Chapter 5.5

1. $A=\begin{bmatrix}1 & -2\\ 1 & 3\end{bmatrix}$, $A-\lambda I=\begin{bmatrix}1-\lambda & -2\\ 1 & 3-\lambda\end{bmatrix}$. $\det(A-\lambda I)=(1-\lambda)(3-\lambda)-(-2)=\lambda^2-4\lambda+5$. Use the quadratic formula to find the eigenvalues: $\lambda=\dfrac{4\pm\sqrt{16-20}}{2}=2\pm i$. Example 2 gives a shortcut for finding one eigenvector, and Example 5 shows how to write the other eigenvector with no effort.

<u>For $\lambda = 2+i$:</u> $A-(2+i)I=\begin{bmatrix}-1-i & -2\\ 1 & 1-i\end{bmatrix}$. The equation $(A-\lambda I)\mathbf{x}=\mathbf{0}$ gives

$$\begin{aligned}(-1-i)x_1-2x_2&=0\\ x_1+(1-i)x_2&=0\end{aligned}$$

As in Example 2, the two equations are equivalent—each determines the same relation between x_1 and x_2. So use the second equation to obtain $x_1=-(1-i)x_2$, with x_2 free. The general solution is $x_2\begin{bmatrix}-1+i\\ 1\end{bmatrix}$, and the vector $\mathbf{v}_1=\begin{bmatrix}-1+i\\ 1\end{bmatrix}$ provides a basis for the eigenspace.

<u>For $\lambda = 2-i$:</u> Let $\mathbf{v}_2=\overline{\mathbf{v}}_1=\begin{bmatrix}-1-i\\ 1\end{bmatrix}$. The remark prior to Example 5 shows that $\mathbf{v}_2$ is automatically an eigenvector for $\overline{2+i}$. In fact, calculations similar to those above would show that $\{\mathbf{v}_2\}$ is a basis for the eigenspace. (In general, for a real matrix A, it can be shown that the set of complex conjugates of the vectors in a basis of the eigenspace for λ is a basis of the eigenspace for $\overline{\lambda}$.)

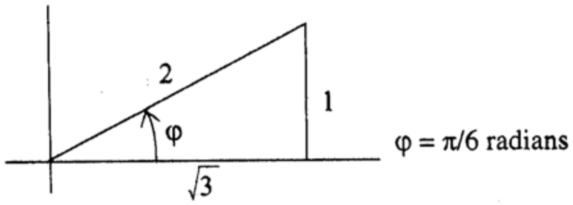
2. $A=\begin{bmatrix}5 & -5\\ 1 & 1\end{bmatrix}$. The characteristic polynomial is $\lambda^2-6\lambda+10$, so the eigenvalues of A are $\lambda=\dfrac{6\pm\sqrt{36-40}}{2}=3\pm i$.

<u>For $\lambda = 3+i$:</u> $A-(3+i)I=\begin{bmatrix}2-i & -5\\ 1 & -2-i\end{bmatrix}$. The equation $(A-(3+i)I)\mathbf{x}=\mathbf{0}$ amounts to $x_1+(-2-i)x_2=0$, so $x_1=(2+i)x_2$ with x_2 free. A basis vector for the eigenspace is thus $\mathbf{v}_1=\begin{bmatrix}2+i\\ 1\end{bmatrix}$.

<u>For $\lambda = 3-i$:</u> A basis vector for the eigenspace is $\mathbf{v}_2=\overline{\mathbf{v}}_1=\begin{bmatrix}2-i\\ 1\end{bmatrix}$.

7. $A=\begin{bmatrix}\sqrt{3} & -1\\ 1 & \sqrt{3}\end{bmatrix}$. From Example 6, the eigenvalues are $\sqrt{3}\pm i$. The scale factor for the transformation $\mathbf{x}\mapsto A\mathbf{x}$ is $r=|\lambda|=\sqrt{(\sqrt{3})^2+1^2}=2$. For the angle of rotation, plot the point $(a,b)=(\sqrt{3},1)$ in the xy-plane and use trigonometry:

$$\varphi=\arctan(b/a)=\arctan(1/\sqrt{3})=\pi/6 \text{ radians.}$$



25. Write $\mathbf{x} = \text{Re } \mathbf{x} + i(\text{Im } \mathbf{x})$, so that $A\mathbf{x} = A(\text{Re } \mathbf{x}) + iA(\text{Im } \mathbf{x})$. Since A is real, so are $A(\text{Re } \mathbf{x})$ and $A(\text{Im } \mathbf{x})$. Thus $A(\text{Re } \mathbf{x})$ is the real part of $A\mathbf{x}$ and $A(\text{Im } \mathbf{x})$ is the imaginary part of $A\mathbf{x}$.

26. **a**. If $\lambda = a - bi$, then $A\mathbf{v} = \lambda\mathbf{v} = (a - bi)(\text{Re } \mathbf{v} + i \text{ Im } \mathbf{v}) = \underbrace{(a \text{ Re } \mathbf{v} + b \text{ Im } \mathbf{v})}_{\text{Re } Av} + i\underbrace{(a \text{ Im } \mathbf{v} - b \text{ Re } \mathbf{v})}_{\text{Im } Av}$. By Exercise 25,

$$A(\text{Re } \mathbf{v}) = \text{Re } A\mathbf{v} = a \text{ Re } \mathbf{v} + b \text{ Im } \mathbf{v}$$
$$A(\text{Im } \mathbf{v}) = \text{Im } A\mathbf{v} = -b \text{ Re } \mathbf{v} + a \text{ Im } \mathbf{v}$$

b. Let $P = \begin{bmatrix} \text{Re } \mathbf{v} & \text{Im } \mathbf{v} \end{bmatrix}$. By (a), $A(\text{Re } \mathbf{v}) = P\begin{bmatrix} a \\ b \end{bmatrix}, A(\text{Im } \mathbf{v}) = P\begin{bmatrix} -b \\ a \end{bmatrix}$. So

$$AP = \begin{bmatrix} A(\text{Re } \mathbf{v}) & A(\text{Im } \mathbf{v}) \end{bmatrix} = \begin{bmatrix} P\begin{bmatrix} a \\ b \end{bmatrix} & P\begin{bmatrix} -b \\ a \end{bmatrix} \end{bmatrix} = P\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = PC.$$

Chapter 5.7

1. From the "eigendata" (eigenvalues and corresponding eigenvectors) given, the eigenfunctions for the differential equation $\mathbf{x}' = A\mathbf{x}$ are $\mathbf{v}_1 e^{4t}$ and $\mathbf{v}_2 e^{2t}$. The general solution of $\mathbf{x}' = A\mathbf{x}$ has the form $c_1\begin{bmatrix}-3\\1\end{bmatrix}e^{4t} + c_2\begin{bmatrix}-1\\1\end{bmatrix}e^{2t}$. The initial condition $\mathbf{x}(0) = \begin{bmatrix}-6\\1\end{bmatrix}$ determines c_1 and c_2:

$c_1\begin{bmatrix}-3\\1\end{bmatrix}e^{4(0)} + c_2\begin{bmatrix}-1\\1\end{bmatrix}e^{2(0)} = \begin{bmatrix}-6\\1\end{bmatrix}$. Solving the system: $\begin{bmatrix}-3 & -1 & -6\\1 & 1 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 5/2\\0 & 1 & -3/2\end{bmatrix}$.

Thus $c_1 = 5/2, c_2 = -3/2,$ and $\mathbf{x}(t) = \frac{5}{2}\begin{bmatrix}-3\\1\end{bmatrix}e^{4t} - \frac{3}{2}\begin{bmatrix}-1\\1\end{bmatrix}e^{2t}$.

5. $A = \begin{bmatrix}7 & -1\\3 & 3\end{bmatrix}$, $\det(A - \lambda I) = \lambda^2 - 10\lambda + 24 = (\lambda - 4)(\lambda - 6) = 0$. Eigenvalues: 4 and 6.

<u>For $\lambda = 4$:</u> $\begin{bmatrix}3 & -1 & 0\\3 & -1 & 0\end{bmatrix} \sim \begin{bmatrix}1 & -1/3 & 0\\0 & 0 & 0\end{bmatrix}$, so $x_1 = (1/3)x_2$ with x_2 free. Take $x_2 = 3$ and $\mathbf{v}_1 = \begin{bmatrix}1\\3\end{bmatrix}$.

<u>For $\lambda = 6$:</u> $\begin{bmatrix}1 & -1 & 0\\3 & -3 & 0\end{bmatrix} \sim \begin{bmatrix}1 & -1 & 0\\0 & 0 & 0\end{bmatrix}$, so $x_1 = x_2$ with x_2 free. Take $x_2 = 1$ and $\mathbf{v}_2 = \begin{bmatrix}1\\1\end{bmatrix}$.

For the initial condition $\mathbf{x}(0) = \begin{bmatrix}3\\2\end{bmatrix}$, find c_1 and c_2 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{x}(0)$:

$\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{x}(0)\end{bmatrix} = \begin{bmatrix}1 & 1 & 3\\3 & 1 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -1/2\\0 & 1 & 7/2\end{bmatrix}$. Thus $c_1 = -1/2, c_2 = 7/2,$ and

$\mathbf{x}(t) = -\frac{1}{2}\begin{bmatrix}1\\3\end{bmatrix}e^{4t} + \frac{7}{2}\begin{bmatrix}1\\1\end{bmatrix}e^{6t}$.

Since both eigenvalues are positive, the origin is a repellor of the dynamical system described by $\mathbf{x}' = A\mathbf{x}$. The direction of greatest repulsion is the line through $\mathbf{v}_2$ and the origin.

7. From Exercise 5, $A=\begin{bmatrix}7 & -1\\ 3 & 3\end{bmatrix}$, with eigenvectors $\mathbf{v}_1=\begin{bmatrix}1\\ 3\end{bmatrix}$ and $\mathbf{v}_2=\begin{bmatrix}1\\ 1\end{bmatrix}$ corresponding to eigenvalues 4 and 6 respectively. To decouple the equation $\mathbf{x}'=A\mathbf{x}$, set $P=[\mathbf{v}_1\ \ \mathbf{v}_2]=\begin{bmatrix}1 & 1\\ 3 & 1\end{bmatrix}$ and let $D=\begin{bmatrix}4 & 0\\ 0 & 6\end{bmatrix}$, so that $A=PDP^{-1}$ and $D=P^{-1}AP$. Substituting $\mathbf{x}(t)=P\mathbf{y}(t)$ into $\mathbf{x}'=A\mathbf{x}$ we have $\dfrac{d}{dt}(P\mathbf{y})=A(P\mathbf{y})=PDP^{-1}(P\mathbf{y})=PD\mathbf{y}$. Since P has constant entries, $\frac{d}{dt}(P\mathbf{y})=P(\frac{d}{dt}(\mathbf{y}))$, so that left-multiplying the equality $P(\frac{d}{dt}(\mathbf{y}))=PD\mathbf{y}$ by P^{-1} yields $\mathbf{y}'=D\mathbf{y}$, or $\begin{bmatrix}y_1'(t)\\ y_2'(t)\end{bmatrix}=\begin{bmatrix}4 & 0\\ 0 & 6\end{bmatrix}\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix}$.

9. $A=\begin{bmatrix}-3 & 2\\ -1 & -1\end{bmatrix}$. An eigenvalue of A is $-2+i$ with corresponding eigenvector $\mathbf{v}=\begin{bmatrix}1-i\\ 1\end{bmatrix}$. The complex eigenfunctions $\mathbf{v}e^{\lambda t}$ and $\overline{\mathbf{v}}e^{\overline{\lambda}t}$ form a basis for the set of all complex solutions to $\mathbf{x}'=A\mathbf{x}$. The general complex solution is $c_1\begin{bmatrix}1-i\\ 1\end{bmatrix}e^{(-2+i)t}+c_2\begin{bmatrix}1+i\\ 1\end{bmatrix}e^{(-2-i)t}$, where c_1 and c_2 are arbitrary complex numbers. To build the general real solution, rewrite $\mathbf{v}e^{(-2+i)t}$ as:

$$\begin{aligned}
\mathbf{v}e^{(-2+i)t} &= \begin{bmatrix}1-i\\ 1\end{bmatrix}e^{-2t}e^{it}=\begin{bmatrix}1-i\\ 1\end{bmatrix}e^{-2t}(\cos t+i\sin t)\\
&=\begin{bmatrix}\cos t-i\cos t+i\sin t-i^2\sin t\\ \cos t+i\sin t\end{bmatrix}e^{-2t}\\
&=\begin{bmatrix}\cos t+\sin t\\ \cos t\end{bmatrix}e^{-2t}+i\begin{bmatrix}\sin t-\cos t\\ \sin t\end{bmatrix}e^{-2t}
\end{aligned}$$

The general real solution has the form $c_1\begin{bmatrix}\cos t+\sin t\\ \cos t\end{bmatrix}e^{-2t}+c_2\begin{bmatrix}\sin t-\cos t\\ \sin t\end{bmatrix}e^{-2t}$, where c_1 and c_2 now are real numbers. The trajectories are spirals because the eigenvalues are complex. The spirals tend toward the origin because the real parts of the eigenvalues are negative.