Chapter 4.3

1. Consider the matrix whose columns are the given set of vectors. This 3×3 matrix is in echelon form, and has 3 pivot positions. Thus by the Invertible Matrix Theorem, its columns are linearly independent and span $\mathbb{R}^3$. So the given set of vectors is a basis for $\mathbb{R}^3$.

2. Since the zero vector is a member of the given set of vectors, the set cannot be linearly independent and thus cannot be a basis for $\mathbb{R}^3$. Now consider the matrix whose columns are the given set of vectors. This 3×3 matrix has only 2 pivot positions. Thus by the Invertible Matrix Theorem, its columns do not span $\mathbb{R}^3$.

8. Consider the matrix whose columns are the given set of vectors. Since the matrix cannot have a pivot in each column, the set cannot be linearly independent and thus cannot be a basis for $\mathbb{R}^3$. The reduced echelon form of this matrix is $\begin{bmatrix} 1 & 0 & 3 & 0 \\ -4 & 3 & -5 & 2 \\ 3 & -1 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -3/2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}$, so the matrix has a pivot in each row. Thus the given set of vectors spans $\mathbb{R}^3$.

10. We find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables by using the reduced echelon form of A: $\begin{bmatrix} 1 & 0 & -5 & 1 & 4 \\ -2 & 1 & 6 & -2 & -2 \\ 0 & 2 & -8 & 1 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 0 & 7 \\ 0 & 1 & -4 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$. So $x_1 = 5x_3 - 7x_5$, $x_2 = 4x_3 - 6x_5$, $x_4 = 3x_5$, with x_3 and x_5 free. So $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix}$, and a basis for Nul A is

$$\left\{ \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

21. **a**. False. The zero vector by itself is linearly dependent. See the paragraph preceding Theorem 4.

b. False. The set $\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ must also be linearly independent. See the definition of a basis.

c. True. See Example 3.

d. False. See the subsection "Two Views of a Basis."

e. False. See the box before Example 9.

23. Let $A = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4]$. Then A is square and its columns span $\mathbb{R}^4$ since $\mathbb{R}^4 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. So its columns are linearly independent by the Invertible Matrix Theorem, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis for $\mathbb{R}^4$.

25. In order for the set to be a basis for H, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ must be a spanning set for H; that is, $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. The exercise shows that H is a subset of $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. but there are vectors in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ which are not in H ($\mathbf{v}_1$ and $\mathbf{v}_3$, for example). So $H \neq \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is not a basis for H.

33. Neither polynomial is a multiple of the other polynomial. So $\{\mathbf{p}_1, \mathbf{p}_2\}$ is a linearly independent set in $\mathbb{P}_3$. Note: $\{\mathbf{p}_1, \mathbf{p}_2\}$ is also a linearly independent set in $\mathbb{P}_2$ since $\mathbf{p}_1$ and $\mathbf{p}_2$ both happen to be in $\mathbb{P}_2$.

Chapter 4.4

1. We calculate that $\mathbf{x} = 5\begin{bmatrix} 3 \\ -5 \end{bmatrix} + 3\begin{bmatrix} -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$.

3. We calculate that $\mathbf{x} = 3\begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix} + (-1)\begin{bmatrix} 4 \\ -7 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ 9 \end{bmatrix}$.

5. The matrix $\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{x} \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -5 \end{bmatrix}$, so $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$.

14. We must find c_1, c_2, and c_3 such that $c_1(1-t^2) + c_2(t-t^2) + c_3(2-2t+t^2) = \mathbf{p}(t) = 3 + t - 6t^2$.
Equating the coefficients of the two polynomials produces the system of equations

$$\begin{array}{rcrcrcr} c_1 & + & & & 2c_3 & = & 3 \\ & & c_2 & - & 2c_3 & = & 1 \\ -c_1 & - & c_2 & + & c_3 & = & -6 \end{array}$$

. We row reduce the augmented matrix for the system of equations to find $\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ -1 & -1 & 1 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$, so $[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix}$.

One may also solve this problem using the coordinate vectors of the given polynomials relative to the standard basis $\{1, t, t^2\}$; the same system of linear equations results.

15. **a**. True. See the definition of the $\mathcal{B}$-coordinate vector.

b. False. See Equation (4).

c. False. $\mathbb{P}_3$ is isomorphic to $\mathbb{R}^4$. See Example 5.

33. The coordinate mapping produces the coordinate vectors (3, 7, 0, 0), (5, 1, 0, –2), (0, 1, –2, 0) and (1, 16, –6, 2) respectively. To determine whether the set of polynomials is a basis for $\mathbb{P}_3$, we investigate whether the coordinate vectors form a basis for $\mathbb{R}^4$. Writing the vectors as the columns of a matrix and row reducing $\begin{bmatrix} 3 & 5 & 0 & 1 \\ 7 & 1 & 1 & 16 \\ 0 & 0 & -2 & -6 \\ 0 & -2 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, we find that the matrix is not row equivalent to I_4. Thus the coordinate vectors do not form a basis for $\mathbb{R}^4$. By the isomorphism between $\mathbb{R}^4$ and $\mathbb{P}_3$, the given set of polynomials does not form a basis for $\mathbb{P}_3$.