Lesson 1

Chapter 1
Linear equations in Linear Algebra

- Systems of Linear Equations
- Row Reduction and Echelon Forms
- Vector Equations
- The Matrix Equation $A\mathbf{x} = \mathbf{b}$
- Solution Sets of Linear Systems
- Linear Independence

1.1 Systems of Linear Equations

$$a_{11} \cdot x_1 + a_{12} \cdot x_2 + a_{13} \cdot x_3 + \cdots + a_{1n} \cdot x_n = b_1$$
$$a_{21} \cdot x_1 + a_{22} \cdot x_2 + a_{23} \cdot x_3 + \cdots + a_{2n} \cdot x_n = b_2$$
$$a_{31} \cdot x_1 + a_{32} \cdot x_2 + a_{33} \cdot x_3 + \cdots + a_{3n} \cdot x_n = b_3$$

…

…

$$a_{m1} \cdot x_1 + a_{m2} \cdot x_2 + a_{m3} \cdot x_3 + \cdots + a_{mn} \cdot x_n = b_m$$

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Quiz: Svar A, B, C, D, E – kan ikke fortrydes!

Ex.1

L1: $2x + y = 60$

L2: $x + 2y = 75$

Gradbøjning:
en matrix, matricen, flere matricer, alle matricerne

Notation

$$A\mathbf{x} = \lambda\mathbf{x} \text{ or } A\vec{x} = \lambda\vec{x}$$

- matrices: CAPITAL LETTERS
- vectors: **bold** or arr$\vec{\text{o}}$w
- scalars: standard letters

Matrix-element

$m \times n$ matrix: m rows, n columns $\qquad A = [a_{ij}]$

Matrix size

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

1.2 Row Reduction and Echelon Forms

$$\begin{bmatrix} 2 & -3 & 5 & 0 & 2 \\ 0 & -3 & -1 & 4 & -8 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$$

Rækkeoperationer

Elementary row operations:

1. Replacement: Replace one row by the sum of itself and a multiple of another row: $a_{ri} \rightarrow \ a_{ri} + \alpha \cdot a_{si};\ \ \mathrm{i} = 1, \ldots, \mathrm{n}$

2. Interchange: Interchange two rows: $a_{ri} \leftrightarrow a_{si}\ , \mathrm{i} = 1, \ldots, \mathrm{n}$

3. Scaling: Multiply all entries in a row by a nonzero constant:

$$a_{ri} \rightarrow \ \alpha \cdot a_{ri};\ \mathrm{i} = 1, \ldots, \mathrm{n};\ \ \alpha \neq 0$$

Often 1. + 3. are used simultaneously:

Replace one row by the sum of a nonzero multiple of itself and a multiple of another row:

$$a_{ri} \rightarrow \ \alpha \cdot a_{ri} + \beta \cdot a_{si};\ \ \mathrm{i} = 1, \ldots, \mathrm{n};\ \alpha \neq 0$$

Rækkeækvivalente

Two matrices are called row equivalent if there is a sequence of elementary row operations that transforms one matrix into the other.
And visa versa – row operations are reversible!

Augmenterede / Udvidede

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Two fundamental questions about a linear system:

1. Is the system consistent → does at least one solution exist?
2. If a solution exist, is it the only one → is the solution unique?

Ex.2

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

$$\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**)

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

$$\begin{bmatrix} 1 & 0 & * & 0 & 0 \\ 0 & 1 & * & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Theorem 1.1 (Uniqueness of the Reduced Echelon Form)

Each matrix is row equivalent to one and only one reduced echelon matrix

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column in A that contains a pivot position.

<u>Original matrix</u>

Pivot positions

<u>Reduced echelon form</u>

Pivot

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot colomns

Pivot colomns

Row Reduction Algorithm

Foreward phase:

1. Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.
2. Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.
3. Use row replacement operations to create zeros in alle positions below the pivot.
4. Ignore the row containing the pivot position and all rows above it. Apply step 1-3 to the submatrix that remains. Repeat the proces until there are no more nonzero rows to modify.

→ The matrix is now in echelon form

Backward phase:

5. Beginning with rightmost pivot and workning upward and to the left, create zeros above each pivot by row replacement operations. If a pivot is not 1, make it 1 by a scaling operation.

→ The matrix is now in reduced echelon form

Theorem 1.2: Existence an Uniqueness Theorem

A linear system is **consistent** if and only if **the rightmost column of the augmented matrix is not a pivot column** – that is, if and only if an echelon form of the augmented matrix has not rows of the form:

$$[0 \quad \dots \quad 0 \quad b] \text{ with } b \neq 0$$

If a linear system is consistent, then the solution set contains either:

i. **A unique solution**, when there are no free variables – that is, **all except the last column are pivot columns**.
ii. **Infinitely many solutions**, when there is a least one free variable – that is, **at least one column besides the last one is not a pivot column**.

1.3 Vector Equations

$$a_1 \cdot \boldsymbol{v_1} + a_2 \cdot \boldsymbol{v_2} + a_3 \cdot \boldsymbol{v_3} + a_4 \cdot \boldsymbol{v_4} = \boldsymbol{b}$$

$$c_1 \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + c_2 \cdot \begin{bmatrix} -4 \\ 2 \\ 2 \end{bmatrix} + c_3 \cdot \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 2 \end{bmatrix}$$

Matrix with only one column (m x 1 matrix) = (column) vector

$$\boldsymbol{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} = (3, -1) \in \mathbb{R}^2$$

OBS: $\begin{bmatrix} 3 \\ -1 \end{bmatrix} \neq \begin{bmatrix} 3 & -1 \end{bmatrix}$

$$\boldsymbol{v} = \begin{bmatrix} -2 \\ 3 \\ 8 \end{bmatrix} = (-2, 3, 8) \in \mathbb{R}^3$$

$$\boldsymbol{w} = \begin{bmatrix} 5 \\ -2 \\ \vdots \\ 4 \end{bmatrix} = (5, -2, \cdots, 4) \in \mathbb{R}^n$$

Vector algebra

$\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$, c and d are scalars

Commutative $$\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u} \tag{1}$$

Associative $$(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w}) \tag{2}$$

Zero vector $$\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u} \tag{3}$$

$$\mathbf{u}+(-\mathbf{u})=-\mathbf{u}+\mathbf{u}=\mathbf{0}, \qquad -\mathbf{u}=(-1)\mathbf{u} \tag{4}$$

$$c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v} \tag{5}$$

$$(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u} \tag{6}$$

$$c(d\mathbf{u})=(cd)\mathbf{u} \tag{7}$$

$$1\mathbf{u}=\mathbf{u} \tag{8}$$

Linear combinations of vectors

If $\boldsymbol{v}_i \in \mathbb{R}^n$ and c_i are scalars for $i = 1, \cdots, p$, then

$$\boldsymbol{y} = c_1 \cdot \boldsymbol{v}_1 + c_2 \cdot \boldsymbol{v}_2 + \cdots + c_p \cdot \boldsymbol{v}_p$$

Is called a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_p$ with weights $c_1, c_2, \cdots, c_p$

Ex.3

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \in \mathbb{R}^3 \qquad \boldsymbol{v}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix} \in \mathbb{R}^3 \qquad \boldsymbol{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix} \in \mathbb{R}^3$$

Linear combination: $$\boldsymbol{y} = 2\boldsymbol{v}_1 - 3\boldsymbol{v}_2 = 2\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} - 3\begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix} + \begin{bmatrix} -15 \\ 39 \\ 9 \end{bmatrix} = \begin{bmatrix} 2-15 \\ -4+39 \\ 6+9 \end{bmatrix} = \begin{bmatrix} -13 \\ 35 \\ 15 \end{bmatrix} \in \mathbb{R}^3$$

Vector equation: $$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 = \boldsymbol{b} \Leftrightarrow \begin{bmatrix} c_1 + 5c_2 \\ -2c_1 - 13c_2 \\ 3c_1 - 3c_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix} \Leftrightarrow \begin{cases} c_1 + 5c_2 = -3 \\ -2c_1 - 13c_2 = 8 \\ 3c_1 - 3c_2 = 1 \end{cases} \longleftarrow$$ *System of Linear Equations*

$$\Leftrightarrow \begin{array}{c} \begin{bmatrix} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{b} \end{matrix} \end{array} \sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & -18 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & -13 & 2 \\ 0 & 0 & -2 \end{bmatrix} \longleftarrow$$ *No solutions*

Vector equation

A vector equation

$$c_1 \cdot \boldsymbol{v}_1 + c_2 \cdot \boldsymbol{v}_2 + \cdots + c_p \cdot \boldsymbol{v}_p = \boldsymbol{b}$$

have the same solution as the linear system whose augmented matrix is

$$[\boldsymbol{v}_1 \quad \boldsymbol{v}_2 \quad \cdots \quad \boldsymbol{v}_p \quad \boldsymbol{b}]$$

A vector $\boldsymbol{b}$ can be generated as a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_p$ if and only if there exist solutions to the linear system corresponding to the above matrix.

Span/Udspænd

Definition of Span

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned** (or **generated**) **by $\mathbf{v}_1, \ldots, \mathbf{v}_p$** . That is $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

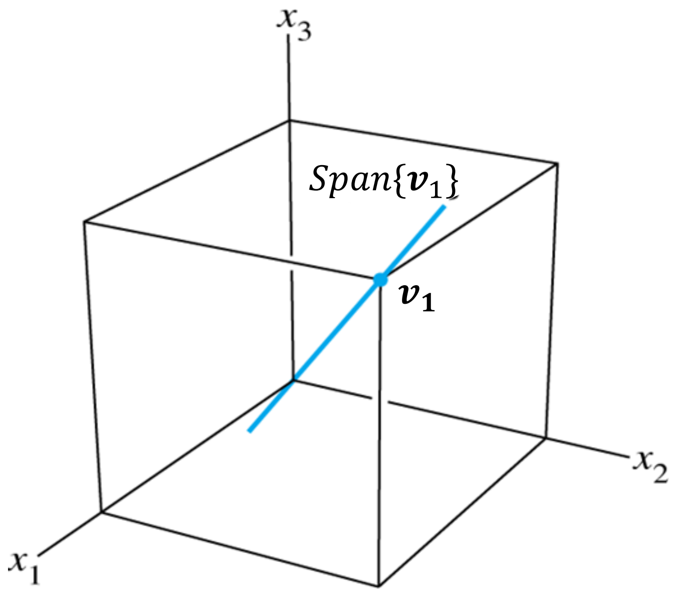
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p$$

with $c_1, \ldots, c_p$ scalars.

$\boxed{c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 = \boldsymbol{b} \;\Leftrightarrow\; \boldsymbol{b} \in Span\{\boldsymbol{v}_1, \boldsymbol{v}_2\} \;\Leftrightarrow \text{Solution to Linear Equation System}}$

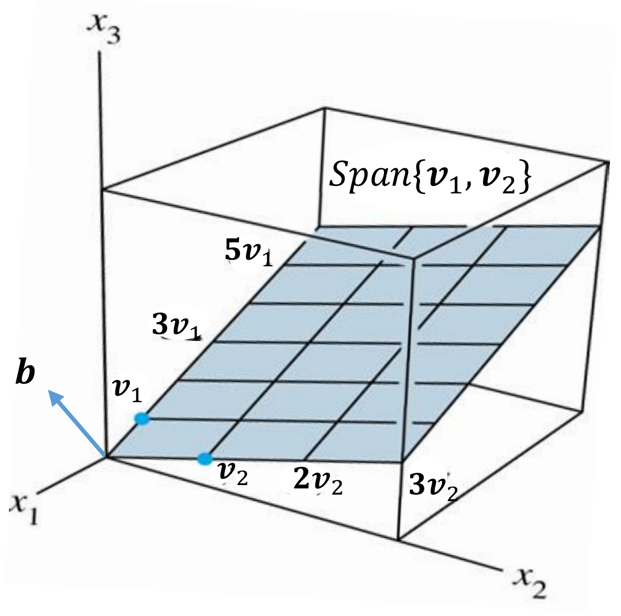
Geometric Description of Span

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \qquad Span\{\boldsymbol{v}_1\} = \text{Straight line in } \mathbb{R}^3$$



$$\boldsymbol{v}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix} \qquad Span\{\boldsymbol{v}_1, \boldsymbol{v}_2\} = \text{Plane in } \mathbb{R}^3$$

$$\boldsymbol{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix} \qquad Span\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{b}\} = \mathbb{R}^3$$



Todays words and concepts

Rækkeoperationer

Matrixelement

Lineære ligningssystemer

Rækkeækvivalent

Rækkereduktion

Augmented/Udvidet matrix

Vektor

Echelon

Reduceret echelon

Pivot-positioner

Matrix

Pivot

Pivot-søjler

Ledende indgang / Leading Entry

Linear kombination

Vektorligning

Span

Eksistens / Consistency

Entydighed / Uniqueness