Rigid-Body Equations of Motion
MECH 642 - *Advanced Dynamics*

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Key Identities

- Recall that

$$\begin{aligned}
\boldsymbol{\omega}_b^{ba} &= \mathbf{S}_b^{ba}(\mathbf{q}^{ba})\dot{\mathbf{q}}^{ba}, \\
\dot{\mathbf{q}}^{ba} &= \boldsymbol{\Gamma}_b^{ba}(\mathbf{q}^{ba})\boldsymbol{\omega}_b^{ba}, \\
\boldsymbol{\Xi}_b^{ba}(\mathbf{q}^{ba})\dot{\mathbf{q}}^{ba} &= \mathbf{0}, \\
\boldsymbol{\Xi}_b^{ba}(\mathbf{q}^{ba})\boldsymbol{\Gamma}_b^{ba}(\mathbf{q}^{ba}) &= \mathbf{0}, \\
\mathbf{S}_b^{ba}(\mathbf{q}^{ba})\boldsymbol{\Gamma}_b^{ba}(\mathbf{q}^{ba}) &= \mathbf{1},
\end{aligned}$$

  where $\mathbf{q}^{ba}$ is a constrained parameterization.
- Also, it can be shown that

$$\left(\dot{\mathbf{S}}_b^{ba} - \frac{\partial \boldsymbol{\omega}_b^{ba}}{\partial \mathbf{q}^{ba}}\right)\boldsymbol{\Gamma}_b^{ba} = -\boldsymbol{\omega}_b^{ba\times}, \quad \frac{\partial(\mathbf{C}_{ba}^{\mathsf{T}}\mathbf{s}_b)}{\partial \mathbf{q}^{ba}}\boldsymbol{\Gamma}_b^{ba} = -\mathbf{C}_{ba}^{\mathsf{T}}\mathbf{s}_b^{\times}, \quad \frac{\partial(\mathbf{C}_{ba}\mathbf{s}_a)}{\partial \mathbf{q}^{ba}}\boldsymbol{\Gamma}_b^{ba} = (\mathbf{C}_{ba}\mathbf{s}_a)^{\times},$$

  where $\mathbf{s}_a \in \mathbb{R}^3$ and $\mathbf{s}_b \in \mathbb{R}^3$ are arbitrary.

Deriving Identities (DCM Case)

- We will derive $\boldsymbol{\omega}_b^{ba} = \mathbf{S}_b^{ba}(\bar{\mathbf{c}}_{ba})\dot{\bar{\mathbf{c}}}_{ba}$, $\dot{\bar{\mathbf{c}}}_{ba} = \boldsymbol{\Gamma}_b^{ba}(\bar{\mathbf{c}}_{ba})\boldsymbol{\omega}_b^{ba}$, and $\mathbf{0} = \boldsymbol{\Xi}_b^{ba}(\bar{\mathbf{c}}_{ba})\dot{\bar{\mathbf{c}}}_{ba}$ when
$$\mathbf{q}^{ba} = \bar{\mathbf{c}}_{ba}, \text{ where } \mathbf{C}_{ba}^{\mathsf{T}} = \begin{bmatrix} \bar{\mathbf{c}}_{ba}^1 & \bar{\mathbf{c}}_{ba}^2 & \bar{\mathbf{c}}_{ba}^3 \end{bmatrix}. \tag{1}$$
- We will first derive an expression of the form $\boldsymbol{\omega}_b^{ba} = \mathbf{S}_b^{ba}(\bar{\mathbf{c}}_{ba})\dot{\bar{\mathbf{c}}}_{ba}$.
- Recall Poisson's equation,
$$\dot{\mathbf{C}}_{ba} = -\boldsymbol{\omega}_b^{ba^\times}\mathbf{C}_{ba}, \quad \text{or} \quad \boldsymbol{\omega}_b^{ba^\times} = -\dot{\mathbf{C}}_{ba}\mathbf{C}_{ba}^{\mathsf{T}}. \tag{2}$$
Using Equation (1) Poisson's Equation in Equation (2) can be written
$$\begin{aligned}
\boldsymbol{\omega}_b^{ba^\times} &= -\dot{\mathbf{C}}_{ba}\mathbf{C}_{ba}^{\mathsf{T}}, \\
\begin{bmatrix} 0 & -\omega_{b3}^{ba} & \omega_{b2}^{ba} \\ \omega_{b3}^{ba} & 0 & -\omega_{b1}^{ba} \\ -\omega_{b2}^{ba} & \omega_{b1}^{ba} & 0 \end{bmatrix} &= -\begin{bmatrix} \dot{\bar{\mathbf{c}}}_{ba}^{1^\mathsf{T}} \\ \dot{\bar{\mathbf{c}}}_{ba}^{2^\mathsf{T}} \\ \dot{\bar{\mathbf{c}}}_{ba}^{3^\mathsf{T}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{c}}_{ba}^1 & \bar{\mathbf{c}}_{ba}^2 & \bar{\mathbf{c}}_{ba}^3 \end{bmatrix} \\
&= -\begin{bmatrix} \dot{\bar{\mathbf{c}}}_{ba}^{1^\mathsf{T}}\bar{\mathbf{c}}_{ba}^1 & \dot{\bar{\mathbf{c}}}_{ba}^{1^\mathsf{T}}\bar{\mathbf{c}}_{ba}^2 & \dot{\bar{\mathbf{c}}}_{ba}^{1^\mathsf{T}}\bar{\mathbf{c}}_{ba}^3 \\ \dot{\bar{\mathbf{c}}}_{ba}^{2^\mathsf{T}}\bar{\mathbf{c}}_{ba}^1 & \dot{\bar{\mathbf{c}}}_{ba}^{2^\mathsf{T}}\bar{\mathbf{c}}_{ba}^2 & \dot{\bar{\mathbf{c}}}_{ba}^{2^\mathsf{T}}\bar{\mathbf{c}}_{ba}^3 \\ \dot{\bar{\mathbf{c}}}_{ba}^{3^\mathsf{T}}\bar{\mathbf{c}}_{ba}^1 & \dot{\bar{\mathbf{c}}}_{ba}^{3^\mathsf{T}}\bar{\mathbf{c}}_{ba}^2 & \dot{\bar{\mathbf{c}}}_{ba}^{3^\mathsf{T}}\bar{\mathbf{c}}_{ba}^3 \end{bmatrix}.
\end{aligned}$$

- Noting that

$$\omega_{b3}^{ba} = \dot{\bar{\mathbf{c}}}_{ba}^{1^\mathsf{T}} \bar{\mathbf{c}}_{ba}^{2} = -\dot{\bar{\mathbf{c}}}_{ba}^{2^\mathsf{T}} \bar{\mathbf{c}}_{ba}^{1},$$
$$\omega_{b2}^{ba} = \dot{\bar{\mathbf{c}}}_{ba}^{1^\mathsf{T}} \bar{\mathbf{c}}_{ba}^{3} = -\dot{\bar{\mathbf{c}}}_{ba}^{3^\mathsf{T}} \bar{\mathbf{c}}_{ba}^{1},$$
$$\omega_{b1}^{ba} = \dot{\bar{\mathbf{c}}}_{ba}^{2^\mathsf{T}} \bar{\mathbf{c}}_{ba}^{3} = -\dot{\bar{\mathbf{c}}}_{ba}^{3^\mathsf{T}} \bar{\mathbf{c}}_{ba}^{2},$$

it follows that

$$\dot{\bar{\mathbf{c}}}_{ba}^{i^\mathsf{T}} \bar{\mathbf{c}}_{ba}^{j} = \begin{cases} 0 & i = j \\ -\dot{\bar{\mathbf{c}}}_{ba}^{j^\mathsf{T}} \bar{\mathbf{c}}_{ba}^{i} & i \neq j \end{cases}, \qquad i, j = 1, 2, 3.$$

- Therefore,

$$\begin{aligned}
\boldsymbol{\omega}_b^{ba} &= \begin{bmatrix} \dot{\bar{\mathbf{c}}}_{ba}^{2^\mathsf{T}} \bar{\mathbf{c}}_{ba}^{3} \\ -\dot{\bar{\mathbf{c}}}_{ba}^{1^\mathsf{T}} \bar{\mathbf{c}}_{ba}^{3} \\ \dot{\bar{\mathbf{c}}}_{ba}^{1^\mathsf{T}} \bar{\mathbf{c}}_{ba}^{2} \end{bmatrix} = \begin{bmatrix} \dot{\bar{\mathbf{c}}}_{ba}^{2^\mathsf{T}} \bar{\mathbf{c}}_{ba}^{3} \\ \dot{\bar{\mathbf{c}}}_{ba}^{3^\mathsf{T}} \bar{\mathbf{c}}_{ba}^{1} \\ \dot{\bar{\mathbf{c}}}_{ba}^{1^\mathsf{T}} \bar{\mathbf{c}}_{ba}^{2} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{c}}_{ba}^{3^\mathsf{T}} \dot{\bar{\mathbf{c}}}_{ba}^{2} \\ \bar{\mathbf{c}}_{ba}^{1^\mathsf{T}} \dot{\bar{\mathbf{c}}}_{ba}^{3} \\ \bar{\mathbf{c}}_{ba}^{2^\mathsf{T}} \dot{\bar{\mathbf{c}}}_{ba}^{1} \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} \mathbf{0} & \bar{\mathbf{c}}_{ba}^{3^\mathsf{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{c}}_{ba}^{1^\mathsf{T}} \\ \bar{\mathbf{c}}_{ba}^{2^\mathsf{T}} & \mathbf{0} & \mathbf{0} \end{bmatrix}}_{\mathbf{S}_b^{ba}(\bar{\mathbf{c}}_{ba})} \underbrace{\begin{bmatrix} \dot{\bar{\mathbf{c}}}_{ba}^{1} \\ \dot{\bar{\mathbf{c}}}_{ba}^{2} \\ \dot{\bar{\mathbf{c}}}_{ba}^{3} \end{bmatrix}}_{\dot{\bar{\mathbf{c}}}_{ba}}, \\
\boldsymbol{\omega}_b^{ba} &= \mathbf{S}_b^{ba}(\bar{\mathbf{c}}_{ba}) \dot{\bar{\mathbf{c}}}_{ba}.
\end{aligned}$$

- Next we will derive an expression of the form $\dot{\bar{\mathbf{c}}}_{ba} = \mathbf{\Gamma}_b^{ba}(\bar{\mathbf{c}}_{ba})\boldsymbol{\omega}_b^{ba}$.
- To begin,

$$\begin{aligned}
\dot{\mathbf{C}}_{ba}^{\mathsf{T}} &= \mathbf{C}_{ba}^{\mathsf{T}}\boldsymbol{\omega}_b^{ba^\times} \\
&= \mathbf{C}_{ba}^{\mathsf{T}}\boldsymbol{\omega}_b^{ba^\times}\mathbf{C}_{ba}\mathbf{C}_{ba}^{\mathsf{T}} \\
&= \left(\mathbf{C}_{ba}^{\mathsf{T}}\boldsymbol{\omega}_b^{ba}\right)^\times \mathbf{C}_{ba}^{\mathsf{T}},
\end{aligned}$$

  which is also equal to

$$\begin{bmatrix} \dot{\bar{\mathbf{c}}}_{ba}^1 & \dot{\bar{\mathbf{c}}}_{ba}^2 & \dot{\bar{\mathbf{c}}}_{ba}^3 \end{bmatrix} = \left(\mathbf{C}_{ba}^{\mathsf{T}}\boldsymbol{\omega}_b^{ba}\right)^\times \begin{bmatrix} \bar{\mathbf{c}}_{ba}^1 & \bar{\mathbf{c}}_{ba}^2 & \bar{\mathbf{c}}_{ba}^3 \end{bmatrix}$$

- Therefore,

$$\begin{aligned}
\dot{\bar{\mathbf{c}}}_{ba}^i &= \left(\mathbf{C}_{ba}^{\mathsf{T}}\boldsymbol{\omega}_b^{ba}\right)^\times \bar{\mathbf{c}}_{ba}^i \\
&= -\bar{\mathbf{c}}_{ba}^{i^\times}\mathbf{C}_{ba}^{\mathsf{T}}\boldsymbol{\omega}_b^{ba} \\
&= -\bar{\mathbf{c}}_{ba}^{i^\times}\begin{bmatrix} \bar{\mathbf{c}}_{ba}^1 & \bar{\mathbf{c}}_{ba}^2 & \bar{\mathbf{c}}_{ba}^3 \end{bmatrix}\boldsymbol{\omega}_b^{ba} \\
&= -\left(\bar{\mathbf{c}}_{ba}^{i^\times}\bar{\mathbf{c}}_{ba}^1\omega_{b1}^{ba} + \bar{\mathbf{c}}_{ba}^{i^\times}\bar{\mathbf{c}}_{ba}^2\omega_{b2}^{ba} + \bar{\mathbf{c}}_{ba}^{i^\times}\bar{\mathbf{c}}_{ba}^3\omega_{b3}^{ba}\right) \\
&= -\begin{bmatrix} \bar{\mathbf{c}}_{ba}^{i^\times}\bar{\mathbf{c}}_{ba}^1 & \bar{\mathbf{c}}_{ba}^{i^\times}\bar{\mathbf{c}}_{ba}^2 & \bar{\mathbf{c}}_{ba}^{i^\times}\bar{\mathbf{c}}_{ba}^3 \end{bmatrix}\boldsymbol{\omega}_b^{ba}, \qquad i = 1, 2, 3.
\end{aligned}$$

- Combining the expressions for $\dot{\bar{\mathbf{c}}}^i_{ba}$, $i = 1, 2, 3$, leads to

$$\underbrace{\begin{bmatrix} \dot{\bar{\mathbf{c}}}^1_{ba} \\ \dot{\bar{\mathbf{c}}}^2_{ba} \\ \dot{\bar{\mathbf{c}}}^3_{ba} \end{bmatrix}}_{\dot{\bar{\mathbf{c}}}_{ba}} = - \begin{bmatrix} \bar{\mathbf{c}}^{1\times}_{ba} \bar{\mathbf{c}}^1_{ba} & \bar{\mathbf{c}}^{1\times}_{ba} \bar{\mathbf{c}}^2_{ba} & \bar{\mathbf{c}}^{1\times}_{ba} \bar{\mathbf{c}}^3_{ba} \\ \bar{\mathbf{c}}^{2\times}_{ba} \bar{\mathbf{c}}^1_{ba} & \bar{\mathbf{c}}^{2\times}_{ba} \bar{\mathbf{c}}^2_{ba} & \bar{\mathbf{c}}^{2\times}_{ba} \bar{\mathbf{c}}^3_{ba} \\ \bar{\mathbf{c}}^{3\times}_{ba} \bar{\mathbf{c}}^1_{ba} & \bar{\mathbf{c}}^{3\times}_{ba} \bar{\mathbf{c}}^2_{ba} & \bar{\mathbf{c}}^{3\times}_{ba} \bar{\mathbf{c}}^3_{ba} \end{bmatrix} \boldsymbol{\omega}^{ba}_b$$

$$= - \begin{bmatrix} \mathbf{0} & \bar{\mathbf{c}}^{1\times}_{ba} \bar{\mathbf{c}}^2_{ba} & \bar{\mathbf{c}}^{1\times}_{ba} \bar{\mathbf{c}}^3_{ba} \\ \bar{\mathbf{c}}^{2\times}_{ba} \bar{\mathbf{c}}^1_{ba} & \mathbf{0} & \bar{\mathbf{c}}^{2\times}_{ba} \bar{\mathbf{c}}^3_{ba} \\ \bar{\mathbf{c}}^{3\times}_{ba} \bar{\mathbf{c}}^1_{ba} & \bar{\mathbf{c}}^{3\times}_{ba} \bar{\mathbf{c}}^2_{ba} & \mathbf{0} \end{bmatrix} \boldsymbol{\omega}^{ba}_b$$

$$= \underbrace{\begin{bmatrix} \mathbf{0} & -\bar{\mathbf{c}}^3_{ba} & \bar{\mathbf{c}}^2_{ba} \\ \bar{\mathbf{c}}^3_{ba} & \mathbf{0} & -\bar{\mathbf{c}}^1_{ba} \\ -\bar{\mathbf{c}}^2_{ba} & \bar{\mathbf{c}}^1_{ba} & \mathbf{0} \end{bmatrix}}_{\boldsymbol{\Gamma}^{ba}_b(\bar{\mathbf{c}}_{ba})} \boldsymbol{\omega}^{ba}_b$$

where $\bar{\mathbf{c}}^1_{ba} = \bar{\mathbf{c}}^{2\times}_{ba} \bar{\mathbf{c}}^3_{ba}$, $\bar{\mathbf{c}}^2_{ba} = \bar{\mathbf{c}}^{3\times}_{ba} \bar{\mathbf{c}}^1_{ba}$, and $\bar{\mathbf{c}}^3_{ba} = \bar{\mathbf{c}}^{1\times}_{ba} \bar{\mathbf{c}}^2_{ba}$ have been used to simplify (which each come from the orthonormality constraint of the DCM).

- Now the the matrix $\boldsymbol{\Xi}_b^{ba}(\bar{\mathbf{c}}_{ba})$ will be derived where $\mathbf{0} = \boldsymbol{\Xi}_b^{ba}(\bar{\mathbf{c}}_{ba})\dot{\bar{\mathbf{c}}}_{ba}$.
- To begin, recall that the constraint $\mathbf{C}_{ba}^{\mathsf{T}}\mathbf{C}_{ba} = \mathbf{1}$, or $\bar{\mathbf{C}}_{ba}\bar{\mathbf{C}}_{ba}^{\mathsf{T}} = \mathbf{1}$, gives

$$\begin{aligned}
\bar{\mathbf{c}}_{ba}^{1\mathsf{T}}\bar{\mathbf{c}}_{ba}^{1} &= 1, & \bar{\mathbf{c}}_{ba}^{1\mathsf{T}}\bar{\mathbf{c}}_{ba}^{2} &= 0, & \bar{\mathbf{c}}_{ba}^{1\mathsf{T}}\bar{\mathbf{c}}_{ba}^{3} &= 0,\\
\bar{\mathbf{c}}_{ba}^{2\mathsf{T}}\bar{\mathbf{c}}_{ba}^{1} &= 0, & \bar{\mathbf{c}}_{ba}^{2\mathsf{T}}\bar{\mathbf{c}}_{ba}^{2} &= 1, & \bar{\mathbf{c}}_{ba}^{2\mathsf{T}}\bar{\mathbf{c}}_{ba}^{3} &= 0,\\
\bar{\mathbf{c}}_{ba}^{3\mathsf{T}}\bar{\mathbf{c}}_{ba}^{1} &= 0, & \bar{\mathbf{c}}_{ba}^{3\mathsf{T}}\bar{\mathbf{c}}_{ba}^{2} &= 0, & \bar{\mathbf{c}}_{ba}^{3\mathsf{T}}\bar{\mathbf{c}}_{ba}^{3} &= 1,
\end{aligned}$$

  where Equation (1) has been used.
- Notice that

$$\begin{aligned}
1 &= \bar{\mathbf{c}}_{ba}^{3\mathsf{T}}\bar{\mathbf{c}}_{ba}^{3}\\
&= \left(\bar{\mathbf{c}}_{ba}^{1^\times}\bar{\mathbf{c}}_{ba}^{2}\right)^{\mathsf{T}}\left(\bar{\mathbf{c}}_{ba}^{1^\times}\bar{\mathbf{c}}_{ba}^{2}\right)\\
&= -\bar{\mathbf{c}}_{ba}^{2^\mathsf{T}}\bar{\mathbf{c}}_{ba}^{1^\times}\bar{\mathbf{c}}_{ba}^{1^\times}\bar{\mathbf{c}}_{ba}^{2}\\
&= \bar{\mathbf{c}}_{ba}^{2^\mathsf{T}}\left(\bar{\mathbf{c}}_{ba}^{1^\mathsf{T}}\bar{\mathbf{c}}_{ba}^{1}\mathbf{1} - \bar{\mathbf{c}}_{ba}^{1}\bar{\mathbf{c}}_{ba}^{1^\mathsf{T}}\right)\bar{\mathbf{c}}_{ba}^{2}\\
&= \bar{\mathbf{c}}_{ba}^{2^\mathsf{T}}\bar{\mathbf{c}}_{ba}^{2} - \bar{\mathbf{c}}_{ba}^{2^\mathsf{T}}\bar{\mathbf{c}}_{ba}^{1}\bar{\mathbf{c}}_{ba}^{1^\mathsf{T}}\bar{\mathbf{c}}_{ba}^{2}\\
&= \bar{\mathbf{c}}_{ba}^{2^\mathsf{T}}\bar{\mathbf{c}}_{ba}^{2}.
\end{aligned}$$

- Thus, need one of $\bar{\mathbf{c}}_{ba}^{2^\mathsf{T}}\bar{\mathbf{c}}_{ba}^{2} = 1$ or $\bar{\mathbf{c}}_{ba}^{3^\mathsf{T}}\bar{\mathbf{c}}_{ba}^{3} = 1$, not both.

- The constraint $\mathbf{C}_{ba}^{\mathsf{T}}\mathbf{C}_{ba} = \mathbf{1}$, or $\bar{\mathbf{C}}_{ba}\bar{\mathbf{C}}_{ba}^{\mathsf{T}} = \mathbf{1}$, and $\det \mathbf{C}_{ba} = +1$ is captured by $\mathbf{\Phi}(\bar{\mathbf{c}}_{ba}) = \mathbf{0}$ where

$$\mathbf{\Phi}(\bar{\mathbf{c}}_{ba}) = \begin{bmatrix} \bar{\mathbf{c}}_{ba}^{1\,\mathsf{T}}\bar{\mathbf{c}}_{ba}^{1} - 1 \\ \bar{\mathbf{c}}_{ba}^{2\,\mathsf{T}}\bar{\mathbf{c}}_{ba}^{2} - 1 \\ \bar{\mathbf{c}}_{ba}^{2\,\mathsf{T}}\bar{\mathbf{c}}_{ba}^{1} \\ \bar{\mathbf{c}}_{ba}^{1\,\times}\bar{\mathbf{c}}_{ba}^{2} - \bar{\mathbf{c}}_{ba}^{3} \end{bmatrix}.$$

- Taking the derivative of $\mathbf{\Phi}(\bar{\mathbf{c}}_{ba}) = \mathbf{0}$ with respect to time gives

$$\begin{aligned} \mathbf{0} &= \dot{\mathbf{\Phi}}(\bar{\mathbf{c}}_{ba}) \\ &= \begin{bmatrix} 2\bar{\mathbf{c}}_{ba}^{1\,\mathsf{T}}\dot{\bar{\mathbf{c}}}_{ba}^{1} \\ 2\bar{\mathbf{c}}_{ba}^{2\,\mathsf{T}}\dot{\bar{\mathbf{c}}}_{ba}^{2} \\ \bar{\mathbf{c}}_{ba}^{2\,\mathsf{T}}\dot{\bar{\mathbf{c}}}_{ba}^{1} + \bar{\mathbf{c}}_{ba}^{1\,\mathsf{T}}\dot{\bar{\mathbf{c}}}_{ba}^{2} \\ \dot{\bar{\mathbf{c}}}_{ba}^{1\,\times}\bar{\mathbf{c}}_{ba}^{2} + \bar{\mathbf{c}}_{ba}^{1\,\times}\dot{\bar{\mathbf{c}}}_{ba}^{2} - \dot{\bar{\mathbf{c}}}_{ba}^{3} \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} 2\bar{\mathbf{c}}_{ba}^{1\,\mathsf{T}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\bar{\mathbf{c}}_{ba}^{2\,\mathsf{T}} & \mathbf{0} \\ \bar{\mathbf{c}}_{ba}^{2\,\mathsf{T}} & \bar{\mathbf{c}}_{ba}^{1\,\mathsf{T}} & \mathbf{0} \\ -\bar{\mathbf{c}}_{ba}^{2\,\times} & \bar{\mathbf{c}}_{ba}^{1\,\times} & -\mathbf{1} \end{bmatrix}}_{\mathbf{\Xi}(\bar{\mathbf{c}}_{ba})} \underbrace{\begin{bmatrix} \dot{\bar{\mathbf{c}}}_{ba}^{1} \\ \dot{\bar{\mathbf{c}}}_{ba}^{2} \\ \dot{\bar{\mathbf{c}}}_{ba}^{3} \end{bmatrix}}_{\dot{\bar{\mathbf{c}}}_{ba}}, \\ \mathbf{0} &= \mathbf{\Xi}_{b}^{ba}(\bar{\mathbf{c}}_{ba})\dot{\bar{\mathbf{c}}}_{ba}. \end{aligned}$$

- It is straightforward to verify that

$$\begin{bmatrix} 2\bar{\mathbf{c}}_{ba}^{1^{\mathsf{T}}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\bar{\mathbf{c}}_{ba}^{2^{\mathsf{T}}} & \mathbf{0} \\ \bar{\mathbf{c}}_{ba}^{2^{\mathsf{T}}} & \bar{\mathbf{c}}_{ba}^{1^{\mathsf{T}}} & \mathbf{0} \\ -\bar{\mathbf{c}}_{ba}^{2^{\times}} & \bar{\mathbf{c}}_{ba}^{1^{\times}} & -\mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{0} & -\bar{\mathbf{c}}_{ba}^{3} & \bar{\mathbf{c}}_{ba}^{2} \\ \bar{\mathbf{c}}_{ba}^{3} & \mathbf{0} & -\bar{\mathbf{c}}_{ba}^{1} \\ -\bar{\mathbf{c}}_{ba}^{2} & \bar{\mathbf{c}}_{ba}^{1} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1-1 \\ \bar{\mathbf{c}}_{ba}^{1^{\times}}\bar{\mathbf{c}}_{ba}^{3} + \bar{\mathbf{c}}_{ba}^{2} & \bar{\mathbf{c}}_{ba}^{2^{\times}}\bar{\mathbf{c}}_{ba}^{3} - \bar{\mathbf{c}}_{ba}^{1} & \mathbf{0} \end{bmatrix}$$

$$= \mathbf{0},$$

that is, $\boldsymbol{\Xi}_b^{ba}(\bar{\mathbf{c}}_{ba})\boldsymbol{\Gamma}_b^{ba}(\bar{\mathbf{c}}_{ba}) = \mathbf{0}$.

- Additionally, by direct matrix multiplication it follows that

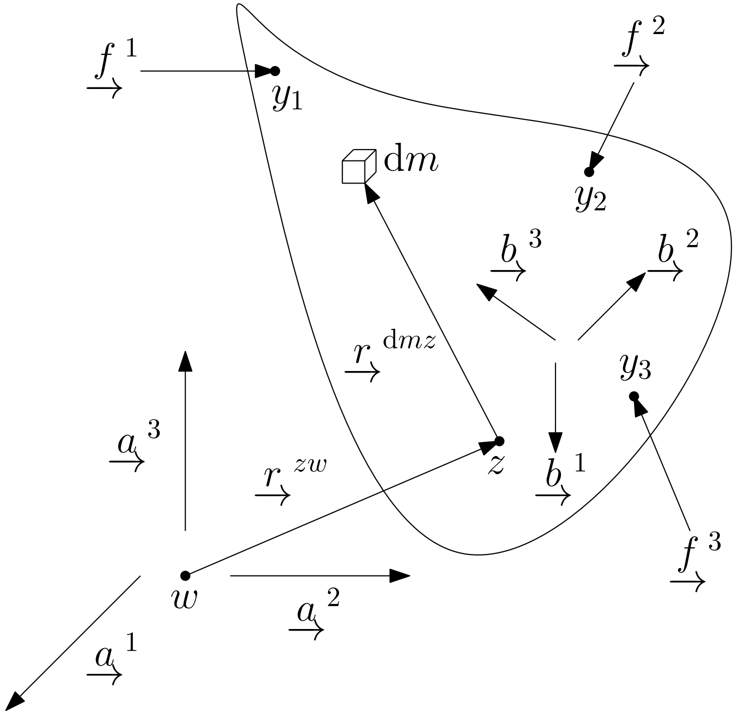
$$\begin{bmatrix} \mathbf{0} & \bar{\mathbf{c}}_{ba}^{3^{\mathsf{T}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{c}}_{ba}^{1^{\mathsf{T}}} \\ \bar{\mathbf{c}}_{ba}^{2^{\mathsf{T}}} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & -\bar{\mathbf{c}}_{ba}^{3} & \bar{\mathbf{c}}_{ba}^{2} \\ \bar{\mathbf{c}}_{ba}^{3} & \mathbf{0} & -\bar{\mathbf{c}}_{ba}^{1} \\ -\bar{\mathbf{c}}_{ba}^{2} & \bar{\mathbf{c}}_{ba}^{1} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{c}}_{ba}^{3^{\mathsf{T}}}\bar{\mathbf{c}}_{ba}^{3} & 0 & -\bar{\mathbf{c}}_{ba}^{3^{\mathsf{T}}}\bar{\mathbf{c}}_{ba}^{1} \\ -\bar{\mathbf{c}}_{ba}^{1^{\mathsf{T}}}\bar{\mathbf{c}}_{ba}^{2} & \bar{\mathbf{c}}_{ba}^{1^{\mathsf{T}}}\bar{\mathbf{c}}_{ba}^{1} & 0 \\ 0 & -\bar{\mathbf{c}}_{ba}^{2^{\mathsf{T}}}\bar{\mathbf{c}}_{ba}^{3} & \bar{\mathbf{c}}_{ba}^{2^{\mathsf{T}}}\bar{\mathbf{c}}_{ba}^{2} \end{bmatrix} = \mathbf{1},$$

that is, $\mathbf{S}_b^{ba}(\bar{\mathbf{c}}_{ba})\boldsymbol{\Gamma}_b^{ba}(\bar{\mathbf{c}}_{ba}) = \mathbf{1}$.

EOM via Lagrange's Equation

- Now onto deriving the EOM of a continuous rigid-body via Lagrange's equation.
- Follow the four steps to success:

1. Kinematics
   1.1) Frames and DCMs,
   2.2) Angular Velocity,
   3.3) Position,
   4.4) Velocity,
   5.5) Constraints.
2. Kinetic Energy, Potential Energy, and the Lagrangian
   1.1) $T_{\mathcal{B}w/a}$,
   2.2) $U_{\mathcal{B}w}$,
   3.3) $L_{\mathcal{B}w/a} = T_{\mathcal{B}w/a} - U_{\mathcal{B}w}$.
3. The Method of Virtual Work (MVW) and the Generalized Forces and Moments (GFM)
4. Lagrange's Equation
   1.1) $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{\mathcal{B}w/a}}{\partial \dot{\mathbf{q}}}\right)$,
   2.2) $\frac{\partial L_{\mathcal{B}w/a}}{\partial \mathbf{q}}$,
   3.3) $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{\mathcal{B}w/a}}{\partial \dot{\mathbf{q}}}\right)^{\mathsf{T}} - \left(\frac{\partial L_{\mathcal{B}w/a}}{\partial \mathbf{q}}\right)^{\mathsf{T}} = \mathbf{\Xi}^{\mathsf{T}}\boldsymbol{\lambda} + \boldsymbol{f}$.



$\underset{\rightarrow}{f}^{1}$
y_1
$\underset{\rightarrow}{f}^{2}$
$\mathrm{d}m$
y_2
$\underset{\rightarrow}{b}^{3}$
$\underset{\rightarrow}{b}^{2}$
$\underset{\rightarrow}{r}^{\mathrm{d}mz}$
y_3
$\underset{\rightarrow}{a}^{3}$
$\underset{\rightarrow}{r}^{zw}$
z
$\underset{\rightarrow}{b}^{1}$
$\underset{\rightarrow}{f}^{3}$
w
$\underset{\rightarrow}{a}^{2}$
$\underset{\rightarrow}{a}^{1}$

Step 1: Kinematics

1.1 $\mathcal{F}_a \rightarrow \mathcal{F}_b$, $\mathbf{C}_{ba}$.

1.2 $\boldsymbol{\omega}_b^{ba} = \mathbf{S}_b^{ba}(\mathbf{q}^{ba})\dot{\mathbf{q}}^{ba}$, $\dot{\mathbf{q}}^{ba} = \boldsymbol{\Gamma}_b^{ba}(\mathbf{q}^{ba})\boldsymbol{\omega}_b^{ba}$.

1.3 $\underrightarrow{r}^{\mathrm{d}m\,w} = \underrightarrow{r}^{\mathrm{d}m\,z} + \underrightarrow{r}^{zw}$ where $\underrightarrow{r}^{zw} = \underrightarrow{\mathcal{F}}_a^{\mathsf{T}}\mathbf{r}_a^{zw}$ and $\underrightarrow{r}^{\mathrm{d}m\,z} = \underrightarrow{\mathcal{F}}_b^{\mathsf{T}}\mathbf{r}_b^{\mathrm{d}m\,z}$.

1.4

$$\begin{aligned}
\underrightarrow{r}^{\mathrm{d}m\,w^{\bullet a}} &= \underrightarrow{r}^{\mathrm{d}m\,z^{\bullet a}} + \underrightarrow{r}^{zw^{\bullet a}} \\
&= \underrightarrow{r}^{zw^{\bullet a}} + \underrightarrow{\omega}^{ba} \times \underrightarrow{r}^{\mathrm{d}m\,z}, \\
\mathbf{v}_a^{\mathrm{d}m\,w/a} &= \mathbf{v}_a^{zw/a} + \mathbf{C}_{ba}^{\mathsf{T}}\boldsymbol{\omega}_b^{ba^\times}\mathbf{r}_b^{\mathrm{d}m\,z}
\end{aligned}$$

1.5 $\boldsymbol{\Xi}_b^{ba}\dot{\mathbf{q}}^{ba} = \mathbf{0}$. Picking $\mathbf{q} = \begin{bmatrix} \mathbf{r}_a^{zw} \\ \mathbf{q}^{ba} \end{bmatrix}$, $\dot{\mathbf{q}} = \begin{bmatrix} \mathbf{v}_a^{zw/a} \\ \dot{\mathbf{q}}^{ba} \end{bmatrix}$, we have

$$\underbrace{\begin{bmatrix} \mathbf{0} & \boldsymbol{\Xi}_b^{ba} \end{bmatrix}}_{\boldsymbol{\Xi}} \begin{bmatrix} \mathbf{v}_a^{zw/a} \\ \dot{\mathbf{q}}^{ba} \end{bmatrix} = \mathbf{0}.$$

Step 2) $L_{\mathcal{B}w/a} = T_{\mathcal{B}w/a} - U_{\mathcal{B}w}$

2.1

$$
\begin{aligned}
T_{\mathcal{B}w/a} &= \frac{1}{2}\int_{\mathcal{B}} \underset{\rightarrow}{v}^{\mathrm{d}mw/a} \cdot \underset{\rightarrow}{v}^{\mathrm{d}mw/a}\,\mathrm{d}m \\
&= \frac{1}{2}\int_{\mathcal{B}} \left(\mathbf{v}_a^{zw/a^{\mathsf{T}}} - \mathbf{r}_b^{\mathrm{d}m\,z^{\mathsf{T}}}\boldsymbol{\omega}_b^{ba^{\times}}\mathbf{C}_{ba}\right)\left(\mathbf{v}_a^{zw/a} + \mathbf{C}_{ba}^{\mathsf{T}}\boldsymbol{\omega}_b^{ba^{\times}}\mathbf{r}_b^{\mathrm{d}m\,z}\right)\mathrm{d}m \\
&= \frac{1}{2}m_{\mathcal{B}}\mathbf{v}_a^{zw/a^{\mathsf{T}}}\mathbf{v}_a^{zw/a} - \mathbf{v}_a^{zw/a^{\mathsf{T}}}\mathbf{C}_{ba}^{\mathsf{T}}\mathbf{c}_b^{\mathcal{B}\,z^{\times}}\boldsymbol{\omega}_b^{ba} + \frac{1}{2}\boldsymbol{\omega}_b^{ba^{\mathsf{T}}}\mathbf{J}_b^{\mathcal{B}\,z}\boldsymbol{\omega}_b^{ba} \\
&= \frac{1}{2}\begin{bmatrix} \mathbf{v}_a^{zw/a^{\mathsf{T}}} & \boldsymbol{\omega}_b^{ba^{\mathsf{T}}} \end{bmatrix} \underbrace{\begin{bmatrix} m_{\mathcal{B}}\mathbf{1} & -\mathbf{C}_{ba}^{\mathsf{T}}\mathbf{c}_b^{\mathcal{B}\,z^{\times}} \\ \mathbf{c}_b^{\mathcal{B}\,z^{\times}}\mathbf{C}_{ba} & \mathbf{J}_b^{\mathcal{B}\,z} \end{bmatrix}}_{\mathbf{M}(\mathbf{q})} \underbrace{\begin{bmatrix} \mathbf{v}_a^{zw/a} \\ \boldsymbol{\omega}_b^{ba} \end{bmatrix}}_{\boldsymbol{\nu}} \\
&= \frac{1}{2}\boldsymbol{\nu}^{\mathsf{T}}\mathbf{M}(\mathbf{q})\boldsymbol{\nu}.
\end{aligned}
$$

Using $\boldsymbol{\omega}_b^{ba} = \mathbf{S}_b^{ba}(\mathbf{q}^{ba})\dot{\mathbf{q}}^{ba}$ can also write

$$T_{\mathcal{B}w/a} =$$

$$\frac{1}{2}\begin{bmatrix} \mathbf{v}_a^{zw/a^\mathsf{T}} & \dot{\mathbf{q}}^{ba^\mathsf{T}} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_b^{ba^\mathsf{T}} \end{bmatrix} \begin{bmatrix} m_{\mathcal{B}}\mathbf{1} & -\mathbf{C}_{ba}^\mathsf{T}\mathbf{c}_b^{\mathcal{B}z^\times} \\ \mathbf{c}_b^{\mathcal{B}z^\times}\mathbf{C}_{ba} & \mathbf{J}_b^{\mathcal{B}z} \end{bmatrix} \underbrace{\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_b^{ba} \end{bmatrix}}_{\mathbf{S}} \underbrace{\begin{bmatrix} \mathbf{v}_a^{zw/a} \\ \dot{\mathbf{q}}^{ba} \end{bmatrix}}_{\dot{\mathbf{q}}}.$$

Thus

$$T_{\mathcal{B}w/a} = \tfrac{1}{2}\boldsymbol{\nu}^\mathsf{T}\mathbf{M}(\mathbf{q})\boldsymbol{\nu} = \tfrac{1}{2}\dot{\mathbf{q}}^\mathsf{T}\mathbf{S}^\mathsf{T}\mathbf{M}(\mathbf{q})\mathbf{S}\dot{\mathbf{q}}.$$

2.2 Assume no potential energy terms, hence $U_{\mathcal{B}w} = 0$.

2.3

$$L_{\mathcal{B}w/a} = T_{\mathcal{B}w/a} = \frac{1}{2}\boldsymbol{\nu}^{\mathsf{T}}\mathbf{M}(\mathbf{q})\boldsymbol{\nu} = \frac{1}{2}\dot{\mathbf{q}}^{\mathsf{T}}\mathbf{S}^{\mathsf{T}}\mathbf{M}(\mathbf{q})\mathbf{S}\dot{\mathbf{q}}.$$

Step 3) MVW and FGM

- $\delta W^i_{\mathcal{B}w} = \underset{\rightarrow}{f}^{i} \cdot \delta \underset{\rightarrow}{r}^{iw} = \mathbf{f}_a^{i^\mathsf{T}} \delta \mathbf{r}_a^{iw}$ (where, for simplicity, $\underset{\rightarrow}{r}^{y_i w} = \underset{\rightarrow}{r}^{iw}$).
-
$$\begin{aligned} \underset{\rightarrow}{r}^{iw} &= \underset{\rightarrow}{r}^{iz} + \underset{\rightarrow}{r}^{zw} \\ &= \underset{\rightarrow}{\mathcal{F}}_a^\mathsf{T} \mathbf{r}_a^{zw} + \underset{\rightarrow}{\mathcal{F}}_b^\mathsf{T} \mathbf{r}_b^{iz}, \\ \mathbf{r}_a^{iw} &= \mathbf{r}_a^{zw} + \mathbf{C}_{ba}^\mathsf{T} \mathbf{r}_b^{iz}, \\ \delta \mathbf{r}_a^{iw} &= \delta \mathbf{r}_a^{zw} + \frac{\partial (\mathbf{C}_{ba}^\mathsf{T} \mathbf{r}_b^{iz})}{\partial \mathbf{q}^{ba}} \delta \mathbf{q}^{ba}. \end{aligned}$$
-
$$\begin{aligned} \delta W^i_{\mathcal{B}w} &= \mathbf{f}_a^{i^\mathsf{T}} \delta \mathbf{r}_a^{iw} \\ &= \mathbf{f}_a^{i^\mathsf{T}} \delta \mathbf{r}_a^{zw} + \mathbf{f}_a^{i^\mathsf{T}} \frac{\partial (\mathbf{C}_{ba}^\mathsf{T} \mathbf{r}_b^{iz})}{\partial \mathbf{q}^{ba}} \delta \mathbf{q}^{ba} \\ &= \underbrace{\begin{bmatrix} \delta \mathbf{r}_a^{zw^\mathsf{T}} & \delta \mathbf{q}^{ba^\mathsf{T}} \end{bmatrix}}_{\delta \mathbf{q}^\mathsf{T}} \underbrace{\begin{bmatrix} \mathbf{f}_a^i \\ \frac{\partial (\mathbf{C}_{ba}^\mathsf{T} \mathbf{r}_b^{iz})}{\partial \mathbf{q}^{ba}}^\mathsf{T} \mathbf{f}_a^i \end{bmatrix}}_{\boldsymbol{f}^i}. \end{aligned}$$

Thus

$$\begin{aligned} \delta W_{\mathcal{B}w} &= \sum_{i=1}^{N_i} \delta W^i_{\mathcal{B}w} \\ &= \delta \mathbf{q}^{\mathsf{T}} \boldsymbol{f}, \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{f} &= \sum_{i=1}^{N_i} \boldsymbol{f}^i \\ &= \begin{bmatrix} \sum_{i=1}^{N_i} \mathbf{f}^i_a \\ \sum_{i=1}^{N_i} \frac{\partial (\mathbf{C}^{\mathsf{T}}_{ba} \mathbf{r}^{iz}_b)}{\partial \mathbf{q}^{ba}}^{\mathsf{T}} \mathbf{f}^i_a \end{bmatrix} \end{aligned}$$

are the generalized forces and moments.

Step 4) Lagrange's Equation

- Generalized coordinates and generalized coordinate rates are $\mathbf{q} = \begin{bmatrix} \mathbf{r}_a^{zw/a} \\ \mathbf{q}^{ba} \end{bmatrix}$ and $\dot{\mathbf{q}} = \begin{bmatrix} \mathbf{v}_a^{zw/a} \\ \dot{\mathbf{q}}^{ba} \end{bmatrix}$, respectively.
- Recall that $L_{\mathcal{B}w/a} = T_{\mathcal{B}w/a} = \frac{1}{2}\boldsymbol{\nu}^{\mathsf{T}}\mathbf{M}(\mathbf{q})\nu = \frac{1}{2}\dot{\mathbf{q}}^{\mathsf{T}}\mathbf{S}^{\mathsf{T}}\mathbf{M}(\mathbf{q})\mathbf{S}\dot{\mathbf{q}}$. Thus

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{\mathcal{B}w/a}}{\partial \dot{\mathbf{q}}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\mathbf{q}}^{\mathsf{T}}\mathbf{S}^{\mathsf{T}}\mathbf{M}(\mathbf{q})\mathbf{S}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{\nu}^{\mathsf{T}}\mathbf{M}(\mathbf{q})\mathbf{S}\right) \\ &= \dot{\boldsymbol{\nu}}^{\mathsf{T}}\mathbf{M}(\mathbf{q})\mathbf{S} + \boldsymbol{\nu}^{\mathsf{T}}\dot{\mathbf{M}}(\mathbf{q})\mathbf{S} + \boldsymbol{\nu}^{\mathsf{T}}\mathbf{M}(\mathbf{q})\dot{\mathbf{S}}, \end{aligned}$$

where

$$\dot{\mathbf{S}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dot{\mathbf{S}}_b^{ba} \end{bmatrix}.$$

- Recall that $\dot{\mathbf{C}}_{ba} = -\boldsymbol{\omega}_b^{ba^\times} \mathbf{C}_{ba}$ and

$$\mathbf{M}(\mathbf{q}) = \begin{bmatrix} m_{\mathcal{B}} \mathbf{1} & -\mathbf{C}_{ba}^{\mathsf{T}} \mathbf{c}_b^{\mathcal{B}\,z^\times} \\ \mathbf{c}_b^{\mathcal{B}\,z^\times} \mathbf{C}_{ba} & \mathbf{J}_b^{\mathcal{B}\,z} \end{bmatrix}.$$

- Thus

$$\begin{aligned} \dot{\mathbf{M}}(\mathbf{q}) &= \begin{bmatrix} \mathbf{0} & -\dot{\mathbf{C}}_{ba}^{\mathsf{T}} \mathbf{c}_b^{\mathcal{B}\,z^\times} \\ \mathbf{c}_b^{\mathcal{B}\,z^\times} \dot{\mathbf{C}}_{ba} & \mathbf{0} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0} & -\mathbf{C}_{ba}^{\mathsf{T}} \boldsymbol{\omega}_b^{ba^\times} \mathbf{c}_b^{\mathcal{B}\,z^\times} \\ -\mathbf{c}_b^{\mathcal{B}\,z^\times} \boldsymbol{\omega}_b^{ba^\times} \mathbf{C}_{ba} & \mathbf{0} \end{bmatrix}. \end{aligned}$$

Write $\frac{\partial L_{\mathcal{B}w/a}}{\partial \mathbf{q}}$ as

$$
\begin{aligned}
\frac{\partial L_{\mathcal{B}w/a}}{\partial \mathbf{q}} &= \begin{bmatrix} \frac{\partial L_{\mathcal{B}w/a}}{\partial \mathbf{r}_a^{zw}} & \frac{\partial L_{\mathcal{B}w/a}}{\partial \mathbf{q}^{ba}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial L_{\mathcal{B}w/a}}{\partial \mathbf{r}_a^{zw}} & \mathbf{0} \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{\partial L_{\mathcal{B}w/a}}{\partial \mathbf{v}^{zw/a}} & \frac{\partial L_{\mathcal{B}w/a}}{\partial \boldsymbol{\omega}_b^{ba}} \end{bmatrix}}_{\frac{\partial L_{\mathcal{B}w/a}}{\partial \boldsymbol{\nu}}} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\partial \boldsymbol{\omega}_b^{ba}}{\partial \mathbf{q}^{ba}} \end{bmatrix} \\
&\quad + \begin{bmatrix} \mathbf{0} & \frac{\hat{\partial} L_{\mathcal{B}w/a}}{\hat{\partial} \mathbf{q}^{ba}}, \end{bmatrix}
\end{aligned}
$$

where we have used the chain rule of differentiation.

The term $\frac{\hat{\partial} L_{\mathcal{B}w/a}}{\hat{\partial} \mathbf{q}^{ba}}$ is the partial derivative of $L_{\mathcal{B}w/a}$ excluding the $\mathbf{q}^{ba}$ dependence of $\boldsymbol{\omega}_b^{ba} = \mathbf{S}_b^{ba}(\mathbf{q}^{ba})\dot{\mathbf{q}}^{ba}$.

- $\frac{\partial L_{\mathcal{B}w/a}}{\partial \mathbf{r}_a^{zw}} = \mathbf{0}$. (This is not always the case, especially when gravity is present.)
- $$\frac{\partial L_{\mathcal{B}w/a}}{\partial \boldsymbol{\nu}} = \boldsymbol{\nu}^{\mathsf{T}}\mathbf{M}(\mathbf{q}).$$
- To compute $\frac{\hat{\partial} L_{\mathcal{B}w/a}}{\hat{\partial} \mathbf{q}^{ba}}$ first recall that

$$L_{\mathcal{B}w/a} = T_{\mathcal{B}w/a} = \tfrac{1}{2} m_{\mathcal{B}} \mathbf{v}_a^{zw/a^{\mathsf{T}}} \mathbf{v}_a^{zw/a} - \mathbf{v}_a^{zw/a^{\mathsf{T}}} \mathbf{C}_{ba}^{\mathsf{T}} \mathbf{c}_b^{\mathcal{B}z^{\times}} \boldsymbol{\omega}_b^{ba} + \tfrac{1}{2} \boldsymbol{\omega}_b^{ba^{\mathsf{T}}} \mathbf{J}_b^{\mathcal{B}z} \boldsymbol{\omega}_b^{ba}.$$

- Neglecting the $\boldsymbol{\omega}_b^{ba}$ terms when computing $\frac{\hat{\partial} L_{\mathcal{B}w/a}}{\hat{\partial} \mathbf{q}^{ba}}$ it follows that

$$\begin{aligned}
\frac{\hat{\partial} L_{\mathcal{B}w/a}}{\hat{\partial} \mathbf{q}^{ba}} &= \frac{\hat{\partial}}{\hat{\partial} \mathbf{q}^{ba}} \left(-\mathbf{v}_a^{zw/a^{\mathsf{T}}} \mathbf{C}_{ba}^{\mathsf{T}} \mathbf{c}_b^{\mathcal{B}z^{\times}} \boldsymbol{\omega}_b^{ba} \right) \\
&= \frac{\hat{\partial}}{\hat{\partial} \mathbf{q}^{ba}} \left(\boldsymbol{\omega}_b^{ba^{\mathsf{T}}} \mathbf{c}_b^{\mathcal{B}z^{\times}} \mathbf{C}_{ba} \mathbf{v}_a^{zw/a} \right) \\
&= \boldsymbol{\omega}_b^{ba^{\mathsf{T}}} \mathbf{c}_b^{\mathcal{B}z^{\times}} \frac{\partial (\mathbf{C}_{ba} \mathbf{v}_a^{zw/a})}{\partial \mathbf{q}^{ba}}
\end{aligned}$$

- Therefore

$$
\begin{aligned}
\frac{\partial L_{\mathcal{B}w/a}}{\partial \mathbf{q}} &= \left[\begin{array}{cc} \frac{\partial L_{\mathcal{B}w/a}}{\partial \mathbf{r}_a^{zw}} & 0 \end{array}\right] + \underbrace{\left[\begin{array}{cc} \frac{\partial L_{\mathcal{B}w/a}}{\partial \mathbf{v}^{zw/a}} & \frac{\partial L_{\mathcal{B}w/a}}{\partial \boldsymbol{\omega}_b^{ba}} \end{array}\right]}_{\frac{\partial L_{\mathcal{B}w/a}}{\partial \boldsymbol{\nu}}} \left[\begin{array}{cc} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\partial \boldsymbol{\omega}_b^{ba}}{\partial \mathbf{q}^{ba}} \end{array}\right] \\
&\quad + \left[\begin{array}{cc} \mathbf{0} & \frac{\hat{\partial} L_{\mathcal{B}w/a}}{\hat{\partial} \mathbf{q}^{ba}} \end{array}\right] \\
&= \boldsymbol{\nu}^{\mathsf{T}} \mathbf{M}(\mathbf{q}) \left[\begin{array}{cc} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\partial \boldsymbol{\omega}_b^{ba}}{\partial \mathbf{q}^{ba}} \end{array}\right] + \left[\begin{array}{cc} \mathbf{0} & \boldsymbol{\omega}_b^{ba^{\mathsf{T}}} \mathbf{c}_b^{\mathcal{B}\,z^{\times}} \frac{\partial (\mathbf{C}_{ba} \mathbf{v}_a^{zw/a})}{\partial \mathbf{q}^{ba}} \end{array}\right].
\end{aligned}
$$

Using Lagrange's Equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{\mathcal{B}w/a}}{\partial \dot{\mathbf{q}}}\right)^{\mathsf{T}} - \left(\frac{\partial L_{\mathcal{B}w/a}}{\partial \mathbf{q}}\right)^{\mathsf{T}} = \boldsymbol{f} + \boldsymbol{\Xi}^{\mathsf{T}}\boldsymbol{\lambda}$$

we (finally) have

$$\begin{aligned}
&\mathbf{S}^{\mathsf{T}}\mathbf{M}(\mathbf{q})\dot{\boldsymbol{\nu}} + \mathbf{S}^{\mathsf{T}}\dot{\mathbf{M}}(\mathbf{q})\boldsymbol{\nu} + \dot{\mathbf{S}}^{\mathsf{T}}\mathbf{M}(\mathbf{q})\boldsymbol{\nu} \\
&- \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\partial \boldsymbol{\omega}_b^{ba}}{\partial \mathbf{q}^{ba}}^{\mathsf{T}} \end{bmatrix} \mathbf{M}(\mathbf{q})\boldsymbol{\nu} - \begin{bmatrix} \mathbf{0} \\ -\frac{\partial(\mathbf{C}_{ba}\mathbf{v}_a^{zw/a})}{\partial \mathbf{q}^{ba}}^{\mathsf{T}} \mathbf{c}_b^{\mathcal{B}\,z^{\times}} \boldsymbol{\omega}_b^{ba} \end{bmatrix} = \boldsymbol{f} + \boldsymbol{\Xi}^{\mathsf{T}}\boldsymbol{\lambda}.
\end{aligned}$$

Okay ... great ... now what?

Can we "get rid of" the partial derivative terms somehow?

- Define

$$\boldsymbol{\Gamma} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Gamma}_b^{ba} \end{bmatrix}.$$

- Note that

$$\mathbf{S}\boldsymbol{\Gamma} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_b^{ba} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Gamma}_b^{ba} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_b^{ba}\boldsymbol{\Gamma}_b^{ba} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}.$$

- The matrix $\boldsymbol{\Gamma}$ is also a mapping between the augmented velocities and the generalized coordinate rates, that is

$$\begin{bmatrix} \mathbf{v}_a^{zw/a} \\ \dot{\mathbf{q}}^{ba} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Gamma}_b^{ba} \end{bmatrix} \begin{bmatrix} \mathbf{v}_a^{zw/a} \\ \boldsymbol{\omega}_b^{ba} \end{bmatrix}.$$

Premultiplying

$$
\mathbf{S}^{\mathsf{T}}\mathbf{M}(\mathbf{q})\dot{\boldsymbol{\nu}} + \mathbf{S}^{\mathsf{T}}\dot{\mathbf{M}}(\mathbf{q})\boldsymbol{\nu} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \left(\dot{\mathbf{S}}_b^{ba^{\mathsf{T}}} - \frac{\partial \boldsymbol{\omega}_b^{ba}}{\partial \mathbf{q}^{ba}}^{\mathsf{T}}\right) \end{bmatrix} \mathbf{M}(\mathbf{q})\boldsymbol{\nu}
+ \begin{bmatrix} \mathbf{0} \\ \frac{\partial(\mathbf{C}_{ba}\mathbf{v}_a^{zw/a})}{\partial \mathbf{q}^{ba}}^{\mathsf{T}} \mathbf{c}_b^{\mathcal{B}\, z^{\times}} \boldsymbol{\omega}_b^{ba} \end{bmatrix} = \boldsymbol{f} + \boldsymbol{\Xi}^{\mathsf{T}}\boldsymbol{\lambda}
$$

by $\boldsymbol{\Gamma}^{\mathsf{T}}$ and using the identities

$$
\boldsymbol{\Gamma}_b^{ba^{\mathsf{T}}}\left(\dot{\mathbf{S}}_b^{ba^{\mathsf{T}}} - \frac{\partial \boldsymbol{\omega}_b^{ba}}{\partial \mathbf{q}^{ba}}^{\mathsf{T}}\right) = \boldsymbol{\omega}_b^{ba^{\times}}, \quad \boldsymbol{\Gamma}_b^{ba^{\mathsf{T}}} \frac{\partial(\mathbf{C}_{ba}\mathbf{v}_a^{zw/a})}{\partial \mathbf{q}^{ba}}^{\mathsf{T}} = -\left(\mathbf{C}_{ba}\mathbf{v}_a^{zw/a}\right)^{\times}
$$

yields ...

$$
\mathbf{M}(\mathbf{q})\dot{\boldsymbol{\nu}} + \dot{\mathbf{M}}(\mathbf{q})\boldsymbol{\nu} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\omega}_b^{ba^\times} \end{bmatrix} \mathbf{M}(\mathbf{q})\boldsymbol{\nu}
$$

$$
+ \begin{bmatrix} \mathbf{0} \\ -\left(\mathbf{C}_{ba}\mathbf{v}_a^{zw/a}\right)^\times \mathbf{c}_b^{\mathcal{B}\,z^\times} \boldsymbol{\omega}_b^{ba} \end{bmatrix} = \boldsymbol{\Gamma}^\mathsf{T}\boldsymbol{f} + \boldsymbol{\Gamma}^\mathsf{T}\boldsymbol{\Xi}^\mathsf{T}\boldsymbol{\lambda}
$$

Well, we removed the partial derivative terms! That's great!

But, what happens to $\boldsymbol{\Gamma}^\mathsf{T}\boldsymbol{\Xi}^\mathsf{T}$? I ... wonder ... if ... herm ... if ... could it be that ...

Recalling that $\boldsymbol{\Xi}_b^{ba}\boldsymbol{\Gamma}_b^{ba} = \mathbf{0}$ we have

$$\boldsymbol{\Xi}\boldsymbol{\Gamma} = \begin{bmatrix} \mathbf{0} & \boldsymbol{\Xi}_b^{ba} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Gamma}_b^{ba} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Xi}_b^{ba}\boldsymbol{\Gamma}_b^{ba} \end{bmatrix} = \mathbf{0}.$$

Thus, $\boldsymbol{\Xi}$ and $\boldsymbol{\Gamma}$ are orthogonal complements.

The $\boldsymbol{\Gamma}^{\mathsf{T}}\boldsymbol{\Xi}^{\mathsf{T}}\boldsymbol{\lambda}$ drops out!

What happens to $\boldsymbol{\Gamma}^{\mathsf{T}}\boldsymbol{f}$ ...

Using the identity $\boldsymbol{\Gamma}_b^{ba^{\mathsf{T}}} \frac{\partial(\mathbf{C}_{ba}^{\mathsf{T}}\mathbf{r}_b^{iz})}{\partial \mathbf{q}^{ba}}^{\mathsf{T}} = \mathbf{r}_b^{iz^{\times}} \mathbf{C}_{ba}$ gives

$$
\begin{aligned}
\boldsymbol{\Gamma}^{\mathsf{T}} \boldsymbol{f} &= \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Gamma}_b^{ba^{\mathsf{T}}} \end{bmatrix} \begin{bmatrix} \sum_{i=1}^{N_i} \mathbf{f}_a^i \\ \sum_{i=1}^{N_i} \frac{\partial(\mathbf{C}_{ba}^{\mathsf{T}}\mathbf{r}_b^{iz})}{\partial \mathbf{q}^{ba}}^{\mathsf{T}} \mathbf{f}_a^i \end{bmatrix} \\
&= \begin{bmatrix} \sum_{i=1}^{N_i} \mathbf{f}_a^i \\ \sum_{i=1}^{N_i} \boldsymbol{\Gamma}_b^{ba^{\mathsf{T}}} \frac{\partial(\mathbf{C}_{ba}^{\mathsf{T}}\mathbf{r}_b^{iz})}{\partial \mathbf{q}^{ba}}^{\mathsf{T}} \mathbf{f}_a^i \end{bmatrix} \\
&= \begin{bmatrix} \sum_{i=1}^{N_i} \mathbf{f}_a^i \\ \sum_{i=1}^{N_i} \mathbf{r}_b^{iz^{\times}} \mathbf{C}_{ba} \mathbf{f}_a^i \end{bmatrix} \qquad \text{(Note that } \ \mathbf{f}_b^i = \mathbf{C}_{ba}\mathbf{f}_a^i.\text{)} \\
&= \begin{bmatrix} \sum_{i=1}^{N_i} \mathbf{f}_a^i \\ \sum_{i=1}^{N_i} \mathbf{m}_b^{iz} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{f}_a^{\mathcal{B}} \\ \mathbf{m}_b^{\mathcal{B}z} \end{bmatrix}.
\end{aligned}
$$

Therefore,

$$\mathbf{M}(\mathbf{q})\dot{\boldsymbol{\nu}} + \dot{\mathbf{M}}(\mathbf{q})\boldsymbol{\nu} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\omega}_b^{ba\times} \end{bmatrix} \mathbf{M}(\mathbf{q})\boldsymbol{\nu} + \begin{bmatrix} \mathbf{0} \\ -\left(\mathbf{C}_{ba}\mathbf{v}_a^{zw/a}\right)^{\times} \mathbf{c}_b^{\mathcal{B}z\times} \boldsymbol{\omega}_b^{ba} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_a^{\mathcal{B}} \\ \mathbf{m}_b^{\mathcal{B}z} \end{bmatrix}$$

Okay, cool. But can we simplify this further!? Yep ...

We can write

$$
\begin{aligned}
\dot{\mathbf{M}}(\mathbf{q})\boldsymbol{\nu} &= \begin{bmatrix} \mathbf{0} & -\mathbf{C}_{ba}^{\mathsf{T}}\boldsymbol{\omega}_b^{ba^\times}\mathbf{c}_b^{\mathcal{B}z^\times} \\ -\mathbf{c}_b^{\mathcal{B}z^\times}\boldsymbol{\omega}_b^{ba^\times}\mathbf{C}_{ba} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{v}_a^{zw/a} \\ \boldsymbol{\omega}_b^{ba} \end{bmatrix} \\
&= \begin{bmatrix} -\mathbf{C}_{ba}^{\mathsf{T}}\boldsymbol{\omega}_b^{ba^\times}\mathbf{c}_b^{\mathcal{B}z^\times}\boldsymbol{\omega}_b^{ba} \\ -\mathbf{c}_b^{\mathcal{B}z^\times}\boldsymbol{\omega}_b^{ba^\times}\mathbf{C}_{ba}\mathbf{v}_a^{zw/a} \end{bmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\omega}_b^{ba^\times} \end{bmatrix}\mathbf{M}(\mathbf{q})\boldsymbol{\nu} &= \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\omega}_b^{ba^\times} \end{bmatrix} \begin{bmatrix} m_{\mathcal{B}}\mathbf{v}_a^{zw/a} - \mathbf{C}_{ba}^{\mathsf{T}}\mathbf{c}_b^{\mathcal{B}z^\times}\boldsymbol{\omega}_b^{ba} \\ \mathbf{c}_b^{\mathcal{B}z^\times}\mathbf{C}_{ba}\mathbf{v}_a^{zw/a} + \mathbf{J}_b^{\mathcal{B}z}\boldsymbol{\omega}_b^{ba} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\omega}_b^{ba^\times}\mathbf{c}_b^{\mathcal{B}z^\times}\mathbf{C}_{ba}\mathbf{v}_a^{zw/a} + \boldsymbol{\omega}_b^{ba^\times}\mathbf{J}_b^{\mathcal{B}z}\boldsymbol{\omega}_b^{ba} \end{bmatrix}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\mathbf{M}(\mathbf{q})\dot{\boldsymbol{\nu}} \\
&+ \left[\begin{array}{c} -\mathbf{C}_{ba}^{\mathsf{T}} \boldsymbol{\omega}_b^{ba^\times} \mathbf{c}_b^{\mathcal{B}z^\times} \boldsymbol{\omega}_b^{ba} \\ -\mathbf{c}_b^{\mathcal{B}z^\times} \boldsymbol{\omega}_b^{ba^\times} \mathbf{C}_{ba} \mathbf{v}_a^{zw/a} + \boldsymbol{\omega}_b^{ba^\times} \mathbf{c}_b^{\mathcal{B}z^\times} \mathbf{C}_{ba} \mathbf{v}_a^{zw/a} + \boldsymbol{\omega}_b^{ba^\times} \mathbf{J}_b^{\mathcal{B}z} \boldsymbol{\omega}_b^{ba} - \left(\mathbf{C}_{ba} \mathbf{v}_a^{zw/a} \right)^\times \mathbf{c}_b^{\mathcal{B}z^\times} \boldsymbol{\omega}_b^{ba} \end{array} \right] \\
&= \left[\begin{array}{c} \mathbf{f}_a^{\mathcal{B}} \\ \mathbf{m}_b^{\mathcal{B}z} \end{array} \right],
\end{aligned}
$$

which simplifies to

$$
\mathbf{M}(\mathbf{q})\dot{\boldsymbol{\nu}} + \underbrace{\left[\begin{array}{c} -\mathbf{C}_{ba}^{\mathsf{T}} \boldsymbol{\omega}_b^{ba^\times} \mathbf{c}_b^{\mathcal{B}z^\times} \boldsymbol{\omega}_b^{ba} \\ \boldsymbol{\omega}_b^{ba^\times} \mathbf{J}_b^{\mathcal{B}z} \boldsymbol{\omega}_b^{ba} \end{array} \right]}_{-\mathbf{f}_{\text{non}}(\mathbf{q},\dot{\mathbf{q}})} = \left[\begin{array}{c} \mathbf{f}_a^{\mathcal{B}} \\ \mathbf{m}_b^{\mathcal{B}z} \end{array} \right],
$$

which finally can be written concisely as

$$
\mathbf{M}(\mathbf{q})\dot{\boldsymbol{\nu}} = \mathbf{f}_{\text{non}}(\mathbf{q}, \dot{\mathbf{q}}) + \left[\begin{array}{c} \mathbf{f}_a^{\mathcal{B}} \\ \mathbf{m}_b^{\mathcal{B}z} \end{array} \right].
$$

Equivalence to Newton-Euler Approach

It would be nice to show that the above is equivalent to what a Newton-Euler approach would give.

We can write

$$\mathbf{M}(\mathbf{q})\dot{\boldsymbol{\nu}} + \begin{bmatrix} -\mathbf{C}_{ba}^{\mathsf{T}}\boldsymbol{\omega}_b^{ba^\times}\mathbf{c}_b^{\mathcal{B}z^\times}\boldsymbol{\omega}_b^{ba} \\ \boldsymbol{\omega}_b^{ba^\times}\mathbf{J}_b^{\mathcal{B}z}\boldsymbol{\omega}_b^{ba} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_a^{\mathcal{B}} \\ \mathbf{m}_b^{\mathcal{B}z} \end{bmatrix},$$

as

$$\begin{bmatrix} m_{\mathcal{B}}\dot{\mathbf{v}}_a^{zw/a} - \mathbf{C}_{ba}^{\mathsf{T}}\mathbf{c}_b^{\mathcal{B}z^\times}\dot{\boldsymbol{\omega}}_b^{ba} - \mathbf{C}_{ba}^{\mathsf{T}}\boldsymbol{\omega}_b^{ba^\times}\mathbf{c}_b^{\mathcal{B}z^\times}\boldsymbol{\omega}_b^{ba} \\ \mathbf{c}_b^{\mathcal{B}z^\times}\mathbf{C}_{ba}\dot{\mathbf{v}}_a^{zw/a} + \mathbf{J}_b^{\mathcal{B}z}\dot{\boldsymbol{\omega}}_b^{ba} + \boldsymbol{\omega}_b^{ba^\times}\mathbf{J}_b^{\mathcal{B}z}\boldsymbol{\omega}_b^{ba} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_a^{\mathcal{B}} \\ \mathbf{m}_b^{\mathcal{B}z} \end{bmatrix}$$

Note that

$$\begin{aligned} \underrightarrow{v}^{zw/a^{\bullet a}} &= \underrightarrow{v}^{zw/a^{\bullet b}} + \underrightarrow{\omega}^{ba} \times \underrightarrow{v}^{zw/a}, \\ \mathbf{C}_{ba}\mathbf{a}_a^{zw/a/a} &= \mathbf{a}_b^{zw/a/b} + \boldsymbol{\omega}_b^{ba^\times}\mathbf{v}_b^{zw/a}. \quad (\star) \end{aligned}$$

Substitution $(\star)$ into the top row and premultiplying by $\mathbf{C}_{ba}$ gives

$$\begin{aligned} m_{\mathcal{B}}\mathbf{C}_{ba}^{\mathsf{T}}\mathbf{a}_b^{zw/a/b} + m_{\mathcal{B}}\mathbf{C}_{ba}^{\mathsf{T}}\boldsymbol{\omega}_b^{ba^\times}\mathbf{v}_b^{zw/a} - \mathbf{C}_{ba}^{\mathsf{T}}\mathbf{c}_b^{\mathcal{B}z^\times}\dot{\boldsymbol{\omega}}_b^{ba} - \mathbf{C}_{ba}^{\mathsf{T}}\boldsymbol{\omega}_b^{ba^\times}\mathbf{c}_b^{\mathcal{B}z^\times}\boldsymbol{\omega}_b^{ba} &= \mathbf{f}_a^{\mathcal{B}}, \\ m_{\mathcal{B}}\mathbf{a}_b^{zw/a/b} + m_{\mathcal{B}}\boldsymbol{\omega}_b^{ba^\times}\mathbf{v}_b^{zw/a} - \mathbf{c}_b^{\mathcal{B}z^\times}\dot{\boldsymbol{\omega}}_b^{ba} - \boldsymbol{\omega}_b^{ba^\times}\mathbf{c}_b^{\mathcal{B}z^\times}\boldsymbol{\omega}_b^{ba} &= \mathbf{f}_b^{\mathcal{B}}. \qquad (\dagger) \end{aligned}$$

Similarly, substitution of $(\star)$ into the bottom row gives

$$\begin{aligned} \mathbf{c}_b^{\mathcal{B}z^\times}\mathbf{a}_b^{zw/a/b} + \mathbf{c}_b^{\mathcal{B}z^\times}\boldsymbol{\omega}_b^{ba^\times}\mathbf{v}_b^{zw/a} + \mathbf{J}_b^{\mathcal{B}z}\dot{\boldsymbol{\omega}}_b^{ba} + \boldsymbol{\omega}_b^{ba^\times}\mathbf{J}_b^{\mathcal{B}z}\boldsymbol{\omega}_b^{ba} &= \mathbf{m}_b^{\mathcal{B}z}, \\ \mathbf{c}_b^{\mathcal{B}z^\times}\mathbf{a}_b^{zw/a/b} - \mathbf{c}_b^{\mathcal{B}z^\times}\mathbf{v}_b^{zw/a^\times}\boldsymbol{\omega}_b^{ba} + \mathbf{J}_b^{\mathcal{B}z}\dot{\boldsymbol{\omega}}_b^{ba} + \boldsymbol{\omega}_b^{ba^\times}\mathbf{J}_b^{\mathcal{B}z}\boldsymbol{\omega}_b^{ba} &= \mathbf{m}_b^{\mathcal{B}z}. \quad (\dagger\dagger) \end{aligned}$$

Together $(\dagger)$ and $(\dagger\dagger)$ can be written as ...

$$\mathbf{M}_b^{\mathcal{B}z}\dot{\boldsymbol{\nu}}_b + \boldsymbol{\nu}_b^{\otimes}\mathbf{M}_b^{\mathcal{B}z}\boldsymbol{\nu}_b = \boldsymbol{f}_b^{\mathcal{B}z},$$

where

$$\begin{aligned}
\mathbf{M}_b^{\mathcal{B}z} &= \begin{bmatrix} m_{\mathcal{B}}\mathbf{1} & -\mathbf{c}_b^{\mathcal{B}z^\times} \\ \mathbf{c}_b^{\mathcal{B}z^\times} & \mathbf{J}_b^{\mathcal{B}z} \end{bmatrix}, \\
\dot{\boldsymbol{\nu}}_b &= \begin{bmatrix} \mathbf{a}_b^{zw/a/b} \\ \boldsymbol{\omega}_b^{ba} \end{bmatrix}, \\
\boldsymbol{\nu}_b &= \begin{bmatrix} \mathbf{v}_b^{zw/a} \\ \boldsymbol{\omega}_b^{ba} \end{bmatrix}, \\
\boldsymbol{\nu}_b^{\otimes} &= \begin{bmatrix} \boldsymbol{\omega}_b^{ba^\times} & \mathbf{0} \\ \mathbf{v}_b^{zw/a^\times} & \boldsymbol{\omega}_b^{ba^\times} \end{bmatrix}, \\
\boldsymbol{f}_b^{\mathcal{B}z} &= \begin{bmatrix} \mathbf{f}_b^{\mathcal{B}} \\ \mathbf{m}_b^{\mathcal{B}z} \end{bmatrix}.
\end{aligned}$$

This is exactly what a Newton-Euler approach gives.

To Summarize

- Identities are key. They lead to three (critical) simplifications.
- The generalized forces and moments *are not* forces and moments. Must premultiply by $\mathbf{\Gamma}^{\mathsf{T}}$ to get the forces and moments.
- Identities lead to motion equation in terms of $\mathbf{r}_a^{zw}$ and $\mathbf{q}^{ba}$, $\mathbf{v}_a^{zw/a}$ and $\boldsymbol{\omega}_b^{ba}$, and $\dot{\mathbf{v}}_a^{zw/a}$ and $\dot{\boldsymbol{\omega}}_b^{ba}$, and not in term of $\mathbf{r}_a^{zw}$ and $\mathbf{q}^{ba}$, $\mathbf{v}_a^{zw/a}$ and $\dot{\mathbf{q}}^{ba}$, and $\dot{\mathbf{v}}_a^{zw/a}$ and $\ddot{\mathbf{q}}^{ba}$.
  - If the motion equation were in terms of $\mathbf{r}_a^{zw}$ and $\mathbf{q}^{ba}$, $\mathbf{v}_a^{zw/a}$ and $\dot{\mathbf{q}}^{ba}$, and $\dot{\mathbf{v}}_a^{zw/a}$ and $\ddot{\mathbf{q}}^{ba}$, they wouldn't be wrong, just the equivalence between the Lagrangian approach and the Newton-Euler approach would not be apparent.
- The strength of this approach is
  1. it hold for any DCM parameterization, or no parameterization of the DCM, and
  2. it easily generalizes to multiple bodies in the presence of both holonomic and nonholonomic constraints.

Notes and References

Material herein is based on [1, 2].

References

[1] J. R. Forbes, "Identities for Deriving Equations of Motion Using Constrained Attitude Parameterizations," *AIAA Journal of Guidance, Control, and Dynamics*, vol. 37, no. 4, pp. 1283–1289, 2014.

[2] A. de Ruiter and J. R. Forbes, "General Identities for Parameterizations of $SO(3)$ with Applications," *ASME Journal of Applied Mechanics*, vol. 81, pp. 071007 (1–16), 2014.