Chapter 1.1

7. Ordinarily, the next step would be to interchange R3 and R4, to put a 1 in the third row and third column. But in this case, the third row of the augmented matrix corresponds to the equation $0\,x_1 + 0\,x_2 + 0\,x_3 = 1$, or simply, $0 = 1$. A system containing this condition has no solution. Further row operations are unnecessary once an equation such as $0 = 1$ is evident.
The solution set is empty.

11. First, swap R1 and R2. Then replace R3 by R3 + (–3)R1. Finally, replace R3 by R3 + (2)R2.

$$\begin{bmatrix} 0 & 1 & 4 & -5 \\ 1 & 3 & 5 & -2 \\ 3 & 7 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 3 & 7 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & -2 & -8 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

The system is inconsistent, because the last row would require that $0 = 2$ if there were a solution.
The solution set is empty.

13. Replace R2 by R2 + (–2)R1. Then interchange R2 and R3. Next replace R3 by R3 + (–2)R2. Then divide R3 by 5. Finally, replace R1 by R1 + (–2)R3.

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 2 & 15 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 5 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$ The solution is (5, 3, –1).

19. $\begin{bmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & h & 4 \\ 0 & 6-3h & -4 \end{bmatrix}$ Write c for $6 - 3h$. If $c = 0$, that is, if $h = 2$, then the system has no solution, because 0 cannot equal –4. Otherwise, when $h \neq 2$, the system has a solution.

23. **a.** True. See the remarks following the box titled “Elementary Row Operations”.

b. False. A 5×6 matrix has five rows.

c. False. The description given applies to a single solution. The solution *set* consists of all possible solutions. Only in special cases does the solution set consist of exactly one solution. Mark a statement True only if the statement is *always* true.

d. True. See the box before Example 2.

24. **a**. True. See the box preceding the subsection titled "Existence and Uniqueness Questions".

b. False. The definition of *row equivalent* requires that there exist a sequence of row operations that transforms one matrix into the other.

c. False. By definition, an inconsistent system has *no* solution.

d. True. This definition of *equivalent systems* is in the second paragraph after equation (2).

Chapter 1.2

1. Reduced echelon form: **a** and **b**. Echelon form: **d**. Not echelon: **c**.

7. $$\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Corresponding system of equations:
$$\begin{aligned} x_1 + 3x_2 \quad &= -5 \\ x_3 &= 3 \end{aligned}$$

The basic variables (corresponding to the pivot positions) are x_1 and x_3. The remaining variable x_2 is free. Solve for the basic variables in terms of the free variable. The general solution is

$$\begin{cases} x_1 = -5 - 3x_2 \\ x_2 \text{ is free} \\ x_3 = 3 \end{cases}$$

9. $$\begin{bmatrix} 0 & 1 & -6 & 5 \\ 1 & -2 & 7 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 7 & -6 \\ 0 & 1 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 4 \\ 0 & 1 & -6 & 5 \end{bmatrix}$$

Corresponding system:
$$\begin{aligned} x_1 \quad - 5x_3 &= 4 \\ x_2 - 6x_3 &= 5 \end{aligned}$$

Basic variables: x_1, x_2; free variable: x_3. General solution: $\begin{cases} x_1 = 4 + 5x_3 \\ x_2 = 5 + 6x_3 \\ x_3 \text{ is free} \end{cases}$

11. $\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4/3 & 2/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Corresponding system:
$$\begin{aligned} x_1 - \frac{4}{3}x_2 + \frac{2}{3}x_3 &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

Basic variable: x_1; free variables x_2, x_3. General solution: $\begin{cases} x_1 = \frac{4}{3}x_2 - \frac{2}{3}x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$

23. Yes. The system is consistent because with three pivots, there must be a pivot in the third (bottom) row of the coefficient matrix. The reduced echelon form cannot contain a row of the form [0 0 0 0 0 1].

24. The system is inconsistent because the pivot in column 5 means that there is a row of the form [0 0 0 0 1] in the reduced echelon form. Since the matrix is the *augmented* matrix for a system, Theorem 2 shows that the system has no solution.

25. If the coefficient matrix has a pivot position in every row, then there is a pivot position in the bottom row, and there is no room for a pivot in the augmented column. So, the system is consistent, by Theorem 2.

26. Since there are three pivots (one in each row), the augmented matrix must reduce to the form

$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix}$ and so $\begin{aligned} x_1 &= a \\ x_2 &= b \\ x_3 &= c \end{aligned}$

No matter what the values of a, b, and c, the solution exists and is unique.

33. For a quadratic polynomial $p(t) = a_0 + a_1t + a_2t^2$ to exactly fit the data (1, 12), (2, 15), and (3, 16), the coefficients a_0, a_1, a_2 must satisfy the systems of equations given in the text. Row reduce the augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 12 \\ 1 & 2 & 4 & 15 \\ 1 & 3 & 9 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 2 & 8 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 13 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The polynomial is $p(t) = 7 + 6t - t^2$.

Chapter 1.3

1. $\mathbf{u}+\mathbf{v}=\begin{bmatrix}-1\\2\end{bmatrix}+\begin{bmatrix}-3\\-1\end{bmatrix}=\begin{bmatrix}-1+(-3)\\2+(-1)\end{bmatrix}=\begin{bmatrix}-4\\1\end{bmatrix}$.

Using the definitions carefully,

$\mathbf{u}-2\mathbf{v}=\begin{bmatrix}-1\\2\end{bmatrix}+(-2)\begin{bmatrix}-3\\-1\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}+\begin{bmatrix}(-2)(-3)\\(-2)(-1)\end{bmatrix}=\begin{bmatrix}-1+6\\2+2\end{bmatrix}=\begin{bmatrix}5\\4\end{bmatrix}$, or, more quickly

$\mathbf{u}-2\mathbf{v}=\begin{bmatrix}-1\\2\end{bmatrix}-2\begin{bmatrix}-3\\-1\end{bmatrix}=\begin{bmatrix}-1+6\\2+2\end{bmatrix}=\begin{bmatrix}5\\4\end{bmatrix}$. The intermediate step is often not written.

5. $x_1\begin{bmatrix}6\\-1\\5\end{bmatrix}+x_2\begin{bmatrix}-3\\4\\0\end{bmatrix}=\begin{bmatrix}1\\-7\\-5\end{bmatrix}$, $\begin{bmatrix}6x_1\\-x_1\\5x_1\end{bmatrix}+\begin{bmatrix}-3x_2\\4x_2\\0\end{bmatrix}=\begin{bmatrix}1\\-7\\-5\end{bmatrix}$, $\begin{bmatrix}6x_1-3x_2\\-x_1+4x_2\\5x_1\end{bmatrix}=\begin{bmatrix}1\\-7\\-5\end{bmatrix}$

$$\begin{aligned}6x_1 - 3x_2 &= 1\\ -x_1 + 4x_2 &= -7\\ 5x_1 \qquad &= -5\end{aligned}$$

11. The question

Is $\mathbf{b}$ a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$?

is equivalent to the question

Does the vector equation $x_1\mathbf{a}_1+x_2\mathbf{a}_2+x_3\mathbf{a}_3=\mathbf{b}$ have a solution?

The equation

$$\underset{\substack{\uparrow\\ \mathbf{a}_1}}{x_1\begin{bmatrix}1\\-2\\0\end{bmatrix}}+\underset{\substack{\uparrow\\ \mathbf{a}_2}}{x_2\begin{bmatrix}0\\1\\2\end{bmatrix}}+\underset{\substack{\uparrow\\ \mathbf{a}_3}}{x_3\begin{bmatrix}5\\-6\\8\end{bmatrix}}=\underset{\substack{\uparrow\\ \mathbf{b}}}{\begin{bmatrix}2\\-1\\6\end{bmatrix}} \qquad (*)$$

has the same solution set as the linear system whose augmented matrix is

$$M=\begin{bmatrix}1&0&5&2\\-2&1&-6&-1\\0&2&8&6\end{bmatrix}$$

Row reduce M until the pivot positions are visible:

$$M\sim\begin{bmatrix}1&0&5&2\\0&1&4&3\\0&2&8&6\end{bmatrix}\sim\begin{bmatrix}1&0&5&2\\0&1&4&3\\0&0&0&0\end{bmatrix}$$

The linear system corresponding to M *has* a solution, so the vector equation (*) has a solution, and therefore $\mathbf{b}$ *is* a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

17. $[\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{b}] = \begin{bmatrix} 1 & -2 & 4 \\ 4 & -3 & 1 \\ -2 & 7 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & 5 & -15 \\ 0 & 3 & h+8 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 3 & h+8 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & h+17 \end{bmatrix}$. The vector $\mathbf{b}$ is in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$ when $h + 17$ is zero, that is, when $h = -17$.

24. **a**. True. See the beginning of the subsection "Vectors in $\mathbb{R}^n$ ".

b. True. Use Fig. 7 to draw the parallelogram determined by $\mathbf{u} - \mathbf{v}$ and $\mathbf{v}$.

c. False. See the first paragraph of the subsection "Linear Combinations".

d. True. See the statement that refers to Fig. 11.

e. True. See the paragraph following the definition of Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

25. **a**. There are only three vectors in the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, and $\mathbf{b}$ is not one of them.

b. There are infinitely many vectors in $W = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. To determine if $\mathbf{b}$ is in W, use the method of Exercise 13.

$$\begin{array}{c} \begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ -2 & 6 & 3 & -4 \end{bmatrix} \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{array} \end{array} \sim \begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 6 & -5 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

The system for this augmented matrix is consistent, so $\mathbf{b}$ is in W.

c. $\mathbf{a}_1 = 1\mathbf{a}_1 + 0\mathbf{a}_2 + 0\mathbf{a}_3$. See the discussion in the text following the definition of Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.