Chapter 7.4

1. Let $A=\begin{bmatrix}1 & 0\\ 0 & -3\end{bmatrix}$. Then $A^TA=\begin{bmatrix}1 & 0\\ 0 & 9\end{bmatrix}$, and the eigenvalues of A^TA are seen to be (in decreasing order) $\lambda_1=9$ and $\lambda_2=1$. Thus the singular values of A are $\sigma_1=\sqrt{9}=3$ and $\sigma_2=\sqrt{1}=1$.

2. Let $A=\begin{bmatrix}-3 & 0\\ 0 & 0\end{bmatrix}$. Then $A^TA=\begin{bmatrix}9 & 0\\ 0 & 0\end{bmatrix}$, and the eigenvalues of A^TA are (in decreasing order) $\lambda_1=9$ and $\lambda_2=0$. Thus the singular values of A are $\sigma_1=\sqrt{9}=3$ and $\sigma_2=\sqrt{0}=0$.

3. Let $A=\begin{bmatrix}2 & 3\\ 0 & 2\end{bmatrix}$. Then $A^TA=\begin{bmatrix}4 & 6\\ 6 & 13\end{bmatrix}$, and the eigenvalues of A^TA are (in decreasing order) $\lambda_1=16$ and $\lambda_2=1$. Thus the singular values of A are $\sigma_1=\sqrt{16}=4$ and $\sigma_2=\sqrt{1}=1$.

4. Let $A=\begin{bmatrix}3 & 0\\ 8 & 3\end{bmatrix}$. Then $A^TA=\begin{bmatrix}73 & 24\\ 24 & 9\end{bmatrix}$, and the eigenvalues of A^TA are seen to be (in decreasing order) $\lambda_1=81$ and $\lambda_2=1$. Thus the singular values of A are $\sigma_1=\sqrt{81}=9$ and $\sigma_2=\sqrt{1}=1$.

5. Let $A=\begin{bmatrix}-2 & 0\\ 0 & 0\end{bmatrix}$. Then $A^TA=\begin{bmatrix}4 & 0\\ 0 & 0\end{bmatrix}$, and the eigenvalues of A^TA are seen to be (in decreasing order) $\lambda_1=4$ and $\lambda_2=0$. Associated unit eigenvectors may be computed: $\lambda_1=4$: $\begin{bmatrix}1\\ 0\end{bmatrix}$, $\lambda_2=0$: $\begin{bmatrix}0\\ 1\end{bmatrix}$. Thus one choice for V is $V=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. The singular values of A are $\sigma_1=\sqrt{4}=2$ and $\sigma_2=\sqrt{0}=0$. Thus the matrix Σ is $\Sigma=\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}$. Next compute $\mathbf{u}_1=\frac{1}{\sigma_1}A\mathbf{v}_1=\begin{bmatrix}-1\\ 0\end{bmatrix}$. Because $A\mathbf{v}_2=\mathbf{0}$, the only column found for U so far is $\mathbf{u}_1$. The other column of U is found by extending $\{\mathbf{u}_1\}$ to an orthonormal basis for $\mathbb{R}^2$. An easy choice is $\mathbf{u}_2=\begin{bmatrix}0\\ 1\end{bmatrix}$. Let $U=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$. Thus

$$A=U\Sigma V^T=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

7. Let $A=\begin{bmatrix}2 & -1\\ 2 & 2\end{bmatrix}$. Then $A^TA=\begin{bmatrix}8 & 2\\ 2 & 5\end{bmatrix}$, and the characteristic polynomial of A^TA is $\lambda^2-13\lambda+36=(\lambda-9)(\lambda-4)$, and the eigenvalues of A^TA are (in decreasing order) $\lambda_1=9$ and $\lambda_2=4$. Associated unit eigenvectors may be computed: $\lambda_1=9$: $\begin{bmatrix}2/\sqrt{5}\\ 1/\sqrt{5}\end{bmatrix}$, $\lambda_2=4$: $\begin{bmatrix}-1/\sqrt{5}\\ 2/\sqrt{5}\end{bmatrix}$. Thus one choice for V is $V=\begin{bmatrix}2/\sqrt{5} & -1/\sqrt{5}\\ 1/\sqrt{5} & 2/\sqrt{5}\end{bmatrix}$. The singular values of A are $\sigma_1=\sqrt{9}=3$ and $\sigma_2=\sqrt{4}=2$. Thus the matrix Σ is $\Sigma=\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}$. Next compute $\mathbf{u}_1=\frac{1}{\sigma_1}A\mathbf{v}_1=\begin{bmatrix}1/\sqrt{5}\\ 2/\sqrt{5}\end{bmatrix}$, $\mathbf{u}_2=\frac{1}{\sigma_2}A\mathbf{v}_2=\begin{bmatrix}-2/\sqrt{5}\\ 1/\sqrt{5}\end{bmatrix}$. Since $\{\mathbf{u}_1,\mathbf{u}_2\}$ is a basis for $\mathbb{R}^2$, let $U=\begin{bmatrix}1/\sqrt{5} & -2/\sqrt{5}\\ 2/\sqrt{5} & 1/\sqrt{5}\end{bmatrix}$. Thus

$$A=U\Sigma V^T=\begin{bmatrix}1/\sqrt{5} & -2/\sqrt{5}\\ 2/\sqrt{5} & 1/\sqrt{5}\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}2/\sqrt{5} & 1/\sqrt{5}\\ -1/\sqrt{5} & 2/\sqrt{5}\end{bmatrix}.$$

9. Let $A = \begin{bmatrix} 3 & -3 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$. Then $A^T A = \begin{bmatrix} 10 & -8 \\ -8 & 10 \end{bmatrix}$, and the characteristic polynomial of $A^T A$ is $\lambda^2 - 20\lambda + 36 = (\lambda - 18)(\lambda - 2)$, and the eigenvalues of $A^T A$ are (in decreasing order) $\lambda_1 = 18$ and $\lambda_2 = 2$. Associated unit eigenvectors may be computed: $\lambda_1 = 18$: $\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, $\lambda_2 = 2$: $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

Thus one choice for V is $V = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$. The singular values of A are $\sigma_1 = \sqrt{18} = 3\sqrt{2}$ and $\sigma_2 = \sqrt{2}$. Thus the matrix Σ is $\Sigma = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$. Next compute $\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$,

$\mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Since $\{\mathbf{u}_1, \mathbf{u}_2\}$ is not a basis for $\mathbb{R}^3$, we need a unit vector $\mathbf{u}_3$ that is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$. The vector $\mathbf{u}_3$ must satisfy the set of equations $\mathbf{u}_1^T \mathbf{x} = 0$ and $\mathbf{u}_2^T \mathbf{x} = 0$. These are equivalent to the linear equations $\begin{matrix} -x_1 + 0x_2 + 0x_3 = 0 \\ 0x_1 + 0x_2 + x_3 = 0 \end{matrix}$, so $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Therefore let

$U = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Thus $A = U \Sigma V^T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

15. **a**. Since A has 2 nonzero singular values, rank $A = 2$.

b. By Example 6, $\{\mathbf{u}_1, \mathbf{u}_2\} = \left\{ \begin{bmatrix} .40 \\ .37 \\ -.84 \end{bmatrix}, \begin{bmatrix} -.78 \\ -.33 \\ -.52 \end{bmatrix} \right\}$ is a basis for Col A and $\{\mathbf{v}_3\} = \left\{ \begin{bmatrix} .58 \\ -.58 \\ .58 \end{bmatrix} \right\}$ is a basis for Nul A.

18. Let $A = U\Sigma V^T = U\Sigma V^{-1}$. Since A is square and invertible, rank $A = n$, and all of the entries on the diagonal of Σ must be nonzero. So $A^{-1} = (U\Sigma V^{-1})^{-1} = V\Sigma^{-1}U^{-1} = V\Sigma^{-1}U^T$.

Chapter 7, Supplementary Exercises

15. **[M]** The reduced SVD of A is $A = U_r D V_r^T$, where

$$U_r = \begin{bmatrix} .966641 & .253758 & -.034804 \\ .185205 & -.786338 & -.589382 \\ .125107 & -.398296 & .570709 \\ .125107 & -.398296 & .570709 \end{bmatrix}, D = \begin{bmatrix} 9.84443 & 0 & 0 \\ 0 & 2.62466 & 0 \\ 0 & 0 & 1.09467 \end{bmatrix},$$

$$\text{and } V_r = \begin{bmatrix} -.313388 & .009549 & .633795 \\ -.313388 & .009549 & .633795 \\ -.633380 & .023005 & -.313529 \\ .633380 & -.023005 & .313529 \\ .035148 & .999379 & .002322 \end{bmatrix}$$

So the pseudoinverse $A^+ = V_r D^{-1} U_r^T$ may be calculated, as well as the solution $\hat{\mathbf{x}} = A^+\mathbf{b}$ for the system $A\mathbf{x} = \mathbf{b}$:

$$A^+ = \begin{bmatrix} -.05 & -.35 & .325 & .325 \\ -.05 & -.35 & .325 & .325 \\ -.05 & .15 & -.175 & -.175 \\ .05 & -.15 & .175 & .175 \\ .10 & -.30 & -.150 & -.150 \end{bmatrix}, \hat{\mathbf{x}} = \begin{bmatrix} .7 \\ .7 \\ -.8 \\ .8 \\ .6 \end{bmatrix}$$

Row reducing the augmented matrix for the system $A^T\mathbf{z} = \hat{\mathbf{x}}$ shows that this system has a solution, so $\hat{\mathbf{x}}$ is in $\text{Col } A^T = \text{Row } A$. A basis for Nul A is $\{\mathbf{a}_1, \mathbf{a}_2\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$, and an arbitrary element of Nul A is $\mathbf{u} = c\mathbf{a}_1 + d\mathbf{a}_2$. One computes that $\|\hat{\mathbf{x}}\| = \sqrt{131/50}$, while $\|\hat{\mathbf{x}} + \mathbf{u}\| = \sqrt{(131/50) + 2c^2 + 2d^2}$. Thus if $\mathbf{u} \neq \mathbf{0}$, $\|\hat{\mathbf{x}}\| < \|\hat{\mathbf{x}} + \mathbf{u}\|$, which confirms that $\hat{\mathbf{x}}$ is the minimum length solution to $A\mathbf{x} = \mathbf{b}$.

16. **[M]** The reduced SVD of A is $A = U_r D V_r^T$, where

$$U_r = \begin{bmatrix} -.337977 & .936307 & .095396 \\ .591763 & .290230 & -.752053 \\ -.231428 & -.062526 & -.206232 \\ -.694283 & -.187578 & -.618696 \end{bmatrix}, D = \begin{bmatrix} 12.9536 & 0 & 0 \\ 0 & 1.44553 & 0 \\ 0 & 0 & .337763 \end{bmatrix},$$

$$\text{and } V_r = \begin{bmatrix} -.690099 & .721920 & .050939 \\ 0 & 0 & 0 \\ .341800 & .387156 & -.856320 \\ .637916 & .573534 & .513928 \\ 0 & 0 & 0 \end{bmatrix}$$

So the pseudoinverse $A^+ = V_r D^{-1} U_r^T$ may be calculated, as well as the solution $\hat{\mathbf{x}} = A^+\mathbf{b}$ for the system $A\mathbf{x} = \mathbf{b}$:

$$A^+ = \begin{bmatrix} .5 & 0 & -.05 & -.15 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & .5 & 1.5 \\ .5 & -1 & -.35 & -1.05 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \hat{\mathbf{x}} = \begin{bmatrix} 2.3 \\ 0 \\ 5.0 \\ -.9 \\ 0 \end{bmatrix}$$

Row reducing the augmented matrix for the system $A^T\mathbf{z} = \hat{\mathbf{x}}$ shows that this system has a solution, so $\hat{\mathbf{x}}$ is in $\operatorname{Col} A^T = \operatorname{Row} A$. A basis for Nul A is $\{\mathbf{a}_1, \mathbf{a}_2\} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, and an arbitrary element of Nul A is $\mathbf{u} = c\mathbf{a}_1 + d\mathbf{a}_2$. One computes that $\|\hat{\mathbf{x}}\| = \sqrt{311/10}$, while $\|\hat{\mathbf{x}} + \mathbf{u}\| = \sqrt{(311/10) + c^2 + d^2}$. Thus if $\mathbf{u} \neq \mathbf{0}$, $\|\hat{\mathbf{x}}\| < \|\hat{\mathbf{x}} + \mathbf{u}\|$, which confirms that $\hat{\mathbf{x}}$ is the minimum length solution to $A\mathbf{x} = \mathbf{b}$.