Lesson 13

Chapter 7
Symmetric Matrices and Quadratic Forms

- Diagonalization of Symmetric Matrices
- Quadratic Forms
- Constrained Optimization
- The Singular Value Decomposition
- Applications to Image Processing and Statistics

7.1 Diagonalization of Symmetric Matrices

Definition:

Symmetric matrix: $A^T = A$ → Symmetric around the main diagonal: $a_{ij} = a_{ji}$

Ex 1

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & 8 \\ 0 & 8 & -7 \end{bmatrix} \quad \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \longleftarrow \text{Symmetric}$$

$$\begin{bmatrix} 1 & -3 \\ 3 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & -4 & 0 \\ -6 & 1 & -4 \\ 0 & -6 & 1 \end{bmatrix} \quad \begin{bmatrix} 5 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix} \longleftarrow \text{Nonsymmetric}$$

En matrix A er symmetrisk, hvis $A = A^T$

Lad nu en matrix A være defineret ved det *ydre produkt* af en vektor a: $A = \mathbf{a}\mathbf{a}^T$

Spørgsmål: Vil A (altid) være symmetrisk?

Beregning:

Definer fire vektorer **a** af dimension 4*1 med tilfældige talværdier, beregn A som ovenfor og kvalificer svaret.

Theorem 5.5, The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors of A.

Step 1: Eigenvalues: $\det(A - \lambda_i I) = 0$ (Characteristic Equation)

Step 2: Eigenvectors: $(A - \lambda_i I)\boldsymbol{v}_i = \boldsymbol{0}$; $\boldsymbol{v}_i \neq \boldsymbol{0}$

Step 3: Diagonalization matrix: $P = [\boldsymbol{v}_1 \ \boldsymbol{v}_2 \ \dots \ \boldsymbol{v}_n]$

Step 4: Diagonal matrix: $D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$

<u>Ex 2</u> $A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix} = PDP^{-1}$???

Step 1: Eigenvalues: $\det(A - \lambda I) = -\lambda^3 + 17\lambda^2 - 90\lambda + 144 = -(\lambda - 8)(\lambda - 6)(\lambda - 3) = 0$

$\Rightarrow \lambda_1 = 8;\ \ \lambda_2 = 6;\ \ \lambda_3 = 3$

Step 2: Eigenvectors: $(A - \lambda_i I)\boldsymbol{v}_i = \boldsymbol{0} \ \Rightarrow\ \lambda_1 = 8\text{:}\ \boldsymbol{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix};\quad \lambda_2 = 6\text{:}\ \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix};\quad \lambda_3 = 6\text{:}\ \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = \boldsymbol{v}_1 \cdot \boldsymbol{v}_3 = \boldsymbol{v}_2 \cdot \boldsymbol{v}_3 = 0$ → **Orthogonal basis** for $\mathbb{R}^3$

$\boldsymbol{u}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix};\quad \boldsymbol{u}_2 = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix};\quad \boldsymbol{u}_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$ → Orthonormal basis of eigenvectors

Ex 2

Step 3: Diagonalization matrix:

Theorem 6.6

$$P = [\boldsymbol{u}_1 \;\; \boldsymbol{u}_2 \;\; \boldsymbol{u}_3] = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$ → Orthogonal matrix → $P^{-1} = P^T = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$

Step 4: Diagonal matrix: $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix} = PDP^{-1} = PDP^T$ → **Orthogonally** diagonalizable ✓

Ex 3

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix} = PDP^{-1} \ ???$$

Step 1: Eigenvalues: $\det(A - \lambda I) = -\lambda^3 + 12\lambda^2 - 21\lambda - 98 = -(\lambda - 7)^2(\lambda + 2) = 0$

$\Rightarrow \lambda_1 = \lambda_2 = 7; \ \ \lambda_3 = -2$

Two-dimensional eigenspace

Step 2: Eigenvectors: $(A - \lambda_i I)\boldsymbol{v}_i = \boldsymbol{0} \ \Rightarrow \ \lambda_1 = \lambda_2 = 7\text{:} \ \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}; \ \ \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}; \quad \lambda_3 = -2\text{:} \ \boldsymbol{v}_3 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$

$\boldsymbol{v}_1 \cdot \boldsymbol{v}_3 = \boldsymbol{v}_2 \cdot \boldsymbol{v}_3 = 0; \quad \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = -1$ → Gram-Schmidt: $\boldsymbol{z}_2 = \boldsymbol{v}_2 - \frac{\boldsymbol{v}_2 \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}\boldsymbol{v}_1 = \begin{bmatrix} -1/2 \\ 2 \\ 1/2 \end{bmatrix}$

$\boldsymbol{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}; \quad \boldsymbol{u}_2 = \begin{bmatrix} -1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}; \quad \boldsymbol{u}_3 = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$ → Orthonormal basis of eigenvectors

Ex 3

Step 3: Diagonalization matrix:

Theorem 6.6

$$P = [\boldsymbol{u}_1 \;\; \boldsymbol{u}_2 \;\; \boldsymbol{u}_3] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \end{bmatrix}$$ → Orthogonal matrix → $$P^{-1} = P^T = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{18} & 4/\sqrt{18} & 1/\sqrt{18} \\ -2/3 & -1/3 & 2/3 \end{bmatrix}$$

Step 4: Diagonal matrix: $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix} = PDP^{-1} = PDP^T$$ → **Orthogonally** diagonalizable ✓

Definition:

Orthogonal matrix   Diagonal matrix

A is orthogonally diagonalizable: $\mathrm{A} = PDP^{-1}$

Theorem 7.1:

A is symmetric $\Rightarrow$ any two eigenvectors from different eigenspaces are orthogonal

Theorem 7.2:

An $n \times n$ matrix A is orthogonal diagonalizable $\Leftrightarrow$ A is symmetric

Numerical note: Using orthogonal matrices generally reduces numerical errors in calculations

Theorem 7.3: The Spectral Theorem for Symmetric Matrices

A $n \times n$ symmetric matrix A has the following properties

- A has n real eigenvalues, counting multiplicities.
- The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.
- The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.
- A is orthogonally diagonalizable.

Spectral decomposition:

Spectrum of A

Let: A a symmetric $n \times n$ matrix with eigenvalues: $\lambda_1, \cdots, \lambda_n$ and corresponding orthononal eigenvectors: $\boldsymbol{u}_1, \cdots, \boldsymbol{u}_n$.

Then:

$$A = PDP^{-1} = PDP^T = [\boldsymbol{u}_1 \ \ldots \ \boldsymbol{u}_n] \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1^T \\ \vdots \\ \boldsymbol{u}_n^T \end{bmatrix} = [\lambda_1 \boldsymbol{u}_1 \ \ldots \ \lambda_n \boldsymbol{u}_n] \begin{bmatrix} \boldsymbol{u}_1^T \\ \vdots \\ \boldsymbol{u}_n^T \end{bmatrix}$$

$$= \lambda_1 \boldsymbol{u}_1 \boldsymbol{u}_1^T + \cdots + \lambda_i \boldsymbol{u}_i \boldsymbol{u}_i^T + \cdots + \lambda_n \boldsymbol{u}_n \boldsymbol{u}_n^T$$

Spectral decomposition of A

Også kaldet:
egenværdi-dekomposition

$n \times n$ matrix
with rank 1

Projection matrix on
subspace spanned by $\boldsymbol{u}_i$:
$(\boldsymbol{u}_i \boldsymbol{u}_i^T)\boldsymbol{x} = proj_{\boldsymbol{u}_i}\boldsymbol{x}$

Ex 4

$$A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

$$\boldsymbol{u}_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \Longrightarrow \boldsymbol{u}_1\boldsymbol{u}_1^T = \begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \Longrightarrow \boldsymbol{u}_2\boldsymbol{u}_2^T = \begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{bmatrix}$$

Spectral decomposition:

$$A = 8\boldsymbol{u}_1\boldsymbol{u}_1^T + 3\boldsymbol{u}_2\boldsymbol{u}_2^T = \begin{bmatrix} 32/5 & 16/5 \\ 16/5 & 8/5 \end{bmatrix} + \begin{bmatrix} 3/5 & -6/5 \\ -6/5 & 12/5 \end{bmatrix} = \begin{bmatrix} 35/5 & 10/5 \\ 10/5 & 20/5 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} \checkmark$$

OPGAVE 1

For den symmetriske matrix A:

$$A = \begin{bmatrix} 1 & 2 & -5 \\ 2 & -3 & 0 \\ -5 & 0 & -1 \end{bmatrix}$$

- Bestem egenværdierne og de tilhørende *ortonormale* egenvektorer
- Udregn de tre matricer i den spektrale dekomposition og opskriv dekompositionen
- Gør prøve: Summerer de tre matricer til A?

- Er dekompositionen PDP^{-1} i øvrigt lig med PDP^{T} ??

7.2 Quadratic Forms

Definition:

Q is a **quadratic form** on $\mathbb{R}^n$ if:

$$Q(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x}$$

where $\boldsymbol{x}$ is a vector in $\mathbb{R}^n$ and A – the **matrix of the quadratic form** – is an $n \times n$ symmetric matrix .

Theorem 7.2

A symmetric $\Rightarrow$ $\exists$ orthogonal matrix P: $P^T AP = \mathrm{D}$, where D is a diagonal matrix.

See Chap. 4.4

Change of variable:

$\boldsymbol{x} = P\boldsymbol{y}$ (or $\boldsymbol{y} = P^{-1}\boldsymbol{x}$) → $\boldsymbol{y}$ is the coordinate vector of $\boldsymbol{x}$ in the orthonormal basis of $\mathbb{R}^n$ determined by the columns of P.

The Principal Axes

"Hovedakse"

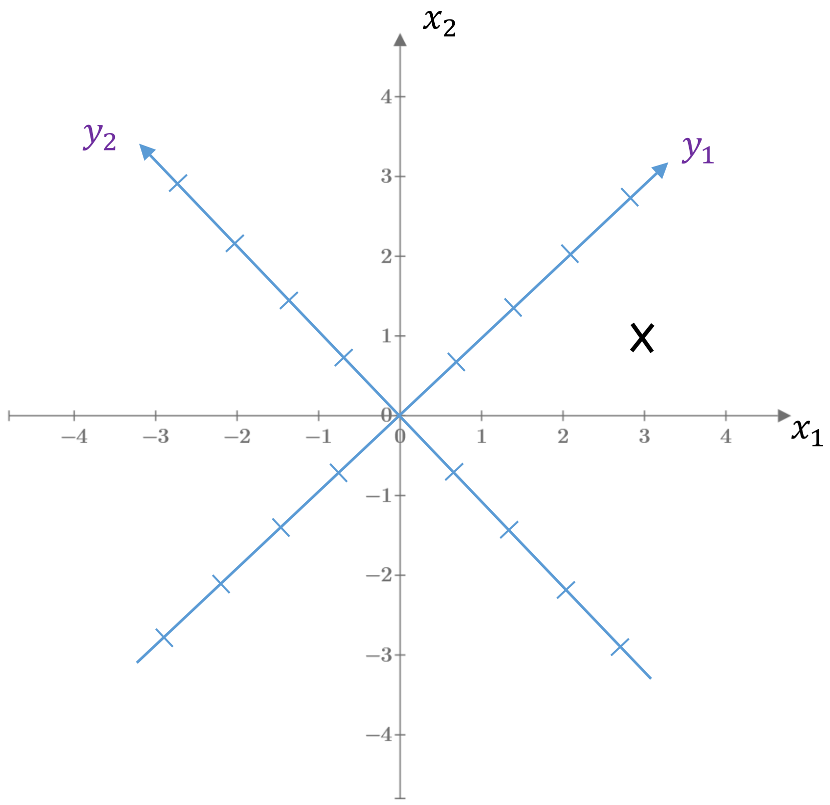
Theorem 7.4: The Principal Axes Theorem:

$$Q(\boldsymbol{x}) = \boldsymbol{x}^T A\boldsymbol{x} = (P\boldsymbol{y})^T A(P\boldsymbol{y}) = \boldsymbol{y}^T(P^T AP)\boldsymbol{y} = \boldsymbol{y}^T D\boldsymbol{y} = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$$

OBS: No cross-product terms!

<u>Ex 6</u> $Q(\boldsymbol{x}) = 5x_1^2 - 4x_1x_2 + 5x_2^2$

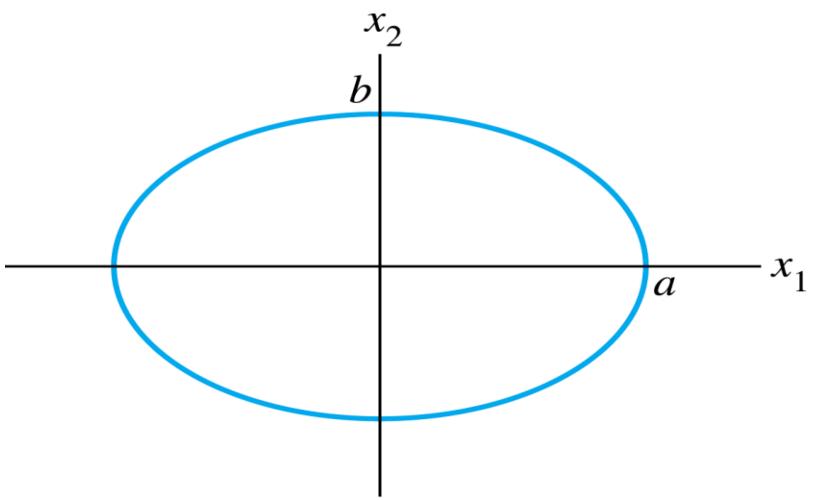
$\rightarrow$ $P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$; $D = \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$; $P^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$



Geometric interpretation of Principal Axes

$\boldsymbol{x}^T A \boldsymbol{x} = 1;\ \ \boldsymbol{x} \in \mathbb{R}^2,\ A$ diagonal 2×2 matrix

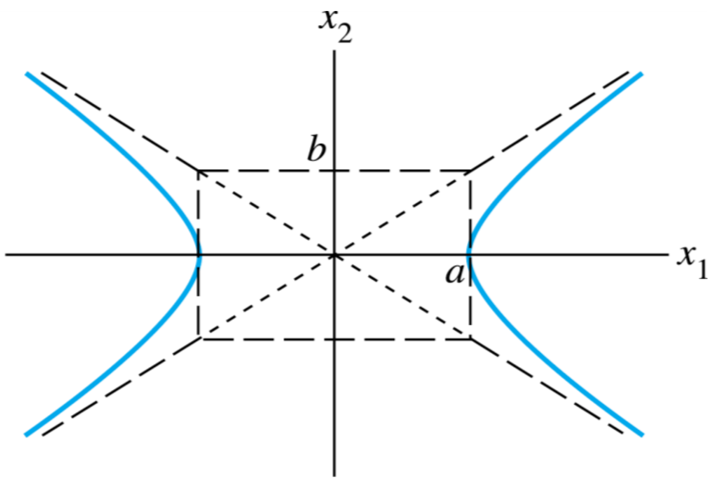
$$A = \begin{bmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{bmatrix}$$



$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1,\ a > b > 0$$

ellipse

$$A = \begin{bmatrix} 1/a^2 & 0 \\ 0 & -1/b^2 \end{bmatrix}$$



$$\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1,\ a > b > 0$$

hyperbola

Geometric interpretation of Principal Axes (Ex 6)

$\boldsymbol{x}^T A \boldsymbol{x} = c$; $\boldsymbol{x} \in \mathbb{R}^2$, $c \in \mathbb{R}$, A symmetric 2×2 matrix

Principal Axes

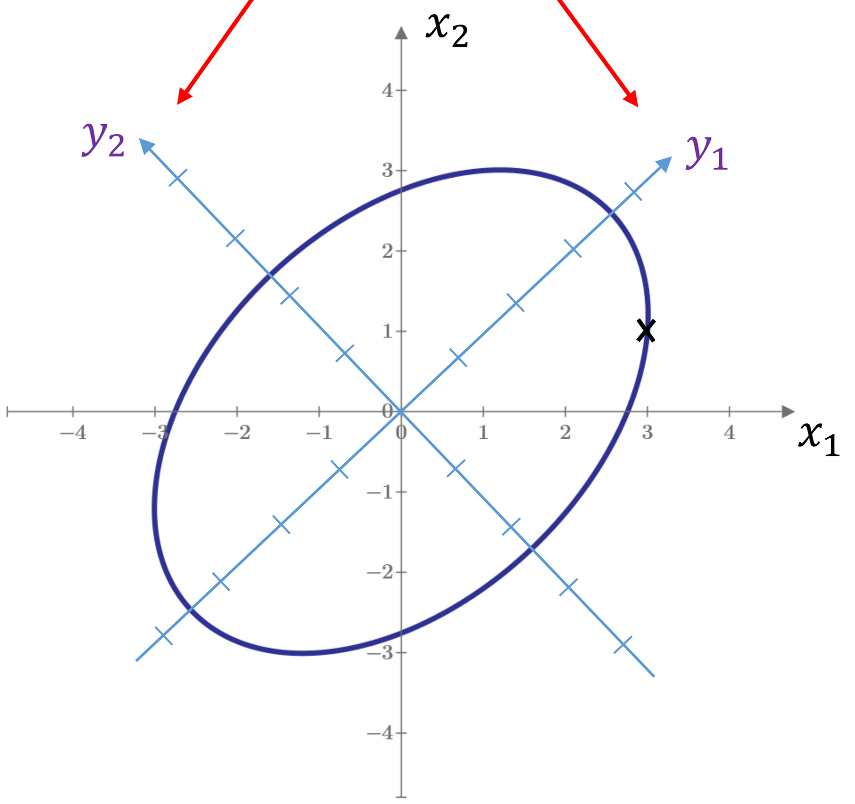
$$A = \begin{bmatrix} a & d \\ d & b \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}; \quad c = 38$$

$$\boldsymbol{x}^T A \boldsymbol{x} = c \quad \Leftrightarrow \quad 5x_1^2 - 4x_1x_2 + 5x_2^2 = 38$$

$$\lambda_1 = 3: \ \boldsymbol{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \lambda_2 = 7: \ \boldsymbol{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$$

$$\boldsymbol{y} = P^{-1}\boldsymbol{x} = \frac{1}{\sqrt{2}}\begin{bmatrix} x_1 + x_2 \\ -x_1 + x_2 \end{bmatrix}$$



$5x_1^2 - 4x_1x_2 + 5x_2^2 = 38$

$3y_1^2 + 7y_2^2 = 38$

$$\boldsymbol{x}^T A \boldsymbol{x} = \boldsymbol{y}^T D \boldsymbol{y} = c \quad \Leftrightarrow \quad 3y_1^2 + 7y_2^2 = 38 \quad \Leftrightarrow \quad \frac{y_1^2}{\left(\sqrt{38/3}\right)^2} + \frac{y_2^2}{\left(\sqrt{38/7}\right)^2} = 1 \longleftarrow \text{Ellipse}$$

Geometric interpretation of Principal Axes

$\boldsymbol{x}^T A \boldsymbol{x} = c;\ \ \boldsymbol{x} \in \mathbb{R}^2,\ \ c \in \mathbb{R},\ \ A$ symmetric 2×2 matrix

$$A = \begin{bmatrix} a & d \\ d & -b \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ -4 & -5 \end{bmatrix};\ \ c = 16$$

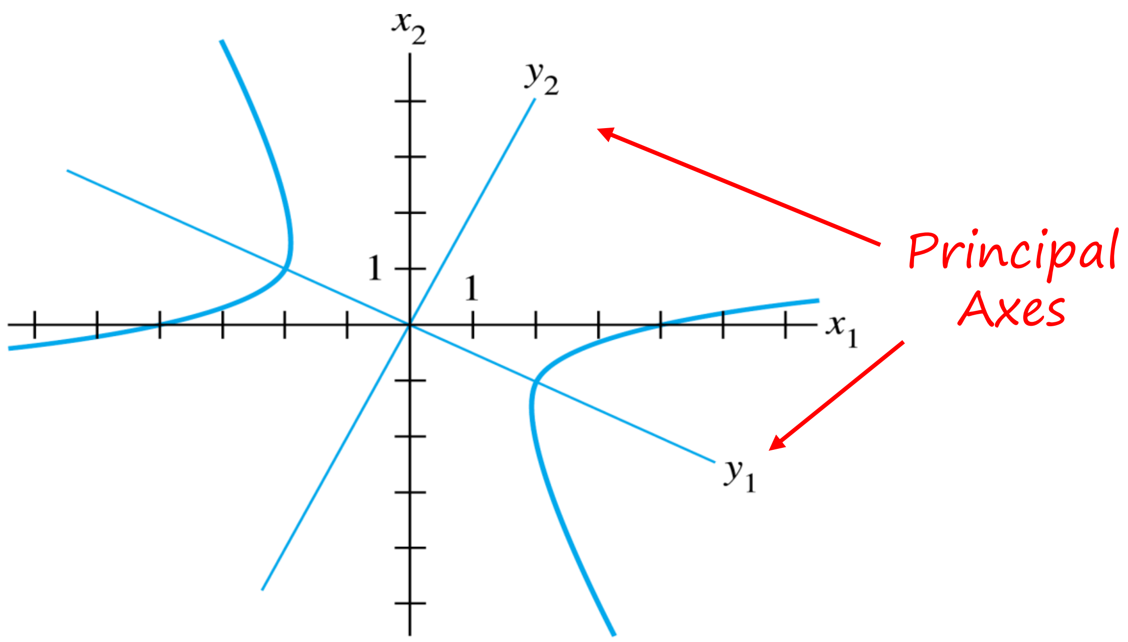
$$\boldsymbol{x}^T A \boldsymbol{x} = c \ \ \Leftrightarrow\ \ x_1^2 - 8x_1x_2 - 5x_2^2 = 16$$

$$\lambda_1 = 3:\ \boldsymbol{u}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad \lambda_2 = -7:\ \boldsymbol{u}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$P = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}$$

$$\boldsymbol{y} = P^{-1}\boldsymbol{x} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2x_1 - x_2 \\ x_1 + 2x_2 \end{bmatrix}$$

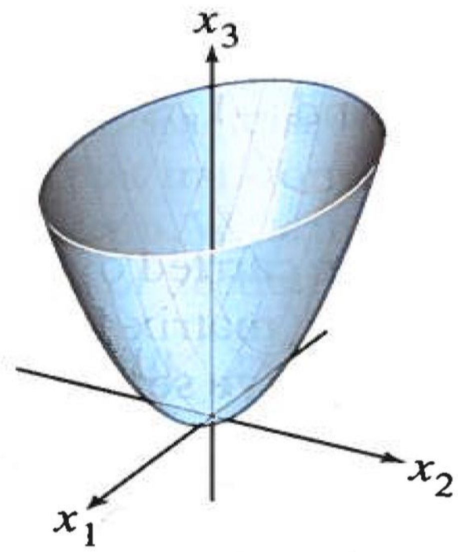
$$\boldsymbol{x}^T A \boldsymbol{x} = \boldsymbol{y}^T D \boldsymbol{y} = c \ \ \Leftrightarrow\ \ 3y_1^2 - 7y_2^2 = 16 \ \Leftrightarrow\ \frac{y_1^2}{\left(4/\sqrt{3}\right)^2} - \frac{y_2^2}{\left(4/\sqrt{7}\right)^2} = 1$$



(b) $x_1^2 - 8x_1x_2 - 5x_2^2 = 16$

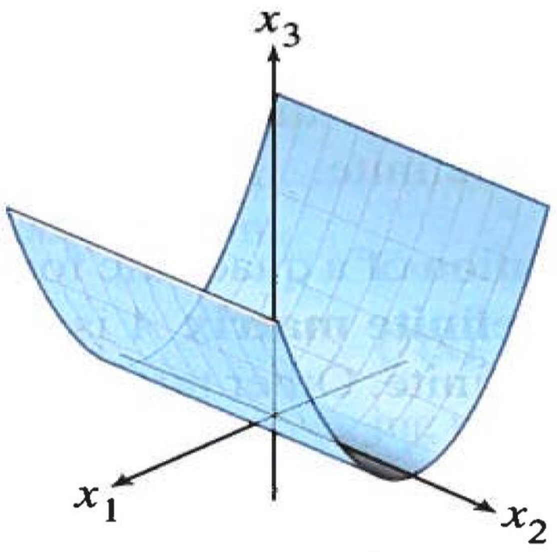
$3y_1^2 - 7y_2^2 = 16$

Classifying Quadratic Forms



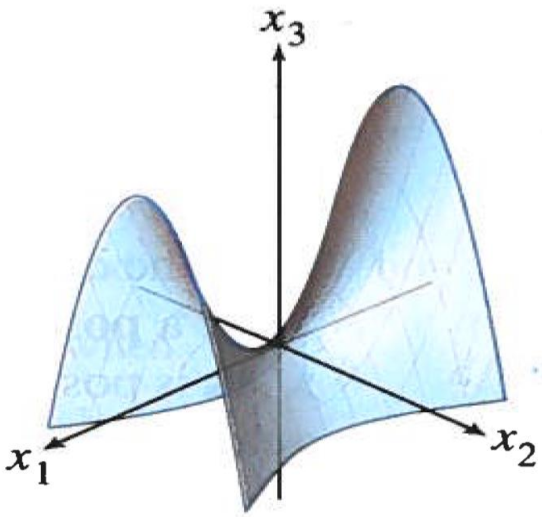
(a) $z = 3x_1^2 + 7x_2^2$

Positive definite:
$Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$



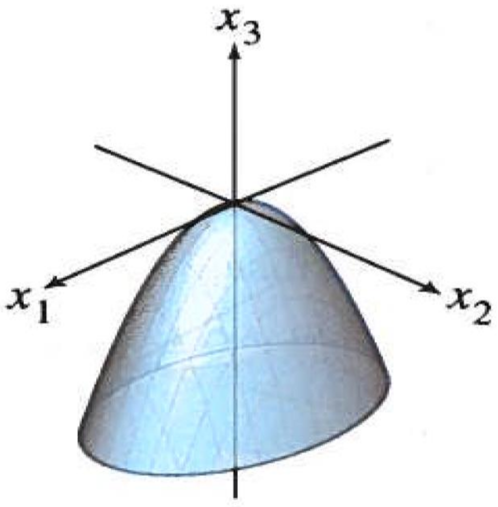
(b) $z = 3x_1^2$

Positive semidefinite:
$Q(\mathbf{x}) \geq 0$ for all $\mathbf{x}$



(c) $z = 3x_1^2 - 7x_2^2$

Indefinite:
$Q(\mathbf{x})$ both positive and negative



(d) $z = -3x_1^2 - 7x_2^2$

Negative definite:
$Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$

Theorem 7.5, Quadratic forms and eigenvalues

Let A be an $n \times n$ symmetric matrix. Then a quadratic form $\mathbf{x}^T A\mathbf{x}$ is:

a: Positive definite if and only if the eigenvalues of A are all positive.

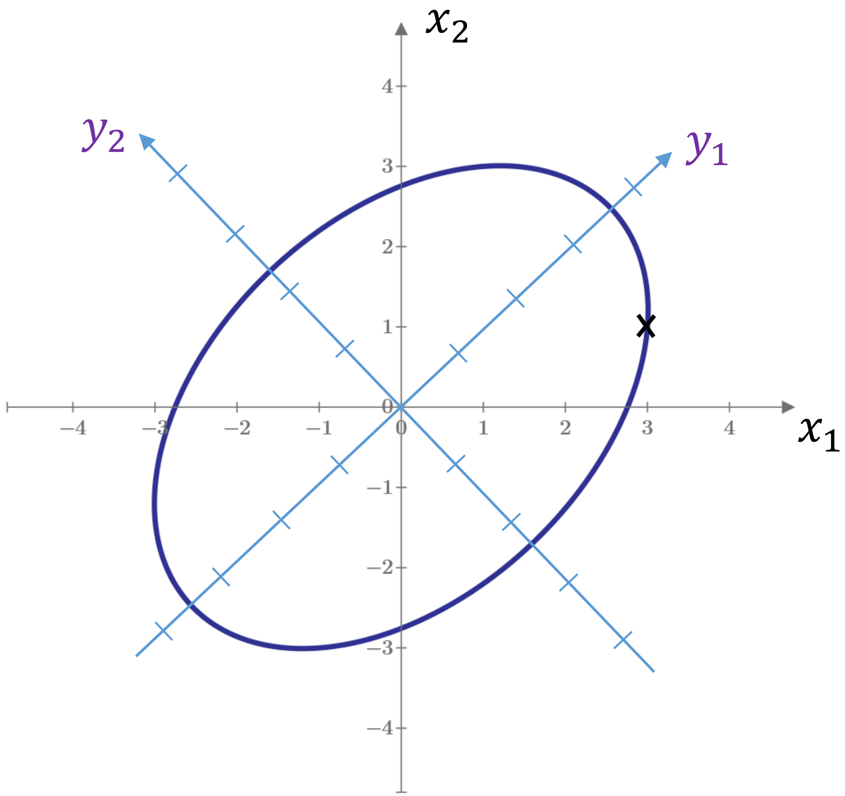
b: Negative definite if and only if the eigenvalues of A are all negative.

c: Indefinite if and only if A has both positive and negative eigenvalues.

Note: some times the term *nonnegative definite* is also used for *positive semidefinite*

Diskussion:

I hvilke tilfælde af beskrivelse af datasæt kunne det være smart at lave koordinattransformation og dermed danne nye hovedakser i koordinatsystemet?



7.3 Constrained Optimization

Constrained optimization

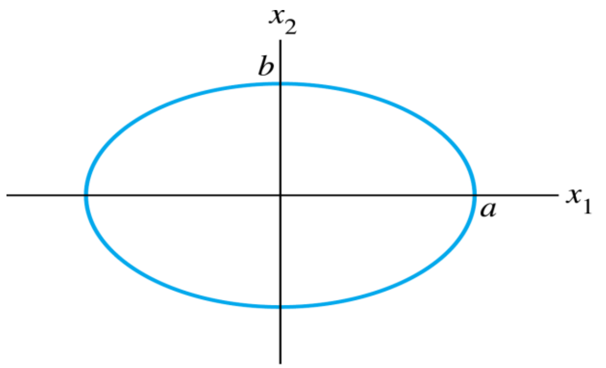
Finding max/min of a quadratic form $Q(\boldsymbol{x})$ for $\boldsymbol{x}$ in some specific (constrained) set. Typically with constrainment:

$$\|\boldsymbol{x}\|^2 = \|\boldsymbol{x}\| = \boldsymbol{x}^T\boldsymbol{x} = x_1^2 + \cdots + x_n^2 = 1$$

Ex 7

$$Q(\boldsymbol{x}) = 3x_1^2 + 7x_2^2 = \boldsymbol{x}^T A\boldsymbol{x}; \qquad A = \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} \rightarrow \begin{cases} \lambda_1 = 7: \ \boldsymbol{u}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \lambda_2 = 3: \ \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{cases}$$

$$\|\boldsymbol{x}\|^2 = \boldsymbol{x}^T\boldsymbol{x} = x_1^2 + x_2^2 = 1$$



$Q(\boldsymbol{x}) = 3x_1^2 + 7x_2^2 = c$

$$\left.\begin{array}{r} Q(\boldsymbol{x}) = 3x_1^2 + 7x_2^2 \ \le \ 7x_1^2 + 7x_2^2 = 7\|\boldsymbol{x}\|^2 = 7 \\ Q(\pm\boldsymbol{u}_1) = Q(0, \pm 1) = 7 \end{array}\right\} \Rightarrow Q_{max} = 7 = \lambda_{max}$$

$$\left.\begin{array}{r} Q(\boldsymbol{x}) = 3x_1^2 + 7x_2^2 \ \ge 3x_1^2 + 3x_2^2 = 3\|\boldsymbol{x}\|^2 = 3 \\ Q(\pm\boldsymbol{u}_2) = Q(\pm 1{,}0) = 3 \end{array}\right\} \Rightarrow Q_{min} = 3 = \lambda_{min}$$

$$\|\boldsymbol{x}\|^2 = \boldsymbol{x}^T\boldsymbol{x} = 1 \ \Rightarrow \ 3 \le Q(\boldsymbol{x}) \le 7 \ \text{(continuous)}$$

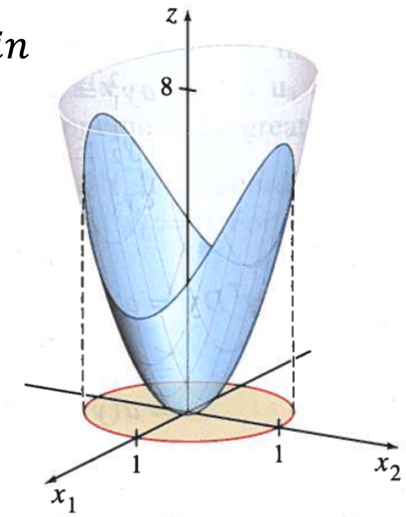


FIGURE 1 $z = 3x_1^2 + 7x_2^2$.

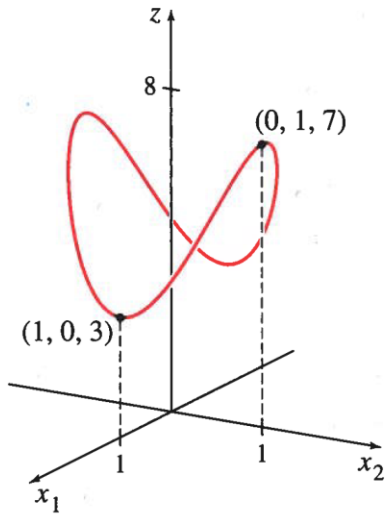


FIGURE 2 The intersection of $z = 3x_1^2 + 7x_2^2$ and the cylinder $x_1^2 + x_2^2 = 1$.

Ex 8

Symmetric matrix Eigenvalues/-vectors

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix} \rightarrow \lambda_1 = 6\text{: } \boldsymbol{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix};\ \lambda_2 = 3\text{: } \boldsymbol{u}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix};\ \lambda_3 = 1\text{: } \boldsymbol{u}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Change of coordinates: $\boldsymbol{x} = P\boldsymbol{y} \Rightarrow \|\boldsymbol{x}\|^2 = \|P\boldsymbol{y}\|^2 = (P\boldsymbol{y})^T(P\boldsymbol{y}) = \boldsymbol{y}^T P^T P\boldsymbol{y} = \boldsymbol{y}^T I\boldsymbol{y} = \boldsymbol{y}^T\boldsymbol{y} = \|\boldsymbol{y}\|^2$

Constrain: $\|\boldsymbol{x}\|^2 = 1 \iff \|\boldsymbol{y}\|^2 = 1$

Diagonalization: $\boldsymbol{x} = P\boldsymbol{y} \Rightarrow \boldsymbol{x}^T A\boldsymbol{x} = \boldsymbol{y}^T D\boldsymbol{y}$, where $\mathrm{D} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow 1 \le \boldsymbol{x}^T A\boldsymbol{x} = \boldsymbol{y}^T D\boldsymbol{y} \le 6$

Constrained limits: $\Rightarrow 1 \le Q(\boldsymbol{x}) = \boldsymbol{x}^T A\boldsymbol{x} = 3x_1^2 + 3x_2^2 + 4x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3 \le 6$

Maximum value/vector: $\Rightarrow Q(\boldsymbol{u}_1) = Q\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = 6$

Minimum value/vector: $\Rightarrow Q(\boldsymbol{u}_3) = Q\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) = 1$

Theorem 7.6

Let A be a symmetric matrix and let m and M be defined as

$$m = \min\{\mathbf{x}^T A\mathbf{x} \ : \ ||\mathbf{x}|| = 1\}, \quad M = \max\{\mathbf{x}^T A\mathbf{x} \ : \ ||\mathbf{x}|| = 1\},$$

Then M is the greatest eigenvalue λ_1 of A and m is the least eigenvalue of A. The value of $\mathbf{x}^T A\mathbf{x}$ is M when $\mathbf{x}$ is a unit eigenvector $\mathbf{u}_1$ corresponding to λ_1. The value of $\mathbf{x}^T A\mathbf{x}$ is m when $\mathbf{x}$ is a unit eigenvector corresponding to m.

For $||\boldsymbol{x}||$ =1: $Q(\boldsymbol{x}) = \boldsymbol{x}^T A\boldsymbol{x}$ will take all values between m and M.

Todays words and concepts

Orthogonal diagonalizable

Symmetric matrix

Matrix of quadratic form

Indefinite

Negative definite

Constrained optimization

The Spectral Theorem

The Principal Axes Theorem

Positive definite

Positive semidefinite

Principal Axes

Spectral decomposition

Quadratic forms

Negative semidefinite