In the grading of the exercises special attention will be paid to check that answers are substantiated and that the procedure of calculations is well documented. When results are achieved using a calculator or a pc it should be noted in the paper. All 6 problems are weighted equally in the grading. Your solution to the problem set can be written in Danish or English as you prefer.

PROBLEM 1.
Let the matrix A and the vector $\mathbf{b}$ be given by

$$A = \begin{bmatrix} 1 & -2 & 2 & 6 \\ 2 & -3 & 4 & 9 \end{bmatrix}, \qquad \text{and } \mathbf{b} = \begin{bmatrix} -6 \\ -8 \end{bmatrix}$$

1. Solve the matrix equation $A\mathbf{x} = \mathbf{b}$.

Let B be an invertible $n \times n$ matrix.

2. Reduce the expression $B^2 B^T B B^{-1} (B^{-1})^T B (B^{-1})^2$ as much as possible and account for the rules used in each step of the reduction.

PROBLEM 2.
Let the matrix A be given by

$$A = \begin{bmatrix} 3-2q & 1 \\ 4 & 3+2q \end{bmatrix},$$

where q is a scalar.

1. Calculate q so that

$$A^2 = \begin{bmatrix} 29 & 6 \\ 24 & 125 \end{bmatrix}$$

PROBLEM 3.
Let a matrix be given by

$$A = \begin{bmatrix} 4 & 5 & 7 \\ 8 & 10 & 14 \\ 4 & 5 & 7 \\ 12 & 20 & 16 \end{bmatrix}$$

1. Determine the dimension of the column space of A, col A.
2. Determine an orthogonal basis for col A.

Let a vector $\mathbf{b}$ be given by

$$\mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

3. Calculate the orthogonal projection of $\mathbf{b}$ onto col A.

PROBLEM 4.
In case 1 cubic splines was introduced. In this problem we will restrict ourselves to work with quadratic splines to reduce the complexity of the calculations somewhat. Completely similar to the cubic splines, the quadratic spline must pass through the given datapoints. Let the following datapoints be given.

x	y
2	1
4	2
6	7

The quadratic spline is defined by

$$y(x) = \begin{cases} a_0 + a_1 x + a_2 x^2 & 2 \leq x \leq 4 \\ b_0 + b_1 x + b_2 x^2 & 4 \leq x \leq 6 \end{cases}$$

The spline function and its derivative must be continuous at the $x = 4$ data point. Further it can be assumed that $y'(2) = 0$.

1. Write down the equations needed to determine a_0, a_1, a_2, b_0, b_1 and b_2.
2. Solve the above equations and write down the full expression for the quadratic spline.

PROBLEM 5.
The difference equation $\mathbf{x}(k+1) = A\mathbf{x}(k)$ has the general solution $\mathbf{x}(k) = A^k \mathbf{x}(0)$. Consider the case where A is a diagonalizable 2×2 matrix with two real eigenvalues, λ_1 and λ_2 and assume that $\mathbf{x}(0) \neq \mathbf{0}$.

1. Explain how different values for the eigenvalues λ_1 and λ_2 influences $\mathbf{x}(k)$ as $k \to \infty$ for the three cases of

   (a) $\lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{1}{2}$ (b) $\lambda_1 = 1, \lambda_2 = 1$ (c) $\lambda_1 = 2, \lambda_2 = 2$

PROBLEM 6.
Consider the vector space consisting of all polynomial functions functions defined on the interval $0 \leq t \leq 1$. Let the inner product between vectors f and g be defined as

$$< f, g >= \int_0^1 f(t)g(t)\, dt$$

Let two vectors be defined as

$$p_1(t) = 1 + t, \qquad p_2(t) = 2 + t^2$$

1. Calculate the distance between $p_1(t)$ and $p_2(t)$.

The Cauchy-Schwarz inequality states that

$$|< f, g >| \leq \|f\| \|g\|$$

where equality only holds if $f = cg$ for some constant c.

2. Use the Cauchy-Schwarz inequality to show that the vectors $p_1(t)$ and $p_2(t)$ are linearly independent.