PROBLEM 1.

Consider the following matrix and vector

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ -2 & 3 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 7 \\ -3 \\ c \end{bmatrix}$$

where c is a scalar.

1. Solve $A\mathbf{x} = \mathbf{b}$ for $c = 1$.

2. Can $A\mathbf{x} = \mathbf{b}$ be solved for any value of c?

PROBLEM 1. Solution

The augmented matrix is written up and row reduced

$$[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 0 & -1 & 1 & -3 \\ -2 & 3 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right].$$

From which it is seen the solution is

$$\mathbf{x} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}.$$

The problem can be solved for any value of c. From the above row reduction is it seen that A has a pivot in all rows and hence $A\mathbf{x} = \mathbf{b}$ can be solved for any $\mathbf{b}$.

PROBLEM 2.

Let the following 2×2 matrix be given

$$A = \begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix}.$$

1. Find the inverse of A using the $[A|I] \sim [I|A^{-1}]$ algorithm from the textbook and show the steps during the row reduction.

Consider another matrix B with the property

$$B^2 - 3B + I = 0.$$

2. Show that $B^{-1} = 3I - B$.

PROBLEM 2. Solution

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right].$$

Thus

$$A^{-1} = \begin{bmatrix} -1 & 2 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

The equation can be rewritten as

$$B^2 - 3B + I = 0 \iff I = 3B - B^2 \iff I = (3I - B)B.$$

From the definition of the inverse matrix $BB^{-1} = B^{-1}B = I$ it is recognized from the above that $B^{-1} = 3I - B$.

PROBLEM 3.

The vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

are a very simple example of a *wavelet* basis. Wavelets are used in e.g. signal and image processing.

1. Show that the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_4$ form a basis for R^4.

Consider the vector $\mathbf{x} = \begin{bmatrix} 4 & -2 & 1 & 5 \end{bmatrix}^T$.

2. Find the coordinates of $\mathbf{x}$ in the wavelet basis.

PROBLEM 3. Solution

To form a basis for $\mathbb{R}^4$ the vectors must be linearly dependent and span $\mathbb{R}^4$. This is checked by forming the matrix $V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix}$ and row reducing.

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since V is row equivalent to the identity matrix, the four vectors are linear independent and span $\mathbb{R}^4$. The vectors therefore form a basis for $\mathbb{R}^4$.

The coordinates of $\mathbf{x}$ in the wavelet basis are the numbers $c_1, \ldots, c_4$ needed to form

$$\mathbf{x} = c_1\mathbf{v}_1 + \ldots c_4\mathbf{v}_4.$$

The coordinates are found by

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 4 \\ 1 & 1 & -1 & 0 & -2 \\ 1 & -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & -1 & 5 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

hence

$$[\mathbf{x}]_{wavelet} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ -2 \end{bmatrix}.$$

PROBLEM 4.

For the statements given below, state whether they are true or false and justify your answer for each statement.

1. The matrix equation $A\mathbf{x} = \mathbf{b}$ with A an $n \times n$ matrix is inconsistent if rank $A < n$.
2. The distance between the vectors $\begin{bmatrix} 4 & 3 & 2 \end{bmatrix}^T$ and $\begin{bmatrix} 3 & 2 & 1 \end{bmatrix}^T$ is $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$.
3. An $n \times n$ matrix A can *not* be diagonalized if two or more of the eigenvalues are identical.

PROBLEM 4. Solution

1. The matrix equation $A\mathbf{x} = \mathbf{b}$ with A an $n \times n$ matrix is inconsistent if rank $A < n$.

This statement is *false*. The matrix equation can be both consistent and inconsistent when rank $A < n$.

2. The distance between the vectors $\begin{bmatrix} 4 & 3 & 2 \end{bmatrix}^T$ and $\begin{bmatrix} 3 & 2 & 1 \end{bmatrix}^T$ is $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$.

This statement is *false*. The distance between two vectors is defined as $\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$ and is $\sqrt{3}$ in this case.

3. An $n \times n$ matrix A can *not* be diagonalized if two or more of the eigenvalues are identical.

This statement is *false*. For an $n \times n$ matrix to be diagonalizable it must have n linearly independent eigenvectors. There are no restrictions on the actual values of the eigenvalues.

PROBLEM 5.

In case 3, Computer Graphics in Automotive Design, homogeneous coordinates were introduced. In this problem, homogeneous coordinates in $\mathbb{R}^2$ are used.

An object $\mathcal{O}$ has the nodes $n_1, n_2, \ldots, n_5$ with coordinates

$$\mathcal{C}_n = \{(1,1),(1,3),(3,3),(3,1),(2,2)\}$$

and adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

1. Sketch the shape of the object, $\mathcal{O}$.
2. Find the matrix T, which translates $\mathcal{O}$, such that it is centered on $(0,0)$.

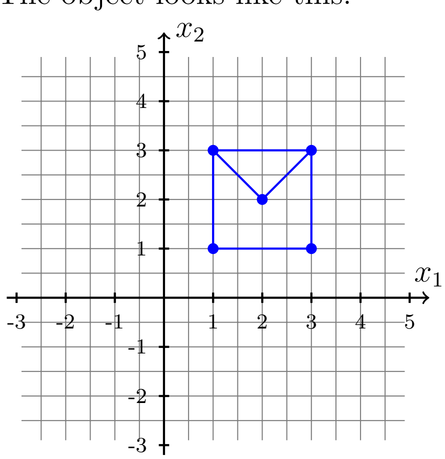
Consider the matrix

$$T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

3. What is the effect of T_2 applied to $\mathcal{O}$?

PROBLEM 5. Solution

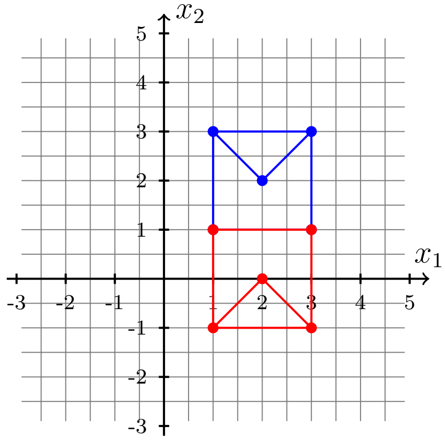
The object looks like this.



To center the object at the origin it must be translated -2 in both the x and y direction. The T-matrix is given by

$$T_= \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

The effect of applying T_2 is to reflect the object across the line $y = 1$ as shown in red here.



PROBLEM 6.

Consider the polynomial space $\mathbb{P}_2$ and let an inner product be defined as

$$< f, g >= \int_0^1 f(x)g(x)dx.$$

Let the following three vectors be given

$$p_1(x) = x + 1, \quad p_2(x) = x - 1, \quad p_3(x) = x^2 + x.$$

1. Show that the three vectors form a basis for $\mathbb{P}_2$.
2. Show that the three vectors are not orthogonal.
3. Use the Gram-Schmidt procedure to construct an orthogonal basis for $\mathbb{P}_2$.

PROBLEM 6. Solution

Using the standard mapping from $\mathbb{P}_2$ to $\mathbb{R}^3$ and writing the polynomials as column vectors the following matrix can be formed and row reduced

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since there are pivots in all three rows the vectors are linear independent and span $\mathbb{P}_2$, hence they form a basis.

The non-orthogonality of the vectors can be shown by calculating the inner products.

$$\begin{aligned}
< p_1(x), p_2(x) > &= \int_0^1 (x+1)(x-1)dx = \int_0^1 (x^2-1)dx = \left[\frac{1}{3}x^3 - x\right]_0^1 = -\frac{2}{3} \\
< p_1(x), p_3(x) > &= \int_0^1 (x+1)(x^2+x)dx = \frac{17}{12} \\
< p_2(x), p_3(x) > &= \int_0^1 (x-1)(x^2+x)dx = -\frac{1}{4}
\end{aligned}$$

Since all inner products are non-zero the vectors are not orthogonal.

The first vector in the new orthogonal basis is $v_1(x) = p_1(x) = x+1$. The second vector is found as

$$\begin{aligned}
v_2(x) &= p_2(x) - \frac{< p_2(x), v_1(x) >}{< v_1(x), v_1(x) >} v_1(x) \\
&= x - 1 - \frac{-\frac{2}{3}}{\frac{7}{3}}(x+1) = \frac{9}{7}x - \frac{5}{7}
\end{aligned}$$

and the third vector as

$$\begin{aligned}
v_3(x) &= p_3(x) - \frac{< p_3(x), v_1(x) >}{< v_1(x), v_1(x) >} v_1(x) - \frac{< p_3(x), v_2(x) >}{< v_2(x), v_2(x) >} v_2(x) \\
&= x^2 + x - \frac{\frac{17}{12}}{\frac{7}{3}}(x+1) - \frac{\frac{13}{84}}{\frac{1}{7}}\left(\frac{9}{7}x - \frac{5}{7}\right) = x^2 - x + \frac{1}{6}
\end{aligned}$$