1. Let X_i be i.i.d $Uniform(0,1)$. We define the sample mean as

$$M_n = \frac{X_1 + X_2 + ... + X_n}{n}.$$

(a) Find $E[M_n]$ and $\text{Var}(M_n)$ as a function of n.

(b) Using the Chebyshev's inequality, find an upper bound on

$$P\left(\left|M_n - \frac{1}{2}\right| \geq \frac{1}{100}\right).$$

(c) Using your bound, show that

$$\lim_{n\to\infty} P\left(\left|M_n - \frac{1}{2}\right| \geq \frac{1}{100}\right) = 0.$$

Solution:

(a)

$$\begin{aligned}
EM_n &= \frac{EX_1 + \cdots + EX_n}{n} \\
&= \frac{nEX_1}{n} \\
&= EX_1 = \frac{1}{2} \\
\text{Var}(M_n) &= \frac{1}{n^2}\sum_{i=1}^{n} \text{Var}(X_i) \\
&= \frac{n\text{Var}X_1}{n^2} \\
&= \frac{\text{Var}(X_1)}{n} \\
&= \frac{\frac{1}{12}}{n} = \frac{1}{12n}
\end{aligned}$$

(b)

$$\begin{aligned}
P\left(\left|M_n - \frac{1}{2}\right| \geq \frac{1}{100}\right) &\leq \frac{\text{Var}(M_n)}{\left(\frac{1}{100}\right)^2} \\
&= \frac{10000}{12n}
\end{aligned}$$

(c)

$$\begin{aligned}
\lim_{n\to\infty} P\left(\left|M_n - \frac{1}{2}\right| \geq \frac{1}{100}\right) &\leq \lim_{n\to\infty} \frac{10000}{12n} = 0 \\
\lim_{n\to\infty} P\left(\left|M_n - \frac{1}{2}\right| \geq \frac{1}{100}\right) &= 0 \quad \text{(since probability is non-negative)}
\end{aligned}$$

2. The number of accidents in a certain city is modeled by a Poisson random variable with average rate of 10 accidents per day. Suppose that the number of accidents in different days are independent. Use the central limit theorem to find the probability that there will be more than 3800 accidents in a certain year. Assume that there are 365 days in a year.

   Solution:

$$\begin{aligned}
Y &= X_1 + X_2 + \cdots + X_n, \quad n = 365 \\
X_i &\sim Poisson(\lambda = 10). \quad \text{Thus:} \quad EX_i = 10 \\
\text{Var}(X_i) &= \lambda = 10 \\
EY &= 365 \times 10 = 3650 \\
\text{Var}(Y) &= 365 \times 10 = 3650 \\
\frac{Y - 3650}{\sqrt{3650}} & \text{ is approximately} \quad N(0,1) \quad \text{(by the CLT)} \\
P(Y \geq 3800) &= P\left(\frac{Y - 3650}{\sqrt{3650}} \geq \frac{3800 - 3650}{\sqrt{3650}}\right) \\
&= 1 - \Phi\left(\frac{3800 - 3650}{\sqrt{3650}}\right) \\
&\approx 1 - \Phi(2.48) \\
&\approx 0.0065
\end{aligned}$$

3. In a communication system, each codeword consists of 1000 bits. Due to the noise, each bit may be received in error with probability 0.1. It is assumed bit errors occur independently. Since error correcting codes are used in this system, each codeword can be decoded reliably if there are less than or equal to 125 errors in the received codeword, otherwise the decoding fails. Using the CLT, find the probability of decoding failure.

Solution: Let $Y = X_1 + X_2 + \cdots + X_n, \quad n = 1000.$

$$\begin{aligned}
X_i &\sim \quad \text{Bernoulli}(p = 0.1) \\
EX_i &= p = 0.1 \\
\text{Var}(X_i) &= p(1-p) = 0.09 \\
EY &= np = 100 \\
\text{Var}(Y) &= np(1-p) = 90
\end{aligned}$$

By the CLT:

$$\frac{Y - EY}{\sqrt{\text{Var}(Y)}} = \frac{Y - 100}{\sqrt{90}} \quad \text{(can be approximated by} \quad N(0,1)). \quad \text{Thus,}$$

$$\begin{aligned}
P(Y > 125) &= P\left(\frac{Y-100}{\sqrt{90}} > \frac{125-100}{\sqrt{90}}\right) \\
&= 1 - \Phi\left(\frac{25}{\sqrt{90}}\right) \\
&\approx 0.0042
\end{aligned}$$

4. 50 students live in a dormitory. The parking lot has the capacity for 30 cars. Each student has a car with probability $\frac{1}{2}$, independently from other students. Use the CLT (with continuity correction) to find the probability that there won't be enough parking spaces for all the cars?

   Solution:

$$\begin{aligned}
Y &= X_1 + X_2 + \cdots + X_{50} \\
X_i &\sim \mathit{Bernoulli}(\frac{1}{2}) \\
EX_i &= \frac{1}{2} \\
\text{Var}(X_i) &= \frac{1}{4} \\
EY &= 50\frac{1}{2} = 25 \\
\text{Var}Y &= \frac{50}{4} = 12.5
\end{aligned}$$

Therefore,

$$\begin{aligned}
P(Y > 30) &= P(Y \geq 31) \\
&= P(Y \geq 30.5) \quad \text{(continuity correction)} \\
&= P\left(\frac{Y - 25}{\sqrt{12.5}} > \frac{30.5 - 25}{\sqrt{12.5}}\right) \\
&\approx 1 - \Phi\left(\frac{5.5}{\sqrt{12.5}}\right) \text{(By CLT)} \\
&\approx 0.06
\end{aligned}$$

5. The amount of time needed for a certain machine to process a job is a random variable with mean $EX_i = 10$ minutes and $\mathrm{Var}(X_i) = 2$ minutes^2. The time needed for different jobs are independent from each other. Find the probability that the machine processes less than or equal to 40 jobs in 7 hours.

   Solution:

$$\begin{aligned}
Y &= X_1 + X_2 + \cdots + X_{40} \\
EX_i &= 10, \mathrm{Var}(X_i) = 2 \\
EY &= 40 \times 10 = 400 \\
\mathrm{Var}(Y) &= 40 \times 2 = 80 \\
P(\text{Less than or equal to 40 jobs in 7 hours}) &= P(Y > 7 \times 60) \\
&= P(Y > 420) \\
&= P\left(\frac{Y - 400}{\sqrt{80}} > \frac{420 - 400}{\sqrt{80}}\right) \\
&\approx 1 - \Phi\left(\frac{20}{\sqrt{80}}\right) \approx 0.0127
\end{aligned}$$

6. You have a fair coin. You toss the coin n times. Let X be the portion of times that you observe heads. How large n has to be so that you are 95% sure that $0.45 \leq X \leq 0.55$? In other words, how large n has to be so that

$$P\big(0.45 \leq X \leq 0.55\big) \geq .95 \ ?$$

Solution:

$$\begin{aligned}
X &= \frac{X_1 + X_2 + \cdots + X_n}{n} \\
&= \frac{Y}{n} \quad \text{where} \quad Y = X_1 + X_2 + \cdots + X_n \\
X_i &\sim Bernoulli\left(\frac{1}{2}\right) \\
EX_i &= \frac{1}{2} \\
\mathrm{Var}(X_i) &= \frac{1}{4} \\
EY &= \frac{n}{2} \\
\mathrm{Var}(Y) &= \frac{n}{4} \\
P(0.45n \le Y \le 0.55n) &= P\left(\frac{0.45n - 0.5n}{\frac{\sqrt{n}}{2}} \le \frac{Y - 0.5n}{\frac{\sqrt{n}}{2}} \le \frac{0.55n - 0.5n}{\frac{\sqrt{n}}{2}}\right) \\
&\approx \Phi(0.1\sqrt{n}) - \Phi(-0.1\sqrt{n}) = 0.95 \\
2\Phi(0.1\sqrt{n}) - 1 &= 0.95 \\
\Phi(0.1\sqrt{n}) &= 0.975 \\
0.1\sqrt{n} &\approx 1.96 \\
n &\ge 385
\end{aligned}$$

7. An engineer is measuring a quantity q. It is assumed that there is a random error in each measurement, so the engineer will take n measurements and reports the average of the measurements as the estimated value of q. Specifically, if Y_i is the value that is obtained in the ith measurement, we assume that

$$Y_i = q + X_i,$$

where X_i is the error in the i'th measurement. We assume that X_i's are i.i.d with $EX_i = 0$ and $\mathrm{Var}(X_i) = 4$ units. The engineer reports the average of measurements

$$M_n = \frac{Y_1 + Y_2 + ... + Y_n}{n}.$$

How many measurements does the engineer need to make until he is 95% sure that the final error is less than 0.1 units? In other words, what should the value of n be such that

$$P\big(q - 0.1 \le M_n \le q + 0.1\big) \ge 0.95 \ ?$$

Solution:

$$\begin{aligned}
EY_i &= q + EX_i = q \\
\mathrm{Var}(Y_i) &= \mathrm{Var}(X_i) = 4 \\
Y &= Y_1 + \cdots + Y_n \quad \text{Thus:} \quad EY = nq \\
\mathrm{Var}(Y) &= n\mathrm{Var}(Y_i) = 4n \\
P(q - 0.1 \le M_n \le q + 0.1) &= P\left(q - 0.1 \le \frac{Y_1 + \cdots + Y_n}{n} \le q + 0.1\right) \\
&= P(qn - 0.1n \le Y \le qn + 0.1n) \\
&= P\left(\frac{qn - 0.1n - nq}{2\sqrt{n}} \le \frac{Y - nq}{2\sqrt{n}} \le \frac{qn + 0.1n - nq}{2\sqrt{n}}\right) \\
&= P\left(-0.05\sqrt{n} \le \frac{Y - nq}{2\sqrt{n}} \le 0.05\sqrt{n}\right) \\
&\approx \Phi(0.05\sqrt{n}) - \Phi(-0.05\sqrt{n}) \\
&= 2\Phi\left(0.05\sqrt{n}\right) - 1 = 0.95 \\
\Phi\left(0.05\sqrt{n}\right) &= 0.975 \\
0.05\sqrt{n} &\ge 1.96 \\
n &\ge 1537
\end{aligned}$$

8. Let X_2, X_3, X_4, $\cdots$ be a sequence of random variables such that

$$F_{X_n}(x) = \begin{cases} \frac{e^{n(x-1)}}{1+e^{n(x-1)}} & x > 0 \\ \\ 0 & \text{otherwise} \end{cases}$$

Show that X_n converges in distribution to $X = 1$.

Solution: For $x > 1$, we have

$$\begin{aligned}
\lim_{n\to\infty} F_{X_n}(x) &= \lim_{n\to\infty} \frac{e^{n(x-1)}}{1 + e^{n(x-1)}} \\
&= 1
\end{aligned}$$

For $0 \le x < 1$,

$$\begin{aligned}
\lim_{n\to\infty} F_{X_n}(x) &= \lim_{n\to\infty} \frac{e^{n(x-1)}}{1 + e^{n(x-1)}} \\
&= 0
\end{aligned}$$

For $x < 0$,

$$F_{X_n}(x) = 0$$

Therefore,

$$\lim_{n\to\infty} F_{X_n}(x) = \begin{cases} 1 & x > 1 \\ \\ 0 & x < 1 \end{cases}$$

Thus,

$$X \xrightarrow{d} 1$$

9. Let $X_2,\ X_3,\ X_4,\ \cdots$ be a sequence of non-negative random variables such that

$$F_{X_n}(x) = \begin{cases} \frac{e^{nx}+xe^n}{e^{nx}+\left(\frac{n+1}{n}\right)e^n} & 0 \le x \le 1 \\ \\ \frac{e^{nx}+e^n}{e^{nx}+\left(\frac{n+1}{n}\right)e^n} & x > 1 \end{cases}$$

Show that X_n converges in distribution to $Uniform(0,1)$.

Solution: Since X_n's are non-negative we have

$$F_{X_n}(x) = 0 \qquad \text{for } x < 0.$$

For $0 < x < 1$,

$$\begin{aligned} \lim_{n\to\infty} F_{X_n}(x) &= \lim_{n\to\infty}\left[\frac{e^{nx}+xe^n}{e^{nx}+\left(\frac{n+1}{n}\right)e^n}\right] \\ &= \lim_{n\to\infty}\frac{xe^n}{\left(\frac{n+1}{n}\right)e^n} \\ &= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)x \\ &= x \end{aligned}$$

For $x > 1$,

$$\begin{aligned} \lim_{F_{X_n}(x)\to\infty} &= \lim_{n\to\infty}\frac{e^{nx}}{e^{nx}} \\ &= 1 \end{aligned}$$

$$\lim_{n\to\infty} F_{X_n}(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 1 \\ x & 0 < x < 1 \end{cases}$$

$$X_n \xrightarrow{d} Uniform(0,1)$$

10. Consider a sequence $\{X_n, n = 1, 2, 3, \cdots\}$ such that

$$X_n = \begin{cases} n & \text{with probability } \frac{1}{n^2} \\ 0 & \text{with probability } 1 - \frac{1}{n^2} \end{cases}$$

Show that

(a) $X_n \xrightarrow{p} 0$.

(b) $X_n \xrightarrow{L^r} 0$, for $r < 2$.

(c) X_n does not converge to 0 in the rth mean for any $r \geq 2$.

(d) $X_n \xrightarrow{a.s.} 0$.

Solution:

(a)

$$P\left(|X_n| > \epsilon\right) = \frac{1}{n^2} \to 0 \quad \text{as} \quad n \to \infty$$
$$X_n \xrightarrow{p} 0$$

(b)

$$\begin{aligned} E|X_n|^r &= \frac{1}{n^2} \cdot n^r + \left(1 - \frac{1}{n^2}\right) \cdot 0 \\ &= n^{r-2} \to 0 \quad \text{for} \quad r < 2 \end{aligned}$$
$$X_n \xrightarrow{L^r} 0 \quad \text{for} \quad r < 2$$

(c)

$$E|X_n|^r = n^{r-2} \to \infty \quad \text{for} \quad r > 2$$

X_n does not converge to 0 in the rth mean for any $r \geq 2$.

(d)

$$\sum_{n=1}^{\infty} P(|X| > \epsilon) \to \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$
$$X_n \xrightarrow{a.s.} 0$$

11. We perform the following random experiment. We put $n \geq 10$ blue balls and n red balls in a bag. We pick 10 balls at random (without replacement) from the bag. Let X_n be the number of blue balls. We perform this experiment for $n = 10, 11, 12, \cdots$. Prove that $X_n \xrightarrow{d} Binomial\left(10, \frac{1}{2}\right)$.

    Solution:

$$P\left(X_n = k\right) = \frac{\begin{pmatrix} n \\ k \end{pmatrix} \cdot \begin{pmatrix} n \\ 10-k \end{pmatrix}}{\begin{pmatrix} 2n \\ 10 \end{pmatrix}} \quad \text{for} \quad k = 0, 1, 2, \cdots, 10$$

    Note that for any fixed k,

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n(n-1)\cdots(n-k+1)}{k!} \to \frac{n^k}{k!}$$

$$\begin{aligned} P(X_n = k) &\xrightarrow{as \quad n \to \infty} \frac{\frac{n^k}{k!}\frac{n^{10-k}}{(10-k)!}}{\frac{(2n)^{10}}{10!}} \\ &= \frac{10!}{k!(10-k)!}\left(\frac{1}{2}\right)^{10} \\ &= \begin{pmatrix} 10 \\ k \end{pmatrix}\left(\frac{1}{2}\right)^{10} \end{aligned}$$

    Thus,

$$\begin{cases} R_{X_n} = \{0, 1, 2, \cdots, 10\} \\ \\ \lim_{n\to\infty} P(X_n = k) = \begin{pmatrix} 10 \\ k \end{pmatrix} \left(\frac{1}{2}\right)^{10} \end{cases}$$

    Therefore,

$$X_n \xrightarrow{d} Binomial(10, \frac{1}{2})$$

12. Find two sequences of random variables $\{X_n, n = 1, 2, \cdots\}$ and $\{Y_n, n = 1, 2, \cdots\}$ such that

$$X_n \xrightarrow{d} X,$$
and
$$Y_n \xrightarrow{d} Y,$$

but $X_n + Y_n$ does not converge in distribution to $X + Y$.

Solution:

- Choose X_n i.i.d. Bernoulli($\frac{1}{2}$).
- Choose Y_n i.i.d. Bernoulli($\frac{1}{2}$).
- Let $X = -Y \sim Bernoulli(\frac{1}{2})$.

Then, $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{d} Y$.

$$Z_n = \begin{cases} 0 & \text{with probability } \frac{1}{4} \\ 1 & \text{with probability } \frac{1}{2} \\ 2 & \text{with probability } \frac{1}{4} \end{cases}$$

But $X + Y = 0$. Thus, $X_n + Y_n$ does not converge to $X + Y$ in distribution.

13. Let X_1, X_2, X_3, $\cdots$ be a sequence of continuous random variable such that

$$f_{X_n}(x) = \frac{n}{2} e^{-n|x|}.$$

Show that X_n converges in probability to 0.

Solution:

$$\begin{aligned} P(|X_n| > \epsilon) &= 2 \int_{\epsilon}^{\infty} f_{X_n}(x) dx \quad \text{(since} \quad f_{X_n}(-x) = f_{X_n}(x)) \\ &= 2 \int_{\epsilon}^{\infty} \frac{n}{2} e^{-nx} dx \\ &= \left[-e^{-nx} \right]_{\epsilon}^{\infty} \\ &= e^{-n\epsilon} \end{aligned}$$

$$\text{Thus,} \quad \lim_{n \to \infty} P(|X_n| > \epsilon) = 0$$

$$X_n \xrightarrow{p} 0$$

14. Let X_1, X_2, X_3, $\cdots$ be a sequence of continuous random variable such that

$$f_{X_n}(x) = \begin{cases} \frac{1}{nx^2} & x > \frac{1}{n} \\ \\ 0 & \text{otherwise} \end{cases}$$

Show that X_n converges in probability to 0.

Solution:

$$\begin{aligned}
P\left(|X_n| > \epsilon\right) &= P\left(X_n > \epsilon\right) \quad (\text{since} \quad X_n > 0) \\
&= \int_{\epsilon}^{\infty} \frac{1}{nx^2} dx \\
&= \left[\frac{-1}{nx}\right]_{\epsilon}^{\infty} \\
&= \frac{1}{n\epsilon} \\
\text{Thus} \quad P\left(|X_n| > \epsilon\right) &\to 0 \quad \text{as} \quad n \to \infty \\
X_n &\xrightarrow{p} 0
\end{aligned}$$

15. Let Y_1, Y_2, Y_3, $\cdots$ be a sequence of i.i.d random variables with mean $EY_i = \mu$ and finite variance $\text{Var}(Y_i) = \sigma^2$. Define the sequence $\{X_n, n = 2, 3, ...\}$ as

$$X_n = \frac{Y_1 Y_2 + Y_2 Y_3 + \cdots Y_{n-1} Y_n + Y_n Y_1}{n}, \qquad \text{for } n = 2, 3, \cdots .$$

Show that $X_n \xrightarrow{p} \mu^2$.

Solution:

$$\begin{aligned}
E[X_n] &= \frac{1}{n}\left[E\left[Y_1 Y_2\right] + E\left[Y_2 Y_3\right] + \cdots + E\left[Y_n Y_1\right]\right] \\
&= \frac{1}{n} \cdot n \cdot EY_1 \cdot EY_2 \\
&= (\mu)^2.
\end{aligned}$$

Also, for $n \geq 3$, we can write

$$\begin{aligned}
\text{Var}(X_n) &= \frac{1}{n^2}\left[n\text{Var}\left(Y_1 Y_2\right) + 2n\text{Cov}\left(Y_1 Y_2, Y_2 Y_3\right)\right] \\
\text{Var}\left(Y_1 Y_2\right) &= E\left[Y_1^2 Y_2^2\right] - \left(E[Y_1 Y_2]\right)^2 \\
&= E\left[Y_1\right]^2 E\left[Y_2\right]^2 - (\mu)^4 \\
&= \left(\sigma^2 + \mu^2\right)\left(\sigma^2 + \mu^2\right) - (\mu)^4 \\
&= \sigma^4 + 2(\mu^2)(\sigma^2) \\
\text{Cov}\left(Y_1 Y_2, Y_2 Y_3\right) &= E\left[Y_1\right] E\left[Y_3\right] E\left[Y_2^2\right] - E\left[Y_1\right] E\left[Y_2\right] E\left[Y_2\right] E\left[Y_3\right] \\
&= \mu^2\left(\mu^2 + \sigma^2\right) - (\mu^4) \\
&= \mu^2 \sigma^2 \\
\text{Therefore} \quad \text{Var}(X_n) &= \frac{1}{n^2}\left[n\sigma^4 + 2n\mu^2\sigma^2 + 2n\mu^2\sigma^2\right] \\
&= \frac{1}{n}\left(\sigma^4 + 2\mu^2\sigma^2 + 2\mu^2\sigma^2\right)
\end{aligned}$$

In particular $\quad \text{Var}(X_n) \to 0 \quad \text{as} \quad n \to \infty$

Now, using Chebyshev's Inequality, we can write

$$P(|X_n - EX_n| > \epsilon) < \frac{\mathrm{Var}(X_n)}{\epsilon^2} \to 0 \quad \text{as} \quad n \to \infty$$
$$P(|X_n - EX_n| > \epsilon) \to 0 \quad \text{as} \quad n \to \infty.$$

Thus,

$$X_n \xrightarrow{p} \mu^2.$$

16. Let $Y_1, Y_2, Y_3, \cdots$ be a sequence of positive i.i.d random variables with $0 < E[\ln Y_i] = \gamma < \infty$. Define the sequence $\{X_n, n = 1, 2, 3, ...\}$ as

$$X_n = (Y_1 Y_2 Y_3 \cdots Y_{n-1} Y_n)^{\frac{1}{n}}, \qquad \text{for } n = 1, 2, 3, \cdots.$$

Show that $X_n \xrightarrow{p} e^{\gamma}$.

Solution: Define:

$$V_n = \ln X_n$$
$$= \frac{1}{n} \sum_{k=1}^{n} \ln Y_i$$

Since $E[\ln Y_i] < \infty$, by the WLLN:

$$\frac{1}{n} \sum_{k=1}^{n} \ln Y_i \xrightarrow{p} E[\ln Y_i] = \gamma$$
$$\text{Thus} \quad V_n \xrightarrow{p} \gamma$$

Since e^x is a continuous function, we conclude (by the continuous mapping Theorem) that

$$e^{V_n} \xrightarrow{p} e^{\gamma}$$
$$\text{But,} \quad e^{V_n} = e^{\ln X_n} = X_n$$
$$\text{Thus,} \quad X_n \xrightarrow{p} e^{\gamma}$$

17. Let $X_1, X_2, X_3, \cdots$ be a sequence of random variable such that

$$X_n \sim Poisson(n\lambda), \qquad \text{for } n = 1, 2, 3, \cdots,$$

where $\lambda > 0$ is a constant. Define a new sequence Y_n as

$$Y_n = \frac{1}{n} X_n, \qquad \text{for } n = 1, 2, 3, \cdots.$$

Show that Y_n converges in mean square to λ, i.e., $Y_n \xrightarrow{m.s.} \lambda$.

Solution:

$$
\begin{aligned}
EY_n &= \frac{1}{n}EX_n = \frac{1}{n} \cdot n\lambda = \lambda \\
E[|Y_n - \lambda|^2] &= E\left[\left|\frac{1}{n}X_n - \lambda\right|^2\right] \\
&= \frac{1}{n^2}E[(X_n - n\lambda)^2] \\
&= \frac{1}{n^2}\mathrm{Var}(X_n) \\
&= \frac{1}{n^2} \cdot n\lambda = \frac{\lambda}{n} \to 0 \quad \text{as} \quad n \to \infty \\
&\text{Thus,} \quad Y_n \xrightarrow{m.s.} \lambda
\end{aligned}
$$

18. Let $\{X_n, n = 1, 2, \cdots\}$ and $\{Y_n, n = 1, 2, \cdots\}$ be two sequences of random variables, defined on the sample space S. Suppose that we know

$$
\begin{aligned}
X_n &\xrightarrow{L^r} X, \\
Y_n &\xrightarrow{L^r} Y.
\end{aligned}
$$

Prove that $X_n + Y_n \xrightarrow{L^r} X + Y$. *Hint:* You may want to use the Minkowski's inequality which states that for two random variables X and Y with finite moments, and $1 \leq p < \infty$, we have

$$
E\left[\left|X + Y\right|^p\right] \leq E\left[|X|^p\right]^{\frac{1}{p}} + E\left[|Y|^p\right]^{\frac{1}{p}}.
$$

Solution:

$$
\begin{aligned}
X_n &\xrightarrow{L^r} X \quad \text{Thus} \quad E\left|X_n - X\right|^r \to 0 \quad \text{as} \quad n \to \infty \\
Y_n &\xrightarrow{L^r} Y \quad \text{Thus} \quad E\left|Y_n - Y\right|^r \to 0 \quad \text{as} \quad n \to \infty \\
E\left[|X_n + Y_n - (X+Y)|^r\right] &= E\left[|X_n - X + Y_n - Y|^r\right] \\
&\leq \left(E\left[|X_n - X|^r\right]\right)^{\frac{1}{r}} + \left(E\left[|Y_n - Y|^r\right]\right)^{\frac{1}{r}} \to 0 + 0 \quad \text{as} \quad n \to \infty \\
X_n + Y_n &\xrightarrow{L^r} X + Y
\end{aligned}
$$

19. Let X_1, X_2, X_3, $\cdots$ be a sequence of random variable such that $X_n \sim Rayleigh(\frac{1}{n})$, i.e.,

$$
f_{X_n}(x) = \begin{cases} n^2 x e^{-\frac{n^2x^2}{2}} & x > 0 \\ 0 & \text{otherwise} \end{cases}
$$

Show that $X_n \xrightarrow{a.s.} 0$.

Solution: Note that:

$$\begin{aligned}
F_{X_n}(x) &= \int_0^x f_n(\alpha) d\alpha \\
&= 1 - e^{-\frac{n^2x^2}{2}} \\
\text{that} \quad P\left(|X_n| > \epsilon\right) &= P\left(X_n > \epsilon\right) \\
&= 1 - P\left(X_n < \epsilon\right) \\
&= e^{-\frac{n^2\epsilon^2}{2}} \\
\text{thus,} \quad \sum_{n=1}^{\infty} P\left(|X_n| > \epsilon\right) &= \sum_{n=1}^{\infty} e^{-\frac{n^2\epsilon^2}{2}} \\
&\le \sum_{n=1}^{\infty} e^{-\frac{n\epsilon^2}{2}} \\
&= \frac{e^{-\frac{\epsilon^2}{2}}}{1 - e^{-\frac{\epsilon^2}{2}}} < \infty \\
X_n &\xrightarrow{a.s.} 0
\end{aligned}$$

20. Let Y_1,Y_2, $\cdots$ be independent random variables, where $Y_n \sim Bernoulli\left(\frac{n}{n+1}\right)$ for $n = 1, 2, 3, \cdots$. We define the sequence $\{X_n, n = 2, 3, 4, \cdots\}$ as

$$X_{n+1} = Y_1 Y_2 Y_3 \cdots Y_n, \qquad \text{for } n = 1, 2, 3, \cdots.$$

Show that $X_n \xrightarrow{a.s.} 0$.

Solution: Let

$$\begin{aligned}
A_m &= \{|X_n - 0| < \epsilon, \quad \text{for all} \quad n \ge m\} \\
&= \{|X_n| < \epsilon, \quad \text{for all} \quad n \ge m\} \\
\{X_n = 0, \quad \text{for all} \quad n \ge m\} &= \{Y_n = 0 \quad \text{for some} \quad n < m\} \\
&\quad (\text{Since} \quad X_n\text{'s are Bernoulli random variables,} \quad \epsilon < 1) \\
P\left(A_m\right) &= 1 - P\left(\{Y_n = 1 \quad \text{for all} \quad n < m\}\right) \\
&= 1 - \left(\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{m-1}{m}\right) \\
&= 1 - \frac{1}{m} \\
\text{so,} \quad \lim_{m \to \infty} P\left(A_m\right) &= 1
\end{aligned}$$

Therefore, $\quad X_n \xrightarrow{a.s.} 0$.