Lesson 5

Chapter 4
Vector Spaces

A: $n \times n$ $matix$

Determinant: $\boldsymbol{det\, A = |A|}$

Cofactor: $C_{ij} = (-1)^{i+j} \cdot det\, A_{ij}$

Sign $(-1)^{i+j}$: $\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$

$$det\, A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{cases} \sum_{j=1}^{n} (-1)^{i+j}\, a_{ij} \cdot det\, A_{ij} = \sum_{j=1}^{n} a_{ij} \cdot C_{ij} & \text{Row (i) expansion} \\ \sum_{i=1}^{n} (-1)^{i+j}\, a_{ij} \cdot det\, A_{ij} = \sum_{i=1}^{n} a_{ij} \cdot C_{ij} & \text{Column (j) expansion} \end{cases}$$

Row operations

Let A be a square matrix

- If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.
- If two rows of A are interchanged to produce B, then $\det B = -\det A$.
- If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

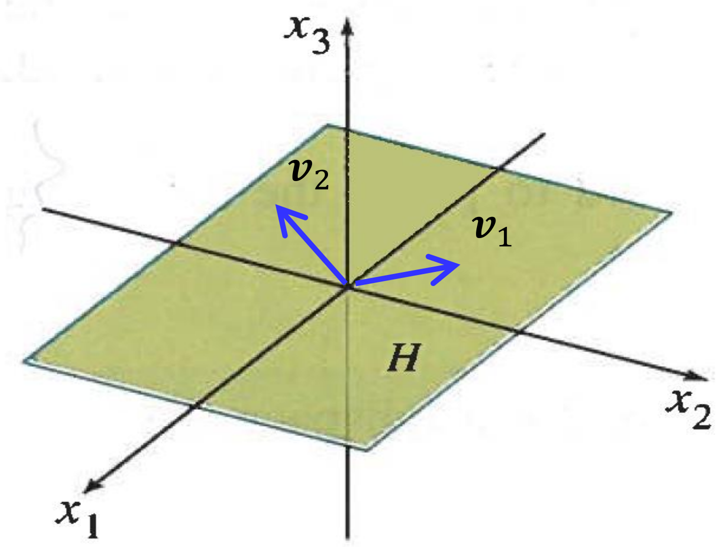
A square matrix A is invertible $\Leftrightarrow$ $det\, A \neq 0$

If A is a square matrix: $det\, A^T = det\, A$

If A and B are $n\ x\ n$ matrices: $det\, AB = det\, A \cdot det\, B$

OBS: But det

4.1 Vector Spaces and Subspaces



Vektor-
rum

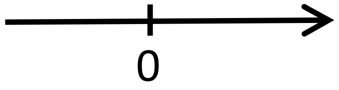
A vector space is a nonempty set V of objects, called *vectors*, on which are defined two operations called: *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in V and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$ is in V. → *Closed under addition*
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{v}+(\mathbf{u}+\mathbf{w})$.
4. There is a **zero** vector in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$. → *Neutral element*
5. For each $\mathbf{u}$ in V there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$ → *Inverse element*
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V. → *Closed under multiplication*
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$. → *Neutral element*

Ex 1 $\mathbb{R}^2$ vector space? $\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2$ $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$

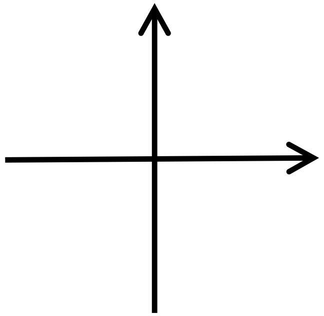
Vector spaces - Examples

$\mathbb{R}$



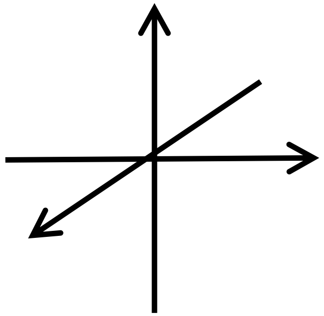
$\boldsymbol{u} = u$

$\mathbb{R}^2$



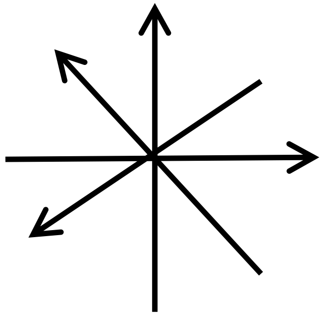
$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$\mathbb{R}^3$



$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

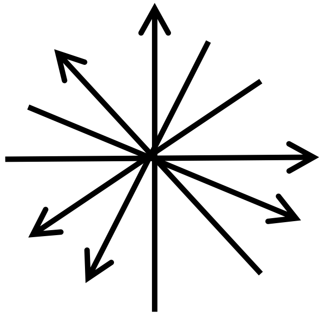
$\mathbb{R}^4$



$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$

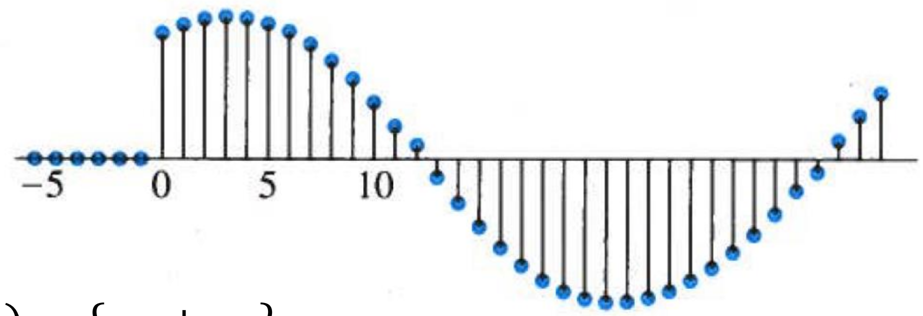
…

$\mathbb{R}^n$



$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_n \end{bmatrix}$

Vector spaces - Examples



$\mathbb{S}$: Discrete-time signals $\{y_k\} = (\cdots, y_{-2}, y_{-1}, y_0, y_1, y_2, \cdots)$

$$\{y_k\} + \{z_k\} = (\cdots, y_{-2} + z_{-2}, y_{-1} + z_{-1}, y_0 + z_0, y_1 + z_1, y_2 + z_2, \cdots) = \{y_k + z_k\}$$

$$\{cy_k\} = (\cdots, cy_{-2}, cy_{-1}, cy_0, cy_1, cy_2, \cdots) = \mathrm{c}\{y_k\}$$

$$\{0\} = (\cdots, 0,0,0,0,0, \cdots)$$

$\mathbb{P}_n$: Polynomials of degree ≤ n $\quad \boldsymbol{p}(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$

$$\boldsymbol{p}(t) + \boldsymbol{q}(t) = (a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n = (\boldsymbol{p} + \boldsymbol{q})(t)$$

$$c\boldsymbol{p}(t) = ca_0 + ca_1 t + ca_2 t^2 + \cdots + ca_n t^n = (c\boldsymbol{p})(t)$$

$$\boldsymbol{0}(t) = 0 + 0t + 0t^2 + \cdots + 0t^n = 0$$

Definition

Underrum

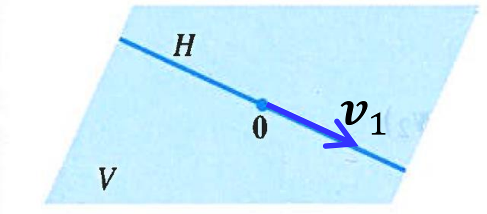
A subspace of a vector space V is a subset H of V that has three properties:

1. The zero vector from V is in H.
2. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.
3. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

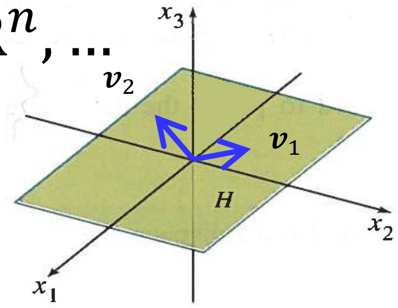
A subspace forms a vector space by itself.

Ex 2 Subspaces

- $\{\mathbf{0}\}$ subspace of $\mathbb{V}$ → Zero-subspace
- $\mathbb{P}_n$ subspace of $\mathbb{P}$ (the vector space of all polynomials)
- $c \cdot \boldsymbol{v}$ (straight line through 0) subspace of $\mathbb{R}^2, \mathbb{R}^3, \ldots \mathbb{R}^n, \ldots$



- $c_1 \cdot \boldsymbol{v}_1 + c_2 \cdot \boldsymbol{v}_2 \quad (\boldsymbol{v}_1 \neq c \cdot \boldsymbol{v}_2)$ (plane through 0) subspace of $\mathbb{R}^3, \mathbb{R}^4, \ldots \mathbb{R}^n, \ldots$



➢ **BUT**: $\mathbb{R}$ is **NOT** a subspace of $\mathbb{R}^2$, $\mathbb{R}^2$ is **NOT** a subspace of $\mathbb{R}^3$, etc.

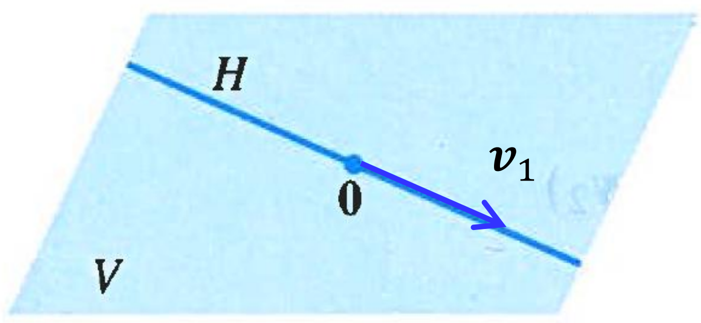
- $\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \epsilon\ \mathbb{R}^2; \quad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \epsilon\ \mathbb{R}^3; \quad \boldsymbol{u} \notin \mathbb{R}^3$
- $\boldsymbol{w} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} \epsilon\ \mathbb{R}^3$ is a subspace of $\mathbb{R}^3$, but: $\boldsymbol{w} \notin \mathbb{R}^2$

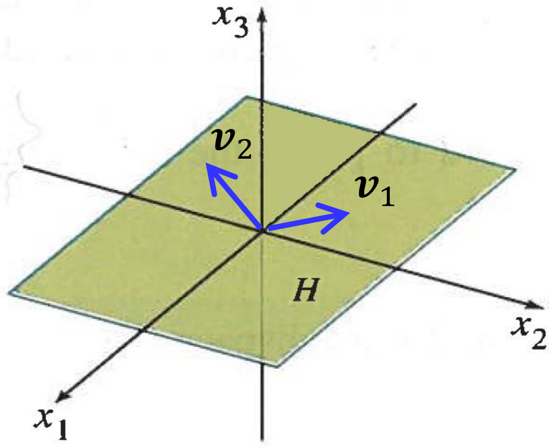
Ex 3

$\boldsymbol{v}_1 \in \mathbb{V}$ $\quad \boldsymbol{v}_2 \in \mathbb{V}$ $\quad \boldsymbol{u}, \boldsymbol{v} \in \mathbb{H} = \text{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ ← Subspace spanned by $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$

Theorem 4.1

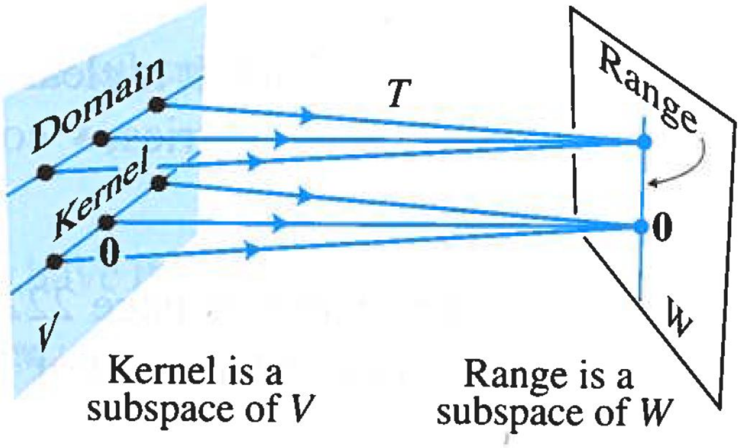
If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.





Important subspaces

4.2 Null Spaces, Column Space and Linear Transformations



Definition

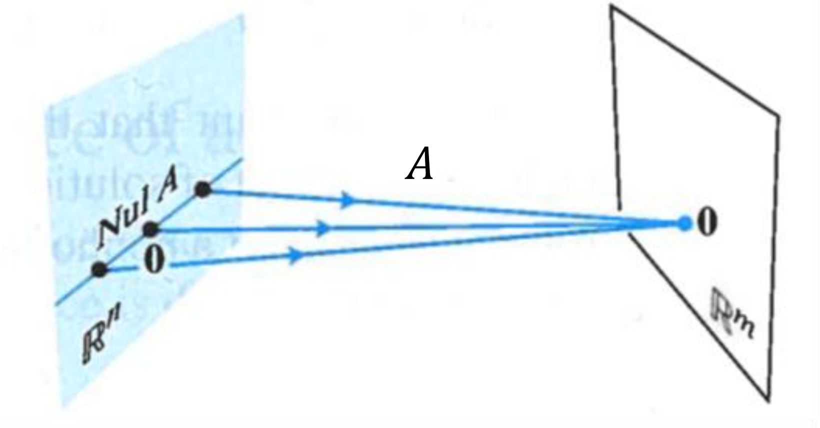
Nul-rum

The null space of a $m \times n$ matrix A, written as Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

$$\text{Nul } A = \{\mathbf{x} | \mathbf{x} \in \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$$

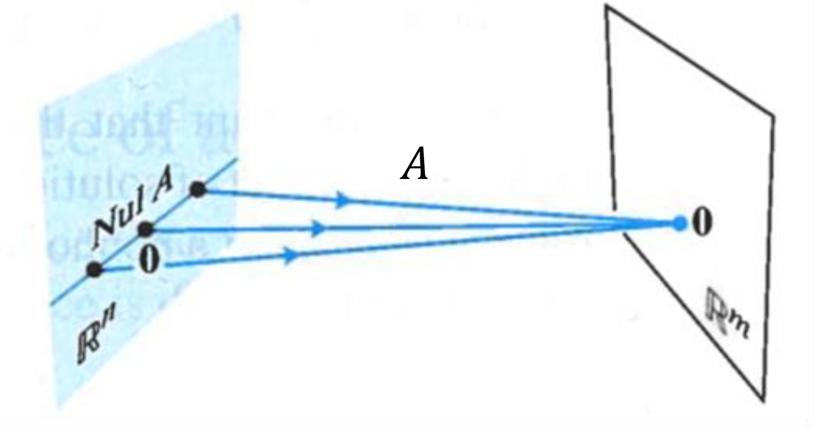
Theorem 4.2

The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.



Proof of Theorem 4.2

Let: $\boldsymbol{v}_1, \boldsymbol{v}_2 \in Nul\ A \iff A\boldsymbol{v}_1 = A\boldsymbol{v}_2 = 0$



Ex 4 $\quad A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \qquad \boldsymbol{x}_1 = \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix} \in Nul\ A\ ? \qquad \boldsymbol{x}_2 = \begin{bmatrix} -10 \\ -6 \\ 4 \end{bmatrix} \in Nul\ A\ ?$

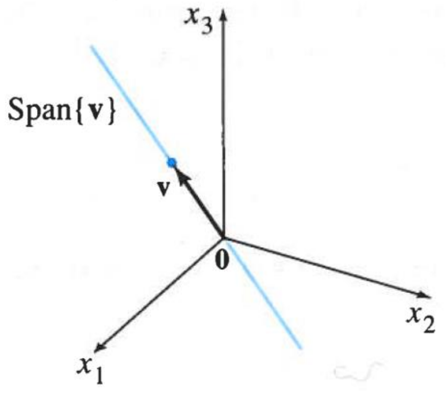
Ex 5 $\quad A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$ $\qquad$ Find all vectors $\boldsymbol{x}$ in $Nul\ A$!

$$A\boldsymbol{x} = \boldsymbol{0} \rightarrow \left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ -5 & 9 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & -6 & -9 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 5/2 & 0 \\ 0 & 1 & 3/2 & 0 \end{array}\right]$$

$$\rightarrow \begin{cases} x_1 + \frac{5}{2}x_3 = 0 \\ x_2 + \frac{3}{2}x_3 = 0 \end{cases} \rightarrow \boldsymbol{x} = \begin{bmatrix} -\frac{5}{2}x_3 \\ -\frac{3}{2}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -\frac{5}{2} \\ -\frac{3}{2} \\ 1 \end{bmatrix} = t \cdot \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = t \cdot \boldsymbol{v}; \quad t \epsilon \mathbb{R}$$

$\rightarrow$ Straight line in $\mathbb{R}^3$

$\rightarrow Nul\ A = Span\{\boldsymbol{v}\}$



Definition

Søjle-rum

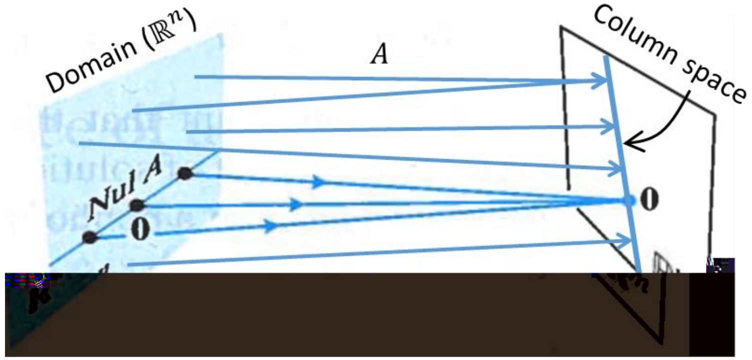
The column space of an $m \times n$ matrix A, written as Col A, is the set of all linear combinations of the columns of A. If $A = [\mathbf{a}_1 \ldots \mathbf{a}_n]$, then

$$\text{Col } A = \text{span}\,\{\mathbf{a}_1, \ldots \mathbf{a}_n\}$$

This can also be written as

$$\text{Col } A = \{\mathbf{b} | \mathbf{b} = A\mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n\}$$ → Range of Ax

Theorem 4.3: Col A of a $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.



Ex 6 $\quad A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} = [\boldsymbol{u}_1 \;\; \boldsymbol{u}_2 \;\; \boldsymbol{u}_3]$

$$Col\ A = span\left(\begin{bmatrix} 1 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right) = x_1\boldsymbol{u}_1 + x_2\boldsymbol{u}_2 + x_3\boldsymbol{u}_3 = \begin{bmatrix} x_1 - 3x_2 - 2x_3 \\ -5x_1 + 9x_2 + x_3 \end{bmatrix} \epsilon\ \mathbb{R}^2$$

→ Subspace of $\mathbb{R}^2$

$\boldsymbol{v}$ in Col A ? $\quad \rightarrow \boldsymbol{v}\ \epsilon\ span\left(\begin{bmatrix} 1 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right) \rightarrow x_1\boldsymbol{u}_1 + x_2\boldsymbol{u}_2 + x_3\boldsymbol{u}_3 = \boldsymbol{v}$

$\rightarrow \begin{bmatrix} x_1 - 3x_2 - 2x_3 \\ -5x_1 + 9x_2 + x_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ have a solution (x_1, x_2, x_3)

$\rightarrow [A|\boldsymbol{v}]$ have a solution (no pivot in the last column)

Definition

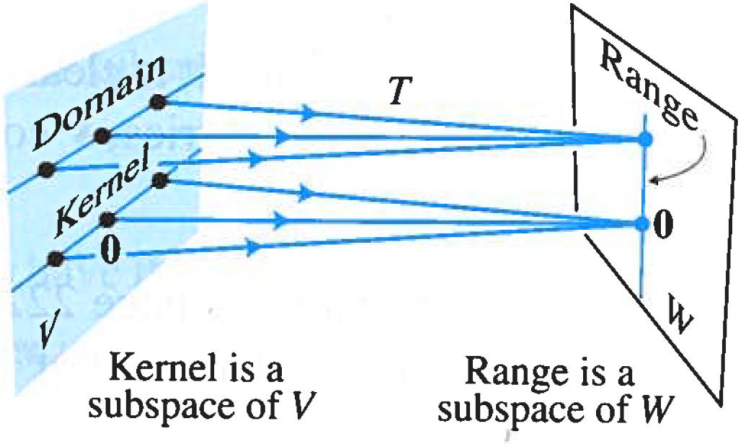
A **linear transformation** from a vector space V to a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W such that

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \qquad \text{for all } \mathbf{u}, \mathbf{v} \text{ in } V \qquad (1)$$

$$T(c\mathbf{u}) = cT(\mathbf{u}) \qquad \text{for all } \mathbf{u} \text{ in V and all scalars } c \qquad (2)$$

Kernen

Kernel/Null space $\boldsymbol{x} \in V$:
$T(\boldsymbol{x}) = \mathbf{0}$



Range/Column space $\boldsymbol{b} \in W$:
$T(\boldsymbol{x}) = \boldsymbol{b}$

Ex 7

$T\left(p(t)\right) = \frac{dp}{dt} + 2p;$ $\quad$ $p(t)\ \epsilon\ \mathbb{V}$ (all real functions on [a,b])

T a linear transfomation?

Todays words and concepts

Vector space

Subspace

Closed under addition

Col A

Inverse element

Null space

Column space

Nul A

Linear Transfomation

Closed under multiplication

Neutral element