

# Linear programming

## Background material

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# Linear program

Linear program, standard form:

$$\begin{aligned} \min_x \quad & c^T x \\ \text{s.t.} \quad & Ax = b, \\ & x \geq 0, \end{aligned}$$

with  $x, c \in \mathbb{R}^n$ ,  $b \in \mathbb{R}^m$ ,  $A \in \mathbb{R}^{m \times n}$ .

Assumptions:  $\text{rank}(A) = m$ ,  $m \leq n$ .

## Standard form

Any linear program can be transformed to the standard form.  
Consider for instance the program

$$\min_x c^T x \text{ s.t. } Ax \geq b.$$

Note that there is no bound constraints on  $x$ .

We first add a vector  $z$  of surplus variables:

$$\begin{aligned} \min_x \quad & c^T x \\ \text{s.t.} \quad & Ax - z = b, \\ & z \geq 0. \end{aligned}$$

We next decompose  $x$  as the difference of two non-negative variables:  $x = x^+ - x^-$ , where  $x^+ = \max\{x, 0\} \geq 0$ , and  $x^- = \max\{-x, 0\} \geq 0$ .

## Standard form (cont'd)

The problem becomes

$$\begin{aligned} \min_x \quad & (c \quad -c \quad 0) \begin{pmatrix} x^+ \\ x^- \\ z \end{pmatrix}, \\ \text{s.t.} \quad & (A \quad -A \quad -I) \begin{pmatrix} x^+ \\ x^- \\ z \end{pmatrix} = b, \begin{pmatrix} x^+ \\ x^- \\ z \end{pmatrix} \geq 0. \end{aligned}$$

The inequalities constraints of the form  $x \leq u$  or  $Ax \leq b$  can be handled by adding slack variables:

$$\begin{aligned} x \leq u &\Leftrightarrow x + w = u, \quad w \geq 0, \\ Ax \leq b &\Leftrightarrow Ax + y = b, \quad y \geq 0. \end{aligned}$$

# Basic solutions

Without loss of generality, suppose that the first  $m$  columns are independent, and form

$$\mathbf{A} = (\mathbf{B} \quad \mathbf{D})$$

*Basic solution:*  $\mathbf{x} = (\mathbf{x}_b \ 0)$ , with  $\mathbf{B}\mathbf{x}_b = \mathbf{b}$ .

*Degenerated basic solution:* if some components of  $\mathbf{x}_b$  are equal to zero.

*Feasible basic solution:* basic solution such that  $\mathbf{A}\mathbf{x} = \mathbf{b}$  and  $\mathbf{x} \geq 0$ .

# Fundamental theorem of linear programming

Consider an LP under standard form, with  $\mathbf{A} \in \mathbb{R}^{m \times n}$ , and  $\text{rank}(\mathbf{A}) = m$ .

- If there exists a feasible solution, then there exists a feasible basic solution.
- If there exists a feasible optimal solution, then there exists a feasible optimal basic solution.

# Duality

- **Primal** program

$$\begin{aligned} \min_{x \geq 0} \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \end{aligned}$$

- **Dual** program

$$\begin{aligned} \max_{\lambda \geq 0} \quad & \lambda^T b \\ \text{s.t.} \quad & A^T \lambda \leq c \end{aligned}$$

- $A \in \mathbb{R}^{m \times n}$ ,  $c, x, \in \mathbb{R}^n$ ,  $\lambda, b \in \mathbb{R}^m$ .

# Duality

- $x$ : primal variables
- $\lambda$ : dual variables
- Dual of the dual:

$$\begin{aligned} \min_x \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$



## Duality: standard form

The program

$$\begin{array}{ll}\min_x & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0\end{array}$$

is equivalent to

$$\begin{array}{ll}\min_x & c^T x \\ \text{s.t.} & Ax \geq b \\ & -Ax \geq -b \\ & x \geq 0\end{array}$$

## Duality: standard form

The dual problem can then be written as

$$\begin{aligned} \max_{u,v} \quad & u^T b - v^T b \\ \text{s.t.} \quad & u^T A - v^T A \leq c^T \\ & u \geq 0 \\ & v \geq 0 \end{aligned}$$

or, with  $\lambda = u - v$ ,

$$\begin{aligned} \max_{\lambda} \quad & \lambda^T b \\ \text{s.t.} \quad & \lambda^T A \leq c^T \end{aligned}$$

Asymmetric form  $\lambda \in \mathbb{R}^m$ .

# Primal-dual conversion

Minimization	Maximization
Constraints $\geq$ $\leq$ $=$	Variables $\geq 0$ $\leq 0$ unconstrained
Variables $\geq 0$ $\leq 0$ unconstrained	Constraints $\leq$ $\geq$ $=$

# Weak duality

(Asymmetric or symmetric form)

If  $x$  and  $\lambda$  are feasible for the primal and the dual, respectively, then

$$c^T x \geq \lambda^T b$$

**Proof.**

For any primal feasible  $x$  and dual feasible  $\lambda$ , we have

$$\lambda^T b \leq \lambda^T A x \leq c^T x.$$



## Corollary

If  $x_0$  and  $\lambda_0$  are feasible for the primal and the dual, respectively, and if

$$c^T x_0 = \lambda_0^T b,$$

then  $x_0$  and  $\lambda_0$  are optimal for their respective problem.

But we have said nothing about the feasibility of one problem with respect to the other one!

## Strong duality

If one of the problems, primal or dual, has a finite optimal solution, the other problem also has a finite optimal solution, and the corresponding values of the objective functions are equal. If one of the problems has an unbounded objective, the other problem has no feasible solution.

If a program is feasible, this does not imply that its dual is unbounded. It can also be unfeasible.

Primal / Dual	Bounded	Unbounded	Unfeasible
Bounded	possible	impossible	impossible
Unbounded	impossible	impossible	possible
Unfeasible	impossible	possible	possible

## Optimality et duality

The optimality can be deduced from KKT conditions.

In LP, we do not need a constraint qualification to apply KKT conditions.

Lagrangian:

$$\mathcal{L} = c^T x - \pi^T (Ax - b) - s^T x.$$

KKT conditions: Lagrangian vectors  $\pi$  and  $s$  s.t.

$$A^T \pi + s = c,$$

$$Ax = b,$$

$$x \geq 0,$$

$$s \geq 0,$$

$$x_i s_i = 0, \quad i = 1, 2, \dots, n \quad (\text{complementarity}).$$

## Dual problem

Let  $(x^*, \pi^*, s^*)$  be a vector satisfying the KKT conditions. We have that

$$c^T x^* = (A^T \pi^* + s^*)^T x^* = (Ax^*)^T \pi^* = b^T \pi^*.$$

It is furthermore easy to show that the (necessary) KKT conditions are sufficient.

Dual problem:

$$\max_{\pi} b^T \pi, \text{ s.t. } A^T \pi \leq c,$$

or, equivalently,

$$\max_{\pi} b^T \pi, \text{ s.t. } A^T \pi + s = c, s \geq 0.$$