In the grading of the exercises special attention will be paid to check that answers are substantiated and that the procedure of calculations is well documented. When results are achieved using a calculator or a pc it should be noted in the paper. All 6 problems are weighted equally in the grading. Your solution to the problem set can be written in Danish or English as you prefer.

PROBLEM 1.

Let the matrix A and the vector $\mathbf{b}$ be given by

$$A = \begin{bmatrix} 4 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & q \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

1. Determine the values of q for which the equation $A\mathbf{x} = \mathbf{b}$ is consistent.

Let B, C and D be invertible $n \times n$ matrices.

2. Solve the following three equations for X.

$$(I)\;\; XBCD = I, \qquad (II)\;\; CXB^{-1} = D, \qquad (III)\;\; XB - X = 2D.$$

PROBLEM 2.

Assume it is requested to find the solution to the homogenous matrix equation $A\mathbf{x} = \mathbf{0}$ for some unknown 4×4 matrix. The augmented matrix has been row reduced and the result is

$$A = \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

1. Find the solution of $A\mathbf{x} = \mathbf{0}$.

PROBLEM 3.

Let the matrix A be given as

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

1. Compute the characteristic equation.

The eigenvalues of A are $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$.

2. By hand, calculate the eigenvectors and find orthogonal bases for the eigenspaces.
3. Write the vector $\mathbf{y} = [1\ 1\ 2]^T$ as a linear combination of the eigenvectors for A.

PROBLEM 4.

For the statements given below, state whether they are true or false and justify your answer for each statement.

1. $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.
2. Every $m \times n$ matrix has exactly m pivots.
3. An $n \times n$ matrix with only real elements can have both real and complex eigenvalues.

PROBLEM 5.

Consider the system $\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$ with matrices

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

and let $\mathbf{x}_0 = \mathbf{0}$.

1. Find the controllability matrix for the system and show that the system $\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$, is controllable.
2. Find control vectors $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{u}_2$ that will force the system to $\mathbf{y} = \begin{bmatrix} 66 \\ 56 \\ 41 \end{bmatrix}$.

PROBLEM 6.

Consider the following set of three equations with two unknowns.

$$\begin{aligned} x_1 - 3x_2 &= 2 \\ 2x_1 - x_2 &= -1 \\ x_1 + x_2 &= 0 \end{aligned}$$

1. Justify that the set of equations do not possess a solution.
2. Find a least squares solution of the system using a pseudoinverse.