PROBLEM 1.

Consider the following set of four vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_4\}$ and the vector $\mathbf{b}$.

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \ \mathbf{a}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{a}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{a}_4 = \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}.$$

1. Solve the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4 = \mathbf{b}$.

Assume that the $\mathbf{a}_4$ vector is removed from the set.

2. Explain whether the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$ can also be solved.

PROBLEM 1. Solution

The first problem is solved by row reduction of the augmented matrix

$$\left[\, A \mid \mathbf{b} \,\right] = \left[\begin{array}{cccc|c} 1 & -1 & 3 & 1 & 4 \\ 2 & 1 & 0 & -4 & -1 \\ 1 & 0 & 1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & -2 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The system is consistent and contains two free variables, x_3 and x_4. The general solution is written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

It can be seen from the row reduced matrix above that $\mathbf{a}_4 = -\mathbf{a}_1 - 2\mathbf{a}_2$ i.e. $\mathbf{a}_4$ is linearly dependent on $\mathbf{a}_1$ and $\mathbf{a}_2$. Therefore, span$\{\mathbf{a}_1, \ldots, \mathbf{a}_4\}$=span$\{\mathbf{a}_1, \ldots, \mathbf{a}_3\}$ and the vector equation can still be solved with $\mathbf{a}_4$ removed. In fact, span$\{\mathbf{a}_1, \ldots, \mathbf{a}_4\}$=span$\{\mathbf{a}_1, \mathbf{a}_2\}$ and $\mathbf{a}_3$ can also be removed.

PROBLEM 2.

Let the following matrix and vector be given.

$$A = \begin{bmatrix} 2 & 2 & -1 & -3 \\ 1 & 2 & -1 & -4 \\ 2 & -1 & 1 & 5 \\ 1 & -2 & 1 & 6 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 3 \\ 5 \\ -8 \\ -9 \end{bmatrix}.$$

1. Determine the rank of the matrix.
2. Compute bases for the column space, row space and null space of the matrix.
3. Determine if $\mathbf{y}$ is in the null space or column space of A.

PROBLEM 2. Solution

The rank is determined by row reducing the matrix and counting the number of pivots (encircled below).

$$\begin{bmatrix} 2 & 2 & -1 & -3 \\ 1 & 2 & -1 & -4 \\ 2 & -1 & 1 & 5 \\ 1 & -2 & 1 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

As seen, rank A=3.

The pivot columns of A form a basis for col A.

$$\text{basis for col } A = \left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

The row columns of the row reduced matrix forms a basis for row A.

$$\text{basis for row } A = \left\{ \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} \right\}.$$

The basis for null A is found by solving $A\mathbf{x} = \mathbf{0}$. This can be done based on the above row reduction, where it is seen that x_4 is a free variable. The solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ 2 \\ -1 \\ 1 \end{bmatrix}.$$

Therefore

$$\text{basis for null } A = \left\{ \begin{bmatrix} -1 \\ 2 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

It can be checked if $\mathbf{y}$ is in the null space by direct computation:

$$A\mathbf{y} = \begin{bmatrix} 51 \\ 57 \\ -52 \\ -69 \end{bmatrix}.$$

As $A\mathbf{y} \neq \mathbf{0}$, $\mathbf{y}$ is not in the null space.

If $\mathbf{y}$ is in the column space of A, it can be written as a linear combination of the basis vectors. This is checked by writing up an augmented matrix containing the three basis vectors and the $\mathbf{y}$ vector and row reducing.

$$\left[\begin{array}{ccc|c} 2 & 2 & -1 & 3 \\ 1 & 2 & -1 & 5 \\ 2 & -1 & 1 & -8 \\ 1 & -2 & 1 & -9 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

From which we see that $\mathbf{y}$ is indeed a linear combination of the basis vectors and hence $\mathbf{y}$ is in the column space of A.

PROBLEM 3.

For the statements given below, state whether they are true or false and justify your answer for each statement.

1. The matrix equation $A\mathbf{x} = \mathbf{b}$ is inconsistent if A has more rows than columns.
2. If a matrix has full rank it is invertible.
3. If a vector space W contains the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ then $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ is also in the vector space.

PROBLEM 3. Solution

The first statement is **false**. The matrix equation can be inconsistent, but is not necessarily so. A counter example where A has more rows than columns, yet the $A\mathbf{x} = \mathbf{b}$ equation is still consistent is

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}.$$

The second statement is **false**. Full rank means the the matrix has the maximum number of pivots possible. However,only square matrices are invertible. An example of a non-invertible matrix with full rank is

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

The third statement is **true**. Linear combinations of vectors from a vector space remain within the vector space. This can be argued using the rules of *closed under multiplication* and *closed under addition* from the textbook. Since a vector space is closed under multiplication, the vectors $c_1\mathbf{v}_1$, $c_2\mathbf{v}_2$ and $c_3\mathbf{v}_3$ are in the vector space as $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are in the vector space. Application of the closed under addition rule gives that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ is in the vector space as $c_1\mathbf{v}_1$ and $c_2\mathbf{v}_2$ are in the vector space. Repeating this argument, $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ is in the vector space as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ and $c_3\mathbf{v}_3$ are in the vector space.

PROBLEM 4.

In the case *Computer Graphics in Automotive Design*, homogeneous coordinates and perspective projection were introduced. Consider a tetrahedron-shaped 3D object described by the coordinate matrix D and adjacency matrix A as

$$D = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

1. Compute the matrix containing the (x^*, y^*) values using $(b, c, d) = (3, 1, 10)$ as center of projection and the xy plane as viewing plane. Sketch the projection of the object.

2. Rotate the object 10° around the y-axis, compute the new coordinates using the same center of projection and xy viewing plane.

PROBLEM 4. Solution

Following the procedure from the case we use

$$P = \begin{bmatrix} 1 & 0 & -\frac{b}{d} & 0 \\ 0 & 1 & -\frac{c}{d} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{d} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -0.3 & 0 \\ 0 & 1 & -0.1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & 1 \end{bmatrix}.$$

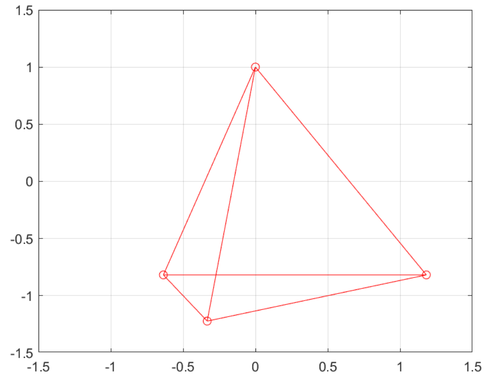
Then PD is computed

$$PD = \begin{bmatrix} 1 & 0 & -0.3 & 0 \\ 0 & 1 & -0.1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -0.3 & -0.7 & 1.3 \\ 1 & -1.1 & -0.9 & -0.9 \\ 0 & 0 & 0 & 0 \\ 1 & 0.9 & 1.1 & 1.1 \end{bmatrix}.$$

Next rows 1 and 2 are each divided by row 4 and rows 3 and 4 are discarded to give the projected data matrix D^*

$$D^* = \begin{bmatrix} 0 & -0.3333 & -0.6364 & 1.1818 \\ 1 & -1.2222 & -0.8182 & -0.8182 \end{bmatrix}.$$

Which is plotted in MATLAB with `gplot`



Rotation about the $y-$axis is done with the rotation matrix R_y defined by

$$R_y = \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

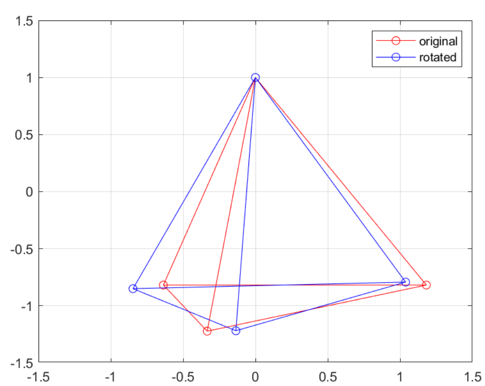
To get the coordinates of the rotated tetrahedron we compute

$$PR_yD = \begin{bmatrix} 0 & 0.0601 & -1.1024 & 0.9822 \\ 1 & -1.0940 & -0.9402 & -0.8718 \\ 0 & 0 & 0 & 0 \\ 1 & 0.9060 & 1.0598 & 1.1282 \end{bmatrix}.$$

As before rows 1 and 2 are each divided by row 4 and rows 3 and 4 are discarded to give the rotated and projected data matrix D_y^*

$$D_y^* = \begin{bmatrix} 0 & -0.1351 & -0.8464 & 1.0384 \\ 1 & -1.2185 & -0.8499 & -0.7924 \end{bmatrix}.$$

Which results in the following plot



PROBLEM 5.

Consider the following 2×2 matrix and vector.

$$A = \begin{bmatrix} 1 & -1 \\ 0.4 & 0.6 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

1. Compute the characteristic polynomial by hand and use it to show that A has complex eigenvalues.

The matrix can be factorized as $A = PCP^{-1}$.

2. Determine P and C.
3. Plot the vectors $\mathbf{x}_0$, $\mathbf{x}_1 = A\mathbf{x}_0$ and $\mathbf{x}_2 = A\mathbf{x}_1$ in the same coordinate system and explain the plot based on the above factorization.

PROBLEM 5. Solution

The characteristic polynomial is given by $\det(A - \lambda I) = 0$. This becomes

$$\begin{vmatrix} 1-\lambda & -1 \\ 0.4 & 0.6-\lambda \end{vmatrix} = \lambda^2 - 1.6\lambda + 1 = 0$$

The solution of this quadratic equation reveals the complex eigenvalues

$$\lambda = \frac{1.6 \pm \sqrt{(-1.6)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = 0.8 \pm 0.6i$$

Theorem 9 from chapter 5 gives P and C.

THEOREM 9 Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ $(b \neq 0)$ and an associated eigenvector $\mathbf{v}$ in $\mathbb{C}^2$. Then

$$A = PCP^{-1}, \quad \text{where} \quad P = [\,\text{Re}\,\mathbf{v} \quad \text{Im}\,\mathbf{v}\,] \quad \text{and} \quad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

First, the eigenvectors are found with MATLAB

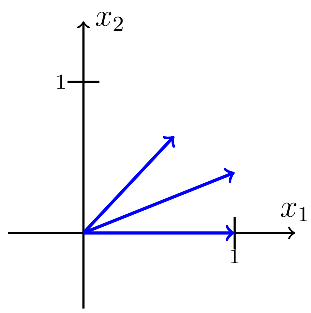
```
>> A=[1 -1;0.4 0.6]

A =

1.0000   -1.0000
0.4000    0.6000

>> [V,D]=eig(A)

V =

0.8452 + 0.0000i   0.8452 + 0.0000i
0.1690 - 0.5071i   0.1690 + 0.5071i

D =

0.8000 + 0.6000i   0.0000 + 0.0000i
0.0000 + 0.0000i   0.8000 - 0.6000i
```

In the second column of D the eigenvalue $a - bi$ is identified, $a = 0.8$, $b = 0.6$. The real and imaginary parts of the corresponding eigenvector in V gives the components in P.

$$P = \begin{bmatrix} 0.8452 & 0.0000 \\ 0.1690 & 0.5071 \end{bmatrix}, \quad C = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}.$$

The three vectors are computed and plotted

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0.4 \end{bmatrix}, \quad \mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 0.6 \\ 0.64 \end{bmatrix}.$$



The mapping $\mathbf{x} \mapsto C\mathbf{x}$ amounts to a rotation and a scaling. With the above numbers the scaling factor is one and the rotation angle is 36.87°. In this case, however, the mapping applied is $\mathbf{x} \mapsto A\mathbf{x}$ or $\mathbf{x} \mapsto PCP^{-1}\mathbf{x}$. This mapping is composed of three steps: 1. a transformation to the eigenvector coordinate system, 2. a rotation and scaling, 3. transformation back to the original coordinate system. The overall effect is that repeated mapping $\mathbf{x}_{n+1} = A\mathbf{x}_n$ will trace out an ellipse.

PROBLEM 6.

Let the following inner product be defined on $\mathbb{R}^3$.

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T W \mathbf{y}.$$

Where W is a diagonal matrix

$$W = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Further, let two vectors be given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}.$$

1. Compute the distance between $\mathbf{v}_1$ and $\mathbf{v}_2$ using the above inner product.
2. Compute the orthogonal projection of $\mathbf{v}_2$ onto $\mathbf{v}_1$ using the above inner product.
3. Show that the symmetry condition $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ holds for this inner product.

PROBLEM 6. Solution

The distance between $\mathbf{v}_1$ and $\mathbf{v}_2$ is computed by

$$\begin{aligned} \operatorname{dist}(\mathbf{v}_1, \mathbf{v}_2) &= \|\mathbf{v}_1 - \mathbf{v}_2\| \\ &= \sqrt{\langle \mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2 \rangle} \\ &= \sqrt{(\mathbf{v}_1 - \mathbf{v}_2)^T W (\mathbf{v}_1 - \mathbf{v}_2)} \\ &= \sqrt{41} \approx 6.40 \end{aligned}$$

The orthogonal projection of $\mathbf{v}_2$ onto $\mathbf{v}_1$ is computed by

$$\operatorname{proj}_{\mathbf{v}_1} \mathbf{v}_2 = \frac{\langle \mathbf{v}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 = \frac{\mathbf{v}_2^T W \mathbf{v}_1}{\mathbf{v}_1^T W \mathbf{v}_1} \mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}.$$

The symmetry condition can be shown be explicitly computing the inner product and utilizing that W is a diagonal matrix, i.e. only W_{11}, W_{22} and W_{33} are non-zero.

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y} \rangle &= \mathbf{x}^T W \mathbf{y} \\ &= x_1 W_{11} y_1 + x_2 W_{22} y_2 + x_3 W_{33} y_3 \\ &= y_1 W_{11} x_1 + y_2 W_{22} x_2 + y_3 W_{33} x_3 \\ &= \mathbf{y}^T W \mathbf{x} \\ &= \langle \mathbf{y}, \mathbf{x} \rangle. \end{aligned}$$