Lesson 2

Chapter 1
Linear equations in Linear Algebra

System of linear equations:

$$\begin{aligned}
&a_{11} \cdot x_1 + a_{12} \cdot x_2 + a_{13} \cdot x_3 + \cdots + a_{1n} \cdot x_n = b_1 \\
&a_{21} \cdot x_1 + a_{22} \cdot x_2 + a_{23} \cdot x_3 + \cdots + a_{2n} \cdot x_n = b_2 \\
&a_{31} \cdot x_1 + a_{32} \cdot x_2 + a_{33} \cdot x_3 + \cdots + a_{3n} \cdot x_n = b_3 \\
&\ldots \\
&\ldots \\
&a_{m1} \cdot x_1 + a_{m2} \cdot x_2 + a_{m3} \cdot x_3 + \cdots + a_{mn} \cdot x_n = b_m
\end{aligned}$$

Augmented matrix:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

Row reduction

Row equivalent

Reduced echelon form:

$$\begin{bmatrix} 1 & 0 & * & 0 & 0 \\ 0 & 1 & * & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Pivot / Pivot columns

No / One / ∞ solutions

Vectors

A vector equation: $c_1 \cdot \boldsymbol{v_1} + c_2 \cdot \boldsymbol{v_2} + \cdots + c_p \cdot \boldsymbol{v_p} = \boldsymbol{b}$

→ have the same solution as the linear system with augmented matrix

$$\begin{bmatrix} \boldsymbol{v_1} & \boldsymbol{v_2} & \cdots & \boldsymbol{v_p} & \boldsymbol{b} \end{bmatrix}$$

$$\boxed{Span\{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_p}\} = c_1 \cdot \boldsymbol{v_1} + c_2 \cdot \boldsymbol{v_2} + \cdots + c_p \cdot \boldsymbol{v_p}}$$

← Linear Combination

1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

$$A\boldsymbol{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \boldsymbol{b}$$

Ex 1

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}$$

Definition

If A is a $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the **product of** A **and** $\mathbf{x}$ denoted by $A\mathbf{x}$ is **the linear combination of the columns of** A **using the corresponding entries in** $\mathbf{x}$ **as weights**; that is,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n$$

Vector in $\mathbb{R}^m$

Note:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix} \text{ and } Ax = \begin{bmatrix} a_{11} \cdot x_1 + a_{12} \cdot x_2 + a_{13} \cdot x_3 + \dots + a_{1n} \cdot x_n \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 + a_{23} \cdot x_3 + \dots + a_{2n} \cdot x_n \\ \vdots \\ a_{m1} \cdot x_1 + a_{m2} \cdot x_2 + a_{m3} \cdot x_3 + \dots + a_{mn} \cdot x_n \end{bmatrix} \in \mathbb{R}^m$$

Matrix equation: $A\boldsymbol{x} = \boldsymbol{b}$

Let a matrix and vector be given by

$$A = \begin{bmatrix} 2 & 8 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Solve the matrix equation: $A\boldsymbol{x} = \boldsymbol{b}$

That is, find $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ so $A\boldsymbol{x} = \boldsymbol{b}$

Ex 2

$$A = \begin{bmatrix} 2 & 8 \\ 1 & 3 \end{bmatrix}; \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \qquad \boldsymbol{b} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Theorem 1.3

If A is an *m x n* matrix with columns $\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots, \boldsymbol{a_n}$, and if $\boldsymbol{b}$ is in $\mathbb{R}^m$ the matrix equation

$$A\boldsymbol{x} = \boldsymbol{b}$$

OBS:
#Columns in A = #Rows in $\boldsymbol{x}$
#Rows in A = #Rows in $\boldsymbol{b}$

has the same solution set as the vector equation

$$x_1 \cdot \boldsymbol{a_1} + x_2 \cdot \boldsymbol{a_2} + \cdots + x_n \cdot \boldsymbol{a_n} = \boldsymbol{b}$$

which, in turn, has the same solution set as the system of linear equations

$$\left\{\begin{array}{c} a_{11} \cdot x_1 + a_{12} \cdot x_2 + a_{13} \cdot x_3 + \cdots + a_{1n} \cdot x_n = b_1 \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 + a_{23} \cdot x_3 + \cdots + a_{2n} \cdot x_n = b_2 \\ \vdots \\ a_{m1} \cdot x_1 + a_{m2} \cdot x_2 + a_{m3} \cdot x_3 + \cdots + a_{mn} \cdot x_n = b_m \end{array}\right\}$$

whose augmented matrix is

$$[\boldsymbol{a_1} \quad \boldsymbol{a_2} \quad \cdots \quad \boldsymbol{a_n} \quad \boldsymbol{b}]$$

Note:

$$x_1 \cdot \boldsymbol{a_1} + x_2 \cdot \boldsymbol{a_2} + \cdots + x_n \cdot \boldsymbol{a_n} = \boldsymbol{b}$$

→ $\boldsymbol{b}$ is a linear combination of the columns of A

→ $\boldsymbol{b} \in Span\{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots, \boldsymbol{a_n}\}$

→ $A\boldsymbol{x} = \boldsymbol{b}$ has solutions if and only if $\boldsymbol{b} \in Span\{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots, \boldsymbol{a_n}\}$

Ex 3 $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix} = [\boldsymbol{a}_1 \;\; \boldsymbol{a}_2 \;\; \boldsymbol{a}_3];$ $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix};$

Har $A\boldsymbol{x} = \boldsymbol{b}$ løsninger for alle $\boldsymbol{b}$?

Theorem 1.4

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

1. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
2. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
3. The columns in A span $\mathbb{R}^m$.
4. A has a pivot position in every row.

Theorem 1.5

$$\boxed{\begin{gathered} A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} \\ A(c\mathbf{u}) = c(A\mathbf{u}) \end{gathered}}$$

Fx.

$$\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ -9 \end{bmatrix} + \begin{bmatrix} -6 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 3 \\ 0 & 1 \end{bmatrix}\left(-3\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right) = -3\begin{bmatrix} 5 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = -3\begin{bmatrix} 23 \\ 1 \end{bmatrix} = \begin{bmatrix} -69 \\ -3 \end{bmatrix}$$

1.5 Solution Sets of Linear Systems

Homogeneous Linear System: $A\boldsymbol{x} = \mathbf{0}$

Inhomogeneous Linear System: $A\boldsymbol{x} = \boldsymbol{b} \neq \mathbf{0}$

Theorem 1.2: Existence an Uniqueness Theorem

A linear system is **consistent** if and only if **the rightmost column of the augmented matrix is not a pivot column** – that is, if and only if an echelon form of the augmented matrix has not rows of the form:

$$[0 \quad \dots \quad 0 \quad b] \text{ with } b \neq 0$$

If a linear system is consistent, then the solution set contains either:

i. **A unique solution**, when there are no free variables – that is, **all except the last column are pivot columns**.
ii. **Infinitely many solutions**, when there is a least one free variable – that is, **at least one column besides the last one is not a pivot column**.

Homogeneous matrix equation: $A\boldsymbol{x} = \mathbf{0} = \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}$

→ Allways the ***trivial*** solution: $\boldsymbol{x} = \mathbf{0}$

→ ***Nontrivial*** solution ($\boldsymbol{x} \neq \mathbf{0}$) if and only if at least one free variable
→ ∞ many solutions

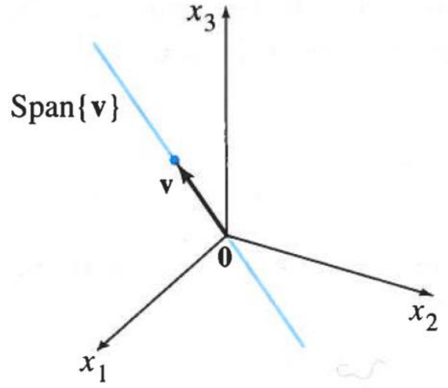
Ex 4: Homogeneous equation $A\boldsymbol{x} = \mathbf{0}$

$$\left\{\begin{matrix} 3x_1 + 5x_2 - 4x_3 = 0 \\ -3x_1 - 2x_2 + 4x_3 = 0 \\ 6x_1 + x_2 - 8x_3 = 0 \end{matrix}\right\} \Leftrightarrow \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Longrightarrow x_1 - \frac{4}{3}x_3 = 0; \;\; x_2 = 0; \;\; x_3 = x_3$$

$$\rightarrow \quad \boldsymbol{x} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} = x_3\boldsymbol{v}$$ → Straight line in $\mathbb{R}^3$



Ex 5: Inhomogeneous equation $A\boldsymbol{x} = \boldsymbol{b}$

$$\begin{cases} 3x_1 + 5x_2 - 4x_3 = 7 \\ -3x_1 - 2x_2 + 4x_3 = -1 \\ 6x_1 + x_2 - 8x_3 = -4 \end{cases} \Leftrightarrow \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 - \frac{4}{3}x_3 = -1 \\ x_2 = 2 \\ x_3 = x_3 = t \end{cases} \Rightarrow \begin{cases} x_1 = -1 + \frac{4}{3} \cdot t \\ x_2 = 2 + 0 \cdot t \\ x_3 = 0 + 1 \cdot t \end{cases}$$

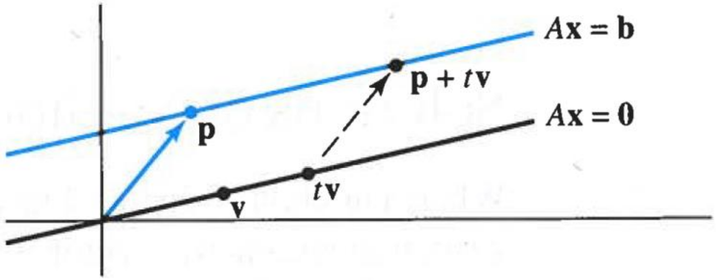
Parametric vector equation

$$\Rightarrow \quad \boldsymbol{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} = \boldsymbol{p} + \boldsymbol{v}_h$$

→ Translated straight line in $\mathbb{R}^3$

Inhomogeneous solution

Homogeneous solution

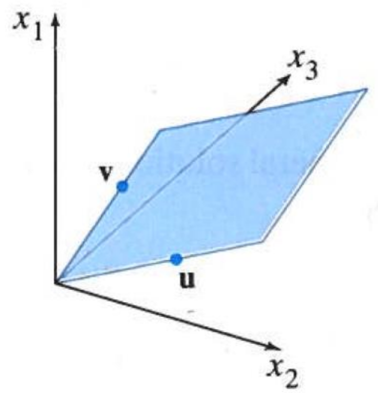


Ex 6: Homogeneous equation $A\boldsymbol{x} = \mathbf{0}$

$$\{10x_1 - 3x_2 - 2x_3 = 0\} \iff \begin{bmatrix}10 & -3 & -2\end{bmatrix} \cdot \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} = \mathbf{0}$$

$$\Rightarrow 10x_1 = 3x_2 + 2x_3 \quad \Rightarrow\ x_1 = \frac{3}{10}x_2 + \frac{2}{10}x_3; \quad x_2, x_3\ \mathit{free}$$

$$\Rightarrow \boldsymbol{x} = \begin{bmatrix}\frac{3}{10}x_2 + \frac{2}{10}x_3\\ x_2\\ x_3\end{bmatrix} = x_2\begin{bmatrix}\frac{3}{10}\\1\\0\end{bmatrix} + x_3\begin{bmatrix}\frac{2}{10}\\0\\1\end{bmatrix} = x_2\boldsymbol{u} + x_3\boldsymbol{v}$$ → Plane in $\mathbb{R}^3$

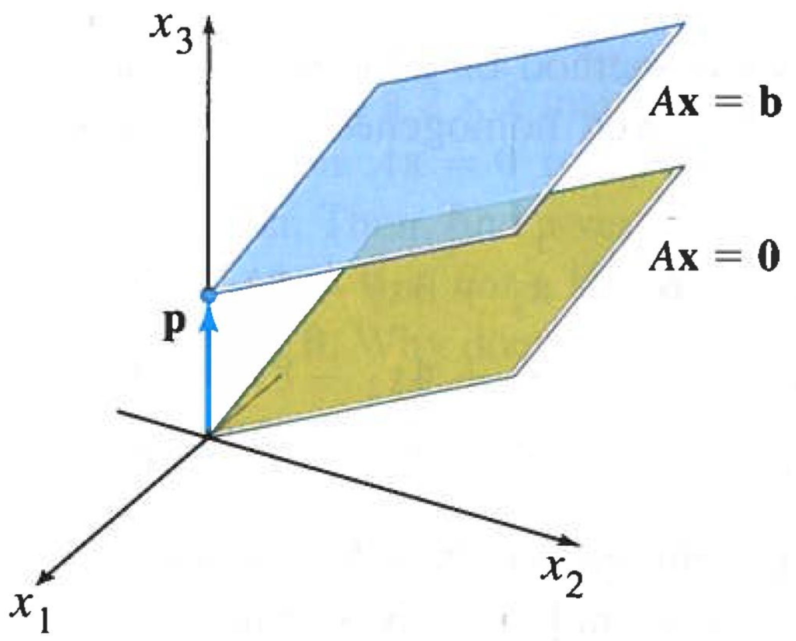


→ Solutions to $A\boldsymbol{x} = \mathbf{0}$: $\quad \boldsymbol{x} \in Span\{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_j}\}$ if j free variables

Null Space

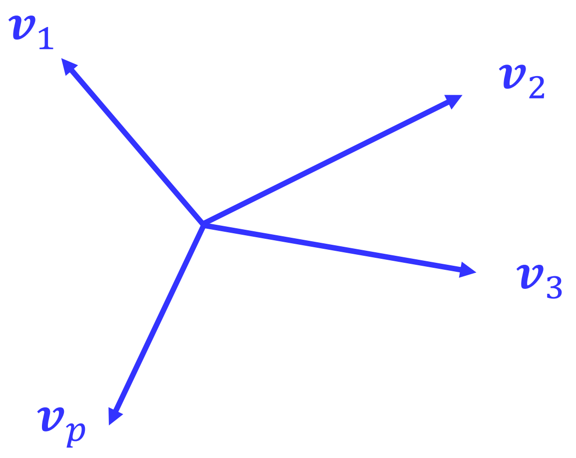
Theorem 1.6

Let $\boldsymbol{p}$ be a solution to the consistent inhomogeneous matrix equation $A\boldsymbol{x} = \boldsymbol{b}$. Then the set of all solutions to $A\boldsymbol{x} = \boldsymbol{b}$ is the set of vectors $\boldsymbol{w} = \boldsymbol{p} + \boldsymbol{v}_h$, where $\boldsymbol{v}_h$ is the solution set of the homogeneous equation $A\boldsymbol{x} = \boldsymbol{0}$.



1.7 Linear Independence

$$c_1 \cdot \boldsymbol{v}_1 + c_2 \cdot \boldsymbol{v}_2 + \cdots + c_p \cdot \boldsymbol{v}_p = \mathbf{0}$$



Homogeneous matrix equation: $A\boldsymbol{x} = \mathbf{0} = \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}$

→ Allways the ***trivial*** solution: $\boldsymbol{x} = \mathbf{0}$

→ ***Nontrivial*** solution ($\boldsymbol{x} \neq \mathbf{0}$) if and only if at least one free variable

→ ∞ many solutions

When are the trivial solution the only solution?

Let $A = [\boldsymbol{v_1} \quad \boldsymbol{v_2} \quad \cdots \quad \boldsymbol{v_p}]$:

$$A\boldsymbol{x} = x_1 \cdot \boldsymbol{v}_1 + x_2 \cdot \boldsymbol{v}_2 + \cdots + x_p \cdot \boldsymbol{v}_p = \mathbf{0} \overset{?}{\Leftrightarrow} x_1 = x_2 = \cdots = x_p = 0$$

Definition

An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be linearly independent if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is said to be linearly dependent if there exist weights $c_1, c_2, \ldots c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = \mathbf{0}$$

Homogeneous matrix equation: $A\boldsymbol{x} = \mathbf{0} = \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}$

→ Allways the ***trivial*** solution: $\boldsymbol{x} = \mathbf{0}$

→ ***Nontrivial*** solution ($\boldsymbol{x} \neq \mathbf{0}$) if and only if at least one free variable

→ ∞ many solutions

When are the trivial solution the only solution?

$A = [\boldsymbol{v_1} \quad \boldsymbol{v_2} \quad \cdots \quad \boldsymbol{v_p}]$:

$A\boldsymbol{x} = x_1 \cdot \boldsymbol{v}_1 + x_2 \cdot \boldsymbol{v}_2 + \cdots + x_p \cdot \boldsymbol{v}_p = \mathbf{0}$ → Only the trivial solution

$\Leftrightarrow$ **The columns of A are linearly independent**

Linearly independence

a) $\boldsymbol{v} \neq \mathbf{0} \;\rightarrow$ Linearly independent

b) $\boldsymbol{v}_1 \neq c \cdot \boldsymbol{v}_2 \;\rightarrow$ Linearly independent

c) $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_p \in \mathbb{R}^n; \;\; p > n \;\rightarrow$ Linearly dependent

$\left[\boldsymbol{v}_1\ \boldsymbol{v}_2 \ldots \boldsymbol{v}_p\ \ \mathbf{0}\right]$ → Max. n pivot (one in each row) → Min. 1 column not a Pivot column

→At least one free parameter

→ $c_1 \cdot \boldsymbol{v_1} + c_2 \cdot \boldsymbol{v_2} + \cdots + c_p \cdot \boldsymbol{v_p} = \mathbf{0}$ has infinitely many (nontrivial) solutions

Theorem 1.7

An indexed set $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_p}\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. That is the set is linearly dependent if

$$\boldsymbol{v_j} = c_1 \cdot \boldsymbol{v_1} + c_2 \cdot \boldsymbol{v_2} + \cdots + c_{j-1} \cdot \boldsymbol{v_{j-1}}$$

for some $1 < j \leq p$ and $\boldsymbol{v_1} \neq \boldsymbol{0}$

Theorem 1.8

If a set contains more vectors than there are entries (rows) in each vector, then the set is linearly dependent. That is, any set $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_p}\}$ in $\mathbb{R}^n$ is lineary dependent if $p > n$.

Theorem 1.9

If a set $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_p}\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

Todays words and concepts

Matrix equation

Homogeneous linear system

Linearly dependent

Multiplication properties

Trivial solution

Non-trivial solution

Inhomogeneous linear system

Linear combination

Linearly independent

Parametric vector equation

Matrix-vector multiplication