1 Standard fit with multiple scattering

1.1 Fit with no material

Let us assume to have a series of plane measurements $\vec{y}^\star$ in locations x_1, x_2, $\cdots$, x_n and a predictive model $y_i = y_i(\vec{\alpha}, x_i)$, where $\vec{\alpha}$ are parameters to be determined by fitting the measurements. The fit is performed minimizing the χ^2:

$$\chi^2 = (\vec{y}(\vec{\alpha}) - \vec{y}^\star)^t D^{-1} (\vec{y}(\vec{\alpha}) - \vec{y}^\star) \tag{1}$$

where D is the digonal matrix containing the squared errors on the $\vec{y}^\star$:

$$D = \begin{pmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \sigma_n^2 \end{pmatrix} \tag{2}$$

The minimiztion can be solved recursively by linearizing the prediction function:

$$\vec{y}(\vec{\alpha}) \simeq \vec{y}(\vec{\alpha}_0) + \frac{\partial \vec{y}}{\partial \vec{\alpha}} (\vec{\alpha} - \vec{\alpha}_0) = \vec{y}_0 + A \delta \vec{\alpha} \tag{3}$$

The derivative of the χ^2 is given by:

$$\frac{1}{2} \frac{\partial \chi^2}{\partial \vec{\alpha}} = A^t D^{-1} (A \delta \vec{\alpha} + \vec{y}_0 - \vec{y}^\star) = 0 \tag{4}$$

that can be solved to give the parameters $\vec{\alpha}$:

$$\delta \vec{\alpha} = (A^t D^{-1} A)^{-1} A^t D^{-1} (\vec{y}^\star - \vec{y}_0) \tag{5}$$

and the parameter covariance matrix, C:

$$\frac{1}{2} \frac{\partial^2 \chi^2}{\partial \vec{\alpha} \partial \vec{\alpha}} = C^{-1} = A^t D^{-1} A \tag{6}$$

1.2 Fit with material in front

Let us now assume that we have a layer of material before all the measurements at location $x_0 < x_i$, $i = 1, 2, \cdots, n$ generating an average multiple scattering angle $\bar{\theta}$. We want to calculate how this affects the covariance matrix of the fit parameters.

Let $\vec{\alpha}$ be the parameters before the scattering layer and $\vec{\alpha}\,'$ those after the scattering layer toward the measurements. The following relation holds:

$$\vec{\alpha}\,' \simeq \vec{\alpha} + \frac{\partial \vec{\alpha}}{\partial \theta} \theta \tag{7}$$

This parameter variation affects the measurements:

$$\vec{y}\,' = \vec{y}(\vec{\alpha}\,') \simeq \vec{y}(\vec{\alpha}) + \frac{\partial \vec{y}}{\partial \vec{\alpha}} \frac{\partial \vec{\alpha}}{\partial \theta} \theta \tag{8}$$

and generates an additional component to the measurement covariance matrix, M:

$$M = \frac{\partial \vec{y}}{\partial \vec{\alpha}}\frac{\partial \vec{\alpha}}{\partial \theta}\bar{\theta}^2 \left(\frac{\partial \vec{y}}{\partial \vec{\alpha}}\frac{\partial \vec{\alpha}}{\partial \theta}\right)^t = A\frac{\partial \vec{\alpha}}{\partial \theta}\bar{\theta}^2 \left(\frac{\partial \vec{\alpha}}{\partial \theta}\right)^t A^t = \vec{u}\vec{u}^t \tag{9}$$

where the derivative of the parameters is taken at the scattering location x_0 and $\vec{u}$ is given by:

$$\vec{u} = A\frac{\partial \vec{\alpha}}{\partial \theta}\bar{\theta} = A\vec{g}\bar{\theta} \tag{10}$$

The measurements error matrix becomes $S = D + M = D + \vec{u}\vec{u}^t$ and can be inverted using Miller's lemma:

$$S^{-1} = D^{-1} - \frac{D^{-1}\vec{u}\vec{u}^t D^{-1}}{1 + \vec{u}^t D^{-1}\vec{u}} \tag{11}$$

Replacing D^{-1} with S^{-1} in eq. (6) we obtain the updated covariance matrix:

$$C^{-1} = A^t S^{-1} A = A^t \left(D^{-1} - \frac{D^{-1}\vec{u}\vec{u}^t D^{-1}}{1 + \vec{u}^t D^{-1}\vec{u}}\right) A = A^t D^{-1} A - \frac{A^t D^{-1}\vec{u}\vec{u}^t D^{-1} A}{1 + \vec{u}^t D^{-1} D D^{-1}\vec{u}} \tag{12}$$

Letting C_0 be the covariance matrix obtained without the material in front we can rewrite eq. (12) as:

$$C^{-1} = C_0^{-1} - \frac{C_0^{-1}\vec{g}\bar{\theta}^2\vec{g}^t C_0^{-1}}{1 + \vec{g}^t C_0^{-1}\vec{g}\,\bar{\theta}^2} = C_0^{-1}\left(C_0 - \frac{\vec{g}\vec{g}^t\,\bar{\theta}^2}{1 + \vec{g}^t C_0^{-1}\vec{g}\,\bar{\theta}^2}\right) C_0^{-1} \tag{13}$$

2 Kalman fit with multiple scattering

In the Kalman approach from outside in the covariance matrix of the parameters is obtained adding one measurement point at a time and finally adding the multiple scattering contribution from the scattering layer. The covariance matrix of the parameters is generated iteratively starting from a large external covariance matrix, C_{n+1} such that $C_{n+1}^{-1} = 0$. Using the notation where C_i is the covariance matrix after adding point x_i, we extract e recursive relation between the C_i by minizing the χ^2:

$$\chi^2 = (\vec{\alpha}_i - \vec{\alpha}_{i+1})^t C_{i+1}(\vec{\alpha}_i - \vec{\alpha}_{i+1}) + \frac{(y_i(\vec{\alpha}_i) - y_i^\star)^2}{\sigma_i^2} \tag{14}$$

where $\vec{\alpha}_{i+1}$ are the parameters determined in the previous iteration. The covariance of $\vec{\alpha}_i$ is given by:

$$C_i^{-1} = \frac{1}{2}\frac{\partial^2 \chi^2}{\partial \vec{\alpha}_i \partial \vec{\alpha}_i} = C_{i+1}^{-1} + \frac{1}{\sigma_i^2}\frac{\partial y_i}{\partial \vec{\alpha}_i}\left(\frac{\partial y_i}{\partial \vec{\alpha}_i}\right)^t \tag{15}$$

After adding all points starting from y_n and ending at y_1, we have the covariance matrix before including the contribution from the multiple scattering:

$$C_1^{-1} = \sum_{i=1}^{n} \frac{1}{\sigma_i^2}\left(\frac{\partial y_i}{\partial \vec{\alpha}}\right)^t \frac{\partial y_i}{\partial \vec{\alpha}} \tag{16}$$

where we have dropped the index i in the parameters assuming to use always the most probable (true) value. We recall that:

$$A = \frac{\partial \vec{y}}{\partial \vec{\alpha}} = \begin{pmatrix} \frac{\partial y_1}{\partial \alpha_1} & \frac{\partial y_1}{\partial \alpha_2} & \cdots & \frac{\partial y_1}{\partial \alpha_p} \\ \frac{\partial y_2}{\partial \alpha_1} & \frac{\partial y_2}{\partial \alpha_2} & \cdots & \frac{\partial y_2}{\partial \alpha_p} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial \alpha_1} & \frac{\partial y_n}{\partial \alpha_2} & \cdots & \frac{\partial y_n}{\partial \alpha_p} \end{pmatrix} = \begin{pmatrix} \frac{\partial y_1}{\partial \vec{\alpha}} \\ \frac{\partial y_2}{\partial \vec{\alpha}} \\ \cdots \\ \frac{\partial y_n}{\partial \vec{\alpha}} \end{pmatrix} \tag{17}$$

therefore:

$$C_1^{-1} = \sum_{i=1}^{n} \frac{1}{\sigma_i^2} \left(\frac{\partial y_i}{\partial \vec{\alpha}} \right)^t \frac{\partial y_i}{\partial \vec{\alpha}} = \sum_{i,j=1}^{n} \left(\frac{\partial y_i}{\partial \vec{\alpha}} \right)^t \frac{\delta_{ij}}{\sigma_i^2} \frac{\partial y_j}{\partial \vec{\alpha}} = A^t D^{-1} A = C_0^{-1} \tag{18}$$

We now add the contribution from the scattering layer:

$$C = C_0 + \frac{\partial \vec{\alpha}}{\partial \theta} \bar{\theta}^2 \left(\frac{\partial \vec{\alpha}}{\partial \theta} \right)^t = C_0 + \vec{g}\vec{g}^t \bar{\theta}^2 = (A^t D^{-1} A)^{-1} + \vec{v}\vec{v}^t \quad \text{where } \vec{v} = \frac{\partial \vec{\alpha}}{\partial \theta} \bar{\theta} \tag{19}$$

The inverse of the covariance matrix is then given by:

$$C^{-1} = A^t D^{-1} A - \frac{A^t D^{-1} A \vec{v}\vec{v}^t A^t D^{-1} A}{1 + \vec{v}^t A^t D^{-1} D D^{-1} A \vec{v}} \tag{20}$$

but since $\vec{u} = A\vec{v}$, the above equation becomes:

$$C^{-1} = A^t D^{-1} A - \frac{A^t D^{-1} \vec{u}\vec{u}^t D^{-1} A}{1 + \vec{u}^t D^{-1} D D^{-1} \vec{u}} \tag{21}$$

that is identical to eq. (12). It is thus demonstrated that the global and Kalman approaches are equivalent.

3 Discussion of the multiple scattering terms

We now calculate the derivative of the parameters with respect to the MS angle in the simplified situation when $D = 0$. We note that:

$$\frac{\partial \vec{\alpha}}{\partial \theta} = \frac{\partial \vec{\alpha}}{\partial \vec{p}} \frac{\partial \vec{p}}{\partial \theta} \tag{22}$$

so we start with the derivative of the parameters with respect to the momentum.
Derivative of D:

$$D = \frac{1}{a}(T - p_\perp), \text{ so: } \frac{\partial D}{\partial \vec{p}} = \frac{1}{a} \left(\frac{\partial T}{\partial \vec{p}} - \frac{\partial p_\perp}{\partial \vec{p}} \right) \tag{23}$$

where

$$T = \sqrt{p_\perp^2 - 2a(xp_y - yp_x) + a^2(x^2 + y^2)} \tag{24}$$

leading to:

$$\frac{\partial D}{\partial \vec{p}} = \frac{1}{a} \left(\frac{p_x + ay}{T} - \frac{p_x}{p_\perp}, \frac{p_y - ax}{T} - \frac{p_y}{p_\perp}, 0 \right) \tag{25}$$

since however $D = 0$ then $T = p_\perp$ the derivative simplifies into:

$$\frac{\partial D}{\partial \vec{p}} = \frac{1}{p_\perp}(y,\, -x,\, 0) \tag{26}$$

Derivative of φ_0:
Recalling that:

$$cos\,\varphi_0 = \frac{p_x + ay}{T} = \frac{p_x + ay}{p_\perp} \text{ and } tan\,\varphi_0 = \frac{p_y - ax}{p_x + ay} \tag{27}$$

we get that:

$$\frac{\partial \varphi_0}{\partial \vec{p}} = \frac{cos^2\,\varphi_0}{p_x + ay}\,(-tan\,\varphi_0,\, 1,\, 0) = \frac{(p_x + ay)^2}{(p_x + ay)p_\perp^2}\left(-\frac{p_y - ax}{p_x + ay},\, 1,\, 0\right) \tag{28}$$

that simplifies into:

$$\frac{\partial \varphi_0}{\partial \vec{p}} = \frac{1}{p_\perp^2}(-p_y + ax,\, p_x + ay,\, 0) = \frac{1}{p_\perp}(-sin\,\varphi_0,\, cos\,\varphi_0,\, 0) \tag{29}$$

Derivative of C:
the derivative of the half curvature $C = a/(2p_\perp$ is:

$$\frac{\partial C}{\partial \vec{p}} = \frac{a}{2}\left(-\frac{p_x}{p_\perp^3},\, -\frac{p_y}{p_\perp^3},\, 0\right) = \frac{a}{2p_\perp^3}\,(-p_x,\, -p_y,\, 0) = \frac{C}{p_\perp^2}\,(-p_x,\, -p_y,\, 0) \tag{30}$$

Derivative of $\lambda = cot\,\theta$:
the derivative of $\lambda = p_z/p_\perp$ is:

$$\frac{\partial \lambda}{\partial \vec{p}} = \left(-\frac{p_x p_z}{p_\perp^3},\, -\frac{p_y p_z}{p_\perp^3},\, \frac{1}{p_\perp}\right) = \frac{1}{p_\perp}\left(-\frac{p_x p_z}{p_\perp^2},\, -\frac{p_y p_z}{p_\perp^2},\, 1\right) \tag{31}$$

Derivative of z_0:
Recall that $z_0 = z - \lambda\, s/(2C) = z - p_z\, s/a$, where s is the phase and that $s = tan^{-1}(p_y/p_x) - \varphi_0$ then:

$$\frac{\partial z_0}{\partial \vec{p}} = \left(\frac{p_z}{a}\left(\frac{p_y}{p_\perp^2} + \frac{\partial \varphi_0}{\partial p_x}\right),\, \frac{p_z}{a}\left(-\frac{p_x}{p_\perp^2} + \frac{\partial \varphi_0}{\partial p_y}\right),\, -s/a\right) = \left(\frac{p_z}{a}\,\frac{a\,x}{p_\perp^2},\, \frac{p_z}{a}\,\frac{a\,y}{p_\perp^2},\, -s/a\right) \tag{32}$$

that simplifies to:

$$\frac{\partial z_0}{\partial \vec{p}} = \left(\frac{\lambda\, x}{p_\perp},\, \frac{\lambda\, y}{p_\perp},\, -s/a\right) \tag{33}$$

Summing everything up:

$$\frac{\partial \vec{\alpha}}{\partial \vec{p}} = \begin{pmatrix} \frac{y}{p_\perp} & -\frac{x}{p_\perp} & 0 \\ -\frac{sin\,\varphi_0}{p_\perp} & \frac{cos\,\varphi_0}{p_\perp} & 0 \\ -\frac{C\,p_x}{p_\perp^2} & -\frac{C\,p_y}{p_\perp^2} & 0 \\ -\frac{p_x p_z}{p_\perp^3} & -\frac{p_y p_z}{p_\perp^3} & \frac{1}{p_\perp} \\ \frac{\lambda\, x}{p_\perp} & \frac{\lambda\, y}{p_\perp} & -s/a \end{pmatrix} = \frac{1}{p_\perp}\begin{pmatrix} y & -x & 0 \\ -sin\,\varphi_0 & cos\,\varphi_0 & 0 \\ -\frac{C\,p_x}{p_\perp} & -\frac{C\,p_y}{p_\perp} & 0 \\ -\frac{p_x p_z}{p_\perp^2} & -\frac{p_y p_z}{p_\perp^2} & 1 \\ \lambda\, x & \lambda\, y & -\frac{s}{2C} \end{pmatrix} \tag{34}$$

The full derivative relative to the transverse multiple scattering angle is then given by:
Transverse

$$\frac{\partial\vec{\alpha}}{\partial\theta} = \frac{\partial\vec{\alpha}}{\partial\vec{p}}\frac{\partial\vec{p}}{\partial\theta} = \frac{p}{p_\perp^2}\begin{pmatrix} y & -x & 0 \\ -sin\,\varphi_0 & cos\,\varphi_0 & 0 \\ -\frac{C\,p_x}{p_\perp} & -\frac{C\,p_y}{p_\perp} & 0 \\ -\frac{p_x p_z}{p_\perp^2} & -\frac{p_y p_z}{p_\perp^2} & 1 \\ \lambda\,x & \lambda\,y & -\frac{s}{2C} \end{pmatrix}\begin{pmatrix} -p_y \\ p_x \\ 0 \end{pmatrix} = \frac{p}{p_\perp^2}\begin{pmatrix} -(x\,p_x + y\,p_y) \\ p_x\,cos\,\varphi_0 + p_y\,sin\,\varphi_0 \\ 0 \\ 0 \\ \lambda(y\,p_x - x\,p_y) \end{pmatrix} \tag{35}$$

This can be simplified taking into account the parametric helix equation as a funtion of the phase s:

$$\frac{\partial\vec{\alpha}}{\partial\theta} = \frac{p}{p_\perp}\begin{pmatrix} -\frac{1}{2C}sin\,s \\ cos\,s \\ 0 \\ 0 \\ \frac{1}{2C}(cos\,s - 1) \end{pmatrix} \tag{36}$$

where we remind that

$$s = 2sin^{-1}\left(C\sqrt{\frac{r^2 - D^2}{1 + 2CD}}\right) = sin^{-1}[2C(x\,cos\,\varphi_0 + y\,sin\,\varphi_0)] \tag{37}$$

when $D = 0$ then:

$$rC = sin\frac{s}{2} \tag{38}$$

In order to estimate the MS effect on an $R\varphi$ measurement at a radius R one has to estimate the derivative of $R\varphi$ with respect to the helix parameters:

$$y = R\varphi = R\varphi_0 + R\,sin^{-1}\left(\frac{RC + (1 + CD)D/R}{1 + 2CD}\right) = R\varphi_0 + R\,sin^{-1}A \tag{39}$$

then
D derivative

$$\frac{\partial y}{\partial D} = \frac{R}{\sqrt{1 - A^2}}\frac{(1 + 2CD)^2/R - 2C[RC + (1 + CD)D/R]}{(1 + 2CD)^2} \tag{40}$$

that becomes for $D = 0$:

$$\frac{\partial y}{\partial D} = \frac{1}{\sqrt{1 - R^2C^2}}(1 - 2R^2C^2) \tag{41}$$

If we let t be the phase at radius R then:

$$\frac{\partial y}{\partial D} = \frac{1 - 2sin^2\frac{t}{2}}{cos\frac{t}{2}} = \frac{cos\,t}{cos\frac{t}{2}} = 2cos\frac{t}{2} - \frac{1}{cos\frac{t}{2}} \simeq 1 \tag{42}$$

The φ_0 **derivative** is easy $\partial y/\partial\varphi_0 = R = sin(t/2)/C$. Finally the
C **derivative:**

$$\frac{\partial y}{\partial C} = \frac{R}{\sqrt{1 - A^2}}\frac{(R + D^2/R)(1 + 2CD) - 2D(RC + (1 + CD)D/R)}{(1 + 2CD)^2} \tag{43}$$

that becomes for $D = 0$:

$$\frac{\partial y}{\partial C} = \frac{R^2}{\sqrt{1 - R^2C^2}} = \frac{sin^2\frac{t}{2}}{C^2 cos\frac{t}{2}} \tag{44}$$

Finally, setting to t the phase at the measurement layer and s at the scattering layer, the MS effect is given by:

$$\frac{\delta\, R\varphi}{\bar{\theta}} = \left[\frac{\partial R\varphi}{\partial \vec{\alpha}}\right]_{\omega=t} \left[\frac{\partial \vec{\alpha}}{\partial \theta}\right]_{\omega=s} = \frac{sin\frac{t}{2}}{C}\, cos\, s - \frac{1}{2C} sin\, s \frac{cos\, t}{cos\frac{t}{2}} \simeq \frac{t - s}{2C} \tag{45}$$