Assignment 1

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The definition of a subspace is the following :

A nonempty subset $\mathcal{S}$ of a vector space $\mathcal{V}$ is a subspace of $\mathcal{V}$ over the field $\mathbb{F}$ iff :

$$\forall \mathbf{v}^1, \mathbf{v}^2 \in \mathcal{S} \Rightarrow \mathbf{v}^1 + \mathbf{v}^2 \in \mathcal{S}$$

$$\forall \mathbf{v}^1 \in \mathcal{S} \Rightarrow \alpha \mathbf{v}^1 \in \mathcal{S} \quad \forall \alpha \in \mathbb{F}$$

Let $\mathbf{r}^1, \mathbf{r}^2 \in \mathcal{R}(\mathbf{A}) \subset \mathbb{R}^m$ and $\alpha \in \mathbb{R}$. Moreover, let $\mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^n$ such that $\mathbf{A}\mathbf{x}^1 = \mathbf{r}^1$ and $\mathbf{A}\mathbf{x}^2 = \mathbf{r}^2$. We have :

$$\mathbf{A}(\mathbf{x}^1 + \mathbf{x}^2) = \mathbf{A}\mathbf{x}^1 + \mathbf{A}\mathbf{x}^2 = \mathbf{r}^1 + \mathbf{r}^2 \Rightarrow \mathbf{r}^1 + \mathbf{r}^2 \in \mathcal{R}(\mathbf{A})$$

$$\mathbf{A}(\alpha \mathbf{x}^1) = \alpha \mathbf{A}\mathbf{x}^1 = \alpha \mathbf{r}^1 \Rightarrow \alpha \mathbf{r}^1 \in \mathcal{R}(\mathbf{A}) \quad \forall \alpha \in \mathbb{R}$$

Therefore, $\mathcal{R}(\mathbf{A})$ is a subspace of $\mathbb{R}^n$ over the field $\mathbb{R}$.

Note: For it to be a subspace over the field $\mathbb{C}$, we would have to consider complex values of $\mathbf{r}, \mathbf{x}$ and α, but it would also be valid.

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a)

$$\begin{aligned} \det \mathbf{v}^\times &= 0 \cdot \det \begin{bmatrix} 0 & -v_1 \\ v_1 & 0 \end{bmatrix} - v_3 \cdot \det \begin{bmatrix} -v_3 & v_2 \\ v_1 & 0 \end{bmatrix} - v_2 \cdot \det \begin{bmatrix} -v_3 & v_2 \\ 0 & -v_1 \end{bmatrix} \\ &= v_3 v_1 v_2 - v_2 v_3 v_1 = 0 \end{aligned}$$

b) $\det \mathbf{v}^\times = 0$ means that $\det \mathbf{v}^\times$ is *not* invertible. In other words, the columns of $\det \mathbf{v}^\times$ are linearly dependent and the nullity of $\det \mathbf{v}^\times$ is greater than 0.

c)

$$\begin{aligned}\mathbf{v}^{\times}\mathbf{x}=\mathbf{0} &\Leftrightarrow \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ &\Leftrightarrow \begin{bmatrix} -v_3x_2+v_2x_3 \\ v_3x_1-v_1x_3 \\ -v_2x_1+v_1x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\end{aligned}$$

We must now consider two cases:

1. $v_1=v_2=v_3=0$ in which case $\mathbf{v}^{\times}$ is the null matrix. Then $\nu(\mathbf{v}^{\times})=3$ and $\mathcal{N}(\mathbf{v}^{\times})=\mathbb{R}^3$.
2. At least one of the v_i, $i=1,2,3$, is different than zero. For simplicity, let's say $v_1\neq 0$ (The two other cases being similar). Therefore, from the previous matrix equation and by defining $x_1/v_1=\alpha$:
$$x_1=\frac{v_1}{v_1}x_1=\alpha v_1, \qquad x_2=\frac{v_2}{v_1}x_1=\alpha v_2, \qquad x_3=\frac{v_3}{v_1}x_1=\alpha v_3$$
In other words, $\mathcal{N}(\mathbf{v}^{\times})=\{\mathbf{x}\in\mathbb{R}^3 \mid \mathbf{x}=\alpha\mathbf{v},\ \alpha\in\mathbb{R}\}$ and $\nu(\mathbf{v}^{\times})=1$

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$$\begin{aligned}\det\mathbf{Q}=\det\begin{bmatrix} q_1^1 & q_1^2 & q_1^3 \\ q_2^1 & q_2^2 & q_2^3 \\ q_3^1 & q_3^2 & q_3^3 \end{bmatrix} &= q_1^1\cdot\det\begin{bmatrix} q_2^2 & q_2^3 \\ q_3^2 & q_3^3 \end{bmatrix}-q_2^1\cdot\det\begin{bmatrix} q_1^2 & q_1^3 \\ q_3^2 & q_3^3 \end{bmatrix}+q_3^1\cdot\det\begin{bmatrix} q_1^2 & q_1^3 \\ q_2^2 & q_2^3 \end{bmatrix} \\ &= q_1^1q_2^2q_3^3-q_1^1q_3^2q_2^3-q_2^1q_1^2q_3^3+q_2^1q_3^2q_1^3+q_3^1q_1^2q_2^3-q_3^1q_2^2q_1^3\end{aligned}$$

$$\mathbf{q}^{1\mathsf{T}}\mathbf{q}^{2\times}\mathbf{q}^3=\begin{bmatrix} q_1^1 & q_2^1 & q_3^1 \end{bmatrix}\begin{bmatrix} 0 & -q_3^2 & q_2^2 \\ q_3^2 & 0 & -q_1^2 \\ -q_2^2 & q_1^2 & 0 \end{bmatrix}\begin{bmatrix} q_1^3 \\ q_2^3 \\ q_3^3 \end{bmatrix}=\begin{bmatrix} q_1^1 & q_2^1 & q_3^1 \end{bmatrix}\begin{bmatrix} q_2^2q_3^3-q_3^2q_2^3 \\ q_3^2q_1^3-q_1^2q_3^3 \\ q_1^2q_2^3-q_2^2q_1^3 \end{bmatrix}$$
$$q_1^1q_2^2q_3^3-q_1^1q_3^2q_2^3+q_2^1q_3^2q_1^3-q_2^1q_1^2q_3^3+q_3^1q_1^2q_2^3-q_3^1q_2^2q_1^3$$

After inspection, we can easily check that all the terms match, thus $\det\mathbf{Q}=\mathbf{q}^{1\mathsf{T}}\mathbf{q}^{2\times}\mathbf{q}^3$.

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Let $\mathbf{A}\in\mathbb{R}^{n\times n}$ such that $\mathbf{A}^{\mathsf{T}}=\mathbf{A}$. Moreover, let $\lambda_i\in\mathbb{C}$ be the eigenvalues of $\mathbf{A}$ and $\mathbf{v}^i$ be the eigenvector associated with the eigenvalue λ_i, $i=1,...,n$. We have:

$$\mathbf{A}\mathbf{v}^i=\lambda_i\mathbf{v}^i \implies \mathbf{v}^{i\mathsf{H}}\mathbf{A}\mathbf{v}^i=\mathbf{v}^{i\mathsf{H}}\lambda_i\mathbf{v}^i=\lambda_i\mathbf{v}^{i\mathsf{H}}\mathbf{v}^i \implies \lambda_i=\frac{\mathbf{v}^{i\mathsf{H}}\mathbf{A}\mathbf{v}^i}{\mathbf{v}^{i\mathsf{H}}\mathbf{v}^i}$$

Therefore:

$$\lambda_i^{\mathsf{H}} = \left(\frac{\mathbf{v}^{i\,\mathsf{H}}\mathbf{A}\mathbf{v}^i}{\mathbf{v}^{i\,\mathsf{H}}\mathbf{v}^i}\right)^{\mathsf{H}} = \frac{\mathbf{v}^{i\,\mathsf{H}}\mathbf{A}^{\mathsf{H}}\mathbf{v}^i}{\mathbf{v}^{i\,\mathsf{H}}\mathbf{v}^i} = \frac{\mathbf{v}^{i\,\mathsf{H}}\mathbf{A}\mathbf{v}^i}{\mathbf{v}^{i\,\mathsf{H}}\mathbf{v}^i} = \lambda_i \implies \lambda_i \in \mathbb{R}$$

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Let's prove that $\mathbf{x} = \mathbf{0}$ is the only solution to the equation $\mathbf{Ax} = \mathbf{0}$ if $\mathbf{A}$ has the mentioned properties.

$$\mathbf{Ax} = \begin{bmatrix} \mathbf{M} & -\mathbf{\Theta}^{\mathsf{T}} \\ -\mathbf{\Theta} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \end{bmatrix} = \begin{bmatrix} \mathbf{M}\mathbf{x}^1 - \mathbf{\Theta}^{\mathsf{T}}\mathbf{x}^2 \\ -\mathbf{\Theta}\mathbf{x}^1 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

From the first component equation, we get:

$$\mathbf{M}\mathbf{x}^1 - \mathbf{\Theta}^{\mathsf{T}}\mathbf{x}^2 = \mathbf{0} \tag{1}$$

Multiplying both sides on the left by ${\mathbf{x}^1}^{\mathsf{T}}$:

$${\mathbf{x}^1}^{\mathsf{T}}\mathbf{M}\mathbf{x}^1 - {\mathbf{x}^1}^{\mathsf{T}}\mathbf{\Theta}^{\mathsf{T}}\mathbf{x}^2 = \mathbf{0} \tag{2}$$

Additionally, taking the transpose of the second component equation:

$$-{\mathbf{x}^1}^{\mathsf{T}}\mathbf{\Theta}^{\mathsf{T}} = \mathbf{0} \tag{3}$$

Substituting (3) into (2), we get:

$${\mathbf{x}^1}^{\mathsf{T}}\mathbf{M}\mathbf{x}^1 = 0 \tag{4}$$

Since $\mathbf{M} > 0$, the only solution to (4) is $\mathbf{x}^1 = \mathbf{0}$. We can now substitute this into (1) and we find:

$$-\mathbf{\Theta}^{\mathsf{T}}\mathbf{x}^2 = \mathbf{0} \tag{5}$$

Since $\mathbf{\Theta}$ is full row rank, $\mathbf{\Theta}^{\mathsf{T}}$ is full (column) rank and the only solution to (5) is $\mathbf{x}^2 = \mathbf{0}$. Therefore, $\mathbf{x} = \mathbf{0}$ is the only solution to the equation $\mathbf{Ax} = \mathbf{0}$. In other words, $\mathbf{A}$ is nonsingular.

Note: For simplicity, I didn't make the distinction between the different null matrices $\mathbf{0}$. Their exact dimension must be determined by looking at the other terms of each equation.

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a) By multiplying the ODE by $\mathbf{M}^{-1}$ on the left and rearranging, we get:

$$\ddot{\mathbf{q}} = -\mathbf{M}^{-1}\mathbf{K}\mathbf{q}$$

Now defining $\mathbf{x}$ as follows:

$$\mathbf{x} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix}$$

We can rewrite the ODE in the first order form:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{M}^{-1}\mathbf{K} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} = \mathbf{A}\mathbf{x}$$

where:

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{M}^{-1}\mathbf{K} & \mathbf{0} \end{bmatrix}$$

b) See the following matlab code.