# Wave function of $^4$He from first principles:

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⇝ Starting point as in Bergstrom:

$$\phi_A = C \cdot \exp\left(-\frac{1}{2} \upsilon_A \cdot \sum_{i=1}^{4} (\vec{x}_i - \vec{R})^2\right)$$

⇝ We transform into the cms system $\vec{R} = 0$

$$\phi_A = C \cdot \exp\left(-\frac{1}{2} \upsilon_A \left(\vec{x}_1^{\,2} + \vec{x}_2^{\,2} + \vec{x}_3^{\,2} + \vec{x}_4^{\,2}\right)\right)$$

⇝ Then we apply the Jacobi variables:

$$\vec{R} = \frac{1}{4}(\vec{x}_1 + \vec{x}_2 + \vec{x}_3 + \vec{x}_4)$$

$$\vec{r}_1 = \vec{x}_2 - \vec{x}_1$$

$$\vec{r}_2 = \vec{x}_3 - \frac{1}{2}(\vec{x}_2 + \vec{x}_1)$$

$$\vec{r}_3 = \vec{x}_4 - \frac{1}{3}(\vec{x}_3 + \vec{x}_2 + \vec{x}_1)$$

⇝ As derived in Mrowczynski, we can verify with Mathematica:

$$\vec{x}_1^{\,2} + \vec{x}_2^{\,2} + \vec{x}_3^{\,2} + \vec{x}_4^{\,2} = 4\vec{R}^{\,2} + \frac{1}{2}\vec{r}_1^{\,2} + \frac{2}{3}\vec{r}_2^{\,2} + \frac{3}{4}\vec{r}_3^{\,2}$$

⇝ For the Jacobian, we obtain:

$$d\vec{R}\, d\vec{r}_1\, d\vec{r}_2\, d\vec{r}_3 = \left| \det \frac{\partial(\vec{R}, \vec{r}_1, \vec{r}_2, \vec{r}_3)}{\partial(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4)} \right| d\vec{x}_1\, d\vec{x}_2\, d\vec{x}_3\, d\vec{x}_4$$

and thus

$$\begin{vmatrix} \frac{\partial \vec{R}}{\partial x_1} & \cdots & \frac{\partial \vec{r}_3}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial \vec{R}}{\partial x_4} & \cdots & \frac{\partial \vec{r}_3}{\partial x_4} \end{vmatrix} = \begin{vmatrix} \frac{1}{4} & -1 & -\frac{1}{2} & -\frac{1}{3} \\ \frac{1}{4} & 1 & -\frac{1}{2} & -\frac{1}{3} \\ \frac{1}{4} & 0 & 1 & -\frac{1}{3} \\ \frac{1}{4} & 0 & 0 & 1 \end{vmatrix} = 1. ✓$$

→ For the normalisation constant, we thus get:

$$\frac{1}{C^2} = \int |\phi_A|^2 \, d\vec{x}_1 \cdots d\vec{x}_4 = \int \exp^{\cancel{2}}\left(-\frac{1}{\cancel{2}} \upsilon_A \left(\frac{1}{2} r_1^2 + \frac{2}{3} r_2^2 + \frac{3}{4} r_3^2\right)\right) dr_1 \, dr_2 \, dr_3$$

$$4\pi r_1^2 dr_1$$

$$= (4\pi)^3 \underbrace{\int r_1^2 \cdot \exp\left(-\frac{1}{2} r_1^2 \upsilon_A\right) dr_1}_{} \cdot \underbrace{\int r_2^2 \cdot \exp\left(-\frac{2}{3} r_2^2 \upsilon_A\right) dr_2}_{} \cdot \underbrace{\int r_3^2 \cdot \exp\left(-\frac{3}{4} r_3^2 \upsilon_A\right) dr_3}_{}$$

$$= (4\pi)^3 \cdot \frac{\sqrt{\pi}}{4} \cdot \frac{1}{\left(\frac{1}{2} \upsilon_A\right)^{3/2}} \quad \frac{\sqrt{\pi}}{4} \cdot \frac{1}{\left(\frac{2}{3} \upsilon_A\right)^{3/2}} \quad \frac{\sqrt{\pi}}{4} \cdot \frac{1}{\left(\frac{3}{4} \upsilon_A\right)^{3/2}}$$

$$= \frac{\cancel{4^3} \cdot \pi^3 \cdot \pi^{3/2}}{\cancel{4^3} \cdot \upsilon_A^{9/2}} \cdot \frac{1}{\left(\frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{3}}{4}\right)^{3/2}}$$

$$= \left(\frac{\pi}{\upsilon_A}\right)^{9/2} \cdot \frac{1}{\left(\frac{1}{4}\right)^{3/2}} \Rightarrow C^2 = \frac{\upsilon_A^{9/2}}{4^{3/2} \cdot \pi^{9/2}}$$

$$\Rightarrow C = \left(\frac{\upsilon_A^3}{4 \cdot \pi^3}\right)^{3/4}.$$

as in SATO/YAZAKI!

For the RMS we obtain accordingly: ~~$r^2_{rms} = \int \sum_i \left(\frac{x_i - \vec{R}}{2}\right)^2 \cdot |\phi_A|^2 dx_1 dx_2 dx_3 dx_4$~~

$$r_{rms}^2 = \frac{1}{4}\int (x_1^2 + x_2^2 + x_3^2 + x_4^2) \cdot |\phi_A|^2 \, dx_1\, dx_2\, dx_3\, dx_4$$

$\frac{1}{4}$ because rms describes the mean deviation

$$= \frac{1}{4} C'^2 \cdot \Bigg( \underbrace{\frac{1}{2} \int r_1^2 \cdot \exp\left(-\frac{1}{2}\upsilon_A r_1^2\right) \cdot \exp\left(-\frac{2}{3}\upsilon_A r_2^2\right) \cdot \exp\left(-\frac{3}{4} r_3^2\right) d\vec{r}_1\, d\vec{r}_2\, d\vec{r}_3}_{1}$$

$$+ \frac{2}{3} \int r_2^2 \cdot \exp(\dots) \cdot \exp(\dots) \cdot \exp(\dots) \cdot d\vec{r}_1\, d\vec{r}_2\, d\vec{r}_3$$

$$+ \frac{3}{4} \int r_3^2 \cdot \exp(\dots) \cdot \exp(\dots) \cdot \exp(\dots) \underbrace{d\vec{r}_1\, d\vec{r}_2\, d\vec{r}_3}_{= (4\pi)^3 \cdot r_1^2 \cdot r_2^2 \cdot r_3^2} \Bigg)$$

We obtain for 1:

$$\frac{1}{2}(4\pi^3) \cdot \underbrace{\int r_1^4 \cdot \exp\left(-\frac{1}{2}\upsilon_A r_1^2\right) dr_1}_{} \cdot \underbrace{\int r_2^2 \cdot \exp\left(-\frac{2}{3}\upsilon_A r_2^2\right) dr_2}_{} \cdot \underbrace{\int r_3^2 \exp\left(-\frac{3}{4} r_3^2\right) dr_3}_{}$$

$$= (4\pi)^3 \cdot \frac{1}{2} \cdot \frac{3\sqrt{\pi}}{8} \cdot \frac{1}{(\frac{1}{2}\upsilon_A)^{5/2}} \cdot \frac{\sqrt{\pi}}{4} \cdot \frac{1}{(\frac{2}{3}\upsilon_A)^{3/2}} \cdot \frac{\sqrt{\pi}}{4} \cdot \frac{1}{(\frac{3}{4}\upsilon_A)^{3/2}}$$

~~$\Rightarrow r_{rms}^2 = a^2$~~ $\Rightarrow r_{rms}^2 = C'^2 (1 + 2 + 3)$

$$\Rightarrow r_{rms}^2 = \frac{1}{4} C'^2 \cdot (4\pi)^3 \cdot \frac{3\pi^{3/2}}{8 \cdot 4^2 \cdot \upsilon_A^{11/2}} \underbrace{\left( \frac{1/2}{(\frac{1}{2})^{5/2} \cdot (\frac{2}{3})^{3/2} \cdot (\frac{3}{4})^{3/2}} + \frac{2/3}{(\frac{1}{2})^{3/2} \cdot (\frac{2}{3})^{5/2} \cdot (\frac{3}{4})^{3/2}} + \frac{3/4}{(\frac{1}{2})^{3/2} (\frac{2}{3})^{3/2} \cdot (\frac{3}{4})^{5/2}} \right)}_{= 24}$$

$$= \frac{1}{4} \frac{\upsilon_A^{9/2} \cdot 4^{\cancel{3}} \cdot \cancel{\pi^3} \cdot 3 \cdot \cancel{\pi^{3/2}} \cdot \cancel{24}\,3}{4^{3/2} \cdot \cancel{\pi^{9/2}} \cdot \cancel{8} \cdot \cancel{4^2} \cdot \upsilon_A^{11/2}}$$

$$= \frac{1}{\upsilon_A} \cdot \frac{1}{4} \cdot \frac{4 \cdot 9}{2^3} = \frac{9}{8\upsilon_A}.$$