Chapter 3.1

1. Expanding along the first row: $\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix} = 3\begin{vmatrix} 3 & 2 \\ 5 & -1 \end{vmatrix} - 0\begin{vmatrix} 2 & 2 \\ 0 & -1 \end{vmatrix} + 4\begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} = 3(-13) + 4(10) = 1$

Expanding along the second column:

$$\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix} = (-1)^{1+2}\cdot 0\begin{vmatrix} 2 & 2 \\ 0 & -1 \end{vmatrix} + (-1)^{2+2}\cdot 3\begin{vmatrix} 3 & 4 \\ 0 & -1 \end{vmatrix} + (-1)^{3+2}\cdot 5\begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix} = 3(-3) - 5(-2) = 1$$

2. Expanding along the first row: $\begin{vmatrix} 0 & 4 & 1 \\ 5 & -3 & 0 \\ 2 & 3 & 1 \end{vmatrix} = 0\begin{vmatrix} -3 & 0 \\ 3 & 1 \end{vmatrix} - 4\begin{vmatrix} 5 & 0 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 5 & -3 \\ 2 & 3 \end{vmatrix} = -4(5) + 1(15+6) = 1$

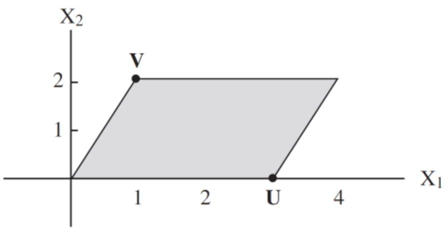
Expanding along the second column:

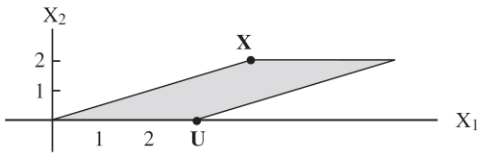
$$\begin{vmatrix} 0 & 4 & 1 \\ 5 & -3 & 0 \\ 2 & 3 & 1 \end{vmatrix} = (-1)^{1+2}\cdot 4\begin{vmatrix} 5 & 0 \\ 2 & 1 \end{vmatrix} + (-1)^{2+2}\cdot(-3)\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} + (-1)^{3+2}\cdot 3\begin{vmatrix} 0 & 1 \\ 5 & 0 \end{vmatrix} = -4(5) - 3(-2) - 3(-5) = 1$$

9. First expand along the third row, then expand along the first row of the remaining matrix:

$$\begin{vmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 3 & 0 & 0 & 0 \\ 8 & 3 & 1 & 7 \end{vmatrix} = (-1)^{3+1}\cdot 3\begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 7 \end{vmatrix} = 3\cdot(-1)^{1+3}\cdot 5\begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix} = 15(1) = 15$$

41. The area of the parallelogram determined by $\mathbf{u} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{u}+\mathbf{v}$, and $\mathbf{0}$ is 6, since the base of the parallelogram has length 3 and the height of the parallelogram is 2. By the same reasoning, the area of the parallelogram determined by $\mathbf{u} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ 2 \end{bmatrix}$, $\mathbf{u}+\mathbf{x}$, and $\mathbf{0}$ is also 6.





Also note that $\det[\mathbf{u} \quad \mathbf{v}] = \det\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = 6$, and $\det[\mathbf{u} \quad \mathbf{x}] = \det\begin{bmatrix} 3 & x \\ 0 & 2 \end{bmatrix} = 6$. The determinant of the matrix whose columns are those vectors which define the sides of the parallelogram adjacent to **0** is equal to the area of the parallelogram

45. **[M]** Answers will vary. The conclusion should be that $\det(A+B) \neq \det A + \det B$, most of the time.

Chapter 3.2

1. Rows 1 and 2 are interchanged, so the determinant changes sign (Theorem 3b.).

2. The row replacement operation does not change the determinant (Theorem 3a.).

5. $\begin{vmatrix} 1 & 5 & -4 \\ -1 & -4 & 5 \\ -2 & -8 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -4 \\ 0 & 1 & 1 \\ 0 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{vmatrix} = -3$

15. $\begin{vmatrix} a & b & c \\ d & e & f \\ 3g & 3h & 3i \end{vmatrix} = 3\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3(7) = 21$

16. $\begin{vmatrix} a & b & c \\ 5d & 5e & 5f \\ g & h & i \end{vmatrix} = 5\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5(7) = 35$

17. $\begin{vmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$

18. $\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -7$

19. $\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2(7) = 14$

20. $\begin{vmatrix} a & b & c \\ d+3g & e+3h & f+3i \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$

25. Since $\begin{vmatrix} 7 & -8 & 7 \\ -4 & 5 & 0 \\ -6 & 7 & -5 \end{vmatrix} = -1 \neq 0$, the columns of the matrix form a linearly independent set.

32. By Theorem 6 $\det A^3 = (\det A)^3$. Since $\det A^3 = 0$, then $(\det A)^3 = 0$. Thus det A = 0, and A is not invertible by Theorem 4.

45. [M] Answers will vary, but will show that $\det A^T A$ always equals 0 while $\det AA^T$ should seldom be zero. To see why $A^T A$ should not be invertible (and thus $\det A^T A = 0$), let A be a matrix with more columns than rows. Then the columns of A must be linearly dependent, so the equation $A\mathbf{x} = \mathbf{0}$ must have a non-trivial solution **x**. Thus $(A^T A)\mathbf{x} = A^T(A\mathbf{x}) = A^T\mathbf{0} = \mathbf{0}$, and the equation $(A^T A)\mathbf{x} = \mathbf{0}$ has a non-trivial solution. Since $A^T A$ is a square matrix, the Invertible Matrix Theorem now says that $A^T A$ is not invertible. Notice that the same argument will not work in general for AA^T, since A^T has more rows than columns, so its columns are not automatically linearly dependent.