Chapter 2.1

1. $-2A=(-2)\begin{bmatrix}2&0&-1\\4&-5&2\end{bmatrix}=\begin{bmatrix}-4&0&2\\-8&10&-4\end{bmatrix}$. Next, use $B-2A=B+(-2A)$:

$$B-2A=\begin{bmatrix}7&-5&1\\1&-4&-3\end{bmatrix}+\begin{bmatrix}-4&0&2\\-8&10&-4\end{bmatrix}=\begin{bmatrix}3&-5&3\\-7&6&-7\end{bmatrix}.$$

The product AC is not defined because the number of columns of A does not match the number of rows of C. $CD=\begin{bmatrix}1&2\\-2&1\end{bmatrix}\begin{bmatrix}3&5\\-1&4\end{bmatrix}=\begin{bmatrix}1\cdot3+2(-1)&1\cdot5+2\cdot4\\-2\cdot3+1(-1)&-2\cdot5+1\cdot4\end{bmatrix}=\begin{bmatrix}1&13\\-7&-6\end{bmatrix}$. For mental computation, the row-column rule is probably easier to use than the definition.

2. $A+2B=\begin{bmatrix}2&0&-1\\4&-5&2\end{bmatrix}+2\begin{bmatrix}7&-5&1\\1&-4&-3\end{bmatrix}=\begin{bmatrix}2+14&0-10&-1+2\\4+2&-5-8&2-6\end{bmatrix}=\begin{bmatrix}16&-10&1\\6&-13&-4\end{bmatrix}$

The expression $3C-E$ is not defined because $3C$ has 2 columns and $-E$ has only 1 column.

$$CB=\begin{bmatrix}1&2\\-2&1\end{bmatrix}\begin{bmatrix}7&-5&1\\1&-4&-3\end{bmatrix}=\begin{bmatrix}1\cdot7+2\cdot1&1(-5)+2(-4)&1\cdot1+2(-3)\\-2\cdot7+1\cdot1&-2(-5)+1(-4)&-2\cdot1+1(-3)\end{bmatrix}=\begin{bmatrix}9&-13&-5\\-13&6&-5\end{bmatrix}$$

The product EB is not defined because the number of columns of E does not match the number of rows of B.

10. $AB=\begin{bmatrix}2&-3\\-4&6\end{bmatrix}\begin{bmatrix}8&4\\5&5\end{bmatrix}=\begin{bmatrix}1&-7\\-2&14\end{bmatrix}$, $AC=\begin{bmatrix}2&-3\\-4&6\end{bmatrix}\begin{bmatrix}5&-2\\3&1\end{bmatrix}=\begin{bmatrix}1&-7\\-2&14\end{bmatrix}$

13. Use the definition of AB written in reverse order: $[A\mathbf{b}_1\ \cdots\ A\mathbf{b}_p]=A[\mathbf{b}_1\ \cdots\ \mathbf{b}_p]$. Thus

$[Q\mathbf{r}_1\ \cdots\ Q\mathbf{r}_p]=QR$, when $R=[\mathbf{r}_1\ \cdots\ \mathbf{r}_p]$.

39. **[M]** The matrix S "shifts" the entries in a vector (a, b, c, d, e) to yield $(b, c, d, e, 0)$. The entries in S^2 result from applying S to the columns of S, and similarly for S^3, and so on. This explains the patterns of entries in the powers of S:

$$S^2=\begin{bmatrix}0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix},\ S^3=\begin{bmatrix}0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix},\ S^4=\begin{bmatrix}0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

S^5 is the 5×5 zero matrix. S^6 is also the 5×5 zero matrix.

40. **[M]** $A^5=\begin{bmatrix}.3318&.3346&.3336\\.3346&.3323&.3331\\.3336&.3331&.3333\end{bmatrix}$, $A^{10}=\begin{bmatrix}.333337&.333330&.333333\\.333330&.333336&.333334\\.333333&.333334&.333333\end{bmatrix}$

The entries in A^{20} all agree with .3333333333 to 9 or 10 decimal places. The entries in A^{30} all agree with .33333333333333 to at least 14 decimal places. The matrices appear to approach the matrix $\begin{bmatrix}1/3&1/3&1/3\\1/3&1/3&1/3\\1/3&1/3&1/3\end{bmatrix}$. Further exploration of this behavior appears in Sections 4.9 and 5.2.

Chapter 2.2

1. $\begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}^{-1} = \dfrac{1}{32-30}\begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}$

5. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, and the solution is

$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -9 \end{bmatrix}$. Thus $x_1 = 7$ and $x_2 = -9$.

9. **a.** True, by definition of *invertible*. **b.** False. See Theorem 6(b).

c. False. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then $ab - cd = 1 - 0 \neq 0$, but Theorem 4 shows that this matrix is not invertible, because $ad - bc = 0$.

d. True. This follows from Theorem 5, which also says that the solution of $A\mathbf{x} = \mathbf{b}$ is unique, for each $\mathbf{b}$.

e. True, by the box just before Example 6.

30. $[A \quad I] = \begin{bmatrix} 5 & 10 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1/5 & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1/5 & 0 \\ 0 & -1 & -4/5 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 1/5 & 0 \\ 0 & 1 & 4/5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7/5 & 2 \\ 0 & 1 & 4/5 & -1 \end{bmatrix}$. $A^{-1} = \begin{bmatrix} -7/5 & 2 \\ 4/5 & -1 \end{bmatrix}$

31. $[A \quad I] = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{bmatrix}$. $A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}$

Chapter 2.3

1. The columns of the matrix $\begin{bmatrix} 5 & 7 \\ -3 & -6 \end{bmatrix}$ are not multiples, so they are linearly independent. By (e) in the IMT, the matrix is invertible. Also, the matrix is invertible by Theorem 4 in Section 2.2 because the determinant is nonzero.

2. The fact that the columns of $\begin{bmatrix} -4 & 6 \\ 6 & -9 \end{bmatrix}$ are multiples is not so obvious. The fastest check in this case may be the determinant, which is easily seen to be zero. By Theorem 4 in Section 2.2, the matrix is not invertible.

3. Row reduction to echelon form is trivial because there is really no need for arithmetic calculations:
$\begin{bmatrix} 5 & 0 & 0 \\ -3 & -7 & 0 \\ 8 & 5 & -1 \end{bmatrix} \sim \begin{bmatrix} 5 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 5 & -1 \end{bmatrix} \sim \begin{bmatrix} 5 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ The 3×3 matrix has 3 pivot positions and hence is invertible, by (c) of the IMT. [Another explanation could be given using the transposed matrix. But see the note below that follows the solution of Exercise 14.]

4. The matrix $\begin{bmatrix} -7 & 0 & 4 \\ 3 & 0 & -1 \\ 2 & 0 & 9 \end{bmatrix}$ obviously has linearly dependent columns (because one column is zero), and so the matrix is not invertible (or singular) by (e) in the IMT.

11. **a.** True, by the IMT. If statement (d) of the IMT is true, then so is statement (b).
b. True. If statement (h) of the IMT is true, then so is statement (e).
c. False. Statement (g) of the IMT is true only for invertible matrices.
d. True, by the IMT. If the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then statement (d) of the IMT is false. In this case, all the lettered statements in the IMT are false, including statement (c), which means that A must have fewer than n pivot positions.
e. True, by the IMT. If A^T is not invertible, then statement (1) of the IMT is false, and hence statement (a) must also be false.

12. **a.** True. If statement (k) of the IMT is true, then so is statement (j).
b. True. If statement (e) of the IMT is true, then so is statement (h).
c. True. See the remark immediately following the proof of the IMT.
d. False. The first part of the statement is not part (i) of the IMT. In fact, if A is any $n\times n$ matrix, the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ into $\mathbb{R}^n$, yet not every such matrix has n pivot positions.
e. True, by the IMT. If there is a $\mathbf{b}$ in $\mathbb{R}^n$ such that the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, then statement (g) of the IMT is false, and hence statement (f) is also false. That is, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ cannot be one-to-one.

15. If A has two identical columns then its columns are linearly dependent. Part (e) of the IMT shows that A cannot be invertible.

16. Part (h) of the IMT shows that a 5×5 matrix cannot be invertible when its columns do not span $\mathbb{R}^5$.