Solution for the ET-ALA exam (Q3-2013)

PROBLEM 1.
Let A be a 2×2 matrix with the property that

$$A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix} \quad \text{and } A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

1. Determine the values of the four elements, a_{11}, a_{12}, a_{21} and a_{22} of A.

PROBLEM 1. Solution
The matrix A is given by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

When written out the two clues therefore turns into

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix} \quad \Longleftrightarrow \quad \begin{array}{rcrl} a_{11} & + & 2a_{12} & = -1 \\ a_{21} & + & 2a_{22} & = 7 \end{array}$$

and

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} \quad \Longleftrightarrow \quad \begin{array}{rcrl} -a_{11} & + & a_{12} & = -2 \\ -a_{21} & + & a_{22} & = -1 \end{array}$$

Which is 4 equations in the 4 unknowns. This can be written as an matrix problem and reduces to

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{12} \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ -2 \\ -1 \end{bmatrix}$$

Writing the problem as an augmented matrix and reducing gives

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 & 7 \\ -1 & 1 & 0 & 0 & -2 \\ 0 & 0 & -1 & 2 & -1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

All elements of A are therefore uniquely determined and we get

$$\underline{\underline{A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}}}$$

PROBLEM 2.
A matrix A is given by

$$A = \begin{bmatrix} 1 & -1 & 3 & 5 \\ -1 & -3 & 1 & -1 \\ 2 & 6 & -2 & 2 \end{bmatrix}$$

1. Determine bases for the null space, column space and row space of A.
2. How many solutions are there to the homogenous equation $A\mathbf{x} = \mathbf{0}$?
3. Find a vector $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ can be solved.

PROBLEM 2. Solution
Finding a basis for the null space amounts to solving the homogenous equation $A\mathbf{x} = \mathbf{0}$. According to the comment on page 201 the solution of $A\mathbf{x} = \mathbf{0}$ automatically produces linearly independent vectors as needed for a basis

$$[A|\mathbf{0}] = \left[\begin{array}{cccc|c} 1 & -1 & 3 & 5 & 0 \\ -1 & -3 & 1 & -1 & 0 \\ 2 & 6 & -2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

From which we identify that x_3 and x_4 are free variables. Rewriting in standard form gives

$$\begin{aligned} x_1 &= -2x_3 - 4x_4 \\ x_2 &= x_3 + x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}$$

which is written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

The basis for Null(A) is therefore

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

From the above row reduction it is seen that the first two columns of A are pivot columns. These two columns therefore form a basis for col(A)

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 6 \end{bmatrix} \right\}$$

Finally the same reduction shows that row 1 and 2 are pivot rows and hence forms a basis for the row space. The basis vectors can be taken from either A or the reduced A with the later containing 'nicer' values. The basis for row(A) is therefore

$$\left\{ \begin{bmatrix} 1 & 0 & 2 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 & -1 \end{bmatrix} \right\}$$

Since two free variables were found in the solution of $A\mathbf{x} = \mathbf{0}$ we have an infinite amount of solutions to this equation.

Any vector $\mathbf{b}$ taken from the column space of A leads to $A\mathbf{x} = \mathbf{b}$ being solvable. The simplest choice is just the first column of A

$$\underline{\underline{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}}}$$

PROBLEM 3.
Consider the matrix equation $A\mathbf{x} = \mathbf{b}$ where A and $\mathbf{b}$ are given by

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & 4 \\ 0 & 1 & -1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

1. The equation $A\mathbf{x} = \mathbf{b}$ can not be solved. Show why.

Two guesses of approximate solutions to $A\mathbf{x} = \mathbf{b}$ are

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

2. Determine which of the two proposed solutions is the best one in the least squares sense.

PROBLEM 3. Solution
That $A\mathbf{x} = \mathbf{b}$ is not solvable is easily shown by forming the augmented matrix and row reducing it

$$[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 2 & 2 & 1 & 1 \\ 2 & -1 & 4 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1.5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

From which we see in the third row that the matrix equation is inconsistent and therefore non-solvable.

The best solution of the two proposed solutions is found by direct calculation of their distances to the desired vector and comparing the result. This gives

$$||A\mathbf{x}_1 - \mathbf{b}|| = 9.6437 \quad \text{and} \quad ||A\mathbf{x}_2 - \mathbf{b}|| = 3$$

As the distance between $A\mathbf{x}_2$ and $\mathbf{b}$ is much smaller than the distance between $A\mathbf{x}_1$ and $\mathbf{b}$, the error in using $\mathbf{x}_2$ as the solution is the smallest and $\mathbf{x}_2$ is therefore the better solution in the least squares sense

PROBLEM 4.
Inner products can be defined with weight functions. One example of this is continuous functions defined on the interval $[a, b]$. Here, the weigthed inner product is given by

$$< f, g >= \int_a^b f(x)g(x)w(x)dx$$

Where $w(x)$ is the weight function. Here we consider the space of all continuous functions $C[0, 1]$ with $w(x) = x$.

Let two vectors be given by

$$f(x) = e^{-x} \qquad \text{and} \qquad g(x) = e^{-2x}$$

1. Show that the two vectors are linearly independent.
2. Determine if the two vectors are orthogonal with respect to the above defined inner product.

Part of the following table of integrals might be useful in the above problem

$$\int e^{ax}dx = \frac{e^{ax}}{a}$$
$$\int xe^{ax}dx = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right)$$
$$\int x^2e^{ax}dx = \frac{e^{ax}}{a}\left(x^2 - \frac{2x}{a} - \frac{2}{a^2}\right)$$

PROBLEM 4. Solution
If the two vectors are to be linearly independent the equation

$$c_1 f(x) + c_2 g(x) = 0$$

Must only have the trivial solution, $c_1 = c_2 = 0$. Rewriting this equation gives

$$f(x) = -\frac{c_2}{c_1} g(x)$$

A nontrivial solution thus implies that $f(x)$ can be found from $g(x)$ by multiplying $g(x)$ by a number c. Since

$$e^{-x} \neq ce^{-2x}$$

the two functions are linearly independent.

The orthogonality is tested by simple calculation of the inner product

$$< f, g >= \int_a^b f(x)g(x)w(x)dx = \int_0^1 e^{-x}e^{-2x}x\,dx = \int_0^1 xe^{-3x}\,dx$$

Since both x and e^{-3x} are positive functions when $0 < x < 1$ we're integrating a function that is non-negative everywhere and the result of performing the integration will therefore be a number larger than zero. Therefore, the two vectors are not orthogonal.

PROBLEM 5.
A set of three coupled differential equations are given by

$$\begin{aligned} x_1'(t) &= 4x_1(t) - x_2(t) + x_3(t) \\ x_2'(t) &= -2x_1(t) + 5x_2(t) + x_3(t) \\ x_3'(t) &= 2x_1(t) - x_2(t) + 3x_3(t) \end{aligned}$$

1. Rewrite as three decoupled differential equations.

PROBLEM 5. Solution
The coupled differential equations can be written in matrix form $\mathbf{x}'(t) = A\mathbf{x}(t)$ with

$$A \begin{bmatrix} 4 & -1 & 1 \\ -2 & 5 & 1 \\ 2 & -1 & 3 \end{bmatrix}$$

The eigenvalues and eigenvectors of A can be found with Matlab

```
A =

     4    -1     1
    -2     5     1
     2    -1     3

>> [P,D]=eig(A)

P =

   -0.5774    0.5774    0.5774
    0.5774    0.5774    0.5774
   -0.5774   -0.5774    0.5774


D =

    6.0000         0         0
         0    2.0000         0
         0         0    4.0000
```

As the 3×3 matrix A has three linearly independent eigenvectors, A can be factorized as $A = PDP^{-1}$.

Inserting the factorization into the differential equation gives

$$\mathbf{x}'(t) = A\mathbf{x}(t) \iff \mathbf{x}'(t) = PDP^{-1}\mathbf{x}(t) \iff P^{-1}\mathbf{x}'(t) = DP^{-1}\mathbf{x}(t)$$

Defining $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$ changes the differential equation into

$$\mathbf{y}'(t) = D\mathbf{y}(t)$$

Since D is a diagonal matrix the three components of $\mathbf{y}$ are decoupled and the differential equation can be written as

$$\begin{aligned} y_1'(t) &= 6y_1(t) \\ y_2'(t) &= 2y_2(t) \\ y_3'(t) &= 4y_3(t) \end{aligned}$$

PROBLEM 6.
In case 3, Computer Graphics in Automotive Design, homogeneous coordinates were introduced. With homogenous coordinates a point (x, y, z) in $\mathbb{R}^3$ is written as a 4-dimensional vector $(x, y, z, 1)$.

1. Explain the purpose of the 1 in $(x, y, z, 1)$.

PROBLEM 6. Solution
This question can be discussed in several ways. The main point is that the introduction of homogenous coordinates allows the transformation of coordinates to be performed as matrix multiplications. For instance with standard (x, y, z) coordinates it is not possible to perform a translation of the coordinates as a multiplication. No matrix can be found that gives

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + \Delta x \\ y + \Delta y \\ z + \Delta z \end{bmatrix}$$

Since the matrix multiplication will mix up x, y and z. However with homogenous coordinates the additional 1 makes translation easy

$$\begin{bmatrix} 1 & 0 & 0 & \Delta x \\ 0 & 1 & 0 & \Delta y \\ 0 & 0 & 1 & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} 1x + 1\Delta x \\ y + \Delta y \\ z + \Delta z \\ 1 \end{bmatrix} = \begin{bmatrix} x + \Delta x \\ y + \Delta y \\ z + \Delta z \\ 1 \end{bmatrix}$$