Chapter 4.5

1. This subspace is $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are not multiples of each other, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and is thus a basis for *H*. Hence the dimension of *H* is 2.

13. The matrix *A* is in echelon form. There are three pivot columns, so the dimension of Col *A* is 3. There are two columns without pivots, so the equation $A\mathbf{x} = \mathbf{0}$ has two free variables. Thus the dimension of Nul *A* is 2.

19. **a**. True. See the box before Example 5.

b. False. The plane must pass through the origin; see Example 4.

c. False. The dimension of $\mathbb{P}_n$ is $n + 1$; see Example 1.

d. False. The set *S* must also have *n* elements; see Theorem 12.

e. True. See Theorem 9.

21. The matrix whose columns are the coordinate vectors of the Hermite polynomials relative to the standard basis $\{1, t, t^2, t^3\}$ of $\mathbb{P}_3$ is $A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$. This matrix has 4 pivots, so its columns are linearly independent. Since their coordinate vectors form a linearly independent set, the Hermite polynomials themselves are linearly independent in $\mathbb{P}_3$. Since there are four Hermite polynomials and dim $\mathbb{P}_3 = 4$, the Basis Theorem states that the Hermite polynomials form a basis for $\mathbb{P}_3$.

22. The matrix whose columns are the coordinate vectors of the Laguerre polynomials relative to the standard basis $\{1, t, t^2, t^3\}$ of $\mathbb{P}_3$ is $A = \begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{bmatrix}$. This matrix has 4 pivots, so its columns are linearly independent. Since their coordinate vectors form a linearly independent set, the Laguerre polynomials themselves are linearly independent in $\mathbb{P}_3$. Since there are four Laguerre polynomials and dim $\mathbb{P}_3$ = 4, the Basis Theorem states that the Laguerre polynomials form a basis for $\mathbb{P}_3$.

24. The coordinates of $\mathbf{p}(t) = 7 - 8t + 3t^2$ with respect to $\mathcal{B}$ satisfy

$$c_1(1) + c_2(1-t) + c_3(2 - 4t + t^2) = 7 - 8t + 3t^2$$

Equating coefficients of like powers of t produces the system of equations

$$\begin{array}{ccccccc} c_1 & + & c_2 & + & 2c_3 & = & 7 \\ & & -c_2 & - & 4c_3 & = & -8 \\ & & & & c_3 & = & 3 \end{array}$$

Solving this system gives $c_1 = 5,\ c_2 = -4,\ c_3 = 3,$ and $[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$.

Chapter 4.6

1. The matrix B is in echelon form. There are two pivot columns, so the dimension of Col A is 2. There are two pivot rows, so the dimension of Row A is 2. There are two columns without pivots, so the equation $A\mathbf{x} = \mathbf{0}$ has two free variables. Thus the dimension of Nul A is 2. A basis for Col A is the pivot columns of A: $\left\{ \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix} \right\}$. A basis for Row A consists of the pivot rows of B: $\{(1,0,-1,5),(0,-2,5,-6)\}$. To find a basis for Nul A row reduce to reduced echelon form: $A \sim \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & 1 & -5/2 & 3 \end{bmatrix}$. The solution to $A\mathbf{x} = \mathbf{0}$ in terms of free variables is $x_1 = x_3 - 5x_4$, $x_2 = (5/2)x_3 - 3x_4$ with x_3 and x_4 free. Thus a basis for Nul A is $\left\{ \begin{bmatrix} 1 \\ 5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

3. The matrix B is in echelon form. There are three pivot columns, so the dimension of Col A is 3. There are three pivot rows, so the dimension of Row A is 3. There are two columns without pivots, so the equation $A\mathbf{x} = \mathbf{0}$ has two free variables. Thus the dimension of Nul A is 2. A basis for Col A is the pivot columns of A: $\left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix} \right\}$. A basis for Row A is the pivot rows of B: $\{(2,-3,6,2,5),(0,0,3,-1,1),(0,0,0,1,3)\}$. To find a basis for Nul A row reduce to reduced echelon form: $A \sim \begin{bmatrix} 1 & -3/2 & 0 & 0 & -9/2 \\ 0 & 0 & 1 & 0 & 4/3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The solution to $A\mathbf{x} = \mathbf{0}$ in terms of free variables is $x_1 = (3/2)x_2 + (9/2)x_5$, $x_3 = -(4/3)x_5$, $x_4 = -3x_5$, with x_2 and x_5 free. Thus a basis for Nul A is $\left\{ \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 9/2 \\ 0 \\ -4/3 \\ -3 \\ 1 \end{bmatrix} \right\}$.

8. Since A has four pivot columns, $\text{rank } A = 4$, and $\dim \text{Nul } A = 6 - \text{rank } A = 6 - 4 = 2$.

No. $\text{Col } A \neq \mathbb{R}^4$. It is true that $\dim \text{Col } A = \text{rank } A = 4$, but Col A is a subspace of $\mathbb{R}^5$.

13. The rank of a matrix A equals the number of pivot positions which the matrix has. If A is either a 7×5 matrix or a 5×7 matrix, the largest number of pivot positions that A could have is 5. Thus the largest possible value for rank A is 5.

19. Yes. Consider the system as $A\mathbf{x}=\mathbf{0}$, where A is a 5×6 matrix. The problem states that $\dim \text{ Nul } A=1$. By the Rank Theorem, $\text{rank } A=6-\dim \text{ Nul } A=5$. Thus $\dim \text{Col } A=\text{rank } A=5$, and since Col A is a subspace of $\mathbb{R}^5$, Col A = $\mathbb{R}^5$. So every vector **b** in $\mathbb{R}^5$ is also in Col A, and $A\mathbf{x}=\mathbf{b}$, has a solution for all **b**.

25. No. Consider the system as $A\mathbf{x}=\mathbf{b}$, where A is a 10×12 matrix. The problem states that $\dim \text{Nul} A=3$. By the Rank Theorem, $\dim\text{Col } A=\text{rank } A=12-\dim\text{Nul } A=9$. Thus Col A will be a proper subspace of $\mathbb{R}^{10}$. Thus there exists a **b** in $\mathbb{R}^{10}$ for which the system $A\mathbf{x}=\mathbf{b}$ is inconsistent, and the system $A\mathbf{x}=\mathbf{b}$ cannot have a solution for all **b**.

27. Since A is an $m\times n$ matrix, Row A is a subspace of $\mathbb{R}^n$, Col A is a subspace of $\mathbb{R}^m$, and Nul A is a subspace of $\mathbb{R}^n$. Likewise since A^T is an $n\times m$ matrix, Row A^T is a subspace of $\mathbb{R}^m$, Col A^T is a subspace of $\mathbb{R}^n$, and Nul A^T is a subspace of $\mathbb{R}^m$. Since $\text{Row } A=\text{Col } A^T$ and $\text{Col } A=\text{Row } A^T$, there are four dinstict subspaces in the list: Row A, Col A, Nul A, and Nul A^T.

Chapter 4.7

1. **a**. Since $\mathbf{b}_1=6\mathbf{c}_1-2\mathbf{c}_2$ and $\mathbf{b}_2=9\mathbf{c}_1-4\mathbf{c}_2$, $[\mathbf{b}_1]_\mathcal{C}=\begin{bmatrix}6\\-2\end{bmatrix}$, $[\mathbf{b}_2]_\mathcal{C}=\begin{bmatrix}9\\-4\end{bmatrix}$, and $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}=\begin{bmatrix}6&9\\-2&-4\end{bmatrix}$.

b. Since $\mathbf{x}=-3\mathbf{b}_1+2\mathbf{b}_2$, $[\mathbf{x}]_\mathcal{B}=\begin{bmatrix}-3\\2\end{bmatrix}$ and $[\mathbf{x}]_\mathcal{C}=\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[x]_\mathcal{B}=\begin{bmatrix}6&9\\-2&-4\end{bmatrix}\begin{bmatrix}-3\\2\end{bmatrix}=\begin{bmatrix}0\\-2\end{bmatrix}$

7. To find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$, row reduce the matrix $[\mathbf{c}_1\quad\mathbf{c}_2\quad\mathbf{b}_1\quad\mathbf{b}_2]$:

$[\mathbf{c}_1\quad\mathbf{c}_2\quad\mathbf{b}_1\quad\mathbf{b}_2]\sim\begin{bmatrix}1&0&-3&1\\0&1&-5&2\end{bmatrix}$. Thus $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}=\begin{bmatrix}-3&1\\-5&2\end{bmatrix}$, and $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}=\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1}=\begin{bmatrix}-2&1\\-5&3\end{bmatrix}$.

13. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} = \{1-2t+t^2, 3-5t+4t^2, 2t+3t^2\}$ and let $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\} = \{1, t, t^2\}$. The $\mathcal{C}$-coordinate vectors of $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ are $[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}, [\mathbf{b}_3]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$. So

$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}$. Let $\mathbf{x} = -1 + 2t$. Then the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ satisfies

$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$. This system may be solved by row reducing its augmented matrix:

$$\begin{bmatrix} 1 & 3 & 0 & -1 \\ -2 & -5 & 2 & 2 \\ 1 & 4 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \text{ so } [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}.$$