Chapter 6.7

1. The inner product is $\langle x, y\rangle = 4x_1y_1 + 5x_2y_2$. Let $\mathbf{x} = (1, 1)$, $\mathbf{y} = (5, -1)$.

a. Since $\|\mathbf{x}\|^2 = \langle x, x\rangle = 9$, $\|\mathbf{x}\| = 3$. Since $\|\mathbf{y}\|^2 = \langle y, y\rangle = 105$, $\|\mathbf{y}\| = \sqrt{105}$. Finally, $|\langle x, y\rangle|^2 = 15^2 = 225$.

b. A vector **z** is orthogonal to **y** if and only if $\langle x, y\rangle = 0$, that is, $20z_1 - 5z_2 = 0$, or $4z_1 = z_2$. Thus all multiples of $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ are orthogonal to **y**.

3. The inner product is $\langle p, q\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$, so $\langle 4+t, 5-4t^2\rangle = 3(1) + 4(5) + 5(1) = 28$.

21. The inner product is $\langle f, g\rangle = \int_0^1 f(t)g(t)dt$. Let $f(t) = 1-3t^2$, $g(t) = t - t^3$. Then $\langle f, g\rangle = \int_0^1 (1-3t^2)(t-t^3)\,dt = \int_0^1 3t^5 - 4t^3 + t\,dt = 0$.

23. The inner product is $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$, so $\langle f, f\rangle = \int_0^1 (1-3t^2)^2\,dt = \int_0^1 9t^4 - 6t^2 + 1\,dt = 4/5$, and $\|f\| = \sqrt{\langle f, f\rangle} = 2/\sqrt{5}$.

25. The inner product is $\langle f, g\rangle = \int_{-1}^1 f(t)g(t)dt$. Then 1 and *t* are orthogonal because $\langle 1, t\rangle = \int_{-1}^1 t\,dt = 0$. So 1 and *t* can be in an orthogonal basis for $\text{Span}\{1, t, t^2\}$. By the Gram-Schmidt process, the third basis element in the orthogonal basis can be $t^2 - \frac{\langle t^2, 1\rangle}{\langle 1, 1\rangle}1 - \frac{\langle t^2, t\rangle}{\langle t, t\rangle}t$. Since $\langle t^2, 1\rangle = \int_{-1}^1 t^2 dt = 2/3$, $\langle 1, 1\rangle = \int_{-1}^1 1\,dt = 2$, and $\langle t^2, t\rangle = \int_{-1}^1 t^3 dt = 0$, the third basis element can be written as $t^2 - (1/3)$. This element can be scaled by 3, which gives the orthogonal basis as $\{1, t, 3t^2 - 1\}$.

Chapter 6.8

1. The weighting matrix W, design matrix X, parameter vector β, and observation vector $\mathbf{y}$ are:

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, X = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 4 \\ 4 \end{bmatrix}.$$

The design matrix X and the observation vector $\mathbf{y}$ are scaled by W:

$$WX = \begin{bmatrix} 1 & -2 \\ 2 & -2 \\ 2 & 0 \\ 2 & 2 \\ 1 & 2 \end{bmatrix}, W\mathbf{y} = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 8 \\ 4 \end{bmatrix}.$$

Further compute . $(WX)^T WX = \begin{bmatrix} 14 & 0 \\ 0 & 16 \end{bmatrix}, (WX)^T W\mathbf{y} = \begin{bmatrix} 28 \\ 24 \end{bmatrix}$ and find that

$\hat{\beta} = ((WX)^T WX)^{-1}(WX)^T W\mathbf{y} = \begin{bmatrix} 1/14 & 0 \\ 0 & 1/16 \end{bmatrix}\begin{bmatrix} 28 \\ 24 \end{bmatrix} = \begin{bmatrix} 2 \\ 3/2 \end{bmatrix}$. Thus the weighted least-squares line is $y = 2 + (3/2)x$.

2. Let X be the original design matrix, and let $\mathbf{y}$ be the original observation vector. Let W be the weighting matrix for the first method. Then $2W$ is the weighting matrix for the second method. The weighted least-squares by the first method is equivalent to the ordinary least-squares for an equation whose normal equation is $(WX)^T WX\hat{\beta} = (WX)^T W\mathbf{y}$, while the second method is equivalent to the ordinary least-squares for an equation whose normal equation is $(2WX)^T(2W)X\hat{\beta} = (2WX)^T(2W)\mathbf{y}$. Since the second equation can be written as $4(WX)^T WX\hat{\beta} = 4(WX)^T W\mathbf{y}$, it has the same solutions as the first equation).

10. Let $f(t) = \begin{cases} 1 & \text{for } 0 \le t < \pi \\ -1 & \text{for } \pi \le t < 2\pi \end{cases}$. The Fourier coefficients for f are:

$\frac{a_0}{2} = \frac{1}{2}\frac{1}{\pi}\int_0^{2\pi} f(t)\,dt = \frac{1}{2\pi}\int_0^{\pi} dt - \frac{1}{2\pi}\int_{\pi}^{2\pi} dt = 0$, and for $k > 0$,

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos kt\,dt = \frac{1}{\pi}\int_0^{\pi}\cos kt\,dt - \frac{1}{\pi}\int_{\pi}^{2\pi}\cos kt\,dt = 0 \text{ and}$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin kt\,dt = \frac{1}{\pi}\int_0^{\pi}\sin kt\,dt - \frac{1}{\pi}\int_{\pi}^{2\pi}\sin kt\,dt = \begin{cases} 4/(k\pi) & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even} \end{cases}.$$

The third-order Fourier approximation to *f* is thus $b_1 \sin t + b_3 \sin 3t = \frac{4}{\pi}\sin t + \frac{4}{3\pi}\sin 3t$.

13. Let *f* and *g* be in $C\,[0, 2\pi]$ and let *m* be a nonnegative integer. Then the linearity of the inner product shows that $\langle (f+g), \cos mt\rangle = \langle f, \cos mt\rangle + \langle g, \cos mt\rangle$ and $\langle (f+g), \sin mt\rangle = \langle f, \sin mt\rangle + \langle g, \sin mt\rangle$.

Dividing these identities respectively by $\langle \cos mt, \cos mt\rangle$ and $\langle \sin mt, \sin mt\rangle$ shows that the Fourier coefficients a_m and b_m for $f + g$ are the sums of the corresponding Fourier coefficients of *f* and of *g*.