Lesson 4

Chapter 3
Determinants

Introduction to Determinants

Properties of Determinants

Matrix multiplication:

A: $m \times n$ *matrix*
B: $n \times p$ *matrix*
AB: $m \times p$ *matrix*

$$AB = A[\mathbf{b}_1\ \mathbf{b}_2\ \dots\ \mathbf{b}_p] = [A\mathbf{b}_1\ A\mathbf{b}_2\ \dots\ A\mathbf{b}_p]$$

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$$

(row i in A multiplied on column j in B)

Let A be an $m \times n$ matrix and let B and C have sizes for which the indicated sums and products are defined.

- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$
- $(B + C)A = BA + CA$
- $r(AB) = (rA)B = A(rB)$
- $I_m A = A = A I_n$

OBS!!! In general:

- $AB \neq BA$ (non-commutating)
- $AB = AC \not\Rightarrow B = C$ (no cancellation)
- $AB = 0 \not\Rightarrow A = 0 \ \vee \ B = 0$

Transponeret

Transposed matrix: $A = \{a_{ij}\} \;\Leftrightarrow\; A^T = \{a_{ji}\}$ (”mirroring” in the diagonal)

Theorem 2.3: Rules for transposing

$$(A^T)^T = A$$
$$(A+B)^T = A^T + B^T$$
$$(rA)^T = rA^T, \quad \forall r \in \mathbb{R}$$
$$(AB)^T = B^T A^T$$

Inverse matrix:

A: $n \times n$ $matrix$

$$\boxed{AA^{-1} = A^{-1}A = I}$$

A invertible $\Leftrightarrow$ A is row equivalent to I $\Leftrightarrow$ $[\,A \mid I\,]$ is row equivalent to $[\,I \mid A^{-1}]$

Rules for inverse matrices:

$$(A^{-1})^{-1} = A$$

$$(AB)^{-1} = B^{-1}A^{-1} \qquad (A \dots YZ)^{-1} = Z^{-1}Y^{-1} \dots A^{-1}$$

$$(A^T)^{-1} = (A^{-1})^T$$

Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.

b. A is row equivalent to the $n \times n$ identity matrix.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has ~~at least one~~ a unique solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

3.1 Introduction to Determinants

$$det\ A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Undermatrix

Definition: $\boldsymbol{Submatrix}\ A_{ij} = Matrix\ A\ deleting\ row\ i\ and\ column\ j$

Matrix:
$$A = \begin{bmatrix} a_{11} & \cdots & a_{1\,j-1} & a_{1j} & a_{1\,j+1} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i-1\,1} & \cdots & a_{i-1\,j-1} & a_{i-1\,j} & a_{i-1\,j+1} & \cdots & a_{i-1\,n} \\ a_{i1} & \cdots & a_{i\,j-1} & a_{ij} & a_{i\,j+1} & \cdots & a_{in} \\ a_{i+1\,1} & \cdots & a_{i+1\,j-1} & a_{i+1\,j} & a_{i+1\,j+1} & \cdots & a_{i+1\,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{m\,j-1} & a_{mj} & a_{m\,j+1} & \cdots & a_{mn} \end{bmatrix}$$
$m \times n\ matrix$

Submatrix:
$$A_{ij} = \begin{bmatrix} a_{11} & \cdots & a_{1\,j-1} & a_{1\,j+1} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i-1\,1} & \cdots & a_{i-1\,j-1} & a_{i-1\,j+1} & \cdots & a_{i-1\,n} \\ a_{i+1\,1} & \cdots & a_{i+1\,j-1} & a_{i+1\,j+1} & \cdots & a_{i+1\,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{m\,j-1} & a_{m\,j+1} & \cdots & a_{mn} \end{bmatrix}$$
$(m-1) \times (n-1)\ matrix$

A: $n \times n$ $matix$

Determinant: $\boldsymbol{det\,A = |A| \in \mathbb{R}}$

$A = [a_{11}]$: $\quad det\,A = |a_{11}| = a_{11}$

+

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}: \quad det\,A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$$

÷

Row (i) expansion

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}: \quad det\,A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{cases} \displaystyle\sum_{j=1}^{n} (-1)^{i+j}\, a_{ij} \cdot det\,A_{ij} \\ \displaystyle\sum_{i=1}^{n} (-1)^{i+j}\, a_{ij} \cdot det\,A_{ij} \end{cases}$$

Column (j) expansion

A: $n \times n$ $matix$

$$\boxed{\textbf{Determinant:} \quad \boldsymbol{det\,A} = |\boldsymbol{A}| \in \mathbb{R}}$$

Cofactor: $C_{ij} = (-1)^{i+j} \cdot det\,A_{ij}$

Sign $(-1)^{i+j}$: $\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$

$$det\,A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{cases} \displaystyle\sum_{j=1}^{n} (-1)^{i+j}\, a_{ij} \cdot det\,A_{ij} = \sum_{j=1}^{n} a_{ij} \cdot C_{ij} & \text{Row } (i) \text{ expansion} \\ \displaystyle\sum_{i=1}^{n} (-1)^{i+j}\, a_{ij} \cdot det\,A_{ij} = \sum_{i=1}^{n} a_{ij} \cdot C_{ij} & \text{Column } (j) \text{ expansion} \end{cases}$$

Ex 1

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & 1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix} \Longrightarrow \det(A) = \begin{cases} 1 \cdot C_{11} + 2 \cdot C_{21} + 3 \cdot C_{31} + 0 \cdot C_{41} & \text{1. søjle} \\ 0 \cdot C_{41} + 4 \cdot C_{42} - 2 \cdot C_{43} + 0 \cdot C_{44} & \text{4. række} \end{cases}$$

Ex 2

$$A = \begin{bmatrix} 3 & 4 & 0 & 8 \\ 0 & -1 & 5 & 6 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\Downarrow$$

$$\det(A) = \begin{vmatrix} 3 & 4 & 0 & 8 \\ 0 & -1 & 5 & 6 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 2 \end{vmatrix}$$

Theorem 3.2: Triangular matrix

If A is a triangular matrix, then $det\ A$ is the product of the entries on the main diagnal of A.

Product symbol

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1\,n-1} & a_{1n} \\ 0 & a_{22} & \cdots & a_{2\,n-1} & a_{2n} \\ 0 & 0 & \cdots & a_{3\,n-1} & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \vdots & a_{n-1\,n-1} & a_{n-1\,n} \\ 0 & 0 & \cdots & 0 & a_{nn} \end{vmatrix} = a_{11} \cdot a_{22} \cdots a_{nn} = \prod_{i=1}^{n} a_{ii}$$

3.2 Properties of Determinants

$$det\ A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

OBS: Numerical note

Calculation of a $n \times n$ determinant by cofactor expansion: $\gtrsim n!$ multiplications

Fx.:

25×25 determinant (very small) →

$25! \approx 1.5 \cdot 10^{25}$ / $1 \cdot 10^{12}$ multiplications pr. second → 500.000 years

→ Faster methods for calculating determinants needed!!!!

Theorem 3.3: Row operations

Let A be a square matrix

- If a multiple of one row of A is added to another row to produce a matrix B, then det$B$=det$A$.
- If two rows of A are interchanged to produce B, then det$B$=-det$A$.
- If one row of A is multiplied by k to produce B, then det$B$=$k\cdot$det$A$.

Ex 3

$$\det(A) = \begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix}$$

Calculating determinants

A^{-1} exist

Product of pivots in U (■)

$$A \sim U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix} \sim I \quad \Rightarrow det\ A = (-1)^r \cdot \det U = (-1)^r \cdot \prod_i u_{ii}$$

Row replacements and r row interchanges – but NO scaling

$$A \sim U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 \end{bmatrix} \cancel{\sim} I \quad \Rightarrow det\ A = (-1)^r \cdot \det U = 0$$

A^{-1} do NOT exist

$\blacksquare = Pivot\ (\neq 0)$

$* = Any\ number\ (could\ also\ be\ 0)$

Theorem 3.4:

$$\boxed{\text{A square matrix } A \text{ is invertible } \Leftrightarrow \ det\, A \neq 0}$$

Theorem 3.5:

$$\boxed{\text{If } A \text{ is a square matrix: } \quad det\, A^T = det\, A}$$

Theorem 3.6:

$$\boxed{\text{If } A \text{ and } B \text{ are } n \times n \text{ matrices: } \quad det\, AB = det\, A \cdot det\, B}$$

OBS: But $det(A + B) \neq det\, A + det\, B$

Ex 4 Is A invertible?

$$A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$$

Ex 5

$$A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$$

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l. A^T is an invertible matrix.

Continuation of the invertible matrix theorem

Let A be an $n \times n$ matrix. Then A is invertible if and only if:

t. The determinant of A is not 0.

Todays words and concepts

Submatrix

Determinant

Cofactor

Row expansion

Column expansion