Final Report - ADR Spacecraft

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I. Introduction

SPACE debris are becoming a real threat to spacial exploration. Indeed, lower earth orbits (LEO) are slowly becoming saturated with loose objects and sending additional satellites or human missions is therefore becoming riskier.

ESA, the European Space Agency, is now dedicating some of its ressources tracking these objects and planning their extraction or destruction. Their program is called *Clean Space Initiative* [1], and it involves ADR, standing for Active Debris Removal. It consists of sending spacecrafts in space that can either send the large and dangerous objects into higher orbits, known as graveyards, or slowing them down in order to initiate their reentry into the atmosphere, which would destroy them. In parallel, they are pressing governments to address this problem in their respective space law.

ESA's first planned ADR mission is called *eDeorbit* and it has given rise to nummerous technical challenges, in particular in the field of robotics and advanced systems of guidance. They are currently exploring two options to catch the abandoned satellites.

1) A spacecraft with a robotic arm that can grip the object.
2) A spacecraft that can throw a net over the object.

The objective of this project is to model a simplified version of the second design and to derive its equations of motion using the notation and the concepts exposed in [2].

II. Configuration of the Problem

The chosen design consists of a cylindrical spacecraft with one main propeller (P1), two auxiliary propellers (P2 & P3), and two reaction wheels (W1 & W2), as shown in Figure 1. Let $\mathcal{F}_e$ be the reference frame attached to the Earth, o a point attached to the ground, g_d the centre of masse of the debris and $\underset{\rightarrow}{v}^{g_d o/e}$ the velocity of g_d relative to o w.r.t. $\mathcal{F}_e$. Let A_d be the axis spanned by $\underset{\rightarrow}{v}^{g_d o/e}$ and passing through g_d. The goal is to place the spacecraft's centre of mass on A_d, at a constant distance from g_d and on the opposite direction of $\underset{\rightarrow}{v}^{g_d o/e}$. Then, W1 and W2 can be used to align A_s with A_d such that the net launcher faces the debris. Once in position, the net launcher (NL) can be activated and the two auxiliary propellers can be used to slow down the debris just enough to initiate its reentry in the atmosphere.

In order to simplify the model, the following assumptions will be made:

- The curvature of the space is negligible compared to characteristic size of the problem, i.e. instead of considering a LEO, both objects are treated like they were in deep space. In this case, gravity is neglected.
- The debris is free from external force, i.e. $\underset{\rightarrow}{v}^{g_d o/e}$ is constant.
- The total mass of the space craft and its inertia properties remain constant throughout the mission, i.e. the mass of the propellant used during the activation of P1, P2 and P3 is negligible.
- The spacecraft fuselage, the two reaction wheels and the debris are modeled as continuous rigid bodies with constant density.

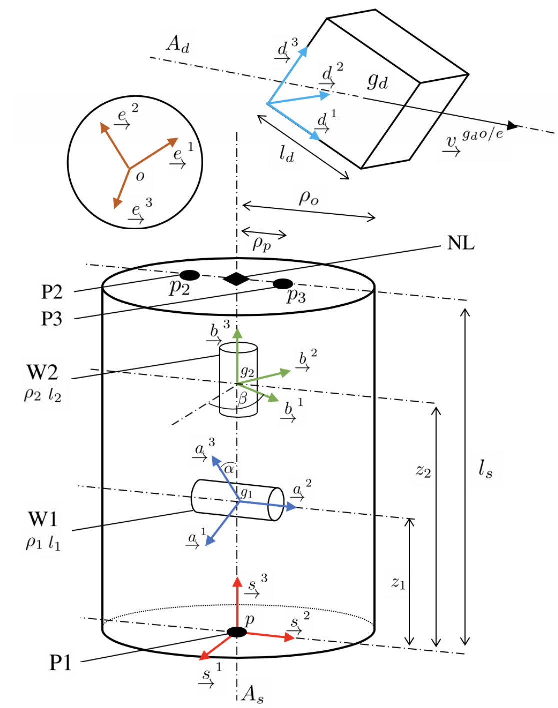

Fig. 1. Spacecraft and debris configuration.

III. Geometrical Parameterization

Let $\mathcal{S}$ and $\mathcal{D}$ denote the spacecraft fuselage and the debris, respectively, and let $\mathcal{A}$ represent the whole system ($\mathcal{S}$+W1+W2+$\mathcal{D}$). The body $\mathcal{S}$ is modeled as a closed end hollow cylinder with a wall thickness t_s, an outer radius ρ_o and a length l_s. Its main propeller P1 is attached to the main body at the point p, at the center of its base. On the top, the two auxiliary propellers P2 and P3 sit diametrically opposite each other, at a distance ρ_p from A_s. The two reaction wheels W1 and W2 are modeled as full cylinders of radius and length ρ_1 and l_1 (respectively ρ_2 and l_2). W1 is oriented radially and W2 axially, in such a way that their centre of mass g_1 and g_2 are both on A_s. Moreover, W1 is aligned with P2 and P3. The length of point g_1 (and g_2) relative to point p is z_1 (respectively z_2). The centre of mass of the wall g_s coincide with the geometrical centre of the spacecraft. As for the debris, it is modeled as a full cube of size l_d.

IV. Reference Frames

For this problem, 5 different reference frames are considerd:

- $\mathcal{F}_e$: An inertial reference frame attached to the earth.
- $\mathcal{F}_s$: The reference frame attached to the body of the spacecraft. $\underset{\rightarrow}{s}^2$ is aligned with W1, $\underset{\rightarrow}{s}^3$ is aligned with A_s and oriented towards the front of the spacecraft. Finally, $\underset{\rightarrow}{s}^1 = \underset{\rightarrow}{s}^2 \times \underset{\rightarrow}{s}^3$.
- $\mathcal{F}_a$: The reference frame attached to W1, obtained by rotating $\mathcal{F}_s$ about $\underset{\rightarrow}{s}^2 = \underset{\rightarrow}{a}^2$. The angle of this rotation is refered to as α.
- $\mathcal{F}_b$: The reference frame attached to W2, obtained by rotating $\mathcal{F}_s$ about $\underset{\rightarrow}{s}^3 = \underset{\rightarrow}{b}^3$. The angle of this rotation is refered to as β.
- $\mathcal{F}_d$: The reference frame attached to the body of the debris and aligned with its edges.

V. Attitude Description

Since this problem is highly three dimensional, all the mutual configurations of the different reference frames are likely to happen. Using attitude parametrization that are not global or have kinematic singularities is risky and might in the end complicate the problem even more. This is why Direction Cosine Matrices will mostly be used. However, Euler Angles will be used to parameterize $\mathbf{C}_{as}$ and $\mathbf{C}_{bs}$ in order to avoid excessive kinematic constraints. In fact, only one parameter is needed to describe each of them (α and β), and it will be shown later that this is a special case in which kinematic singularities are impossible. Moreover, this parameterization is global and its non uniqueness is not a problem since the exact values of α and β are of poor value.

VI. Direction Cosine Matrices

For this problem, the following 7 DCMs and their transpose will be used:

$$\mathbf{C}_{es} = \mathbf{C}_{se}^{\mathsf{T}} = \begin{bmatrix} \mathbf{s}_e^1 & \mathbf{s}_e^2 & \mathbf{s}_e^3 \end{bmatrix}, \tag{1}$$
$$\mathbf{C}_{ea} = \mathbf{C}_{ae}^{\mathsf{T}} = \begin{bmatrix} \mathbf{a}_e^1 & \mathbf{a}_e^2 & \mathbf{a}_e^3 \end{bmatrix}, \tag{2}$$
$$\mathbf{C}_{eb} = \mathbf{C}_{be}^{\mathsf{T}} = \begin{bmatrix} \mathbf{b}_e^1 & \mathbf{b}_e^2 & \mathbf{b}_e^3 \end{bmatrix}, \tag{3}$$
$$\mathbf{C}_{ed} = \mathbf{C}_{de}^{\mathsf{T}} = \begin{bmatrix} \mathbf{d}_e^1 & \mathbf{d}_e^2 & \mathbf{d}_e^3 \end{bmatrix}, \tag{4}$$
$$\mathbf{C}_{sd} = \mathbf{C}_{ds}^{\mathsf{T}} = \begin{bmatrix} \mathbf{d}_s^1 & \mathbf{d}_s^2 & \mathbf{d}_s^3 \end{bmatrix}, \tag{5}$$
$$\mathbf{C}_{as} = \mathbf{C}_{sa}^{\mathsf{T}} = \mathbf{C}_2(\alpha), \tag{6}$$
$$\mathbf{C}_{bs} = \mathbf{C}_{sb}^{\mathsf{T}} = \mathbf{C}_3(\beta), \tag{7}$$

where $\mathbf{C}_2(\alpha)$ and $\mathbf{C}_3(\beta)$ are given by (42) and (43).

VII. Angular Velocities

Let $\mathbf{q}^{xy}$ be the column matrix whose components are the columns of $\mathbf{C}_{xy}^{\mathsf{T}}$. In particular,

$$\mathbf{q}^{se} \triangleq \begin{bmatrix} \mathbf{s}_e^1 \\ \mathbf{s}_e^2 \\ \mathbf{s}_e^3 \end{bmatrix}, \quad \mathbf{q}^{ae} \triangleq \begin{bmatrix} \mathbf{a}_e^1 \\ \mathbf{a}_e^2 \\ \mathbf{a}_e^3 \end{bmatrix}, \quad \mathbf{q}^{be} \triangleq \begin{bmatrix} \mathbf{b}_e^1 \\ \mathbf{b}_e^2 \\ \mathbf{b}_e^3 \end{bmatrix},$$
$$\mathbf{q}^{de} \triangleq \begin{bmatrix} \mathbf{d}_e^1 \\ \mathbf{d}_e^2 \\ \mathbf{d}_e^3 \end{bmatrix}, \qquad \mathbf{q}^{ds} \triangleq \begin{bmatrix} \mathbf{d}_s^1 \\ \mathbf{d}_s^2 \\ \mathbf{d}_s^3 \end{bmatrix}.$$

In this case, the angular velocities can be expressed in term of the parameters as follows:

$$\begin{aligned}
\boldsymbol{\omega}_s^{se} &= \mathbf{S}_s^{se}\dot{\mathbf{q}}^{se}, \\
\boldsymbol{\omega}_a^{ae} &= \mathbf{S}_a^{ae}\dot{\mathbf{q}}^{ae}, \\
\boldsymbol{\omega}_b^{be} &= \mathbf{S}_b^{be}\dot{\mathbf{q}}^{be}, \\
\boldsymbol{\omega}_d^{de} &= \mathbf{S}_d^{de}\dot{\mathbf{q}}^{de}, \\
\boldsymbol{\omega}_d^{ds} &= \mathbf{S}_d^{ds}\dot{\mathbf{q}}^{ds}, \\
\boldsymbol{\omega}_a^{as} &= \mathbf{S}_a^{as}\dot{\alpha}, \\
\boldsymbol{\omega}_b^{bs} &= \mathbf{S}_b^{bs}\dot{\beta},
\end{aligned}$$

where the matrices $\mathbf{S}_x^{xy}$ are derived in [3] and given by equations (44) to (50). Similarly, the inverse relations are

$$\begin{aligned}
\dot{\mathbf{q}}^{se} &= \boldsymbol{\Gamma}_s^{se}\boldsymbol{\omega}_s^{se}, \\
\dot{\mathbf{q}}^{ae} &= \boldsymbol{\Gamma}_a^{ae}\boldsymbol{\omega}_a^{ae}, \\
\dot{\mathbf{q}}^{be} &= \boldsymbol{\Gamma}_b^{be}\boldsymbol{\omega}_b^{be}, \\
\dot{\mathbf{q}}^{de} &= \boldsymbol{\Gamma}_d^{de}\boldsymbol{\omega}_d^{de}, \\
\dot{\mathbf{q}}^{ds} &= \boldsymbol{\Gamma}_d^{ds}\boldsymbol{\omega}_d^{ds}, \\
\dot{\alpha} &= \boldsymbol{\Gamma}_a^{as}\boldsymbol{\omega}_a^{as}, \\
\dot{\beta} &= \boldsymbol{\Gamma}_b^{bs}\boldsymbol{\omega}_b^{bs},
\end{aligned}$$

where the matrices $\boldsymbol{\Gamma}_x^{xy}$ are also derived in [3] and are given by equations (51) to (57). Lastly, let $\mathbf{S}$ and $\boldsymbol{\Gamma}$ be defined as follows:

$$\mathbf{S} \triangleq \operatorname{diag}\left\{\mathbf{1}, \mathbf{S}_s^{se}, \mathbf{1}, \mathbf{S}_a^{ae}, \mathbf{1}, \mathbf{S}_b^{be}, \mathbf{1}, \mathbf{S}_d^{de}\right\}, \tag{8}$$
$$\boldsymbol{\Gamma} \triangleq \operatorname{diag}\left\{\mathbf{1}, \boldsymbol{\Gamma}_s^{se}, \mathbf{1}, \boldsymbol{\Gamma}_a^{ae}, \mathbf{1}, \boldsymbol{\Gamma}_b^{be}, \mathbf{1}, \boldsymbol{\Gamma}_d^{de}\right\}. \tag{9}$$

VIII. Components parameterization

Due to the cylindrical symmetry of the spacecraft and the reaction wheels, the components of the physical vectors resolved in frame $\mathcal{F}_s$, $\mathcal{F}_a$ and $\mathcal{F}_b$ will be parametrized using the cylindrical coordinates. Let $\mathrm{d}m_s$, $\mathrm{d}m_1$ and $\mathrm{d}m_2$ be material elements of the spacecraft wall, of W1 and of W2, respectively. The parametrization of the components of $\mathbf{r}_s^{\mathrm{d}m_s p}$, $\mathbf{r}_a^{\mathrm{d}m_1 g_1}$ and $\mathbf{r}_b^{\mathrm{d}m_2 g_2}$ will be done accordingly to Figure 2.

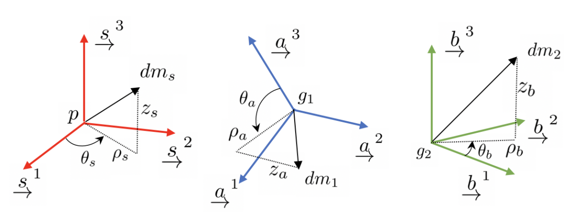

Fig. 2. Component parametrization in $\mathcal{F}_s$, $\mathcal{F}_a$ and $\mathcal{F}_b$.

As for the debris, the position of its material elements $\mathrm{d}m_d$ relative to g_d resolved in $\mathcal{F}_d$ will be described using a cartesian coordinate system. More precisely, x_d, y_d and z_d

for the components along $\underrightarrow{d}^1$, $\underrightarrow{d}^2$ and $\underrightarrow{d}^3$, respectively. In matrix form:

$$\mathbf{r}_s^{dm_sp} = \begin{bmatrix} \rho_s\cos(\theta_s) \\ \rho_s\sin(\theta_s) \\ z_s \end{bmatrix}, \quad \mathbf{r}_a^{dm_1g_1} = \begin{bmatrix} \rho_a\cos(\theta_a) \\ z_a \\ \rho_a\sin(\theta_a) \end{bmatrix},$$
$$\mathbf{r}_b^{dm_2g_2} = \begin{bmatrix} \rho_b\cos(\theta_b) \\ \rho_b\sin(\theta_b) \\ z_b \end{bmatrix}, \quad \mathbf{r}_d^{dm_dg_d} = \begin{bmatrix} x_d \\ y_d \\ z_d \end{bmatrix}.$$

IX. State of the System

This problem has 14 degrees of freedom (6 for the spacecraft, 1 for each reaction wheel and 6 for the debris). Therefore, in order to have a well defined and observable problem, at least 28 independent measurables are needed (14 positions and 14 velocities). The following assumptions will be made:

- The earth has a radar system that can determine $\mathbf{r}_e^{g_do}$, $\mathbf{v}_e^{g_do/e}$, $\mathbf{r}_e^{po}$ and $\mathbf{v}_e^{po/e}$. Moreover, it communicates with the spacecraft sensors in order to determine $\mathbf{C}_{es}$. Lastly, it can compute $\boldsymbol{\omega}_e^{se}$.
- The spacecraft system knows α, β, $\dot{\alpha}$ and $\dot{\beta}$.
- The spacecraft has sensors that can observe the debris' behavior, i.e. $\mathbf{C}_{sd}$ and $\boldsymbol{\omega}_s^{ds}$ (It is assumed that the radars on earth are too far to observe directly the orientation and the spin of the debris).

It is useful to note that $\boldsymbol{\omega}_e^{se} = \boldsymbol{\omega}_s^{se}$ and $\boldsymbol{\omega}_s^{ds} = \boldsymbol{\omega}_d^{ds}$. In addition, let the column matrix $\bar{\mathbf{x}}$ represent the measurable state of the system, whose components are simply the measurables.

On the other hand, in order to be able to treat the dynamic of each body separately, one can define the generalized coordinate $\mathbf{q}$, the augmented velocities $\boldsymbol{\nu}$ and the reduced augmented velocities $\hat{\boldsymbol{\nu}}$ as follows:

$$\mathbf{q} \triangleq \begin{bmatrix} \mathbf{r}_e^{po} \\ \mathbf{q}^{se} \\ \mathbf{r}_e^{g_1o} \\ \mathbf{q}^{ae} \\ \mathbf{r}_e^{g_2o} \\ \mathbf{q}^{be} \\ \mathbf{r}_e^{g_do} \\ \mathbf{q}^{de} \end{bmatrix}, \quad \boldsymbol{\nu} \triangleq \begin{bmatrix} \mathbf{v}_e^{po/e} \\ \boldsymbol{\omega}_s^{se} \\ \mathbf{v}_e^{g_1o/e} \\ \boldsymbol{\omega}_a^{ae} \\ \mathbf{v}_e^{g_2o/e} \\ \boldsymbol{\omega}_b^{be} \\ \mathbf{v}_e^{g_do/e} \\ \boldsymbol{\omega}_d^{de} \end{bmatrix}, \quad \hat{\boldsymbol{\nu}} \triangleq \begin{bmatrix} \mathbf{v}_e^{po/e} \\ \boldsymbol{\omega}_s^{se} \\ \dot{\alpha} \\ \dot{\beta} \\ \mathbf{v}_e^{g_do/e} \\ \boldsymbol{\omega}_d^{de} \end{bmatrix}.$$

The relations between these three quantities are

$$\dot{\mathbf{q}} = \boldsymbol{\Gamma}\boldsymbol{\nu} \tag{10}$$

and

$$\boldsymbol{\nu} = \boldsymbol{\Pi}\hat{\boldsymbol{\nu}} \tag{11}$$

where $\boldsymbol{\Gamma}$ and $\boldsymbol{\Pi}$ are given by equations (9) and (58), respectively. Lastly, one can define the augmented state of the system $\mathbf{x}$ as follows:

$$\mathbf{x} \triangleq \begin{bmatrix} \mathbf{q} \\ \hat{\nu} \end{bmatrix}.$$

The relation between the measurable state and the augmented state is given by

$$\mathbf{x} = \boldsymbol{\Sigma}(\bar{\mathbf{x}}), \tag{12}$$

where $\boldsymbol{\Sigma}(\bar{\mathbf{x}})$ is defined by (59). Here is a quick explanation why these different quantities are useful: $\mathbf{q}$ and $\dot{\mathbf{q}}$ are the generalized coordinates used in the Lagrange's Equation. However, expressing the kinetic energy directly in function of $\mathbf{q}$ and $\dot{\mathbf{q}}$ can be tydious. This is why $\boldsymbol{\nu}$ is introduced. Once the equations of motion are found in function of $\boldsymbol{\nu}$, one can use $\hat{\boldsymbol{\nu}}$ to perform the Null Space step so that the constraints disappear. Finally, one can use $\mathbf{x}$ to transform the problem into a first order ODE that can be numerically integrated with the initial conditions $\mathbf{x}_0 = \boldsymbol{\Sigma}(\bar{\mathbf{x}}_0)$ where $\bar{\mathbf{x}}_0$ are the measurables at time $t = 0$.

X. Position

One can now express the position, velocity and acceleration of each point relative to o w.r.t. $\mathcal{F}_e$ and in functions of the measurables and their time derivative.

$$\underrightarrow{r}^{dm_so} = \underrightarrow{\mathcal{F}}_e^{\mathsf{T}}\left(\mathbf{r}_e^{po} + \mathbf{C}_{es}\mathbf{r}_s^{dm_sp}\right), \tag{13}$$
$$\underrightarrow{r}^{dm_1o} = \underrightarrow{\mathcal{F}}_e^{\mathsf{T}}\left(\mathbf{r}_e^{po} + \mathbf{C}_{es}\mathbf{1}_3 z_1 + \mathbf{C}_{es}\mathbf{C}_{as}^{\mathsf{T}}\mathbf{r}_a^{dm_1g_1}\right), \tag{14}$$
$$\underrightarrow{r}^{dm_2o} = \underrightarrow{\mathcal{F}}_e^{\mathsf{T}}\left(\mathbf{r}_e^{po} + \mathbf{C}_{es}\mathbf{1}_3 z_2 + \mathbf{C}_{es}\mathbf{C}_{bs}^{\mathsf{T}}\mathbf{r}_b^{dm_2g_2}\right), \tag{15}$$
$$\underrightarrow{r}^{dm_do} = \underrightarrow{\mathcal{F}}_e^{\mathsf{T}}\left(\mathbf{r}_e^{g_do} + \mathbf{C}_{es}\mathbf{C}_{sd}\mathbf{r}_d^{dm_dg_d}\right). \tag{16}$$

XI. Velocity

$$\underrightarrow{v}^{dm_so/e} = \underrightarrow{\mathcal{F}}_e^{\mathsf{T}}\left(\mathbf{v}_e^{po/e} + \boldsymbol{\omega}_e^{se\times}\mathbf{C}_{es}\mathbf{r}_s^{dm_sp}\right), \tag{17}$$
$$\begin{aligned}\underrightarrow{v}^{dm_1o/e} = \underrightarrow{\mathcal{F}}_e^{\mathsf{T}}\Big(&[\mathbf{C}_{es}\mathbf{1}_2\dot{\alpha} + \boldsymbol{\omega}_e^{se}]^{\times}\,\mathbf{C}_{es}\mathbf{C}_{as}^{\mathsf{T}}\mathbf{r}_a^{dm_1g_1} \\ &+ \boldsymbol{\omega}_e^{se\times}\mathbf{C}_{es}\mathbf{1}_3 z_1 + \mathbf{v}_e^{po/e}\Big),\end{aligned} \tag{18}$$
$$\begin{aligned}\underrightarrow{v}^{dm_2o/e} = \underrightarrow{\mathcal{F}}_e^{\mathsf{T}}\Big(&\Big[\mathbf{C}_{es}\mathbf{1}_3\dot{\beta} + \boldsymbol{\omega}_e^{se}\Big]^{\times}\,\mathbf{C}_{es}\mathbf{C}_{bs}^{\mathsf{T}}\mathbf{r}_b^{dm_2g_2} \\ &+ \boldsymbol{\omega}_e^{se\times}\mathbf{C}_{es}\mathbf{1}_3 z_2 + \mathbf{v}_e^{po/e}\Big),\end{aligned} \tag{19}$$
$$\begin{aligned}\underrightarrow{v}^{dm_do/e} = \underrightarrow{\mathcal{F}}_e^{\mathsf{T}}\Big(&\big[\mathbf{C}_{es}\boldsymbol{\omega}_s^{ds} + \boldsymbol{\omega}_e^{se}\big]^{\times}\,\mathbf{C}_{es}\mathbf{C}_{sd}\mathbf{r}_d^{dm_dg_d} \\ &+ \mathbf{v}_e^{g_do/e}\Big).\end{aligned} \tag{20}$$

XII. Acceleration

$$\begin{aligned}\underrightarrow{a}^{dm_so/e} = \underrightarrow{\mathcal{F}}_e^{\mathsf{T}}\Big(&\dot{\mathbf{v}}_e^{po/e} + \dot{\boldsymbol{\omega}}_e^{se\,\times}\,\mathbf{C}_{es}\mathbf{r}_s^{dm_sp} \\ &+ \boldsymbol{\omega}_e^{se\times}\boldsymbol{\omega}_e^{se\times}\,\mathbf{C}_{es}\mathbf{r}_s^{dm_sp}\Big),\end{aligned} \tag{21}$$
$$\begin{aligned}\underrightarrow{a}^{dm_1o/e} = \underrightarrow{\mathcal{F}}_e^{\mathsf{T}}\Big(&\big[\mathbf{C}_{es}\mathbf{1}_2\ddot{\alpha} + \dot{\boldsymbol{\omega}}_e^{se} - \mathbf{C}_{es}(\mathbf{1}_2\dot{\alpha})^{\times}\boldsymbol{\omega}_e^{se}\big]^{\times}\ldots \\ &\mathbf{C}_{es}\mathbf{C}_{as}^{\mathsf{T}}\mathbf{r}_a^{dm_1g_1} + [\mathbf{C}_{es}\mathbf{1}_2\dot{\alpha} + \boldsymbol{\omega}_e^{se}]^{\times}\ldots \\ &[\mathbf{C}_{es}\mathbf{1}_2\dot{\alpha} + \boldsymbol{\omega}_e^{se}]^{\times}\,\mathbf{C}_{es}\mathbf{C}_{as}^{\mathsf{T}}\mathbf{r}_a^{dm_1g_1} \\ &+ \big[\dot{\boldsymbol{\omega}}_e^{se\,\times} + \boldsymbol{\omega}_e^{se\times}\boldsymbol{\omega}_e^{se\times}\big]\,\mathbf{C}_{es}\mathbf{1}_3 z_1 \\ &+ \dot{\mathbf{v}}_e^{po/e}\Big),\end{aligned} \tag{22}$$
$$\begin{aligned}\underrightarrow{a}^{dm_2o/e} = \underrightarrow{\mathcal{F}}_e^{\mathsf{T}}\Big(&\Big[\mathbf{C}_{es}\mathbf{1}_3\ddot{\beta} + \dot{\boldsymbol{\omega}}_e^{se} - \mathbf{C}_{es}(\mathbf{1}_3\dot{\beta})^{\times}\boldsymbol{\omega}_e^{se}\Big]^{\times}\ldots \\ &\mathbf{C}_{es}\mathbf{C}_{bs}^{\mathsf{T}}\mathbf{r}_b^{dm_2g_2} + \Big[\mathbf{C}_{es}\mathbf{1}_3\dot{\beta} + \boldsymbol{\omega}_e^{se}\Big]^{\times}\ldots \\ &\Big[\mathbf{C}_{es}\mathbf{1}_3\dot{\beta} + \boldsymbol{\omega}_e^{se}\Big]^{\times}\,\mathbf{C}_{es}\mathbf{C}_{bs}^{\mathsf{T}}\mathbf{r}_b^{dm_2g_2} \\ &+ \big[\dot{\boldsymbol{\omega}}_e^{se\,\times} + \boldsymbol{\omega}_e^{se\times}\boldsymbol{\omega}_e^{se\times}\big]\,\mathbf{C}_{es}\mathbf{1}_3 z_2\end{aligned}$$

$$+\dot{\mathbf{v}}_e^{po/e}\Big), \tag{23}$$

$$\underset{\rightarrow}{a}^{dm_d o/e} = \underset{\rightarrow}{\mathcal{F}}_e^{\mathsf{T}}\Big(\Big[\mathbf{C}_{es}\dot{\boldsymbol{\omega}}_s^{ds} + \dot{\boldsymbol{\omega}}_e^{se} - \left\{\mathbf{C}_{es}\boldsymbol{\omega}_s^{ds}\right\}^\times \boldsymbol{\omega}_e^{se}\Big]^\times \dots$$
$$\mathbf{C}_{es}\mathbf{C}_{sd}\mathbf{r}_d^{dm_d g_d}$$
$$+\left[\mathbf{C}_{es}\boldsymbol{\omega}_s^{ds} + \boldsymbol{\omega}_e^{se}\right]^\times \left[\mathbf{C}_{es}\boldsymbol{\omega}_s^{ds} + \boldsymbol{\omega}_e^{se}\right]^\times \dots$$
$$\mathbf{C}_{es}\mathbf{C}_{sd}\mathbf{r}_d^{dm_d g_d} + \dot{\mathbf{v}}_e^{g_d o/e}\Big). \tag{24}$$

XIII. Approach

In order to derive the equations of motion, the Lagrangian approach will be used. It has the advantage of not involving any internal force. Moreover, it allows to treat each body of the system separately. Their motions are then coupled together by simply introducing attitude and collocation constraints, which is precisely what is done in the end part of this report.

XIV. Mass Properties

In order to shorten the notation, let $\boldsymbol{F}^{\mathcal{B}}(\mathbf{x})$ denote the integration of the quantity $\mathbf{x}$ (scalar or matrix) over the body $\mathcal{B}$. In particular, using the the parameters defined in section III, $\boldsymbol{F}^{\mathcal{S}}(\mathbf{x})$, $\boldsymbol{F}^{W1}(\mathbf{x})$, $\boldsymbol{F}^{W2}(\mathbf{x})$ and $\boldsymbol{F}^{\mathcal{D}}(\mathbf{x})$ are given by equations (60) to (63). Let V_s, V_1, V_2 and V_d be the volume of $\mathcal{S}$, W1, W2 and $\mathcal{D}$, respectively. They are simply given by:

$$V_s = \boldsymbol{F}^{\mathcal{S}}(1), \quad V_1 = \boldsymbol{F}^{W1}(1), \quad V_2 = \boldsymbol{F}^{W2}(1), \quad V_d = \boldsymbol{F}^{\mathcal{D}}(1).$$

Therefore, since the densities are assumed to be constant, the corresponding masses become:

$$m_s = \sigma_s V_s, \quad m_1 = \sigma_1 V_1, \quad m_2 = \sigma_2 V_2, \quad m_d = \sigma_d V_d.$$

The relevant first moments of mass of each body are given by:

$$\underset{\rightarrow}{c}^{\mathcal{S}p} = m_s \underset{\rightarrow}{r}^{g_s p} = \begin{bmatrix} 0 & 0 & \frac{1}{2}m_s l_s \end{bmatrix} \underset{\rightarrow}{\mathcal{F}}_s,$$
$$\underset{\rightarrow}{c}^{W1g_1} = \underset{\rightarrow}{c}^{W2g_2} = \underset{\rightarrow}{c}^{\mathcal{D}g_d} = \underset{\rightarrow}{0}.$$

Similarly, using the following identities:

$$\underset{\rightarrow}{J}^{\mathcal{B}z} = \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \mathbf{J}_b^{\mathcal{B}z} \underset{\rightarrow}{\mathcal{F}}_b, \quad \mathbf{J}_b^{\mathcal{B}z} = -\int_{\mathcal{B}} \mathbf{r}_b^{\mathrm{d}mz^\times} \mathbf{r}_b^{\mathrm{d}mz^\times}\,\mathrm{d}m,$$

one can compute the second moments of mass of each body resolved in their respective body frame. In particular,

$$\mathbf{J}_s^{\mathcal{S}p} = \boldsymbol{F}^{\mathcal{S}}(-\sigma_s \mathbf{r}_s^{\mathrm{d}m_s p^\times} \mathbf{r}_s^{\mathrm{d}m_s p^\times}),$$
$$\mathbf{J}_a^{W1g_1} = \boldsymbol{F}^{W1}(-\sigma_1 \mathbf{r}_a^{\mathrm{d}m_1 g_1^\times} \mathbf{r}_a^{\mathrm{d}m_1 g_1^\times}),$$
$$\mathbf{J}_b^{W2g_2} = \boldsymbol{F}^{W2}(-\sigma_2 \mathbf{r}_b^{\mathrm{d}m_2 g_2^\times} \mathbf{r}_b^{\mathrm{d}m_2 g_2^\times}),$$
$$\mathbf{J}_d^{\mathcal{D}g_d} = \boldsymbol{F}^{\mathcal{D}}(-\sigma_d \mathbf{r}_d^{\mathrm{d}m_d g_d^\times} \mathbf{r}_d^{\mathrm{d}m_d g_d^\times}),$$

where the argument of each body integration is given explicitly by (64), (65), (66) and (67), respectivley. This allows to define the following mass matrices:

$$\mathbf{M}^{\mathcal{S}p} \triangleq \begin{bmatrix} m_s\mathbf{1} & -\mathbf{C}_{es}\mathbf{c}_s^{\mathcal{S}p^\times} \\ \mathbf{c}_s^{\mathcal{S}p^\times}\mathbf{C}_{es}^{\mathsf{T}} & \mathbf{J}_s^{\mathcal{S}p} \end{bmatrix},$$
$$\mathbf{M}^{W1g_1} \triangleq \begin{bmatrix} m_1\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_a^{W1g_1} \end{bmatrix},$$
$$\mathbf{M}^{W2g_2} \triangleq \begin{bmatrix} m_2\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_b^{W2g_2} \end{bmatrix},$$
$$\mathbf{M}^{\mathcal{D}g_d} \triangleq \begin{bmatrix} m_d\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_d^{\mathcal{D}g_d} \end{bmatrix},$$

Finally, one can define the mass matrix of the whole system as

$$\mathbf{M} \triangleq \mathrm{diag}\left\{\mathbf{M}^{\mathcal{S}p}, \mathbf{M}^{W1g_1}, \mathbf{M}^{W2g_2}, \mathbf{M}^{\mathcal{D}g_d}\right\}. \tag{25}$$

XV. Control

The spacecraft has three linear actuators (P1, P2 and P3) and two rotary actuators (W1 and W2). The direction of the thrust produced by the propellers is assumed to remain perpendicular to the surface of the spacecraft, and directed towards it (no pull). Let p, p_2 and p_3 be the points where the actuators P1, P2 and P3 are fixed to the spacecraft, respectively. Under these assumptions,

$$\underset{\rightarrow}{f}^{p} = \begin{bmatrix} 0 & 0 & f^{P1} \end{bmatrix} \underset{\rightarrow}{\mathcal{F}}_s,$$
$$\underset{\rightarrow}{f}^{p_2} = \begin{bmatrix} 0 & 0 & -f^{P2} \end{bmatrix} \underset{\rightarrow}{\mathcal{F}}_s,$$
$$\underset{\rightarrow}{f}^{p_3} = \begin{bmatrix} 0 & 0 & -f^{P3} \end{bmatrix} \underset{\rightarrow}{\mathcal{F}}_s.$$

Additionally, let $\underset{\rightarrow}{\tau}^{W1\mathcal{S}}$ and $\underset{\rightarrow}{\tau}^{W2\mathcal{S}}$ be the torques applied on W1 and W2 by the actuators placed on the spacecraft wall. From Newton's third law, the torques applied on the wall by the reaction wheels, $\underset{\rightarrow}{\tau}^{\mathcal{S}W1}$ and $\underset{\rightarrow}{\tau}^{\mathcal{S}W2}$, are simply given by

$$\underset{\rightarrow}{\tau}^{\mathcal{S}W1} = -\underset{\rightarrow}{\tau}^{W1\mathcal{S}}, \quad \underset{\rightarrow}{\tau}^{\mathcal{S}W2} = -\underset{\rightarrow}{\tau}^{W2\mathcal{S}}.$$

Resolving these physical vectors in the body frames yields

$$\underset{\rightarrow}{\tau}^{W1\mathcal{S}} = \begin{bmatrix} 0 & \tau^{W1} & 0 \end{bmatrix} \underset{\rightarrow}{\mathcal{F}}_a,$$
$$\underset{\rightarrow}{\tau}^{\mathcal{S}W1} = \begin{bmatrix} 0 & -\tau^{W1} & 0 \end{bmatrix} \underset{\rightarrow}{\mathcal{F}}_s,$$
$$\underset{\rightarrow}{\tau}^{W2\mathcal{S}} = \begin{bmatrix} 0 & 0 & \tau^{W2} \end{bmatrix} \underset{\rightarrow}{\mathcal{F}}_b,$$
$$\underset{\rightarrow}{\tau}^{\mathcal{S}W2} = \begin{bmatrix} 0 & 0 & -\tau^{W2} \end{bmatrix} \underset{\rightarrow}{\mathcal{F}}_s.$$

Therefore, the behavior of the system only depends on the initial configuration and the following controllable (time-dependent) quantity:

$$\mathfrak{f} \triangleq \begin{bmatrix} f^{P1} & f^{P2} & f^{P3} & \tau^{W1} & \tau^{W2} \end{bmatrix}^{\mathsf{T}}.$$

In this case, the generalized forces and moments with respect to $\mathbf{q}$ are given by

$$\boldsymbol{f} = \mathbf{B}\mathfrak{f}, \quad \mathbf{B} \triangleq \begin{bmatrix} \mathbf{B}_s \\ \mathbf{B}_a \\ \mathbf{B}_b \\ \mathbf{0} \end{bmatrix}, \tag{26}$$

where $\mathbf{B}_s$, $\mathbf{B}_a$ and $\mathbf{B}_b$ are given by (72), (73) and (74), respectively.

XVI. Constraints

There are four kinematic constraints (related to the DCMs), two collocation constraints (related to g_1 and g_2) and two attitude constraints (related to $\boldsymbol{\omega}_s^{as}$ and $\boldsymbol{\omega}_b^{bs}$). The kinematic constraints can be expressed as follows:

$$\boldsymbol{\Phi}_{kin}(\mathbf{q}) \stackrel{!}{=} \mathbf{0} \tag{27}$$

where $\mathbf{\Phi}_{kin}(\mathbf{q})$ is derived in [3] and given by (68). Additionally, the Pfaffian form of (27) is given by

$$\mathbf{\Xi}^{kin}\dot{\mathbf{q}} = \mathbf{0}, \tag{28}$$

where $\mathbf{\Xi}^{kin}$ is defined by (69). The two collocation constraints are the following:

$$\underset{\rightarrow}{r}^{g_1 o} \stackrel{!}{=} \underset{\rightarrow}{r}^{g_1 p} + \underset{\rightarrow}{r}^{po}, \tag{29}$$

$$\underset{\rightarrow}{r}^{g_2 o} \stackrel{!}{=} \underset{\rightarrow}{r}^{g_2 p} + \underset{\rightarrow}{r}^{po}. \tag{30}$$

Taking the time derivative of (29) and (30) w.r.t. $\mathcal{F}_e$ and using the Transport theorem, one can obtain directly the Pfaffian form of the collocation constraints:

$$\mathbf{\Xi}^{col}\dot{\mathbf{q}} = \mathbf{0}, \tag{31}$$

where $\mathbf{\Xi}^{col}$ is given by (70). Moreover, the attitude constraints are stated as follows:

$$\omega_{a1}^{as} = \omega_{a3}^{as} \stackrel{!}{=} 0, \tag{32}$$

$$\omega_{b1}^{bs} = \omega_{b2}^{bs} \stackrel{!}{=} 0, \tag{33}$$

and the corresponding Pfaffian form is

$$\mathbf{\Xi}^{att}\dot{\mathbf{q}} = \mathbf{0}, \tag{34}$$

where $\mathbf{\Xi}^{att}$ is given by (71). Finally, all the constraints are arranged in a matrix format:

$$\mathbf{\Xi} \triangleq \begin{bmatrix} \mathbf{\Xi}^{kin} \\ \mathbf{\Xi}^{col} \\ \mathbf{\Xi}^{att} \end{bmatrix}. \tag{35}$$

XVII. Equations of Motion

Since potential energies are neglected, one can simply compute the Lagrangian of the system as follows:

$$L_{\mathcal{A}o/e} = T_{\mathcal{A}o/e} = \tfrac{1}{2}\boldsymbol{\nu}^{\mathsf{T}}\mathbf{M}\boldsymbol{\nu}. \tag{36}$$

Additionally, the general form of the Lagrange's Equation is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{\mathcal{A}o/e}}{\partial \dot{\mathbf{q}}}\right)^{\mathsf{T}} - \left(\frac{\partial L_{\mathcal{A}o/e}}{\partial \mathbf{q}}\right)^{\mathsf{T}} = \boldsymbol{f} + \mathbf{\Xi}^{\mathsf{T}}\boldsymbol{\lambda}. \tag{37}$$

Substituting (36) into (37) and using the the derivation shown in [3], [4], the equation of motion can be written as

$$\mathbf{S}^{\mathsf{T}}\mathbf{M}\dot{\boldsymbol{\nu}} + \mathbf{S}^{\mathsf{T}}\dot{\mathbf{M}}\boldsymbol{\nu} + \dot{\mathbf{S}}^{\mathsf{T}}\mathbf{M}\boldsymbol{\nu} - \mathbf{\Delta}^{\mathsf{T}}\mathbf{M}\boldsymbol{\nu} - \mathbf{a}_{non} = \mathbf{B}\mathfrak{f} + \mathbf{\Xi}^{\mathsf{T}}\boldsymbol{\lambda} \tag{38}$$

where

$$\mathbf{\Delta} \triangleq \operatorname{diag}\left\{\mathbf{\Delta}_s, \mathbf{\Delta}_a, \mathbf{\Delta}_b, \mathbf{\Delta}_d\right\},$$

$$\mathbf{\Delta}_s \triangleq \operatorname{diag}\left\{\mathbf{0}, \frac{\partial \boldsymbol{\omega}_s^{se}}{\partial \mathbf{q}^{se}}\right\}, \qquad \mathbf{\Delta}_a \triangleq \operatorname{diag}\left\{\mathbf{0}, \frac{\partial \boldsymbol{\omega}_a^{ae}}{\partial \mathbf{q}^{ae}}\right\},$$

$$\mathbf{\Delta}_b \triangleq \operatorname{diag}\left\{\mathbf{0}, \frac{\partial \boldsymbol{\omega}_b^{be}}{\partial \mathbf{q}^{be}}\right\}, \qquad \mathbf{\Delta}_d \triangleq \operatorname{diag}\left\{\mathbf{0}, \frac{\partial \boldsymbol{\omega}_d^{de}}{\partial \mathbf{q}^{de}}\right\},$$

$$\mathbf{a}_{non} \triangleq \begin{bmatrix} \mathbf{a}_{non,s}^{\mathsf{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}^{\mathsf{T}},$$

$$\mathbf{a}_{non,s} \triangleq \begin{bmatrix} \mathbf{0} \\ -\frac{\partial (\mathbf{C}_{se}\mathbf{v}_e^{po/e})^{\mathsf{T}}}{\partial \mathbf{q}^{se}} \mathbf{c}_s^{\mathcal{S}p\times}\boldsymbol{\omega}_s^{se} \end{bmatrix}.$$

XVIII. Null Space Method

Premultiplying (38) on the left by $\mathbf{\Pi}^{\mathsf{T}}\mathbf{\Gamma}^{\mathsf{T}}$ and using the following identities:

$$\mathbf{\Gamma}^{\mathsf{T}}\mathbf{S}^{\mathsf{T}} = \mathbf{1}, \qquad \mathbf{\Pi}^{\mathsf{T}}\mathbf{\Gamma}^{\mathsf{T}}\mathbf{\Xi}^{\mathsf{T}} = \mathbf{0},$$

$$\boldsymbol{\nu} = \mathbf{\Pi}\hat{\boldsymbol{\nu}}, \qquad \mathbf{\Gamma}^{\mathsf{T}}\left(\mathbf{S}^{\mathsf{T}} - \mathbf{\Delta}^{\mathsf{T}}\right) = \mathbf{\Omega},$$

$$\mathbf{\Omega} \triangleq \operatorname{diag}\left\{\mathbf{0}, \boldsymbol{\omega}_s^{se\times}, \mathbf{0}, \boldsymbol{\omega}_a^{ae\times}, \mathbf{0}, \boldsymbol{\omega}_b^{be\times}, \mathbf{0}, \boldsymbol{\omega}_d^{de\times}\right\},$$

one can rewrite the equations of motion concisely as

$$\hat{\mathbf{M}}\dot{\hat{\boldsymbol{\nu}}} = \hat{\boldsymbol{f}}_{non} + \hat{\boldsymbol{f}}, \tag{39}$$

where

$$\hat{\mathbf{M}} \triangleq \mathbf{\Pi}^{\mathsf{T}}\mathbf{M}\mathbf{\Pi},$$

$$\hat{\boldsymbol{f}}_{non} \triangleq \left(-\mathbf{\Pi}^{\mathsf{T}}\mathbf{M}\dot{\mathbf{\Pi}} - \mathbf{\Pi}^{\mathsf{T}}\dot{\mathbf{M}}\mathbf{\Pi} - \mathbf{\Pi}^{\mathsf{T}}\mathbf{\Omega}\mathbf{M}\mathbf{\Pi}\right)\hat{\boldsymbol{\nu}} + \mathbf{\Pi}^{\mathsf{T}}\mathbf{\Gamma}^{\mathsf{T}}\mathbf{a}_{non},$$

$$\hat{\boldsymbol{f}} \triangleq \mathbf{\Pi}^{\mathsf{T}}\mathbf{\Gamma}^{\mathsf{T}}\mathbf{B}\mathfrak{f}.$$

The full expressions of $\dot{\mathbf{M}}$ and $\dot{\mathbf{\Pi}}$ are given by (75) and (76), respectively. Lastly, additional simplifications can be made using

$$\hat{\mathbf{a}}_{non,s} \triangleq \operatorname{diag}\{\mathbf{1}, \mathbf{\Gamma}_s^{se\mathsf{T}}\}\mathbf{a}_{non,s} = \begin{bmatrix} \mathbf{0} \\ -\left(\mathbf{C}_{es}^{\mathsf{T}}\mathbf{v}_e^{po/e}\right)^{\times}\mathbf{c}_s^{\mathcal{S}p\times}\boldsymbol{\omega}_s^{se} \end{bmatrix}.$$

XIX. Nummerical Integration

In order to integrate (39), one can use MATLAB and the *ode45* built-in function. This function allows to integrate a system of first order differential equations defined as follows:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t)), \quad \mathbf{x}(0) = \mathbf{x}_0. \tag{40}$$

For this problem, $\mathbf{x}(t)$ is the augmented state of the system and $\mathbf{f}(t, \mathbf{x}(t))$ is given by

$$\mathbf{f}(t, \mathbf{x}(t)) \triangleq \begin{bmatrix} \mathbf{\Gamma}(\mathbf{q})\mathbf{\Pi}(\mathbf{q})\hat{\boldsymbol{\nu}} \\ \hat{\mathbf{M}}^{-1}(\mathbf{q})\left(\hat{\boldsymbol{f}}_{non}(\mathbf{q}, \dot{\mathbf{q}}) + \hat{\boldsymbol{f}}(\mathfrak{f})\right) \end{bmatrix}. \tag{41}$$

Note: $\dot{\mathbf{q}} = \mathbf{\Gamma}\mathbf{\Pi}\hat{\boldsymbol{\nu}}$.

XX. Simulation Result

In order to verify that the equations of motion are correctly derived, one can simulate the system[1] with no external force and make sure that the laws of conservation are satisfied within the numerical accuracy of the integration. In particular, Figure 3 shows that the energy is conserved for a particular set of non-trivial initial conditions (given in the Appendix, Table I), which is a good indicator of an accurate dynamic analysis. In order to make sure that the error introduced is not systematic, one can integrate the same system with a different tolerance and then verify that the error also varies accordingly. Figure 4 and 3 represents the kinetic energy of the same system with the same initial conditions. However, they were created using two different sets of simulation data that were integrated using a tolerance of 10^{-6} and of 10^{-8}, respectively. As expected, both the absolute and relative deviations of the kinetic energy vary proportionally to the numerical tolerance. Lastly, Figure 5 represents the evolution of the kinetic energy of each body with respect to time and also the transfer of energy happening between the reactions wheels and the spacecraft fuselage.

[1]The geometrical parameters used for the simulation are given in the Appendix, Table II.

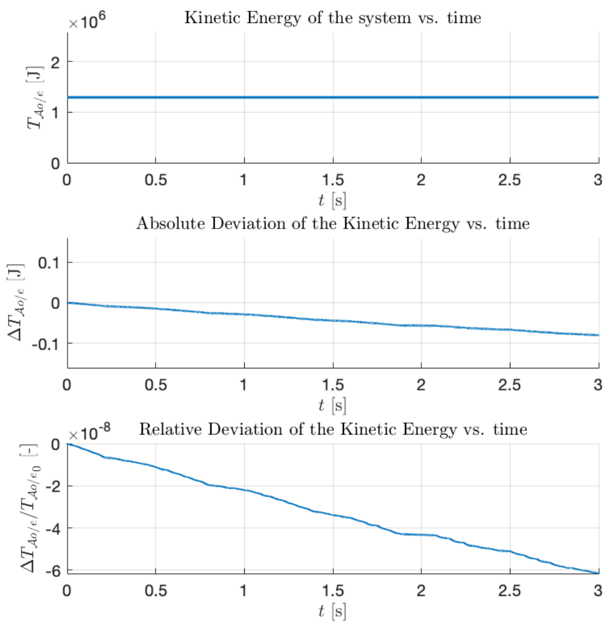

Fig. 3. Kinetic Energy of the system over time with non-trivial initial conditions and no external forces, integrated with an absolute and relative tolerance of 10^{-6}.

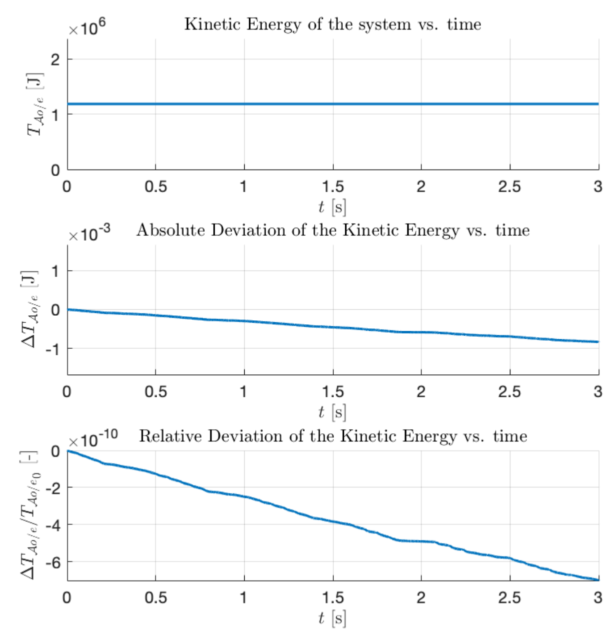

Fig. 4. Kinetic Energy of the system over time with non-trivial initial conditions and no external forces, integrated with an absolute and relative tolerance of 10^{-8}.

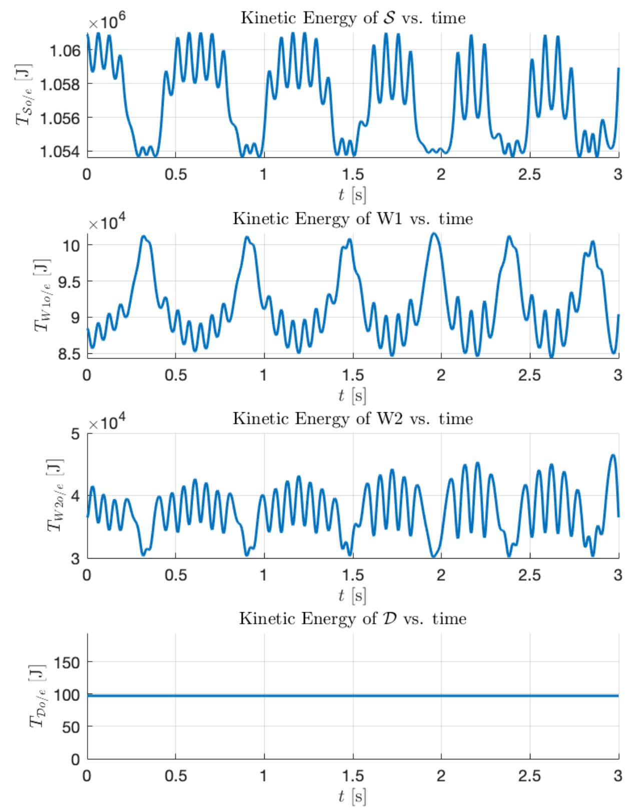

Fig. 5. Kinetic Energy of each body over time with non-trivial initial conditions and no external forces, integrated with an absolute and relative tolerance of 10^{-8}.

XXI. Conclusion

The results obtained are in accordance with what one can expect for such a problem. Additional verifications implying non-zero external forces could be made in order to make sure that the forces are correctly implemented. Once this is done, the next step would be to create a controller that drives the state of the system towards the desired state (i.e. the optimal state for the net launch). For this, one could also introduce an observer to take into consideration the inaccuracies introduced by the sensors. A more advanced analysis would also consider the variation of the propellant mass and the gravity acting on the bodies. The derivations presented in this report also illustrates the advantages of the Lagrangian approach to Dynamics. In particular, it allows to derive directly the equation of motions in a matrix format, which can be implemented efficiently in MATLAB. In addition, the null space method simplifies greatly the problem since the Lagrange multipliers do not have to be implicitly computed.

References

[1] (2016, April) The European Space Agency website. [Online]. Available: www.esa.int/Our_Activities/Space_Engineering_Technology/Clean_Space
[2] J. Forbes, "A systematic approach to dynamics," January 2019.
[3] J. R. Forbes. Slides 8 : Rigid-body equations of motion.
[4] ——. Slides 9 : Constrained rigid-bodies.

Appendix

$$\mathbf{C}_2(\alpha) \triangleq \begin{bmatrix} \cos(\alpha) & 0 & -\sin(\alpha) \\ 0 & 1 & 0 \\ \sin(\alpha) & 0 & \cos(\alpha) \end{bmatrix} \tag{42}$$

$$\mathbf{C}_3(\beta) \triangleq \begin{bmatrix} \cos(\beta) & \sin(\beta) & 0 \\ -\sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{43}$$

$$\mathbf{S}_s^{se} \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{s}_e^{3\mathsf{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{s}_e^{1\mathsf{T}} \\ \mathbf{s}_e^{2\mathsf{T}} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{44}$$

$$\mathbf{S}_a^{ae} \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{a}_e^{3\mathsf{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{a}_e^{1\mathsf{T}} \\ \mathbf{a}_e^{2\mathsf{T}} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{45}$$

$$\mathbf{S}_b^{be} \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{b}_e^{3\mathsf{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{b}_e^{1\mathsf{T}} \\ \mathbf{b}_e^{2\mathsf{T}} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{46}$$

$$\mathbf{S}_d^{de} \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{d}_e^{3\mathsf{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{d}_e^{1\mathsf{T}} \\ \mathbf{d}_e^{2\mathsf{T}} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{47}$$

$$\mathbf{S}_d^{ds} \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{d}_s^{3\mathsf{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{d}_s^{1\mathsf{T}} \\ \mathbf{d}_s^{2\mathsf{T}} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{48}$$

$$\mathbf{S}_a^{as} \triangleq \mathbf{1}_2 \tag{49}$$

$$\mathbf{S}_b^{bs} \triangleq \mathbf{1}_3 \tag{50}$$

$$\boldsymbol{\Gamma}_s^{se} \triangleq \begin{bmatrix} \mathbf{0} & -\mathbf{s}_e^3 & \mathbf{s}_e^2 \\ \mathbf{s}_e^3 & \mathbf{0} & -\mathbf{s}_e^1 \\ -\mathbf{s}_e^2 & \mathbf{s}_e^1 & \mathbf{0} \end{bmatrix} \tag{51}$$

$$\boldsymbol{\Gamma}_a^{ae} \triangleq \begin{bmatrix} \mathbf{0} & -\mathbf{a}_e^3 & \mathbf{a}_e^2 \\ \mathbf{a}_e^3 & \mathbf{0} & -\mathbf{a}_e^1 \\ -\mathbf{a}_e^2 & \mathbf{a}_e^1 & \mathbf{0} \end{bmatrix} \tag{52}$$

$$\boldsymbol{\Gamma}_b^{be} \triangleq \begin{bmatrix} \mathbf{0} & -\mathbf{b}_e^3 & \mathbf{b}_e^2 \\ \mathbf{b}_e^3 & \mathbf{0} & -\mathbf{b}_e^1 \\ -\mathbf{b}_e^2 & \mathbf{b}_e^1 & \mathbf{0} \end{bmatrix} \tag{53}$$

$$\boldsymbol{\Gamma}_d^{de} \triangleq \begin{bmatrix} \mathbf{0} & -\mathbf{d}_e^3 & \mathbf{d}_e^2 \\ \mathbf{d}_e^3 & \mathbf{0} & -\mathbf{d}_e^1 \\ -\mathbf{d}_e^2 & \mathbf{d}_e^1 & \mathbf{0} \end{bmatrix} \tag{54}$$

$$\boldsymbol{\Gamma}_d^{ds} \triangleq \begin{bmatrix} \mathbf{0} & -\mathbf{d}_s^3 & \mathbf{d}_s^2 \\ \mathbf{d}_s^3 & \mathbf{0} & -\mathbf{d}_s^1 \\ -\mathbf{d}_s^2 & \mathbf{d}_s^1 & \mathbf{0} \end{bmatrix} \tag{55}$$

$$\boldsymbol{\Gamma}_a^{as} = \mathbf{1}_2^{\mathsf{T}} \tag{56}$$

$$\boldsymbol{\Gamma}_b^{bs} = \mathbf{1}_3^{\mathsf{T}} \tag{57}$$

$$\boldsymbol{\Pi} \triangleq \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & -\mathbf{C}_{es}\mathbf{r}_s^{g_1 p \times} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{as} & \mathbf{S}_a^{as} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & -\mathbf{C}_{es}\mathbf{r}_s^{g_2 p \times} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{bs} & \mathbf{0} & \mathbf{S}_b^{bs} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \tag{58}$$

$$\boldsymbol{\Sigma}(\bar{\mathbf{q}}) \triangleq \begin{bmatrix} \mathbf{r}_e^{po} \\ \mathbf{q}^{se} \\ \mathbf{r}_e^{po} + \mathbf{C}_{es}\mathbf{r}_s^{g_1 p} \\ \mathbf{C}_{es}\mathbf{C}_2^{\mathsf{T}}(\alpha)\mathbf{1}_1 \\ \mathbf{C}_{es}\mathbf{C}_2^{\mathsf{T}}(\alpha)\mathbf{1}_2 \\ \mathbf{C}_{es}\mathbf{C}_2^{\mathsf{T}}(\alpha)\mathbf{1}_3 \\ \mathbf{r}_e^{po} + \mathbf{C}_{es}\mathbf{r}_s^{g_2 p} \\ \mathbf{C}_{es}\mathbf{C}_3^{\mathsf{T}}(\beta)\mathbf{1}_1 \\ \mathbf{C}_{es}\mathbf{C}_3^{\mathsf{T}}(\beta)\mathbf{1}_2 \\ \mathbf{C}_{es}\mathbf{C}_3^{\mathsf{T}}(\beta)\mathbf{1}_3 \\ \mathbf{r}_e^{g_d o} \\ \mathbf{C}_{es}\mathbf{C}_{sd}\mathbf{1}_1 \\ \mathbf{C}_{es}\mathbf{C}_{sd}\mathbf{1}_2 \\ \mathbf{C}_{es}\mathbf{C}_{sd}\mathbf{1}_3 \\ \mathbf{v}_e^{po/e} \\ \boldsymbol{\omega}_s^{se} \\ \dot{\alpha} \\ \dot{\beta} \\ \mathbf{v}_e^{g_d o} \\ \boldsymbol{\omega}_d^{ds} + \mathbf{C}_{sd}^{\mathsf{T}}\boldsymbol{\omega}_s^{se} \end{bmatrix} \tag{59}$$

$$\begin{aligned} \boldsymbol{F}^{\mathcal{S}}(\mathbf{x}) \triangleq & \int_0^{\rho_o}\int_0^{2\pi}\int_0^{t_s} \mathbf{x}\,\rho_s\,\mathrm{d}z_s\,\mathrm{d}\theta_s\,\mathrm{d}\rho_s \\ & + \int_{(\rho_o - t_s)}^{\rho_o}\int_0^{2\pi}\int_{t_s}^{(l_s - t_s)} \mathbf{x}\,\rho_s\,\mathrm{d}z_s\,\mathrm{d}\theta_s\,\mathrm{d}\rho_s \\ & + \int_0^{\rho_o}\int_0^{2\pi}\int_{(l_s - t_s)}^{l_s} \mathbf{x}\,\rho_s\,\mathrm{d}z_s\,\mathrm{d}\theta_s\,\mathrm{d}\rho_s \end{aligned} \tag{60}$$

$$\boldsymbol{F}^{W1}(\mathbf{x}) \triangleq \int_0^{\rho_1}\int_0^{2\pi}\int_{-\frac{l_1}{2}}^{\frac{l_1}{2}} \mathbf{x}\,\rho_a\,\mathrm{d}z_a\,\mathrm{d}\theta_a\,\mathrm{d}\rho_a \tag{61}$$

$$\boldsymbol{F}^{W2}(\mathbf{x}) \triangleq \int_0^{\rho_2}\int_0^{2\pi}\int_{-\frac{l_2}{2}}^{\frac{l_2}{2}} \mathbf{x}\,\rho_b\,\mathrm{d}z_b\,\mathrm{d}\theta_b\,\mathrm{d}\rho_b, \tag{62}$$

$$\boldsymbol{F}^{\mathcal{D}}(\mathbf{x}) \triangleq \int_{-\frac{l_d}{2}}^{\frac{l_d}{2}}\int_{-\frac{l_d}{2}}^{\frac{l_d}{2}}\int_{-\frac{l_d}{2}}^{\frac{l_d}{2}} \mathbf{x}\,\mathrm{d}z_d\,\mathrm{d}y_d\,\mathrm{d}x_d. \tag{63}$$

$$\left(\mathbf{r}_s^{\mathrm{d}m_s p^\times}\right)^2 = \begin{bmatrix} -z_s^2 - \rho_s^2 s_{\theta_s}^2 & \rho_s^2 s_{\theta_s} c_{\theta_s} & z_s\rho_s c_{\theta_s} \\ \rho_s^2 s_{\theta_s} c_{\theta_s} & -z_s^2 - \rho_s^2 c_{\theta_s}^2 & z_s\rho_s s_{\theta_s} \\ z_s\rho_s c_{\theta_s} & z_s\rho_s s_{\theta_s} & -\rho_s^2 \end{bmatrix} \tag{64}$$

$$\left(\mathbf{r}_a^{\mathrm{d}m_1 g_1^\times}\right)^2 = \begin{bmatrix} -z_a^2 - \rho_a^2 s_{\theta_a}^2 & z_a\rho_a c_{\theta_a} & \rho_a^2 s_{\theta_a} c_{\theta_a} \\ z_a\rho_a c_{\theta_a} & -\rho_a^2 & z_a\rho_a s_{\theta_a} \\ \rho_a^2 s_{\theta_a} c_{\theta_a} & z_a\rho_a s_{\theta_a} & -z_a^2 - \rho_a^2 c_{\theta_a}^2 \end{bmatrix} \tag{65}$$

$$\left(\mathbf{r}_b^{\mathrm{d}m_2 g_2^\times}\right)^2 = \begin{bmatrix} -z_b^2 - \rho_b^2 s_{\theta_b}^2 & \rho_b^2 s_{\theta_b} c_{\theta_b} & z_b\rho_b c_{\theta_b} \\ \rho_b^2 s_{\theta_b} c_{\theta_b} & -z_b^2 - \rho_b^2 c_{\theta_b}^2 & z_b\rho_b s_{\theta_b} \\ z_b\rho_b c_{\theta_b} & z_b\rho_b s_{\theta_b} & -\rho_b^2 \end{bmatrix} \tag{66}$$

$$\left(\mathbf{r}_d^{\mathrm{d}m_d g_d^\times}\right)^2 = \begin{bmatrix} -z_d^2 - y_d^2 & x_d y_d & x_d z_d \\ x_d y_d & -z_d^2 - x_d^2 & y_d z_d \\ x_d z_d & y_d z_d & -x_d^2 - y_d^2 \end{bmatrix} \tag{67}$$

$$\boldsymbol{\Phi}_{kin}(\mathbf{q}) \triangleq \begin{bmatrix} \boldsymbol{\Phi}_{se}(\mathbf{q}^{se}) \\ \boldsymbol{\Phi}_{ae}(\mathbf{q}^{ae}) \\ \boldsymbol{\Phi}_{be}(\mathbf{q}^{be}) \\ \boldsymbol{\Phi}_{de}(\mathbf{q}^{de}) \end{bmatrix} \tag{68}$$

$$\boldsymbol{\Phi}_{se}(\mathbf{q}^{se}) \triangleq \begin{bmatrix} \mathbf{s}_e^{1\mathsf{T}}\mathbf{s}_e^1 - 1 \\ \mathbf{s}_e^{2\mathsf{T}}\mathbf{s}_e^2 - 1 \\ \mathbf{s}_e^{2\mathsf{T}}\mathbf{s}_e^1 \\ \mathbf{s}_e^{1\times}\mathbf{s}_e^2 - \mathbf{s}_e^3 \end{bmatrix} \qquad \boldsymbol{\Phi}_{ae}(\mathbf{q}^{ae}) \triangleq \begin{bmatrix} \mathbf{a}_e^{1\mathsf{T}}\mathbf{a}_e^1 - 1 \\ \mathbf{a}_e^{2\mathsf{T}}\mathbf{a}_e^2 - 1 \\ \mathbf{a}_e^{2\mathsf{T}}\mathbf{a}_e^1 \\ \mathbf{a}_e^{1\times}\mathbf{a}_e^2 - \mathbf{s}_e^3 \end{bmatrix}$$

$$\boldsymbol{\Phi}_{be}(\mathbf{q}^{be}) \triangleq \begin{bmatrix} \mathbf{b}_e^{1\mathsf{T}}\mathbf{b}_e^1 - 1 \\ \mathbf{b}_e^{2\mathsf{T}}\mathbf{b}_e^2 - 1 \\ \mathbf{b}_e^{2\mathsf{T}}\mathbf{b}_e^1 \\ \mathbf{b}_e^{1\times}\mathbf{b}_e^2 - \mathbf{s}_e^3 \end{bmatrix} \quad \boldsymbol{\Phi}_{de}(\mathbf{q}^{de}) \triangleq \begin{bmatrix} \mathbf{d}_e^{1\mathsf{T}}\mathbf{d}_e^1 - 1 \\ \mathbf{d}_e^{2\mathsf{T}}\mathbf{d}_e^2 - 1 \\ \mathbf{d}_e^{2\mathsf{T}}\mathbf{d}_e^1 \\ \mathbf{d}_e^{1\times}\mathbf{d}_e^2 - \mathbf{d}_s^3 \end{bmatrix}$$

$$\boldsymbol{\Xi}^{kin} \triangleq \begin{bmatrix} \mathbf{0} & \boldsymbol{\Xi}_s^{se} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\Xi}_a^{ae} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\Xi}_b^{be} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\Xi}_d^{de} \end{bmatrix} \tag{69}$$

$$\boldsymbol{\Xi}_s^{se} \triangleq \begin{bmatrix} 2\mathbf{s}_e^{1\mathsf{T}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\mathbf{s}_e^{2\mathsf{T}} & \mathbf{0} \\ \mathbf{s}_e^{2\mathsf{T}} & \mathbf{s}_e^{1\mathsf{T}} & \mathbf{0} \\ -\mathbf{s}_e^{2\times} & \mathbf{s}_e^{1\times} & -\mathbf{1} \end{bmatrix} \quad \boldsymbol{\Xi}_a^{ae} \triangleq \begin{bmatrix} 2\mathbf{a}_e^{1\mathsf{T}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\mathbf{a}_e^{2\mathsf{T}} & \mathbf{0} \\ \mathbf{a}_e^{2\mathsf{T}} & \mathbf{a}_e^{1\mathsf{T}} & \mathbf{0} \\ -\mathbf{a}_e^{2\times} & \mathbf{a}_e^{1\times} & -\mathbf{1} \end{bmatrix}$$

$$\boldsymbol{\Xi}_b^{be} \triangleq \begin{bmatrix} 2\mathbf{b}_e^{1\mathsf{T}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\mathbf{b}_e^{2\mathsf{T}} & \mathbf{0} \\ \mathbf{b}_e^{2\mathsf{T}} & \mathbf{b}_e^{1\mathsf{T}} & \mathbf{0} \\ -\mathbf{b}_e^{2\times} & \mathbf{b}_e^{1\times} & -\mathbf{1} \end{bmatrix} \quad \boldsymbol{\Xi}_d^{de} \triangleq \begin{bmatrix} 2\mathbf{d}_e^{1\mathsf{T}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\mathbf{d}_e^{2\mathsf{T}} & \mathbf{0} \\ \mathbf{d}_e^{2\mathsf{T}} & \mathbf{d}_e^{1\mathsf{T}} & \mathbf{0} \\ -\mathbf{d}_e^{2\times} & \mathbf{d}_e^{1\times} & -\mathbf{1} \end{bmatrix}$$

$$\boldsymbol{\Xi}^{col} \triangleq \begin{bmatrix} \mathbf{1} & \boldsymbol{\Xi}_{1,2}^{col} & -\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \boldsymbol{\Xi}_{2,2}^{col} & \mathbf{0} & \mathbf{0} & -\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{70}$$

$$\boldsymbol{\Xi}_{1,2}^{col} \triangleq -\mathbf{C}_{es}\mathbf{r}_s^{g_1 p\times}\mathbf{S}_s^{se}, \qquad \boldsymbol{\Xi}_{2,2}^{col} \triangleq -\mathbf{C}_{es}\mathbf{r}_s^{g_2 p\times}\mathbf{S}_s^{se}.$$

$$\boldsymbol{\Xi}^{att} \triangleq \begin{bmatrix} \mathbf{0} & \boldsymbol{\Xi}_{1,2}^{att} & \mathbf{0} & \boldsymbol{\Xi}_{1,4}^{att} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Xi}_{2,2}^{att} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\Xi}_{2,6}^{att} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{71}$$

$$\boldsymbol{\Xi}_{1,2}^{att} \triangleq -\begin{bmatrix} \mathbf{1}_1^{\mathsf{T}} \\ \mathbf{1}_3^{\mathsf{T}} \end{bmatrix}\mathbf{C}_{as}\mathbf{S}_s^{se}, \qquad \boldsymbol{\Xi}_{1,4}^{att} \triangleq \begin{bmatrix} \mathbf{1}_1^{\mathsf{T}} \\ \mathbf{1}_3^{\mathsf{T}} \end{bmatrix}\mathbf{S}_a^{ae},$$

$$\boldsymbol{\Xi}_{2,2}^{att} \triangleq -\begin{bmatrix} \mathbf{1}_1^{\mathsf{T}} \\ \mathbf{1}_2^{\mathsf{T}} \end{bmatrix}\mathbf{C}_{bs}\mathbf{S}_s^{se}, \qquad \boldsymbol{\Xi}_{2,6}^{att} \triangleq \begin{bmatrix} \mathbf{1}_1^{\mathsf{T}} \\ \mathbf{1}_2^{\mathsf{T}} \end{bmatrix}\mathbf{S}_b^{be}.$$

$$\mathbf{B}_s \triangleq \begin{bmatrix} \mathbf{s}_e^3 & -\mathbf{s}_e^3 & -\mathbf{s}_e^3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{s}_e^2 \\ \mathbf{0} & \rho_s\mathbf{s}_e^3 & -\rho_s\mathbf{s}_e^3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -l_s\mathbf{s}_e^3 & -l_s\mathbf{s}_e^3 & -\mathbf{s}_e^1 & \mathbf{0} \end{bmatrix} \tag{72}$$

$$\mathbf{B}_a \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{a}_e^1 & \mathbf{0} \end{bmatrix} \tag{73}$$

$$\mathbf{B}_b \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{b}_e^2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{74}$$

$$\dot{\mathbf{M}} = \operatorname{diag}\{\dot{\mathbf{M}}^{\mathcal{S}p}, \mathbf{0}, \mathbf{0}, \mathbf{0}\} \tag{75}$$

$$\dot{\mathbf{M}}^{\mathcal{S}p} = \begin{bmatrix} \mathbf{0} & -\mathbf{C}_{es}\boldsymbol{\omega}_s^{se\times}\mathbf{c}_s^{\mathcal{S}p\times} \\ -\mathbf{c}_s^{\mathcal{S}p\times}\boldsymbol{\omega}_s^{se\times}\mathbf{C}_{es}^{\mathsf{T}} & \mathbf{0} \end{bmatrix}$$

$$\dot{\boldsymbol{\Pi}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dot{\boldsymbol{\Pi}}_{3,2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dot{\boldsymbol{\Pi}}_{4,2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dot{\boldsymbol{\Pi}}_{5,2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dot{\boldsymbol{\Pi}}_{6,2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{76}$$

$$\dot{\boldsymbol{\Pi}}_{3,2} = -\mathbf{C}_{es}\boldsymbol{\omega}_s^{se\times}\mathbf{r}_s^{g_1 p\times} \qquad \dot{\boldsymbol{\Pi}}_{4,2} = -\boldsymbol{\omega}_a^{as\times}\mathbf{C}_{as}$$

$$\dot{\boldsymbol{\Pi}}_{5,2} = -\mathbf{C}_{es}\boldsymbol{\omega}_s^{se\times}\mathbf{r}_s^{g_2 p\times} \qquad \dot{\boldsymbol{\Pi}}_{6,2} = -\boldsymbol{\omega}_b^{bs\times}\mathbf{C}_{bs}$$

TABLE I
INITIAL CONDITIONS

Measurable	Value
$\mathbf{r}_e^{po/e}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ [m]
$\mathbf{C}_{es}$	$\mathbf{1}$
α	0 [rad]
β	0 [rad]
$\mathbf{r}_e^{g_d o/e}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ [m]
$\mathbf{C}_{sd}$	$\mathbf{1}$
$\mathbf{v}_e^{po/e}$	$\begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$ [m/ sec]
$\boldsymbol{\omega}_s^{se}$	$\begin{bmatrix} 0 \\ \pi \\ 10\pi \end{bmatrix}$ [rad/ sec]
$\dot{\alpha}$	100π [rad/ sec]
$\dot{\beta}$	50π [rad/ sec]
$\mathbf{v}_e^{g_d o/e}$	$\begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$ [m/ sec]
$\boldsymbol{\omega}_d^{ds}$	$\begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$ [rad/ sec]

TABLE II
GEOMETRICAL PARAMETERS

Paramter	Value
t_s	0.03 [m]
ρ_o	1 [m]
l_s	3 [m]
ρ_p	0.5 [m]
σ_s	2700 [kg m^{-3}]
l_1	0.1 [m]
ρ_1	0.25 [m]
z_1	1 [m]
σ_1	2700 [kg m^{-3}]
l_2	0.1 [m]
ρ_2	0.25 [m] [m/ sec]
z_2	2 [m]
σ_2	2700 [kg m^{-3}]
l_d	0.2 [m]
σ_d	2700 [kg m^{-3}]