

Constructive Vector Fields for Path Following in Matrix Lie Groups

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Motivation

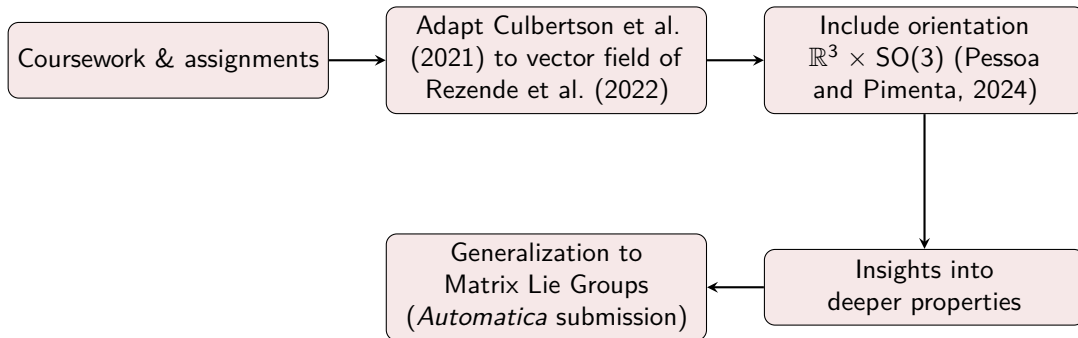
- Generalize the vector field strategy in Rezende et al. (2022) to allow more motion possibilities, including rotations;
- Gain deeper insight into vector field properties through generalization;
- Facilitate path following for systems with both translational and rotational motion, such as omnidirectional UAVs and robotic manipulators.



Contributions

- Development of a novel vector field guidance strategy applicable to systems with an inherent matrix Lie group structure;
- Implementation framework for $SE(3)$ systems, providing all necessary tools for practical application of the proposed strategy;
- Validation through kinematic simulations in $SE(3)$ and $SO^+(3, 1)$, demonstrating the theoretical results and their practical implications;
- Design of an adaptive control strategy for collaborative simulations in $\mathbb{R}^3 \times SO(3)$, where the vector field guidance strategy generates reference velocities for dynamic control.

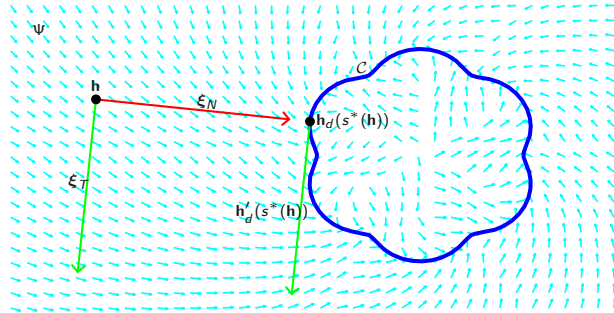
Background



Vector field in Euclidean space

The vector field strategy in Rezende et al. (2022) is based on a parametric curve representation and is characterized by:

- A distance function D ;
- $\dot{D} = \nabla D \xi = -\xi_N^\top \xi$;
- Tangent component depends only on the curve;
- Normal and tangent components are orthogonal;
- Absence of local minima outside the curve;
- Vector field
$$\Psi(\mathbf{h}) = k_N(D)\xi_N(\mathbf{h}) + k_T(D)\xi_T(\mathbf{h}).$$



Lie Groups and Lie algebras

Lie group G

Manifolds with group structure. The group operation and inverse map are continuous and smooth.

E.g.: set of rotation matrices $SO(3)$.

Lie algebra \mathfrak{g}

Tangent space of G at the identity.

E.g.: skew-symmetric matrices for $SO(3)$.

Exponential map

Maps elements of \mathfrak{g} to G . For matrix Lie groups:
$$\exp(\mathbf{A}) = \sum_{i=0}^{\infty} \frac{\mathbf{A}^i}{i!} = \mathbf{X} \in G.$$

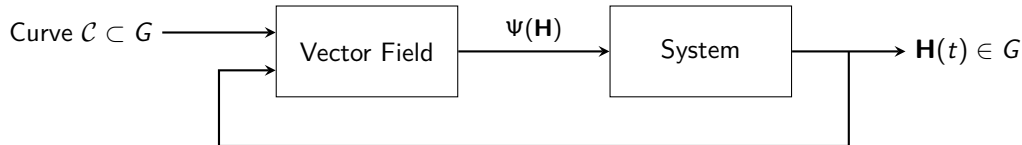
Not always surjective.

\mathcal{S} map

Linear map from \mathbb{R}^m to \mathfrak{g} , relating velocities to tangent space elements.

E.g.: angular velocities \rightarrow skew-symmetric matrices in $\mathfrak{so}(3)$.

Formulation



We assume the system model:

$$\dot{\mathbf{H}}(t) = \mathcal{S}(\boldsymbol{\xi}(t))\mathbf{H}(t), \quad \mathbf{H} \in G, \quad \boldsymbol{\xi} \in \mathbb{R}^m,$$

The vector field is given by:

$$\boldsymbol{\psi}(\mathbf{H}) \triangleq k_N(\mathbf{H})\boldsymbol{\xi}_N(\mathbf{H}) + k_T(\mathbf{H})\boldsymbol{\xi}_T(\mathbf{H}).$$

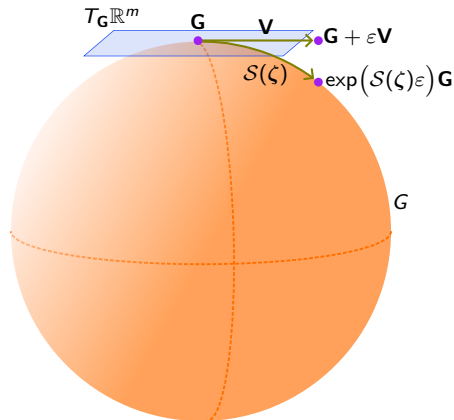
Gradient in Lie groups

The L operator, $L[f] : G \rightarrow \mathbb{R}^{1 \times m}$, acts as a gradient while respecting Lie group constraints.

For any scalar function $f : G \rightarrow \mathbb{R}$, it is implicitly defined as:

L operator

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left(f(\exp(S(\zeta)\varepsilon)\mathbf{G}) - f(\mathbf{G}) \right) = L[f](\mathbf{G})\zeta \quad \forall \zeta \in \mathbb{R}^m$$
$$\left. \frac{d}{d\varepsilon} \left(f(\exp(S(\zeta)\varepsilon)\mathbf{G}) \right) \right|_{\varepsilon=0} = L[f](\mathbf{G})\zeta \quad \forall \zeta \in \mathbb{R}^m$$



Distances

The vector field formulation requires two distance functions:

EE-distance \hat{D}

Measures the distance between two Lie group elements.

Positive definite: $\hat{D}(\mathbf{V}, \mathbf{W}) \geq 0$, $\hat{D}(\mathbf{V}, \mathbf{W}) = 0 \iff \mathbf{V} = \mathbf{W}$;

Differentiability: at least once differentiable in both arguments almost everywhere.

E.g.: for exponential Lie groups, an EE-distance is given by

$$\hat{D}(\mathbf{V}, \mathbf{W}) = \|\log(\mathbf{V}^{-1}\mathbf{W})\|_F.$$

EC-distance D

Measures the distance between a Lie group element and a curve:

$$D(\mathbf{H}) \triangleq \min_{\mathbf{Y} \in \mathcal{C}} \hat{D}(\mathbf{H}, \mathbf{Y}) = \min_{s \in [0,1]} \hat{D}(\mathbf{H}, \mathbf{H}_d(s)).$$

Components and Properties

Normal and tangent components:

$$\xi_N(\mathbf{H}) \triangleq -L_{\mathbf{V}}[\hat{D}](\mathbf{H}, \mathbf{H}_d(s^*))^\top$$

$$\xi_T(\mathbf{H}) \triangleq \mathcal{S}^{-1}\left(\frac{d\mathbf{H}_d}{ds}(s^*)\mathbf{H}_d(s^*)^{-1}\right)$$

Components are orthogonal if

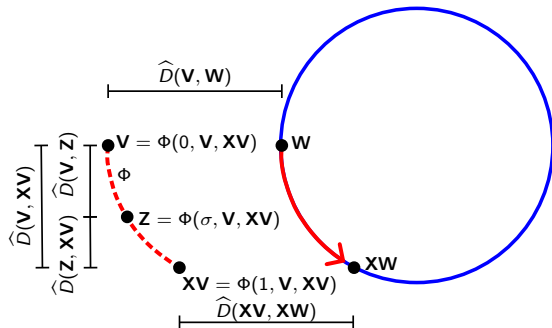
Left-invariant distance

$$\hat{D}(\mathbf{XV}, \mathbf{XW}) = \hat{D}(\mathbf{V}, \mathbf{W}) \forall \mathbf{V}, \mathbf{W}, \mathbf{X} \in G.$$

Absence of local minima outside the curve if

Chainable distance

$$\hat{D}(\mathbf{V}, \mathbf{W}) = \hat{D}(\mathbf{V}, \phi_\sigma) + \hat{D}(\phi_\sigma, \mathbf{W}).$$



Convergence Proof Sketch

Let \hat{D} be a left-invariant and chainable EE-distance, then:

$$\dot{D} = L_{\mathbf{v}}[\hat{D}]\xi + (L_{\mathbf{w}}[\hat{D}]\xi_T) \frac{ds^*}{dt}.$$

By the optimality condition of D , we have

$$\dot{D} = L_{\mathbf{v}}[\hat{D}]\xi = -\xi_N^\top \xi.$$

If the system follows the vector field, $\xi = \Psi$, then

$$\dot{D} = -\xi_N^\top (k_N \xi_N + k_T \xi_T) = -k_N \|\xi_N\|^2 \leq 0.$$

Circulation relies on proper parametrization.

Kinematic control in SE(3)

The distance function has a simpler expression. Let

$$\mathbf{V}^{-1}\mathbf{W} = \begin{bmatrix} \mathbf{Q} & \mathbf{u} \\ \mathbf{0} & 1 \end{bmatrix},$$

then

$$\hat{D}(\mathbf{V}, \mathbf{W}) = \|\log(\mathbf{V}^{-1}\mathbf{W})\|_F = \sqrt{2\theta^2 + \mathbf{u}^\top \bar{\mathbf{X}} \mathbf{u}},$$

where

$$\begin{aligned} \theta &= \text{atan2} \left(\frac{1}{2\sqrt{2}} \|\mathbf{Q} - \mathbf{Q}^\top\|_F, \frac{1}{2} (\text{tr}(\mathbf{Q}) - 1) \right), \\ \bar{\mathbf{X}} &= (1 - 2\beta_0)\mathbf{I} + \beta_0(\mathbf{Q} + \mathbf{Q}^\top), \\ \beta_0 &= \frac{2 - 2\cos\theta - \theta^2}{4(1 - \cos\theta)^2}. \end{aligned}$$

Kinematic control in SE(3)

Vector field gains:

$$k_N(\mathbf{H}) = \tanh(20D(\mathbf{H})),$$
$$k_T(\mathbf{H}) = 1 - 0.5 \tanh(D(\mathbf{H})).$$

Lie algebra basis reflecting the mechanical twist:

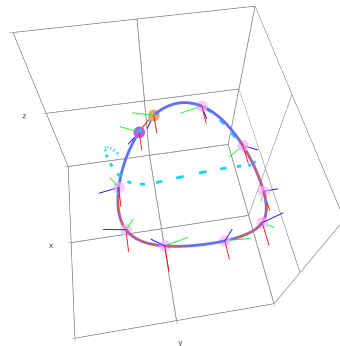
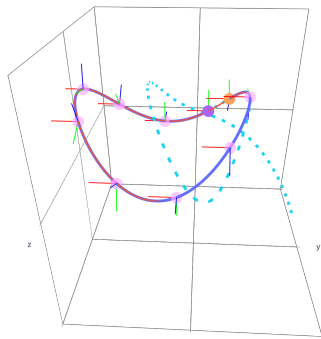
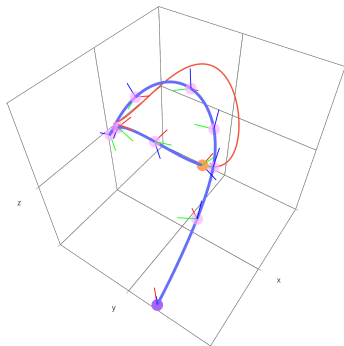
$$\mathcal{S}(\xi) = \begin{bmatrix} 0 & -\xi_6 & \xi_5 & \xi_1 \\ \xi_6 & 0 & -\xi_4 & \xi_2 \\ -\xi_5 & \xi_4 & 0 & \xi_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The normal and tangent components were numerically approximated.
The system was integrated using:

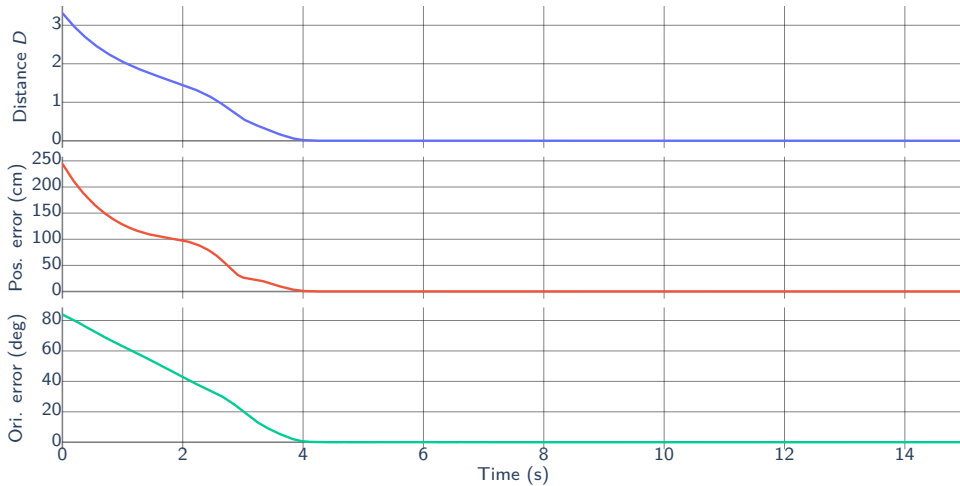
$$\mathbf{H}(t + \Delta t) \approx \exp\left(\mathcal{S}(\Psi(\mathbf{H}))\Delta t\right)\mathbf{H}(t),$$

with a time step of $\Delta t = 1 \cdot 10^{-2}$ s.

Trajectory



Distances



Kinematic control in $SO^+(3, 1)$

EE-distance:

$$\hat{D}(\mathbf{V}, \mathbf{W}) = \|\log(\mathbf{V}^{-1}\mathbf{W})\|_F$$

Vector field gains:

$$k_N(\mathbf{H}) = \tanh(1000D(\mathbf{H}))$$

$$k_T(\mathbf{H}) = 0.5 \left(1 - \tanh(100D(\mathbf{H})) \right).$$

Lie algebra isomorphism:

$$S(\xi) = \begin{bmatrix} 0 & -\xi_3 & \xi_2 & \xi_4 \\ \xi_3 & 0 & -\xi_1 & \xi_5 \\ -\xi_2 & \xi_1 & 0 & \xi_6 \\ \xi_4 & \xi_5 & \xi_6 & 0 \end{bmatrix}.$$

Parameterized curve:

$$\mathbf{H}_d(s) = \begin{bmatrix} \gamma(s) & 0 & 0 & -\gamma(s)v(s) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma(s)v(s) & 0 & 0 & \gamma(s) \end{bmatrix},$$

$$\text{where } v(s) = 0.9 + \frac{0.09}{2}(\cos(2\pi s) + 1),$$

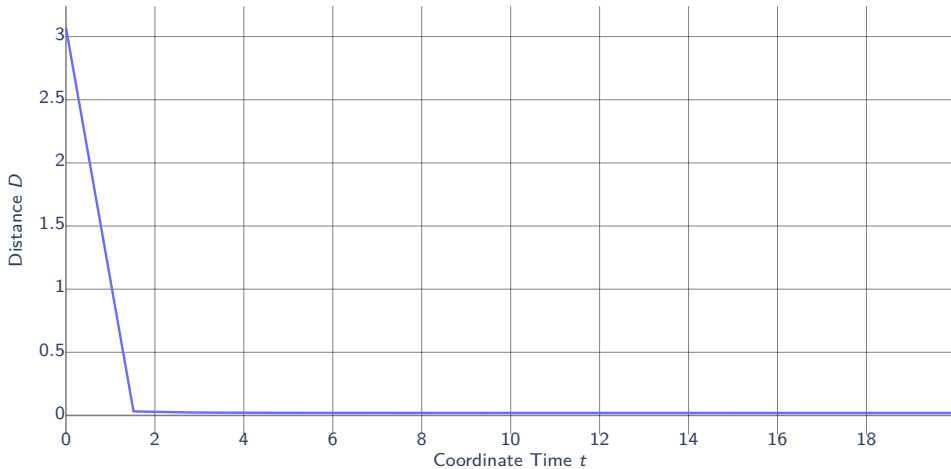
$$\gamma(s) = \frac{1}{\sqrt{1-v(s)^2}}.$$

Initial condition:

$$\mathbf{H}(0) = \exp(S(\xi_0)),$$

$$\text{where } \xi_0 = [0 \ 0 \ 0 \ 0.7 \ 0 \ 0]^\top.$$

Distance



Collaborative Manipulation

Each agent can measure the pose and velocity of the object's measurement point:

$$\text{pose: } \chi = (\mathbf{p}, \mathbf{R}) \in \mathbb{R}^3 \times \text{SO}(3);$$

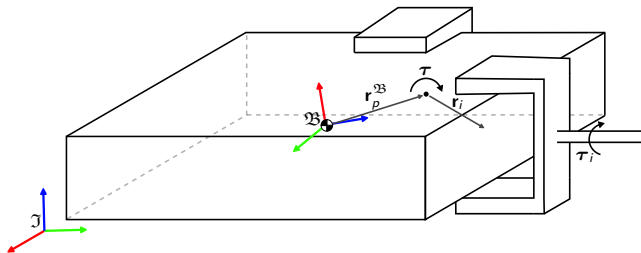
$$\text{velocity: } \dot{\chi} = [\dot{\mathbf{p}}^\top, \dot{\boldsymbol{\omega}}^\top]^\top \in \mathbb{R}^6;$$

$$\text{acceleration: } \ddot{\chi} = [\ddot{\mathbf{p}}^\top, \ddot{\boldsymbol{\omega}}^\top]^\top \in \mathbb{R}^6.$$

System model:

$$\boldsymbol{\tau} = \mathbf{M}(\chi)\ddot{\chi} + \mathbf{C}(\chi, \dot{\chi})\dot{\chi} + \mathbf{g},$$

Unknowns: $m, \mathbb{I}_{\text{cm}}^{\mathcal{B}}, \mathbf{r}_p, \mathbf{r}_i$



Vector Field

Equivalent matrix Lie group: ISE(3).

Let

$$\mathbf{V}^{-1}\mathbf{W} = \begin{bmatrix} \mathbf{Q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{u} \\ \mathbf{0} & \mathbf{0} & 1 \end{bmatrix} \in \text{ISE}(3), \mathbf{Q} \in \text{SO}(3), \mathbf{u} \in \mathbb{R}^3,$$

then the EE-distance is computed as:

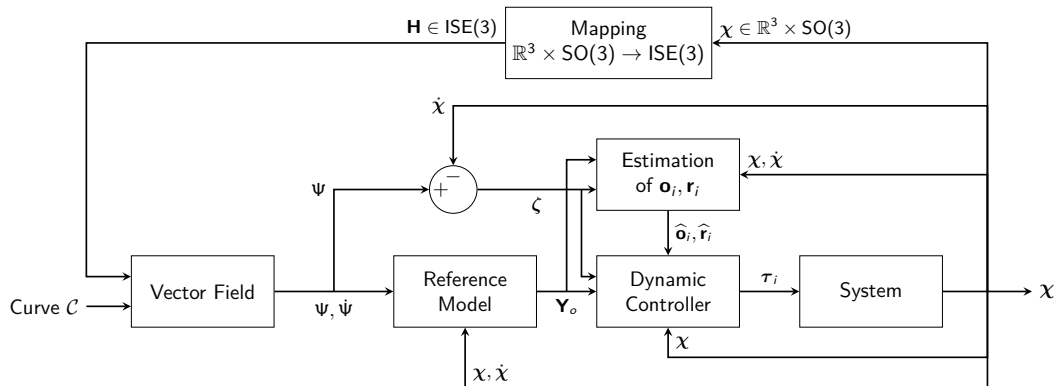
$$\hat{D}(\mathbf{V}, \mathbf{W}) = \|\log(\mathbf{V}^{-1}\mathbf{W})\|_F = \sqrt{2\theta^2 + \|\mathbf{u}\|^2}.$$

Isomorphism:

$$\mathcal{S}(\xi) = \begin{bmatrix} \hat{S}(\omega) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{v} \\ \mathbf{0} & \mathbf{0} & 0 \end{bmatrix},$$

where $\xi = [\mathbf{v}^\top \ \omega^\top]^\top$.

Control

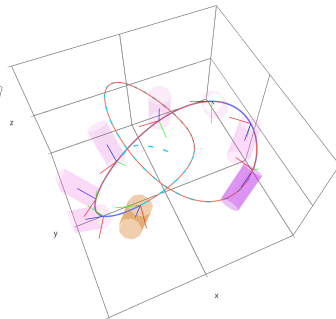
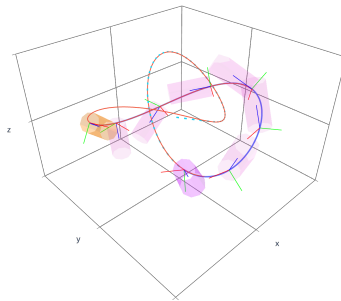
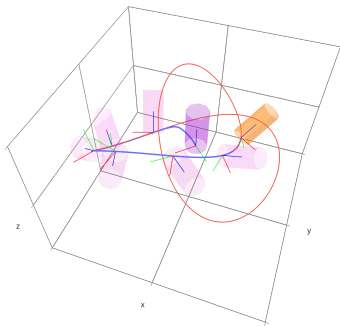


Reference Model:

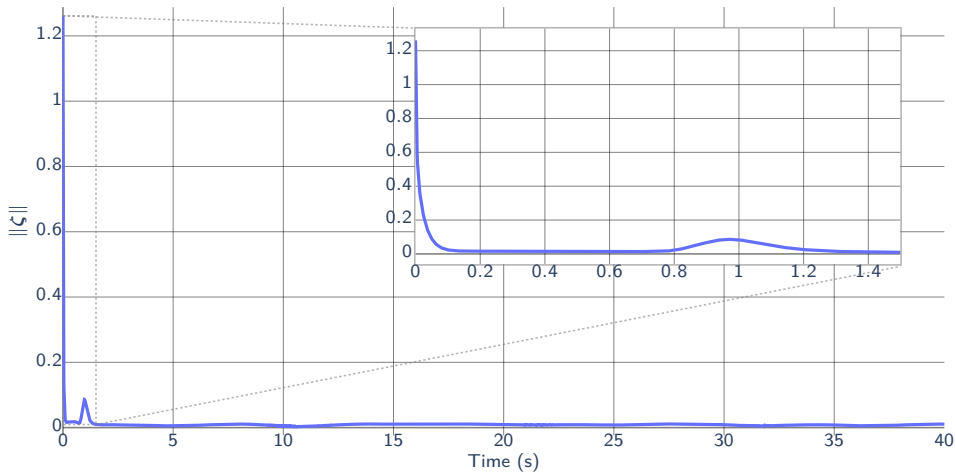
$$\alpha_i (\mathbf{M}(\chi) \dot{\psi} + \mathbf{C}(\chi, \dot{\chi}) \psi + \mathbf{g}) = \mathbf{Y}_o \mathbf{o}_i \quad \tau_i = \mathbf{G}(\chi, \hat{\mathbf{r}}_i)^{-1} \eta_i, \quad \dot{\hat{\mathbf{o}}}_i = -\mathbf{\Gamma}_o \mathbf{Y}_o^\top (\chi, \dot{\chi}, \psi, \dot{\psi}) \zeta,$$

$$\eta_i = \mathbf{Y}_o \hat{\mathbf{o}}_i - \mathbf{K}_D \zeta, \quad \dot{\hat{\mathbf{r}}}_i = -\mathbf{\Gamma}_r \mathbf{Y}_r^\top (\eta_i, \chi) \zeta.$$

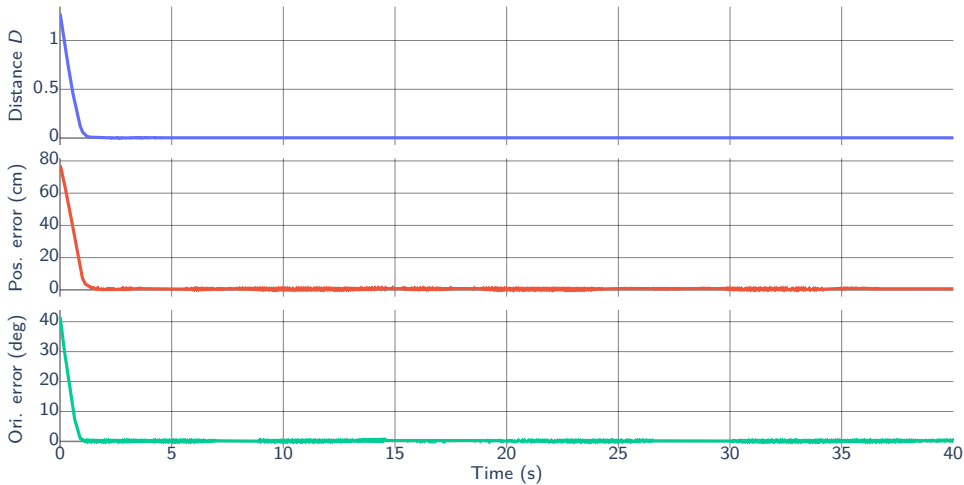
Trajectory



Velocity error



Distance



Conclusion and Future Work

Conclusions:

- The proposed vector field strategy successfully guides systems in different Lie groups to track a desired curve;
- the approach demonstrates the flexibility of using different distance metrics in Euclidean space;
- the vector field can function as a high-level controller, providing velocity references for a lower-level dynamic controller.

Future works include:

- Handling time-varying and self-intersecting curves;
- exploring simpler distance functions;
- improving optimization algorithms;
- incorporating nonholonomic constraints;
- validating the approach in real-world systems.

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