Lesson 3

Chapter 2
Matrix Algebra

▸ Matrix Operations ▸ The Inverse of a Matrix ▸ Characterizations of Invertible Matrices

Matrix-vector multiplication

Definition

If A is a $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the **product of A and x** denoted by $A\mathbf{x}$ is **the linear combination of the columns of A using the corresponding entries in x as weights**; that is,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n$$

Vector in $\mathbb{R}^m$

Note:

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{bmatrix} \text{ and } A\boldsymbol{x} = \begin{bmatrix} a_{11} \cdot x_1 + a_{12} \cdot x_2 + a_{13} \cdot x_3 + \cdots + a_{1n} \cdot x_n \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 + a_{23} \cdot x_3 + \cdots + a_{2n} \cdot x_n \\ \vdots \\ a_{m1} \cdot x_1 + a_{m2} \cdot x_2 + a_{m3} \cdot x_3 + \cdots + a_{mn} \cdot x_n \end{bmatrix} \in \mathbb{R}^{\boldsymbol{m}}$$

Linear Equations

Solutions to:

- The matrix equation: $A\boldsymbol{x} = \boldsymbol{b}$
- The vector equation: $x_1 \cdot \boldsymbol{a_1} + x_2 \cdot \boldsymbol{a_2} + \cdots + x_n \cdot \boldsymbol{a_n} = \boldsymbol{b}$
- The system of linear equations:

$$\left\{ \begin{matrix} a_{11} \cdot x_1 + a_{12} \cdot x_2 + a_{13} \cdot x_3 + \cdots + a_{1n} \cdot x_n = b_1 \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 + a_{23} \cdot x_3 + \cdots + a_{2n} \cdot x_n = b_2 \\ \vdots \\ a_{m1} \cdot x_1 + a_{m2} \cdot x_2 + a_{m3} \cdot x_3 + \cdots + a_{mn} \cdot x_n = b_m \end{matrix} \right\}$$

- The augmented matrix $[\boldsymbol{a_1} \quad \boldsymbol{a_2} \quad \cdots \quad \boldsymbol{a_n} \quad \boldsymbol{b}]$

are the same!

Linearly dependence/indenpendence

Definition

An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be linearly independent if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is said to be linearly dependent if there exist weights $c_1, c_2, \ldots c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = \mathbf{0}$$

2.1 Matrix Operations

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

Matrix-notation

Column j

Diagonal entries a_{ii}

m x n matrix

$$A = [\boldsymbol{a_1}\ \boldsymbol{a_2} \cdots \boldsymbol{a_n}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} = \{a_{ij}\} \quad \begin{matrix} i = 1,\ldots,m \\ j = 1,\ldots,n \end{matrix}$$

Row i

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Square (n x n) matrix

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Diagonal matrix

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Identity matrix

$$0 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Zero matrix

Ex 1

$$A = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}$$

Sum of matrices: $C = A + B \;\Leftrightarrow\; \{c_{ij}\} = \{a_{ij} + b_{ij}\}$ $\quad i = 1,..,m; \quad j = 1,..,n$

Requirement: A and B (and C) equal size ($m\ x\ n$)

Scale multiplication: $C = r \cdot A \;\Leftrightarrow\; \{c_{ij}\} = \{r \cdot a_{ij}\}$ $\quad i = 1,..,m; \quad j = 1,..,n$

Theorem 2.1: Summation and scalar multiplication properties

A, B and C: All same size; r and s: Scalars

a) $A + B = B + A$

b) $(A + B) + C = A + (B + C)$

c) $A + 0 = A$

d) $r(A + B) = rA + rB$

e) $(r + s)A = rA + sA$

f) $r(sA) = (rs)A$

Ex 2

$$A = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} = [\boldsymbol{a}_1 \;\; \boldsymbol{a}_2] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} = [\boldsymbol{b}_1 \;\; \boldsymbol{b}_2 \;\; \boldsymbol{b}_3] = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$A \cdot B = A \cdot [\boldsymbol{b}_1 \;\; \boldsymbol{b}_2 \;\; \boldsymbol{b}_3] = [A\boldsymbol{b}_1 \;\; A\boldsymbol{b}_2 \;\; A\boldsymbol{b}_3] = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \end{bmatrix}$$

$$A \cdot B = \begin{array}{r} \\ \text{1. row in A} \\ \\ \text{2. row in A} \end{array} \begin{array}{c} \begin{array}{ccc} \text{1. column in B} & \text{2. column in B} & \text{3. column in B} \end{array} \\ \begin{bmatrix} A_{1*}B_{*1} & A_{1*}B_{*2} & A_{1*}B_{*3} \\ \\ A_{2*}B_{*1} & A_{2*}B_{*2} & A_{2*}B_{*3} \end{bmatrix} \end{array} = \{(AB)_{ij} = (row\ i\ in\ A) \cdot (column\ j\ in\ B)\}$$

Ex 2

$$A = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}$$

Definition

Matrix multiplication

OBS:
#columns in A
= #rows in B

If A is an $m \times n$ matrix and B is an $n \times p$ matrix with columns $\mathbf{b}_1, \ldots, \mathbf{b}_p$ then the product AB is the $m \times p$ matrix whose colums are $[A\mathbf{b}_1 \ \ldots \ A\mathbf{b}_p]$. That is,

$$AB = A[\mathbf{b}_1 \ \mathbf{b}_2 \ \ldots \ \mathbf{b}_p] = [A\mathbf{b}_1 \ A\mathbf{b}_2 \ \ldots \ A\mathbf{b}_p].$$

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

(row i in A multiplied on column j in B)

The columns in AB are linear combinations of A's columns with weigths given by the corresponding columns in B.

Ex 3

$$A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$$

Theorem 2.2: Matrix multiplication properties

Let A be an $m \times n$ matrix and let B and C have sizes for which the indicated sums and products are defined.

- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$
- $(B + C)A = BA + CA$
- $r(AB) = (rA)B = A(rB)$
- $I_m A = A = A I_n$

OBS!!! In general:

- $AB \neq BA$ (non-commutating)
- $AB = AC \nRightarrow B = C$ (no cancellation)
- $AB = 0 \nRightarrow A = 0 \ \vee \ B = 0$

Discuss with your neighbour:

- Assume that column 2 in matrix B consist of only zeros. What does that imply for the product AB?

- Assume that the first two columns of the matrix B are identical. What does that imply for the product AB?

- Assume that the third column of matrix A consist of only zeros. What does that imply for the product AB?

- Assume that the third row of matrix A consist of only zeros. What does that imply for the product AB?

Transponeret

Transposed matrix: $A = \{a_{ij}\} \Leftrightarrow A^T = \{a_{ji}\}$ (”mirroring” in the diagonal)

$Row\ i \leftrightarrow Column\ i$

Explain to your neighbour how you transpose a matrix. What is A^T when A is given by

$$A = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 7 & 4 & -3 & 3 \\ 1 & 0 & -1 & 3 \end{bmatrix}$$

Transponeret

Transposed matrix: $A = \{a_{ij}\} \quad \Leftrightarrow \quad A^T = \{a_{ji}\}$ (”mirroring” in the diagonal)

$Row\ i \ \leftrightarrow Column\ i$

Theorem 2.3: Rules for transposing

$$(A^T)^T = A$$
$$(A+B)^T = A^T + B^T$$
$$(rA)^T = rA^T, \quad \forall r \in \mathbb{R}$$
$$(AB)^T = B^T A^T$$

Ex 4

$$A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$$

Ex 3 → $AB = \begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix} \qquad BA = \begin{bmatrix} 10 & 2 \\ 29 & -2 \end{bmatrix}$

2.2 The Inverse of a Matrix

$$A^{-1}A = AA^{-1} = I$$

Ex 5

$$A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \qquad C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$$

Definition:

An $n \times n$ matrix is invertible, if there exist a matrix C with the properties

$$AC = I \quad \text{and } CA = I,$$

where I is the $n \times n$ identity matrix. The matrix C is the inverse matrix of A and is denoted by A^{-1}. Hence

$$AA^{-1} = A^{-1}A = I.$$

If the inverse matrix A^{-1} exist, it is unique.

Theorem 2.6: Rules for inverse matrices:

$$(A^{-1})^{-1} = A$$

$$(AB)^{-1} = B^{-1}A^{-1} \qquad (A \dots YZ)^{-1} = Z^{-1}Y^{-1} \dots A^{-1}$$

$$(A^T)^{-1} = (A^{-1})^T$$

Ex 6

$$2x_1 + 5x_2 = 4$$

$$-3x_1 - 7x_2 = 2$$

Ex 7 $\quad [A \mid I] = \begin{bmatrix} 2 & 5 & 1 & 0 \\ -3 & -7 & 0 & 1 \end{bmatrix}$

2 × 2 matrix: $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Theorem 2.7

A invertible $\Leftrightarrow$ A is row equivalent to I $\Leftrightarrow$ $[\,A \mid I\,]$ is row equivalent to $\left[\,I \mid A^{-1}\right]$

Algoritm for finding A^{-1}:

- Row reduce the augmented matrix $[\,A \mid I\,]$
- If A is row reduced to I, then $[\,A \mid I\,]$ is row equivalent to $\left[\,I \mid A^{-1}\right]$
- Otherwise A does not have an inverse

If A^{-1} does not exist, A is called a **singular matrix**

2.3 Characterizations of Invertible Matrices

The Inverse Matrix Theorem

Theorem 2.8: Let A be a square n x n matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.
b. A is row equivalent to the $n \times n$ identity matrix.
c. A has n pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
h. The columns of A span $\mathbb{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix C such that $CA = I$.
k. There is an $n \times n$ matrix D such that $AD = I$.
l. A^T is an invertible matrix.

Theorem 2.8: Let A be a square $n \ x \ n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.
b. A is row equivalent to the $n \times n$ identity matrix.
c. A has n pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
g. The equation $A\mathbf{x} = \mathbf{b}$ has ~~at least one~~ a unique solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
h. The columns of A span $\mathbb{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix C such that $CA = I$.
k. There is an $n \times n$ matrix D such that $AD = I$.
l. A^T is an invertible matrix.

Todays words and concepts

Square matrix

Singular matrix

Non-commutating

Transposed matrix

Inverse matrix

Matrix size

Matrix multiplicaton

Sum of matrices

Matrix operations

Scalar multiplicaton

Diagonal matrix

No cancellation

Identity matrix

Zero matrix

The Inverse Matrix Theorem I

Diagonal entry