarrow \text{Br}(X))$ and $\text{Br}_1(X) := \ker(\text{Br } X \rightarrow \text{Br } \overline{X})$. For an abelian group A and an integer $d > 0$, we let $A[d]$ denote the d -torsion subgroup of A . When d is prime, let $A\{d\}$ denote the d -primary part $\varprojlim_n A[d^n]$ of A .

2. GLOBAL CASE

Let X be a smooth, proper, geometrically integral variety over a number field k . Throughout this section, we fix $f: Y \rightarrow X$ to be a G -torsor for some linear algebraic group G over k and a Y smooth k -variety. In this section, we define the sets $Z_0^d(X(\mathbb{A}_k))^f$ and $Z_0^d(X(\mathbb{A}_k))^{\text{et}, \text{Br}}$ for 0-cycles analogous to the corresponding sets for rational points, namely $X(\mathbb{A}_k)^f$ and $X(\mathbb{A}_k)^{\text{et}, \text{Br}}$. In order to extend the definitions that arise in the context of rational points, one needs to keep careful track of residue fields and degrees. This leads us to the following collection of definitions.

Definition 2.1. Let k be a number field and let \bar{k} be a fixed separable closure of k . We let

$$\mathfrak{F}_k := \{K \subset \bar{k} : K \text{ is a finite field extension of } k\}.$$

Recall that if $z := \sum n_y y \in Z_0(Y)$, the *pushforward* of z is

$$f_*(z) = \sum_{y \in Y} n_y f_*(y) = \sum_{y \in Y} n_y [k(y) : k(f(y))] f(y) \in Z_0(X).$$

Definition 2.2. We define the map

$$\mathbf{f}_*: \coprod_{\substack{S \subset \mathfrak{F}_k \\ S \neq \emptyset \text{ finite}}} \coprod_{\substack{(T_K \subset H^1(K, G))_{K \in S} \\ T_K \neq \emptyset \text{ finite for all } K \in S}} \prod_{K \in S} \prod_{\tau \in T_K} Z_0(Y_K^\tau) \rightarrow Z_0(X)$$

as follows. Let

$$z := \left(\left(\sum_{y \in Y_K^\tau} n_y y \right)_{\tau \in T_K} \right)_{K \in S}.$$

Then $\mathbf{f}_*(z)$ is the 0-cycle on X given by

$$\begin{aligned} \mathbf{f}_*(z) &= \sum_{K \in S} \sum_{\tau \in T_K} \sum_{y \in Y_K^\tau} n_y (s_K \circ f_K^\tau)_*(y) \\ &= \sum_{K \in S} \sum_{\tau \in T_K} \sum_{y \in Y_K^\tau} n_y [K(y) : k(s_K \circ f_K^\tau(y))] (s_K \circ f_K^\tau)(y), \end{aligned}$$

where $f_K^\tau: Y_K^\tau \rightarrow X_K$ and $s_K: X_K \rightarrow X$ are the natural morphisms.

The map \mathbf{f}_* is compatible with degrees of 0-cycles in the following way.

Lemma 2.3. Fix an integer d . Then

$$\mathbf{f}_*: \coprod_{S \subset \mathfrak{F}_k} \coprod_{(T_K \subset H^1(K, G))_{K \in S}} \prod_{\substack{((\Delta_\tau)_{\tau \in T_K})_{K \in S}: \\ \Delta_\tau \in \mathbb{Z} \text{ for all } \tau, \\ \sum_{K \in S} [K:k] \sum_{\tau \in T_K} \Delta_\tau = d}} \prod_{K \in S} \prod_{\tau \in T_K} Z_0^{\Delta_\tau}(Y_K^\tau) \rightarrow Z_0^d(X).$$

Proof. Let $z := \left(\left(\sum_{y \in Y_K^\tau} n_y y \right)_{\tau \in T_K} \right)_{K \in S}$ be an element in the domain of \mathbf{f}_* . Then,

$$\mathbf{f}_*(z) = \sum_{K \in S} \sum_{\tau \in T_K} \sum_{y \in Y_K^\tau} n_y [K(y) : k(s_K \circ f_K^\tau(y))] (s_K(f_K^\tau(y))).$$

Therefore,

$$\begin{aligned} \deg(\mathbf{f}_*(z)) &= \sum_{K \in S} \sum_{\tau \in T_K} \sum_{y \in Y_K^\tau} n_y [K(y) : k] \\ &= \sum_{K \in S} \sum_{\tau \in T_K} \sum_{y \in Y_K^\tau} n_y [K(y) : K] [K : k] \\ &= \sum_{K \in S} \sum_{\tau \in T_K} [K : k] \sum_{y \in Y_K^\tau} n_y [K(y) : K] \end{aligned}$$

$$\begin{aligned}
&= \sum_{K \in S} \sum_{\tau \in T_K} [K : k] \Delta_\tau \\
&= d,
\end{aligned}$$

where the third equality follows from the fact that the inner summation represents the degree of the 0-cycle $\sum_{y \in Y_K^\tau} n_y y$, which is by definition Δ_τ , and the final equality is by definition of the integers Δ_τ . \square

Definition 2.4. Let $d \in \mathbb{Z}$. The degree d global descent set of X associated to f is

$$Z_0^d(X)^f := \bigcup_{\substack{S \subset \mathfrak{F}_k \\ S \neq \emptyset \text{ finite}}} \mathbf{f}_* \left(\prod_{\substack{(T_K \subset H^1(K, G))_{K \in S} \\ T_K \neq \emptyset \text{ finite for all } K \in S}} \prod_{\substack{((\Delta_\tau)_{\tau \in T_K})_{K \in S} : \\ \Delta_\tau \in \mathbb{Z} \text{ for all } \tau, \\ \sum_{K \in S} [K : k] \sum_{\tau \in T_K} \Delta_\tau = d}} \prod_{K \in S} \prod_{\tau \in T_K} Z_0^{\Delta_\tau}(Y_K^\tau) \right).$$

Proposition 2.5. *There is an equality of sets, $Z_0^d(X)^f = Z_0^d(X)$.*

Proof. The forward containment is the content of Lemma 2.3. For the reverse inclusion, let $z := \sum_{x \in X} n_x x \in Z_0^d(X)$, and suppose that $\{x_1, \dots, x_r\} := \text{supp}(z)$. By the classical properties of torsors for rational points (see [Sko01, §5.3]), for each x_i we take any closed point x'_i of $X_{k(x_i)}$ projecting to x_i , and we have the partition

$$x'_i \in X(k(x_i)) = \coprod_{\sigma \in H^1(k(x_i), G)} f_{k(x_i)}^\sigma \left(Y_{k(x_i)}^\sigma(k(x_i)) \right).$$

Hence there exists some $\sigma_i \in H^1(k(x_i), G)$ such that x_i lifts to some $y_i \in Y_{k(x_i)}^{\sigma_i}(k(x_i))$, which we fix for each x_i and which we can consider as closed points on $Y_{k(x_i)}^{\sigma_i}$. Note that $k(x_i)(y_i) = k(x_i)$, since $k(x_i)(y_i) \subset k(x_i)$ as y_i is a $k(x_i)$ -point. So, if we consider the morphism

$$s_{k(x_i)} \circ f_{k(x_i)}^{\sigma_i} : Y_{k(x_i)}^{\sigma_i} \rightarrow X_{k(x_i)} \rightarrow X,$$

then the pushforward of $y_i \in Z_0(Y_{k(x_i)}^{\sigma_i})$ is

$$\begin{aligned}
(s_{k(x_i)} \circ f_{k(x_i)}^{\sigma_i})_*(y_i) &= [k(x_i)(y_i) : k(s_{k(x_i)} \circ f_{k(x_i)}^{\sigma_i}(y_i))] x_i \\
&= [k(x_i)(y_i) : k(x_i)] x_i \\
&= x_i \in Z_0(X).
\end{aligned}$$

Consider the set

$$M := \{(k(x_i), \sigma_i) : i = 1, \dots, r\} = \{(K_1, \tau_1), \dots, (K_s, \tau_s)\},$$

where $K_i \in \{k(x_1), \dots, k(x_r)\}$ and $\tau_i \in \{\sigma_1, \dots, \sigma_r\}$. We partition the set $\{x_1, \dots, x_r\}$ as follows. The points x_i and x_j with $i \neq j$ belong to the same partition set $P_{(K, \tau)}$ if and only if $K = k(x_i) = k(x_j)$ and $\tau = \sigma_i = \sigma_j$. We write

$$\{x_1, \dots, x_r\} = \bigsqcup_{i=1}^s P_{(K_i, \tau_i)}.$$

For each $(K_i, \tau_i) \in M$, define

$$\Delta_{(K_i, \tau_i)} := \sum_{x \in P_{(K_i, \tau_i)}} n_x [K_i(y) : K_i] = \sum_{x \in P_{(K_i, \tau_i)}} n_x [K_i(x) : K_i],$$

and recall that y denotes the fixed lift of x and the n_x are the coefficients on the closed points in the 0-cycle $z \in Z_0^d(X)$. Then, for each (K_i, τ_i) , we have

$$\sum_{x \in P_{(K_i, \tau_i)}} n_x y \in Z_0^{\Delta_{(K_i, \tau_i)}}(Y_{K_i}^{\tau_i}).$$

Let $S := \{K_1, \dots, K_s\}$. For each $K \in S$, we let $T_K := \{\tau : (K, \tau) \in M\}$. Consider the tuple of 0-cycles

$$w := \left(\sum_{x \in P_{(K_i, \tau_i)}} n_x y \right)_{i \in \{1, \dots, s\}} = \left(\left(\sum_{x \in P_{(K, \tau)}} n_x y \right)_{\tau \in T_K} \right)_{K \in S}.$$

We claim that $\mathbf{f}_*(w) = z$. Indeed, we have

$$\mathbf{f}_*(w) = \sum_{i=1}^s \sum_{x \in T_{(K_i, \tau_i)}} n_x (s_{K_i} \circ f_{K_i}^{\tau_i})_*(y) = \sum_{x \in \text{supp}(z)} n_x x = z.$$

Moreover, we claim that w is in

$$\prod_{\substack{S \subset \mathfrak{F}_k \\ S \neq \emptyset \text{ finite}}} \prod_{\substack{(T_K \subset H^1(K, G))_{K \in S} \\ T_K \neq \emptyset \text{ finite for all } K \in S}} \prod_{\substack{((\Delta_\tau)_{\tau \in T_K})_{K \in S} \\ \sum_{K \in S} [K:k] \sum_{\tau \in T_K} \Delta_\tau = d}} \prod_{K \in S} \prod_{\tau \in T_K} Z_0^{\Delta_\tau}(Y_K^\tau)$$

To show this, we check that the condition on the degrees $\Delta_{(K_i, \tau_i)}$ holds. We have

$$\begin{aligned} \sum_{K \in S} \sum_{\tau \in T_K} \Delta_{(K, \tau)} [K : k] &= \sum_{i=1}^s \Delta_{(K_i, \tau_i)} [K_i : k] \\ &= \sum_{i=1}^s \sum_{x \in P_{(K_i, \tau_i)}} n_x [K_i(x) : K_i] [K_i : k] \\ &= \sum_{j=1}^r n_{x_j} [k(x_j) : k] \\ &= \deg z \\ &= d \end{aligned}$$

Hence the inclusion $Z_0^d(X)^f \supset Z_0^d(X)$ holds and thus $Z_0^d(X)^f = Z_0^d(X)$. \square

Remark 2.6. The subset of $Z_0^1(X)^f$ given by

$$Z_0^1(X)_k^{f, \text{eff}} := \mathbf{f}_* \left(\prod_{\substack{T_k \subset H^1(k, G) \\ T_k \neq \emptyset \text{ finite}}} \prod_{\substack{(\Delta_\tau)_{\tau \in T_k} : \\ \Delta_\tau \in \mathbb{Z} \text{ for all } \tau, \\ \sum_{\tau \in T_k} \Delta_\tau = 1}} \prod_{\tau \in T_k} Z_0^{\Delta_\tau, \text{eff}}(Y^\tau) \right),$$

where we have taken $S = \{k\}$ in Definition 2.4, is equal to $\prod_{\tau \in H^1(k, G)} f^\tau(Y^\tau(k))$. Indeed, any effective 0-cycle in $Z_0^{\Delta_\tau, \text{eff}}(Y^\tau)$ which is not identically zero must have degree $\Delta_\tau > 0$. Thus for $\Delta_\tau \leq 0$, we have $Z_0^{\Delta_\tau, \text{eff}}(Y^\tau) = \emptyset$. (Strictly speaking, when $\Delta_\tau = 0$ we need to also remove the identically zero 0-cycle; we will be a bit imprecise and ignore this minor issue.) But the only way in which $\sum_{\tau \in T_k} \Delta_\tau = 1$ for $\Delta_\tau > 0$ is if $T_k = \{\tau\}$ for some $\tau \in H^1(k, G)$ and $\Delta_\tau = 1$. Hence, we have

$$Z_0^1(X)_k^{f, \text{eff}} = \mathbf{f}_* \left(\prod_{\tau \in H^1(k, G)} Z_0^{1, \text{eff}}(Y^\tau) \right).$$

One can check that $Z_0^{1, \text{eff}}(Y^\tau) = Y^\tau(k)$. The result then follows by the definition of \mathbf{f}_* .

3. LOCAL CASE

In this section, we extend the definitions from the previous section to the local setting. For any variety V over a number field k and any integer d , the set of adelic 0-cycles of degree d of V is the subset $Z_0^d(V_{\mathbb{A}_k})$

of $\prod_{v \in \Omega_k} Z_0^d(V_{k_v})$ of 0-cycles $(z_v)_v$ such that, for all but finitely many $v \in \Omega_k$, we have that z_v extends to a 0-cycle over some model $\mathcal{V} \rightarrow \text{Spec}(\mathcal{O}_{k_v})$ of $V_{k_v} \rightarrow \text{Spec}(k_v)$. If V is proper, then $Z_0^d(V_{\mathbb{A}_k}) = \prod_{v \in \Omega_k} Z_0^d(V_{k_v})$.

Let X be a smooth, proper, geometrically integral variety over a number field k . For any integer d and any place $v \in \Omega_k$, we briefly recall how the natural map $\text{res}_v : Z_0^d(X) \rightarrow Z_0^d(X_{k_v})$ is defined. For $z := \sum_{x \in X} n_x x \in Z_0^d(X)$, we let

$$\text{res}_v(z) := \sum_{x \in X} \sum_{w \in \Omega_{k(x)} : w|v} n_x(x)_w,$$

where $(x)_w \in X(k(x)_w)$ is the image of x under the natural inclusion $X(k(x)) \rightarrow X(k(x)_w)$. For any $x \in X$, we have by e.g. [Neu99, Ch. II, Cor. 8.4] that

$$\sum_{w \in \Omega_{k(x)} : w|v} [k(x)_w : k_v] = [k(x) : k]$$

and thus

$$\deg(z_v) = \sum_{x \in X} \sum_{w \in \Omega_{k(x)} : w|v} n_x [k(x)_w : k_v] = \sum_{x \in X} n_x [k(x) : k] = \deg(z).$$

Definition 3.1. Let X be a smooth, proper, geometrically integral variety over a number field k . Let $f : Y \rightarrow X$ be a G -torsor over X , where G is a linear algebraic group over k . We define the map

$$\mathbf{f}_{*,\text{ad}} : \prod_{\substack{S \subset \mathfrak{F}_k \\ S \neq \emptyset \text{ finite}}} \prod_{\substack{(T_K \subset H^1(K, G))_{K \in S} \\ T_K \neq \emptyset \text{ finite for all } K \in S}} \prod_{K \in S} \prod_{\tau \in T_K} Z_0(Y_{\mathbb{A}_K}^\tau) \rightarrow Z_0(X_{\mathbb{A}_k})$$

as follows. We first observe that

$$Z_0(Y_{\mathbb{A}_K}^\tau) = \prod_{w \in \Omega_K} Z_0(Y_{K_w}^\tau) = \prod_{v \in \Omega_k} \prod_{w \in \Omega_K : w|v} Z_0(Y_{K_w}^\tau).$$

Hence, an element \tilde{y} in the domain of $\mathbf{f}_{*,\text{ad}}$ can be written as

$$\begin{aligned} \tilde{y} &:= \left(\left(\left(\left(\sum_{y \in Y_{K_w}^\tau} n_y y \right)_{w \in \Omega_K : w|v} \right)_{v \in \Omega_k} \right)_{\tau \in T_K} \right)_{K \in S} \\ &= \left(\left(\left(\left(\sum_{y \in Y_{K_w}^\tau} n_y y \right)_{w \in \Omega_K : w|v} \right)_{\tau \in T_K} \right)_{K \in S} \right)_{v \in \Omega_k}, \end{aligned}$$

and we define

$$\mathbf{f}_{*,\text{ad}}(\tilde{y}) := \left(\sum_{K \in S} \sum_{\tau \in T_K} \sum_{w \in \Omega_K : w|v} \sum_{y \in Y_{K_w}^\tau} n_y (s_{K_w} \circ f_{K_w}^\tau)_*(y) \right)_{v \in \Omega_k}$$

where $f_{K_w}^\tau : Y_{K_w}^\tau \rightarrow X_{K_w}$ and $s_{K_w} : X_{K_w} \rightarrow X_{k_v}$ are the natural maps.

Lemma 3.2. Let X be a smooth, proper, geometrically integral variety over a number field k . Let $f : Y \rightarrow X$ be a torsor under some linear algebraic group G over k . Fix $d \in \mathbb{Z}$. Then,

$$\mathbf{f}_{*,\text{ad}} : \prod_{S \subset \mathfrak{F}_k} \prod_{(T_K \subset H^1(K, G))_{K \in S}} \prod_{\substack{((\Delta_\tau)_{\tau \in T_K})_{K \in S} : \\ \Delta_\tau \in \mathbb{Z} \text{ for all } \tau, \\ \sum_{K \in S} [K:k] \sum_{\tau \in T_K} \Delta_\tau = d}} \prod_{K \in S} \prod_{\tau \in T_K} Z_0^{\Delta_\tau}(Y_{\mathbb{A}_K}^\tau) \rightarrow Z_0^d(X_{\mathbb{A}_k}).$$

Proof. The proof follows in a similar manner to that of Lemma 2.3. Let \tilde{y} be in the domain of $\mathbf{f}_{*,\text{ad}}$. Then, for each $v \in \Omega_k$, we have

$$\deg(\mathbf{f}_{*,\text{ad}}(\tilde{y})_v) = \sum_{K \in S} \sum_{\tau \in T_K} \sum_{w \in \Omega_K : w|v} \sum_{y \in Y_{K_w}^\tau} n_y [K_w(y) : k_v]$$

$$\begin{aligned}
&= \sum_{K \in S} \sum_{\tau \in T_K} \sum_{w \in \Omega_K : w|v} \sum_{y \in Y_{K_w}^\tau} n_y[K_w(y) : K_w][K_w : k_v] \\
&= \sum_{K \in S} \sum_{\tau \in T_K} \sum_{w \in \Omega_K : w|v} [K_w : k_v] \sum_{y \in Y_{K_w}^\tau} n_y[K_w(y) : K_w] \\
&= \sum_{K \in S} \sum_{\tau \in T_K} \Delta_\tau \sum_{w \in \Omega_K : w|v} [K_w : k_v] \\
&= \sum_{K \in S} \sum_{\tau \in T_K} \Delta_\tau [K : k] \\
&= d,
\end{aligned}$$

where the fourth equality follows from the fact that the inner summation represents the degree of the 0-cycle $\sum_{y \in Y_K^\tau} n_y y$ which is by definition Δ_τ , the fifth equality comes from the fact that $\sum_{w \in \Omega_K : w|v} [K_w : k_v] = [K : k]$, and the final equality comes from the definition of the integers Δ_τ . \square

Definition 3.3. Let X be a smooth, proper, geometrically integral variety over a number field k . Let $f : Y \rightarrow X$ be a G -torsor over X , where G is a linear algebraic group over k . Let $d \in \mathbb{Z}$. For any given finite non-empty set $S \subset \mathfrak{F}_k$, we define the **degree d f -descent set of X associated to S** by

$$Z_0^d(X_{\mathbb{A}_k})_S^f := \mathbf{f}_{*,\text{ad}} \left(\prod_{\substack{(T_K \subset H^1(K,G))_{K \in S} \\ T_K \neq \emptyset \text{ finite for all } K \in S}} \prod_{\substack{((\Delta_\tau)_{\tau \in T_K})_{K \in S} : \\ \Delta_\tau \in \mathbb{Z} \text{ for all } \tau, \\ \sum_{K \in S} [K:k] \sum_{\tau \in T_K} \Delta_\tau = d}} \prod_{K \in S} \prod_{\tau \in T_K} Z_0^{\Delta_\tau}(Y_{\mathbb{A}_K}^\tau) \right).$$

We define the **degree d f -descent set of X** by

$$Z_0^d(X_{\mathbb{A}_k})^f := \bigcup_{\substack{S \subset \mathfrak{F}_k \\ S \neq \emptyset \text{ finite}}} Z_0^d(X_{\mathbb{A}_k})_S^f.$$

More generally, if \mathcal{G} is a set of k -isomorphism classes of linear algebraic groups over k , then the **degree d \mathcal{G} -descent set of X** is defined as

$$Z_0^d(X_{\mathbb{A}_k})^\mathcal{G} := \bigcup_{\substack{S \subset \mathfrak{F}_k \\ S \neq \emptyset \text{ finite}}} \bigcap_{F \in \mathcal{G}} \bigcap_{[f:Y \rightarrow X] \in H^1(X,F)} Z_0^d(X_{\mathbb{A}_k})_S^f$$

Remark 3.4. If we set $d = 1$, $S = \{k\}$ in Definition 3.3, and we restrict to effective 0-cycles only, then $Z_0^{1,\text{eff}}(X_{\mathbb{A}_k})_k^f = X(\mathbb{A}_k)^f$. Indeed, the proof follows the same argument as that of Remark 2.6.

Definition 3.5. Let X be a smooth, proper, geometrically integral variety over a number field k . Let $f : Y \rightarrow X$ be a G -torsor over X , where G is a linear algebraic group over k . Let $d \in \mathbb{Z}$. For any given finite non-empty set $S \subset \mathfrak{F}_k$, we define the **degree d f -Brauer set of X associated to S** by

$$Z_0^d(X_{\mathbb{A}_k})_S^{f,\text{Br}} := \mathbf{f}_{*,\text{ad}} \left(\prod_{\substack{(T_K \subset H^1(K,G))_{K \in S} \\ T_K \neq \emptyset \text{ finite for all } K \in S}} \prod_{\substack{((\Delta_\tau)_{\tau \in T_K})_{K \in S} : \\ \Delta_\tau \in \mathbb{Z} \text{ for all } \tau, \\ \sum_{K \in S} [K:k] \sum_{\tau \in T_K} \Delta_\tau = d}} \prod_{K \in S} \prod_{\tau \in T_K} Z_0^{\Delta_\tau}(Y_{\mathbb{A}_K}^\tau)^{\text{Br}} \right).$$

We define the degree d f -Brauer set of X by

$$Z_0^d(X_{\mathbb{A}_k})^{f,\text{Br}} := \bigcup_{\substack{S \subset \mathfrak{F}_k \\ S \neq \emptyset \text{ finite}}} \mathbf{f}_{*,\text{ad}} \left(\prod_{\substack{(T_K \subset H^1(K,G))_{K \in S} \\ T_K \neq \emptyset \text{ finite for all } K \in S}} \prod_{\substack{((\Delta_\tau)_{\tau \in T_K})_{K \in S}: \\ \Delta_\tau \in \mathbb{Z} \text{ for all } \tau, \\ \sum_{K \in S} [K:k] \sum_{\tau \in T_K} \Delta_\tau = d}} \prod_{K \in S} \prod_{\tau \in T_K} Z_0^{\Delta_\tau}(Y_{\mathbb{A}_K}^\tau)^{\text{Br}} \right).$$

The degree d étale-Brauer set of X is defined by

$$Z_0^d(X_{\mathbb{A}_k})^{\text{et},\text{Br}} := \bigcup_{\substack{S \subset \mathfrak{F}_k \\ S \neq \emptyset \text{ finite}}} \bigcap_{G \text{ finite lin. alg. } k\text{-group}} \bigcap_{[f:Y \rightarrow X] \in H^1(X,G)} Z_0^d(X_{\mathbb{A}_k})_S^{f,\text{Br}}.$$

Proposition 3.6. *Let X be a smooth, proper, geometrically integral variety over a number field k . Let $f : Y \rightarrow X$ be a G -torsor over X , where G is a linear algebraic group over k . Then $Z_0^d(X_{\mathbb{A}_k})^{f,\text{Br}} \subset Z_0^d(X_{\mathbb{A}_k})^{\text{et},\text{Br}}$. In particular, $Z_0^d(X_{\mathbb{A}_k})^{\text{et},\text{Br}} \subset Z_0^d(X_{\mathbb{A}_k})^{\text{Br}}$.*

Proof. Let $x \in Z_0^d(X_{\mathbb{A}_k})^{f,\text{Br}}$. This means that there exists some

$$\tilde{y} := \left(\left(\left(\sum_{y \in Y_{K_w}^\tau} n_y y \right)_{w \in \Omega_K} \right)_{\tau \in T_K} \right)_{K \in S} \in \prod_{K \in S} \prod_{\tau \in T_K} Z_0^{\Delta_\tau}(Y_{\mathbb{A}_K}^\tau)^{\text{Br}}$$

for some finite $S \neq \emptyset$, some tuple of non-empty finite sets $(T_K \subset H^1(K, G))_{K \in S}$, and some tuple of integers $((\Delta_\tau)_{\tau \in T_K})_{K \in S}$ with $\sum_{K \in S} [K : k] \sum_{\tau \in T_K} \Delta_\tau = d$, such that

$$x = \mathbf{f}_{*,\text{ad}}(\tilde{y}) = \left(\sum_{K \in S} \sum_{\tau \in T_K} \sum_{w \in \Omega_K : w|v} \sum_{y \in Y_{K_w}^\tau} n_y (s_{K_w} \circ f_{K_w}^\tau)_*(y) \right)_{v \in \Omega_k}.$$

In particular, we remark that, for any $\tau \in T_K$, we have that, for any $\beta \in \text{Br}(Y_K^\tau)$,

$$\sum_{w \in \Omega_K} \sum_{y \in Y_{K_w}^\tau} n_y \text{inv}_w \text{Cor}_{K_w(y)/K_w}(\beta(y)) = 0.$$

We need to show that $x \in Z_0^d(X_{\mathbb{A}_k})^{\text{Br}}$. Let $\gamma \in \text{Br } X$. Fix $K \in S$ and $\tau \in T_K$. Let $\gamma_v := \text{res}_{k_v/k}(\gamma) \in \text{Br}(X_{k_v})$, $\alpha := s_{K_w}^*(\gamma_v) \in \text{Br}(X_{K_w})$, and $(f_{K_w}^\tau)^*(\alpha) \in \text{Br}(Y_{K_w}^\tau)$. Then

$$\begin{aligned} 0 &= \sum_{w \in \Omega_K} \sum_{y \in Y_{K_w}^\tau} n_y \text{inv}_w \text{Cor}_{K_w(y)/K_w}((f_{K_w}^\tau)^*(\alpha)(y)) \\ &= \sum_{w \in \Omega_K} \sum_{y \in Y_{K_w}^\tau} n_y \text{inv}_w \text{Cor}_{K_w(f_{K_w}^\tau(y))/K_w} \left(\text{Cor}_{K_w(y)/K_w}(f_{K_w}^\tau(y)) ((f_{K_w}^\tau)^*(\alpha)(y)) \right) \\ &= \sum_{w \in \Omega_K} \sum_{y \in Y_{K_w}^\tau} n_y [K_w(y) : K_w(f_{K_w}^\tau(y))] \text{inv}_w \text{Cor}_{K_w(f_{K_w}^\tau(y))/K_w}(\alpha(f_{K_w}^\tau(y))) \\ &= \sum_{v \in \Omega_k} \sum_{w \in \Omega_K : w|v} \sum_{y \in Y_{K_w}^\tau} n_y [K_w(y) : K_w(f_{K_w}^\tau(y))] \text{inv}_w \text{Cor}_{K_w(f_{K_w}^\tau(y))/K_w}((s_{K_w}^*(\gamma_v))(f_{K_w}^\tau(y))) \\ &= \sum_{v \in \Omega_k} \sum_{w \in \Omega_K : w|v} \sum_{y \in Y_{K_w}^\tau} n_y [K_w(y) : k_v(s_{K_w}(f_{K_w}^\tau(y)))] \text{inv}_v \text{Cor}_{k_v(s_{K_w}(f_{K_w}^\tau(y)))/k_v}(\gamma_v(s_{K_w}(f_{K_w}^\tau(y)))) , \end{aligned}$$

where in the third equality we have used restriction-corestriction for the extension $K_w(y)/K_w(f_{K_w}^\tau(y))$, in the fifth equality we have used the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Br}(\mathrm{Spec}(K_w(f_{K_w}^\tau(y)))) & \xrightarrow{\mathrm{Cor}_{K_w(f_{K_w}^\tau(y))/K_w}} & \mathrm{Br}(\mathrm{Spec}(K_w)) & \xrightarrow{\mathrm{inv}_w} & \mathbb{Q}/\mathbb{Z} \\
\downarrow \mathrm{Cor}_{K_w(f_{K_w}^\tau(y))/k_v(s_{K_w}(f_{K_w}^\tau(y)))} & & \downarrow \mathrm{Cor}_{K_w/k_v} & & \downarrow = \\
\mathrm{Br}(\mathrm{Spec}(k_v(s_{K_w}(f_{K_w}^\tau(y)))) & \xrightarrow{\mathrm{Cor}_{k_v(s_{K_w}(f_{K_w}^\tau(y)))/k_v}} & \mathrm{Br}(\mathrm{Spec}(k_v)) & \xrightarrow{\mathrm{inv}_v} & \mathbb{Q}/\mathbb{Z}
\end{array}$$

together with restriction-corestriction for the extension $K_w(f_{K_w}^\tau(y))/k_v(s_{K_w}(f_{K_w}^\tau(y)))$. But then, when considering $\langle \mathbf{f}_{*,\mathrm{ad}}(y), \gamma \rangle_{BM}$, we get

$$\begin{aligned}
\langle \mathbf{f}_{*,\mathrm{ad}}(\tilde{y}), \gamma \rangle_{BM} &= \sum_{K \in S} \sum_{\tau \in T_K} \sum_{v \in \Omega_k} \sum_{w \in \Omega_K : w|v} \sum_{y \in Y_{K_w}^\tau} n_y [K_w(y) : k_v(s_{K_w}(f_{K_w}^\tau(y)))] \mathrm{inv}_v \mathrm{Cor}_{k_v(s_{K_w}(f_{K_w}^\tau(y)))/k_v} (\gamma_v(s_{K_w}(f_{K_w}^\tau(y)))) \\
&= \sum_{K \in S} \sum_{\tau \in T_K} 0 \\
&= 0,
\end{aligned}$$

that is, $x = \mathbf{f}_{*,\mathrm{ad}}(\tilde{y}) \in Z_0^d(X_{\mathbb{A}_k})^{\mathrm{Br}}$, as required. \square

4. EXTENDING LIANG'S STRATEGY TO TORSORS

4.1. Chow groups and weak approximation for 0-cycles. Let X be a smooth, proper, geometrically integral variety over a number field k . The *Chow group* $\mathrm{CH}_0(X)$ of X is the quotient of $Z_0(X)$ by the subgroup generated by all 0-cycles of the form $f_*(\mathrm{div}_C(g))$, for all $f : C \rightarrow X$ proper morphisms over k from normal integral k -curves C and for all $g \in k(C)^\times$. In other words, $\mathrm{CH}_0(X)$ is the quotient of $Z_0(X)$ by the subgroup of 0-cycles rationally equivalent to zero.

Recall that, for any $d \in \mathbb{Z}$, we say that X *satisfies weak approximation for 0-cycles of degree d* if the following condition holds: for any positive integer n and for any finite subset $S \subset \Omega_k$ of places of k , if $(z_v)_v \in Z_0^d(X_{\mathbb{A}_k})$, then there exists a global 0-cycle $z_{n,S} \in Z_0^d(X)$ such that $z_{n,S}$ and z_v have the same image in $\mathrm{CH}_0(X_{k_v})/n$ for any $v \in S$. We can also refine the notion of weak approximation by replacing the set $Z_0^d(X_{\mathbb{A}_k})$ by some potentially smaller set $Z_0^d(X_{\mathbb{A}_k})^\omega$ still containing $Z_0^d(X)$, where ω is some “obstruction” (e.g. $\omega = \mathrm{Br}$, or $\omega = \mathrm{etBr}$, or $\omega = f$ for some F -torsor $f : Y \rightarrow X$).

Definition 4.1. Let X be a smooth, proper, geometrically integral variety over a number field k . Let $d \in \mathbb{Z}$. Let ω be some “obstruction”. We say that *the ω obstruction is the only one for weak approximation for 0-cycles of degree d on X* if the following condition holds: for any positive integer n and for any finite subset $S \subset \Omega_k$ of places of k , if $(z_v)_v \in Z_0^d(X_{\mathbb{A}_k})^\omega$, then there exists a global 0-cycle $z_{n,S} \in Z_0^d(X)$ such that $z_{n,S}$ and z_v have the same image in $\mathrm{CH}_0(X_{k_v})/n$ for any $v \in S$.

Finally, we recall that if $f : Y \rightarrow X$ is a proper morphism of varieties over k , then the map $f_* : Z_0(Y) \rightarrow Z_0(X)$ induces a map of Chow groups $f_* : \mathrm{CH}_0(Y) \rightarrow \mathrm{CH}_0(X)$.

4.2. Extending Liang's strategy to torsors. In [Lia13, Theorem 3.2.1] Liang proves, under certain geometric assumptions on the k -variety X , that, if the Brauer-Manin obstruction is the only one for weak approximation of K -rational points on X_K for any finite extension K/k , then the Brauer-Manin obstruction is the only one for weak approximation of 0-cycles of degree 1 on X .

Recall that the *unramified Brauer group* $\mathrm{Br}_{nr}(Y) = \mathrm{Br}_{nr}(k(Y)/k)$ of a smooth, geometrically integral variety Y over k is the subgroup of $\mathrm{Br}(k(Y))$ defined by the intersection of the images of the natural maps $\mathrm{Br}(A) \hookrightarrow \mathrm{Br}(k(Y))$ for all discrete valuation rings A with field of fractions $k(Y)$ such that $k \subset A$. The unramified Brauer group is a birational invariant that can be used even when no smooth, projective model of Y is available. We note that when Y is proper, $\mathrm{Br}_{nr}(Y) = \mathrm{Br}(Y)$. For further details, see e.g., [CTS21, Chapter 6].

Theorem 4.2. *Let X be a smooth, proper, geometrically integral variety over a number field k . Let $f : Y \rightarrow X$ be (respectively, a proper map which is) an F -torsor for some linear algebraic group F over k and with Y geometrically integral. Let d be any integer. Assume that*

- (i) *for any finite extension K/k and any $\tau \in H^1(K, F)$, the quotient $\mathrm{Br}_{nr}(Y_K^\tau)/\mathrm{Br}_0(Y_K^\tau)$ is finite, and there exists a finite extension K'_τ of K so that for all finite extensions L of K linearly disjoint from K'_τ over K , the homomorphism induced by restriction*

$$\mathrm{res}_{L/K} : \mathrm{Br}_{nr}(Y_K^\tau)/\mathrm{Br}_0(Y_K^\tau) \rightarrow \mathrm{Br}_{nr}(Y_L^\tau)/\mathrm{Br}_0(Y_L^\tau)$$

is surjective;

- (ii) *for any finite extension L/k , we have that $X_L(\mathbb{A}_L)^{f_L, \mathrm{Br}_{nr}} \neq \emptyset$ if and only if $X(L) \neq \emptyset$ (respectively, if $X_L(\mathbb{A}_L)^{f_L, \mathrm{Br}_{nr}} \neq \emptyset$, then weak approximation holds for X_L).*

Then f -descent with unramified Brauer obstruction is the only obstruction to the Hasse principle (respectively, weak approximation) for 0-cycles of degree d on X .

Proof. We give a proof for weak approximation; the proof for the Hasse principle is similar. Fix a positive integer n and a finite subset $S \subset \Omega_k$. Fix a closed point $\tilde{x} \in X$. Since we are proving weak approximation, we note that f is proper, by hypothesis.

Let $(z_v)_{v \in \Omega_k} \in Z_0^d(X_{\mathbb{A}_k})^{f, \mathrm{Br}}$. Then, by definition, there exist a non-empty finite set S' of finite extensions of k , a non-empty finite set $T_K \subset H_{\mathrm{et}}^1(K, F)$ for each $K \in S'$, a tuple of integers $((\Delta_\tau)_{\tau \in T_K})_{K \in S'}$ satisfying $\sum_{K \in S'} [K : k] \sum_{\tau \in T_K} \Delta_\tau = d$, and adelic 0-cycles $(\tilde{z}_w^\tau)_{w \in \Omega_K} \in Z_0^{\Delta_\tau}(Y_{\mathbb{A}_K}^\tau)^{\mathrm{Br}}$ with

$$\mathbf{f}_{*, \mathrm{ad}}(((\tilde{z}_w^\tau)_{w \in \Omega_K})_{\tau \in T_K})_{K \in S'} = (z_v)_{v \in \Omega_k}.$$

For each τ , we follow the proof of [Lia13, Theorem 3.2.1], with the following modifications. Liang translates arithmetic information on a proper variety Z to arithmetic information on $Z \times \mathbb{P}^1$ in part via the isomorphism of $\mathrm{Br}(Z \times \mathbb{P}^1) \xrightarrow{\sim} \mathrm{Br}(Z)$. In our setting, Y_K^τ need not be proper, so we use the unramified Brauer group instead since it is a stably-birational invariant [CTS21, Cor. 6.2.10], hence $\mathrm{Br}_{nr}(Y_K^\tau \times \mathbb{P}_K^1) \simeq \mathrm{Br}_{nr}(Y_K^\tau)$.

Without loss of generality, we can further enlarge the set S in the following way. For each $K \in S'$ and $\tau \in T_K$, since we are assuming that the quotient $\mathrm{Br}_{nr}(Y_K^\tau)/\mathrm{Br}_0(Y_K^\tau)$ is finite, we can fix a complete finite set $\mathfrak{R}_{(K, \tau)} \subset \mathrm{Br}_{nr}(Y_K^\tau)$ of representatives for $\mathrm{Br}_{nr}(Y_K^\tau)/\mathrm{Br}_0(Y_K^\tau)$. Hence, we can find a finite set $S_{0, K, \tau} \subset \Omega_K$ and an integral model \mathcal{Y}_K^τ of Y_K^τ over $\mathrm{Spec}(\mathcal{O}_{K, S_{0, K, \tau}})$ such that all the representatives in $\mathfrak{R}_{(K, \tau)}$ actually come from elements of $\mathrm{Br}(\mathcal{Y}_K^\tau)$. Moreover, we recall that $(\tilde{z}_w^\tau)_{w \in \Omega_K} \in Z_0^{\Delta_\tau}(Y_{\mathbb{A}_K}^\tau)$, so that, by definition of adelic 0-cycles, there are only finitely many $w \in \Omega_K$ (depending on the chosen integral model for Y_K^τ) for which \tilde{z}_w^τ is not integral. Hence, by enlarging $S_{0, K, \tau}$ further if necessary, we can assume that all the places $w \in \Omega_K$ for which \tilde{z}_w^τ is not integral (with respect to the model \mathcal{Y}_K^τ) are contained in $S_{0, K, \tau}$. We then enlarge S by including all the (finitely many) places $v \in \Omega_k$ below the (finitely many) places $w \in S_{0, K, \tau}$, for each $K \in S'$ and $\tau \in T_K$. After enlarging S as above if needed, it follows that, for any $K \in S'$ and any $\tau \in T_K$, if $w \in \Omega_K$ is not above any place of S , then the evaluation $\beta(\tilde{z}_w^\tau)$ is actually in $\mathrm{Br}(\mathcal{O}_{K_w}) = 0$, for each $\beta \in \mathfrak{R}_{(K, \tau)}$.

Following the proof by Liang and using assumption (i), for each $K \in S'$ and each $\tau \in T_K$ we can then construct a finite extension K_τ/K with

$$[K_\tau : K] \equiv \Delta_\tau \pmod{n \cdot [k(\tilde{x}) : k]}$$

and an adelic point $(y_{w'}^\tau)_{w' \in \Omega_{K_\tau}} \in Y^\tau(\mathbb{A}_{K_\tau})^{\mathrm{Br}_{nr}}$ with $\sum_{w' \in \Omega_{K_\tau} : w'|w} r_{K_\tau, w'/K_w}(y_{w'}^\tau)$ sufficiently close to \tilde{z}_w^τ for all places w above S , where $r_{K_\tau, w'/K_w} : Y_{K_\tau, w'}^\tau \rightarrow Y_{K_w}^\tau$. In particular, for all the places $w \in \Omega_K$ above a place in S , the 0-cycles $\sum_{w' \in \Omega_{K_\tau} : w'|w} r_{K_\tau, w'/K_w}(y_{w'}^\tau)$ and \tilde{z}_w^τ have the same image in $\mathrm{CH}_0(Y_{K_w}^\tau)/n$, say

$$\left[\sum_{w' \in \Omega_{K_\tau} : w'|w} r_{K_\tau, w'/K_w}(y_{w'}^\tau) \right] = [\tilde{z}_w^\tau] + n\mu_w^\tau$$

for some $\mu_w^\tau \in \mathrm{CH}_0(Y_{K_w}^\tau)$.

We note that if Y_K^τ is proper, since $\mathrm{Br} Y_K^\tau = \mathrm{Br}_{nr}(Y_K^\tau)$, the proof of the existence of the adelic point above becomes identical to the original proof in [Lia13, Theorem 3.2.1].

Now, let $x_{w'}^\tau := f_{K_\tau}^\tau(y_{w'}^\tau)$. Then $(x_{w'}^\tau)_{w'} \in X(\mathbb{A}_{K_\tau})^{f_{K_\tau}^\tau, \mathrm{Br}}$. By assumption (ii), there exists a K_τ -rational point $\tilde{x}^\tau \in X(K_\tau)$ such that $(\tilde{x}^\tau)_{w'} = x_{w'}^\tau$ for all w' above S .

Consider the global 0-cycle $z := (\sum_K \sum_\tau s_{K_\tau}(\tilde{x}^\tau)) + \lambda n \tilde{x}$ for some $\lambda \in \mathbb{Z}$ such that $z \in Z_0^d(X)$. Note that such λ exists by construction, since

$$\deg \left(\sum_{K \in S'} \sum_{\tau \in T_K} s_{K_\tau}(\tilde{x}^\tau) \right) = \sum_{K \in S'} \sum_{\tau \in T_K} [K_\tau : k] \equiv \sum_{K \in S'} \sum_{\tau \in T_K} \Delta_\tau [K : k] \equiv d \pmod{n \cdot [k(\tilde{x}) : k]}.$$

We claim that the 0-cycle z has the same image as z_v in $\mathrm{CH}_0(X_{k_v})/n$ for all $v \in S$. Indeed, for each $v \in S$, under the natural map $Z_0^d(X) \rightarrow Z_0^d(X_{k_v})$ we send

$$z \mapsto \mathrm{res}_v(z) := \left(\sum_{K \in S'} \sum_{\tau \in T_K} \sum_{w' \in \Omega_{K_\tau} : w'|v} s_{K_{\tau,w'}}((\tilde{x}^\tau)_{w'}) \right) + \lambda n \sum_{w \in \Omega_{k(\tilde{x})} : w|v} (\tilde{x})_w \in Z_0^d(X_{k_v})$$

and, working in $\mathrm{CH}_0(X_{k_v})$, we have that

$$\begin{aligned} [\mathrm{res}_v(z)] &= \left[\sum_K \sum_\tau \sum_{w' \in \Omega_{K_\tau} : w'|v} s_{K_{\tau,w'}}((\tilde{x}^\tau)_{w'}) \right] + \lambda n \left[\sum_{w \in \Omega_{k(\tilde{x})} : w|v} (\tilde{x})_w \right] \\ &\equiv \left[\sum_K \sum_\tau \sum_{w' \in \Omega_{K_\tau} : w'|v} s_{K_{\tau,w'}}((\tilde{x}^\tau)_{w'}) \right] \\ &\equiv \left[\sum_K \sum_\tau \sum_{w' \in \Omega_{K_\tau} : w'|v} s_{K_{\tau,w'}}(x_{w'}^\tau) \right] \\ &\equiv \left[\sum_K \sum_\tau \sum_{w' \in \Omega_{K_\tau} : w'|v} s_{K_{\tau,w'}}(f_{K_\tau}^\tau(y_{w'}^\tau)) \right] \\ &\equiv \left[\sum_K \sum_\tau \sum_{w' \in \Omega_{K_\tau} : w'|v} s_{K_w}(f_K^\tau(r_{K_{\tau,w'}/K_w}(y_{w'}^\tau))) \right] \\ &\equiv \left[\sum_K \sum_\tau \sum_{w \in \Omega_K : w|v} \sum_{w' \in \Omega_{K_\tau} : w'|w} (s_{K_w} \circ f_K^\tau)_*(r_{K_{\tau,w'}/K_w}(y_{w'}^\tau)) \right] \\ &\equiv \sum_K \sum_\tau \sum_{w \in \Omega_K : w|v} (s_{K_w} \circ f_K^\tau)_* \left(\left[\sum_{w' \in \Omega_{K_\tau} : w'|w} r_{K_{\tau,w'}/K_w}(y_{w'}^\tau) \right] \right) \\ &\equiv \sum_K \sum_\tau \sum_{w \in \Omega_K : w|v} (s_{K_w} \circ f_K^\tau)_*([\tilde{z}_w^\tau] + n\mu_w^\tau) \\ &\equiv \sum_K \sum_\tau \sum_{w \in \Omega_K : w|v} (s_{K_w} \circ f_K^\tau)_*([\tilde{z}_w^\tau]) \\ &\equiv \left[\sum_K \sum_\tau \sum_{w \in \Omega_K : w|v} (s_{K_w} \circ f_K^\tau)_*(\tilde{z}_w^\tau) \right] \\ &\equiv [z_v] \pmod{n\mathrm{CH}_0(X_{k_v})}, \end{aligned}$$

where in the fifth equality we have used the commutative diagram

$$\begin{array}{ccc} Y_{K_{\tau,w'}}^\tau & \xrightarrow{r_{K_{\tau,w'}/K_w}} & Y_{K_w}^\tau \\ \downarrow f_{K_\tau}^\tau & & \downarrow f_K^\tau \\ X_{K_{\tau,w'}} & \xrightarrow{s_{K_{\tau,w'}/K_w}} & X_{K_w} \\ \downarrow s_{K_{\tau,w'}} & & \downarrow s_{K_w} \\ X_{k_v} & \xrightarrow{=} & X_{k_v}. \end{array}$$

and where in the seventh and tenth equalities we have used the fact that proper pushforward maps of sets of 0-cycles induce maps of Chow groups. \square

Remark 4.3. In order to use Liang's strategy in the proof of Theorem 4.2, whenever Brauer groups appear, we need to assume that the varieties involved are geometrically integral. (If no Brauer groups appear, then Liang's strategy would work even without such assumptions, and the statement of the theorem would be greatly simplified.) We remark, however, that the following construction is available. Let $f : Y \rightarrow X$ be a torsor for some *finite* linear algebraic group F over k , with X smooth, proper, geometrically integral over k and Y not necessarily geometrically integral. Then, by e.g., [Dem09, Lemme 4], there exists a $\sigma \in H_{\text{et}}^1(k, F)$, a finite linear algebraic group F_0 over k together with a morphism $\rho : F_0 \rightarrow F^\sigma$, an F_0 -torsor $f_0 : Y_0 \rightarrow X$ with Y_0 geometrically integral, and a morphism $\psi : Y_0 \rightarrow Y^\sigma$ such that the following diagram commutes:

$$\begin{array}{ccc} Y_0 \times_k F_0 & \xrightarrow{\psi \times \rho} & Y^\sigma \times_k F^\sigma \\ \downarrow & & \downarrow \\ Y_0 & \xrightarrow{\psi} & Y^\sigma \\ & \searrow & \swarrow \\ & X & \end{array}$$

Remark 4.4. In Theorem 4.2, we consider only one torsor $f : Y \rightarrow X$ at a time. It is a much harder problem to deal with multiple torsors simultaneously (as it could happen when trying to compute, for example, the étale-Brauer obstruction), since the combinatorial compatibilities between the degrees of the fields constructed in the proof of Theorem 4.2 become much more difficult to enforce and check – at least, in full generality.

In the case that X has a closed point with degree a power of a prime p , we may relax the assumptions of Theorem 4.2.

Proposition 4.5. *Let X be a smooth, proper, geometrically integral variety over a number field k and suppose there exists a closed point \tilde{x} of X of degree $[k(\tilde{x}) : k] = p^r$, for some prime p . Let $f : Y \rightarrow X$ be an F -torsor for a linear algebraic group F over k and Y geometrically integral. Let d be an integer coprime to p . Assume that*

- (i) *for any finite extension K/k and any $\tau \in H^1(K, F)$, the quotient $\text{Br}_{nr}(Y_K^\tau) / \text{Br}_0(Y_K^\tau)$ is finite, and there exists a finite extension K'_τ of K so that for all finite extensions L of K linearly disjoint from K'_τ over K and with $\gcd([L : K], p) = 1$, the homomorphism induced by restriction*

$$\text{res}_{L/K} : \text{Br}_{nr}(Y_K^\tau) / \text{Br}_0(Y_K^\tau) \rightarrow \text{Br}_{nr}(Y_L^\tau) / \text{Br}_0(Y_L^\tau)$$

is surjective.

- (ii) *for any finite extension L/k of degree coprime to p , we have that $X_L(\mathbb{A}_L)^{f_L, \text{Br}_{nr}} \neq \emptyset$ if and only if $X(L) \neq \emptyset$.*

Then $Z_0^d(\mathbb{A}_k)^{f, \text{Br}} \neq \emptyset \implies Z_0^1(X) \neq \emptyset$.

Proof. Let $(z_v)_v \in Z_0^d(X_{\mathbb{A}_k})^{f, \text{Br}_{nr}}$. Then there exist a non-empty finite set S of field extensions of k , non-empty finite sets $T_K \subset H_{\text{et}}^1(K, F)$ for each $K \in S$, and a tuple $((\Delta_\tau)_{\tau \in T_K})_{K \in S}$ of integers satisfying $\sum_{K \in S} \sum_{\tau \in T_K} [K : k] \Delta_\tau = d$, and 0-cycles $(z_w^\tau)_{w \in \Omega_K} \in Z_0^{\Delta_\tau}(Y_{\mathbb{A}_K}^\tau)^{\text{Br}}$ for all $\tau \in T_K$ and all $K \in S$ such that

$$\mathbf{f}_{*, \text{ad}} \left(\left(((z_w^\tau)_{w \in \Omega_K})_{\tau \in T_K} \right)_{K \in S} \right) = (z_v)_v.$$

Since $\sum \Delta_\tau [K : k] = d$ and $\gcd(d, p) = 1$, there exists some τ such that $\gcd(\Delta_\tau [K : k], p) = 1$ and hence $\gcd(\Delta_\tau, p) = 1$. Under assumption (i), by applying a similar strategy as in the proof of [Lia13, Theorem 3.2.1] to the corresponding torsor Y_K^τ we obtain an extension L_τ/K of degree $[L_\tau : K] \equiv \Delta_\tau \pmod{p}$ and adelic point of $(y'_w)_w \in Y_{L_\tau}^\tau(\mathbb{A}_{L_\tau})^{\text{Br}_{nr}(Y_K^\tau)}$. Our assumption (i) guarantees that $(y'_w)_w \in Y_{L_\tau}^\tau(\mathbb{A}_{L_\tau})^{\text{Br}_{nr}(Y_{L_\tau}^\tau)}$ and thus that $f^\tau((y'_w)_w) \in X_{L_\tau}(\mathbb{A}_{L_\tau})^{f^\tau, \text{Br}_{nr}}$. Since L_τ/k has degree $[L_\tau : k] = [L_\tau : K][K : k]$ which is coprime to p , our assumption (ii) yields a rational point $x \in X(L_\tau)$, which can be viewed as a closed point

on X of degree coprime to p . Hence, by taking a suitable linear combination of x and \tilde{x} , we get a 0-cycle of degree 1 on X . \square

Remark 4.6. If in the statement of Proposition 4.5 we consider weak approximation instead, it is unlikely that the above strategy of proof extends to this setting.

Remark 4.7. See Theorem 5.6 for a result which uses similar ideas as those in the proof of Proposition 4.5 in the context of (twisted) Kummer varieties.

5. APPLICATIONS

5.1. Enriques surfaces. In this section, we study the arithmetic behaviour of 0-cycles on Enriques surfaces using our newly defined obstruction sets and some recent results on K3 surfaces. There is indeed a well-known relationship between Enriques surfaces and K3 surfaces: any Enriques surface X over a number field k can be realised as the quotient of a K3 surface Y by a fixed-point-free involution (see [Bea96, Prop III.17]). In other words, for any Enriques surface Y over k , we have a $\mathbb{Z}/2\mathbb{Z}$ -torsor $f : X \rightarrow Y$ with X a K3 surface over k .

Conjecturally, the qualitative arithmetic behaviour of rational points on K3 surfaces is completely determined by the Brauer-Manin obstruction.

Conjecture 5.1 (Skorobogatov, [Sko09b]). *The Brauer-Manin obstruction is the only obstruction to the Hasse principle and weak approximation of rational points on K3 surfaces over number fields.*

Some recent evidence towards Conjecture 5.1 includes [CTSSD98], [SSD05], [IS15], [HS16]. In [Ier21], Ieronymou used Liang's strategy to prove, conditionally on Skorobogatov's conjecture, that the Brauer-Manin obstruction also completely determines the qualitative arithmetic of 0-cycles on K3 surfaces.

Theorem 5.2 ([Ier21, Theorem 1.2]). *Let Y be a K3 surface over a number field k and fix an integer d . Suppose that Conjecture 5.1 holds. Then, for any positive integer n , if $(z_v)_v \in Z_0^d(Y_{\mathbb{A}_k})^{\text{Br}}$ then there exists a global 0-cycle $z_n \in Z_0^d(Y)$ such that z_n and z_v have the same image in $\text{CH}_0(Y_{k_v})/n$ for all $v \in \Omega_k$.*

By using Ieronymou's result and exploiting the K3 coverings of Enriques surfaces, we are able to study the arithmetic behaviour of 0-cycles on Enriques surfaces.

Theorem 5.3. *Let X be an Enriques surface over a number field k and let $f : Y \rightarrow X$ be a K3 covering of X , i.e. a $\mathbb{Z}/2\mathbb{Z}$ -torsor over X with Y a K3 surface. Assume that Conjecture 5.1 is true. Let $d \in \mathbb{Z}$. Then, for any positive integer n , if $(z_v)_v \in Z_0^d(X_{\mathbb{A}_k})^{f, \text{Br}}$ then there exists a global 0-cycle $z_n \in Z_0^d(X)$ such that z_n and $(z_v)_v$ have the same image in $\text{CH}_0(X_{k_v})/n$ for all $v \in \Omega_k$.*

Proof. Fix a positive integer n . If $(z_v)_v \in Z_0^d(X_{\mathbb{A}_k})^{f, \text{Br}}$, then by definition there exist a non-empty finite set S of field extensions of k , non-empty finite sets $T_K \subset H_{\text{et}}^1(K, \mathbb{Z}/2\mathbb{Z})$ for each $K \in S$, and a tuple $((\Delta_\tau)_{\tau \in T_K})_{K \in S}$ of integers satisfying $\sum_{K \in S} \sum_{\tau \in T_K} [K : k] \Delta_\tau = d$, and 0-cycles $(z_w^\tau)_{w \in \Omega_K} \in Z_0^{\Delta_\tau}(Y_{\mathbb{A}_K}^\tau)^{\text{Br}}$ for all $\tau \in T_K$ and all $K \in S$ such that

$$\mathbf{f}_{*, \text{ad}} \left(\left(((z_w^\tau)_{w \in \Omega_K})_{\tau \in T_K} \right)_{K \in S} \right) = (z_v)_v.$$

Under the assumption that Conjecture 5.1 is true, from Theorem 5.2 we deduce that there exist global 0-cycles $z_\tau \in Z_0^{\Delta_\tau}(Y_K^\tau)$ such that z_τ and $(z_w^\tau)_{w \in \Omega_K}$ have the same image in $\text{CH}_0(Y_{K_w}^\tau)/n$ for all $w \in \Omega_K$, for all $\tau \in T_K$ and all $K \in S$.

Let $z := \mathbf{f}_* \left(\left((z_\tau)_{\tau \in T_K} \right)_{K \in S} \right)$. By Lemma 2.3, $z \in Z_0^d(X)$. Moreover, it is not hard to see that z and z_v must have the same image in $\text{CH}_0(X_{k_v})/n$ for all $v \in \Omega_k$. Indeed, for all $w \in \Omega_K$, for all $\tau \in T_K$ and $K \in S'$, working at the level of Chow groups we have

$$[z_w^\tau] = [\text{res}_w(z_\tau)] + n\lambda_w^\tau$$

for some $\lambda_w^\tau \in \text{CH}_0(Y_{K_w}^\tau)$, where $\text{res}_w(z_\tau)$ is the image of z_τ under the natural map $Z_0^{\Delta\tau}(Y_K^\tau) \rightarrow Z_0^{\Delta\tau}(Y_{K_w}^\tau)$. Then, using the fact that proper pushforward maps at the level of sets of 0-cycles induce maps of Chow groups, for every $v \in \Omega_k$ we have

$$\begin{aligned} [z_v] &= \sum_{K \in S} \sum_{\tau \in T_K} \sum_{w \in \Omega_K : w|v} (s_K \circ f_K^\tau)_* ([z_w^\tau]) \\ &= \sum_{K \in S} \sum_{\tau \in T_K} \sum_{w \in \Omega_K : w|v} (s_K \circ f_K^\tau)_* ([\text{res}_w(z_\tau)] + n\lambda_w^\tau) \\ &= [\text{res}_v(z)] + \sum_{K \in S} \sum_{\tau \in T_K} \sum_{w \in \Omega_K : w|v} (s_K \circ f_K^\tau)_* (n\lambda_w^\tau) \\ &= [\text{res}_v(z)] + n \sum_{K \in S} \sum_{\tau \in T_K} \sum_{w \in \Omega_K : w|v} (s_K \circ f_K^\tau)_* (\lambda_w^\tau), \end{aligned}$$

where $\text{res}_v(z)$ is the image of z under the natural map $Z_0^d(X) \rightarrow Z_0^d(X_v)$. Since

$$\sum_{K \in S} \sum_{\tau \in T_K} \sum_{w \in \Omega_K : w|v} (s_K \circ f_K^\tau)_* (\text{CH}_0(Y_{K_w}^\tau)) \subset \text{CH}_0(X_{k_v}),$$

we are done. \square

5.2. (Twisted) Kummer varieties as torsors. Let A be an abelian variety over a number field k of dimension ≥ 2 . Let $\sigma := [T \rightarrow \text{Spec } k] \in H_{\text{et}}^1(k, A[2])$. Under the natural morphism $H_{\text{et}}^1(k, A[2]) \rightarrow H_{\text{et}}^1(k, A)$, we have that σ gives rise to a 2-covering $\rho : V \rightarrow A$, where V has the structure of a k -torsor under A of period dividing 2. The involution $[-1] : A \rightarrow A$, fixing $A[2]$, induces an involution $\iota : V \rightarrow V$ fixing $T = \rho^{-1}(0_A)$. Let $\tilde{V} \rightarrow V$ be the blow-up of V at T . Then the involution ι induces an involution $\tilde{\iota} : \tilde{V} \rightarrow \tilde{V}$ fixing the exceptional divisors of the blow-up. The quotient $\tilde{V}/\tilde{\iota}$ is called the *(twisted) Kummer variety associated to A and σ* . In what follows, we sometimes omit the references to A and σ and just talk about *(twisted) Kummer varieties*.

Lemma 5.4. *Let Y be a (twisted) Kummer variety over a number field k . Then Y has a 0-cycle of degree a power of 2.*

Proof. Since Y is a (twisted) Kummer variety, it admits a double cover by a smooth, proper variety Z which is birational to a torsor V under an abelian variety of dimension g of period $P(V)$ dividing 2. If $I(V)$ denotes the index of V , that is, the gcd of the degrees of all closed points on V , then the following divisibility relation between the period and index is well-known:

$$P(V) \mid I(V) \mid P(V)^{2g}.$$

In particular, $I(V)$ must be a power of 2. Since, for k a number field, the index is a birational invariant of smooth varieties, it follows that $I(V) = I(Z)$. Hence, by definition of the index, Z has a 0-cycle of degree $I(Z)$, a power of 2. By pushing forward this 0-cycle from Z to Y , we obtain a 0-cycle on Y of degree a power of 2, as required. \square

Given that there is a close relationship between (twisted) Kummer varieties and k -torsors under abelian varieties, and since, conditionally on the finiteness of the relevant Tate-Shafarevich group, the (algebraic) Brauer-Manin obstruction is the only one for the existence of rational points on k -torsors under abelian varieties (see e.g., [Man71, Cre20]), it is natural to ask the following question (and to possibly expect a positive answer).

Question 5.5. *Let X be a (twisted) Kummer variety over a number field k . Is it true that, for any finite extension L/k of odd degree, $X(\mathbb{A}_L)^{\text{Br}} \neq \emptyset$ implies $X(L) \neq \emptyset$?*

For some evidence towards a positive answer to Question 5.5, see for example [SSD05], [HS16], and [Har19].

Theorem 5.6. *Let X be a smooth, proper, geometrically integral variety over a number field k . Let $f : Y \rightarrow X$ be a torsor under some linear algebraic group F over k , where Y is a (twisted) Kummer variety over k . Let $d \in \mathbb{Z}$ be odd. Assume that Question 5.5 has a positive answer. Then $Z_0^d(X_{\mathbb{A}_k})^{f, \text{Br}\{2\}} \neq \emptyset$ implies $Z_0^d(X) \neq \emptyset$.*

Proof. Let $(z_v)_v \in Z_0^d(X_{\mathbb{A}_k})^{f, \text{Br}\{2\}}$. Then, by definition, there exist a non-empty finite set S of field extensions of k , non-empty finite sets $T_K \subset H_{\text{et}}^1(K, F)$ for each $K \in S$, and a tuple $((\Delta_\tau)_{\tau \in T_K})_{K \in S}$ of integers satisfying $\sum_{K \in S} \sum_{\tau \in T_K} [K : k] \Delta_\tau = d$, and 0-cycles $(z_w^\tau)_{w \in \Omega_K} \in Z_0^{\Delta_\tau}(Y_{\mathbb{A}_K}^\tau)^{\text{Br}\{2\}}$ for all $\tau \in T_K$ and all $K \in S$ such that

$$\mathbf{f}_{*, \text{ad}} \left(\left((z_w^\tau)_{w \in \Omega_K} \right)_{\tau \in T_K} \right)_{K \in S} = (z_v)_v.$$

Since d is odd, it follows that there exists some $K \in S$ and some $\tau \in T_K$ such that $\gcd([K : k] \Delta_\tau, 2) = 1$. By the proof of [Lia13, Theorem 3.2.1], we can construct a finite extension K_τ/K with $[K_\tau : K] \equiv \Delta_\tau \pmod{2}$ such that $Y_{K_\tau}^\tau(\mathbb{A}_{K_\tau})^{\text{Br}(Y_K^\tau)\{2\}} \neq \emptyset$. But, by construction, $[K_\tau : K]$ is odd. Hence, by [BN21, Lemma 7.1] we know that actually $Y_{K_\tau}^\tau(\mathbb{A}_{K_\tau})^{\text{Br}(Y_K^\tau)\{2\}} = Y_{K_\tau}^\tau(\mathbb{A}_{K_\tau})^{\text{Br}(Y_{K_\tau}^\tau)\{2\}}$. Since $Y_{K_\tau}^\tau$ is a (twisted) Kummer variety, [CV18, Theorem 5.10] then yields that $Y_{K_\tau}^\tau(\mathbb{A}_{K_\tau})^{\text{Br}(Y_{K_\tau}^\tau)} \neq \emptyset$. By assumption, this implies that there exists some rational point $y \in Y_{K_\tau}^\tau(K_\tau)$, which can be viewed as a closed point of degree $[K_\tau : K]$ on Y_K^τ . Since Y_K^τ is a (twisted) Kummer variety, we know by Lemma 5.4 that Y_K^τ has a global 0-cycle of degree a power of 2. Hence, by Bézout's theorem, there exists a global 0-cycle $\tilde{z}_\tau \in Z_0^1(Y_K^\tau)$ and thus a global 0-cycle $z \in Z_0^{[K:k]}(X)$. If $\#S = 1$, then $[K : k] \mid d$ and we are done. If $\#S > 1$, then, since $\sum_{K' \in S} \sum_{\tau' \in T_{K'}} [K' : k] \Delta_{\tau'} = d$, it follows that

$$\gcd([K : k], [K' : k] \text{ for } K' \in S - \{K\}) \mid d.$$

In particular, since $[K : k]$ is odd, it follows that

$$(5.1) \quad \gcd([K : k], 2^{t_{K'}} [K' : k] \text{ for } K' \in S - \{K\}) \mid d$$

for any integers $t_{K'} \geq 0$. For any $K' \in S - \{K\}$, fix some $\tau' \in T_{K'}$. Then, by considering the (twisted) Kummer variety $Y_{K'}^{\tau'}$ and, by Lemma 5.4, a global 0-cycle y' of degree a power of 2 on $Y_{K'}^{\tau'}$, we get that the pushforward $(f_{K'}^{\tau'})_*(y')$ is a global 0-cycle of degree a power of 2 on $X_{K'}$, and thus a global 0-cycle x' of degree $2^{t_{K'}} [K' : k]$ on X , for some $t_{K'} \geq 0$. Hence, by (5.1), we can take an appropriate linear combination of the 0-cycles x' (for each $K' \in S - \{K\}$) and z we obtain a 0-cycle of degree d on X , as required. \square

Remark 5.7. By using [BN21], one can also consider the more general case of torsors under arbitrary finite products of (twisted) Kummer varieties, K3 surfaces, and geometrically rationally connected varieties over some number field.

Remark 5.8. As already mentioned in the introduction, the recent preprint [Ier22] by Ieronymou should remove some of the conditions in Theorem 5.6, namely we can get a statement for any $d \in \mathbb{Z}$ and for the full Brauer groups, with a proof similar to that of Theorem 5.3. We have preferred to leave the statement of Theorem 5.6 in its current form as its proof could potentially be used in other contexts where one has only limited information about the arithmetic of rational points (e.g. when one only knows local-to-global principles for rational points with 2-primary Brauer groups).

5.3. Universal torsors and torsors under tori. Recall that if X is a variety over k and $f : Y \rightarrow X$ is a G -torsor over X for some linear algebraic group G over k of multiplicative type, then the *type of the torsor* $f : Y \rightarrow X$ is the map

$$\lambda : \widehat{G} \rightarrow \text{Pic } \overline{X}$$

which associates to any character $\chi \in \widehat{G} = \widehat{G}(\overline{k})$ the class of the pushforward $\chi_*(\overline{Y}) \rightarrow \overline{X}$ in $H^1(\overline{X}, \mathbb{G}_m) = \text{Pic } \overline{X}$ (see [Sko01, Lemma 2.3.1]), where $\widehat{G} := \text{Hom}_{k\text{-groups}}(G, \mathbb{G}_m) = \widehat{G}(\overline{k})$ is the module of characters of G . If, moreover, $\text{Pic } \overline{X}$ is finitely generated as a \mathbb{Z} -module, then we say that $f : Y \rightarrow X$ is a *universal torsor* if the type map $\lambda : \widehat{G} \rightarrow \text{Pic } \overline{X}$ is an isomorphism of $\text{Gal}(\overline{k}/k)$ -modules. One useful feature of universal torsors is that they are really defined geometrically, implying that if $g : W \rightarrow X$ is a universal torsor for X under some linear algebraic group G over k , then $g_K : W_K \rightarrow X_K$ is also a universal torsor for X_K (under G_K) for any finite extension K/k . Universal torsors satisfy many nice properties, in the context of rational points. For example, we have the following.

Theorem 5.9 ([Sko01, Theorem 6.1.2]). *Let X be a variety over k with $\bar{k}[X]^\times = \bar{k}^\times$ and $\text{Pic } \bar{X}$ finitely generated as a \mathbb{Z} -module. Assume that $g : W \rightarrow X$ is a universal torsor for X . Then $X(\mathbb{A}_k)^g = X(\mathbb{A}_k)^{\text{Br}_1}$.*

In this section, we leverage some of our knowledge of universal torsors in the context of rational points to deduce some information in the context of 0-cycles.

Theorem 5.10. *Let X be a smooth, proper, geometrically integral variety over k with $\text{Pic } \bar{X}$ finitely generated as a \mathbb{Z} -module. Suppose that a universal torsor $g : W \rightarrow X$ under some group G of multiplicative type over k exists. Then, for any integer $d \in \mathbb{Z}$, for any positive integer n , and for any finite subset $S' \subset \Omega_k$ of places of k , we have that*

- (1) $\mathbb{Z}_0^d(X_{\mathbb{A}_k})^g \neq \emptyset$ implies $\mathbb{Z}_0^d(X_{\mathbb{A}_k})^{\text{Br}_1} \neq \emptyset$;
- (2) if, moreover, $\text{Br}_1(X)/\text{Br}_0(X)$ is finite, then $(z_v)_v \in \mathbb{Z}_0^d(X_{\mathbb{A}_k})^{\text{Br}_1}$ implies that there exists some $(u_v)_v \in \mathbb{Z}_0^d(X_{\mathbb{A}_k})^g$ such that z_v and u_v have the same image in $\text{CH}_0(X_{k_v})/n$ for all $v \in S'$.

Proof. (1) Let $(z_v)_v \in \mathbb{Z}_0^d(X_{\mathbb{A}_k})^g$. Then, by definition, there exist a non-empty finite set S of field extensions of k , non-empty finite sets $T_K \subset H_{\text{et}}^1(K, G)$ for each $K \in S$, and a tuple $((\Delta_\tau)_{\tau \in T_K})_{K \in S}$ of integers satisfying $\sum_{K \in S} \sum_{\tau \in T_K} [K : k] \Delta_\tau = d$, and 0-cycles $(z_{v'}^\tau)_{v' \in \Omega_K} \in Z_0^{\Delta_\tau}(W_{\mathbb{A}_K}^\tau)$ for all $\tau \in T_K$ and all $K \in S$ such that

$$\mathbf{g}_{*,\text{ad}} \left(\left((z_{v'}^\tau)_{v' \in \Omega_K} \right)_{\tau \in T_K} \right)_{K \in S} = (z_v)_v.$$

For each $\tau \in T_K$ (for each $K \in S$), we can fix a closed point $y_\tau \in W_K^\tau$, with $F_\tau := K(y_\tau)$ and degree say $\delta_{y_\tau} := [F_\tau : K]$. Then, arguing as in [Lia13, Theorem 3.2.1], we can construct an extension L_τ/K of degree

$$[L_\tau : K] \equiv \Delta_\tau \pmod{n\delta_{y_\tau}},$$

say $[L_\tau : K] = \Delta_\tau + \lambda_\tau n \delta_{y_\tau}$ for some integer λ_τ , and an adelic point $(m_w)_{w \in \Omega_{L_\tau}} \in W^\tau(\mathbb{A}_{L_\tau})$ with $\sum_{w \in \Omega_{L_\tau}} r_{L_\tau, w/K_{v'}}(m_w)$ sufficiently close (in the sense of [Lia13]) to $z_{v'}^\tau$, all the places $v' \in \Omega_K$ above the places in S' , where $r_{L_\tau, w/K_{v'}} : W_{L_\tau, w}^\tau \rightarrow W_{K_{v'}}^\tau$ is the natural map. In particular, $(x_w)_{w \in \Omega_{L_\tau}} := (g_{L_\tau}^\tau(m_w))_{w \in \Omega_{L_\tau}} \in X(\mathbb{A}_{L_\tau})^{g_{L_\tau}}$. But $g_{L_\tau} : W_{L_\tau} \rightarrow X_{L_\tau}$ is again a universal torsor (under G_{L_τ}). Hence, Theorem 5.9 yields that $(x_w)_{w \in \Omega_{L_\tau}} \in X(\mathbb{A}_{L_\tau})^{\text{Br}_1(X_{L_\tau})}$. Similarly, $(\tilde{x}_{w'}^{(\tau)})_{w' \in \Omega_{F_\tau}} := (g_{F_\tau}^\tau((y_\tau)_{w'}))_{w' \in \Omega_{F_\tau}} \in X(\mathbb{A}_{F_\tau})^{\text{Br}_1(X_{F_\tau})}$.

Consider the adelic 0-cycle on X whose v -adic component is given by

$$u_v := \sum_{K \in S} \sum_{\tau \in T_K} \sum_{v' \in \Omega_K : v'|v} \left(\sum_{w \in \Omega_{L_\tau} : w|v'} s_{L_\tau, w/k_v}(x_w^{(\tau)}) - \lambda_\tau n \sum_{w' \in \Omega_{F_\tau} : w'|v'} s_{F_\tau, w'/k_v}(\tilde{x}_{w'}^{(\tau)}) \right),$$

where $s_{L_\tau, w/k_v} : X_{L_\tau, w} \rightarrow X_{k_v}$ and $s_{F_\tau, w'/k_v} : X_{F_\tau, w'} \rightarrow X_{k_v}$ are the natural maps, with degree

$$\begin{aligned} \deg(u_v) &= \sum_{K \in S} \sum_{\tau \in T_K} \sum_{v' \in \Omega_K : v'|v} \left(\sum_{w \in \Omega_{L_\tau} : w|v'} [(L_\tau)_w : k_v] - \lambda_\tau n \sum_{w' \in \Omega_{F_\tau} : w'|v'} [(F_\tau)_{w'} : k_v] \right) \\ &= \sum_{K \in S} \sum_{\tau \in T_K} ([L_\tau : K] - \lambda_\tau n [F_\tau : K]) \sum_{v' \in \Omega_K : v'|v} [(K)_{v'} : k_v] \\ &= \sum_{K \in S} [K : k] \sum_{\tau \in T_K} ([L_\tau : K] - \lambda_\tau n [F_\tau : K]) \\ &= \sum_{K \in S} [K : k] \sum_{\tau \in T_K} \Delta_\tau \\ &= d. \end{aligned}$$

We claim that $(u_v)_v \in Z_0^d(X_{\mathbb{A}_k})^{\text{Br}_1}$. Indeed, for any $\alpha \in \text{Br}_1(X)$,

$$\begin{aligned} & \sum_{v \in \Omega_k} \sum_{K \in S} \sum_{\tau \in T_K} \sum_{v' \in \Omega_K : v'|v} \left[\sum_{w \in \Omega_{L_\tau} : w|v'} \text{inv}_v \left(\text{cores}_{(L_\tau)_w/k_v} \left(\alpha \left(x_w^{(\tau)} \right) \right) \right) \right. \\ & \quad \left. - \lambda_\tau n \sum_{w' \in \Omega_{F_\tau} : w'|v'} \text{inv}_v \left(\text{cores}_{(F_\tau)_{w'}/k_v} \left(\alpha \left(\tilde{x}_{w'}^{(\tau)} \right) \right) \right) \right] \\ &= \left[\sum_{v \in \Omega_k} \sum_{K \in S} \sum_{\tau \in T_K} \sum_{v' \in \Omega_K : v'|v} \sum_{w \in \Omega_{L_\tau} : w|v'} \text{inv}_v \left(\text{cores}_{(L_\tau)_w/k_v} \left(\alpha \left(x_w^{(\tau)} \right) \right) \right) \right] \\ & \quad - \lambda_\tau n \left[\sum_v \sum_{K \in S} \sum_{\tau \in T_K} \sum_{v' \in \Omega_K : v'|v} \sum_{w' \in \Omega_{F_\tau} : w'|v'} \text{inv}_v \left(\text{cores}_{(F_\tau)_{w'}/k_v} \left(\alpha \left(\tilde{x}_{w'}^{(\tau)} \right) \right) \right) \right] \end{aligned}$$

Now,

$$\begin{aligned}
& \sum_{v \in \Omega_k} \sum_{K \in S} \sum_{\tau \in T_K} \sum_{v' \in \Omega_K : v'|v} \sum_{w \in \Omega_{L_\tau} : w|v'} \text{inv}_v \left(\text{cores}_{(L_\tau)_w/k_v} \left(\alpha \left(x_w^{(\tau)} \right) \right) \right) \\
&= \sum_{v \in \Omega_k} \sum_{K \in S} \sum_{\tau \in T_K} \sum_{v' \in \Omega_K : v'|v} \sum_{w \in \Omega_{L_\tau} : w|v'} \text{inv}_w \left(\alpha \left(x_w^{(\tau)} \right) \right) \\
&= \sum_{K \in S} \sum_{\tau \in T_K} \sum_{w \in \Omega_{L_\tau}} \text{inv}_w \left(\alpha \left(x_w^{(\tau)} \right) \right) \\
&= \sum_{K \in S} \sum_{\tau \in T_K} 0 \\
&= 0,
\end{aligned}$$

where in the first equality we have used the commutative diagram

$$\begin{array}{ccc}
\text{Br}(\text{Spec}((L_\tau)_w)) & \xrightarrow{\text{inv}_w} & \mathbb{Q}/\mathbb{Z} \\
\downarrow \text{Cor}_{(L_\tau)_w/k_v} & & \downarrow = \\
\text{Br}(\text{Spec}(k_v)) & \xrightarrow{\text{inv}_v} & \mathbb{Q}/\mathbb{Z}
\end{array}$$

and in third equality we have used the fact that $(x_w^\tau)_{w \in \Omega_{L_\tau}} \in X(\mathbb{A}_{L_\tau})^{\text{Br}_1(X_{L_\tau})}$. Similarly, one can show that

$$\sum_{v \in \Omega_k} \sum_{K \in S} \sum_{\tau \in T_K} \sum_{v' \in \Omega_K : v'|v} \sum_{w' \in \Omega_{F_\tau} : w'|v'} \text{inv}_v \left(\text{cores}_{(F_\tau)_{w'}/k_v} \left(\alpha \left(\tilde{x}_{w'}^{(\tau)} \right) \right) \right) = 0.$$

Hence, $(u_v)_v \in Z_0^d(X_{\mathbb{A}_k})^{\text{Br}_1}$, as required. (We remark that, moreover, if $\sum_{w \in \Omega_{L_\tau} : w|v'} r_{L_\tau, w/K_{v'}}(m_w)$ being sufficiently close to $z_{v'}^\tau$ for all $\tau \in T_K$, all $K \in S$, and all $v \in S'$ were enough to ensure that $\sum_{K \in S} \sum_{\tau \in T_K} \sum_{v' \in \Omega_K : v'|v} \sum_{w \in \Omega_{L_\tau} : w|v'} s_{L_\tau, w/k_v}(x_w^{(\tau)})$ is close enough to z_v , then it would follow immediately that u_v and z_v have the same image in $\text{CH}_0(X_{k_v})/n$ for all $v \in S'$.)

(2) The assumption that $\text{Br}_1 X / \text{Br}_0 X$ is finite can only be true when $\text{Pic } \overline{X}$ is torsion-free: since $H_{\text{et}}^3(k, \overline{k}^\times) = 0$ for any number field k and $\overline{k}^\times[X] = \overline{k}^\times$ by our assumptions on X , by the long exact sequence coming from the Hochschild-Serre spectral sequence

$$E_2^{p,q} := H_{\text{et}}^p(k, H_{\text{et}}^q(\overline{X}, \mathbb{G}_m)) \Rightarrow H_{\text{et}}^{p+q}(X, \mathbb{G}_m)$$

we know that

$$\text{Br}_1 X / \text{Br}_0 X = H_{\text{et}}^1(k, \text{Pic } \overline{X}),$$

and, by Kummer theory, $H_{\text{et}}^1(k, \text{Pic } \overline{X})$ is infinite as soon as $\text{Pic } \overline{X}$ has non-trivial torsion. Let K/k be a finite Galois extension such that $\text{Gal}(\overline{k}/K)$ acts trivially on $\text{Pic } \overline{X}$. Then, following for example the proof of [Lia13, Prop 3.1.1], we have that, for any finite extension l/k linearly disjoint from K over k , the natural restriction map

$$\text{res}_{l/k} : \text{Br}_1 X / \text{Br}_0 X \rightarrow \text{Br}_1(X_l) / \text{Br}_0(X_l)$$

is an isomorphism.

Let $(z_v)_v \in Z_0^d(X_{\mathbb{A}_k})^{\text{Br}_1}$. We fix a closed point $x \in X$ of degree, say, $\delta_x := [k(x) : k]$. In particular, $((x)_w)_{w \in \Omega_{k(x)}} \in X(\mathbb{A}_{k(x)})^{\text{Br}_1(X_{k(x)})}$. Since $g_{k(x)} : W_{k(x)} \rightarrow X_{k(x)}$ is still a universal torsor, Theorem 5.9 yields $((x)_w)_{w \in \Omega_{k(x)}} \in X(\mathbb{A}_{k(x)})^{g_{k(x)}}$.

Following [Lia13], we can construct a finite extension l/k , linearly disjoint from K and $k(x)$ over k , such that $[l : k] \equiv d \pmod{n\delta_x}$, say $[l : k] = d + \lambda n\delta_x$ for some $\lambda \in \mathbb{Z}$, and an adelic point $(\tilde{x}_{w'})_{w' \in \Omega_l} \in X(\mathbb{A}_l)^{\text{Br}_1(X_l)} = X(\mathbb{A}_l)^{g_l}$ with $\sum_{w' \in \Omega_l : w'|v} s_{l_{w'}/k_v}(\tilde{x}_{w'})$ sufficiently close (in the sense of [Lia13]) to z_v for each place $v \in S'$, where $s_{l_{w'}/k_v} : X_{l_{w'}} \rightarrow X_{K_{v'}}$ is the natural map. Note that $X(\mathbb{A}_l)^{\text{Br}_1(X_l)} = X(\mathbb{A}_l)^{g_l}$ by the fact that $g_l : W_l \rightarrow X_l$ is still a universal torsor together with Theorem 5.9.

Consider the adelic 0-cycle on X whose v -adic component is given by

$$u_v := \sum_{w' \in \Omega_l : w'|v} s_{l_{w'}/k_v}(\tilde{x}_{w'}) - \lambda n \sum_{w \in \Omega_{k(x)} : w|v} s_{k(x)_w/k_v}((x)_w)$$

and has degree

$$\begin{aligned} \deg(z_v) &= \sum_{w' \in \Omega_l : w'|v} [l_{w'} : k_v] - \lambda n \sum_{w \in \Omega_{k(x) : w|v}} [k(x)_w : k_v] \\ &= [l : k] - \lambda n [k(x) : k] \\ &= d. \end{aligned}$$

We claim that $(u_v) \in Z_0^d(X_{\mathbb{A}_k})^g$. Indeed, take $S := \{l, k(x)\}$ (noticing that l and $k(x)$ are linearly disjoint over k by construction); take $T_l := \{\tau\} \subset H^1(l, G)$ and $T_{k(x)} := \{\sigma\} \subset H^1(k(x), G)$, where τ is any twist such that there exists some $(\tilde{y}_{w'})_{w'} \in W^\tau(\mathbb{A}_l)$ above $(\tilde{x}_{w'})_{w'}$, and σ is any twist such that there exists some $(y_w)_w \in W^\sigma(\mathbb{A}_{k(x)})$ above $((x)_w)_w$; take $\Delta_\tau = 1$ and $\Delta_\sigma = -\lambda n$. Then $(\tilde{y}_{w'})_{w'} \in Z_0^1(W_{\mathbb{A}_l}^\tau)$ and $(-\lambda n y_w)_w \in Z_0^{-\lambda}(W_{\mathbb{A}_{k(x)}}^\sigma)$, with

$$\mathbf{g}_{*,\text{ad}}(((\tilde{y}_{w'})_{w'}, (-\lambda n y_w)_w)) = \left(\sum_{w' \in \Omega_l : w'|v} s_{l_{w'}/k_v}(\tilde{x}_{w'}) - \lambda n \sum_{w \in \Omega_{k(x) : w|v}} s_{k(x)_w/k_v}((x)_w) \right)_v = (u_v)_v$$

and

$$\Delta_\tau \cdot [l : k] + \Delta_\sigma \cdot \delta_x = [l : k] - \lambda n \delta_x = d,$$

as required. Finally, it is easy to check that the u_v and z_v have the same image in $\text{CH}_0(X_{k_v})/n$ for all $v \in S'$. \square

Remark 5.11. If we further assume that g is proper (e.g., when G is finite), then the first statement of Theorem 5.10 can be replaced by the more precise statement

- (1) if $(z_v) \in Z_0^d(X_{\mathbb{A}_k})^g$ then there is some $(z'_v) \in Z_0^d(X_{\mathbb{A}_k})^{\text{Br}_1}$ such that z_v and z'_v have the same image in $\text{CH}_0(X_{k_v})/n$, for all $v \in S'$.

Finally, we consider torsors under tori. In the rational points setting, Harpaz and Wittenberg have proved the following nice result.

Theorem 5.12 ([HW20, Théorème 2.1]). *Let X be a smooth, geometrically integral variety over k . Let $f : Y \rightarrow X$ be a torsor under a k -torus T . Let $A \subset \text{Br } X$ be the inverse image of $\text{Br}_{nr}(Y) \subset \text{Br } Y$ under $f^* : \text{Br } X \rightarrow \text{Br } Y$. Then*

$$X(\mathbb{A}_k)^A \subset \bigcup_{\sigma \in H^1(k, T)} f^\sigma(Y^\sigma(\mathbb{A}_k)^{\text{Br}_{nr}}).$$

A very similar proof to that of Theorem 5.10 (2), together with Theorem 5.12, yields the following immediate statement for 0-cycles in the spirit of Theorem 5.12, albeit with the usual restrictions on the Brauer groups.

Theorem 5.13. *Let X be a smooth, proper, geometrically integral variety over k . Let $f : Y \rightarrow X$ be a torsor under a k -torus T . Assume that $\text{Br } X / \text{Br}_0 X$ is finite and that there is some finite extension F/k such that $\text{res}_{l/k} : \text{Br } X / \text{Br}_0 X \rightarrow \text{Br}(X_l) / \text{Br}_0(X_l)$ is surjective for all finite extensions l/k linearly disjoint from F over k . Then, for any integer $d \in \mathbb{Z}$, for any positive integer n , and for any finite subset $S' \subset \Omega_k$ of places of k , we have that $(z_v)_v \in Z_0^d(X_{\mathbb{A}_k})^{\text{Br}}$ implies that there exists some $(u_v)_v \in Z_0^d(X_{\mathbb{A}_k})^{f, \text{Br}_{nr}}$ such that z_v and u_v have the same image in $\text{CH}_0(X_{k_v})/n$ for all $v \in S'$.*

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THE AMERICAN UNIVERSITY OF PARIS, 5 BOULEVARD DE LA TOUR-MAUBOURG, 75007 PARIS, FRANCE

Email address: `fbalestrieri@aup.edu`

BUCKNELL UNIVERSITY, DEPARTMENT OF MATHEMATICS, LEWISBURG, PA 17837, USA

Email address: `jsb047@bucknell.edu`