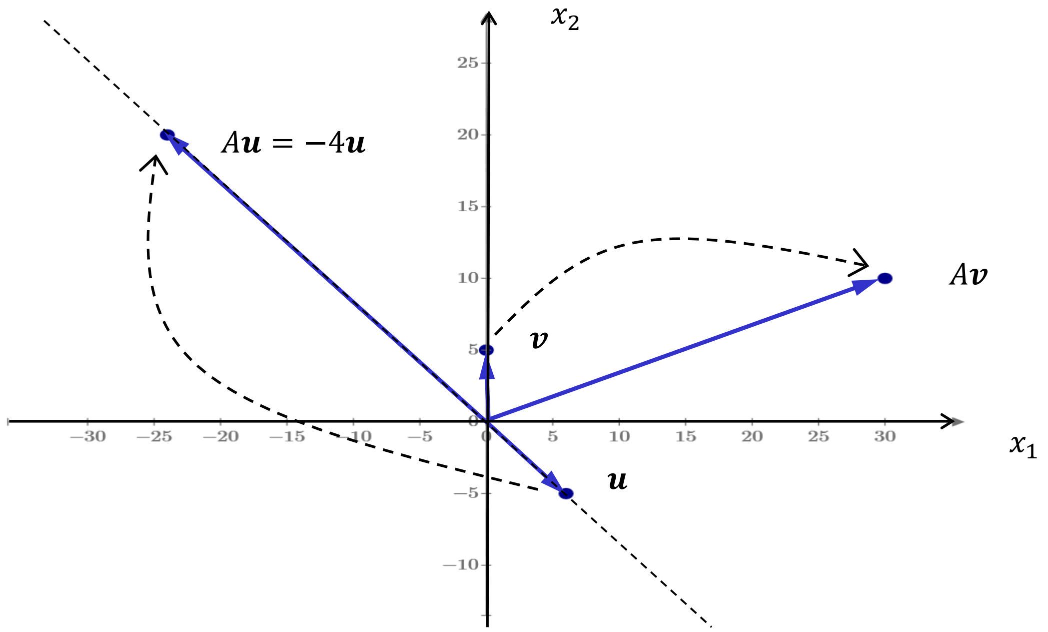
Lesson 8

Chapter 5
Eigenvectors and Eigenvalues

5.1 Eigenvectors and Eigenvalues

$$A\boldsymbol{x} = \lambda\boldsymbol{x}$$

Ex 1 $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ $\boldsymbol{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ $\boldsymbol{v} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$



Definition:

Egenvektor

OBS!!!

OBS!!!

Egenværdi

An eigenvector of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an eigenvalue of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.

<u>Ex 2</u> $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ $\lambda = -4$ eigenvalue, $\boldsymbol{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ eigenvector

$\lambda = 7$ eigenvalue ??

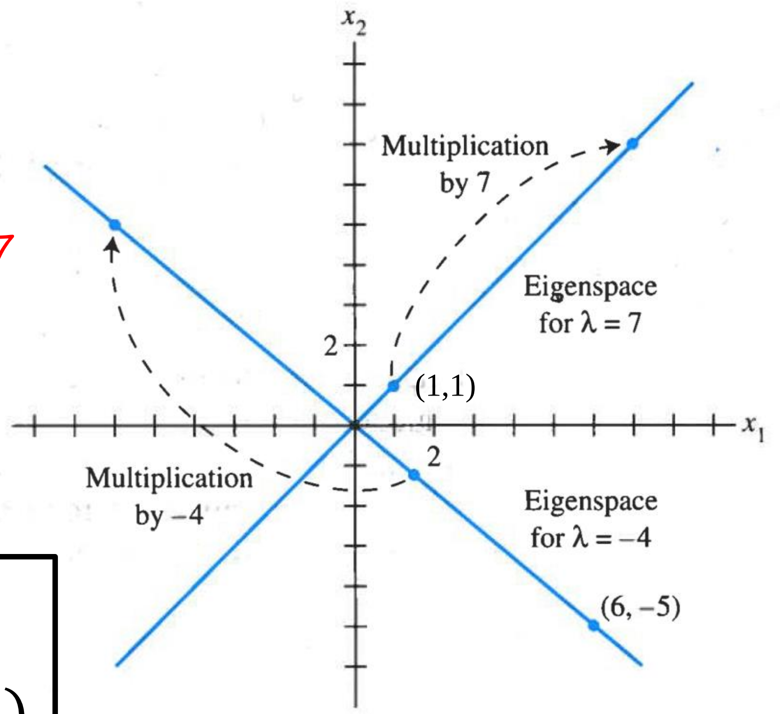
$A\boldsymbol{x} = 7\boldsymbol{x} \Rightarrow (A - 7I)\boldsymbol{x} = 0 \Rightarrow \left(\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}\right)\boldsymbol{x} = 0 \Rightarrow \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}\boldsymbol{x} = 0$ non-trivial solutions?

$\rightarrow \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{cases} x_1 - x_2 = 0 \\ 0 = 0 \end{cases} \Rightarrow \boldsymbol{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Eigenvector for λ=7

$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+6 \\ 5+2 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Eigenspace for λ=7: $c\begin{bmatrix} 1 \\ 1 \end{bmatrix}$



λ eigenvalue $\Leftrightarrow (A - \lambda I)\boldsymbol{x} = 0$ has non-trivial solutions

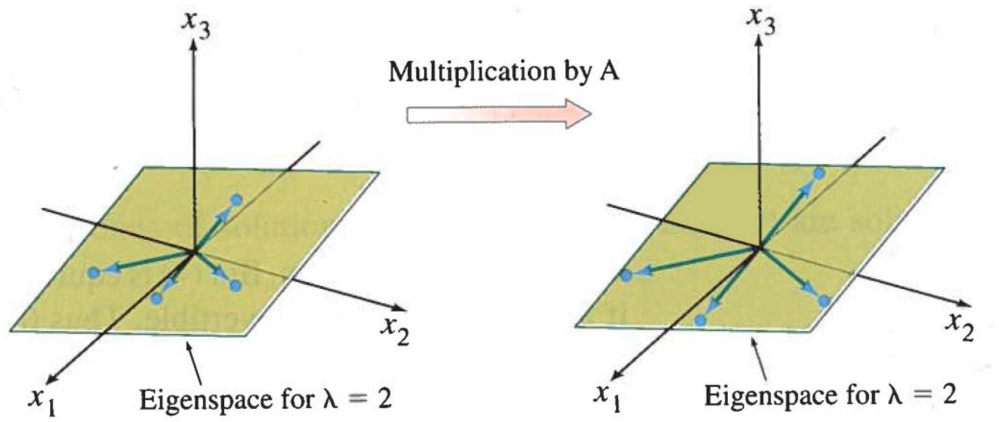
$\rightarrow$ Eigenspace of A corresponding to eigenvalue $\lambda = Nul(A - \lambda I)$

Ex 3 $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ $\lambda = 2$ eigenvalue → Eigenvector/-space?

$A\boldsymbol{x} = 2\boldsymbol{x} \Rightarrow (A - 2I)\boldsymbol{x} = 0 \rightarrow \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow$ 2 free variables: $x_1 = \frac{1}{2}x_2 - 3x_3$

$$\Rightarrow \boldsymbol{x} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \{Eigenspace\ for\ \lambda = 2\} = span\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$\boldsymbol{x} = 3\boldsymbol{v}_1 + \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix} \Rightarrow A\boldsymbol{x} = \begin{bmatrix} -6+6 \\ 6+6 \\ -6+8 \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \\ 2 \end{bmatrix} = 2\boldsymbol{x}$$



Eigenspaces

Egenrum

- The set of solutions of $(A - \lambda I)\mathbf{x} = \mathbf{0}$ is the null space of $A - \lambda I$
- This is also called the eigenspace of A corresponding to λ
- There is an eigenspace for each eigenvalue λ
- An eigenspace can be multidimensional

Theorem 5.2:

If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that corresponds to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

Proof of Theorem 5.2

Assume: $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_r\}$ linearly dependent

v_{p+1} *eigenvector*

$\Downarrow$

$\exists p < r$: $\boldsymbol{v}_{p+1} = c_1\boldsymbol{v}_1 + \cdots + c_p\boldsymbol{v}_p \neq \boldsymbol{0} \implies \lambda_{p+1}\boldsymbol{v}_{p+1} = \lambda_{p+1}c_1\boldsymbol{v}_1 + \cdots + \lambda_{p+1}c_p\boldsymbol{v}_p$ (1)

$\Downarrow$

$A\boldsymbol{v}_{p+1} = Ac_1\boldsymbol{v}_1 + \cdots + Ac_p\boldsymbol{v}_p \implies \lambda_{p+1}\boldsymbol{v}_{p+1} = \lambda_1 c_1\boldsymbol{v}_1 + \cdots + \lambda_p c_p\boldsymbol{v}_p$ (2)

(1) − (2) $\implies \boldsymbol{0} = (\lambda_{p+1} - \lambda_1)c_1\boldsymbol{v}_1 + \cdots + (\lambda_{p+1} - \lambda_p)c_p\boldsymbol{v}_p$

But: $\left.\begin{array}{l}\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_p\} \text{ linearly independent} \\ \lambda_{p+1} \neq \lambda_1, \cdots, \lambda_p\end{array}\right\} \implies c_1 = \cdots = c_p = 0 \implies \boldsymbol{v}_{p+1} = \boldsymbol{0}$ ÷

$\implies$ Assumption incorrect $\implies \{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_r\}$ linearly independent $\implies$ Theorem 5.2

Eigenvector: $\boldsymbol{v} \neq \mathbf{0}$ Eigenvalue: $\lambda = 0$?

λ eigenvalue $\Leftrightarrow$ $(A - \lambda I)\boldsymbol{x} = 0$ has non-trivial solutions

$\Downarrow$

0 eigenvalue $\Leftrightarrow$ $A\boldsymbol{x} = 0$ has non-trivial solutions $\Leftrightarrow$ A is not invertible $\Leftrightarrow$ $det\ A = 0$

$\rightarrow$ {Eigenspace of A correspong to $\lambda = 0$} $= Nul\ A$

Diagonal matrix: $D = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}$

λ eigenvalue $\Leftrightarrow$ $(D - \lambda I)\boldsymbol{x} = 0$ has non-trivial solutions $\Leftrightarrow$ $(D - \lambda I)$ is not invertible

$\Leftrightarrow$ $det(D - \lambda I) = (a_1 - \lambda) \cdot \cdots \cdot (a_n - \lambda) = 0$ $\Leftrightarrow$ $\lambda_1 = a_1, \dots, \lambda_n = a_n$

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.

b. A is row equivalent to the $n \times n$ identity matrix.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has ~~at least one~~ a unique solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

The Invertible Matrix Theorem - continued

Let A be a square $n \times n$ matrix. Then the following statements are equivalent.
That is, for a given A, the statements are either all true or all false.

m. The columns of A form a basis of $\mathbb{R}^n$.

n. Col $A = \mathbb{R}^n$.

o. dim Col $A = n$.

p. Rank $A = n$.

q. Nul $A = \{\mathbf{0}\}$.

r. dim Nul A=0.

New → s. The number 0 is not a eigenvalue of A.

t. The determinant of A is not 0.

OBS!
Due to (l): A invertible $\Leftrightarrow A^T$ invertible
and Row A = Col A^T:

- All statement could also be stated for A^T
- All statements on Col A could also be stated on Row A

5.2 The Characteristic Equation

$$det(A - \lambda I) = 0$$

How to find Eigenvalues: $Ax = \lambda x \Leftrightarrow (A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ have non-tvivial solutions

$n \times n$ matrix

$\Updownarrow$

$(A - \lambda I)$ not inversible

$\Updownarrow$

$det(A - \lambda I) = 0$

$\downarrow$

Polynomial of degree n in λ

$\downarrow$

Exactly n (complex) roots (solutions for λ)

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{bmatrix}$$

$$det(A - \lambda I) = (1-\lambda)(2-\lambda) - 30 = 0$$

The Characteristic Equation

$\Downarrow$

$$\lambda^2 - 3\lambda - 28 = 0$$

The Characteristic Polynomial

$\Downarrow$

$$\lambda = \frac{3 \pm \sqrt{9 - 4 \cdot 1 \cdot (-28)}}{2} = \begin{cases} 7 \\ -4 \end{cases}$$

5.3 Diagonalization

$$D = \begin{bmatrix} d_{11} & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ 0 & 0 & d_{33} & 0 \\ 0 & 0 & 0 & d_{44} \end{bmatrix} = P^{-1}AP$$

Definition: A and B similar $\Leftrightarrow$ $A = PBP^{-1}$

Theorem 5.4:

If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues.

Definition: Diagonaliserbar

A are **diagonalizable** if A is similar to an diagonal matrix D:
$A = PDP^{-1}$, where D is an $n \times n$ diagonal matrix

$\hookrightarrow D = P^{-1}AP$

Ex 4 Diagonalization $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

$$P = \begin{bmatrix} 6 & 1 \\ -5 & 1 \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{6+5}\begin{bmatrix} 1 & -1 \\ 5 & 6 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 1 & -1 \\ 5 & 6 \end{bmatrix}$$

Eigenvectors of A

$$P^{-1}AP = \frac{1}{11}\begin{bmatrix} 1 & -1 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 6 & 1 \\ -5 & 1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 1 & -1 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} -24 & 7 \\ 20 & 7 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} -44 & 0 \\ 0 & 77 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & 7 \end{bmatrix} = D$$

Eigenvalues -4 and 7 = Eigenvalues of A

Ex 5 Powers of diagonal matrix

$$D = \begin{bmatrix} -4 & 0 \\ 0 & 7 \end{bmatrix} \Rightarrow D^2 = \begin{bmatrix} -4 & 0 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} -4 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} (-4)^2 & 0 \\ 0 & 7^2 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 49 \end{bmatrix}$$

$$\Rightarrow D^k = \begin{bmatrix} (-4)^k & 0 \\ 0 & 7^k \end{bmatrix}$$

$$D = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix} \Longrightarrow D^k = \begin{bmatrix} a_1^k & 0 & \cdots & 0 \\ 0 & a_2^k & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_n^k \end{bmatrix}$$

Ex 6 Powers of diagonalizable matrix

$$A = PDP^{-1} \Rightarrow A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$$
$$\vdots$$
$$\Rightarrow A^k = PDP^{-1}PDP^{-1} \cdots PDP^{-1} = PD^kP^{-1}$$

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \qquad P = \begin{bmatrix} 6 & 1 \\ -5 & 1 \end{bmatrix} \qquad P^{-1} = \frac{1}{11}\begin{bmatrix} 1 & -1 \\ 5 & 6 \end{bmatrix} \qquad D = \begin{bmatrix} -4 & 0 \\ 0 & 7 \end{bmatrix}$$

$$A^3 = PD^3P^{-1} = \frac{1}{11}\begin{bmatrix} 6 & 1 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} (-4)^3 & 0 \\ 0 & 7^3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 5 & 6 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 6 & 1 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} -64 & 0 \\ 0 & 343 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 5 & 6 \end{bmatrix}$$
$$= \begin{bmatrix} 121 & 222 \\ 185 & 158 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}^3$$

Theorem 5.5, The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors of A.

Theorem 5.6:

An $n \times n$ matrices A with n distinct eigenvalues is diagonalizable.

Ex 7 $\quad A = \begin{bmatrix} 1 & 3 & 3 \\ -2 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \to PDP^{-1}$???

Step 1: Eigenvalues: $\det(A - \lambda I) = 0 \quad \Longrightarrow -\lambda^3 - 3\lambda^2 + 4 = -(\lambda - 1)(\lambda + 2)^2 = 0$

$$\Longrightarrow \lambda = 1, -2, -2$$

Step 2: Eigenvectors: $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}; \quad \boldsymbol{x} \neq \boldsymbol{0}$

$$\lambda = 1\text{:}\ (A - I)\boldsymbol{x} = \boldsymbol{0} \ \to \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \ \boldsymbol{x} = x_3 \begin{array}{c} \boldsymbol{v}_1 \\ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \end{array}$$

$$\lambda = -2\text{:}\ (A + 2I)\boldsymbol{x} = \boldsymbol{0} \ \to \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \ \boldsymbol{x} = x_2 \begin{array}{c} \boldsymbol{v}_2 \\ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \end{array} + x_3 \begin{array}{c} \boldsymbol{v}_3 \\ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{array}$$

Ex 7

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -2 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \to PDP^{-1} \;\; ???$$

Step 3: Diagonalization matrix: $P = [\boldsymbol{v}_1 \quad \boldsymbol{v}_2 \quad \boldsymbol{v}_3] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Step 4: Diagonal matrix: $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Step 5: Kontrol: $AP = \begin{bmatrix} 1 & 3 & 3 \\ -2 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$

$$PD = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} = AP \quad \checkmark$$

Todays words and concepts

Eigenspace

Eigenvector

Characteristic Polynomial

Diagonalization

Eigenvalue

Similarity

Diagonalization matrix

Characteristic Equation