Solution for the ET-ALA reexam (Q3-2014)

PROBLEM 1.
Let the matrix A and the vector $\mathbf{b}$ be given by

$$A = \begin{bmatrix} 1 & -2 & 2 & 6 \\ 2 & -3 & 4 & 9 \end{bmatrix}, \qquad \text{and } \mathbf{b} = \begin{bmatrix} -6 \\ -8 \end{bmatrix}$$

1. Solve the matrix equation $A\mathbf{x} = \mathbf{b}$.

Let B be an invertible $n \times n$ matrix.

2. Reduce the expression $B^2B^TBB^{-1}(B^{-1})^TB(B^{-1})^2$ as much as possible and account for the rules used in each step of the reduction.

PROBLEM 1. Solution
The first problem is solved by writing the augmented matrix and row reducing it.

$$\left[\begin{array}{cccc|c} 1 & -2 & 2 & 6 & -6 \\ 2 & -3 & 4 & 9 & -8 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & -3 & 4 \end{array}\right]$$

The reduced system is consistent and from the third and fourth column it is seen that x_3 and x_4 are free variables. From the augmented matrix the solution can be written in parametric form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

The expression is reduced as follows

$$\begin{aligned} &B^2B^TBB^{-1}(B^{-1})^TB(B^{-1})^2 = B^2B^TI(B^{-1})^TB(B^{-1})^2 = \\ &B^2B^T(B^{-1})^TB(B^{-1})^2 = B^2B^T(B^T)^{-1}BB^{-1}B^{-1} = \\ &B^2IIB^{-1} = B \end{aligned}$$

Where the rules $BB^{-1} = I$, $(B^{-1})^T = (B^T)^{-1}$, $(B^{-1})^2 = B^{-1}B^{-1}$ was used together with the fact that an I can be inserted and removed at will.

PROBLEM 2.
Let the matrix A be given by

$$A = \begin{bmatrix} 3-2q & 1 \\ 4 & 3+2q \end{bmatrix},$$

where q is a scalar.

1. Calculate q so that

$$A^2 = \begin{bmatrix} 29 & 6 \\ 24 & 125 \end{bmatrix}$$

PROBLEM 2. Solution
First A^2 is calculated as

$$A^2 = \begin{bmatrix} (2q-3)^2+4 & 6 \\ 24 & (2q+3)^2+4 \end{bmatrix}$$

By equating the a_{11} and the a_{22} entries the following two equations emerges

$$\begin{aligned} (2q-3)^2+4 &= 29 \\ (2q+3)^2+4 &= 125 \end{aligned}$$

Which gives

$$\begin{aligned} (2q-3)^2 &= 25 \\ (2q+3)^2 &= 121 \\ &\Updownarrow \\ 2q-3 &= \pm 5 \\ 2q+3 &= \pm 11 \\ &\Updownarrow \\ q &= \frac{3 \pm 5}{2} \\ q &= \frac{-3 \pm 11}{2} \\ &\Updownarrow \\ q &= \begin{cases} 4 \\ -1 \end{cases} \\ q &= \begin{cases} 4 \\ -7 \end{cases} \end{aligned}$$

From the two solutions it is seen that only $\underline{\underline{q=4}}$ is a solution to both equations.

PROBLEM 3.
Let a matrix be given by

$$A = \begin{bmatrix} 4 & 5 & 7 \\ 8 & 10 & 14 \\ 4 & 5 & 7 \\ 12 & 20 & 16 \end{bmatrix}$$

1. Determine the dimension of the column space of A, col A.
2. Determine an orthogonal basis for col A.

Let a vector $\mathbf{b}$ be given by

$$\mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

3. Calculate the orthogonal projection of $\mathbf{b}$ onto col A.

PROBLEM 3. Solution
The dimension of the column space is found by counting the number of pivots after row reducing the matrix

$$A = \begin{bmatrix} 4 & 5 & 7 \\ 8 & 10 & 14 \\ 4 & 5 & 7 \\ 12 & 20 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since we have 2 pivots in the first two columns, the dimension of the column space equals 2.
The column space is spanned by the pivot columns. These two columns are clearly not orthogonal $\mathbf{a}_1 \cdot \mathbf{a}_2 \neq 0$, however, the Gram-Schmidt algorithm can be used to form an orthogonal basis. This gives

$$\mathbf{u}_1 = \mathbf{a}_1 = \begin{bmatrix} 4 \\ 8 \\ 4 \\ 12 \end{bmatrix}, \qquad \mathbf{u}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 = \begin{bmatrix} -1 \\ -2 \\ -1 \\ 2 \end{bmatrix}$$

Finally, the orthogonal projection is calculated using The Orthogonal Decomposition Theorem as

$$\hat{\mathbf{b}} = \frac{\mathbf{b} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{b} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2$$

Using Matlab to perform the dot products and multiplications the result becomes

$$\hat{\mathbf{b}} = \begin{bmatrix} 4/3 \\ 8/3 \\ 4/3 \\ 2 \end{bmatrix}$$

PROBLEM 4.

In case 1 cubic splines was introduced. In this problem we will restrict ourselves to work with quadratic splines to reduce the complexity of the calculations somewhat. Completely similar to the cubic splines, the quadratic spline must pass through the given datapoints. Let the following datapoints be given.

x	y
2	1
4	2
6	7

The quadratic spline is defined by

$$y(x) = \begin{cases} a_0 + a_1 x + a_2 x^2 & 2 \le x \le 4 \\ b_0 + b_1 x + b_2 x^2 & 4 \le x \le 6 \end{cases}$$

The spline function and its derivative must be continuous at the $x = 4$ data point. Further it can be assumed that $y'(2) = 0$.

1. Write down the equations needed to determine a_0, a_1, a_2, b_0, b_1 and b_2.
2. Solve the above equations and write down the full expression for the quadratic spline.

PROBLEM 4. Solution

Inserting the values at the endpoints of the specified intervals gives the first four equations

$$\begin{aligned} &(x=2) \quad a_0 + 2a_1 + 4a_2 = 1 && (1) \\ &(x=4) \quad a_0 + 4a_1 + 16a_2 = 2 && (2) \\ &(x=4) \quad b_0 + 4b_1 + 16b_2 = 2 && (3) \\ &(x=6) \quad b_0 + 6b_1 + 36b_2 = 7 && (4) \end{aligned}$$

The derivative of the spline is

$$y'(x) = \begin{cases} a_1 + 2a_2 x & 2 \le x \le 4 \\ b_1 + 2b_2 x & 4 \le x \le 6 \end{cases}$$

Since the derivative must be continuous at $x = 4$ the fifth equation becomes

$$a_1 + 8a_2 = b_1 + 8b_2 \iff a_1 + 8a_2 - b_1 - 8b_2 = 0 \tag{5}$$

The sixth equation is obtained by setting $y'(2) = 0$.

$$a_1 + 4a_2 = 0 \tag{6}$$

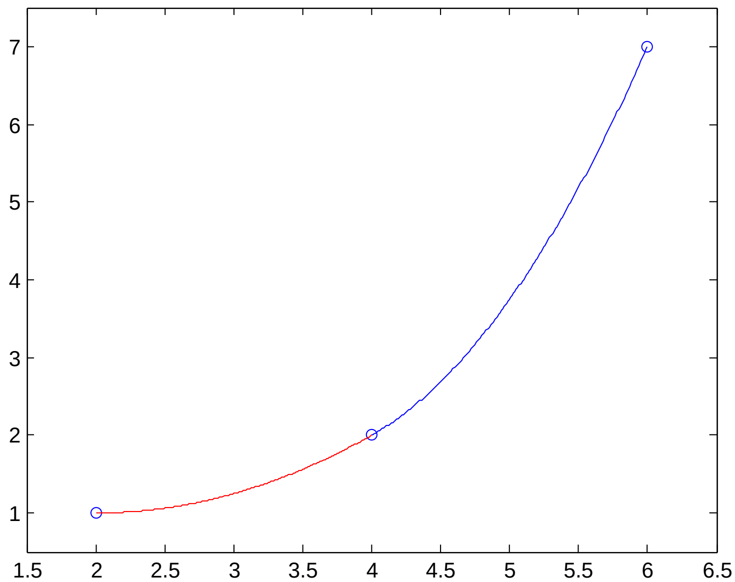
The six equations are written as an augmented matrix with columns $a_0, \ldots, b_2$ and row reduced.

$$\left[\begin{array}{cccccc|c} 1 & 2 & 4 & 0 & 0 & 0 & 1 \\ 1 & 4 & 16 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 & 16 & 2 \\ 0 & 0 & 0 & 1 & 6 & 36 & 7 \\ 0 & 1 & 8 & 0 & -1 & -8 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0.25 \\ 0 & 0 & 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0.75 \end{array}\right]$$

And the spline is therefore given by

$$y(x) = \begin{cases} 2 + -x + 0.25x^2 & 2 \leq x \leq 4 \\ 10 + -5x + 0.75x^2 & 4 \leq x \leq 6 \end{cases}$$

A plot of the spline is shown below



PROBLEM 5.

The difference equation $\mathbf{x}(k+1) = A\mathbf{x}(k)$ has the general solution $\mathbf{x}(k) = A^k\mathbf{x}(0)$. Consider the case where A is a diagonalizable 2×2 matrix with two real eigenvalues, λ_1 and λ_2 and assume that $\mathbf{x}(0) \neq \mathbf{0}$.

1. Explain how different values for the eigenvalues λ_1 and λ_2 influences $\mathbf{x}(k)$ as $k \to \infty$ for the three cases of

   (a) $\lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{1}{2}$ (b) $\lambda_1 = 1, \lambda_2 = 1$ (c) $\lambda_1 = 2, \lambda_2 = 2$

PROBLEM 5. Solution

The key to understanding the limiting behaviour of $\mathbf{x}(k)$ lies in the diagonalizability of A, i.e. $A = PDP^{-1}$. We have that

$$\begin{aligned}\mathbf{x}(k) &= A^k\mathbf{x}(0) = PDP^{-1}\ PDP^{-1} \dots\ PDP^{-1}\mathbf{x}(0)\\ &= PD^kP^{-1}\mathbf{x}(0)\end{aligned}$$

The factor $P^{-1}\mathbf{x}(0)$ amounts to a transformation of the initial vector into eigenvector coordinates. The factor D^k is the important part. For the 2×2 diagonal matrix we have

$$D^k = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^k = \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}$$

For the three different cases we get

- $\lambda_1 = \frac{1}{2}$, $\lambda_2 = -\frac{1}{2}$. Since both $\left(\frac{1}{2}\right)^k$ and $\left(-\frac{1}{2}\right)^k$ tends to zero as $k \to \infty$ the matrix $D^k \to \mathbf{0}$. Therefore $\mathbf{x}(k) \to \mathbf{0}$ for $k \to \infty$.
- $\lambda_1 = \lambda_2 = 1$. In this case D^k is the identity matrix for all values of k. Therefore $PD^kP^{-1} = PIP^{-1} = PP^{-1} = I$ and $\mathbf{x}(k) = \mathbf{x}(0)$ for all k.
- $\lambda_1 = \lambda_2 = 2$. In this case λ_1^k and λ_2^k will diverge to infinity as k increases. $\mathbf{x}(k)$ will diverge correspondingly.

PROBLEM 6.
Consider the vector space consisting of all polynomial functions functions defined on the interval $0 \leq x \leq 1$. Let the inner product between vectors f and g be defined as

$$< f, g >= \int_0^1 f(t)g(t)\, dt$$

Let two vectors be defined as

$$p_1(t) = 1 + t, \qquad p_2(t) = 2 + t^2$$

1. Calculate the distance between $p_1(t)$ and $p_2(t)$.

The Cauchy-Schwarz inequality states that

$$|< f, g >| \leq \|f\|\|g\|$$

where equality only holds if $f = cg$ for some constant c.

2. Use the Cauchy-Schwarz inequality to show that the vectors $p_1(t)$ and $p_2(t)$ are linearly independent.

PROBLEM 6. Solution
The distance between two vector is defined as $\|p_2(t) - p_1(t)\| = \sqrt{< p_2(t) - p_1(t), p_2(t) - p_1(t) >}$. As $p_2(t) - p_1(t) = t^2 - t + 1$ the distance becomes

$$\|p_2(t) - p_1(t)\| = \sqrt{\int_0^1 (t^2 - t + 1)^2 dt} = \sqrt{0.7} = 0.8367$$

The problem is solved by calculating the left hand and the right hand sign and checking whether the equal sign holds (linear dependence) or not (linearly independence). For the left hand side

$$| < 1 + t, 2 + t^2 > | = \left| \int_0^1 \big(t^3 + t^2 + 2t + 2\big)\, dt \right| = 3.5833$$

For the right hand side

$$\|1+t\| = \sqrt{\int_0^1 (t^2 + 2t + 1)\, dt} = 1.5275, \qquad \|2+t^2\| = \sqrt{\int_0^1 (t^4 + 4t^2 + 4)\, dt} = 2.3523,$$

Since $1.5275 \cdot 2.3523 \approx 3.5931$ we have

$$3.5833 < 3.5931$$

Thus the inequality strictly holds, hence $p_1(t) \neq cp_2(t)$ and the vectors are therefore linearly independent.