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PLANETS ORBITING EVOLVING BINARY STARS

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Subtitle

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ABSTRACT

Short summary of the contents in English... a great guide by Kent Beck how to write good abstracts can be found here:

https://plg.uwaterloo.ca/~migod/research/beckOOPSLA.html

We have seen that computer programming is an art,
because it applies accumulated knowledge to the world,
because it requires skill and ingenuity, and especially
because it produces objects of beauty.

— **knuth:1974 (knuth:1974)**

ACKNOWLEDGMENTS

Put your acknowledgments here.

Many thanks to everybody who already sent me a postcard!

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1 Members of GuIT (Gruppo Italiano Utilizzatori di TeX e LaTeX)

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1 INTRODUCTION

In this chapter, I introduce the main topic of the thesis, review the observational evidence of planets around binary stars, and review the basic theory of binary stellar evolution in order to motivate the problem.

1.1 HISTORY

-write about exoplanets detection in recent decades, mention first circumbinary planet around a pulsar -plot by Fabrycky or similar of planets detected so far

1.2 BINARY STARS AND TYPES OF PLANETARY ORBITS

- define circumbinary planets, mention types of orbits - mention planet formation around binaries - define orbital elements and include figure

1.3 EXOPLANET DETECTION TECHNIQUES

- briefly describe detection techniques with a focus on transits nad TTVs

1.4 BINARY STELLAR EVOLUTION

- everything about stellar evolution goes into this chapter

1.4.1 *Evolution on the main sequence*

- describe Roche Lobe surfaces and the Lagrange points

1.4.2 *Post main sequence evolution and the common envelope*

- branches of stellar evolution, roche lobe overflow, mass loss and mass transfer - don't mention sweeping resonances, just mention that as the binary evolves bad things can happen to the planets

1.5 TWO POPULATIONS OF CIRCUMBINARY PLANETS

- describe the MS and post-MS observations of CBPs - tables and plots of basic properties - mention Zorotovich and Schreiber paper - mention theories about the origin of the PCE planets

2 THEORETICAL BACKGROUND

In this chapter I will review the most important parts of the theory of planetary dynamics. Section 2.1 describes the two–body problem which introduces planet orbits and orbital elements. In section 2.2 I review the basic theory of a different formulation of mechanics called *Hamiltonian mechanics*. The Hamiltonian formalism and its tools will be necessary for the development of an analytical model of resonance capture in chapter 3. In **??** I describe a useful toy model for studying mean–motion resonances – the pendulum. Finally, in **??** I review the theory of the three–body problem which forms the basis for subsequent chapters.

The second chapter deals with analytic models of mean–motion resonances (MMRs for short). This concept is the key part of the thesis because the overlap of MMRs and the passage of planetary or stellar bodies through them are crucial for the determination of the stability of planetary bodies. Here I develop a new analytical model of a 6:1 resonance in the case where the inner two bodies are comparable in mass. The 6:1 MMR is the first important resonance encountered by circumbinary planets similar to the observed MS population once the binary starts evolving off the main–sequence. I use this model to predict the eccentricity kick to a planet orbiting the evolving binary as the stars approach each other due to tidal forces.

2.1 THE TWO–BODY PROBLEM

2.1.1 *The orbit equation and its solution*

If we are to attack the problem of three gravitionally interacting bodies in a circumbinary system, we first need to understand a simpler problem – that of two massive bodies. This problem is often called the *Kepler problem*. Consider two bodies with masses m_1 and m_2 and position vectors[1] $\mathbf{r}_1$ and $\mathbf{r}_2$ relative to a fixed origin O in inertial space. The forces acting on the two bodies are given by

$$\mathbf{F}_1 = Gm_1m_2\frac{\hat{\mathbf{r}}}{r^3} = m_1\ddot{\mathbf{r}}_1 \tag{1}$$

$$\mathbf{F}_2 = -Gm_1m_2\frac{\hat{\mathbf{r}}}{r^3} = m_2\ddot{\mathbf{r}}_2 \tag{2}$$

where $G = 6.672\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$ is the gravitational constant, $\hat{\mathbf{r}}$ is the unit vector pointing from $\mathbf{r}_1$ to $\mathbf{r}_2$ and r is the magnitude of the relative seperation vector $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. The double dots denote second time

1 Throughout the thesis, vector quantities will be written with a bold–face font

derivatives. From the definition of $\mathbf{r}$ and the equations of motion, it follows that

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1 = -G(m_1 + m_2)\frac{\hat{\mathbf{r}}}{r^3} \tag{3}$$

which can be rewritten as

$$\ddot{\mathbf{r}} + \mu\frac{\hat{\mathbf{r}}}{r^3} = 0 \tag{4}$$

where $\mu = G(m_1 + m_2)$. If we take a cross product of eq. (4) with $\mathbf{r}\times$ from the left-hand side and using the fact that $\mathbf{r} \times \mathbf{r} = 0$, we obtain

$$\mathbf{r} \times \ddot{\mathbf{r}} = 0 \tag{5}$$

This can be integrated to get

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h} \tag{6}$$

where $\mathbf{h}$ is a constant vector perpendicular to the plane spanned by $\mathbf{r} \times \dot{\mathbf{r}}$. $\mathbf{h}$ is in fact the angular momentum per unit mass. Thus, the motion of m_2 relative to m_1 is confined to a plane perpendicular to $\mathbf{h}$. Since the motion is in a plane, we can simplify the problem further by transfering to polar coordinated (r, θ) centered at m_1, we then have the following relations between vectors available in any vector calculus book

$$\mathbf{r} = r\hat{\mathbf{r}} \tag{7}$$

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta} \tag{8}$$

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\hat{\theta} \tag{9}$$

By substituting these transformations into eq. (6) we get

$$\mathbf{h} = r^2\dot{\theta}\hat{\mathbf{z}} \tag{10}$$

where $\hat{\mathbf{z}}$ is perpendicular to the orbital plane.

By inserting the expression for $\ddot{\mathbf{r}}$ (eq. (9)) into eq. (4) and considering only the component (the $\hat{\theta}$ component just says that h is constant which we know already), we have the following scalar equation

$$\ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2} = 0 \tag{11}$$

To solve this differential equation, we use the substitution $r = u^{-1}$ and eq. (10). From the chain rule, it follows

$$\dot{r} = -h\frac{du}{d\theta} \tag{12}$$

$$\ddot{r} = -h^2u^2\frac{d^2u}{d\theta^2} \tag{13}$$

And finally, we have

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} \tag{14}$$

Equation (14) is called the *orbit equation*. This second order differential equation has the solution (after transforming back to r)

$$r = \frac{h^2/\mu}{1 + e\cos(\theta - \omega)} \tag{15}$$

where the *eccentricity* e and the *argument of pericentre* ω are the two constants of integration. Equation (15) defines a *conic section* curve in 2D space. Depending on the eccentricity, it can either be an ellipse for $e < 1$ corresponding to a closed orbit, or a hyperbola for $e > 1$ corresponding to an unbound orbit. The special case $e = 1$ defines a parabola but is of little physical significance since any particular orbit is highly unlikely to have eccentricity exactly zero. Similarly $e = 0$ defines a circular orbit which is of more interest since most stable planetary orbits are very nearly circular. Figure 1 shows an

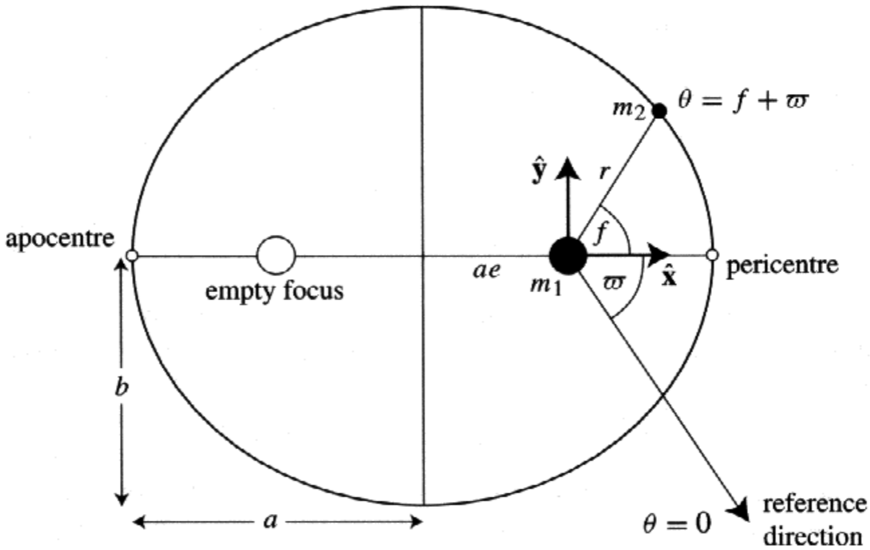

Figure 1: An elliptical orbit. The mass m_1 sits in one focus and m_2 orbits around it. The postion of m_2 on the ellipse is specified by two angles, the true anomaly f and the argument of pericentre ω. Only f varies in the two–body problem, ω stays fixed in the absence of an external perturbation. Figure taken from Murray and Dermott (1999).

elliptical orbit in two dimensional space. The mass m_2 orbits around m_1 which is located in one of the focci of the ellipse. The position of m_2 at each moment in time is described by the 2π-periodic angle $f = \theta - omega$ called the *true anomaly*. The angle ω is specified relative to an arbitrary reference direction and it is constant throughout the motion. $f = 0$ corresponds to the closest approach of m_2 to m_1, this point on orbit is called the *pericentre*. Conversely, the point furthest away from m_2 at $f = \pi$ is called the *apocentre* . The *semi-major axis* of the ellipse a is given by

$$a = \frac{h^2}{\mu}\frac{1+e}{1-e} \tag{16}$$

One can easily derive (ex. Murray and Dermott, 1999) *Kepler's third law* which says that

$$T^2 = \frac{4\pi^2}{\mu}a^3 \tag{17}$$

where T is the orbital period of m_2 around m_1. We also define the so-called *mean motion* n, as

$$n = \frac{2\pi}{T} \tag{18}$$

The mean motion is the average angular frequency of the periodic motion. It is constant in the two–body problem but in general varies when additional bodies are present.

If we multiply eq. (4) by $\dot{\mathbf{r}}$ and use the expressions for $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$ from eq. (9), we obtain the following constant of motion

$$\frac{1}{2}v^2 - \frac{\mu}{r} = C \tag{19}$$

where $v^2 = \dot{\mathbf{r}} \cdot \dot{\mathbf{r}}$ is velocity squared and C is a constant of motion, the energy per unit mass. It can be shown (Murray and Dermott, 1999) that C is given by

$$C = -\frac{\mu}{2a} \tag{20}$$

thus, of a closed orbit in the two–body problem depends only on the semi-major axis.

2.1.2 *The mean and eccentric anomaly*

By solving the orbit equation, we have established that the mass m_2 orbits around m_1 in an ellipse if $e < 1$. However, it is not immediately clear how to explicitly solve for the time dependance of r and f and thus determine the position of m_2 at any given time. It is obvious that f and r vary non-linearly with time for $e \neq 0$. For reasons which will become apparent later, we would like to construct an angle which varies linearly with time. One such angle is the *mean anomaly* M defined as

$$M = n(t - \tau) \tag{21}$$

where τ is the *time of pericentre passage* and is constant. M incrases linearly with time at a rate equal to the mean motion. At $t = \tau$ we have $M = f = 0$ and at $t = \tau + /2$ $M = f = \pi$, thus at the pericentre and apocentre M matches with f. The angle M has no obvious geometricall significance but we can define another angle which does. Figure 2 shows the orbital ellipse with semi-major axis a together with a circumscribed circle of radius a concentric with the ellipse. A line perpendicular to the semi-major axis of the ellipse interesects two points, one on the orbit and one on the circumscribed circle. We define the *eccentric anomaly* E to be the angle between the semi-major axis of the ellipse and the intersected point on the circle. Again, we have $E = M = 0$ at $f = 0$ and $E = M = \pi$ at $f = \pi$. From geometry

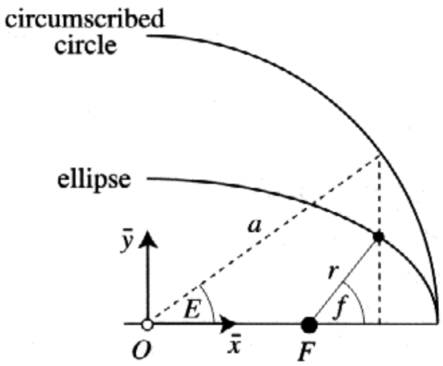

Figure 2: A geometrical description of the eccentric anomaly E.

one can show that the following relation between the angles f and E is satisfied

$$\cos f = \frac{\cos E - e}{1 - e \cos E} \tag{22}$$

Thus, there is a one-to-one correspondance between f and E. To locate the location of the body on its orbit at time t, we need a relationship between E and M. This relation ship is called the *Kepler's equation* and is given by (Murray and Dermott, 1999)

$$M = E - e \sin E \tag{23}$$

A solution to this equation enables us to locate the body on its orbit at any given time. The procedure is as follows

1. At a particular time t find M from eq. (21)
2. Solve the Kepler's equation for E
3. Use eq. (22) to find f

Kepler's equation is transcendental in E and therefore it cannot be solved directly.

Finally, we define one last angle λ called the *mean longitude* as

$$\lambda = M + \omega \tag{24}$$

Since it is derived from M, it does not have a geometrical interpretation. All longitudes are defined with respect to a common, arbitrary reference point.

2.1.3 *Orbit in an inertial frame*

So far we have derived a solution for the *relative* motion of m_2 with respect to m_1, we now turn to the description of the orbit in a non-accelerating *inertial frame*. It is not difficult to show that the the masses

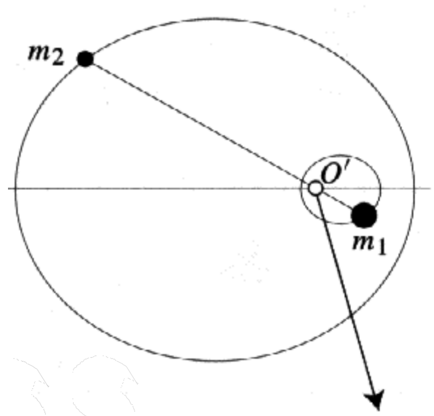

Figure 3: The motion of m_2 and m_1 with respect to their centre of mass O'.

m_1 and m_2 again orbit in a conic section around their centre of mass with the same period T as before. Figure 3 shows the orbits with respect to the centre of mass. We need not worry about the motion of the centre of mass itself because of a result from elementary mechanics which says that the centre of mass of a collection of particles always moves at constant velocity in a straight line and is therefore a valid inertial reference frame. The conic sections of the orbits relative to the centre of mass are reduced in scale by mass factors, as follows

$$a_1 = \frac{m_2}{m_1 + m_2} a \quad a_2 = \frac{m_1}{m_1 + m_2} a \tag{25}$$

where a_1 is the semi-major axis of the orbit of m_1 around O' and a_2 is the semi-major axis of the orbit of m_2 around O'.

The *total* angular momentum of the system is given by

$$L = \frac{m_1 m_2}{m_1 + m_2} \sqrt{\mu a (1 - e^2)} \tag{26}$$

and the total orbital energy is

$$E = -G \frac{m_1 m_2}{2a} \tag{27}$$

The energy of a Keplerian orbit depends only on the semi-major axis and the angular momentum depends on both the semi-major axis and the eccentricity. In particular, if the semi-major axis is constant the only way to change the eccentricity is by changing the angular momentum. This simple fact is the essence of so-called secular interactions described in section 2.4. The angular momentum is largest for a circular orbit.

2.1.4 *Orbit in three-dimensional space*

We have determined that the bodies in the two–body problem move on on an ellipse in inertial space. The orientation of that ellipse stays fixed for all time if no external bodies are present. If there are also other bodies in the system however, the orbit no longer stays fixed, both its shape and orientation change in three-dimensional space. Because of that it is useful to define the orientation of the orbit in 3D space relative to a fixed reference plane. Figure 4 shows the orbit in

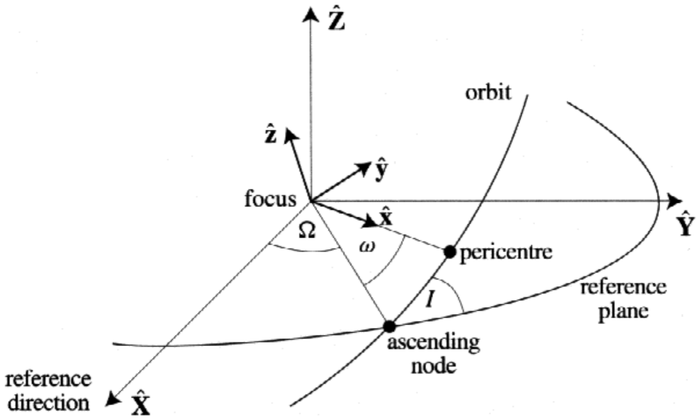

Figure 4: A keplerian orbit in 3D space.

3D Cartesian coordinate system, the reference plane is taken to be the $X - Y$ plane. The orbital ellipse intersects the reference plane in two point. In order to define its orientation relative to fixed axes we have to choose one. Independent on wheater the orbiting body is moving around the ellipse in a clockwise or counter-clockwise direction, the body will pass through the reference plane *from below* (where below is the $-Z$ direction) at one of the two points. We call this point the *ascending node* and choose it as a reference. The angle from the ascending node to the X axis is then called the *longitude of the ascending node* and is denoted by Ω. The angle between the plane of the ellipse and the reference plane is called the *inclination* of the orbit and is defined in the range $0 \leqslant I \leqslant \pi$. Thus, we have completely described the orbit in 3D space. However, it is useful to define another angle called the *longitude of pericentre.* is not really a true angle since it is defined as a sum of angles in two seperate planes. When the inclination is zero (the orbit is co-planar with the reference plane) $= \omega$.

It can be shown that each there is a one-to-one correspondance between a set of Cartesian positions and velocities (x, y, z, v_x, v_y, v_z) of a given massive particle and an *instantaneous* Keplerian orbit defined by $(a, e, I, \Omega, \omega, f)$ with respect to another massive particle. This why it is still usefull to talk about orbits even when we are dealing with a system of multiple bodies. Altough those bodies won't stay on a fixed

Keplerian orbit for all time, at any given time we can still define an instantaneous Keplerian orbit.

Most bodies in stable systems change their orbital elements slowly when exchanging energy and angular momentum with other bodies, thus it is often more useful to use the orbital elements as a set of coordinates instead of the Cartesian coordinates.

2.2 A BRIEF REVIEW OF HAMILTONIAN MECHANICS

2.2.1 *Hamilton's equations*

The two–body prolem could have been solved equally well using a different formulation of mechanics called *Hamiltonian mechanics* after William Rowan Hamilton (1805-1865). Hamiltonian mechanics is equivalent to Newtonian mechanics but is often more suitable for certain types of problems and the concept of a Hamiltonian function is a lot more general than Newton's second law of mechanics.

In Newtonian mechanics the full description of a dynamical system consisting of N particles is obtained by solving a system of second order differential equations of the form

$$\dot{\mathbf{p}}_i = \mathbf{F}_i \tag{28}$$

where $\dot{\mathbf{p}}_i = m_i \dot{\mathbf{r}}_i$ is the momentum of the i-th particle, m_i is its mass and $\mathbf{r}_i$ its position vector relative to an origin of an inertial reference system. This constitutes a system of 3N second order differential equations for the positions vectors $\mathbf{r}_i$.

In the Hamiltonian formalism a dynamical system is described by a function of *generalized coordinates* $\mathbf{q}$ and *momenta* $\mathbf{p}$ called the *Hamiltonian* $\mathcal{H}(\mathbf{q}, \mathbf{p})$. Each pair (q_i, p_i) constitutes a single *degree of freedom*, it is said to be *conjugate* . The time evolution of these coordinates and momenta $(\mathbf{q}, \mathbf{p})$, collectively known as the *phase space* is given by *Hamilton's equations*

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \quad \dot{p}_i = \frac{\partial \mathcal{H}}{\partial q_i} \tag{29}$$

Instead of 3N second order differential equations for a system of N particles , we now have 6N *coupled first-ordered* equations. Once the initial conditions $(\mathbf{q}_0, \mathbf{p}_0)$ are specified, the solution of Hamilton's equations defines a *unique* trajectory in a 6N dimensional phase space. The coordinate pair $(\mathbf{q}, \mathbf{p})$ is said to be *canonical* if the coordinates satisfy Hamilton's equations. The main advantage of the Hamiltonian formalism compared to other formalisms of mechanics is the ability to easily transform to different choices of $(\mathbf{q}, \mathbf{p})$ as long as the new set of coordinates is also canonical. There are no other strict requirements on the new coordinates, the momentum p_i coincides with the real momentum $m_i \dot{q}_i$ only in Cartesian coordinates.

We can arbitrarily scale the momentum and the coordinate by a constant factor, for example $p \to \eta p \quad q \to \nu q,$, as long as we also rescale the time. This is obvious from the form of eq. (29) because

$$\frac{\partial \mathcal{H}}{\partial(\eta p)} = \frac{dq}{d(\eta t)} \tag{30}$$

and similary for the coordinate scaling. Transformations of this type are known as *scale transformations* and they are simply a reflection of the fact that the equations of motion should be invariant to the changes of units.

2.2.2 *Integrable Hamiltonians*

The Hamiltonian formalism is useful for finding conserved quantities. If a generalized coordinate q_i does not appear in the Hamiltonian the the corresponding momentum conjugate p_i is a conserved quantity

$$\dot{p}_i = \frac{\partial \mathcal{H}}{\partial q_i} = 0 \tag{31}$$

If the motion is in a fixed potential, the Hamiltonian is equal to the total energy of the system E.

If a particular hamiltonian $\mathcal{H}$ can be reduced to a form where it depends only on the momenta, that is

$$\mathcal{H}(\mathbf{p}) \tag{32}$$

then the momenta $\mathbf{p}$ are conserved and the system is said to be *integrable*. An integrable system with n degrees of freedom has n constants of motion. From Hamilton's equations, it follows that the time evolution of the coordinates is simply

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} = \omega_i \tag{33}$$

that is, all of the coordinates evolve linearly in time with constant frequencies ω_i which depend only on the momenta

$$q_i(t) = \omega_i t \tag{34}$$

Integrable system are very rare and it is not immediately clear if a given Hamiltonian can be reduced to an integrable form.

Systems of the form

$$\mathcal{H} = \mathcal{H}_0(\mathbf{p}) + \epsilon \mathcal{H}_1(\mathbf{q}) \tag{35}$$

where $\mathcal{H}_0$ is an integrable Hamiltonian, $\mathcal{H}_0$ is a perturbation and ϵ is a small parameter are said to be nearly integrable and in general they display chaotic behaviour .However, if ϵ is small enough most solutions still lie in the region of phase space allowed by the solutions of $\mathcal{H}_0$.

Systems with a single degree of freedom are always integrable and the Hamiltonian itself is a conserved quantity (i.e. the energy is conserved). The trajectory is defined completely by the value of energy. They are often used as an approximation for a generally more complex system. Obtaining a single degree of freedom Hamiltonian for a resonance is a major goal in chapter 3.

The Keplerian Hamiltonian for the two–body problem is completely integrable. It can be written as

$$\mathcal{H}_k = -\frac{\mu^2 \mu^*}{2\Lambda^2} \tag{36}$$

where $\mu^* = m_1 m_2/(m_1 + m_2)$ and Λ is the generalized momentum conjugate to the orbital element coordinate λ, the mean anomaly. The original Keplerian Hamiltonian has 6 degrees of freedom, the conservation of total linear and total angular momentum vectors and the conservation of total energy give 7 constants of motion. There is also a hidden symmetry in the problem that we haven't mentioned. One can show that one of the three components of the *Runge-Lenz* vector is also a constant of motion, which bring the total to 8. The Runge-Lenz vector is a vector pointing in the direction of periastron, defined by

$$\mathbf{e} = \dot{\mathbf{r}} \times (\mathbf{r} \times \dot{\mathbf{r}})/(G(m_1 + m_2)) - \hat{\mathbf{r}} \tag{37}$$

Its magnitude is equal to the eccentricity of the orbit. We see that the number of constants of motion over-determines the problem. In fact, the two extra constants of motion are responsible for the fact that that the relative motion is not only restricted to a conic curve but it is also a conic section in the inertial (centre of mass) frame. Due to this fact the orbit is also fixed in 3D space.

2.2.3 *Fixed points*

Given a Hamiltonian $\mathcal{H}(q, p)$ with a single degree of freedom its *fixed points* are solutions of

$$\dot{p} = 0 \quad \dot{q} = 0 \tag{38}$$

Those are points at which there is no motion in phase space. From Hamilton's equations, it follows that their location is given by

$$\frac{\partial \mathcal{H}}{\partial p} = \frac{\partial \mathcal{H}}{\partial q} = 0 \tag{39}$$

Given a fixed point (q_0, p_0), we would like to derive the equations of motion in its vicinity. If a test particle in its vicinity moves on a trajectory away from the fixed point, the point is said to be *unstable*.

Conversely, if it remains near the fixed point for all time, the point is said to be *stable*. We can expand the Hamiltonian near the fixed point in a Taylor series

$$\mathcal{H}(\tilde{q},\tilde{p}) = \mathcal{H}(q_0,p_0) + \frac{\partial^2\mathcal{H}(q_0,p_0)}{\partial q^2}\frac{\tilde{q}^2}{2} + \frac{\partial^2\mathcal{H}(q_0,p_0)}{\partial p^2}\frac{\tilde{p}^2}{2} + \frac{\partial^2\mathcal{H}(q_0,p_0)}{\partial q \partial p}\tilde{q}\tilde{p} \tag{40}$$

where $\tilde{q} = q - q_0$ and $\tilde{p} = p - p_0$. We can write this more succinctly in vectorial form as

$$\mathcal{H}(\tilde{q},\tilde{p}) = \frac{1}{2}\mathbf{x}^{\tau}\mathbf{M}\mathbf{x} \tag{41}$$

where $\mathbf{x} = (\tilde{q},\tilde{p})$ and τ denotes the transpose operation. $\mathbf{M}$ is called the *Hessian matrix* and is given by

$$\mathbf{M} = \begin{pmatrix} \frac{\partial^2\mathcal{H}}{\partial q^2} & \frac{\partial^2\mathcal{H}}{\partial q \partial p} \\ \frac{\partial^2\mathcal{H}}{\partial p \partial q} & \frac{\partial^2\mathcal{H}}{\partial p^2} \end{pmatrix} \tag{42}$$

The matrix $\mathbf{M}$ is evaluated at the fixed points (q_0, p_0). It is symmetric which means that it has two real eigenvalues and two eigenvectors. It can be shown that after diagonalizing this matrix, the Hamiltonian assumes the form

$$\mathcal{H}(q,p) = \frac{1}{2}(\lambda_1 q^2 + \lambda_2 p^2) \tag{43}$$

where λ_1 and λ_2 are the eigenvalues. If the eigenvalues are both negative or both positive (i.e. the determinant is positive) the system undergoes harmonic oscillations (librations) about the fixed point with frequency

$$\omega = \sqrt{\lambda_{12}} \tag{44}$$

and we say the fixed point is a *center*. If one of the eigenvalues is negative and the other positive (the determinant is negative) the system is diverging exponentially away from the fixed point in the direction of one *eigenvector* and heading towards the fixed point along the direction of the other eigenvector. The fixed point is unstable and we call it a *saddle* .

2.2.4 *Canonical transformations*

A *canonical transormation* is a coordinate transformation from a set $(\mathbf{q},\mathbf{p})$ to $(Q(\mathbf{q},\mathbf{p}), P(\mathbf{q},\mathbf{p})$ which preserves the form of Hamilton's equations. It can be shown that canonical transformations satisfy the Poisson brackets

$$\{P_i, P_j\} = 0 \quad \{Q_i, Q_j\} = 0 \quad \{Q_i, P_j\} = \delta_{i,j} \tag{45}$$

where $\delta_{i,j}$ is the *Kronecker delta* symbol and the Poisson bracket is defined as

$$\{f, g\} = \sum_{i=0}^{N} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \tag{46}$$

where the sum goes over all the degrees of freedom. The question now is how to easily construct transformations which are canonical. The answer lies in the form of *generating functions*. Consider a function $F_1(q, Q)$ of the old and new coordinates and let

$$p_i = \frac{\partial F_1}{\partial q_i} \tag{47}$$

After inverting, this equation defines a new coordinate $Q_i = Q_i(q, p)$. One can show that the new momentum is then given by

$$P_i = -\frac{\partial F_1}{\partial Q_i} \tag{48}$$

Thus we have found a way to construct a canonical transformation to new coordinates. We choose the new coordinates Q_i, however, the requirement that the new coordinates form a conjugate pair restricts are freedom to choose the new momentum as well. The function $F_1(q, Q)$ is called the *generating function of first kind* . There are three additional kinds of generating functions, the possibilities are listed in table 1. Which kind of the generating function is the best depends on the

Generating function	Derivatives	
$F_1(q, Q)$	$p_i = \frac{\partial F_1}{\partial Q_i}$	$P_i = \frac{\partial F_1}{\partial q_i}$
$F_2(q, P)$	$p_i = \frac{\partial F_2}{\partial q_i}$	$Q_i = \frac{\partial F_2}{\partial P_i}$
$F_3(p, Q)$	$q_i = -\frac{\partial F_3}{\partial p_i}$	$P_i = -\frac{\partial F_3}{\partial Q_i}$
$F_4(p, P)$	$q_i = -\frac{\partial F_4}{\partial p_i}$	$Q_i = \frac{\partial F_4}{\partial P_i}$

Table 1: Different kinds of generating functions.

problem at hand.

2.3 THE PENDULUM

A simple single degree of freedom Hamiltonian useful for the study of resonance is that of the pendulum. The Hamiltonian has the form

$$\mathcal{H}(\phi, p) = \frac{1}{2}p^2 - \omega_0^2 \cos\phi \tag{49}$$

Where ω_0 is the frequency of oscillations. By solving Hamilton's equations, we obtain

$$\dot{\phi} = p \tag{50}$$

$$\dot{p} = -\omega_0^2 \sin\phi \tag{51}$$

Combining the two equations, we obtain a single equation of motion

$$\ddot{\phi} + \omega_0^2 \sin\phi = 0 \tag{52}$$

We see that in the limit of small ϕ the motion is equivalent to that of the harmonic oscillator oscillating with the frequency ω_0. The fixed points are located at $(\phi, p) = (\pm k\pi, 0)$ (where k is an interger) and there is a single stable center point located at $(\phi,p) = (\pi, 0)$. It is sufficient to study the two fixed points $(0, 0)$ and $(\pi, 0)$ since the motion is periodic. For the fixed point at $(0, 0)$, we have

$$\mathbf{M} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{53}$$

and we see that this point is a stable center. For the other point at $(\pi, 0)$, we have

$$\mathbf{M} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{54}$$

The point is a an unstable saddle point. Figure 5 shows the level

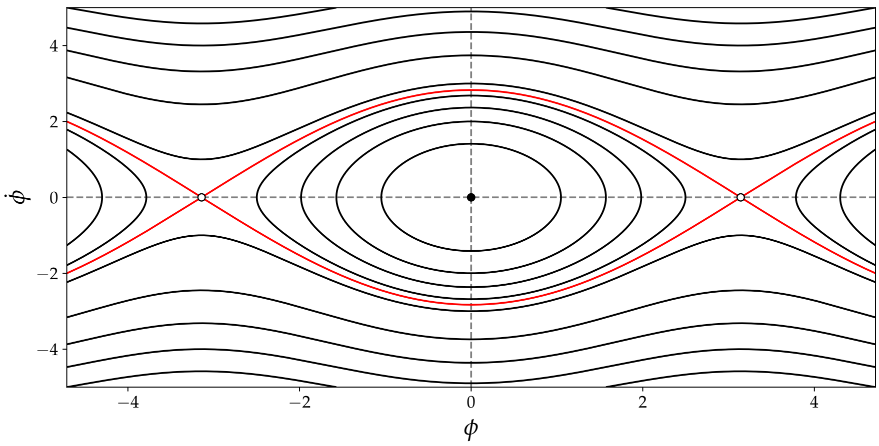

Figure 5: The phase space of a pendulum. There red curve denotes the separatrix filled circles denote stable fixed points, open circles denote unstable fixed points.

curves (curves of constant $\mathcal{H}$) of the pendulum. Given a specific value of the energy, the system stays on one of the curves for all time. The motion around the fixed point $(0, 0)$ in between the fixed points at $-\pi$ and π is said to be *libratory*. There is also an entirely different type of motion where ϕ is unbounded, these trajectories are called *circulatory*. The curve which passes through the unstable points separates the two regimes and is called the *separatrix*. As seen from the figure[2],

2 This can also be shown rigorously by deriving the expression for ω_{lib} as a function of maximum ϕ.

the oscillation period increases from the initial small amplitude value of $2\pi/\omega_0$ to infinity as the separatrix is approached. Once the separatrix is crossed the motion is unbounded. This steep dependance of the librational period on the distance to the separatrix is responsible for chaos in weakly interacting non-linear systems. The concepts presented in this section will be important for the study of resonance in the three–body problem.

2.4 THE THREE–BODY PROBLEM

2.4.1 *The disturbing function*

Finally, we move to the problem of three gravitationally interacting massive bodies suchs as the circumbinary system consisting of two stars and an outer planet. The three–body problem is famously not integrable, many great mathematicians such as Newton and Poincaré have tried and failed to find and exact solution. However, it is still possible to do a perturbative analysis in the case when one of the bodies is only weakly interacting with the other two. The following discussion largely follows Mardling (2013).

A stable hierarchical system of three bodies naturally divides into two orbits composed of an "inner binary" and an "outer binary". We will work in *Jacobi coordinates* which are defined such that in a hierarchical system consisting of N bodies, the Nth body is defined in a coordinate system whose origin is the centre of mass of the previous $N-1$ bodies. Figure 6 shows a system of three masses m_1, m_2 and m_3 in Jacobi coordinates. The position vector $\mathbf{r}$ points from m_1 to m_2 and it is the same vector as in our previous analysis of the two–body problem. The position vector $\mathbf{R}$ points from the centre of mass of the inner binary consisting of m_1 and m_2 to the outer mass m_3. Thus, at any given moment we can define two Keplerian orbits, one for the inner binary and one for the outer binary. Since the three–body problem is not integrable the orbits will in general no longer be fixed and will change their orbital elements with time. This choice of coordinates obviously fails in the case of crossing orbits since the hiararchy loses its meaning, however, a system with crossing orbits is inherently unstable and not the subject of our interests.

When the system is stable the inner and outer orbits interact only weakly by means of an interacting potential called the *disturbing function*. The disturbing function can be written as an infinite Fourier series of angles called the *resonance angles*, it is responsible for the exchange of energy and angular momentum between the two orbits. Each resonance angle is a linear superposition of all angles in the system. A resonance angles can either circulate or librate in exactly the same way as in the pendulum model described in the previous section. If a particular resonance angle is librating, we say that the

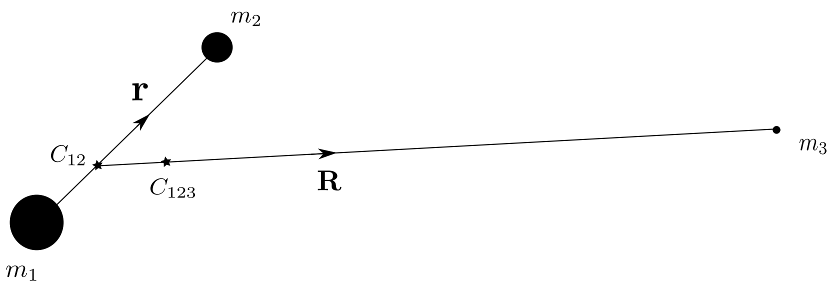

Figure 6: A system of three massive bodies in Jacobi coordinates. The point C_{12} denotes the centre of mass of the inner two bodies and the point C_{123} that of the whole system.

system is in *resonance*. The various resonance angles can mutually interact, under certain conditions a system can exisist in two neighbouring resonant states where two resonant angles librate at a similar period. This is known as *resonance overlap* and it leads to chaotic behaviour as the resonance angle approaches the unstable fixed points (Instability and Oscillator, 1969). The resonance overlap criterion is then a good indication of wheather the system is stable or not.

We can write the equations of motions as

$$\begin{aligned} m_1\ddot{\mathbf{r}}_1 &= \frac{Gm_1m_2}{r_{12}^2}\hat{\mathbf{r}}_{12} + \frac{Gm_1m_3}{r_{13}^2}\hat{\mathbf{r}}_{13} \\ m_2\ddot{\mathbf{r}}_2 &= -\frac{Gm_1m_2}{r_{12}^2}\hat{\mathbf{r}}_{12} + \frac{Gm_2m_3}{r_{23}^2}\hat{\mathbf{r}}_{23} \\ m_3\ddot{\mathbf{r}}_3 &= -\frac{Gm_1m_3}{r_{13}^2}\hat{\mathbf{r}}_{13} - \frac{Gm_2m_3}{r_{23}^2}\hat{\mathbf{r}}_{23} \end{aligned} \tag{55}$$

where the position vectors r_i point from the centre of mass of system C_{123} to the mass m_i and $\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$. Equation (55) constitues a system with 9 degrees of freedom. Again we have the 7 conserved quantities due to momentum and energy conservation, however, there is no analogue of the Runge-Lenz vector in the three–body problem. Therefore, the system is not completely integrable, in fact, it admits *chaotic* solutions – solutions which are extremely sensitive to small variations in the inital conditions. There are two kinds of stability. One is called *Lagrange stability* and the other *Hill instability*. The latter happens due to close approaches of two bodies, the former does not require close approaches. In this chapter we are primarily interested in Lagrange instability since scattering events are difficult to handle analytically and only become important after the onset of Lagrange instability in circumbinary systems, because the circumbinary planet is initally far away from the inner stellar binary.

We start by rewriting eq. (55) in Jacobi coordinates and define the vectors $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ and $\mathbf{R} = (m_{123}/m_{12})\mathbf{r}_3$ where $\mathbf{r}$ is the same relative position vector from the two–body problem and $\mathbf{R}$ points from the centre of mass C_{12} to m_3. We can now rewrite eq. (55) as

$$\mu_i \ddot{\mathbf{r}} + \frac{Gm_1m_2}{r^2}\hat{\mathbf{r}} = \frac{\partial \mathcal{R}}{\partial \mathbf{r}} \tag{56}$$

$$\mu_o \ddot{\mathbf{R}} + \frac{Gm_{12}m_3}{R^2}\hat{\mathbf{R}} = \frac{\partial \mathcal{R}}{\partial \mathbf{R}} \tag{57}$$

where $R = |\mathbf{R}|$, $\mu_i = m_1m_2/m_{12}$ and $\mu_o = m_{12}m_3/m_{123}$ and

$$\mathcal{R} = -\frac{Gm_{12}m_3}{R} + \frac{Gm_2m_3}{|\mathbf{R} - \beta_1\mathbf{r}|} + \frac{Gm_1m_3}{|\mathbf{R} + \beta_2\mathbf{r}|} \tag{58}$$

is the disturbing function with $\beta_i = m_i/m_{12}$, $i = 1, 2$. We use the subscripts i and o to denote quantities defined with respoct to the inner and outer orbits respectively. The notation $\partial/\partial\mathbf{r}$ refers to the gradient with respect to the spherical polar coordinates (r, θ_1, ϕ_1) associated with the position od body 2 relative to the centre of mass C_{12}, similarly, $\partial/\partial\mathbf{R}$ for the position of body 3 with coordinates (R, θ_o, ϕ_o) relative to the same origin. All information about the mutual interaction of the inner and outer orbits is contained in $\mathcal{R}$[3]. In the limit when the inner two masses coalesce ($r/R \to 0$) or the mass ratio between the planet mass m_3 and the total binary mass m_{12} goes to zero ($m_3/m12 \to 0$) $\mathcal{R}$ vanishes and the two orbits no longer interact with each other. The total energy (or the Hamiltonian) is given by

$$\mathcal{H} = \mathcal{H}_i + \mathcal{H}_o - \mathcal{R} \tag{59}$$

where

$$\mathcal{H}_i = \frac{1}{2}\mu_i\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} - \frac{Gm_1m_2}{r}, \quad \mathcal{H}_o = \frac{1}{2}\mu_o\dot{\mathbf{R}}\cdot\dot{\mathbf{R}} - \frac{Gm_{12}m_3}{R} \tag{60}$$

are the Keplerian Hamiltonians corresponding to the inner and outer orbits and the disturbing function has a role of interaction energy. The two Keplerian Hamiltonians individually are fully integrable as mentioned previously, the complete Hamiltonian is not.

Given eq. (59), one can calculate the Hamilton's equations of motion. Those are equivalent to eq. (57), except that they are first order in time. If those equations are are then expressed in terms of the inner and outer orbital elements, they are called *Lagrange's planetary equations*. The derivation is presented in ex. Brouwer and Clemence, 1961, here we merely state the result for coplanar systems

$$\begin{aligned}
\dot{e} &= -\frac{s(1-s)}{\mu n a^2 e}\frac{\partial \mathcal{R}}{\partial \lambda} - \frac{s}{\mu n a^2 e}\frac{\partial \mathcal{R}}{\partial \varpi} \\
\dot{\varpi} &= \frac{s}{\mu n a^2 e}\frac{\partial \mathcal{R}}{\partial e} \\
\dot{\epsilon} &= -\frac{2}{\mu n a}\frac{\partial \mathcal{R}}{\partial a} + \frac{s(1-s)}{\mu n a^2 e}\frac{\partial \mathcal{R}}{\partial e}
\end{aligned} \tag{61}$$

3 Historically, most versions of the expanded disturbing function have units of energy per unit mass. The expansion presented in RM2013 has units of energy

where the subscripts of the orbital elements are either i or o for the inner and outer orbits respectively. $s = \sqrt{1-e^2}$ and ϵ is the mean longitude at $t = t_0$ and is related to the mean longitude by $\int_0^t n dt$ (see. Mardling, 2013, for details). There is no unique general solution for eq. (61), best one can do is to expand $\mathcal{R}$ in a series and consider a few dominant terms in certain cases.

We start by expanding the last two terms in eq. (58) using a well known[4] expansion in terms of *spherical harmonics*.

$$\frac{1}{|\mathbf{R}-\beta_s \mathbf{r}|} = \frac{1}{R}\sum_{l=0}^{\infty}\frac{4\pi}{2l+1}\left(\frac{\beta_s r}{R}\right)^l \sum_{m=-l}^{l} Y_{lm}(\theta_i,\phi_i)Y^*_{lm}(\theta_o,\phi_o) \quad (62)$$

Y_{lm} is a spherical harmonic[5] of *degree l* and *order m* with Y^*_{lm} being its complex conjugate. Using eq. (62) the disturbing function $\mathcal{R}$ becomes

$$\mathcal{R} = G\mu_i m_3 \sum_{l=2}^{\infty}\sum_{m=-l}^{l}\left(\frac{4\pi}{2l+1}\right)\mathcal{M}_l\left(\frac{r^l}{R^{l+1}}\right)Y_{lm}(\theta_i,\phi_i)Y^*_{lm}(\theta_o,\phi_o) \quad (63)$$

where $\mathcal{M}_l$ is a mass factor given by

$$\mathcal{M}_l = \frac{m_1^{l-1}+(-1)^l m_2^{l-1}}{m_{12}^{l-1}} \quad (64)$$

Equation (63) is a spherical harmonic expansion with coefficients proportional to $(r/R)^l$. In order for this series to converge, we require that r/R is a small number. This is satisfied for all circumbinary planets because as will be shown later, there is an inner instability region outside of the stellar binary and the only stable orbits exist further outside the stellar binary's orbit. A spherical harmonic is defined by

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\mathcal{P}_l^m(\cos\theta)e^{im\phi} \quad (65)$$

where $\mathcal{P}_l^m(\theta,\phi)$ is the *associated Legendre function* (Jackson, 2007).

Next, we focus on coplanar systems (mutual inclination between orbits is assumed to be zero) and thus we take $\theta_i = \theta_o = \pi/2$ (the motion is restricted to the $x-y$ plane), $\phi_i = f_i + \varpi_i$ and $\phi_o = f_o + \varpi_o$ where f_i and f_o are the true anomalies of the outer and inner orbits respectively. We obtain

$$\mathcal{R} = g\mu_i m_3 \sum_{l=2}^{\infty}\sum_{m=-l,2}^{l}\frac{1}{2}c^2_{lm}\mathcal{M}_l e^{im(\varpi_i-\varpi_o)}\left(r^l e^{imf_i}\right)\left(\frac{e^{-imf_o}}{R^{l+1}}\right) \quad (66)$$

4 Expressions involving a difference between two vectors such as $1/|\mathbf{r}-\mathbf{r}'|$ occur in all kinds of problems in physics.

5 Spherical harmonics are special functions which form a complete set of orthogonal functions on the sphere, any function defined in terms of spherical polar coordinates can be expanded in an infinite series of spherical harmonics as $f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_l^m Y_l^m(\theta,\phi)$. The series converges if the coefficients f_m^l decay in l sufficiently rapidly.

where

$$c_{lm}^2 = \frac{8\pi}{2l+1}\left[Y_{lm}(\pi/2,0)\right] = c_{l-m}^2 \tag{67}$$

and the notation $m = -l, 2$ in the summation over m means that the summation is taken in steps of 2. The first few values for c_{lm} can be found in Mardling, 2013. For stable systems the expressions in the last two brackets of eq. (66) are nearly periodic and can therefore be expanded in a *Fourier series* of the inner and outer mean anomalies $M_i = n_i t + M_i(0)$ and $M_o = n_o t + M_o(0)$, where n_i, n_o are the mean motions associated with the inner and outer orbits and $M_i(0), M_o(0)$ are their values at $t = 0$. The result is

$$r^l e^{imf_i} \overset{\mathcal{F}}{=} a_i^l \sum_{n=-\infty}^{\infty} X_n^{l,m}(e_i) e^{inM_i} \tag{68}$$

and

$$\frac{r^{-imf_o}}{R^{l+1}} \overset{\mathcal{F}}{=} a_o^{-(l+1)} \sum_{n'=-\infty}^{\infty} X_{n'}^{-(l+1),m}(e_o) e^{-n'M_o} \tag{69}$$

where the $\mathcal{F}$ above the equality sign denotes Fourier expansion and the *Fourier coefficients* are given by

$$\begin{aligned} X_n^{l,m}(e_i) &= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a_i}\right)^l e^{imf_i} e^{inM_i} dM_i \\ &= \frac{1}{2\pi}\int_0^{2\pi} r^{l+1} e^{imf_i} e^{-inM_i} dE_i = \mathcal{O}(e_i^{|m-n|}) \end{aligned} \tag{70}$$

and

$$\begin{aligned} X_{n'}^{-(l+1),m}(e_o) &= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{R}{a_o}\right)^{-(l+1)} e^{-imf_o} e^{in'M_o} dM_o \\ &= \frac{1}{2\pi}\int_0^{2\pi} R^{-l} e^{-imf_o} e^{in'M_o} dE_o = \mathcal{O}(e_o^{|m-n'|}) \end{aligned} \tag{71}$$

are called the *Hansen coefficients* and the notion $\mathcal{O}()$ refers to the order of the leading terms. Plugging eq. (68) and eq. (69) into eq. (66), we obtain (Mardling, 2013)

$$\mathcal{R} = \sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty} \mathcal{R}_{mnn'} \cos\phi_{mnn'} \tag{72}$$

where

$$\phi_{mnn'} = nM_i - n'M_o + m(\varpi_i - \varpi_o) \tag{73}$$

is the *harmonic angle*,

$$\mathcal{R}_{mnn'} = \frac{G\mu_i m_3}{a_o}\sum_{l=l_{\min},2}^{\infty} \zeta_m c_{lm}^2 \mathcal{M}_l \alpha^l X_n^{l,m}(e_i) X_n^{-(l+1),m\prime}(e_o) \tag{74}$$

is the *harmonic coefficient* associated with the harmonic angle $\phi_{mnn'}$, $\alpha = a_i/a_o$ is the semi-major axis ratio, and the factor ζ_m is defined as

$$\zeta_m = \begin{cases} 1/2 & m = 0 \\ 1 & \text{otherwise} \end{cases} \tag{75}$$

The summation over l stars at

$$l_{min} = \begin{cases} 2 & m = 0 \\ 3 & m = 1 \\ m & m \geqslant 2 \end{cases} \tag{76}$$

The series contains three independent indices associated with the each harmonic coefficient since there are three independent frequencies in the problem. The indices n and n' are associated with the two mean motions and the index m is associated with the change in the relative orientation of the orbits called the *rate of apsidial advance* $\dot{\varpi}_i - \dot{\varpi}_o$. The terms corresponding to $l = 2$ are called *quadropole* terms, those with $l = 3$ are *octopole* terms and so on. The harmonic angle can also be written in terms of mean longitudes $\lambda_i = M_i + \varpi_i$ and $\lambda_o = M_o + \varpi_o$ as

$$\phi_{mnn'} = n\lambda_i - n'\lambda_o + (m-n)\varpi_i - (m-n')\varpi_o \tag{77}$$

The harmonic angle should be invariant to the rotation of the coordinate axes, since such a rotation changes all longitude angles by the same amount, their coefficients should add up to zero. This property is called the *d'Alembert relation* (Murray and Dermott, 1999). A short inspection of eq. (77) shows that this is indeed satisfied for all possible coefficients.

At this point, it is useful to review what has been done. We have started with the equations of motion for the three–body problem and expressed them in terms of the Keplerian motion of masses m_1 and m_2 (the inner orbit) , the motion of mass m_3 around the centre of mass of the inner two masses (the outer orbit), and an interaction term between the two orbits called the disturbing function. The disturbing function can be written as an infinite series of spherical harmonics whose coefficients depend on r/R. We then impose the condition of coplanar orbits and rewrite the spherical harmonics in terms of exponentials containing the three angles in the problem. Finally we expand the terms with the exponentials in an infinite Fourier series. We end up with an expression for $\mathcal{R}$ which is a triple infinite Fourier cosine series whose coefficients contain another infinite series in α, the semi-major axis ratio, which reflects the original expansion in r/R. The dependence on the eccentricity is contained only in the Hansen coefficients, which can *in principle* be calculated exactly. The question now is what is the significance of the various terms in eq. (72).

2.4.2 *Secular dynamics*

There are two different kinds of harmonic angles, those which include the mean longitudes which necessarily vary on an orbital timescale, and those which don't include the mean longitudes but contain only the angles which vary on a slower timescale, such as $\varpi_i - \varpi_o$ and the inclination angles in non-coplanar systems. As we will be shown in **??**, in many cases one can ignore the disturbing function terms involving the fast-varying mean longitudes and avarage them over the orbital period, this is due to the fact that for such terms ϕ is slowly varying and therefore $\cos\phi$ becomes significant. . In practice, the averaging[6] is achieved by simply retaining only the terms with $n = n' = 0$ in eq. (72). The resulting disturbing function is called the *secular* [7] distrubing function. It is given by

$$\tilde{\mathcal{R}} = \sum_{m=0}^{\infty} \tilde{\mathcal{R}}_m \cos[m(\varpi_i - \varpi_o)] \tag{78}$$

where

$$\tilde{\mathcal{R}}_m = \frac{G\mu_i m_3}{a_o} \sum_{l=l_{\min},2}^{\infty} \zeta_m c_{lm}^2 \mathcal{M}_l \alpha^l X_0^{l,m}(e_i) X_0^{-(l+1),m}(e_o) \tag{79}$$

Closed-form expressions exist for Hansen coefficients when $n = n' = 0$. Up the octopole order, the secular disturbing function becomes (Mardling, 2013)

$$\begin{aligned}\tilde{\mathcal{R}} = \frac{G\mu_i m_3}{a_o} \Bigg[&\frac{1}{4}\left(\frac{a_i}{a_o}\right)^2 \frac{1+\frac{3}{2}e_i^2}{(1-e_o^2)^{3/2}} \\ &- \frac{15}{16}\left(\frac{a_i}{a_o}\right)^3 \left(\frac{m_1-m_2}{m_{12}}\right) \frac{e_i e_o (1+\frac{3}{4}e_i^2)}{(1-e_o^2)^{5/2}} \cos(\varpi_i - \varpi_o) \Bigg]\end{aligned} \tag{80}$$

It turns out that a purely secular coplanar three–body problem involving only terms up to octopole order is fully integrable (ex. Murray and Dermott, 1999) and hence it does not admit chaotic solutions.

One can show (Murray and Dermott, 1999) that by using the Lagrange planetary equations together with eq. (80) expanded to second order in eccentricity, it is possible to obtain a unique solution. The solution is best expressed in the following coordinates

$$h_i = e_i \sin\varpi_i \quad k_i = e_i \cos\varpi_i \tag{81}$$

$$h_o = e_o \sin\varpi_i \quad k_o = e_o \cos\varpi_o \tag{82}$$

6 One can show (Murray and Dermott, 1999) that the averaging over the mean longitudes is equivalent to considering the dynamics of rings made up by spreading the orbiting masses around their orbits. The procedure is called *Gauss's averaging method*, is shows us that secular interactions between planets are equivalent to interactions between massive rings because the specific location of given mass along its orbit doesn't matter on long timescales.

7 The word secular comes from Latin *seculum* meaning century, or long period.

and the Lagrange planetary equations reduce to form

$$\begin{pmatrix} \dot{h}_i \\ \dot{h}_o \end{pmatrix} = \mathbf{A} \begin{pmatrix} k_i \\ k_o \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \dot{k}_i \\ \dot{k}_o \end{pmatrix} = -\mathbf{A} \begin{pmatrix} h_i \\ h_o \end{pmatrix} \tag{83}$$

where the matrix $\mathbf{A}$ contains constant factors. This system can be solved as an eigenvalue problem, the solutions are

$$\begin{aligned} h_i &= \sum_{l=1}^{2} e_{il} \sin(g_l t + \beta_l) \quad & k_i &= \sum_{l=1}^{2} e_{il} \cos(g_l t + \beta_l) \\ h_o &= \sum_{l=1}^{2} e_{ol} \sin(g_l t + \beta_l) \quad & k_o &= \sum_{l=1}^{2} e_{ol} \cos(g_l t + \beta_l) \end{aligned} \tag{84}$$

where the frequencies g are the eigenvalues of the matrix $\mathbf{A}$ with e_{ol} and e_{il} the components of two corresponding eigenvectors. These are determined from the inital conditions together with the phases β_l. Equation (84) is known as the *Laplace-Lagrange secular solution*. As expected, the solution does not depend on the mean longitudes since we are neglecting them in the secular disturbing function. The solution gives the variation of the eccentricities and pericentres[8]of the two orbits as a function of time and it implies that the orbits are stable for all time within the limits of the secular approximation (terms with mean longitudes can be avaraged out) and the eccentricity expansion to second order. The Laplace-Lagrange holds not only in the three body case but also in the N-body case and it successfuly reproduces most aspects of the secular dynamics of the Solar System.

Free and Forced elements

In section 2.4.2 we have shown that it is possible to find an exact solution for the evolution of slowly-varying orbital elements in the three–body problem. We can use this solution to study the dynamics of an outer massless particle perturbed by the other three bodies. Let the orbital elements of the massless particle be $(a', \lambda', e', I' = 0, \varpi', \Omega' = 0)$. Through a derivation similar to similar to that described in section 2.4.2 (see ch. 7 sec. 4 of Murray and Dermott, 1999) we obtain the following solution for h' and k'

$$h' = e_{\text{free}} \sin(At + \beta) + h_0(t) \quad k' = e_{\text{free}} \cos(At + \beta) + k_0(t) \tag{85}$$

8 In fact, the Laplace-Lagrange secular solution is also valid in the case of inclined system in which there are two additional coordinates $p = I \sin \Omega$ and $q = I \cos \Omega$.

where e_{free}, β and A are constants determined by the initial conditions and the functions $h_0(t)$ and $k_0(t)$ are given by

$$h_0(t) = -\sum_{l=1}^{2} \frac{\nu_l}{A - g_l} \sin(g_l t + \beta_l) \tag{86}$$

$$k_0(t) = -\sum_{l=1}^{2} \frac{\nu_l}{A - g_l} \cos(g_l t + \beta_l) \tag{87}$$

whereν_l, g_l and β_l are again constants depending on the initial conditions, such as the mass ratios and (fixed) semi-major axes. The solution is best described by plotting eq. (85) in the plane $(e' \cos \varpi', e' \sin \varpi')$. Figure 7 shows the geometrical interpretation for the motion of the

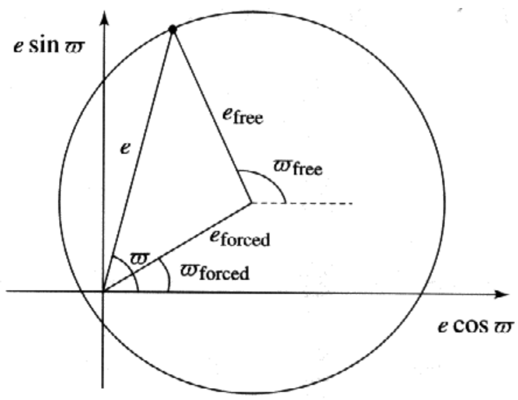

Figure 7: The geometric relationship between the free and forced eccentricities and longitudes. Figure taken from Murray and Dermott, 1999.

test particle. The point in the plane represents a certain (h', k') value, which defines a vector pointing from the origin to the point with magnitude e' and it makes an angle ϖ' with the x axis. This vector can be thought of as a vector sum of two vectors. One points from the origin to the point (h_0, k_0) with magnitude e_{forced} called the *forced eccentricity* at an angle ϖ_{forced} called the *forced longitude of pericentre*, the other pointing from (h_0, k_0) to the point (k, h) with magnitude e_{free} called the *free eccentricity* at an angle $\varpi_{free} = At + \beta$ called the *free longitude of pericentre*.

Thus, the particle's motion can be thought of as a motion around a circle centered at (h_0, k_0) at rate constant rate A where the point (h_0, k_0) itself moves in a complicated path determined by the Laplace-Lagrange solution for the three massive bodies.

2.4.3 *Resonant dynamics*

Consider the harmonic angle defined in eq. (77), $\cos\phi_{mnn'}$ becomes large when $\phi_{mnn'}$ is a slowly varying angle, in other words, $\dot{\phi}_{mnn'} \approx 0$. Differentiating eq. (77) with respect to time, we have

$$\dot{\phi}_{mnn'} = n n_i - n' n_o + (m-n)\dot{\varpi}_i - (m-n')\dot{\varpi}_o \approx 0 \tag{88}$$

where $n_i = \dot{\lambda}_i$ and $n_o = \dot{\lambda}_o$ are the two mean motions. We can rewrite eq. (88) as

$$\frac{n_i + (\frac{m}{n} - 1)\dot{\varpi}_i}{n_o + (\frac{m}{n'} - 1)\dot{\varpi}_o} = \frac{n'}{n} \tag{89}$$

Since the rates of pericentre precession $\dot{\varpi}_i$ and $\dot{\varpi}_o$ are small compared to the mean motions, we require approximately

$$\frac{n_i}{n_o} = \frac{n'}{n} \tag{90}$$

Thus, the harmonic angle is slowly varying if there is an interger ratio (since n and n' are interger coefficients in a Fourier series) between the mean motions of the inner and outer orbits, or equivalently, there is an interger ratio between the periods since $n = 2\pi/P$. If this requirement is satisfied, we say that the mean motions are *commensurate*. If eq. (89) is satisfied we say that the system is in an $n' : n$ *mean motion resonance*. The difference $n' - n$ is called the *resonance order*. The effect of the precession of pericentres is to shift the location of a mean motion resonance (MMR for short) from an exact mean motion commensurability.

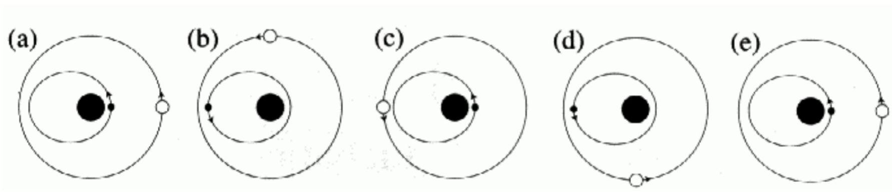

Figure 8: The relative positions of Jupiter (white circle) and an asteroid (small filled circle) in a $2:1$ mean motion resonance. Figure taken from Murray and Dermott, 1999.

An MMR is best understood by considering the geometry of the orbits. Figure 8 shows an example of a $2:1$ mean motion resonance of Jupiter with an inner asteroid. If both bodies start at $_i = \varpi_o = 0$ in *conjuction*[9] (panel (a) in fig. 8) and we neglect the pericentre precession, they will experience a conjuction again once the asteroid has gone twice around its orbit (panel (d) in fig. 8). Conjunctions happen independently of wheather or not there is a resonant relationship

9 A conjuction is defined by $\lambda_i = \lambda_o$

between the mean motion, the crucial point is that in the case of an MMR, the conjuctions happen *at the same position on the orbit*. Another aspect of MMRs is visible in fig. 8, if one of the bodies starts at apocentre and the other at pericentre the conjuctions – points of closest approach always happen when the bodies are as far away from each other as possible. Thus, the effect of the 2 : 1 MMR in this case is to prevent close encounters and the resonance acts as a stablizing agent. Conversely, if bodies started at pericentre the conjuctions would happen at a point where the two orbits are closest to each other and the resonance would act to destabilize the system via close encounters between the bodies. We see that capture into MMR can be either beneficial or destructive for a planetary system, the exact outcome depending on the geometry of the orbits at the point of resonant capture.

In the case when the pericentre precession cannot be neglected the picture is somewhat more complicated, the conjuctions occur at the almost same relative position in the two orbits, but not necessarily also at the same position in inertial space.

A consequence of eqs. (70) and (71) defining the Hansen coefficients is that the combined power exponent of the inner and outer eccentricites grows with the resonance order. The eccentricity dependance for a given term in the disturbing function is of the form $\mathcal{O}(e_i^{|m-n|}e_o^{|m-n'|})$. The combined exponent is then

$$|m-n|+|m-n'| = \begin{cases} n'-n, & n \leqslant m \leqslant n' \\ |2m-n-n'|, & \text{otherwise} \end{cases} \tag{91}$$

The term in the distrubing function expansion corresponding to a specific $n' : n$ MMR with the lowest combined order in eccentricity is the one with $n \leqslant m \leqslant n'$ and its combined exponent is equal to the resonance order. Thus, we have the result that the higher the resonance order of a particular MMR, the 'stronger' is the corresponding dominant term in the disturbing function expansion. Mardling, 2013 refers to the resonant terms which satisfy as the *principal resonances* or *principal harmonics* of an $n' : n$ MMR. For example, for the 2 : 1 MMR there are two principal resonances with harmonic angles $\phi_{112} = \lambda_i - 2\lambda_o + \varpi_o$ and $\phi_{212} = \lambda_i - 2\lambda_o + \varpi_i$.

From eq. (74) it follows that $\mathcal{R}_{mnn'} = \mathcal{O}(\alpha^m)$, $m \geqslant 2$, $\mathcal{R}_{0nn'} = \mathcal{O}(\alpha^2)$ and $\mathcal{R}_{1nn'} = \mathcal{O}(\alpha^3)$. Thus, unless $e_o \ll e_i$, the principal resonance with $m = n$ provides the largest contribution to the disturbing function, unless $n = 1$, in which case the $m = 2$ harmonic makes gives the largest contribution.

2.4.3.1 *Resonance widths*

In general, the harmonic angle $\phi_{mnn'}$ will circulate in the manner similar to that of a pendulum, unless the system is sufficiently close to a period commensurability in which case it can librate. . The harmonic angle can librate even if the system is not at exact resonance because the orbits exchange energy and the period ratio changes slightly after every outer orbit. If $\phi_{mnn'}$ is librating then $\oint \cos\phi_{mnn'} d\phi_{mnn'} \neq 0$ where the integral is taken over one libration cycle. A natural question to ask then is just how close to a period commensurability we have to be in order for a specific harmonic angle to start librating. This is referred to as the *resonance width*i. Just as in the case of a pendulum described in section 2.3, the libration period depends on the distance from the border between the libration region and the circulation region, i.e. the separatrix.

In order to calculate the resonance width, Mardling, 2013 derives a pendulum like differential equation for the harmonic angle $\phi_{mnn'}$. Given a pendulum equation it straightforward to calculate the dimensionless width of the resonance in units of period ratio.

Assuming $\dot{\varpi}_i \ll n_o$ and $\dot{\varpi}_i \ll n_o$ eq. (88) is approximately

$$\dot{\phi}_{mnn'} = nn_i - n'n_o \tag{92}$$

Consider the case of an $N : 1$ resonant term with $m = 2$ (the dominant term for an $N : 1$ principal resonance. We want to derive an equation of the form

$$\dot{\phi}_N = -\omega_N^2 \sin\phi_N \tag{93}$$

where $\phi_N \equiv \phi_N$ is determined by the parameters of the system. Once we know ϕ_N we can determine the libration criterion from the equation of the pendulum separatrix **??**

$$\dot{\phi}_N = \pm 2\omega_N \cos\left(\frac{\phi_N}{2}\right) \tag{94}$$

Libration occurs if $\dot{\phi}_N < 2\omega_N$. We start rewriting eq. (92) as

$$\ddot{\phi}_N = n_o \left(\frac{n_i}{n_o}\frac{\dot{n}_o}{n_i} - N\frac{\dot{n}_o}{n_o}\right) = -\frac{3}{2}n_o \left(\frac{n_i}{n_o}\frac{\dot{a}_i}{a_i} - N\frac{\dot{a}_o}{a_o}\right) \tag{95}$$

where we have used Kepler's third law to replace the mean motions with semi-major axes. We can then use Lagrange's planetary equations together with the dominant disturbing function term for $N : 1$ resonance to calculate $\dot{a}_i$ and $\dot{a}_o$. The result is

$$\begin{aligned} \ddot{\phi}_N &= -\omega_N^2 \sin\phi_N \\ &= \frac{9}{4}n_o^2 \left[\frac{m_3}{m_{123}} + N^{2/3}\left(\frac{m_{12}}{m_{123}}\right)^{2/3}\left(\frac{m_1 m_2}{m_{12}^2}\right)\right] X_1^{2,2}(e_i) X_N^{-3,2}(e_o) \sin\phi_N \end{aligned} \tag{96}$$

The harmonic angle ϕ_N librates around 0 because $X_1^{2,2}(e_i) < 0$ and $X_N^{-3,2}(e_o) > 0$ for all e_i and e_o and thus there is a minus sign in front of ω_N^2. If it weren't for the minus sign, the angle would librate around π (NEED REFERENCE). The libration condition is then

$$\dot{\phi}_N = n_o \left(\frac{n_i}{n_o} - N \right) < 2\omega_N \tag{97}$$

Thus, the distance from resonance N in units of inner mean motion within which the harmonic angle is librating is given by

$$\begin{aligned} \sigma &= \frac{n_i}{n_o} - N = \frac{2\omega_n}{n_o} \\ &= 3 \left[\frac{m_3}{m_{123}} + N^{2/3} \left(\frac{m_{12}}{m_{123}} \right)^{2/3} \left(\frac{m_1 m_2}{m_{12}^2} \right) \right]^{1/2} \sqrt{X_1^{2,2}(e_i) X_N^{-3,2}(e_o)} \end{aligned} \tag{98}$$

where σ is the dimensionless distance from resonance. The Hansen coefficient can be calculated to arbitrary order using a series expansion of the integrals 70 and 71. One can show that$\lim_{e_o \to 0} \sigma = 0$ for $N \geqslant 3$, that is, the widths of high N resonances are only significant if e_0 is not very small. We also have $\lim_{e_i \to 0} \sigma = 0$ which implies that the resonance widths become infinitely narrow for circular inner orbits. However, the situation when $e_i = 0$ is physically unrealistic since some eccentricity is always induced secularly (see section 2.4.2). We will use eq. (98) in **??** together with numerical techniques to predict the stability of circumbinary systems.

So far, we have managed the disentangle the three body problem by by expanding the interaction potential term in a Fourier series and classifying the behaviour of different terms in the expansions. We have defined what it means for a system to be in a mean motion resonance and derived a useful expression which enables us to calculate the location of the regions of instability due to resonance overlap. However, we haven't mentioned how the system got into resonance in the first place.

The main subject of this thesis is what happens during *resonance passage* as n_i/n_o grows due to tidal decay of the stellar binary. In order to describe the physics behind resonant passage, we need to use Hamiltonian mechanics from section 2.2. Ideally, we would like to have a single degree of freedom Hamiltonian which describes the passage of whichever resonance is important for circumbinary planets. In the next section we will show that it is always possible to construct such a Hamiltonian by considering a single dominant resonant term in the disturbing function. Considering more than one terms at the time usually leads to a Hamiltonian with more degrees of freedom.

2.5 THE SMALL DIVISOR PROBLEM

We have mentioned that in order to study resonance capture, we have to reduce the Hamiltonian 59 to a single degree of freedom, which can only be done if we isolate a single resonant term. How do we know that by considering a single dominant term we can still capture the major aspects of the dynamics of resonance passage? Certainly this assumption has to fail if the system is in a region of resonance overlap. We can make this approximation more rigorous in the following way.

Consider a Hamiltonian of the form

$$\mathcal{H}(\mathbf{J},\boldsymbol{\theta}) = \mathcal{H}_0(\mathbf{J}) + \epsilon\cos(\mathbf{k}\cdot\boldsymbol{\theta}) \tag{99}$$

where ϵ is a small parameter, $(\mathbf{J},\boldsymbol{\theta})$ for a conjugate pair of variables where $\mathbf{J}$ is the momentum vector and $\boldsymbol{\theta}$ is the coordinate vector, $\mathbf{k}$ is a vector whose elements are intergers. This Hamiltonian has the form of eq. (59) where we have isolted a paricular term in $\mathcal{R}$. We transform the coordinates by means of a canonical transformation $(\mathbf{J},\boldsymbol{\theta}) \to (\mathbf{I},\boldsymbol{\theta}')$ generated by

$$F_2(\theta,\mathbf{I}) = \mathbf{I}\cdot\boldsymbol{\theta} - \frac{\epsilon}{\mathbf{k}\cdot\boldsymbol{\omega}}\sin\left(\mathbf{k}\cdot\boldsymbol{\theta}\right) \tag{100}$$

Where $\boldsymbol{\omega}$ is just an unknown vector at this point. Table 1 then give the relations between new and old coordinates

$$\frac{\partial F_2}{\partial \mathbf{I}} = \boldsymbol{\theta} = \boldsymbol{\theta}' \tag{101}$$

$$\frac{\partial F_2}{\partial \boldsymbol{\theta}} = \mathbf{I} - \frac{\epsilon\mathbf{k}}{\mathbf{k}\boldsymbol{\omega}}\cos(k\cdot\boldsymbol{\theta}) = \mathbf{J} \tag{102}$$

Inserting the new coordinates in eq. (99), we we have

$$\mathcal{H}(\mathbf{I},\boldsymbol{\theta}') = \mathcal{H}_0\left(\mathbf{I} - \frac{\epsilon\mathbf{k}}{\mathbf{k}\cdot\boldsymbol{\omega}}\cos(k\cdot\boldsymbol{\theta}')\right) + \epsilon\cos(\mathbf{k}\cdot\boldsymbol{\theta}') \tag{103}$$

If we expand $\mathcal{H}_0$ to first order in the small parameter ϵ, we have

$$\mathcal{H}(\mathbf{I},\boldsymbol{\theta}') = \mathcal{H}_0(\mathbf{I}) - \frac{\epsilon\nabla\mathcal{H}_0(\mathbf{I})\cdot\mathbf{k}}{\mathbf{k}\cdot\boldsymbol{\omega}}\cos(\mathbf{k}\cdot\boldsymbol{\theta}') + \epsilon\cos(\mathbf{k}\cdot\boldsymbol{\theta}') \tag{104}$$

If we take $\boldsymbol{\omega}$ to be equal to $\nabla\mathcal{H}_0(\mathbf{I})$ the $\epsilon\cos(\mathbf{k}\cdot\boldsymbol{\theta}')$ term is cancelled. Note that if $\nabla\mathcal{H}_0(\mathbf{I})$, it is a function of $\mathbf{I}$ and we should have taken that into account when taking the derivative of F_2, however, to first order in ϵ, the transformation is correct.

Here we have considered only a cosine term but the procedure is equally valid for any number of terms since their general form is the same. The procedure relies crucially on the last step in which we expand $\mathcal{H}_0$ around $\mathbf{I}$ which is only allowed if $\mathbf{k}\cdot\boldsymbol{\omega}$ is not small (since ϵ is small). If however $\mathbf{k}\cdot\boldsymbol{\omega}$ is small which happens if the perturbation term is commensurate, the expansion fails because the new momenta are not close to the old ones. This is known as the *small* *divisor problem*. Thus, any attempt to remove a commensurate term in the disturbing function by means of a canonical transformation fails. Any non-commensurate term however, can be consistently removed.

Therefore, as long a single resonant term in the distrubing function dominates, and all the other terms are non-commensurate, we can consider only a single term in Hamiltonian 59. It remains to show that such a Hamiltonian can then be reduced to a single degree of freedom.

2.6 REDUCTION TO A SINGLE DEGREE OF FREEDOM

We start with the Hamiltonian

3 RESONANT PASSAGE OF THE 6:1 MEAN MOTION RESONANCE

After reviewing the theory required for studying resonant capture in the framework of a three–body problem with arbitrary mass ratios, we turn to the construction of a one dimensional Hamiltonian for a particular mean motion resonance relevant to circumbinary planets. The question is, which resonances are relevant?

Looking at the circumbinary planets orbiting main–sequence stars, we see that all of them are located outside of the 5 : 1 MMR with the stellar binary. This is due to the fact that the inner regions in period space are mostly unstable because of resonance overlap as will be shown in **??**. Thus, as the stellar binary evolves and the period ratio $P_o/P_i = n_i/n_o$ grows, the first major resonance that will be encountered is the 6 : 1 resonance, which is 5th order. More distant resonances such as 7 : 1 and 8 : 1 might be important as well, however, since resonance 'strength' drops off with resonance order, their effects will be less important.

3.1 THE 6:1 MEAN MOTION RESONANCE

A 6 : 1 MMR is defined by the labels $n' = 6$, $n = 1$ in the disturbing function expansion in eq. (72). Table 2 lists all of the harmonic angles associated with the principal harmonics of a 6 : 1 MMR together with the leading order of the harmonic coefficient in the semi-major axis ratio α and the inner and outer eccentricities. As expected, the sum of the exponent powers of e_i and e_o is equal to 5 which is the order of the resonance.

Harmonic angle $\phi_{mnn'}$	Leading order of $\mathcal{R}_{mnn'}$
$\phi_{116} = \lambda_i - 6\lambda_o + 5\varpi_o$	$\mathcal{O}(\alpha^3, e_i^0, e_o^5)$
$\phi_{216} = \lambda_i - 6\lambda_o + \varpi_i + 4\varpi_o$	$\mathcal{O}(\alpha^2, e_i^1, e_o^4)$
$\phi_{316} = \lambda_i - 6\lambda_o + 2\varpi_i + 3\varpi_o$	$\mathcal{O}(\alpha^3, e_i^2, e_o^3)$
$\phi_{416} = \lambda_i - 6\lambda_o + 3\varpi_i + 2\varpi_o$	$\mathcal{O}(\alpha^4, e_i^3, e_o^2)$
$\phi_{516} = \lambda_i - 6\lambda_o + 4\varpi_i + \varpi_o$	$\mathcal{O}(\alpha^5, e_i^4, e_o^1)$
$\phi_{616} = \lambda_i - 6\lambda_o + 5\varpi_i$	$\mathcal{O}(\alpha^6, e_i^5, e_o^5)$

Table 2: Harmonic angles associated with the principal harmonics of a 6 : 1 mean motion resonance (those with $n \leqslant m \leqslant n'$).

In general we would have to consider all of the angles in table 2 because near a $6:1$ commensurability they will all librate and it is not possible to remove them via a canonical transformation because because of the small divisor problem described in section 2.5. There is however a single harmonic angle associated with the dominant term, the one with $m = 2$, its harmonic coefficient is proportional to α^2 while all the other harmonic coefficients are higher order in α. The difference between the dominant term and the next one is of order $\mathcal{O}(\alpha)$ in absolute value. For a $6:1$ resonance, $\alpha = a_i/a_o \approx 6^{-2/3} = 0.3$, discarding the other terms is thus not an ideal approximation but it is the one we have to make in order to get to an integrable Hamiltonian.

After isolating only the $\mathcal{R} = \mathcal{R}_{216}\cos\phi_{216}$ term in the disturbing function, the Hamiltonian 59 becomes

$$\mathcal{H} = \mathcal{H}_k + \mathcal{R} \tag{105}$$

where

$$\mathcal{H}_k = \mathcal{H}_i + \mathcal{H}_o = -G\frac{m_1 m_2}{a_i} - G\frac{m_{12} m_3}{a_o} \tag{105}$$

is the Keplerian part of the Hamiltonian which depends only on the semi-major axes and

$$\mathcal{R} = \frac{3}{4}\frac{G\mu_i m_3}{a_o}\left(\frac{a_i}{a_o}\right)^2 X_1^{2,2}(e_i)\, X_6^{-3,2}(e_o)\cos(\lambda_i - 6\lambda_o + \varpi_i + 4\varpi_o) \tag{105}$$

is the single resonant distrubing function term. Hamiltonian 3.1 is written in terms of the orbital elements (λ, a, e, ϖ) which do not form a canonically conjugate set of variables. Proceed to rewrite the Hamiltonian in so called *Poincaré variables* which are often used in Celestial mechanics and do form a canonically conjugate set of variables. The Poincaré variables are defined in terms of the orbital elements as

$$\begin{aligned} \lambda_i &= \lambda_i & \Lambda_i &= \mu_i\sqrt{G m_{12} a_i} \\ \gamma_i &= -\omega_i & \Gamma_i &= \mu_i\sqrt{G m_{12} a_i}\left(1 - \sqrt{1 - e_i^2}\right) \end{aligned} \tag{105}$$

and similary for the outer orbit with $m_{12} \to m_{123}$ and the index o for the orbital elements. Remembering eq. (26) for the angular momentum of a Keplerian orbit, we see that the Poincaré Lambda-s correspond to the angular momentum for a circular outer and inner orbit respectively. The Gammas are then the differences between two-body angular momenta for a circular and elliptical orbit[1] We can then solve

1 In secular interactions the sum of all Γ elements of the system is called the *angular momentum deficit* (AMD). It is a conserved quantity because the semi-major axes are constant and the total angular momentum is conserved. Laskar, 1997 has shown that the Solar System is AMD unstable in the sense that if say all planets except, say, Venus attained maximum angular momentum (corresponding to a circular orbit), Venus's eccentricity would inrease enough for crossing orbits to occur.

the system 3.1 for the orbital elements in terms of Poincaré variables, the result is

$$
\begin{aligned}
a_i &= \frac{\Lambda_i^2}{G\mu_i^2 m_{12}} \\
e_i &= \frac{1}{\Lambda_i}\sqrt{\Lambda_i^2 - (\Gamma_i - \Lambda_i)^2} \\
\varpi_i &= -\gamma_i
\end{aligned}
\tag{105}
$$

and again similarly for the outer orbital elements. We have taken the positive root of e_i since eccentricity is defined to be positive. The Hamiltonian 3.1 expressed in terms of the new variables is then

$$
\begin{aligned}
\mathcal{H} = &-G^2\frac{\mu_i^3 m_{12}^2}{2\Lambda_i^2} - G^2\frac{\mu_o^3 m_{123}^2}{2\Lambda_o^2} \\
&- \frac{3}{4}G^2\frac{\mu_o^6}{\mu_i^3}\frac{m_{123}^3 m_3}{m_{12}^2}\frac{1}{\Lambda_o^2}\left(\frac{\Lambda_i}{\Lambda_o}\right)^4 X_1^{2,2}(\Lambda_i)\, X_6^{-3,2}(\Lambda_o)\cos(\lambda_i - 6\lambda_o - \gamma_i - 4\gamma_o)
\end{aligned}
\tag{105}
$$

We can simplify the Hamiltonian 3.1 by changing to dimensionless units. This is achieved by scaling all masses, lengths and time by a constant factor, as follows

$$
\hat{m} = \frac{m}{m'} \quad \hat{a} = \frac{a}{a'} \quad \hat{t} = \frac{t}{t'} \tag{105}
$$

where m stands for any quantity with the dimension of mass in section 3.1 and a stands for any quantity with the dimension of length. Time is not present explicitely in section 3.1. The hats denote the fact that the new variables are dimensionless. Plugging in the rescaled variables in the Hamiltonian (and taking out G, which has dimensions, from the definition of the Poincaré momenta) we obtain a Hamiltonian of the form

$$
\mathcal{H} = \frac{Gm'^2}{a'}\hat{\mathcal{H}} \tag{105}
$$

where $\hat{\mathcal{H}}$ is now dimensionless and exactly the same as section 3.1 except with G factored out and all variables with a physical dimension given hats. The factor Gm'^2/a' has dimensions of energy (as it should) and we can multiply the Hamiltonian $\mathcal{H}$ by its inverse to obtain a dimensionless Hamiltonian. Multiplying any Hamiltonian by a constant factor is equivalent to rescaling the time by that same factor[2]. We then choose the scaling factors which give the simplest Hamiltonian, a natural choice is $m' = m_{12}$ as a unit of mass, $\tilde{a}_i$ (the reason for the use of $willbecomeclearinthenextparagraph)asaunitofsemi-$

2 This follows from Hamilton's equations. Since $\frac{\partial}{\partial p}(a\mathcal{H}) = \frac{dq}{dt}$ where a is some constant, it follows that $\frac{\partial \mathcal{H}}{\partial p} = \frac{dq}{d(at)}$.

$majoraxis, and 1/n_i$ as a unit of time. Thus we can set everywhere $m_{12} = \tilde{a}_i = \tilde{a}_o = 1$. The dimensionless Hamiltonian is then given by

$$\begin{aligned}\mathcal{H} = &-\frac{\mu_i^3}{2\Lambda_i^2} - \frac{\mu_o^3}{2\Lambda_o^2} \\ &-\frac{3}{4}\frac{\mu_o^6}{\mu_i^3}\frac{m_3}{\Lambda_o^2}\left(\frac{\Lambda_i}{\Lambda_o}\right)^4 X_1^{2,2}(\Gamma_i)\, X_6^{-3,2}(\Gamma_o)\cos(\lambda_i - 6\lambda_o - \gamma_i - 4\gamma_o)\end{aligned} \tag{105}$$

where we have omitted all the hats for clarity and we have used the approximation $m_{123} \approx m_{12}$, since the most massive planets we will be considering are $m_3 \approx 10^{-3} m_{12}$ which is negligible.

In order to reduce the Hamiltonian 3.1, to a form which resembles that of the pendulum, we have to expand it in a Taylor series around a location of an exact resonance. Since we are interested in the dynamics in the vicinity of the resonance, this is a valid expansion. We choose to expand the Keplerian part to *second order* and the resonant term to *zeroth order* around $\Lambda_i = \tilde{\Lambda}_i$ and $\Lambda_o = \tilde{\Lambda}_o$, where 'tilde' denotes Poincaré momenta evaluated at exact resonance[3]

$$\begin{aligned}\tilde{\Lambda}_i &= m_1 m_2 \sqrt{\tilde{a}_i/\tilde{a}_i} = m_1 m_2 = \mu_i \\ \tilde{\Lambda}_o &= \mu_o \sqrt{\tilde{a}_o/\tilde{a}_i} = 6^{1/3}\mu_o\end{aligned} \tag{105}$$

where we have used Kepler's third law to evaluate the outer semi-major axis at the location of 6 : 1 MMR. The Hamiltonian 3.1 becomes

$$\begin{aligned}\mathcal{H} = &\frac{\mu_i^3}{2\tilde{\Lambda}_i^3}(\Lambda_i - \tilde{\Lambda}_i) - \frac{3}{2}\frac{\mu_i^3}{\tilde{\Lambda}_i^4}(\Lambda_i - \tilde{\Lambda}_i)^2 + \frac{\mu_o^3}{2\tilde{\Lambda}_o^3}(\Lambda_o - \tilde{\Lambda}_o) - \frac{3}{2}\frac{\mu_o^3}{\tilde{\Lambda}_o^4}(\Lambda_o - \tilde{\Lambda}_o)^2 \\ &-\frac{3}{4}\frac{\mu_o^6}{\mu_i^3}\frac{m_3}{\tilde{\Lambda}_o^2}\left(\frac{\tilde{\Lambda}_i}{\tilde{\Lambda}_o}\right)^4 X_1^{2,2}(\Gamma_i)\, X_6^{-3,2}(\Gamma_o)\cos(\lambda_i - 6\lambda_o - \gamma_i - 4\gamma_o)\end{aligned} \tag{105}$$

where we have ignored constant terms because Hamilton's equations are invariant to an addition of a constant to the Hamiltonian. We then define new momenta $J_i = \Lambda_i - \tilde{\Lambda}_i$ and $J_o = \Lambda_o - \tilde{\Lambda}_o$ which are shifted from Λ_i and λ_o by a constant (it is easy to check that the canonical form is preserved).

Finally, we turn to the Hansen coefficients which we have kept in symbolic form so far. There are no closed-form solutions for the integrals eqs. (70) and (71) which define the coefficients. It is however possible to obtain a series solution in eccentricities using a computer

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3 There is a subtlety here concerning the definition of Γ_i and Γ_o which is worth pointing out. After expanding the resonant term about the resonance location to zeroth order in $\tilde{\Lambda}_i$ and $\tilde{\Lambda}_o$, we also choose to neglect the variation of a_i and a_o in Γ_i and Γ_o because it is negligible in the Keplerian term. We thus have $\Gamma_i = \tilde{\Lambda}_i\left(1 - \sqrt{1 - e_i^2}\right)$ and $\Gamma_o = \tilde{\Lambda}_o\left(1 - \sqrt{1 - e_o^2}\right)$.

algebra system. Mardling, 2013 provides `Wolfram Mathematica` (*Mathematica, Version 11.1*) code for a series solution in eccentricities. The integral for the inner Hansen coefficient is given by eq. (70) as

$$X_1^{2,2}(e_i) = \frac{1}{2\pi}\int_0^{2\pi} (1 - e_i \cos E_i)^{l+1} \left[\cos(mf_i) + i\sin(mf_i)\right] \\ \left[\cos(nM_i) - i\sin(nM_i)\right] dE_i \tag{105}$$

where we have used the relation $(r/a_i) = 1 - e_i \cos E_i$ and wrote the exponentials in complex number form by using the Euler's identity. We do the same with $X_6^{-3,2}(e_o)$. Next, we replace the trigonometric terms and the mean anomaly by using eq. (22) and **??**. Finally, we expand the integrand in the inner coefficient in a Taylor series around e_i, and similarly around e_o for the outer coefficient.

We can only use the lowest order approximation for the Hansen coefficinets accurate to order $\mathcal{O}(e_i^3, e_o^6)$, otherwise the Hamiltonian becomes too complicated later on. In order to establish the domain of validity of this approximation, we plot the Hansen coefficients as functions of eccentricity. fig. 9 shows that the first order approximation is valid up to about $e_i \leqslant 3$ for the inner eccentricity, and $e_o \leqslant 0.4$ for the outer. This is a fairly limiting assumption, however, the vast majority of the observe circumbinary systems are comfortably in the low eccentricity regime so it remains justified. Finally, after convert-

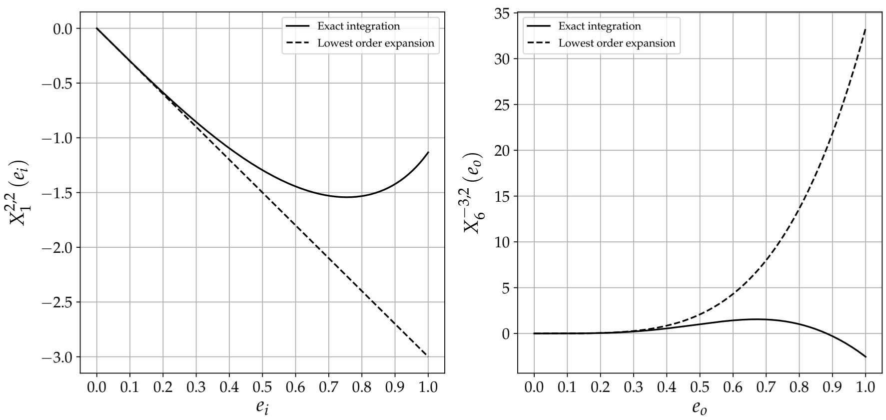


Figure 9: Lowest order expansion term for the Hansen coefficients (dashed curves) compared to exact integration.

ing the series approximation to Poincaré momenta, the Hamiltonian 3.1 becomes

$$\mathcal{H} = \frac{\mu_i^3}{2\tilde{\Lambda}_i^3} J_i - \frac{3}{2}\frac{\mu_i^3}{\tilde{\Lambda}_i^4} J_i^2 + \frac{\mu_o^3}{2\tilde{\Lambda}_o^3} + J_o - \frac{3}{2}\frac{\mu_o^3}{\tilde{\Lambda}_o^4} J_o^2 \\ + \frac{4797\sqrt{2}}{16} m_3 \frac{\mu_o^6}{\mu_i^3}\frac{\tilde{\Lambda}_i^{\frac{7}{2}}}{\tilde{\Lambda}_o^8} \sqrt{\Gamma_i}\Gamma_o^2 \cos(\lambda_i - 6\lambda_o + -\gamma_i - 4\gamma_o) \tag{105}$$

3.2 REDUCTION TO A SINGLE DEGREE OF FREEDOM

The Hamiltonian 3.1 has four degrees of freedom. We would like to find a canonical transformation which reduces the number of degrees of freedom to one. It is known from one of the first Hamiltonian models of resonance (Henrard and Lemaitre, 1983) that a suitable canonical transformation has the harmonic angle as a position coordinate. Inspired by this fact, we choose a canonical transformation to coordinates $(\theta_1, \theta_2, \theta_3, \theta_4; \Theta_1, \Theta_2, \Theta_3, \Theta_4)$ generated by

$$F_2 = -(\lambda_i - 6\lambda_o - \gamma_i - 4\gamma_o)\Theta_1 + \lambda_i\Theta_2 + \lambda_o\Theta_3 + \gamma_i\Theta_4 \qquad (105)$$

From table 1, it follows that

$$\begin{aligned} J_1 &= \frac{\partial F_2}{\partial \lambda_i} = -\Theta_1 + \Theta_2 & \theta_1 &= \frac{\partial F_2}{\partial \Theta_1} = -(\lambda_i - 6\lambda_o - \gamma_i - 4\gamma_o) \\ J_2 &= \frac{\partial F_2}{\partial \lambda_o} = 6\Theta_1 + \Theta_3 & \theta_2 &= \frac{\partial F_2}{\partial \Theta_2} = \lambda_i \\ \Gamma_i &= \frac{\partial F_2}{\partial \gamma_i} = \Theta_1 + \Theta_4 & \theta_3 &= \frac{\partial F_2}{\partial \Theta_3} = \lambda_o \\ \Gamma_o &= \frac{\partial F_2}{\partial \gamma_o} = 4\Theta_1 & \theta_4 &= \frac{\partial F_2}{\partial \Theta_4} = \gamma_i \end{aligned} \qquad (105)$$

We can then easily solve for the new momenta in terms of old momenta

$$\begin{aligned} \Theta_1 &= \frac{1}{4}\Gamma_o \\ \Theta_2 &= J_1 + \frac{1}{4}\Gamma_o \\ \Theta_3 &= J_2 - \frac{3}{2}\Gamma_o \\ \Theta_4 &= \Gamma_i - \frac{1}{4}\Gamma_o \end{aligned} \qquad (105)$$

and the Hamiltonian expressed in terms of the new coordinates is

$$\begin{aligned} \mathcal{H} = &\left(-\frac{3 \cdot 6^{2/3}}{2m_3} - \frac{3}{2\mu_i}\right)\Theta^2 + \left(\frac{3\Theta_2}{\mu_i} - \sqrt[3]{\frac{9}{2}}\frac{\Theta_3}{m_3}\right)\Theta \\ &+ \frac{533 \cdot 2^{5/6}\sqrt[3]{3}}{24}\frac{\sqrt{\mu_i}}{m_3}\Theta^2\sqrt{\Theta + \Theta_4}\cos(\theta) \end{aligned} \qquad (105)$$

Where we have taken $\theta_1 \equiv \theta$ and $\Theta_1 \equiv \Theta$. The resulting Hamiltonian depends only on one coordinate θ with Θ its momentum conjugate and is therefore a fully integrable single degree of freedom Hamiltonian. From Hamilton's equations, it follows that $\dot{\Theta}_2 = \dot{\Theta}_3 = \dot{\Theta}_4 = 0$, that is, Θ_1, Θ_2 and Θ_3 are constants of motion.

The Hamiltonian 3.2 depends on many parameters which are constants, we wish to reduce the number of parameters to a smallest set of linearly independent parameters. We proceed by rewriting section 3.2 as

$$\mathcal{H} = \alpha\Theta^2 + \beta\Theta + \epsilon\Theta^2\sqrt{\Theta+\Theta_4}\cos\theta \tag{105}$$

where

$$\begin{aligned}
\alpha &= -\frac{3\cdot 6^{2/3}}{2m_3} - \frac{3}{2\mu_i} \\
\beta &= \frac{3\Theta_2}{\mu_i} - \sqrt[3]{\frac{9}{2}\frac{\Theta_3}{m_3}} \\
\epsilon &= \frac{533\cdot 2^{5/6}\sqrt[3]{3}}{24}\frac{\sqrt{\mu_i}}{m_3}
\end{aligned} \tag{105}$$

α and ϵ depend purely on the mass ratios $\mu_i = m_1 m_2$ and m_3, the ϵ parameter depends on Θ_2 and Θ_3 which in turn depend on the distance to the resonance. In order to further reduce the number of parameters, we scale the momentum Θ by means of a simple scale transformation $\Theta \to \eta\Theta$, where η is a constant factor to be determined; remembering that we also have to scale time by the same factor . Hamiltonian 3.2 becomes

$$\mathcal{H} = \eta^2\alpha\Theta^2 + \eta\beta\Theta + \eta^{5/2}\epsilon\Theta^2\sqrt{\Theta + \frac{\Theta_4}{\eta}}\cos\theta) \tag{105}$$

We then choose the scaling parameter η such that the coefficients in front of the first and the last term become equal, that is, we require that

$$\eta^2\alpha = \eta^{5/2}\epsilon \tag{105}$$

It follows that

$$\eta = \left(\frac{\alpha}{\epsilon}\right)^2 \tag{105}$$

and the Hamiltonian is

$$\mathcal{H} = \frac{\alpha^5}{\epsilon^4}\Theta^2 + \beta\frac{\alpha^2}{\epsilon^2}\Theta + \frac{\alpha^5}{\epsilon^4}\Theta^2\sqrt{\Theta + \frac{\Theta_4}{\eta}}\cos\theta \tag{105}$$

We can now multiply the hamiltonian by the dimensionless factor $\frac{\epsilon^4}{\alpha^5}$ which corresponds to rescaling the time again. The final Hamiltonian then has the form

$$\mathcal{H} = \Theta^2 - \delta\Theta + \Theta^2\sqrt{\Theta + c}\cos\theta \tag{105}$$

where the two constants δ and c are given by

$$\delta = -\frac{\beta\epsilon^2}{\alpha^3} \tag{105}$$

$$c = \left(\frac{\epsilon}{\alpha}\right)^2\Theta_4 = \left(\frac{\epsilon}{\alpha}\right)^2\left(\Gamma_i - \frac{1}{4}\Gamma_o\right) \tag{105}$$

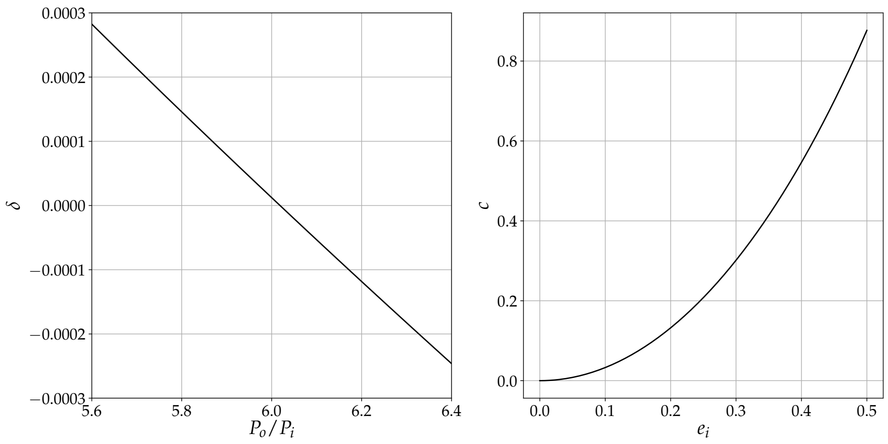


Figure 10: (a) Hamiltonian parameter δ on the period ratio $\mu_i = 0.5$, $m_3 = 10^{-3}$, $e_i = 0.2$, $e_o = 0.05$. (b)Hamiltonian parameter c as a function of the inner eccentricity for $\mu_i = 0.5$, $m_3 = 10^{-3}$, $e_o = 0.05$.

We have thus managed to reduce the $6:1$ resonant Hamiltonian to the simplest possible form, a one degree of freedom Hamiltonian with two parameters. We now turn to studying its structure. In order to gain insight into the dependance of the two parameters δ and c, we plot them in fig. 10. We see that δ is a measure of proximity to the exact commensurability since it reaches zero at $P_o/P_i \approx 6$. In the case of an evolving circumbinary system δ monotonically decreases from a positive value to a negative value or equivalently, a changing period ratio can be seen as varying δ. Thus, we can model the passage through the resonance through δ.

The c parameter by definition stays constant as the period ratio varies because the Poincaré momenta Γ_i and Γ_o in section 3.2 are evaluated at exact resonance. More interesting is the dependence of c on the inner eccentricity, shown in panel (b) of fig. 10. We see that $c > 0$ for all e_i for a particular value of e_o. In fact, the plot shown in (b) stays almost exactly the same for all reasonable values of e_o. Negative values of c occur only for extremely small eccentricities which never really occur because some eccentricity is always induced due to secular interactions. In what follows, we therefore consider only the case $c > 0$.

The fixed points of the Hamiltonian are given by Hamilton's equations as

$$\frac{\partial \mathcal{H}}{\partial \Theta} = \frac{\partial \mathcal{H}}{\partial \theta} = 0 \tag{105}$$

We obtain

$$-\Theta^2 \sqrt{\Theta + c} \sin\theta = 0 \tag{105}$$

$$\frac{\Theta^2}{2\sqrt{\Theta + c}} \cos\theta + 2\Theta\sqrt{\Theta + c} + 2\Theta - \delta = 0 \tag{105}$$

The only non trivial solution for the first equation is $\theta = \{0, \pi\}$ for $\theta \in \{0, 2\pi\}$. The second equation then becomes

$$(-1)^s \frac{\Theta^2}{2\sqrt{\Theta + c}} + 2\Theta\sqrt{\Theta + c}2\Theta - \delta = 0 \tag{105}$$

where $s = \{0, 1\}$. There is no analytic solution for section 3.2, we can gain insight into the possible solution by searching for the roots graphically. We define $R = \sqrt{2\Theta}$ and rewrite section 3.2 as

$$f(\delta, R) = g(R, c) \tag{105}$$

where

$$f(\delta, R) = -\delta + R^2 \tag{105}$$

and

$$g(R, c) = (-1)^s \frac{R^4}{8\sqrt{\frac{1}{2}R^2 + c}} + R^2\sqrt{\frac{1}{2}R^2 + c} \tag{105}$$

Figure 11 shows a plot of $f(\delta, R)$ and $g(R, c)$ for a fixed value of c.

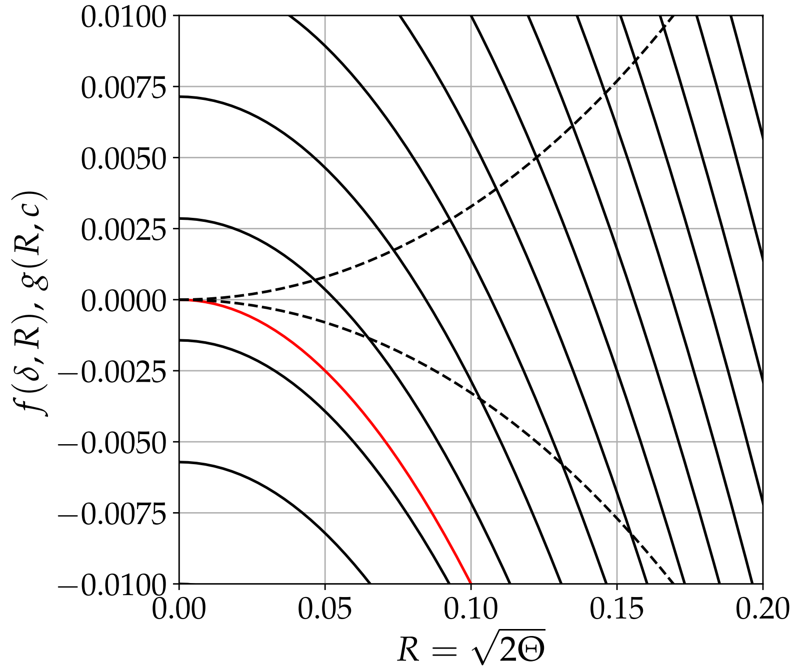


Figure 11: A graphical solution to the equation defining the fixed points of the Hamiltonian 3.2. The solid lines are plots of $f(\delta, R)$ for various values of δ. The dashed lines are plots of the function $g(R, c)$ for a fixed positive value of $c = 0.1$.

Each solid line corresponds to a different value of delta. The intersections of f and g are the roots of section 3.2. There are two solutions for $\delta > 0$ and no solutions for $\delta < 0$. At $\delta = 0$ and $\Phi = 0$ a double root of section 3.2 either appears or disappears depending on wheater zero is approached from above or below. Thus, we conclude that there are two distinct behaviours of the Hamiltonian depending on the sign of δ, $\delta = 0$ is then called a *bifurcation point* of the Hamiltonian 3.2. [4]

The properties of Hamiltonian 3.2 are most easily seen in phase space plots for different values of δ. We plot the Hamiltonian in so called Poincaré Cartesian variables, defined as

$$\begin{aligned} x &= \sqrt{2\Theta}\cos\theta \\ y &= \sqrt{2\Theta}\sin\theta \end{aligned} \tag{105}$$

This would be a standard polar to Cartesian transformation were it not for the square roots. The square roots are necessary if the transformation is to be canonical (Ferraz-Mello, 2007), and the factor of 2 is for convenience. The reason we use these coordinates is because the coordinates (θ, Θ) are singular at the origin $\Theta = 0$ since θ becomes ill defined. The Hamiltonian in the new coordinates is then

$$\mathcal{H} = \frac{1}{4}\left(x^2 + y^2\right) - \frac{1}{2}\delta\left(x^2 + y^2\right) + \frac{1}{8}x\left(x^2 + y^2\right)^{3/2}\sqrt{4c + 2x^2 + 2y^2} \tag{105}$$

Because the coordinates (θ, Θ) are ill-defined at the orgin, when solving section 3.2 for the fixed points, we missed a third fixed point at the origin. It is easy to see by writing down the Hamilton's equations in (x, y) coordinates that the origin $(0, 0)$ is always a fixed point, independent of the value of δ.

4 In the theory of dynamical systems, a bifurcation occurs when a small change in a certain parameter of a system causes a sudden *topological* change in its behaviour.

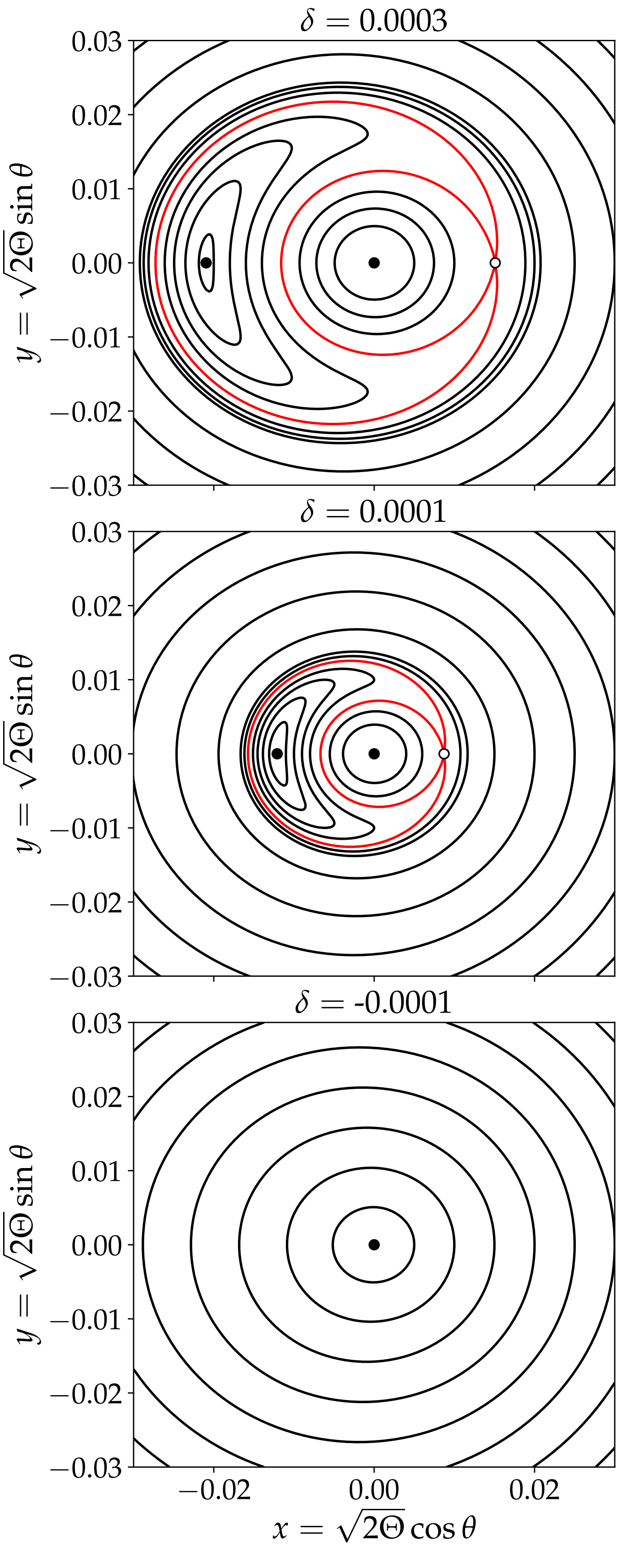
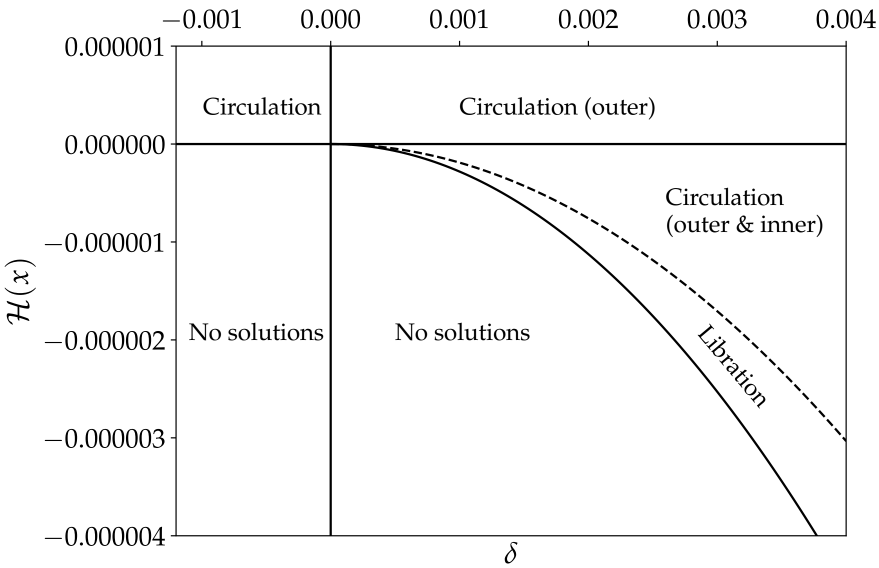


Figure 12: Phase space portraits of the Hamiltonian 3.2 for different values of δ. The red curve is the separatrix. Filled circles are stable fixed points and the open circle is the unstable saddle point.



−0.001
0.000
0.001
0.002
0.003
0.004
0.000001
0.000000
−0.000001
−0.000002
−0.000003
−0.000004
$\mathcal{H}(x)$
δ
Circulation
Circulation (outer)
Circulation
(outer & inner)
No solutions
No solutions
Libration

Figure 13

4 N-BODY SIMULATIONS OF CIRCUMBINARY PLANETS UP TO COMMON ENVELOPE PHASE

While analytic models can give important insight into the physics of divergent resonance passage in a circumbinary planetary system they cannot compare to the full solution of the equations of motion. Here I describe a different approach to the problem, by means of direct N-body simulations coupled with simulations of the stellar binary. These simulations provide a clear picture of the dynamical evolution of the system.

4.1 THE N-BODY PROBLEM

4.2 REBOUND - AN OPEN SOURCE N-BODY INTEGRATOR

- mention integrators, what does order stand for - symplectic vs non-symplectic integrators - describe IAS15 in more detail - mention reproducibility and simulation archive - MEGNO as a criterion for stabilility (not sure this should be here)

4.3 BINARY_C - A BINARY STELLAR EVOLUTION CODE

- describe what it does, mention it's built on BSE - Python interface to the C library

4.4 STABILITY OF OBSERVED CIRCUMBINARY PLANETS ON THE MAIN SEQUENCE

- MEGNO maps of Kepler planets together with resonance bubbles

4.5 STELLAR EVOLUTION TRAJECTORIES

4.6 INTIAL CONDITIONS

5 RESULTS

Here I describe the results of both the analytic approach and the N-body simulations.

5.1 SIMULATIONS WITH A SINGLE PLANET

5.2 SIMULATIONS WITH TWO PLANETS

5.3 PLANETS INITIALLY AT RESONANCE

6 CONCLUSIONS

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DECLARATION

Put your declaration here.

Rijeka, Croatia, July 2017

Fran Bartolić

COLOPHON

This document was typeset using the typographical look-and-feel `classicthesis` developed by André Miede. The style was inspired by Robert Bringhurst's seminal book on typography "*The Elements of Typographic Style*". `classicthesis` is available for both LaTeX and LyX:

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