Chapter 6
Orthogonality and least squares

Or how to find a nearly solution of inconsistent systems

Et lille overblik so far…

Produkt
Determinant
- Invers matrix
- Egenværdier og egenvektorer
  - Skift af koordinatsystem (rotationer mm.)
  - Diagonalisering
    - De-kobling af dynamiske systemer
    - Dekompositioner (ex. QR og SVD)

$$A\boldsymbol{x} = \lambda\boldsymbol{x} \Leftrightarrow (A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$$ have non-tvivial solutions

The Characteristic Equation $\longrightarrow$ $det(A - \lambda I) = 0$

$\uparrow$

The Characteristic Polynomial $\longrightarrow$ Polynomial of degree n in λ

$\downarrow$

Exactly n (complex) eigenvalues (roots) λ

For a <u>real</u> matrix:

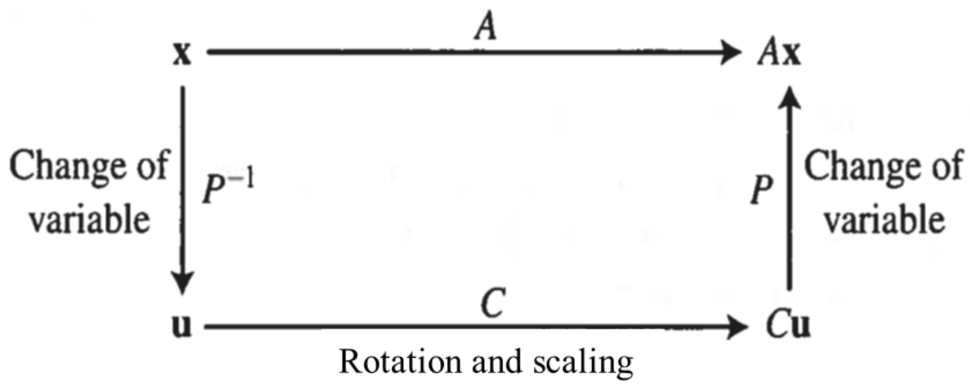
- Complex eigenvalues and -vectors come in pairs: $\begin{cases} \lambda = a \pm j \cdot b \\ \boldsymbol{v} = Re(\boldsymbol{v}) \pm j \cdot Im(\boldsymbol{v}) \end{cases}$

Theorem 5.9

Let A be a real 2 x 2 matrix with a complex eigenvalue $\lambda = a + j \cdot b \; (b \neq 0)$ and an associated eigenvector $\boldsymbol{v} = Re(\boldsymbol{v}) + j \cdot Im(\boldsymbol{v})$ in $\mathbb{C}^2$. Then:

$$A = PCP^{-1}$$

where $P = [Re(\boldsymbol{v}_1) \quad Im(\boldsymbol{v}_1)]$ and $C = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{bmatrix}$



rotationsmatrix

Note, that eventhough the theorem only speaks about one eigenvalue and one eigenvector we actually know two eigenvalues and two eigenvectors as $\lambda_2 = {\lambda_1}^*$ and $\boldsymbol{v}_2 = {\boldsymbol{v}_1}^*$

Decoupling a dynamic system: $\boldsymbol{x}'(t) = A\boldsymbol{x}(t)$

Eigenvalues (λ_i) /-vectors ($\boldsymbol{v}_i$) /-functions for A: $\boldsymbol{v}_1 e^{\lambda_1 t}, \cdots, \boldsymbol{v}_n e^{\lambda_n t}$

Change-of-variable matrix: $P = [\boldsymbol{v}_1 \ \cdots \ \boldsymbol{v}_n]$

Decoupled system

Solution in eigen-function basis

Diagonal matrix: $D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$; $A = PDP^{-1} \implies \boldsymbol{y}' = D\boldsymbol{y}$ where $\boldsymbol{y}(t) = P^{-1}\boldsymbol{x}(t)$

Initial value: $\boldsymbol{x}(0) = \boldsymbol{x}_0 \Rightarrow \boldsymbol{y}(0) = P^{-1}\boldsymbol{x}(0) = P^{-1}\boldsymbol{x}_0 = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Solution in the original system:

Solution in eigen-function basis

eigenvalue

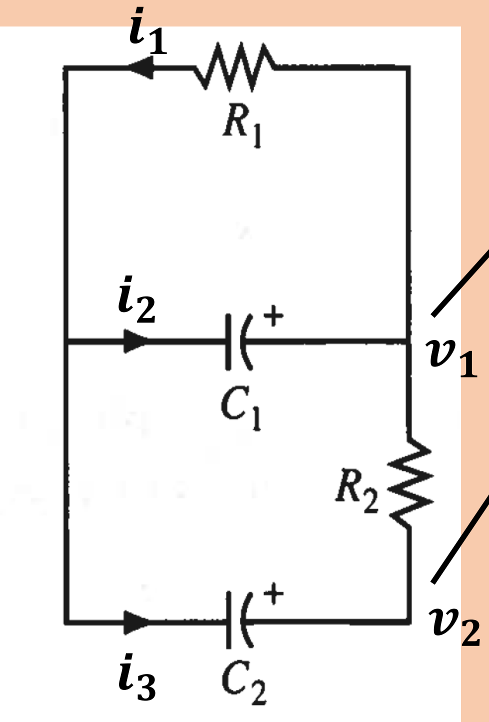
$$\boldsymbol{x}(t) = P\boldsymbol{y}(t) = [\boldsymbol{v}_1 \ \cdots \ \boldsymbol{v}_n]\begin{bmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix} = c_1\boldsymbol{v}_1 e^{\lambda_1 t} + \cdots + c_n \boldsymbol{v}_n e^{\lambda_n t} = \sum_{i=1}^{n} c_i \boldsymbol{v}_i e^{\lambda_i t}$$

eigenvector

Dynamic systems

Time developing

Ex fra L9



KCL

$$\left.\begin{aligned} -C_1\frac{dv_1}{dt} &= \frac{v_1-v_2}{R_2}+\frac{v_1}{R_1} \\ -C_2\frac{dv_2}{dt} &= \frac{v_2-v_1}{R_2} \end{aligned}\right\}$$

Two coupled linear 1. order diffential equations

$$\Rightarrow \left.\begin{aligned} \frac{dv_1}{dt} &= -\frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)v_1+\frac{1}{R_2C_1}v_2 \\ \frac{dv_2}{dt} &= \frac{1}{R_2C_2}v_1-\frac{1}{R_2C_2}v_2 \end{aligned}\right\}$$

Knudepunktsligninger

Husk strømmen gennem en kondensator

$I_C = C*dv_C/dt$

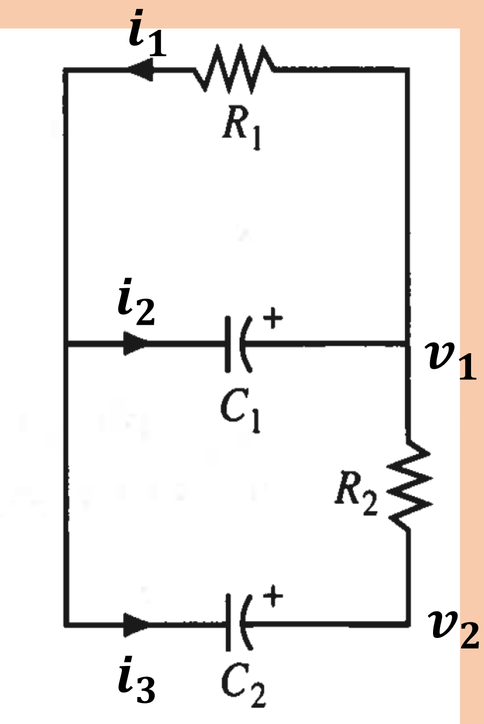
$$\Rightarrow \begin{bmatrix} \frac{dv_1}{dt} \\ \frac{dv_2}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) & \frac{1}{R_2C_1} \\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix} \cdot \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$$

$$\Rightarrow \quad \boldsymbol{x}'(t) = A\boldsymbol{x}(t)$$

System of coupled linear 1. order diffential equations

Ex fra L9

$$R_1 = 1\Omega \quad R_2 = 2\Omega \quad C_1 = 1F \quad C_2 = 0.5F \quad v_1(0) = 5V \quad v_2(0) = 4V$$



$$\begin{bmatrix} \dfrac{dv_1}{dt} \\ \dfrac{dv_2}{dt} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{C_1}\left(\dfrac{1}{R_1} + \dfrac{1}{R_2}\right) & \dfrac{1}{R_2C_1} \\ \dfrac{1}{R_2C_2} & -\dfrac{1}{R_2C_2} \end{bmatrix} \cdot \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} -\dfrac{3}{2} & \dfrac{1}{2} \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -\dfrac{3}{2} - \lambda & \dfrac{1}{2} \\ 1 & -1 - \lambda \end{vmatrix} = \lambda^2 + \frac{5}{2}\lambda + 1 = 0 \quad \Rightarrow \quad \lambda = \begin{cases} -½ \\ -2 \end{cases}$$

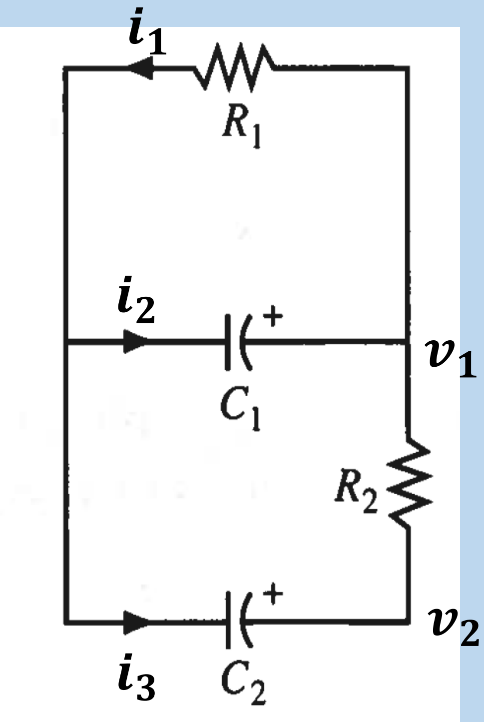
Eigenvectors: $\lambda_1 = -\frac{1}{2} \Rightarrow \boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\lambda_2 = -2 \Rightarrow \boldsymbol{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$P = [\boldsymbol{u}_1 \ \boldsymbol{u}_2] = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} -½ & 0 \\ 0 & -2 \end{bmatrix} \qquad \boldsymbol{y}(t) = P^{-1}\boldsymbol{v}(t) = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{3}v_1(t) + \frac{1}{3}v_2(t) \\ -\frac{2}{3}v_1(t) + \frac{1}{3}v_2(t) \end{bmatrix}$$

$$\boldsymbol{y}' = \begin{bmatrix} -½ & 0 \\ 0 & -2 \end{bmatrix}\boldsymbol{y} \Rightarrow \begin{cases} y_1' = -½ y_1 \\ y_2' = -2y_2 \end{cases} \Rightarrow \boldsymbol{y}(t) = \begin{bmatrix} c_1 e^{-½t} \\ c_2 e^{-2t} \end{bmatrix} = \begin{bmatrix} 3e^{-½t} \\ -2e^{-2t} \end{bmatrix} \qquad \begin{bmatrix} c_1 \\ c_1 \end{bmatrix} = \boldsymbol{y}(0) = P^{-1}\boldsymbol{v}(0) = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$\Rightarrow \boldsymbol{v}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = P\boldsymbol{y}(t) = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3e^{-½t} \\ -2e^{-2t} \end{bmatrix} = \begin{bmatrix} 3e^{-½t} + 2e^{-2t} \\ 6e^{-½t} - 2e^{-2t} \end{bmatrix}$$

OPGAVE 1



”rigtige” værdier

$R_1 = 4.7k\Omega$

$R_2 = 22k\Omega$

$C_1 = 1\mu F$

$C_2 = 47nF$

- Bestem talværdierne i koefficientmatricen A
- Find determinanten for A
- Løs λ-polynomiet og find derved egenværdierne
- Bestem de to tilhørende egenvektorer
- Opstil matricerne P og D, og find den inverse af P
- Bestem den transformerede spændingsvektor (t)
- Find transformerede begyndelsesbetingelser (0)
- Indsæt (0) samt λ_1 og λ_2 i egenfunktionerne
- Transformer (t) tilbage til (t) vha. P

- Plot v_1 og v_2 som funktion af tiden

Begyndelsesbetingelser:

$v_1(0) = 10V \qquad v_2(0) = 12V$

6.1 Inner Product, Length and Orthogonality

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}^T \boldsymbol{v}$$

Prik-produkt

$$\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$$

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}^T \boldsymbol{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i \in \mathbb{R}$$

Rules calculating ________________:

1. $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$
2. $(\boldsymbol{u} + \boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{w}$
3. $(c\boldsymbol{u}) \cdot \boldsymbol{v} = \boldsymbol{u} \cdot (c\boldsymbol{v})$
4. $\boldsymbol{u} \cdot \boldsymbol{u} \geq 0$ and $\boldsymbol{u} \cdot \boldsymbol{u} = 0 \Leftrightarrow \boldsymbol{u} = 0$

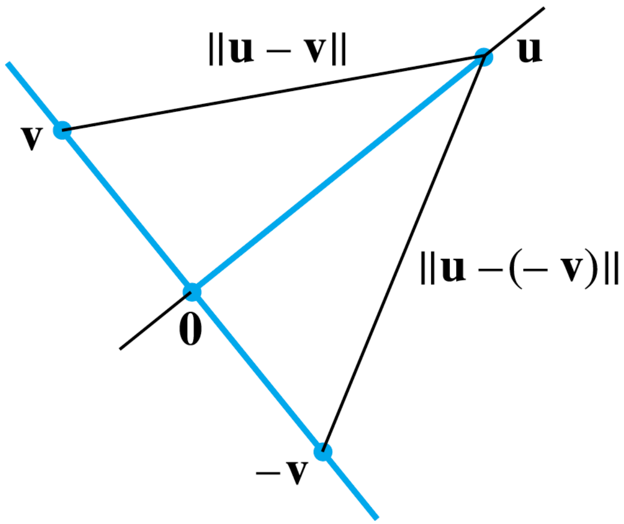
- The (or length) of a vector is defined as: $\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{v_1^2 + \cdots + v_n^2}$
- A vector $\boldsymbol{v}$ is (unit vector) if: $\|\boldsymbol{v}\| = 1$
- The between two vectors is defined as:

$$dist(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\| = \sqrt{(u_1 - v_1)^2 + \cdots + (u_n - v_n)^2}$$

- The θ between two vectors is given by: $\boldsymbol{u} \cdot \boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\|\cos(\theta)$

Definition

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.



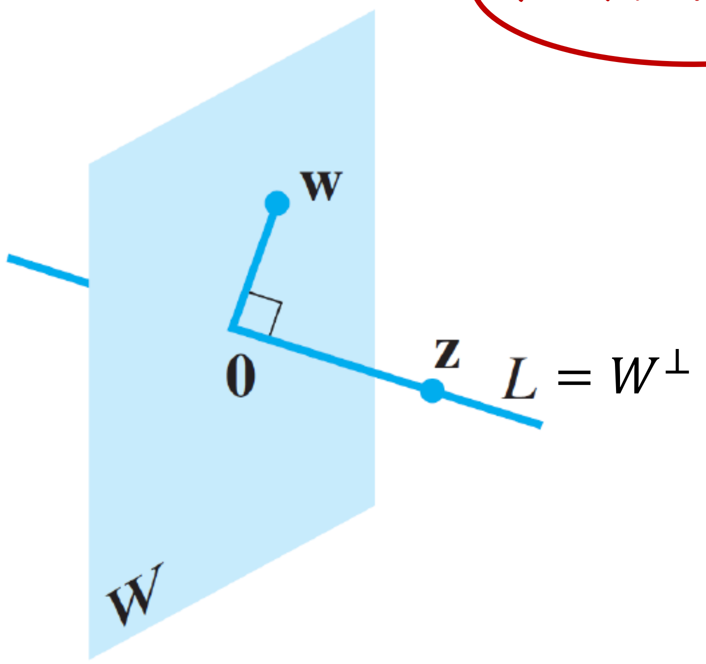
Pythagoras: $\boldsymbol{u}$ and $\boldsymbol{v}$ orthogonal $\Leftrightarrow ||\boldsymbol{u}+\boldsymbol{v}||^2 = ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2$

Definition

Let W be a subspace of $\mathbb{R}^n$. The set of all vectors in $\mathbb{R}^n$ which are orthogonal to all vectors in W is called the *orthogonal complement to W* and is denoted $W^\perp$.

W vinkelret

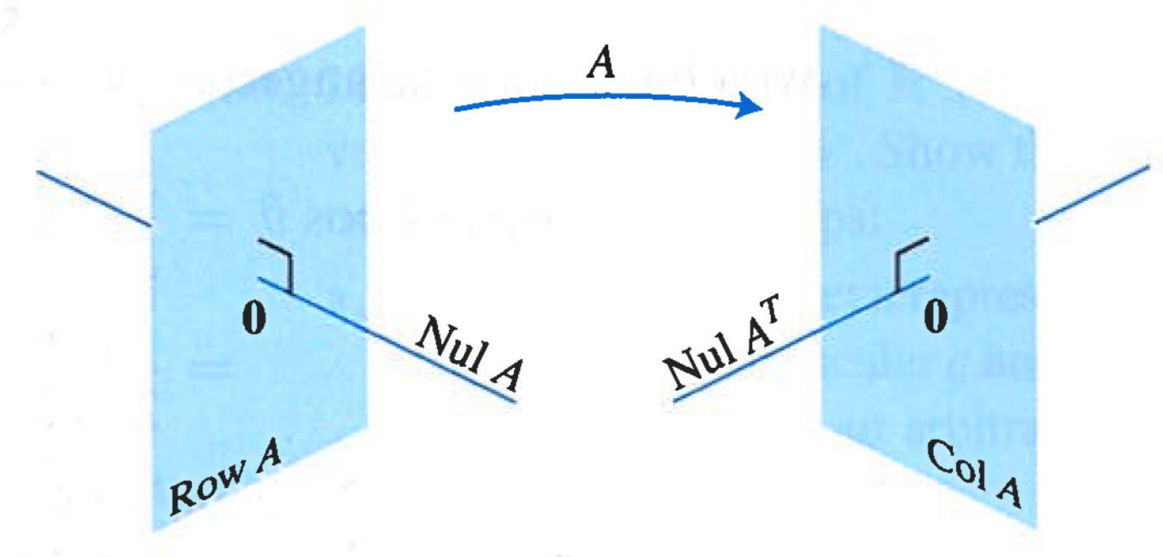
Ortogonale/vinkelrette komplement



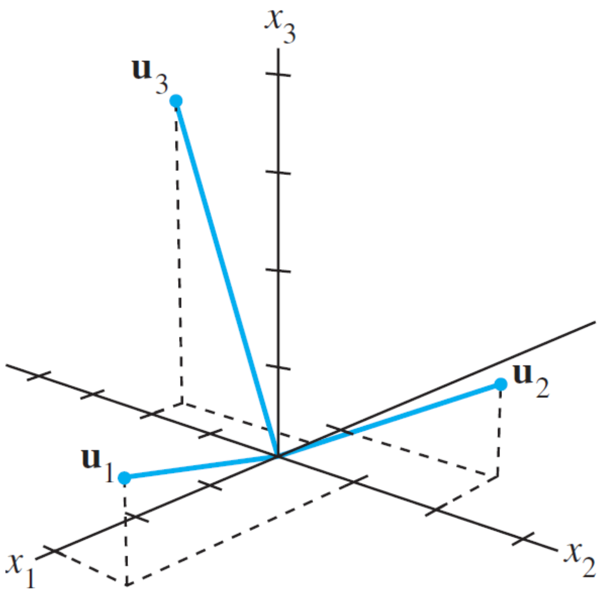
1. A vector is in $W^\perp$ if and only if is orthogonal to every vector in a set that spans W.
2. $W^\perp$ is a subspace of $\mathbb{R}^n$

Let A be an m x n matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

$$(Row\ A)^{\perp} = Nul\ A \quad \text{and} \quad (Col\ A)^{\perp} = Nul\ A^T$$



6.2 Orthogonal Sets



Definitions

A set of vectors $S = \{\boldsymbol{u}_1 \ \cdots \ \boldsymbol{u}_n\}$ in $\mathbb{R}^n$ with $\boldsymbol{u}_i \perp \boldsymbol{u}_j$ for all $i \neq j$

That is: $\boldsymbol{u}_i \cdot \boldsymbol{u}_j = 0$ for all $i \neq j$

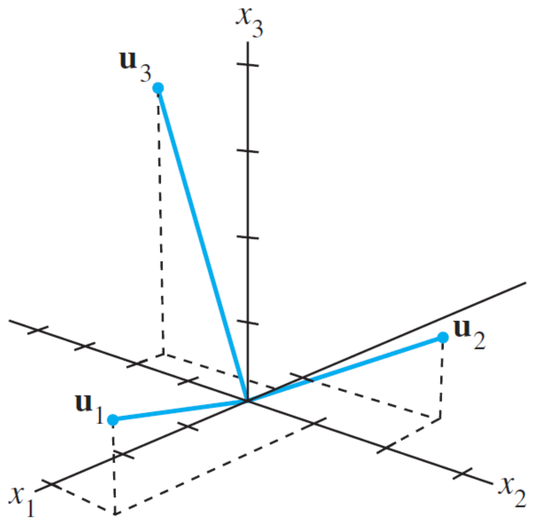
An orthogonal set of vectors $S = \{\boldsymbol{u}_1 \ \cdots \ \boldsymbol{u}_n\}$ in $\mathbb{R}^n$ with $\|\boldsymbol{u}_i\| = 1$ for all $\mathrm{i} = 1, \dots, n$.

That is: $\boldsymbol{u}_i \cdot \boldsymbol{u}_j = \delta_{ij} = \begin{cases} 1 & if \;\; i = j \\ 0 & if \;\; i \neq j \end{cases}$

Ex 2

$$\boldsymbol{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \qquad \boldsymbol{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$$

> *Diskussion:*
>
> Udgør vektorerne u_1, u_2 og u_3 et ortogonalt set?
> - Og hvis ja, er det da også orto-*normalt*?



Theorem 6.4:

$S = \{\boldsymbol{u}_1 \ \cdots \ \boldsymbol{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$

$\Longrightarrow S$ is a linearly independent basis for $\text{span}\{S\}$

Theorem 6.5:

Basis that is also an orthogonal set

Let $\{\boldsymbol{u}_1 \ \cdots \ \boldsymbol{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$.

For each $\boldsymbol{y}$ in W, the weights in the linear combination

$$\boldsymbol{y} = c_1\boldsymbol{u}_1 + \cdots + c_p\boldsymbol{u}_p$$

are given by

$$c_j = \frac{\boldsymbol{y} \cdot \boldsymbol{u}_j}{\boldsymbol{u}_j \cdot \boldsymbol{u}_j}; \quad (j = 1, \cdots, p)$$

Ex 3

$$\boldsymbol{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \qquad \boldsymbol{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix} \qquad \boldsymbol{y} = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$$

$$\|\boldsymbol{u}_1\| = \sqrt{11} \qquad \|\boldsymbol{u}_2\| = \sqrt{6} \qquad \|\boldsymbol{u}_3\| = \sqrt{33/2}$$

$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ → Orthogonal set

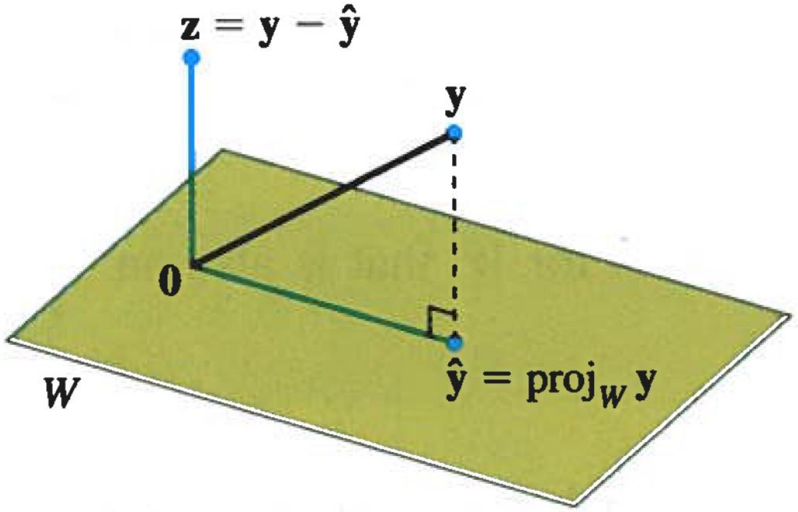
Beregning:

Bestem koefficienterne i linearkombinationen

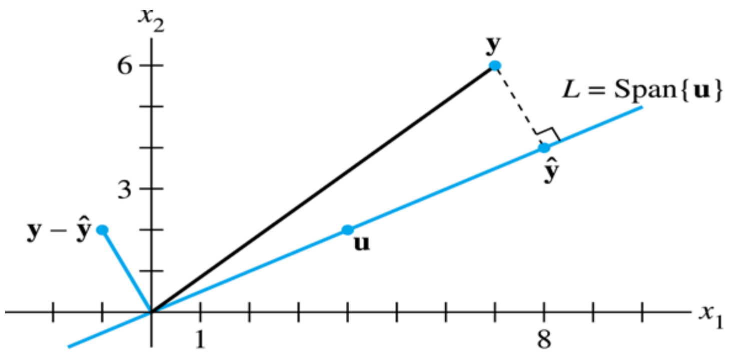
$= c_1 \ {}_1 + c_2 \ {}_2 + c_3 \ {}_3$

$c_1 = 1$
$c_2 = -2$
$c_3 = -2$

6.3 Orthogonal Projections



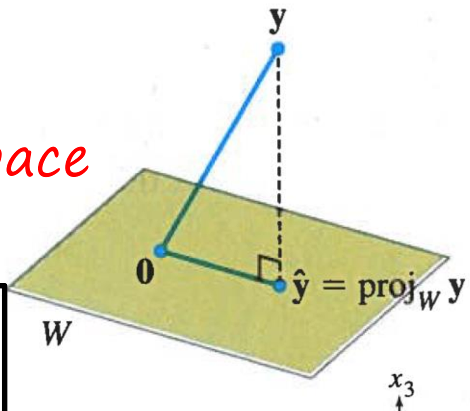
Orthogonal projection



$c_{\boldsymbol{u}}$

$$\boxed{\hat{\boldsymbol{y}} = \frac{\boldsymbol{y} \cdot \boldsymbol{u}}{\boldsymbol{u} \cdot \boldsymbol{u}} \boldsymbol{u} = \frac{\|\boldsymbol{y}\| \|\boldsymbol{u}\| cos(\theta)}{\|\boldsymbol{u}\|^2} \boldsymbol{u} = \|\boldsymbol{y}\| cos(\theta) \boldsymbol{e}_u = proj_L \boldsymbol{y}}$$

Projection of $\boldsymbol{y}$ on line L spanned by $\boldsymbol{u}$



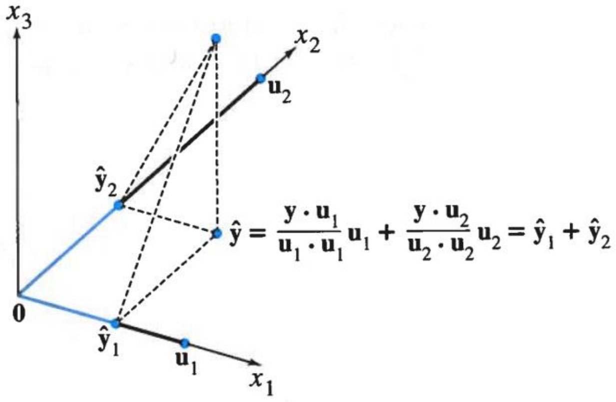
Projection of $\boldsymbol{y}$ on subspace W spanned by $\boldsymbol{u}_1, \cdots, \boldsymbol{u}_p$

$$\boxed{\hat{\boldsymbol{y}} = \frac{\boldsymbol{y} \cdot \boldsymbol{u}_1}{\boldsymbol{u}_1 \cdot \boldsymbol{u}_1} \boldsymbol{u}_1 + \cdots + \frac{\boldsymbol{y} \cdot \boldsymbol{u}_p}{\boldsymbol{u}_p \cdot \boldsymbol{u}_p} \boldsymbol{u}_p = proj_W \boldsymbol{y}}$$

where $W = Span\{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_p\}$

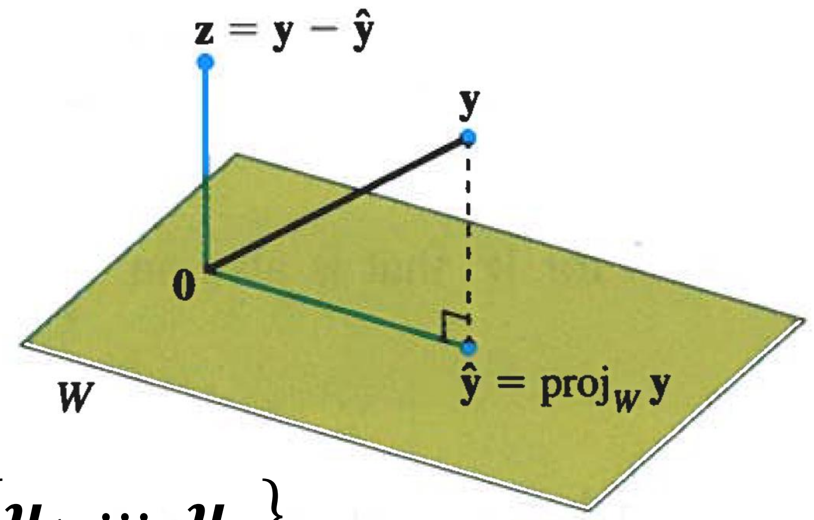
Orthogonal set

The projection of $\boldsymbol{y}$ in the direction defined by $\boldsymbol{u}_p$



Theorem 6.8 - The Orthogonal Decomposition Theorem

$$\boldsymbol{y} = c_1\boldsymbol{u}_1 + \cdots + c_n\boldsymbol{u}_n = \widehat{\boldsymbol{y}} + \boldsymbol{z} \quad \epsilon\ \mathbb{R}^n$$



where

$$\widehat{\boldsymbol{y}} = \underbrace{\frac{\boldsymbol{y}\cdot\boldsymbol{u}_1}{\boldsymbol{u}_1\cdot\boldsymbol{u}_1}}_{c_1}\boldsymbol{u}_1 + \cdots + \underbrace{\frac{\boldsymbol{y}\cdot\boldsymbol{u}_p}{\boldsymbol{u}_p\cdot\boldsymbol{u}_p}}_{c_p}\boldsymbol{u}_p = proj_W\boldsymbol{y} \ \in W = Span\{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_p\}$$

$$\boldsymbol{z} = \boldsymbol{y} - \widehat{\boldsymbol{y}} = \underbrace{\frac{\boldsymbol{y}\cdot\boldsymbol{u}_{p+1}}{\boldsymbol{u}_{p+1}\cdot\boldsymbol{u}_{p+1}}}_{c_{p+1}}\boldsymbol{u}_{p+1} + \cdots + \underbrace{\frac{\boldsymbol{y}\cdot\boldsymbol{u}_n}{\boldsymbol{u}_n\cdot\boldsymbol{u}_n}}_{c_n}\boldsymbol{u}_n = proj_{W^\perp}\boldsymbol{y} \in W^\perp = Span\{\boldsymbol{u}_{p+1}, \cdots, \boldsymbol{u}_n\}$$

$$\boldsymbol{z} = \boldsymbol{y} - \widehat{\boldsymbol{y}} \perp \widehat{\boldsymbol{y}}$$

Basis representation

Let $\{\boldsymbol{b}_1 \ \cdots \ \boldsymbol{b}_p\}$ be an non-orthogonal basis for a subspace W of $\mathbb{R}^n$:

$$\boldsymbol{y} = c_1\boldsymbol{b}_1 + \cdots + c_p\boldsymbol{b}_p; \quad c_j: \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & \vdots & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}$$

Hard work

Let $\{\boldsymbol{u}_1 \ \cdots \ \boldsymbol{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$:

$$\boldsymbol{y} = c_1\boldsymbol{u}_1 + \cdots + c_p\boldsymbol{u}_p; \quad c_j = \frac{\boldsymbol{y} \cdot \boldsymbol{u}_j}{\boldsymbol{u}_j \cdot \boldsymbol{u}_j}$$

Fair

Let $\{\boldsymbol{e}_{\boldsymbol{u}_1} \ \cdots \boldsymbol{e}_{\boldsymbol{u}_p}\}$ be an orthonormal basis for a subspace W of $\mathbb{R}^n$:

$$\boldsymbol{y} = c_1\boldsymbol{e}_{\boldsymbol{u}_1} + \cdots + c_p\boldsymbol{e}_{\boldsymbol{u}_p}; \quad c_j = \boldsymbol{y} \cdot \boldsymbol{e}_{\boldsymbol{u}_j} = \|\boldsymbol{y}\| cos(\theta_j)$$

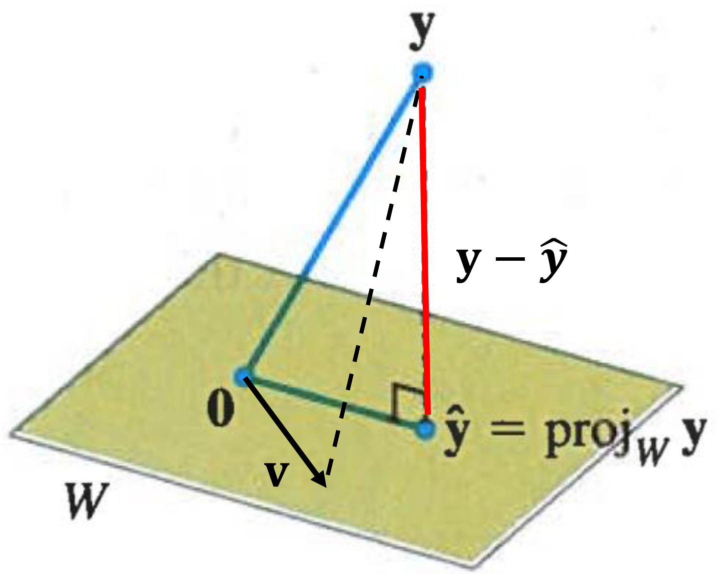
Easy

Theorem 6.9: The Best Approximation Theorem

Let W be a subspace of $\mathbb{R}^n$, let $\mathbf{y}$ be any vector in $\mathbb{R}^n$, and let $\hat{\mathbf{y}}$ be the orthogonal projection of $\mathbf{y}$ onto W. Then $\hat{\mathbf{y}}$ is the closest point in W to $\mathbf{y}$ in the sense that

$$||\mathbf{y} - \hat{\mathbf{y}}|| < ||\mathbf{y} - \mathbf{v}||$$

for all $\mathbf{v}$ in W distinct from $\hat{\mathbf{y}}$.



Theorem 6.10

If $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthonormal basis for a subspace W of $\mathbb{R}^n$ then

$$\text{proj}_W \mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + \ \ldots \ + (\mathbf{y} \cdot \mathbf{u}_p)\mathbf{u}_p$$

If $U = [\mathbf{u}_1 \ \ldots \ \mathbf{u}_p]$, then

$$\text{proj}_W \mathbf{y} = UU^T\mathbf{y} \qquad \text{for all } \mathbf{y} \text{ in } \mathbb{R}^n$$

OPGAVE 2

Matricen A her er tæt på at være ortogonal

- Hvad vil det sige, at en matrix er ortogonal?
- ”Reparer” på A, således at den bliver ortogonal
  hint: kig på elementet med indeks 3,2

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Todays words and concepts

Orthogonal

Inner product

Length

Dot product

Orthonormal

Orthonormal basis

Norm

Best approximation

Orthogonal complement

Orthogonal set

Orthogonal matrix

Distance

Orthogonal projection

Orthogonal decomposition