Solution for the ET-ALA reexam (Q1-2014)

PROBLEM 1.
Let the matrix A and the vector $\mathbf{b}$ be given by

$$A = \begin{bmatrix} 4 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & q \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

1. Determine the values of q for which the equation $A\mathbf{x} = \mathbf{b}$ is consistent.

Let B, C and D be invertible $n \times n$ matrices.

2. Solve the following three equations for X.

$$(I)\ \ XBCD = I, \qquad (II)\ \ CXB^{-1} = D, \qquad (III)\ \ XB - X = 2D.$$

PROBLEM 1. Solution
The augmented matrix is written down and row reduced

$$\left[\begin{array}{ccc|c} 4 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 3 & 1 & q & 2 \end{array}\right] \sim\sim \left[\begin{array}{ccc|c} 0 & -2 & -3 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & -2 & q-3 & 2 \end{array}\right] \sim\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 3 & -1 \\ 0 & -2 & q-3 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 3 & -1 \\ 0 & 0 & q & 1 \end{array}\right]$$

From the bottom row is is seen that the system is consistent for $q \neq 0$.

The equations are solved as follows

$(I)\ \ XBCD = I \iff X = ID^{-1}C^{-1}B^{-1} = D^{-1}C^{-1}B^{-1}$

$(II)\ \ CXB^{-1} = D \iff X = C^{-1}DB$

$(III)\ \ XB - X = 2D \iff XB - XI = 2D \iff X(B - I) = 2D \iff X = 2D(B - I)^{-1}$

Where $B - I$ is assumed invertible in the last step.

PROBLEM 2.

Assume it is requested to find the solution to the homogenous matrix equation $A\mathbf{x} = \mathbf{0}$ for some unknown 4×4 matrix. The augmented matrix has been row reduced and the result is

$$A = \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

1. Find the solution of $A\mathbf{x} = \mathbf{0}$.

PROBLEM 2. Solution

From the matrix it is seen that x_1, x_3 and x_4 are free variables and $x_2 = 0$. The solution can thus be written in parametric form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

PROBLEM 3.
Let the matrix A be given as

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

1. Compute the characteristic equation.

The eigenvalues of A are $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$.

2. By hand, calculate the eigenvectors and find orthogonal bases for the eigenspaces.
3. Write the vector $\mathbf{y} = [1\ 1\ 2]^T$ as a linear combination of the eigenvectors for A.

PROBLEM 3. Solution
The characteristic equation is given by $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -\lambda & -1 & -1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} =$$

$$(-1)^{1+1}(-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} + (-1)^{1+2}(-1)\begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix} + (-1)^{1+3}(-1)\begin{vmatrix} 1 & 2-\lambda \\ 1 & 1 \end{vmatrix} =$$

$$-\lambda^3 + 4\lambda^2 - 5\lambda + 2 = 0$$

For the double eigenvalue $\lambda = 1$ the eigenvectors are found as the solutions to $[A - 1I|\mathbf{0}]$

$$\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The solution to this problem is the two vectors

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The two vectors are clearly not orthogonal. A new orthogonal vector $\mathbf{q}_2$ can be found with Gram-Schmidt.

$$\mathbf{q}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

For the double eigenvalue $\lambda = 2$ the eigenvector becomes

$$\left[\begin{array}{ccc|c} -2 & -1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The solution is the vector

$$\mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

It is easily checked that $\mathbf{v}_3$ is orthogonal to $\mathbf{v}_1$ and $\mathbf{q}_2$. There are two eigenspaces for A: One 2-dimensional with $\lambda_1 = \lambda_2 = 1$ and a basis of orthogonal eigenvectors $\mathbf{v}_1$ and $\mathbf{q}_2$. The second eigenspace is 1-dimensional with $\lambda_3 = 2$ and the basis is eigenvector $\mathbf{v}_3$.

The problem is solved by writing down augmented matrix $[\mathbf{v}_1\mathbf{q}_2\mathbf{v}_3|\mathbf{y}]$ and reducing it to find the weigths.

$$\left[\begin{array}{ccc|c} -1 & -\frac{1}{2} & -1 & 1 \\ 1 & -\frac{1}{2} & 1 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

thus

$$\mathbf{y} = -4\mathbf{v}_1 - 2\mathbf{q}_2 + 4\mathbf{v}_3$$

PROBLEM 4.
For the statements given below, state whether they are true or false and justify your answer for each statement.

1. $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.
2. Every $m \times n$ matrix has exactly m pivots.
3. An $n \times n$ matrix with only real elements can have both real and complex eigenvalues.

PROBLEM 4. Solution
Statement 1 is **false** by construction. $\mathbb{R}^2$ is built up of vectors with two elements whereas $\mathbb{R}^3$ is built up of vectors with three elements and there is no connection between $\mathbb{R}^2$ and $\mathbb{R}^3$.

Statement 2 is **false**. The maximum number of pivots in an $m \times n$ matrix is the smaller of the numbers m and n. An example of a $m \times n$ matrix with less than m pivots is the following 3×2 matrix with only 1 pivot

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Statement 3 is **true**. The characteristic equation can have complex roots even when all elements in the matrix are real numbers. The complex eigenvalues will appear in complex conjugate pairs. An example of a matrix with only real elements, yet complex eigenvalues is

```
>> A=[3 7;-1 3]

A =

     3     7
    -1     3

>> eig(A)

ans =

   3.0000 + 2.6458i
   3.0000 - 2.6458i
```

PROBLEM 5.
Consider the system $\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$ with matrices

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

and let $\mathbf{x}_0 = \mathbf{0}$.

1. Find the controllability matrix for the system and show that the system $\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$, is controllable.

2. Find control vectors $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{u}_2$ that will force the system to $\mathbf{y} = \begin{bmatrix} 66 \\ 56 \\ 41 \end{bmatrix}$.

PROBLEM 5. Solution
The controllability matrix $M$is given by

$$M = [B\ AB\ A^2B] = \begin{bmatrix} 1 & 5 & 23 \\ 2 & 5 & 19 \\ 0 & 4 & 15 \end{bmatrix}.$$

By row reducing it is easily checked that M is row equivalent with the identity matrix and the system is therefore controllable.

The control vectors are found by solving the following system of equations.

$$[B\ AB\ A^2B \mid \mathbf{y}] = \left[\begin{array}{ccc|c} 1 & 5 & 23 & 66 \\ 2 & 5 & 19 & 56 \\ 0 & 4 & 15 & 41 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

The solution is therefore $\mathbf{u}_0 = 3$, $\mathbf{u}_1 = -1$ and $\mathbf{u}_2 = 2$.

PROBLEM 6.

Consider the following set of three equations with two unknowns.

$$\begin{aligned} x_1 - 3x_2 &= 2 \\ 2x_1 - x_2 &= -1 \\ x_1 + x_2 &= 0 \end{aligned}$$

1. Justify that the set of equations do not possess a solution.
2. Find a least squares solution of the system using a pseudoinverse.

PROBLEM 6. Solution

The problem is equivalent to an $A\mathbf{x} = \mathbf{b}$ problem with

$$A = \begin{bmatrix} 1 & -3 \\ 2 & -1 \\ 1 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

When the augmented matrix is row reduced

$$[A|\mathbf{b}] = \left[\begin{array}{cc|c} 1 & -3 & 2 \\ 2 & -1 & -1 \\ 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

it is seen that the system is inconsistent and there is therefore not a solution.
The pseudoinverse can be found with Matlab, either with `pinv` or with a singular value decomposition

```
>> A=[1 -3;2 -1;1 1]

A =  1     -3
     2     -1
     1      1

>> [U,S,V]=svd(A)

U = 0.8551   -0.2980    0.4243
    0.5073    0.6501   -0.5657
   -0.1072    0.6989    0.7071

S = 3.6355         0
         0    1.9450
         0         0

V = 0.4848    0.8746
   -0.8746    0.4848
```

Following the notation from the book we have $r = 2$, and `Ur=U(:,1:2)`, `D=S(1:2,1:2)`, `Vr=V` and $A^+ = V_r D^{-1} U_r^T$. This gives

$$A^+ = \begin{bmatrix} -0.02 & 0.36 & 0.30 \\ -0.28 & 0.04 & 0.20 \end{bmatrix}$$

The least squares solution of $A\mathbf{x} = \mathbf{b}$ is $\hat{\mathbf{x}} = A^+\mathbf{b} = \begin{bmatrix} -0.4 \\ -0.6 \end{bmatrix}$