Homotopy-Based Path Planning Using Smooth Signed Distances

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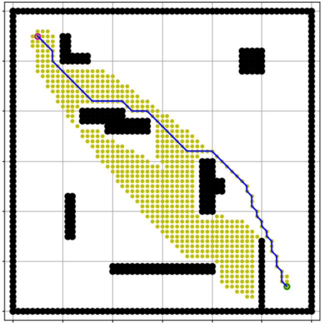
Classical Path Planning Methods



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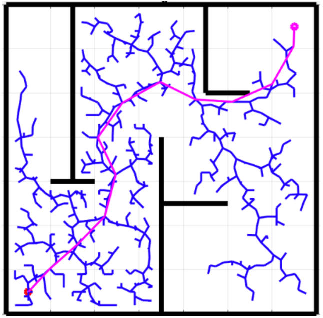
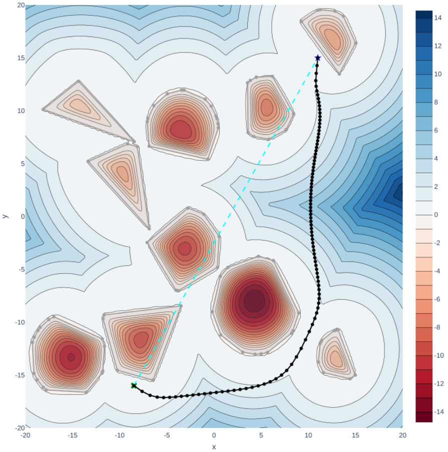


RRT

Motivation

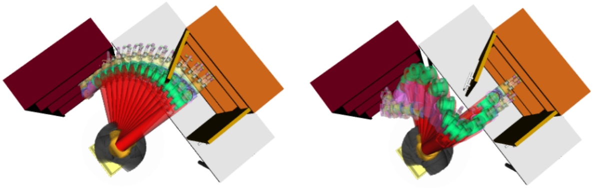
- Traditional planners often produce jagged or suboptimal paths that require post-processing;
- Optimizing paths via deformation ensures smoother, optimal, and more feasible paths;
- Signed distance functions provide continuous, differentiable measures of proximity to obstacles;
- Gradient descent on such functions allows for intuitive and efficient path updates.



Deformed smooth path

CHOMP (Covariant Hamiltonian Optimization for Motion Planning)

- starts with an initial trajectory (e.g., straight line in joint or task space);
- defines a cost functional with two terms:
  - smoothness cost (e.g., velocity or acceleration penalties);
  - obstacle cost based on a precomputed cost map;
- performs functional gradient descent to iteratively minimize the total cost;
- ensure gradient steps preserve the shape and timing of the trajectory (covariant updates);
- outputs a smooth, collision-free trajectory.



Generalized Point to Set Distance

Definition (GP2SD)

Let $\mathcal{S} \subset \mathbb{R}^n$ be a convex set and $f_{\mathcal{S}} : \mathbb{R}^n \to \mathbb{R}$. Then f is said a *generalized point-to-set distance (GP2SD)* if it has the following properties:

i $f_{\mathcal{S}}(\mathbf{p})$ is twice differentiable in $\mathbf{p}$;

ii $f_{\mathcal{S}}(\mathbf{p}) \geq 0 \, \forall \, \mathbf{p} \in \mathbb{R}^n$ and $f_{\mathcal{S}}(\mathbf{p}) = 0 \iff \mathbf{p} \in \mathcal{S}$;

iii the hessian $\frac{\partial^2 f}{\partial \mathbf{p} \partial \mathbf{p}^\top}(\mathbf{p})$ is positive for all $\mathbf{p} \notin \mathcal{S}$;

iv the term $\left(\frac{\partial^2 f}{\partial \mathbf{p} \partial \mathbf{p}^\top}(\mathbf{p}) - \mathbf{I}_n\right)$ is negative definite.

Furthermore, if the hessian of $f_{\mathcal{S}}$ is only positive semidefinite for $\mathbf{p} \notin \mathcal{S}$, we denote it a *weak GP2SD (WGP2SD)*.

Basic Generalized Point to Set Distance

Definition (BGP2SD)

Let $\Phi : \mathbb{R} \to \mathbb{R}_+$ be a scalar function. It is called a *basic GP2SD (BGP2SD)* if it satisfies all the properties of a GP2SD for $\mathcal{S} = \mathbb{R}_-$.

Let $\gamma > 1$, our BGP2SD is defined as

$$\Phi(s) = \begin{cases} \gamma \log\left(\cosh\left(\frac{s}{\gamma}\right)\right), & s > 0 \\ 0, & s \leq 0 \end{cases}$$

Signed Generalized Point to Set Distance

Definition (SGP2SD)

A *signed GP2SD (SGP2SD)* $d_{\mathcal{S}} : \mathbb{R}^n \to \mathbb{R}$ is a function composed by an *outer distance* $d_{\mathcal{S}}^{+} : \mathbb{R}^n \to \mathbb{R}_{+}$ and an *inner distance* $d_{\mathcal{S}}^{-} : \mathbb{R}^n \to \mathbb{R}_{-}$ such that $d_{\mathcal{S}}(\mathbf{p}) = d_{\mathcal{S}}^{+}(\mathbf{p}) + d_{\mathcal{S}}^{-}(\mathbf{p})$. The outer and inner distances have the following properties

i $d_{\mathcal{S}}^{+}(\mathbf{p})$ is a GP2SD;

ii $d_{\mathcal{S}}^{-}(\mathbf{p}) < 0 \, \forall \, \mathbf{p} \in \mathring{\mathcal{S}}$ and $d_{\mathcal{S}}^{-}(\mathbf{p}) = 0 \, \forall \, \mathbf{p} \notin \mathring{\mathcal{S}}$.

iii $d_{\mathcal{S}}^{-}(\mathbf{p}) \in C^2$.

Outer and Inner distances

Let $\mathcal{S}$ be a polyhedron, then $\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$.
We adopt the following distances

$$d^+_{\mathcal{S}}(\mathbf{p}) = \frac{1}{m}\sum_{i=1}^{m} \Phi\left(\mathbf{a}_i^\top \mathbf{p} - \mathbf{b}_i\right).$$

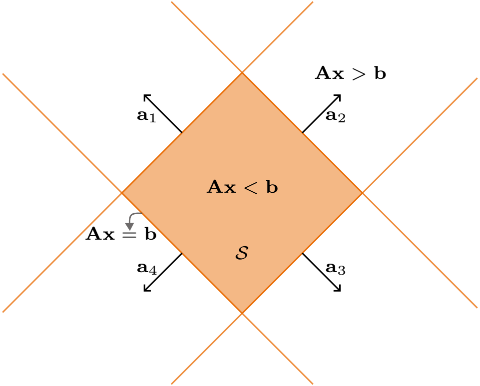
$$d^-_{\mathcal{S}}(\mathbf{p}) = -\left(\frac{1}{m}\sum_{i=1}^{m} \Phi\left(\mathbf{b}_i - \mathbf{a}_i^\top \mathbf{p}\right)\right)^{-\frac{1}{r}}\Bigg)^{-r}$$

$$d_{\mathcal{S}}(\mathbf{p}) = \Gamma\left(d^+_{\mathcal{S}}(\mathbf{p})\right) + d^-_{\mathcal{S}}(\mathbf{p}),$$

where

$$\Gamma\left(f(\mathbf{p})\right) = \varepsilon\rho(\mathbf{p}) + \sqrt{\varepsilon^2\rho(\mathbf{p})^2 + (1-2\varepsilon)f(\mathbf{p})^2}$$

$$\rho(\mathbf{p}) = \frac{1}{2}\left(\|\mathbf{p} - \mathbf{c}\|^2 - R^2\right)$$



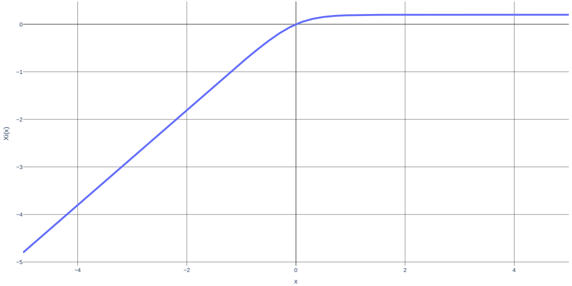
Total Distance

Let $\mathcal{O} = \{\mathcal{S}^1, \mathcal{S}^2, \dots, \mathcal{S}^\ell\}$ be the set of all ℓ convex obstacles in the environment. The signed distance to all obstacles is computed as

$$D_{\mathcal{O}}(\mathbf{p}) = \sum_{\mathcal{S}^i \in \mathcal{O}} \Xi\Big(\Gamma\big(d^+_{\mathcal{S}^i}(\mathbf{p})\big) + \eta d^-_{\mathcal{S}^i}(\mathbf{p})\Big),$$

where Ξ acts as a smooth saturation:

$$\Xi(x) = -\frac{1}{\alpha}\log\left(\frac{1+e^{-\alpha x}}{2}\right), \quad \alpha \in \mathbb{R}_+$$



Optimization Problem

A path $\mathcal{P} = (\mathbf{p}^1, \mathbf{p}^2, \ldots, \mathbf{p}^N)$ is an ordered sequence of N points in $\mathbb{R}^n$.

The following problem maximizes the total distance between a point and obstacles in the environment, while minimizing the path length:

$$\min_{\mathbf{p}^i \in \mathcal{P}} \sum_{i=1}^{N} -D_{\mathcal{O}}(\mathbf{p}^i) + \frac{\zeta}{2} \sum_{i=1}^{N-1} \left\| \mathbf{p}^{i+1} - \mathbf{p}^i \right\|^2$$

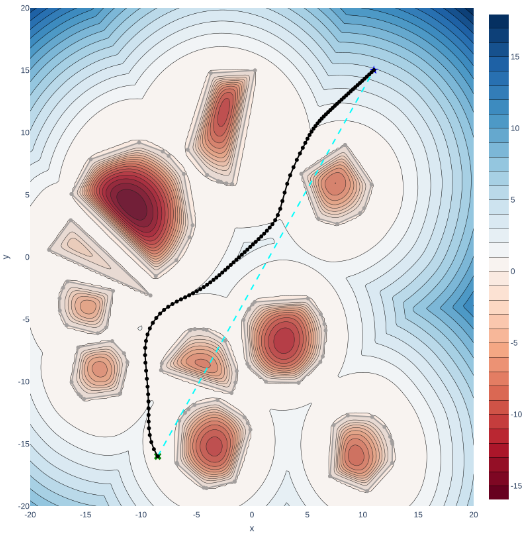
s.t.:

$$\mathbf{p}^1 = \mathbf{q}_0,$$
$$\mathbf{p}^N = \mathbf{q}_d,$$

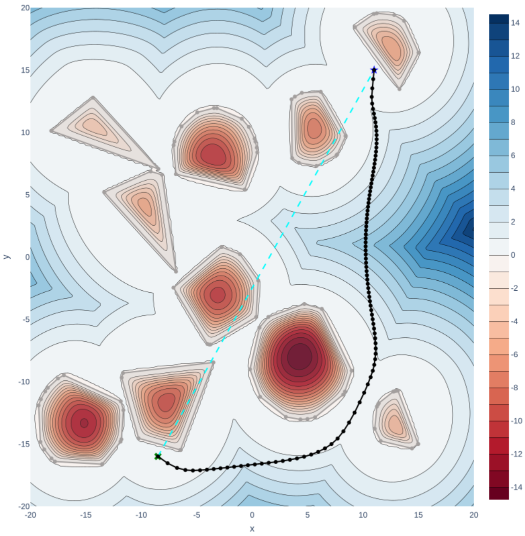
Input: $\mathcal{P}_{\text{init}} = (\mathbf{p}^1, \ldots, \mathbf{p}^N)$
Output: $\mathcal{P}_{\text{final}}$

$\mathbf{q}_0 \leftarrow \mathbf{p}^1$
$\mathbf{q}_d \leftarrow \mathbf{p}^N$
$\mathcal{P}_{\text{final}} \leftarrow \mathcal{P}_{\text{init}}$
while $\exists \mathbf{p}^i \in \mathcal{P}_{\text{final}}$ such that $D_{\mathcal{O}}(\mathbf{p}^i) < \delta$ **do**
  $\mathcal{P} \leftarrow \{\mathbf{q}_0\}$
  for $k \leftarrow 2$ to $N-1$ **do**
    $\mathbf{w}^k \leftarrow \zeta(2\mathbf{p}^k - \mathbf{p}^k - \mathbf{p}^{k+1})$
    $\mathbf{v}^k \leftarrow \sqrt{|D_{\mathcal{O}}(\mathbf{p}^k)|} \nabla D_{\mathcal{O}}(\mathbf{p}^k)^\top$
    $\mathbf{p}^k_{\text{final}} \leftarrow \mathbf{p}^k - \mathbf{v}^k + \mathbf{w}^k$
    $\mathcal{P} \leftarrow \mathcal{P} \cup \{\mathbf{p}^k_{\text{final}}\}$
  end for
  $\mathcal{P} \leftarrow \mathcal{P} \cup \{\mathbf{q}_d\}$
  $\mathcal{P}_{\text{final}} \leftarrow \mathcal{P}$
end while

First map

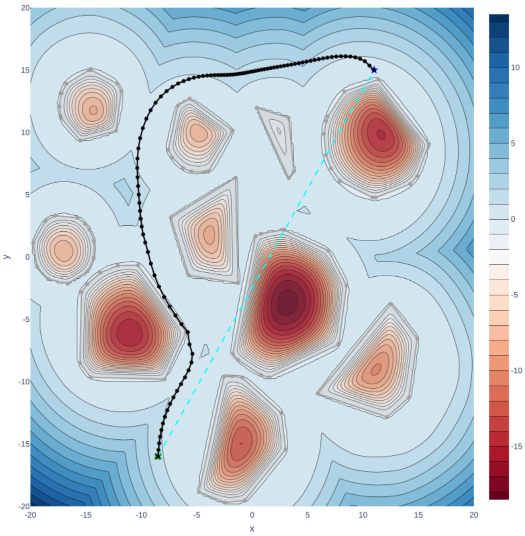


Second map



20
15
10
5
0
-5
-10
-15
-20
y
-20
-15
-10
-5
0
5
10
15
20
x
14
12
10
8
6
4
2
0
-2
-4
-6
-8
-10
-12
-14

Third map



Conclusion and Future Work

Conclusions:

- proposed a smooth signed distance function and its gradient behaves as expected;
- strategy performs well, having generated collision-free paths for each situation simulated.

Future works include:

- defining a smooth minimum function for scalars in general;
- exploring non-polyhedral constraints;
- defining the distance for non-convex obstacles;
- investigating the strategy in the configuration space of manipulators.