Lesson 6

Chapter 4
Vector Spaces

Vektor-
rum

A vector space is a nonempty set V of objects, called *vectors*, on which are defined two operations called: *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in V and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$ is in V. → *Closed under addition*
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{v}+(\mathbf{u}+\mathbf{w})$.
4. There is a **zero** vector in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$. → *Neutral element*
5. For each $\mathbf{u}$ in V there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$ → *Inverse element*
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V. → *Closed under multiplication*
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$. → *Neutral element*

Definition

Underrum

A subspace of a vector space V is a subset H of V that has three properties:

1. The zero vector from V is in H.
2. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.
3. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

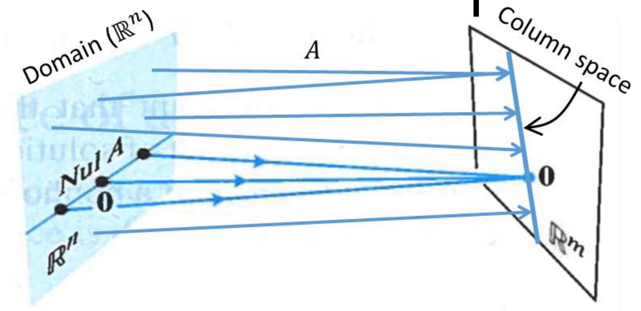
A subspace forms a vector space by itself.

Definition

Nul-rum

The null space of a $m \times n$ matrix A, written as Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

$$\text{Nul } A = \{\mathbf{x} | \mathbf{x} \in \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$$



Definition

Søjle-rum

The column space of an $m \times n$ matrix A, written as Col A, is the set of all linear combinations of the columns of A. If $A = [\mathbf{a}_1 \ldots \mathbf{a}_n]$, then

$$Col\ A = Span\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\} = \{\boldsymbol{b} | \boldsymbol{b} = A\boldsymbol{x},\ \forall \boldsymbol{x} \epsilon \mathbb{R}^m\}$$

4.3 Linearly Independent Sets; Bases

$$\mathbb{H} = \boldsymbol{Span}\{\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_p\}$$

$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_p\} \in \mathbb{V}$:

Linear independent:

$c_1 \cdot \boldsymbol{v}_1 + c_2 \cdot \boldsymbol{v}_2 + \cdots + c_p \cdot \boldsymbol{v}_p = \mathbf{0}$ $\Rightarrow$ Only trivial solution (all $c_i = 0$)

Linear dependent:

$c_1 \cdot \boldsymbol{v}_1 + c_2 \cdot \boldsymbol{v}_2 + \cdots + c_p \cdot \boldsymbol{v}_p = \mathbf{0}$ $\Rightarrow$ Non-trivial solution exist (at least one $c_i \neq 0$)

Theorem 4.4

An indexed set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$ is linearly dependent if and only if some $\mathbf{v}_j$ with $j > 1$ is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots \mathbf{v}_{j-1}$.

Ex 1

$$\left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\} \epsilon\ \mathbb{R}^2$$

$$\{\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{p}_4\}\ \epsilon\ \mathbb{P} \qquad \boldsymbol{p}_1(\mathrm{t}) = 1; \quad \boldsymbol{p}_2(t) = t; \quad \boldsymbol{p}_3(t) = 4 - t; \quad \boldsymbol{p}_4(t) = t^2 - t;$$

Definition of **basis**:

Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p\}$ in V is a **basis** for H if

- $\mathcal{B}$ is a linearly independent set, and → *no unnecessary vectors*
- the subspace spanned by $\mathcal{B}$ coincides with H; that is, $H =$Span$\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p\}$.

→ *Smallest possible spanning set*
→ *Largest possible linear independent spanning set*

or

A basis for V is a linearly independent set of vectors that spans V.

$$e_1 = \begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix};\ e_2 = \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix};\ e_3 = \begin{bmatrix}0\\0\\1\\\vdots\\0\end{bmatrix};\ \cdots;\ e_n = \begin{bmatrix}0\\0\\0\\\vdots\\1\end{bmatrix}$$

→ **Standard basis** for $\mathbb{R}^n$

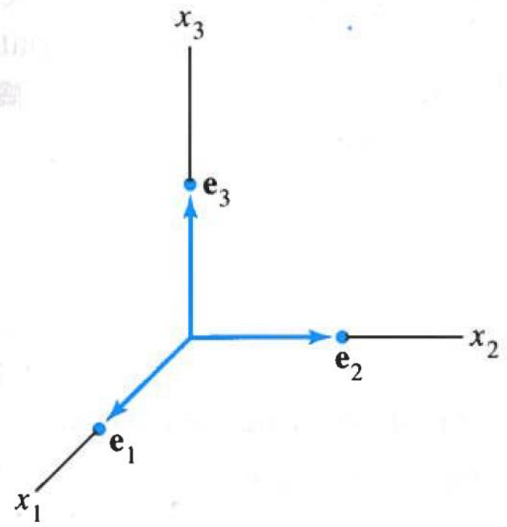


FIGURE 1
The standard basis for $\mathbb{R}^3$.

$$S = \{1, t, t^2, \cdots, t^n\}$$

→ **Standard basis** for $\mathbb{P}_n$

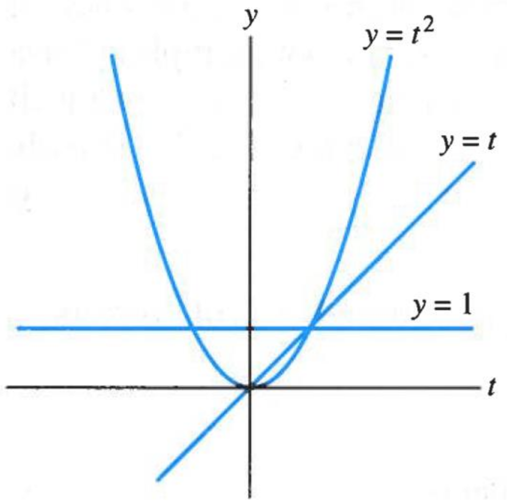


FIGURE 2
The standard basis for $\mathbb{P}_2$.

Ex 2

$$\left\{\begin{bmatrix}3\\0\\-6\end{bmatrix}, \begin{bmatrix}-4\\1\\7\end{bmatrix}, \begin{bmatrix}-2\\1\\5\end{bmatrix}\right\}$$ basis for $\mathbb{R}^3$?

→ Linearly independent ? $c_1 \cdot \boldsymbol{v}_1 + c_2 \cdot \boldsymbol{v}_2 + c_3 \cdot \boldsymbol{v}_3 = \boldsymbol{0} \;\Rightarrow\; A\boldsymbol{x} = \boldsymbol{0}$

$$\Rightarrow \begin{bmatrix}3 & -4 & -2 & 0\\0 & 1 & 1 & 0\\-6 & 7 & 5 & 0\end{bmatrix} \sim \begin{bmatrix}3 & -4 & -2 & 0\\0 & 1 & 1 & 0\\0 & -1 & 1 & 0\end{bmatrix} \sim \begin{bmatrix}3 & -4 & -2 & 0\\0 & 1 & 1 & 0\\0 & 0 & 2 & 0\end{bmatrix}$$

$\Rightarrow \boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$ linearly independent!

$$\boldsymbol{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \in \mathbb{R}^3: \quad \boldsymbol{x} = c_1 \cdot \boldsymbol{v}_1 + c_2 \cdot \boldsymbol{v}_2 + c_3 \cdot \boldsymbol{v}_3 \;\Rightarrow\; A\boldsymbol{c} = \boldsymbol{x} \;\Rightarrow\; \boldsymbol{c} = A^{-1}\boldsymbol{x}$$

A an invertible $n \times n$ matrix: $A = [\boldsymbol{a_1}\ \boldsymbol{a_2}\ \cdots\ \boldsymbol{a_n}]$

$\Rightarrow \{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots, \boldsymbol{a_n}\}$ is a basis for $\mathbb{R}^n$

by Theorem 2.8 (Invertible Matrix Theorem):

a. A is an invertible matrix.

e. The columns of A form a linearly independent set.

h. The columns of A span $\mathbb{R}^n$.

Theorem 4.5, The Spanning Set Theorem

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set in V, and let $H =$Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

- If one of the vectors in S - say, $\mathbf{v}_k$ - is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.
- If $H \neq \{\mathbf{0}\}$ some subset of S is a basis for H.

Ex 3

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 5 & 10 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for $Nul\ A$:

$$A\boldsymbol{x} = \boldsymbol{0} \implies \begin{cases} x_1 - 2x_2 - x_4 + 3x_5 = 0 \\ \\ x_3 + 2x_4 - 2x_5 = 0 \end{cases} \implies \begin{cases} x_1 = 2x_2 + x_4 - 3x_5 \\ x_2 = x_2 \\ x_3 = -2x_4 + 2x_5 \\ x_4 = x_4 \\ x_5 = x_5 \end{cases}$$

$$\implies \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \sum_{x_i\, free} x_i \cdot \boldsymbol{v}_i$$

$\implies$ $\boxed{\text{Basis for } Nul\ A = \{\boldsymbol{v}_2, \boldsymbol{v}_4, \boldsymbol{v}_5\} = \{\boldsymbol{v}_i\}_{x_i\, free}}$ (Linearly independent set)

Ex 4

$$A=\begin{array}{c}\begin{bmatrix}-3 & 6 & -1 & 1 & -7\\ 1 & -2 & 2 & 3 & -1\\ 2 & -4 & 5 & 8 & -4\end{bmatrix}\\ \begin{matrix}\boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5\end{matrix}\end{array} \sim \begin{bmatrix}1 & -2 & 2 & 3 & -1\\ 0 & 0 & 1 & 2 & -2\\ 0 & 0 & 5 & 10 & -10\end{bmatrix} \sim \begin{array}{c}\begin{bmatrix}1 & -2 & 0 & -1 & 3\\ 0 & 0 & 1 & 2 & -2\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}\\ \begin{matrix}\boldsymbol{b}_1 & \boldsymbol{b}_2 & \boldsymbol{b}_3 & \boldsymbol{b}_4 & \boldsymbol{b}_5\end{matrix}\end{array} = B$$

Basis for $Col\ A$:

$A \sim B \ \Rightarrow\ A\boldsymbol{x}=\mathbf{0}$ and $B\boldsymbol{x}=\mathbf{0}$ have equal solutions $\boldsymbol{x}$

$\Downarrow$

$$\left.\begin{array}{l} x_1\cdot \boldsymbol{a}_1 + x_2\cdot\boldsymbol{a}_2+\cdots+x_5\cdot\boldsymbol{a}_5=\mathbf{0} \\ \\ x_1\cdot \boldsymbol{b}_1 + x_2\cdot\boldsymbol{b}_2+\cdots+x_5\cdot\boldsymbol{b}_5=\mathbf{0}\end{array}\right\} \Rightarrow \text{Same linearly dependent/independent column vectors in } A \text{ and } B$$

$\{\boldsymbol{b_1},\boldsymbol{b_3}\}$ linearly independent (pivot columns); $\quad \boldsymbol{b_2}=-2\boldsymbol{b_1},\quad \boldsymbol{b_4}=-\boldsymbol{b_1}+2\boldsymbol{b_3},\quad \boldsymbol{b_5}=-3\boldsymbol{b_1}-2\boldsymbol{b_3}$

$\Downarrow$

$\{\boldsymbol{a_1},\boldsymbol{a_3}\}$ linearly independent (pivot columns) $\quad \boldsymbol{a_2}=-2\boldsymbol{a_1},\quad \boldsymbol{a_4}=-\boldsymbol{a_1}+2\boldsymbol{a_3},\quad \boldsymbol{a_5}=-3\boldsymbol{a_1}-2\boldsymbol{a_3}$

$\Downarrow$

$Col\ A = Span\{\boldsymbol{a_1},\boldsymbol{a_2},\boldsymbol{a_3},\boldsymbol{a_4},\boldsymbol{a_5}\}=Span\{\boldsymbol{a_1},\boldsymbol{a_3}\} \neq Span\{\boldsymbol{b_1},\boldsymbol{b_3}\}$

$\Downarrow$

$\boxed{\text{Basis for } Col\ A=\{\boldsymbol{a_1},\boldsymbol{a_3}\}=\{\boldsymbol{a_i}\}_{\boldsymbol{pivot\ columns\ in\ A}}}$

Basis for $Nul\ A$ (see chap. 4.2 ex.3):

Linearly independent

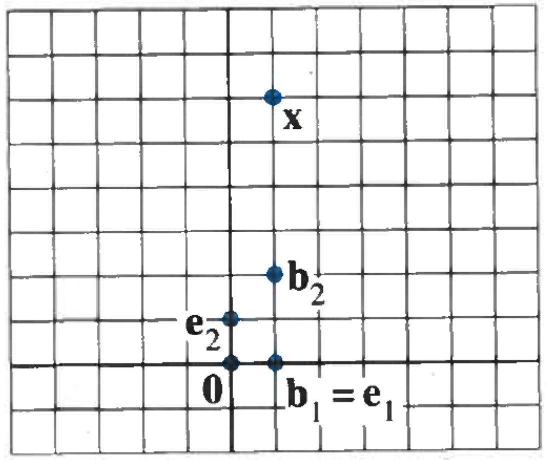
$$A\boldsymbol{x} = \boldsymbol{0} \rightarrow [A\ \ \boldsymbol{0}] \rightarrow \text{Reduced echelon form} \rightarrow \boldsymbol{x} = \sum_{free\ variables} x_i \boldsymbol{v}_i\ \epsilon\ Nul\ A$$

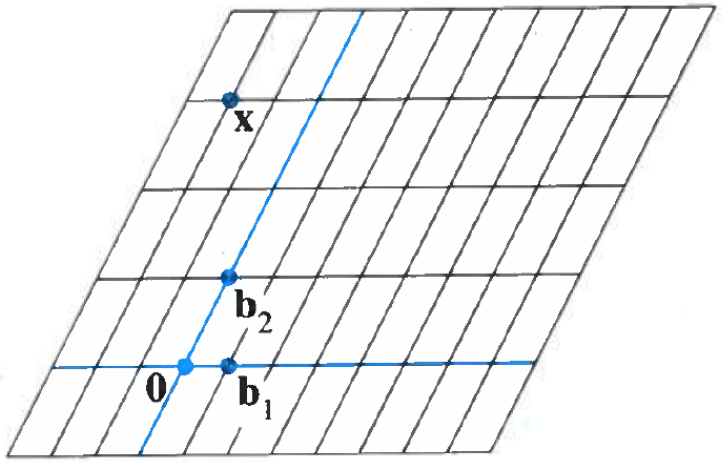
$$\rightarrow\ \{\boldsymbol{v}_i\}\ basis\ for\ Nul\ A$$

Basis for Col A :

The pivot columns of a matrix A form a basis for Col A.

4.4 Coordinate Systems





Theorem 4.7, The Unique Representation Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exist a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n.$$

Definition

Suppose $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis for V and $\mathbf{x}$ is in V. The **coordinates of x relative to the basis** $\mathcal{B}$ (or the $\mathcal{B}$**-coordinates of x**) are the weights $c_1, c_2, \ldots c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$. Hence

$$\boldsymbol{x} \mapsto [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Coordinate vector of $\boldsymbol{x}$ (relative to $\mathcal{B}$)

Coordinate mapping (koordinat afbilding)

Ex 5

Basis for $\mathbb{R}^2$: $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ $\quad \mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \longleftarrow$ Standard basis for $\mathbb{R}^2$

$$[\boldsymbol{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \Longrightarrow \boldsymbol{x} = -2 \cdot \boldsymbol{b}_1 + 3 \cdot \boldsymbol{b}_2 = -2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix} = 1 \cdot \boldsymbol{e}_1 + 6 \cdot \boldsymbol{e}_2 = [\boldsymbol{x}]_{\mathcal{E}}$$

$$P_{\mathcal{B}} = [\boldsymbol{b}_1 \;\; \boldsymbol{b}_2] = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \Longrightarrow \boldsymbol{x} = [\boldsymbol{x}]_{\mathcal{E}} = P_{\mathcal{B}}[\boldsymbol{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

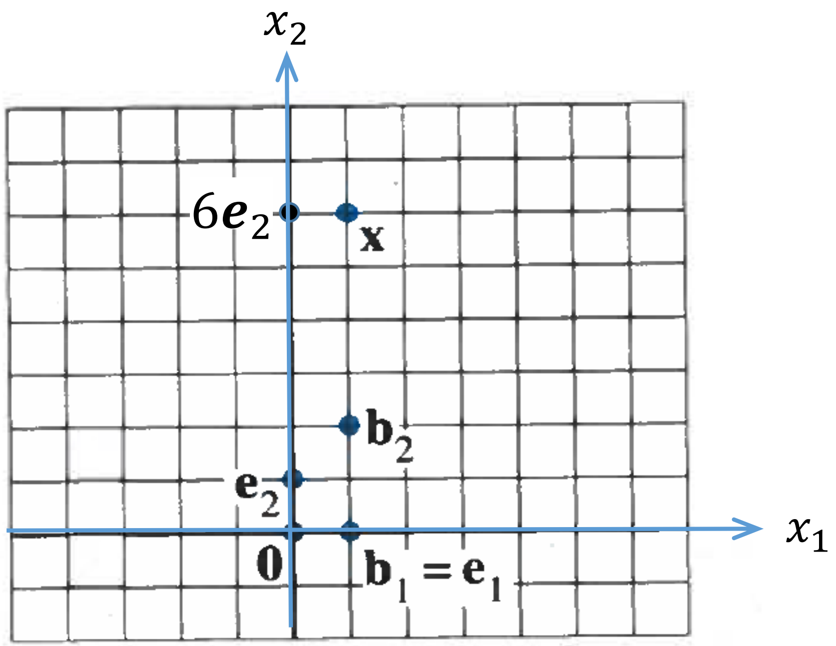


FIGURE 1 Standard graph paper.

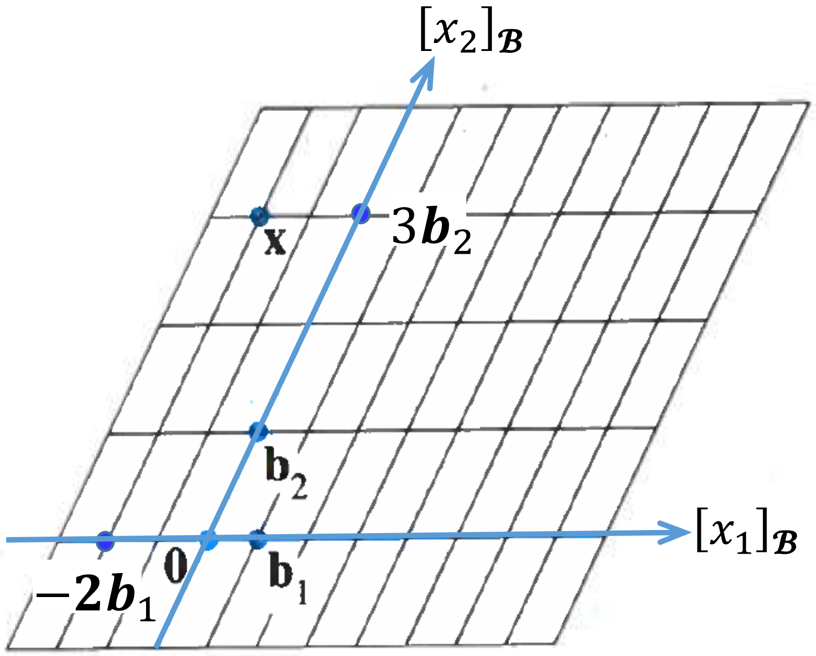


FIGURE 2 $\mathcal{B}$-graph paper.

Ex 6

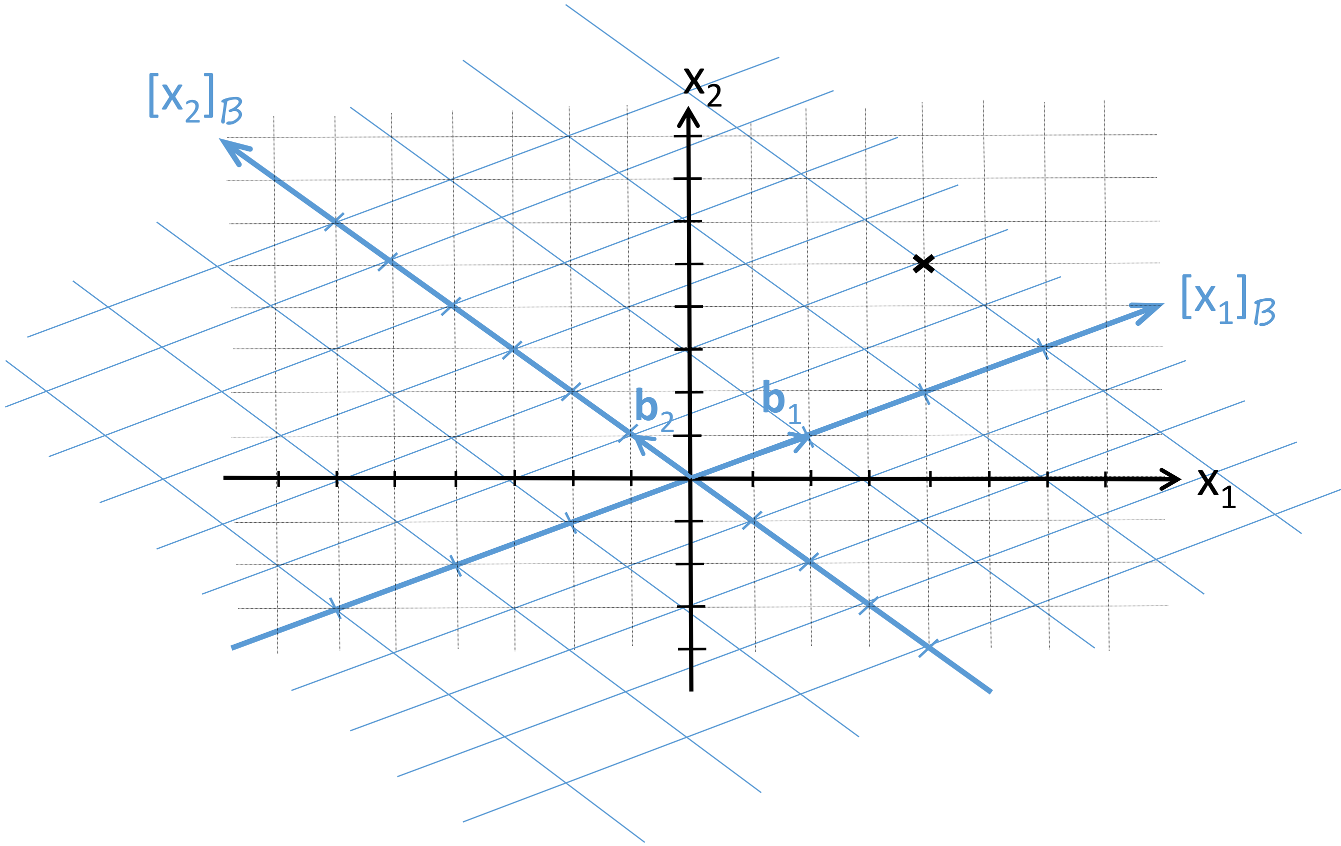
Basis for $\mathbb{R}^2$: $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ $\quad \boldsymbol{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \Rightarrow [\boldsymbol{x}]_{\mathcal{B}} = ?$

$$\boldsymbol{x} = c_1 \cdot \boldsymbol{b}_1 + c_2 \cdot \boldsymbol{b}_2$$

$$\Longrightarrow \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & 4 \\ 1 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\Longrightarrow \boldsymbol{x} = 3\boldsymbol{b}_1 + 2\boldsymbol{b}_2$$

$$\Longrightarrow [\boldsymbol{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$



Coordinates in basis $\mathcal{B}$

Let $\mathcal{B} = \{\boldsymbol{b_1}, \boldsymbol{b_2}, \cdots, \boldsymbol{b_n}\}$ be a basis of $\mathbb{R}^n$ and $[\boldsymbol{x}]_{\mathcal{B}} = [c_1\ c_2\ \cdots\ c_n]^T$

Change-of-coordinate matrix: $P_{\mathcal{B}} = [\boldsymbol{b}_1\ \boldsymbol{b}_2\ \cdots\ \boldsymbol{b}_n]$ ⟵ Invertible according to Inverse Matrix Theorem

Change-of-coordinates from $\mathcal{B}$ to standard basis of $\mathbb{R}^n$:

$$\boldsymbol{x} = c_1 \cdot \boldsymbol{b}_1 + c_2 \cdot \boldsymbol{b}_2 + \cdots + c_n \cdot \boldsymbol{b}_n = P_{\mathcal{B}}[\boldsymbol{x}]_{\mathcal{B}}$$

Change-of-coordinates from standard basis of $\mathbb{R}^n$ to $\mathcal{B}$:

$$\boldsymbol{x} \longmapsto {P_{\mathcal{B}}}^{-1}\boldsymbol{x} = [\boldsymbol{x}]_{\mathcal{B}}$$

⟵ One-to-one linear transformation from $\mathbb{R}^n$ onto $\mathbb{R}^n$

Coordinate mapping

Ex 6 revised

Basis for $\mathbb{R}^2$: $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ $\quad \boldsymbol{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \Rightarrow [\boldsymbol{x}]_{\boldsymbol{\mathcal{B}}} = ?$

$$P_{\boldsymbol{\mathcal{B}}} = [\boldsymbol{b}_1 \ \ \boldsymbol{b}_2] = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \Longrightarrow P_{\boldsymbol{\mathcal{B}}}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

$$[\boldsymbol{x}]_{\boldsymbol{\mathcal{B}}} = P_{\boldsymbol{\mathcal{B}}}^{-1}\boldsymbol{x} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 9 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\boldsymbol{x} = [\boldsymbol{x}]_{\mathcal{E}} = P_{\boldsymbol{\mathcal{B}}}[\boldsymbol{x}]_{\boldsymbol{\mathcal{B}}} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

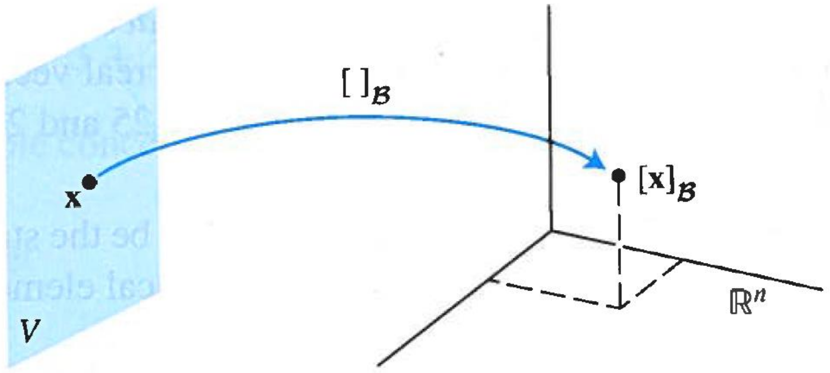
Let $\mathcal{B} = \{\boldsymbol{b_1}, \boldsymbol{b_2}, \cdots, \boldsymbol{b_n}\}$ be a basis for a vector space V

and $\boldsymbol{x}$ a vector in V:

$$\boldsymbol{x} = c_1 \cdot \boldsymbol{b}_1 + c_2 \cdot \boldsymbol{b}_2 + \cdots + c_n \cdot \boldsymbol{b}_n.$$

The coordinate mapping:

$$\boldsymbol{x} \longmapsto [x]_{\mathcal{B}} = [c_1\ c_2\ \cdots\ c_n]^T$$



is a <u>one-to-one linear transformation</u> from V onto $\mathbb{R}^n$

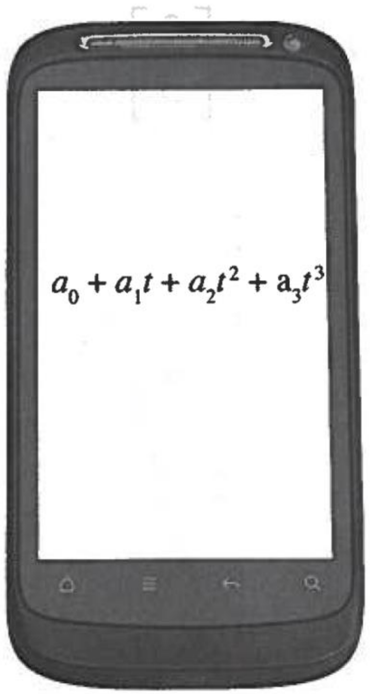
Isomorphism from vector space V onto vector space W ($\mathbb{R}^n$) $\longrightarrow$ *Calculations in V $\updownarrow$ Calculations in W*

$\mathcal{B} = \{1, t, t^2, t^3\}$ standard basis of $\mathbb{P}_3$

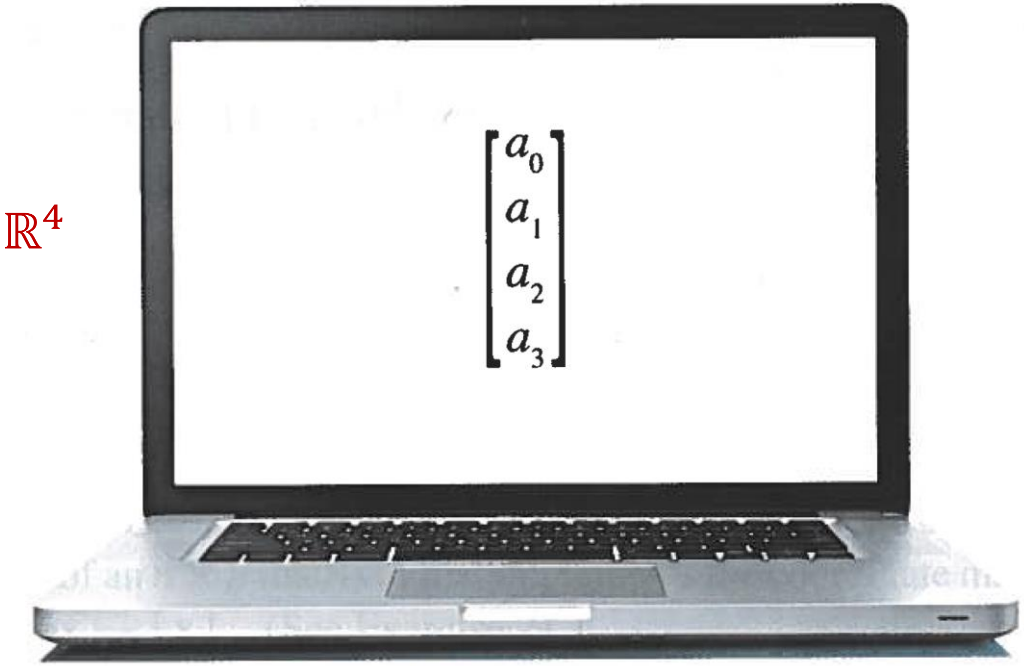
$$\boldsymbol{p}(t) = a_0 \cdot 1 + a_1 \cdot t + a_2 \cdot t^2 + a_3 \cdot t^3 \in \mathbb{P}_3$$

$$\boldsymbol{p}(t) \mapsto [\boldsymbol{p}(t)]_{\mathcal{B}} = [a_0 \;\; a_1 \;\; a_2 \;\; a_3]^T \in \mathbb{R}^4$$

Coordinate mapping / Isomorphism of $\mathbb{P}_3$ onto $\mathbb{R}^4$



Isomorphism of $\mathbb{P}_3$ onto $\mathbb{R}^4$



Todays words and concepts

Isomorphism

Standard basis

Unique Representation

Basis

Coordinates

Coordinate system