2019spring_exam

In [1]:

```
load('../etala_utilities.sage')
```

2019 Spring Exam¶

Problem 1¶

Consider the following matrix and vector

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 0 & 6 \\ -1 & 2 & -4 \\ 1 & -2 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ -12 \\ 12 \end{bmatrix}.$$

1. Solve $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$.
2. Is it possible to find a vector $\mathbf{b}$ so $A\mathbf{x} = \mathbf{b}$ cannot be solved?

1.¶

In order to solve the equations, the augmented matrices are formed and put into row reduced echelon form

In [2]:

```
A = Matrix(SR, 4, 3, [2, 1, 3, 3, 0, 6, -1, 2, -4, 1, -2, 4])
b = vector(SR, [4, 12, -12, 12])
A0 = A.augment(vector(SR, 4), subdivide=True); show_var()
Ab = A.augment(b, subdivide=True); show_var()
A0_rref = A0.rref(); show_var()
Ab_rref = Ab.rref(); show_var()
```

for the first equation, this yields the equations:
$x_1 + 2x_3 = 0 \Leftrightarrow x_1 = -2x_3$
$x_2 - x_3 = 0 \Leftrightarrow x_2 = x_3$
which leads to the general solution:
$\bs x = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} $

for the second equation, this yields the equations:
$x_1 + 2x_3 = 4 \Leftrightarrow x_1 = 4 - 2x_3$
$x_2 - x_3 = -4 \Leftrightarrow x_2 = -4 + x_3$
which leads to the general solution:
$\bs x = \begin{bmatrix} 4 \\ -4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} $

2.¶

Since the column space of A is the set of all $\bs b$ such that $A\bs x = \bs b$ has a solution, it is only possible if Col(A) doesn't span $\Bbb R^4$. Since this would require at least four vectors, and A only has three columns, **it is possible to find a $\bs b$ such that $A\bs x = \bs b$ doesn't have a solution**.

Problem 2¶

Let four vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{b}$ be given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 7 \\ 2 \\ 7 \end{bmatrix}.$$

1. Show that the three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for $\mathbb{R}^3$.
2. Express $\mathbf{b}$ in the new basis.

1.¶

By theorem 12 in chapter 4, any set of 3 linearly independent vectors in $\Bbb R^3$ forms a basis for $\Bbb R^3$. Therefore if the set $\left\{\bs v_1, \bs v_2, \bs v_3 \right\}$ is linearly independent.
To check this the three vectors are concatenated to form a matrix. By the invertible matrix theorem, the formed matrix will have a pivot in every row if and only if the columns are linearly independent.

In [3]:

```
P = Matrix(3, 3, [1, 0, 2, 0, 1, 1, -3, 2, 2]); show_var()
P_rref = P.rref(); show_var()
```

since there are three pivots, **the set $\left\{\bs v_1, \bs v_2, \bs v_3 \right\}$ forms a basis for $\Bbb R^3$**.

2.¶

By theorem 15 in chapter 4, the matrix P above is the matrix for change of basis from the basis formed by $\left\{\bs v_1, \bs v_2, \bs v_3 \right\}$ to $\Bbb R^3$

In [4]:

```
b = vector([7, 2, 7])
b_new_basis = P.inverse()*b; show_var()
```

Problem 3¶

Consider the following matrix and vector

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -1 & 0 & 1 \\ 2 & -2 & 2 \\ 1 & 2 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -2 \\ 3 \\ 0 \\ -5 \end{bmatrix}.$$

1. Show that the matrix equation $A\mathbf{x} = \mathbf{b}$ is inconsistent.
2. Find the least-squares solution of $A\mathbf{x} = \mathbf{b}$.

In a number of applications, sparse solutions, i.e. solutions where most elements are zero, are desired. Consider the two sparse vectors

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathbf{x}_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}.$$

3. Determine whether $\mathbf{x}_1$ or $\mathbf{x}_2$ is the better solution of $A\mathbf{x} = \mathbf{b}$ in the least-squares sense.

1.¶

To check the system for consistency, the augmented matrix is row reduced

In [5]:

```
A = Matrix(4, 3, [1, 3, 4, -1, 0, 1, 2, -2, 2, 1, 2, -1])
b = vector([-2, 3, 0, -5])
Ab_aug = A.augment(b)
Ab_rref = Ab_aug.rref(); show_var()
```

Since the last row corresponds to $0 = 1$ **the system is inconsistent.**

2.¶

In order to find the least squares solution, the pseudo inverse of A is determined. Since the columns of A are linearly independent (as can be seen in the rref(A) above), by theorem 14 in chapter 6, the least squares solution can be determined as $\boldsymbol{\hat{x}} = \left(A^TA\right)^{-1}A^T \bs b$

In [6]:

```
x_hat = (A.T*A).inverse()*A.T*b; show_var()
```

3.¶

In order to determine which x is the better solution, the lengths of the residual vectors are compared

In [7]:

```
x_1 = vector([2, 0, 0])
x_2 = vector([0, 2, 0])
e_x_1 = (b - A*x_1).norm(); show_var()  # could be the other way around, but the norm is the
e_x_2 = (b - A*x_2).norm(); show_var()  # here too
```

Since $\vert \epsilon_{x_1} \vert $ is smaller than $\vert \epsilon_{x_2} \vert $, x_1 is a better solution in the least squares sense.

Problem 4¶

For the statements given below, state whether they are true or false and justify your answer for each statement.

1. A square, upper-triangular matrix with non-zero elements on the diagonal is invertible.
2. If a matrix A has an eigenvalue λ, then $c\lambda$, with c a scalar, is also an eigenvalue.
3. The matrix $\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}$ is positive semidefinite.

1.¶

True
By scaling each row in the described matrix by the value on the diagonal of the given row, a row reduced echelon form with ones along the diagonal must result. By the invertible matrix theorem, such a matrix is invertible

2.¶

False
This would lead to an infinite number of eigenvalues. It is however true that if matrix A has an *eigenvector* $\bs v$, then $c\bsv$ is also an eigenvector of A

3.¶

False
To test this, determine the eigenvalues of the matrix:

In [8]:

```
A = Matrix(2, 2, [-3, 3, 3, -3])
A.eigenvalues()
```

Out[8]:

```
[0, -6]
```

Since the eigenvalues are <= 0, the matrix is negative semi-definite

Problem 5¶

In the case *Computer Graphics in Automotive Design*, homogeneous coordinates and rotation matrices were introduced. In this problem we are not concerned with homogeneous coordinates, but only work with standard coordinates. In $\mathbb{R}^2$ a rotation of the vector $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ by an angle, θ about the origin is obtained by multiplying the following rotation matrix with $\mathbf{x}$.

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

1. Show that $R(\theta)$ is an orthogonal matrix.
2. Argue that $R(2\theta) = R^2(\theta)$.
3. Compute $R^2(\theta)$ and use this result to find formulas for $\cos(2\theta)$ and $\sin(2\theta)$ expressed by $\cos\theta$ and $\sin\theta$.

1.¶

In order to show that $R(\theta)$ is orthogonal, the inner product of the columns must be 0

In [9]:

```
var('theta')
R_theta = Matrix(SR, 2, 2, [cos(theta), -sin(theta), sin(theta), cos(theta)]); show_var()
```

```
inner_p_R = R_theta.column(0).inner_product(R_theta.column(1)); show_var()
```

And so **$R(\theta)$ is orthogonal**.

2.¶

In order to show this, the two sides of the equation are calculated algebraically:

In [10]:

```
R_theta_sq = R_theta^2; show_var()
R_theta_double = Matrix(SR, 2, 2, [cos(2*theta), -sin(2*theta), sin(2*theta), cos(2*theta)])
```

The following trigonometric identities, then show how these are equal:
$ cos(2x) = cos(x)^2 - sin(x)^2$ (cell 1,1 and 2,2)
$ sin(2x) = 2 sin(x) cos(x) $ (cell 1,2 and 2,1)

3.¶

$R^2(\theta) has already been calculated above.
extracting the trigonometric

Problem 6¶

A special and somewhat rare class of square matrices are called skew-symmetric. A general 3×3 skew-symmetric matrix has this form

$$\begin{bmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{bmatrix},$$

where a_1, a_2 and a_3 are scalars. One particular use of 3×3 skew-symmetric matrices sometimes encountered in mechanical engineering is as a way of expressing the vector cross product as a matrix multiplication.

1. For a symmetric matrix $A = A^T$. What is the corresponding relation for skew-symmetric matrices?

Consider the set of all 3×3 skew-symmetric matrices, here denoted as $\mathbb{S}^{3\times 3}$.

2. Show that $\mathbb{S}^{3\times 3}$ forms a vector space.

1.¶

Since the two halves are negatives of each other, $A = -A^T$ for a skew-symmetric matrix.

2.¶

To test that $\Bbb S^{3\times3}$ is a vector space, the fact that addition and multiplication by a scalar is defined, plus the 10 axioms of vector spaces must be shown to be true.

Firstly note that addition and multiplication by scalars is defined for all matrices, so the elements of $\Bbb S^{3\times3}$ are valid vectors.
The fact that the vector space is closed under addition, can be shown by calculating the algebraic sum of two arbitrary 3x3 skew-symmetrix matrices:

In [11]:

```
var('a_1, a_2, a_3, b_1, b_2, b_3')
A = Matrix(SR, 3, 3, [0, a_1, a_2, -a_1, 0, a_3, -a_2, -a_3, 0])
B = Matrix(SR, 3, 3, [0, b_1, b_2, -b_1, 0, b_3, -b_2, -b_3, 0])
show(A, ' + ', B, ' = ', A+B)
```

Since the resulting matrix is also a skew-symmetric matrix, the vector space is closed under addition

axioms 2, 3, 7, 8, and 9 all follow from the rules for matrix algebra (see theorem 1 chapter 2)

since $-0 = 0$, the three by three zero matrix is in $\Bbb S^{3\times3}$. Adding the zero matrix to another matrix results in the same element, and so $0^{3\times3}$ is a valid neutral element and axiom 4 is fulfilled

In [12]:

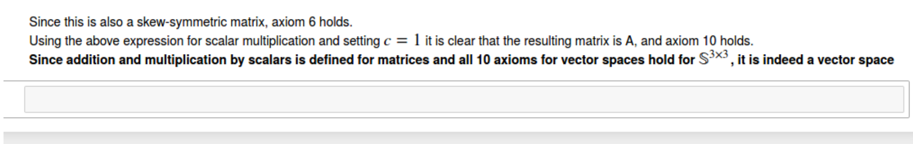
```
show(A + -A)
```

The above clearly shows axiom 5.
To show axiom 6, the general expression for scalar multiplication is shown:

In [13]:

```
var('c')
show(c*A)
```

Since this is also a skew-symmetric matrix, axiom 6 holds.
Using the above expression for scalar multiplication and setting $c = 1$ it is clear that the resulting matrix is A, and axiom 10 holds.
Since addition and multiplication by scalars is defined for matrices and all 10 axioms for vector spaces hold for $\Bbb S^{3\times3}$, it is indeed a vector space


Since this is also a skew-symmetric matrix, axiom 6 holds.
Using the above expression for scalar multiplication and setting $c = 1$ it is clear that the resulting matrix is A, and axiom 10 holds.
Since addition and multiplication by scalars is defined for matrices and all 10 axioms for vector spaces hold for $\mathbb{S}^{3\times3}$, it is indeed a vector space

In [ ]: