In the grading of the exercises special attention will be paid to check that answers are substantiated and that the procedure of calculations is well documented. When results are achieved using a calculator or a pc it should be noted in the paper. All 6 problems are weighted equally in the grading. Your solution to the problem set can be written in Danish or English as you prefer.

PROBLEM 1.

Consider the following matrix and vector

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 0 & 6 \\ -1 & 2 & -4 \\ 1 & -2 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ -12 \\ 12 \end{bmatrix}.$$

1. Solve $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$.
2. Is it possible to find a vector $\mathbf{b}$ so $A\mathbf{x} = \mathbf{b}$ cannot be solved?

PROBLEM 2.

Let four vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{b}$ be given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 7 \\ 2 \\ 7 \end{bmatrix}.$$

1. Show that the three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for $\mathbb{R}^3$.
2. Express $\mathbf{b}$ in the new basis.

PROBLEM 3.

Consider the following matrix and vector

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -1 & 0 & 1 \\ 2 & -2 & 2 \\ 1 & 2 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -2 \\ 3 \\ 0 \\ -5 \end{bmatrix}.$$

1. Show that the matrix equation $A\mathbf{x} = \mathbf{b}$ is inconsistent.
2. Find the least-squares solution of $A\mathbf{x} = \mathbf{b}$.

In a number of applications, sparse solutions, i.e. solutions where most elements are zero, are desired. Consider the two sparse vectors

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathbf{x}_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}.$$

3. Determine whether $\mathbf{x}_1$ or $\mathbf{x}_2$ is the better solution of $A\mathbf{x} = \mathbf{b}$ in the least-squares sense.

PROBLEM 4.

For the statements given below, state whether they are true or false and justify your answer for each statement.

1. A square, upper-triangular matrix with non-zero elements on the diagonal is invertible.
2. If a matrix A has an eigenvalue λ, then $c\lambda$, with c a scalar, is also an eigenvalue.
3. The matrix $\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}$ is positive semidefinite.

PROBLEM 5.

In the case *Computer Graphics in Automotive Design*, homogeneous coordinates and rotation matrices were introduced. In this problem we are not concerned with homogeneous coordinates, but only work with standard coordinates. In $\mathbb{R}^2$ a rotation of the vector $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ by an angle, θ about the origin is obtained by multiplying the following rotation matrix with $\mathbf{x}$.

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

1. Show that $R(\theta)$ is an orthogonal matrix.
2. Argue that $R(2\theta) = R^2(\theta)$.
3. Compute $R^2(\theta)$ and use this result to find formulas for $\cos(2\theta)$ and $\sin(2\theta)$ expressed by $\cos\theta$ and $\sin\theta$.

PROBLEM 6.

A special and somewhat rare class of square matrices are called skew-symmetric. A general 3×3 skew-symmetric matrix has this form

$$\begin{bmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{bmatrix},$$

where a_1, a_2 and a_3 are scalars. One particular use of 3×3 skew-symmetric matrices sometimes encountered in mechanical engineering is as a way of expressing the vector cross product as a matrix multiplication.

1. For a symmetric matrix $A = A^T$. What is the corresponding relation for skew-symmetric matrices?

Consider the set of all 3×3 skew-symmetric matrices, here denoted as $\mathbb{S}^{3\times 3}$.

2. Show that $\mathbb{S}^{3\times 3}$ forms a vector space.