Case 4 - Error-Detecting and Error-Correcting Codes

Question 1

- US Postal Service codes.

a) The bar code translate into 346836. The sum of the first five digits is 24, when the final 6 is added the sum is 30 and the code is ok.

b) The bar code translate into 018679. The sum of the first five digits is 22, when the final 9 is added the sum is 31 and the code is not ok.

c) The bar code translate into 207425. The sum of the first five digits is 15, when the final 5 is added the sum is 20 and the code is ok.

Question 2

- Consider the following vectors in $\mathbb{Z}_2^4$.

$$\mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Compute the following

a) $\mathbf{a} + \mathbf{b}$

b) $\mathbf{c} - \mathbf{b} + \mathbf{a}$

The computations are

$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{c} - \mathbf{b} + \mathbf{a} = \mathbf{c} + \mathbf{b} + \mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

In the last calculation it was used that subtraction is exactly the same as addition in $\mathbb{Z}_2$.

Question 3

- Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be as in Question 2. Is the set $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ linearly independent or linearly dependent?

The linear independence is determined in the usual manner, i.e. the set is linearly independent if we only have the trivial solution to the equation $x_1\mathbf{a} + x_2\mathbf{b} + x_3\mathbf{c} = \mathbf{0}$. This is checked by writing up the appropriate matrix and row reducing using $\mathbb{Z}_2$ arithmetic.

$$\left[\begin{array}{ccc|c} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{0} \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

From the above it is seen that we only have the trivial solution and the set is therefore linearly independent.

Question 4

- Find a basis for the column space, a basis for the null space and the rank of

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

To solve this problem we row reduce B

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Columns 1, 2 and 4 of the row reduced matrix contains pivots. The basis for col B therefore consists of these columns in B.

$$\text{basis for col } B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Rank $B = 3$. It is the number of pivots (or the number of vectors in the basis for the column space). To find a basis for the null space we solve $B\mathbf{x} = \mathbf{0}$

$$\left[\ B \mid \mathbf{0}\ \right] = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

We see that x_3 is a free variable. The equations corresponds to

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 + x_3 &= 0 \\ x_3 &\text{ is free} \\ x_4 &= 0 \end{aligned}$$

In parametric form we get (remember in Z_2 arithmetic, $x_1 + x_3 = 0 \iff x_1 = -x_3 \iff x_1 = x_3$).

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

The basis for null B therefore consist of the vector

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Which is easily checked by multiplication with B

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Question 5

- Encode the following messages using the Hamming (7,4) code.

  a) 1001

  b) 0011

  c) 0101

Encoding the messages corresponds to multiplying the message vector (as a 4×1 column vector) with the A matrix. The results are

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\0&1&1&1\\1&0&1&1\\1&1&0&1\end{bmatrix}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}=\begin{bmatrix}1\\0\\0\\1\\1\\0\\0\end{bmatrix},\quad \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\0&1&1&1\\1&0&1&1\\1&1&0&1\end{bmatrix}\begin{bmatrix}0\\0\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\1\\1\\0\\0\\1\end{bmatrix},\quad \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\0&1&1&1\\1&0&1&1\\1&1&0&1\end{bmatrix}\begin{bmatrix}0\\1\\0\\1\end{bmatrix}=\begin{bmatrix}0\\1\\0\\1\\0\\1\\0\end{bmatrix}.$$

Question 6

- Each of the following messages has been received, and each had been encoded using the Hamming (7,4) code. During transmission at most one element in the vector was changed. Either determine that no error was made in transmission, or find the error made in transmission and correct it.

  a) 0101101

  b) 1000011

  c) 0010111

  d) 0101010

  e) 0111100

  f) 1001101

  g) 1010010

  h) 1110111

This problem is solved by following the example in the note. Each vector is multiplied by H and the result is analyzed

$$\begin{bmatrix}0&0&0&1&1&1&1\\0&1&1&0&0&1&1\\1&0&1&0&1&0&1\end{bmatrix}\begin{bmatrix}0\\1\\0\\1\\1\\0\\1\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}.$$

The output is recognized as $\mathbf{h}_4$. Therefore the fourth bit of 0101101 is wrong and the vector should rightfully read 0100101. As a check

$$\begin{bmatrix}0&0&0&1&1&1&1\\0&1&1&0&0&1&1\\1&0&1&0&1&0&1\end{bmatrix}\begin{bmatrix}0\\1\\0\\{\color{red}0}\\1\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

The computations for the remaining vectors are done similarly. The results are:

b)	1000011	no error
c)	0010111	error i bit 7
d)	0101010	no error
e)	0111100	no error
f)	1001101	error in bit 7
g)	1010010	error in bit 4
h)	1110111	error in bit 4