Lesson 9

Chapter 5
Eigenvectors and Eigenvalues

Definition:

An eigenvector of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an eigenvalue of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.

Eigenspaces

- The set of solutions of $(A - \lambda I)\mathbf{x} = \mathbf{0}$ is the null space of $A - \lambda I$
- This is also called the eigenspace of A corresponding to λ
- There is an eigenspace for each eigenvalue λ
- An eigenspace can be multidimensional

Theorem 5.5, The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors of A.

5.5 Complex Eigenvalues

$$\lambda = a + j \cdot b$$

Eigenvalue: $A\boldsymbol{x} = \lambda\boldsymbol{x} \Leftrightarrow (A - \lambda I)\boldsymbol{x} = \mathbf{0}$ have non-tvivial solutions

$n \times n$ matrix

$$\Updownarrow$$

$(A - \lambda I)$ not inversible

$$\Updownarrow$$

The Characteristic Equation $\longrightarrow$ $det(A - \lambda I) = 0$

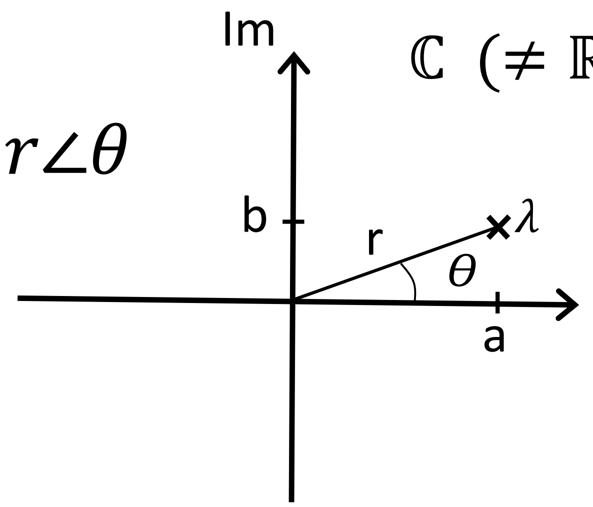
The Characteristic Polynomial $\longrightarrow$ Polynomial of degree n in λ

Exactly n (complex) eigenvalues (roots) λ

$\boldsymbol{x} \in \mathbb{C}^n$ the corresponding (complex) eigenvector

Complex numbers:

$\mathbb{C}\ (\neq \mathbb{R}^2)$



$$\lambda \in \mathbb{C}: \quad \lambda = a + j \cdot b = r \cdot cos(\theta) + j \cdot r \cdot sin(\theta) = r\angle\theta$$

$Re(\lambda) = a = r \cdot cos(\theta) \quad Im(\lambda) = b = r \cdot sin(\theta)$

Modulus: $|\lambda| = r = \sqrt{a^2 + b^2}$

Argument: $\mathrm{Arg}(\lambda) = \theta = arctg\left(\frac{b}{a}\right)\ (\pm\pi)$

Complex vectors:

$$\boldsymbol{v} \in \mathbb{C}^2: \qquad \boldsymbol{v} = \begin{bmatrix} a_1 + j \cdot b_1 \\ a_2 + j \cdot b_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + j \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \underset{\in \mathbb{R}^2}{Re(\boldsymbol{v})} + j \cdot \underset{\in \mathbb{R}^2}{Im(\boldsymbol{v})}$$

Discuss with your neighbour

The general expression for a complex vector is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad x_i \in \mathbb{C}$$

For some particular n, does the set of all n-dimensional complex vectors

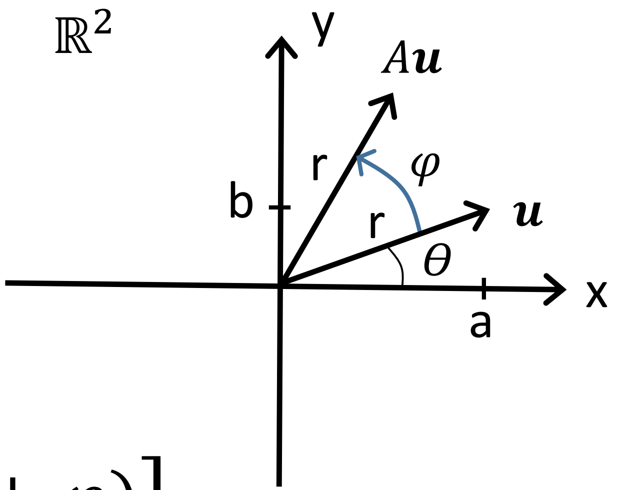
$$\{\mathbf{x} | \mathbf{x} \in \mathbb{C}^n\}$$

form a vector space?

$$A = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix}$$

Rotate angle φ

$$\boldsymbol{u} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r \cdot \cos(\theta) \\ r \cdot \sin(\theta) \end{bmatrix}$$



$$A\boldsymbol{u} = \begin{bmatrix} r \cdot \cos(\theta)\cos(\varphi) - r \cdot \sin(\theta)\sin(\varphi) \\ r \cdot \cos(\theta)\sin(\varphi) + r \cdot \sin(\theta)\cos(\varphi) \end{bmatrix} = \begin{bmatrix} r \cdot \cos(\theta + \varphi) \\ r \cdot \sin(\theta + \varphi) \end{bmatrix}$$

u rotated angle φ

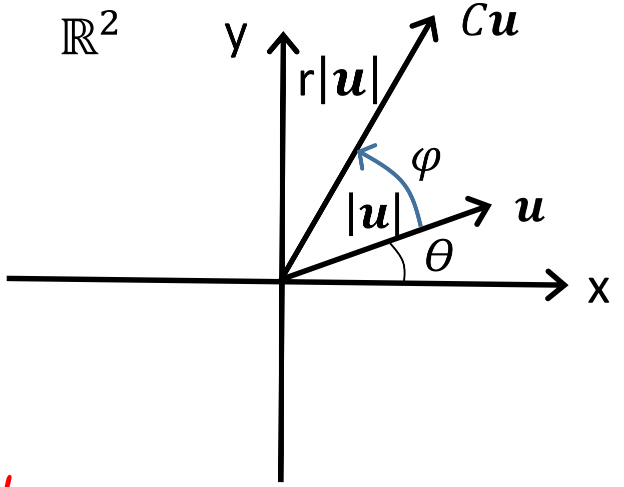
$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r \begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix} \qquad a, b \in \mathbb{R}; \quad (a, b) \neq (0,0)$$

$$r = \sqrt{a^2 + b^2}$$

$$= r \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix}$$

Scaling r

Rotate angle φ



Ex 2

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix} \quad a, b \in \mathbb{R}; \quad b \neq 0$$

$$r = \sqrt{a^2 + b^2}; \quad \varphi = \arctan\left(\frac{b}{a}\right)$$

Eigenvalues: $det(C - \lambda I) = \begin{vmatrix} a-\lambda & -b \\ b & a-\lambda \end{vmatrix} = (a-\lambda)^2 + b^2 = \lambda^2 - 2a\lambda + a^2 + b^2 = 0$

$$\Rightarrow \lambda = \frac{2a \pm \sqrt{4a^2 - 4(a^2 + b^2)}}{2} = a \pm jb \,\epsilon\, \mathbb{C}$$

Eigenvectors: $(C - \lambda I)\boldsymbol{x} = \boldsymbol{0}$

$\lambda_1 = a + jb$: $C - \lambda_1 I = \begin{bmatrix} -jb & -b \\ b & -jb \end{bmatrix} \sim \begin{bmatrix} -jb & -b \\ 0 & 0 \end{bmatrix} \rightarrow -jbx_1 - bx_2 = 0 \Rightarrow \boldsymbol{x} = \begin{bmatrix} x_1 \\ -jx_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -j \end{bmatrix} = x_1 \boldsymbol{v}_1;$

$$\rightarrow \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -j \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - j \begin{bmatrix} 0 \\ 1 \end{bmatrix} \epsilon\, \mathbb{C}^2$$

$\lambda_2 = a - jb$: $C - \lambda_2 I = \begin{bmatrix} jb & -b \\ b & jb \end{bmatrix} \sim \begin{bmatrix} jb & -b \\ 0 & 0 \end{bmatrix} \rightarrow jbx_1 - bx_2 = 0 \Rightarrow \boldsymbol{x} = \begin{bmatrix} x_1 \\ jx_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ j \end{bmatrix} = x_1 \boldsymbol{v}_2;$

$= \lambda_1^*$

$$\rightarrow \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ j \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + j \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \boldsymbol{v}_1^* \,\epsilon\, \mathbb{C}^2$$

__Ex 3__ $A = \begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix}$

Eigenvalues: $det(A - \lambda I) = \begin{vmatrix} 0.5-\lambda & -0.6 \\ 0.75 & 1.1-\lambda \end{vmatrix} = (0.5-\lambda)(1.1-\lambda) + 0.75 \cdot 0.6 = \lambda^2 - 1.6\lambda + 1 = 0$

$$\Rightarrow \lambda = \frac{1.6 \pm \sqrt{1.6^2 - 4}}{2} = 0.8 \pm j0.6 \in \mathbb{C} = \begin{cases} \lambda_1 \\ \lambda_2 = \lambda_1^* \end{cases}$$

Eigenvectors: $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$

$$\lambda_1 = 0.8 + j0.6: \; A - \lambda_1 I = \begin{bmatrix} -0.3 - j0.6 & -0.6 \\ 0.75 & 0.3 - j0.6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0.4 - j0.8 \\ 0 & 0 \end{bmatrix} \rightarrow x_1 + (0.4 - j0.8)x_2 = 0$$

$$\Rightarrow \boldsymbol{x} = x_1 \begin{bmatrix} -2 + 4j \\ 5 \end{bmatrix} = x_1 \boldsymbol{v}_1 \rightarrow \boldsymbol{v}_1 = \begin{bmatrix} -2 + 4j \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} + j \begin{bmatrix} 4 \\ 0 \end{bmatrix} \in \mathbb{C}^2$$

$$\lambda_2 = 0.8 - j0.6: \; A - \lambda_2 I = \begin{bmatrix} -0.3 + j0.6 & -0.6 \\ 0.75 & 0.3 + j0.6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0.4 + j0.8 \\ 0 & 0 \end{bmatrix} \rightarrow x_1 + (0.4 + j0.8)x_2 = 0$$

$$\Rightarrow \boldsymbol{x} = x_1 \begin{bmatrix} -2 - 4j \\ 5 \end{bmatrix} = x_1 \boldsymbol{v}_2 \rightarrow \boldsymbol{v}_2 = \boldsymbol{v}_1^* = \begin{bmatrix} -2 - 4j \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} + j \begin{bmatrix} -4 \\ 0 \end{bmatrix} \in \mathbb{C}^2$$

$A = \{a_{ij}\};\ \ a_{ij} \in \mathbb{R}$:

$$A\boldsymbol{x} = \lambda\boldsymbol{x} \Longrightarrow \overline{A\boldsymbol{x}} = \overline{\lambda\boldsymbol{x}} \Longrightarrow A\overline{\boldsymbol{x}} = \bar{\lambda}\overline{\boldsymbol{x}} \Longrightarrow \begin{cases} \bar{\lambda} = \lambda^*\ eigenvalue \\ \overline{\boldsymbol{x}} = \boldsymbol{x}^* eigenvector \end{cases}$$

For a <u>real</u> matrix:

- Complex eigenvalues and –vectors comes in pairs: $\begin{cases} \lambda = a \pm j \cdot b \\ \boldsymbol{v} = Re(\boldsymbol{v}) \pm j \cdot Im(\boldsymbol{v}) \end{cases}$

Ex 4

$$A = \begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix} \qquad \lambda_1 = 0.8 + j \cdot 0.6 \qquad \boldsymbol{v}_1 = \begin{bmatrix} -2+4j \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} + j \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 0.8 - j \cdot 0.6 \qquad \boldsymbol{v}_2 = \begin{bmatrix} -2-4j \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} + j \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

$$P = [Re(\boldsymbol{v}_1) \;\; Im(\boldsymbol{v}_1)] = \begin{bmatrix} -2 & 4 \\ 5 & 0 \end{bmatrix} \qquad \Longrightarrow \; P^{-1} = \frac{1}{-20} \begin{bmatrix} 0 & -4 \\ -5 & -2 \end{bmatrix}$$

$$C = P^{-1}AP = \frac{1}{-20} \begin{bmatrix} 0 & -4 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 5 & 0 \end{bmatrix}$$

C og A similary

$$= \frac{1}{-20} \begin{bmatrix} 0 & -4 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 4 & 3 \end{bmatrix}$$

$$= \frac{1}{-20} \begin{bmatrix} -16 & -12 \\ 12 & -16 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix} = \begin{bmatrix} Re(\lambda_1) & Im(\lambda_1) \\ -Im(\lambda_1) & Re(\lambda_1) \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Scaling and rotation

Ex 4 fortsat

$$A = \begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix}$$

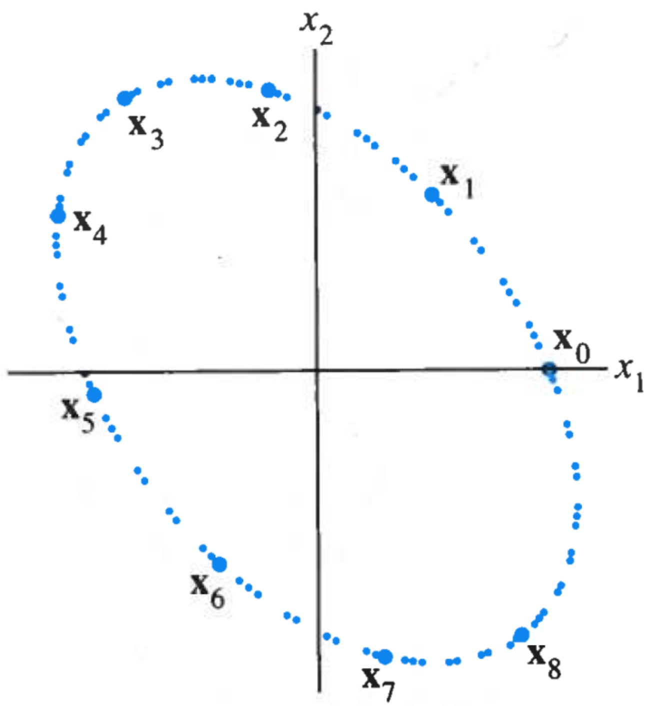
$$\lambda_1 = 0.8 + j \cdot 0.6 \qquad \boldsymbol{v}_1 = \begin{bmatrix} -2+4j \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} + j \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 0.8 - j \cdot 0.6 \qquad \boldsymbol{v}_2 = \begin{bmatrix} -2-4j \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} + j \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

$$P = [Re(\boldsymbol{v}_1) \;\; Im(\boldsymbol{v}_1)] \qquad C = P^{-1}AP = \begin{bmatrix} Re(\lambda_1) & Im(\lambda_1) \\ -Im(\lambda_1) & Re(\lambda_1) \end{bmatrix}$$

$$\Longrightarrow A = PCP^{-1}$$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbf{x} \arrow[r, "A"] \arrow[d, "P^{-1}", "\text{Change of variable}"'] & A\mathbf{x} \\
\mathbf{u} \arrow[r, "C", "\text{Rotation and scaling}"'] & C\mathbf{u} \arrow[u, "P"', "\text{Change of variable}"]
\end{tikzcd}$$



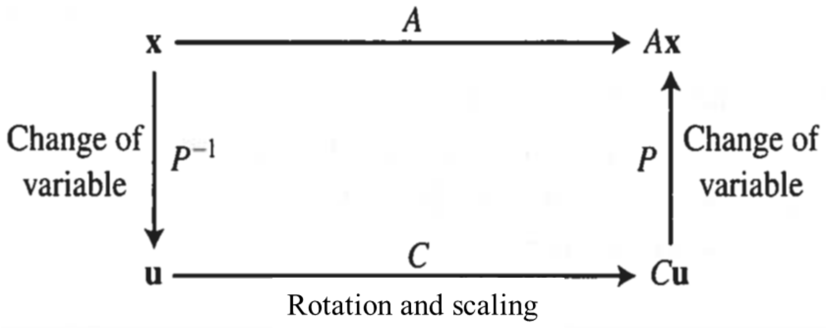
$$\boldsymbol{x}_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}; \;\; \boldsymbol{x}_1 = A\boldsymbol{x}_0; \;\; \boldsymbol{x}_2 = A\boldsymbol{x}_1; \; \cdots ; \;\; \boldsymbol{x}_{n+1} = A\boldsymbol{x}_n; \; \cdots$$

Theorem 5.9

OBS!

Let A be a real 2 x 2 matrix with a complex eigenvalue $\lambda = a - j \cdot b \; (b \neq 0)$ and an associated eigenvector $\boldsymbol{v} = Re(\boldsymbol{v}) + j \cdot Im(\boldsymbol{v})$ in $\mathbb{C}^2$.
Then:

$$A = PCP^{-1} \quad \text{where} \;\; P = [Re(\boldsymbol{v}) \;\; Im(\boldsymbol{v})] \;\; \text{and} \;\; C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$



Note, that eventhough the theorem only speaks about one eigenvalue and one eigenvector we actually know two eigenvalues and two eigenvectors as $\lambda_2 = \lambda_1^*$ and $\boldsymbol{v}_2 = \boldsymbol{v}_1^*$

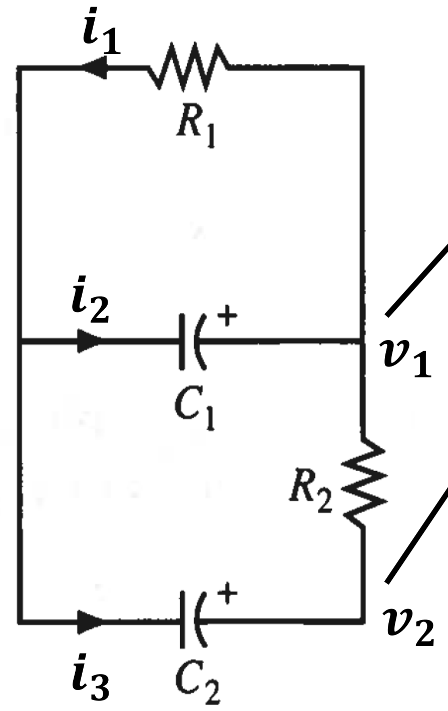
5.7 Applications to Differential Equations

$$\boldsymbol{x}'(t) = A\boldsymbol{x}(t);$$

Dynamic systems

Time developing

Ex 5.7.1



KCL

$$\left.\begin{aligned} -C_1\frac{dv_1}{dt} &= \frac{v_1 - v_2}{R_2} + \frac{v_1}{R_1} \\ -C_2\frac{dv_2}{dt} &= \frac{v_2 - v_1}{R_2} \end{aligned}\right\}$$

Two coupled linear 1. order diffential equations

$$\Rightarrow \left.\begin{aligned} \frac{dv_1}{dt} &= -\frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)v_1 + \frac{1}{R_2C_1}v_2 \\ \frac{dv_2}{dt} &= \frac{1}{R_2C_2}v_1 - \frac{1}{R_2C_2}v_2 \end{aligned}\right\}$$

$$\Rightarrow \begin{bmatrix} \frac{dv_1}{dt} \\ \frac{dv_2}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) & \frac{1}{R_2C_1} \\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix} \cdot \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$$

$$\Rightarrow \quad \boldsymbol{x}'(t) = A\boldsymbol{x}(t)$$

System of coupled linear 1. order diffential equations

<u>System of coupled linear 1. order diffential equations</u> (dynamic system):

$$\left.\begin{aligned} x_1' &= a_{11}x_1 + \cdots a_{1n}x_n \\ x_2' &= a_{21}x_1 + \cdots a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + \cdots a_{nn}x_n \end{aligned}\right\} \rightarrow \boldsymbol{x}'(t) = A\boldsymbol{x}(t); \quad \boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}; \ \boldsymbol{x}'(t) = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n' t) \end{bmatrix}; \ A = \begin{bmatrix} a_{11} \cdots a_{11} \\ \vdots \ \cdots \ \vdots \\ a_{n1} \cdots a_{n1} \end{bmatrix}$$

Linear:

$\boldsymbol{u}$ and $\boldsymbol{v}$ solutions ($\boldsymbol{u}' = A\boldsymbol{u}$ and $\boldsymbol{v}' = A\boldsymbol{v}$)

$\Longrightarrow (c_1\boldsymbol{u} + c_2\boldsymbol{v})' = c_1\boldsymbol{u}' + c_2\boldsymbol{v}' = c_1A\boldsymbol{u} + c_2A\boldsymbol{v} = A(c_1\boldsymbol{u} + c_2\boldsymbol{v}) \ \Longrightarrow \ (c_1\boldsymbol{u} + c_2\boldsymbol{v})$ solution

Fundamental set of solutions:

- n linearly independent functions → Basis set of solutions
- Any solution is a unique linear combination of the fundamental set → Infinitely many
- The solution set is an n-dimensional vector space of functions

Initial value problem: $\boldsymbol{x}(0) = \boldsymbol{x}_0$ → Unique function (solution) $\boldsymbol{x}(t)$

For the system of n coupled 1. order differential equations described by

$$\mathbf{x}' = A\mathbf{x},$$

there exists a fundamental set of n linearly independent solutions. These solutions form a basis for the set of all solutions to the differential equations. An arbitrary solution of $\mathbf{x}' = A\mathbf{x}$ can be written as a linear combination of the fundamental solutions.

Decoupling a dynamic system: $\boldsymbol{x}'(t) = A\boldsymbol{x}(t)$

Eigenvalues (λ_i) /-vectors ($\boldsymbol{v}_i$) /-functions for A: $\boldsymbol{v}_1 e^{\lambda_1 t}, \cdots, \boldsymbol{v}_n e^{\lambda_n t}$

Change-of-variable matrix: $P = [\boldsymbol{v}_1 \ \cdots \ \boldsymbol{v}_n]$

Diagonal matrix: $D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$; $A = PDP^{-1}$

Coordinate vector of $\boldsymbol{x}(t)$ relative to the eigenvector basis

Change-of-variable: $\boldsymbol{y}(t) = P^{-1}\boldsymbol{x}(t) \iff \boldsymbol{x}(t) = P\boldsymbol{y}(t)$

Differential equations: $\boldsymbol{x}' = A\boldsymbol{x} \Longrightarrow P\boldsymbol{y}' = \dfrac{dP\boldsymbol{y}}{dt} = AP\boldsymbol{y} = PDP^{-1}P\boldsymbol{y} = PD\boldsymbol{y}$

$$\Longrightarrow P^{-1}P\boldsymbol{y}' = P^{-1}PD\boldsymbol{y} \Longrightarrow \boldsymbol{y}' = D\boldsymbol{y}$$

Decoupling a dynamic system: $\boldsymbol{x}'(t) = A\boldsymbol{x}(t)$

Simple (uncoupled) differential equations

Decoupled system

$$\boldsymbol{y}' = D\boldsymbol{y} \implies \begin{bmatrix} y_1' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \implies \begin{cases} y_1' = \lambda_1 y_1 \\ \quad\vdots \\ y_n' = \lambda_n y_n \end{cases} \implies \begin{cases} y_1(t) = c_1 e^{\lambda_1 t} \\ \quad\vdots \\ y_n(t) = c_n e^{\lambda_n t} \end{cases} \implies \boldsymbol{y}(t) = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix}$$

Solution in eigenfunction basis

Initial value: $\boldsymbol{x}(0) = \boldsymbol{x}_0 \implies \boldsymbol{y}(0) = P^{-1}\boldsymbol{x}(0) = P^{-1}\boldsymbol{x}_0 = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Solution in the original system:

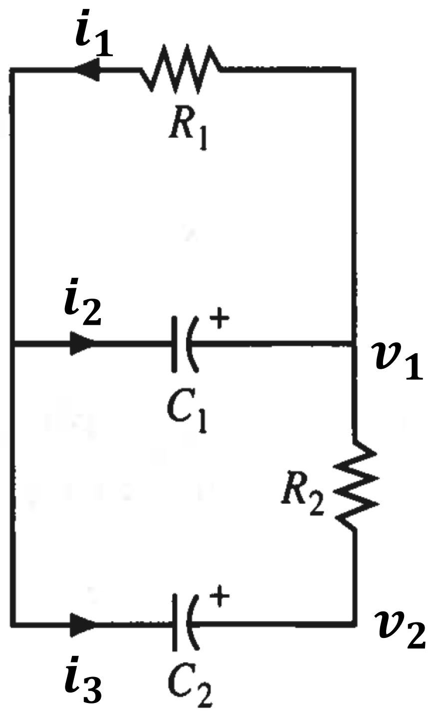
eigenvalue

$$\boldsymbol{x}(t) = P\boldsymbol{y}(t) = [\boldsymbol{v}_1 \ \cdots \ \boldsymbol{v}_n] \begin{bmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix} = c_1 \boldsymbol{v}_1 e^{\lambda_1 t} + \cdots + c_n \boldsymbol{v}_n e^{\lambda_n t} = \sum_{i=1}^{n} c_i \boldsymbol{v}_i e^{\lambda_i t}$$

eigenvector

Ex 5.7.1

$$R_1 = 1\Omega \quad R_2 = 2\Omega \quad C_1 = 1F \quad C_2 = 0.5F \quad v_1(0) = 5V \quad v_2(0) = 4V$$



$$\begin{bmatrix} \dfrac{dv_1}{dt} \\ \dfrac{dv_2}{dt} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{C_1}\left(\dfrac{1}{R_1}+\dfrac{1}{R_2}\right) & \dfrac{1}{R_2C_1} \\ \dfrac{1}{R_2C_2} & -\dfrac{1}{R_2C_2} \end{bmatrix} \cdot \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} -\dfrac{3}{2} & \dfrac{1}{2} \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} -\dfrac{3}{2}-\lambda & \dfrac{1}{2} \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 + \frac{5}{2}\lambda + 1 = 0 \implies \lambda = \begin{cases} -½ \\ -2 \end{cases}$$

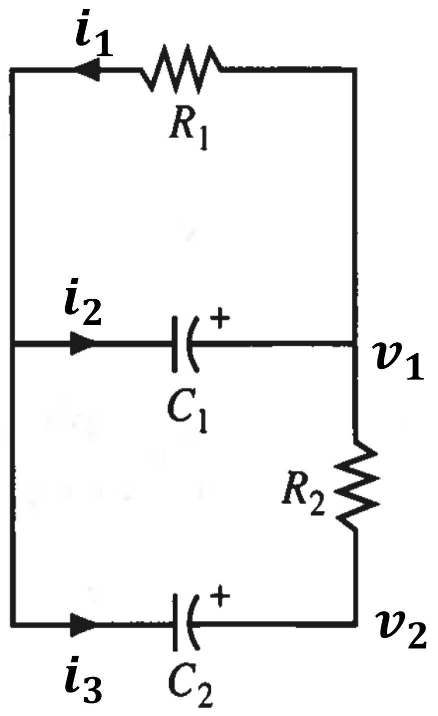
Eigenvectors: $\lambda_1 = -\frac{1}{2} \implies \boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\lambda_2 = -2 \implies \boldsymbol{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$P = [\boldsymbol{u}_1\ \boldsymbol{u}_2] = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} -½ & 0 \\ 0 & -2 \end{bmatrix} \qquad \boldsymbol{y}(t) = P^{-1}\boldsymbol{v}(t) = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{3}v_1(t) + \frac{1}{3}v_2(t) \\ -\frac{2}{3}v_1(t) + \frac{1}{3}v_2(t) \end{bmatrix}$$

$$\boldsymbol{y}' = \begin{bmatrix} -½ & 0 \\ 0 & -2 \end{bmatrix}\boldsymbol{y} \implies \begin{cases} y_1' = -½ y_1 \\ y_2' = -2y_2 \end{cases} \implies \boldsymbol{y}(t) = \begin{bmatrix} c_1 e^{-½t} \\ c_2 e^{-2t} \end{bmatrix} = \begin{bmatrix} 3e^{-½t} \\ -2e^{-2t} \end{bmatrix} \qquad \begin{bmatrix} c_1 \\ c_1 \end{bmatrix} = \boldsymbol{y}(0) = P^{-1}\boldsymbol{v}(0) = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$\implies \boldsymbol{v}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = P\boldsymbol{y}(t) = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3e^{-½t} \\ -2e^{-2t} \end{bmatrix} = \begin{bmatrix} 3e^{-½t} + 2e^{-2t} \\ 6e^{-½t} - 2e^{-2t} \end{bmatrix}$$

Ex 5.7.1



$R_1 = 1\Omega \quad R_2 = 2\Omega \quad C_1 = 1F \quad C_2 = 0.5F \quad v_1(0) = 5V \quad v_2(0) = 4V$

$$\begin{bmatrix} \frac{dv_1}{dt} \\ \frac{dv_2}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) & \frac{1}{R_2 C_1} \\ \frac{1}{R_2 C_2} & -\frac{1}{R_2 C_2} \end{bmatrix} \cdot \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -\frac{3}{2} - \lambda & \frac{1}{2} \\ 1 & -1 - \lambda \end{vmatrix} = \lambda^2 + \frac{5}{2}\lambda + 1 = 0 \implies \lambda = \begin{cases} -½ \\ -2 \end{cases}$$

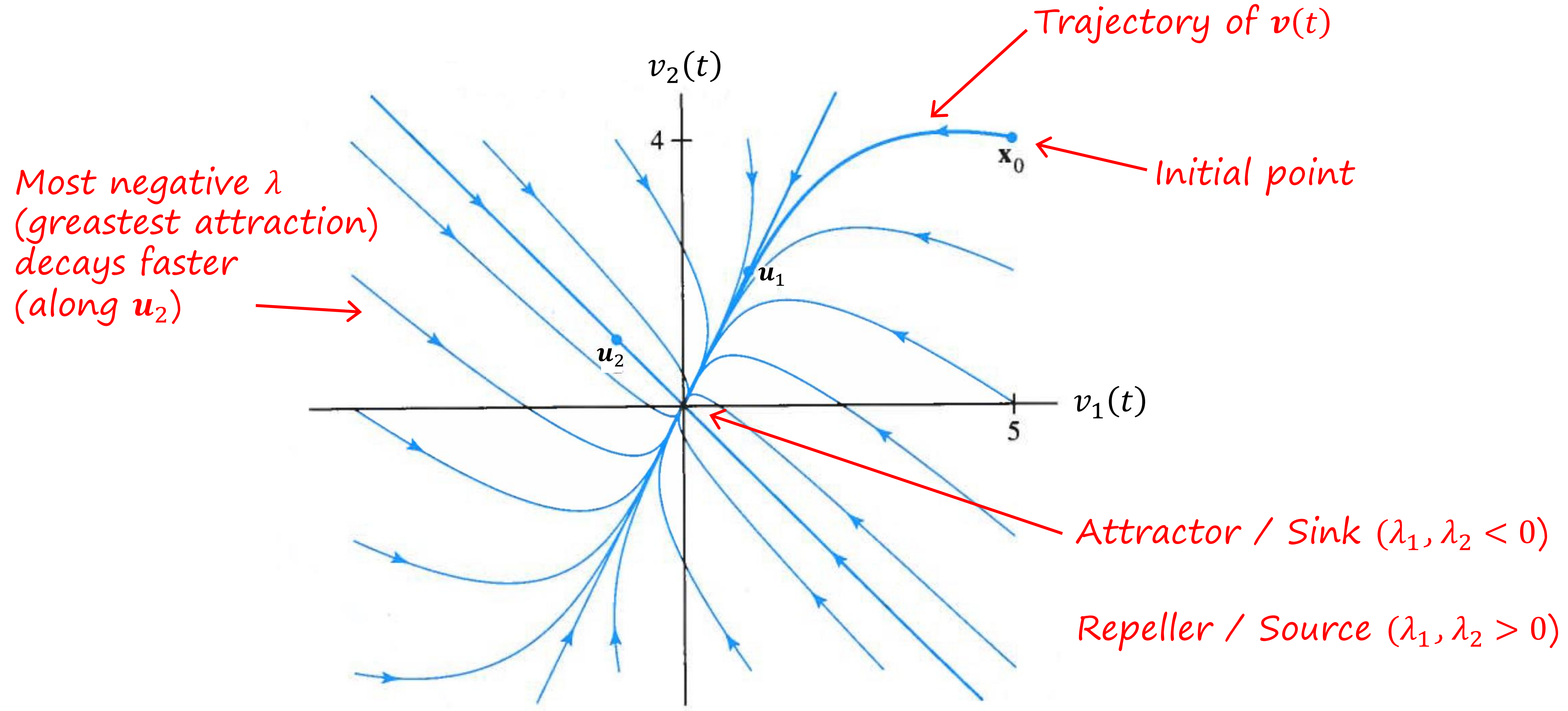
Eigenvectors: $\lambda_1 = -\frac{1}{2} \implies \boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\lambda_2 = -2 \implies \boldsymbol{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Eigenfunctions: $\boldsymbol{u}_1 e^{\lambda_1 t} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-\frac{1}{2}t}$ and $\boldsymbol{u}_2 e^{\lambda_2 t} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$

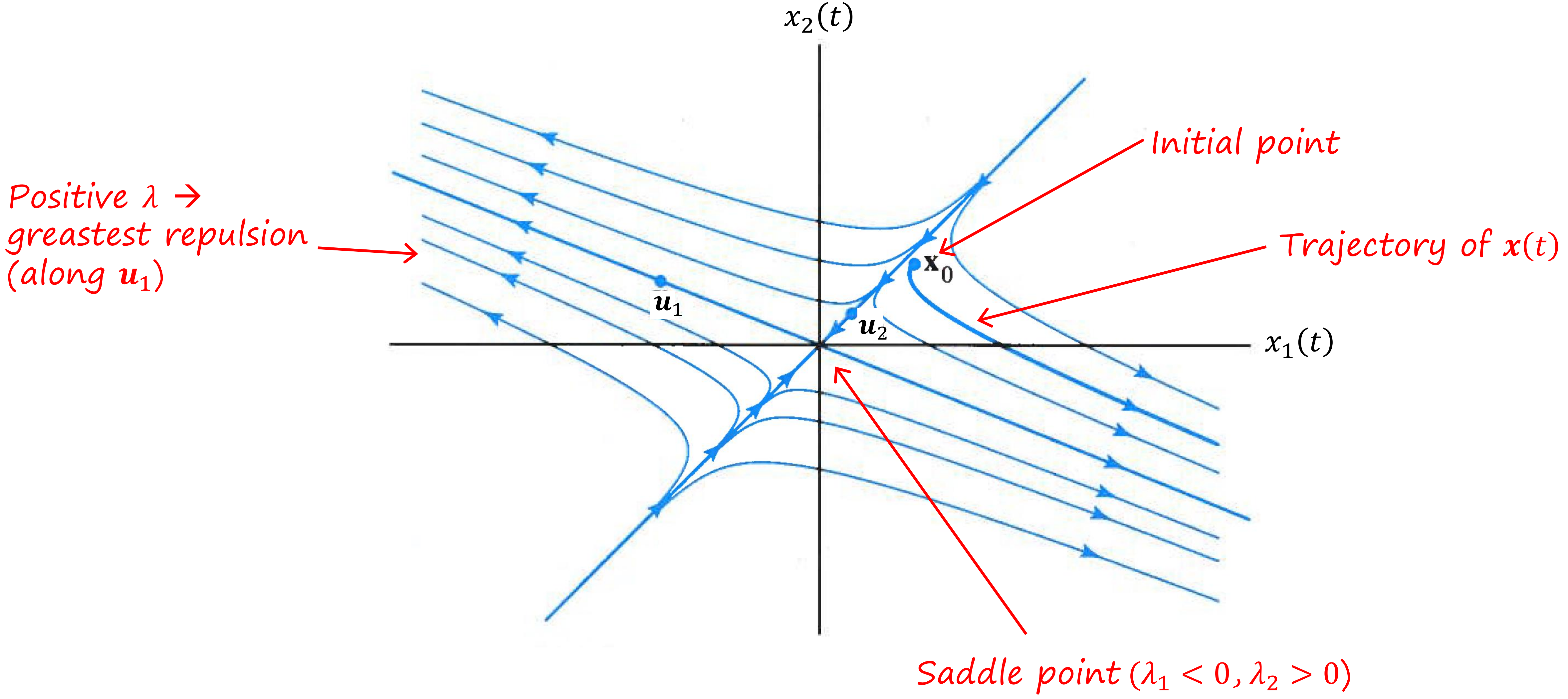
$$\boldsymbol{v}(t) = \sum_{i=1}^{n} c_i \boldsymbol{u}_i e^{\lambda_i t} = c_1 \boldsymbol{u}_1 e^{\lambda_1 t} + c_2 \boldsymbol{u}_2 e^{\lambda_2 t} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-\frac{1}{2}t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} = \begin{bmatrix} c_1 e^{-\frac{1}{2}t} - c_2 e^{-2t} \\ 2c_1 e^{-\frac{1}{2}t} + c_2 e^{-2t} \end{bmatrix}$$

$$\boldsymbol{v}(0) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \implies \begin{cases} c_1 = 3 \\ c_2 = -2 \end{cases} \qquad \implies \quad \boldsymbol{v}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-\frac{1}{2}t} - 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} = \begin{bmatrix} 3e^{-\frac{1}{2}t} + 2e^{-2t} \\ 6e^{-\frac{1}{2}t} - 2e^{-2t} \end{bmatrix}$$

$$\boldsymbol{v}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = 3\underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\boldsymbol{u}_1} e^{-\frac{1}{2}t} - 2\underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\boldsymbol{u}_2} e^{-2t} = \begin{bmatrix} 3e^{-\frac{1}{2}t} + 2e^{-2t} \\ 6e^{-\frac{1}{2}t} - 2e^{-2t} \end{bmatrix}$$



$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \underbrace{\begin{bmatrix} -5 \\ 2 \end{bmatrix}}_{\boldsymbol{u}_1} e^{6t} + c_2 \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\boldsymbol{u}_2} e^{-t} = \begin{bmatrix} -5c_1 e^{6t} + c_2 e^{-t} \\ 2c_1 e^{6t} + c_2 e^{-t} \end{bmatrix}$$



Complex eigenvalues: $\boldsymbol{x}'(t) = A\boldsymbol{x}(t)$;

$\lambda = a + jb$ eigenvalue for A; $\boldsymbol{v}$ the corresponding (complex) eigenvector

$$\boldsymbol{x}(t) = \boldsymbol{v}e^{\lambda t} = \big(Re(\boldsymbol{v}) + j \cdot Im(\boldsymbol{v})\big) \cdot e^{(a+jb)t} = \big(Re(\boldsymbol{v}) + j \cdot Im(\boldsymbol{v})\big) \cdot e^{at}e^{jbt}$$

$$= \big(Re(\boldsymbol{v}) + j \cdot Im(\boldsymbol{v})\big) \cdot e^{at}(\cos(bt) + j \cdot \sin(bt))$$

$$= e^{at}(Re(\boldsymbol{v}) \cdot \cos(bt) - Im(\boldsymbol{v}) \cdot \sin(bt)) + j \cdot e^{at}(Re(\boldsymbol{v}) \cdot \sin(bt) + Im(\boldsymbol{v}) \cdot \cos(bt))$$

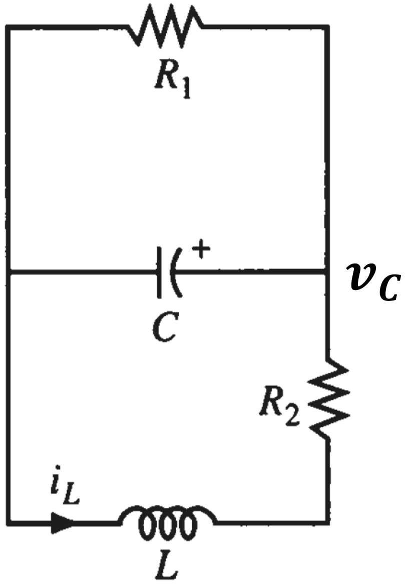
$$= Re\big(\boldsymbol{x}(t)\big) + j \cdot Im\big(\boldsymbol{x}(t)\big)$$

<u>Real</u> solutions to $\boldsymbol{x}'(t) = A\boldsymbol{x}(t)$:

$$\left.\begin{aligned} \boldsymbol{y}_1(t) &= e^{at}(Re(\boldsymbol{v}) \cdot \cos(bt) - Im(\boldsymbol{v}) \cdot \sin(bt)) \\ \boldsymbol{y}_2(t) &= e^{at}(Re(\boldsymbol{v}) \cdot \sin(bt) + Im(\boldsymbol{v}) \cdot \cos(bt)) \end{aligned}\right\} \Longrightarrow \boldsymbol{y}(t) = c_1\boldsymbol{y}_1(t) + c_2\boldsymbol{y}_2(t)$$

Ex 5.7.3

$R_1 = 5\Omega \quad R_2 = 0.8\Omega \quad C_1 = 0.1F \quad L = 0.4H \qquad i_L(0) = 3A \qquad v_C(0) = 3V$



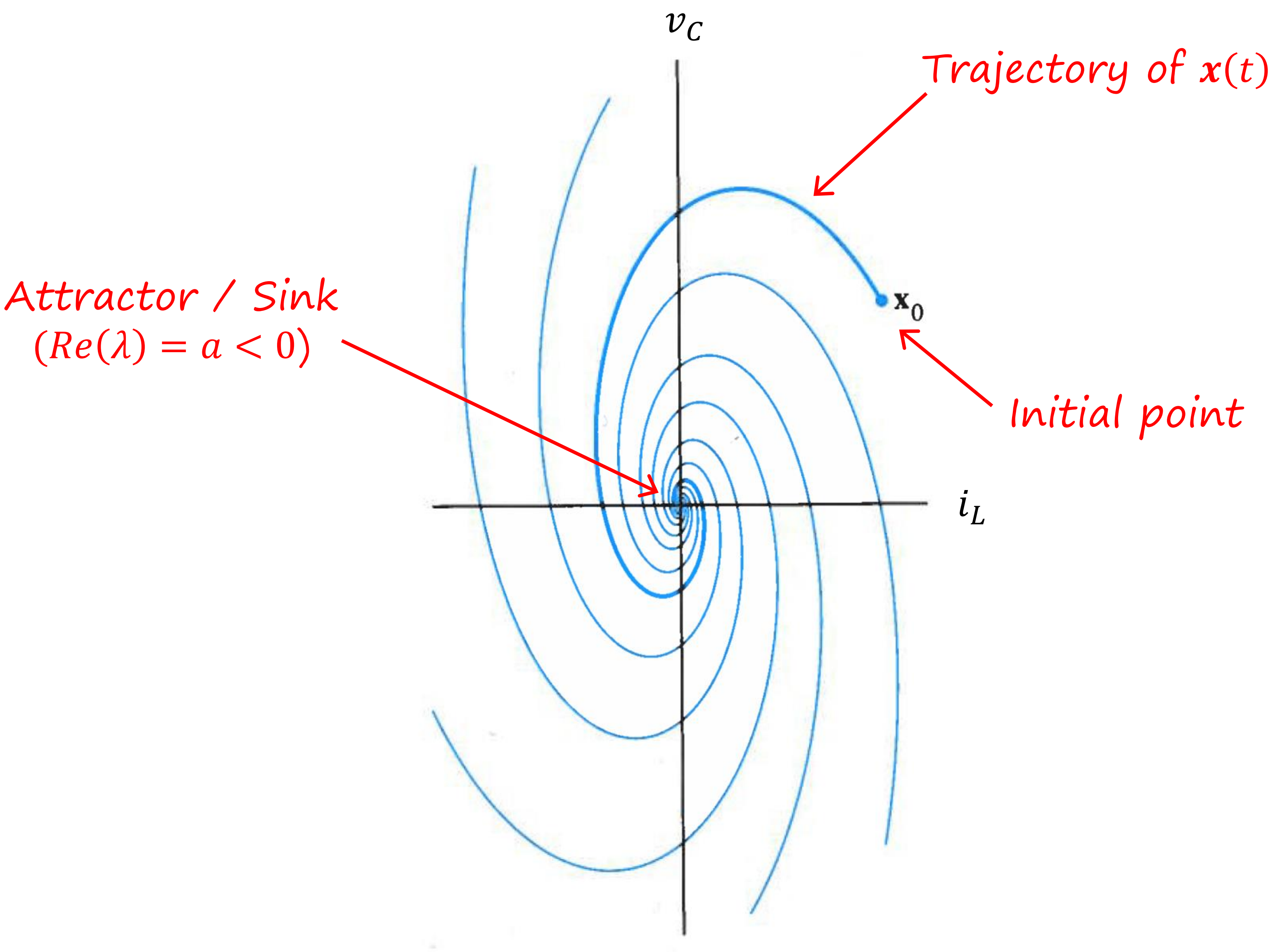
$$\left.\begin{aligned} -L\frac{di_L}{dt} &= R_2 i_L + v_C \\ C\frac{dv_C}{dt} &= i_L - \frac{v_C}{R_1} \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \frac{di_L}{dt} &= -\frac{R_2}{L}i_L - \frac{1}{L}v_C \\ \frac{dv_C}{dt} &= \frac{1}{C}i_L - \frac{1}{R_1 C}v_C \end{aligned}\right\} \Rightarrow \begin{bmatrix} i_L' \\ v_C' \end{bmatrix} = \begin{bmatrix} \frac{R_2}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{R_1 C} \end{bmatrix} \cdot \begin{bmatrix} i_L \\ v_C \end{bmatrix} = \begin{bmatrix} -2 & -2.5 \\ 10 & -2 \end{bmatrix} \cdot \begin{bmatrix} i_L \\ v_C \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & -2.5 \\ 10 & -2-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 29 = 0 \;\Rightarrow\; \lambda = -2 \pm 5j$$

Eigenvector: $\boldsymbol{v} = \begin{bmatrix} j \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + j\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

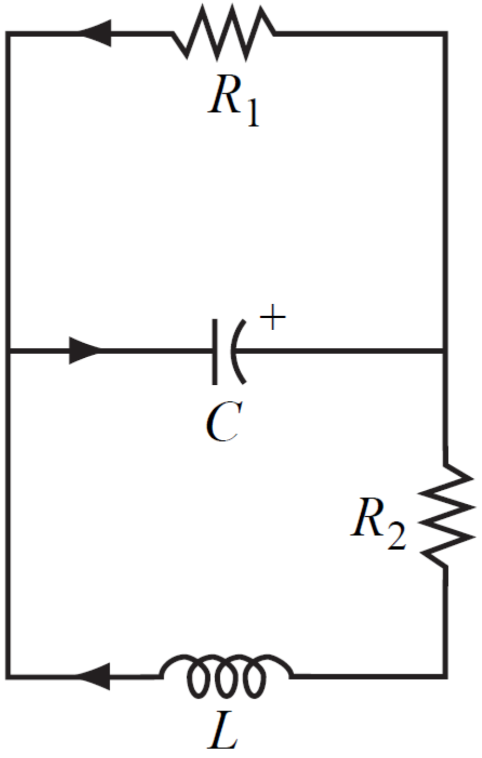
$$\left.\begin{aligned} \boldsymbol{y}_1(t) &= e^{-2t}\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\cos(5t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin(5t)\right) \\ \boldsymbol{y}_2(t) &= e^{at}(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\sin(5t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos(5t)) \end{aligned}\right\} \Rightarrow \begin{bmatrix} i_L \\ v_C \end{bmatrix} = c_1\boldsymbol{y}_1(t) + c_2\boldsymbol{y}_2(t) = c_1 e^{-2t}\begin{bmatrix} -\sin(5t) \\ 2\cos(5t) \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} \cos(5t) \\ 2\sin(5t) \end{bmatrix}$$

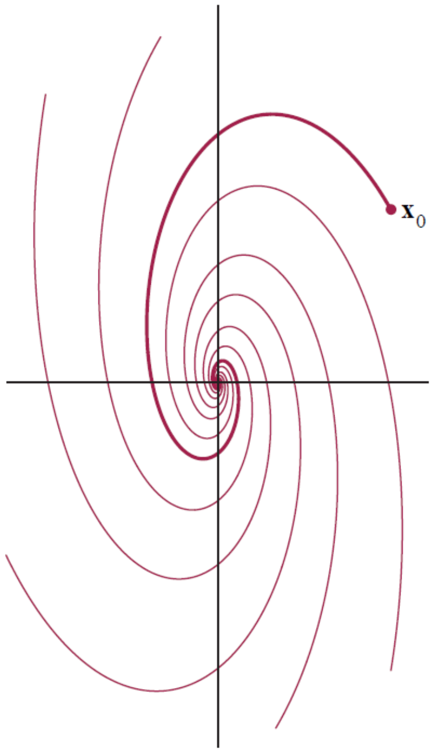
$$\begin{bmatrix} i_L(0) \\ v_C(0) \end{bmatrix} = c_1\begin{bmatrix} 0 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \;\Rightarrow\; \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 3 \end{bmatrix} \;\Rightarrow\; \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}\sin(5t) + 3\cos(5t) \\ 3\cos(5t) + 6\sin(5t) \end{bmatrix} e^{-2t}$$


v_C
Trajectory of $x(t)$
Attractor / Sink
$(Re(\lambda) = a < 0)$
$\mathbf{x}_0$
Initial point
i_L

Discuss with your neighbour:

Consider figure 4 and figure 5 in example 3 on page 334





Why does it make physical sense that the solutions 'spirals' inwards to zero?

Todays words and concepts

Dynamic system

Repeller

Sink

Complex vector

Complex eigenvalue

Rotation

Attractor

Decoupled system

Saddle point

Scaling

Trajectory

Source

Coupled differential equations