Lesson 7

Chapter 4
Vector Spaces

Definition of basis

An indexed set of vectors $\mathcal{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_p\}$ is a **basis** for a vector space / subspace V if:

- $\mathcal{B}$ is a linearly independent set → *no unnecessary vectors*
- $V = span\{\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_p\}$ → $\begin{cases} Any\ \ \boldsymbol{v} \in V: \ \ \boldsymbol{v} = c_1 \cdot \boldsymbol{b}_1 + \cdots + c_p \cdot \boldsymbol{b}_p \\ Any\ \ c_1 \cdot \boldsymbol{b}_1 + \cdots + c_p \cdot \boldsymbol{b}_p \in V \ \ \rightarrow \ \ \boldsymbol{b}_1, \dots, \boldsymbol{b}_p \in V \end{cases}$

If a vector space V has a basis of n vectors $\mathcal{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n\}$ then:

- Any set in V containing more than n vectors must be linearly independent → If $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_p\} \epsilon \boldsymbol{V}$ is linearly independent, then $p \leq n$

- Every basis of V consists of exactly n vectors

↑

If $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_n\}$ and $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_m\}$ both are linearly independent basis of V, then $m \leq n$ and $n \leq m$ by the above

4.5 The Dimension of a Vector Space

$\dim V = n$

Definition of the dimension of a vector space

The dimension of the vector space V, $\dim V$, is the number of vectors in a basis of V.

- The dimension of the zero vector space $\{\mathbf{0}\}$ is 0.
- If V is not spanned by a finite set of vectors, V is said to be *infinite-dimensional*.

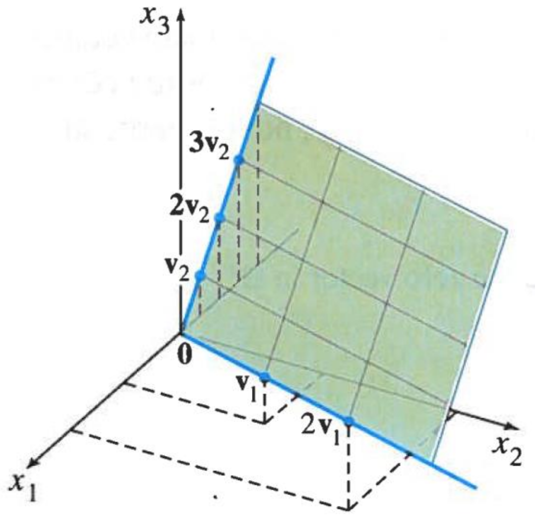
dim V → an intrinsic property of space V independent on basis

Examples:

$\dim \mathbb{R}^n = n$ $\quad \dim \mathbb{P}_2 = \dim \{1, t, t^2\} = 3$ $\quad \dim \mathbb{P}_n = n+1$ $\quad \dim \mathbb{P} = \infty$

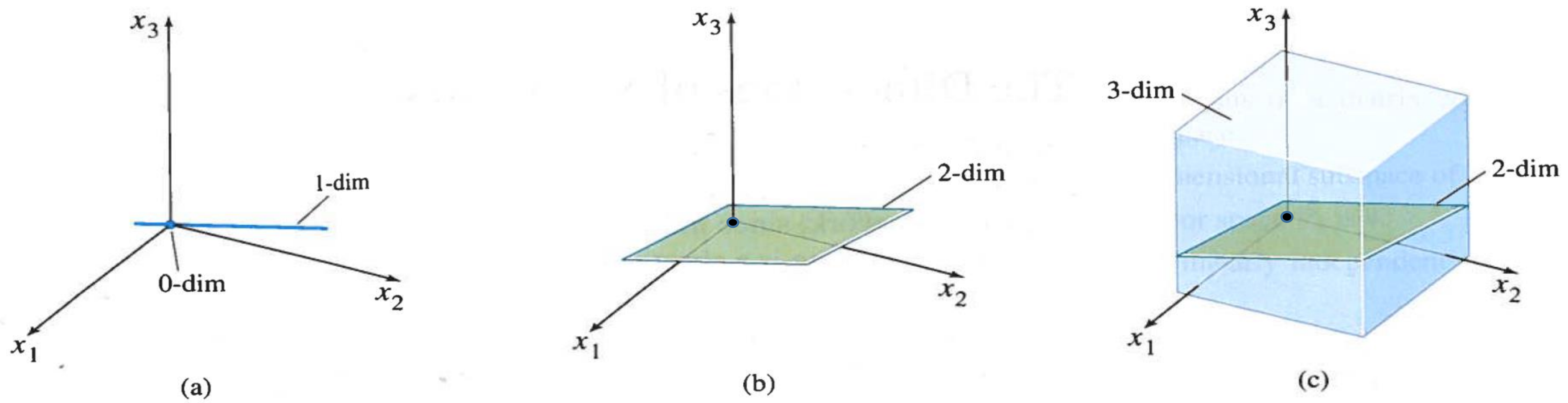
<u>Ex 1</u>

$$H_1 = span\left\{\begin{bmatrix}3\\6\\2\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$$



$$H_2 = span\left\{\begin{bmatrix}a-3b+6c\\5a+4d\\b-2c-d\\5d\end{bmatrix}\right\}$$

Subspaces of $\mathbb{R}^n$:



Theorem 4.12, The Basis Theorem

Let V be a p-dimensional vector space ($p \geq 1$).

- Any linearly independent set of exactly p elements in V is automatically a basis for V.
- Any set of exactly p elements that spans V is automatically a basis for V.

Sometimes linearly independence is easier to verify than spanning – and vice versa

Ex 2 dim Col A ? dim Nul A ?

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 & 1 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 & 1 \\ 2 & -4 & 5 & 8 & -4 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 & 1 \\ 0 & 0 & 1 & 2 & -2 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -9 & 8 \end{bmatrix}$$

Theorem 4.6: Pivot columns basis for Col A $\Rightarrow$ $\dim \text{Col A} = \#\text{pivot} = 3 \le \#\text{rows}$

#vectors in basis for Nul A $=$ #free parameters in $A\boldsymbol{x} = \mathbf{0}$

$=$ #non-pivot columns in A

$$\Rightarrow \dim \text{Nul A} = \#\text{non-pivot columns} = 4 = \#\text{columns} - \#\text{pivot} \ge \#\text{columns} - \#\text{rows}$$

4.6 Rank

Let A be an $m \times n$ matrix: $A = \begin{bmatrix} -2 & 5 & -3 \\ 1 & 0 & 3 \end{bmatrix}$

$$A = [\boldsymbol{a}_1 \;\; \boldsymbol{a}_2 \;\; \cdots \;\; \boldsymbol{a}_n] = \begin{bmatrix} \boldsymbol{b}_1{}^T \\ \boldsymbol{b}_2{}^T \\ \vdots \\ \boldsymbol{b}_m{}^T \end{bmatrix} \rightarrow \begin{cases} \boldsymbol{a}_i \in \mathbb{R}^{\boldsymbol{m}} \text{ are the column vectors of } A \\ \boldsymbol{b}_i \in \mathbb{R}^{\boldsymbol{n}} \text{ are the row vectors of } A \end{cases}$$

Definitions:

Column space: $\mathrm{Col}\, A = \mathrm{Span}\{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots, \boldsymbol{a_n}\} = \mathrm{Row}\, A^T$

Row space: $\mathrm{Row}\, A = \mathrm{Span}\{\boldsymbol{b_1}, \boldsymbol{b_2}, \cdots, \boldsymbol{b_m}\} = \mathrm{Col}\, A^T$

Null space: $\mathrm{Nul}\, A = \mathrm{Span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_p}\}; A\boldsymbol{v_i} = \boldsymbol{0}$

Ex 3

$$A = \begin{array}{c}\begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \\ \begin{array}{ccccc} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 \end{array}\end{array} \sim \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{array}{cc}\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} \boldsymbol{b}_1^T \\ \boldsymbol{b}_2^T \\ \boldsymbol{b}_3^T \\ \boldsymbol{b}_4^T \end{array}\end{array} = B$$

$$Col\ A = span\{Pivot\ columns\} = span\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_4\}$$

$$A \sim B \quad \Rightarrow Row\ A = Row\ B = span\{non\text{-}zero\ rows\ in\ B\} = span\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$$

Theorem 4.13:

- If $A \sim B$ (row equivalent), then $\mathrm{Row}\ A = \mathrm{Row}\ B$
- If B is in echelon form, the nonzero rows of B form a basis for $\mathrm{Row}\ B$ (and $\mathrm{Row}\ A$)

Ex 3

$$A = \begin{array}{c} \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \\ \begin{array}{ccccc} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 \end{array} \end{array} \sim \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{array}{cc} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} \boldsymbol{b}_1^T \\ \boldsymbol{b}_2^T \\ \boldsymbol{b}_3^T \\ \boldsymbol{b}_4^T \end{array} \end{array} = B$$

$$Col\ A = span\{Pivot\ columns\} = span\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_4\}$$

$$A \sim B \ \Rightarrow Row\ A = Row\ B = span\{non\text{–}zero\ rows\ in\ B\} = span\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$$

$$\Rightarrow dim\ Col\ A = dim\ Row\ A = \#pivot\ = rank\ A$$

$$Nul\ A\!:\ A\boldsymbol{x} = \boldsymbol{0} \sim B\boldsymbol{x} = \boldsymbol{0} \ \Rightarrow \begin{cases} x_1 + x_3 + x_5 = 0 \\ x_2 - 2x_3 + 3x_5 = 0 \\ x_4 - 5x_5 = 0 \end{cases} \Rightarrow \ \boldsymbol{x} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}$$

$$\Rightarrow\ Nul\ A = span\{\boldsymbol{v}_1, \boldsymbol{v}_2\} \quad \Rightarrow \ dim\ Nul\ A = \#non\text{–}pivot\ columns$$

Important and useful theorem

Theorem 4.14: The Rank Theorem

Let A be an $m \times n$ matrix. Then:

- $$\begin{aligned} \operatorname{rank} A &= \dim \operatorname{Col} A = \dim \operatorname{Row} A \\ &= \text{Number of pivot positions in } A \end{aligned}$$

- $$\operatorname{rank} A + \dim \operatorname{Nul} A = n$$

  (#pivot-columns + #nonpivot-columns = #columns in A)

Theorem 2.8: Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.
b. A is row equivalent to the $n \times n$ identity matrix.
c. A has n pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
g. The equation $A\mathbf{x} = \mathbf{b}$ has ~~at least one~~ a unique solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
h. The columns of A span $\mathbb{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix C such that $CA = I$.
k. There is an $n \times n$ matrix D such that $AD = I$.
l. A^T is an invertible matrix.

The Invertible Matrix Theorem

m. The columns of A form a basis of $\mathbb{R}^n$.

n. Col $A = \mathbb{R}^n$.

o. dim Col $A = n$.

p. Rank $A = n$.

q. Nul $A = \{\mathbf{0}\}$.

r. dim Nul A=0.

OBS!
Due to (l): A invertible $\Leftrightarrow$ A^T invertible
and Row A = Col A^T:

- All statement could also be stated for A^T
- All statements on Col A could also be stated on Row A

OBS: Numerical Note

Many of the discussed algorithms usefull for:

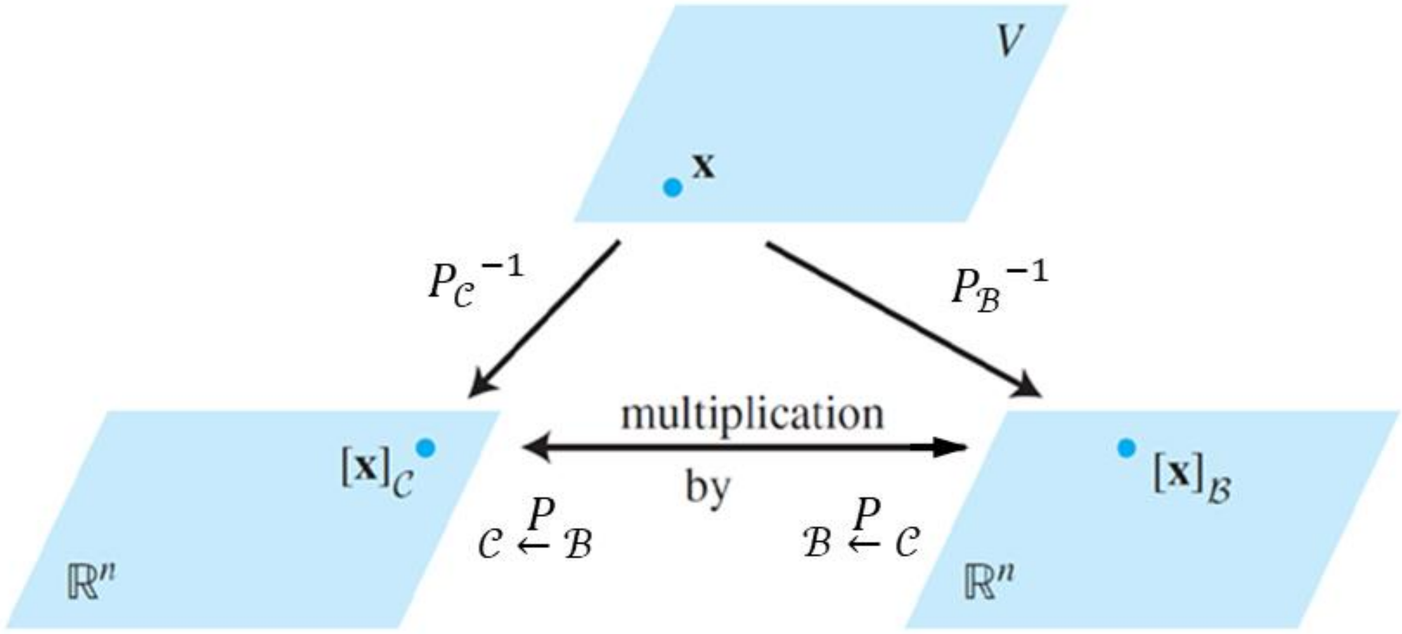
- understanding the concepts
- making simple calculations by hand

Large-scale real-life problems → computer calculations:

- these algorithms ineffective
- computer-roundings could change the apparent rank of the matrix – and thereby the result

Fx: $\begin{bmatrix} 5 & 7 \\ 5 & x \end{bmatrix} \rightarrow \begin{cases} x \equiv 7 & \rightarrow rank = 1 \\ x \not\equiv 7 & \rightarrow rank = 2 \end{cases}$

4.7 Change of Basis



From Chapter 4.4: Coordinate Systems

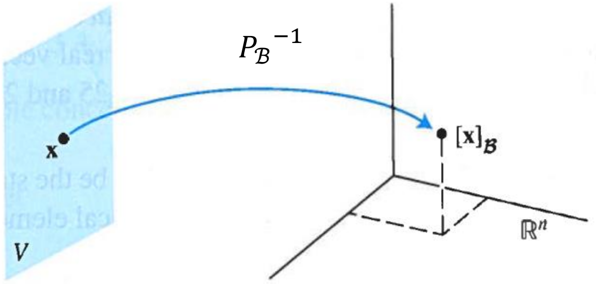
Let $\mathcal{B} = \{\boldsymbol{b_1}, \boldsymbol{b_2}, \cdots, \boldsymbol{b_n}\}$ be a basis for a vector space V (defining a coordinate system in V),

and $\boldsymbol{x}$ a vector in V: $\quad \boldsymbol{x} = c_1 \cdot \boldsymbol{b}_1 + c_2 \cdot \boldsymbol{b}_2 + \cdots + c_n \cdot \boldsymbol{b}_n = P_{\mathcal{B}}[\boldsymbol{x}]_{\mathcal{B}}$

where

Invertible according to
Inverse Matrix Theorem

Change-of-coordinate matrix: $P_{\mathcal{B}} = [\boldsymbol{b}_1 \ \boldsymbol{b}_2 \ \cdots \ \boldsymbol{b}_n]$



and

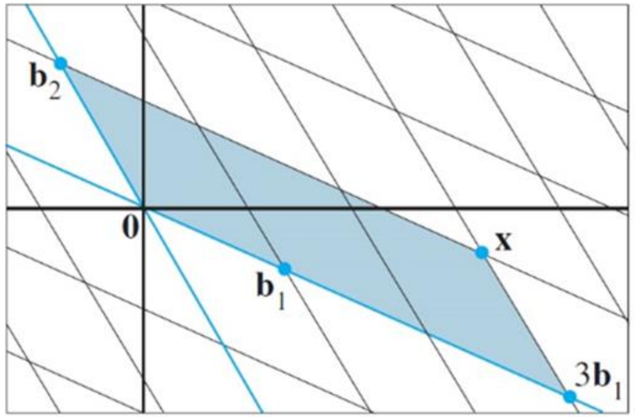
Coordinate mapping: $[\boldsymbol{x}]_{\mathcal{B}} = P_{\mathcal{B}}{}^{-1}\boldsymbol{x} = [c_1 \ c_2 \ \cdots \ c_n]^T$

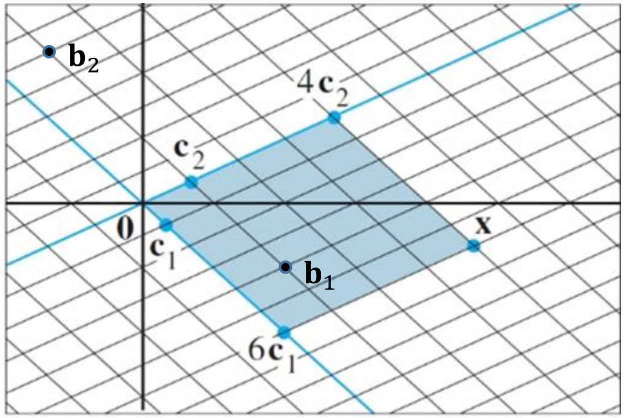
Coordinate mapping: $\mathcal{E} \curvearrowright \mathcal{B}$

Coordinates in basis $\mathcal{B}$

Ex 4

Vector space V:





Bases of V: $\mathcal{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ $\mathcal{C} = \{\boldsymbol{c}_1, \boldsymbol{c}_2\}$

$$\boldsymbol{b}_1 = 4\boldsymbol{c}_1 + \boldsymbol{c}_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}_{\mathcal{C}} \qquad \boldsymbol{b}_2 = -6\boldsymbol{c}_1 + \boldsymbol{c}_2 = \begin{bmatrix} -6 \\ 1 \end{bmatrix}_{\mathcal{C}}$$

$$\boldsymbol{x} = 3\boldsymbol{b}_1 + \boldsymbol{b}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} ? \\ ? \end{bmatrix}_{\mathcal{C}}$$

Theorem 4.15: Change-of-coordinate matrix

Let $\mathcal{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_n\}$ and $\mathcal{C} = \{\boldsymbol{c}_1, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_n\}$ be bases of a vector space V.

Then there is a unique $n \times n$ matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ such that

$$[\boldsymbol{x}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [\boldsymbol{x}]_{\mathcal{B}}$$

The columns of $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$.

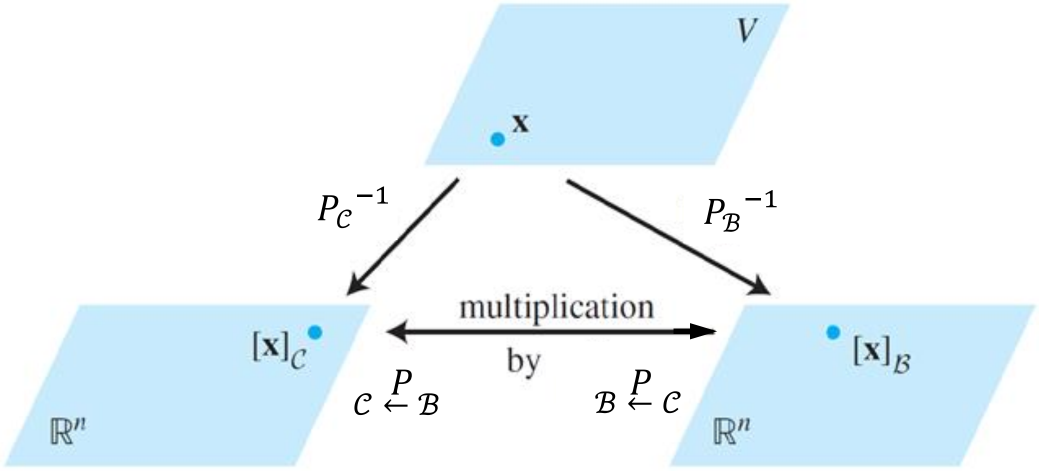
That is: $$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \left[[\boldsymbol{b}_1]_{\mathcal{C}} \;\; [\boldsymbol{b}_2]_{\mathcal{C}} \;\; \dots \;\; [\boldsymbol{b}_n]_{\mathcal{C}} \right]$$

$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ is invertible (the columns are linearly independent), so:

$$\left(\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}\right)^{-1}[\boldsymbol{x}]_{\mathcal{C}} = [\boldsymbol{x}]_{\mathcal{B}}$$

where

$$\left(\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \begin{bmatrix}[\boldsymbol{c}_1]_{\mathcal{B}} & [\boldsymbol{c}_2]_{\mathcal{B}} & \dots & [\boldsymbol{c}_n]_{\mathcal{B}}\end{bmatrix}$$



Ex 5

Bases of V: $\mathcal{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2\} = \left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}\right\}$ $\quad$ $\mathcal{C} = \{\boldsymbol{c}_1, \boldsymbol{c}_2\} = \left\{\begin{bmatrix}0\\3\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}\right\}$

$$\boldsymbol{b}_1 = \boldsymbol{c}_1 + \boldsymbol{c}_2 = \begin{bmatrix}1\\1\end{bmatrix}_{\mathcal{C}}; \quad \boldsymbol{b}_2 = \boldsymbol{c}_1 + 2\boldsymbol{c}_2 = \begin{bmatrix}1\\2\end{bmatrix}_{\mathcal{C}}$$

Change-of-coordinate matrix: $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \big[[\boldsymbol{b}_1]_{\mathcal{C}}\ \ [\boldsymbol{b}_2]_{\mathcal{C}}\big] = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}$

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} = \begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix} = \big[[\boldsymbol{c}_1]_{\mathcal{B}}\ \ [\boldsymbol{c}_2]_{\mathcal{B}}\big]$$

$$\boldsymbol{c}_1 = \begin{bmatrix}2\\-1\end{bmatrix}_{\mathcal{B}} = 2\boldsymbol{b}_1 - \boldsymbol{b}_2; \qquad \boldsymbol{c}_2 = \begin{bmatrix}-1\\1\end{bmatrix}_{\mathcal{B}} = -\boldsymbol{b}_1 + \boldsymbol{b}_2$$

$$\boldsymbol{x} = \begin{bmatrix}3\\-1\end{bmatrix}_{\mathcal{B}} = 3\boldsymbol{b}_1 - \boldsymbol{b}_2 = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\boldsymbol{x}]_{\mathcal{B}} = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}3\\-1\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}2\\1\end{bmatrix}_{\mathcal{C}} = 2\boldsymbol{c}_1 + \boldsymbol{c}_2 = \begin{bmatrix}1\\5\\2\end{bmatrix}$$

$$\boldsymbol{y} = \begin{bmatrix}-2\\1\end{bmatrix}_{\mathcal{C}} = -2\boldsymbol{c}_1 + \boldsymbol{c}_2 = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}[\boldsymbol{y}]_{\mathcal{C}} = \begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix}_{\mathcal{C}} = \begin{bmatrix}-5\\3\end{bmatrix}_{\mathcal{C}} = -5\boldsymbol{b}_1 + 3\boldsymbol{b}_2 = \begin{bmatrix}1\\-7\\-2\end{bmatrix}$$

Ex 6

Bases of $\mathbb{R}^2$: $\mathcal{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2\} = \left\{\begin{bmatrix} -9 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ -1 \end{bmatrix}\right\}$ $\quad$ $\mathcal{C} = \{\boldsymbol{c}_1, \boldsymbol{c}_2\} = \left\{\begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \end{bmatrix}\right\}$

Standard basis of $\mathbb{R}^2$: $\mathcal{E} = \{\boldsymbol{e}_1, \boldsymbol{e}_2\} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

Change-of-coordinate matrix: $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = (\underset{\mathcal{C} \leftarrow \mathcal{E}}{P})(\underset{\mathcal{E} \leftarrow \mathcal{B}}{P}) = (\underset{\mathcal{E} \leftarrow \mathcal{C}}{P})^{-1}(\underset{\mathcal{E} \leftarrow \mathcal{B}}{P}) = P_{\mathcal{C}}{}^{-1} P_{\mathcal{B}}$

$$P_{\mathcal{B}} = \underset{\mathcal{E} \leftarrow \mathcal{B}}{P} = [\boldsymbol{b}_1 \;\; \boldsymbol{b}_2] = \begin{bmatrix} -9 & -5 \\ 1 & -1 \end{bmatrix}$$

(Alternative method: $[\boldsymbol{c}_1 \;\; \boldsymbol{c}_2 \mid \boldsymbol{b}_1 \;\; \boldsymbol{b}_2] \sim [\, I \mid \underset{\mathcal{C} \leftarrow \mathcal{B}}{P}]$)

$$P_{\mathcal{C}} = \underset{\mathcal{E} \leftarrow \mathcal{C}}{P} = [\boldsymbol{c}_1 \;\; \boldsymbol{c}_2] = \begin{bmatrix} 1 & 3 \\ -4 & -5 \end{bmatrix} \quad \Rightarrow \quad P_{\mathcal{C}}{}^{-1} = \underset{\mathcal{C} \leftarrow \mathcal{E}}{P} = \begin{bmatrix} 1 & 3 \\ -4 & -5 \end{bmatrix}^{-1} = \frac{1}{7}\begin{bmatrix} -5 & -3 \\ 4 & 1 \end{bmatrix}$$

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \frac{1}{7}\begin{bmatrix} -5 & -3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} -9 & -5 \\ 1 & -1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 42 & 28 \\ -35 & -21 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$$

$$[\boldsymbol{b}_1]_{\mathcal{C}} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}_{\mathcal{C}} = 6\boldsymbol{c}_1 - 5\boldsymbol{c}_2 = 6\begin{bmatrix} 1 \\ -4 \end{bmatrix} - 5\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$$

Ditto $[\boldsymbol{b}_2]_{\mathcal{C}}$

Todays words and concepts

The Rank Theorem

Dimension

Full rank

dim V

Rank

Finite-dimensional

Row space

The Basis Theorem

Infinite-dimensional

Row A

Change-of-coordinate matrix