

Chapter 1.4

1. The matrix-vector product $A\mathbf{x}$ is not defined because the number of columns (2) in the 3×2 matrix

$$\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix} \text{ does not match the number of entries (3) in the vector } \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}.$$

2. The matrix-vector product $A\mathbf{x}$ is not defined because the number of columns (1) in the 3×1 matrix

$$\begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix} \text{ does not match the number of entries (2) in the vector } \begin{bmatrix} 5 \\ -1 \end{bmatrix}.$$

$$3. A\mathbf{x} = \begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = 2 \begin{bmatrix} 6 \\ -4 \\ 7 \end{bmatrix} - 3 \begin{bmatrix} 5 \\ -3 \\ 6 \end{bmatrix} = \begin{bmatrix} 12 \\ -8 \\ 14 \end{bmatrix} + \begin{bmatrix} -15 \\ 9 \\ -18 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}, \text{ and}$$

$$A\mathbf{x} = \begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \cdot 2 + 5 \cdot (-3) \\ (-4) \cdot 2 + (-3) \cdot (-3) \\ 7 \cdot 2 + 6 \cdot (-3) \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}$$

$$4. A\mathbf{x} = \begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 8 \\ 5 \end{bmatrix} + 1 \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8+3-4 \\ 5+1+2 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}, \text{ and}$$

$$A\mathbf{x} = \begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \cdot 1 + 3 \cdot 1 + (-4) \cdot 1 \\ 5 \cdot 1 + 1 \cdot 1 + 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

9. The system has the same solution set as the vector equation

$$x_1 \begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

and this equation has the same solution set as the matrix equation

$$\begin{bmatrix} 3 & 1 & -5 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

11. To solve $A\mathbf{x} = \mathbf{b}$, row reduce the augmented matrix $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{b}]$ for the corresponding linear system:

$$\begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ -2 & -4 & -3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 0 \\ 0 & \textcircled{1} & 0 & -3 \\ 0 & 0 & \textcircled{1} & 1 \end{bmatrix}$$

The solution is $\begin{cases} x_1 = 0 \\ x_2 = -3 \\ x_3 = 1 \end{cases}$. As a vector, the solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$.

21. Row reduce the matrix $[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$ to determine whether it has a pivot in each row.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 1 \\ 0 & \textcircled{1} & 0 \\ 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix $[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$ does not have a pivot in each row, so the columns of the matrix do not span \mathbb{R}^4 , by Theorem 4. That is, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ does not span \mathbb{R}^4 .

23. a. False. See the paragraph following equation (3). The text calls $A\mathbf{x} = \mathbf{b}$ a *matrix equation*.
 b. True. See the box before Example 3.
 c. False. See the warning following Theorem 4.
 d. True. See Example 4.
 e. True. See parts (c) and (a) in Theorem 4.
 f. True. In Theorem 4, statement (a) is false if and only if statement (d) is also false.
25. By definition, the matrix-vector product on the left is a linear combination of the columns of the matrix, in this case using weights $-3, -1$, and 2 . So $c_1 = -3$, $c_2 = -1$, and $c_3 = 2$.
31. A 3×2 matrix has three rows and two columns. With only two columns, A can have at most two pivot columns, and so A has at most two pivot positions, which is not enough to fill all three rows. By Theorem 4, the equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all \mathbf{b} in \mathbb{R}^3 . Generally, if A is an $m \times n$ matrix with $m > n$, then A can have at most n pivot positions, which is not enough to fill all m rows. Thus, the equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all \mathbf{b} in \mathbb{R}^3 .
32. A set of three vectors in cannot span \mathbb{R}^4 . Reason: the matrix A whose columns are these three vectors has four rows. To have a pivot in each row, A would have to have at least four columns (one for each pivot), which is not the case. Since A does not have a pivot in every row, its columns do not span \mathbb{R}^4 , by Theorem 4. In general, a set of n vectors in \mathbb{R}^m cannot span \mathbb{R}^m when n is less than m .

$$\begin{aligned}
40. \text{ [M]} \quad & \begin{bmatrix} 8 & 11 & -6 & -7 & 13 \\ -7 & -8 & 5 & 6 & -9 \\ 11 & 7 & -7 & -9 & -6 \\ -3 & 4 & 1 & 8 & 7 \end{bmatrix} \sim \begin{bmatrix} 8 & 11 & -6 & -7 & 13 \\ 0 & 13/8 & -1/4 & -1/8 & 19/8 \\ 0 & -65/8 & 5/4 & 5/8 & -191/8 \\ 0 & 65/8 & -5/4 & 43/8 & 95/8 \end{bmatrix} \\
& \sim \begin{bmatrix} 8 & 11 & -6 & -7 & 13 \\ 0 & 13/8 & -1/4 & -1/8 & 19/8 \\ 0 & 0 & 0 & 0 & -12 \\ 0 & 0 & 0 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{8} & 11 & -6 & -7 & 13 \\ 0 & \textcircled{13/8} & -1/4 & -1/8 & 19/8 \\ 0 & 0 & 0 & \textcircled{6} & 0 \\ 0 & 0 & 0 & 0 & \textcircled{-12} \end{bmatrix}
\end{aligned}$$

The original matrix has a pivot in every row, so its columns span \mathbb{R}^4 , by Theorem 4.

42. [M] Examine the calculations in Exercise 40. The third column of the original matrix, say A , is not a pivot column. Let C be the matrix formed by deleting column 3 of A , let B be the echelon form obtained from A , and let D be the matrix obtained by deleting column 3 of B . The sequence of row operations that reduces A to B also reduces C to D . Since D is in echelon form, it shows that C has a pivot position in each row. Therefore, the columns of C span \mathbb{R}^4 .

It is possible to delete column 1 or 2 of A instead of column 3. (See the remark for Exercise 41.)

However, only *one* column can be deleted. If two or more columns were deleted from A , the resulting matrix would have fewer than four columns, so it would have fewer than four pivot positions. In such a case, not every row could contain a pivot position, and the columns of the matrix would not span \mathbb{R}^4 , by Theorem 4.

Chapter 1.5

1. Reduce the augmented matrix to echelon form and circle the pivot positions. If a column of the *coefficient* matrix is not a pivot column, the corresponding variable is free and the system of equations has a nontrivial solution. Otherwise, the system has *only* the trivial solution.

$$\begin{bmatrix} 2 & -5 & 8 & 0 \\ -2 & -7 & 1 & 0 \\ 4 & 2 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 12 & -9 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{2} & -5 & 8 & 0 \\ 0 & \textcircled{-12} & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The variable x_3 is free, so the system has a nontrivial solution.

3. $\begin{bmatrix} -3 & 5 & -7 & 0 \\ -6 & 7 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{-3} & 5 & -7 & 0 \\ 0 & \textcircled{-3} & 15 & 0 \end{bmatrix}$. The variable x_3 is free; the system has nontrivial solutions. An alert student will realize that row operations are unnecessary. With only two equations, there can be at most two basic variables. One variable *must* be free. Refer to Exercise 31 in Section 1.2

$$5. \begin{bmatrix} 1 & 3 & 1 & 0 \\ -4 & -9 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & -5 & 0 \\ 0 & \textcircled{1} & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\textcircled{x_1} - 5x_3 = 0$$

$$\textcircled{x_2} + 2x_3 = 0. \text{ The variable } x_3 \text{ is free, } x_1 = 5x_3, \text{ and } x_2 = -2x_3.$$

$$0 = 0$$

In parametric vector form, the general solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$.

$$7. \begin{bmatrix} 1 & 3 & -3 & 7 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 9 & -8 & 0 \\ 0 & \textcircled{1} & -4 & 5 & 0 \end{bmatrix}. \quad \begin{array}{l} \textcircled{x_1} + 9x_3 - 8x_4 = 0 \\ \textcircled{x_2} - 4x_3 + 5x_4 = 0 \end{array}$$

The basic variables are x_1 and x_2 , with x_3 and x_4 free. Next, $x_1 = -9x_3 + 8x_4$, and $x_2 = 4x_3 - 5x_4$. The general solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 + 8x_4 \\ 4x_3 - 5x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 \\ 4x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 8x_4 \\ -5x_4 \\ 0 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

11.

$$\begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & -2 & 0 & 0 & 7 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 0 & 0 & 0 & 5 & 0 \\ \textcircled{0} & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & \textcircled{0} & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & \textcircled{0} & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \textcircled{x_1} - 4x_2 + 5x_6 &= 0 \\ \textcircled{x_3} - x_6 &= 0 \\ \textcircled{x_5} - 4x_6 &= 0 \\ 0 &= 0 \end{aligned} \quad \text{The basic variables are } x_1, x_3, \text{ and } x_5. \text{ The remaining variables are}$$

free. In particular, x_4 is free (and not zero as some may assume). The solution is $x_1 = 4x_2 - 5x_6$, $x_3 = x_6$, $x_5 = 4x_6$, with x_2 , x_4 , and x_6 free. In parametric vector form,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4x_2 - 5x_6 \\ x_2 \\ x_6 \\ x_4 \\ 4x_6 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_6 \\ 0 \\ x_6 \\ 0 \\ 4x_6 \\ x_6 \end{bmatrix} = x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{u} & \mathbf{v} & \mathbf{w} \end{array}$$

23. a. True. See the first paragraph of the subsection titled “Homogeneous Linear Systems”.
- b. False. The equation $A\mathbf{x} = \mathbf{0}$ gives an *implicit* description of its solution set. See the subsection entitled “Parametric Vector Form”.
- c. False. The equation $A\mathbf{x} = \mathbf{0}$ *always* has the trivial solution. The box before Example 1 uses the word *nontrivial* instead of *trivial*.
- d. False. The line goes through \mathbf{p} parallel to \mathbf{v} . See the paragraph that precedes Fig. 5.
- e. False. The solution set could be *empty*! The statement (from Theorem 6) is true only when there exists a vector \mathbf{p} such that $A\mathbf{p} = \mathbf{b}$.

29. a. When A is a 3×3 matrix with three pivot positions, the equation $A\mathbf{x} = \mathbf{0}$ has no free variables and hence has no nontrivial solution.
- b. With three pivot positions, A has a pivot position in each of its three rows. By Theorem 4 in Section 1.4, the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every possible \mathbf{b} . The term “possible” in the exercise means that the only vectors considered in this case are those in \mathbb{R}^3 , because A has three rows.
30. a. When A is a 3×3 matrix with two pivot positions, the equation $A\mathbf{x} = \mathbf{0}$ has two basic variables and one free variable. So $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.
- b. With only two pivot positions, A cannot have a pivot in every row, so by Theorem 4 in Section 1.4, the equation $A\mathbf{x} = \mathbf{b}$ cannot have a solution for every possible \mathbf{b} (in \mathbb{R}^3).
31. a. When A is a 3×2 matrix with two pivot positions, each column is a pivot column. So the equation $A\mathbf{x} = \mathbf{0}$ has no free variables and hence no nontrivial solution.
- b. With two pivot positions and three rows, A cannot have a pivot in every row. So the equation $A\mathbf{x} = \mathbf{b}$ cannot have a solution for every possible \mathbf{b} (in \mathbb{R}^3), by Theorem 4 in Section 1.4.
32. a. When A is a 2×4 matrix with two pivot positions, the equation $A\mathbf{x} = \mathbf{0}$ has two basic variables and two free variables. So $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.
- b. With two pivot positions and only two rows, A has a pivot position in every row. By Theorem 4 in Section 1.4, the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every possible \mathbf{b} (in \mathbb{R}^2).
33. Look at $x_1 \begin{bmatrix} -2 \\ 7 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -6 \\ 21 \\ -9 \end{bmatrix}$ and notice that the second column is 3 times the first. So suitable values

for x_1 and x_2 would be 3 and -1 respectively. (Another pair would be 6 and -2 , etc.) Thus $\mathbf{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ satisfies $A\mathbf{x} = \mathbf{0}$.

Chapter 1.7

1. Use an augmented matrix to study the solution set of $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{0}$ (*), where \mathbf{u} , \mathbf{v} , and \mathbf{w} are

the three given vectors. Since $\begin{bmatrix} 5 & 7 & 9 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & -6 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{5} & 7 & 9 & 0 \\ 0 & \textcircled{2} & 4 & 0 \\ 0 & 0 & \textcircled{4} & 0 \end{bmatrix}$, there are no free variables. So

the homogeneous equation (*) has only the trivial solution. The vectors are linearly independent.

9. a. The vector \mathbf{v}_3 is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ if and only if the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{v}_3$ has a solution. To find out, row reduce $[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$, considered as an augmented matrix:

$$\begin{bmatrix} 1 & -3 & 5 \\ -3 & 9 & -7 \\ 2 & -6 & h \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -3 & 5 \\ 0 & 0 & \textcircled{8} \\ 0 & 0 & h-10 \end{bmatrix}$$

At this point, the equation $0 = 8$ shows that the original vector equation has no solution. So \mathbf{v}_3 is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ for *no* value of h .

- b. For $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to be linearly independent, the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ must have only the trivial solution. Row reduce the augmented matrix $[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{0}]$:

$$\begin{bmatrix} 1 & -3 & 5 & 0 \\ -3 & 9 & -7 & 0 \\ 2 & -6 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 5 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & h-10 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -3 & 5 & 0 \\ 0 & 0 & \textcircled{8} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For every value of h , x_2 is a free variable, and so the homogeneous equation has a nontrivial solution. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set for all h .

15. The set is linearly dependent, by Theorem 8, because there are four vectors in the set but only two entries in each vector.
16. The set is linearly dependent because the second vector is $3/2$ times the first vector.
17. The set is linearly dependent, by Theorem 9, because the list of vectors contains a zero vector.
18. The set is linearly dependent, by Theorem 8, because there are four vectors in the set but only two entries in each vector.
19. The set is linearly independent because neither vector is a multiple of the other vector. [Two of the entries in the first vector are -4 times the corresponding entry in the second vector. But this multiple does not work for the third entries.]
20. The set is linearly dependent, by Theorem 9, because the list of vectors contains a zero vector.
21. a. False. A homogeneous system *always* has the trivial solution. See the box before Example 2.
 b. False. See the warning after Theorem 7.
 c. True. See Fig. 3, after Theorem 8.
 d. True. See the remark following Example 4.

$$23. \begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}$$

$$24. \begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$25. \begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$41. [\mathbf{M}] \ A = \begin{bmatrix} 8 & -3 & 0 & -7 & 2 \\ -9 & 4 & 5 & 11 & -7 \\ 6 & -2 & 2 & -4 & 4 \\ 5 & -1 & 7 & 0 & 10 \end{bmatrix} \sim \begin{bmatrix} 8 & -3 & 0 & -7 & 2 \\ 0 & 5/8 & 5 & 25/8 & -19/4 \\ 0 & 1/4 & 2 & 5/4 & 5/2 \\ 0 & 7/8 & 7 & 35/8 & 35/4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 8 & -3 & 0 & -7 & 2 \\ 0 & 5/8 & 5 & 25/8 & -19/4 \\ 0 & 0 & 0 & 0 & 22/5 \\ 0 & 0 & 0 & 0 & 77/5 \end{bmatrix} \sim \begin{bmatrix} \textcircled{8} & -3 & 0 & -7 & 2 \\ 0 & \textcircled{5/8} & 5 & 25/8 & -19/4 \\ 0 & 0 & 0 & 0 & \textcircled{22/5} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns of A are 1, 2, and 5. Use them to form $B = \begin{bmatrix} 8 & -3 & 2 \\ -9 & 4 & -7 \\ 6 & -2 & 4 \\ 5 & -1 & 10 \end{bmatrix}$.

Other likely choices use columns 3 or 4 of A instead of 2: $\begin{bmatrix} 8 & 0 & 2 \\ -9 & 5 & -7 \\ 6 & 2 & 4 \\ 5 & 7 & 10 \end{bmatrix}, \begin{bmatrix} 8 & -7 & 2 \\ -9 & 11 & -7 \\ 6 & -4 & 4 \\ 5 & 0 & 10 \end{bmatrix}$.

Actually, any set of three columns of A that includes column 5 will work for B , but the concepts needed to prove that are not available now. (Column 5 is not in the two-dimensional subspace spanned by the first four columns.)