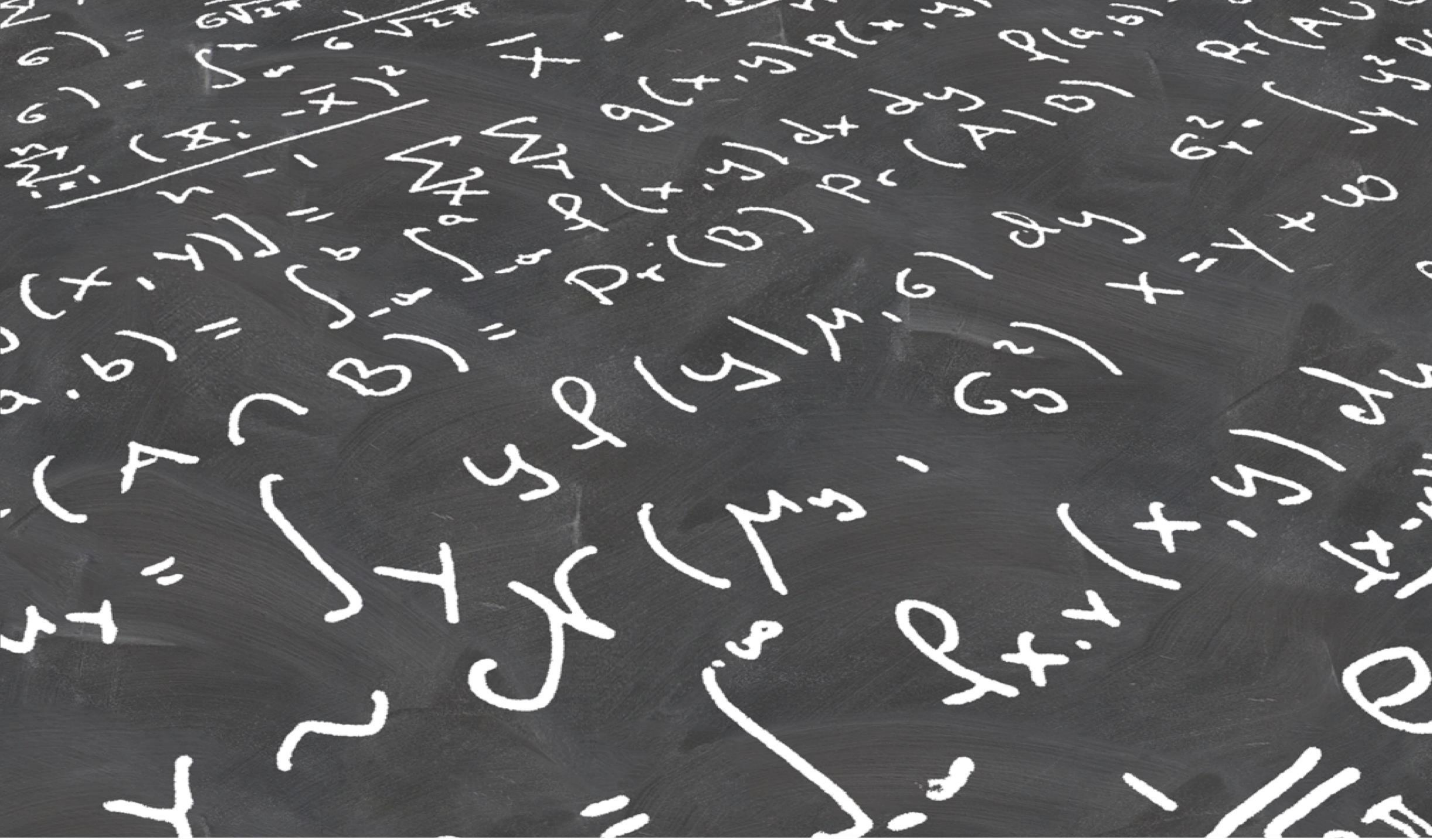

2. Probability Theory and Combinatorics

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Agenda for Today

- Repetition from last time
- Bayesian probability calculations and total probability
- Bernoulli trials
- Combinatorics
- An experiment

Basic Probability

- Probability theory tells us what is in the sample given nature

- Basic Axions:

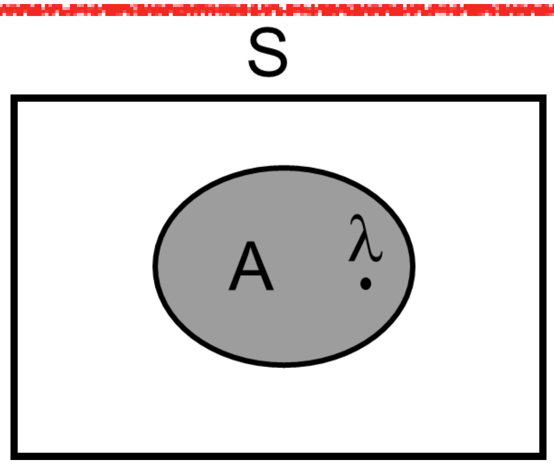
Axion 1: $0 \le Pr(\mathrm{A}) \le 1$

Axion 2: $Pr(S) = 1$

Axion 3: If $A_1 \cap A_2 = \emptyset$ then

$$Pr(A_1 \cup A_2) = Pr(A_1) + Pr(A_2)$$

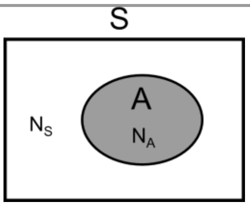
S: Sample space
A: Event
λ: Sample point



- Cardinality (size): $|A| = N_A \rightarrow \begin{cases} \text{Finite/Infinite} \\ \text{Countable/Uncountable} \end{cases}$

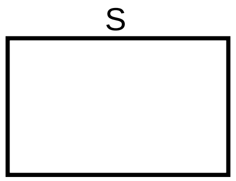
Basic Probability

- Relative frequency: $Pr(A) = \frac{N_A}{N_S} = \frac{|A|}{|S|}$

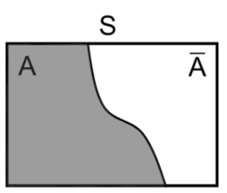


- The certain/universal set S: $Pr(S) = 1$

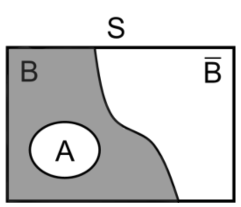
- The empty/null set ∅ : $|\emptyset| = 0$; $Pr(\emptyset) = 0$



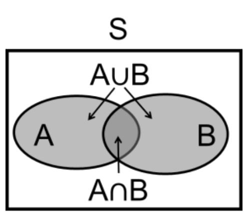
- Complement: $Pr(\bar{A}) = 1 - Pr(A)$



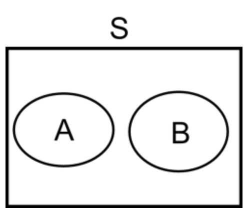
- Exclusive: $Pr(\bar{A} \cap B) = Pr(B) - Pr(A)$ if $A \subset B$



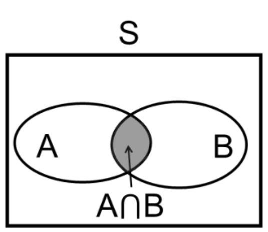
- Union: $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$



- Mutually Exclusive: $Pr(A \cap B) = 0$



- Independence: $Pr(A \cap B) = Pr(A) \cdot Pr(B)$



Conditional Probability *(Betingede sandsynligheder)*

- We write a conditional probability as:

$$\boxed{Pr(A|B)}$$

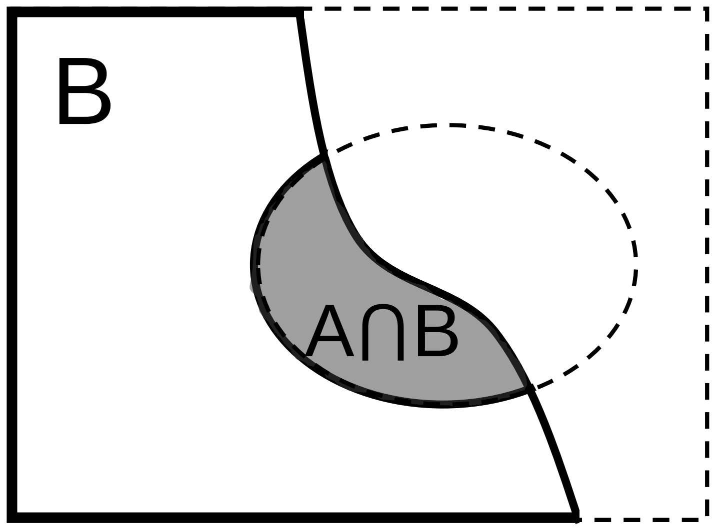
"A given B"

- This means that if the event B has already happened, what is the probability of the event A.
- Reduction of the sample space (possible events) from S to B

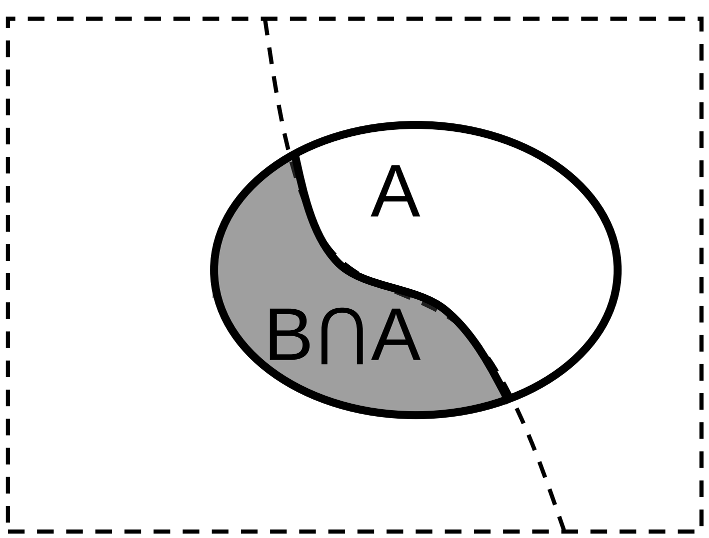
Example:

- From a population, I have selected a female.
- What is the chance that the selected person is below 1.6 m in height?

Conditional Probabilities – Bayes Rule



$$\Pr(A|B) = \frac{N_{A\cap B}}{N_B} = \frac{N_{A\cap B}/N_S}{N_B/N_S} = \frac{\Pr(A\cap B)}{\Pr(B)}$$



$$\Pr(B|A) = \frac{N_{B\cap A}}{N_A} = \frac{N_{B\cap A}/N_S}{N_A/N_S} = \frac{\Pr(B\cap A)}{\Pr(A)}$$

Very important!

Bayes Rule

- We can write Bayes rule for two events as:

$$\boxed{Pr(A \cap B) = Pr(A|B) \cdot Pr(B) = Pr(B|A) \cdot Pr(A)}$$

or

$$\boxed{Pr(A|B) = \frac{Pr(B|A) \cdot Pr(A)}{Pr(B)} = \frac{Pr(A \cap B)}{Pr(B)}}$$

Bayes rule

Notice:

- We can extend this rule to multiple events:

$$\boxed{Pr(A \cap B \cap C) = Pr(C|A \cap B) \cdot Pr(B|A) \cdot Pr(A)}$$

- If A and B independent:

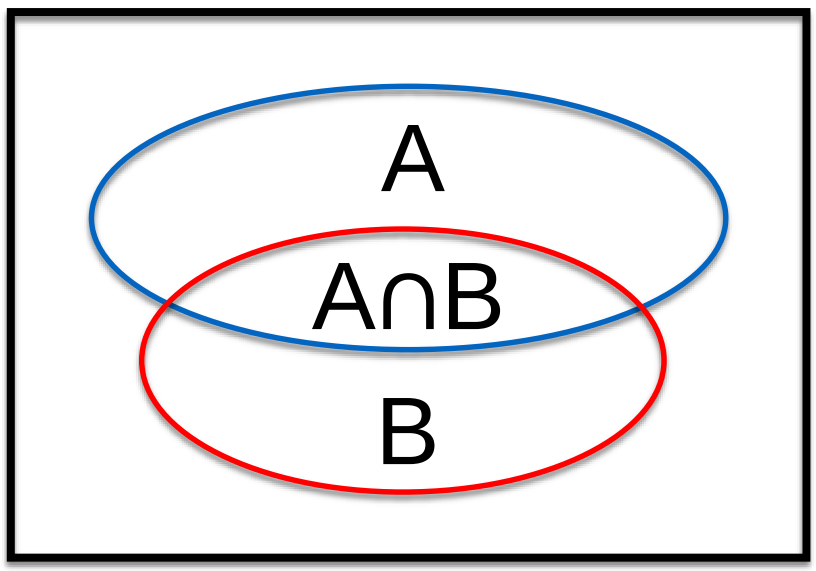
$$\boxed{Pr(A|B) = Pr(A) \quad \text{and} \quad Pr(B|A) = Pr(B)}$$

- <u>Joint</u> events are not the same as <u>conditional</u> events

$(A \cap B)$ $\qquad\qquad$ $(A|B)$

Conditional Probabilities – Café Example

Café



A = Coffee B = Cake

$$\Pr(A) = 0{,}70$$

$$\Pr(B) = 0{,}40$$

$$\Pr(A \cap B) = 0{,}20$$

Both coffee <u>and</u> cake

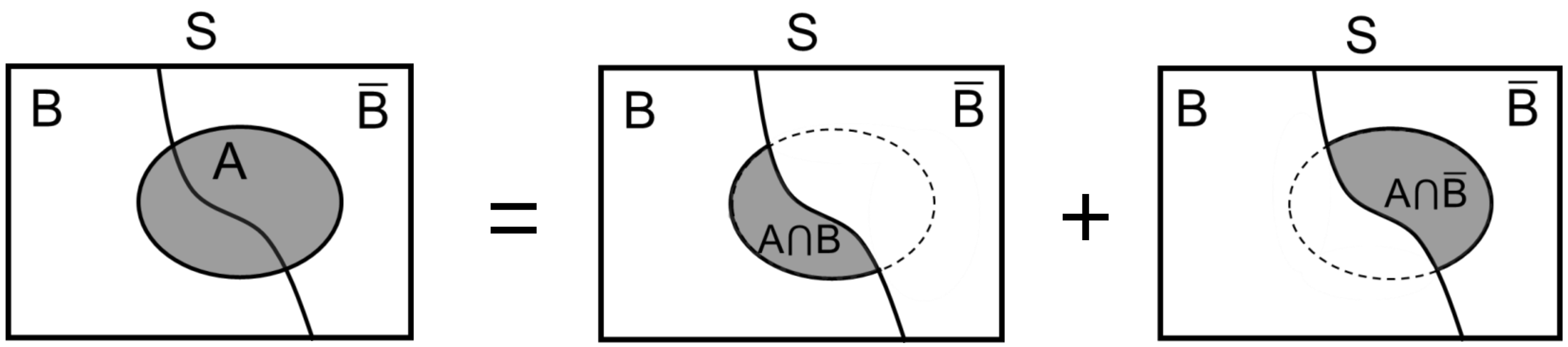
$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)} = \frac{0{,}20}{0{,}40} = \frac{1}{2}$$

Coffee <u>given</u> cake

$$Pr(B|A) = \frac{Pr(B \cap A)}{Pr(A)} = \frac{Pr(A|B) \cdot Pr(B)}{Pr(A)} = \frac{½ \cdot 0{,}40}{0{,}70} = \frac{2}{7} = 0{,}286$$

Cake <u>given</u> coffee

Conditional Probabilities – Total Probability



$$\Pr(A) = \frac{N_A}{N_S} = \frac{N_{A\cap B}}{N_S} + \frac{N_{A\cap\bar{B}}}{N_S} = \frac{N_{A\cap B}}{N_B}\cdot\frac{N_B}{N_S} + \frac{N_{A\cap\bar{B}}}{N_{\bar{B}}}\cdot\frac{N_{\bar{B}}}{N_S}$$

$$\boxed{\begin{aligned}\Pr(A) &= \Pr(A\cap B) + \Pr\left(A\cap\bar{B}\right)\\ &= \Pr(A|B)\cdot\Pr(B) + \Pr\left(A|\bar{B}\right)\cdot\Pr(\bar{B})\end{aligned}}$$

Conditional Probabilities - Example

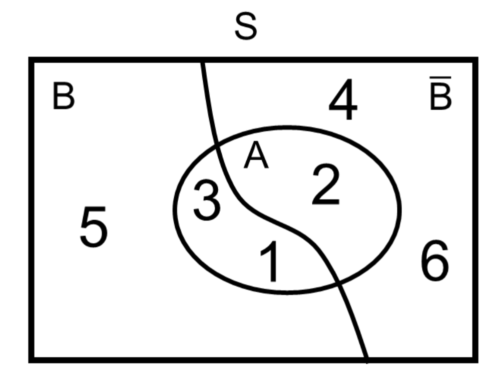
Rolling a dice:

Sample space: $S=\{1,2,3,4,5,6\}$

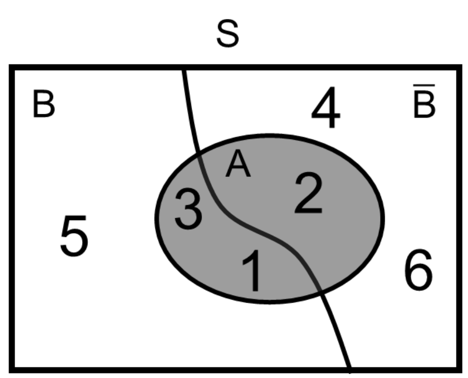
Events: $A=\{1,2,3\}$

$B=\{1,3,5\}$

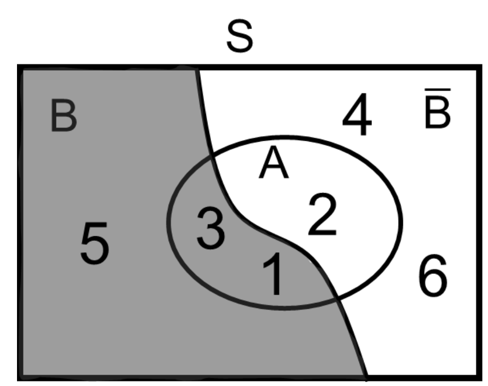
$\bar{B}=\{2,4,6\}$



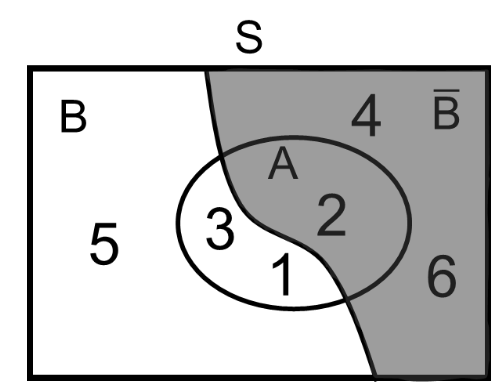
Venn diagram



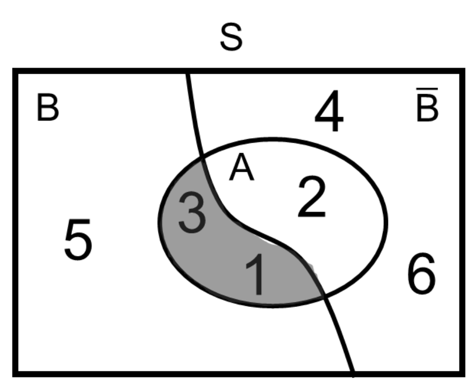
$$\Pr(A) = \frac{N_A}{N_S} = \frac{3}{6} = \frac{1}{2}$$



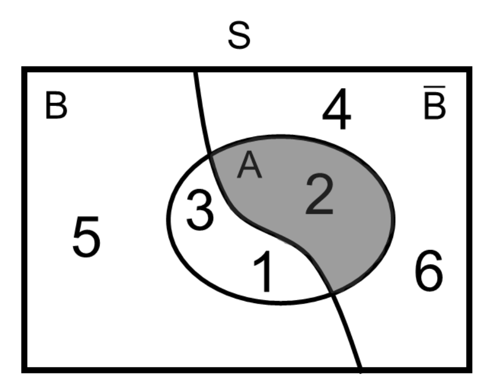
$$\Pr(B) = \frac{N_B}{N_S} = \frac{3}{6} = \frac{1}{2}$$



$$\Pr(\bar{B}) = \frac{N_{\bar{B}}}{N_S} = \frac{3}{6} = \frac{1}{2}$$

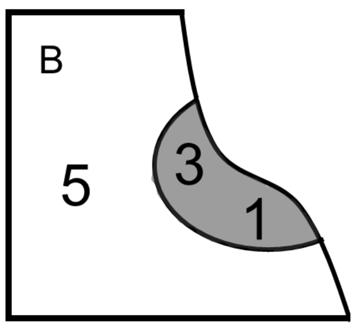


$$\Pr(A \cap B) = \frac{N_{A \cap B}}{N_S} = \frac{2}{6} = \frac{1}{3}$$



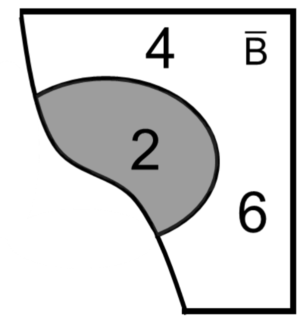
$$\Pr(A \cap \bar{B}) = \frac{N_{A \cap \bar{B}}}{N_S} = \frac{1}{6}$$

Conditional Probabilities - Example



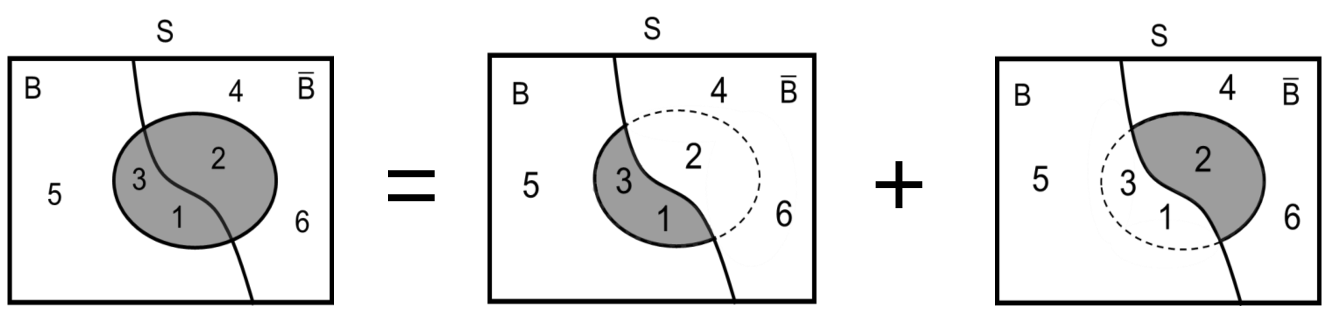
$$\Pr(A|B) = \frac{N_{A\cap B}}{N_B} = \frac{2}{3}$$

$$= \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{1/3}{1/2} = \frac{2}{3}$$



$$\Pr\left(A|\bar{B}\right) = \frac{N_{A\cap\bar{B}}}{N_{\bar{B}}} = \frac{1}{3}$$

$$= \frac{\Pr(A \cap \bar{B})}{\Pr(\bar{B})} = \frac{1/6}{1/2} = \frac{2}{6} = \frac{1}{3}$$



$$\Pr(A) = \frac{N_A}{N_S} = \frac{N_{A\cap B}}{N_S} + \frac{N_{A\cap\bar{B}}}{N_S} = \frac{N_{A\cap B}}{N_B}\cdot\frac{N_B}{N_S} + \frac{N_{A\cap\bar{B}}}{N_{\bar{B}}}\cdot\frac{N_{\bar{B}}}{N_S}$$

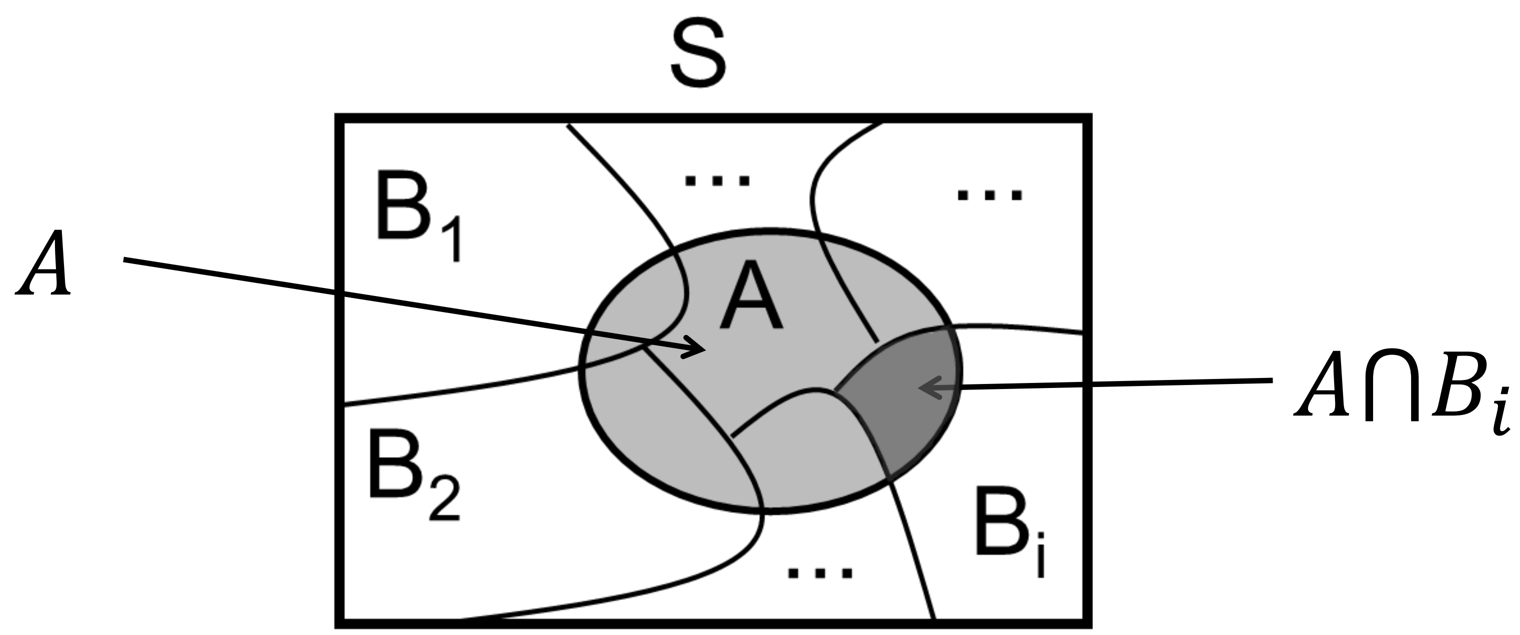
$$= \frac{3}{6} = \frac{2}{6} + \frac{1}{6} = \frac{2}{3}\cdot\frac{3}{6} + \frac{1}{3}\cdot\frac{3}{6} = \frac{1}{2}$$

$$\Pr(A) = \Pr(A \cap B) + \Pr\left(A \cap \bar{B}\right) = \Pr(A|B)\cdot\Pr(B) + \Pr\left(A|\bar{B}\right)\cdot\Pr\left(\bar{B}\right) = \frac{2}{3}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2} = \frac{3}{6} = \frac{1}{2}$$

Total Probability

We sometime call it the marginal

- Pr(A) of an event is the total probability of that event.

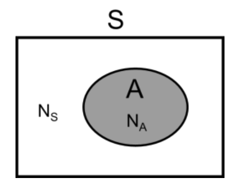


$$Pr(A) = Pr(A \cap B_1) + Pr(A \cap B_2) + \cdots + Pr(A \cap B_i) + \cdots$$
$$= Pr(A|B_1) \cdot Pr(B_1) + Pr(A|B_2) \cdot Pr(B_2) + \cdots$$

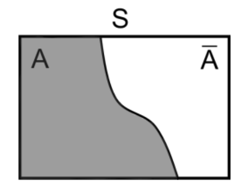
where the B_i's are mutually exclusive ($B_i \cap B_j = \varnothing$ for $i \neq j$) and $S = B_1 \cup B_2 \cup \cdots \cup B_i \cup \cdots$

Summary of Probability

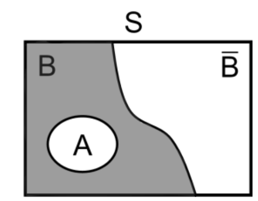
Relative frequency: $Pr(A) = \frac{N_A}{N_S}$



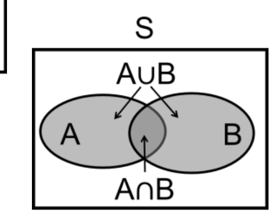
Complement: $Pr(\bar{A}) = 1 - Pr(A)$



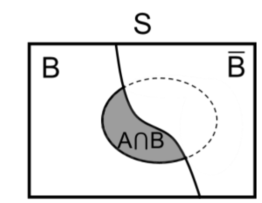
Exclusive: $Pr(\bar{A} \cap B) = Pr(B) - Pr(A)$ if $A \subset B$



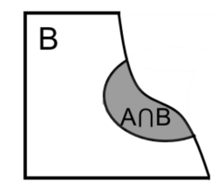
Union: $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$



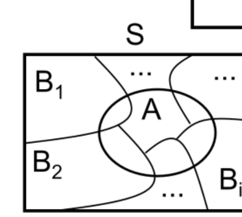
Joint: $Pr(A \cap B) = Pr(A|B) \cdot Pr(B) = Pr(B|A) \cdot Pr(A)$



Conditional: $Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$ if $Pr(B) \neq 0$



Total probability: $Pr(A) = \sum_{i=1}^{n} Pr(A|B_i) \cdot Pr(B_i)$



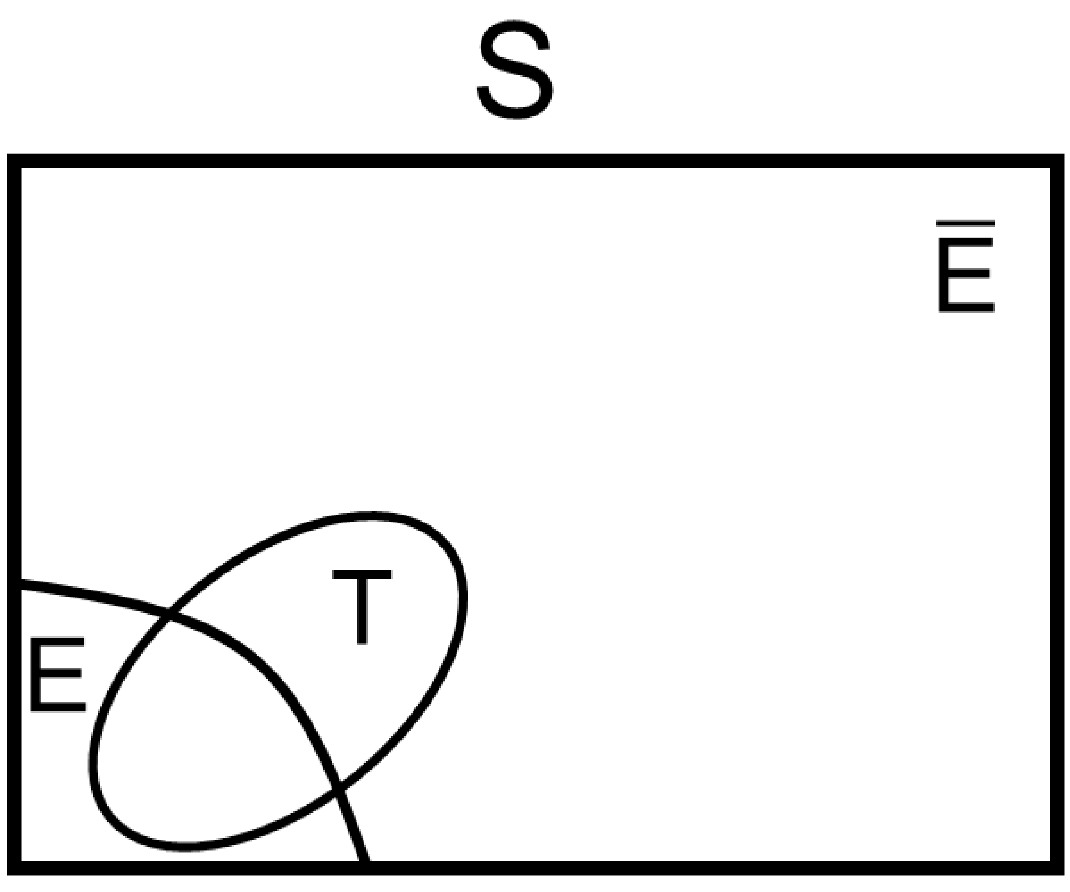
Bayes rule: $Pr(B|A) = \frac{Pr(A|B) \cdot Pr(B)}{Pr(A)}$

Bayes formula: $Pr(B_i|A) = \frac{Pr(A|B_i) \cdot Pr(B_i)}{\sum_{i=1}^{n} Pr(A|B_i) \cdot Pr(B_i)}$

Independence: $Pr(A \cap B) = Pr(A) \cdot Pr(B)$

Example: Ebola Test

- Event E: Patient are infectious with Ebola.
- Event T: The Ebola test is positive.



Example: Ebola Test

- **Prior**: What are the probability of a patient having Ebola?

$$Pr(E)$$

- **Likelihood**: What are the probability of a positive test given infectious with Ebola? Or of a negative test given not infectious with Ebola?

$Pr(T|E)$ Sensitivity $\qquad$ $Pr(\bar{T}|\bar{E})$ Specificity

- **Posterior**: What are the probability of being infectious given that a test is positive?

$$Pr(E|T)$$

Example: Ebola Test – Prior knowlegde

- **Prior**: What are the probability of a patient having ebola?

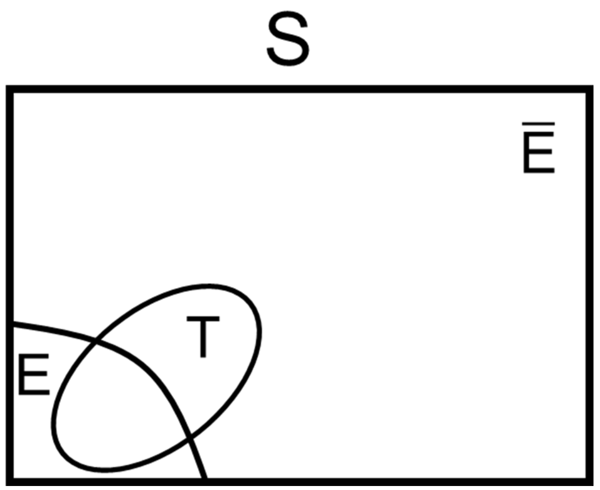
Complement of E

$$Pr(E) = 0,01 \qquad Pr(\bar{E}) = 1 - 0,01 = 0,99$$

- **Likelihood**: What are the probabilities of the tests?

$$Pr(T|E) = 0,9$$ Sensitivity

$$Pr(\bar{T}|\bar{E}) = 0,8$$ Specificity



Example: Ebola Test — Type I and II Error

- **Complement (Errors)**:

  What are the probability of a patient having a positive test without being infectious?

$$Pr\left(T\middle|\bar{E}\right) = 1 - Pr\left(\bar{T}\middle|\bar{E}\right) = 1 - 0{,}8 = 0{,}2$$

Type I Error

  What are the probability of a patient having a negative test being infectious?

$$Pr\left(\bar{T}\middle|E\right) = 1 - Pr(T|E) = 1 - 0{,}9 = 0{,}1$$

Type II Error

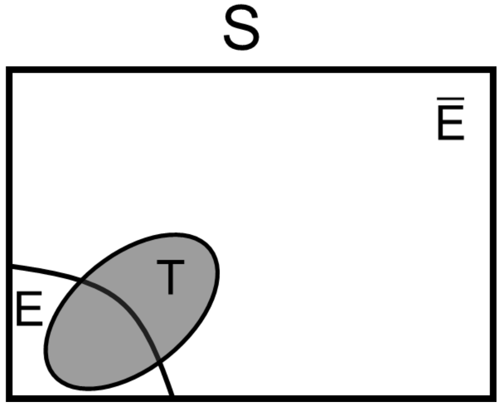
Tests and Types of Errors

- We can classify testing with two outcomes as:

Given / Result	**Disease (True)**	**No disease (False)**
Positive test	Sensitivity	Type I Error
Negative test	Type II Error	Specificity

Example: Ebola Test — Total Probability

- **Total Probability with the Sum Rule**: What are the probability of a patient having a positive test?



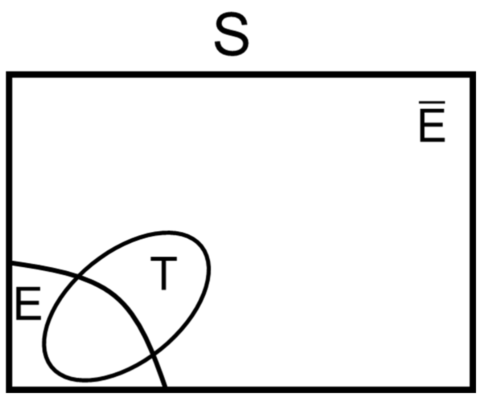
$$Pr(T) = Pr(T \cap E) + Pr(T \cap \bar{E})$$

- **The Product Rule**: We can with Bayes rule find

$$\begin{aligned} Pr(T) &= Pr(T|E)\,Pr(E) + Pr(T|\bar{E})\,Pr\left(\bar{E}\right) \\ &= 0{,}9 \cdot 0{,}01 + 0{,}2 \cdot 0{,}99 \\ &= 0{,}09 + 0{,}198 \\ &= 0{,}207 \end{aligned}$$

Ebola Example — Posterior

- **We have**: We now know the probabilities:



$$Pr(E) = 0{,}01$$ ← Prior

$$Pr(T) = 0{,}207$$ ← Total probability

$$Pr(T|E) = 0{,}9$$ ← Likelihood (Sensitivity)

$$Pr(\bar{T}|\bar{E}) = 0{,}8$$ ← Likelihood (Specificity)

$$Pr(T|\bar{E}) = 0{,}2$$ ← Type I Error

$$Pr(\bar{T}|E) = 0{,}1$$ ← Type II Error

Ebola Example — Posterior

Bayes rule

- What are the probability of being infectious given that a test is positive?

$$Pr(E|T) = \frac{Pr(T|E)Pr(E)}{Pr(T)} = \frac{0{,}9 \cdot 0{,}01}{0{,}207} = 0{,}043$$

- What are the probability of <u>not</u> being infectious given that a test is positive?

$$Pr(\bar{E} \mid T) = 1 - Pr(E|T) = 0{,}957$$

- What are the probability of <u>not</u> being infectious given a negative test?

$$Pr(\bar{E}|\bar{T}) = \frac{Pr(\bar{T} \mid \bar{E})Pr(\bar{E})}{Pr(\bar{T})} = \frac{0{,}8 \cdot 0{,}99}{0{,}793} = 0{,}999$$

- What are the probability of being infectious given that a test is negative?

$$Pr(\mathrm{E} \mid \bar{T}) = 1 - Pr(\bar{E}|\bar{T}) = 0{,}001$$

Ebola Example — Conclusion

- If the test is negative, it is allmost certain (99,9%) that you’re not being infectious:

$$Pr\left(\bar{E}\middle|\bar{T}\right) = 0{,}999$$

- If the test is positive, there is still only a small risk (4,3%) that you actually are being infectious:

$$Pr(E|T) = 0{,}043$$

- The test will only catch 90% of the patients having ebola; ie. 10% of the ebola infected patients will not be caught:

$$Pr(T|E) = 0{,}90 \qquad Pr\left(\bar{T}\middle|E\right) = 0{,}1$$

The Bernoulli Trial

- Two possible outcomes:

Bernoulli trial

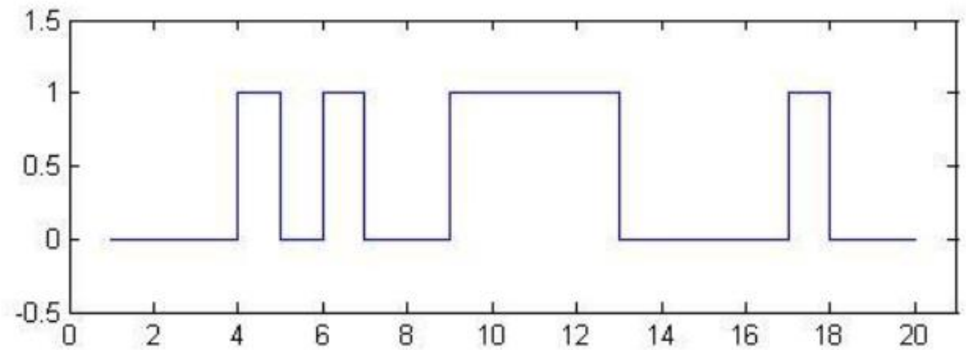
  - **1 = ”Success”** : $\Pr(B = 1) = p$
  - **0 = ”Failure”** : $\Pr(B = 0) = 1 - p = q$

Examples:

- Flip a coin
- Sample space for the experiment is: {Head (1), Tail (0)}

S

- Digital noise
- Sample space: {1, 0}



The Binomial Distribution

- We have n repeated trials.
- Each trial has two possible outcomes

Bernoulli trial

  - **Success** — probability p
  - **Failure** — probability q=1-p

- What is the probability of having k successes out of n trials?
- We write this question as:

$$\boxed{Pr_n(k) = \frac{n!}{k!\,(n-k)!} p^k q^{n-k} = \binom{n}{k} p^k q^{n-k}}$$

- Faculty: $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$

  $0! = 1$

Binomial Coefficient

Definition: The binomial coefficient is defined as:

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

Number of ways to select k objects out of a collection of n objects

Example: Out of 10 children, what is the probability that exactly 2 are girls?

$$Pr_n(k) = \frac{n!}{k!(n-k)!} p^k q^{n-k}$$
$$= \frac{10!}{2!(10-2)!} (0,5)^2 (1-0,5)^{10-2} = 0,044$$

Combinatorics

- Take an object from a collection of n objects.
- Repeat the test k times.

Lotto:
Unordered without replacement

Types of Experiments:

- With or without replacement
- Ordered or unordered

Joker:
Ordered with replacement

Example:

What is the probability that if I have two children that the oldest is a girl and the youngest is a boy?

- Ordered.
- With replacement.

Ordered with Replacement

- Take an object from a collection of n objects.
- **Put it back** each time.
- Repeat the test k times.
- **The sequence** of the objects **matters**.

- The number of combinations is: n^k
  - ➢ Each trial has n possible outcomes
  - ➢ All the trials are independent

Joker:
10^7 = 10.000.000

Ordered without Replacement

- Take an object from a collection of n objects.
- **Do not** put it back each time.
- Repeat the test k times.
- **The sequence** of the objects **matters**.

- The number of combinations is:

$$ {}_nP_k = P_k^n = \frac{n!}{(n-k)!} = \mathrm{n} \cdot (n-1) \ldots (n-k+1) $$

> The 1st trial has n possible outcomes, the 2nd trial has n-1 possible outcomes, … , the k'th trial has n-k+1 possible outcomes

Unordered without Replacement

- Take an object from a collection of n objects.
- **Do not** put it back each time.
- Repeat the test k times.
- **The sequence** of the objects **do not matter**.

- The number of combinations is:

Lotto:

$$\frac{36!}{7!\,29!} = 8.347.680$$

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

➢ The k ordered draws can be shuffled in k! different ways (sequences)

Unordered with Replacement

- Take an object from a collection of n objects.
- **Put it back** each time.
- Repeat the test k times.
- **The sequence** of the objects **do not matter**.
- The number of combinations is:

$$\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!\,(n-1)!}$$

  - Each time we draw an object, we should replace an object (except for the last draw). This correspond to we start with n+k-1 object and draw k objects unordered without replacement.

- Equal to the number of solutions to the equation:
  $x_1 + x_2 + \cdots x_n = k$, where $x_i \in \{0,1,2,\dots,k\}$

Summary of Combinatorics

- We can summarise the number of possible outcomes of k trials, sampled from a set of n objects.

		Replacement	
		With	Without
Sam-pling	Ordered	n^k	$P_k^n = \frac{n!}{(n-k)!}$
	Unordered	$\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!\,(n-1)!}$	$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

Words and Concepts to Know

Prior

Binomial coefficient

Type I Error

Sampling

Bayes rule

Unordered

Replacement

Specificity

Likelihood

Total probability

Combinatorics

Sensitivity

Bernoulli Trial

Posterior

Ordered

Type II Error

Binomial distribution

Conditional probability