Constructive Vector Fields for Path Following in Matrix Lie Groups

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UFMG

Motivation

- Generalize the vector field strategy in Rezende et al. (2022) to allow more motion possibilities, including rotations;
- Gain deeper insight into vector field properties through generalization;
- Facilitate path following for systems with both translational and rotational motion, such as omnidirectional UAVs and robotic manipulators.

Contributions

- Development of a novel vector field guidance strategy applicable to systems with an inherent matrix Lie group structure;
- Implementation framework for $\mathsf{SE}(3)$ systems, providing all necessary tools for practical application of the proposed strategy;
- Validation through kinematic simulations in $\mathsf{SE}(3)$ and $\mathsf{SO}^+(3,1)$, demonstrating the theoretical results and their practical implications;
- Design of an adaptive control strategy for collaborative simulations in $\mathbb{R}^3 \times \mathsf{SO}(3)$, where the vector field guidance strategy generates reference velocities for dynamic control.

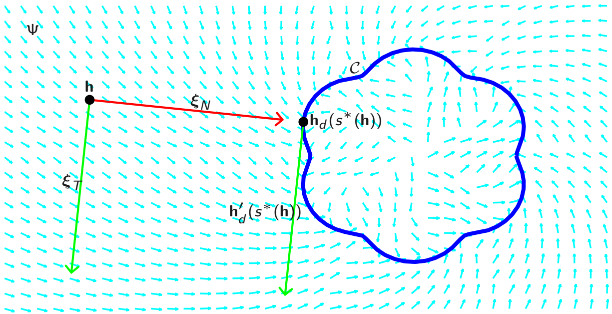
Background

Coursework & assignments → Adapt Culbertson et al. (2021) to vector field of Rezende et al. (2022) → Include orientation $\mathbb{R}^3 \times SO(3)$ (Pessoa and Pimenta, 2024) → Insights into deeper properties → Generalization to Matrix Lie Groups (*Automatica* submission)

Vector field in Euclidean space

The vector field strategy in Rezende et al. (2022) is based on a parametric curve representation and is characterized by:

- A distance function D;
- $\dot{D} = \nabla D \boldsymbol{\xi} = -\boldsymbol{\xi}_N^\top \boldsymbol{\xi}$;
- Tangent component depends only on the curve;
- Normal and tangent components are orthogonal;
- Absence of local minima outside the curve;
- Vector field $\Psi(\mathbf{h}) = k_N(D)\boldsymbol{\xi}_N(\mathbf{h}) + k_T(D)\boldsymbol{\xi}_T(\mathbf{h})$.



Lie Groups and Lie algebras

Lie group G

Manifolds with group structure. The group operation and inverse map are continuous and smooth.

E.g.: set of rotation matrices SO(3).

Lie algebra $\mathfrak{g}$

Tangent space of G at the identity.

E.g.: skew-symmetric matrices for SO(3).

Exponential map

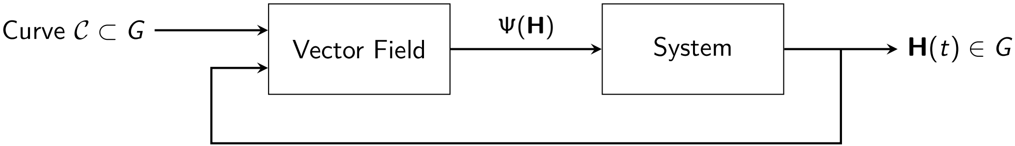
Maps elements of $\mathfrak{g}$ to G. For matrix Lie groups: $\exp(\mathbf{A}) = \sum_{i=0}^{\infty} \frac{\mathbf{A}^n}{n!} = \mathbf{X} \in G$.
Not always surjective.

$\mathcal{S}$ map

Linear map from $\mathbb{R}^m$ to $\mathfrak{g}$, relating velocities to tangent space elements.

E.g.: angular velocities $\rightarrow$ skew-symmetric matrices in $\mathfrak{so}(3)$.

Formulation



We assume the system model:

$$\dot{\mathbf{H}}(t) = \mathcal{S}\big(\xi(t)\big)\mathbf{H}(t), \quad \mathbf{H} \in G, \ \xi \in \mathbb{R}^m,$$

The vector field is given by:

$$\Psi\left(\mathbf{H}\right) \triangleq k_N(\mathbf{H})\xi_N(\mathbf{H}) + k_T(\mathbf{H})\xi_T(\mathbf{H}).$$

Gradient in Lie groups

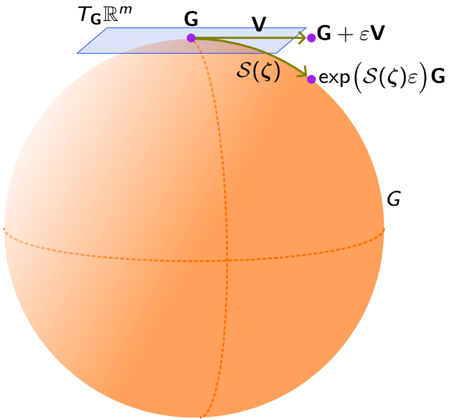
The L operator, $\mathrm{L}[f] : G \to \mathbb{R}^{1\times m}$, acts as a gradient while respecting Lie group constraints.

For any scalar function $f : G \to \mathbb{R}$, it is implicitly defined as:

L operator

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(f\left(\exp\left(\mathcal{S}(\zeta)\,\varepsilon\right)\mathbf{G}\right) - f\left(\mathbf{G}\right)\right) = \mathrm{L}[f](\mathbf{G})\zeta \;\forall\, \zeta \in \mathbb{R}^m$$

$$\frac{d}{d\varepsilon}\left(f\left(\exp\left(\mathcal{S}(\zeta)\,\varepsilon\right)\mathbf{G}\right)\right)\Big|_{\varepsilon=0} = \mathrm{L}[f](\mathbf{G})\zeta \;\forall\, \zeta \in \mathbb{R}^m$$



Distances

The vector field formulation requires two distance functions:

EE-distance $\widehat{D}$

Measures the distance between two Lie group elements.

Positive definite: $\widehat{D}(\mathbf{V},\mathbf{W}) \geq 0,\ \widehat{D}(\mathbf{V},\mathbf{W}) = 0 \iff \mathbf{V} = \mathbf{W}$;

Differentiability: at least once differentiable in both arguments almost everywhere.

E.g.: for exponential Lie groups, an EE-distance is given by

$$\widehat{D}(\mathbf{V},\mathbf{W}) = \left\|\log(\mathbf{V}^{-1}\mathbf{W})\right\|_F.$$

EC-distance D

Measures the distance between a Lie group element and a curve:

$$D(\mathbf{H}) \triangleq \min_{\mathbf{Y}\in\mathcal{C}} \widehat{D}(\mathbf{H},\mathbf{Y}) = \min_{s\in[0,1]} \widehat{D}\big(\mathbf{H},\mathbf{H}_d(s)\big).$$

Components and Properties

Normal and tangent components:

$$\xi_N(\mathbf{H}) \triangleq -\mathrm{L}_{\mathbf{V}}[\widehat{D}]\Big(\mathbf{H}, \mathbf{H}_d(s^*)\Big)^\top$$

$$\xi_T(\mathbf{H}) \triangleq \mathcal{S}^{-1}\left(\frac{d\mathbf{H}_d}{ds}(s^*)\mathbf{H}_d(s^*)^{-1}\right)$$

Components are orthogonal if

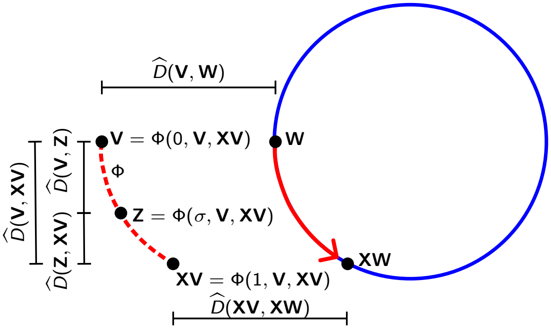
Left-invariant distance

$$\widehat{D}(\mathbf{XV}, \mathbf{XW}) = \widehat{D}(\mathbf{V}, \mathbf{W}) \,\forall\, \mathbf{V}, \mathbf{W}, \mathbf{X} \in G.$$

Absence of local minima outside the curve if

Chainable distance

$$\widehat{D}(\mathbf{V}, \mathbf{W}) = \widehat{D}(\mathbf{V}, \Phi_\sigma) + \widehat{D}(\Phi_\sigma, \mathbf{W}).$$



Convergence Proof Sketch

Let $\widehat{D}$ be a left-invariant and chainable EE-distance, then:

$$\dot{D} = \mathsf{L}_{\mathsf{V}}[\widehat{D}]\xi + \left(\mathsf{L}_{\mathsf{W}}[\widehat{D}]\xi_T\right)\frac{ds^*}{dt}.$$

By the optimality condition of D, we have

$$\dot{D} = \mathsf{L}_{\mathsf{V}}[\widehat{D}]\xi = -\xi_N^\top \xi.$$

If the system follows the vector field, $\xi = \Psi$, then

$$\dot{D} = -\xi_N^\top(k_N\xi_N + k_T\xi_T) = -k_N\|\xi_N\|^2 \leq 0.$$

Circulation relies on proper parametrization.

Kinematic control in SE(3)

The distance function has a simpler expression. Let

$$\mathbf{V}^{-1}\mathbf{W} = \begin{bmatrix} \mathbf{Q} & \mathbf{u} \\ \mathbf{0} & 1 \end{bmatrix},$$

then

$$\widehat{D}(\mathbf{V}, \mathbf{W}) = \|\log(\mathbf{V}^{-1}\mathbf{W})\|_F = \sqrt{2\theta^2 + \mathbf{u}^\top \bar{\mathbf{X}} \mathbf{u}},$$

where

$$\theta = \operatorname{atan2}\left(\frac{1}{2\sqrt{2}}\|\mathbf{Q} - \mathbf{Q}^\top\|_F,\ \frac{1}{2}\left(\operatorname{tr}(\mathbf{Q}) - 1\right)\right),$$

$$\bar{\mathbf{X}} = (1 - 2\beta_0)\mathbf{I} + \beta_0\left(\mathbf{Q} + \mathbf{Q}^\top\right),$$

$$\beta_0 = \frac{2 - 2\cos\theta - \theta^2}{4(1 - \cos\theta)^2}.$$

Kinematic control in SE(3)

Vector field gains:

$$k_N(\mathbf{H}) = \tanh\big(20D(\mathbf{H})\big),$$
$$k_T(\mathbf{H}) = 1 - 0.5\tanh\big(D(\mathbf{H})\big).$$

Lie algebra basis reflecting the mechanical twist:

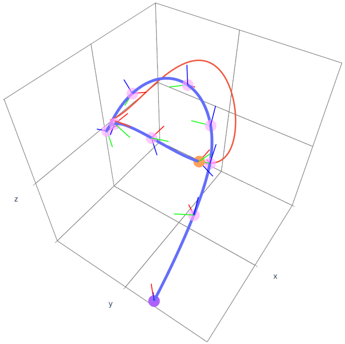
$$\mathcal{S}(\boldsymbol{\xi}) = \begin{bmatrix} 0 & -\xi_6 & \xi_5 & \xi_1 \\ \xi_6 & 0 & -\xi_4 & \xi_2 \\ -\xi_5 & \xi_4 & 0 & \xi_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

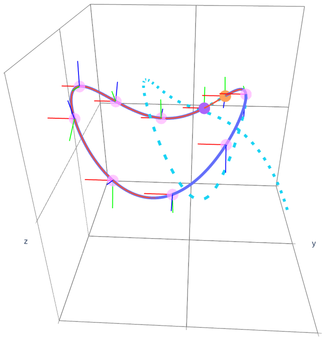
The normal and tangent components were numerically approximated.
The system was integrated using:

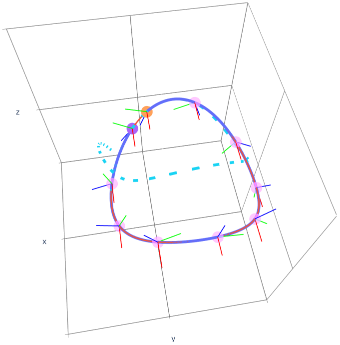
$$\mathbf{H}(t + \Delta t) \approx \exp\Big(\mathcal{S}\big(\Psi(\mathbf{H})\big)\Delta t\Big)\mathbf{H}(t),$$

with a time step of $\Delta t = 1 \cdot 10^{-2}$ s.

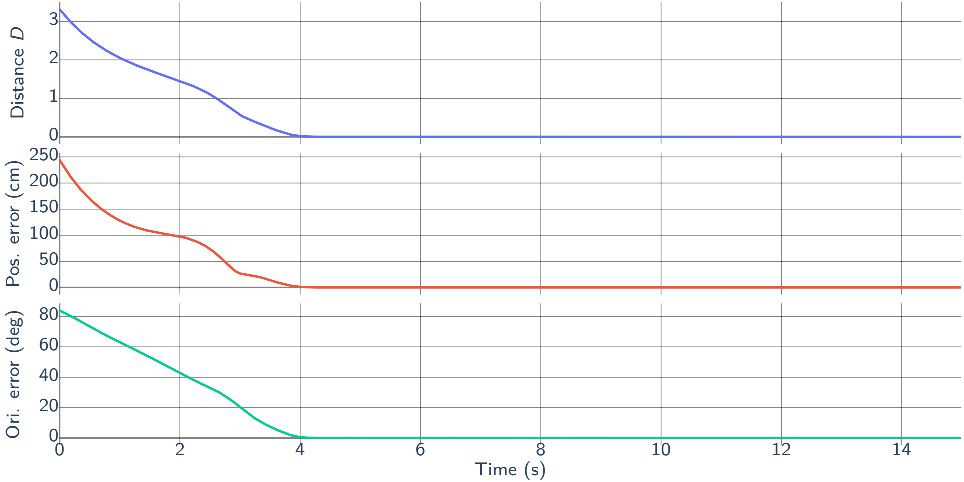
Trajectory







Distances



Distance D
3
2
1
0
Pos. error (cm)
250
200
150
100
50
0
Ori. error (deg)
80
60
40
20
0
0
2
4
6
8
10
12
14
Time (s)

Kinematic control in $\mathrm{SO}^+(3,1)$

EE-distance:

$$\widehat{D}(\mathbf{V},\mathbf{W}) = \left\|\log(\mathbf{V}^{-1}\mathbf{W})\right\|_F$$

Vector field gains:

$$k_N(\mathbf{H}) = \tanh\big(1000D(\mathbf{H})\big)$$
$$k_T(\mathbf{H}) = 0.5\Big(1 - \tanh\big(100D(\mathbf{H})\big)\Big).$$

Lie algebra isomorphism:

$$\mathcal{S}(\boldsymbol{\xi}) = \begin{bmatrix} 0 & -\xi_3 & \xi_2 & \xi_4 \\ \xi_3 & 0 & -\xi_1 & \xi_5 \\ -\xi_2 & \xi_1 & 0 & \xi_6 \\ \xi_4 & \xi_5 & \xi_6 & 0 \end{bmatrix}.$$

Parameterized curve:

$$\mathbf{H}_d(s) = \begin{bmatrix} \gamma(s) & 0 & 0 & -\gamma(s)v(s) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma(s)v(s) & 0 & 0 & \gamma(s) \end{bmatrix},$$

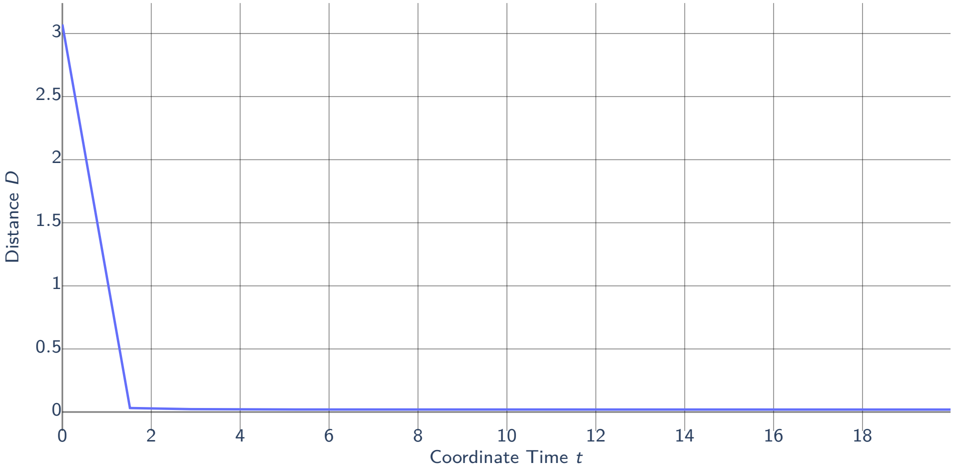
where $v(s) = 0.9 + \frac{0.09}{2}(\cos(2\pi s) + 1)$,
$\gamma(s) = \frac{1}{\sqrt{1-v(s)^2}}$.

Initial condition:

$$\mathbf{H}(0) = \exp\big(\mathcal{S}(\boldsymbol{\xi}_0)\big),$$

where $\boldsymbol{\xi}_0 = [0\ 0\ 0\ 0.7\ 0\ 0]^\top$.

Distance



Collaborative Manipulation

Each agent can measure the pose and velocity of the object's measurement point:

pose: $\chi = (\mathbf{p}, \mathbf{R}) \in \mathbb{R}^3 \times SO(3)$;

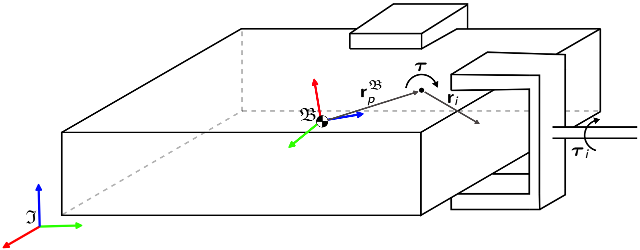
velocity: $\dot{\chi} = \left[\dot{\mathbf{p}}^\top, \boldsymbol{\omega}^\top\right]^\top \in \mathbb{R}^6$;

acceleration: $\ddot{\chi} = \left[\ddot{\mathbf{p}}^\top, \dot{\boldsymbol{\omega}}^\top\right]^\top \in \mathbb{R}^6$.

System model:

$$\boldsymbol{\tau} = \mathbf{M}(\chi)\ddot{\chi} + \mathbf{C}(\chi, \dot{\chi})\dot{\chi} + \mathbf{g},$$

Unknowns: $m, \mathbb{I}_{cm}^{\mathfrak{B}}, \mathbf{r}_p, \mathbf{r}_i$



Vector Field

Equivalent matrix Lie group: ISE(3).
Let

$$\mathbf{V}^{-1}\mathbf{W} = \begin{bmatrix} \mathbf{Q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{u} \\ \mathbf{0} & \mathbf{0} & 1 \end{bmatrix} \in \text{ISE}(3)\,,\ \mathbf{Q} \in \text{SO}(3),\ \mathbf{u} \in \mathbb{R}^3,$$

then the EE-distance is computed as:

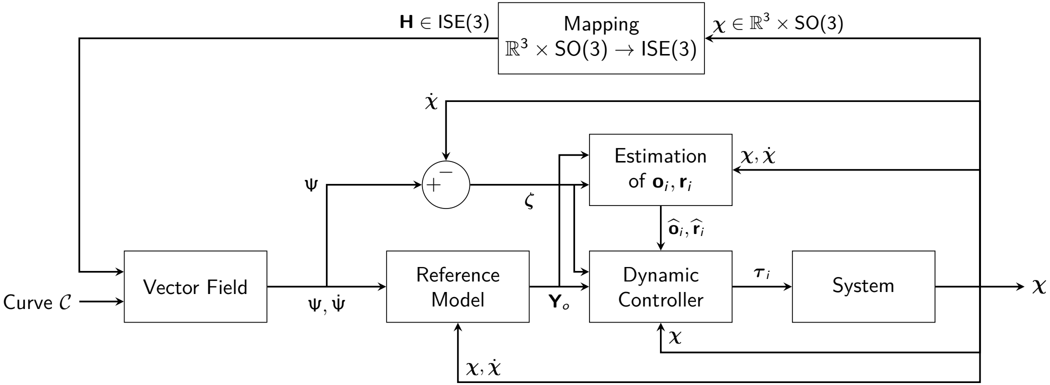
$$\widehat{D}(\mathbf{V},\mathbf{W}) = \left\|\log(\mathbf{V}^{-1}\mathbf{W})\right\|_F = \sqrt{2\theta^2 + \|\mathbf{u}\|^2}.$$

Isomorphism:

$$\mathcal{S}(\xi) = \begin{bmatrix} \widehat{\mathcal{S}}(\omega) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{v} \\ \mathbf{0} & \mathbf{0} & 0 \end{bmatrix},$$

where $\xi = [\mathbf{v}^\top\ \omega^\top]^\top$.

Control



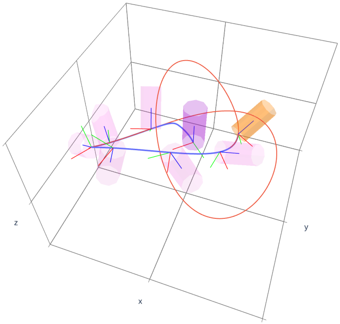
Reference Model:

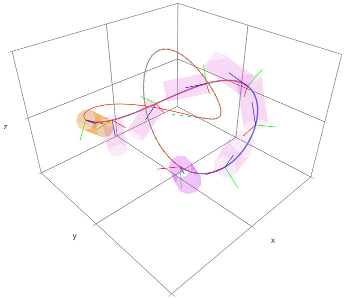
$$\alpha_i \left(\mathbf{M}(\chi)\dot{\Psi} + \mathbf{C}(\chi, \dot{\chi})\Psi + \mathbf{g}\right) = \mathbf{Y}_o \mathbf{o}_i$$

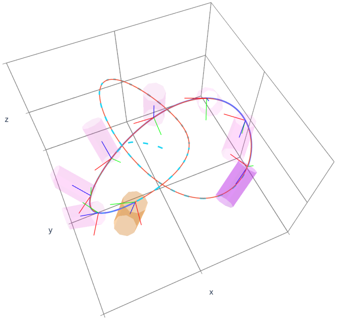
$$\boldsymbol{\tau}_i = \mathbf{G}(\chi, \hat{\mathbf{r}}_i)^{-1}\boldsymbol{\eta}_i, \quad \dot{\hat{\mathbf{o}}}_i = -\boldsymbol{\Gamma}_o \mathbf{Y}_o^\top \left(\chi, \dot{\chi}, \Psi, \dot{\Psi}\right)\zeta,$$

$$\boldsymbol{\eta}_i = \mathbf{Y}_o \hat{\mathbf{o}}_i - \mathbf{K}_D \zeta, \quad \dot{\hat{\mathbf{r}}}_i = -\boldsymbol{\Gamma}_r \mathbf{Y}_r^\top \left(\boldsymbol{\eta}_i, \chi\right)\zeta.$$

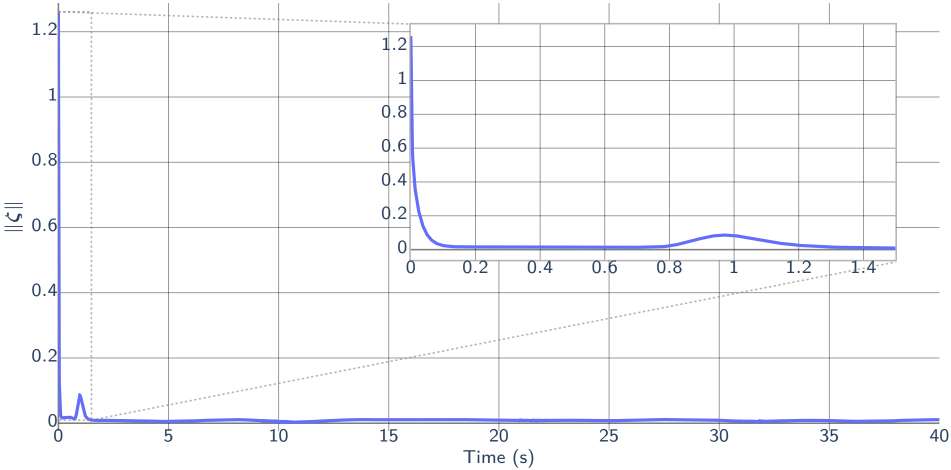
Trajectory



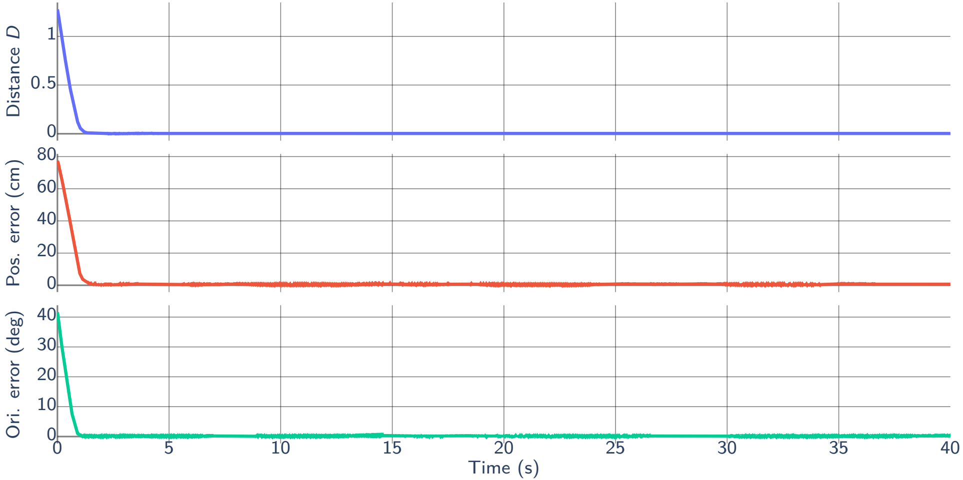




Velocity error



Distance


Distance D
1
0.5
0
Pos. error (cm)
80
60
40
20
0
Ori. error (deg)
40
30
20
10
0
0 5 10 15 20 25 30 35 40
Time (s)

Conclusion and Future Work

Conclusions:

- The proposed vector field strategy successfully guides systems in different Lie groups to track a desired curve;
- the approach demonstrates the flexibility of using different distance metrics in Euclidean space;
- the vector field can function as a high-level controller, providing velocity references for a lower-level dynamic controller.

Future works include:

- Handling time-varying and self-intersecting curves;
- exploring simpler distance functions;
- improving optimization algorithms;
- incorporating nonholonomic constraints;
- validating the approach in real-world systems.

References

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