1. Suppose that the universal set S is defined as $S = \{1, 2, \cdots, 10\}$ and $A = \{1, 2, 3\}$, $B = \{x \in S : 2 \leq x \leq 7\}$, and $C = \{7, 8, 9, 10\}$.
   (a) Find $A \cup B$
   (b) Find $(A \cup C) - B$
   (c) Find $\bar{A} \cup (B - C)$
   (d) Do A, B, and C form a partition of S?

   Solution:

   (a)

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$$

   (b)

$$\begin{aligned} A \cup C &= \{1, 2, 3, 7, 8, 9, 10\} \\ B &= \{2, 3, \cdots, 7\} \\ \text{thus:} \quad (A \cup C) - B &= \{1, 8, 9, 10\} \end{aligned}$$

   (c)

$$\begin{aligned} \bar{A} &= \{4, 5, \cdots, 10\} \\ B - C &= \{2, 3, 4, 5, 6\} \\ \text{thus:} \quad \bar{A} \cup (B - C) &= \{2, 3, \cdots, 10\} \end{aligned}$$

   (d) No, since they are not disjoint. For example,

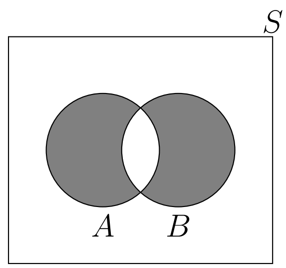
$$A \cap B = \{2, 3\} \neq \varnothing$$

2. When working with real numbers, our universal set is $\mathbb{R}$. Find each of the following sets.
   (a) $[6, 8] \cup [2, 7)$
   (b) $[6, 8] \cap [2, 7)$
   (c) $[0, 1]^c$
   (d) $[6, 8] - (2, 7)$

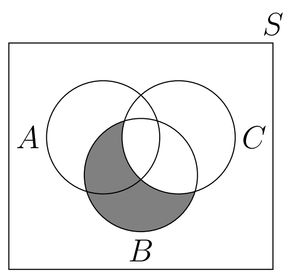
   Solution:

(a) $[2, 8]$
(b) $[6, 7)$
(c) $(-\infty, 0) \cup (1, \infty)$
(d) $[7, 8]$

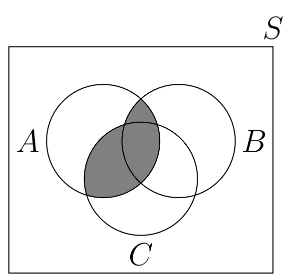
3. For each of the following Venn diagrams, write the set denoted by the shaded area.
(a)



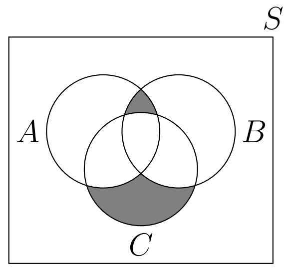
(b)



(c)



(d)



Solution: Note that there are generally several ways to represent each of the sets, so the answers to this question are not unique.

(a) $(A - B) \cup (B - A)$
(b) $B - C$
(c) $(A \cap B) \cup (A \cap C)$
(d) $(C - A - B) \cup ((A \cap B) - C)$

4. A coin is tossed twice. Let S be the set of all possible pairs that can be observed, i.e., $S = \{H, T\} \times \{H, T\} = \{(H, H), (H, T), (T, H), (T, T)\}$. Write the following sets by listing their elements.
(a) A: The first coin toss results in head.
(b) B: At least one tail is observed.
(c) C: The two coin tosses result in different outcomes.

Solution:

(a) A={(H,H),(H,T)}.
(b) B={(H,T),(T,H),(T,T)}.
(c) C={(H,T),(T,H)}.

5. * Let $A = \{1, 2, \cdots, 100\}$. For any $i \in \mathbb{N}$, Define A_i as the set of numbers in A that are divisible by i. For example:
$A_2 = \{2, 4, 6, \cdots, 100\}$
$A_3 = \{3, 6, 9, \cdots, 99\}$
(a) Find $|A_2|$,$|A_3|$,$|A_4|$,$|A_5|$.
(b) Find $|A_2 \cup A_3 \cup A_5|$.

Solution:

(a) $|A_2| = 50$, $|A_3| = 33$, $|A_4| = 25$, $|A_5| = 20$.
Note that in general:
$|A_i| = \lfloor \frac{100}{i} \rfloor$, where $\lfloor x \rfloor$ is the largest integer less than or equal to x. (b) By the inclusion-exclusion principle:

$$\begin{aligned}|A_2 \cup A_3 \cup A_5| = &|A_2| + |A_3| + |A_5| \\ &- |A_2 \cap A_3| - |A_2 \cap A_5| - |A_3 \cap A_5| \\ &+ |A_2 \cap A_3 \cap A_5|.\end{aligned}$$

We have:

$$\begin{aligned}|A_2| &= 50 \\ |A_3| &= 33 \\ |A_5| &= 20 \\ |A_2 \cap A_3| &= |A_6| = 16 \\ |A_2 \cap A_5| &= |A_{10}| = 10 \\ |A_3 \cap A_5| &= |A_{15}| = 6 \\ |A_2 \cap A_3 \cap A_5| &= |A_{30}| = 3 \\ |A_2 \cup A_3 \cup A_5| &= 50 + 33 + 20 \\ &- 16 - 10 - 6 \\ &+ 3 = 74\end{aligned}$$

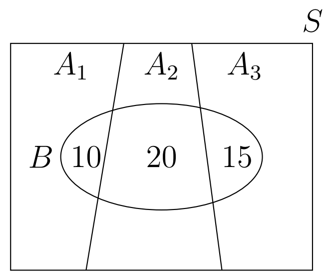
6. Suppose that A_1, A_2, A_3 form a partition of the universal set S. Let B be an arbitrary set. Assume that we know:

$$\begin{aligned}|B \cap A_1| &= 10 \\ |B \cap A_2| &= 20 \\ |B \cap A_3| &= 15\end{aligned}$$

Find $|B|$.

Solution:

It is useful to look at the venn diagram.



We see that in fact the sets $B \cap A_1$, $B \cap A_2$, and $B \cap A_3$ form a partition of B. Therefore

$$
\begin{aligned}
|B| &= |B \cap A_1| + |B \cap A_2| + |B \cap A_3| \\
&= 10 + 20 + 15 \\
&= 45
\end{aligned}
$$

7. Determine whether each of the following sets is countable or uncountable.
(a) $A = \{1, 2, \cdots, 10^{10}\}$.
(b) $B = \{a + b\sqrt{2} | \quad a, b \in \mathbb{Q}\}$.
(c) $C = \{(X, Y) \in \mathbb{R}^2 | \quad x^2 + y^2 \leq 1\}$.

Solution:

(a) A is <u>countable</u> because it is a finite set.
(b) B is <u>countable</u> becausewe can create a list with all the elements.
We have shown previously (refer to Figure 1.13 in the book) that if we can write any set B in the form of

$$B = \bigcup_i \bigcup_j \{q_{ij}\},$$

where indices i and j belong to some countable sets, that set in this form is countable.

For this case we can write

$$B = \bigcup_{i \in \mathbb{Q}} \bigcup_{j \in \mathbb{Q}} \{a_i + b_j\sqrt{2}\}.$$

So, we can replace q_{ij} by $a_i + b_j\sqrt{2}$.

(c) C is <u>uncountable</u>. To see this note that for all $x \in [0, 1]$ then $(x, 0) \in C$.

8. * Let $A_n = \left[0, \frac{n-1}{n}\right) = \{x \in \mathbb{R} | \quad 0 \leq x < \frac{n-1}{n}\}$, for $n = 2, 3, \cdots$. Define

$$A = \bigcup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup A_3 \cdots .$$

Find A.

Solution:

By definition of the union:

$$A = \{x | x \in A_n \quad \text{for some} \quad n\}$$

We claim $A = [0, 1)$.
First note that for all n, $A_n \subset [0, 1)$. Therefore

$$A = \bigcup_{n=1}^{\infty} A_n \subset [0, 1).$$

Next, we show $[0, 1) \subset A$: let $x \subset [0, 1)$. Choose m such that $\frac{1}{m} < 1 - x$. Then $x < \frac{m-1}{m}$, so $x \in [0, \frac{m-1}{m})$
Therefore $x \in A_m$. We conclude

$$x \in \bigcup_{n=1}^{\infty} A_n.$$

9. * Let $A_n = [0, \frac{1}{n}) = \{x \in \mathbb{R} | \quad 0 \leq x < \frac{1}{n}\}$ for $n = 1, 2, \cdots$. Define

$$A_n = \bigcap_{n=1}^{\infty} A_n = A_1 \cap A_2 \cap \cdots$$

Find A.

Solution:

By definition of the intersection

$$A = \{x | x \in A_n \quad \text{for } \underline{\text{all}} \quad n = 1, 2, \cdots\}$$

We claim $A = \{0\}$.
First note that $0 \in A_n$ for all $n = 1, 2, \cdots$. Thus $\{0\} \subset A$.
Next we show that A does not have any other elements. Since $A_n \subset [0, 1)$ then $A \subset [0, 1)$. Let $x \in (0, 1)$. Choose $n > \frac{1}{x}$ then $\frac{1}{n} < x$. Thus $x \notin A_n$ and this results in $x \notin A$.

10. * In this problem our goal is to show that sets that are not in the form of intervals may also be uncountable. In particular, consider the set A defined as the set of all subsets of $\mathbb{N}$:

$$A = \{B : B \subset \mathbb{N}\}$$

We usually denote this set by $A = 2^{\mathbb{N}}$.
(a)Show that $2^{\mathbb{N}}$ is in one-to-one correspondence with the set of all (infinite) binary sequences:

$$C = \{b_1, b_2, b_3, \cdots | \quad b_i \in \{0, 1\}\}$$

(b) Show that C is in one-to-one correspondence with $[0, 1]$. From (a) and (b) we conclude that the set $2^{\mathbb{N}}$ is uncountable.

Solution:

(a) Consider a subset B of (N). This subset can be identified by the following. Sequence $a_1, a_2, \cdots$:.

$$a_i = \begin{cases} 1 & \text{if} \quad i \in \mathbb{N} \\ 0 & \text{else} \end{cases}$$

Thus, for any set $B \subset \mathbb{N}$ we have a unique binary sequence and for any binary sequence we have a unique set $B \subseteq \mathbb{N}$.
(b) Any $x \in [0, 1]$ can be uniquely identified by its binary expansion:

$$x = \sum_{i=1}^{\infty} b_i 2^{-i}.$$

Therefore, [0,1] is in one-to-one correspondence with the set of all unending binary sequences.

11. ** Show the set $[0, 1)$ is uncountable. That is you can never provide a list in the form of $\{a_1, a_2, a_3, \cdots\}$ that contains all the elements in $[0, 1)$.

Solution:

Note that any $x \in [0, 1)$ can be written in its binary expansion:

$$x = 0.b_1b_2b_3 \cdots$$

where $b_i \in \{0, 1\}$. Now suppose that $\{a_1, a_2, a_3, \cdots\}$ is a list containing all $x \in [0, 1)$. For example:

$$\begin{aligned}
a_1 &= 0.\boxed{1}0101101001\cdots \\
a_2 &= 0.0\boxed{0}0110110111\cdots \\
a_3 &= 0.00\boxed{1}101001001\cdots \\
a_4 &= 0.100\boxed{1}001111001\cdots
\end{aligned}$$

Now, we find a number $a \in [0, 1)$ that does not belong to the list. Consider a such that the k^{th} bit of a is the complement of the k^{th} bit of a_k. For example, for the above list, a would be

$$a = 0.0100\cdots$$

We see that $a \notin \{a_1, a_2, \cdots\}$.

12. Recall that

$$\begin{aligned}
\{H, T\}^3 &= \{H, T\} \times \{H, T\} \times \{H, T\} \\
&= \{(H, H, H), (H, H, T), \cdots, (T, T, T)\}.
\end{aligned}$$

Consider the following function

$$f : \{H, T\}^3 \longrightarrow \mathbb{N} \cup \{0\}.$$

Defined as

$$f(x) = \text{the number of H's in x.}$$

For example:

$$f(HTH) = 2.$$

(a) Determine the domain and co-domain for f.
(b) Find range of f:Range(f).
(c) If we know $f(x) = 2$, what can we say about x?

Solution:

(a) By definition:
Domain: $\{H, T\}^3$
Co-domain: $\mathbb{N} \cup \{0\}$
(b) Since $0 \leq f(x) \leq 3$ for all x. We conclude:

$$Range(f) = \{0, 1, 2, 3\}. \tag{1}$$

(c) If $f(x) = 2$ then

$$x \in \{(H, H, T), (H, T, H), (T, H, H)\}.$$

13. Two teams A and B play a soccer match, and we are interested in the winner. The sample space can be defined as:

$$S = \{a, b, d\}$$

where a shows the outcome that A wins, b shows the outcome that B wins, and d shows the outcome that they draw. Suppose that we know that (1) the probability that A wins is $P(a) = P(\{a\}) = 0.5$ (2) the probability of a draw is $P(d) = P(\{d\}) = 0.25$.
(a) Find the probability that B wins.
(b) Find the probability that B wins or a draw occurs.

Solution:

$$\begin{aligned} P(a) + P(b) + P(d) &= 1 \\ P(a) &= 0.5 \\ P(d) &= 0.25 \end{aligned}$$

Therefore $P(b) = 0.25$.
(b)

$$\begin{aligned} P(\{b, d\}) &= P(b) + P(d) \\ &= 0.5 \end{aligned}$$

14. * Let A and B be two events such that:

$$P(A) = 0.4, P(B) = 0.7, P(A \cup B) = 0.9$$

(a) Find $P(A \cap B)$.
(b) Find $P(A^c \cap B)$.
(c) Find $P(A - B)$.
(d) Find $P(A^c - B)$.
(e) Find $P(A^c \cup B)$.
(f) Find $P(A \cap (B \cup A^c))$.

Solution:

We have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

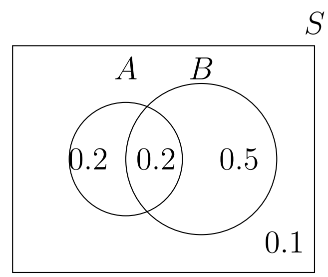
Thus,

$$0.9 = 0.7 + 0.4 - P(A \cap B)$$

which results in:

$$P(A \cap B) = 0.2.$$

A Venn diagram is useful here:



(b)

$$\begin{aligned} P(A^c \cap B) &= P(B - A) \\ &= P(B) - P(B \cap A) \\ &= 0.7 - 0.2 \\ &= 0.5 \end{aligned}$$

(c)

$$P(A - B) = 0.2$$

(d)

$$\begin{aligned} P(A^c - B) &= P(A^c \cap B^c) \\ &= P((A \cup B)^c) \\ &= 1 - P(A \cup B) \\ &= 1 - 0.9 \\ &= 0.1 \end{aligned}$$

(e)

$$\begin{aligned} P(A^c \cup B) &= P(A^c) + P(B) - P(A^c \cap B) \\ &= 1 - P(A) + P(B) - P(B - A) \\ &= 1 - 0.4 + 0.7 - 0.5 \\ &= 0.8 \end{aligned}$$

(f)

$$\begin{aligned}
P(A \cap (B \cup A^c)) &= P((A \cap B) \cup (A \cap A^c)) \quad \text{(distributive law)} \\
&= P((A \cap B) \cup \varnothing) \quad \text{(since} \quad A \cap A^c = \varnothing) \\
&= P(A \cap B) \\
&= 0.2
\end{aligned}$$

15. * I roll a fair die twice and obtain two numbers. X_1 = result of the first roll, X_2 = result of the second roll.
(a) Find the probability that $X_2 = 4$.
(b) Find the probability that $X_1 + X_2 = 7$.
(c) Find the probability that $X_1 \neq 2$ and $X_2 \geq 4$.

Solution:

The sample space has 36 elements.

$$\begin{aligned}
S = \{&(1,1),(1,2),\cdots,(1,6), \\
&(2,1),(2,2),\cdots,(2,6), \\
&\vdots \\
&(6,1),(6,2),\cdots,(6,6)\}
\end{aligned}$$

(a) The event $X_2 = 4$ can be represented by the set.

$$A = \{(1,4),(2,4),(3,4),(4,4),(5,4),(6,4)\}$$

Thus

$$P(A) = \frac{|A|}{|S|} = \frac{6}{36} = \frac{1}{6}$$

(b)

$$\begin{aligned}
B &= \{(x_1,x_2)|x_1 + x_2 = 7\} \\
&= \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}
\end{aligned}$$

Therefore

$$P(B) = \frac{|6|}{36} = \frac{1}{6}$$

(c)

$$\begin{aligned}
C &= \{(X_1, X_2) | X_1 \neq 2, X_2 \geq 4\} \\
&= \{(1,4), (1,5), (1,6), \\
&(3,4), (3,5), (3,6), \\
&(4,4), (4,5), (4,6), \\
&(5,4), (5,5), (5,6), \\
&(6,4), (6,5), (6,6)\}
\end{aligned}$$

Therefore

$$|C| = 15$$

Which results in:

$$P(C) = \frac{15}{36} = \frac{5}{12}$$

16. Consider a random experiment with a sample space.

$$S = \{1, 2, 3, \cdots\}.$$

Suppose that we know:

$$P(k) = P(\{k\}) = \frac{c}{3^k} \quad \text{for} \quad k = 1, 2, \cdots$$

where c is a constant number.
(a) Find c.
(b) Find $P(\{2, 4, 6\})$.
(c) Find $P(\{3, 4, 5, \cdots\})$.

Solution:

(a) We must have

$$\sum_{k=1}^{\infty} P(k) = 1.$$

Thus

$$\begin{aligned}
1 &= \sum_{k=1}^{\infty} \frac{c}{3^k} \\
&= c \cdot \sum_{k=1}^{\infty} \frac{1}{3^k} \\
&= c \cdot \frac{\frac{1}{3}}{1-\frac{1}{3}} \\
&= \frac{c}{2}
\end{aligned}$$

Thus

$$c = 2$$

(b)

$$\begin{aligned}
P(\{2,4,6\}) &= P(2) + P(4) + P(6) \\
&= 2(\frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6}) \\
&= 2 \cdot \frac{3^4 + 3^2 + 1}{3^6} \\
&= 2 \cdot \frac{91}{3^6} \approx 0.25
\end{aligned}$$

(c)

$$\begin{aligned}
P(\{3,4,5,\cdots\}) &= \sum_{k=3}^{\infty} P(k) = 2 \sum_{k=3}^{\infty} \frac{1}{3^k} \\
&= 2 \cdot \frac{\frac{1}{3^3}}{1-\frac{1}{3}} \\
&= \frac{1}{9}
\end{aligned}$$

Another way is to write:

$$\begin{aligned}
P(\{3,4,5,\cdots\}) &= 1 - P(\{1,2\}) \\
&= 1 - 2 \cdot (\frac{1}{3} + \frac{1}{9}) \\
&= 1 - \frac{2 \cdot 4}{9} \\
&= \frac{1}{9}
\end{aligned}$$

17. Four teams A, B, C, and D compete in a tournament. Teams A and B have the same chance of winning the tournament. Team C is twice as likely to win the tournament as team D. The probability that either team A or team C wins the tournament is 0.6. Find the probabilities of each team winning the tournament.

Solution:

We have

$$\begin{cases} P(A) = P(B) \\ P(C) = 2P(D) \\ P(A \cup C) = 0.6 & \text{thus} \quad P(A) + P(C) = 0.6 \\ P(A) + P(B) + P(C) + P(D) = 1 \end{cases}$$

which results in

$$\begin{cases} P(A) = P(B) = P(D) = 0.2 \\ P(C) = 0.4 \end{cases}$$

18. Let T be the time needed to complete a job at a certain factory. By using the historical data, we know that

$$P(T \leq t) = \begin{cases} \frac{1}{16}t^2 & \text{for} \quad 0 \leq t \leq 4 \\ 1 & \text{for} \quad t \geq 4 \end{cases}$$

(a) Find the probability that the job is completed in less than one hour, i.e., find $P(T \leq 1)$.
(b) Find the probability that the job needs more than 2 hours.
(c) Find the probability that $1 \leq T \leq 3$.

Solution:

(a)

$$\begin{aligned} P(T \leq 1) &= \frac{1}{16} \cdot 1 \\ &= \frac{1}{16}. \end{aligned}$$

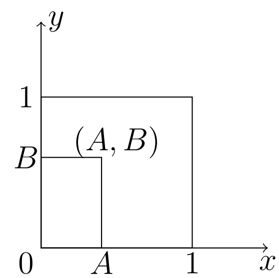
(b)

$$\begin{aligned} P(T \geq 2) &= 1 - P(T < 2) \\ &= 1 - \frac{1}{16} \cdot 4 \\ &= 1 - \frac{1}{4} \\ &= \frac{3}{4} \end{aligned}$$

(c)

$$\begin{aligned} P(1 \leq T \leq 3) &= P(T \leq 3) - P(T \leq 1) \\ &= \frac{9}{16} - \frac{1}{16} \\ &= \frac{1}{2} \end{aligned}$$

19. * You choose a point (A, B) uniformly at random in the unit square $\{(x, y) : 0 \leq x, y \leq 1\}$.



What is the probability that the equation

$$AX^2 + X + B = 0$$

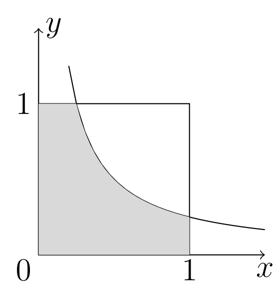
has real solutions?

Solution:

The equation has real roots if and only if:

$$1 - 4AB > 0 \quad \text{i.e.} \quad AB < \frac{1}{4}.$$

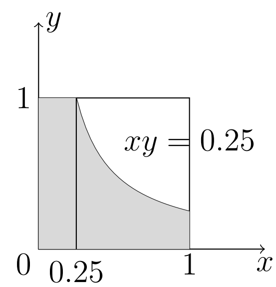
This area is shown here:



Since (A, B) is uniformly chosen in the square we can say that probability of having <u>real roots</u> is

$$\begin{aligned} P(R) &= \frac{\text{area of the shaded region}}{\text{area of the square}} \\ &= \frac{\text{area of the shaded region}}{1} \end{aligned}$$

To find the area of the shaded region we can set up the following integral:



$$\begin{aligned} Area &= \frac{1}{4} + \int_{\frac{1}{4}}^{1} \frac{1}{4x} dx \\ &= \frac{1}{4} + \frac{1}{4} [\ln(x)]_{\frac{1}{4}}^{1} \\ &= \frac{1}{4} + \frac{1}{4} \ln 4 \end{aligned}$$

20. ** (continuity of probability)
(a) Let $A_1, A_2, A_3, \cdots$ be a sequence of increasing events, that is

$$A_1 \subset A_2 \subset A_3 \subset \cdots$$

Show that

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P(A_i)$$

(b) Using part(a), show that if $A_1, A_2, \cdots$ is a decreasing sequence of events, i.e.,

$$A_1 \supset A_2 \supset A_3 \supset \cdots$$

then

$$P\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P(A_i)$$

Solution:

If $A_1 \subset A_2 \subset \cdots$, define the sequence of events $B_1, B_2, \cdots$ as follows,

$$\begin{aligned}
B_1 &= A_1 \\
B_2 &= A_2 - A_1 \\
B_3 &= A_3 - A_2 \\
&\vdots \\
B_i &= A_i - A_{i-1} \\
&\vdots
\end{aligned}$$

Then:
(1) B_i's are disjoint.
(2) $\bigcup_{i=1}^{n} B_i = A_n$
(3) $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$.
Thus

$$\begin{aligned}
P\left(\bigcup_{i=1}^{\infty} A_i\right) &= P\left(\bigcup_{i=1}^{\infty} B_i\right) \\
&= \sum_{i=1}^{\infty} P(B_i) \quad \text{(since} B_i\text{'s are disjoint)} \\
&= \lim_{n\to\infty}\left(\sum_{i=1}^{n} P(B_i)\right) \quad \text{(by definition of infinite sums)} \\
&= \lim_{n\to\infty}[P\left(\bigcup_{i=1}^{n} B_i\right)] \quad \text{(since} B_i\text{'s are disjoint)} \\
&= \lim_{n\to\infty} P(A_n)
\end{aligned}$$

(b) Consider the sequence $A_1^c, A_2^c, A_3^c, \cdots$ then $A_1^c \subset A_2^c \subset A_3^c \subset \cdots$. Thus by part

(a)

$$
\begin{aligned}
P\left(\bigcup_{i=1}^{\infty} A_i^c\right) &= \lim_{i\to\infty} P(A_i^c) \\
1 - P\left(\bigcap_{i=1}^{\infty} A_i\right) &= 1 - \lim_{i\to\infty} P(A_i) \\
\text{thus} \quad P\left(\bigcap_{i=1}^{\infty} A_i\right) &= \lim_{i\to\infty} P(A_i).
\end{aligned}
$$

21. ** (continuity of probability) For any sequence of events $A_1, A_2, A_3, \cdots$. Prove

$$
\begin{aligned}
P\left(\bigcup_{i=1}^{\infty} A_i\right) &= \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} A_i\right) \\
P\left(\bigcap_{i=1}^{\infty} A_i\right) &= \lim_{n\to\infty} P\left(\bigcap_{i=1}^{n} A_i\right)
\end{aligned}
$$

Solution:

Define the new sequence $B_1, B_2, \cdots$ as

$$
\begin{aligned}
B_1 &= A_1 \\
B_2 &= A_2 - A_1 \\
B_3 &= A_3 - (A_1 \cup A_2) \\
&\vdots \\
B_i &= A_i - \left(\bigcup_{j=1}^{i-1} A_j\right)
\end{aligned}
$$

Then we have:
(a) B_i's are disjoint.
(b) $\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i$.
(c) $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$.

Then we can write:

$$\begin{aligned}
P\left(\bigcup_{i=1}^{\infty} A_i\right) &= P\left(\bigcup_{i=1}^{\infty} B_i\right) \\
&= \sum_{i=1}^{\infty} P(B_i) \quad (Bi\text{'s are disjoint}) \\
&= \lim_{n\to\infty}\left(\sum_{i=1}^{n} P(B_i)\right) \quad \text{(definition of infinite sum)} \\
&= \lim_{n\to\infty}\left[P\left(\bigcup_{i=1}^{n} B_i\right)\right] \quad (Bi\text{'s are disjoint}) \\
&= \lim_{n\to\infty}\left[P\left(\bigcup_{i=1}^{n} A_i\right)\right]
\end{aligned}$$

To prove the second part apply the result of the first part to $A_1^c, A_2^c, \cdots$.

22. Suppose that of all the customers at a coffee shop:

-70% purchase a cup of coffee.
-40% purchase a piece of cake.
-20% purchase both a cup of coffee and a piece of cake.
Given that a randomly chosen customer has purchased a piece of cake, what is the probability that he/she has also purchased a cup of coffee?

Solution:

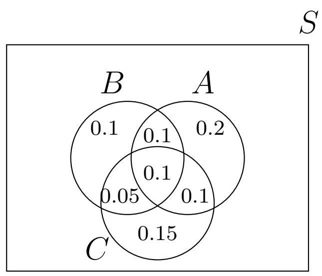
We know

$$\begin{aligned}
P(A) &= 0.7 \\
P(B) &= 0.4 \\
P(A \cap B) &= 0.2
\end{aligned}$$

Therefore:

$$\begin{aligned}
P(A|B) &= \frac{P(A \cap B)}{P(B)} \\
&= \frac{0.2}{0.4} \\
&= \frac{1}{2}
\end{aligned}$$

23. Let A, B, and C be three events with probabilities given:



a) Find $P(A|B)$
b) Find $P(C|B)$
c) Find $P(B|A \cup C)$
d) Find $P(B|A, C) = P(B|A \cap C)$

Solution:

(a)

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{0.2}{0.35} \\ &= \frac{4}{7} \end{aligned}$$

(b)

$$\begin{aligned} P(C|B) &= \frac{P(C \cap B)}{P(B)} \\ &= \frac{0.15}{0.35} \\ &= \frac{3}{7} \end{aligned}$$

(c)

$$\begin{aligned} P(B|A \cup C) &= \frac{P(B \cap (A \cup C))}{P(A \cup C)} \\ &= \frac{0.1 + 0.1 + 0.05}{0.2 + 0.1 + 0.1 + 0.1 + 0.5 + 0.05} \\ &= \frac{0.25}{0.7} \\ &= \frac{5}{14} \end{aligned} \tag{2}$$

(d)

$$\begin{aligned} P(B|A, C) &= \frac{P(B \cap A \cap C)}{P(A \cap C)} \\ &= \frac{0.1}{0.2} \\ &= \frac{1}{2} \end{aligned}$$

24. A **real** number X is selected uniformly at random in the continuous interval $[0, 10]$. (For example, X could be 3.87.)
(a) Find $P\,(2 \leq X \leq 5)$.
(b) Find $P\,(X \leq 2|X \leq 5)$.
(c) Find $P\,(3 \leq X \leq 8|X \geq 4)$.

Solution:

Since X is selected uniformly at random we conclude that:

$$P(a \leq X \leq b) = (b - a) \times c \quad \text{for} \quad 0 \leq a \leq b \leq 10$$

where c is a constant. Since $P(0 \leq X \leq 10) = 1$. We conclude $c = \frac{1}{10}$. Therefore

$$P(a \leq X \leq b) = \frac{b - a}{10} \quad \text{for} \quad 0 \leq a \leq b \leq 10$$

(a)

$$\begin{aligned} P(2 \leq X \leq 5) &= \frac{5 - 2}{10} \\ &= 0.3 \end{aligned}$$

(b)

$$
\begin{aligned}
P[(X \leq 2)|(X \leq 5)] &= \frac{P[(X \leq 2) \cap (X \leq 5)]}{P[X \leq 5]} \\
&= \frac{P[(X \leq 2) \quad \text{and} \quad (X \leq 5)]}{P[X \leq 5]} \\
&= \frac{P[X \leq 2]}{[X \leq 5]} \\
&= \frac{P[0 \leq X \leq 2]}{P[0 \leq X \leq 5]} \\
&= \frac{\frac{2-0}{10}}{\frac{5-0}{10}} \\
&= 0.4
\end{aligned}
$$

(c)

$$
\begin{aligned}
P[3 \leq X \leq 8|X \geq 4] &= \frac{P[(3 \leq X \leq 8) \quad \text{and} \quad (X \geq 4)]}{P[X \geq 4]} \\
&= \frac{P[4 \leq X \leq 8]}{P[4 \leq X \leq 10]} \\
&= \frac{\frac{8-4}{10}}{\frac{10-4}{10}} \\
&= \frac{4}{6} = \frac{2}{3}.
\end{aligned}
$$

25. A professor thinks students who live on campus are more likely to get As in the probability course. To check this theory, the professor combines the data from the past few years:

    (a) 600 students have taken the course.
    (b) 120 students have got As.
    (c) 200 students lived on campus.
    (d) 80 students lived off campus and got As.

    Does this data suggest that "getting an A" and "living on campus" are dependent or independent?

    Solution:

let C be the event that a random student lives on campus and A be the event that he/she gets an A. We have:

$$\begin{aligned}
P(A) &\approx \frac{120}{600} = \frac{1}{5} \\
P(C) &\approx \frac{200}{600} = \frac{1}{3} \\
P(A \cap C^c) &\approx \frac{80}{600} = \frac{2}{15} \\
P(A \cap C) &= P(A) - P(A \cap C^c) \\
&= \frac{1}{5} - \frac{2}{15} \\
&= \frac{1}{15}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{15} &= P(A \cap C) \\
&= P(A).P(C)
\end{aligned}$$

The data suggests that A and C are independent.

26. I roll a dice n times, $n \in \mathbb{N}$. Find the probability that numbers 1 and 6 are both observed at least once.

Solution: Let A_1 be the event that the number 1 is observed at least once and let A_6 be the event that 6 is observed at least once. Then, we are interested in probability $P(A_1 \cap A_6)$. We can write

$$P(A_1 \cap A_6) = P(A_1) + P(A_6) - P(A_1 \cup A_6).$$

Then

$$\begin{aligned}
P(A_1) &= 1 - P(A_1^c) \\
&= 1 - P(\text{No} \quad 1s) \\
&= 1 - (\frac{5}{6})^n.
\end{aligned}$$

Similarly, $P(A_6) = 1 - (\frac{5}{6})^n$. Finally,

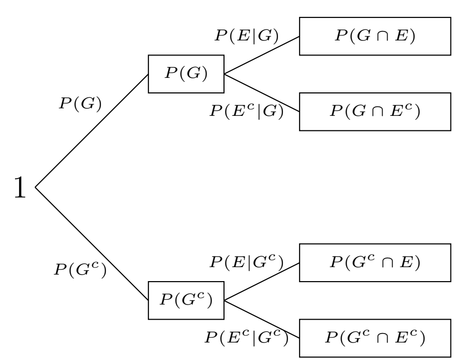
$$\begin{aligned} P(A_1 \cup A_6) &= 1 - P(A_1^c \cap A_6^c) \\ &= 1 - P(\text{No} \quad 1s \quad \text{and No} \quad 6s) \\ &= 1 - (\frac{4}{6})^n \\ &= 1 - (\frac{2}{3})^n. \end{aligned}$$

Thus, we have

$$\begin{aligned} P(A_1 \cap A_6) &= P(A_1) + P(A_6) - P(A_1 \cup A_6) \\ &= 2\left[1 - \left(\frac{5}{6}\right)^n\right] - 1 + \left(\frac{2}{3}\right)^n. \end{aligned}$$

27. Consider a communication system. At any given time, the communication channel is in good condition with probability 0.8 and is in bad condition with probability 0.2. An error occurs in a transmission with probability 0.1 if the channel is in good condition and with probability 0.3 if the channel is in bad condition. Let G be the event that the channel is in good condition and E be the event that there is an error in transmission.

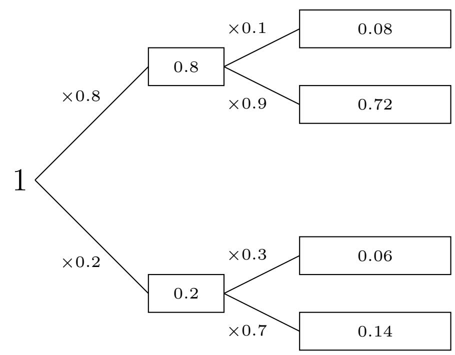
(a) Complete the following tree diagram:



(b) Using the tree find $P(E)$.
(c) Using the tree find $P(G|E^c)$.

Solution:

(a)



(b)

$$
\begin{aligned}
P(E) &= P(G \cap E) + P(G^c \cap E) \\
&= 0.08 + 0.06 \\
&= 0.14
\end{aligned}
$$

(c)

$$
\begin{aligned}
P(G|E^c) &= \frac{P(G \cap E^c)}{P(E^c)} \\
&= \frac{0.72}{1 - 0.14} \\
&= \frac{0.72}{0.86} \\
&\approx 0.84
\end{aligned}
$$

28. * In a factory there are 100 units of a certain product, 5 of which are defective. We pick three units from the <u>100</u> units at random. What is the probability that exactly one of them is defective?

Solution:

Let D_i be the event that the i^{th} chosen unit is defective, for $i = 1, 2, 3$. We are interested in

$$P(A) = P\left((D_1^c \cap D_2^c \cap D_3) \cup (D_1^c \cap D_2 \cap D_3^c) \cup (D_1 \cap D_2^c \cap D_3^c)\right)$$

We have:

$$\begin{aligned}
P(D_1^c \cap D_2^c \cap D_3) &= P(D_1^c) \cdot P(D_2^c|D_1^c) \cdot P(D_3|D_1^c \cap D_2^c) \\
&= \frac{95}{100} \cdot \frac{94}{99} \cdot \frac{5}{98} \\
P(D_1^c \cap D_2 \cap D_3^c) &= \frac{95}{100} \cdot \frac{5}{99} \cdot \frac{94}{98} \\
P(D_1 \cap D_2^c \cap D_3^c) &= \frac{5}{100} \cdot \frac{95}{99} \cdot \frac{94}{98}
\end{aligned}$$

Therefore:

$$\begin{aligned}
P(A) &= \frac{5 \cdot 3 \cdot 95 \cdot 94}{100 \cdot 99 \cdot 98} \\
&\approx 0.14
\end{aligned}$$

29. Reliability:
Real-life systems often are composed of several components. For example, a system may consist of two components that are connected in parallel as shown in Figure 1. When the system's components are connected in parallel, the system works if <u>at least one</u> of the components is functional. The components might also be connected in series as shown in Figure 1. When the system's components are connected in series, the system works if <u>all</u> of the components are functional.

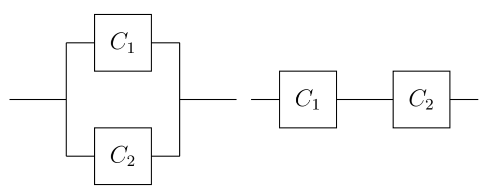
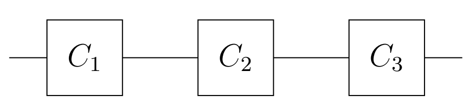
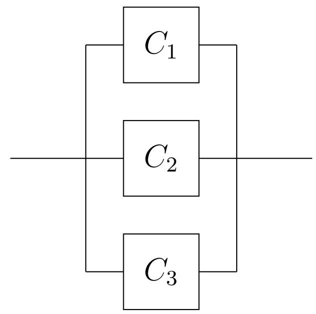


Figure 1: In left figure, Components C_1 and C_2 are connected in parallel. The system is functional if at least one of the C_1 and C_2 is functional. In right figure, Components C_1 and C_2 are connected in series. The system is functional only if both C_1 and C_2 are functional.

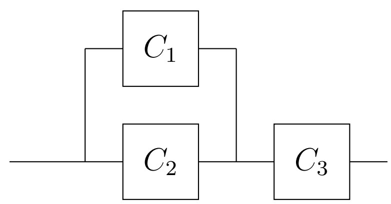
For each of the following systems, find the probability that the system is functional. Assume that component k is functional with probability P_k independent of other components.
a)



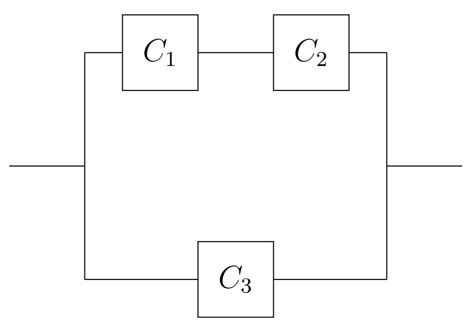
b)



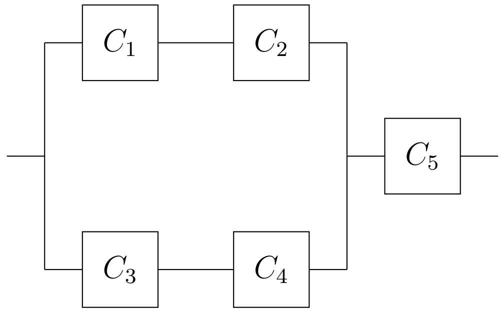
c)



d)



e)



Solution:

Let A_k be the event that the k^{th} component is functional and let A be the event that the whole system is functional.

a)

$$\begin{aligned}P(A) &= P(A_1 \cap A_2 \cap A_3)\\ &= P(A_1)\cdot P(A_2)\cdot P(A_3) \quad \text{(since } A_i\text{s are independent)}\\ &= P_1P_2P_3\end{aligned}$$

b)

$$\begin{aligned}P(A) &= P(A_1 \cup A_2 \cup A_3)\\ &= 1 - P(A_1^c \cap A_2^c \cap A_3^c) \quad \text{(Demorgan's law)}\\ &= 1 - P(A_1^c)P(A_2^c)P(A_3^c) \quad \text{(since } A_i\text{s are independent)}\\ &= 1-(1-P_1)(1-P_2)(1-P_3).\end{aligned}$$

c)

$$\begin{aligned}P(A) &= P((A_1 \cup A_2)\cap A_3)\\ &= P(A_1 \cup A_2)\cdot P(A_3) \quad \text{(since } A_i\text{s are independent)}\\ &= [1-P(A_1^c \cap A_2^c)]\cdot P(A_3)\\ &= [1-(1-P_1)(1-P_2)]P_3\end{aligned}$$

d)

$$\begin{aligned}P(A) &= P\left[(A_1 \cap A_2)\cup A_3\right]\\ &= 1 - P\left((A_1\cap A_2)^c\right)\cdot P(A_3^c) \quad \text{(since } A_i\text{s are independent)}\\ &= 1-(1-P(A_1)\cdot P(A_2))\,(1-P(A_3))\\ &= 1-(1-P_1P_2)(1-P_3)\end{aligned}$$

e)

$$
\begin{aligned}
P(A) &= P\left[((A_1 \cap A_2) \cup (A_3 \cap A_4)) \cap A_5\right] \\
&= P\left((A_1 \cap A_2) \cup (A_3 \cap A_4)\right) \cdot P(A_5) \quad \text{(since } A_i\text{s are independent)} \\
&= \left[1 - (1 - P(A_1 \cap A_2)) \cdot (1 - P(A_3 \cap A_4))\right] P_5 \quad \text{(parallel links)} \\
&= \left[1 - (1 - P_1 P_2)(1 - P_3 P_4)\right] P_5
\end{aligned}
$$

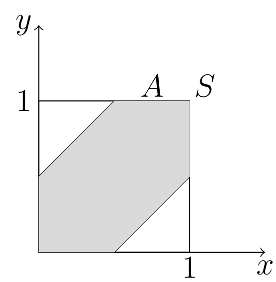
30. You choose a point (X, Y) uniformly at random in the unit square.

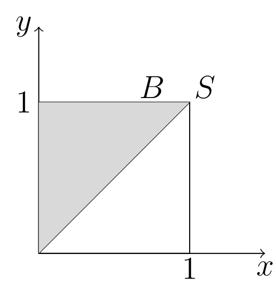
$$S = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1, 0 \le y \le 1\}$$

Let A be the event $\{(x, y) \in S : |x - y| \le \frac{1}{2}\}$ and B be the event $\{(x, y) \in S : y \ge x\}$.
(a) Show sets A and B in the x-y plane.
(b) Find $P(A)$ and $P(B)$.
(c) Are A and B independent?

Solution:

(a)





(b)

$$
\begin{aligned}
P(A) &= \frac{\text{Area of } A}{\text{Area of } S} \\
&= \frac{3}{4} \\
P(B) &= \frac{\text{Area of } B}{\text{Area of } S} \\
&= \frac{1}{2}
\end{aligned}
$$

(c)

$$
\begin{aligned}
P(A \cap B) &= \frac{\text{Area of } (A \cap B)}{\text{Area of } S} \\
&= \frac{3}{8} \\
&= P(A) \cdot P(B) \\
&\text{thus } A \text{ and } B \text{ are independent.}
\end{aligned}
$$

31. One way to design a spam filter is to look at the words in an email. In particular, some words are more frequent in spam emails. Suppose that we have the following information:

    (a) 50% of emails are spam.

    (b) 1% of spam emails contain the word "refinance".

    (c) 0.001% of non-spam emails contain the word "refinance".

    Suppose that an email is checked and found out to contain the word refinance. What is the probability that the email is a spam?

    Solution:

    Let S be the event that an email is a spam and let R be the event that the email contains the word "refinance". Then,

$$
\begin{aligned}
P(S) &= \frac{1}{2} \\
P(R|S) &= \frac{1}{100} \\
P(R|S^c) &= \frac{1}{100000}
\end{aligned}
$$

Then,

$$\begin{aligned} P(S|R) &= \frac{P(R|S)P(S)}{P(R)} \\ &= \frac{P(R|S)P(S)}{P(R|S)P(S) + P(R|S^c)P(S^c)} \\ &= \frac{\frac{1}{100} \times \frac{1}{2}}{\frac{1}{100} \times \frac{1}{2} + \frac{1}{100000} \times \frac{1}{2}} \\ &\approx 0.999 \end{aligned}$$

32. * You would like to go from point A to point B in Figure 2. There are 5 bridges on different branches of the river as shown in Figure 2.

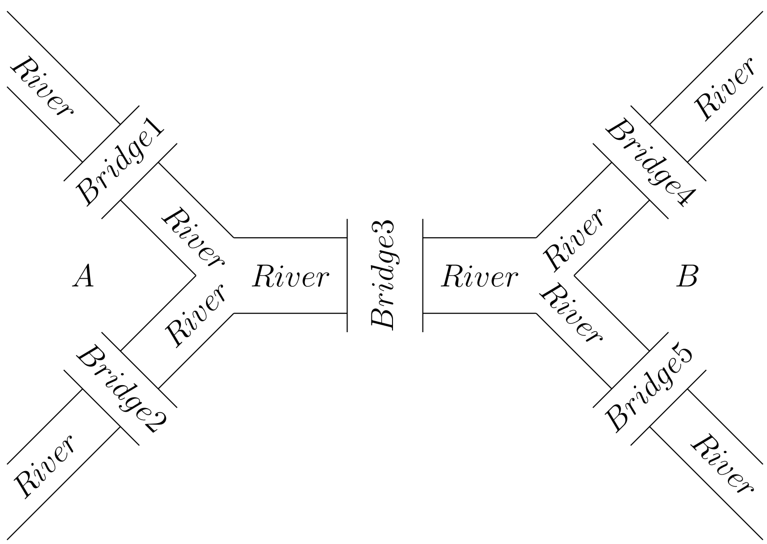


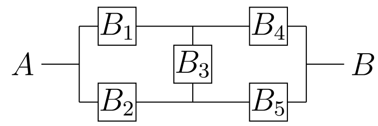
Figure 2: Problem 32

Bridge i is open with probability P_i, $i = 1, 2, 3, 4, 5$. Let A be the event that there is a path from A to B and let B_k be the event that k^{th} bridge is open.

(a) Find $P(A)$.

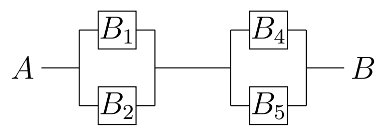
(b) Find $P(B_3|A)$.

Solution:



$$P(A) = P(A|B_3)P(B_3) + P(A|B_3^c)P(B_3^c)$$

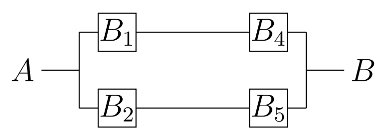
Given B_3:



Thus,

$$P(A|B_3) = [1 - (1 - P_1)(1 - P_2)][1 - (1 - P_4)(1 - P_5)]$$

Given B_3^c:



Thus,

$$P(A|B_3^c) = 1 - (1 - P_1P_4)(1 - P_2P_5)$$

We conclude:

$$\begin{aligned} P(A) &= [1 - (1 - P_1)(1 - P_2)][1 - (1 - P_4)(1 - P_5)]P_3 \\ &\quad + (1 - P_3)[1 - (1 - P_1P_4)(1 - P_2P_5)] \end{aligned}$$

(b)

$$\begin{aligned} P(B_3|A) &= \frac{P(A|B_3)P(B_3)}{P(A)} \quad \text{(Bayes rule)} \\ &= \frac{P(A|B_3)P(B_3)}{P(A|B_3)P(B_3) + P(A|B_3^c)P(B_3^c)} \\ &= \frac{[1 - (1 - P_1)(1 - P_2)][1 - (1 - P_4)(1 - P_5)]P_3}{[1 - (1 - P_1)(1 - P_2)][1 - (1 - P_4)(1 - P_5)]P_3 + (1 - P_3)[1 - (1 - P_1P_4)(1 - P_2P_5)]} \end{aligned}$$

33. * (The Monte Hall Problem [1]) You are in a game show, and the host gives you the choice of three doors. Behind one door is a car and behind the others are goats. You pick a door, say door 1. The host who knows what is behind the doors opens a different door and reveals a goat (the host can always open such a door because there is only one door behind which is a car). The host then asks you: "Do you want to switch?" The question is, is it to your advantage to switch your choice?

$$\boxed{1} \qquad \boxed{2} \qquad \boxed{Goat}$$

Solution:

Yes, if you switch, your chance of winning the car is $\frac{2}{3}$. Let $\underline{W}$ be the event that you win the car if you switch. Let C_i be the event that the car is behind door i, for $i = 1, 2, 3$. Then $P(C_i) = \frac{1}{3} \quad i = 1, 2, 3$.
Then,

$$\begin{aligned}
P(W) &= \sum_{i=1}^{3} P(W|C_i)P(C_i) \\
&= P(W|C_1)P(C_1) + P(W|C_2)P(C_2) + P(W|C_3)P(C_3) \\
&= 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} \\
&= \frac{2}{3}
\end{aligned}$$

34. * I toss a fair die twice, and obtain two numbers X and Y. Let A be the event that $X = 2$, B be the event that $X + Y = 7$, and C be the event that $Y = 3$.

    (a) Are A and B independent?
    (b) Are A and C independent?
    (c) Are B and C independent?
    (d) Are A, B, and C are independent?

Solution:

[1] http://en.wikipedia.org/wiki/Monty_Hall_problem

$$\begin{aligned}
A &= \{(2,1),\cdots,(2,6)\} \\
B &= \{(1,6),(2,5),\cdots,(6,1)\} \\
C &= \{(1,3),(2,3),(3,3),\cdots,(6,3)\}
\end{aligned}$$

(a)

$$A \cap B = \{(2,5)\}$$

Thus,

$$\begin{aligned}
P(A \cap B) &= \frac{1}{36} \\
P(A) &= \frac{1}{6} \\
P(B) &= \frac{1}{6} \\
P(A \cap B) &= P(A) \cdot P(B) \quad \text{(A and B are independent)}
\end{aligned}$$

(b)

$$A \cap C = \{(2,3)\}$$

Therefore,

$$\begin{aligned}
P(A \cap C) &= \frac{1}{36} \\
&= P(A).P(C) \\
&= \frac{1}{6} \cdot \frac{1}{6}
\end{aligned}$$

(c)

$$\begin{aligned}
B \cap C &= \{(4,7)\} \\
P(B \cap C) &= \frac{1}{36} \\
&= \frac{1}{6} \cdot \frac{1}{6} \\
&= P(B) \cdot P(C) \quad \text{(B,C are independent)}
\end{aligned}$$

(d)

$$\begin{aligned} A \cap B \cap C &= \varnothing \\ P(A \cap B \cap C) &= 0 \\ &\neq P(A) \cdot P(B) \cdot P(C) \quad \text{(A,B,C are \underline{NOT} independent)} \end{aligned}$$

35. * You and I play the following game: I toss a coin repeatedly. The coin is unfair and $P(H) = p$. The game ends the first time that two consecutive heads (HH) or two consecutive tails (TT) are observed. I win if (HH) is observed and you win if (TT) is observed. Given that I won the game find the probability that the first coin toss resulted in head?

Solution:

Let A be the event that I win.

$$
\begin{aligned}
P(A) &= P(A|H)P(H) + P(A|T)P(T) \\
P(A|H) &: \quad \text{the probability that I win given that the result of the first coin toss is a head.} \\
A|H &: HH, HTHH, HTHTHH, \cdots \\
P(A|H) &= p + pqp + (pq)^2 p + \cdots \\
&= p[1 + pq + \cdots] \\
&= \frac{p}{1-pq} \\
A|T &: THH, THTHH, THTHTHH, \cdots \\
P(A|T) &= p^2 + p(1-p)p^2 + \cdots \\
&= p^2[1 + pq + (pq)^2 + \cdots] \\
&= \frac{p^2}{1-pq} \\
P(A) &= P(A|H)P(H) + P(A|T)P(T) \\
&= \frac{p^2}{1-pq} + \frac{p^2 q}{1-pq} \\
&= \frac{p^2(1+q)}{1-pq} \\
P(H|A) &= \frac{P(A|H)P(H)}{P(A)} \\
&= \frac{\frac{p^2}{1-pq}}{\frac{p^2}{1-pq}(1+q)} \\
&= \frac{1}{1+q} \\
&= \frac{1}{2-p}
\end{aligned}
$$

36. * A box contains two coins: a regular coin and one fake two headed coin (P(H)=1). I choose a coin at random and toss it n times. If the first n coin tosses result in heads, what is the probability that the $(n+1)^{th}$ coin toss will also result in heads?

Solution:

Define the following events:

$$
\begin{aligned}
A_n: &\quad \text{The first } n \text{ coin tosses result in heads.}\\
H_{n+1}: &\quad \text{The } (n+1)^{th} \text{ coin toss results in head.}\\
C: &\quad \text{The regular coin has been selected.}
\end{aligned}
$$

$$P(H_{n+1}) = P(H_{n+1}|C)P(C) + P(H_{n+1}|C^c)P(C^c)$$

$$P(H_{n+1}|A_n) = P(H_{n+1}|C, A_n)P(C|A_n) + P(H_{n+1}|C^c, A_n)P(C^c|A_n) \quad \text{(conditioning on } A_n)$$

given C (or C^c), A_n and H_{n+1} are independent; Thus,

$$
\begin{aligned}
P(H_{n+1}|A_n) &= P(H_{n+1}|C)P(C|A_n) + P(H_{n+1}|C^c)P(C^c|A_n)\\
P(C|A_n) &= \frac{P(A_n|C)\cdot P(C)}{P(A_n)}\\
&= \frac{P(A_n|C)\cdot P(C)}{P(A_n|C)\cdot P(C) + P(A_n|C^c)\cdot P(C^c)}\\
&= \frac{\frac{1}{2}^n \cdot \frac{1}{2}}{\frac{1}{2}^n \cdot \frac{1}{2} + \frac{1}{2}}\\
&= \frac{1}{2^n + 1}\\
P(C^c|A_n) &= 1 - P(C|A_n)\\
&= \frac{2^n}{2^n+1}
\end{aligned}
$$

Thus:

$$
\begin{aligned}
P(H_{n+1}|A_n) &= \frac{1}{2}\cdot\frac{1}{2^n+1} + 1\cdot\frac{2^n}{2^n+1}\\
&= \frac{2^{n+1}+1}{2^{n+1}+2}
\end{aligned}
$$

37. * A family has n children, $n \geq 2$. What is the probability that all children are girls, given that at least one of them is a girl?

Solution:

The sample space has 2^n elements,

$$S = \{(G, G, \cdots, G), (G, \cdots, B), \cdots, (B, B, \cdots, B)\}.$$

Let A be the event that all the children are girls, then

$$A = \{(G, G, \cdots, G)\}.$$

thus

$$P(A) = \frac{1}{2^n}.$$

Let B be the event that at least one child is a girl, then:

$$\begin{aligned} B &= S - \{(B, \cdots, B)\} \\ |B| &= 2^n - 1 \\ P(B) &= \frac{2^n - 1}{2^n} \end{aligned} \tag{3}$$

Then

$$\begin{aligned} A \cap B &= A \\ P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{P(A)}{P(B)} \\ &= \frac{\frac{1}{2^n}}{\frac{2^n-1}{2^n}} \\ &= \frac{1}{2^n - 1} \end{aligned}$$

Note: If we let $n = 2$, we obtain $P(A|B) = \frac{1}{3}$ which is the same as Example 17 in the text.

38. * A family has n children, $n \geq 2$. We know that they have a daughter named Lilia. What is the probability that all their children are girls? Here you can assume that if a child is a girl, her name will be Lilia with probability $\alpha \ll 1$ independently from other children's names.

Solution:

Let L be the event that the family has at least one child named Lilia. Let A be the event that all the children are girls and let B be the event that at least one child is a girl. Then,

$$\begin{aligned} P(L|A) &= 1 - (1 - \alpha)^n \\ P(L) &= 1 - [1 - \frac{\alpha}{2}]^n \\ P(A) &= \frac{1}{2^n} \end{aligned}$$

So, we have:

$$
\begin{aligned}
P(A|L) &= \frac{P(L|A)P(A)}{P(L)} \\
&= \frac{\frac{[1-(1-\alpha)^n]}{2^n}}{1-[1-\frac{\alpha}{2}]^n}
\end{aligned}
$$

$$
\text{as} \quad \alpha \to 0 \quad (1-\alpha)^n \approx 1 - n\alpha
$$

$$
(1-\alpha/2)^n \approx 1 - \frac{n\alpha}{2}
$$

Thus,

$$
\begin{aligned}
P(A|L) &\approx \frac{1}{2^n} \cdot \frac{n\alpha}{\frac{n\alpha}{2}} \\
&= \frac{1}{2^{n-1}}
\end{aligned}
$$

Letting $n = 2 \cdots$ (solved problem 7).

39. * A family has n children. We pick one of them at random and find out that she is a girl. What is the probability that all their children are girls?

Solution:

Let Gr be the event that a randomly chosen child is a girl. Let A be the event that all the children are girls. Then,

$$
\begin{aligned}
P(Gr|A) &= 1 \\
P(A) &= \frac{1}{2^n} \\
P(Gr) &= \frac{1}{2}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
P(A|Gr) &= \frac{P(Gr|A)P(A)}{P(Gr)} \\
&= \frac{1 \cdot \frac{1}{2^n}}{\frac{1}{2}} \\
&= \frac{1}{2^{n-1}}
\end{aligned}
$$