Chapter 7.1

1. Since $A = \begin{bmatrix} 3 & 5 \\ 5 & -7 \end{bmatrix} = A^T$, the matrix is symmetric.

2. Since $A = \begin{bmatrix} 3 & -5 \\ -5 & -3 \end{bmatrix} = A^T$, the matrix is symmetric.

3. Since $A = \begin{bmatrix} 2 & 3 \\ 4 & 4 \end{bmatrix} \neq A^T$, the matrix is not symmetric.

4. Since $A = \begin{bmatrix} 0 & 8 & 3 \\ 8 & 0 & -4 \\ 3 & 2 & 0 \end{bmatrix} \neq A^T$, the matrix is not symmetric.

5. Since $A = \begin{bmatrix} -6 & 2 & 0 \\ 2 & -6 & 2 \\ 0 & 2 & -6 \end{bmatrix} = A^T$, the matrix is symmetric.

6. Since A is not a square matrix $A \neq A^T$ and the matrix is not symmetric.

7. Let $P = \begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}$, and compute that $P^T P = \begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$. Since P is a square matrix, P is orthogonal and $P^{-1} = P^T = \begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}$.

8. Let $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, and compute that $P^T P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I_2 \neq I_2$. Thus P is not orthogonal.

9. Let $P = \begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix}$, and compute that $P^T P = \begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix}\begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$. Since P is a square matrix, P is orthogonal and $P^{-1} = P^T = \begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix}$.

10. Let $P = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$, and compute that

$$P^T P = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$ Since P is a square matrix,

P is orthogonal and $P^{-1} = P^T = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$.

13. Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Then the characteristic polynomial of A is

$(3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 8 = (\lambda-4)(\lambda-2)$, so the eigenvalues of A are 4 and 2. For $\lambda = 4$, one computes that a basis for the eigenspace is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which can be normalized to get $\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. For $\lambda = 2$ one computes that a basis for the eigenspace is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, which can be normalized to get $\mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. Let $P = [\mathbf{u}_1 \quad \mathbf{u}_2] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$. Then P orthogonally diagonalizes A, and $A = PDP^{-1}$.

23. Let $A = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}$. Since each row of A sums to 2, $A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A with corresponding eigenvalue $\lambda = 2$. The eigenvector may be normalized to get $\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$. For $\lambda = 5$, one computes that a basis for the eigenspace is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$, so $\lambda = 5$ is an eigenvalue of A. This basis may be converted via orthogonal projection to an orthogonal basis $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$ for the eigenspace, and these vectors can be normalized to get $\mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$. Let $P = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3] = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. Then P orthogonally diagonalizes A, and $A = PDP^{-1}$.

25. **a**. True. See Theorem 2 and the paragraph preceding the theorem.

b. True. This is a particular case of the statement in Theorem 1, where **u** and **v** are nonzero.

c. False. There are n real eigenvalues (Theorem 3), but they need not be distinct (Example 3).

d. False. See the paragraph following formula (2), in which each **u** is a unit vector.

29. Since A is orthogonally diagonalizable, $A = PDP^{-1}$, where P is orthogonal and D is diagonal. Since A is invertible, $A^{-1} = (PDP^{-1})^{-1} = PD^{-1}P^{-1}$. Notice that D^{-1} is a diagonal matrix, so A^{-1} is orthogonally diagonalizable.

Chapter 7.2

2. **a.** $\mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 3x_1^2 + 2x_2^2 + 4x_1x_2 + 2x_2x_3$

b. When $\mathbf{x} = \begin{bmatrix} -2 \\ -1 \\ 5 \end{bmatrix}$, $\mathbf{x}^T A\mathbf{x} = 3(-2)^2 + 2(-1)^2 + 4(-2)(-1) + 2(-1)(5) = 12.$

c. When $\mathbf{x} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, $\mathbf{x}^T A\mathbf{x} = 3(1/\sqrt{2})^2 + 2(1/\sqrt{2})^2 + 4(1/\sqrt{2})(1/\sqrt{2}) + 2(1/\sqrt{2})(1/\sqrt{2}) = 11/2.$

5. **a.** The matrix of the quadratic form is $\begin{bmatrix} 3 & -3 & 4 \\ -3 & 2 & -2 \\ 4 & -2 & -5 \end{bmatrix}$.

b. The matrix of the quadratic form is $\begin{bmatrix} 0 & 3 & 2 \\ 3 & 0 & -5 \\ 2 & -5 & 0 \end{bmatrix}$.

7. The matrix of the quadratic form is $A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$. The eigenvalues of A are 6 and –4. An eigenvector for $\lambda = 6$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which may be normalized to $\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. An eigenvector for $\lambda = -4$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, which may be normalized to $\mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. Then $A = PDP^{-1}$, where $P = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ and $D = \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix}$. The desired change of variable is $\mathbf{x} = P\mathbf{y}$, and the new quadratic form is $\mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T P^T AP\mathbf{y} = \mathbf{y}^T D\mathbf{y} = 6y_1^2 - 4y_2^2$.

9. The matrix of the quadratic form is $A = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}$. The eigenvalues of A are 6 and 2, so the quadratic form is positive definite. An eigenvector for $\lambda = 6$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, which may be normalized to $\mathbf{u}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. An eigenvector for $\lambda = 2$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which may be normalized to $\mathbf{u}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. Then $A = PDP^{-1}$, where $P = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ and $D = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}$. The desired change of variable is $\mathbf{x} = P\mathbf{y}$, and the new quadratic form is
$\mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T P^T AP\mathbf{y} = \mathbf{y}^T D\mathbf{y} = 6y_1^2 + 2y_2^2$.

21. **a**. True. See the definition before Example 1, even though a nonsymmetric matrix could be used to compute values of a quadratic form.

b. True. See the paragraph following Example 3.

c. True. The columns of P in Theorem 4 are eigenvectors of A. See the Diagonalization Theorem in Section 5.3.

d. False. $Q(\mathbf{x}) = 0$ when $\mathbf{x} = \mathbf{0}$.

e. True. See Theorem 5(a).

f. True. See the Numerical Note after Example 6.

Chapter 7.3

1. The matrix of the quadratic form on the left is $A = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 6 & -2 \\ 0 & -2 & 7 \end{bmatrix}$. The equality of the quadratic forms implies that the eigenvalues of A are 9, 6, and 3. An eigenvector may be calculated for each eigenvalue and normalized: $\lambda = 9: \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$, $\lambda = 6: \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$, $\lambda = 3: \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$. A desired change of variable is $\mathbf{x} = P\mathbf{y}$, where $P = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$.

3. a. By Theorem 6, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$ is the greatest eigenvalue λ_1 of A. By Exercise 1, $\lambda_1 = 9$.

b. By Theorem 6, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$ occurs at a unit eigenvector $\mathbf{u}$ corresponding to the greatest eigenvalue λ_1 of A. By Exercise 1, $\mathbf{u} = \pm\begin{bmatrix} -1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$.

c. By Theorem 7, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraints $\mathbf{x}^T\mathbf{x} = 1$ and $\mathbf{x}^T\mathbf{u} = 0$ is the second greatest eigenvalue λ_2 of A. By Exercise 1, $\lambda_2 = 6$.

7. The eigenvalues of the matrix of the quadratic form are $\lambda_1 = 2$, $\lambda_2 = -1$, and $\lambda_3 = -4$. By Theorem 6, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$ occurs at a unit eigenvector $\mathbf{u}$ corresponding to the greatest eigenvalue λ_1 of A. One may compute that $\begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 2$, so $\mathbf{u} = \pm\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$.

9. This is equivalent to finding the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$. By Theorem 6, this value is the greatest eigenvalue λ_1 of the matrix of the quadratic form. The matrix of the quadratic form is $A = \begin{bmatrix} 7 & -1 \\ -1 & 3 \end{bmatrix}$, and the eigenvalues of A are $\lambda_1 = 5+\sqrt{5},\ \lambda_2 = 5-\sqrt{5}$. Thus the desired constrained maximum value is $\lambda_1 = 5+\sqrt{5}$.

11. Since $\mathbf{x}$ is an eigenvector of A corresponding to the eigenvalue 3, $A\mathbf{x} = 3\mathbf{x}$, and $\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T(3\mathbf{x}) =$ $3(\mathbf{x}^T\mathbf{x}) = 3\|\mathbf{x}\|^2 = 3$ since $\mathbf{x}$ is a unit vector.