Homotopy-Based Path Planning Using Smooth Signed Distances

Felipe Bartelt

***Abstract*— This paper presents a novel path planning strategy based on generalized smooth signed distance functions. The proposed method deforms an initial straight-line path into a collision-free path by performing gradient descent on a cost functional that combines obstacle avoidance and path length. The path is iteratively adjusted to maximize the distance between the path and surrounding obstacles while preserving the initial and final configurations. Simulation results in randomly generated cluttered environments demonstrate that the method produces safe and smooth paths with few iterations.**

***Index Terms*— path planning, curve deformation, signed distance function, collision avoidance, gradient descent.**

I. Introduction

PATH planning is a fundamental component of autonomous robotic systems, directly impacting safety, efficiency, and adaptability in dynamic environments. A key challenge lies in generating collision-free trajectories that are not only feasible but also smooth and responsive to changes in the environment. Traditional sampling-based methods, such as Rapidly-exploring Random Trees (RRT) and Probabilistic Roadmaps (PRM), provide effective global exploration but often yield paths that are suboptimal in terms of smoothness or clearance. To address this, a growing body of research has explored optimization-based techniques that refine an initial path by minimizing cost functionals defined over smooth paths.

Among such approaches, the Elastic Bands framework [1] bridges planning and control by modeling paths as elastic structures subject to virtual forces, allowing continuous deformation in response to environmental changes. Building on these ideas, CHOMP (Covariant Hamiltonian Optimization for Motion Planning) [2] introduced a trajectory optimization framework that minimizes a combination of smoothness and obstacle cost using covariant gradient descent. Subsequent works such as STOMP [3], TrajOpt [4], and GPMP2 [5] have further refined these concepts by introducing stochastic sampling, sequential convex optimization, and Gaussian process representations, respectively. These methods exemplify a shift toward continuous optimization and deformation-based techniques that leverage cost functions or distance fields to enforce safety and efficiency in high-dimensional planning.

This paper proposes a novel trajectory planning method that utilizes generalized smooth signed distance functions to apply gradient-based deformations. Under this methodology, a straight line connecting the initial and final configurations is iteratively deformed so that no point along the path lies inside any obstacle. The deformation process maximizes the distance between the path and surrounding obstacles while ensuring that the trajectory respects a predefined safety margin. Additionally, path length is minimized, promoting efficient and natural-looking motions.

Simulations demonstrate that the proposed strategy performs effectively across a range of planning scenarios, consistently producing safe, collision-free paths within a small number of iterations. The results highlight the method's practicality, efficiency, and potential for integration into real-time robotic systems. The contributions of this work are: 1) definition of a smooth signed distance function, 2) a hotopy-based path planning algorithm.

II. Generalized distances

In this work we will adopt $\mathbb{R}_+$ as the nonnegative real numbers, $\mathbb{R}_-$ as the nonpositive real numbers. Every vector in this work must be regarded as a column vector, and the gradient of a function f, represented by ∇f, is always a row vector. The norm $\|\cdot\|$ represents the Euclidean norm. The symbol λ will always denote eigenvalues, so $\lambda(\mathbf{A})$ represents the eigenvalues of a matrix $\mathbf{A}$, $\lambda_i(\mathbf{A})$ the i^{th} eigenvalue of $\mathbf{A}$, $\lambda_{\min}(\mathbf{A})$ and $\lambda_{\max}(\mathbf{A})$ the minimum and maximum eigenvalues of $\mathbf{A}$, respectively. For brevity, we also borrow some set notation more common to topology: $\mathring{\mathcal{S}}$ represents the interior of a set $\mathcal{S}$, i.e., the union of all open subsets of $\mathcal{S}$; $\partial\mathcal{S}$ denotes the boundary of a set $\mathcal{S}$; and $\bar{\mathcal{S}}$ denotes de closure of $\mathcal{S}$, which is equal to $\mathring{\mathcal{S}} \cup \partial\mathcal{S}$. The complement of a set $\mathcal{S}$ is represented by $\mathcal{S}^c$. The symbol $\blacksquare$ denotes the end of proofs, while the symbol $\square$ denotes the end of definitions.

In this section we introduce concepts of smooth distances in order to define a signed smooth distance for which we can compute a gradient at every point, such that a initial path can be deformed into a collision-free final path. We start with the concept of a *generalized point to set distance*.

Definition 2.1 (GP2SD): Let $\mathcal{S} \subset \mathbb{R}^n$ be a convex set and $f_\mathcal{S} : \mathbb{R}^n \to \mathbb{R}$. Then f is said a *generalized point-to-set distance (GP2SD)* if it has the following properties:

- i $f_\mathcal{S}(\mathbf{p})$ is twice differentiable in $\mathbf{p}$;
- ii $f_\mathcal{S}(\mathbf{p}) \geq 0\, \forall\, \mathbf{p} \in \mathbb{R}^n$ and $f_\mathcal{S}(\mathbf{p}) = 0 \iff \mathbf{p} \in \mathcal{S}$;
- iii the hessian $\frac{\partial^2 f}{\partial \mathbf{p} \partial \mathbf{p}^\top}(\mathbf{p})$ is positive definite for all $\mathbf{p} \notin \mathcal{S}$;
- iv the term $\left(\frac{\partial^2 f}{\partial \mathbf{p} \partial \mathbf{p}^\top}(\mathbf{p}) - \mathbf{I}_n\right)$ is negative definite.

Furthermore, if the hessian of $f_\mathcal{S}$ is only positive semidefinite for $\mathbf{p} \notin \mathcal{S}$—meaning it lacks Property 2.1.iii—we denote it a *weak GP2SD (WGP2SD)*. $\square$

The following function allows an easy procedure to transform WGP2SDs into GP2SDs.

Definition 2.2 (Bulging function): Let $\mathcal{S} \subset \mathbb{R}^n$ be a convex set, and let $f_{\mathcal{S}} : \mathbb{R}^n \to \mathbb{R}$ be a WGP2SD. Let $\mathcal{B}$ be a closed ball with radius R centered at $\mathbf{c}$ such that $\mathcal{S} \subseteq \mathring{\mathcal{B}}$ and $\partial\mathcal{B} \cap \mathcal{S} = \varnothing$. Define the function $\rho(\mathbf{p}) = \frac{1}{2}\big(\|\mathbf{p}-\mathbf{c}\|^2 - R^2\big)$. Then, for any fixed small $\varepsilon > 0$, the *bulging function* $\Gamma : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\Gamma(\mathbf{p}) = \varepsilon\rho(\mathbf{p}) + \sqrt{\varepsilon^2\rho(\mathbf{p})^2 + (1-2\varepsilon)f(\mathbf{p})^2}.$$

□

We now prove that the bulging function maintains the properties of WGP2SDs and enforce that their hessian are positive definite outside the set $\mathcal{S}$.

Lemma 2.3: Let $\mathcal{S} \subset \mathbb{R}^n$ be a convex set, and let $f \triangleq f_{\mathcal{S}} : \mathbb{R}^n \to \mathbb{R}$ be a WGP2SD. Let $\Gamma : \mathbb{R}^n \to \mathbb{R}$ the bulging function, then the composition $\Gamma \circ f$ is a GP2SD.

Proof: We prove each property separately. Note that $\rho(\mathbf{p})$ is positive for any point $\mathbf{p}$ outside the ball $\mathcal{B}$, since $f_{\mathcal{S}}$ is a WGP2SD, $\Gamma\big(f(\mathbf{p})\big) > 0$ for every point $\mathbf{p}$ outside $\mathcal{B}$. Take $\mathbf{p} = \mathbf{x}$ such that $\|\mathbf{x}-\mathbf{c}\| = R$, which implies $\rho(\mathbf{p}) = 0$, but since $\partial\mathcal{B}\cap\mathcal{S} = \varnothing$, $f(\mathbf{p})$ is positive and so is $\Gamma\big(f(\mathbf{p})\big)$. For any point $\mathbf{p}$ inside $\mathcal{S}$, the function $f(\mathbf{p}) = 0$, thus $\Gamma\big(f(\mathbf{p})\big) = \varepsilon\rho(\mathbf{p}) + \sqrt{\varepsilon^2\rho(\mathbf{p})^2}$. In this case, $\rho(\mathbf{p})$ is negative and the function composition evaluates to 0. We then conclude that $\Gamma\big(f(\mathbf{p})\big) \geq 0$ and $\Gamma\big(f(\mathbf{p})\big) = 0 \iff \mathbf{p} \in \mathcal{S}$, thus Property 2.1.ii holds.

To show Properties 2.1.i and 2.1.iii we first derive the hessian of the bulging function:

$$\begin{aligned}\frac{\partial^2\Gamma}{\partial\mathbf{p}\partial\mathbf{p}^\top} =& \varepsilon\mathbf{I} - \frac{1}{4\beta^{\frac{3}{2}}}\Big(2\varepsilon^2\rho\nabla\rho^\top + 2(1-2\varepsilon)f\nabla f^\top\Big)\nabla\beta \\ &+ \frac{1}{2\sqrt{\beta}}\Bigg(2\varepsilon^2\nabla\rho^\top\nabla\rho + 2\varepsilon^2\mathbf{I} \\ &+ 2(1-2\varepsilon)\bigg(\nabla f^\top\nabla f + f\frac{\partial^2 f}{\partial\mathbf{p}\partial\mathbf{p}^\top}\bigg)\Bigg),\end{aligned}$$

where functional dependencies were omitted for readability. Substituting the expression of $\nabla\beta$ and expanding the terms results in

$$\begin{aligned}\frac{\partial^2\Gamma}{\partial\mathbf{p}\partial\mathbf{p}^\top} =& \varepsilon\bigg(1 + \frac{\varepsilon}{\sqrt{\beta}}\bigg)\mathbf{I} + \frac{\varepsilon^2}{\sqrt{\beta}}\Big(1 - \frac{\varepsilon^2\rho^2}{\beta}\Big)\nabla\rho^\top\nabla\rho \\ &+ \frac{(1-2\varepsilon)}{\sqrt{\beta}}\bigg(1 - \frac{(1-2\varepsilon)f^2}{\beta}\bigg)\nabla f^\top\nabla f \\ &+ \frac{(1-2\varepsilon)}{\sqrt{\beta}}f\frac{\partial^2 f}{\partial\mathbf{p}\partial\mathbf{p}^\top}.\end{aligned} \tag{1}$$

Note that, since f is twice differentiable and both the gradients of ρ and f do not have singularities, then Γ is twice differentiable. Furthermore, since β and ε are positive, the first term is positive definite. Since f is positive definite and its hessian is positive semidefinite for all points outside $\mathcal{S}$, the last term is also positive semidefinite outside $\mathcal{S}$. For the remaining terms, first note that $\mathbf{x}\mathbf{x}^\top$ is positive semidefinite for any $\mathbf{x} \in \mathbb{R}^n$, then it suffices to show that the terms within parentheses are positive. We begin with the second term:

$$1 - \frac{\varepsilon^2\rho(\mathbf{p})^2}{\beta(\mathbf{p})} = 1 - \frac{\varepsilon^2\rho(\mathbf{p})^2}{\varepsilon^2\rho(\mathbf{p})^2 + (1-2\varepsilon)f(\mathbf{p})^2},$$

which is clearly positive. Similarly, the for the third term we have

$$1 - \frac{(1-2\varepsilon)f(\mathbf{p})^2}{\beta(\mathbf{p})} = 1 - \frac{(1-2\varepsilon)f(\mathbf{p})^2}{\varepsilon^2\rho(\mathbf{p})^2 + (1-2\varepsilon)f(\mathbf{p})^2},$$

which is also positive. It follows that the hessian of $\Gamma\big(f(\mathbf{p})\big)$ is definite positive for all points $\mathbf{p} \notin \mathcal{S}$ and Property 2.1.iii holds.

Since ε is a small positive number, take the limit of (1) as $\varepsilon \to 0^+$:

$$\begin{aligned}\lim_{\varepsilon\to0^+}\frac{\partial^2\Gamma}{\partial\mathbf{p}\partial\mathbf{p}^\top} &= \frac{1}{\sqrt{f}}\bigg(1 - \frac{f^2}{f^2}\bigg)\nabla f^\top\nabla f + \frac{1}{\sqrt{f^2}}f\frac{\partial^2 f}{\partial\mathbf{p}\partial\mathbf{p}^\top} \\ &= \frac{\partial^2 f}{\partial\mathbf{p}\partial\mathbf{p}^\top}(\mathbf{p}),\end{aligned}$$

which implies $\frac{\partial^2\Gamma}{\partial\mathbf{p}\partial\mathbf{p}^\top} - \mathbf{I}$ when ε tends to 0^+. This shows Property 2.1.iv. ■

In order to define our GP2SD, we first define a simpler structure, which simplifies its definition:

Definition 2.4 (BGP2SD): Let $\Phi_{\mathcal{S}} : \mathbb{R} \to \mathbb{R}_+$ be a scalar function. It is called a *basic GP2SD (BGP2SD)* if it satisfies all the properties of a GP2SD (Definition 2.1) for $\mathcal{S} = \mathbb{R}_-$. □

Throughout this work, we will use the BGP2SD in the following lemma.

Lemma 2.5: Let $\gamma > 1$, then the function $\Phi : \mathbb{R} \to \mathbb{R}_+$ defined as

$$\Phi(s) = \gamma\log\bigg(\cosh\Big(\frac{s}{\gamma}\Big)\bigg) \tag{2}$$

is a BGP2SD.

Proof: We prove every property separately. The first and second derivative of $\Phi(s)$ are given by

$$\frac{d}{ds}\Phi(s) = \tanh\Big(\frac{s}{\gamma}\Big), \tag{3}$$

$$\frac{d^2}{ds^2}\Phi(s) = \frac{1}{\gamma}\operatorname{sech}^2\Big(\frac{s}{\gamma}\Big). \tag{4}$$

Since both have no singularities, Property 2.1.i holds.

Given that γ is positive and that $\operatorname{sech}(x)$ is only zero when $x \to \pm\infty$, then the second derivative of Φ in (4) is positive definite for all $s \in (0,\infty)$ and Property 2.1.iii is satisfied for $\mathcal{S} = \mathbb{R}_-$.

The maximum value of $\operatorname{sech}^2(x)$ is attained at $x = 0$, for which $\operatorname{sech}^2(0) = 1$. Thus $\frac{1}{\gamma}\operatorname{sech}^2(\frac{s}{\gamma}) < 1\,\forall s \in \mathbb{R}, \gamma > 1$. This implies that Property 2.1.iv is satisfied.

Note that $\cosh(\frac{s}{\gamma})$ is zero if and only if[1] $s = 0$. Now, suppose that there exists z such that $\Phi(z)$ is negative. This implies that $\cosh(\frac{z}{\gamma}) < 1$, but $\frac{d}{ds}\Phi(s) = \tanh(\frac{s}{\gamma}) = 0$ implies $s = 0$. Since the second derivative is positive, $s = 0$ is the minimum of the function, and $\cosh(0) = 1$, so there cannot exist a z such that $\Phi(z)$ is negative, thus Property 2.1.ii also holds. ■

We now define a signed GP2SD by decomposing the distance function into two parts, an *outer distance* and a *inner distance*. This distance then outputs a negative number for

[1]The domain of the function is the real line, so complex roots can be discarded.

every point inside the set, a positive number for any point outside it, and zero for points on the boundary.

Definition 2.6 (SGP2SD): A *signed GP2SD (SGP2SD)* $d_{\mathcal{S}} : \mathbb{R}^n \to \mathbb{R}$ is a function composed by an *outer distance* $d^+_{\mathcal{S}} : \mathbb{R}^n \to \mathbb{R}_+$ and an *inner distance* $d^-_{\mathcal{S}} : \mathbb{R}^n \to \mathbb{R}_-$ such that $d_{\mathcal{S}}(\mathbf{p}) = d^+_{\mathcal{S}}(\mathbf{p}) + d^-_{\mathcal{S}}(\mathbf{p})$. The outer and inner distances have the following properties

i $d^+_{\mathcal{S}}(\mathbf{p})$ is a GP2SD;
ii $d^-_{\mathcal{S}}(\mathbf{p}) < 0\,\forall\,\mathbf{p} \in \mathring{\mathcal{S}}$ and $d^-_{\mathcal{S}}(\mathbf{p}) = 0\,\forall\,\mathbf{p} \notin \mathring{\mathcal{S}}$.
iii $d^-_{\mathcal{S}}(\mathbf{p}) \in C^2$.

Furthermore, if the outside distance is only a WGP2SD, we call the resulting function a *weak SGP2SD (WSGP2SD).* □

Before presenting our outer and distance functions, we introduce functions that measure how far a point is from feasibility or unfeasibility[2], meaning how far they are from entering a set and exiting a set, respectively.

Definition 2.7: Let $\mathcal{S} \subset \mathbb{R}^n$ be a convex set. A function $F : \mathbb{R}^n \to \mathbb{R}_+$ is called a *feasibility distance* for $\mathcal{S}$ if:

i $F(\mathbf{p}) = 0\,\forall\,\mathbf{p} \notin \mathring{\mathcal{S}}$;
ii $F(\mathbf{p}) > 0\,\forall\,\mathbf{p} \in \mathring{\mathcal{S}}$.

This function quantifies how far a point is from becoming unfeasible (i.e., from exiting $\mathcal{S}$). □

Definition 2.8: Let $\mathcal{S} \subset \mathbb{R}^n$ be a convex set. A function $U : \mathbb{R}^n \to \mathbb{R}_+$ is called a *unfeasibility distance* for $\mathcal{S}$ if it is a feasibility distance for $\mathcal{S}^c$, the complement of $\mathcal{S}$. This function quantifies how far a point is from becoming feasible (i.e., from entering $\mathcal{S}$) □

Let a set $\mathcal{S}$ be defined by the intersection of m convex sets, i.e., $\mathcal{S} = \bigcap_{i=1}^m \mathcal{S}_i$. Let $U_i : \mathbb{R}^n \to \mathbb{R}_+$ be the respective unfeasible distance for each set $\mathcal{S}_i$. Let $\Phi : \mathbb{R} \to \mathbb{R}_+$ be a BGP2SD. Then, a weak outer distance function is given by

$$d^+_{\mathcal{S}}(\mathbf{p}) = \frac{1}{m}\sum_{i=1}^{m} \Phi\big(U_i(\mathbf{p})\big). \tag{5}$$

Note that, if a point $\mathbf{p}$ lies outside any $\mathcal{S}_i$ then—consequently outside $\mathcal{S}$—the distance $d^+_{\mathcal{S}}(\mathbf{p})$ is positive. On the other hand, if $\mathbf{p}$ lies inside every $\mathcal{S}_i$, thus an interior or boundary point of $\mathcal{S}$, then $d^+_{\mathcal{S}}(\mathbf{p}) = 0$.

Analogously, let $F_i : \mathbb{R}^n \to \mathbb{R}_+$ be the respective feasible distance for each set $\mathcal{S}_i$. Let $r \in \mathbb{R}$ be a positive number, then our inner distance function is

$$d^-_{\mathcal{S}}(\mathbf{p}) = -\left(\frac{1}{m}\sum_{i=1}^{m} \Phi\big(F_i(\mathbf{p})\big)^{-\frac{1}{r}}\right)^{-r}, \tag{6}$$

which is known as the Hölder mean [6]. In this case, if a point $\mathbf{p}$ is unfeasible for any $\mathcal{S}_i$, then $d^-_{\mathcal{S}}(\mathbf{p}) = 0$, otherwise—if the point lies inside the set—the distance is a negative number. Also note that, as r goes to zero, the value of $d^-_{\mathcal{S}}(\mathbf{p})$ tends to the minimum value of $\Phi(F_i)$.

We now show that under minor assumptions on the feasibility and unfeasibility distances, the outer in (5) and inner distances defined in (6) generate a WSGP2SD.

Lemma 2.9: Let $\Phi : \mathbb{R} \to \mathbb{R}_+$ be a BGP2SD as in (2). Let $\mathcal{S} \subset \mathbb{R}^n$ be a convex set such that $\mathcal{S} = \bigcap_{i=1}^m \mathcal{S}_i$, where the m sets $\mathcal{S}_i$ are convex. Let $U_i : \mathbb{R}^n \to \mathbb{R}_+$ be an unfeasibility distance for each set $\mathcal{S}_i$ with additional properties:

i $U_i \in C^2\ \forall\, i = 1, \ldots, m$;
ii $\frac{\partial^2 U_i}{\partial \mathbf{p}\partial \mathbf{p}^\top} \geq 0\ \forall\,\mathbf{p} \notin \mathcal{S}, i = 1, \ldots, m$;
iii $\lambda_{\max}(\nabla U_i(\mathbf{p})^\top \nabla U_i(\mathbf{p})) + \lambda_{\max}(\frac{\partial^2 U_i}{\partial \mathbf{p}\partial \mathbf{p}^\top}(\mathbf{p})) < 1\ \forall\,\mathbf{p} \in \mathbb{R}^n, i = 1, \ldots, m$.

Then $d^+_{\mathcal{S}} : \mathbb{R}^n \to \mathbb{R}_+$ as defined as in (5) is a WGP2SD.

Proof: By the definition of the unfeasible distance, Property 2.1.ii holds.

The gradient of the outer distance is given by

$$\nabla d^+_{\mathcal{S}}(\mathbf{p}) = \frac{1}{m}\sum_{i=1}^{m} \frac{\partial \Phi}{\partial U_i}(U_i)\nabla U_i(\mathbf{p}),$$

and the hessian can then be derived as

$$\begin{aligned}\frac{\partial^2 d^+_{\mathcal{S}}}{\partial \mathbf{p}\partial \mathbf{p}^\top}(\mathbf{p}) = \frac{1}{m}\sum_{i=1}^{m}\Bigg\{ &\frac{\partial^2 \Phi}{\partial U_i^2}\big(U_i(\mathbf{p})\big)\nabla U_i(\mathbf{p})^\top \nabla U_i(\mathbf{p}) \\ &+ \frac{\partial \Phi}{\partial U_i}\big(U_i(\mathbf{p})\big)\frac{\partial^2 U_i}{\partial \mathbf{p}\partial \mathbf{p}^\top}(\mathbf{p})\Bigg\}.\end{aligned} \tag{7}$$

Since Φ is a BGP2SD, if U_i is twice differentiable, then Property 2.1.i holds as well.

Given that $\nabla U_i(\mathbf{p})^\top \nabla U_i(\mathbf{p})$ is always positive semidefinite, and that the hessian of Φ is positive definite, the first term within brackets in (7) is positive semidefinite. Now, note that by definition $U_i(\mathbf{p}) > 0$ for all $\mathbf{p} \notin \mathcal{S}$, and that the derivative of Φ is $\tanh(U_i(\mathbf{p}))$. This implies that $\frac{\partial \Phi}{\partial U_i}\big(U_i(\mathbf{p})\big)$ is always positive, thus if the hessian of $U_i(\mathbf{p})$ is positive semidefinite for all $\mathbf{p} \notin \mathcal{S}$, then the hessian of $d^+_{\mathcal{S}}(\mathbf{p})$ is positive semidefinite for all $\mathbf{p} \notin \mathcal{S}$, a weak version of Property 2.1.iii.

In order to show Property 2.1.iv, note that

$$\begin{aligned}\frac{\partial^2 d^+_{\mathcal{S}}}{\partial \mathbf{p}\partial \mathbf{p}^\top}(\mathbf{p}) - \mathbf{I} &< 0 \\ \implies \mathbf{v}^\top\left(\frac{\partial^2 d^+_{\mathcal{S}}}{\partial \mathbf{p}\partial \mathbf{p}^\top}(\mathbf{p}) - \mathbf{I}\right)\mathbf{v} &< 0,\ \forall\,\mathbf{v} \neq \mathbf{0}, \mathbf{v} \in \mathbb{R}^n \\ \left\|\frac{\partial^2 d^+_{\mathcal{S}}}{\partial \mathbf{p}\partial \mathbf{p}^\top}(\mathbf{p})\mathbf{v}\right\|^2 &< \|\mathbf{v}\|^2,\ \forall\,\mathbf{v} \neq \mathbf{0}, \mathbf{v} \in \mathbb{R}^n.\end{aligned}$$

Taking the square root of both sides and dividing by $\|\mathbf{v}\|$, since it is different than $\mathbf{0}$ implies

$$\begin{aligned}&\frac{\left\|\frac{\partial^2 d^+_{\mathcal{S}}}{\partial \mathbf{p}\partial \mathbf{p}^\top}(\mathbf{p})\mathbf{v}\right\|}{\|\mathbf{v}\|} < 1,\ \forall\,\mathbf{v} \neq \mathbf{0}, \mathbf{v} \in \mathbb{R}^n \\ &\implies \max_{\|\mathbf{v}\| \neq 0} \frac{\left\|\frac{\partial^2 d^+_{\mathcal{S}}}{\partial \mathbf{p}\partial \mathbf{p}^\top}(\mathbf{p})\mathbf{v}\right\|}{\|\mathbf{v}\|} < 1 \\ &\implies \lambda_{\max}\left(\frac{\partial^2 d^+_{\mathcal{S}}}{\partial \mathbf{p}\partial \mathbf{p}^\top}(\mathbf{p})\right) < 1.\end{aligned}$$

[2]These names were inspired by linear programming concepts, for which feasible points are points inside a polyhedron.

Now, since $\frac{\partial^2\Phi}{\partial U_i^2}\big(U_i(\mathbf{p})\big) < 1$ and $\frac{\partial\Phi}{\partial U_i}\big(U_i(\mathbf{p})\big) < 1$ we have[3]

$$\lambda_{\max}\left(\frac{\partial^2 d_{\mathcal{S}}^+}{\partial\mathbf{p}\partial\mathbf{p}^\top}(\mathbf{p})\right) \le \frac{1}{m}\sum_{i=1}^{m}\Bigg\{\lambda_{\max}\Big(\nabla U_i(\mathbf{p})^\top\nabla U_i(\mathbf{p})\Big) + \lambda_{\max}\left(\frac{\partial^2 U_i}{\partial\mathbf{p}\partial\mathbf{p}^\top}(\mathbf{p})\right)\Bigg\}.$$

Thus, if $\lambda_{\max}(\nabla U_i(\mathbf{p})^\top\nabla U_i(\mathbf{p})) + \lambda_{\max}(\frac{\partial^2 U_i}{\partial\mathbf{p}\partial\mathbf{p}^\top}(\mathbf{p})) < 1$, Property 2.1.iv is satisfied. ■

III. Problem Statement

The considered path planning problem will be formulated as an optimization problem. For that, some prior considerations are needed. First, let $\mathbf{q}_0 \in \mathbb{R}^n$ be the initial configuration of a robot, and let $\mathbf{q}_d \in \mathbb{R}^n$ be the final configuration, which the robot needs to achieve. We assume that both $\mathbf{q}_0$ and $\mathbf{q}_d$ do not collide with the environment. Furthermore, we also assume that there always exists a collision-free path between the initial and final configurations. We now define what we consider a path in this work.

Definition 3.1 (Path): A path $\mathcal{P} = (\mathbf{p}^1, \mathbf{p}^2, \ldots, \mathbf{p}^N)$ is an ordered sequence of N points in $\mathbb{R}^n$, where the indices denote traversal order: $\mathbf{p}^i$ is visited before $\mathbf{p}^j$ if and only if $i < j$. □

Let $\mathcal{O} = \{\mathcal{S}^1, \mathcal{S}^2, \ldots, \mathcal{S}^\ell\}$ be the set of all ℓ convex obstacles in the environment. In this work, we consider only polyhedral obstacles, meaning each obstacle is described by the intersection of m_i half-spaces

$$\mathcal{S}^i = \bigcap_{j=1}^{m_i}\mathcal{S}_j^i \,\forall\, i = 1, \ldots, \ell,$$

where each $\mathcal{S}_j^i$ is a half-space in $\mathbb{R}^n$. Equivalently, we write

$$\mathcal{S}^i = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{A}^{(i)}\mathbf{x} \le \mathbf{b}^{(i)}\} \,\forall\, i = 1, \ldots, \ell,$$

where $\mathbf{A}^{(i)} \in \mathbb{R}^{m_i\times n}$ and $\mathbf{b}^{(i)} \in \mathbb{R}^{m_i}$ define the constraint matrices of the i^{th} polyhedron. For notational simplicity, we omit the superscripts on $\mathbf{A}$ and $\mathbf{b}$ when unambiguous.

Let $\mathbf{s}^{(i)} \triangleq \mathbf{s}^{(i)}(\mathbf{p}) = \mathbf{b}^{(i)} - \mathbf{A}^{(i)}\mathbf{p}$ for all $\mathcal{S}^i$, thus for each half-space $\mathcal{S}_j^i$ we define the feasibility and unfeasibility distances as:

$$F_j^i(\mathbf{p}) = \begin{cases}\mathbf{b}_j - \mathbf{a}_j^\top\mathbf{p}, & \mathbf{s}_j^{(i)} > 0.\\ 0, & \mathbf{s}_j^{(i)} \le 0,\end{cases} \tag{8}$$

$$U_j^i(\mathbf{p}) = \begin{cases}\mathbf{a}_j^\top\mathbf{p} - \mathbf{b}_j, & \mathbf{s}_j^{(i)} < 0,\\ 0, & \mathbf{s}_j^{(i)} \ge 0.\end{cases} \tag{9}$$

Note that we ommited the superscripts $\cdot^{(i)}$ on $\mathbf{a}_j$ and $\mathbf{b}_j$. Without loss of generality, we assume that the rows of $\mathbf{A}^{(i)}$ are normalized. Computing both the gradient and hessian of the unfeasibility distance renders

$$\begin{aligned}\nabla U_j^i(\mathbf{p}) &= \mathbf{a}_j^\top,\\ \frac{\partial^2 U_j^i}{\partial\mathbf{p}\partial\mathbf{p}^\top}(\mathbf{p}) &= \mathbf{0},\end{aligned}$$

and since the rows of $\mathbf{A}^{(i)}$ are normalized, we can invoke Lemma 2.9 and show that the outer distance in (5) is a WGP2SD for polyhedral constraints.

With these definitions, we can now formulate a signed distance function for all obstacles as follows

$$D_{\mathcal{O}}(\mathbf{p}) = \sum_{\mathcal{S}^i\in\mathcal{O}} \Xi\Big(\Gamma\big(d_{\mathcal{S}^i}^+(\mathbf{p})\big) + \eta d_{\mathcal{S}^i}^-(\mathbf{p})\Big), \tag{10}$$

where $\eta \in \mathbb{R}_+$ is used to control the penalization of negative distances—it helps avoiding that the sum of the outer distances are larger in magnitude than the inner distance—the outer and inner distances are precisely the ones defined in (5) and (6), respectively, and Γ is the bulging function of Definition 2.2. The function $\Xi : \mathbb{R} \to \mathbb{R}$ acts as a smooth positive saturation function, meaning it saturates only positive values, and is defined as:

$$\Xi(x) = -\frac{1}{\alpha}\log\left(\frac{1 + e^{-\alpha x}}{2}\right), \tag{11}$$

where $\alpha \in \mathbb{R}_+$ controls the value of the saturation. We now prove that the saturated bulged outer distance maintains most propperties of a GP2SD:

Lemma 3.2: Let $\mathcal{S} \subset \mathbb{R}^n$ be a convex set, and let $f_{\mathcal{S}} : \mathbb{R}^n \to \mathbb{R}_+$ be a GP2SD. Given a positive scalar α, the saturated function $\Xi \circ f$ is almost a GP2SD, lacking only Property 2.1.iii.

Proof: For every point $\mathbf{p} \in \mathcal{S}$, the value of $f_{\mathcal{S}}(\mathbf{p})$ is zero, which implies

$$\Xi\big(f_{\mathcal{S}}(\mathbf{p})\big) = -\frac{1}{\alpha}\log(1) = 0.$$

For every point $\mathbf{p} \notin \mathcal{S}$, $f_{\mathcal{S}}(\mathbf{p})$ is a positive value, which implies $\exp(-\alpha f_{\mathcal{S}}(\mathbf{p})) \in (0,1)$ and thus the argument of the logarithm is a positive number less than 1, which results in a negative number. However, since we multiply this result by a negative constant, the final value is a positive number. Hence, Property 2.1.ii is maintained.

The second derivative of this saturation function can be expressed as

$$\frac{d^2\Xi}{dx^2}(x) = -\frac{1}{4}\alpha\,\mathrm{sech}^2\Big(\frac{\alpha x}{2}\Big),$$

which is bounded by 1 and Property 2.1.iv still holds. Since this second derivative has no singularities, Property 2.1.i holds as well. ■

In order to maximize the distance between points in the path and the obstacles, while minimizing the path length, we formulate the following optimization problem

$$\begin{aligned}\min_{\mathbf{p}^i\in\mathcal{P}}\ & \sum_{i=1}^{N} -D_{\mathcal{O}}(\mathbf{p}^i) + \frac{\zeta}{2}\sum_{i=1}^{N-1}\left\|\mathbf{p}^{i+1} - \mathbf{p}^i\right\|^2\\ \text{s.t.:}\ & \\ & \mathbf{p}^1 = \mathbf{q}_0,\\ & \mathbf{p}^N = \mathbf{q}_d,\end{aligned} \tag{12}$$

where the constraints force the initial and final points to be the initial and final configuration, respectively. The parameter $\zeta \in \mathbb{R}_+$ weights the importance of minimizing path length.

[3] Here we used the facts that $\mathrm{tr}(\mathbf{A}+\mathbf{B}) = \mathrm{tr}(\mathbf{A}) + \mathrm{tr}(\mathbf{B})$, $\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n}\lambda_i(\mathbf{A})$, and $\lambda(-\mathbf{A}) = -\lambda(\mathbf{A})$.

Algorithm 1 Gradient-descent strategy for problem (12)

Input: $\mathcal{P}_{\text{init}} = (\mathbf{p}^1, \dots, \mathbf{p}^N)$
Output: $\mathcal{P}_{\text{final}}$
$\mathbf{q}_0 \leftarrow \mathbf{p}^1$
$\mathbf{q}_d \leftarrow \mathbf{p}^N$
$\mathcal{P}_{\text{final}} \leftarrow \mathcal{P}_{\text{init}}$
while $\exists \mathbf{p}^i \in \mathcal{P}_{\text{final}}$ such that $D_{\mathcal{O}}(\mathbf{p}^i) < \delta$ **do**
  $\mathcal{P} \leftarrow \{\mathbf{q}_0\}$
  for $k \leftarrow 2$ to $N-1$ **do**
    $\mathbf{w}^k \leftarrow \zeta(2\mathbf{p}^k - \mathbf{p}^k - \mathbf{p}^{k+1})$
    $\mathbf{v}^k \leftarrow \sqrt{|D_{\mathcal{O}}(\mathbf{p}^k)|}\nabla D_{\mathcal{O}}(\mathbf{p}^k)^\top$
    $\mathbf{p}^k_{\text{final}} \leftarrow \mathbf{p}^k - \mathbf{v}^k + \mathbf{w}^k$
    $\mathcal{P} \leftarrow \mathcal{P} \cup \{\mathbf{p}^k_{\text{final}}\}$
  end for
  $\mathcal{P} \leftarrow \mathcal{P} \cup \{\mathbf{q}_d\}$
  $\mathcal{P}_{\text{final}} \leftarrow \mathcal{P}$
end while

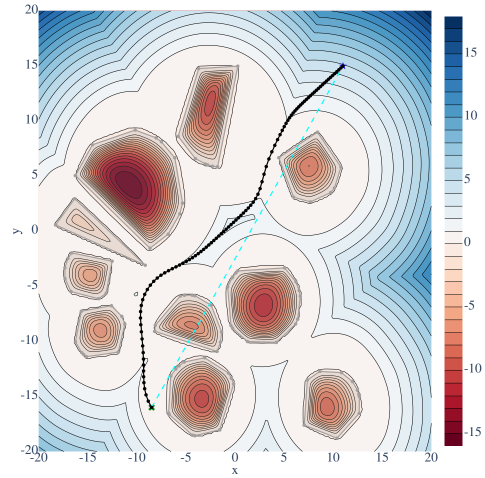
Instead of using a specialized solver, we adopt a gradient descent-based approach with some minor tweaks. We initialize a initial path $\mathcal{P}_{\text{init}}$ as a straight line connecting the initial configuration to the desired configuration, i.e., $\mathcal{P}_{\text{init}} = \left\{\sigma_i \mathbf{q}_0 + (1-\sigma_i)\mathbf{q}_d \,|\, \sigma_i = \frac{i-1}{N-1}, i \in \{1, \dots, N\}\right\}$. Each point $\mathbf{p}^k$, except the initial and final ones in the path—which ensures the constraints of (12)—gets updated based on a sum of the gradient of the path length functional, and $-\sqrt{|D_{\mathcal{O}}(\mathbf{p}^k)|}\nabla D_{\mathcal{O}}(\mathbf{p}^k)^\top$. This process is repeated until there exists no point $\mathbf{p}^k$ in the path such that $D(\mathbf{p}^k) < \delta$, which is a parameter of the strategy. This strategy is illustrated in Algorithm 1.

IV. Results and discussion

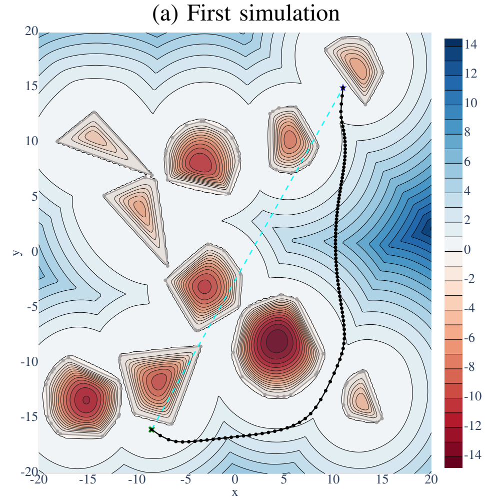
In order to test our strategy we perform three simulations using the strategy in Algorithm 1. We generated three random 2-dimensional maps such that each map has 10 polygons each with v_i vertices, where v_i was uniformly distributed within $\{3, \dots, 15\}$. No polygons intersect with each other and the map guarantees that the initial and final configuration lie outside every polygon. The three maps can be seen in Fig. 1.

The initial configuration was set as $\mathbf{q}_0 = [-8.5 \ -16]^\top$ and the desired configuration as $\mathbf{q}_d = [11 \ 15]^\top$. For the BGP2SD in (2) we chose $\gamma = 1.1$, the smoothing parameter r in the inner distance (6) was taken as $r = 0.8$. Regarding the bulging function in (2.2), we set $\mathbf{c}_i$ as the mean of the vertices of the i^{th} polygon, the radius R was set as the maximum Euclidean distance between the vertices and $\mathbf{c}_i$ times 1.001, and we chose $\varepsilon = 0.05$. The saturation value in (11) was set as $\alpha = \frac{\log(2)}{0.2}$, which implies that the function tends to 0.2 as $x \to \infty$. The weight parameter in (10) was set as $\eta = 4$, and the weight parameter for minimizing the path length in (12) was set as $\zeta = 0.5$. We set $\delta = 2$, pursuing a final path for which every point is at least $0.2\,\mathrm{m}$ away from every obstacle. We also stop the optimization after 500 iterations if this condition has not been achieved.

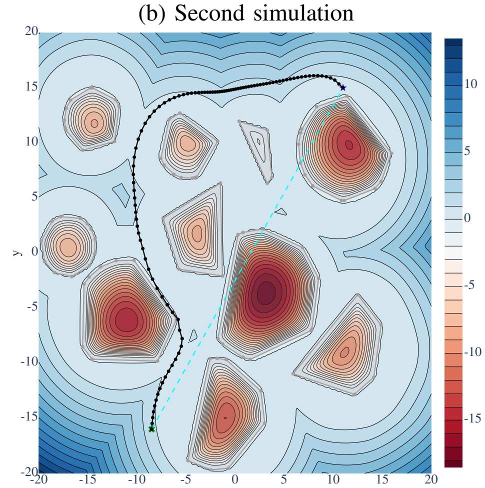
The first simulation, depicted in Fig. 1a, was the easiest scenario. For this case we noted that the initial behavior of the



(a) First simulation



(b) Second simulation



(c) Third simulation

Fig. 1: Resulting planned path in black for three different random maps. Obstacles are represented as gray lines and points. The initial path is shown as a cyan dashed line. Initial and desired configurations are represented by a green cross and blue star, respectively. The contour map represents the minimum distance at each point.

deformation was not similar to the final path obtained. The segment near the initial configuration tended to push points inside the obstacles toward the right side, which would render a risky path that would traverse between obstacles twice. However, after some iterations, while trying to maximize the distance—the outer distance in this case—and minimizing the path length, the points end up getting pushed to the left side, achieving the final path shown.

The second simulation, depicted in Fig. 1b, is harder than the previous. However, in this scenario, the initial behavior of the deformation is towards the same direction of the final path. In this case, all behaviors renders similar deformations, points inside obstacles are pushed to the right side, collision free points are pushed away from obstacles also to the right side, and the path minimization ensures a collision free path by making the path dense in points.

The third simulation, depicted in Fig. 1c, is the hardest scenario. Here, although the final path is collision-free, it is too close to one of the obstacles. This is a consequence of the total distance functional, which penalizes the sum of the distance to every obstacle. If this functional was defined as the maximization of the minimum distance, it would avoid traversing near the obstacle boundary. However, this would require defining a smooth minimum function that handles positive and negative numbers. Furthermore, we note that in this case traversal between obstacles is also avoided by the path minimization functional.

Furthermore, maximum iteration limit was reached in every scenario, which is an issue also related to our total distance functional.

V. Conclusion

In this work we proposed a smooth signed distance function and used its gradient to deform a initial path into a final collision-free path. We used a gradient descent approach that tries to maximize the total distance between points in the path and environment obstacles, as well as minimizing the path length, while ensuring that the initial and final point in the path remain equal to the initial and final configuration, respectively.

We performed three simulations in three randomly generated cluttered environments. The results in these show that our strategy performs well, having generated collision-free paths for each situation. We noted however, that the definition of the total distance functional is not optimal. Although it induces safe paths by maximizing the distance between every obstacle, there are situations when maximizing the total sum is not representative enough for the objective. Ideally, this functional should be changed to maximizing the minimum distance between a point and each obstacle, in a smooth manner.

Future work will explore defining a smooth version of the minimum function between scalars in general—allowing comparison between positive and negative numbers—which will improve the deformation behavior to generate safe collision-free paths. As this work only deals with convex obstacles, we also pursue a way to compute the deformation for nonconvex obstacles. Although the distance functions were proposed for general n-dimensional spaces, it is also of interest to investigate how to utilize our approach in more general spaces, such as a robotic manipulator configuration space.

References

[1] S. Quinlan and O. Khatib, "Elastic bands. Connecting path planning and control," *Proceedings - IEEE International Conference on Robotics and Automation*, vol. 2, pp. 802–807, 1993.

[2] N. Ratliff, M. Zucker, J. Andrew Bagnell, and S. Srinivasa, "CHOMP: Gradient optimization techniques for efficient motion planning," *Proceedings - IEEE International Conference on Robotics and Automation*, pp. 489–494, 2009.

[3] M. Kalakrishnan, S. Chitta, E. Theodorou, P. Pastor, and S. Schaal, "STOMP: Stochastic trajectory optimization for motion planning," *Proceedings - IEEE International Conference on Robotics and Automation*, pp. 4569–4574, 2011.

[4] J. Schulman, Y. Duan, J. Ho, A. Lee, I. Awwal, H. Bradlow, J. Pan, S. Patil, K. Goldberg, and P. Abbeel, "Motion planning with sequential convex optimization and convex collision checking," *The International Journal of Robotics Research*, vol. 33, pp. 1251–1270, aug 2014.

[5] J. Dong, M. Mukadam, F. Dellaert, and B. Boots, "Motion planning as probabilistic inference using gaussian processes and factor graphs," in *Proceedings of Robotics: Science and Systems*, (AnnArbor, Michigan), June 2016.

[6] S. Sýkora, *Mathematical Means and Averages: Basic Properties*, vol. III. Castano Primo, Italy: Stan's Library, 2003.