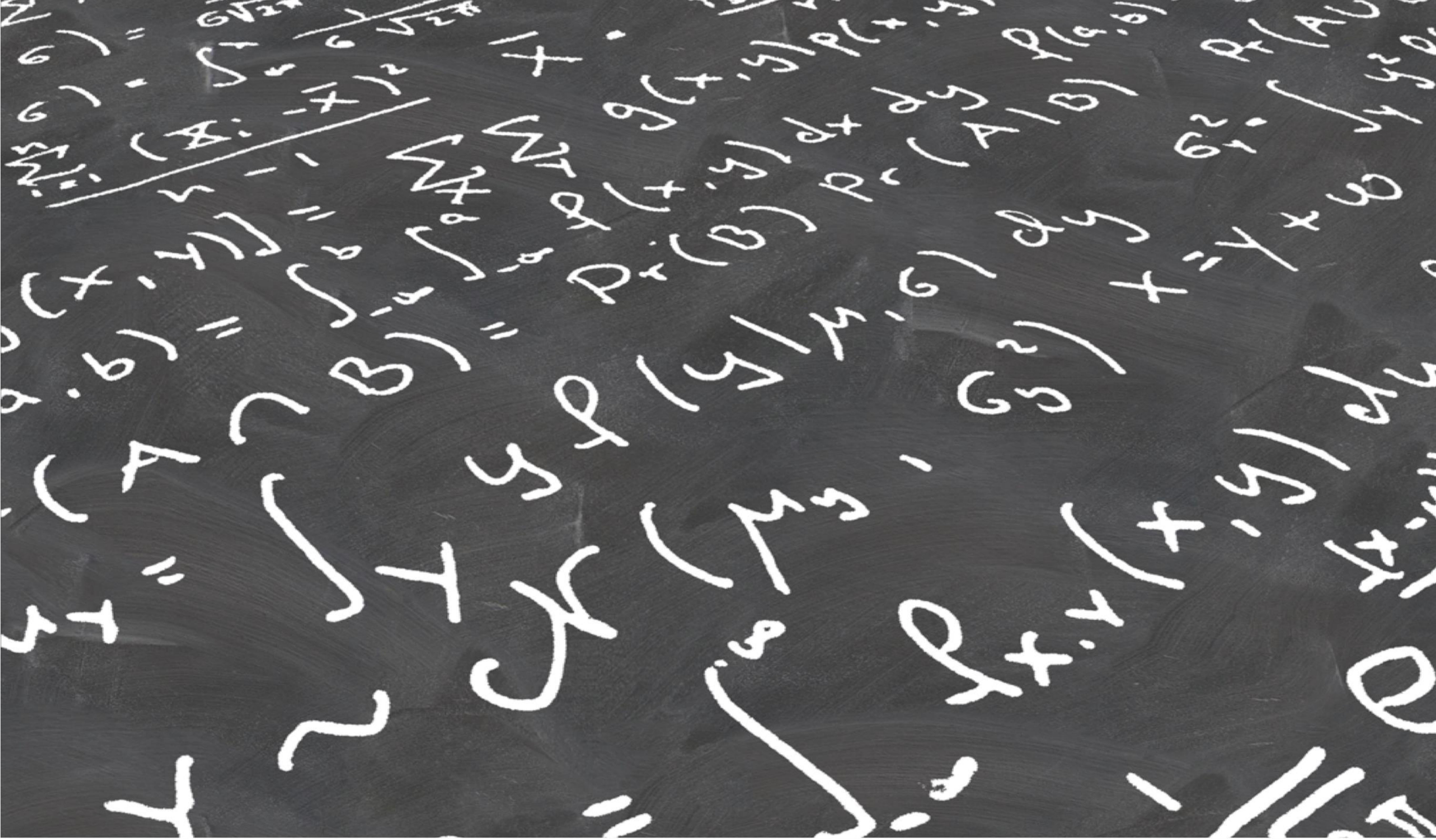

1. Introduction to Probability Theory

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Todays Content

- Introduction to Probability Theory
- Definitions, concepts and notation
- Relative Frequency Approach
- Set theory
- Basic Axions on probabilities

Random Phenomena

- Communication systems transfer information from one place to another as a sequence of 1's and 0's called bits.
- This transmission is often affected by noise, and the information corrupted.
- The figure shows that the transmitted and received bits are different.

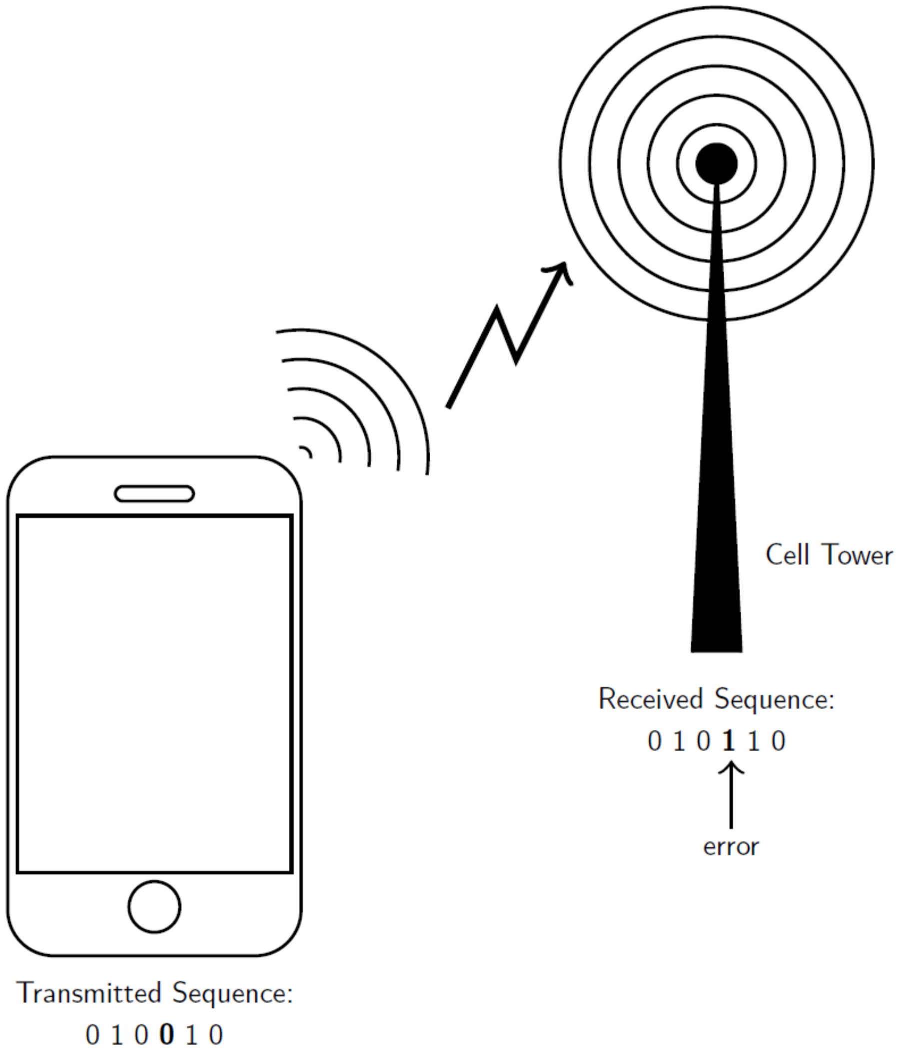
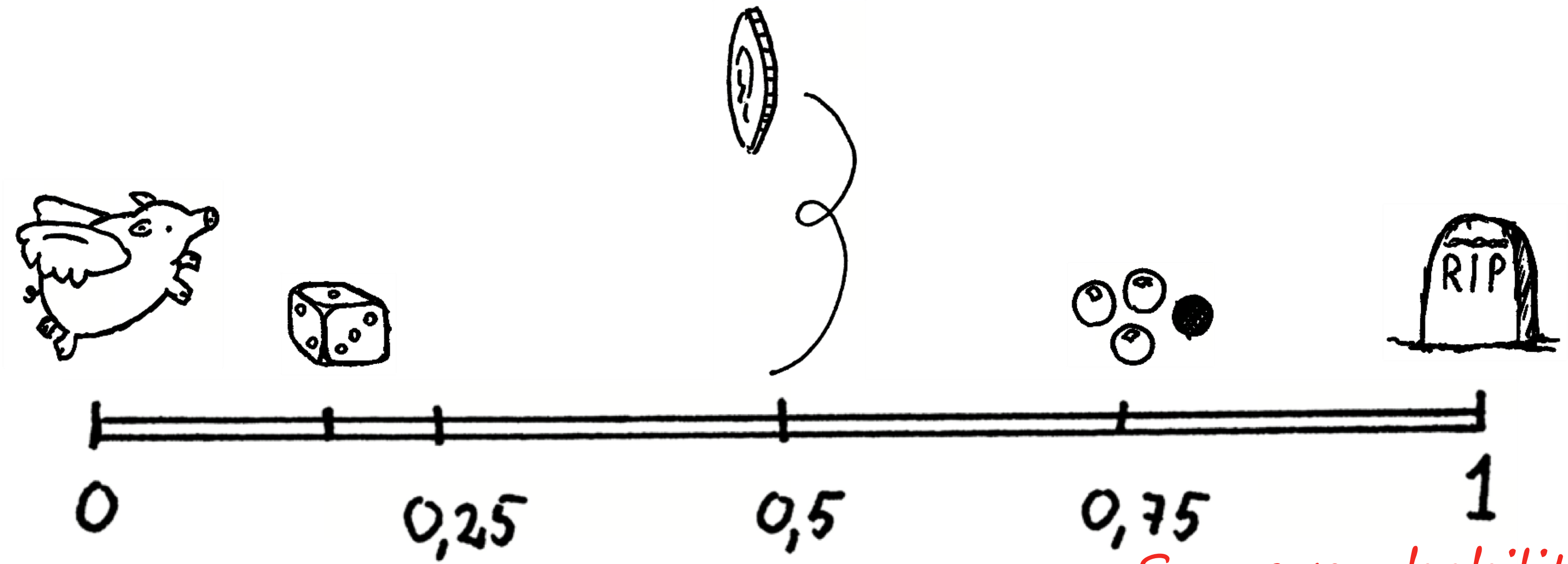


Figure: Transmission of data from a cell phone to a cell tower.

Random Phenomena

- Such errors affect the quality of our transmission and need to be minimized.
- Noise is a random phenomena and we do not know which bits will be affected before transmission.
- Probability theory is used extensively in the design of communication systems to
  a. Understand the behavior of noise.
  b. Take measures in the system to correct errors.

Probability Line



Some probabilities you know

- All probabilities are numbers between 0 and 1.
- In percentage, between 0% to 100%.

Words to Know

- Experiment/trial (*Forsøg/test*) Roll a dice
- Sample space (*Udfaldsrum*) S={1,2,3,4,5,6}
- Sample point (*Bestemt udfald*) a={4}
- Event (*Hændelse*) A={2,4,6} (even number)
  - Elementary event Event that has <u>one</u> possible outcome
  - Joined event Event that has <u>many</u> possible outcomes
  - Simultaneous event Event with <u>two or more sub trials</u>

Basic Axions of Probability

- The probability of a event (A) (collection of sample points) is between 0 and 1.
- All sample points of a probability space (S) sums up to 1.
- The probability of the union of disjoint events A_1 and A_2 is the sum of the probability of each event.

➢ Basic Axions of Probability:

Axion 1: $0 \leq Pr(A) \leq 1$

Axion 2: $Pr(S) = 1$

Axion 3: If $A_1 \cap A_2 = \emptyset$ then

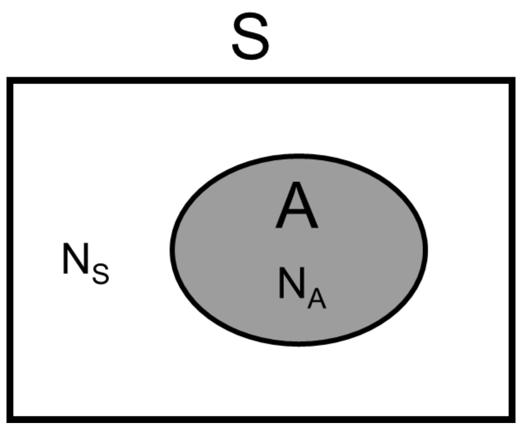
$$Pr(A_1 \cup A_2) = Pr(A_1) + Pr(A_2)$$

Relative Frequency Approach

- The number of times event A occurs: N_A

- The number of times that all events occur (sample space):

$$N_S = N_A + N_B + N_C + \ldots$$

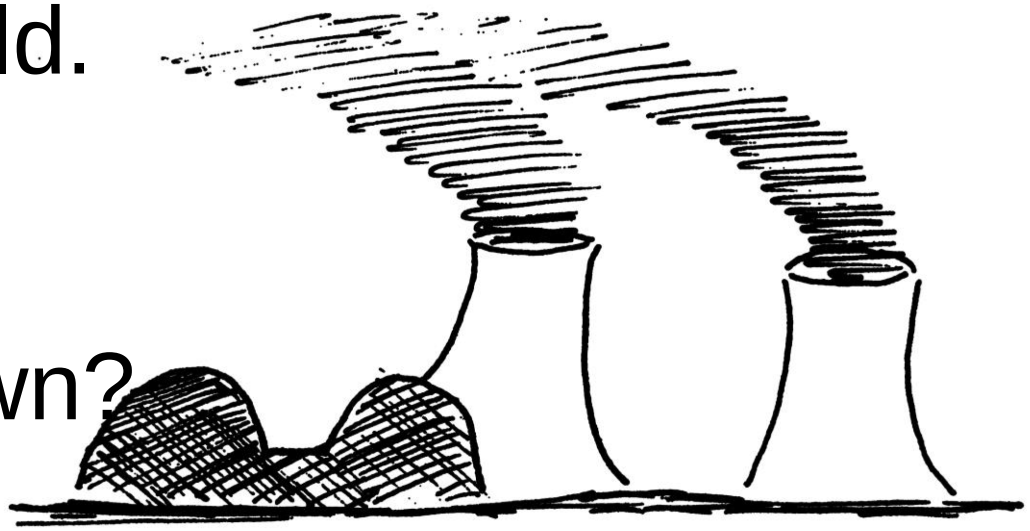


- Then we have the relative frequency:

$$\boxed{\Pr(A) = \frac{N_A}{N}}$$

Risk of a Meltdown

- There are 437 reactors in the world.
- ~153M operating reactor hours.
- ~Four reactor meltdowns.
- What are the chance of a meltdown?



$$\frac{4}{153M} \text{ pr. reactor pr. hour}$$

$$\sum_{n=1}^{437} \frac{4}{153M} = \frac{1}{87600} \text{ pr. hour}$$

$$\frac{24*365}{87600} = \frac{1}{10} \text{ pr. year}$$

→ Be carefull: Small samples, small probabilities, circumstances, etc.
→ Very uncertain: If number of reactors > 4370 → $\Pr(\text{Meltdown}) > 1$

Set Theory (*Mængdelære*)

A set:

- A collection of things.
- Elements of sets are not ordered.

Name of set

Some more elements

$$E = \{\alpha_1, \ldots, \alpha_n\}$$

Element

Examples:

- The set of all persons in a drug trial group.
- The number of cars i DK.
- All colours.
- Natural numbers $\mathbb{N} = \{1,2,3,\ldots\}$ / Real numbers $\mathbb{R} =]-\infty;+\infty[$.
- All digital messages on a transmission line

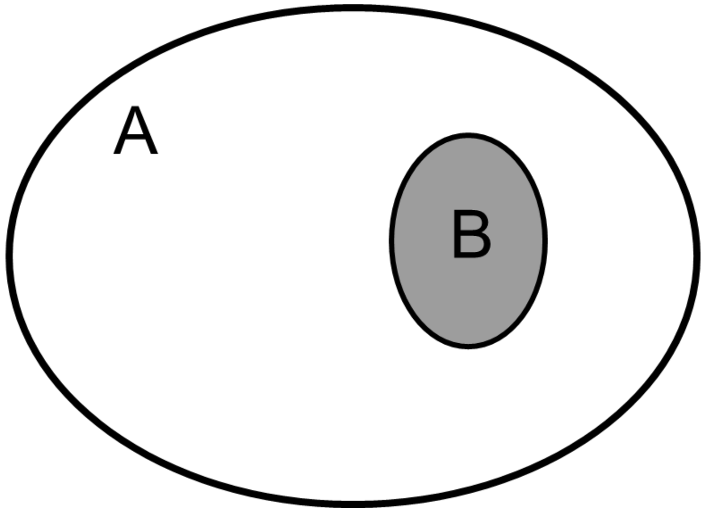
A Subset to a Set (*Delmængder*)

- A subset is any set, where all elements are included in the original set

Notation:

B is a subset to A:

$$B \subset A$$



Example:

For a set $A = \{blue, red, green\}$

we have a subset $B \subset A$ if B is in A

e.g. $\{blue, red\}, \{blue, red, green\}$, $\{green\}$, $\{\}$ But: $\{blue, yellow\} \not\subset A$

The Empty Set (*Den tomme mængde*)

- The empty set is always a subset of any set.
- This corresponds to the impossible event.

$$\varnothing = \{\}$$ The null set

- The probability of the impossible event is 0.

Example:

- The set of boys in an all girlschool.
- The change of pigs growing wings and fly.
- To get an 8 when rolling a dice.

Cardinality *(kardinalitet)*

- The cardinality (size) $|A|$ of a set A is the number of element in the set N_A
- A set can be finite: $|A| < \infty$ or infinite: $|A| = \infty$
- A set is countable: ''Tællelig''
  a) If $|A| < \infty$ (finite set)
     or
  b) One-to-one correspondance with natural numbers $\mathbb{N}$
- Otherwise the set is uncountable ''Utællelig''

Cardinality *(kardinalitet)*

Examples:

- Throw of a dice: S={1,2,3,4,5,6}
  Events: A={6}, B={2,4,6}

$|S| = 6$
$|A| = 1$ Finite sets
$|B| = 3$

- A = {All cars in DK}: $|A| < \infty$ Finite set

- A = {All prime numbers (2, 3, 5, 7, …)}: $|A| = \infty$

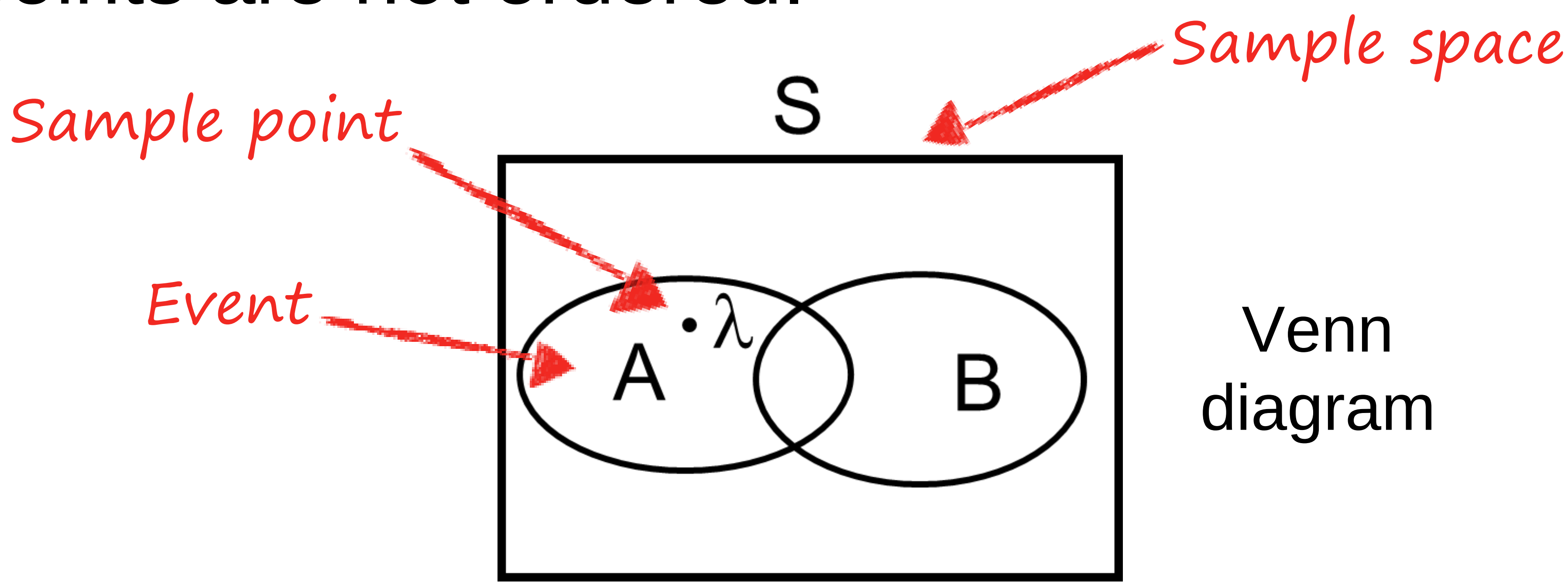
Infinite,
countable set

- A = {x∈ℝ|-3<x≤3} =]-3;3]: $|A| = \infty$

Infinite,
uncountable set

Venn Diagram

- An elementary event is one sample point λ.
- Events (A, B) are collections of sample points.
- Sample space S is the collection of <u>all possible sample points</u>; Pr(S) = 1
- Sample points are not ordered.



Venn diagram

Example:

Throw of a dice:

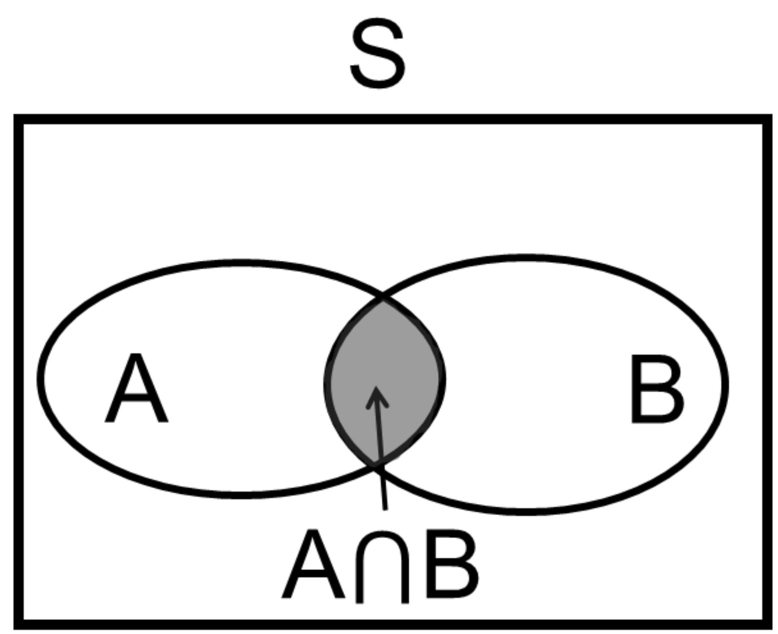
Possible outcomes: 1,2,3,4,5,6 $\rightarrow$ S={1,2,3,4,5,6}

Events: A={1,2,3} and B={2,4,6}; $A \subset S$; $B \subset S$

Joint Events (*Fællesmængde*)

- The intersection $A \cap B$ are the common elements of the events A and B
- $A \cap B$ means A and B.

Venn diagram

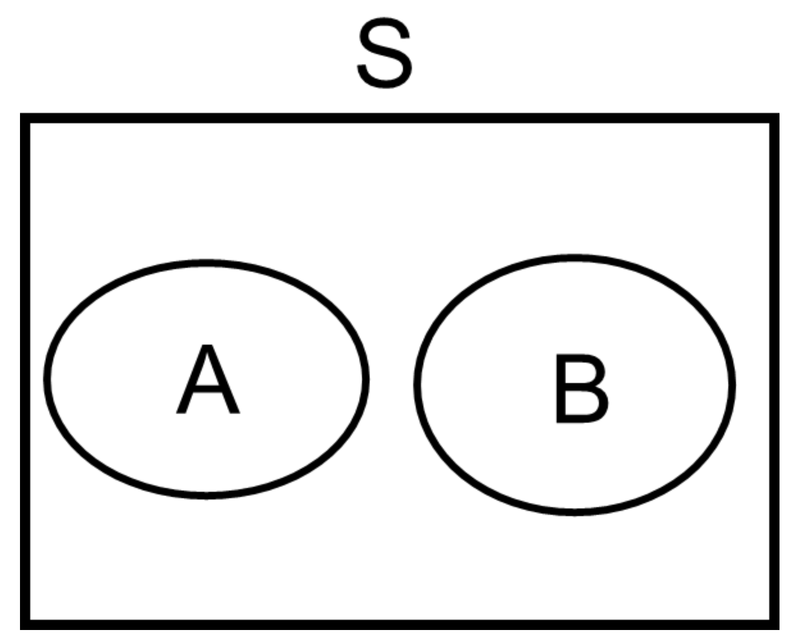


Example:

- Event A is the event of VW cars i DK
- Event B is the event of red cars in DK
- The intersection of the events is all red VW in DK.

Mutually Exclusive (Disjoint) Events (*Disjunkte*)

- The sets of A and B are disjoint if: $A \cap B = \emptyset$



$A \cap B = \emptyset$

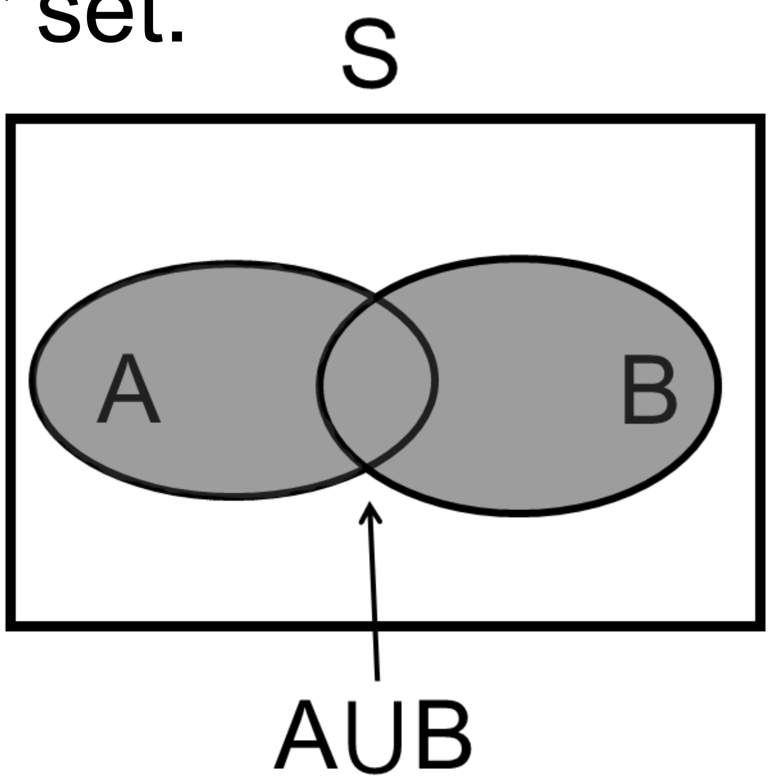
Example:

- Event A: The car is a Ferrari.
- Event B: The car is blue.
- As everybody know:
  A Ferrari is red – otherwise it is not a real Ferrari!

Union of Events (*Foreningsmængde*)

- The union of events $A \cup B$ are all the events in one set ‘plus’ the events in the other set.
- $A \cup B$ means A <u>or</u> B.
- $A \cup B = A + B - A \cap B$
- If A and B finite:

  $|A \cup B| = |A| + |B| - |A \cap B|$

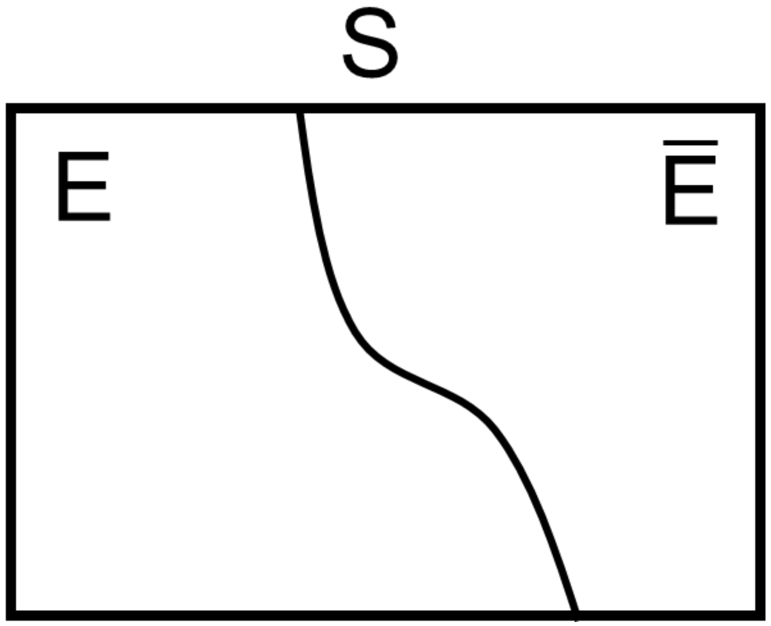


Example:

- Event A is the event of VW cars i DK
- Event B is the event of red cars in DK
- The union of the events are all VW (independent of colour) and all red cars (independent of brand) in DK.

The Complement Event (*Komplementær*)

Notation: $S \setminus E = \overline{E}$ ''not-E''



Notice:

$$E \cup \bar{E} = S$$

The certain event

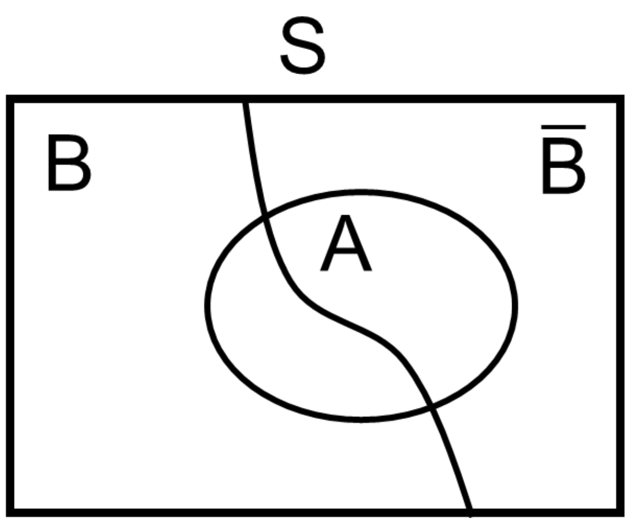
$$E \cap \bar{E} = \varnothing$$

The impossible event

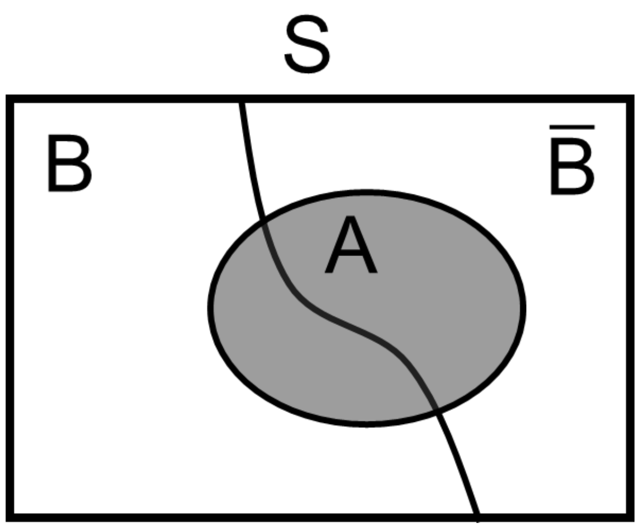
Example:

- The complement of having a disease is not having a disease

Probability of joint events

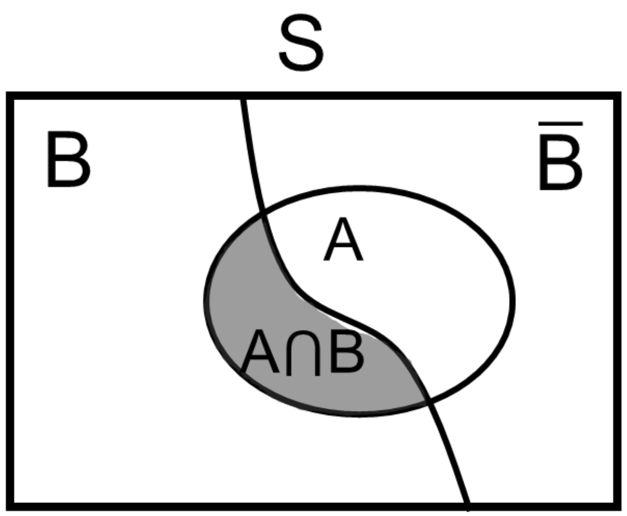


Venn diagram

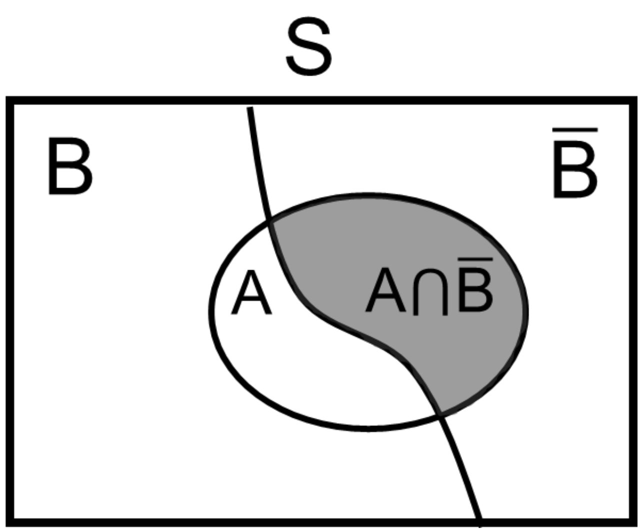


$$\Pr(A) = \frac{N_A}{N_S} = \frac{|A|}{|S|}$$

OBS:
All sample points should have the same a priori probability



$$\Pr(A \cap B) = \frac{N_{A \cap B}}{N_S} = \frac{|A \cap B|}{|S|}$$



$$\Pr\left(A \cap \bar{B}\right) = \frac{N_{A \cap \bar{B}}}{N_S} = \frac{\left|A \cap \bar{B}\right|}{|S|}$$

Independence (*Uafhængighed*)

- We define that two events are independent if and only if:

$$\boxed{Pr(A \cap B) = Pr(A) \cdot Pr(B)}$$

Notice:

- This does not apply if the events A and B are dependent.

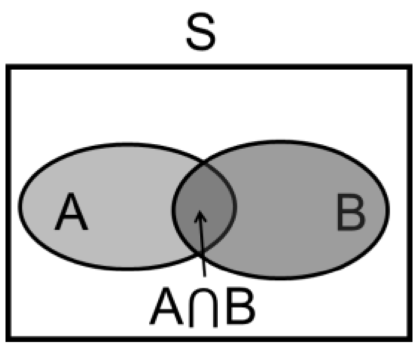
Example:

- Two throws with a dice
- The gender of two siblings

Probabilities of a Union of Event

- We can calculate the probability of a union of events:

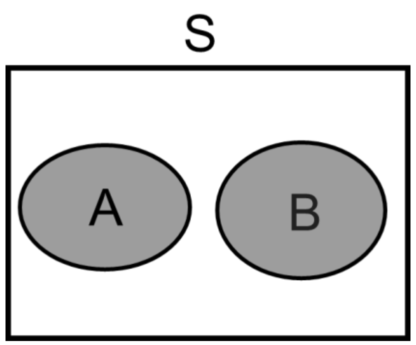
$$\boxed{Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)}$$



Notice:

- If the events are mutually exclusive ($A \cap B = \emptyset$):

Axiom 3 $$\boxed{Pr(A \cup B) = Pr(A) + Pr(B)}$$



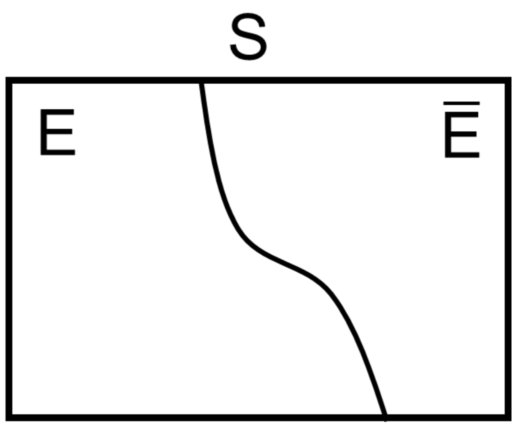
Probabilities of Complement Events

- We can write some rules for the probabilities of a complement event

$$Pr(E \cup \bar{E}) = Pr(S) = 1$$

$$Pr(E) + Pr(\bar{E}) = Pr(S) = 1$$

$$\boxed{Pr(E) = 1 - Pr(\bar{E})}$$



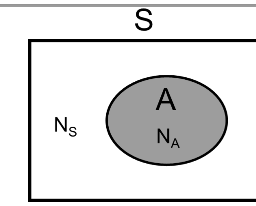
Example:

- The probability of not hitting 2 eyes on dice.

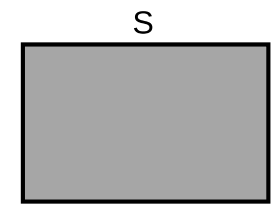
$$Pr(\{1,3,4,5,6\}) = 1 - Pr(\{2\}) = 1 - \tfrac{1}{6} = \tfrac{5}{6}$$

Summary of Probability

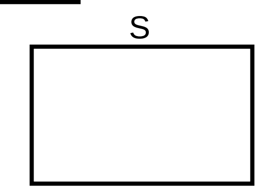
- Relative frequency: $Pr(A) = \frac{N_A}{N_S} = \frac{|A|}{|S|}$



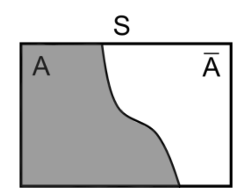
- The certain/universal set S: $\mathrm{Pr}(S) = 1$



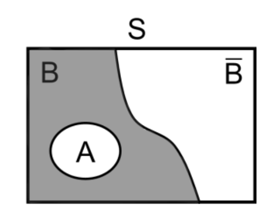
- The empty/null set ∅ : $|\emptyset| = 0$; $\mathrm{Pr}(\emptyset) = 0$



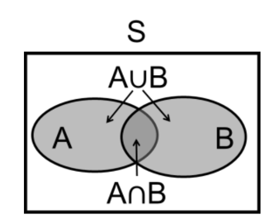
- Complement: $Pr(\bar{A}) = 1 - Pr(A)$



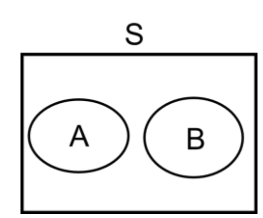
- Exclusive: $Pr(\bar{A} \cap B) = Pr(B) - Pr(A)$ if $A \subset B$



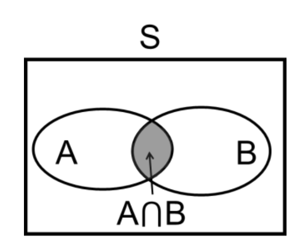
- Union: $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$



- Mutually Exclusive: $Pr(A \cap B) = 0$



- Independence: $Pr(A \cap B) = Pr(A) \cdot Pr(B)$



Words and Concepts to Know

Experiment/Trial

Set

Mutually Exclusive/Disjoint

Sample space

Event

Union

Complement/not

Sample point

Intersection

Subset

Independence

Empty set/Null set

Relative frequency

Joint events