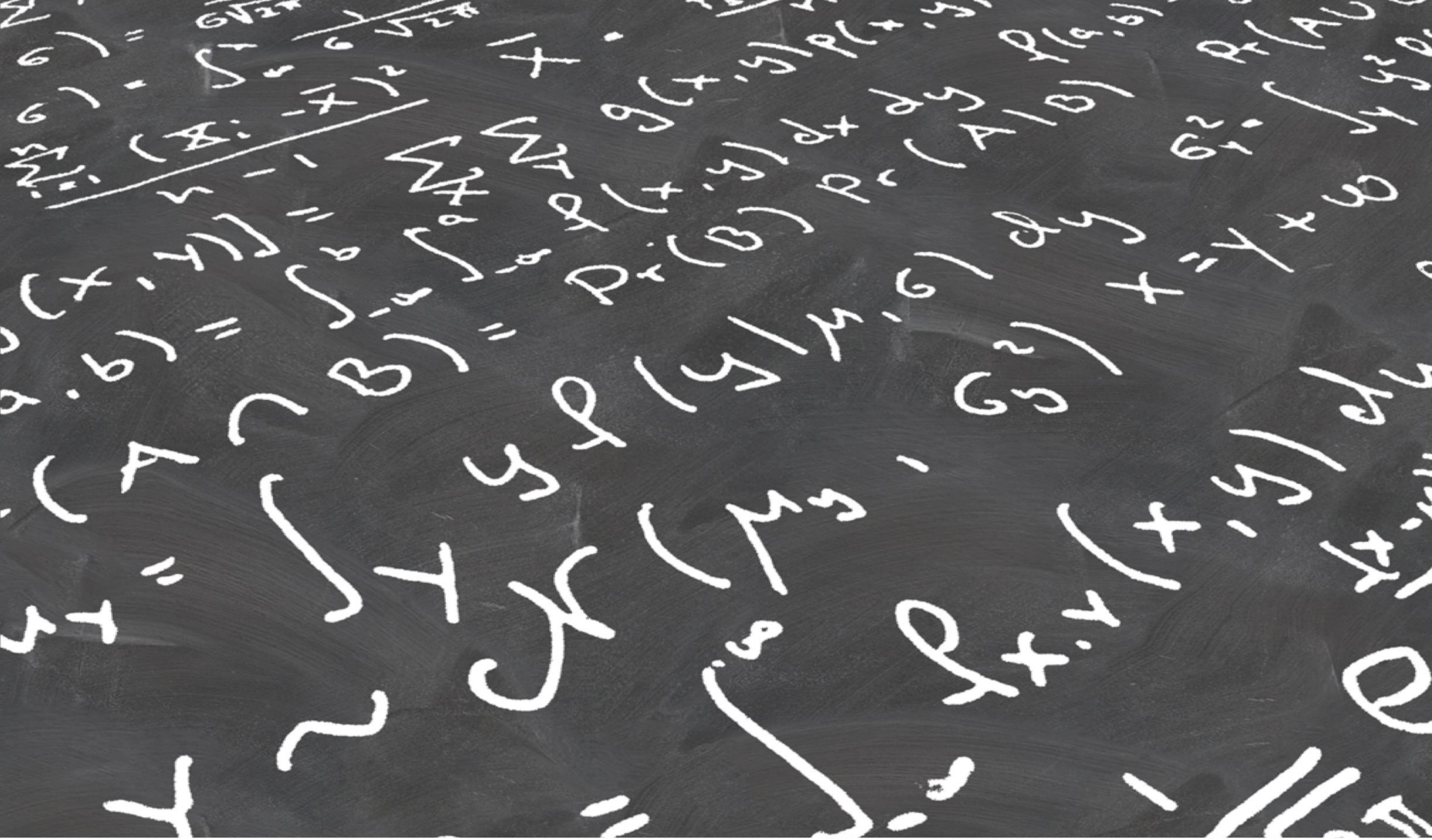

6. Introduction to Stochastic Processes

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Lars Mandrup

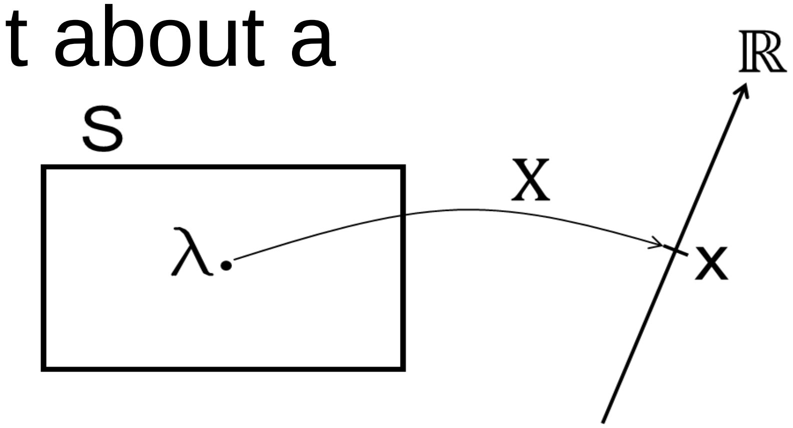
Agenda for Today

- Repetition from last time
  - Random Variables
- Stochastic Processes
  - Definition
  - Stationarity (WSS, SSS)
  - Ergodic Processes

Also just called a random variables

Stochastic Random Variables

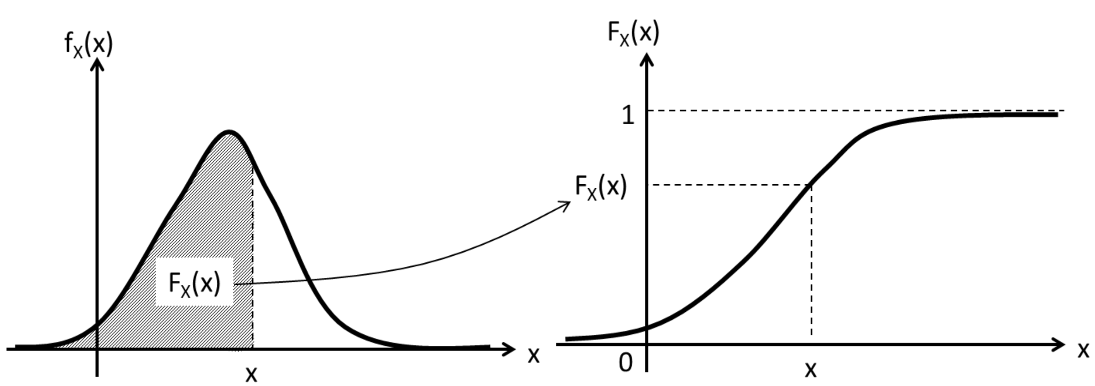
- A random variable tells something important about a stochastic experiment.
- Can be discrete or continous



- Probability density function (pdf):

$$Pr(a \leq X \leq b) = \int_a^b f_X(x)\, dx \qquad f_X(x) \geq 0 \qquad \int_{-\infty}^{\infty} f_X(x) dx = 1$$

- Cumulative distribution function (cdf):



$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du = Pr(X \leq x)$$

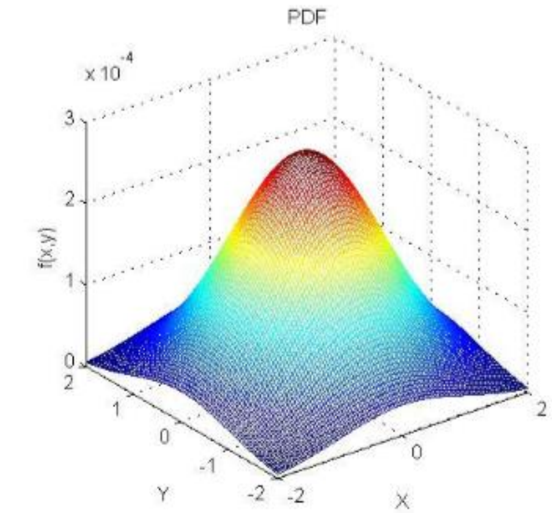
$$0 \leq F_X(x) \leq 1$$

$$\lim_{x \to -\infty} F_X(x) = 0 \qquad \lim_{x \to \infty} F_X(x) = 1$$

Two Simultaneous Continuous Random Variables

Joint (Simultaneous) pdf: $f_{X,Y}(x,y) \geq 0$ $\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1$

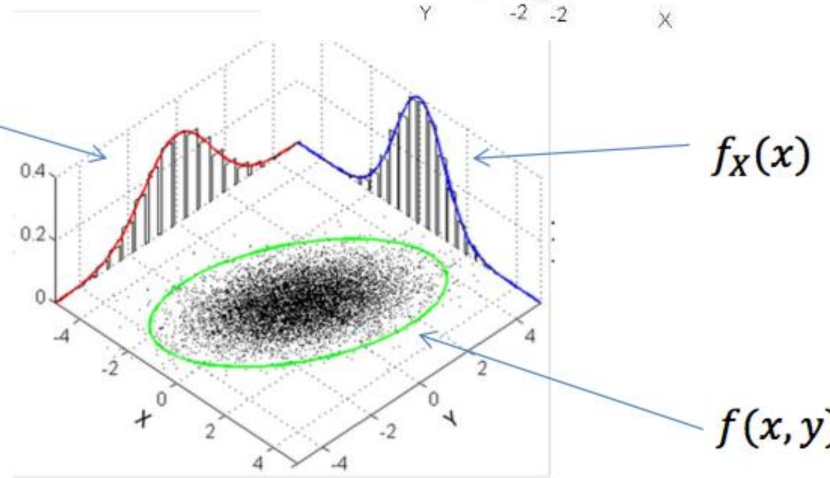
$$Pr\big((a \leq X \leq b) \cap (c \leq Y \leq d)\big) = \int_c^d \int_a^b f_{X,Y}(x,y)dxdy$$



Marginals:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$



Cumulative Distribution Function cdf:

cdf $\quad F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(x,y)dxdy = Pr(X \leq x \wedge Y \leq y)$

pdf $\quad f_{X,Y}(x,y) = \dfrac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$

Two Random Variables

Two random variables: X and Y

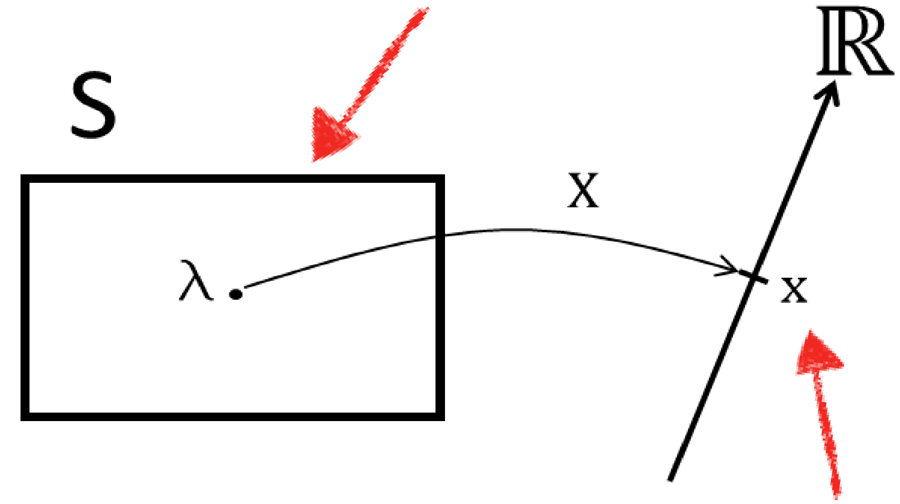
- Simultaneous pdf: $f_{X,Y}(x, y)$ $\rightarrow f_X(x) \cdot f_Y(y)$ if <u>independent</u>
- Marginal pdf: $f_X(x)$ and $f_Y(y)$
- Conditional pdf: $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$ $\rightarrow$ Bayes rule
- Simultaneous cdf: $F_{X,Y}(x, y) = Pr(X \leq x \wedge Y \leq y)$
- Correlation: $corr(X, Y) = E[XY]$
- Covariance: $cov(X, Y) = E[XY] - E[X]E[Y]$
- Correlation coefficient: $\rho = \frac{E[XY]-E[X]E[Y]}{\sigma_X \cdot \sigma_Y}$
- Sum: $\mathrm{Z} = X + Y$ $\rightarrow f_Z(z) = f_X(z) * f_Y(z)$ if <u>independent</u>
- Expectation: $\mathrm{E}[\mathrm{Z}] = E[X] + E[Y]$
- Variance: $\mathrm{Var}[\mathrm{Z}] = Var[X] + Var[Y]$ if <u>independent</u>
  $\mathrm{Var}[\mathrm{Z}] = Var[X] + Var[Y] + 2cov(X, Y)$ if <u>dependent</u>

Stochastic Processes

Stochastic Variables

- Sample space for stochastic experiment

Sample space for
stochastic experiment



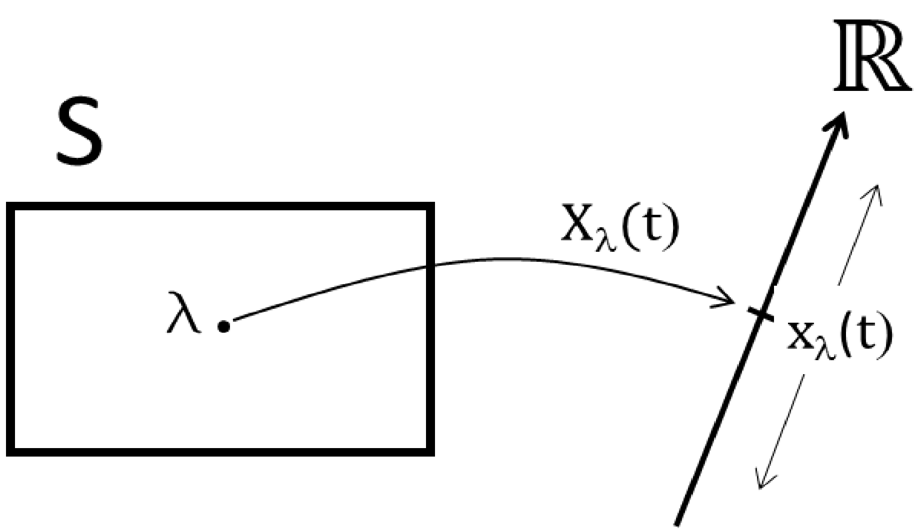
Sample space for
stochastic variable

Time dependent

Stochastic Processes (signals)

- Sample space for stochastic experiment
- Random events that develops in time

Sample space for
stochastic experiment



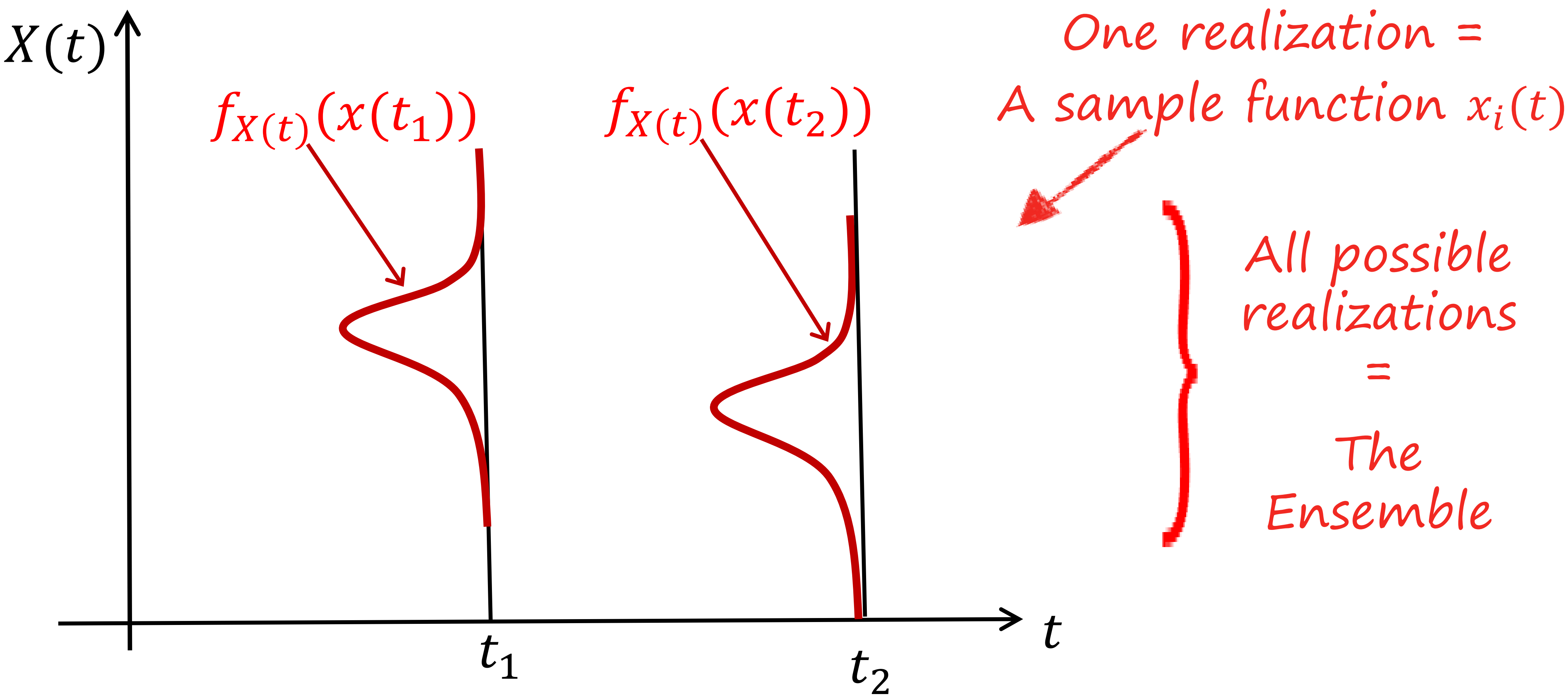
Sample space for
stochastic proces

Also called Random Signals

Stochastic Processes

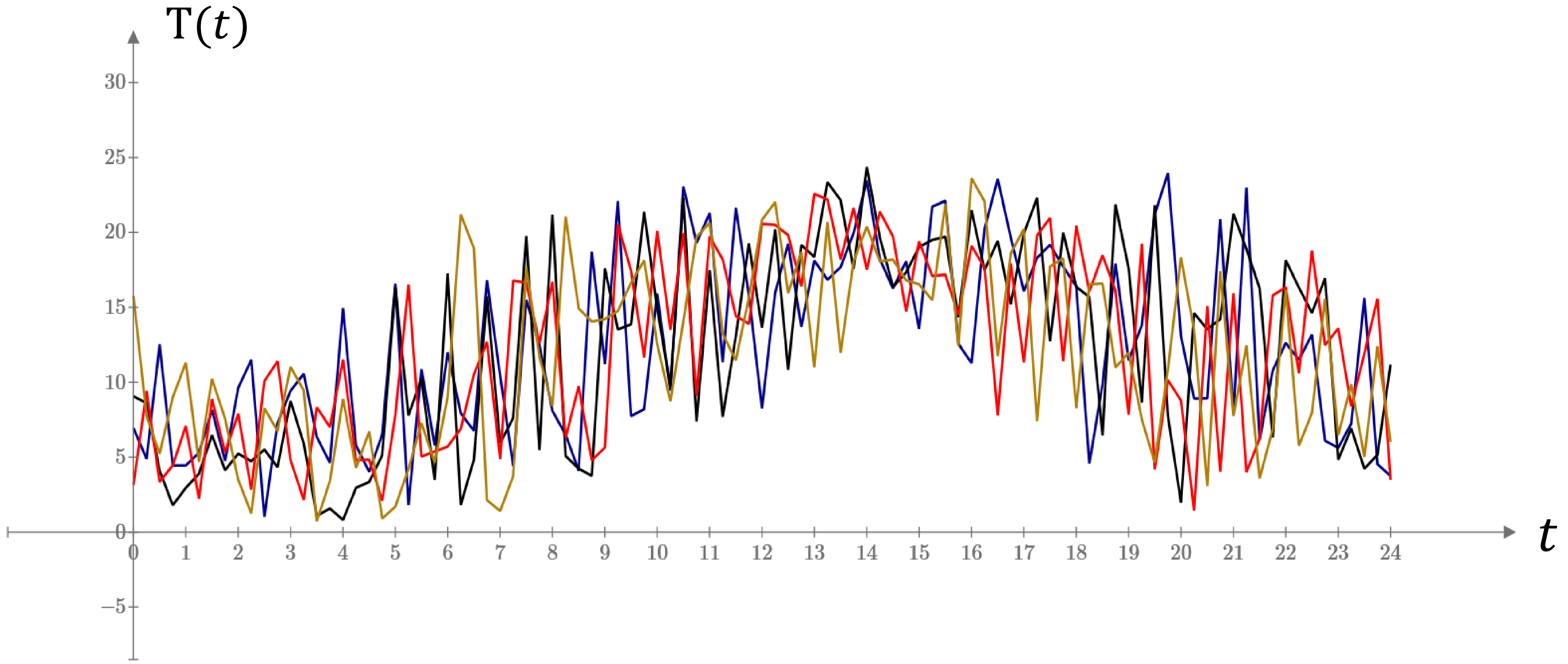
<u>A stochastic process X:</u>

- a collection of time-dependent random (continuous or discrete-valued) variables $x_i(t)$
- at a fixed time $X(t_1)$ is a random variable with a cdf: $f_{X(t)}(x(t_1))$



Stochastic Processes – Outdoor Temperature

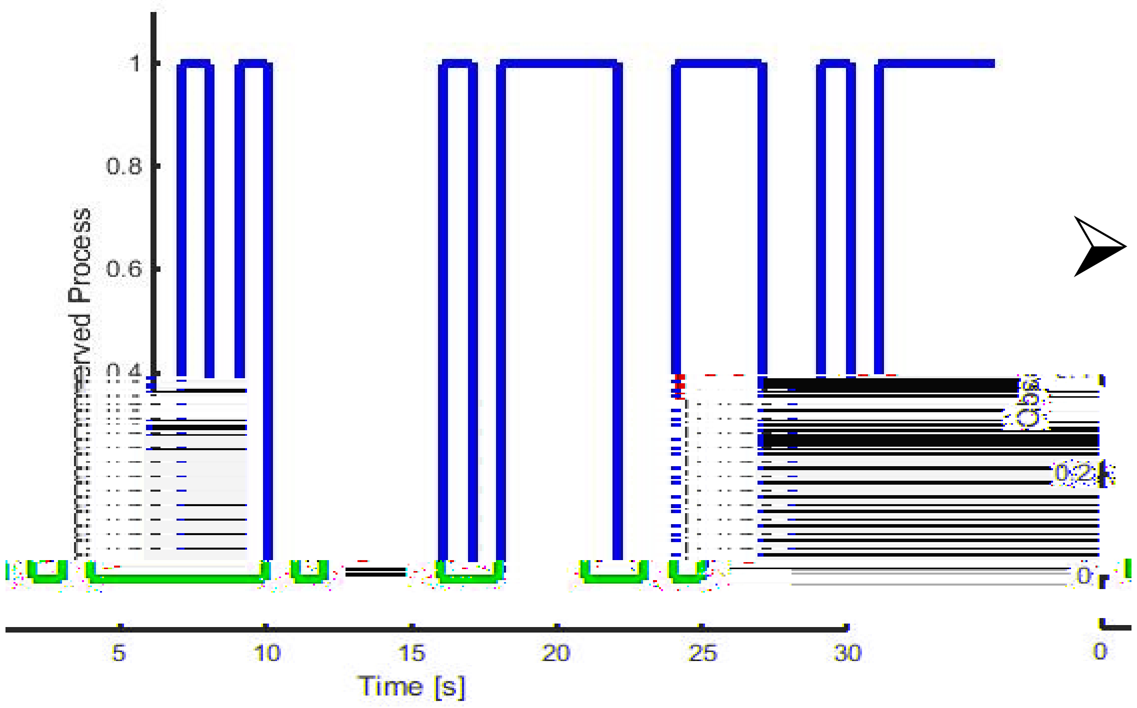
- Outdoor temperature $\mathrm{T}(t)$
  - Continuous-valued: $\mathrm{T}(t) \in [-100, 100]$
  - Continuous-time: $t \in [0, 24]$



Stochastic Processes – Random Binary (digital) Signal

- Bernoulli process (flip a coin)
- A random sequence of H (1) and T (0): THHTHTT…
- A sequence of i.i.d. Bernoulli trials

One realization of a Bernoulli Proces = A sample function x(t)



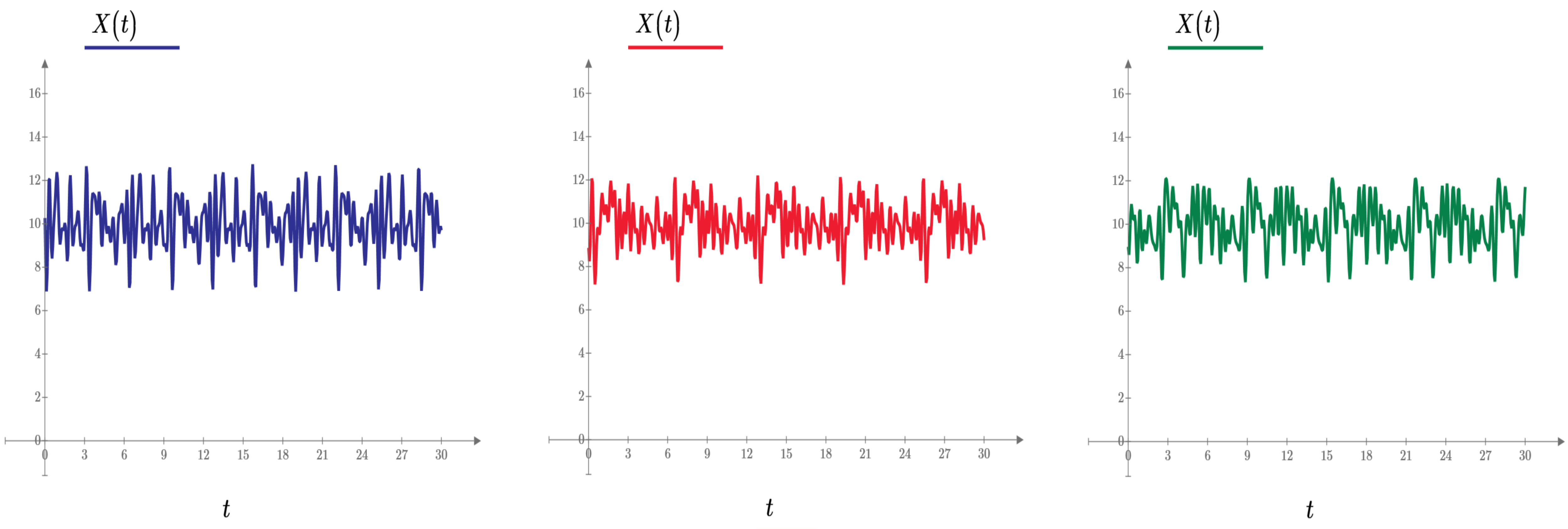
➢ Digital noise $X(t)$
- Discrete-valued: $X(t) \in \{0,1\}$
- Discrete-time: $t \in \{0, \Delta T, 2\Delta T, \dots, n\Delta T, \dots\}$

Stochastic Processes – Random Signals

Additive Noisemodel

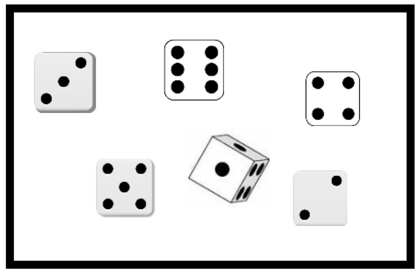
$$observed\ signal = signal\ +\ noise$$

Three Realizations of the Stochastic Process

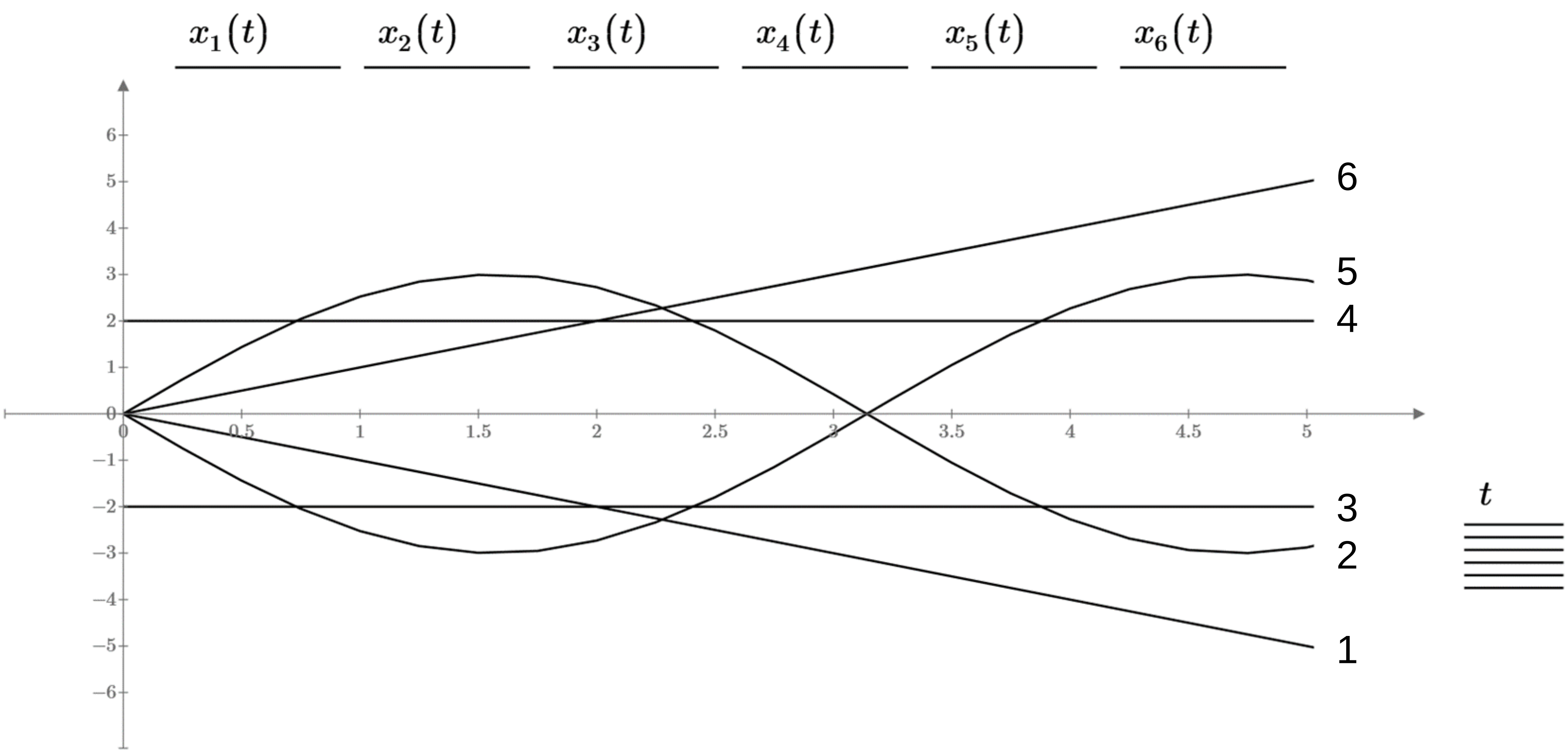


Stochastic Processes – Example

S



$X(t)$: $x_1(t) = -t \quad x_2(t) = 3\sin(t)$
$x_3(t) = -2 \quad x_4(t) = 2$
$x_5(t) = -3\sin(t) \quad x_6(t) = t$



Stochastic Processes

Definitions:

- A stochastic process is a <u>time dependent</u> stochastic variable:

Continuous-time

$$X(t)$$

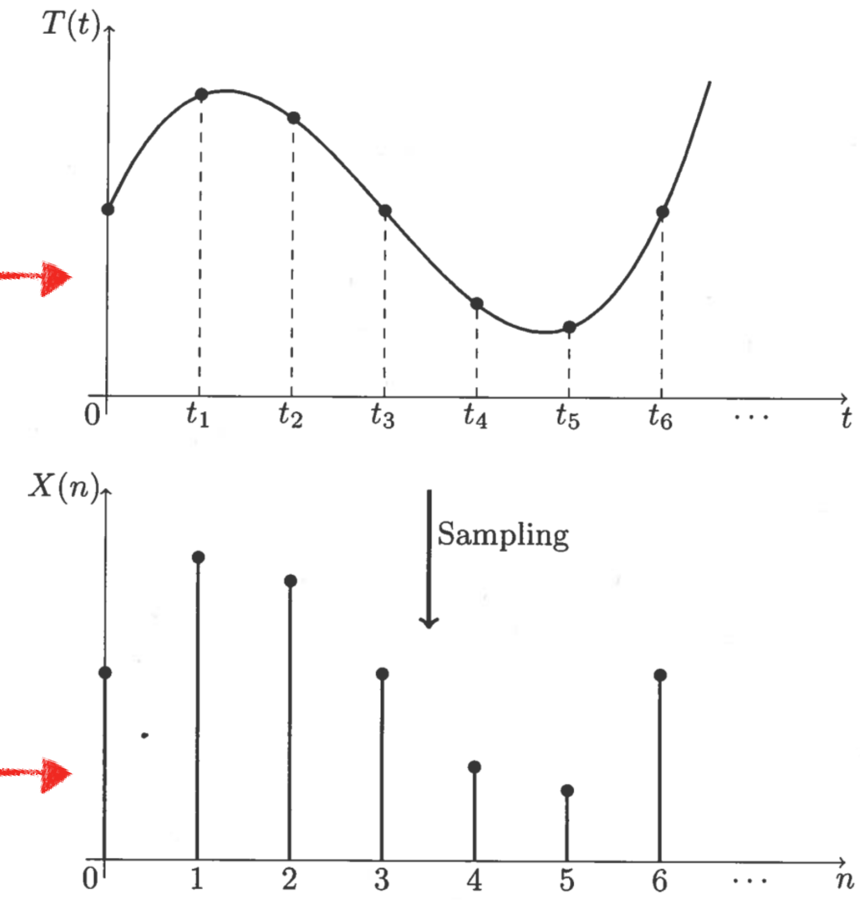
- A discrete stochastic process is given by:

time

$$X[n] = X(nT)$$

where n is an integer.

Discrete-time



Notice:

- When we measure/sample a signal from a stochastic process, we observe only one <u>realization</u> of the process

Sample Functions – Realizations – Ensemble

Definition:

- A <u>Sample Function</u> $x(t)$ is a <u>realization</u> of a stochastic process X
- The <u>Ensemble</u> of the Stochastic Process is the collection of <u>all possible realizations</u> $x(t)$ of the Stochastic Process X

Example:

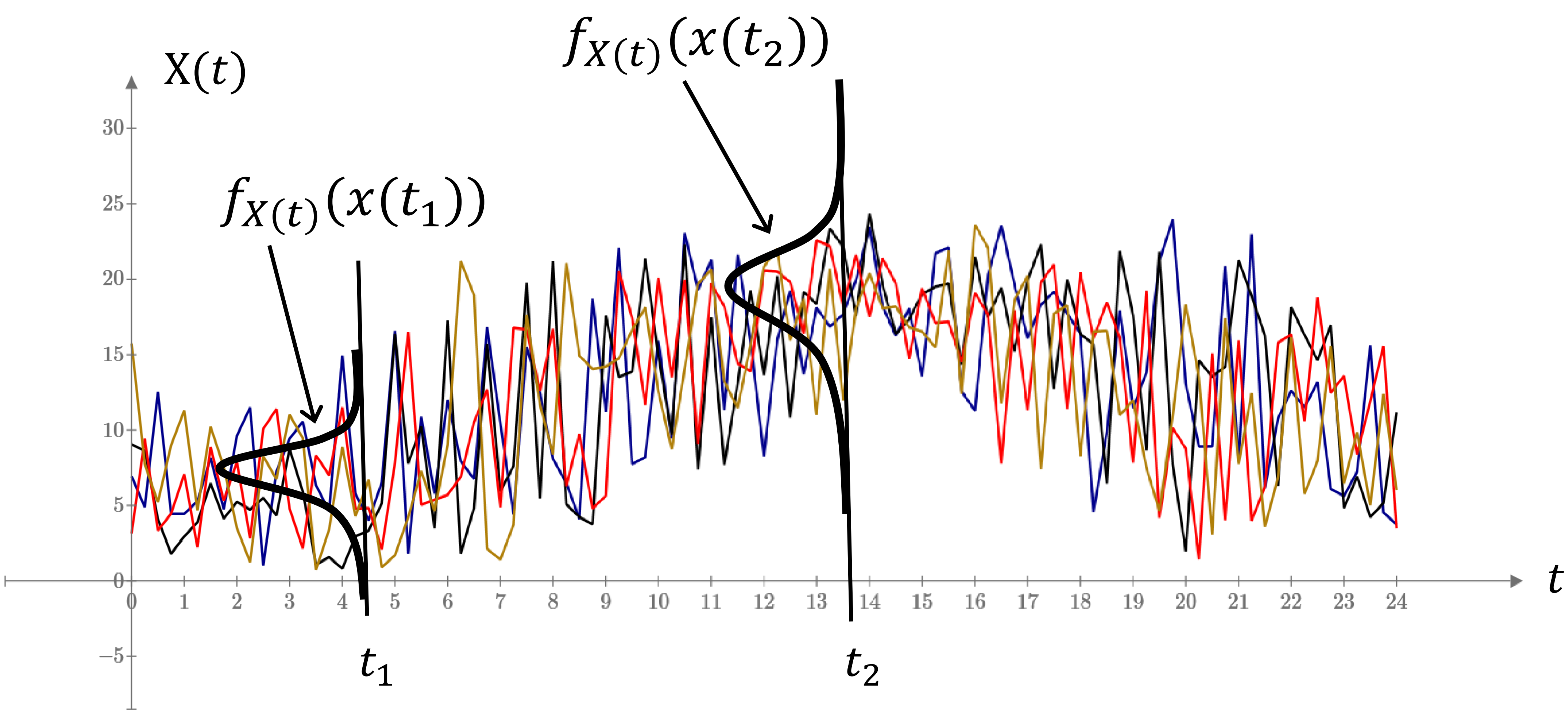
- A coin is thrown every minute: H = head, T = tail
- One realization of the stochastic signal is: $HTHT$
- The Ensemble of the stochastic signals is:

$HTHT, HHTT, TTHH, THTH, THHT, TTHT, HHHH\ldots$

Time Dependent Probability Functions

- Probability density function (pdf): $f_{X(t)}(x(t))$
- Cumulative distribution function (cdf):

$$F_{X(t)}\big(x(t)\big) = Pr\big(X(t) \le x(t)\big) = \int_{-\infty}^{x(t)} f_{X(t)}(x(t))\, dx(t)$$

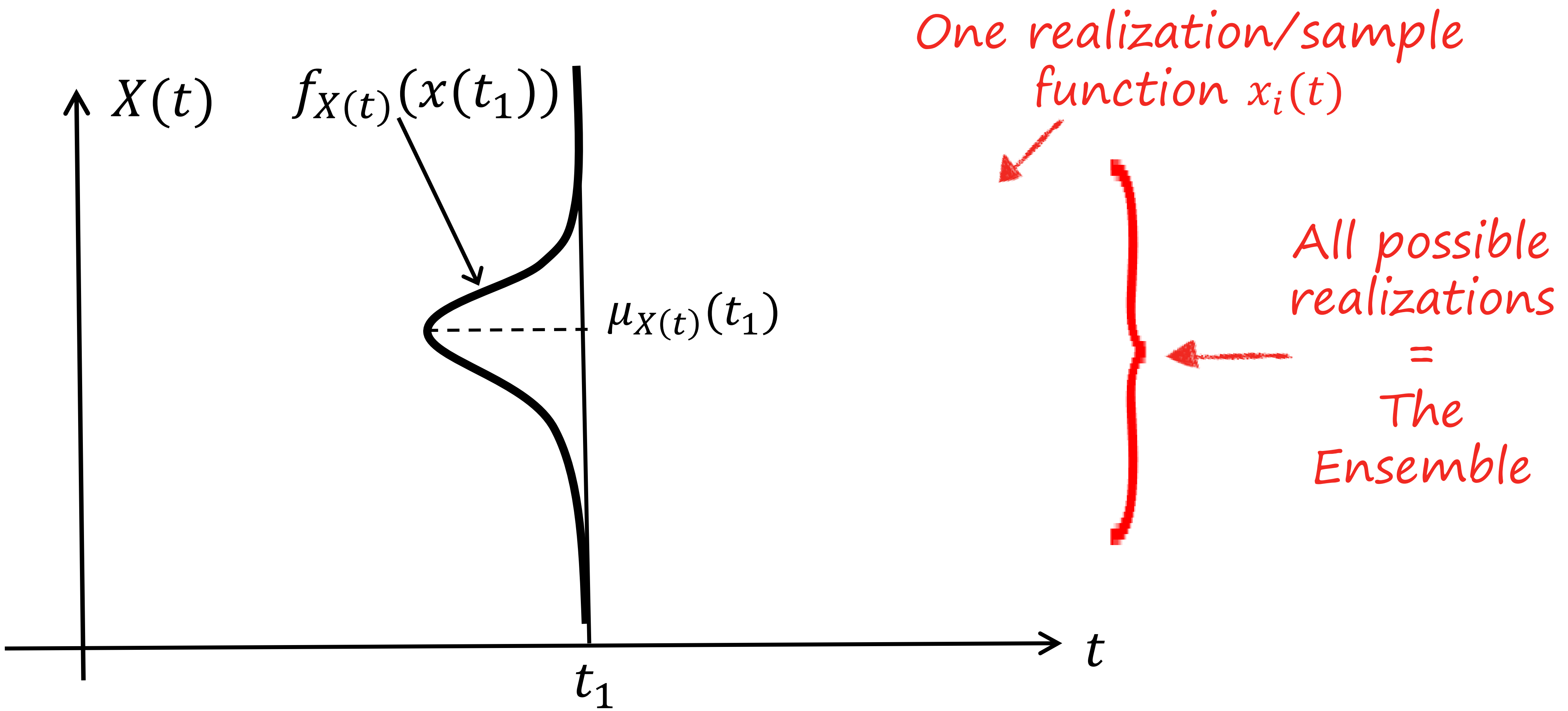


Ensemble mean

- The mean value function:

$$\mu_{X(t)}(t) = E[X(t)] = \int_{-\infty}^{\infty} x(t) f_{X(t)}(x(t))\, dx(t)$$

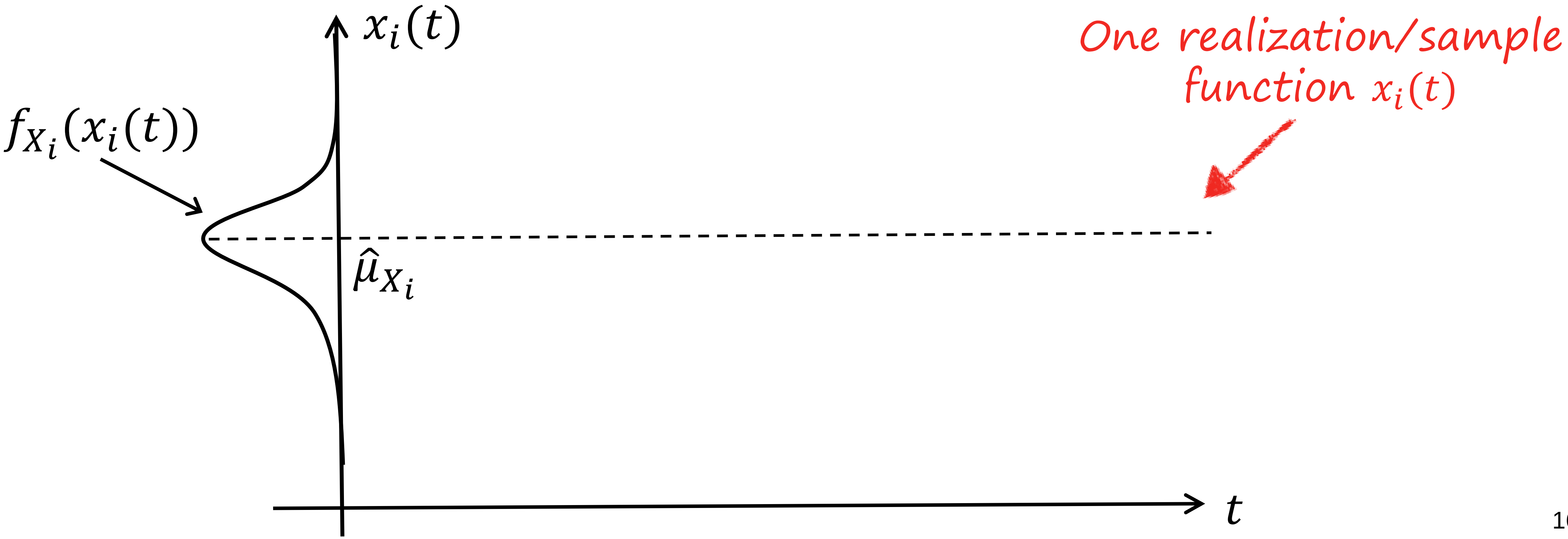
- The mean of <u>all possible realizations</u> to time t



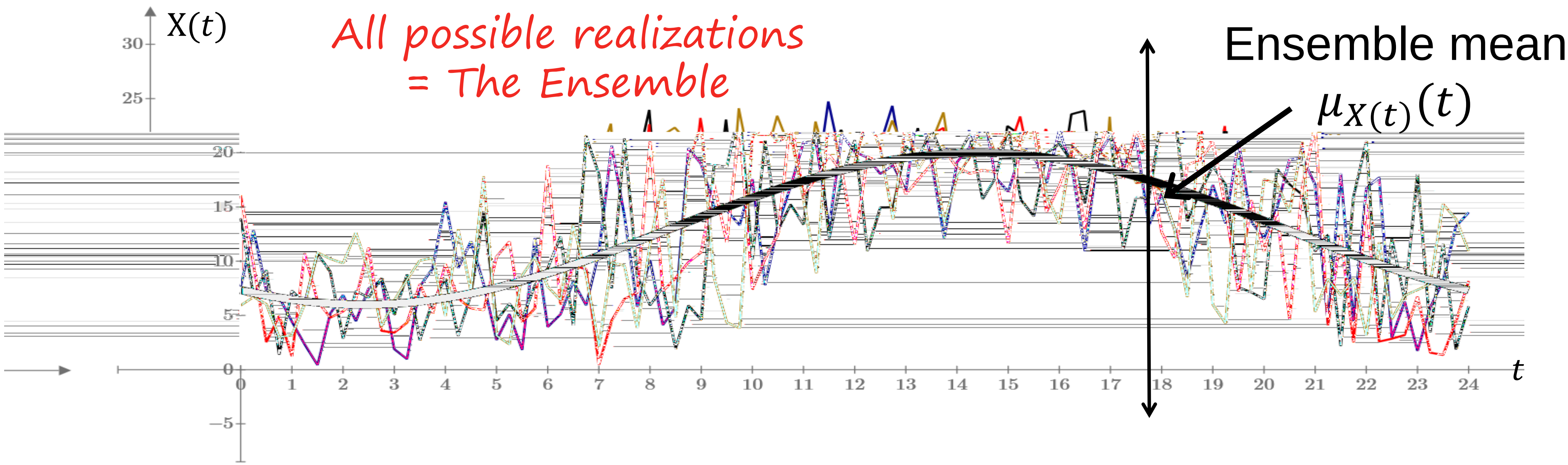
Temporal Mean

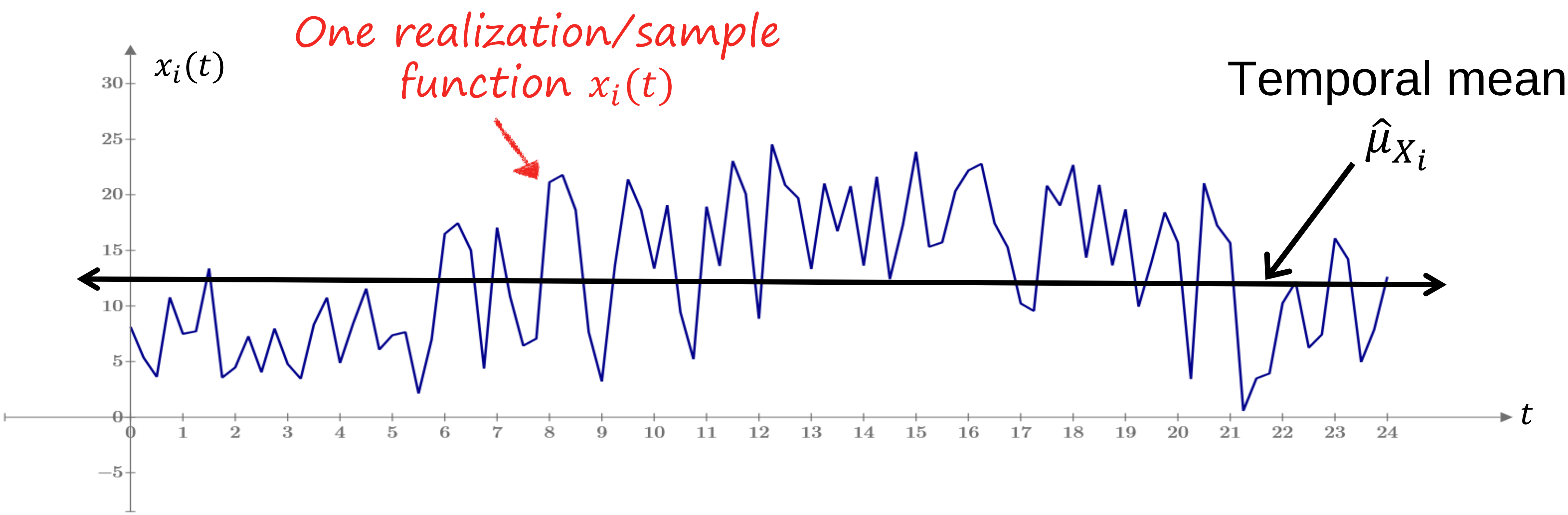
- The time average for one realization of the stochastic process
- The temporal mean can differ from the ensemble mean

$$\hat{\mu}_{X_i} = \langle X_i \rangle_T = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x_i(t)\, dt \quad \left(\text{or } \lim_{T\to\infty} \frac{1}{T} \int_0^T x_i(t)\, dt\right)$$



Stochastic Processes – Ensemble/Temporal Mean



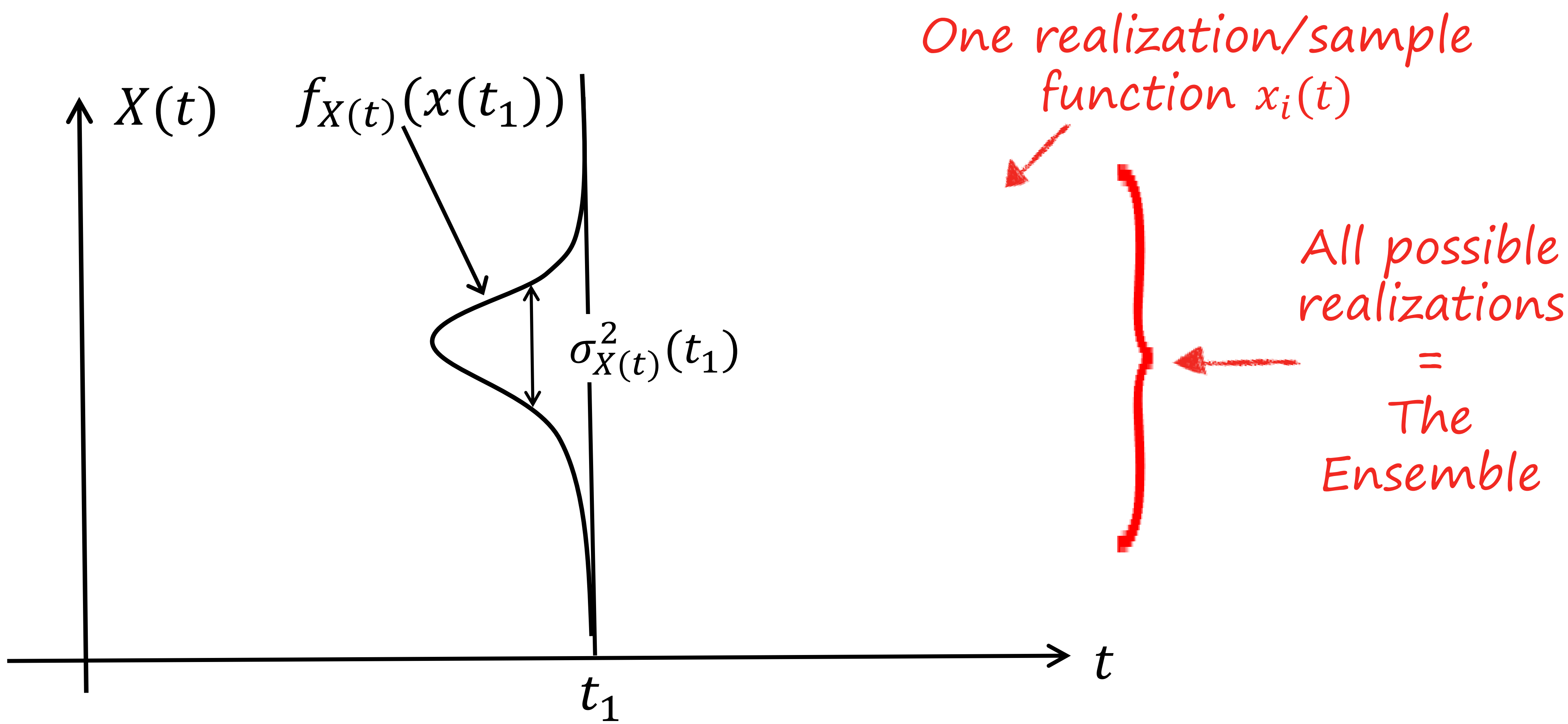


Ensemble Variance

- The variance function:

$$Var\big(X(t)\big) = \sigma^2_{X(t)}(t) = E[\big(X(t) - \mu_{X(t)}(t)\big)^2]$$

- The variance of <u>all possible realizations</u> to time t



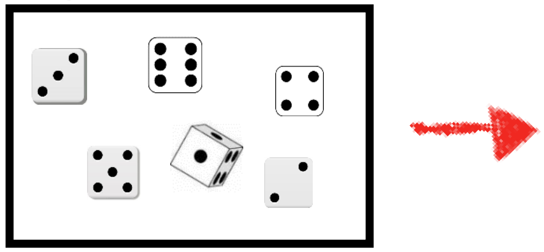
Temporal Variance

- The variance over time for <u>one realization</u> of the stochastic process
- The temporal variance can differ from the ensemble variance

$$\hat{\sigma}_{X_i}^2 = \left\langle X_i^2 \right\rangle_T - \langle X_i \rangle_T^{\ 2} = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left(x_i(t)^2 - \hat{\mu}_{X_i}^{\ 2} \right) dt = Var(X_i)$$

Stochastic Processes – Example

S



$X(t)$: $x_1(t) = -t \quad x_2(t) = 3\sin(t)$
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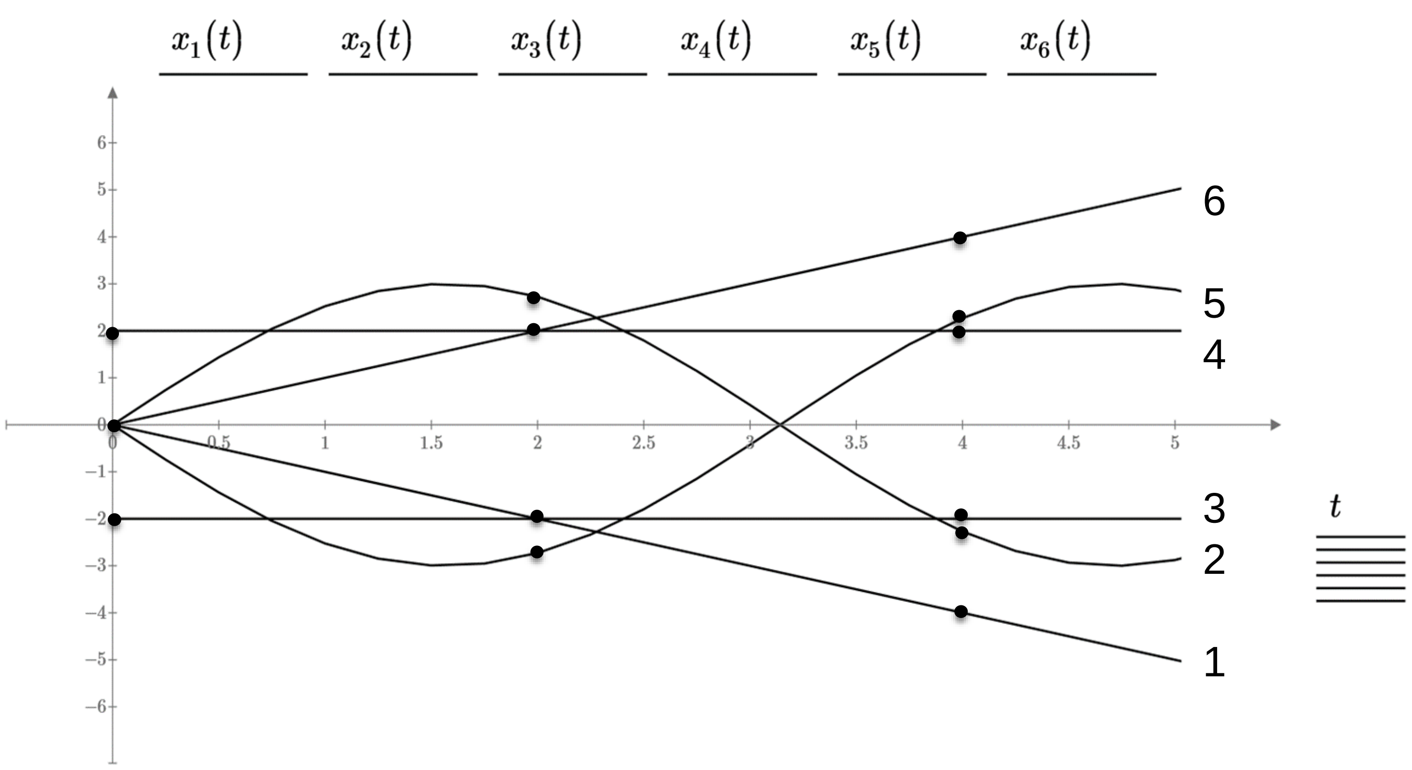


$X(0) = \{-2, 0, 2\}$

$X(2) = \{-2.7, -2, 2, 2.7\}$

$X(4) = \{-4, -2.3, -2, 2, 2.3, 4\}$

$\Pr(X(0) = 0) = 2/3$

$\Pr(X(2) = 2) = 1/3$

$\Pr(X(4) = -4) = 1/6$

Ensemble: $\mu_{X(t)}(t) = E[X(t)] = 0 \qquad Var\big(X(t)\big) = \sigma^2_{X(t)}(t) = \frac{1}{3}(t^2 + 9sin^2(t) + 4)$

Temporale: $\hat{\mu}_{X_2} = 0 \qquad \hat{\mu}_{X_3} = -2 \qquad \hat{\sigma}^2_{X_2} = 4.5 \qquad \hat{\sigma}^2_{X_3} = 0$

Strict Sense Stationary

Stationarity in the Strict Sense (SSS)

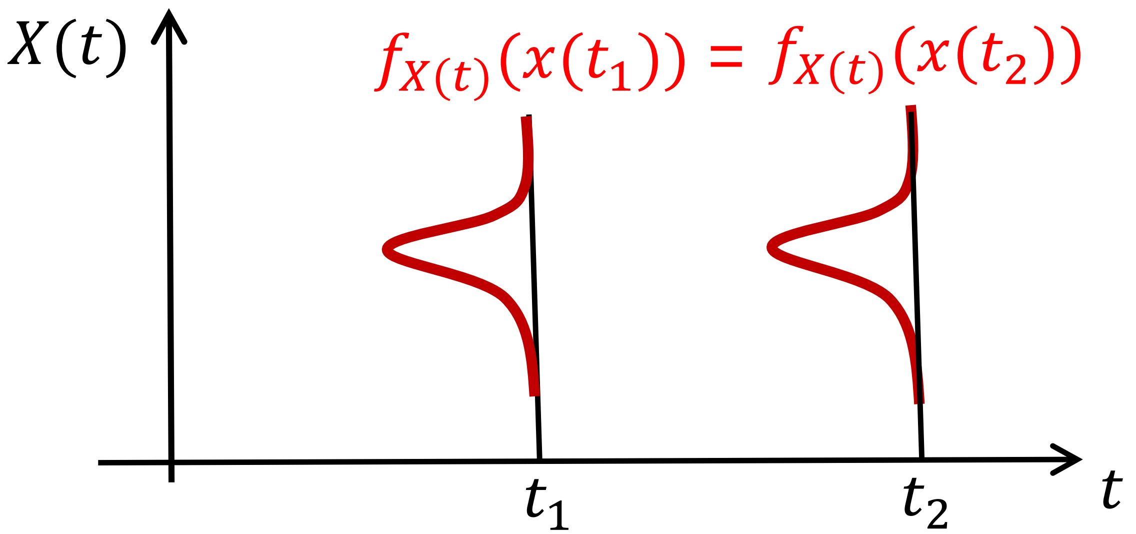
Difficult to test in reality

- The density function $f_{X(t)}\big(x(t)\big)$ do not change with time
  - For all choices of t_1 and Δt_1, the marginal pdf:

$$f_{X(t_1)}\big(x(t_1)\big) = f_{X(t_1+\Delta t_1)}\big(x(t_1+\Delta t_1)\big)$$

  - For all choices of t_1, t_2 and Δt, the simultaneous pdf:

$$f_{X(t_1),X(t_2)}\big(x(t_1),x(t_2)\big) = f_{X(t_1+\Delta t),X(t_2+\Delta t)}\big(x(t_1+\Delta t),x(t_2+\Delta t)\big)$$



Wide Sense Stationary

Stationarity in the Wide Sense (WSS)

Can be tested

- Ensemble mean is a constant

$$\mu_X(t) = E[X(t)] = \mu_X$$ - independent of time

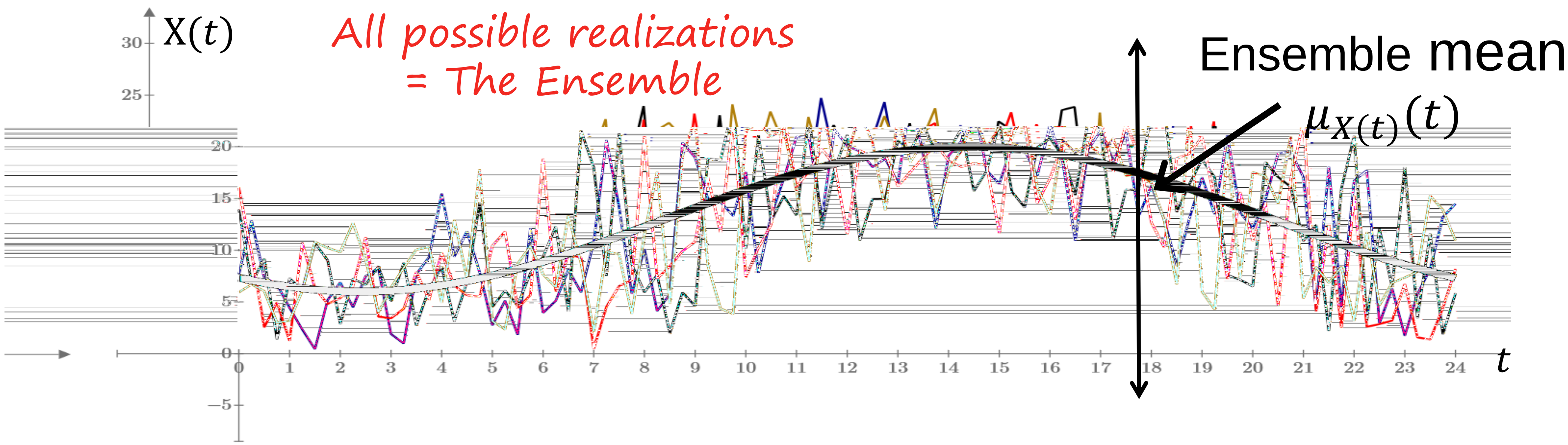
- Ensemble variance is a constant

$$\sigma_X^2(t) = E\left[X(t)^2\right] - E[X(t)]^2 = \sigma_X^2$$ - independent of time

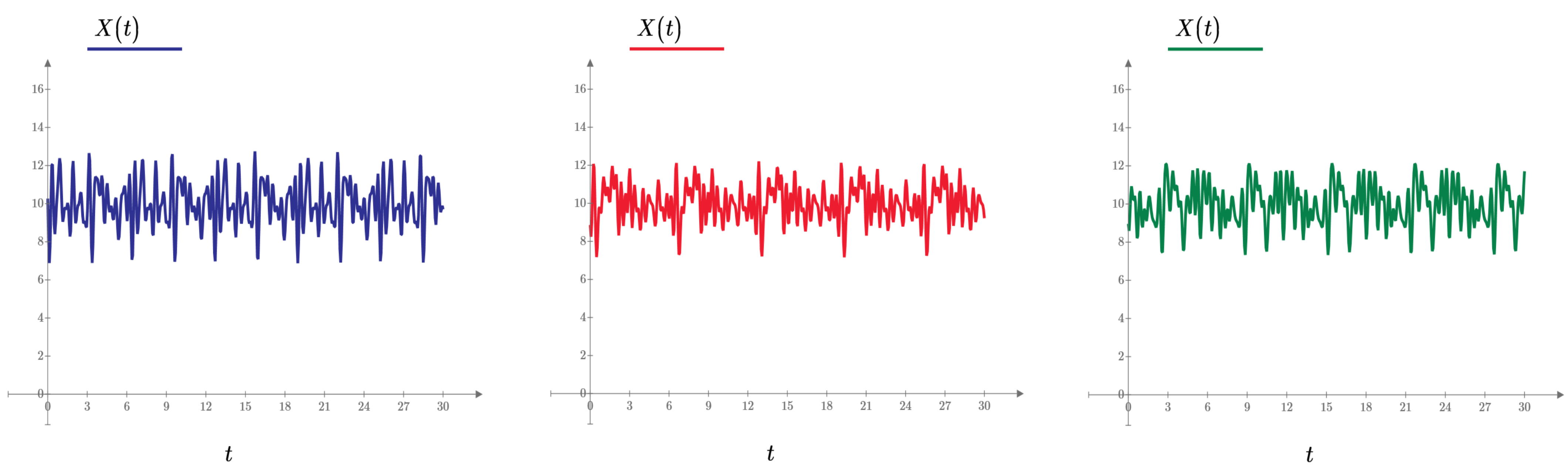
➢ Autocorrelation depends only on the time difference $\tau = t_2 - t_1$

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)] = R_{XX}(\tau)$$ - independent of time

Wide Sense Stationary (WSS) – Examples



÷ WSS



✓ WSS

Ergodicity

- We can say something about the properties of the stochastic process in general based on one sample function (realization), as long as we have observed it for long enough.

Example:

- An i.i.d Gaussian noise stream

Ergodicity

- If ensemble averaging is equivalent to temporal averaging:

$$\mu_X(t) = \bar{X}(t) = \int_{-\infty}^{\infty} x(t) f_X\big(x(t)\big)\, dx(t) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x_i(t)\, dt = \langle X_i \rangle_T = \hat{\mu}_{X_i}$$

- For any moment:

In practice: n=2 (Variance)

$$\overline{X^n} = \int_{-\infty}^{\infty} x^n f_X(x)\, dx = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x_i^n\,(t)\, dt$$

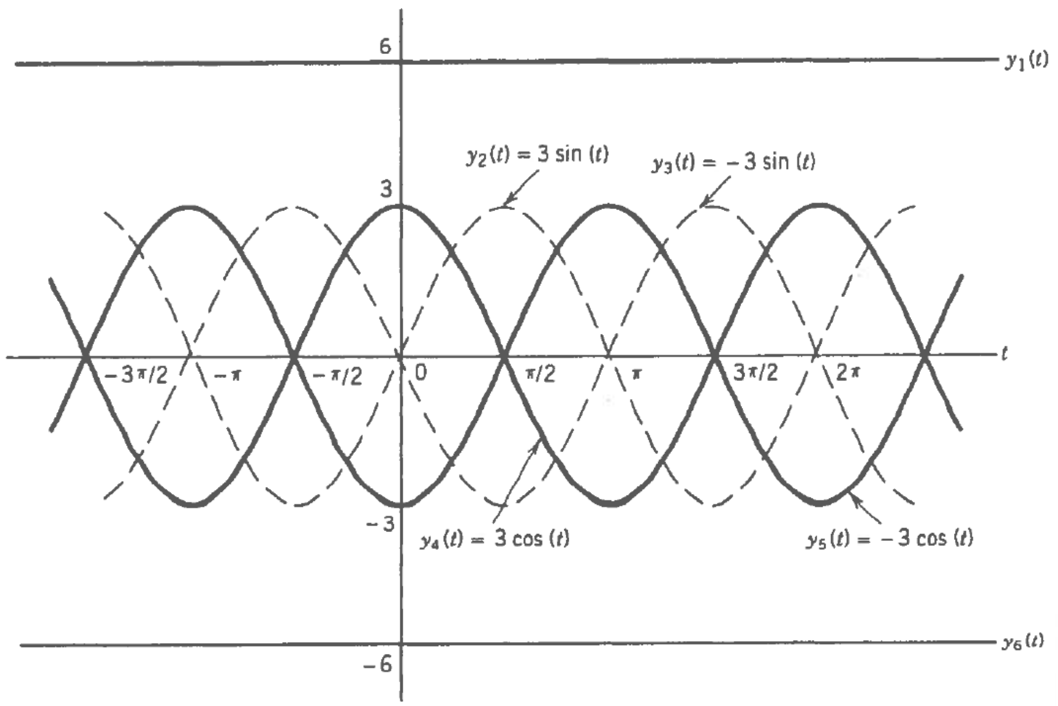
One (any) realization

Ensemple (WSS)

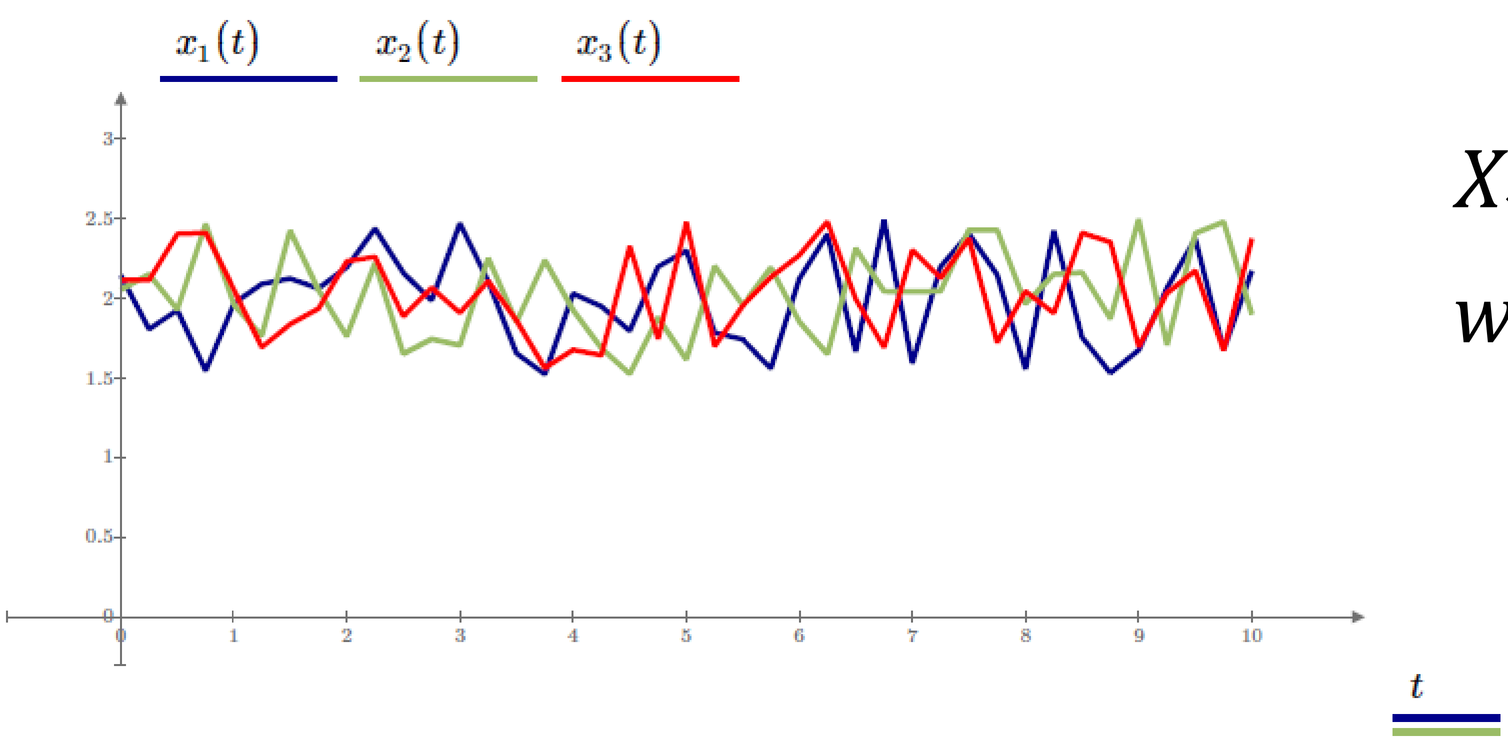
$$\left.\begin{aligned} \hat{\mu}_{X_i} &= \mu_X \\ \hat{\sigma}_{X_i}^2 &= \sigma_X^2 \end{aligned}\right\} \rightarrow Ergodic$$

All information is achieved with one measurement (realization)

WSS and Ergodicity – Examples



÷ SSS
✓ WSS
÷ Ergodic



$$X_n(t) = 2 + w_n(t)$$

$$w_n(t) \sim \mathcal{U}(-0{,}5; 0{,}5)$$

✓ WSS
✓ Ergodic

Stochastic Processes (signals)

Additive Noisemodel

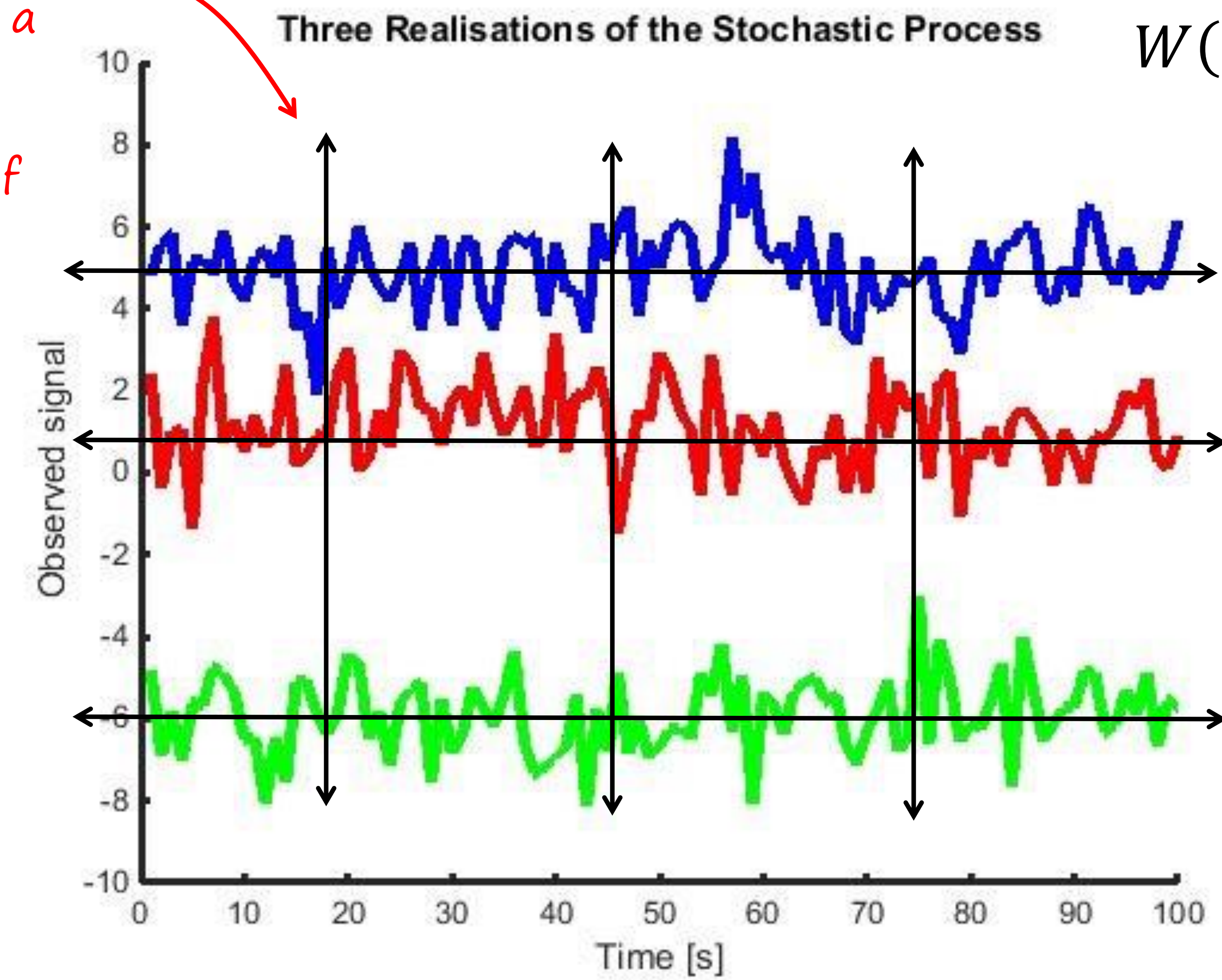
$$\text{observed signal} = \text{signal} + \text{noise} \quad = S + W(t)$$

$$S \sim \mathcal{U}(-6, +6)$$

$$W(t) \sim \mathcal{N}(0,2)$$

Ensemble mean and variance (to a specific time).

If indepentent of time: WSS



Time average and variance of each realization.

If equal (for all realizations): Ergodic

WSS ✓

Ergodic ÷

Realizations / Samples – Example

Discrete stochastic process:

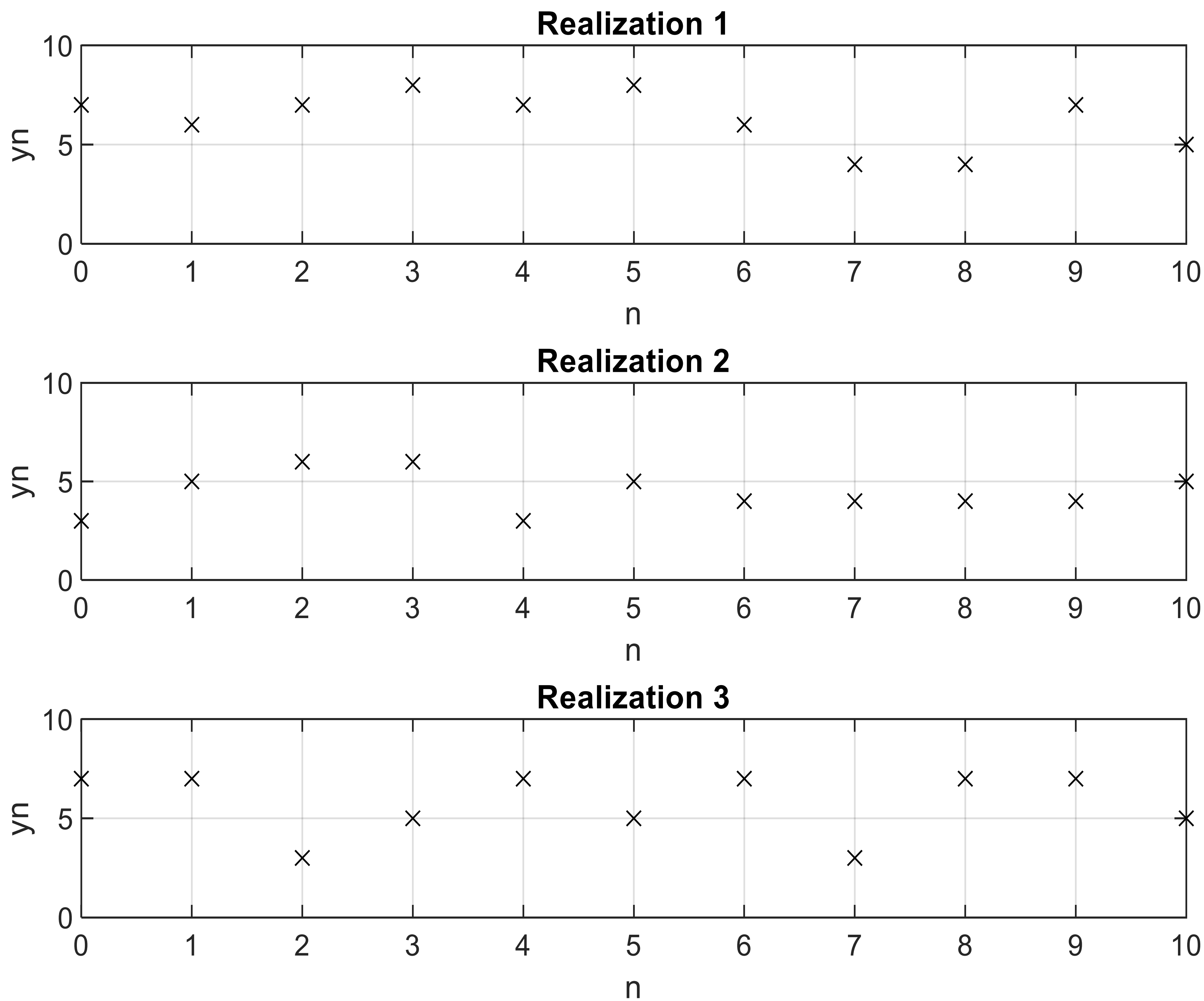
$Y(n) = X + W(n);$

$X \sim \mathcal{B}(10,0.5)$
$W(n) \sim \mathcal{U}[-2,2]$

3 realizations
11 samples
(n=0,..,10)

WSS ✓

Ergodic ÷



Realizations / Samples – Example

Discrete stochastic process:

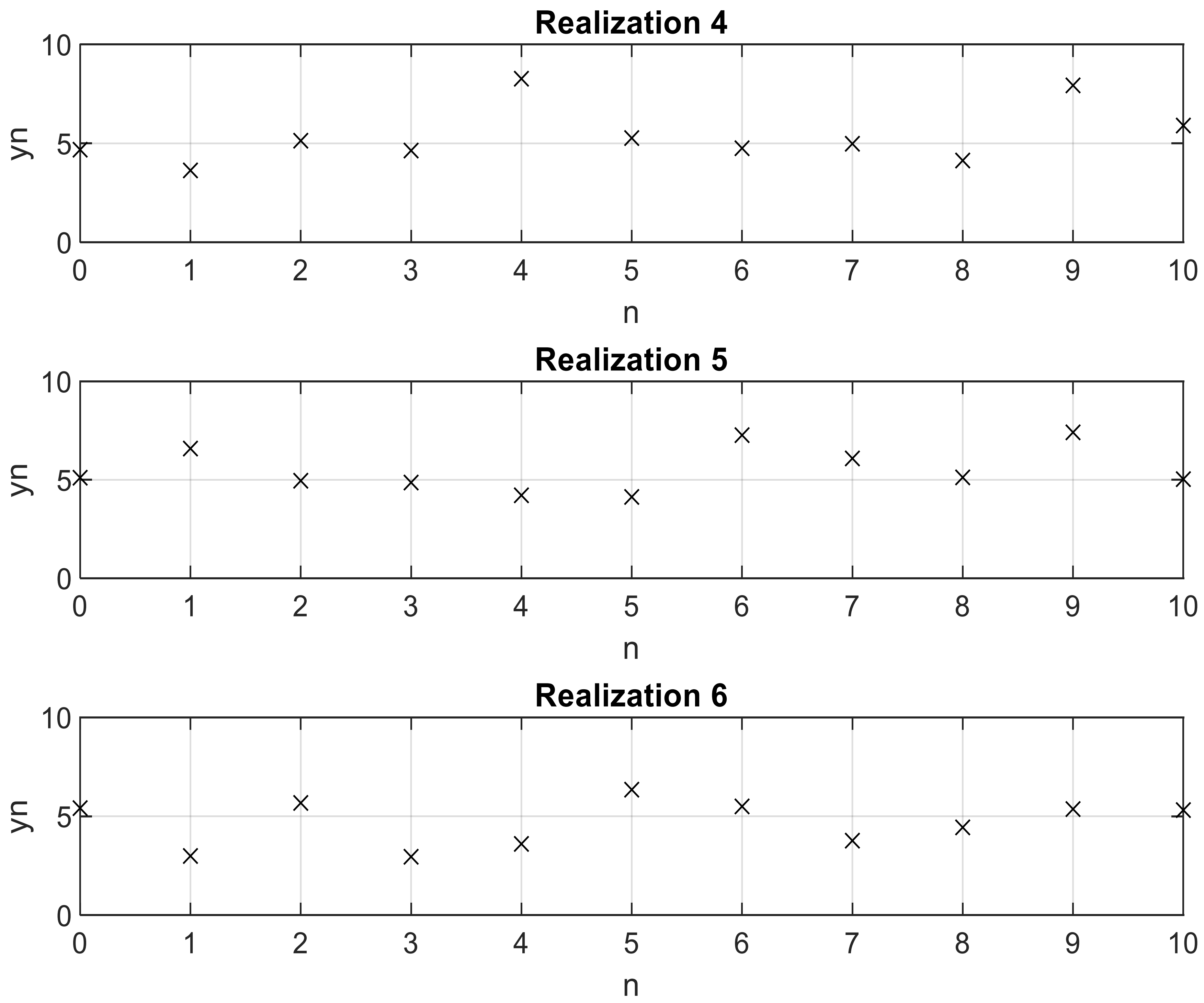
$Y(n) = W(n);$

$W(n) \sim \mathcal{N}(5,2)$

3 realizations
11 samples
(n=0,..,10)

WSS ✓

Ergodic ✓



Realizations / Samples – Example

Continuous stochastic process:

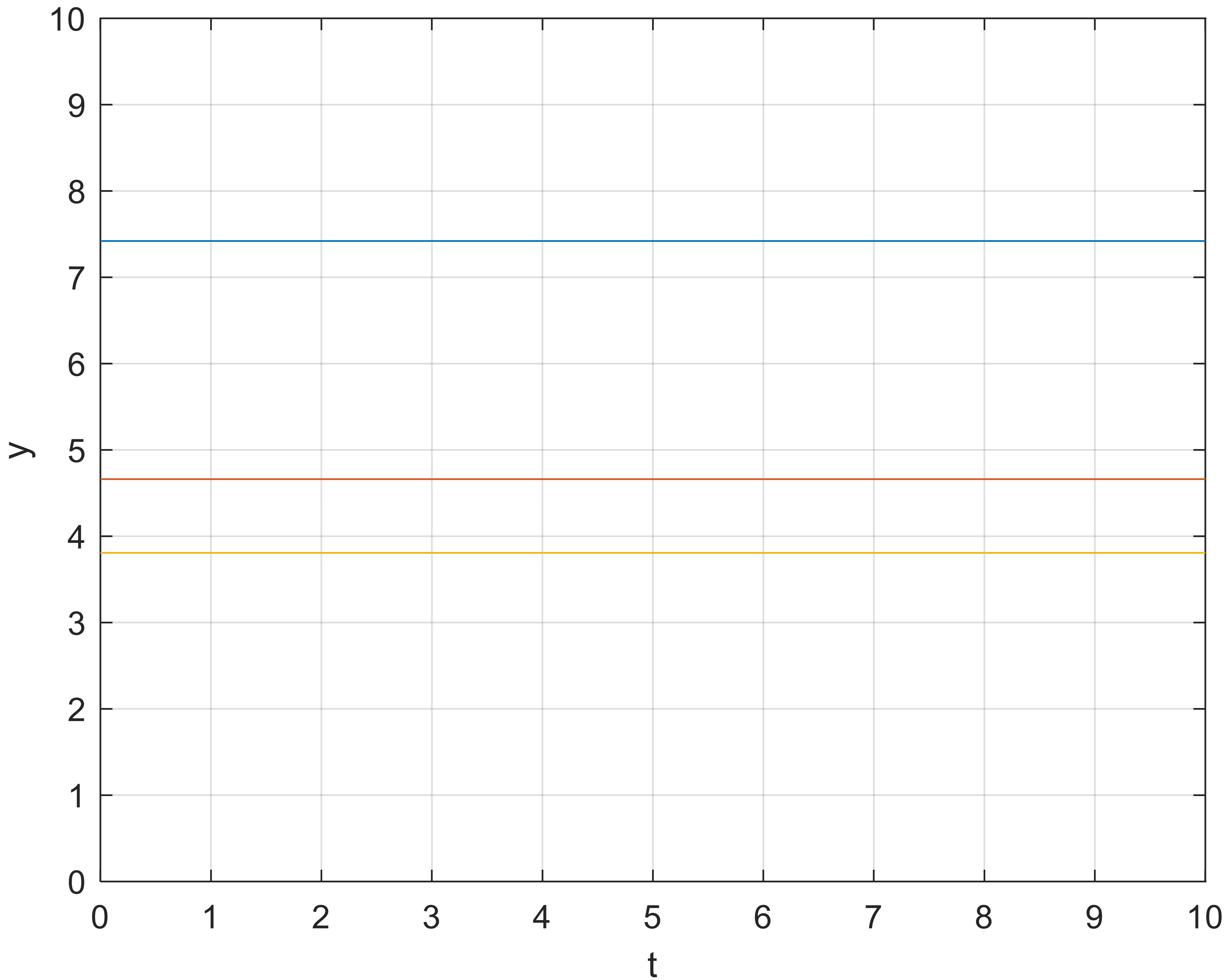
$Y(t) = W;$

$W \sim \mathcal{N}(5,2)$

3 realizations
$0 \leq t \leq 10$

WSS ✓

Ergodic ÷



Realizations / Samples – Example

Continuous stochastic process:

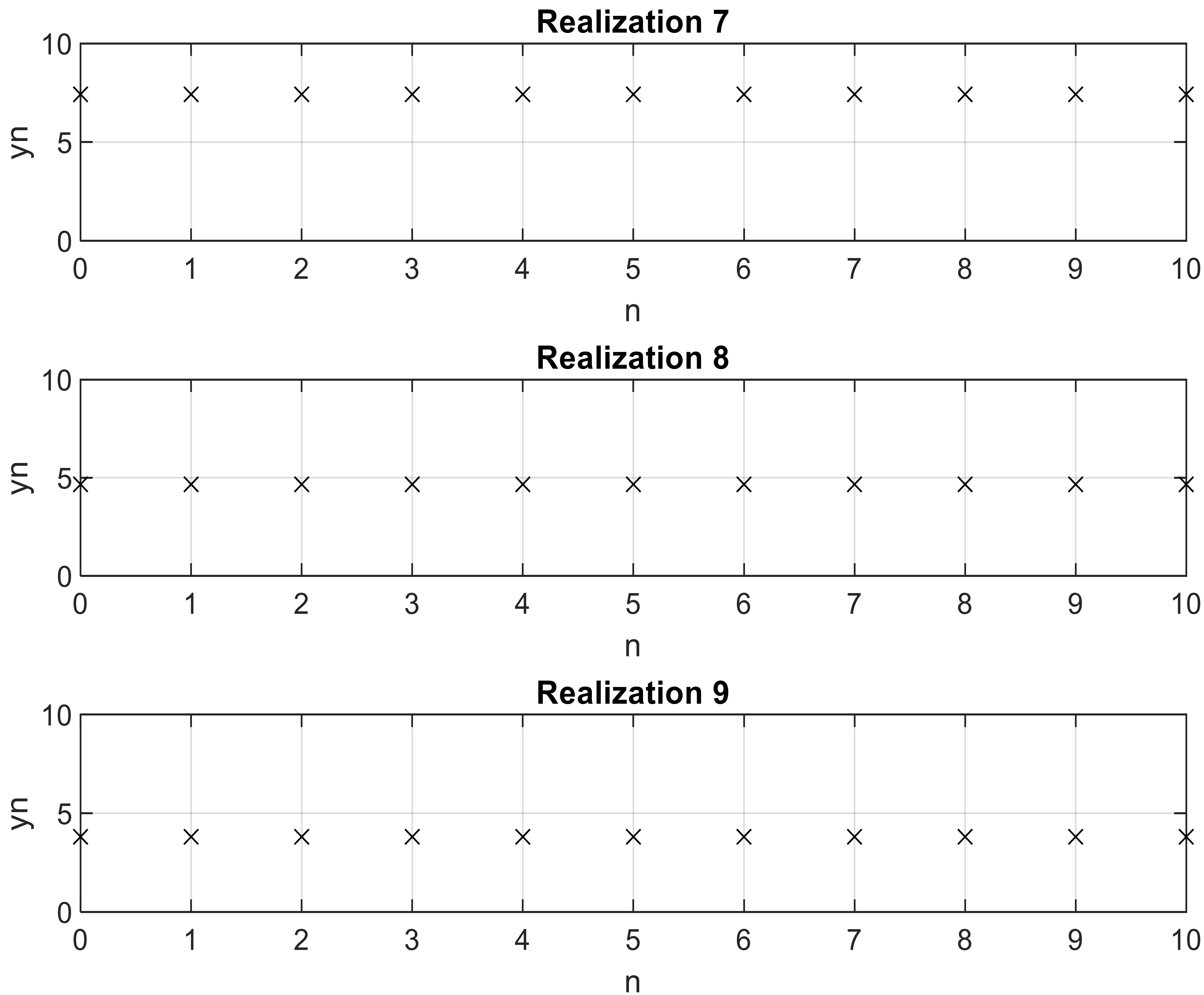
$Y(t) = W;$

$W \sim \mathcal{N}(5,2)$

3 realizations
11 samples
(n=0,..,10)

WSS ✓

Ergodic ÷



Realizations / Samples – Example

Continuous
stochastic process:

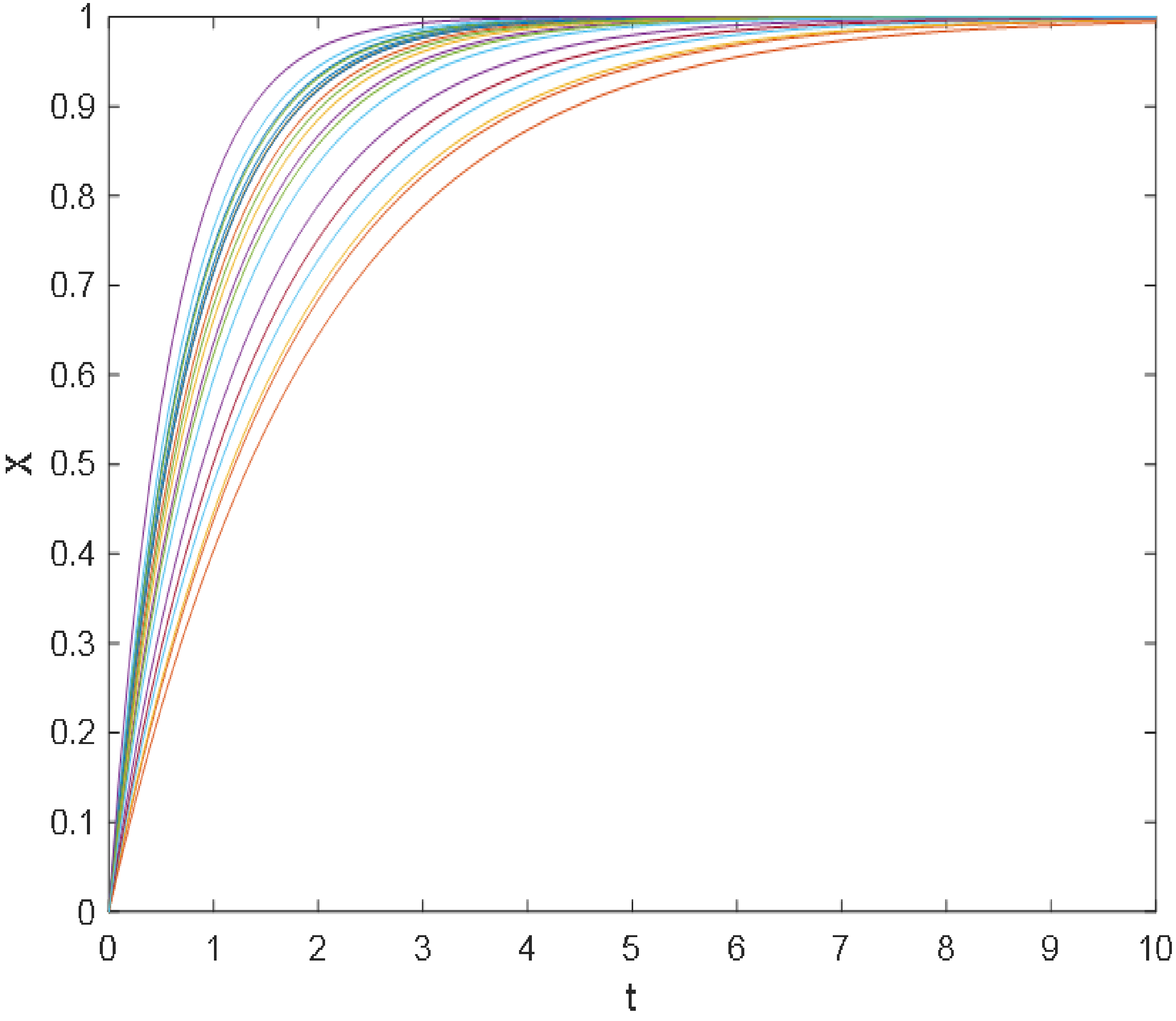
$$X(t) = A(1 - e^{-k \cdot t});$$

$$A = 1;\ k \sim \mathcal{N}(1{,}0.4)$$

20 realizations
$0 \leq t \leq 10$

WSS ÷

Ergodic ÷



Realizations / Samples – Example

Continuous stochastic process:

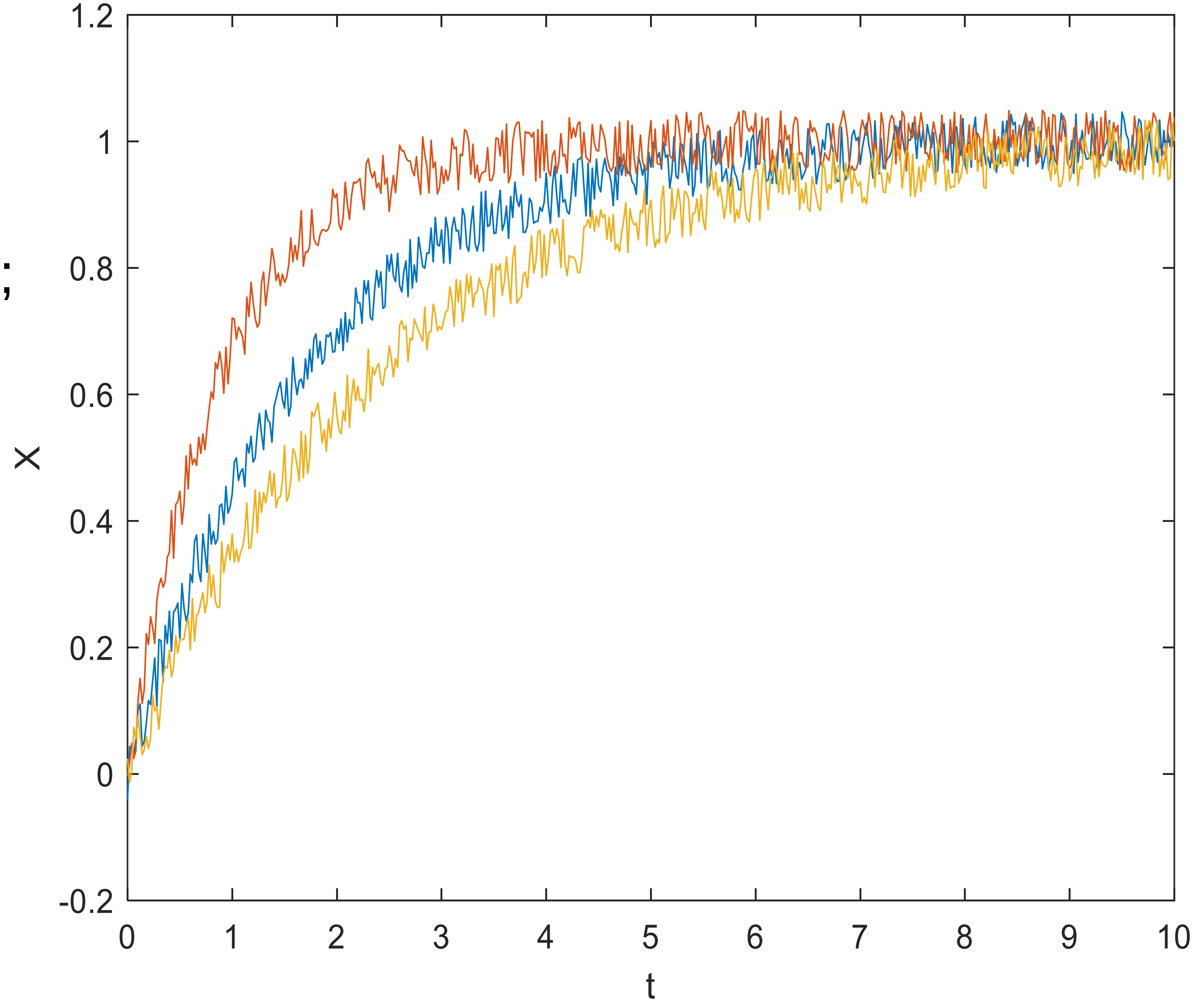
$X(t) = A(1 - e^{-k \cdot t}) + w(t);$

$A = 1;\ k \sim \mathcal{N}(1, 0.4);$
$w(t) \sim \mathcal{U}(-0.1, 0.1)$

3 realizations
$0 \leq t \leq 10$

WSS ÷

Ergodic ÷



Realizations / Samples – Example

Continuous stochastic process:

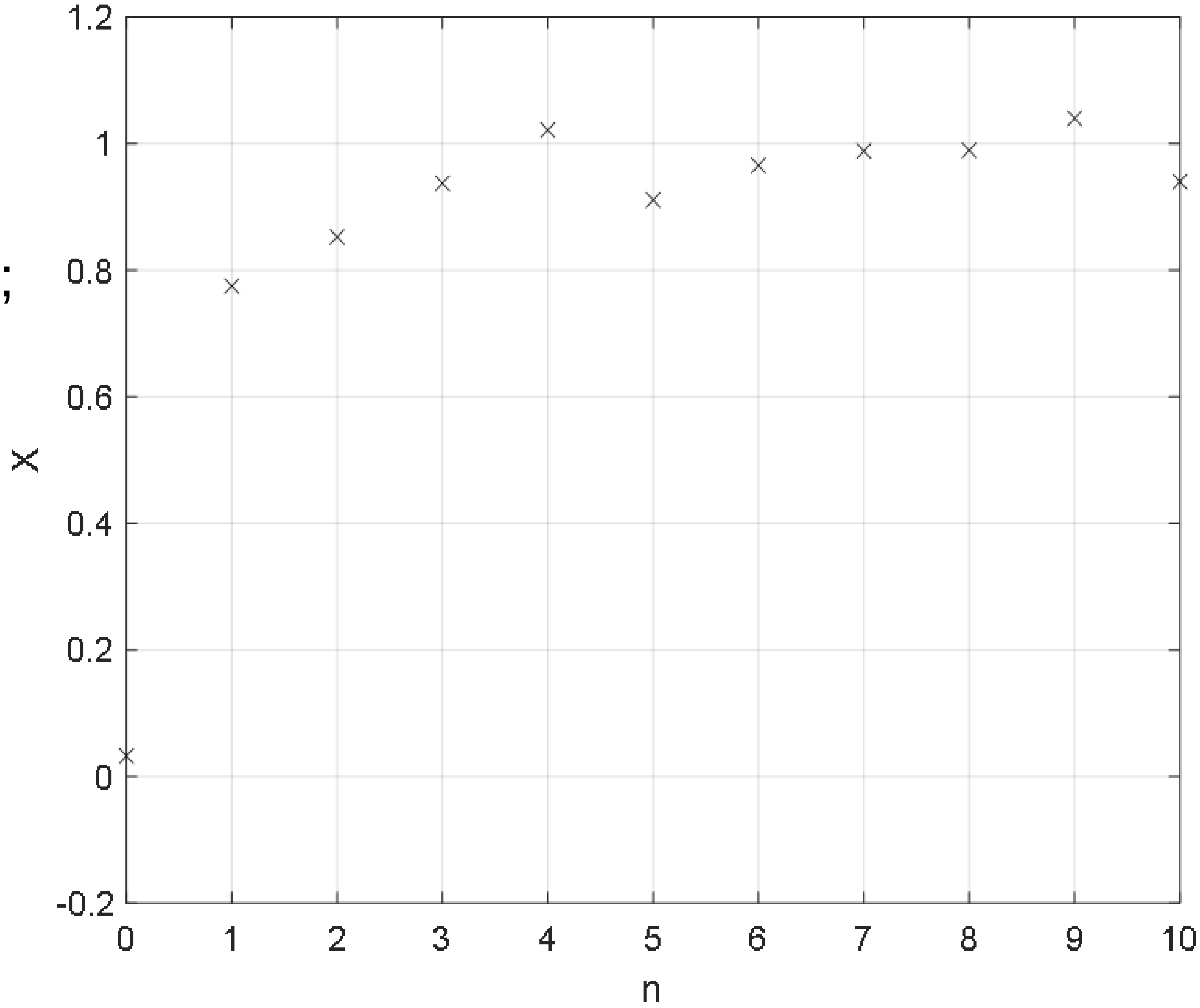
$$X(t) = A(1 - e^{-k \cdot t}) + w(t);$$

$$A = 1;\ k \sim \mathcal{N}(1,0.4);$$
$$w(t) \sim \mathcal{U}(-0.1,0.1)$$

1 realization
11 samples
(n=0,..,10)

WSS ÷

Ergodic ÷



Words and Concepts to Know

Stochastic Processes

Ensemple variance

SSS

Temporal variance

Stationarity

Random signal

Descrete-time

Continuous-valued

Ensemple mean

WSS

Ergodicity

Continuous-time

Strict Sense Stationary

Descrete-valued

Realization

Temporal mean

Wide Sense Stationary