

# Tools for analysis of randomized algorithms

Felix Blix Everberg

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## 1 Linearity of expectation

This theorem states that since the expectation of a random variable is a linear function, the expectation of the sum of a set of random variables is equal to the sum of their individual expectations. That is:

**Theorem 1** (*Linearity of Expectation*): Let  $X_1, \dots, X_n$  be random variables, then

$$\mathbb{E} \left[ \sum_i X_i \right] = \sum_i \mathbb{E}[X_i]$$

- Crucial for working with sums.
- Does NOT require independence.

## 2 Markov's Inequality

Markov's inequality bounds the probability that a non-negative random variable exceeds its expectation by a certain amount.

**Theorem 2** (*Markov's Inequality*): Let  $X$  be a non-negative random variable, then for any  $t > 0$

$$\Pr[X \geq t] \leq \frac{\mathbb{E}[X]}{t}$$

- The most basic inequality for bounding variables.
- Can also be used to bound the probability of  $X$  being smaller than a value, by phrasing it in terms of the negated event, i.e.  $1 - \Pr[x > t]$ .
- Used to prove Chebyshev's inequality.
- Can also be written with  $>$  and  $<$  rather than  $\geq$  and  $\leq$ .

### 3 Chebyshev's Inequality

*This inequality bounds the probability that the realization a random variable  $X$  diverges from it's mean by more than a threshold  $k$ .*

**Theorem 3** (Chebyshev's Inequality): *Let  $X$  be a random variable with finite mean  $\mu$  and finite variance  $\sigma^2$ . let  $k \in \mathbb{R}_{++}$  then*

$$\Pr [|X - \mu| \geq k] \leq \frac{\sigma^2}{k^2}$$

- *Easy way to bound divergence from expectation if variance is known.*
- *Directly derived from Markov's Inequality.*

### 4 Chernoff Bound (Hoeffding's Inequality)

*The Chernoff inequalities provide tighter bounds than Markov's Inequality for a variable constituting the sum of a set of bounded, independent random variables, but also imposes a larger constraint on the treated variable.*

**Theorem 4** (Chernoff Bound/Hoeffding's Inequality): *Let  $X_1, \dots, X_n$  be independent random variables in  $[0, 1]$  and let  $X = \sum_{i=1}^n X_i$ , then for any  $0 < \delta < 1$  and any  $\mu \leq \mathbb{E}[X]$ :*

$$\Pr [X < (1 - \delta) \mu] < e^{-\delta^2 \mu / 2}$$

*and for any  $0 < \delta < 1$  and any  $\mu \geq \mathbb{E}[X]$ :*

$$\Pr [X > (1 + \delta) \mu] < e^{-\delta^2 \mu / 3}$$

*and finally for any  $\delta \geq 1$  and any  $\mu \geq \mathbb{E}[X]$ :*

$$\Pr [X > (1 + \delta) \mu] < \left( \frac{e^\delta}{(1 + \delta)^{1 + \delta}} \right)^\mu$$

- *Tighter bounds but requires variables to be bounded and independent.*
- *The bounding can often be coerced by normalization or similar.*
- *The above can be written with  $\leq$  and  $\geq$  rather than  $<$  and  $>$*

## 5 Union Bound

*The union bound simply states that the probability that any one of a set of events occur is bounded by the sum of probabilities of the individual events. This is intuitive in that it is simply assuming that none of the events overlap, which is an obvious worst case.*

**Theorem 5** (Union Bound): Let  $E_1, \dots, E_n$  be events, then

$$\Pr \left[ \bigcup_i E_i \right] \leq \sum_i \Pr[E_i]$$

- Usefull for easily bounding the probability of a set of possible bad outcomes
- Does NOT require independence