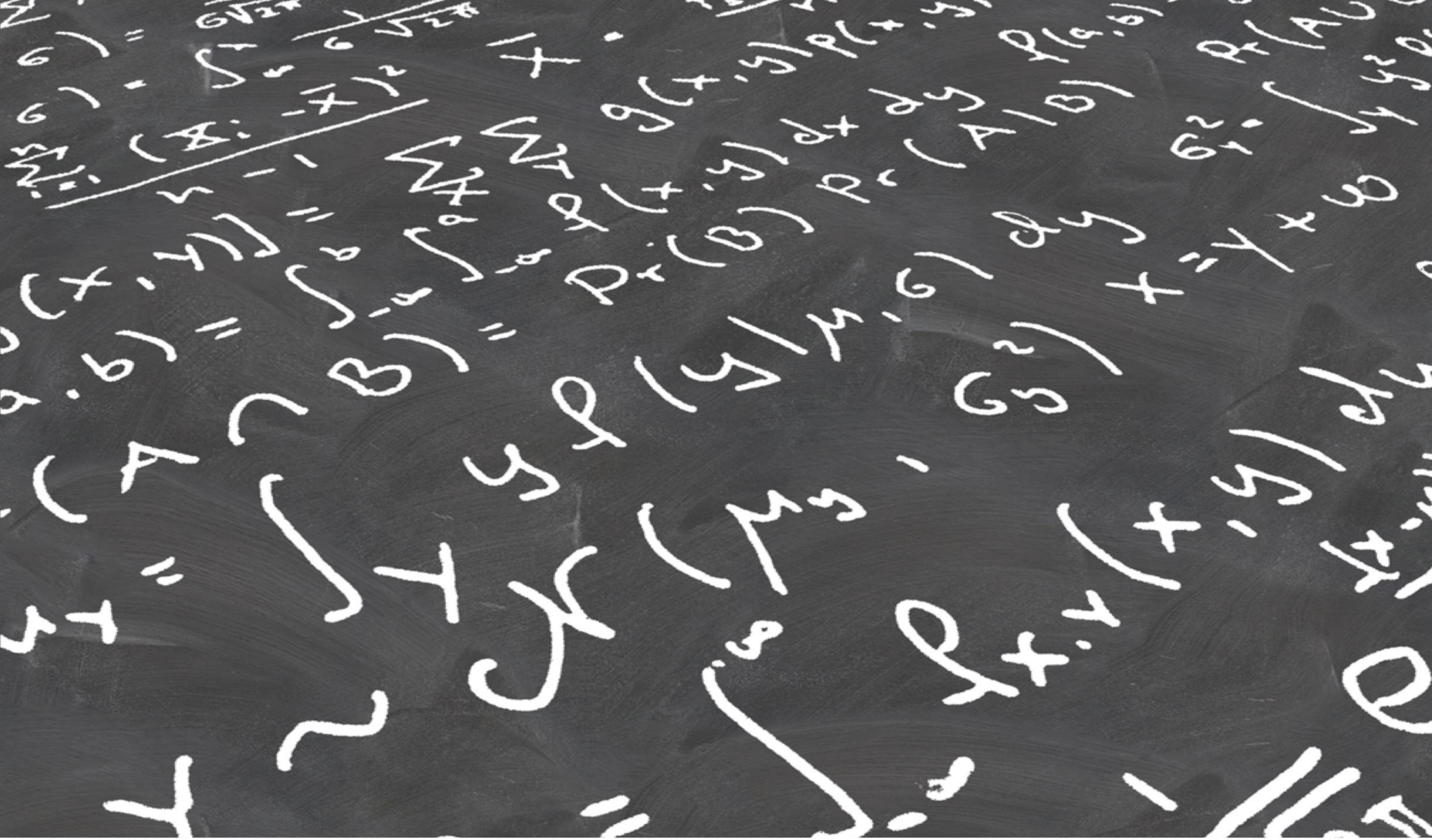

4. Continuous Random Variables

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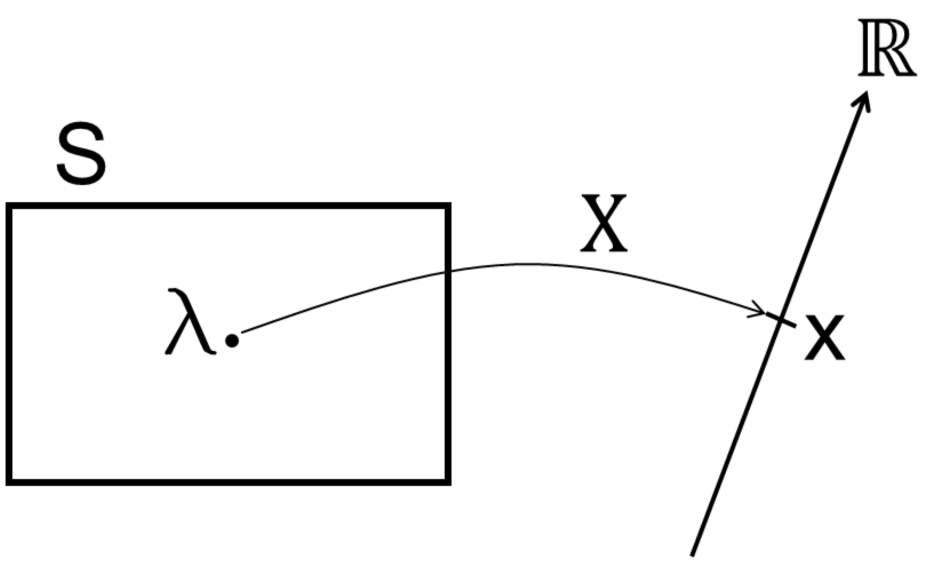
Agenda for Today

- Repetition from last time
  - Discrete Random Variables
- Continuous Random Variables
- Continuous Random Distributions
- Mixed Random Variables
- Dirac’s Delta Function

Also just called a random variables

Stochastic Random Variables

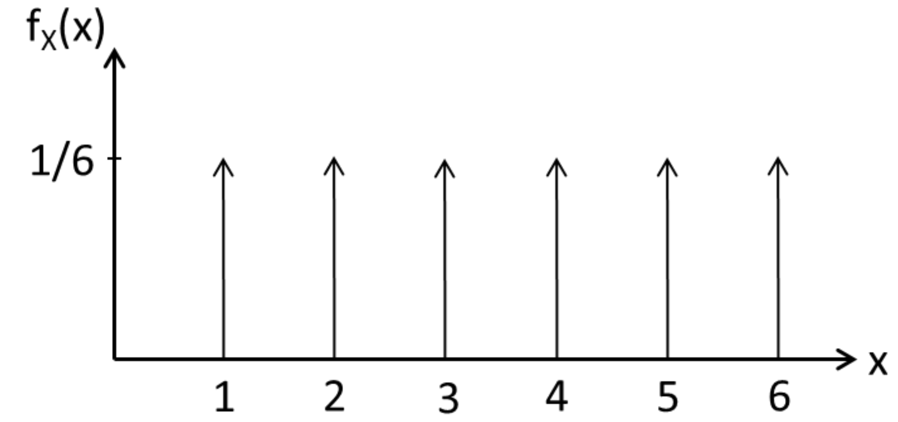
- A random variable tells something important about a stochastic experiment.
- Can be discrete (R_X= range of X, countable) or continuous (R_X = range of X, uncountable)



Discrete Stochastic Variables

- Probability Mass Function (pmf):

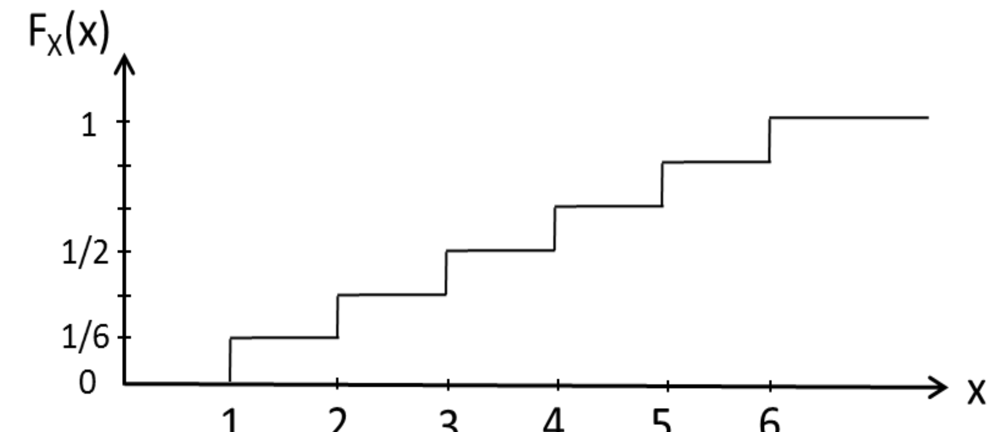
$$\boxed{f_X(x) = \begin{cases} Pr(X = x_i) & for\ X = x_i \\ 0 & otherwise \end{cases}}$$



- $0 \le f_X(x) \le 1, \qquad \sum_{i=1}^{n} f_X(x_i) = \sum_{i=1}^{n} Pr(X = x_i) = 1$

- Cumulative Distribution Function (pmf):

$$\boxed{F_X(x) = Pr(X \le x) = \sum_{x_i=-\infty}^{x_i \le x} f_X(x_i)}$$



- $0 \le F_X(x) \le 1, \quad \lim_{x \to -\infty} F_X(x) = 0, \quad \lim_{x \to \infty} F_X(x) = 1$

- Steps $= \ Pr(X = x_i) = f_X(x_i), \qquad F_X(x_2) - F_X(x_1) = Pr(x_1 < X \le x_2)$

Mean, Variance and Standard deviation

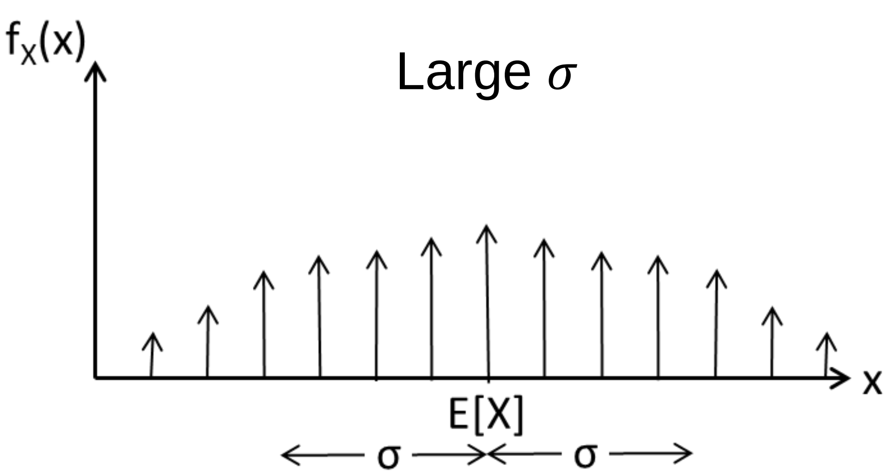
- The <u>mean</u> or the <u>expectation</u> of a discrete random variable X

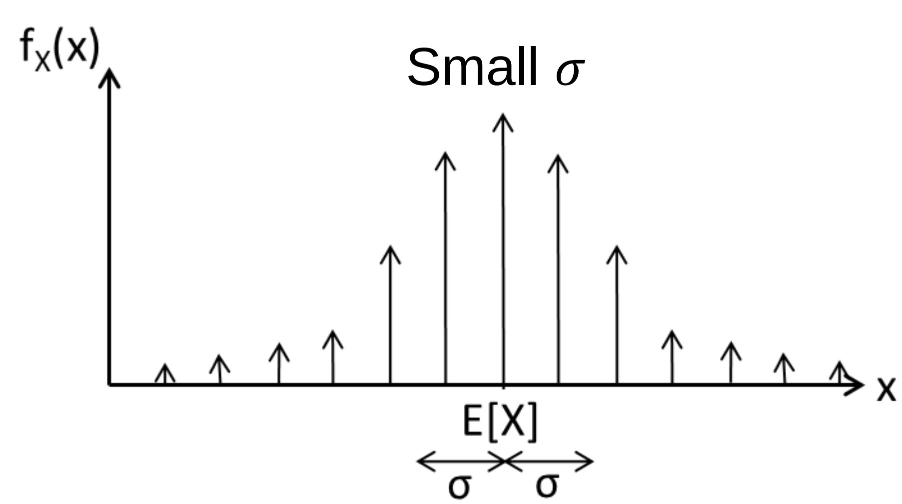
$$E[X] = \mu_X = \bar{X} = \sum_{i=1}^{n} x_i f_X(x_i)$$

Variance and standard deviation tells of the spreading of the data

- The <u>variance</u> σ^2 and the <u>standard deviation</u> σ of a random variable X

$$Var(X) = \sigma_X^2 = E\left[X^2\right] - E[X]^2$$

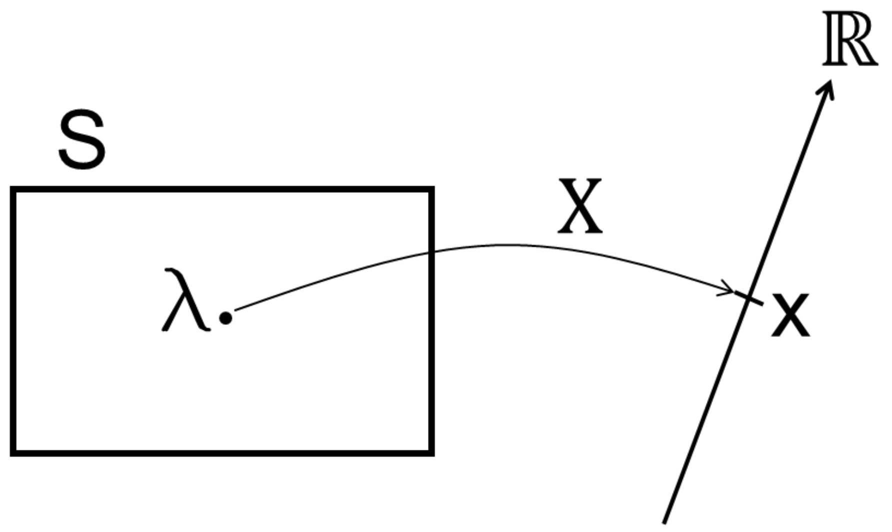




Also called a continous stochastic variable

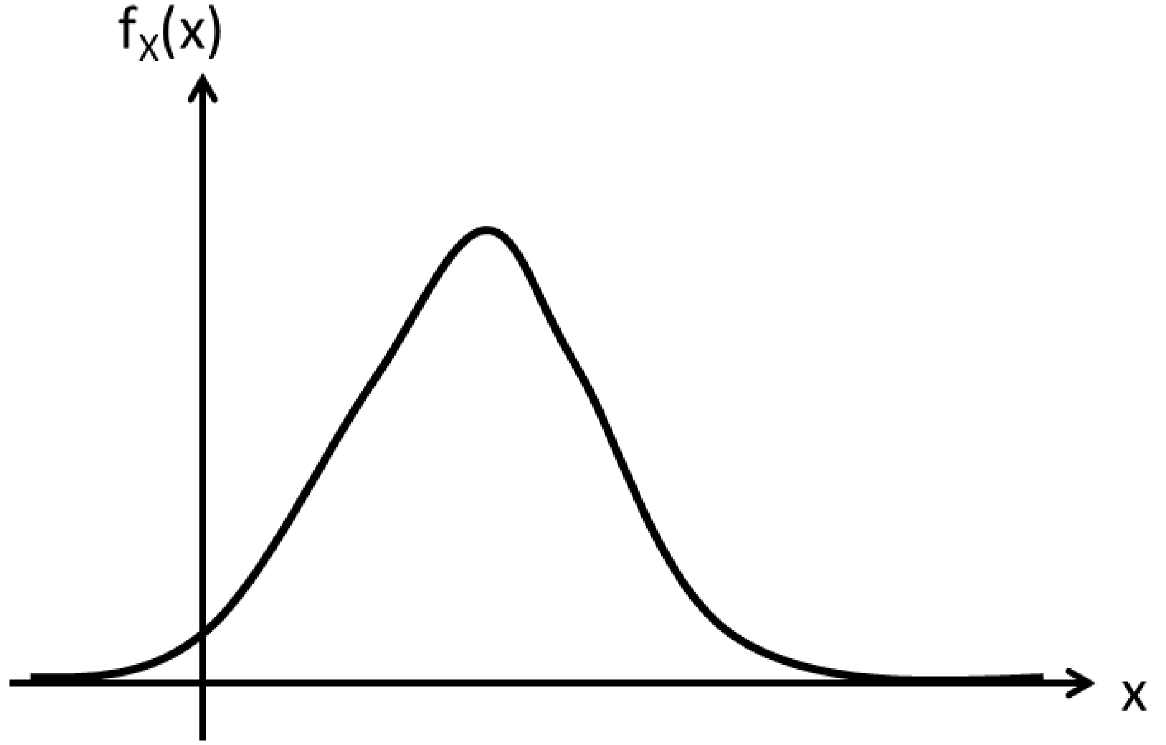
Continuous Random Variables

- We define a stochastic variable X
- X is continuous on $\mathbb{R}$ (R_X = range of X, uncountable)



- Fx. The exact value R of a resistor

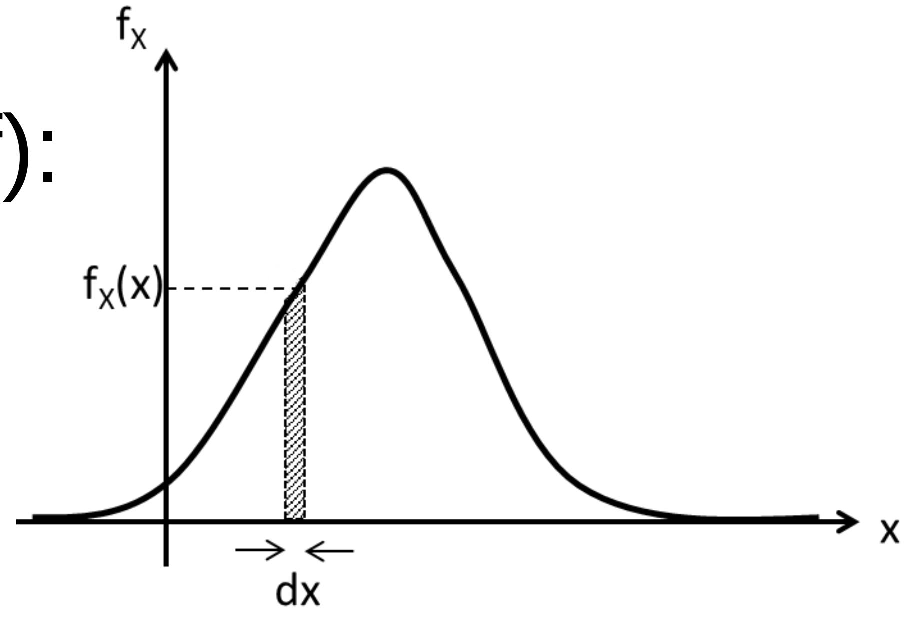
- X is defined by a density function $f_X(x)$



Continuous Random Variables

- The <u>probability density function</u> (pdf):

$$Pr(x < X < x + dx) = f_X(x)dx$$



- The probability of one exact value of the variable is always zero:

$$Pr(X = x) = 0$$

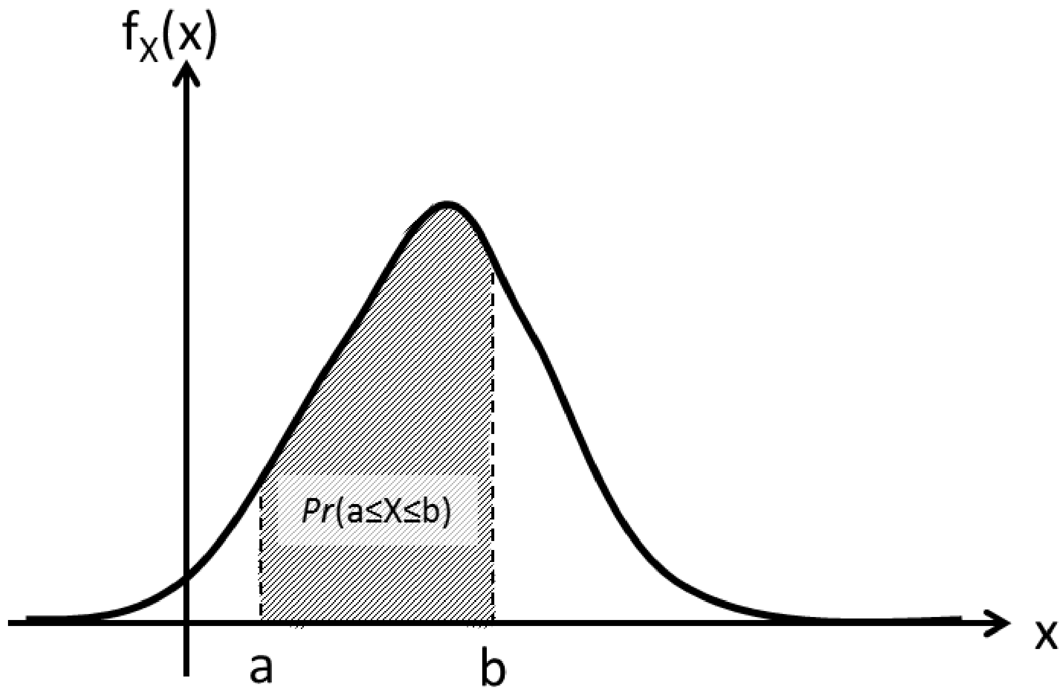
- Discrete → Continuous stochastic variable

$$\sum \rightarrow \int$$

Probability Density Function (pdf)

- We define a probability density function (pdf): $f_X(x)$

$$\boxed{Pr(a \le X \le b) = \int_a^b f_X(x)\,dx}$$



Properties: $f_X(x) \ge 0$

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

Total probability is 1.

Notice: $f_X(x) > 1$ is possible

$$Pr(X = x) = 0$$

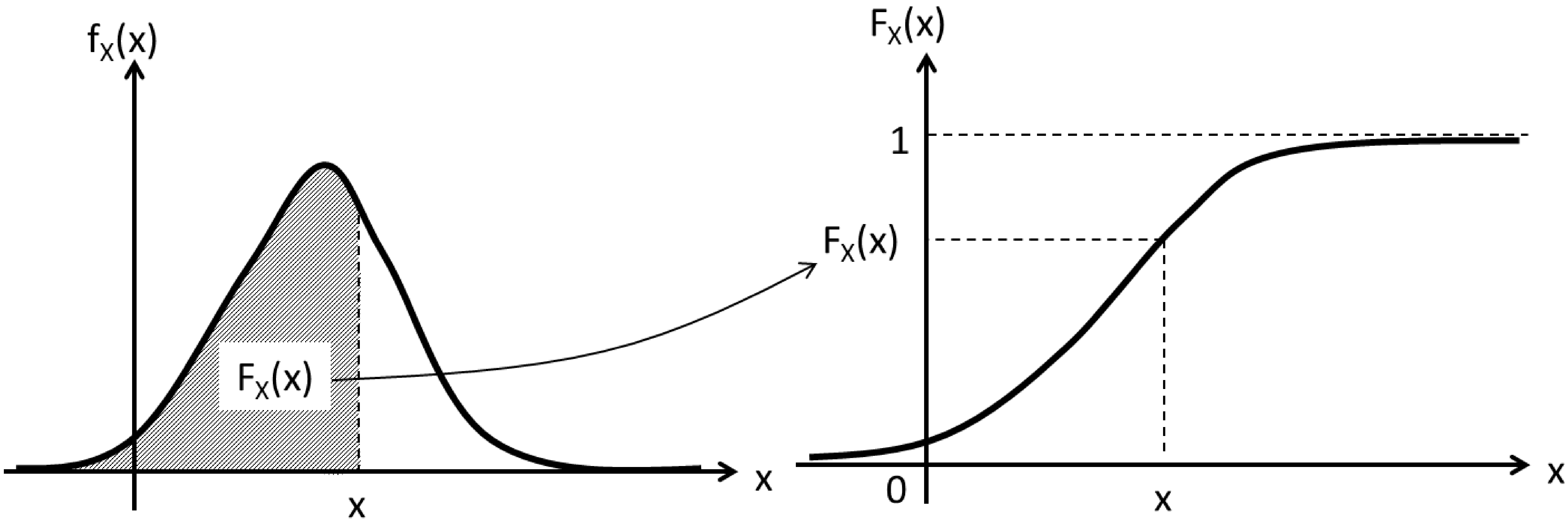
$$Pr(a < X < b) = Pr(a \le X < b) = Pr(a < X \le b) = Pr(a \le X \le b)$$

Cumulative Distribution Function (cdf)

- We define a <u>cumulative distribution function</u> (cdf): $F_X(x)$

Accumulates the probabilities from minus infinite to x.

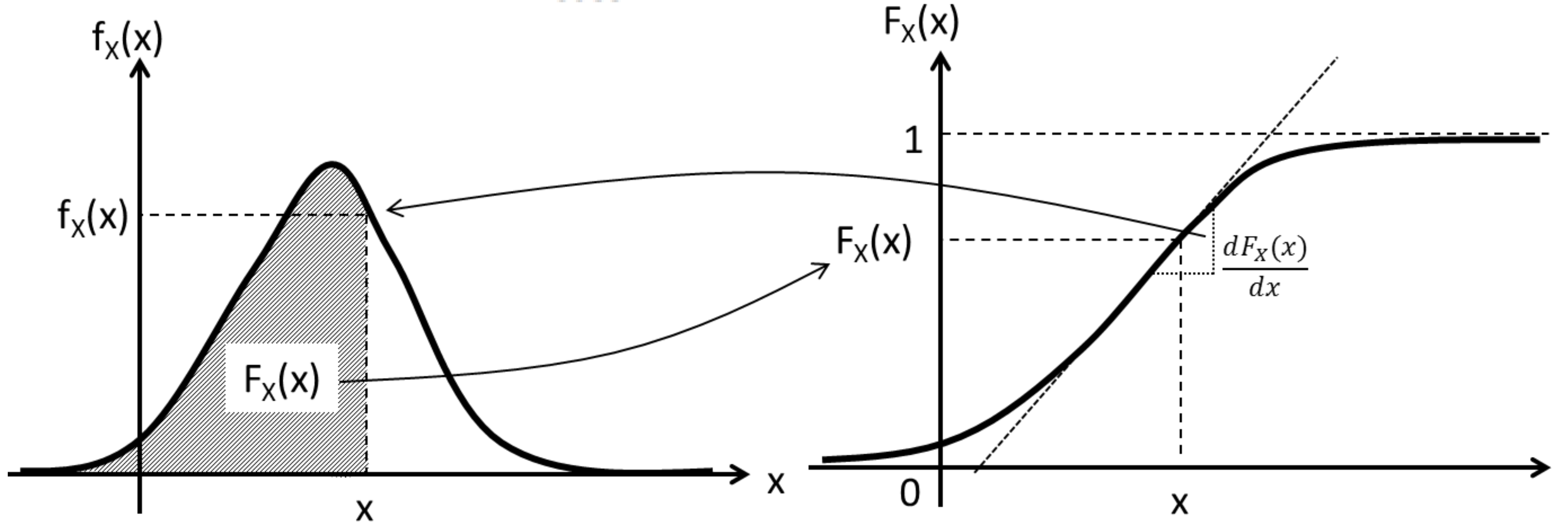
$$\boxed{F_X(x) = \int_{-\infty}^{x} f_X(u)\,du = Pr(X \leq x)}$$



The cdf and pdf contains the same information.

Cumulative Distribution Function (cdf)

- From pdf to cdf: $$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du = Pr(X \leq x)$$
- From cdf to pdf: $$f_X(x) = \frac{dF_X(x)}{dx}$$



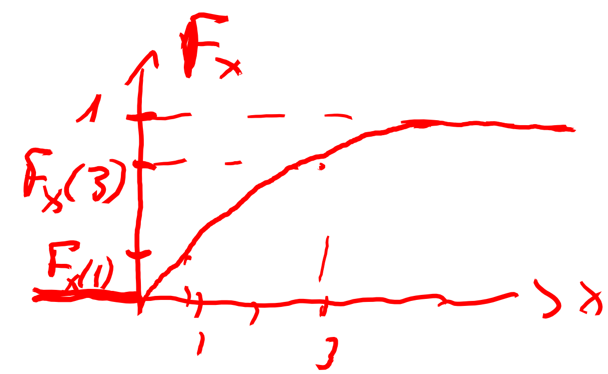
Properties:

- $0 \leq F_X(x) \leq 1$
- $F_X(x)$ is always non−decreasing and continuous
- $Pr(a \leq X \leq b) = \int_a^b f_X(x)dx = F_X(b) - F_X(a)$
- $Pr(X > x) = 1 - Pr(X \leq x) = 1 - F_X(x)$

Example

$$f_X(x) = \begin{cases} c \cdot e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases} \longrightarrow \begin{cases} f_X(x) \geq 0 \Rightarrow c \geq 0 \\ \int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^{\infty} c\, e^{-x}\,dx = \left[-c\, e^{-x}\right]_0^{\infty} = 0 - (-c \cdot 1) = c = \underline{1} \end{cases}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = \begin{cases} 0 & x < 0 \\ \int_0^x e^{-x}\,dx = \left[-e^{-x}\right]_0^x = -e^{-x} + 1 = 1 - e^{-x} & x \geq 0 \end{cases}$$



$$\Pr(1 \leq X \leq 3) = \int_1^3 f_X(x)\,dx = \int_1^3 e^{-x}\,dx = \left[-e^{-x}\right]_1^3 = -e^{-3} + e^{-1} = 0.318$$

$$= F_X(3) - F_X(1) = \cancel{1} - e^{-3} - (\cancel{1} - e^{-1}) = e^{-1} - e^{-3}$$

Also called the Expectation

Mean Value of a Continuous Random Variable

- The mean value is the expectation of X:

$$\boxed{EX = E[X] = \bar{X} = \mu_X = \int_{-\infty}^{\infty} x \cdot f_X(x)dx}$$

Example:

- The value of 5% 1kΩ resistors.

$$EX = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^{\infty} x e^{-x} dx = \left[-(x+1)e^{-x}\right]_0^{\infty} = 0 + 1 \cdot e^{-0} = 1$$

Definition of Expectation

- We define the expectation of $g(X)$ with respect to a pdf $f_X(x)$ as the integral:

$$\boxed{E[g(X)] = \overline{g(X)} = \int_{-\infty}^{\infty} g(x) \cdot f_X(x) dx}$$

LOTUS – Law Of The Unconscious Statistician

Example:

- DC voltage with a noise-signal.

$$E[X^2] = \int_0^\infty x^2 \cdot e^{-x} dx = 2$$

Expectation

- Linear function: $g(X) = aX + b$

$$E[aX + b\ ] = \int_{-\infty}^{\infty} (ax + b) \cdot f_X\,(x)dx = a \cdot E[X] + b$$

- Square function: $g(X) = X^2$

$$E[g(X)] = E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 \cdot f_X\,(x)dx \neq \left(\int_{-\infty}^{\infty} x \cdot f_X\,(x)dx\right)^2 = E[X]^2$$

- Linear sum: $\mathrm{Z} = aX + bY$

$$EZ = E[Z] = \mu_Z = E[aX + bY] = a \cdot E[X] + b \cdot E[Y]$$

Variance and standard deviation tells of the spreading of the data

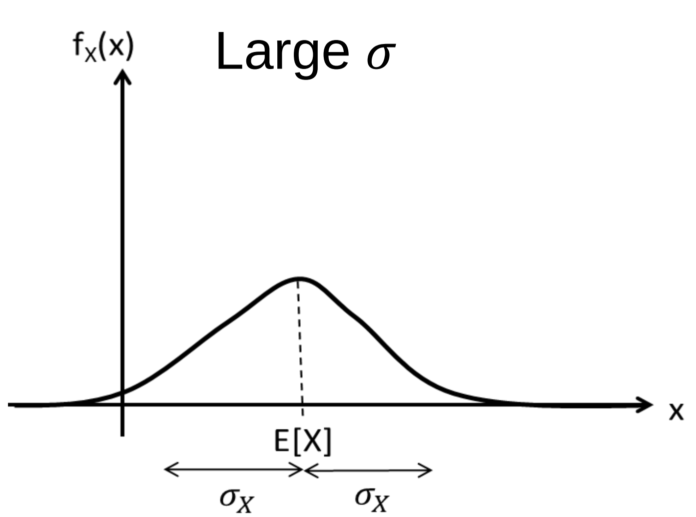
Variance and standard deviation

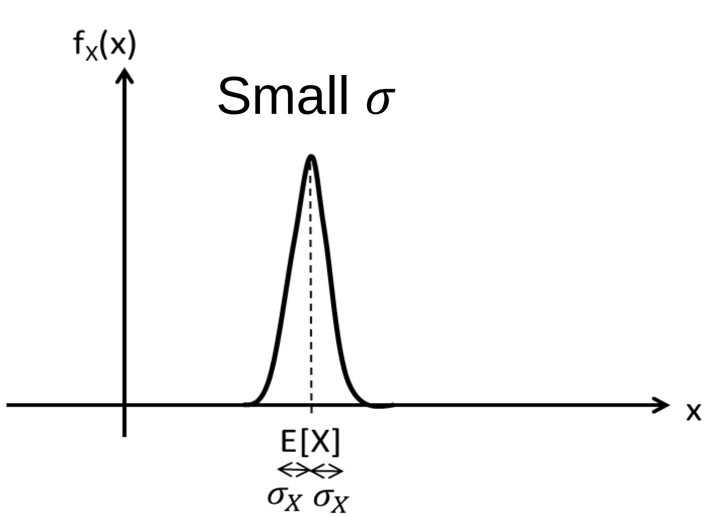
- The variance of a continuous random variable X :

$$\boxed{Var(X) = \sigma_X^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 \cdot f_X(x)dx = E[X^2] - E[X]^2}$$

- The standard deviation σ is the square root of the variance (same unit as X):

$$\boxed{SD(X) = \sigma_X = \sqrt{\sigma_X^2}}$$





Definition of Variance

- We define the variance of $g(X)$ with respect to a pdf $f_X(x)$ as the integral:

$$\boxed{\begin{aligned} Var\big(g(\mathrm{X})\big) &= \int_{-\infty}^{\infty}\big(g(x) - \overline{g(x)}\big)^2 \cdot f_X(x)dx \\ &= E\big[g(X)^2\big] - E[g(X)]^2 \end{aligned}}$$

LOTUS – Law Of The Unconscious Statistician

$$Var(x) = E[x^2] - Ex^2$$
$$= 2 - 1^2 = 1 \Rightarrow SD(x) = \sqrt{1} = 1$$

Variance

- Linear function: $g(X) = aX + b$

$$Var(aX + b) = E\left[(aX + b)^2\right] - E[aX + b]^2$$

$$= \int_{-\infty}^{\infty} (ax + b)^2 \cdot f_X\,(x)dx - (a \cdot E[X] + b\ )^2$$

$$= a^2\left(E[X^2] - E[X]^2\right) = a^2 \cdot Var(X)$$

- Linear sum: $\mathrm{Z} = aX + bY$ (X and Y independent)

$$Var(Z) = \sigma_Z^2 = Var(aX + bY) = a^2 \cdot Var(X) + b^2 \cdot Var(Y)$$

Discrete vs Continuous Random Variables

Discrete (Σ)	Continuous (∫)
CDF: $F_X(x) = \Pr(X \le x) = \sum_{x_i \le x} f_X(x_i)$ non-decreasing step-function	CDF: $F_X(x) = \Pr(X \le x) = \int_{-\infty}^{x} f_X\,(x)dx$ non-decreasing continuous function
PMF: $f_X(x) = \begin{cases} Pr(X = x_i) & for\ x = x_i \in R_X \\ 0 & otherwise \end{cases}$	PDF: $f_X(x) = \frac{dF_X(x)}{dx}$
$E[X] = EX = \mu_X = \bar{X} = \sum_{x_{k \in R_X}} x_k \cdot f_X(x_k)$	$E[X] = EX = \mu_X = \bar{X} = \int_{-\infty}^{\infty} x \cdot f_X(x)dx$
$E[g(X)] = \sum_{x_{k \in R_X}} g(x_k) \cdot f_X(x_k)$	$E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)dx$
$Var(X) = \sigma_X^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 f_X(x_i)$ $= E[X^2] - E[X]^2$	$Var(\mathrm{X}) = \sigma_X^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 \cdot f_X\,(x)dx$ $= E[X^2] - E[X]^2$

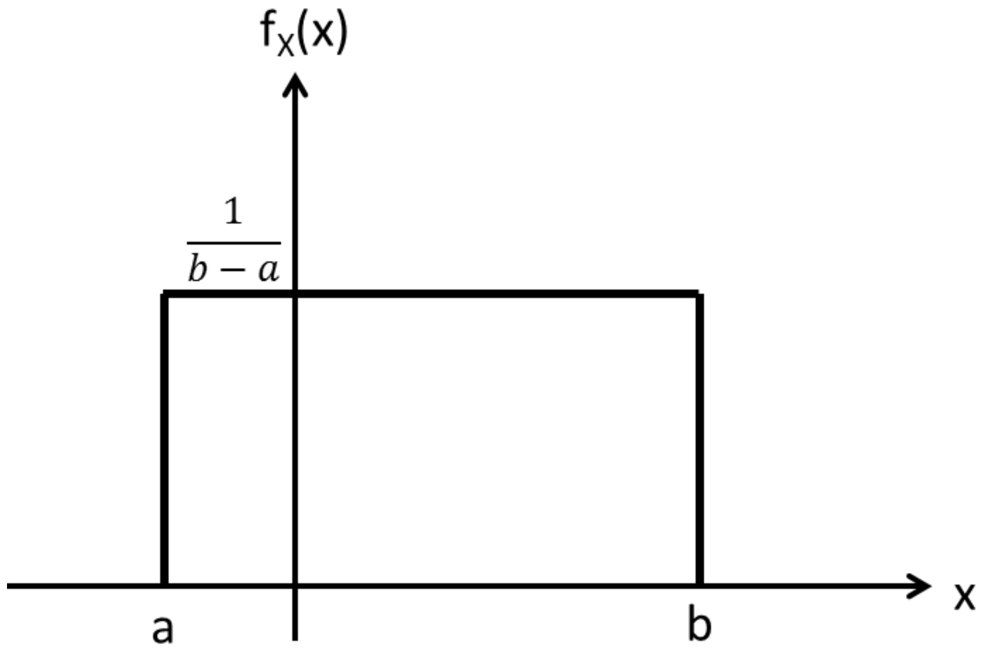
Uniform Distribution (continuous)

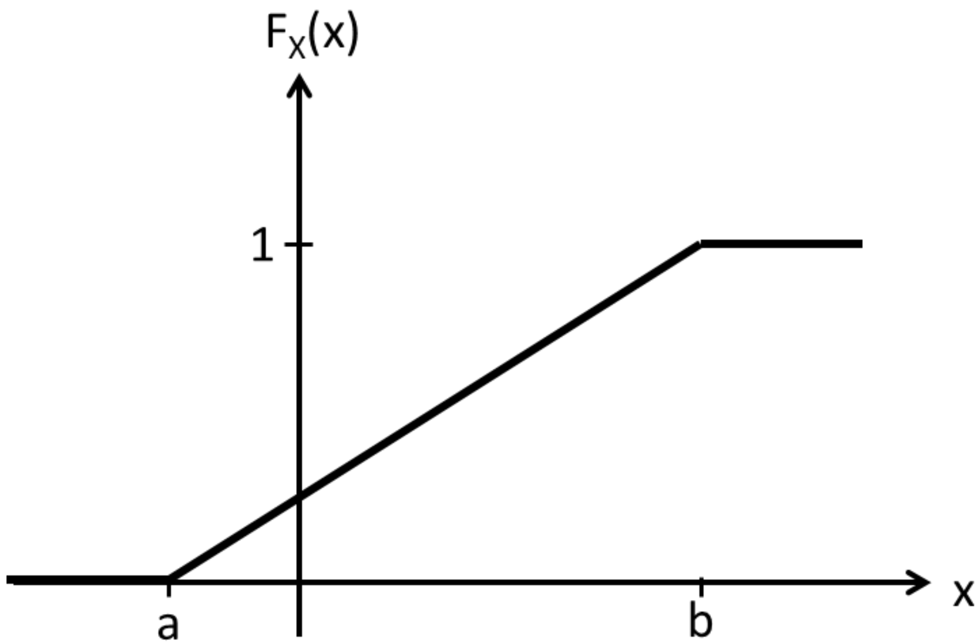
- $\mathcal{U}$(a,b)
- Mean value: $\mu = \frac{a+b}{2}$
- Variance: $\sigma^2 = \frac{1}{12}(b-a)^2$
- pdf: $f_X(x) = \begin{cases} \frac{1}{b-a} & for\ a \le x \le b \\ 0 & otherwise \end{cases}$

  Mathcad: dunif(x,a,b)

- cdf: $F_X(x) = \begin{cases} 0 & for\ x \le a \\ \frac{x-a}{b-a} & for\ a \le x \le b \\ 1 & for\ x \ge b \end{cases}$

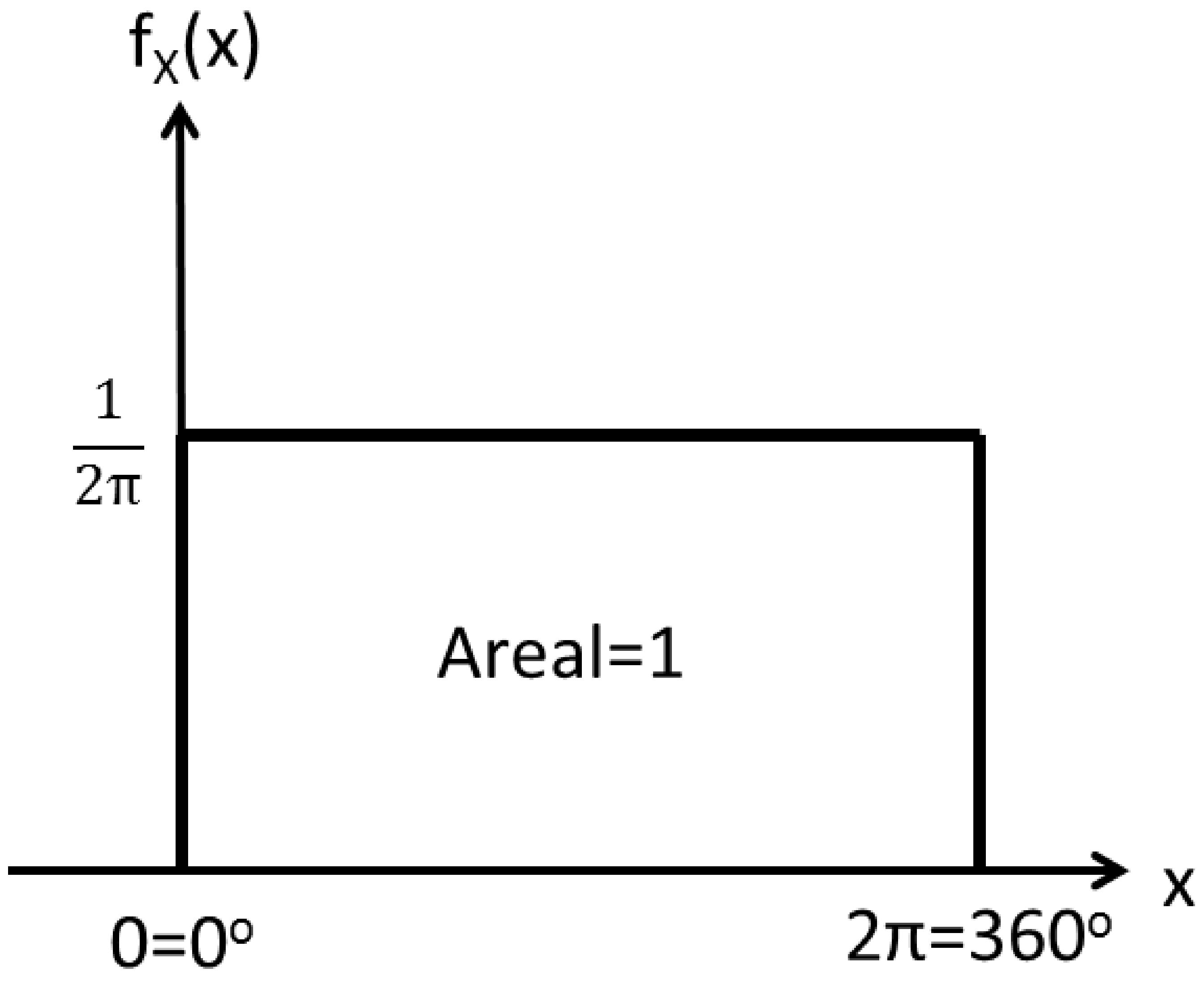
  Mathcad: punif(x,a,b)



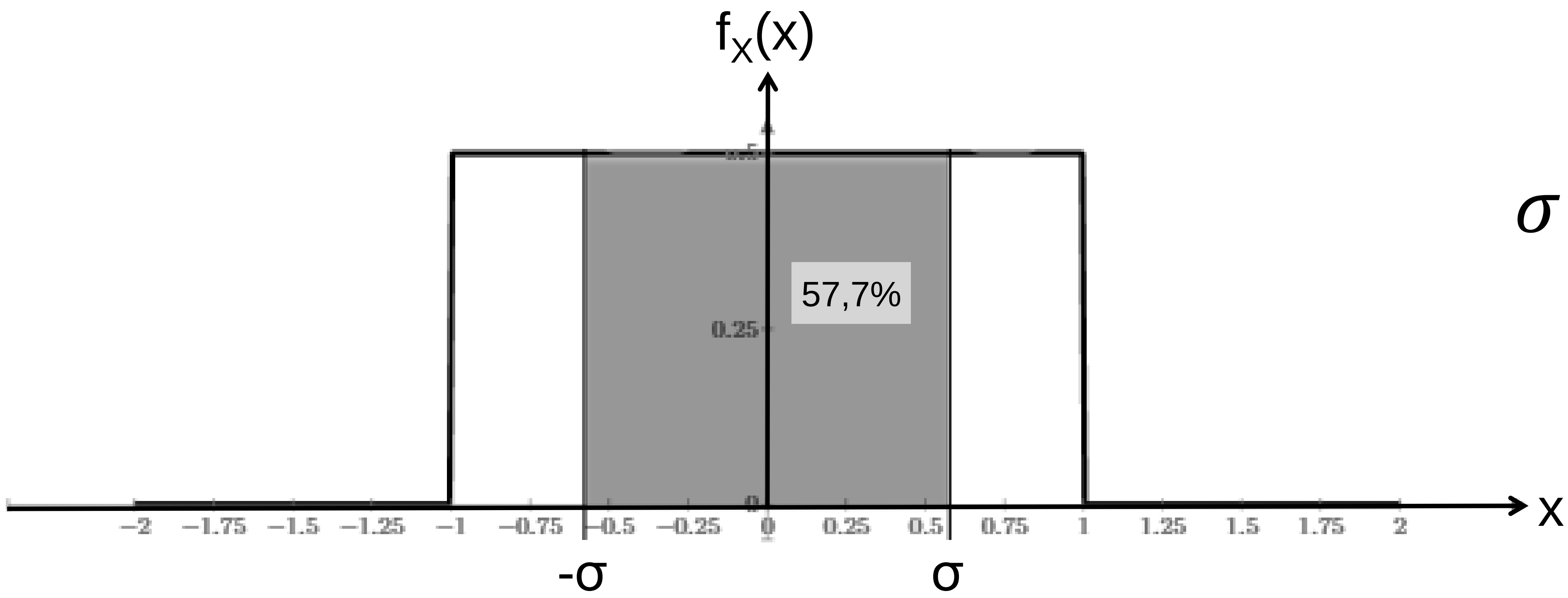


Uniform Distribution — Example

- A phase noise is uniformly distributed.



Uniform Distribution: Standard deviation



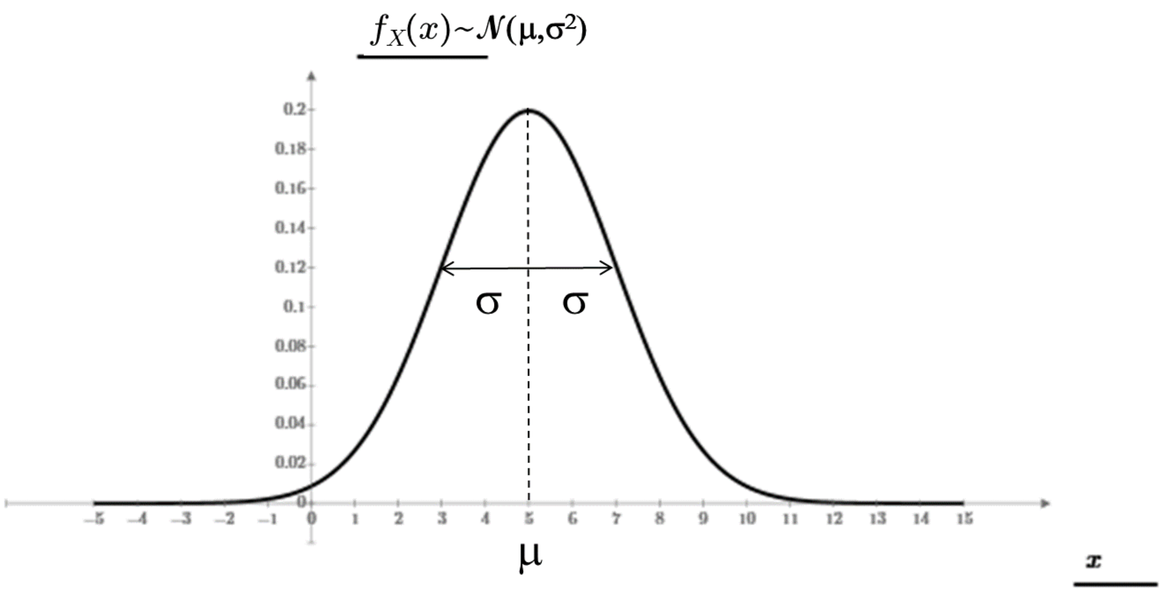
$$\sigma = \frac{b-a}{\sqrt{12}}$$

$$\Pr(|X-\mu| \leq \sigma) = 57{,}7\%$$

$$\Pr(|X-\mu| \leq 2\sigma) = 100\%$$

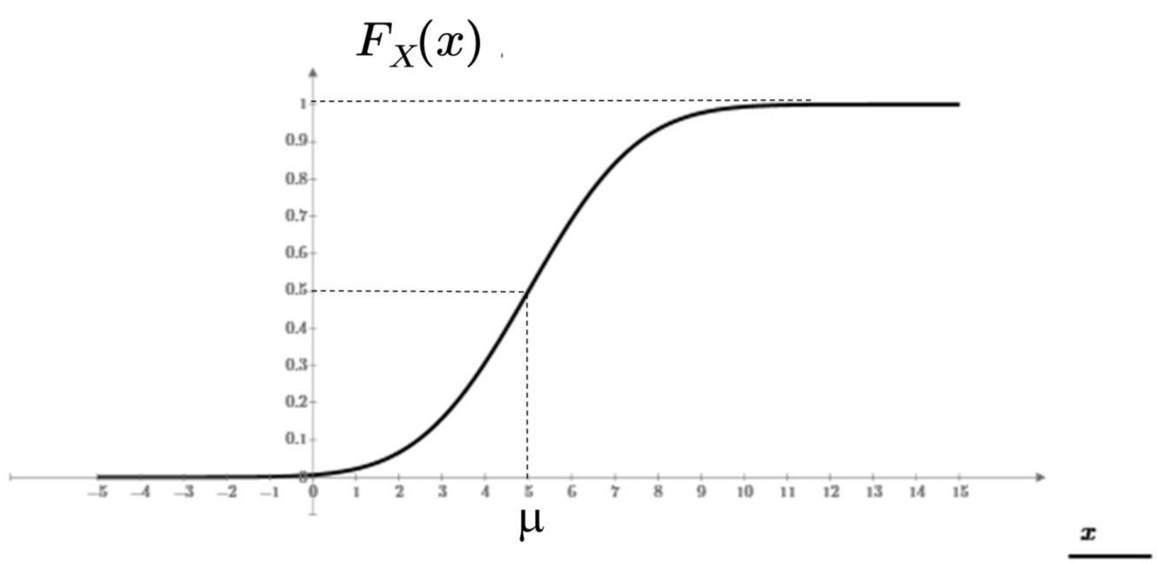
Gaussian Distribution = Normal Distribution

- $\mathcal{N}(\mu,\sigma^2)$
- Mean value: μ
- Variance: σ^2



- pdf: $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

- cdf: $F_X(x) = \frac{1}{2}\left[1 + \mathrm{erf}(\frac{x-\mu}{\sigma\sqrt{2}})\right]$



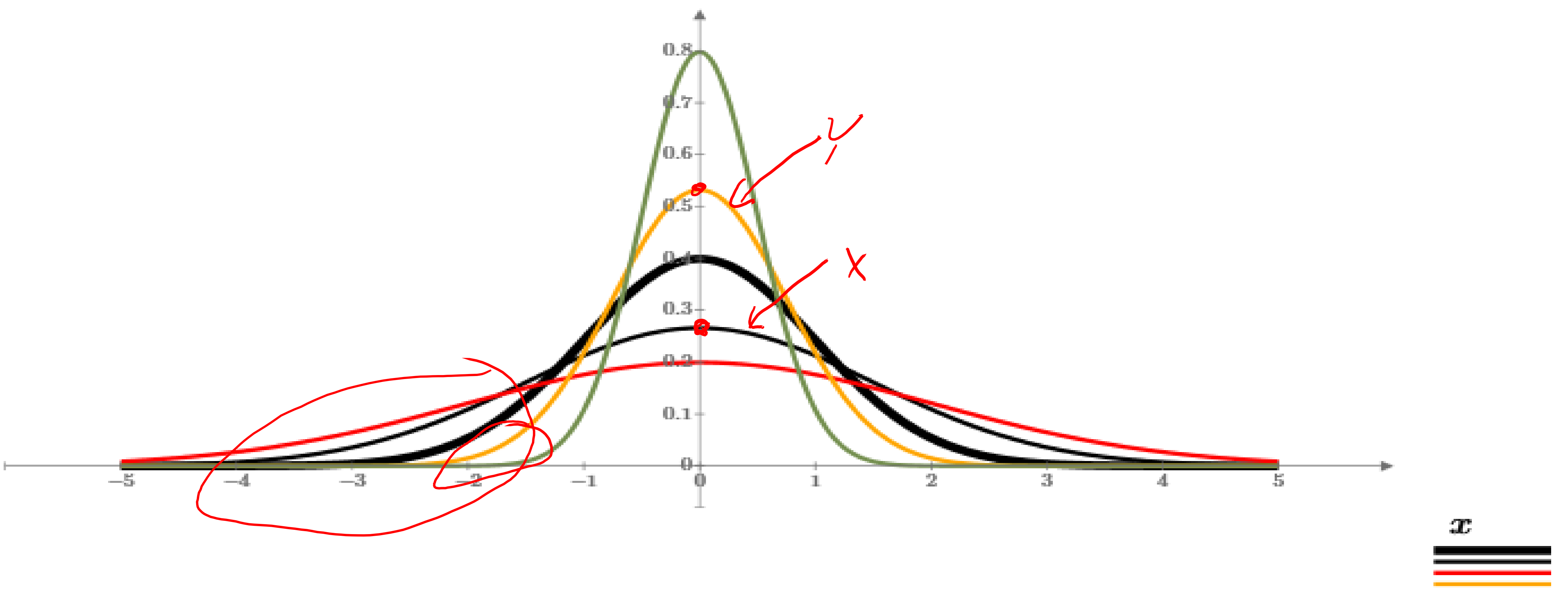
No closed expression for the cdf

erf = error-function: $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$

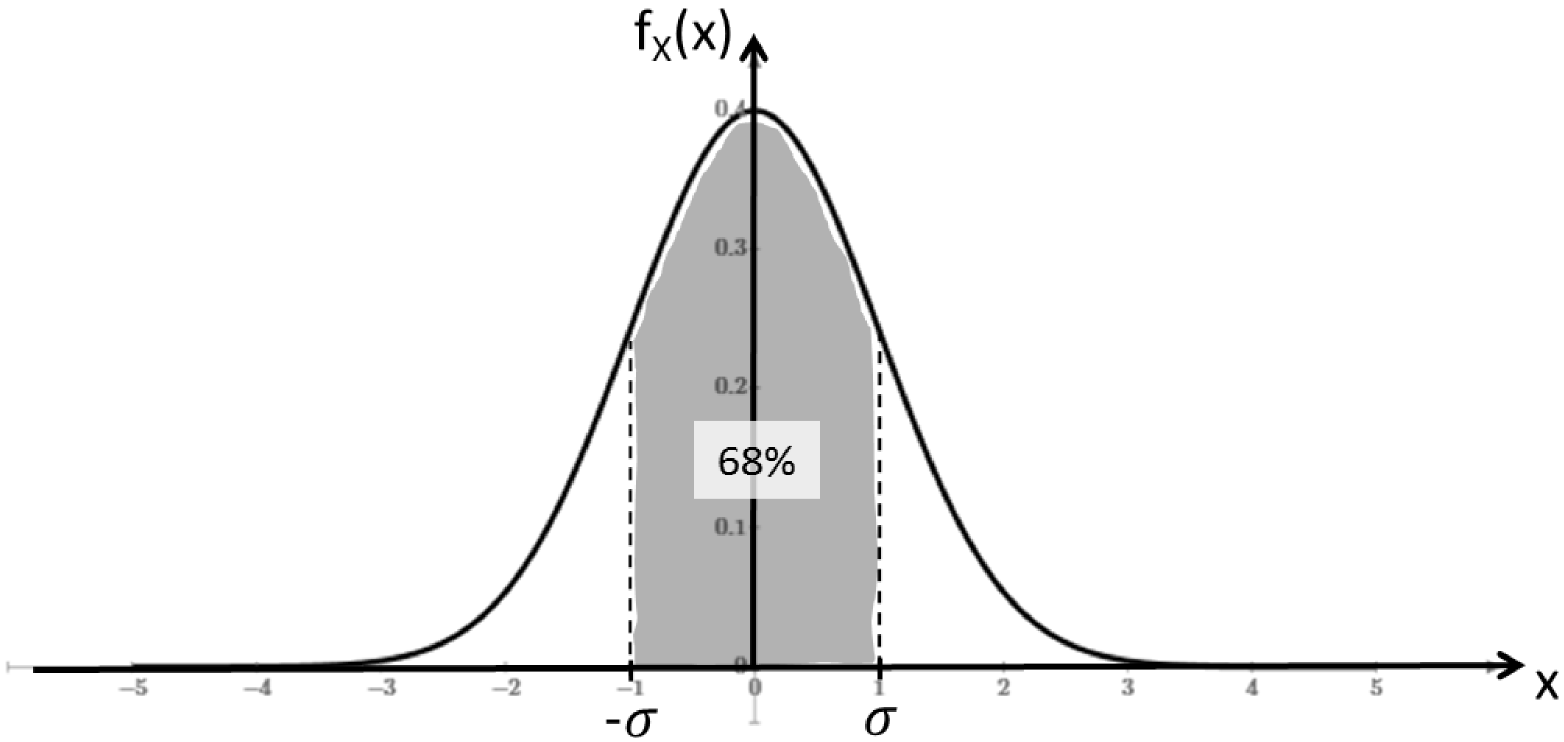
Gaussian Distribution = Normal Distribution

- Maximum probability density at the mean value μ
- The standard deviation (variance) σ determines the form (width and hight)

$$f_X(x,\sigma) \sim \mathcal{N}(0,\sigma^2)$$



Normal Distribution: Standard Deviation



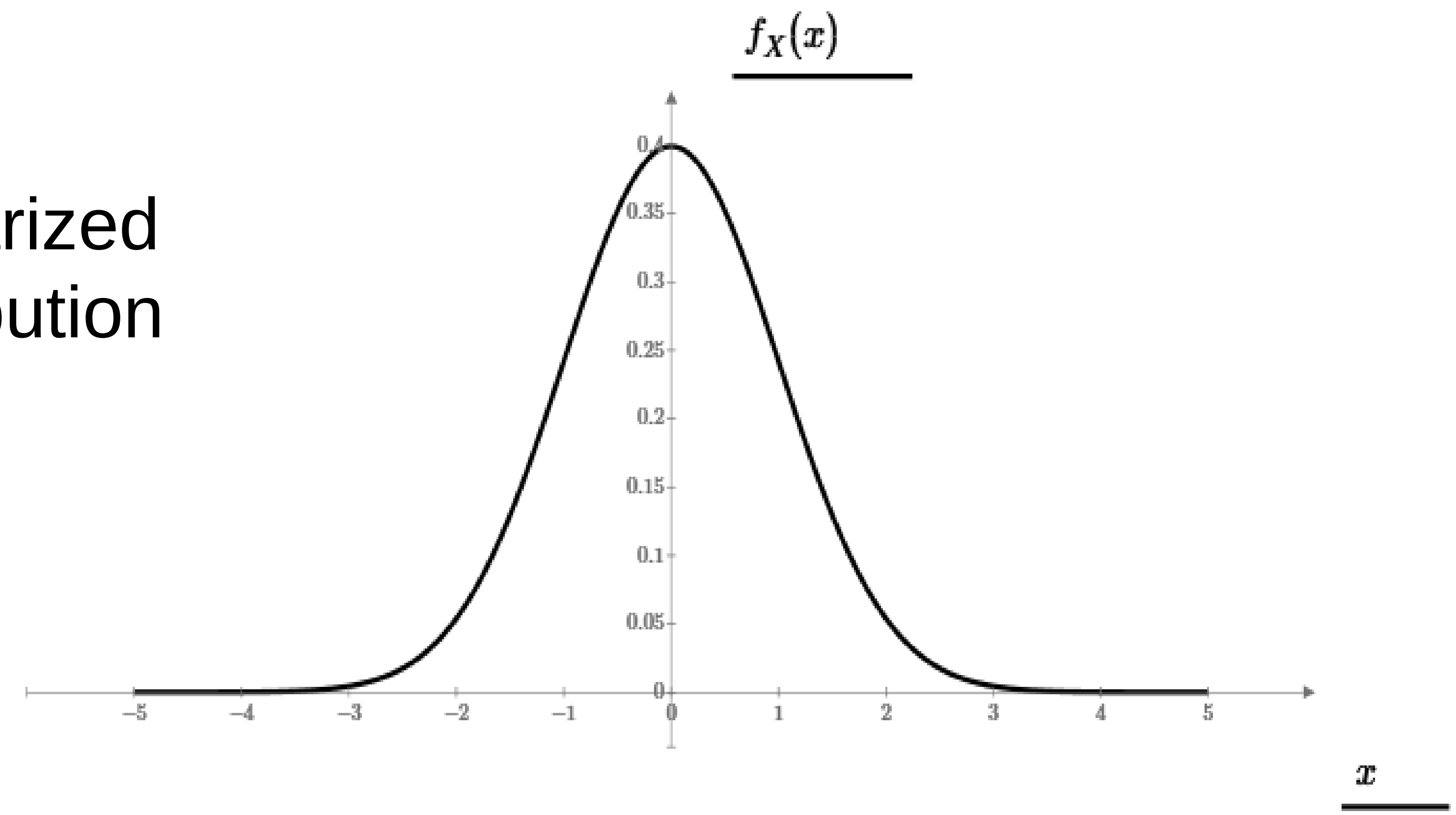
$$\Pr(|X-\mu| \leq \sigma) = 68{,}3\%$$

$$\Pr(|X-\mu| \leq 2\sigma) = 95{,}4\%$$

$$\Pr(|X-\mu| \leq 3\sigma) = 99{,}7\%$$

Gaussian Distribution = Normal Distribution

$\mathcal{N}(0,1)$
→ the standarized
normal distribution



- A lot of things in nature are Gaussian distributed
  - Fx. Examination marks
- Central Limit Theorem → Gaussian distrubution

Linear Transformation of Normal Distribution

- Linear transformation of normal random variable:
  - $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y = a \cdot X + b$
  - Then: $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$

    where $\mu_Y = a \cdot \mu_X + b$ and $\sigma_Y^2 = a^2 \cdot \sigma_X^2$

- Especially:
  - $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ and $X \sim \mathcal{N}(0,1)$
  - Then: $Y = \sigma_Y \cdot X + \mu_Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$
  - And: $X = \frac{Y - \mu_Y}{\sigma_Y} \sim \mathcal{N}(0,1)$

Se video!

Gaussian Distribution = Normal Distribution

- Beregninger med normalfordelinger: Tabelopslag, Matlab og Mathcad:
- $X \sim \mathcal{N}(\mu, \sigma^2) \;\rightarrow\; Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$ (Standard Normal Distribution)
- $F_X(x) = Pr(X \leq x) = Pr\left(Z \leq \frac{x-\mu}{\sigma}\right) = F_Z(z) = \Phi(z)$ hvor $z = \frac{x-\mu}{\sigma}$
- $\Phi(z) = Pr(Z \leq z)$
- $\Phi(-z) = 1 - \Phi(z)$

$Tabel$ 1 ("$Statistik\ og\ Sandsynlighedsregning$")

- Matlab:
  - $\Pr(X \leq x) = F_X(x) = normcdf(x, \mu, \sigma)$
  - $\Pr(Z \leq z) = F_Z(z) = normcdf(z, 0, 1) = normcdf(z)$

Obs: Standard deviation

- Mathcad Prime:
  - $\Pr(X \leq x) = F_X(x) = pnorm(x, \mu, \sigma)$
  - $\Pr(Z \leq z) = F_Z(z) = \Phi(z) = pnorm(z, 0, 1)$

Obs: Standard deviation

Very important!

i.i.d.: Independent and Identically distributed

- We define that for series of random variables that is taken from the <u>same distribution</u> (identically distributed), and are sampled <u>independent</u> of each other, that they are i.i.d.

  i.i.d. = Independent and Identically distributed

- i.i.d. is a very important characteristic in stochastic variable processing and statistics

Central Limit Theorem

- Let $X_1, X_2, \dots, X_n$ be i.i.d. random variables with mean μ and variance σ^2
- Let $\bar{X}$ be the random variable (average):

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

- Then in the limit: $n \to \infty$ we have that: $\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$

i.e. in the limit $\bar{X}$ will be normally distributed with mean = μ and variance = $\frac{\sigma^2}{n}$.

The variance is reduced with a factor 1/n

Central Limit Theorem

- Let $X_1, X_2, \dots, X_n$ be i.i.d. random variables with mean μ and variance σ^2
- Let X be the random variable:

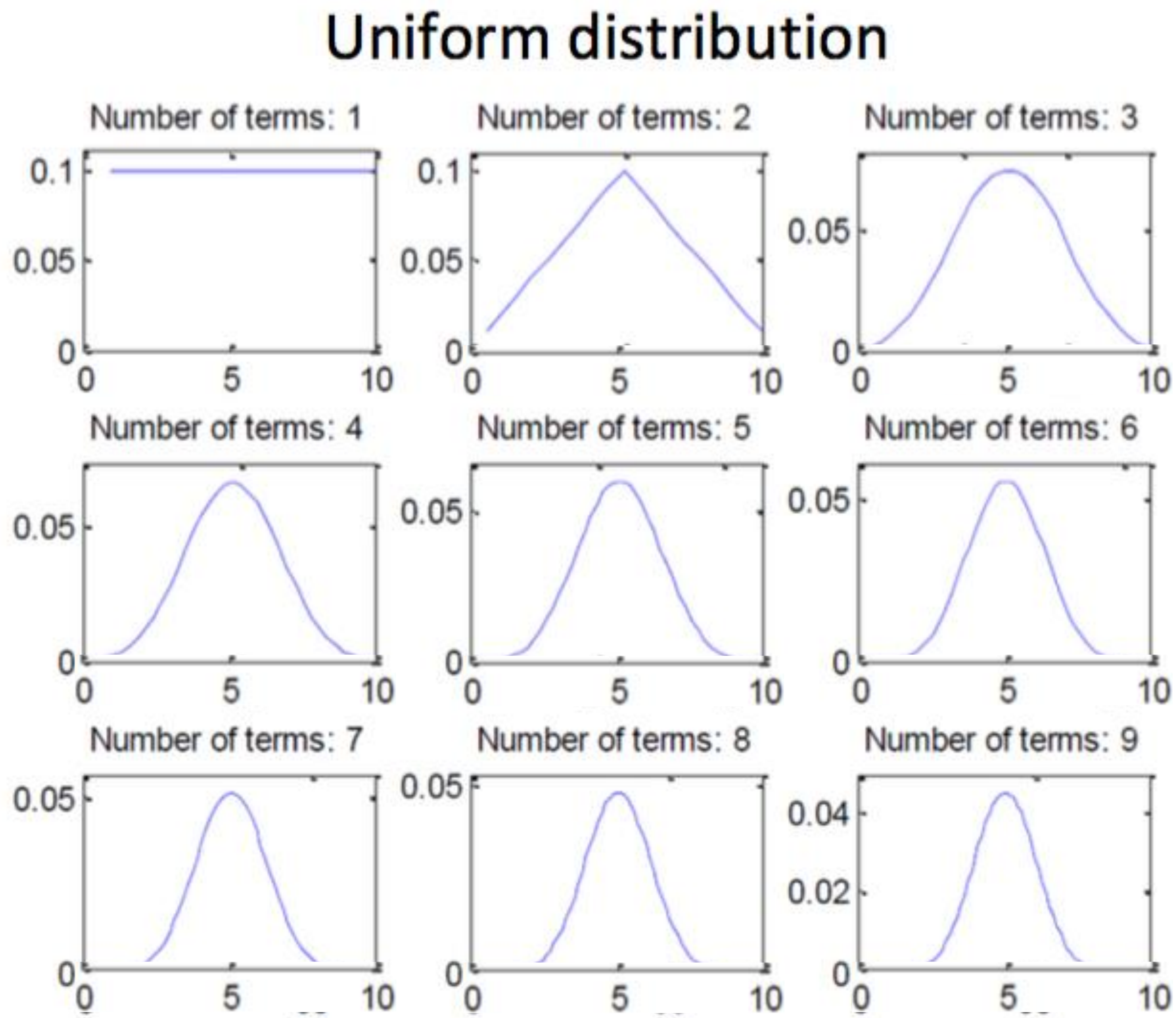
$$X = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i - \mu}{\sqrt{\sigma^2/_n}}$$

- Then in the limit: $n \to \infty$ we have that: $X \sim \mathcal{N}(0,1)$

  i.e. in the limit X will be normally distributed with mean = 0 and variance = 1 (standard normal distributed).

Sum of Random Variables

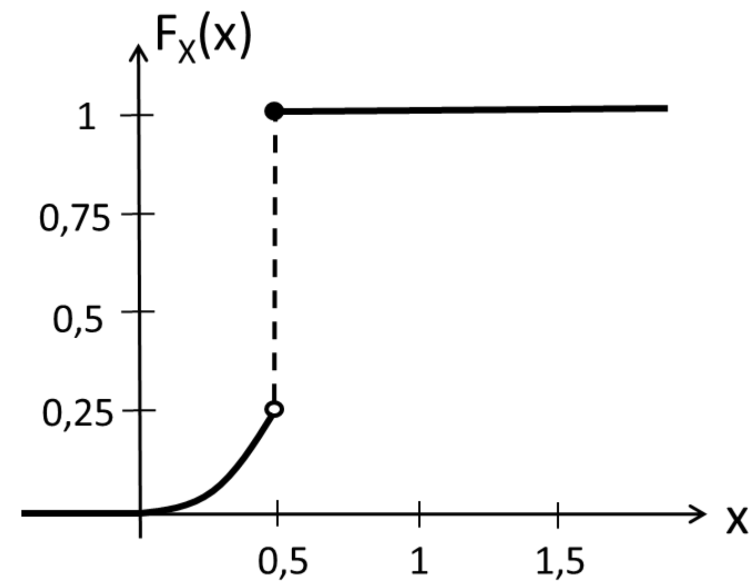
- The random variables are i.i.d and taken from the same uniform distribution.



Mixed Random Variables

- A CDF of a random variable X is given by:

$$F_X(x) = \begin{cases} 0 & for\ x < 0 \\ x^2 & for\ 0 \le x < ½ \\ 1 & for\ x \ge ½ \end{cases}$$

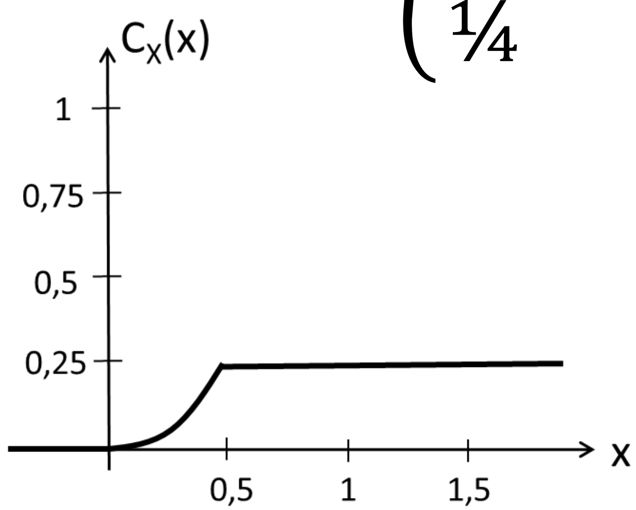


Called a Mixed Random Variable

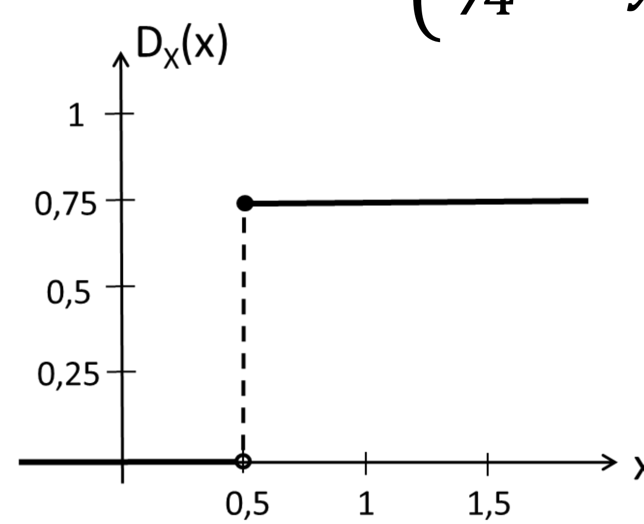
- X is neither discrete or continuous: $F_X(x) = C(x) + D(x)$

$$C(x) = \begin{cases} 0 & for\ x < 0 \\ x^2 & for\ 0 \le x < ½ \\ ¼ & for\ x \ge ½ \end{cases}$$

$$D(x) = \begin{cases} 0 & x < ½ \\ ¾ & x \ge ½ \end{cases}$$



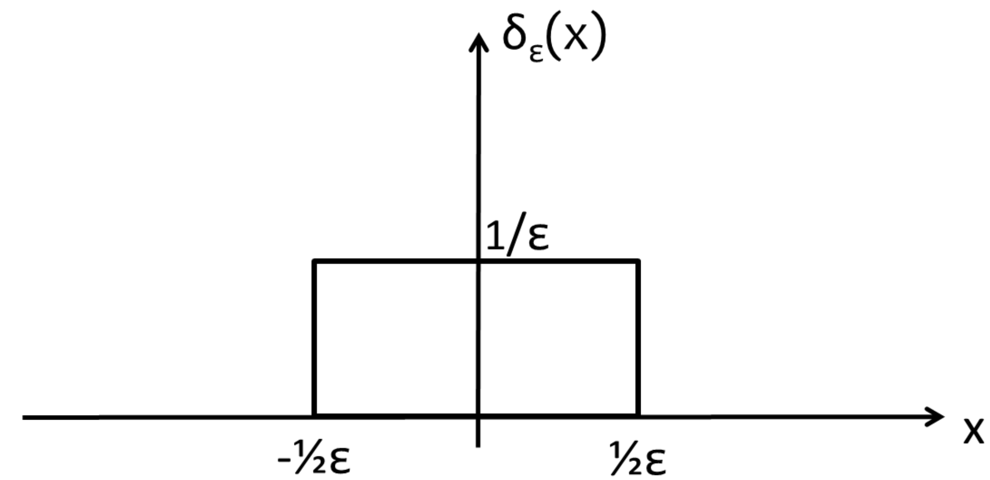
Continuous



Discrete

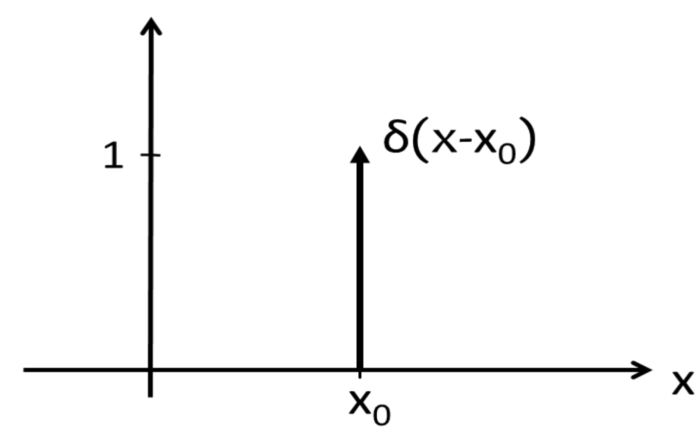
Dirac's Delta Function

- $\delta_\varepsilon(x) = \begin{cases} 1/\varepsilon & -½\varepsilon \le x \le ½\varepsilon \\ 0 & otherwise \end{cases}$
  - $\int_{-\infty}^{\infty} \delta_\varepsilon(x)dx = \frac{1}{\varepsilon} \cdot \varepsilon = 1$



Dirac's Delta Function

- $\delta(x) = \lim_{\varepsilon \to 0} \delta_\varepsilon(x) = \begin{cases} \infty & x = 0 \\ 0 & otherwise \end{cases}$
  - $\int_{-\infty}^{\infty} \delta(x)dx = 1$
  - $\int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = \int_{x_0-\varepsilon}^{x_0+\varepsilon} f(x)\delta(x-x_0)dx = f(x_0)$
  - $\delta(x) = \frac{du(x)}{dx}$ where $u(x)$ is the unit step function



Mixed Random Variables

- CDF of a mixed random variable X : $F_X(x) = C(x) + D(x)$
  - $C(x)$ continuous with $\frac{dC(x)}{dx} = c(x)$, $\mathrm{C}(-\infty) = 0$ and $\mathrm{C}(\infty) < 1$
  - $D(x) = \sum_{x_i \in R_X} Pr(X = x_i) \cdot u(x - x_i)$

- PDF of a mixed random variable X :

The generalized PDF

  - $\boxed{f_X(x) = c(x) + \sum_{x_i \in R_X} Pr(X = x_i) \cdot \delta(x - x_i)}$

- $\lim_{x \to \infty} F_X(x) = \int_{-\infty}^{\infty} c(x)\, dx + \int_{-\infty}^{\infty} \sum_{x_i \in R_X} Pr(X = x_i) \cdot \delta(x - x_i)\, dx$

$$= \mathrm{C}(\infty) + \sum_{x_i \in R_X} Pr(X = x_i) \cdot u(x - x_i) = 1$$

Words and Concepts to Know

Probability density function

Central Limit Theorem

Continuous random variable

Uniform distribution

Gaussian distribution

pdf

Histogram

Independent and Identical Distributed

Normal distribution

Dirac's delta function

i.i.d.

Mixed random variable

Generalized pdf

Exponential distribution

Standard Normal distribution