PROBLEM 1.

Consider the following matrix and vector

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 0 & 6 \\ -1 & 2 & -4 \\ 1 & -2 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ -12 \\ 12 \end{bmatrix}.$$

1. Solve $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$.
2. Is it possible to find a vector $\mathbf{b}$ so $A\mathbf{x} = \mathbf{b}$ cannot be solved?

PROBLEM 1. Solution

The problem is solved most easily by first considering $A\mathbf{x} = \mathbf{b}$ and solving this matrix equation using row reduction of the augmented matrix

$$\left[\, A \mid \mathbf{b} \,\right] = \left[\begin{array}{ccc|c} 2 & 1 & 3 & 4 \\ 3 & 0 & 6 & 12 \\ -1 & 2 & -4 & -12 \\ 1 & -2 & 4 & 12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The system is consistent and contains a free variable, x_3. The general solution is written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

The solution of $A\mathbf{x} = \mathbf{0}$ can be found directly from the above as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

As the solution of $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$ are identical, with the exception that the non-zero $\mathbf{b}$ translates the solution the $A\mathbf{x} = \mathbf{0}$ as discussed in chapter 1.5.

It is indeed possible to find a $\mathbf{b}$ so $A\mathbf{x} = \mathbf{b}$ can't be solved. The A matrix is 4×3. As seen from the above row reduction the matrix contains only two pivots, hence the columns span a 2-dimensional subspace of $\mathbb{R}^4$. Any vector $\mathbf{b}$ that is not an element of this subspace will make $A\mathbf{x} = \mathbf{b}$ inconsistent. One example of such a vector is

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Which is easily verified by row reduction.

PROBLEM 2.

Let four vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{b}$ be given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 7 \\ 2 \\ 7 \end{bmatrix}.$$

1. Show that the three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for $\mathbb{R}^3$.
2. Express $\mathbf{b}$ in the new basis.

PROBLEM 2. Solution

The three vectors form a basis if they are linearly independent and span $\mathbb{R}^3$. To check this, the solutions of the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ is computed by writing up a augmented matrix and row reducing.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ -3 & 2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

As we find only the trivial solution, the vectors are linearly independent. Further, the augmented matrix has pivots in all rows and and hence the vectors span $\mathbb{R}^3$. Therefore, the three vectors form a basis for R^3.

The vector $\mathbf{b}$ is expressed in the new basis by finding the weigths c_1, c_2 and c_3 in the vector equation $c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3 = \mathbf{b}$. Again the solutin can be found by a row reduction

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & 1 & 2 \\ -3 & 2 & 2 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{array}\right].$$

Thus $c_1 = -1$, $c_2 = -2$ and $c_3 = 4$. Therefore, the vector $\mathbf{b}$ in the new basis is given by

$$[\mathbf{b}]_{\mathbb{B}} = \begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix}.$$

PROBLEM 3.

Consider the following matrix and vector

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -1 & 0 & 1 \\ 2 & -2 & 2 \\ 1 & 2 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -2 \\ 3 \\ 0 \\ -5 \end{bmatrix}.$$

1. Show that the matrix equation $A\mathbf{x} = \mathbf{b}$ is inconsistent.
2. Find the least-squares solution of $A\mathbf{x} = \mathbf{b}$.

In a number of applications, sparse solutions, i.e. solutions where most elements are zero, are desired. Consider the two sparse vectors

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathbf{x}_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}.$$

3. Determine whether $\mathbf{x}_1$ or $\mathbf{x}_2$ is the better solution of $A\mathbf{x} = \mathbf{b}$ in the least-squares sense.

PROBLEM 3. Solution

The consistency of $A\mathbf{x} = \mathbf{b}$ is checked by row reduction of the augmented matrix

$$\left[\, A \mid \mathbf{b} \,\right] = \left[\begin{array}{ccc|c} 1 & 3 & 4 & -2 \\ -1 & 0 & 1 & 3 \\ 2 & -2 & 2 & 0 \\ 1 & 2 & -1 & -5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

From the last row it is evident that the matrix equation is inconsistent.

The least-squares solution is computed by $\hat{x} = (A^T A)^{-1} A^T \mathbf{b}$. This is conviniently done with MATLAB and the result is

$$\hat{\mathbf{x}} = \begin{bmatrix} -1.9978 \\ -1.1242 \\ 0.8514 \end{bmatrix}.$$

The best solution is found by computing the norms $\|A\mathbf{x}_1 - \mathbf{b}\|$ and $\|A\mathbf{x}_2 - \mathbf{b}\|$. Using the `norm` command in MATLAB the numbers become

$$\|A\mathbf{x}_1 - \mathbf{b}\| = 10.2956 \quad \text{and} \quad \|A\mathbf{x}_2 - \mathbf{b}\| = 13.0384$$

From which it is seen that $\mathbf{x}_1$ is the better "solution" of $A\mathbf{x} = \mathbf{b}$.

PROBLEM 4.

For the statements given below, state whether they are true or false and justify your answer for each statement.

1. A square, upper-triangular matrix with non-zero elements on the diagonal is invertible.
2. If a matrix A has an eigenvalue λ, then $c\lambda$, with c a scalar, is also an eigenvalue.
3. The matrix $\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}$ is positive semidefinite.

PROBLEM 4. Solution

The first statement is **true**. As an example consider a 4×4 upper triangular matrix with non-zero diagonal elements. Such a matrix looks like

$$\begin{bmatrix} x & y & y & y \\ 0 & x & y & y \\ 0 & 0 & x & y \\ 0 & 0 & 0 & x \end{bmatrix},$$

where the x's denotes non-zero values and the y's can take on any numerical. Clearly, the matrix is row equivalent to the identity matrix and therefore invertible. The example given here uses a 4×4 matrix, but the argument holds for any upper triangular $n \times n$ matrix.

The second statement is **false**. Eigenvalues are unique. It it eigenvectors that are scalable, i.e. if $\mathbf{v}$ is a eigenvector of A, the $c\mathbf{v}$ is also an eigenvector of A corresponding to the same eigenvalue.

The third statement is **false**. If the eigenvalues are computed, e.g. with `eig` in MATLAB the result is $\lambda_1 = -6$ and $\lambda_2 = 0$. Therefore, the correct classification of the matrix is *negative semidefinite.*

PROBLEM 5.

In the case *Computer Graphics in Automotive Design*, homogeneous coordinates and rotation matrices were introduced. In this problem we are not concerned with homogeneous coordinates, but only work with standard coordinates. In $\mathbb{R}^2$ a rotation of the vector $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ by an angle, θ about the origin is obtained by multiplying the following rotation matrix with $\mathbf{x}$.

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

1. Show that $R(\theta)$ is an orthogonal matrix.
2. Argue that $R(2\theta) = R^2(\theta)$.
3. Compute $R^2(\theta)$ and use this result to find formulas for $\cos(2\theta)$ and $\sin(2\theta)$ expressed by $\cos\theta$ and $\sin\theta$.

PROBLEM 5. Solution

The matrix is orthogonal i.e. has orthonormal columns, if $R(\theta)^T R(\theta) = I$. To check this we compute

$$\begin{aligned} R(\theta)^T R(\theta) &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{bmatrix} \end{aligned}$$

When the identity $\cos^2\theta + \sin^2\theta = 1$ is applied, it is seen that $R(\theta)$ is indeed an orthogonal matrix.

$R(2\theta)$ will rotate the vector $\mathbf{x}$ through the angle 2θ. Obviously, this is the same as rotating $\mathbf{x}$ two times through the angle θ, therefore $R(2\theta) = R^2(\theta)$.

From the definition of the rotation matrix we have

$$R(2\theta) = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}.$$

$R^2(\theta)$ is computed as

$$\begin{aligned} R(\theta)R(\theta) &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \end{aligned}$$

By equating the elements of $R(2\theta)$ and $R^2(\theta)$ we find that

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

and

$$\sin 2\theta = 2\sin\theta\cos\theta$$

PROBLEM 6.

A special and somewhat rare class of square matrices are called skew-symmetric. A general 3×3 skew-symmetric matrix has this form

$$\begin{bmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{bmatrix},$$

where a_1, a_2 and a_3 are scalars. One particular use of 3×3 skew-symmetric matrices sometimes encountered in mechanical engineering is as a way of expressing the vector cross product as a matrix multiplication.

1. For a symmetric matrix $A = A^T$. What is the corresponding relation for skew-symmetric matrices?

Consider the set of all 3×3 skew-symmetric matrices, here denoted as $\mathbb{S}^{3\times 3}$.

2. Show that $\mathbb{S}^{3\times 3}$ forms a vector space.

PROBLEM 6. Solution

By transposing the skew-symmetric matrix we get

$$\begin{bmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & -a_1 & -a_2 \\ a_1 & 0 & -a_3 \\ a_2 & a_3 & 0 \end{bmatrix} = - \begin{bmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{bmatrix}$$

Hence $A = -A^T$ for a skew-symmetric matrix.

To show that $\mathbb{S}^{3\times 3}$ forms a vector space, it suffices to show that $\mathbb{S}^{3\times 3}$ contains the zero vector and is closed under multiplication and addition.

The matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is clearly a skew-symmetric 3×3 matrix and hence the set contains a zero vector.

An arbitrary vector in $\mathbb{S}^{3\times 3}$ is given by

$$A = \begin{bmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{bmatrix}$$

Multiplying such a matrix by a scalar α gives

$$\alpha A = \begin{bmatrix} 0 & \alpha a_1 & \alpha a_2 \\ -\alpha a_1 & 0 & \alpha a_3 \\ -\alpha a_2 & -\alpha a_3 & 0 \end{bmatrix}$$

which is still an element of $\mathbb{S}^{3\times 3}$, thus the set is closed under multiplication.

Adding two skew-symmetric 3×3 matrices gives

$$A + B = \begin{bmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b_1 & b_2 \\ -b_1 & 0 & b_3 \\ -b_2 & -b_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a_1 + b_1 & a_2 + a_2 \\ -(a_1 + b_1 & 0 & a_3 + b_3 \\ -(a_2 + b_2) & -(a_3 + b_3) & 0 \end{bmatrix}$$

which is again a skew-symmetric 3×3 matrix and the set is therefore also closed under addition. As all three rules are fulfilled, it can be concluded that $\mathbb{S}^{3 \times 3}$ forms a vector space.