Lista 1 - Sistemas Nebulosos

Felipe Bartelt de Assis Pessoa - 2016026841

Q1

Involução

$$\overline{\overline{A}} = A,\ \mu_A(x) = \begin{cases} 1, & sse\ x \in A \\ 0, & c.c \end{cases}$$

$$\mu_{\overline{A}} = 1 - \mu_A$$

$$\mu_{\overline{\overline{A}}} = 1 - \mu_{\overline{A}} = 1 - (1 - \mu_A) = \mu_A$$

$$\therefore \overline{\overline{A}} = A$$

$\blacksquare$

Absorção

$$A \cup (A \cap B) = A,\ \mu_A(x) = \begin{cases} 1, & sse\ x \in A \\ 0, & c.c \end{cases}$$

$$\mu_{A \cap B} = \min(\mu_A, \mu_B)$$

$$\mu_{A \cup (A \cap B)} = \max\left(\mu_A, \min(\mu_A, \mu_B)\right)$$

$$\text{Suponha } \min(\mu_A, \mu_B) = \mu_A \implies \mu_{A \cup (A \cap B)} = \max(\mu_A,\ \mu_A) = \mu_A$$

$$\text{Suponha } \min(\mu_A, \mu_B) = \mu_B \implies \mu_{A \cup (A \cap B)} = \max(\mu_A,\ \mu_B) = \mu_A,\ \text{uma vez}$$

$$\text{que } \min(\mu_A,\ \mu_B) = \max(\mu_A, \mu_B) \iff \mu_A = \mu_B$$

$\blacksquare$

Contradição

$$A \cap \overline{A} = \emptyset,\ \mu_A(x) = \begin{cases} 1, & sse\ x \in A \\ 0, & c.c \end{cases}$$

$$\mu_{A \cap \overline{A}} = \min(\mu_A,\ \mu_{\overline{A}}) = \min(\mu_A,\ 1 - \mu_A)$$

$$\text{Suponha que } \mu_A = 1 \text{ ou } \mu_A = 0$$

$$\text{Se } \mu_A = 1 \implies \min(1, 1-1) = \min(1, 0) = 0$$

$$\text{Se } \mu_A = 0 \implies \min(0, 1-0) = \min(0, 1) = 0$$

$$\implies \mu_{A \cap \overline{A}}(x) = 0\ \forall\ x \in X$$

$$\therefore x \notin A \cap \overline{A}\ \forall\ x \in X \implies A \cap \overline{A} = \emptyset$$

$\blacksquare$

De Morgan

$$\overline{A \cup B} = \overline{A} \cap \overline{B},\ \mu_A(x) = \begin{cases} 1, & sse\ x \in A \\ 0, & c.c \end{cases}$$

$$\mu_{\overline{A \cup B}} = 1 - \mu_{A \cup B} = 1 - \max(\mu_A,\ \mu_B)$$

$$\mu_{\overline{A} \cap \overline{B}} = \min(1 - \mu_A,\ 1 - \mu_B)$$

$$\text{Suponha que } \mu_A = 1 \text{ ou } \mu_B = 1 \implies \max(\mu_A, \mu_B) = 1 \implies \mu_{\overline{A \cup B}} = 0,$$

$$\text{ainda } \min(1 - \mu_A, 1 - \mu_B) = 0 \therefore \mu_{\overline{A \cup B}} = \mu_{\overline{A} \cap \overline{B}}$$

$$\text{Suponha que } \mu_A = \mu_B = 0 \implies \max(\mu_A, \mu_B) = 0 \implies \mu_{\overline{A \cup B}} = 1,$$

$$\text{ainda } \min(1 - \mu_A, 1 - \mu_B) = 1 \therefore \mu_{\overline{A \cup B}} = \mu_{\overline{A} \cap \overline{B}}$$

■

Q2

Involução

$$\overline{\overline{A}} = A,\ \mu_A(x) : X \to [0, 1], x \in X$$

$$\mu_{\overline{A}} = 1 - \mu_A$$

$$\mu_{\overline{\overline{A}}} = 1 - \mu_{\overline{A}} = 1 - (1 - \mu_A) = \mu_A$$

$$\therefore \overline{\overline{A}} = A$$

■

Absorção

$$A \cup (A \cap B) = A,\ \mu_A(x) : X \to [0, 1], x \in X$$

$$\mu_{A \cap B} = \min(\mu_A, \mu_B)$$

$$\mu_{A \cup (A \cap B)} = \max\left(\mu_A, \min(\mu_A, \mu_B)\right)$$

$$\text{Suponha } \min(\mu_A, \mu_B) = \mu_A \implies \mu_{A \cup (A \cap B)} = \max(\mu_A,\ \mu_A) = \mu_A$$

$$\text{Suponha } \min(\mu_A, \mu_B) = \mu_B \implies \mu_{A \cup (A \cap B)} = \max(\mu_A,\ \mu_B) = \mu_A, \text{ uma vez}$$

$$\text{que } \min(\mu_A,\ \mu_B) = \max(\mu_A, \mu_B) \iff \mu_A = \mu_B$$

■

Contradição

$$A \cap \overline{A} = \emptyset,\ \mu_A(x) : X \to [0,1], x \in X$$
$$\mu_{A\cap\overline{A}} = \min(\mu_A,\ \mu_{\overline{A}}) = \min(\mu_A,\ 1-\mu_A)$$
$$\text{Suponha que } \mu_A \geq 0.5 \implies \min(\mu_A,\ 1-\mu_A) = 1-\mu_A$$
$$\text{Suponha que } \mu_A < 0.5 \implies \min(\mu_A,\ 1-\mu_A) = \mu_A$$
$$\implies \exists\, x \in X \text{ t.q } A \cap \overline{A}(x) \neq \emptyset$$
$$\text{Ainda, se } \mu_A \notin \{0,1\} \implies A \cap \overline{A} \neq \emptyset$$

$\blacksquare$

De Morgan

$$\overline{A \cup B} = \overline{A} \cap \overline{B},\ \mu_A(x) : X \to [0,1], x \in X$$
$$\mu_{\overline{A\cup B}} = 1 - \mu_{A\cup B} = 1 - \max(\mu_A,\ \mu_B)$$
$$\mu_{\overline{A}\cap\overline{B}} = \min(1-\mu_A,\ 1-\mu_B)$$
$$\text{Suponha } \mu_A \leq \mu_B \implies \mu_{\overline{A\cup B}} = 1 - \max(\mu_A,\ \mu_B) = 1 - (\mu_B),$$
$$\text{por outro lado } 1-\mu_B \leq 1-\mu_A \implies \mu_{\overline{A}\cap\overline{B}} = \min(1-\mu_A,\ 1-\mu_B) = 1-\mu_B.$$
$$\text{Suponha } \mu_A > \mu_B \implies \mu_{\overline{A\cup B}} = 1 - \max(\mu_A,\ \mu_B) = 1 - (\mu_A),$$
$$\text{por outro lado } 1-\mu_B > 1-\mu_A \implies \mu_{\overline{A}\cap\overline{B}} = \min(1-\mu_A,\ 1-\mu_B) = 1-\mu_A.$$
$$\mu_{\overline{A\cup B}} = \mu_{\overline{A}\cap\overline{B}} \therefore \overline{A \cup B} = \overline{A} \cap \overline{B}$$

$\blacksquare$

Q3

$$N(a) = \frac{1-a}{1+sa}, s \in (-1, \infty)$$

- Axioma N1: $N(0) = 1,\ N(1) = 0$

$$N(0) = \frac{1-0}{1+0s} = 1$$
$$N(1) = \frac{1-1}{1+s} = 0$$

- Axioma N2: $N(a) \geq N(b),\ se\ a \leq b$

$$\text{Suponha } a \leq b.$$
$$N(a) = \frac{1-a}{1+sa},\ N(b) = \frac{1-b}{1+sb},\ s \in (-1, \infty)$$
$$\text{Suponha também } N(a) < N(b)$$
$$\frac{1-a}{1+sa} < \frac{1-b}{1+sb}$$
$$(1-a)(1+sb) < (1-b)(1+sa)$$
$$1 - asb + sb - a < 1 - asb + sa - b$$
$$b(s+1) < a(s+1),\ s > -1 \text{ por definição}$$
$$\therefore b < a.$$
$$\text{O que contradiz a premissa inicial e, portanto, se } a \leq b \implies N(a) \geq N(b)$$
$$\blacksquare$$

- Axioma N4: $N\left(N(a)\right) = a$

$$N(a) = \frac{1-a}{1+sa} \implies N(N(a)) = \frac{1 - \frac{1-a}{1+sa}}{1 + s\frac{1-a}{1+sa}}$$
$$N(N(a)) = \frac{1 + sa - 1 + a}{1 + sa + s - sa} = \frac{a(s+1)}{1+s} = a, \text{ já que } s > -1$$

Q4

$$S(a,b) = a + b - ab$$

- Axioma S1: $S(0,0) = 0, S(a,0) = S(0,a) = a$

$$S(0,0) = 0 + 0 - (0 \cdot 0) = 0$$
$$S(a,0) = a + 0 - (a \cdot 0) = a$$
$$S(0,a) = 0 + a - (0 \cdot a) = a$$

- Axioma S2: $S(a,b) \leq S(c,d),\ se\ a \leq c, b \leq d$

$$\text{Suponha } a \leq c, b \leq d. \text{ Ainda, suponha } S(a,b) > S(c,d)$$
$$S(a,b) > S(c,d)$$
$$a + b - ab > c + d - cd \qquad (1)$$

$$\text{Como } a,b,c,d \in [0,1]:$$
$$a \leq c \implies ab \leq cb$$
$$b \leq d \implies cb \leq cd$$
$$\therefore ab \leq cb \leq cd$$
$$\text{Ainda,}$$
$$a \leq c \implies a - c \leq 0$$
$$b \leq d \implies b - d \leq 0$$
$$\implies a + b - c - d \leq 0$$
$$\therefore a + b \leq c + d$$
$$\text{Dado } S:[0,1]\times[0,1] \mapsto [0,1], \text{tem-se:}$$
$$0 \leq a \leq 1 \implies 0 \leq 1 - a$$
$$0 \leq b \leq 1 \implies 0 \leq 1 - b$$
$$0 \leq (1-a)(1-b)$$
$$0 \leq 1 - a - b + ab$$
$$\therefore a + b \leq ab + 1. \text{ Resultado análogo para } c,d. \text{ Assim}$$
$$a + b - ab - 1 \leq 0,\ c + d - cd - 1 \leq 0$$
$$a + b - ab - 1 - c - d + cd + 1 \leq 0$$
$$a + b - ab \leq c + d - cd$$
$$\text{Uma contradição com (1), demonstrando a veracidade do axioma para o } S(a,b) \text{ em questão}$$
$$\blacksquare$$

- Axioma S3: $S(a,b) = S(b,a)$

$$S(a,b) = a + b - ab$$
$$S(b,a) = b + a - ba$$
$$a,b \in \mathbb{R} \implies ba = ab$$
$$\therefore S(a,b) = S(b,a)$$

- Axioma S4: $S(a,S(b,c)) = S(S(a,b),c)$

$$S(a,S(b,c)) = a + (b + c - cb) - a\,(b + c - cb) = a(1 - b - c + cb) + b + c - cb$$
$$S(S(a,b),c) = (a + b - ab) + c - (a + b - ab)c = a(1 - b - c + cb) + b + c - cb$$
$$\therefore S(a,S(b,c)) = S(S(a,b),c)$$

Q5

$$S(a,b) = \min(1, a+b)$$

- Axioma S1: $S(0,0) = 0, S(a,0) = S(0,a) = a$

$$S(0,0) = \min(1, 0+0) = 0$$
$$S(a,0) = \min(1, a+0) = a$$
$$S(0,a) = \min(1, 0+a) = a$$

- Axioma S2: $S(a,b) \leq S(c,d), \ se \ a \leq c, b \leq d$

$$\text{Suponha } a \leq c, b \leq d. \text{ Ainda, suponha } S(a,b) > S(c,d)$$
$$S(a,b) > S(c,d)$$
$$\implies \min(1, a+b) > \min(1, c+d) \quad (1)$$

$$\text{Tem-se}$$
$$a \leq c \implies a - c \leq 0$$
$$b \leq d \implies b - d \leq 0$$
$$\implies a + b - c - d \leq 0$$
$$\therefore a + b \leq c + d \quad (2)$$

$$\bullet \text{ Se } \ c + d < 1 \implies \min(1, c+d) = c + d,$$
$$\text{como } a + b \leq c + d < 1, \text{ segue que } a + b < 1 \implies \min(1, a+b) = a + b.$$
$$\text{Por } (2), \text{ nota-se que, para } c + d < 1, \ \min(1, a+b) \leq \min(1, c+d), \text{ o que contradiz } (1).$$
$$\bullet \text{ Se } c + d \geq 1 \implies \min(1, c+d) = 1. \text{ E há duas possibilidades para } \min(1, a+b):$$
$$\text{Caso } a + b < 1 \implies \min(1, a+b) = a + b. \text{ Nesse caso, } \min(1, a+b) < \min(1, c+d)$$
$$\text{Caso } 1 \leq a + b < c + d \implies \min(1, a+b) = 1 \therefore \min(1, a+b) = \min(1, c+d).$$
$$\text{Em todos os casos, vê-se uma contradição com } (1), \text{ portanto segue o axioma}$$

■

Axioma S3: $S(a,b) = S(b,a)$

$$S(a,b) = \min(1, a+b)$$
$$S(b,a) = \min(1, b+a)$$
$$\therefore S(a,b) = S(b,a)$$

- Axioma S4: $S(a, S(b,c)) = S(S(a,b), c)$

$$S(a, S(b,c)) = \min(1, a + \min(1, b + c)) = \min(1, a + x)$$
$$S(S(a,b), c) = \min(1, \min(1, a + b) + c) = \min(1, y + c)$$
$$\text{Se } \ a + x \geq 1 \ \wedge \ y + c \geq 1, \ \text{o mínimo de ambos são iguais a 1.}$$
$$\text{Suponha } a + x < 1 \ ou \ y + c < 1$$
$$\text{Se } a + x < 1 \implies \min(1, a + x) = a + x = a + \min(1, b + c) = a + b + c \ \vee \ a + 1,$$
$$\text{como } a + x < 1, \ a + 1 \text{ não é solução possível, já que implica } a + 1 < 1 \therefore a < 0 \notin [0,1],$$
$$\text{o que não condiz com a forma de conjuntos nebulosos, portanto } a + x > 1 \implies x = b + c,$$
$$\therefore S(a, S(b,c)) = a + b + c < 1 \implies a + b < 1,$$
$$\text{segue então } S(S(a,b), c) = \min(1, \min(1, a + b) + c) = \min(1, a + b + c) = a + b + c$$
$$\therefore S(a, S(b,c)) = S(S(a,b), c)$$
$$\text{Analogamente, se } y + c < 1 \implies \min(1, y + c) = y + c = \min(1, a + b) + c = a + b + c \ \vee \ 1 + c$$
$$\therefore S(S(a,b), c) = a + b + c < 1 \implies b + c < 1$$
$$\text{segue } S(a, S(b,c)) = \min(1, a + \min(1, b + c)) = \min(1, a + b + c) = a + b + c$$
$$\therefore S(S(a,b), c) = S(a, S(b,c))$$

$\blacksquare$

Q6

$$T(a,b) = ab; \ a, b \in [0,1]$$

- Aximoa T1: $T(0,0) = 0, \ T(a,1) = T(1,a) = a$

$$T(0,0) = 0 \cdot 0 = 0$$
$$T(a,1) = a \cdot 1 = a$$
$$T(1,a) = 1 \cdot a = a$$
$$\therefore T(1,a) = T(a,1)$$

- Axioma T2: $T(a,b) \leq T(c,d)$ se $a \leq c, \ b \leq d$

$$\text{Suponha } a \leq c \ \wedge \ b \leq d$$
$$\implies a \cdot b \leq c \cdot b \ \wedge \ c \cdot b \leq c \cdot d$$
$$\implies ab \leq cb \leq cd$$
$$\therefore T(a,b) = ab \leq cd = T(c,d)$$

$\blacksquare$

- Axioma T3: $T(a,b) = T(b,a)$

$$T(a,b) = ab$$
$$T(b,a) = ba$$
$$\text{Como há comutatividade no corpo } \mathbb{R}, \ \text{segue } T(a,b) = T(b,a)$$

- Axioma T4: $T(a, T(b,c)) = T(T(a,b), c)$

$$T(a, T(b,c)) = T(a, bc) = abc$$
$$T(T(a,b), c) = T(ab, c) = abc$$
$$\therefore T(a, T(b,c)) = T(T(a,b), c)$$

Q7

$$T(a,b) = \max(0, a+b-1);\ a, b \in [0,1]$$

- Aximoa T1: $T(0,0) = 0,\ T(a,1) = T(1,a) = a$

$$T(0,0) = \max(0, 0+0-1) = 0$$
$$T(a,1) = \max(0, a+1-1) = a$$
$$T(1,a) = \max(0, 1+a-1) = a$$
$$\therefore T(1,a) = T(a,1)$$

- Axioma T2: $T(a,b) \leq T(c,d)$ se $a \leq c,\ b \leq d$

$$\text{Suponha } a \leq c\ \wedge\ b \leq d$$
$$\implies a+b \leq c+d$$
$$\text{Suponha }\ c+d \leq 1 \implies a+b \leq 1 \therefore \max(0, c+d-1) = 0\ \wedge\ \max(0, a+b-1) = 0$$
$$\therefore T(a,b) = T(c,d)$$
$$\text{Suponha } c+d > 1 \implies \max(0, c+d-1) = c+d-1,$$
$$\text{se } a+b \leq 1 \implies \max(0, a+b-1) = 0 < 1 < c+d-1 \therefore T(a,b) < T(c,d),$$
$$\text{se } 1 < a+b \leq c+d \implies \max(0, a+b-1) = a+b-1,$$
$$\text{como } a+b \leq c+d \implies a+b-1 \leq c+d-1$$
$$\therefore T(a,b) = \max(0, a+b-1) \leq \max(0, c+d-1) = T(c,d)$$

$\blacksquare$

- Axioma T3: $T(a,b) = T(b,a)$

$$T(a,b) = \max(0, a+b-1)$$
$$T(b,a) = \max(0, b+a-1)$$
$$\therefore T(a,b) = T(b,a)$$

- Axioma T4: $T(a, T(b,c)) = T(T(a,b), c)$

$$T(a,T(b,c)) = \max(0, a + \max(0, b+c-1) - 1) = \max(0, a + x - 1)$$
$$T(T(a,b),c) = \max(0, \max(0, a+b-1) + c - 1) = \max(0, y + c - 1)$$
$$\text{Como os conjuntos são nebulosos, } a,b,c \in [0,1],\ \text{assim } b + c \leq 2\ \wedge\ a + b \leq 2$$
$$\implies b + c - 1 \leq 1\ \wedge\ a + b - 1 \leq 1$$
$$\therefore a + x - 1 \leq a \leq 1\ \wedge\ y + c - 1 \leq c \leq 1$$

$$\text{Suponha } a + x - 1 \leq 0\ \wedge\ y + c - 1 \leq 0.$$
$$\text{Então é óbvio que } T(a,T(b,c)) = 0 = T(T(a,b),c)$$

$$\text{Suponha } a + x - 1 > 0\ \vee\ y + c - 1 > 0.$$
$$\text{Se } a + x - 1 > 0 \implies \max(0, a+x-1) = a + x - 1 = a + 0 - 1\ \vee\ a + b + c - 2,$$
$$\text{porém } a + x - 1 = a - 1 \implies a > 1 \notin [0,1],\ \text{portanto a única solução possível é}$$
$$a + x - 1 = a + b + c - 2 > 0 \implies a + b + c > 2$$
$$\therefore T(a,T(b,c)) = a + b + c - 2,$$
$$\text{uma vez que } a+b+c > 2 \text{ implica que todos os conjuntos são não nulos e dois deles}$$
$$\text{são representados por 1, ou seja, é impossível que } a + b - 1 \leq 0,\ \text{já que } a + b > 1,$$
$$\text{então } T(T(a,b),c) = \max(0, a+b+c-2),\ \text{porém } a + b + c > 2$$
$$\implies T(T(a,b),c) = a + b + c - 2 = T(a,T(b,c))$$

$$\text{O resultado é análogo para } y + c - 1 > 0$$
$$\therefore T(a,T(b,c)) = T(T(a,b),c)$$

■

Q8

$$T(a,b) = N\left(S\left(N\left(a\right), N\left(b\right)\right)\right),$$
$$S(a,b) = a + b - ab,$$
$$N(a) = 1 - a$$
$$T(a,b) = ab$$

Tomando como dado $S(a,b)$:

$$S(a,b) = a + b - ab$$
$$N(a) = 1 - a,\ N(b) = 1 - b$$
$$T(a,b) = N(S(1-a, 1-b)) = N(1 - a + 1 - b - ((1-a)(1-b)))$$
$$= N(2 - a - b - (1 + ab - a - b)) = N(1 - ab)$$
$$\therefore T(a,b) = N(1-ab) = 1 - (1 - ab) = ab$$

■

Tomando como dado $T(a,b)$:

$$T(a,b) = ab$$
$$N(a) = 1 - a,\ N(b) = 1 - b$$
$$T(a,b) = ab = N(S(1-a, 1-b)) = 1 - S(1-a, 1-b)$$
$$\implies S(1-a, 1-b) = 1 - ab$$
$$\text{Tomando } \ a = 1 - a \ \wedge \ b = 1 - b :$$
$$S(a,b) = 1 - (1-a)(1-b) = 1 - (1 - a - b + ab) = a + b - ab$$
$$\blacksquare$$