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Case Study: Space Flight and Control Systems

In this case study, the design of engineering control systems (such as the one in Figure 1 on page 207 of your text) is studied. Special attention is paid to how concepts from Chapter 4 may be used in this analysis.

In this Figure, each box represents some process (which could be a piece of equipment, a computer program, a measuring device, etc.) which takes in an input signal and produces an output signal. These signals are represented by arrows in the diagram. The crossed circles are called summing junctions; signals are combined at these points for delivery to some process as input. For example, the commanded pitch is summed with input from the inertial measuring unit, which produces a pitch rate that is used by process K_1 as input. Notice that this system uses **feedback**; that is, the final result (pitch) is used as input to the system.

In order to build a mathematical model for systems such as in Figure 1, first assume that the output of the system is some vector $\mathbf{x}$. In Figure 1, $\mathbf{x}$ would be a vector representing the pitch angle of the shuttle's nose cone. Also assume (for the time being) that the system has no external inputs. Thus if $\mathbf{x}_0$ is the initial pitch of the nose cone, the system will generate a new pitch $\mathbf{x}_1$ from the input $\mathbf{x}_0$, and then a new pitch $\mathbf{x}_2$ will be generated from $\mathbf{x}_1$, and so on, with each $\mathbf{x}_k$ giving rise to a new $\mathbf{x}_{k+1}$. The final assumption is that the process by which $\mathbf{x}_k$ becomes $\mathbf{x}_{k+1}$ is a linear transformation and that it is the same for all times k. Thus the system may be modelled by the equation

$$\mathbf{x}_{k+1} = A\mathbf{x}_k \quad \text{for } k = 0, 1, 2, \ldots$$

given $\mathbf{x_0}$ as the initial state of the system.

Example 1: If $A = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, $\mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$, $\mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} 0 \\ 12 \end{bmatrix}$, etc.

Example 2: If $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & -2 \\ 2 & 2 & 4 \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, then $\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} 5 \\ -3 \\ 10 \end{bmatrix}$, $\mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 22 \\ -12 \\ 44 \end{bmatrix}$, $\mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} 98 \\ -54 \\ 196 \end{bmatrix}$, etc.

Notice that in each example the outputs from the system seem to be careening out of control. Of course, this type of behavior should be expected since the system is being left alone: no external inputs have been allowed. In practice, external inputs are certainly allowed. Consider the situation in Figure 1: all of the internal inputs may help, but the pilot (whether human or computer) must input what ultimate pitch is desired. Thus the pilot is attempting to control this system; the pilot is trying to get the system to deliver the desired output vector.

Mathematically, assume that there is a sequence of external input vectors, $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{u}_2$, ... and a matrix B so that

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k \quad \text{for } k = 0, 1, 2, \ldots \tag{1}$$

Assume further that A is an $n \times n$ matrix, B is a $n \times m$ matrix, that $\mathbf{x}_k$ is in $\mathbb{R}^n$ for $k = 0, 1, 2, \ldots$ and that $\mathbf{u}_k$ is in $\mathbb{R}^m$ for $k = 0, 1, 2, \ldots$. Also assume that the initial vector is $\mathbf{x}_0 = \mathbf{0}$. The sequence $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{u}_2$, ... is called a sequence of controls for the system.

Example 1 (cont.) Suppose that the matrix $B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is given. Notice that $\mathbf{u}_k \in \mathbb{R}^1$ since B has only one column, so the sequence of controls is really a sequence of numbers in this case. For example, if $\mathbf{u}_0 = \mathbf{u}_1 = 1$, then

$$\mathbf{x}_1 = A\mathbf{x}_0 + B\mathbf{u}_0 = B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\mathbf{x}_2 = A\mathbf{x}_1 + B\mathbf{u}_1 = A\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$$

Example 2 (cont.) Suppose that the matrix $B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is given. Again the matrix B has only one column, so the sequence of controls is a sequence of numbers. For example, if $\mathbf{u}_0 = 1$ and $\mathbf{u}_1 = -1$, then

$$\mathbf{x}_1 = A\mathbf{x}_0 + B\mathbf{u}_0 = B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\mathbf{x}_2 = A\mathbf{x}_1 + B\mathbf{u}_1 = A\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \\ 10 \end{bmatrix}$$

The sequence of controls and the matrix B are used to force the system to deliver the desired output within a short number of steps. In the shuttle example, this means that a combination of external control and internal feedback ought to produce a desired pitch within as few iterations of the system as possible. In linear algebra terms, a target vector $\mathbf{y}$ in $\mathbb{R}^n$ is given for the output. A sequence of controls $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{u}_2$, ... is imposed so that $\mathbf{x}_k = \mathbf{y}$ for some time k. Ideally $\mathbf{y}$ should be **any** vector in $\mathbb{R}^n$. Thus the following definition is given.

Definition: If a system has the form

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k, \qquad \mathbf{x}_0 = \mathbf{0}$$

where A is an $n \times n$ matrix and B is a $n \times m$ matrix, and $\mathbf{y}$ is any vector in $\mathbb{R}^n$, then the pair of matrices (A, B) is **controllable** if there exists a sequence of controls $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{u}_2$, ... so that $\mathbf{x}_m = \mathbf{y}$ at some time m.

In order to determine whether a pair is controllable, closely examine the sequence $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$, etc. Before beginning, here is a useful fact about the number of steps needed to produce the target vector $\mathbf{y}$.

Fact: Let

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k, \quad \mathbf{x}_0 = \mathbf{0}$$

where A is an $n \times n$ matrix and B is an $n \times m$ matrix, and $\mathbf{y}$ is any vector in $\mathbb{R}^n$ Then the pair (A, B) is controllable if and only if there exists a sequence of controls $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{u}_2$, $\ldots$ so that $\mathbf{x}_m = \mathbf{y}$ at some time $m \leq n$. That is, the pair is controllable if and only if the system can forced to any output vector in at most n steps.

The system of equatiuons may be solved iteratively:

$$\begin{aligned}
\mathbf{x}_0 &= \mathbf{0} \\
\mathbf{x}_1 &= A\mathbf{x}_0 + B\mathbf{u}_0 = B\mathbf{u}_0 \\
\mathbf{x}_2 &= A\mathbf{x}_1 + B\mathbf{u}_1 = AB\mathbf{u}_0 + B\mathbf{u}_1 \\
\mathbf{x}_3 &= A\mathbf{x}_2 + B\mathbf{u}_2 = A(AB\mathbf{u}_0 + B\mathbf{u}_1) + B\mathbf{u}_2 \\
&= A^2B\mathbf{u}_0 + AB\mathbf{u}_1 + B\mathbf{u}_2 \\
\vdots \; &\vdots \qquad \vdots \\
\mathbf{x}_k &= A^{k-1}B\mathbf{u}_0 + A^{k-2}B\mathbf{u}_1 + A^{k-3}B\mathbf{u}_2 + \ldots + AB\mathbf{u}_{k-2} + B\mathbf{u}_{k-1} \\
\vdots \; &\vdots \qquad \vdots \\
\mathbf{x}_n &= A^{n-1}B\mathbf{u}_0 + A^{n-2}B\mathbf{u}_1 + A^{n-3}B\mathbf{u}_2 + \ldots + AB\mathbf{u}_{n-2} + B\mathbf{u}_{n-1}
\end{aligned}$$

By the Fact above, the pair in Equation (1) is controllable if and only if each target vector $\mathbf{y}$ in $\mathbb{R}^n$ is one of the vectors $\mathbf{x}_0$, $\mathbf{x}_1$, $\ldots \mathbf{x}_n$ for a suitable control sequence $\mathbf{u}_0$, $\mathbf{u}_1$, $\ldots \mathbf{u}_n$. Notice that each of the vectors $\mathbf{x}_k$ is a linear combination of the columns of the matrices B, AB, A^2B, $\ldots$, $A^{n-1}B$ because each term in the sum for $\mathbf{x}_k$ is one of these matrices times a vector $\mathbf{u}_j$ of weights.

All of the columns of B, AB, A^2B, $\ldots$, $A^{n-1}B$ may be placed into one large matrix by constructing the partitioned matrix

$$M = \begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix} \tag{2}$$

Then each target vector $\mathbf{y}$ must be a linear combination of the columns of M. Since this should be true for all $\mathbf{y}$ in $\mathbb{R}^n$, the columns of M must span $\mathbb{R}^n$. This explains the following result.

Result 1: A system of the form in Equation (1) with $\mathbf{x}_0 = \mathbf{0}$ is controllable if and only if each vector $\mathbf{y}$ in $\mathbb{R}^n$ is a linear combination of the columns of the matrix M in Equation (2). Equivalently, the pair is controllable if and only if the columns of M span $\mathbb{R}^n$.

The matrix M is clearly fundamental to the analysis of the controllability of the system; it is called the **controllability matrix** for the system.

Definition: If A is an $n \times n$ matrix and B is an $n \times m$ matrix, then the $n \times nm$ matrix

$$M = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

is called the **controllability matrix** of the system $\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$, $k = 0, 1, 2, \dots$.

Example 1 (cont.): Since A is 2×2 and B is 2×1, the controllability matrix M for this system is

$$M = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 7 \end{bmatrix}$$

Example 2 (cont.): Since A is 3×3 and B is 3×1, the controllability matrix M for this system is

$$M = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 6 & 26 \\ 1 & -4 & -14 \\ 2 & 12 & 52 \end{bmatrix}$$

By Theorem 4 in Section 1.4, the system in Equation (1) with $\mathbf{x}_0 = \mathbf{0}$ will be controllable if the controllability matrix M has a pivot in each row; that is, if the number of pivots in M is n. In Section 4.6, it is shown that the **rank** of a matrix is the number of pivots it possesses; thus the following result is proven:

Result 2: A pair (A, B) (where A is an $n \times n$ matrix and B is an $n \times m$ matrix) is controllable if and only if the rank of the controllability matrix M is n.

Example 1 (cont.): Since

$$M = \begin{bmatrix} 1 & 0 \\ 3 & 7 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

the rank of M is 2 and the system is controllable.

Example 2 (cont.): Since

$$M = \begin{bmatrix} 1 & 6 & 26 \\ 1 & -4 & -14 \\ 2 & 12 & 52 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

the rank of M is $2 < 3$ and thus the system is not controllable.

Example 3: Consider the same matrix A as in Example 2, but now let

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix}.$$

Since

$$AB = \begin{bmatrix} 6 & 9 \\ -4 & -7 \\ 12 & 18 \end{bmatrix} \quad \text{and} \quad A^2B = \begin{bmatrix} 26 & 38 \\ -14 & -20 \\ 52 & 76 \end{bmatrix},$$

the controllability matrix M for this system is

$$M = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 1 & 6 & 9 & 26 & 38 \\ 1 & 2 & -4 & -7 & -14 & -20 \\ 2 & 3 & 12 & 18 & 52 & 76 \end{bmatrix}.$$

Since M is row equivalent to

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{3}{5} & 2 & \frac{16}{5} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{8}{5} & 4 & \frac{29}{5} \end{bmatrix},$$

the rank of M is 3 and the system $\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$, $\mathbf{x}_0 = \mathbf{0}$ is controllable.

Once it is determined that a pair is controllable, the exact sequence of controls which will be needed to produce a desired target vector $\mathbf{y}$ may be computed.

Example 1 (cont.): Let the target vector be $\mathbf{y} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. This system is controllable, so it should be possible to specify $\mathbf{u}_0$ and $\mathbf{u}_1$ with

$$\mathbf{y} = AB\mathbf{u}_0 + B\mathbf{u}_1$$

This system of equations has augmented matrix

$$\begin{bmatrix} B & AB & \mathbf{y} \end{bmatrix} = \begin{bmatrix} M & \mathbf{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 3 & 7 & 5 \end{bmatrix}$$

which reduces to

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -\frac{4}{7} \end{bmatrix}$$

Thus the controls $\mathbf{u}_1 = 3$ and $\mathbf{u}_0 = -\frac{4}{7}$ will drive the system to the vector $\mathbf{y} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

Example 4: Consider the system with matrices

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 6 & 2 \\ 3 & 9 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}.$$

In Question 4 you will be asked to confirm that the controllability matrix for this system is

$$M = \begin{bmatrix} 1 & 0 & 8 & 2 & 80 & 20 \\ 2 & 1 & 16 & 4 & 160 & 40 \\ 1 & -1 & 24 & 6 & 240 & 60 \end{bmatrix}$$

and that this system is controllable. Given the output vector $\mathbf{y} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$, the sequence of controls which will drive the system to $\mathbf{y}$ are needed. To compute the controls, form the system of equations

$$\mathbf{y} = A^2B\mathbf{u}_0 + AB\mathbf{u}_1 + B\mathbf{u}_2$$

Note that the vectors $\mathbf{u}_0 = \begin{bmatrix} u_{01} \\ u_{02} \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix}$, and $\mathbf{u}_2 = \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix}$ are in $\mathbb{R}^2$, thus the 6 entries u_{01}, u_{02}, u_{11}, u_{12}, u_{21}, and u_{22} must be found to solve this problem. The system of equations has augmented matrix

$$\begin{bmatrix} B & AB & A^2B & \mathbf{y} \end{bmatrix} = \begin{bmatrix} M & \mathbf{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 8 & 2 & 80 & 20 & 2 \\ 2 & 1 & 16 & 4 & 160 & 40 & 3 \\ 1 & -1 & 24 & 6 & 240 & 60 & 0 \end{bmatrix}$$

To emphasize the fact that the unknowns in this system are u_{01}, u_{02}, u_{11}, u_{12}, u_{21}, and u_{22}, this augmented matrix may be expressed as a system of equations:

$$\begin{array}{rcrcrcrcrcrcl} u_{21} & & & + & 8u_{11} & + & 2u_{12} & + & 80u_{01} & + & 20u_{02} & = & 2 \\ 2u_{21} & + & u_{22} & + & 16u_{11} & + & 4u_{12} & + & 160u_{01} & + & 40u_{02} & = & 3 \\ u_{21} & - & u_{22} & + & 24u_{11} & + & 6u_{12} & + & 240u_{01} & + & 60u_{02} & = & 0 \end{array}$$

Since the augmented matrix reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \frac{7}{2} \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{4} & 10 & \frac{5}{2} & -\frac{3}{16} \end{bmatrix}$$

the reduced system of equations

$$\begin{array}{rcrcrcrcl} & & & & & & u_{21} & = & \frac{7}{2} \\ & & & & & & u_{22} & = & -1 \\ u_{11} & + & \frac{1}{4}u_{12} & + & 10u_{01} & + & \frac{5}{2}u_{02} & = & -\frac{3}{16} \end{array}$$

is produced. So u_{12}, u_{01}, and u_{02} are the free variables; for simplicity's sake, let them all equal 0. Thus it has been found that $\mathbf{u}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} -\frac{3}{16} \\ 0 \end{bmatrix}$, and $\mathbf{u}_2 = \begin{bmatrix} \frac{7}{2} \\ -1 \end{bmatrix}$ is a set of controls which will drive the system to the point $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$.

Questions:

1. Confirm that the controllability matrices for Examples 1 and 2 are as given above.

2. Consider the system with matrices

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 4 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

   Determine whether the pair (A, B) is controllable.

3. If B is invertible, what can be said about the controllability of the pair (A, B)?

4. Confirm that the controllability matrix in Example 4 is as given above, and that the system is controllable.

5. Consider the system with matrices

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 4 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

   a) Find the controllability matrix for this system.

   b) Show that system $\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$, $\mathbf{x}_0 = \mathbf{0}$ is controllable.

   c) Find controls $\mathbf{u}_0$, $\mathbf{u}_1$, and $\mathbf{u}_2$ which will drive this system to the point $\begin{bmatrix} -7 \\ 3 \\ 1 \end{bmatrix}$.

6. Consider the system with matrices

$$A = \begin{bmatrix} 1 & 3 & -2 \\ -2 & -6 & 4 \\ 5 & 15 & -10 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 1 & 0 \end{bmatrix}$$

a) Find the controllability matrix for this system.

b) Show that system $\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$, $\mathbf{x}_0 = \mathbf{0}$ is controllable.

c) Find controls $\mathbf{u}_0$, $\mathbf{u}_1$, and $\mathbf{u}_2$ which will drive this system to the point $\begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix}$.

d) Suppose that $\mathbf{u}_0 = (1, 1)$. Can the system still be driven to $\begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix}$? If so, find controls $\mathbf{u}_1$ and $\mathbf{u}_2$ to make this happen.