Chapter 6.1

1. Since $\mathbf{u}=\begin{bmatrix}-1\\2\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}4\\6\end{bmatrix}$, $\mathbf{u}\cdot\mathbf{u}=(-1)^2+2^2=5$, $\mathbf{v}\cdot\mathbf{u}=4(-1)+6(2)=8$, and $\dfrac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}=\dfrac{8}{5}$.

13. Since $\mathbf{x}=\begin{bmatrix}10\\-3\end{bmatrix}$ and $\mathbf{y}=\begin{bmatrix}-1\\-5\end{bmatrix}$, $\|\mathbf{x}-\mathbf{y}\|^2=[10-(-1)]^2+[-3-(-5)]^2=125$ and dist $(\mathbf{x},\mathbf{y})=\sqrt{125}=5\sqrt{5}$.

16. Since $\mathbf{u}\cdot\mathbf{v}=12(2)+(3)(-3)+(-5)(3)=0$, $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

19. **a**. True. See the definition of $\|\mathbf{v}\|$.

b. True. See Theorem 1(c).

c. True. See the discussion of Figure 5.

d. False. Counterexample: $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$.

e. True. See the box following Example 6.

26. Theorem 2 in Chapter 4 may be used to show that W is a subspace of $\mathbb{R}^3$, because W is the null space of the 1×3 matrix $\mathbf{u}^T$. Geometrically, W is a plane through the origin.

30. **a**. If $\mathbf{z}$ is in $W^\perp$, $\mathbf{u}$ is in W, and c is any scalar, then $(c\mathbf{z})\cdot\mathbf{u}=c(\mathbf{z}\cdot\mathbf{u})=c0=0$. Since $\mathbf{u}$ is any element of W, $c\mathbf{z}$ is in $W^\perp$.

b. Let $\mathbf{z}_1$ and $\mathbf{z}_2$ be in $W^\perp$. Then for any $\mathbf{u}$ in W, $(\mathbf{z}_1+\mathbf{z}_2)\cdot\mathbf{u}=\mathbf{z}_1\cdot\mathbf{u}+\mathbf{z}_2\cdot\mathbf{u}=0+0=0$. Thus $\mathbf{z}_1+\mathbf{z}_2$ is in $W^\perp$.

c. Since $\mathbf{0}$ is orthogonal to every vector, $\mathbf{0}$ is in $W^\perp$. Thus $W^\perp$ is a subspace.

Chapter 6.2

1. Since $\begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix} = 2 \neq 0,$ the set is not orthogonal.

4. Since $\begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} = 0,$ the set is orthogonal.

7. Since $\mathbf{u}_1 \cdot \mathbf{u}_2 = 12 - 12 = 0$, $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set. Since the vectors are non-zero, $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent by Theorem 4. Two such vectors in $\mathbb{R}^2$ automatically form a basis for $\mathbb{R}^2$. So $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for $\mathbb{R}^2$. By Theorem 5,

$$\mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{x} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 = 3\mathbf{u}_1 + \frac{1}{2}\mathbf{u}_2$$

12. Let $\mathbf{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. The orthogonal projection of **y** onto the line through **u** and the origin is the orthogonal projection of **y** onto **u**, and this vector is $\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = -\frac{2}{5}\mathbf{u} = \begin{bmatrix} 2/5 \\ -6/5 \end{bmatrix}$.

13. The orthogonal projection of **y** onto **u** is $\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = -\frac{13}{65}\mathbf{u} = \begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix}$. The component of **y** orthogonal to **u** is $\mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 14/5 \\ 8/5 \end{bmatrix}$. Thus $\mathbf{y} = \hat{\mathbf{y}} + (\mathbf{y} - \hat{\mathbf{y}}) = \begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix} + \begin{bmatrix} 14/5 \\ 8/5 \end{bmatrix}$.

21. Let $\mathbf{u} = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{20} \\ -1/\sqrt{20} \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. Since $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w} = 0$, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is an orthogonal set. Also, $\|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u} = 1$, $\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v} = 1$, and $\|\mathbf{w}\|^2 = \mathbf{w} \cdot \mathbf{w} = 1$, so $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is an orthonormal set.

35. **[M]** One can compute that $A^T A = 100 I_4$. Since the off-diagonal entries in $A^T A$ are zero, the columns of A are orthogonal.

Chapter 6.3

1. The vector in $\text{Span}\{\mathbf{u}_4\}$ is $\dfrac{\mathbf{x}\cdot\mathbf{u}_4}{\mathbf{u}_4\cdot\mathbf{u}_4}\mathbf{u}_4 = \dfrac{72}{36}\mathbf{u}_4 = 2\mathbf{u}_4 = \begin{bmatrix}10\\-6\\-2\\2\end{bmatrix}$. Since

$\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + \dfrac{\mathbf{x}\cdot\mathbf{u}_4}{\mathbf{u}_4\cdot\mathbf{u}_4}\mathbf{u}_4$, the vector $\mathbf{x} - \dfrac{\mathbf{x}\cdot\mathbf{u}_4}{\mathbf{u}_4\cdot\mathbf{u}_4}\mathbf{u}_4 = \begin{bmatrix}10\\-8\\2\\0\end{bmatrix} - \begin{bmatrix}10\\-6\\-2\\2\end{bmatrix} = \begin{bmatrix}0\\-2\\4\\-2\end{bmatrix}$ is in

$\text{Span}\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$.

7. Since $\mathbf{u}_1\cdot\mathbf{u}_2 = 5+3-8=0$, $\{\mathbf{u}_1,\mathbf{u}_2\}$ is an orthogonal set. By the Orthogonal Decomposition Theorem, $\hat{\mathbf{y}} = \dfrac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \dfrac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 = 0\mathbf{u}_1 + \dfrac{2}{3}\mathbf{u}_2 = \begin{bmatrix}10/3\\2/3\\8/3\end{bmatrix}, \mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix}-7/3\\7/3\\7/3\end{bmatrix}$ and $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^{\perp}$.

13. Note that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. By the Best Approximation Theorem, the closest point in $\text{Span}\{\mathbf{v}_1,\mathbf{v}_2\}$ to $\mathbf{z}$ is $\hat{\mathbf{z}} = \dfrac{\mathbf{z}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 + \dfrac{\mathbf{z}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2 = \dfrac{2}{3}\mathbf{v}_1 - \dfrac{7}{3}\mathbf{v}_2 = \begin{bmatrix}-1\\-3\\-2\\3\end{bmatrix}$.

17. **a.** $U^TU = \begin{bmatrix}1&0\\0&1\end{bmatrix}$, $UU^T = \begin{bmatrix}8/9&-2/9&2/9\\-2/9&5/9&4/9\\2/9&4/9&5/9\end{bmatrix}$.

b. Since $U^TU = I_2$, the columns of U form an orthonormal basis for W, and by Theorem 10

$$\text{proj}_W\mathbf{y} = UU^T\mathbf{y} = \begin{bmatrix}8/9&-2/9&2/9\\-2/9&5/9&4/9\\2/9&4/9&5/9\end{bmatrix}\begin{bmatrix}4\\8\\1\end{bmatrix} = \begin{bmatrix}2\\4\\5\end{bmatrix}.$$

21. **a.** True. See the calculations for $\mathbf{z}_2$ in Example 1 or the box after Example 6 in Section 6.1.

b. True. See the Orthogonal Decomposition Theorem.

c. False. See the last paragraph in the proof of Theorem 8, or see the second paragraph after the statement of Theorem 9.

d. True. See the box before the Best Approximation Theorem.

e. True. Theorem 10 applies to the column space W of U because the columns of U are linearly independent and hence form a basis for W.