Assignment 4 - Problem 1

Frédéric Berdoz
260867318

1

a)   First,

$$
\mathbf{C}_{ba} = \mathbf{C}_3(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.
$$

The position of p relative to w is given by:

$$
\underset{\rightarrow}{r}{}^{pw} = \boldsymbol{\mathcal{F}}_{\rightarrow b}^{\mathsf{T}} \mathbf{r}_b^{pw} = \boldsymbol{\mathcal{F}}_{\rightarrow a}^{\mathsf{T}} \mathbf{C}_{ba}^{\mathsf{T}} \mathbf{r}_b^{pw} = \boldsymbol{\mathcal{F}}_{\rightarrow a}^{\mathsf{T}} \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} x_b \\ 0 \\ 0 \end{bmatrix} = \boldsymbol{\mathcal{F}}_{\rightarrow a}^{\mathsf{T}} \underbrace{\begin{bmatrix} x_b \cos(\theta) \\ x_b \sin(\theta) \\ 0 \end{bmatrix}}_{\mathbf{r}_a^{pw}}
$$

Thus, the velocity of p relative to w w.r.t. $\mathcal{F}_a$ is:

$$
\underset{\rightarrow}{v}{}^{pw/a} = \underset{\rightarrow}{r}{}^{pw \, \bullet a} = \left(\boldsymbol{\mathcal{F}}_{\rightarrow a}^{\mathsf{T}} \mathbf{r}_a^{pw} \right)^{\bullet a} = \boldsymbol{\mathcal{F}}_{\rightarrow a}^{\mathsf{T}} \dot{\mathbf{r}}_a^{pw} = \boldsymbol{\mathcal{F}}_{\rightarrow a}^{\mathsf{T}} \underbrace{\begin{bmatrix} \dot{x}_b \cos(\theta) - x_b \sin(\theta)\dot{\theta} \\ \dot{x}_b \sin(\theta) + x_b \cos(\theta)\dot{\theta} \\ 0 \end{bmatrix}}_{\mathbf{v}_a^{pw/a}}
$$

Finally, the acceleration of p relative to w w.r.t. $\mathcal{F}_a$ is:

$$
\begin{aligned}
\underset{\rightarrow}{a}{}^{pw/a} &= \underset{\rightarrow}{v}{}^{pw/a \, \bullet a} = \left(\boldsymbol{\mathcal{F}}_{\rightarrow a}^{\mathsf{T}} \mathbf{v}_a^{pw/a} \right)^{\bullet a} = \boldsymbol{\mathcal{F}}_{\rightarrow a}^{\mathsf{T}} \dot{\mathbf{v}}_a^{pw/a} \\
&= \boldsymbol{\mathcal{F}}_{\rightarrow a}^{\mathsf{T}} \underbrace{\begin{bmatrix} \ddot{x}_b \cos(\theta) - 2\dot{x}_b \sin(\theta)\dot{\theta} - x_b \cos(\theta)\dot{\theta}^2 - x_b \sin(\theta)\ddot{\theta} \\ \ddot{x}_b \sin(\theta) + 2\dot{x}_b \cos(\theta)\dot{\theta} - x_b \sin(\theta)\dot{\theta}^2 + x_b \cos(\theta)\ddot{\theta} \\ 0 \end{bmatrix}}_{\mathbf{a}_a^{pw/a}} \\
&= \boldsymbol{\mathcal{F}}_{\rightarrow b}^{\mathsf{T}} \mathbf{C}_{ba} \mathbf{a}_a^{pw/a} = \boldsymbol{\mathcal{F}}_{\rightarrow b}^{\mathsf{T}} \underbrace{\begin{bmatrix} \ddot{x}_b - x_b \dot{\theta}^2 \\ 2\dot{x}_b \dot{\theta} + x_b \ddot{\theta} \\ 0 \end{bmatrix}}_{\mathbf{a}_b^{pw/a}}
\end{aligned}
$$

Let $\underset{\rightarrow}{f}^{pt} = \begin{bmatrix} f_{b1}^{pt} & f_{b2}^{pt} & f_{b3}^{pt} \end{bmatrix} \underset{\rightarrow}{\mathcal{F}}_b$ be the reaction force of the table, $\underset{\rightarrow}{f}^{pg} = \begin{bmatrix} 0 & 0 & -mg \end{bmatrix} \underset{\rightarrow}{\mathcal{F}}_b$ the gravitational force applied on p and $\underset{\rightarrow}{f}^{ps} = \begin{bmatrix} -kx_b & 0 & 0 \end{bmatrix} \underset{\rightarrow}{\mathcal{F}}_b$ the spring force. Using the fact that the cut in the table is frictionless, i.e. $f_{b1}^{pt} = 0$, the total force acting on p becomes:

$$\underset{\rightarrow}{f}^p = \underset{\rightarrow}{f}^{pt} + \underset{\rightarrow}{f}^{pg} + \underset{\rightarrow}{f}^{ps} = \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \underbrace{\begin{bmatrix} -kx_b \\ f_{b2}^{pt} \\ f_{b3}^{pt} - mg \end{bmatrix}}_{\mathbf{f}_b^p}$$

Since $\mathcal{F}_a$ is an inertial frame and w is unforced, we can use Newton's Second Law to derive the differential equation that describes the motion of p:

$$m \underset{\rightarrow}{a}^{pw/a} = \underset{\rightarrow}{f}^p$$

$$m \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \mathbf{a}_b^{pw/a} = \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \mathbf{f}_b^p$$

$$m \mathbf{a}_b^{pw/a} = \mathbf{f}_b^p$$

$$m \begin{bmatrix} \ddot{x}_b - x_b\dot{\theta}^2 \\ 2\dot{x}_b\dot{\theta} + x_b\ddot{\theta} \\ 0 \end{bmatrix} = \begin{bmatrix} -kx_b \\ f_{b2}^{pt} \\ f_{b3}^{pt} - mg \end{bmatrix} \tag{1}$$

We can see straight away that $f_{b3}^{pt} = mg$.

b) Using the fact that $\dot{\theta}$ is constant (i.e. $\ddot{\theta} = 0$) and rearranging, (1) can be rewritten as follows:

$$\ddot{x}_b + \left(\frac{k}{m} - \dot{\theta}^2\right)x_b = 0 \tag{2}$$

$$2\dot{x}_b\dot{\theta} - \frac{f_{b2}^{pt}}{m} = 0 \tag{3}$$

The general solution to (2), knowing that $\frac{k}{m} > \dot{\theta}^2$, is given by:

$$x_b(t) = A\cos(\omega t) + B\sin(\omega t), \quad A, B \in \mathbb{R}, \quad \omega = \sqrt{\frac{k}{m} - \dot{\theta}^2} \tag{4}$$

c) First, using the initial condition $x_b(0) = 0.1$, it yields:

$$x_b(0) = A = 0.1$$

. Secondly, using $\dot{x}_b(0) = 0$:

$$\dot{x}_b(0) = [-A\omega\sin(\omega t) + B\omega\cos(\omega t)]_{t=0} = B\omega = 0.$$

Therefore:

$$B = 0$$

and (4) can be rewritten as follows:

$$x_b(t) = 0.1 \cos(\omega t) \text{ [m]}, \quad \omega = \sqrt{\frac{k}{m} - \dot{\theta}^2}. \tag{5}$$

Lastly,

$$\dot{x}_b = -0.1\omega \sin(\omega t) \text{ [m/s]}, \tag{6}$$

$$\ddot{x}_b = -0.1\omega^2 \cos(\omega t) \text{ [m/s}^2]. \tag{7}$$

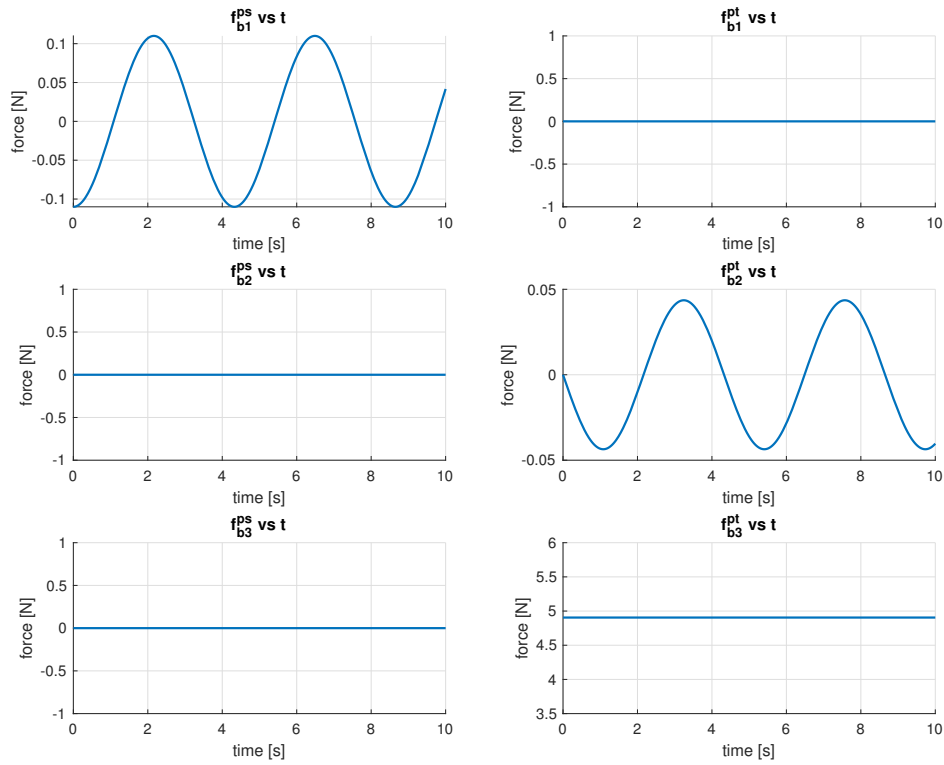Using MATLAB, we obtain the following plots:

Figure 1: $\mathbf{f}_b^{pt}$ and $\mathbf{f}_b^{ps}$ vs time.

Problem 1 (c)

```matlab
close all; clc; clear all;
% Parameters
k=1.1;
m=0.5;
dtheta=0.3;
g=9.81;
xb0=0.1;
dxb0=0;
w=sqrt(k/m-dtheta^2);

% Functions
xb=@(t) xb0*cos(w*t);
dxb=@(t) -xb0*w*sin(w*t);
ddxb=@(t) -xb0*w^2*cos(w*t);

a_b1_pwa=@(t) ddxb(t)-xb(t)*dtheta^2;
a_b2_pwa=@(t) 2*dxb(t)*dtheta;

f_b1_ps=@(t) -k*xb(t);
f_b2_ps=@(t) 0;
f_b3_ps=@(t) 0;

f_b3_pg=-m*g;

f_b1_pt=@(t) 0;
f_b2_pt=@(t) m*a_b2_pwa(t);
f_b3_pt=@(t) m*g;

% Plots
tspan=[0 10];
figure('Name','Assignment 1 - Problem 1 c)');
title('Assignment 1 - Problem 1 c)')
hold on
subplot(3,2,1)
title('f_{b1}^{ps} vs t');
hold on
grid on
fplot(f_b1_ps,tspan,'DisplayName','f_{b1}^{ps}','LineWidth',1.5)
xlabel('time [s]'); ylabel('force [N]')

subplot(3,2,3)
title('f_{b2}^{ps} vs t');
hold on
grid on
fplot(f_b2_ps,tspan,'DisplayName','f_{b2}^{ps}','LineWidth',1.5)
xlabel('time [s]'); ylabel('force [N]')

subplot(3,2,5)
title('f_{b3}^{ps} vs t');
hold on
```

```matlab
grid on
fplot(f_b3_ps,tspan,'DisplayName','f_{b3}^{ps}','LineWidth',1.5)
xlabel('time [s]'); ylabel('force [N]')

subplot(3,2,2)
title('f_{b1}^{pt} vs t');
hold on
grid on
fplot(f_b1_pt,tspan,'DisplayName','f_{b1}^{pt}','LineWidth',1.5)
xlabel('time [s]'); ylabel('force [N]')

subplot(3,2,4)
title('f_{b2}^{pt} vs t');
hold on
grid on
fplot(f_b2_pt,tspan,'DisplayName','f_{b2}^{pt}','LineWidth',1.5)
xlabel('time [s]'); ylabel('force [N]')

subplot(3,2,6)
title('f_{b3}^{pt} vs t');
hold on
grid on
fplot(f_b3_pt,tspan,'DisplayName','f_{b3}^{pt}','LineWidth',1.5)
xlabel('time [s]'); ylabel('force [N]')
```

Published with MATLAB® R2018b