Project Dynamics - ADR Spacecraft

Frédéric Berdoz, 260867318

I. Approach

In order to derive the equations of motion of the system, the Lagrangian approach will be used. Each body will be treated separately and the equations of motion will be coupled using collocation and attitude constraints. Moreover, as it is assumed to be in deep space, gravity will be neglected, i.e. $\underrightarrow{g} = \underrightarrow{0}$ (see Proposal).

II. Integration over the Bodies

Let $\mathcal{S}$ denote the continuous rigid body of the spacecraft wall, $\mathcal{D}$ the debris and $\mathcal{A}$ the whole system ($\mathcal{S}$+W1+W2+$\mathcal{D}$). In order to shorten the notation, let $\boldsymbol{F}^{\mathcal{B}}(\mathbf{x})$ denote the integration of the quantity $\mathbf{x}$ (scalar or matrix) over the body $\mathcal{B}$. In particular, using the the parameters defined in the Project Kinematics[1]:

$$\begin{aligned}
\boldsymbol{F}^{\mathcal{S}}(\mathbf{x}) \triangleq & \int_0^{\rho_o}\int_0^{2\pi}\int_0^{t_s} \mathbf{x}\,\rho_s\,\mathrm{d}z_s\,\mathrm{d}\theta_s\,\mathrm{d}\rho_s \\
&+ \int_{(\rho_o-t_s)}^{\rho_o}\int_0^{2\pi}\int_{t_s}^{(l_s-t_s)} \mathbf{x}\,\rho_s\,\mathrm{d}z_s\,\mathrm{d}\theta_s\,\mathrm{d}\rho_s \\
&+ \int_0^{\rho_o}\int_0^{2\pi}\int_{(l_s-t_s)}^{l_s} \mathbf{x}\,\rho_s\,\mathrm{d}z_s\,\mathrm{d}\theta_s\,\mathrm{d}\rho_s,
\end{aligned}$$

$$\boldsymbol{F}^{W1}(\mathbf{x}) \triangleq \int_0^{\rho_1}\int_0^{2\pi}\int_{-\frac{l_1}{2}}^{\frac{l_1}{2}} \mathbf{x}\,\rho_a\,\mathrm{d}z_a\,\mathrm{d}\theta_a\,\mathrm{d}\rho_a,$$

$$\boldsymbol{F}^{W2}(\mathbf{x}) \triangleq \int_0^{\rho_2}\int_0^{2\pi}\int_{-\frac{l_2}{2}}^{\frac{l_2}{2}} \mathbf{x}\,\rho_b\,\mathrm{d}z_b\,\mathrm{d}\theta_b\,\mathrm{d}\rho_b,$$

$$\boldsymbol{F}^{\mathcal{D}}(\mathbf{x}) \triangleq \int_{-\frac{l_d}{2}}^{\frac{l_d}{2}}\int_{-\frac{l_d}{2}}^{\frac{l_d}{2}}\int_{-\frac{l_d}{2}}^{\frac{l_d}{2}} \mathbf{x}\,\mathrm{d}z_d\,\mathrm{d}y_d\,\mathrm{d}x_d.$$

III. Mass Properties

As mentioned in the Proposal, the total mass of the spacecraft is assumed to remain constant. Moreover, the density of each body is constant over itself. Therefore, let σ_s, σ_1, σ_2 and σ_d be the density of $\mathcal{S}$, W1, W2 and $\mathcal{D}$, respectively. The corresponding masses have already been defined as m_s, m_1, m_2 and m_d. Additionally, let V_s, V_1, V_2 and V_d be the corresponding volumes, given by:

$$V_s = \boldsymbol{F}^{\mathcal{S}}(1), \quad V_1 = \boldsymbol{F}^{W1}(1), \quad V_2 = \boldsymbol{F}^{W2}(1), \quad V_d = \boldsymbol{F}^{\mathcal{D}}(1).$$

Therefore, the masses simply become:

$$m_s = \sigma_s V_s, \quad m_1 = \sigma_1 V_1, \quad m_2 = \sigma_2 V_2, \quad m_d = \sigma_d V_d.$$

The relevant first moments of mass of each body are given by:

$$\begin{aligned}
\underrightarrow{c}^{\mathcal{S}p} &= m_s \underrightarrow{r}^{g_s p} = \begin{bmatrix} 0 & 0 & \frac{1}{2}m_s l_s \end{bmatrix} \underrightarrow{\mathcal{F}}_s, \\
\underrightarrow{c}^{W1g_1} &= \underrightarrow{c}^{W2g_2} = \underrightarrow{c}^{\mathcal{D}g_d} = \underrightarrow{0}.
\end{aligned}$$

Similarly, using the following identities:

$$\underrightarrow{J}^{\mathcal{B}z} = \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \mathbf{J}_b^{\mathcal{B}z} \underrightarrow{\mathcal{F}}_b, \quad \mathbf{J}_b^{\mathcal{B}z} = -\int_{\mathcal{B}} \mathbf{r}_b^{\mathrm{d}mz^\times} \mathbf{r}_b^{\mathrm{d}mz^\times}\,\mathrm{d}m,$$

one can compute the second moments of mass of each body resolved in their respective body frame. In particular,

$$\begin{aligned}
\mathbf{J}_s^{\mathcal{S}p} &= \boldsymbol{F}^{\mathcal{S}}(-\sigma_s \mathbf{r}_s^{\mathrm{d}m_s p^\times} \mathbf{r}_s^{\mathrm{d}m_s p^\times}), \\
\mathbf{J}_a^{W1g_1} &= \boldsymbol{F}^{W1}(-\sigma_1 \mathbf{r}_a^{\mathrm{d}m_1 g_1^\times} \mathbf{r}_a^{\mathrm{d}m_1 g_1^\times}), \\
\mathbf{J}_b^{W2g_2} &= \boldsymbol{F}^{W2}(-\sigma_2 \mathbf{r}_b^{\mathrm{d}m_2 g_2^\times} \mathbf{r}_b^{\mathrm{d}m_2 g_2^\times}), \\
\mathbf{J}_d^{\mathcal{D}g_d} &= \boldsymbol{F}^{\mathcal{D}}(-\sigma_d \mathbf{r}_d^{\mathrm{d}m_d g_d^\times} \mathbf{r}_d^{\mathrm{d}m_d g_d^\times}),
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{r}_s^{\mathrm{d}m_s p^\times} \mathbf{r}_s^{\mathrm{d}m_s p^\times} &= \begin{bmatrix} -z_s^2 - \rho_s^2 s_{\theta_s}^2 & \rho_s^2 s_{\theta_s} c_{\theta_s} & z_s \rho_s c_{\theta_s} \\ \rho_s^2 s_{\theta_s} c_{\theta_s} & -z_s^2 - \rho_s^2 c_{\theta_s}^2 & z_s \rho_s s_{\theta_s} \\ z_s \rho_s c_{\theta_s} & z_s \rho_s s_{\theta_s} & -\rho_s^2 \end{bmatrix}, \\
\mathbf{r}_a^{\mathrm{d}m_1 g_1^\times} \mathbf{r}_a^{\mathrm{d}m_1 g_1^\times} &= \begin{bmatrix} -z_a^2 - \rho_a^2 s_{\theta_a}^2 & z_a \rho_a c_{\theta_a} & \rho_a^2 s_{\theta_a} c_{\theta_a} \\ z_a \rho_a c_{\theta_a} & -\rho_a^2 & z_a \rho_a s_{\theta_a} \\ \rho_a^2 s_{\theta_a} c_{\theta_a} & z_a \rho_a s_{\theta_a} & -z_a^2 - \rho_a^2 c_{\theta_a}^2 \end{bmatrix}, \\
\mathbf{r}_b^{\mathrm{d}m_2 g_2^\times} \mathbf{r}_b^{\mathrm{d}m_2 g_2^\times} &= \begin{bmatrix} -z_b^2 - \rho_b^2 s_{\theta_b}^2 & \rho_b^2 s_{\theta_b} c_{\theta_b} & z_b \rho_b c_{\theta_b} \\ \rho_b^2 s_{\theta_b} c_{\theta_b} & -z_b^2 - \rho_b^2 c_{\theta_b}^2 & z_b \rho_b s_{\theta_b} \\ z_b \rho_b c_{\theta_b} & z_b \rho_b s_{\theta_b} & -\rho_b^2 \end{bmatrix}, \\
\mathbf{r}_d^{\mathrm{d}m_d g_d^\times} \mathbf{r}_d^{\mathrm{d}m_d g_d^\times} &= \begin{bmatrix} -z_d^2 - y_d^2 & x_d y_d & x_d z_d \\ x_d y_d & -z_d^2 - x_d^2 & y_d z_d \\ x_d z_d & y_d z_d & -x_d^2 - y_d^2 \end{bmatrix}.
\end{aligned}$$

This allows to define the following mass matrices:

$$\mathbf{M}^{\mathcal{S}p} \triangleq \begin{bmatrix} m_s\mathbf{1} & -\mathbf{C}_{es}\mathbf{c}_s^{\mathcal{S}p^\times} \\ \mathbf{c}_s^{\mathcal{S}p^\times}\mathbf{C}_{es}^{\mathsf{T}} & \mathbf{J}_s^{\mathcal{S}p} \end{bmatrix}, \quad \mathbf{M}^{W1g_1} \triangleq \begin{bmatrix} m_1\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_a^{W1g_1} \end{bmatrix},$$

$$\mathbf{M}^{W2g_2} \triangleq \begin{bmatrix} m_2\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_b^{W2g_2} \end{bmatrix}, \quad \mathbf{M}^{\mathcal{D}g_d} \triangleq \begin{bmatrix} m_d\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_d^{\mathcal{D}g_d} \end{bmatrix},$$

$$\mathbf{M} \triangleq \mathrm{diag}\left\{\mathbf{M}^{\mathcal{S}p}, \mathbf{M}^{W1g_1}, \mathbf{M}^{W2g_2}, \mathbf{M}^{\mathcal{D}g_d}\right\}.$$

[1]In the Project Kinematics, the outer radius of the spacecraft was denoted by ρ_s. However, in order to prevent any confusion with the parameterization of $\mathbf{r}_s^{\mathrm{d}m_s p}$, ρ_o will be used instead.

IV. Control

The spacecraft has three linear actuators (P1, P2 and P3) and two rotary actuators (W1 and W2). The direction of the thrust produced by the propellers is assumed to remain perpendicular to the surface of the spacecraft, and directed towards it (no pull). Let p, p_2 and p_3 be the points where the actuators P1, P2 and P3 are fixed to the spacecraft, respectively. Under these assumptions,

$$\underset{\rightarrow}{f}^{p} = \begin{bmatrix} 0 & 0 & f^{P1} \end{bmatrix} \underset{\rightarrow}{\mathcal{F}}_{s},$$
$$\underset{\rightarrow}{f}^{p_2} = \begin{bmatrix} 0 & 0 & -f^{P2} \end{bmatrix} \underset{\rightarrow}{\mathcal{F}}_{s},$$
$$\underset{\rightarrow}{f}^{p_3} = \begin{bmatrix} 0 & 0 & -f^{P3} \end{bmatrix} \underset{\rightarrow}{\mathcal{F}}_{s}.$$

Additionally, let $\underset{\rightarrow}{\tau}^{W1\mathcal{S}}$ and $\underset{\rightarrow}{\tau}^{W2\mathcal{S}}$ be the torques applied on W1 and W2 by the actuators placed on the spacecraft wall. From Newton's third law, the torques applied on the wall by the reaction wheels, $\underset{\rightarrow}{\tau}^{\mathcal{S}W1}$ and $\underset{\rightarrow}{\tau}^{\mathcal{S}W2}$, are simply given by

$$\underset{\rightarrow}{\tau}^{\mathcal{S}W1} = -\underset{\rightarrow}{\tau}^{W1\mathcal{S}}, \quad \underset{\rightarrow}{\tau}^{\mathcal{S}W2} = -\underset{\rightarrow}{\tau}^{W2\mathcal{S}}.$$

Resolving these physical vectors in the body frames yields

$$\underset{\rightarrow}{\tau}^{W1\mathcal{S}} = \begin{bmatrix} 0 & \tau^{W1} & 0 \end{bmatrix} \underset{\rightarrow}{\mathcal{F}}_{a},$$
$$\underset{\rightarrow}{\tau}^{\mathcal{S}W1} = \begin{bmatrix} 0 & -\tau^{W1} & 0 \end{bmatrix} \underset{\rightarrow}{\mathcal{F}}_{s},$$
$$\underset{\rightarrow}{\tau}^{W2\mathcal{S}} = \begin{bmatrix} 0 & 0 & \tau^{W2} \end{bmatrix} \underset{\rightarrow}{\mathcal{F}}_{b},$$
$$\underset{\rightarrow}{\tau}^{\mathcal{S}W2} = \begin{bmatrix} 0 & 0 & -\tau^{W2} \end{bmatrix} \underset{\rightarrow}{\mathcal{F}}_{s}.$$

Therefore, the behavior of the system only depends on the initial configuration and the following controllable (time-dependent) quantity:

$$\mathfrak{f} \triangleq \begin{bmatrix} f^{P1} & f^{P2} & f^{P3} & \tau^{W1} & \tau^{W2} \end{bmatrix}^{\mathsf{T}}.$$

V. Generalized Coordinates

The chosen set of generalized coordinates $\mathbf{q}$ is given by

$$\mathbf{q} \triangleq \begin{bmatrix} \mathbf{q}^{\mathcal{S}} \\ \mathbf{q}^{W1} \\ \mathbf{q}^{W2} \\ \mathbf{q}^{\mathcal{D}} \end{bmatrix} \triangleq \begin{bmatrix} \mathbf{r}_e^{po} \\ \mathbf{q}^{se} \\ \mathbf{r}_e^{g_1 o} \\ \mathbf{q}^{ae} \\ \mathbf{r}_e^{g_2 o} \\ \mathbf{q}^{be} \\ \mathbf{r}_e^{g_d o} \\ \mathbf{q}^{de} \end{bmatrix},$$

where

$$\mathbf{q}^{se} \triangleq \begin{bmatrix} \mathbf{s}_e^1 \\ \mathbf{s}_e^2 \\ \mathbf{s}_e^3 \end{bmatrix}, \quad \mathbf{q}^{ae} \triangleq \begin{bmatrix} \mathbf{a}_e^1 \\ \mathbf{a}_e^2 \\ \mathbf{a}_e^3 \end{bmatrix},$$

$$\mathbf{q}^{be} \triangleq \begin{bmatrix} \mathbf{b}_e^1 \\ \mathbf{b}_e^2 \\ \mathbf{b}_e^3 \end{bmatrix}, \quad \mathbf{q}^{de} \triangleq \begin{bmatrix} \mathbf{d}_e^1 \\ \mathbf{d}_e^2 \\ \mathbf{d}_e^3 \end{bmatrix}.$$

In addition, the selected augmented velocities and reduced augmented velocities of the system are given by

$$\boldsymbol{\nu} \triangleq \begin{bmatrix} \mathbf{v}_e^{po/e} \\ \boldsymbol{\omega}_s^{se} \\ \mathbf{v}_e^{g_1 o/e} \\ \boldsymbol{\omega}_a^{ae} \\ \mathbf{v}_e^{g_2 o/e} \\ \boldsymbol{\omega}_b^{be} \\ \mathbf{v}_e^{g_d o/e} \\ \boldsymbol{\omega}_d^{de} \end{bmatrix}, \quad \hat{\boldsymbol{\nu}} \triangleq \begin{bmatrix} \mathbf{v}_e^{po/e} \\ \boldsymbol{\omega}_s^{se} \\ \dot{\alpha} \\ \dot{\beta} \\ \mathbf{v}_e^{g_d o/e} \\ \boldsymbol{\omega}_d^{de} \end{bmatrix}.$$

The relation between $\dot{\mathbf{q}}$ and $\boldsymbol{\nu}$ is given by

$$\dot{\mathbf{q}} = \boldsymbol{\Gamma}\boldsymbol{\nu},$$

where

$$\boldsymbol{\Gamma} \triangleq \operatorname{diag}\left\{\mathbf{1}, \boldsymbol{\Gamma}_s^{se}, \mathbf{1}, \boldsymbol{\Gamma}_a^{ae}, \mathbf{1}, \boldsymbol{\Gamma}_b^{be}, \mathbf{1}, \boldsymbol{\Gamma}_d^{de}\right\},$$

and the relation between $\boldsymbol{\nu}$ and $\hat{\boldsymbol{\nu}}$ is

$$\boldsymbol{\nu} = \boldsymbol{\Pi}\hat{\boldsymbol{\nu}},$$

where

$$\boldsymbol{\Pi} \triangleq \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \boldsymbol{\Pi}_{3,2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Pi}_{4,2} & \boldsymbol{\Pi}_{4,3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \boldsymbol{\Pi}_{5,2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Pi}_{6,2} & \mathbf{0} & \boldsymbol{\Pi}_{6,4} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix},$$

$$\boldsymbol{\Pi}_{3,2} \triangleq -\mathbf{C}_{es}\mathbf{r}_s^{g_1 p \times}, \qquad \boldsymbol{\Pi}_{4,2} \triangleq \mathbf{C}_{as},$$
$$\boldsymbol{\Pi}_{4,3} \triangleq \mathbf{S}_a^{as}, \qquad \boldsymbol{\Pi}_{5,2} \triangleq -\mathbf{C}_{es}\mathbf{r}_s^{g_2 p \times},$$
$$\boldsymbol{\Pi}_{6,2} \triangleq \mathbf{C}_{bs}, \qquad \boldsymbol{\Pi}_{6,4} \triangleq \mathbf{S}_b^{bs}.$$

VI. Constraints

There are four kinematic constraints, two collocation constraints and two attitude constraints. The kinematic constraints, related to the DCMs, can be expressed as follows [1]:

$$\boldsymbol{\Phi}_{kin}(\mathbf{q}) \triangleq \begin{bmatrix} \boldsymbol{\Phi}_{se}(\mathbf{q}^{se}) \\ \boldsymbol{\Phi}_{ae}(\mathbf{q}^{ae}) \\ \boldsymbol{\Phi}_{be}(\mathbf{q}^{be}) \\ \boldsymbol{\Phi}_{de}(\mathbf{q}^{de}) \end{bmatrix} \stackrel{!}{=} \mathbf{0}.$$

where

$$\boldsymbol{\Phi}_{se}(\mathbf{q}^{se}) \triangleq \begin{bmatrix} \mathbf{s}_e^{1\mathsf{T}}\mathbf{s}_e^1 - 1 \\ \mathbf{s}_e^{2\mathsf{T}}\mathbf{s}_e^2 - 1 \\ \mathbf{s}_e^{2\mathsf{T}}\mathbf{s}_e^1 \\ \mathbf{s}_e^{1\times}\mathbf{s}_e^2 - \mathbf{s}_e^3 \end{bmatrix}, \quad \boldsymbol{\Phi}_{ae}(\mathbf{q}^{ae}) \triangleq \begin{bmatrix} \mathbf{a}_e^{1\mathsf{T}}\mathbf{a}_e^1 - 1 \\ \mathbf{a}_e^{2\mathsf{T}}\mathbf{a}_e^2 - 1 \\ \mathbf{a}_e^{2\mathsf{T}}\mathbf{a}_e^1 \\ \mathbf{a}_e^{1\times}\mathbf{a}_e^2 - \mathbf{s}_e^3 \end{bmatrix},$$

$$\boldsymbol{\Phi}_{be}(\mathbf{q}^{be}) \triangleq \begin{bmatrix} \mathbf{b}_e^{1\mathsf{T}}\mathbf{b}_e^1 - 1 \\ \mathbf{b}_e^{2\mathsf{T}}\mathbf{b}_e^2 - 1 \\ \mathbf{b}_e^{2\mathsf{T}}\mathbf{b}_e^1 \\ \mathbf{b}_e^{1\times}\mathbf{b}_e^2 - \mathbf{s}_e^3 \end{bmatrix}, \quad \boldsymbol{\Phi}_{de}(\mathbf{q}^{de}) \triangleq \begin{bmatrix} \mathbf{d}_e^{1\mathsf{T}}\mathbf{d}_e^1 - 1 \\ \mathbf{d}_e^{2\mathsf{T}}\mathbf{d}_e^2 - 1 \\ \mathbf{d}_e^{2\mathsf{T}}\mathbf{d}_e^1 \\ \mathbf{d}_e^{1\times}\mathbf{d}_e^2 - \mathbf{d}_s^3 \end{bmatrix}.$$

Additionally, the Pfaffian form of the kinematic constraints is given by

$$\boldsymbol{\Xi}^{kin}\dot{\mathbf{q}} = \mathbf{0},$$

where

$$\Xi^{kin} \triangleq \begin{bmatrix} \mathbf{0} & \Xi_s^{se} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \Xi_a^{ae} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \Xi_b^{be} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \Xi_d^{de} \end{bmatrix},$$

$$\Xi_s^{se} \triangleq \begin{bmatrix} 2\mathbf{s}_e^{1\mathsf{T}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\mathbf{s}_e^{2\mathsf{T}} & \mathbf{0} \\ \mathbf{s}_e^{2\mathsf{T}} & \mathbf{s}_e^{1\mathsf{T}} & \mathbf{0} \\ -\mathbf{s}_e^{2\times} & \mathbf{s}_e^{1\times} & -\mathbf{1} \end{bmatrix}, \quad \Xi_a^{ae} \triangleq \begin{bmatrix} 2\mathbf{a}_e^{1\mathsf{T}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\mathbf{a}_e^{2\mathsf{T}} & \mathbf{0} \\ \mathbf{a}_e^{2\mathsf{T}} & \mathbf{a}_e^{1\mathsf{T}} & \mathbf{0} \\ -\mathbf{a}_e^{2\times} & \mathbf{a}_e^{1\times} & -\mathbf{1} \end{bmatrix},$$

$$\Xi_b^{be} \triangleq \begin{bmatrix} 2\mathbf{b}_e^{1\mathsf{T}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\mathbf{b}_e^{2\mathsf{T}} & \mathbf{0} \\ \mathbf{b}_e^{2\mathsf{T}} & \mathbf{b}_e^{1\mathsf{T}} & \mathbf{0} \\ -\mathbf{b}_e^{2\times} & \mathbf{b}_e^{1\times} & -\mathbf{1} \end{bmatrix}, \quad \Xi_d^{de} \triangleq \begin{bmatrix} 2\mathbf{d}_e^{1\mathsf{T}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\mathbf{d}_e^{2\mathsf{T}} & \mathbf{0} \\ \mathbf{d}_e^{2\mathsf{T}} & \mathbf{d}_e^{1\mathsf{T}} & \mathbf{0} \\ -\mathbf{d}_e^{2\times} & \mathbf{d}_e^{1\times} & -\mathbf{1} \end{bmatrix}.$$

The two collocation constraints are the following:

$$\underset{\rightarrow}{r}^{g_1 o} \overset{!}{=} \underset{\rightarrow}{r}^{g_1 p} + \underset{\rightarrow}{r}^{po},$$
$$\underset{\rightarrow}{r}^{g_2 o} \overset{!}{=} \underset{\rightarrow}{r}^{g_2 p} + \underset{\rightarrow}{r}^{po}.$$

Taking the time derivative w.r.t. $\mathcal{F}_e$ and using the Transport theorem, one can obtain directly the Pfaffian form of the collocation constraints:

$$\Xi^{col}\dot{\mathbf{q}} = \mathbf{0},$$

where

$$\Xi^{col} \triangleq \begin{bmatrix} \mathbf{1} & \Xi_{1,2}^{col} & -\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \Xi_{2,2}^{col} & \mathbf{0} & \mathbf{0} & -\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\Xi_{1,2}^{col} \triangleq -\mathbf{C}_{es}\mathbf{r}_s^{g_1 p\times}\mathbf{S}_s^{se}, \qquad \Xi_{2,2}^{col} \triangleq -\mathbf{C}_{es}\mathbf{r}_s^{g_2 p\times}\mathbf{S}_s^{se}.$$

Additionally, the attitude constraints are stated as follows:

$$\omega_{a1}^{as} = \omega_{a3}^{as} \overset{!}{=} 0,$$

$$\omega_{b1}^{bs} = \omega_{b2}^{bs} \overset{!}{=} 0,$$

and the corresponding Pfaffian form is

$$\Xi^{att}\dot{\mathbf{q}} = \mathbf{0},$$

where

$$\Xi^{att} \triangleq \begin{bmatrix} \mathbf{0} & \Xi_{1,2}^{att} & \mathbf{0} & \Xi_{1,4}^{att} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Xi_{2,2}^{att} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \Xi_{2,6}^{att} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\Xi_{1,2}^{att} \triangleq -\begin{bmatrix} \mathbf{1}_1^{\mathsf{T}} \\ \mathbf{1}_3^{\mathsf{T}} \end{bmatrix}\mathbf{C}_{as}\mathbf{S}_s^{se}, \qquad \Xi_{1,4}^{att} \triangleq \begin{bmatrix} \mathbf{1}_1^{\mathsf{T}} \\ \mathbf{1}_3^{\mathsf{T}} \end{bmatrix}\mathbf{S}_a^{ae},$$
$$\Xi_{2,2}^{att} \triangleq -\begin{bmatrix} \mathbf{1}_1^{\mathsf{T}} \\ \mathbf{1}_2^{\mathsf{T}} \end{bmatrix}\mathbf{C}_{bs}\mathbf{S}_s^{se}, \qquad \Xi_{2,6}^{att} \triangleq \begin{bmatrix} \mathbf{1}_1^{\mathsf{T}} \\ \mathbf{1}_2^{\mathsf{T}} \end{bmatrix}\mathbf{S}_b^{be}.$$

Finally, all the constraints are arranged in a matrix format:

$$\Xi \triangleq \begin{bmatrix} \Xi^{kin} \\ \Xi^{col} \\ \Xi^{att} \end{bmatrix}.$$

VII. Generalized Forces and Moments

The generalized forces and moments can be written as follows:

$$\boldsymbol{f} \triangleq \begin{bmatrix} \boldsymbol{f}_s \\ \boldsymbol{f}_a \\ \boldsymbol{f}_b \\ \boldsymbol{f}_d \end{bmatrix}.$$

In order to find $\boldsymbol{f}_s$, one must first find the virtual work done on $\mathcal{S}$ by the external torques:

$$\delta W_{\mathcal{S}}^{\tau} = \underset{\rightarrow}{\tau}^{\mathcal{S}W1} \cdot \delta \underset{\rightarrow}{\gamma}^{\mathcal{S}} + \underset{\rightarrow}{\tau}^{\mathcal{S}W2} \cdot \delta \underset{\rightarrow}{\gamma}^{\mathcal{S}},$$

where

$$\delta \underset{\rightarrow}{\gamma}^{\mathcal{S}} = \begin{bmatrix} \delta\gamma_{e1}^{\mathcal{S}} & \delta\gamma_{e2}^{\mathcal{S}} & \delta\gamma_{e3}^{\mathcal{S}} \end{bmatrix} \underset{\rightarrow}{\mathcal{F}}_e$$

is a virtual angular displacement of $\mathcal{S}$. A simple graphical analysis yields

$$\delta\gamma_{e1}^{\mathcal{S}} = \mathbf{s}_e^{3\mathsf{T}}\delta\mathbf{s}_e^2, \quad \delta\gamma_{e2}^{\mathcal{S}} = \mathbf{s}_e^{1\mathsf{T}}\delta\mathbf{s}_e^3, \quad \delta\gamma_{e1}^{\mathcal{S}} = \mathbf{s}_e^{2\mathsf{T}}\delta\mathbf{s}_e^1,$$

and the virtual work $\delta W_{\mathcal{S}}^{\tau}$ becomes

$$\delta W_{\mathcal{S}}^{\tau} = \boldsymbol{f}_s^{\tau\mathsf{T}}\delta\mathbf{q}^{\mathcal{S}}, \quad \boldsymbol{f}_s^{\tau} = \begin{bmatrix} \mathbf{0} \\ -\tau^{W2}\mathbf{s}_e^2 \\ \mathbf{0} \\ -\tau^{W1}\mathbf{s}_e^1 \end{bmatrix}.$$

On the other hand, the generalized forces and moments due to the forces are given by

$$\boldsymbol{f}_s^{f\mathsf{T}} = \underset{\rightarrow}{f}^{p} \cdot \frac{\partial \underset{\rightarrow}{r}^{po}}{\partial \mathbf{q}^{\mathcal{S}}} + \underset{\rightarrow}{f}^{p_2} \cdot \frac{\partial \underset{\rightarrow}{r}^{p_2 o}}{\partial \mathbf{q}^{\mathcal{S}}} + \underset{\rightarrow}{f}^{p_3} \cdot \frac{\partial \underset{\rightarrow}{r}^{p_3 o}}{\partial \mathbf{q}^{\mathcal{S}}}.$$

Resolving every physical vector in the $\mathcal{F}_e$ frame yields

$$\mathbf{f}_e^{p} = \mathbf{s}_e^3 f^{P1}, \quad \mathbf{f}_e^{p_2} = -\mathbf{s}_e^3 f^{P2}, \quad \mathbf{f}_e^{p_3} = -\mathbf{s}_e^3 f^{P3},$$
$$\mathbf{r}_e^{p_2 o} = \mathbf{r}_e^{po} - \rho_p\mathbf{s}_e^2 + l_s\mathbf{s}_e^3, \quad \mathbf{r}_e^{p_3 o} = \mathbf{r}_e^{po} + \rho_p\mathbf{s}_e^2 + l_s\mathbf{s}_e^3,$$

and $\boldsymbol{f}_s^f$ becomes

$$\boldsymbol{f}_s^f = \begin{bmatrix} (f^{P1} - f^{P2} - f^{P3})\mathbf{s}_e^3 \\ \mathbf{0} \\ \rho_s(f^{P2} - f^{P3})\mathbf{s}_e^3 \\ -l_s(f^{P2} + f^{P3})\mathbf{s}_e^3 \end{bmatrix}.$$

Therefore, $\boldsymbol{f}_s$ can be written as follows:

$$\boldsymbol{f}_s = \boldsymbol{f}_s^f + \boldsymbol{f}_s^{\tau} = \mathbf{B}_s\mathfrak{f},$$

$$\mathbf{B}_s \triangleq \begin{bmatrix} \mathbf{s}_e^3 & -\mathbf{s}_e^3 & -\mathbf{s}_e^3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{s}_e^2 \\ \mathbf{0} & \rho_s\mathbf{s}_e^3 & -\rho_s\mathbf{s}_e^3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -l_s\mathbf{s}_e^3 & -l_s\mathbf{s}_e^3 & -\mathbf{s}_e^1 & \mathbf{0} \end{bmatrix}.$$

With a similar analysis,

$$\boldsymbol{f}_a = \mathbf{B}_a\mathfrak{f}, \quad \boldsymbol{f}_b = \mathbf{B}_b\mathfrak{f}, \quad \boldsymbol{f}_d = \mathbf{0},$$

where

$$\mathbf{B}_a \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{a}_e^1 & \mathbf{0} \end{bmatrix}, \quad \mathbf{B}_b \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{b}_e^2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Lastly, the general forces and moments of the whole system can be written concisely as

$$\boldsymbol{f} = \mathbf{B}\mathfrak{f}, \quad \mathbf{B} \triangleq \begin{bmatrix} \mathbf{B}_s \\ \mathbf{B}_a \\ \mathbf{B}_b \\ \mathbf{0} \end{bmatrix}.$$

VIII. Equations of motion

Since potential energies are neglected, one can simply compute the Lagrangian of the system as follows:

$$L_{\mathcal{A}o/e} = T_{\mathcal{A}o/e} = \tfrac{1}{2}\boldsymbol{\nu}^{\mathsf{T}}\mathbf{M}\boldsymbol{\nu}. \tag{1}$$

Additionally, the general form of the Lagrange's Equation is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{\mathcal{A}o/e}}{\partial \dot{\mathbf{q}}}\right)^{\mathsf{T}} - \left(\frac{\partial L_{\mathcal{A}o/e}}{\partial \mathbf{q}}\right)^{\mathsf{T}} = \boldsymbol{f} + \boldsymbol{\Xi}^{\mathsf{T}}\boldsymbol{\lambda}. \tag{2}$$

Substituting (1) into (2) and using the the derivation shown in [1], [2], the equation of motion can be written as

$$\mathbf{S}^{\mathsf{T}}\mathbf{M}\dot{\boldsymbol{\nu}} + \mathbf{S}^{\mathsf{T}}\dot{\mathbf{M}}\boldsymbol{\nu} + \dot{\mathbf{S}}^{\mathsf{T}}\mathbf{M}\boldsymbol{\nu} - \boldsymbol{\Delta}^{\mathsf{T}}\mathbf{M}\boldsymbol{\nu} - \mathbf{a}_{non} = \mathbf{B}\mathfrak{f} + \boldsymbol{\Xi}^{\mathsf{T}}\boldsymbol{\lambda} \tag{3}$$

where

$$\mathbf{S} \triangleq \operatorname{diag}\left\{\mathbf{S}_s, \mathbf{S}_a, \mathbf{S}_b\mathbf{S}_d\right\},$$

$$\mathbf{S}_s \triangleq \operatorname{diag}\left\{\mathbf{1}, \mathbf{S}_s^{se}\right\}, \qquad \mathbf{S}_a \triangleq \operatorname{diag}\left\{\mathbf{1}, \mathbf{S}_a^{ae}\right\},$$
$$\mathbf{S}_b \triangleq \operatorname{diag}\left\{\mathbf{1}, \mathbf{S}_b^{be}\right\}, \qquad \mathbf{S}_d \triangleq \operatorname{diag}\left\{\mathbf{1}, \mathbf{S}_d^{de}\right\},$$

$$\boldsymbol{\Delta} \triangleq \operatorname{diag}\left\{\boldsymbol{\Delta}_s, \boldsymbol{\Delta}_a, \boldsymbol{\Delta}_b, \boldsymbol{\Delta}_d\right\},$$

$$\boldsymbol{\Delta}_s \triangleq \operatorname{diag}\left\{\mathbf{0}, \frac{\partial \boldsymbol{\omega}_s^{se}}{\partial \mathbf{q}^{se}}\right\}, \qquad \boldsymbol{\Delta}_a \triangleq \operatorname{diag}\left\{\mathbf{0}, \frac{\partial \boldsymbol{\omega}_a^{ae}}{\partial \mathbf{q}^{ae}}\right\},$$
$$\boldsymbol{\Delta}_b \triangleq \operatorname{diag}\left\{\mathbf{0}, \frac{\partial \boldsymbol{\omega}_b^{be}}{\partial \mathbf{q}^{be}}\right\}, \qquad \boldsymbol{\Delta}_d \triangleq \operatorname{diag}\left\{\mathbf{0}, \frac{\partial \boldsymbol{\omega}_d^{de}}{\partial \mathbf{q}^{de}}\right\},$$

$$\mathbf{a}_{non} \triangleq \begin{bmatrix} \mathbf{a}_{non,s}^{\mathsf{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}^{\mathsf{T}},$$

$$\mathbf{a}_{non,s} \triangleq \begin{bmatrix} \mathbf{0} \\ -\frac{\partial(\mathbf{C}_{se}\mathbf{v}_e^{po/e})^{\mathsf{T}}}{\partial \mathbf{q}^{se}}\mathbf{c}_s^{\mathcal{S}p\times}\boldsymbol{\omega}_s^{se} \end{bmatrix}.$$

IX. Null Space method

Premultiplying (3) on the left by $\boldsymbol{\Pi}^{\mathsf{T}}\boldsymbol{\Gamma}^{\mathsf{T}}$ and using the following identities:

$$\boldsymbol{\Gamma}^{\mathsf{T}}\mathbf{S}^{\mathsf{T}} = \mathbf{1}, \qquad \boldsymbol{\Pi}^{\mathsf{T}}\boldsymbol{\Gamma}^{\mathsf{T}}\boldsymbol{\Xi}^{\mathsf{T}} = \mathbf{0},$$
$$\boldsymbol{\nu} = \boldsymbol{\Pi}\hat{\boldsymbol{\nu}}, \qquad \boldsymbol{\Gamma}^{\mathsf{T}}\left(\mathbf{S}^{\mathsf{T}} - \boldsymbol{\Delta}^{\mathsf{T}}\right) = \boldsymbol{\Omega},$$
$$\boldsymbol{\Omega} \triangleq \operatorname{diag}\left\{\mathbf{0}, \boldsymbol{\omega}_s^{se\times}, \mathbf{0}, \boldsymbol{\omega}_a^{ae\times}, \mathbf{0}, \boldsymbol{\omega}_b^{be\times}, \mathbf{0}, \boldsymbol{\omega}_d^{de\times}\right\},$$

one can rewrite the equations of motion concisely as

$$\hat{\mathbf{M}}\dot{\hat{\boldsymbol{\nu}}} = \hat{\boldsymbol{f}}_{non} + \hat{\boldsymbol{f}}, \tag{4}$$

where

$$\hat{\mathbf{M}} \triangleq \boldsymbol{\Pi}^{\mathsf{T}}\mathbf{M}\boldsymbol{\Pi},$$
$$\hat{\boldsymbol{f}}_{non} \triangleq \left(-\boldsymbol{\Pi}^{\mathsf{T}}\mathbf{M}\dot{\boldsymbol{\Pi}} - \boldsymbol{\Pi}^{\mathsf{T}}\dot{\mathbf{M}}\boldsymbol{\Pi} - \boldsymbol{\Pi}^{\mathsf{T}}\boldsymbol{\Omega}\mathbf{M}\boldsymbol{\Pi}\right)\hat{\boldsymbol{\nu}} + \boldsymbol{\Pi}^{\mathsf{T}}\boldsymbol{\Gamma}^{\mathsf{T}}\mathbf{a}_{non},$$
$$\hat{\boldsymbol{f}} \triangleq \boldsymbol{\Pi}^{\mathsf{T}}\boldsymbol{\Gamma}^{\mathsf{T}}\mathbf{B}\mathfrak{f}.$$

Lastly, additional simplifications can be made using

$$\dot{\mathbf{M}} = \operatorname{diag}\{\dot{\mathbf{M}}^{\mathcal{S}p}, \mathbf{0}, \mathbf{0}, \mathbf{0}\},$$

$$\dot{\mathbf{M}}^{\mathcal{S}p} = \begin{bmatrix} \mathbf{0} & -\mathbf{C}_{es}\boldsymbol{\omega}_s^{se\times}\mathbf{c}_s^{\mathcal{S}p\times} \\ -\mathbf{c}_s^{\mathcal{S}p\times}\boldsymbol{\omega}_s^{se\times}\mathbf{C}_{es}^{\mathsf{T}} & \mathbf{0} \end{bmatrix},$$

$$\hat{\mathbf{a}}_{non,s} \triangleq \operatorname{diag}\{\mathbf{1}, \boldsymbol{\Gamma}_s^{se\mathsf{T}}\}\mathbf{a}_{non,s} = \begin{bmatrix} \mathbf{0} \\ -\left(\mathbf{C}_{es}^{\mathsf{T}}\mathbf{v}_e^{po/e}\right)^{\times}\mathbf{c}_s^{\mathcal{S}p\times}\boldsymbol{\omega}_s^{se} \end{bmatrix},$$

and

$$\dot{\boldsymbol{\Pi}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dot{\boldsymbol{\Pi}}_{3,2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dot{\boldsymbol{\Pi}}_{4,2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dot{\boldsymbol{\Pi}}_{5,2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dot{\boldsymbol{\Pi}}_{6,2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

where

$$\dot{\boldsymbol{\Pi}}_{3,2} = -\mathbf{C}_{es}\boldsymbol{\omega}_s^{se\times}\mathbf{r}_s^{g_1p\times}, \qquad \dot{\boldsymbol{\Pi}}_{4,2} = -\boldsymbol{\omega}_a^{as\times}\mathbf{C}_{as},$$
$$\dot{\boldsymbol{\Pi}}_{5,2} = -\mathbf{C}_{es}\boldsymbol{\omega}_s^{se\times}\mathbf{r}_s^{g_2p\times}, \qquad \dot{\boldsymbol{\Pi}}_{6,2} = -\boldsymbol{\omega}_b^{bs\times}\mathbf{C}_{bs}.$$

X. Nummerical Integration

In order to integrate (4), one can use MATLAB and the *ode45* built-in function. This function allows to integrate a system of first order differential equations defined as follows:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t)). \tag{5}$$

For this problem, $\mathbf{x}(t)$ and $\mathbf{f}(t, \mathbf{x}(t))$ are given by

$$\mathbf{x}(t) \triangleq \begin{bmatrix} \mathbf{q}(t) \\ \hat{\boldsymbol{\nu}}(t) \end{bmatrix} \quad \text{and} \quad \mathbf{f}(t, \mathbf{x}(t)) \triangleq \begin{bmatrix} \boldsymbol{\Gamma}(\mathbf{q})\boldsymbol{\Pi}(\mathbf{q})\hat{\boldsymbol{\nu}} \\ \hat{\mathbf{M}}^{-1}\left(\hat{\boldsymbol{f}}_{non}(\mathbf{q}, \dot{\mathbf{q}}) + \hat{\boldsymbol{f}}(\mathfrak{f})\right) \end{bmatrix}.$$

Note: $\dot{\mathbf{q}} = \boldsymbol{\Gamma}\boldsymbol{\Pi}\hat{\boldsymbol{\nu}}$.

XI. Initial Configuration

Let $\mathbf{x}_0$ denote the initial conditions of (5). Since the components of $\mathbf{x}_0$ are dependent, it is important to make sure they are compatible. In order to do so, one must express the augmented state of the system $\mathbf{x}$ in function of the measurables[2] $\bar{\mathbf{q}}$

[2]The measurables are also not independent. They will of course be compatible if the are truly measured but in the case of a simulation, one must make sure that $\mathbf{s}_e^i$ and $\mathbf{d}_s^i$, $i = 1, 2, 3$, are indeed the columns of DCMs ($\mathbf{C}_{es}$ and $\mathbf{C}_{sd}$, respectively).

introduced in the Project Kinematics, i.e. $\mathbf{x} = \boldsymbol{\Sigma}(\bar{\mathbf{q}})$ where

$$\boldsymbol{\Sigma}(\bar{\mathbf{q}}) \triangleq \begin{bmatrix} \mathbf{r}_e^{po} \\ \mathbf{q}^{se} \\ \mathbf{r}_e^{po} + \mathbf{C}_{es}\mathbf{r}_s^{g_1 p} \\ \mathbf{C}_{es}\mathbf{C}_2^{\mathsf{T}}(\alpha)\mathbf{1}_1 \\ \mathbf{C}_{es}\mathbf{C}_2^{\mathsf{T}}(\alpha)\mathbf{1}_2 \\ \mathbf{C}_{es}\mathbf{C}_2^{\mathsf{T}}(\alpha)\mathbf{1}_3 \\ \mathbf{r}_e^{po} + \mathbf{C}_{es}\mathbf{r}_s^{g_2 p} \\ \mathbf{C}_{es}\mathbf{C}_3^{\mathsf{T}}(\beta)\mathbf{1}_1 \\ \mathbf{C}_{es}\mathbf{C}_3^{\mathsf{T}}(\beta)\mathbf{1}_2 \\ \mathbf{C}_{es}\mathbf{C}_3^{\mathsf{T}}(\beta)\mathbf{1}_3 \\ \mathbf{r}_e^{g_d o} \\ \mathbf{C}_{es}\mathbf{C}_{sd}\mathbf{1}_1 \\ \mathbf{C}_{es}\mathbf{C}_{sd}\mathbf{1}_2 \\ \mathbf{C}_{es}\mathbf{C}_{sd}\mathbf{1}_3 \\ \mathbf{v}_e^{po/e} \\ \boldsymbol{\omega}_s^{se} \\ \dot{\alpha} \\ \dot{\beta} \\ \mathbf{v}_e^{g_d o} \\ \boldsymbol{\omega}_d^{ds} + \mathbf{C}_{sd}^{\mathsf{T}}\boldsymbol{\omega}_s^{se} \end{bmatrix}.$$

Lastly, $\mathbf{x}_0 = \boldsymbol{\Sigma}(\bar{\mathbf{q}}_0)$ where $\bar{\mathbf{q}}_0 = \bar{\mathbf{q}}\big|_{t=0}$.

XII. Validation

In order to verify that the equations of motion are correctly derived, one can simulate the system with no external force and make sure that the laws of conservation are satisfied within the numerical accuracy of the integration. In particular, Figure 1 shows that the conservation of energy of the system is verified for a particular set of non-trivial initial conditions, which is a good indicator of an accurate dynamic analysis.

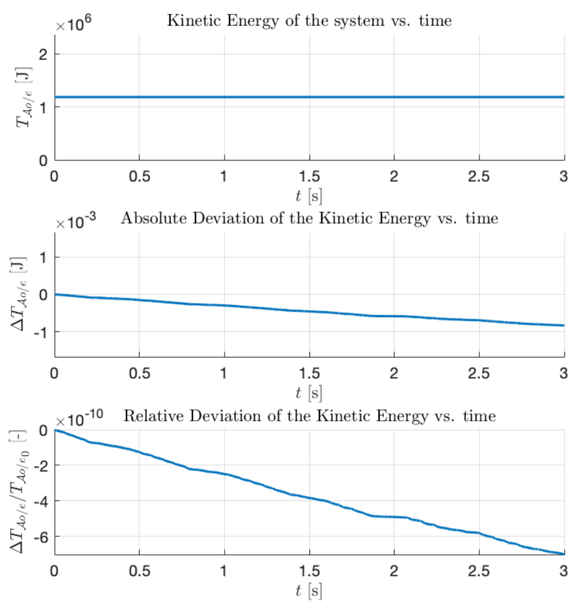

Fig. 1. Kinetic Energy of the system over time with non-trivial initial conditions and no external forces.

Appendix
Complement to the Kinematic Analysis

In this dynamics analysis, some DCMs that have not been properly introduced in the Project Kinematics were used. In particular:

$$\mathbf{C}_{es} = \begin{bmatrix} \mathbf{s}_e^1 & \mathbf{s}_e^2 & \mathbf{s}_e^3 \end{bmatrix}, \qquad \mathbf{C}_{ea} = \begin{bmatrix} \mathbf{a}_e^1 & \mathbf{a}_e^2 & \mathbf{a}_e^3 \end{bmatrix},$$
$$\mathbf{C}_{eb} = \begin{bmatrix} \mathbf{b}_e^1 & \mathbf{b}_e^2 & \mathbf{b}_e^3 \end{bmatrix}, \qquad \mathbf{C}_{ed} = \begin{bmatrix} \mathbf{d}_e^1 & \mathbf{d}_e^2 & \mathbf{d}_e^3 \end{bmatrix}.$$

In addition, to complete the kinematic analysis, one must find the relation between the angular velocities and the parameters (or simply the components) of the DCMs. These relations are given by [1, p. 4]:

$$\boldsymbol{\omega}_s^{se} = \mathbf{S}_s^{se}\begin{bmatrix} \dot{\mathbf{s}}_e^1 \\ \dot{\mathbf{s}}_e^2 \\ \dot{\mathbf{s}}_e^3 \end{bmatrix}, \qquad \mathbf{S}_s^{se} \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{s}_e^{3\mathsf{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{s}_e^{1\mathsf{T}} \\ \mathbf{s}_e^{2\mathsf{T}} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$
$$\boldsymbol{\omega}_a^{ae} = \mathbf{S}_a^{ae}\begin{bmatrix} \dot{\mathbf{a}}_e^1 \\ \dot{\mathbf{a}}_e^2 \\ \dot{\mathbf{a}}_e^3 \end{bmatrix}, \qquad \mathbf{S}_a^{ae} \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{a}_e^{3\mathsf{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{a}_e^{1\mathsf{T}} \\ \mathbf{a}_e^{2\mathsf{T}} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$
$$\boldsymbol{\omega}_b^{be} = \mathbf{S}_b^{be}\begin{bmatrix} \dot{\mathbf{b}}_e^1 \\ \dot{\mathbf{b}}_e^2 \\ \dot{\mathbf{b}}_e^3 \end{bmatrix}, \qquad \mathbf{S}_b^{be} \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{b}_e^{3\mathsf{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{b}_e^{1\mathsf{T}} \\ \mathbf{b}_e^{2\mathsf{T}} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$
$$\boldsymbol{\omega}_d^{de} = \mathbf{S}_d^{de}\begin{bmatrix} \dot{\mathbf{d}}_e^1 \\ \dot{\mathbf{d}}_e^2 \\ \dot{\mathbf{d}}_e^3 \end{bmatrix}, \qquad \mathbf{S}_d^{de} c \begin{bmatrix} \mathbf{0} & \mathbf{d}_e^{3\mathsf{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{d}_e^{1\mathsf{T}} \\ \mathbf{d}_e^{2\mathsf{T}} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$
$$\boldsymbol{\omega}_a^{as} = \mathbf{S}_a^{as}\dot{\alpha}, \qquad \mathbf{S}_a^{as} \triangleq \mathbf{1}_2,$$
$$\boldsymbol{\omega}_b^{bs} = \mathbf{S}_b^{bs}\dot{\beta}, \qquad \mathbf{S}_b^{bs} \triangleq \mathbf{1}_3.$$

And the inverse relations are [1, p. 6]:

$$\begin{bmatrix} \dot{\mathbf{s}}_e^1 \\ \dot{\mathbf{s}}_e^2 \\ \dot{\mathbf{s}}_e^3 \end{bmatrix} = \boldsymbol{\Gamma}_s^{se}\boldsymbol{\omega}_s^{se}, \qquad \boldsymbol{\Gamma}_s^{se} \triangleq \begin{bmatrix} \mathbf{0} & -\mathbf{s}_e^3 & \mathbf{s}_e^2 \\ \mathbf{s}_e^3 & \mathbf{0} & -\mathbf{s}_e^1 \\ -\mathbf{s}_e^2 & \mathbf{s}_e^1 & \mathbf{0} \end{bmatrix},$$
$$\begin{bmatrix} \dot{\mathbf{a}}_e^1 \\ \dot{\mathbf{a}}_e^2 \\ \dot{\mathbf{a}}_e^3 \end{bmatrix} = \boldsymbol{\Gamma}_a^{ae}\boldsymbol{\omega}_a^{ae}, \qquad \boldsymbol{\Gamma}_a^{ae} \triangleq \begin{bmatrix} \mathbf{0} & -\mathbf{a}_e^3 & \mathbf{a}_e^2 \\ \mathbf{a}_e^3 & \mathbf{0} & -\mathbf{a}_e^1 \\ -\mathbf{a}_e^2 & \mathbf{a}_e^1 & \mathbf{0} \end{bmatrix},$$
$$\begin{bmatrix} \dot{\mathbf{b}}_e^1 \\ \dot{\mathbf{b}}_e^2 \\ \dot{\mathbf{b}}_e^3 \end{bmatrix} = \boldsymbol{\Gamma}_b^{be}\boldsymbol{\omega}_b^{be}, \qquad \boldsymbol{\Gamma}_b^{be} \triangleq \begin{bmatrix} \mathbf{0} & -\mathbf{b}_e^3 & \mathbf{b}_e^2 \\ \mathbf{b}_e^3 & \mathbf{0} & -\mathbf{b}_e^1 \\ -\mathbf{b}_e^2 & \mathbf{b}_e^1 & \mathbf{0} \end{bmatrix},$$
$$\begin{bmatrix} \dot{\mathbf{d}}_e^1 \\ \dot{\mathbf{d}}_e^2 \\ \dot{\mathbf{d}}_e^3 \end{bmatrix} = \boldsymbol{\Gamma}_d^{de}\boldsymbol{\omega}_d^{de}, \qquad \boldsymbol{\Gamma}_d^{de} \triangleq \begin{bmatrix} \mathbf{0} & -\mathbf{d}_e^3 & \mathbf{d}_e^2 \\ \mathbf{d}_e^3 & \mathbf{0} & -\mathbf{d}_e^1 \\ -\mathbf{d}_e^2 & \mathbf{d}_e^1 & \mathbf{0} \end{bmatrix},$$
$$\dot{\alpha} = \boldsymbol{\Gamma}_a^{as}\boldsymbol{\omega}_a^{as}, \qquad \boldsymbol{\Gamma}_a^{as} = \mathbf{1}_2^{\mathsf{T}},$$
$$\dot{\beta} = \boldsymbol{\Gamma}_b^{bs}\boldsymbol{\omega}_b^{bs}, \qquad \boldsymbol{\Gamma}_b^{bs} = \mathbf{1}_3^{\mathsf{T}}.$$

References

[1] J. R. Forbes. Slides 8 : Rigid-body equations of motion.
[2] ——. Slides 9 : Constrained rigid-bodies.