Chapter 6.4

1. Set $\mathbf{v}_1 = \mathbf{x}_1$ and compute that $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - 3\mathbf{v}_1 = \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}$. Thus an orthogonal basis for W is $\left\{ \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} \right\}$.

3. Set $\mathbf{v}_1 = \mathbf{x}_1$ and compute that $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - \dfrac{1}{2}\mathbf{v}_1 = \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix}$. Thus an orthogonal basis for W is $\left\{ \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix} \right\}$.

7. Since $\|\mathbf{v}_1\| = \sqrt{30}$ and $\|\mathbf{v}_2\| = \sqrt{27/2} = 3\sqrt{6}/2$, an orthonormal basis for W is

$$\left\{ \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} \right\} = \left\{ \begin{bmatrix} 2/\sqrt{30} \\ -5/\sqrt{30} \\ 1/\sqrt{30} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\}.$$

13. Since A and Q are given, $R = Q^T A = \begin{bmatrix} 5/6 & 1/6 & -3/6 & 1/6 \\ -1/6 & 5/6 & 1/6 & 3/6 \end{bmatrix} \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ 0 & 6 \end{bmatrix}$.

17. **a.** False. Scaling was used in Example 2, but the scale factor was nonzero.

b. True. See (1) in the statement of Theorem 11.

c. True. See the solution of Example 4.

24. **[M]** Call the columns of the matrix $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$ and perform the Gram-Schmidt process on these vectors:

$$\mathbf{v}_1 = \mathbf{x}_1$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - (-1)\mathbf{v}_1 = \begin{bmatrix} 3 \\ 3 \\ -3 \\ 0 \\ 3 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \mathbf{x}_3 - \left(-\frac{1}{2}\right)\mathbf{v}_1 - \left(-\frac{4}{3}\right)\mathbf{v}_2 = \begin{bmatrix} 6 \\ 0 \\ 6 \\ 6 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_4 = \mathbf{x}_4 - \frac{\mathbf{x}_4 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_4 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \frac{\mathbf{x}_4 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\mathbf{v}_3 = \mathbf{x}_4 - \frac{1}{2}\mathbf{v}_1 - (-1)\mathbf{v}_2 - \left(-\frac{1}{2}\right)\mathbf{v}_3 = \begin{bmatrix} 0 \\ 5 \\ 0 \\ 0 \\ -5 \end{bmatrix}$$

Thus an orthogonal basis for W is $\left\{ \begin{bmatrix} -10 \\ 2 \\ -6 \\ 16 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 6 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 0 \\ 0 \\ -5 \end{bmatrix} \right\}$.

25. **[M]** The columns of Q will be normalized versions of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ found in Exercise 24. Thus

$$Q = \begin{bmatrix} -1/2 & 1/2 & 1/\sqrt{3} & 0 \\ 1/10 & 1/2 & 0 & 1/\sqrt{2} \\ -3/10 & -1/2 & 1/\sqrt{3} & 0 \\ 4/5 & 0 & 1/\sqrt{3} & 0 \\ 1/10 & 1/2 & 0 & -1/\sqrt{2} \end{bmatrix}, R = Q^T A = \begin{bmatrix} 20 & -20 & -10 & 10 \\ 0 & 6 & -8 & -6 \\ 0 & 0 & 6\sqrt{3} & -3\sqrt{3} \\ 0 & 0 & 0 & 5\sqrt{2} \end{bmatrix}$$

Chapter 6.5

1. To find the normal equations and to find $\hat{\mathbf{x}}$, compute

$$A^T A = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix}; \ A^T \mathbf{b} = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \end{bmatrix}.$$

a. The normal equations are $(A^T A)\mathbf{x} = A^T \mathbf{b}$: $\begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \end{bmatrix}$.

b. Compute $\hat{\mathrm{x}} = (A^T A)^{-1} A^T \mathbf{b} = \begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix}^{-1} \begin{bmatrix} -4 \\ 11 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 22 & 11 \\ 11 & 6 \end{bmatrix} \begin{bmatrix} -4 \\ 11 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 33 \\ 22 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

3. To find the normal equations and to find $\hat{\mathbf{x}}$, compute

$= \begin{bmatrix} 6 \\ -6 \end{bmatrix}$.

$$A^T A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix}; \ A^T \mathbf{b} = \begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix}$$

a. The normal equations are $(A^T A)\mathbf{x} = A^T \mathbf{b}$: $\begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$.

b. Compute $\hat{\mathbf{x}} = (\mathrm{A}^T \mathrm{A})^{-1} \mathrm{A}^T \mathbf{b} = \begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ -6 \end{bmatrix} = \frac{1}{216} \begin{bmatrix} 42 & -6 \\ -6 & 6 \end{bmatrix} \begin{bmatrix} 6 \\ -6 \end{bmatrix}$.

$$= \frac{1}{216} \begin{bmatrix} 288 \\ -72 \end{bmatrix} = \begin{bmatrix} 4/3 \\ -1/3 \end{bmatrix}$$

9. a. Because the columns $\mathbf{a}_1$ and $\mathbf{a}_2$ of A are orthogonal, the method of Example 4 may be used to find $\hat{\mathbf{b}}$, the orthogonal projection of **b** onto Col A:

$$\hat{\mathbf{b}} = \frac{\mathbf{b} \cdot \mathbf{a}_1}{\mathbf{a}_1 \cdot \mathbf{a}_1} \mathbf{a}_1 + \frac{\mathbf{b} \cdot \mathbf{a}_2}{\mathbf{a}_2 \cdot \mathbf{a}_2} \mathbf{a}_2 = \frac{2}{7} \mathbf{a}_1 + \frac{1}{7} \mathbf{a}_2 = \frac{2}{7} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + \frac{1}{7} \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

b. The vector $\hat{\mathbf{x}}$ contains the weights which must be placed on $\mathbf{a}_1$ and $\mathbf{a}_2$ to produce $\hat{\mathbf{b}}$. These weights are easily read from the above equation, so $\hat{\mathbf{x}} = \begin{bmatrix} 2/7 \\ 1/7 \end{bmatrix}$.

17. **a**. True. See the beginning of the section. The distance from $A\mathbf{x}$ to $\mathbf{b}$ is $\| A\mathbf{x} - \mathbf{b} \|$.

b. True. See the comments about equation (1).

c. False. The inequality points in the wrong direction. See the definition of a least-squares solution.

d. True. See Theorem 13.

e. True. See Theorem 14.

Chapter 6.6

5. If two data points have different x-coordinates, then the two columns of the design matrix X cannot be multiples of each other and hence are linearly independent. By Theorem 14 in Section 6.5, the normal equations have a unique solution.

8. a. The model that produces the correct least-squares fit is $\mathbf{y} = X\beta + \epsilon$ where

$$X = \begin{bmatrix} x_1 & x_1^2 & x_1^3 \\ \vdots & \vdots & \vdots \\ x_n & x_n^2 & x_n^3 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}, \text{and } \epsilon = \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{bmatrix}$$

b. [M] For the given data, $X = \begin{bmatrix} 4 & 16 & 64 \\ 6 & 36 & 216 \\ 8 & 64 & 512 \\ 10 & 100 & 1000 \\ 12 & 144 & 1728 \\ 14 & 196 & 2744 \\ 16 & 256 & 4096 \\ 18 & 324 & 5832 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1.58 \\ 2.08 \\ 2.5 \\ 2.8 \\ 3.1 \\ 3.4 \\ 3.8 \\ 4.32 \end{bmatrix}$, so

$\hat{\beta} = (X^T X)^{-1} X^T \mathbf{y} = \begin{bmatrix} .5132 \\ -.03348 \\ .001016 \end{bmatrix}$, and the least-squares curve is

$y = .5132x - .03348x^2 + .001016x^3$.

10. **a**. The model that produces the correct least-squares fit is $\mathbf{y} = X\beta + \epsilon$ where

$$X = \begin{bmatrix} e^{-.02(10)} & e^{-.07(10)} \\ e^{-.02(11)} & e^{-.07(11)} \\ e^{-.02(12)} & e^{-.07(12)} \\ e^{-.02(14)} & e^{-.07(14)} \\ e^{-.02(15)} & e^{-.07(15)} \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 21.34 \\ 20.68 \\ 20.05 \\ 18.87 \\ 18.30 \end{bmatrix}, \beta = \begin{bmatrix} M_A \\ M_B \end{bmatrix}, \text{and } \epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{bmatrix},$$

b. [M] One computes that (to two decimal places) $\hat{\beta} = \begin{bmatrix} 19.94 \\ 10.10 \end{bmatrix}$, so the desired least-squares equation is $y = 19.94e^{-.02t} + 10.10e^{-.07t}$.

11. **[M]** The model that produces the correct least-squares fit is $\mathbf{y} = X\beta + \epsilon$ where

$$X = \begin{bmatrix} 1 & 3\cos .88 \\ 1 & 2.3\cos 1.1 \\ 1 & 1.65\cos 1.42 \\ 1 & 1.25\cos 1.77 \\ 1 & 1.01\cos 2.14 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 3 \\ 2.3 \\ 1.65 \\ 1.25 \\ 1.01 \end{bmatrix}, \beta = \begin{bmatrix} \beta \\ e \end{bmatrix}, \text{and } \epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{bmatrix}.$$

One computes that (to two decimal places) $\hat{\beta} = \begin{bmatrix} 1.45 \\ .811 \end{bmatrix}$. Since $e = .811 < 1$ the orbit is an ellipse. The equation $r = \beta / (1 - e\cos\vartheta)$ produces $r = 1.33$ when $\vartheta = 4.6$.