Case 3 - Space Flight

Question 1

- Confirm that the controllability matrices for Examples 1 and 2 are as given above.

From example one we have

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

With the controllability matrix M defined as

$$M = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

where A is $n \times n$ and B is $n \times m$. In example 1, $n = 2$ and we get

$$M = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 7 \end{bmatrix}.$$

In example 2, $n = 3$ and the controllability matrix becomes

$$M = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 6 & 26 \\ 1 & -4 & -14 \\ 2 & 12 & 52 \end{bmatrix}.$$

Question 2

- Consider the system with matrices

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 4 \\ 1 & 2 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

  Determine whether the pair (A, B) is controllable.

Result 2 from the note states: A pair (A, B) (where A is an $n \times n$ matrix and B is an $n \times m$ matrix) is controllable if and only if the rank of the controllability matrix M is n. The controllability matrix is therefore calculated and row reduced. As $n = 3$ this matrix becomes

$$M = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 3 & 15 \\ 0 & 6 & 36 \\ 1 & 3 & 21 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

From the above, it is seen that M has 3 pivots and hence rank M=3=n. The system is therefore controllable. Another way to state this is that M has full rank.

Question 3

- If B is invertible, what can be said about the controllability of the pair (A, B)?

If B is invertible, this matrix must be square and it is row equivalent to the identity matrix. The controllability matrix can thus be calculated and row reduced to give

$$M = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix} \sim \begin{bmatrix} I & * & * & \dots & * \end{bmatrix}$$

where $*$ indicates unknown numbers. Due to the identity matrix at the beginning of the row reduced controllability matrix, the controllability matrix has pivots in all rows, rank $M = n$ and the system is controllable.

Question 4

- Confirm that the controllability matrix in Example 4 is as given above, and that the system is controllable.

From Example 4

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 6 & 2 \\ 3 & 9 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}.$$

From A we have $n = 3$ and M becomes

$$M = \begin{bmatrix} 1 & 0 & 8 & 2 & 80 & 20 \\ 2 & 1 & 16 & 4 & 160 & 40 \\ 1 & -1 & 24 & 6 & 240 & 60 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.25 & 10 & 2.5 \end{bmatrix}.$$

From the row reduction it is seen that rank $M = 3 = n$ and the system is controllable.

Question 5

- Consider the system with matrices

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 4 \\ 0 & 0 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

  a) Find the controllability matrix for this system.
  b) Show that system $\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$, $\mathbf{x}_0 = \mathbf{0}$ is controllable.
  c) Find controls $\mathbf{u}_0$, $\mathbf{u}_1$ and $\mathbf{u}_2$ which will drive this system to the point $\begin{bmatrix} -7 \\ 3 \\ 1 \end{bmatrix}$.

The controllability matrix is

$$M = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 5 \\ 1 & 2 & 4 \end{bmatrix}.$$

By row reducing it is found that $M \sim I$ and the system is controllable.

The controls are found by setting up the appropriate $M\mathbf{u} = \mathbf{y}$ problem and solving it

$$\left[\, M \mid \mathbf{y} \,\right] = \left[\begin{array}{ccc|c} 1 & 2 & 2 & -7 \\ 1 & 1 & 5 & 3 \\ 1 & 2 & 4 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -19 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right].$$

From which we find that $\underline{\underline{\mathbf{u}_0 = 4, \mathbf{u}_1 = 2,\ \mathbf{u}_2 = -19}}$.

Question 6

- Consider the system with matrices

$$A = \begin{bmatrix} 1 & 3 & -2 \\ -2 & -6 & 4 \\ 5 & 15 & -10 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 1 & 0 \end{bmatrix}$$

  a) Find the controllability matrix for this system.
  b) Show that system $\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$, $\mathbf{x}_0 = \mathbf{0}$ is controllable.
  c) Find controls $\mathbf{u}_0$, $\mathbf{u}_1$ and $\mathbf{u}_2$ which will drive this system to the point $\begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix}$.

d) Suppose that $\mathbf{u}_0 = (1,\ 1)$. Can the system still be driven to $\begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix}$? If so, find controls $\mathbf{u}_1$ and $\mathbf{u}_2$ to make this happen.

The controllability matrix and its row reduced form are

$$M = \begin{bmatrix} 1 & 1 & 5 & -2 & -75 & 30 \\ 2 & -1 & -10 & 4 & 150 & -60 \\ 1 & 0 & 25 & -10 & -375 & 150 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -0.4 & -15 & 6 \end{bmatrix}.$$

Since the controllability matrix has full rank, the system is controllable.

The controls are found by solving the $M\mathbf{u} = \mathbf{y}$ problem.

$$\left[\ M \mid \mathbf{y}\ \right] = \left[\begin{array}{cccccc|c} 1 & 1 & 5 & -2 & -75 & 30 & 4 \\ 2 & -1 & -10 & 4 & 150 & -60 & -1 \\ 1 & 0 & 25 & -10 & -375 & 150 & -3 \end{array}\right] \sim \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0.75 \\ 0 & 1 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & -0.4 & -15 & 6 & -0.15 \end{array}\right]$$

From the above we see that $u_{21} = 0.75$ and $u_{22} = 4$. From the last line of the matrix we see that u_{12}, u_{01} and u_{02} are free variables and related to u_{11} as

$$u_{11} - 0.4u_{12} - 15u_{01} + 6u_{00} = -0.15$$

The simplest solution is to set the free variables to zero and hence the controls become

$$\mathbf{u}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} -0.15 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0.75 \\ 4 \end{bmatrix}.$$

The situation with $\mathbf{u}_0 = (1,\ 1)$ corresponds to locking two of the three free variables. The above equation then becomes

$$u_{11} - 0.4u_{12} - 15 \cdot 1 + 6 \cdot 1 = -0.15 \iff u_{11} - 0.4u_{12} = 8.85$$

Similar to above, the easy choice is to fix $u_{12} = 0$ and we get $u_{11} = 8.85$. The control vectors in this case is thus

$$\mathbf{u}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 8.85 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0.75 \\ 4 \end{bmatrix}$$

It can easily be checked that this solution is correct

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{x}_1 = A\mathbf{x}_0 + B\mathbf{u}_0 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{x}_2 = A\mathbf{x}_1 + B\mathbf{u}_1 = \begin{bmatrix} 11.85 \\ 11.7 \\ 23.85 \end{bmatrix}$$

$$\mathbf{x}_3 = A\mathbf{x}_2 + B\mathbf{u}_2 = \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix} = \mathbf{y}.$$