Randomized Algorithms

Ioannis Caragiannis (this time) and Kasper Green Larsen

Practical issues

- When: Tuesdays, 8-11am
- Where: 5510-104 Lille Auditorium
- 3 projects in groups of three
- Oral exam

Grading rules:

- To pass the course, you need to pass the projects and the oral exam
- The projects can affect your grade in the oral exam by one point (up or down)

This lecture

- Quick recap: randomization, useful inequalities
- Computing a (global) cut of minimum size in a graph
- A randomized algorithm
- Analysis
- Implications and improvements

Randomization

The elephant in the room

- Randomized algorithms use **random coins**, **dice**, **card shuffling**, etc



- For example, the code of a randomized algorithm implementation will typically have a line like this:
- `if (coin_toss() == HEADS) {…}`

Usual assumptions

Basic operation:

- Access to **fair coins** (Pr[HEADS]=Pr[TAILS]=1/2)

More complicated operations:

- **Random selection** among a finite set of items
- Access to a **random permutation** of elements
- Selection of a **random point** in the interval [0,1]
- Selection from a finite or infinite set according to a **non-uniform probability distribution**

Note: there are important **implementation issues** that we most of the time ignore

Main characteristics

Deterministic algorithms: performs the very **same steps** in **any execution** on the **same input**

Randomized algorithms do not!

- They may produce **different outputs** in different executions
- Their **running time** may not always be the same
- In other words, their output, their running time, the amount of space they use are **random variables**

With the **analysis of randomized algorithms**, our aim is to understand these random variables

Randomized algorithms: Why do we want them?

- Sometimes randomization is absolutely **necessary**
- They are usually **simple**
- Work **well on average** or **with high probability**
- Sometimes, the give insights to the design of better deterministc algorithms (**derandomization**)

Analysis tools

Useful (in)equalities

- Linearity of expectation
- Markov’s inequality
- Chernoff bounds (sharp concentration bounds)
- Union bound

Today: use of a very simple fact

- An algorithm is successful on an input with probability at least t
- After (at least) $\frac{\ln n}{t}$ executions, the algorithm will be successful at least once with probability at least $1 - 1/n$
- $\Pr[\text{success}] = 1 - (1-t)^{\frac{\ln n}{t}} \geq 1 - (e^{-t})^{\frac{\ln n}{t}} = 1 - 1/n$

$$1 - t \leq e^{-t}$$

The minimum s-t cut problem in graphs

Cut problem in graphs

- **Minimum (global) cut**: Given a graph, compute a set of edges of minimum cardinality whose removal disconnects the graph
- **Minimum s-t cut**: Given a graph and two nodes s and t, compute a set of edges whose removal disconnect node s from node t

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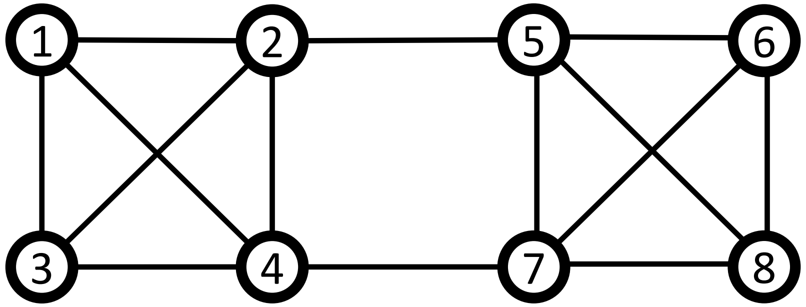

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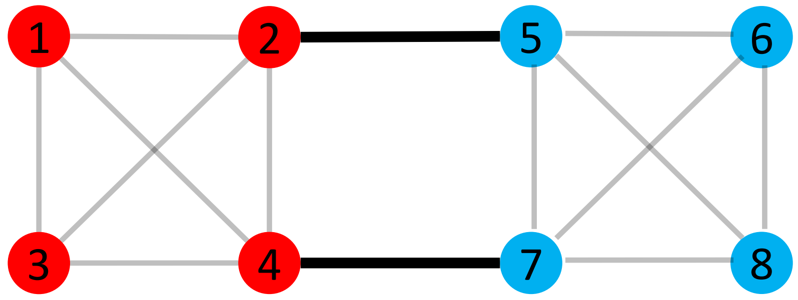

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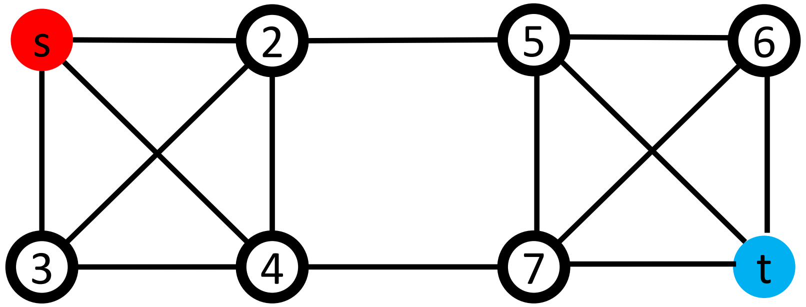

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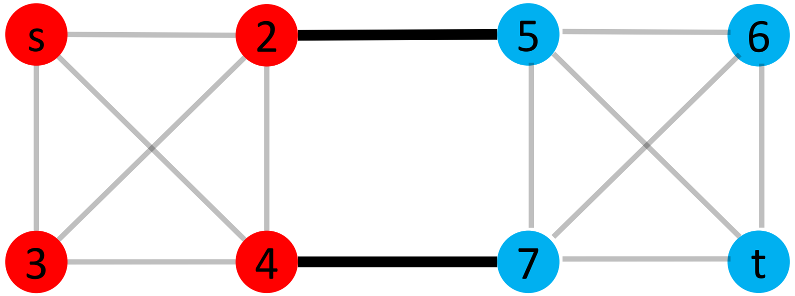

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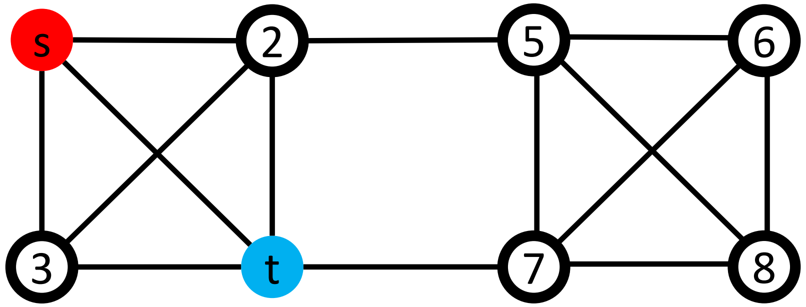

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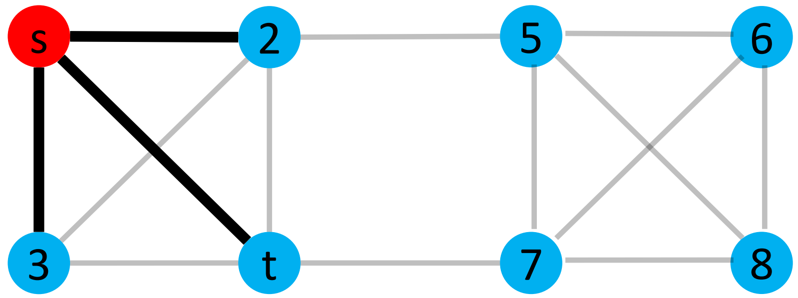

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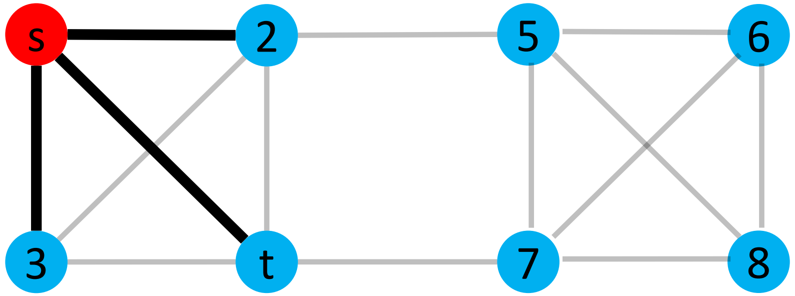
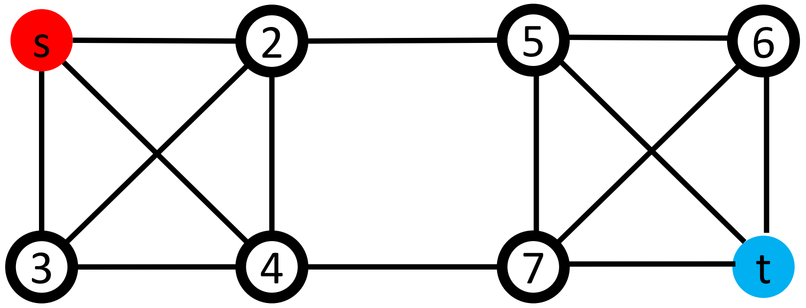


- The input graph is undirected, unweighted, and may contain multiple parallel edges

Algorithms

- Size of **minimum s-t cut** = value of **maximum s-t flow**
- Algorithms for computing maximum flows from node s to node t
- **Ford-Fulkerson**: compute augmenting paths in a **residual network**
- Several implementations: e.g., **Edmonds-Karp** algorithm has runnning time $O(n|E|^2)$, there are several **improvements**



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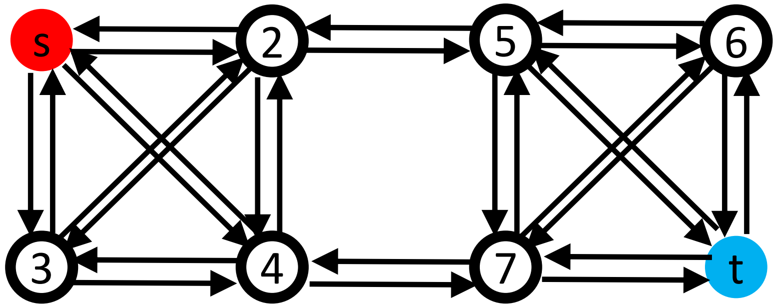

residual network

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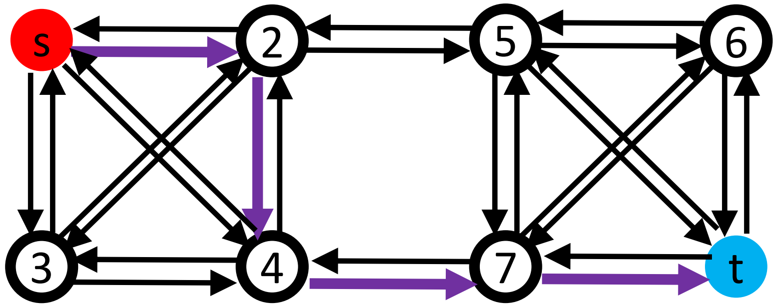

compute augmenting paths

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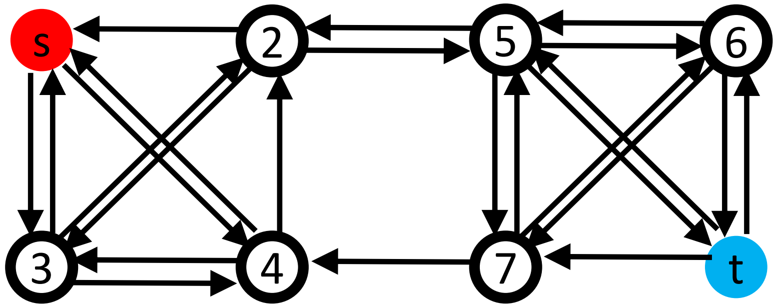

remove path from network

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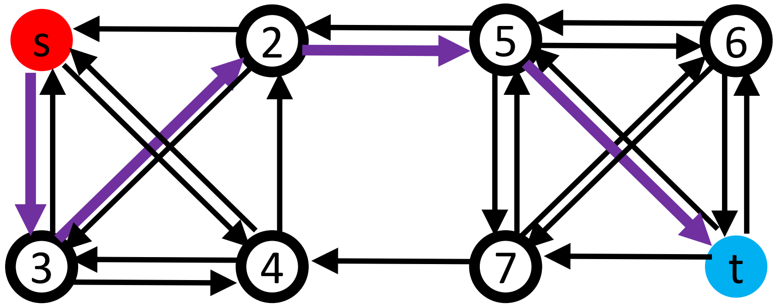

and repeat …

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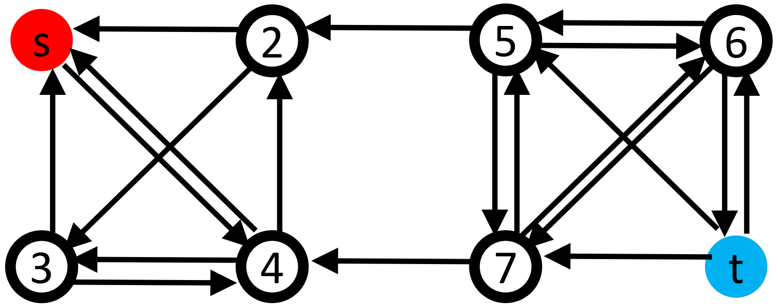

until it is not possible anymore

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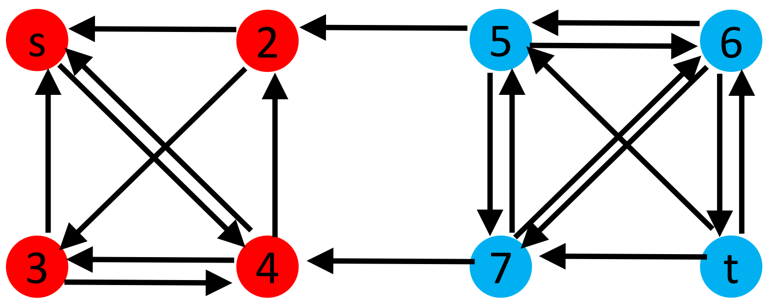

then, BFS computation identifies the two sides of the cut

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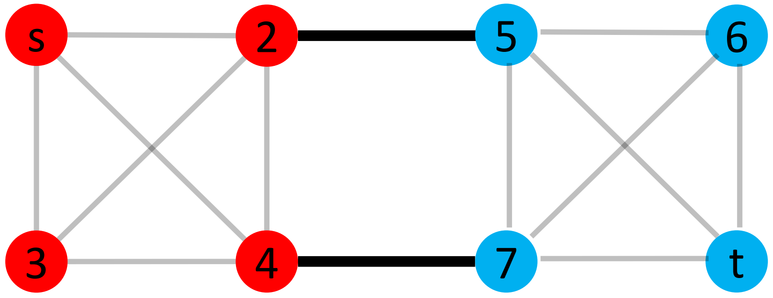

minimum s-t cut

Computing a (global) cut of minimum size

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- Fix a node s
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- For every selection of t, run the **Edmonds-Karp algorithm**
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running time: $O(n|E|^2)$

$n - 1$ executions

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- Overall running time: $O(n^2|E|^2)$

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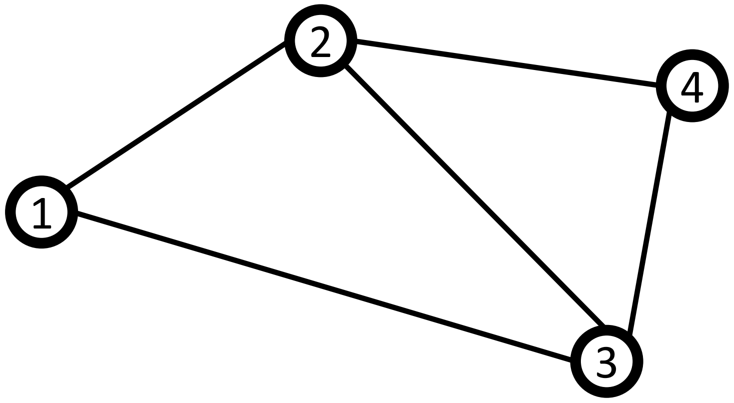
A randomized algorithm

Contractions

- Input: a (multi)graph $G = (V, E)$
- A **contraction** of nodes u and v results in a **new graph** G'
- The nodes u and v are **merged** into a new node uv in G'
- The edges between u and v **disappear**
- Edges (u, w) and (v, w) in G become edges (uv, w) in G'



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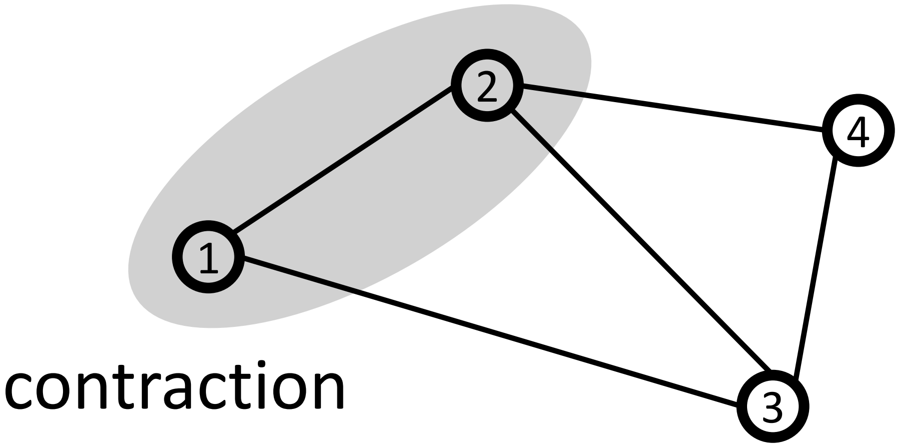

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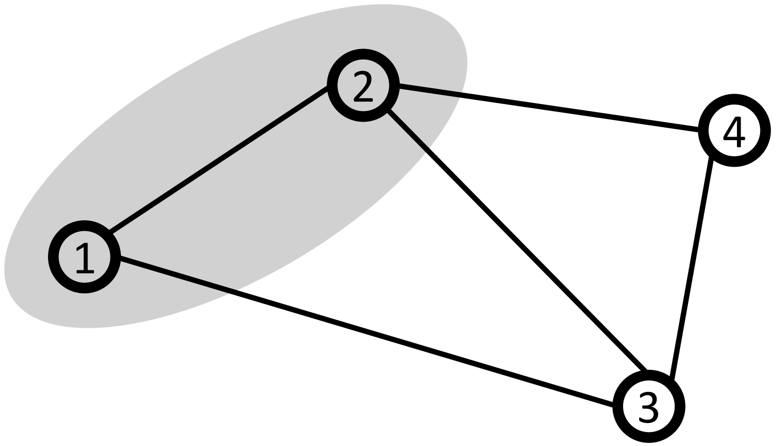

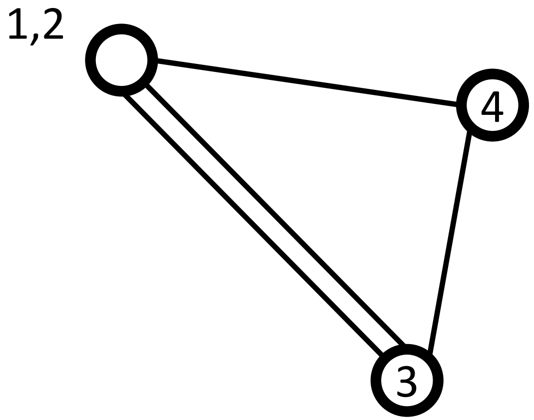

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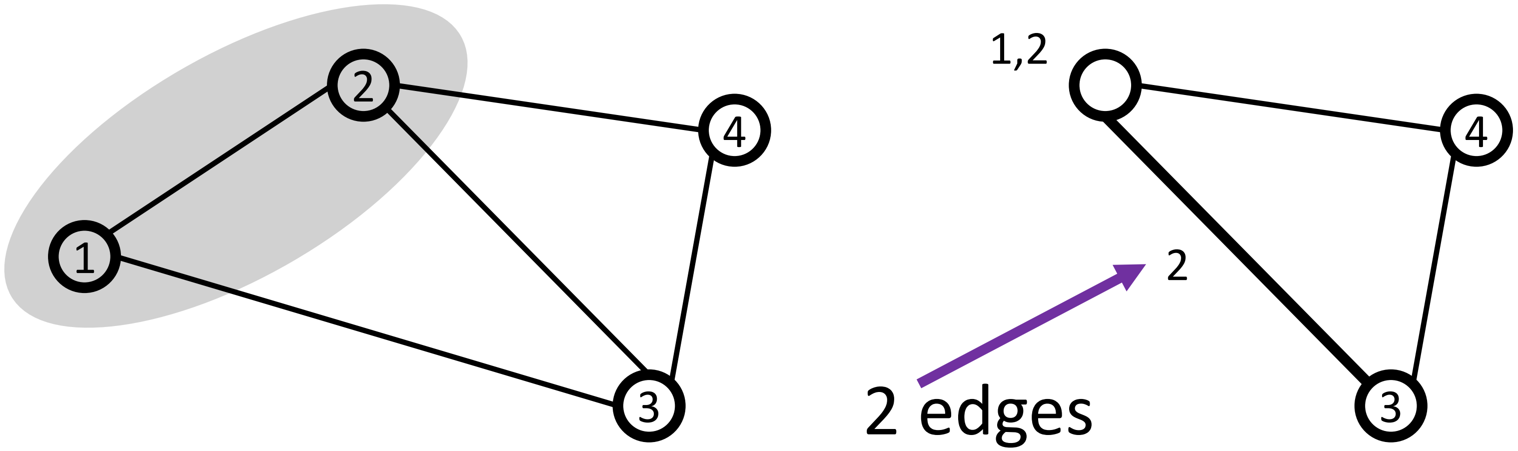

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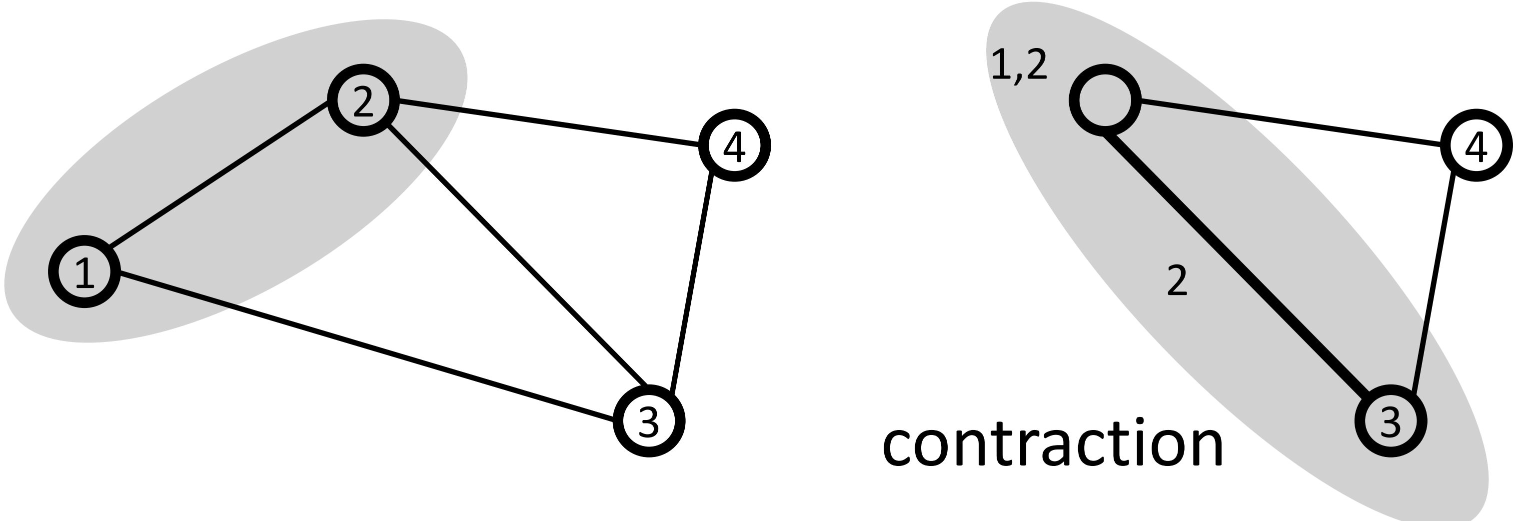

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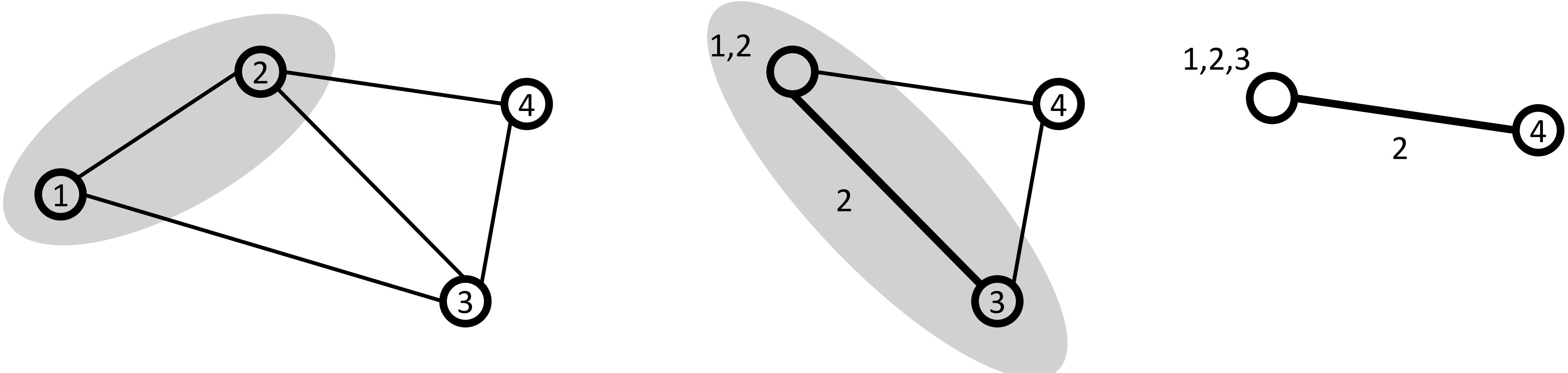
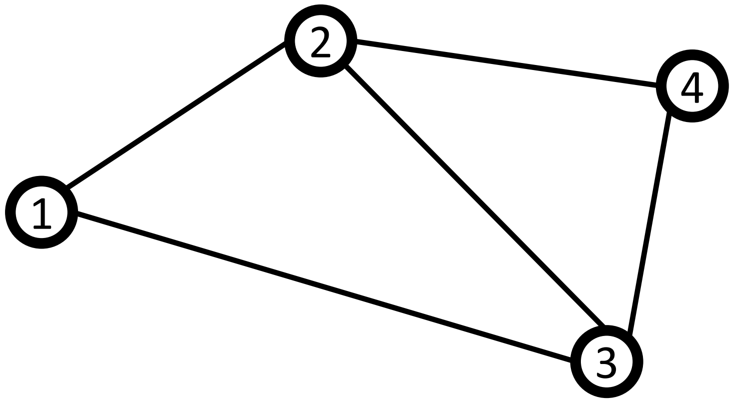
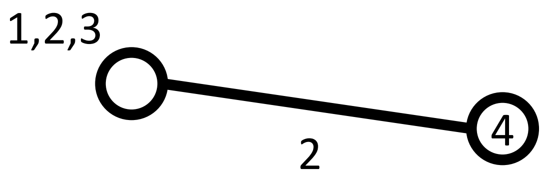


Contractions

- What kind of **information** do we see in the contracted graph?





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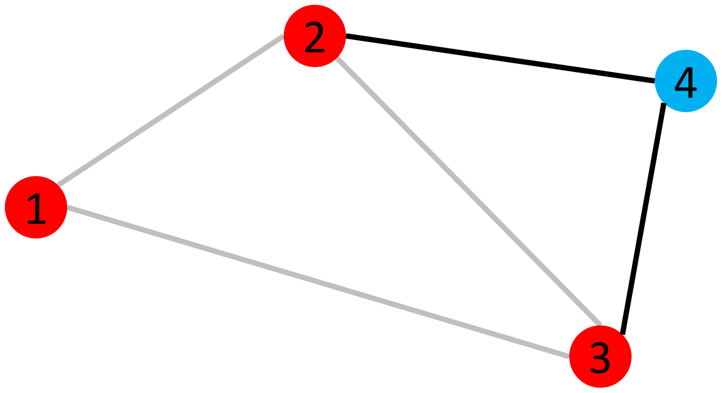
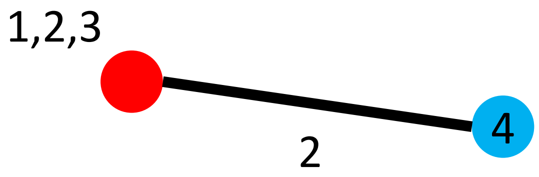
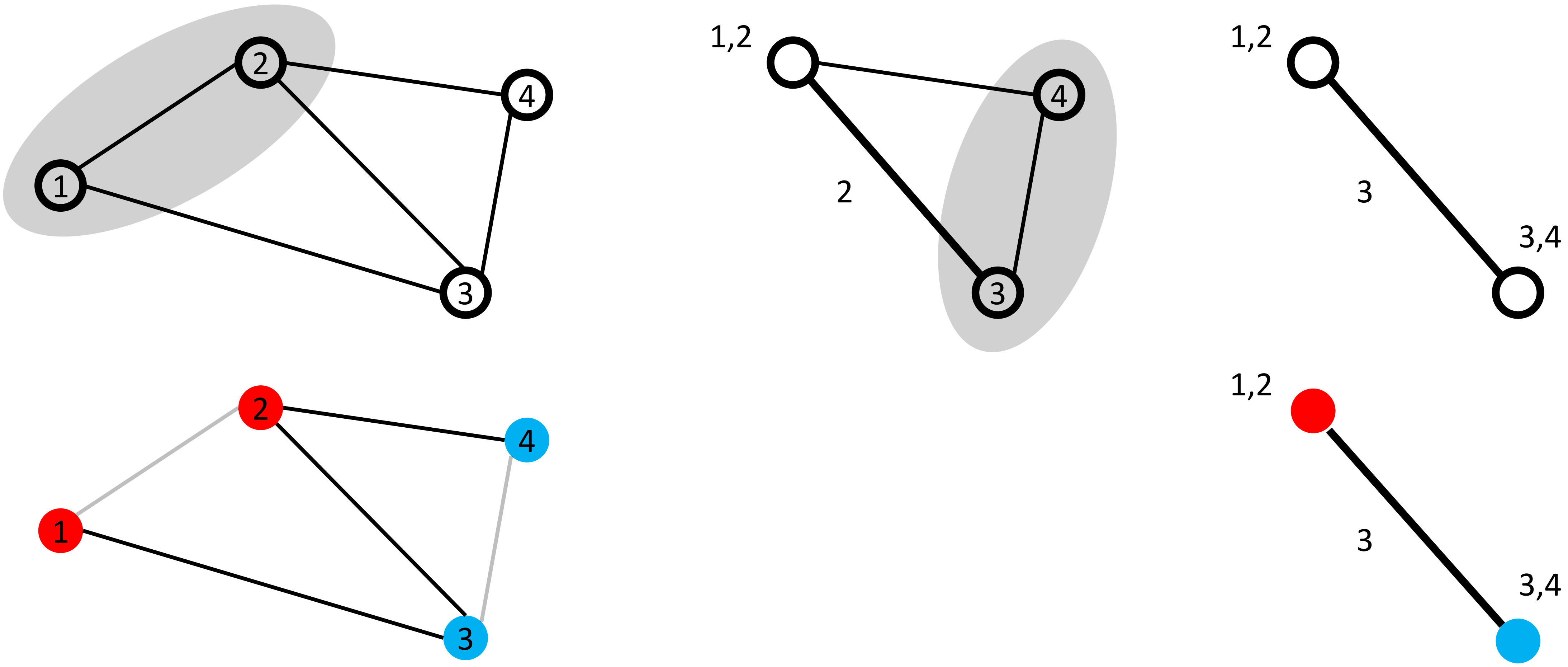




- Number of edges between the sets of nodes {1,2,3} and {4}

Another sequence of contractions



The contraction algorithm

Repeat until just two nodes remain

- Pick an edge uniformly at random and contract its endpoints

Return the set of edges between the two nodes

The contraction algorithm: an example

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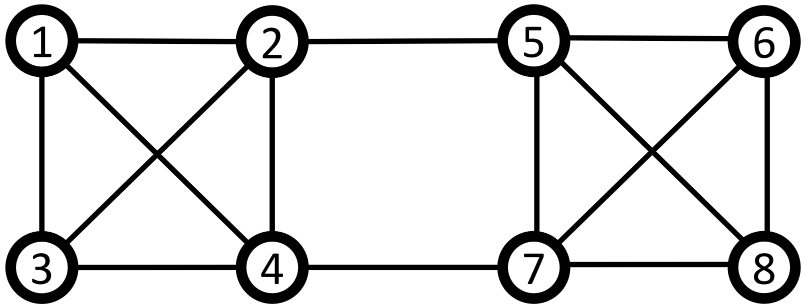

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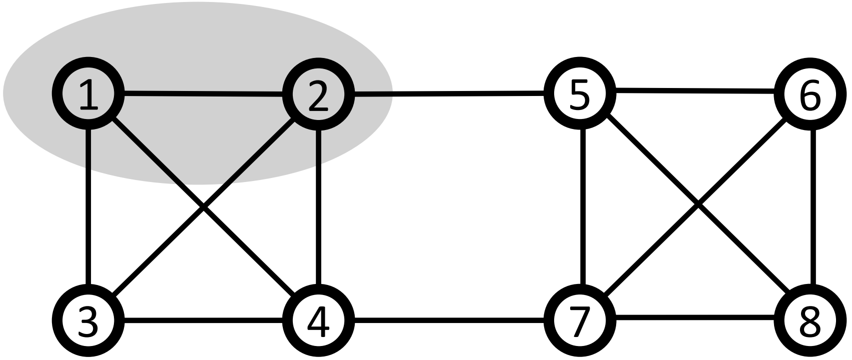

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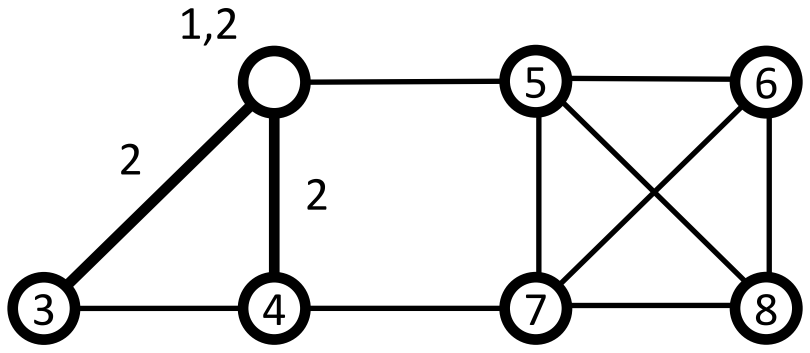

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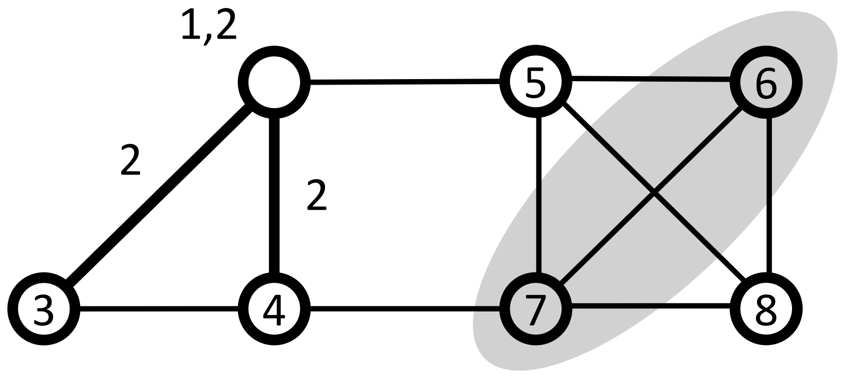

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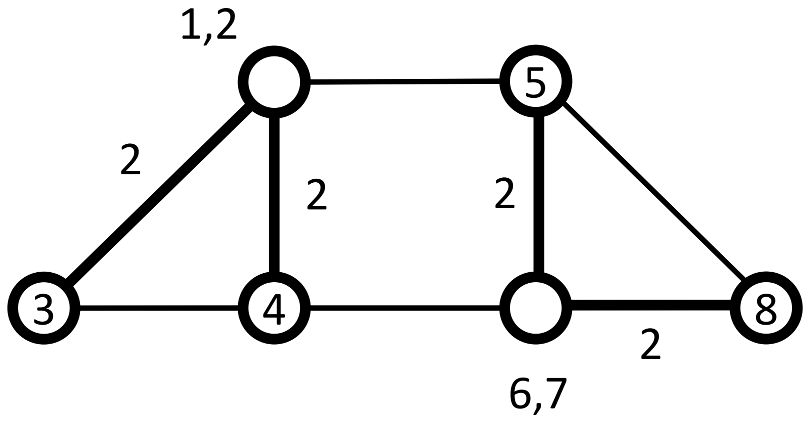

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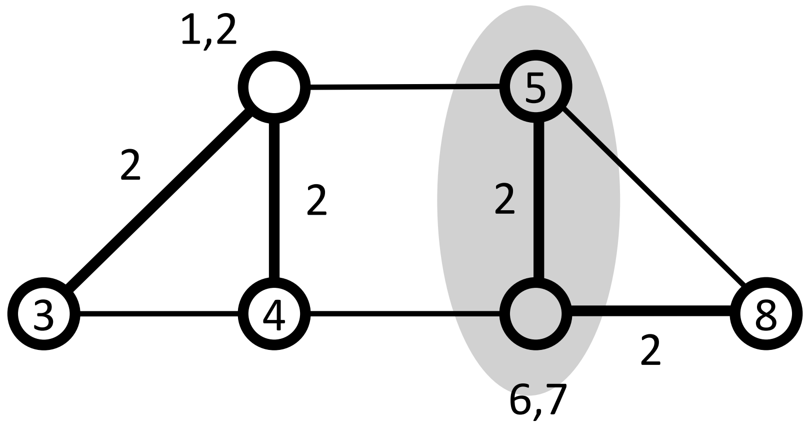

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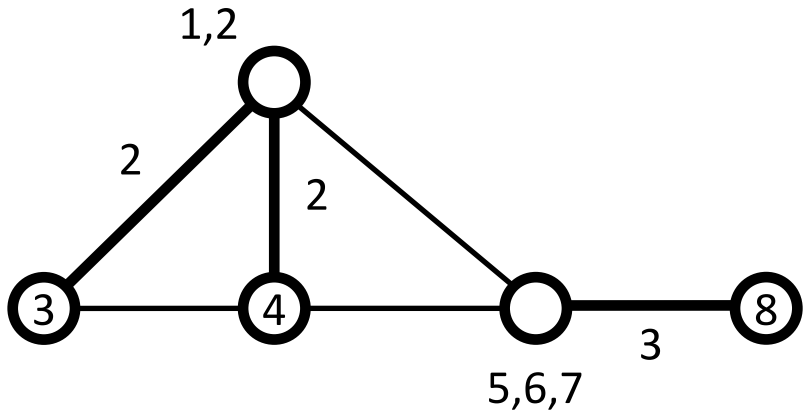

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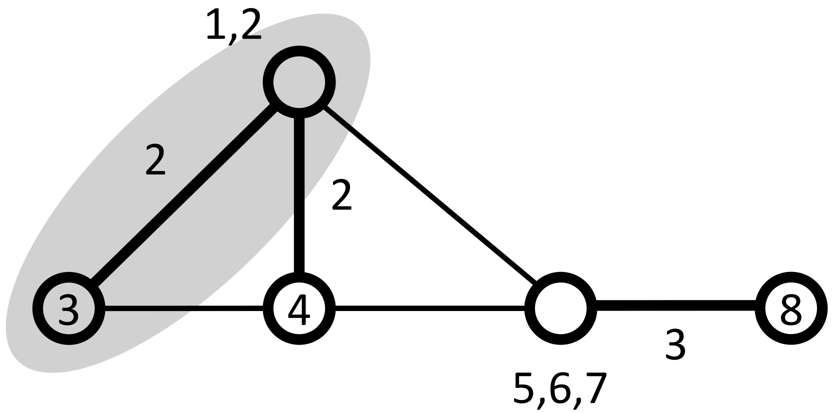

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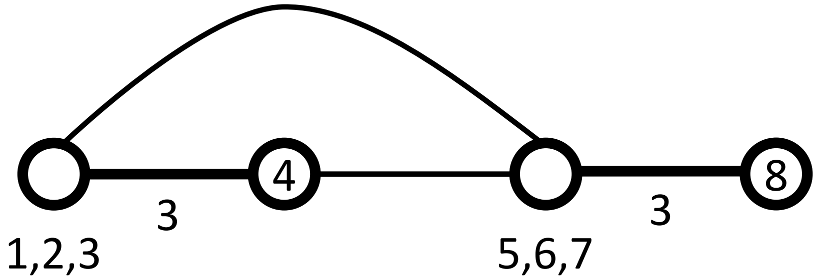

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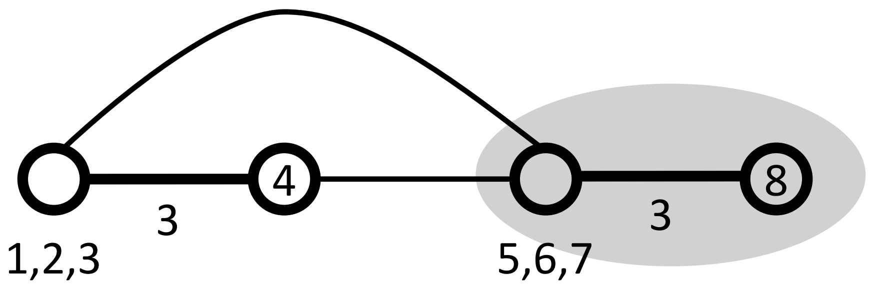

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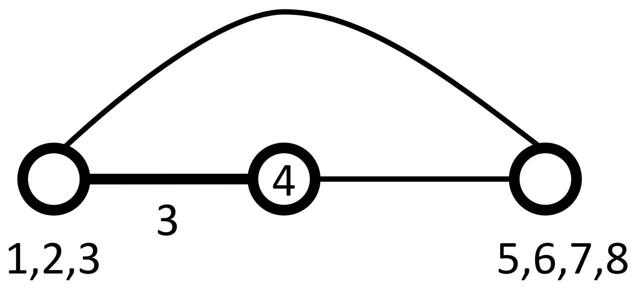

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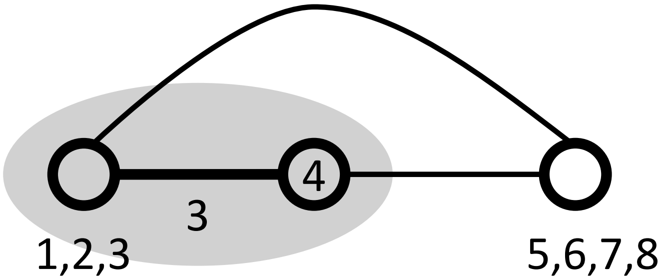

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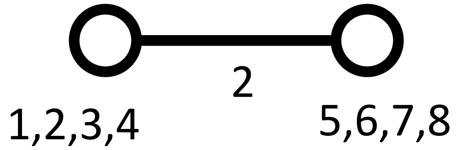

the minimum cut was found ☺

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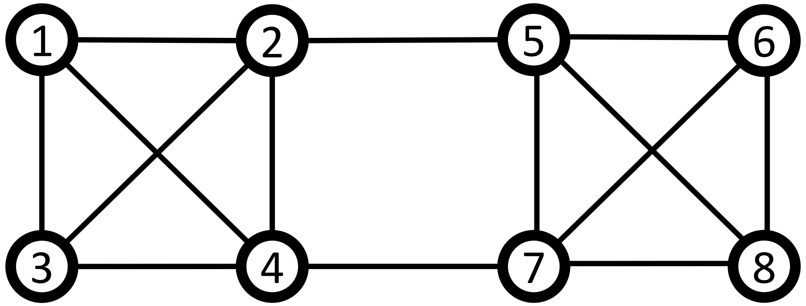

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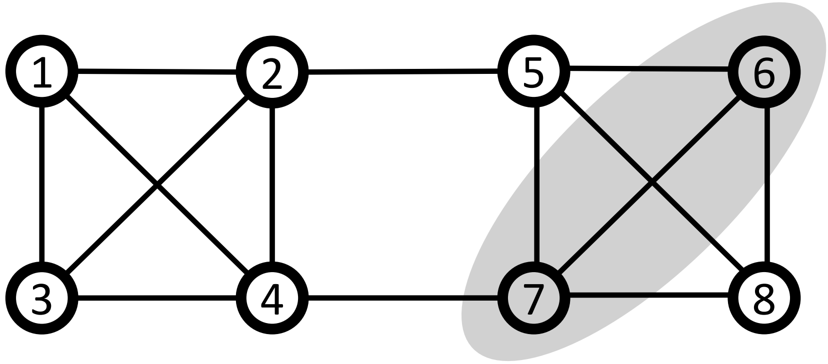

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Repeat until just two nodes remain

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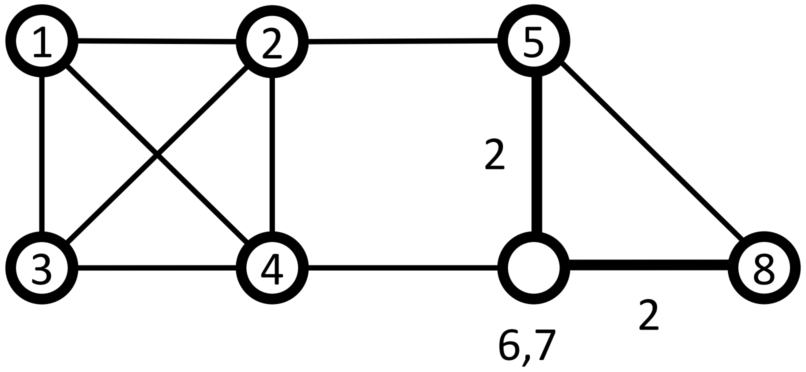

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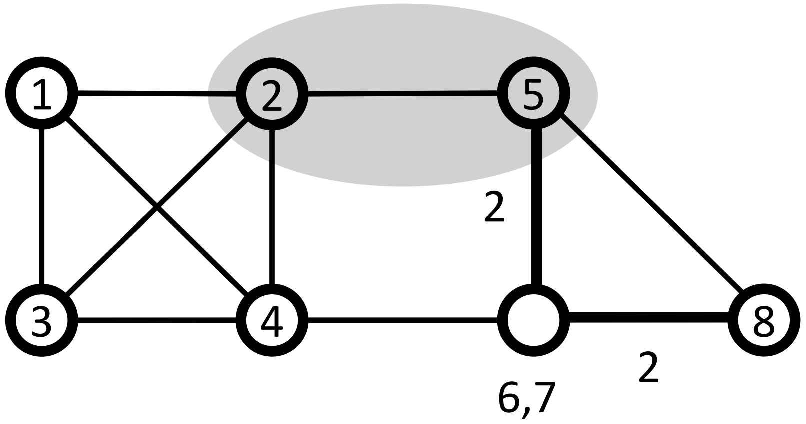

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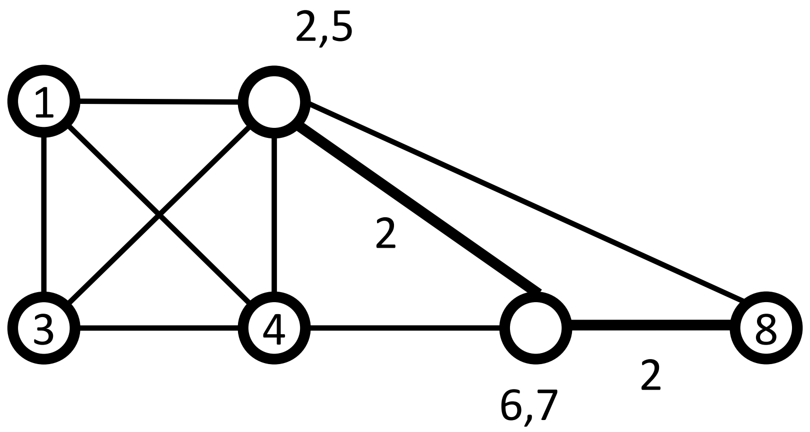

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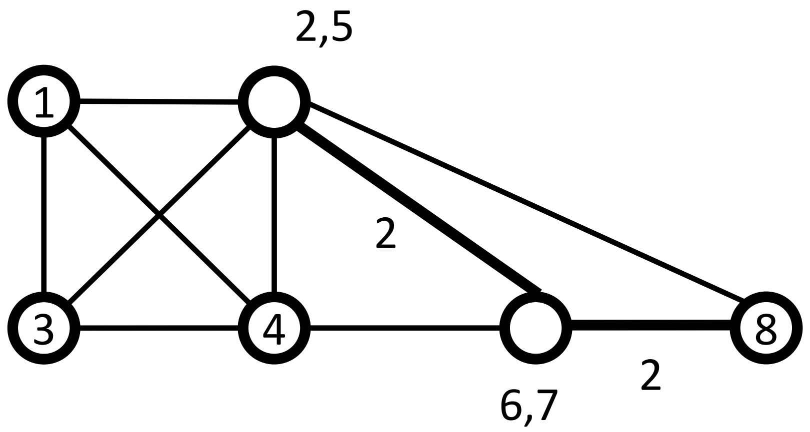

no chance to find the minimum cut ☹

Analysis of the contraction algorithm

The algorithm has running time $O(n^2)$

- Time $O(\log n)$ to select uniformly at random per step (not very important)
- Linear time to update the node/edge information after a contraction
- $n - 2$ contractions in total

Analysis of the contractio

Much better than the
Edmonds-Karp algorithm

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Analysis of the contraction algorithm (contd.)

What is the prob. that the algorithm is successful?

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Fix a cut C of minimum size

All edges in C will have survived at the end if none of them is selected for the $n-2$ contractions

Analysis of the contraction algorithm (contd.)

What is the prob. that the algorithm is successful?

Fix a cut C of minimum size

All edges in C will have survived at the end if none of them is selected for the $n-2$ contractions

Event $E_i =$ some edge of C is selected for the i-th contraction

Event $S_i =$ all edges of C have survived after the i-th contraction

$$\Pr[S_{n-2}] = \Pr[\overline{E_1}] \cdot \Pr[\overline{E_2} | \overline{E_1}] \cdot \Pr[\overline{E_3} | \overline{E_1} \cap \overline{E_2}] \cdots \Pr[\overline{E_{n-2}} | \overline{E_1} \cap \overline{E_2} \ldots \cap \overline{E_{n-3}}]$$

Analysis of the contraction algorithm (contd.)

Bounding the probability $\Pr[\overline{E_i} | \overline{E_1} \cap \overline{E_2} \ldots \cap \overline{E_{i-1}}]$

At step i:

- The edges of C still appear in the graph
- The graph has $n - i + 1$ nodes

No node has degree less than $|C|$; otherwise, C would not be of minimum size

Hence, the number of edges is at least $(n - i + 1)|C|/2$; recall the property $\sum_{v \in V} \deg v = 2|E|$

Then, the probability $\Pr[E_i | \overline{E_1} \cap \overline{E_2} \ldots \cap \overline{E_{i-1}}]$ is at most $\frac{|C|}{\#\text{edges}} \leq \frac{2}{n-i+1}$

Analysis of the contraction algorithm (contd.)

Recap:

The probability of success of the contraction algorithm is

$\Pr[S_{n-2}] = \Pr[\overline{E_1}] \cdot \Pr[\overline{E_2}|\overline{E_1}] \cdot \Pr[\overline{E_3}|\overline{E_1} \cap \overline{E_2}] \cdots \Pr[\overline{E_{n-2}}|\overline{E_1} \cap \overline{E_2} \ldots \cap \overline{E_{n-3}}]$

where $\Pr[\overline{E_i}|\overline{E_1} \cap \overline{E_2} \ldots \cap \overline{E_{i-1}}] \geq 1 - \frac{2}{n-i+1}$

Analysis of the contraction algorithm (contd.)

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Hence,

$\Pr[S_{n-2}] \geq \left(1 - \frac{2}{n}\right) \cdot \left(1 - \frac{2}{n-1}\right) \cdot \left(1 - \frac{2}{n-2}\right) \cdots \left(1 - \frac{2}{5}\right) \cdot \left(1 - \frac{2}{4}\right) \cdot \left(1 - \frac{2}{3}\right)$

$= \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \ldots \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3}$

Analysis of the contraction algorithm (contd.)

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$$= \frac{\cancel{n-2}}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{\cancel{n-2}} \cdots \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3}$$

Analysis of the contraction algorithm (contd.)

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Analysis of the contraction algorithm (contd.)

- The algorithm has running time $O(n^2)$ and is successful with prob. at least $2/n(n-1)$
- Run the algorithm $O(n^2 \log n)$ times and return the best cut
- Then, the algorithm computes the minimum cut with prob. at least $1 - 1/n$ in time $O(n^4 \log n)$

Analysis of the contraction algorithm (contd.)

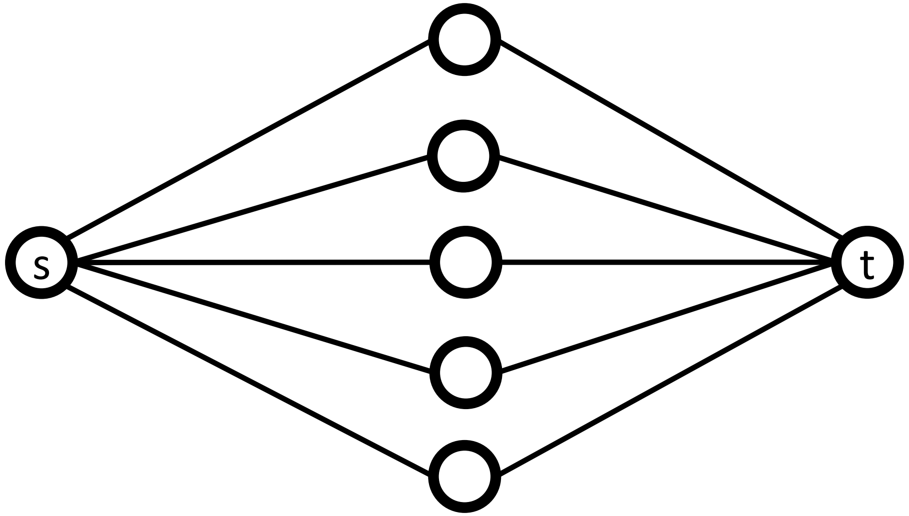
- The algorithm has running time … h, at least $2/n(n-1)$
- Run the algorithm $O(\,$…
- Then, the algorithm … at least $1 - 1/n$ in time $O(n^4 \log n)$

Still better than the Edmonds-Karp algorithm for not very sparse graphs

Counting the number of minimum cuts in graphs

How many cuts are there in a graph?

- Too many minimum s-t cuts



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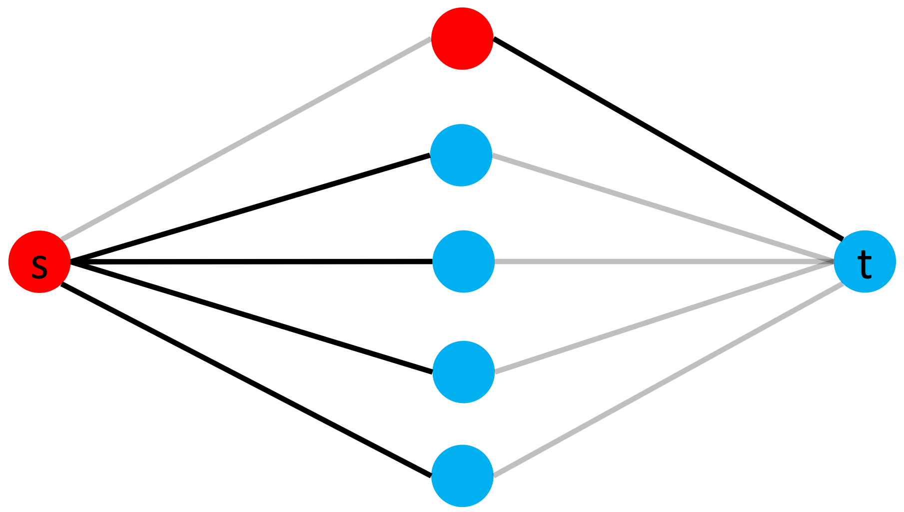

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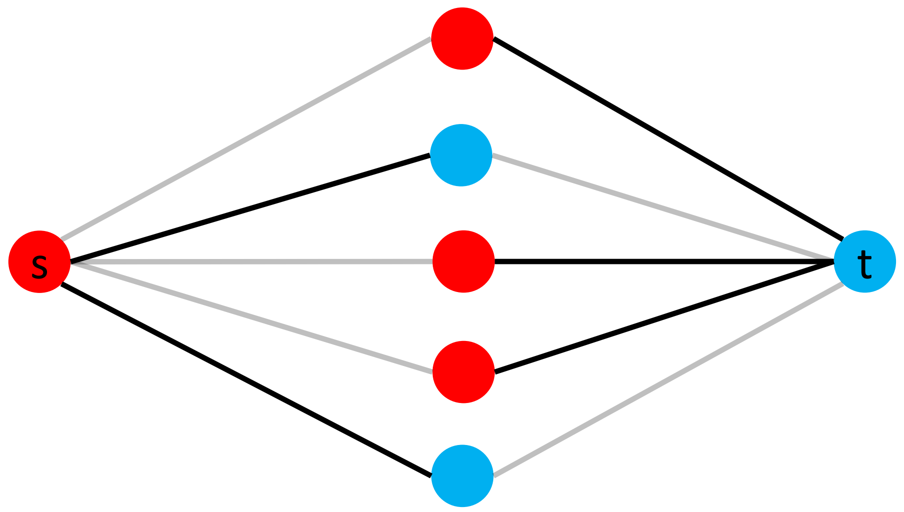

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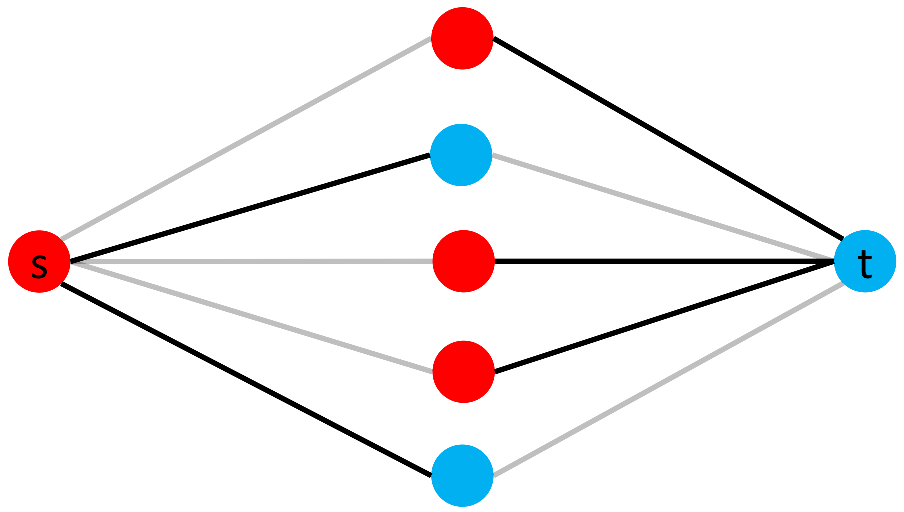

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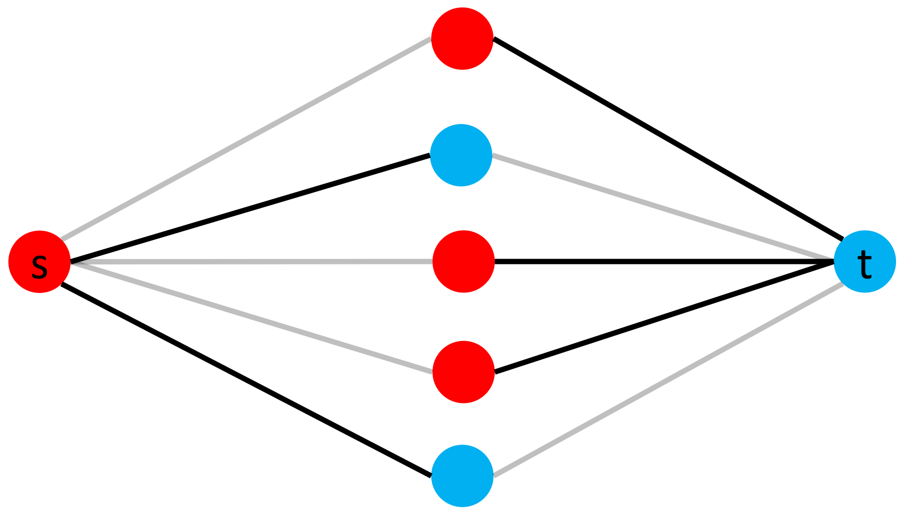

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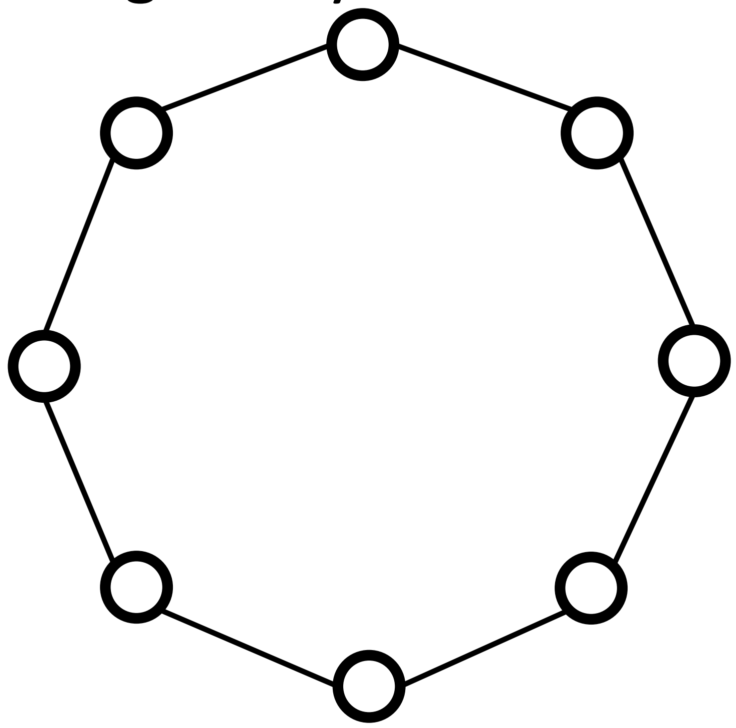

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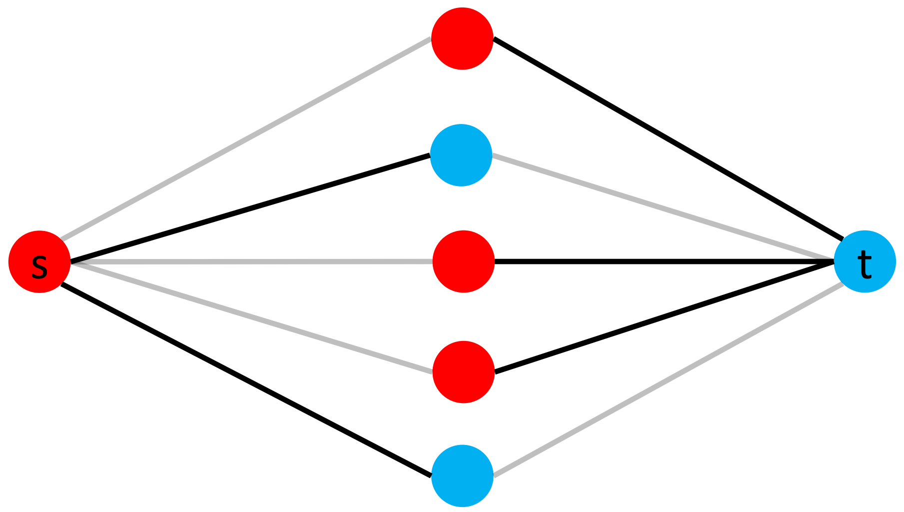

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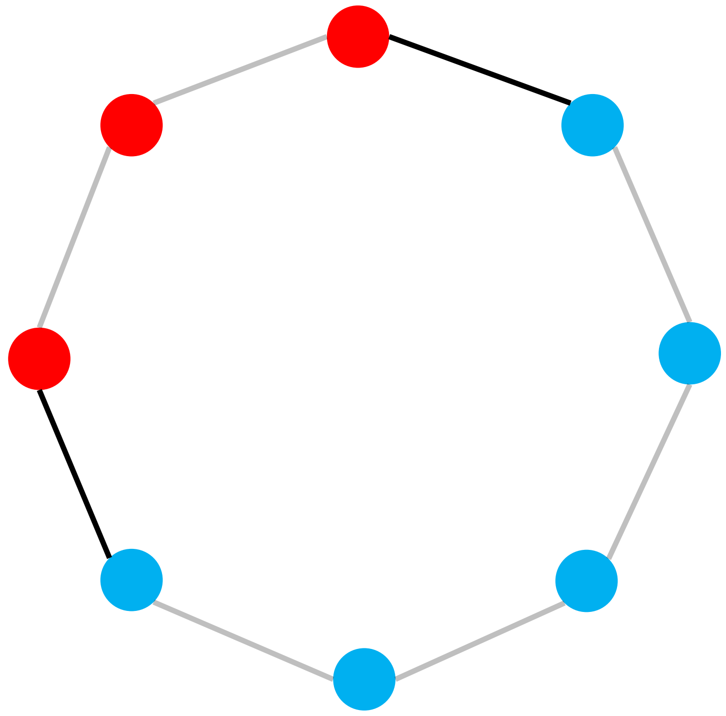

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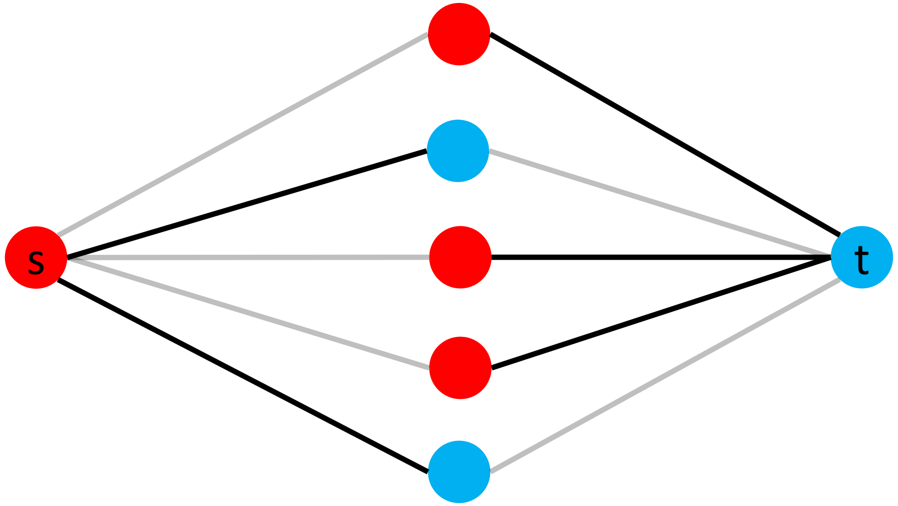

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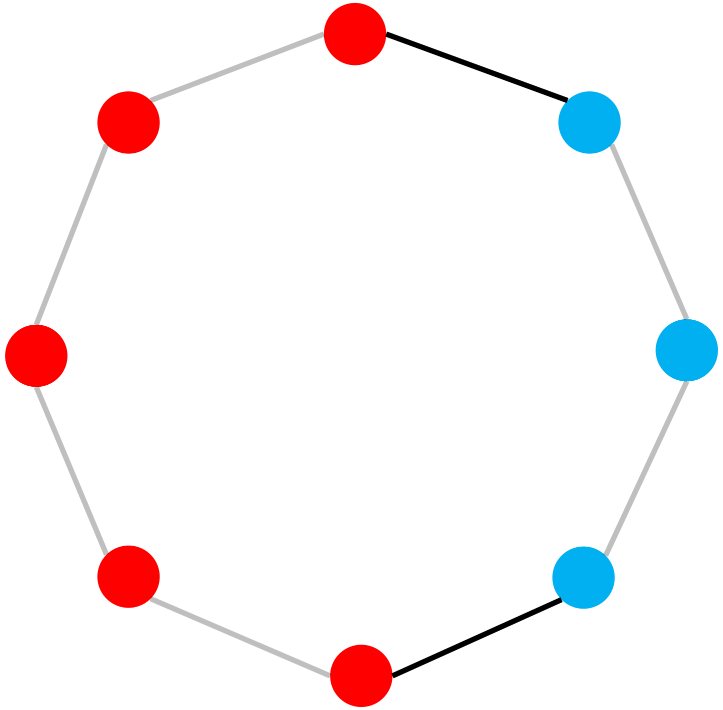

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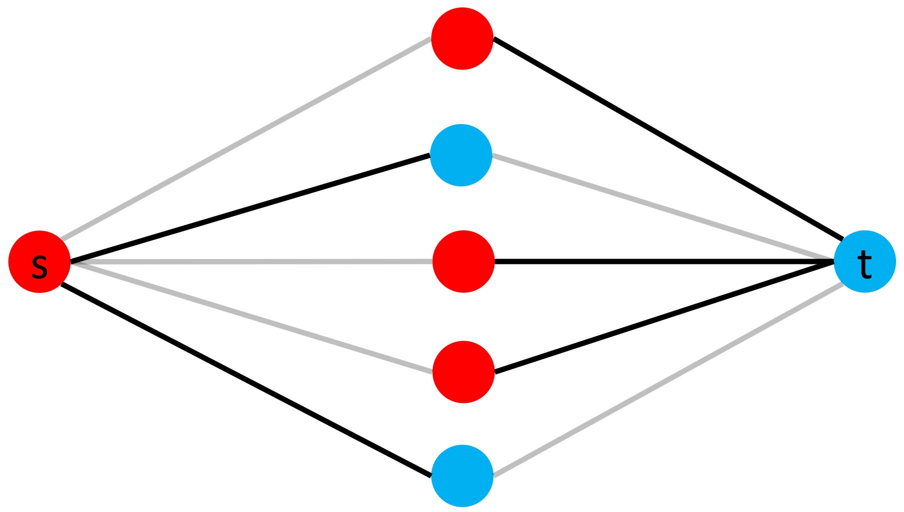

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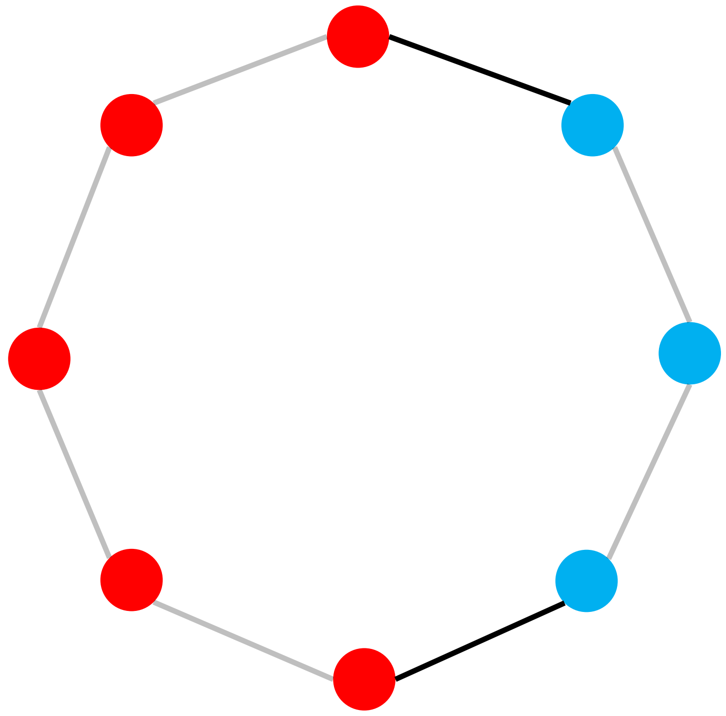

- $\frac{n(n-1)}{2}$ here. How worse can it be?

A graph-theoretic implication

Theorem: Any graph has at most $n(n-1)/2$ distinct global cuts of minimum size

A graph-theoretic implication

<u>Theorem</u>: Any graph has at most $n(n-1)/2$ distinct global cuts of minimum size

- Can we use our result for the contraction algorithm?

A graph-theoretic implication

Theorem: Any graph has at most $n(n-1)/2$ distinct global cuts of minimum size

- Can we use our result for the contraction algorithm?
- Recall that **we fixed a particular minimum cut** and proved that the prob. that it is returned by the contraction algorithm (in one execution) is at least $2/n(n-1)$
- Denote by C_i the event that the i-th minimum cut is returned in an execution by the contraction algorithm
- Hence, $\mathbf{Pr[C_i] \geq 2/n(n-1)}$
- **How many i's** can we have?

A graph-theoretic implication

Theorem: Any graph has at most $n(n-1)/2$ distinct global cuts of minimum size

A trivial argument:

- We have a **random process** that **selects a card** from a subset of the deck
- The selection can be arbitrary but we know that any spade is selected with probability at least t
- Then, the number of **spades** cannot exceed $1/t$

Proof idea: apply the argument with the **contraction algorithm** as the random process, the **cards as outcomes** of the contraction algorithm, and the **minimum cuts** as the spades

Improving the contraction algorithm

A crucial observation

- The probability that the minimum cut C has survived the first contractions is very high

$$\Pr[S_{n-2}] \geq \underbrace{\left(1-\frac{2}{n}\right)\cdot\left(1-\frac{2}{n-1}\right)\cdot\left(1-\frac{2}{n-2}\right)}_{\text{large factors}}\cdots\underbrace{\left(1-\frac{2}{4}\right)\cdot\left(1-\frac{2}{3}\right)}_{\text{small factors}}$$

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- $\Pr[S_{n-l}] \geq \left(1-\frac{2}{n}\right)\cdot\left(1-\frac{2}{n-1}\right)\cdot\left(1-\frac{2}{n-2}\right)\cdot \quad \cdot\left(1-\frac{2}{l+2}\right)\cdot\left(1-\frac{2}{l+1}\right)$

$$= \frac{n-2}{n}\cdot\frac{n-3}{n-1}\cdot\frac{n-4}{n-2}\cdot \quad \cdot\frac{l}{l+2}\cdot\frac{l-1}{l+1}$$

i.e., $\Pr[S_{n-l}]$ is constant if $l = \Omega(n)$

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i.e., $\Pr[S_{n-l}]$ is constant if $l = \Omega(n)$

A crucial observation

- $\Pr[S_{n-l}]$ is constant if $l = \Omega(n)$
- E.g., $\boxed{\text{if } l \approx \frac{n}{\sqrt{2}} \text{ then } \Pr[S_{n-l}] \geq 1/2}$
- So, much fewer (e.g., $O(\log n)$) executions of the first $n - l$ contractions suffice to make almost certain that C has survived in some of them
- Instead, these steps are executed in all the $O(n^2 \log n)$ executions of the contraction algorithm

- Question: how can we **modify the algorithm** to avoid the execution of these steps?

A faster algorithm

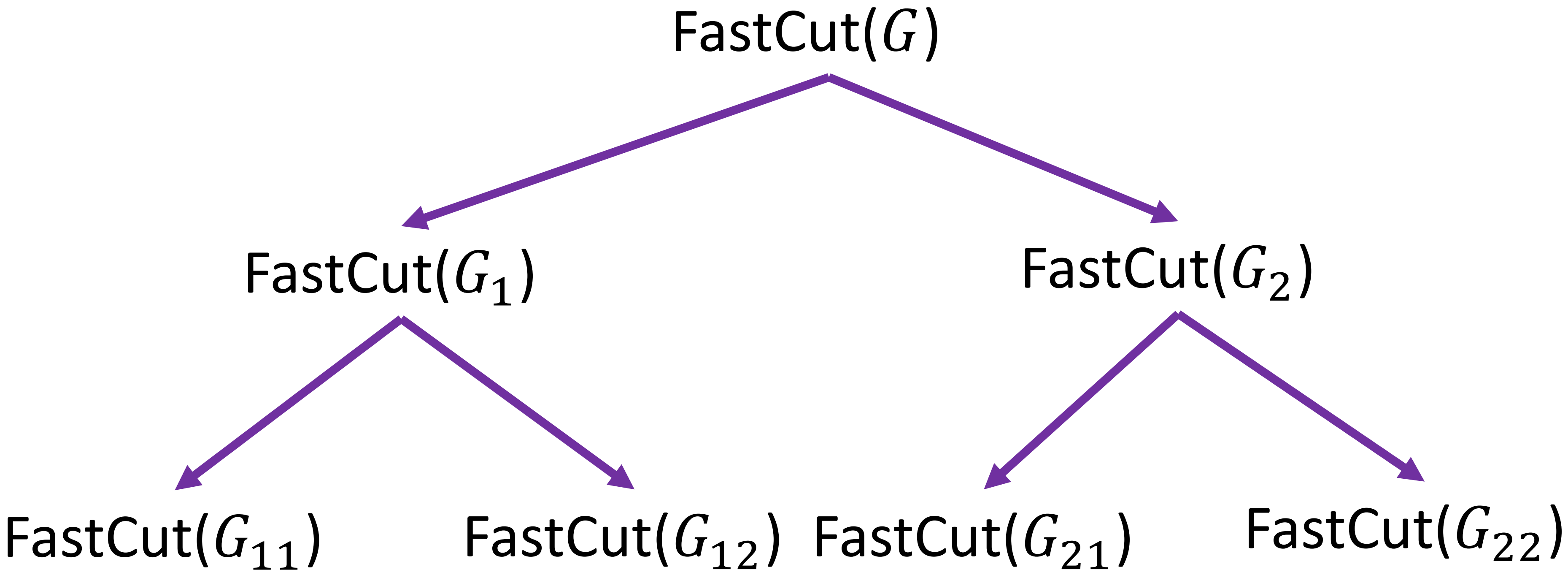
FastCut (multigraph G)

- If $n \leq 6$, compute the minimum cut via **exhaustive search** and return
- $G_1 :=$ a graph obtained after $n - n/\sqrt{2}$ (random) contractions
- $G_2 :=$ a graph obtained after $n - n/\sqrt{2}$ (random) contractions

$$\left.\begin{array}{l} \bullet\ X_1 := \text{FastCut}\,(G_1) \\ \bullet\ X_2 := \text{FastCut}\,(G_2) \end{array}\right\} \textbf{recursive calls}$$

- Return the minimum cut between X_1 and X_2

Analysis of FastCut



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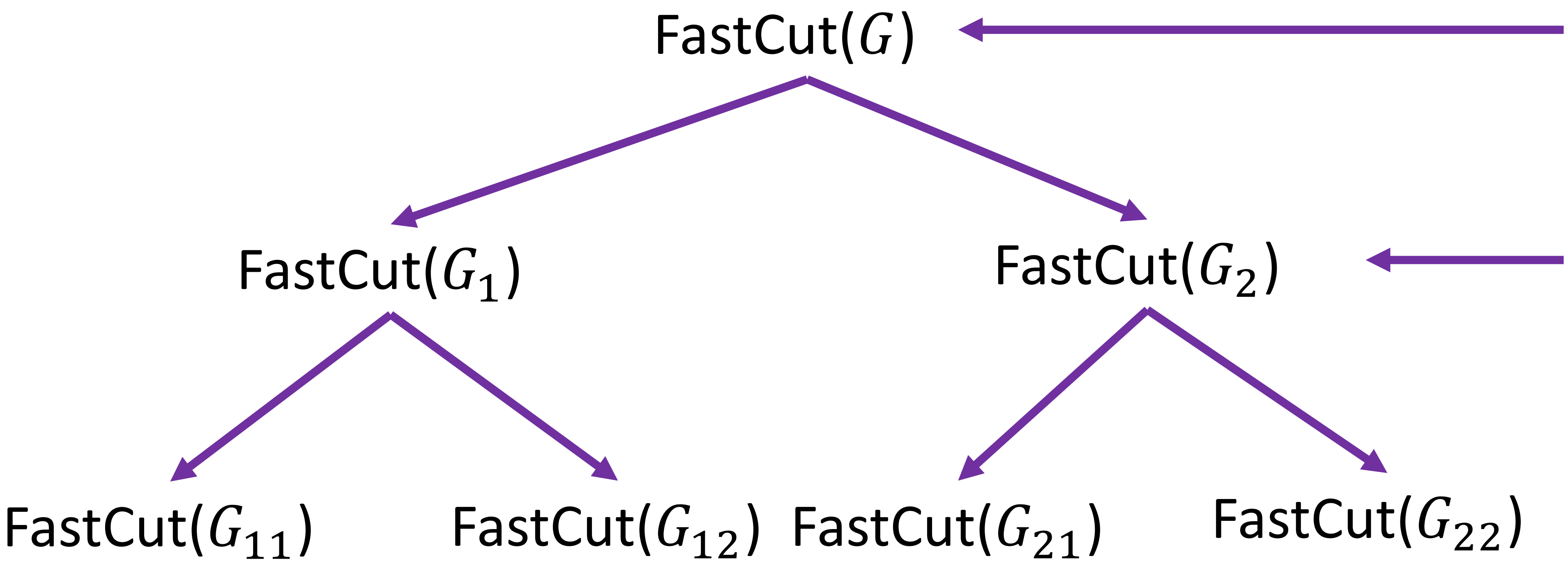
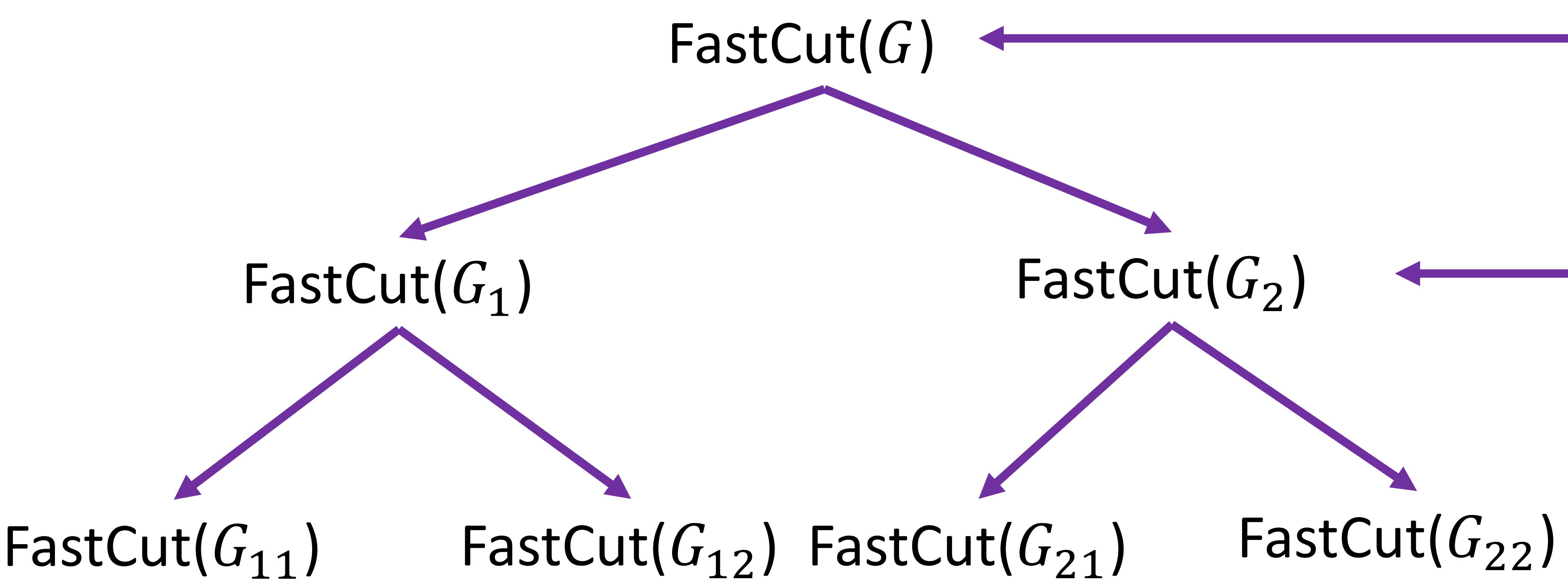


$O(n^2)$ steps for the contractions

2 problems of size $n/\sqrt{2}$ each

- $T(n) = 2T\left(\frac{n}{\sqrt{2}}\right) + O(n^2)$
- I.e., $T(n) = O(n^2 \log n)$

Analysis of FastCut



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So, the running time is **higher** than that of the contraction algorithm

How do we gain?

Analysis of FastCut (contd.)

- <u>Theorem</u>: FastCut finds a minimum cut with prob. at least $c/\log n$
- $P(n) =$ prob. that a call of FastCut with an n-node input is successful

$$P(n) \geq 1 - \left(1 - \frac{1}{2} \cdot P\left(\frac{n}{\sqrt{2}}\right)\right)^2$$

Analysis of FastCut (contd.)

- Theorem: FastCut finds a minimum cut with prob. at least $c/\log n$
- $P(n) =$ prob. that a call of FastCut with an n-node input is successful

$$P(n) \geq 1 - \overbrace{\left(1 - \tfrac{1}{2} \cdot \underbrace{P\left(\tfrac{n}{\sqrt{2}}\right)}_{\text{one successful recursive call}}\right)^2}^{\text{not successful}}$$

two independent contractions

Analysis of FastCut (contd.)

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two independent contractions

- Yields $P(n) = O(1/\log n)$ (why?)

Analysis of FastCut (contd.)

- <u>Theorem</u>: FastCut finds a minimum cut with prob. at least $c/\log n$
- $P(n)$ = prob. that a call of FastCut with an n-node input is successful

$$P(n) \geq 1 - \overbrace{\left(1 - \underset{\text{two independent contractions}}{\frac{1}{2}} \cdot \underbrace{P\left(\frac{n}{\sqrt{2}}\right)}_{\text{one successful recursive call}}\right)^2}^{\text{not successful}}$$

- Yields $P(n) = O(1/\log n)$ (why?)
- By repeating FastCut $O(\log^2 n)$ times, it will find the minimum cut with prob. at least $1 - 1/n$ in one of them
- Overall running time $\boxed{O(n^2 \log^3 n)}$

The solution of $P(n) \geq 1 - \left(1 - \frac{1}{2} \cdot P\left(\frac{n}{\sqrt{2}}\right)\right)^2$ satisfies $P(n) = O(1/\log n)$

- Set $Q(h) = P\left(2^{h/2}\right)$
- The inequality becomes $Q(h) \geq Q(h-1) - \frac{Q(h-1)^2}{4}$
- It suffices to show inductively that $Q(h) \geq \frac{1}{h+1}$
- Observe that the RHS of the inequality is increasing in $Q(h-1)$ (why?)
- Since, by the induction hypothesis, $Q(h-1) \geq 1/h$, the inequality implies that $Q(h) \geq 1/h - \frac{(1/h)^2}{4} \geq \frac{1}{h} - \frac{1}{h(h+1)} = \frac{1}{h+1}$ as desired
- The inequality (and the base case) follows since $n \geq 6$ and, hence, $h \geq 3$

References

- the contraction algorithm is due to Karger (SODA 1993)
- FastCut is due to Karger and Stein (JACM 1996)

Last slide

- Quick recap: randomization, useful inequalities
- Computing a (global) cut of minimum size in a graph
- A randomized algorithm
- Analysis
- Implications and improvements