Assignment 2

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1

From the definition of $\underset{\rightarrow}{u}$ and $\underset{\rightarrow}{v}$, we have:

$$\mathbf{u}_a = \begin{bmatrix} 4 \\ u_{a2} \\ -2 \end{bmatrix}, \qquad \mathbf{v}_a = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

a) Using the definition of the dot product:

$$\underset{\rightarrow}{u} \cdot \underset{\rightarrow}{v} = \mathbf{u}_a^{\mathsf{T}} \mathbf{v}_a = 4 + 3u_{a2} - 10 = -6 + 3u_{a2} \overset{!}{=} 12$$

Solving for u_{a2}:

$$u_{a2} = \frac{12 + 6}{3} = 6$$

b) Using the definition of the cross product:

$$\underset{\rightarrow}{u} \times \underset{\rightarrow}{v} = \underset{\rightarrow}{\mathcal{F}}_a^{\mathsf{T}} \mathbf{u}_a^{\times} \mathbf{v}_a = \underset{\rightarrow}{\mathcal{F}}_a^{\mathsf{T}} \begin{bmatrix} 0 & 2 & u_{a2} \\ -2 & 0 & -4 \\ -u_{a2} & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \underset{\rightarrow}{\mathcal{F}}_a^{\mathsf{T}} \begin{bmatrix} 6 + 5u_{a2} \\ -22 \\ -u_{a2} + 12 \end{bmatrix} \overset{!}{=} \underset{\rightarrow}{\mathcal{F}}_a^{\mathsf{T}} \begin{bmatrix} 16 \\ -22 \\ 10 \end{bmatrix}$$

Solving for u_{a2}, we have the following two equations:

$$u_{a2} = \frac{16 - 6}{5} = 2$$

$$u_{a2} = -(10 - 12) = 2$$

Therefore, $u_{a2} = 2$ is the unique solution.

2

$$\det \boldsymbol{\omega}_a^\times = \det \begin{bmatrix} 0 & -\omega_{a3} & \omega_{a2} \\ \omega_{a3} & 0 & -\omega_{a1} \\ -\omega_{a2} & \omega_{a1} & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \omega_{a3} \\ -\omega_{a2} \end{bmatrix}^\mathsf{T} \begin{bmatrix} -\omega_{a3} \\ 0 \\ \omega_{a1} \end{bmatrix}^\times \begin{bmatrix} \omega_{a2} \\ -\omega_{a1} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \omega_{a3} & -\omega_{a2} \end{bmatrix} \begin{bmatrix} 0 & -\omega_{a1} & 0 \\ \omega_{a1} & 0 & \omega_{a3} \\ 0 & -\omega_{a3} & 0 \end{bmatrix} \begin{bmatrix} \omega_{a2} \\ -\omega_{a1} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \omega_{a3} & -\omega_{a2} \end{bmatrix} \begin{bmatrix} \omega_{a1}^2 \\ \omega_{a1}\omega_{a2} \\ \omega_{a3}\omega_{a1} \end{bmatrix} = \omega_{a3}\omega_{a1}\omega_{a2} - \omega_{a2}\omega_{a3}\omega_{a1} = 0$$

Therefore, $\boldsymbol{\omega}_a^\times$ is not invertible.

3

a) By inspection, we can see that:

$$r_{b1} = r_{a1} = 1, \quad r_{b2} = r_{a3} = 4, \quad r_{b3} = -r_{a2} = -3$$

Therefore, $\mathbf{r}_b = \begin{bmatrix} 1 & 4 & -3 \end{bmatrix}^\mathsf{T}$

b) $\mathcal{F}_b$ is obtained by rotating $\mathcal{F}_a$ about $\underrightarrow{a}^1$ by 90° .

$$\mathbf{C}_{ba} = \underrightarrow{\mathcal{F}}_b \cdot \underrightarrow{\mathcal{F}}_a^\mathsf{T} = \begin{bmatrix} \underrightarrow{b}^1 \\ \underrightarrow{b}^2 \\ \underrightarrow{b}^3 \end{bmatrix} \cdot \begin{bmatrix} \underrightarrow{a}^1 & \underrightarrow{a}^2 & \underrightarrow{a}^3 \end{bmatrix} = \begin{bmatrix} \underrightarrow{b}^1 \cdot \underrightarrow{a}^1 & \underrightarrow{b}^1 \cdot \underrightarrow{a}^2 & \underrightarrow{b}^1 \cdot \underrightarrow{a}^3 \\ \underrightarrow{b}^2 \cdot \underrightarrow{a}^1 & \underrightarrow{b}^2 \cdot \underrightarrow{a}^2 & \underrightarrow{b}^2 \cdot \underrightarrow{a}^3 \\ \underrightarrow{b}^3 \cdot \underrightarrow{a}^1 & \underrightarrow{b}^3 \cdot \underrightarrow{a}^2 & \underrightarrow{b}^3 \cdot \underrightarrow{a}^3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

where the dot products where found by looking at Figure 1.

c)

$$\mathbf{r}_b = \mathbf{C}_{ba}\mathbf{r}_a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$$

This corresponds to the solution found in a).

4

$$
\begin{aligned}
\det \mathbf{Q} = \det \begin{bmatrix} 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{\sqrt{3}}{2} \end{bmatrix}^{\times} \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = -1
\end{aligned}
$$

In conclusion, $\mathbf{Q}$ is not a valid direction cosine matrix because its determinant is different than $+1$.

5

From the definition of the cross product:

$$\underrightarrow{u} \times \underrightarrow{v} = \underrightarrow{\mathcal{F}}_a^{\mathsf{T}} \mathbf{u}_a^{\times} \mathbf{v}_a = \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \mathbf{u}_b^{\times} \mathbf{v}_b$$

Multiplying on the left by $\underrightarrow{\mathcal{F}}_b$ (dot product), we get:

$$\underrightarrow{\mathcal{F}}_b \cdot \underrightarrow{\mathcal{F}}_a^{\mathsf{T}} \mathbf{u}_a^{\times} \mathbf{v}_a = \underrightarrow{\mathcal{F}}_b \cdot \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \mathbf{u}_b^{\times} \mathbf{v}_b \tag{1}$$

Recall:

$$\underrightarrow{\mathcal{F}}_b \cdot \underrightarrow{\mathcal{F}}_a^{\mathsf{T}} = \mathbf{C}_{ba} \tag{2}$$

$$\underrightarrow{\mathcal{F}}_b \cdot \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} = \mathbf{1} \tag{3}$$

Substituting (2) and (3) into (1), we obtain:

$$\mathbf{C}_{ba} \mathbf{u}_a^{\times} \mathbf{v}_a = \mathbf{u}_b^{\times} \mathbf{v}_b \tag{4}$$

Now, from the definition of the DCM:

$$\mathbf{v}_a = \mathbf{C}_{ab} \mathbf{v}_b = \mathbf{C}_{ba}^{\mathsf{T}} \mathbf{v}_b \tag{5}$$

$$\mathbf{u}_b = \mathbf{C}_{ba} \mathbf{u}_a \tag{6}$$

Substituting (5) and (6) into (4), we obtain:

$$\mathbf{C}_{ba} \mathbf{u}_a^{\times} \mathbf{C}_{ba}^{\mathsf{T}} \mathbf{v}_b = (\mathbf{C}_{ba} \mathbf{u}_a)^{\times} \mathbf{v}_b \tag{7}$$

Since (7) must be valid for any $\underrightarrow{v}$ (hence any $\mathbf{v}_b$):

$$\mathbf{C}_{ba} \mathbf{u}_a^{\times} \mathbf{C}_{ba}^{\mathsf{T}} = (\mathbf{C}_{ba} \mathbf{u}_a)^{\times} \quad \square$$

6

a) We have:

$$\underset{\rightarrow}{T} = \underset{\rightarrow}{\mathcal{F}}_a^\top \mathbf{T}_a \underset{\rightarrow}{\mathcal{F}}_a = \underset{\rightarrow}{\mathcal{F}}_b^\top \mathbf{T}_b \underset{\rightarrow}{\mathcal{F}}_b \tag{8}$$

Taking the dot product on the left with $\underset{\rightarrow}{\mathcal{F}}_b$ and on the right with $\underset{\rightarrow}{\mathcal{F}}_b^\top$:

$$\underbrace{\underset{\rightarrow}{\mathcal{F}}_b \cdot \underset{\rightarrow}{\mathcal{F}}_a^\top}_{\mathbf{C}_{ba}} \mathbf{T}_a \underbrace{\underset{\rightarrow}{\mathcal{F}}_a \cdot \underset{\rightarrow}{\mathcal{F}}_b^\top}_{\mathbf{C}_{ab}} = \underbrace{\underset{\rightarrow}{\mathcal{F}}_b \cdot \underset{\rightarrow}{\mathcal{F}}_b^\top}_{\mathbf{1}} \mathbf{T}_b \underbrace{\underset{\rightarrow}{\mathcal{F}}_b \cdot \underset{\rightarrow}{\mathcal{F}}_b^\top}_{\mathbf{1}}$$

Where we used the definition of the direction cosine matrix. Rewriting both sides:

$$\mathbf{C}_{ba}\mathbf{T}_a\mathbf{C}_{ab} = \mathbf{T}_b \quad \square$$

b) Using (8) and $\underset{\rightarrow}{u} = \underset{\rightarrow}{\mathcal{F}}_a^\top \mathbf{u}_a$, we find:

$$\underset{\rightarrow}{T} \cdot \underset{\rightarrow}{u} = \underset{\rightarrow}{\mathcal{F}}_b^\top \mathbf{T}_b \underbrace{\underset{\rightarrow}{\mathcal{F}}_b \cdot \underset{\rightarrow}{\mathcal{F}}_a^\top}_{\mathbf{C}_{ba}} \mathbf{u}_a = \underset{\rightarrow}{\mathcal{F}}_b^\top \mathbf{T}_b \mathbf{C}_{ba} \mathbf{u}_a \quad \square$$

7

a)

$$\begin{aligned}
(\lambda - 1)(\lambda^2 + \lambda(1 - \mathrm{tr}\mathbf{C}_{qp}) + 1) &= (\lambda - 1)\lambda^2 + (\lambda - 1)(\lambda - \lambda\mathrm{tr}\mathbf{C}_{qp}) + (\lambda - 1) \\
&= \lambda^3 - \lambda^2 + \lambda^2 - \lambda^2\mathrm{tr}\mathbf{C}_{qp} - \lambda + \lambda\mathrm{tr}\mathbf{C}_{qp} + \lambda - 1 \\
&= \lambda^3 - \lambda^2\mathrm{tr}\mathbf{C}_{qp} + \lambda\mathrm{tr}\mathbf{C}_{qp} - 1 \\
&= 0 \quad \square
\end{aligned}$$

One obvious solution of this equation is $\lambda_1 = +1$, which is therefore also an eigenvalue of $\mathbf{C}_{qp}$.

b) Recall that the trace is a linear operator, i.e. $\forall \alpha, \beta \in \mathbb{C}, \forall \mathbf{A}, \mathbf{B} \in \mathbb{C}^{n \times n}$, $n \in \mathbb{N}$:

$$\mathrm{tr}(\alpha\mathbf{A} + \beta\mathbf{B}) = \alpha\mathrm{tr}\mathbf{A} + \beta\mathrm{tr}\mathbf{B}$$

Therefore, we can express the trace of $\mathbf{C}_{qp}$ as follows:

$$\mathrm{tr}\mathbf{C}_{qp} = \cos\phi\,\mathrm{tr}(\mathbf{1}) + (1 - \cos\phi)\mathrm{tr}(\mathbf{a}\mathbf{a}^\top) - \sin\phi\,\mathrm{tr}(\mathbf{a}^\times) \tag{9}$$

We can now compute the trace of each element.

- Trivially:

$$\mathrm{tr}(\mathbf{1}) = 3 \tag{10}$$

- Let $\mathbf{a} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}^\mathsf{T}$, therefore:
$$\mathbf{a}\mathbf{a}^\mathsf{T} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} = \begin{bmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_2a_1 & a_2^2 & a_2a_3 \\ a_3a_1 & a_3a_2 & a_3^2 \end{bmatrix}$$
By inspection:
$$\mathrm{tr}(\mathbf{a}\mathbf{a}^\mathsf{T}) = a_1^2 + a_2^2 + a_3^2 = \|\underset{\rightarrow}{a}\|_2 = 1 \tag{11}$$

- Taking the same generic notation for $\mathbf{a}$:
$$\mathbf{a}^\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$
By inspection:
$$\mathrm{tr}(\mathbf{a}^\times) = 0 \tag{12}$$

Substituting (10), (11) and (12) into (9):
$$\mathrm{tr}\mathbf{C}_{qp} = 3\cos\phi + (1 - \cos\phi) = 1 + 2\cos\phi \quad \square$$
Using this result and the fact that $\mathbf{C}_{qp} \in \mathrm{SO}(3)$, we have the two following conditions:
$$\det\mathbf{C}_{qp} = \lambda_1\lambda_2\lambda_3 = 1 \tag{13}$$
$$\mathrm{tr}\mathbf{C}_{qp} = \lambda_1 + \lambda_2 + \lambda_3 = 1 + 2\cos\phi \tag{14}$$
Since we have already found that $\lambda_1 = 1$, (13) and (14) become:
$$\lambda_2\lambda_3 = 1 \tag{15}$$
$$\lambda_2 + \lambda_3 = 2\cos\phi \tag{16}$$
From (16), we find:
$$\lambda_3 = 2\cos\phi - \lambda_2 \tag{17}$$
Substituting (17) into (15):
$$\lambda_2(2\cos\phi - \lambda_2) = 1 \tag{18}$$
We can now rearrange (18) as follows:
$$\lambda_2^2 - 2\cos\phi\lambda_2 + 1 = 0 \tag{19}$$
This is simply a quadratic equation that we can solve as follows:
$$\lambda_2 = \cos\phi \pm \sqrt{\cos^2\phi - 1} = \cos\phi \pm \sqrt{(-1)(1-\cos^2\phi)}$$
$$= \cos\phi \pm \sqrt{-1}\sqrt{\sin^2\phi} = \cos\phi \pm i\sin\phi = e^{\pm i\phi}$$
Substituting this solution into (15):
$$e^{\pm i\phi}\lambda_3 = 1 \quad \Rightarrow \quad \lambda_3 = \frac{1}{e^{\pm i\phi}} = e^{\mp i\phi}$$
In other words, λ_2 is the complex conjugate of λ_3 and vice versa. It doesn't matter which one has the positive sign in the exponent.

8

First, let's derive some useful properties:

$$\mathbf{a}^\times\mathbf{a}^\times = \mathbf{a}\mathbf{a}^\top - \mathbf{a}^\top\mathbf{a}\mathbf{1} = \mathbf{a}\mathbf{a}^\top - \mathbf{1}$$

$$(\mathbf{a}\mathbf{a}^\top - \mathbf{1})^2 = \mathbf{a}\overbrace{\mathbf{a}^\top\mathbf{a}}^{=1}\mathbf{a}^\top - 2\mathbf{a}\mathbf{a}^\top + \mathbf{1} = \mathbf{1} - \mathbf{a}\mathbf{a}^\top$$

$$(\mathbf{a}\mathbf{a}^\top - \mathbf{1})^3 = (\mathbf{a}\mathbf{a}^\top - \mathbf{1})^2(\mathbf{a}\mathbf{a}^\top - \mathbf{1}) = (\mathbf{1} - \mathbf{a}\mathbf{a}^\top)(\mathbf{a}\mathbf{a}^\top - \mathbf{1}) = \mathbf{a}\mathbf{a}^\top - \mathbf{1}$$

$$\vdots$$

$$(\mathbf{a}\mathbf{a}^\top - \mathbf{1})^n = (-1)^n(\mathbf{1} - \mathbf{a}\mathbf{a}^\top), \quad n \in \mathbb{N} \setminus \{0\}$$

Moreover:

$$\mathbf{a}^\times(\mathbf{1} - \mathbf{a}\mathbf{a}^\top) = \mathbf{a}^\times - \overbrace{\mathbf{a}^\times\mathbf{a}}^{=\mathbf{0}}\mathbf{a}^\top = \mathbf{a}^\times$$

Now, let's expand the expression for $e^{-\phi\mathbf{a}^\times}$ using the definition of the matrix exponential and the former properties:

$$\begin{aligned}
e^{-\phi\mathbf{a}^\times} &= \sum_{k=0}^{\infty}\frac{(-\phi)^k}{k!}(\mathbf{a}^\times)^k \\
&= \sum_{l=0}^{\infty}\frac{\overbrace{(-1)^{2l}}^{=1}\phi^{2l}}{(2l)!}(\mathbf{a}^\times)^{2l} + \sum_{m=0}^{\infty}\frac{\overbrace{(-1)^{2m+1}}^{=-1}\phi^{2m+1}}{(2m+1)!}(\mathbf{a}^\times)^{2m+1} \\
&= \sum_{l=0}^{\infty}\frac{\phi^{2l}}{(2l)!}(\overbrace{\mathbf{a}^\times\mathbf{a}^\times}^{=\mathbf{a}\mathbf{a}^\top-\mathbf{1}})^l - \sum_{m=0}^{\infty}\frac{\phi^{2m+1}}{(2m+1)!}\mathbf{a}^\times(\overbrace{\mathbf{a}^\times\mathbf{a}^\times}^{=\mathbf{a}\mathbf{a}^\top-\mathbf{1}})^m \\
&= \mathbf{1} + \sum_{l=1}^{\infty}\frac{\phi^{2l}}{(2l)!}\overbrace{(\mathbf{a}\mathbf{a}^\top - \mathbf{1})^l}^{=(-1)^l(\mathbf{1}-\mathbf{a}\mathbf{a}^\top)} - \left(\phi\mathbf{a}^\times + \sum_{m=1}^{\infty}\frac{\phi^{2m+1}}{(2m+1)!}\mathbf{a}^\times\overbrace{(\mathbf{a}\mathbf{a}^\top - \mathbf{1})^m}^{=(-1)^m(\mathbf{1}-\mathbf{a}\mathbf{a}^\top)}\right) \\
&= \mathbf{1} + \overbrace{\sum_{l=1}^{\infty}\frac{(-1)^l\phi^{2l}}{(2l)!}}^{=\cos\phi-1}(\mathbf{1} - \mathbf{a}\mathbf{a}^\top) - \left(\phi\mathbf{a}^\times + \sum_{m=1}^{\infty}\frac{(-1)^m\phi^{2m+1}}{(2m+1)!}\overbrace{\mathbf{a}^\times(\mathbf{1} - \mathbf{a}\mathbf{a}^\top)}^{=\mathbf{a}^\times}\right) \\
&= \mathbf{1} + (\cos\phi - 1)(\mathbf{1} - \mathbf{a}\mathbf{a}^\top) - \overbrace{\sum_{m=0}^{\infty}\frac{(-1)^m\phi^{2m+1}}{(2m+1)!}}^{=\sin\phi}\mathbf{a}^\times \\
&= \cos\phi\mathbf{1} + (1 - \cos\phi)\mathbf{a}\mathbf{a}^\top - \sin\phi\mathbf{a}^\times \quad \square
\end{aligned}$$