Assignment 5

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1

a) Step 1.1 is already done. Therefore, let's carry out steps 1.2 to 1.5.

$$\begin{aligned}
\underrightarrow{\omega}^{ca} &= \underrightarrow{\omega}^{cb} + \underrightarrow{\omega}^{ba} \\
&= \underrightarrow{\mathcal{F}}_c^{\mathsf{T}} \mathbf{1}_2 \dot{\theta} + \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \mathbf{1}_3 \dot{\phi} \\
&= \underrightarrow{\mathcal{F}}_c^{\mathsf{T}} \left(\mathbf{1}_2 \dot{\theta} + \mathbf{C}_{cb} \mathbf{1}_3 \dot{\phi} \right) \\
&= \underrightarrow{\mathcal{F}}_c^{\mathsf{T}} \begin{bmatrix} -\sin(\theta)\dot{\phi} \\ \dot{\theta} \\ \cos(\theta)\dot{\phi} \end{bmatrix},
\end{aligned}$$

$$\boldsymbol{\omega}_c^{ca\times} = \begin{bmatrix} 0 & -\cos(\theta)\dot{\phi} & \dot{\theta} \\ \cos(\theta)\dot{\phi} & 0 & \sin(\theta)\dot{\phi} \\ -\dot{\theta} & -\sin(\theta)\dot{\phi} & 0 \end{bmatrix},$$

$$\underrightarrow{r}^{pw} = \underrightarrow{\mathcal{F}}_c^{\mathsf{T}} \begin{bmatrix} 0 \\ 0 \\ l \end{bmatrix},$$

$$\begin{aligned}
\underrightarrow{v}^{pw/a} &= \underrightarrow{r}^{pw\,\bullet a} \\
&= \underbrace{\underrightarrow{r}^{pw\,\bullet c}}_{\underrightarrow{0}} + \underrightarrow{\omega}^{ca} \times \underrightarrow{r}^{pw} \\
&= \underrightarrow{\mathcal{F}}_c^{\mathsf{T}} \boldsymbol{\omega}_c^{ca\times} \mathbf{r}_c^{pw} \\
&= \underrightarrow{\mathcal{F}}_c^{\mathsf{T}} \begin{bmatrix} 0 & -\cos(\theta)\dot{\phi} & \dot{\theta} \\ \cos(\theta)\dot{\phi} & 0 & \sin(\theta)\dot{\phi} \\ -\dot{\theta} & -\sin(\theta)\dot{\phi} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ l \end{bmatrix} \\
&= \underrightarrow{\mathcal{F}}_c^{\mathsf{T}} \begin{bmatrix} l\dot{\theta} \\ l\sin(\theta)\dot{\phi} \\ 0 \end{bmatrix}.
\end{aligned}$$

Since we don't have constraints, we can now compute the different energies, i.e. steps 2.1 to 2.3:

$$
\begin{aligned}
T_{pw/a} &= \tfrac{1}{2} m \underset{\rightarrow}{v}^{pw/a} \cdot \underset{\rightarrow}{v}^{pw/a} \\
&= \tfrac{1}{2} m \mathbf{v}_c^{pw/a\,\mathsf{T}} \mathbf{v}_c^{pw/a} \\
&= \tfrac{1}{2} m \begin{bmatrix} l\dot{\theta} & l\sin(\theta)\dot{\phi} & 0 \end{bmatrix} \begin{bmatrix} l\dot{\theta} \\ l\sin(\theta)\dot{\phi} \\ 0 \end{bmatrix} \\
&= \tfrac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2). \\
U_{pw} &= -m \underset{\rightarrow}{g} \cdot \underset{\rightarrow}{r}^{pw} \\
&= -m \begin{bmatrix} 0 & 0 & -g \end{bmatrix} \begin{bmatrix} l\cos(\phi)\sin(\theta) \\ l\sin(\phi)\sin(\theta) \\ l\cos(\theta) \end{bmatrix} \\
&= mgl\cos(\theta). \\
L_{pw/a} &= T_{pw/a} - U_{pw} \\
&= \tfrac{1}{2} m l^2 (\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2) - mgl\cos(\theta).
\end{aligned}
$$

Since no external force is applied, we have

$$
\boldsymbol{f} \triangleq \begin{bmatrix} \mathsf{f}_\phi \\ \mathsf{f}_\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{1}
$$

Applying steps 4.1 and 4.2, we get:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{pw/a}}{\partial \dot{\mathbf{q}}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\begin{bmatrix} \frac{\partial L_{pw/a}}{\partial \dot{\phi}} & \frac{\partial L_{pw/a}}{\partial \dot{\theta}} \end{bmatrix}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\begin{bmatrix} ml^2\sin^2(\theta)\dot{\phi} & ml^2\dot{\theta} \end{bmatrix}\right) \\
&= \begin{bmatrix} ml^2\left\{2\sin(\theta)\cos(\theta)\dot{\theta}\dot{\phi} + \sin^2(\theta)\ddot{\phi}\right\} & ml^2\ddot{\theta} \end{bmatrix} \\
&= \begin{bmatrix} ml^2\left\{\sin(2\theta)\dot{\theta}\dot{\phi} + \sin^2(\theta)\ddot{\phi}\right\} & ml^2\ddot{\theta} \end{bmatrix}, \qquad (2) \\
\frac{\partial L_{pw/a}}{\partial \mathbf{q}} &= \begin{bmatrix} \frac{\partial L_{pw/a}}{\partial \phi} & \frac{\partial L_{pw/a}}{\partial \theta} \end{bmatrix} \\
&= \begin{bmatrix} 0 & ml^2\dot{\phi}^2\sin(\theta)\cos(\theta) + mgl\sin(\theta) \end{bmatrix} \\
&= \begin{bmatrix} 0 & \tfrac{1}{2}ml^2\dot{\phi}^2\sin(2\theta) + mgl\sin(\theta) \end{bmatrix}. \qquad (3)
\end{aligned}
$$

Finally, since our generalized coordinates are unconstrained, we can apply the Lagrange's Equation for an Unconstrained Particle, i.e.

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{pw/a}}{\partial \dot{\mathbf{q}}}\right)^{\mathsf{T}} - \left(\frac{\partial L_{pw/a}}{\partial \mathbf{q}}\right)^{\mathsf{T}} = \boldsymbol{f}. \tag{4}
$$

Substituting (1), (2) and (3) into (4), we obtain the following set of differential equations:

$$\begin{aligned} ml^2\left\{\sin(2\theta)\dot{\theta}\dot{\phi} + \sin^2(\theta)\ddot{\phi}\right\} &= 0, \\ ml^2\ddot{\theta} - \tfrac{1}{2}ml^2\dot{\phi}^2\sin(2\theta) - mgl\sin(\theta) &= 0, \end{aligned}$$

which can be further simplified as follows:

$$\begin{aligned} \sin^2(\theta)\ddot{\phi} + \sin(2\theta)\dot{\theta}\dot{\phi} &= 0, \\ \ddot{\theta} - \tfrac{1}{2}\dot{\phi}^2\sin(2\theta) - \frac{g}{l}\sin(\theta) &= 0. \end{aligned}$$

To verify partly this result, we can see that if we set $\dot{\phi} = \ddot{\phi} = 0$ and if we use $\alpha = \pi - \theta$, we obtain the following differential equation:

$$\ddot{\alpha} + \frac{g}{l}\sin(\alpha) = 0,$$

which is the equation describing the motion of a simple gravity pendulum of length l.

b) Following the same procedure as in a):

$$\begin{aligned}
\underset{\rightarrow}{r}^{pw} &= \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a^{\mathsf{T}} \begin{bmatrix} x_a \\ y_a \\ \sqrt{\ell^2 - x_a^2 - y_a^2} \end{bmatrix}, \\
\underset{\rightarrow}{v}^{pw/a} &= \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a^{\mathsf{T}} \begin{bmatrix} \dot{x}_a \\ \dot{y}_a \\ -\frac{x_a\dot{x}_a + y_a\dot{y}_a}{\sqrt{\ell^2 - x_a^2 - y_a^2}} \end{bmatrix}, \\
T_{pw/a} &= \tfrac{1}{2}m\underset{\rightarrow}{v}^{pw/a} \cdot \underset{\rightarrow}{v}^{pw/a} \\
&= \tfrac{1}{2}m\left(\dot{x}_a^2 + \dot{y}_a^2 + \frac{(x_a\dot{x}_a + y_a\dot{y}_a)^2}{\ell^2 - x_a^2 - y_a^2}\right), \\
U_{pw} &= -m\underset{\rightarrow}{g} \cdot \underset{\rightarrow}{r}^{pw} \\
&= -m\begin{bmatrix} 0 & 0 & -g \end{bmatrix}\begin{bmatrix} x_a \\ y_a \\ \sqrt{\ell^2 - x_a^2 - y_a^2} \end{bmatrix} \\
&= mg\sqrt{\ell^2 - x_a^2 - y_a^2}, \\
L_{pw/a} &= T_{pw/a} - U_{pw} \\
&= \tfrac{1}{2}m\left(\dot{x}_a^2 + \dot{y}_a^2 + \frac{(x_a\dot{x}_a + y_a\dot{y}_a)^2}{\ell^2 - x_a^2 - y_a^2}\right) - mg\sqrt{\ell^2 - x_a^2 - y_a^2}, \\
\boldsymbol{f} &\triangleq \begin{bmatrix} \mathsf{f}_{x_a} \\ \mathsf{f}_{y_a} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{pw/a}}{\partial \dot{\mathbf{q}}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\left[\begin{array}{cc} \frac{\partial L_{pw/a}}{\partial \dot{x_a}} & \frac{\partial L_{pw/a}}{\partial \dot{y_a}} \end{array}\right]\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\left[\begin{array}{cc} \frac{1}{2}m\left\{2\dot{x}_a + 2x_a\frac{x_a\dot{x}_a+y_a\dot{y}_a}{\ell^2-x_a^2-y_a^2}\right\} & \frac{1}{2}m\left\{2\dot{y}_a + 2y_a\frac{x_a\dot{x}_a+y_a\dot{y}_a}{\ell^2-x_a^2-y_a^2}\right\} \end{array}\right]\right) \\
&= m\left[\begin{array}{c} \ddot{x}_a + \frac{2x_a\dot{x}_a^2+\dot{x}_a\dot{y}_a y_a+x_a^2\ddot{x}_a+x_a\dot{y}_a^2+x_a y_a\ddot{y}_a}{\ell^2-x_a^2-y_a^2} + 2x_a\frac{(x_a\dot{x}_a+y_a\dot{y}_a)^2}{(\ell^2-x_a^2-y_a^2)^2} \\ \ddot{y}_a + \frac{2y_a\dot{y}_a^2+\dot{y}_a\dot{x}_a x_a+y_a^2\ddot{y}_a+y_a\dot{x}_a^2+y_a x_a\ddot{x}_a}{\ell^2-x_a^2-y_a^2} + 2y_a\frac{(x_a\dot{x}_a+y_a\dot{y}_a)^2}{(\ell^2-x_a^2-y_a^2)^2} \end{array}\right]^{\mathsf{T}}, \\
\frac{\partial L_{pw/a}}{\partial \mathbf{q}} &= \left[\begin{array}{cc} \frac{\partial L_{pw/a}}{\partial x_a} & \frac{\partial L_{pw/a}}{\partial y_a} \end{array}\right] \\
&= m\left[\begin{array}{c} \frac{gx_a}{\sqrt{\ell^2-x_a^2-y_a^2}} + \dot{x}_a\frac{x_a\dot{x}_a+y_a\dot{y}_a}{\ell^2-x_a^2-y_a^2} + x_a\frac{(x_a\dot{x}_a+y_a\dot{y}_a)^2}{(\ell^2-x_a^2-y_a^2)^2} \\ \frac{gy_a}{\sqrt{\ell^2-x_a^2-y_a^2}} + \dot{y}_a\frac{x_a\dot{x}_a+y_a\dot{y}_a}{\ell^2-x_a^2-y_a^2} + y_a\frac{(x_a\dot{x}_a+y_a\dot{y}_a)^2}{(\ell^2-x_a^2-y_a^2)^2} \end{array}\right]^{\mathsf{T}}.
\end{aligned}
$$

Finally, substituting everything into (4) and rearranging:

$$
\begin{aligned}
\ddot{x}_a + \frac{x_a\dot{x}_a^2 + x_a^2\ddot{x}_a + x_a\dot{y}_a^2 + x_a y_a\ddot{y}_a}{\ell^2 - x_a^2 - y_a^2} + x_a\frac{(x_a\dot{x}_a + y_a\dot{y}_a)^2}{(\ell^2 - x_a^2 - y_a^2)^2} - \frac{gx_a}{\sqrt{\ell^2 - y_a^2 - y_a^2}} &= 0, \\
\ddot{y}_a + \frac{y_a\dot{y}_a^2 + y_a^2\ddot{y}_a + y_a\dot{x}_a^2 + y_a x_a\ddot{x}_a}{\ell^2 - x_a^2 - y_a^2} + y_a\frac{(x_a\dot{x}_a + y_a\dot{y}_a)^2}{(\ell^2 - x_a^2 - y_a^2)^2} - \frac{gy_a}{\sqrt{\ell^2 - y_a^2 - y_a^2}} &= 0.
\end{aligned}
$$

c) As in a) and b):

$$
\begin{aligned}
\underset{\rightarrow}{r}^{pw} &= \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a^{\mathsf{T}}\left[\begin{array}{c} x_a \\ y_a \\ z_a \end{array}\right], \\
\underset{\rightarrow}{v}^{pw/a} &= \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a^{\mathsf{T}}\left[\begin{array}{c} \dot{x}_a \\ \dot{y}_a \\ \dot{z}_a \end{array}\right], \\
T_{pw/a} &= \frac{1}{2}m\,\underset{\rightarrow}{v}^{pw/a}\cdot\underset{\rightarrow}{v}^{pw/a} = \frac{1}{2}m\left(\dot{x}_a^2 + \dot{y}_a^2 + \dot{z}_a^2\right), \\
U_{pw} &= -m\,\underset{\rightarrow}{g}\cdot\underset{\rightarrow}{r}^{pw} = -m\left[\begin{array}{ccc} 0 & 0 & -g \end{array}\right]\left[\begin{array}{c} x_a \\ y_a \\ z_a \end{array}\right] = mgz_a, \\
L_{pw/a} &= T_{pw/a} - U_{pw} = \frac{1}{2}m\left(\dot{x}_a^2 + \dot{y}_a^2 + \dot{z}_a^2\right) - mgz_a, \\
\boldsymbol{f} &\triangleq \left[\begin{array}{c} \mathsf{f}_{x_a} \\ \mathsf{f}_{y_a} \\ \mathsf{f}_{z_a} \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right].
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{pw/a}}{\partial \dot{\mathbf{q}}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\left[\begin{array}{ccc} \frac{\partial L_{pw/a}}{\partial \dot{x}_a} & \frac{\partial L_{pw/a}}{\partial \dot{y}_a} & \frac{\partial L_{pw/a}}{\partial \dot{z}_a} \end{array}\right]\right) = \left[\begin{array}{ccc} m\ddot{x}_a & m\ddot{y}_a & m\ddot{z}_a \end{array}\right], \\
\frac{\partial L_{pw/a}}{\partial \mathbf{q}} &= \left[\begin{array}{ccc} \frac{\partial L_{pw/a}}{\partial x_a} & \frac{\partial L_{pw/a}}{\partial y_a} & \frac{\partial L_{pw/a}}{\partial z_a} \end{array}\right] = \left[\begin{array}{ccc} 0 & 0 & -mg \end{array}\right].
\end{aligned}
$$

However, we know have a holonomic constraint:

$$\phi(\mathbf{q}) = \mathbf{q}^\mathsf{T}\mathbf{q} - \ell^2 = x_a^2 + y_a^2 + z_a^2 - \ell^2 = 0.$$

Defining the quantities

$$\begin{aligned}
\boldsymbol{\Xi} &\triangleq \begin{bmatrix} \frac{\partial\phi(\mathbf{q})}{\partial x_a} & \frac{\partial\phi(\mathbf{q})}{\partial y_a} & \frac{\partial\phi(\mathbf{q})}{\partial z_a} \end{bmatrix} = \begin{bmatrix} 2x_a & 2y_a & 2z_a \end{bmatrix}, \\
\Xi_t &\triangleq \frac{\partial\phi(\mathbf{q})}{\partial t} = 0,
\end{aligned}$$

we can write the rate form of the constraint as follows:

$$\boldsymbol{\Xi}\dot{\mathbf{q}} + \Xi_t = 2x_a\dot{x}_a + 2y_a\dot{y}_a + 2z_a\dot{z}_a = 0. \tag{5}$$

This gives us the first differential equation of motion. To get the others, we apply the Lagrange's Equation for a Constraint System of Particles (λ is a scalar since we only have 1 contraint):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{pw/a}}{\partial\dot{\mathbf{q}}}\right)^\mathsf{T} - \left(\frac{\partial L_{pw/a}}{\partial\mathbf{q}}\right)^\mathsf{T} = \boldsymbol{f} + \lambda\boldsymbol{\Xi}^\mathsf{T}. \tag{6}$$

After substitution, we obtain three additional differential equations:

$$m\ddot{x}_a = 2\lambda x_a, \tag{7}$$
$$m\ddot{y}_a = 2\lambda y_a, \tag{8}$$
$$m\ddot{z}_a + mg = 2\lambda z_a. \tag{9}$$

Together, (7), (8), (9) and (5) describe the motion of p.

(i)

Equations (7), (8) and (9) can be rewritten in matrix form:

$$\mathbf{M}\ddot{\mathbf{q}} - \lambda\boldsymbol{\Xi}^\mathsf{T} = \mathbf{f}_{\text{non}}, \tag{10}$$

where

$$\mathbf{M} = m\mathbf{1}, \qquad \boldsymbol{\Xi}^\mathsf{T} = \begin{bmatrix} 2x_a \\ 2y_a \\ 2z_a \end{bmatrix}, \qquad \mathbf{f}_{\text{non}} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix}.$$

Moreover, taking the time derivative of (5),

$$\boldsymbol{\Xi}\ddot{\mathbf{q}} + \dot{\boldsymbol{\Xi}}\mathbf{q} + \dot{\Xi}_t = \mathbf{0}. \tag{11}$$

We can combine (10) and (11) in matrix form to obtain

$$\begin{bmatrix} \mathbf{M} & -\boldsymbol{\Xi}^\mathsf{T} \\ -\boldsymbol{\Xi} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{\text{non}} \\ \dot{\boldsymbol{\Xi}}\mathbf{q} + \dot{\Xi}_t \end{bmatrix}.$$

I can be shown that the left-hand side matrix is nonsingular, and therefore the equation can be solved for $\begin{bmatrix} \ddot{\mathbf{q}} & \lambda \end{bmatrix}^\mathsf{T}$:

$$\begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{M} & -\mathbf{\Xi}^\mathsf{T} \\ -\mathbf{\Xi} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{f}_{\text{non}} \\ \dot{\mathbf{\Xi}}\mathbf{q} + \dot{\Xi}_t \end{bmatrix} \triangleq \begin{bmatrix} \mathbf{f}_{\ddot{\mathbf{q}}}(\mathbf{q}, \dot{\mathbf{q}}) \\ \mathsf{f}_\lambda(\mathbf{q}, \dot{\mathbf{q}}) \end{bmatrix}.$$

This gives us a direct equation for the Lagrange multiplier λ. Plus, in order to solve for $\mathbf{q}$ versus time, we can numerically integrate the equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),$$

where

$$\mathbf{x} \triangleq \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix}, \qquad \mathbf{f}(\mathbf{x}) \triangleq \begin{bmatrix} \dot{\mathbf{q}} \\ \mathbf{f}_{\ddot{\mathbf{q}}}(\mathbf{q}, \dot{\mathbf{q}}) \end{bmatrix}.$$

(ii)

Writing

$$\begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} = \begin{bmatrix} l\cos(\phi)\sin(\theta) \\ l\sin(\phi)\sin(\theta) \\ l\cos(\theta) \end{bmatrix}$$

and taking the time derivative on both sides, we obtain the following:

$$\begin{aligned} \dot{\mathbf{q}} &= \begin{bmatrix} \dot{x}_a \\ \dot{y}_a \\ \dot{z}_a \end{bmatrix} \\ &= \begin{bmatrix} l\{\cos(\phi)\cos(\theta)\dot{\theta} - \sin(\phi)\sin(\theta)\dot{\phi}\} \\ l\{\sin(\phi)\cos(\theta)\dot{\theta} + \cos(\phi)\sin(\theta)\dot{\phi}\} \\ -l\sin(\theta)\dot{\theta} \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} -l\sin(\phi)\sin(\theta) & l\cos(\phi)\cos(\theta) \\ l\cos(\phi)\sin(\theta) & l\sin(\phi)\cos(\theta) \\ 0 & -l\sin(\theta) \end{bmatrix}}_{\mathbf{\Upsilon}} \underbrace{\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \end{bmatrix}}_{\dot{\mathring{\mathbf{q}}}}. \end{aligned} \tag{12}$$

Using this definition, we have

$$\dot{\mathbf{\Upsilon}} = \begin{bmatrix} -l\{\cos(\phi)\sin(\theta)\dot{\phi} + \cos(\theta)\sin(\phi)\dot{\theta}\} & -l\{\cos(\theta)\sin(\phi)\dot{\phi} + \cos(\phi)\sin(\theta)\dot{\theta}\} \\ l\{-\sin(\phi)\sin(\theta)\dot{\phi} + \cos(\phi)\cos(\theta)\dot{\theta}\} & l\{\cos(\phi)\cos(\theta)\dot{\phi} - \sin(\phi)\sin(\theta)\dot{\theta}\} \\ 0 & -l\cos(\theta)\dot{\theta} \end{bmatrix}.$$

Substituting (12) into (10) and premultiplying by $\mathbf{\Upsilon}^\mathsf{T}$, it yields

$$\mathbf{\Upsilon}^\mathsf{T}\mathbf{M}\mathbf{\Upsilon}\ddot{\mathring{\mathbf{q}}} + \mathbf{\Upsilon}^\mathsf{T}\mathbf{M}\dot{\mathbf{\Upsilon}}\dot{\mathring{\mathbf{q}}} = \mathbf{\Upsilon}^\mathsf{T}\mathbf{f}_{\text{non}}, \tag{13}$$

where we used the fact that $\boldsymbol{\Upsilon}^\mathsf{T}\boldsymbol{\Xi}^\mathsf{T} = \mathbf{0}$. Developing (13) and simplifying, we get

$$\begin{bmatrix} ml^2\sin^2(\theta) & 0 \\ 0 & ml^2 \end{bmatrix}\begin{bmatrix} \ddot{\phi} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} ml^2\sin(\theta)\cos(\theta)\dot{\theta} & ml^2\sin(\theta)\cos(\theta)\dot{\phi} \\ -ml^2\sin(\theta)\cos(\theta)\dot{\phi} & 0 \end{bmatrix}\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ mgl\sin(\theta) \end{bmatrix}.$$

Using the trigonometric identity $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$ and dividing on both side by ml^2, we obtain the same two differential equations as in a):

$$\begin{aligned} \sin^2(\theta)\ddot{\phi} + \sin(2\theta)\dot{\theta}\dot{\phi} &= 0, \\ \ddot{\theta} - \tfrac{1}{2}\dot{\phi}^2\sin(2\theta) - \frac{g}{l}\sin(\theta) &= 0. \end{aligned}$$

2

a) First, the DCM and the angular velocity between $\mathcal{F}_a$ and $\mathcal{F}_b$ are given by:

$$\begin{aligned} \mathbf{C}_{ba} = \mathbf{C}_{bq}\mathbf{C}_{qa} = \mathbf{C}_2(\theta)\mathbf{C}_3(\gamma) &= \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}\begin{bmatrix} \cos(\gamma) & \sin(\gamma) & 0 \\ -\sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta)\cos(\gamma) & \cos(\theta)\sin(\gamma) & -\sin(\theta) \\ -\sin(\gamma) & \cos(\gamma) & 0 \\ \sin(\theta)\cos(\gamma) & \sin(\theta)\sin(\gamma) & \cos(\theta) \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \underset{\rightarrow}{\omega}^{ba} = \underset{\rightarrow}{\omega}^{bq} + \underset{\rightarrow}{\omega}^{qa} &= \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b\mathbf{1}_2\dot{\theta} + \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a\mathbf{1}_3\dot{\gamma} \\ &= \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b\left(\mathbf{1}_2\dot{\theta} + \mathbf{C}_{ba}\mathbf{1}_3\dot{\gamma}\right) \\ &= \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b\begin{bmatrix} -\dot{\gamma}\sin(\theta) \\ \dot{\theta} \\ \dot{\gamma}\cos(\theta) \end{bmatrix}. \end{aligned}$$

Next,

$$\begin{aligned} \underset{\rightarrow}{r}^{pw} &= \underset{\rightarrow}{r}^{pc} + \underset{\rightarrow}{r}^{cw} \\ &= \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b\mathbf{1}_3 r + \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_a\mathbf{r}_a^{cw}, \\ \underset{\rightarrow}{v}^{pw/a} = \underset{\rightarrow}{r}^{pw\bullet a} &= \underset{\rightarrow}{r}^{pc\bullet a} + \underbrace{\underset{\rightarrow}{r}^{cw\bullet a}}_{\underset{\rightarrow}{0}} = \underbrace{\underset{\rightarrow}{r}^{pc\bullet b}}_{\underset{\rightarrow}{0}} + \underset{\rightarrow}{\omega}^{ba}\times\underset{\rightarrow}{r}^{pc} \\ &= \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b\left(\boldsymbol{\omega}_b^{ba\times}\mathbf{r}_b^{pc}\right) \\ &= \underset{\rightarrow}{\boldsymbol{\mathcal{F}}}_b\begin{bmatrix} \dot{\theta}r \\ r\dot{\gamma}\sin(\theta) \\ 0 \end{bmatrix}. \end{aligned}$$

The kinetic and potential energy become

$$\begin{aligned}
T_{pw/a} &= \tfrac{1}{2}m \underrightarrow{v}^{pw/a} \cdot \underrightarrow{v}^{pw/a} \\
&= \tfrac{1}{2}m \mathbf{v}_b^{pw/a\,\mathsf{T}} \mathbf{v}_b^{pw/a} \\
&= \tfrac{1}{2}m \begin{bmatrix} \dot\theta r & r\dot\gamma\sin(\theta) & 0 \end{bmatrix} \begin{bmatrix} \dot\theta r \\ r\dot\gamma\sin(\theta) \\ 0 \end{bmatrix} \\
&= \tfrac{1}{2}m \left(\dot\theta^2 r^2 + r^2\dot\gamma^2\sin^2(\theta)\right) \\
&= \tfrac{1}{2} \underbrace{\begin{bmatrix} \dot\theta & \dot\gamma \end{bmatrix}}_{\dot{\mathbf{q}}^\mathsf{T}} \underbrace{\begin{bmatrix} mr^2 & 0 \\ 0 & mr^2\sin^2(\theta) \end{bmatrix}}_{\mathbf{M}(\mathbf{q})} \underbrace{\begin{bmatrix} \dot\theta \\ \dot\gamma \end{bmatrix}}_{\dot{\mathbf{q}}}.
\end{aligned}$$

$$\begin{aligned}
U_{pw} &= -m \underrightarrow{g} \cdot \underrightarrow{r}^{pw} \\
&= mg\left(\mathbf{C}_{ba}\mathbf{1}_3\right)^\mathsf{T}\left(\mathbf{r}_b^{pc} + \mathbf{C}_{ba}\mathbf{r}_a^{cw}\right) \\
&= mg\left(\mathbf{1}_3^\mathsf{T}\mathbf{C}_{ba}^\mathsf{T}\mathbf{r}_b^{pc} + \mathbf{1}_3^\mathsf{T}\mathbf{C}_{ba}^\mathsf{T}\mathbf{C}_{ba}\mathbf{r}_a^{cw}\right) \\
&= mg\left(r\mathbf{1}_3^\mathsf{T}\mathbf{C}_{ba}^\mathsf{T}\mathbf{1}_3 + \mathbf{1}_3^\mathsf{T}\mathbf{r}_a^{cw}\right) \\
&= mgr\cos(\theta) + mg\mathbf{1}_3^\mathsf{T}\mathbf{r}_a^{cw}.
\end{aligned}$$

And finally,

$$L_{pw/a} = T_{pw/a} - U_{pw} = \tfrac{1}{2}m\left(\dot\theta^2 r^2 + r^2\dot\gamma^2\sin^2(\theta)\right) - mgr\cos(\theta) - mg\mathbf{1}_3^\mathsf{T}\mathbf{r}_a^{cw}.$$

b)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{pw/a}}{\partial \dot{\mathbf{q}}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T_{pw/a}}{\partial \dot{\mathbf{q}}}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\mathbf{q}}^\mathsf{T}\mathbf{M}(\mathbf{q})\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\begin{bmatrix} mr^2\dot\theta & mr^2\dot\gamma\sin^2(\theta) \end{bmatrix}\right) \\
&= \begin{bmatrix} mr^2\ddot\theta & mr^2(\ddot\gamma\sin^2(\theta) + 2\dot\gamma\sin(\theta)\cos(\theta)\dot\theta) \end{bmatrix}, \qquad (14)
\end{aligned}$$

$$\frac{\partial L_{pw/a}}{\partial \mathbf{q}} = \begin{bmatrix} mr^2\dot\gamma^2\sin(\theta)\cos(\theta) + mgr\sin(\theta) & 0 \end{bmatrix}. \qquad (15)$$

Since the generalized coordinates are not constrained and no external forces is applied on the particle, the Lagrange's Equation can be written as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{pw/a}}{\partial \dot{\mathbf{q}}}\right)^\mathsf{T} - \frac{\partial L_{pw/a}}{\partial \mathbf{q}}^\mathsf{T} = \mathbf{0}. \qquad (16)$$

Substituting (14) and (15) into (16), we obtain

$$\begin{bmatrix} mr^2\ddot{\theta} - mr^2\dot{\gamma}^2\sin(\theta)\cos(\theta) - mgr\sin(\theta) \\ mr^2(\ddot{\gamma}\sin^2(\theta) + 2\dot{\gamma}\sin(\theta)\cos(\theta)\dot{\theta}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which can be rewritten as follows

$$\underbrace{\begin{bmatrix} mr^2 & 0 \\ 0 & mr^2\sin^2(\theta) \end{bmatrix}}_{\mathbf{M}(\mathbf{q})} \underbrace{\begin{bmatrix} \ddot{\theta} \\ \ddot{\gamma} \end{bmatrix}}_{\ddot{\mathbf{q}}} = \underbrace{\begin{bmatrix} \frac{1}{2}mr^2\dot{\gamma}^2\sin(2\theta) + mgr\sin(\theta) \\ -mr^2\dot{\gamma}\dot{\theta}\sin(2\theta) \end{bmatrix}}_{\mathbf{f}_{non}(\mathbf{q},\dot{\mathbf{q}})}$$

c) Let's define $\mathbf{x}$ as follows:

$$\mathbf{x} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix}, \qquad \dot{\mathbf{x}} = \begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix}$$

so that we can describe the particle's motion with the following first order DE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} \dot{\mathbf{q}} \\ \mathbf{M}^{-1}(\mathbf{q})\mathbf{f}_{non}(\mathbf{q}, \dot{\mathbf{q}}) \end{bmatrix}$$

See the following MATLAB code for the numerical integration and the verification. The mechanical energy of the system should remain constant (neglecting the numerical errors).