Chapter 5.1

1. The number 2 is an eigenvalue of A if and only if the equation $A\mathbf{x} = 2\mathbf{x}$ has a nontrivial solution. This equation is equivalent to $(A-2I)\mathbf{x} = 0$. Compute $A-2I = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. The columns of A are obviously linearly dependent, so $(A-2I)\mathbf{x} = 0$ has a nontrivial solution, and so 2 is an eigenvalue of A.

5. Is $A\mathbf{x}$ a multiple of $\mathbf{x}$? Compute $\begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. So $\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ is an eigenvector of A for the eigenvalue 0.

9. For $\lambda = 1$: $A - 1I = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 2 & 0 \end{bmatrix}$. The augmented matrix for $(A-I)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} 4 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$. Thus $x_1 = 0$ and x_2 is free. The general solution of $(A-I)\mathbf{x} = \mathbf{0}$ is $x_2\mathbf{e}_2$, where $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and so $\mathbf{e}_2$ is a basis for the eigenspace corresponding to the eigenvalue 1.

For $\lambda = 5$: $A - 5I = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & -4 \end{bmatrix}$. The equation $(A-5I)\mathbf{x} = \mathbf{0}$ leads to $2x_1 - 4x_2 = 0$, so that $x_1 = 2x_2$ and x_2 is free. The general solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. So $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a basis for the eigenspace.

21. **a**. False. The equation $A\mathbf{x} = \lambda\mathbf{x}$ must have a *nontrivial* solution.
b. True. See the paragraph after Example 5.
c. True. See the discussion of equation (3).
d. True. See Example 2 and the paragraph preceding it. Also, see the Numerical Note.
e. False. See the warning after Example 3.

31. Suppose T reflects points across (or through) a line that passes through the origin. That line consists of all multiples of some nonzero vector $\mathbf{v}$. The points on this line do not move under the action of A. So $T(\mathbf{v}) = \mathbf{v}$. If A is the standard matrix of T, then $A\mathbf{v} = \mathbf{v}$. Thus $\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue 1. The eigenspace is Span $\{\mathbf{v}\}$. Another eigenspace is generated by any nonzero vector $\mathbf{u}$ that is perpendicular to the given line. (Perpendicularity in $\mathbf{R}^2$ should be a familiar concept even though orthogonality in $\mathbf{R}^n$ has not been discussed yet.) Each vector $\mathbf{x}$ on the line through $\mathbf{u}$ is transformed into the vector $-\mathbf{x}$. The eigenvalue is -1.

35. Using the figure in the exercise, plot $T(\mathbf{u})$ as $2\mathbf{u}$, because **u** is an eigenvector for the eigenvalue 2 of the standard matrix A. Likewise, plot $T(\mathbf{v})$ as $3\mathbf{v}$, because **v** is an eigenvector for the eigenvalue 3. Since T is linear, the image of **w** is $T(\mathbf{w}) = T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.

Chapter 5.2

1. $A = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$, $A - \lambda I = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{bmatrix}$. The characteristic polynomial is $\det(A - \lambda I) = (2-\lambda)^2 - 7^2 = 4 - 4\lambda + \lambda^2 - 49 = \lambda^2 - 4\lambda - 45$. In factored form, the characteristic equation is $(\lambda - 9)(\lambda + 5) = 0$, so the eigenvalues of A are 9 and -5.

9. $\det(A - \lambda I) = \det\begin{bmatrix} 1-\lambda & 0 & -1 \\ 2 & 3-\lambda & -1 \\ 0 & 6 & 0-\lambda \end{bmatrix}$. From the special formula for 3×3 determinants, the characteristic polynomial is

$$\begin{aligned}\det(A-\lambda I) &= (1-\lambda)(3-\lambda)(-\lambda) + 0 + (-1)(2)(6) - 0 - (6)(-1)(1-\lambda) - 0 \\ &= (\lambda^2 - 4\lambda + 3)(-\lambda) - 12 + 6(1-\lambda) \\ &= -\lambda^3 + 4\lambda^2 - 3\lambda - 12 + 6 - 6\lambda \\ &= -\lambda^3 + 4\lambda^2 - 9\lambda - 6\end{aligned}$$

(This polynomial has one irrational zero and two imaginary zeros.) Another way to evaluate the determinant is to interchange rows 1 and 2 (which reverses the sign of the determinant) and then make one row replacement:

$$\det\begin{bmatrix} 1-\lambda & 0 & -1 \\ 2 & 3-\lambda & -1 \\ 0 & 6 & 0-\lambda \end{bmatrix} = -\det\begin{bmatrix} 2 & 3-\lambda & -1 \\ 1-\lambda & 0 & -1 \\ 0 & 6 & 0-\lambda \end{bmatrix}$$

$$= -\det\begin{bmatrix} 2 & 3-\lambda & -1 \\ 0 & 0 + (.5\lambda - .5)(3-\lambda) & -1 + (.5\lambda - .5)(-1) \\ 0 & 6 & 0-\lambda \end{bmatrix}$$. Next, expand by cofactors down the first column. The quantity above equals

$$-2\det\begin{bmatrix} (.5\lambda - .5)(3-\lambda) & -.5 - .5\lambda \\ 6 & -\lambda \end{bmatrix} = -2[(.5\lambda - .5)(3-\lambda)(-\lambda) - (-.5 - .5\lambda)(6)]$$

$$= (1-\lambda)(3-\lambda)(-\lambda) - (1+\lambda)(6) = (\lambda^2 - 4\lambda + 3)(-\lambda) - 6 - 6\lambda = -\lambda^3 + 4\lambda^2 - 9\lambda - 6$$

25. Example 5 of Section 4.9 showed that $A\mathbf{v}_1 = \mathbf{v}_1$, which means that $\mathbf{v}_1$ is an eigenvector of A corresponding to the eigenvalue 1.

a. Since A is a 2×2 matrix, the eigenvalues are easy to find, and factoring the characteristic polynomial is easy when one of the two factors is known.

$\det\begin{bmatrix} .6-\lambda & .3 \\ .4 & .7-\lambda \end{bmatrix} = (.6-\lambda)(.7-\lambda)-(.3)(.4) = \lambda^2 - 1.3\lambda + .3 = (\lambda-1)(\lambda-.3)$. The eigenvalues are 1 and .3.

For the eigenvalue .3, solve $(A-.3I)\mathbf{x} = \mathbf{0}$: $\begin{bmatrix} .6-.3 & .3 & 0 \\ .4 & .7-.3 & 0 \end{bmatrix} = \begin{bmatrix} .3 & .3 & 0 \\ .4 & .4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Here $x_1 + x_2 = 0$, with x_2 free. The general solution is not needed. Set $x_2 = 1$ to find an eigenvector $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. A suitable basis for $\mathbb{R}^2$ is $\{\mathbf{v}_1, \mathbf{v}_2\}$.

b. Write $\mathbf{x}_0 = \mathbf{v}_1 + c\mathbf{v}_2$: $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix} + c\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. By inspection, c is $-1/14$. (The value of c depends on how $\mathbf{v}_2$ is scaled.)

c. For $k = 1, 2, \ldots,$ define $\mathbf{x}_k = A^k\mathbf{x}_0$. Then $\mathbf{x}_1 = A(\mathbf{v}_1 + c\mathbf{v}_2) = A\mathbf{v}_1 + cA\mathbf{v}_2 = \mathbf{v}_1 + c(.3)\mathbf{v}_2$, because $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors. Again

$\mathbf{x}_2 = A\mathbf{x}_1 = A(\mathbf{v}_1 + c(.3)\mathbf{v}_2) = A\mathbf{v}_1 + c(.3)A\mathbf{v}_2 = \mathbf{v}_1 + c(.3)(.3)\mathbf{v}_2$. Continuing, the general pattern is $\mathbf{x}_k = \mathbf{v}_1 + c(.3)^k\mathbf{v}_2$. As k increases, the second term tends to $\mathbf{0}$ and so $\mathbf{x}_k$ tends to $\mathbf{v}_1$.

Chapter 5.3

1. $P = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, A = PDP^{-1}$, and $A^4 = PD^4P^{-1}$. We compute

$P^{-1} = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}, D^4 = \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}$, and $A^4 = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 226 & -525 \\ 90 & -209 \end{bmatrix}$.

5. By the Diagonalization Theorem, eigenvectors form the columns of the left factor, and they correspond respectively to the eigenvalues on the diagonal of the middle factor.

$\lambda = 5: \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; \lambda = 1: \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$.

7. Since A is triangular, its eigenvalues are obviously ± 1.

For $\lambda = 1$: $A - 1I = \begin{bmatrix} 0 & 0 \\ 6 & -2 \end{bmatrix}$. The equation $(A-1I)\mathbf{x} = \mathbf{0}$ amounts to $6x_1 - 2x_2 = 0$, so $x_1 = (1/3)x_2$ with x_2 free. The general solution is $x_2 \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

For $\lambda = -1$: $A + 1I = \begin{bmatrix} 2 & 0 \\ 6 & 0 \end{bmatrix}$. The equation $(A+1I)\mathbf{x} = \mathbf{0}$ amounts to $2x_1 = 0$, so $x_1 = 0$ with x_2 free. The general solution is $x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and a basis vector for the eigenspace is $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

From $\mathbf{v}_1$ and $\mathbf{v}_2$ construct $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$. Then set $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively.

11. The eigenvalues of A are given to be 1, 2, and 3.

For $\lambda = 3$: $A - 3I = \begin{bmatrix} -4 & 4 & -2 \\ -3 & 1 & 0 \\ -3 & 1 & 0 \end{bmatrix}$, and row reducing $\begin{bmatrix} A-3I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 0 & -1/4 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_3 \begin{bmatrix} 1/4 \\ 3/4 \\ 1 \end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$.

For $\lambda = 2$: $A - 2I = \begin{bmatrix} -3 & 4 & -2 \\ -3 & 2 & 0 \\ -3 & 1 & 1 \end{bmatrix}$, and row reducing $\begin{bmatrix} A-2I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 0 & -2/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_3 \begin{bmatrix} 2/3 \\ 1 \\ 1 \end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$.

For $\lambda = 1$: $A - I = \begin{bmatrix} -2 & 4 & -2 \\ -3 & 3 & 0 \\ -3 & 1 & 2 \end{bmatrix}$, and row reducing $\begin{bmatrix} A-1I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and a basis vector for the eigenspace is $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

From $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ construct $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 1 \\ 4 & 3 & 1 \end{bmatrix}$. Then set $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

23. A is diagonalizable because you know that five linearly independent eigenvectors exist: three in the three-dimensional eigenspace and two in the two-dimensional eigenspace. Theorem 7 guarantees that the set of all five eigenvectors is linearly independent.