In the grading of the exercises special attention will be paid to check that answers are substantiated and that the procedure of calculations is well documented. When results are achieved using a calculator or a pc it should be noted in the paper. All 6 problems are weighted equally in the grading. Your solution to the problem set can be written in Danish or English as you prefer.

PROBLEM 1.

Consider the following set of four vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_4\}$ and the vector $\mathbf{b}$.

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \ \mathbf{a}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{a}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{a}_4 = \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}.$$

1. Solve the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4 = \mathbf{b}$.

Assume that the $\mathbf{a}_4$ vector is removed from the set.

2. Explain whether the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$ can also be solved.

PROBLEM 2.

Let the following matrix and vector be given.

$$A = \begin{bmatrix} 2 & 2 & -1 & -3 \\ 1 & 2 & -1 & -4 \\ 2 & -1 & 1 & 5 \\ 1 & -2 & 1 & 6 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 3 \\ 5 \\ -8 \\ -9 \end{bmatrix}.$$

1. Determine the rank of the matrix.
2. Compute bases for the column space, row space and null space of the matrix.
3. Determine if $\mathbf{y}$ is in the null space or column space of A.

PROBLEM 3.

For the statements given below, state whether they are true or false and justify your answer for each statement.

1. The matrix equation $A\mathbf{x} = \mathbf{b}$ is inconsistent if A has more rows than columns.
2. If a matrix has full rank it is invertible.
3. If a vector space W contains the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ then $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ is also in the vector space.

PROBLEM 4.

In the case *Computer Graphics in Automotive Design*, homogeneous coordinates and perspective projection were introduced. Consider a tetrahedron-shaped 3D object described by the coordinate matrix D and adjacency matrix A as

$$D = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

1. Compute the matrix containing the (x^*, y^*) values using $(b, c, d) = (3, 1, 10)$ as center of projection and the xy plane as viewing plane. Sketch the projection of the object.
2. Rotate the object $10°$ around the y-axis, compute the new coordinates using the same center of projection and xy viewing plane.

PROBLEM 5.

Consider the following 2×2 matrix and vector.

$$A = \begin{bmatrix} 1 & -1 \\ 0.4 & 0.6 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

1. Compute the characteristic polynomial by hand and use it to show that A has complex eigenvalues.

The matrix can be factorized as $A = PCP^{-1}$.

2. Determine P and C.
3. Plot the vectors $\mathbf{x}_0$, $\mathbf{x}_1 = A\mathbf{x}_0$ and $\mathbf{x}_2 = A\mathbf{x}_1$ in the same coordinate system and explain the plot based on the above factorization.

PROBLEM 6.

Let the following inner product be defined on $\mathbb{R}^3$.

$$\langle \mathbf{x},\mathbf{y} \rangle = \mathbf{x}^T W \mathbf{y}.$$

Where W is a diagonal matrix

$$W = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Further, let two vectors be given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}.$$

1. Compute the distance between $\mathbf{v}_1$ and $\mathbf{v}_2$ using the above inner product.
2. Compute the orthogonal projection of $\mathbf{v}_2$ onto $\mathbf{v}_1$ using the above inner product.
3. Show that the symmetry condition $\langle \mathbf{x},\mathbf{y} \rangle = \langle \mathbf{y},\mathbf{x} \rangle$ holds for this inner product.