## Chapter 1.4

**1**. The matrix-vector product $A\mathbf{x}$ is not defined because the number of columns (2) in the 3×2 matrix $\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix}$ does not match the number of entries (3) in the vector $\begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}$.

**2**. The matrix-vector product $A\mathbf{x}$ is not defined because the number of columns (1) in the 3×1 matrix $\begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$ does not match the number of entries (2) in the vector $\begin{bmatrix} 5 \\ -1 \end{bmatrix}$.

**3**. $A\mathbf{x} = \begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = 2\begin{bmatrix} 6 \\ -4 \\ 7 \end{bmatrix} - 3\begin{bmatrix} 5 \\ -3 \\ 6 \end{bmatrix} = \begin{bmatrix} 12 \\ -8 \\ 14 \end{bmatrix} + \begin{bmatrix} -15 \\ 9 \\ -18 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}$, and

$$A\mathbf{x} = \begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 6\cdot 2 + 5\cdot(-3) \\ (-4)\cdot 2 + (-3)\cdot(-3) \\ 7\cdot 2 + 6\cdot(-3) \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}$$

**4**. $A\mathbf{x} = \begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1\cdot\begin{bmatrix} 8 \\ 5 \end{bmatrix} + 1\cdot\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 1\cdot\begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8+3-4 \\ 5+1+2 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$, and

$$A\mathbf{x} = \begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8\cdot 1 + 3\cdot 1 + (-4)\cdot 1 \\ 5\cdot 1 + 1\cdot 1 + 2\cdot 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

**9**. The system has the same solution set as the vector equation

$$x_1\begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

and this equation has the same solution set as the matrix equation

$$\begin{bmatrix} 3 & 1 & -5 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

**11.** To solve $A\mathbf{x} = \mathbf{b}$, row reduce the augmented matrix $[\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{a}_3\ \ \mathbf{b}]$ for the corresponding linear system:

$$\begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ -2 & -4 & -3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The solution is $\begin{cases} x_1 &= 0 \\ x_2 &= -3 \\ x_3 &= 1 \end{cases}$. As a vector, the solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$.

**21.** Row reduce the matrix $[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3]$ to determine whether it has a pivot in each row.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix $[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3]$ does not have a pivot in each row, so the columns of the matrix do not span $\mathbb{R}^4$, by Theorem 4. That is, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ does not span $\mathbb{R}^4$.

**23. a.** False. See the paragraph following equation (3). The text calls $A\mathbf{x} = \mathbf{b}$ a *matrix equation*.

**b.** True. See the box before Example 3.

**c.** False. See the warning following Theorem 4.

**d.** True. See Example 4.

**e.** True. See parts (c) and (a) in Theorem 4.

**f.** True. In Theorem 4, statement (a) is false if and only if statement (d) is also false.

**25.** By definition, the matrix-vector product on the left is a linear combination of the columns of the matrix, in this case using weights –3, –1, and 2. So $c_1 = -3$, $c_2 = -1$, and $c_3 = 2$.

**31.** A 3×2 matrix has three rows and two columns. With only two columns, $A$ can have at most two pivot columns, and so $A$ has at most two pivot positions, which is not enough to fill all three rows. By Theorem 4, the equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all $\mathbf{b}$ in $\mathbb{R}^3$. Generally, if $A$ is an $m\times n$ matrix with $m > n$, then $A$ can have at most $n$ pivot positions, which is not enough to fill all $m$ rows. Thus, the equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all $\mathbf{b}$ in $\mathbb{R}^3$.

**32.** A set of three vectors in cannot span $\mathbb{R}^4$. Reason: the matrix $A$ whose columns are these three vectors has four rows. To have a pivot in each row, $A$ would have to have at least four columns (one for each pivot), which is not the case. Since $A$ does not have a pivot in every row, its columns do not span $\mathbb{R}^4$, by Theorem 4. In general, a set of $n$ vectors in $\mathbb{R}^m$ cannot span $\mathbb{R}^m$ when $n$ is less than $m$.

**40.** **[M]** $\begin{bmatrix} 8 & 11 & -6 & -7 & 13 \\ -7 & -8 & 5 & 6 & -9 \\ 11 & 7 & -7 & -9 & -6 \\ -3 & 4 & 1 & 8 & 7 \end{bmatrix} \sim \begin{bmatrix} 8 & 11 & -6 & -7 & 13 \\ 0 & 13/8 & -1/4 & -1/8 & 19/8 \\ 0 & -65/8 & 5/4 & 5/8 & -191/8 \\ 0 & 65/8 & -5/4 & 43/8 & 95/8 \end{bmatrix}$

$$\sim \begin{bmatrix} 8 & 11 & -6 & -7 & 13 \\ 0 & 13/8 & -1/4 & -1/8 & 19/8 \\ 0 & 0 & 0 & 0 & -12 \\ 0 & 0 & 0 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 8 & 11 & -6 & -7 & 13 \\ 0 & 13/8 & -1/4 & -1/8 & 19/8 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & -12 \end{bmatrix}$$

The original matrix has a pivot in every row, so its columns span $\mathbb{R}^4$, by Theorem 4.

**42.** **[M]** Examine the calculations in Exercise 40. The third column of the original matrix, say $A$, is not a pivot column. Let $C$ be the matrix formed by deleting column 3 of $A$, let $B$ be the echelon form obtained from $A$, and let $D$ be the matrix obtained by deleting column 3 of $B$. The sequence of row operations that reduces $A$ to $B$ also reduces $C$ to $D$. Since $D$ is in echelon form, it shows that $C$ has a pivot position in each row. Therefore, the columns of $C$ span $\mathbb{R}^4$.

It is possible to delete column 1or 2 of $A$ instead of column 3. (See the remark for Exercise 41.) However, only *one* column can be deleted. If two or more columns were deleted from $A$, the resulting matrix would have fewer than four columns, so it would have fewer than four pivot positions. In such a case, not every row could contain a pivot position, and the columns of the matrix would not span $\mathbb{R}^4$, by Theorem 4.

## Chapter 1.5

**1**. Reduce the augmented matrix to echelon form and circle the pivot positions. If a column of the *coefficient* matrix is not a pivot column, the corresponding variable is free and the system of equations has a nontrivial solution. Otherwise, the system has *only* the trivial solution.

$$\begin{bmatrix} 2 & -5 & 8 & 0 \\ -2 & -7 & 1 & 0 \\ 4 & 2 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 12 & -9 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The variable $x_3$ is free, so the system has a nontrivial solution.

**3**. $\begin{bmatrix} -3 & 5 & -7 & 0 \\ -6 & 7 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} -3 & 5 & -7 & 0 \\ 0 & -3 & 15 & 0 \end{bmatrix}$. The variable $x_3$ is free; the system has nontrivial solutions. An alert student will realize that row operations are unnecessary. With only two equations, there can be at most two basic variables. One variable *must* be free. Refer to Exercise 31 in Section 1.2

**5.** $$\begin{bmatrix} 1 & 3 & 1 & 0 \\ -4 & -9 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 \quad - 5x_3 &= 0 \\ x_2 + 2x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

The variable $x_3$ is free, $x_1 = 5x_3$, and $x_2 = -2x_3$.

In parametric vector form, the general solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$.

**7.** $\begin{bmatrix} 1 & 3 & -3 & 7 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & -8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{bmatrix}$. $\begin{aligned} x_1 \quad + 9x_3 - 8x_4 &= 0 \\ x_2 - 4x_3 + 5x_4 &= 0 \end{aligned}$

The basic variables are $x_1$ and $x_2$, with $x_3$ and $x_4$ free. Next, $x_1 = -9x_3 + 8x_4$, and $x_2 = 4x_3 - 5x_4$. The general solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 + 8x_4 \\ 4x_3 - 5x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 \\ 4x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 8x_4 \\ -5x_4 \\ 0 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

**11.** $\begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & -2 & 0 & 0 & 7 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$\begin{aligned} x_1 - 4x_2 \qquad\qquad + 5x_6 &= 0 \\ x_3 \qquad - x_6 &= 0 \\ x_5 - 4x_6 &= 0 \\ 0 &= 0 \end{aligned}$$

. The basic variables are $x_1$, $x_3$, and $x_5$. The remaining variables are free. In particular, $x_4$ is free (and not zero as some may assume). The solution is $x_1 = 4x_2 - 5x_6$, $x_3 = x_6$, $x_5 = 4x_6$, with $x_2$, $x_4$, and $x_6$ free. In parametric vector form,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4x_2 - 5x_6 \\ x_2 \\ x_6 \\ x_4 \\ 4x_6 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_6 \\ 0 \\ x_6 \\ 0 \\ 4x_6 \\ x_6 \end{bmatrix} = x_2 \underset{\substack{\uparrow \\ \mathbf{u}}}{\begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} + x_4 \underset{\substack{\uparrow \\ \mathbf{v}}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}} + x_6 \underset{\substack{\uparrow \\ \mathbf{w}}}{\begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{bmatrix}}$$

**23.** **a**. True. See the first paragraph of the subsection titled "Homogeneous Linear Systems".

**b**. False. The equation $A\mathbf{x} = \mathbf{0}$ gives an *implicit* description of its solution set. See the subsection entitled "Parametric Vector Form".

**c**. False. The equation $A\mathbf{x} = \mathbf{0}$ *always* has the trivial solution. The box before Example 1 uses the word *nontrivial* instead of *trivial*.

**d**. False. The line goes through $\mathbf{p}$ parallel to $\mathbf{v}$. See the paragraph that precedes Fig. 5.

**e**. False. The solution set could be *empty*! The statement (from Theorem 6) is true only when there exists a vector $\mathbf{p}$ such that $A\mathbf{p} = \mathbf{b}$.

**29.** **a**. When $A$ is a 3×3 matrix with three pivot positions, the equation $A\mathbf{x} = \mathbf{0}$ has no free variables and hence has no nontrivial solution.

**b**. With three pivot positions, $A$ has a pivot position in each of its three rows. By Theorem 4 in Section 1.4, the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every possible $\mathbf{b}$. The term "possible" in the exercise means that the only vectors considered in this case are those in $\mathbb{R}^3$, because $A$ has three rows.

**30.** **a**. When $A$ is a 3×3 matrix with two pivot positions, the equation $A\mathbf{x} = \mathbf{0}$ has two basic variables and one free variable. So $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

**b**. With only two pivot positions, $A$ cannot have a pivot in every row, so by Theorem 4 in Section 1.4, the equation $A\mathbf{x} = \mathbf{b}$ cannot have a solution for every possible $\mathbf{b}$ (in $\mathbb{R}^3$).

**31.** **a**. When $A$ is a 3×2 matrix with two pivot positions, each column is a pivot column. So the equation $A\mathbf{x} = \mathbf{0}$ has no free variables and hence no nontrivial solution.

**b**. With two pivot positions and three rows, $A$ cannot have a pivot in every row. So the equation $A\mathbf{x} = \mathbf{b}$ cannot have a solution for every possible $\mathbf{b}$ (in $\mathbb{R}^3$), by Theorem 4 in Section 1.4.

**32.** **a**. When $A$ is a 2×4 matrix with two pivot positions, the equation $A\mathbf{x} = \mathbf{0}$ has two basic variables and two free variables. So $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

**b**. With two pivot positions and only two rows, $A$ has a pivot position in every row. By Theorem 4 in Section 1.4, the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every possible $\mathbf{b}$ (in $\mathbb{R}^2$).

**33.** Look at $x_1\begin{bmatrix}-2\\7\\-3\end{bmatrix}+x_2\begin{bmatrix}-6\\21\\-9\end{bmatrix}$ and notice that the second column is 3 times the first. So suitable values for $x_1$ and $x_2$ would be 3 and –1 respectively. (Another pair would be 6 and –2, etc.) Thus $\mathbf{x}=\begin{bmatrix}3\\-1\end{bmatrix}$ satisfies $A\mathbf{x} = \mathbf{0}$.

## Chapter 1.7

**1**. Use an augmented matrix to study the solution set of $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{0}$ (*), where **u**, **v**, and **w** are the three given vectors. Since $\begin{bmatrix} 5 & 7 & 9 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & -6 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 5 & 7 & 9 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$, there are no free variables. So the vectors are linearly independent(*) ... the homogeneous equation (*) has only the trivial solution.

**9**. **a**. The vector $\mathbf{v}_3$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ if and only if the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{v}_3$ has a solution. To find out, row reduce $[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3]$, considered as an augmented matrix:

$$\begin{bmatrix} 1 & -3 & 5 \\ -3 & 9 & -7 \\ 2 & -6 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & h-10 \end{bmatrix}$$

At this point, the equation 0 = 8 shows that the original vector equation has no solution. So $\mathbf{v}_3$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ for *no* value of $h$.

**b**. For $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to be linearly independent, the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ must have only the trivial solution. Row reduce the augmented matrix $[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3 \;\; \mathbf{0}]$:

$$\begin{bmatrix} 1 & -3 & 5 & 0 \\ -3 & 9 & -7 & 0 \\ 2 & -6 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 5 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & h-10 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 5 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For every value of $h$, $x_2$ is a free variable, and so the homogeneous equation has a nontrivial solution. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set for all $h$.

**15**. The set is linearly dependent, by Theorem 8, because there are four vectors in the set but only two entries in each vector.

**16**. The set is linearly dependent because the second vector is 3/2 times the first vector.

**17**. The set is linearly dependent, by Theorem 9, because the list of vectors contains a zero vector.

**18**. The set is linearly dependent, by Theorem 8, because there are four vectors in the set but only two entries in each vector.

**19**. The set is linearly independent because neither vector is a multiple of the other vector. [Two of the entries in the first vector are – 4 times the corresponding entry in the second vector. But this multiple does not work for the third entries.]

**20**. The set is linearly dependent, by Theorem 9, because the list of vectors contains a zero vector.

**21**. **a**. False. A homogeneous system *always* has the trivial solution. See the box before Example 2.

**b**. False. See the warning after Theorem 7.

**c**. True. See Fig. 3, after Theorem 8.

**d**. True. See the remark following Example 4.

**23.** $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}$ **24.** $\begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **25.** $\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

**41.** **[M]** $A = \begin{bmatrix} 8 & -3 & 0 & -7 & 2 \\ -9 & 4 & 5 & 11 & -7 \\ 6 & -2 & 2 & -4 & 4 \\ 5 & -1 & 7 & 0 & 10 \end{bmatrix} \sim \begin{bmatrix} 8 & -3 & 0 & -7 & 2 \\ 0 & 5/8 & 5 & 25/8 & -19/4 \\ 0 & 1/4 & 2 & 5/4 & 5/2 \\ 0 & 7/8 & 7 & 35/8 & 35/4 \end{bmatrix}$

$$\sim \begin{bmatrix} 8 & -3 & 0 & -7 & 2 \\ 0 & 5/8 & 5 & 25/8 & -19/4 \\ 0 & 0 & 0 & 0 & 22/5 \\ 0 & 0 & 0 & 0 & 77/5 \end{bmatrix} \sim \begin{bmatrix} 8 & -3 & 0 & -7 & 2 \\ 0 & 5/8 & 5 & 25/8 & -19/4 \\ 0 & 0 & 0 & 0 & 22/5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns of $A$ are 1, 2, and 5. Use them to form $B = \begin{bmatrix} 8 & -3 & 2 \\ -9 & 4 & -7 \\ 6 & -2 & 4 \\ 5 & -1 & 10 \end{bmatrix}$.

Other likely choices use columns 3 or 4 of $A$ instead of 2: $\begin{bmatrix} 8 & 0 & 2 \\ -9 & 5 & -7 \\ 6 & 2 & 4 \\ 5 & 7 & 10 \end{bmatrix}, \begin{bmatrix} 8 & -7 & 2 \\ -9 & 11 & -7 \\ 6 & -4 & 4 \\ 5 & 0 & 10 \end{bmatrix}$.

Actually, any set of three columns of $A$ that includes column 5 will work for $B$, but the concepts needed to prove that are not available now. (Column 5 is not in the two-dimensional subspace spanned by the first four columns.)