Assignment 4

Frédéric Berdoz
260867318

1

a) First,

$$\mathbf{C}_{ba} = \mathbf{C}_3(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The position of p relative to w is given by:

$$\underset{\rightarrow}{r}^{pw} = \underset{\rightarrow}{\mathcal{F}}_b^\mathsf{T} \mathbf{r}_b^{pw} = \underset{\rightarrow}{\mathcal{F}}_a^\mathsf{T} \mathbf{C}_{ba}^\mathsf{T} \mathbf{r}_b^{pw} = \underset{\rightarrow}{\mathcal{F}}_a^\mathsf{T} \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}^\mathsf{T} \begin{bmatrix} x_b \\ 0 \\ 0 \end{bmatrix} = \underset{\rightarrow}{\mathcal{F}}_a^\mathsf{T} \underbrace{\begin{bmatrix} x_b \cos(\theta) \\ x_b \sin(\theta) \\ 0 \end{bmatrix}}_{\mathbf{r}_a^{pw}}$$

Thus, the velocity of p relative to w w.r.t. $\mathcal{F}_a$ is:

$$\underset{\rightarrow}{v}^{pw/a} = \underset{\rightarrow}{r}^{pw\,\bullet a} = \left(\underset{\rightarrow}{\mathcal{F}}_a^\mathsf{T} \mathbf{r}_a^{pw}\right)^{\bullet a} = \underset{\rightarrow}{\mathcal{F}}_a^\mathsf{T} \dot{\mathbf{r}}_a^{pw} = \underset{\rightarrow}{\mathcal{F}}_a^\mathsf{T} \underbrace{\begin{bmatrix} \dot{x}_b \cos(\theta) - x_b \sin(\theta)\dot{\theta} \\ \dot{x}_b \sin(\theta) + x_b \cos(\theta)\dot{\theta} \\ 0 \end{bmatrix}}_{\mathbf{v}_a^{pw/a}}$$

Finally, the acceleration of p relative to w w.r.t. $\mathcal{F}_a$ is:

$$\begin{aligned} \underset{\rightarrow}{a}^{pw/a} &= \underset{\rightarrow}{v}^{pw/a\,\bullet a} = \left(\underset{\rightarrow}{\mathcal{F}}_a^\mathsf{T} \mathbf{v}_a^{pw/a}\right)^{\bullet a} = \underset{\rightarrow}{\mathcal{F}}_a^\mathsf{T} \dot{\mathbf{v}}_a^{pw/a} \\ &= \underset{\rightarrow}{\mathcal{F}}_a^\mathsf{T} \underbrace{\begin{bmatrix} \ddot{x}_b \cos(\theta) - 2\dot{x}_b \sin(\theta)\dot{\theta} - x_b \cos(\theta)\dot{\theta}^2 - x_b \sin(\theta)\ddot{\theta} \\ \ddot{x}_b \sin(\theta) + 2\dot{x}_b \cos(\theta)\dot{\theta} - x_b \sin(\theta)\dot{\theta}^2 + x_b \cos(\theta)\ddot{\theta} \\ 0 \end{bmatrix}}_{\mathbf{a}_a^{pw/a}} \\ &= \underset{\rightarrow}{\mathcal{F}}_b^\mathsf{T} \mathbf{C}_{ba} \mathbf{a}_a^{pw/a} = \underset{\rightarrow}{\mathcal{F}}_b^\mathsf{T} \underbrace{\begin{bmatrix} \ddot{x}_b - x_b \dot{\theta}^2 \\ 2\dot{x}_b \dot{\theta} + x_b \ddot{\theta} \\ 0 \end{bmatrix}}_{\mathbf{a}_b^{pw/a}} \end{aligned}$$

Let $\underset{\rightarrow}{f}^{pt} = \begin{bmatrix} f_{b1}^{pt} & f_{b2}^{pt} & f_{b3}^{pt} \end{bmatrix} \underset{\rightarrow}{\mathcal{F}}_b$ be the reaction force of the table, $\underset{\rightarrow}{f}^{pg} = \begin{bmatrix} 0 & 0 & -mg \end{bmatrix} \underset{\rightarrow}{\mathcal{F}}_b$ the gravitational force applied on p and $\underset{\rightarrow}{f}^{ps} = \begin{bmatrix} -kx_b & 0 & 0 \end{bmatrix} \underset{\rightarrow}{\mathcal{F}}_b$ the spring force. Using the fact that the cut in the table is frictionless, i.e. $f_{b1}^{pt} = 0$, the total force acting on p becomes:

$$\underset{\rightarrow}{f}^{p} = \underset{\rightarrow}{f}^{pt} + \underset{\rightarrow}{f}^{pg} + \underset{\rightarrow}{f}^{ps} = \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \underbrace{\begin{bmatrix} -kx_b \\ f_{b2}^{pt} \\ f_{b3}^{pt} - mg \end{bmatrix}}_{\mathbf{f}_b^p}$$

Since $\mathcal{F}_a$ is an inertial frame and w is unforced, we can use Newton's Second Law to derive the differential equation that describes the motion of p:

$$m \underset{\rightarrow}{a}^{pw/a} = \underset{\rightarrow}{f}^{p}$$

$$m\underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \mathbf{a}_b^{pw/a} = \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \mathbf{f}_b^p$$

$$m\mathbf{a}_b^{pw/a} = \mathbf{f}_b^p$$

$$m \begin{bmatrix} \ddot{x}_b - x_b\dot{\theta}^2 \\ 2\dot{x}_b\dot{\theta} + x_b\ddot{\theta} \\ 0 \end{bmatrix} = \begin{bmatrix} -kx_b \\ f_{b2}^{pt} \\ f_{b3}^{pt} - mg \end{bmatrix} \tag{1}$$

We can see straight away that $f_{b3}^{pt} = mg$.

b) Using the fact that $\dot{\theta}$ is constant (i.e. $\ddot{\theta} = 0$) and rearranging, (1) can be rewritten as follows:

$$\ddot{x}_b + (\frac{k}{m} - \dot{\theta}^2)x_b = 0 \tag{2}$$

$$2\dot{x}_b\dot{\theta} - \frac{f_{b2}^{pt}}{m} = 0 \tag{3}$$

The general solution to (2), knowing that $\frac{k}{m} > \dot{\theta}^2$, is given by:

$$x_b(t) = A\cos(\omega t) + B\sin(\omega t), \quad A, B \in \mathbb{R}, \quad \omega = \sqrt{\frac{k}{m} - \dot{\theta}^2} \tag{4}$$

c) First, using the initial condition $x_b(0) = 0.1$, it yields:

$$x_b(0) = A = 0.1$$

. Secondly, using $\dot{x}_b(0) = 0$:

$$\dot{x}_b(0) = [-A\omega\sin(\omega t) + B\omega\cos(\omega t)]_{t=0} = B\omega = 0.$$

Therefore:

$$B = 0$$

and (4) can be rewritten as follows:

$$x_b(t) = 0.1\cos(\omega t)\ [\mathrm{m}], \quad \omega = \sqrt{\frac{k}{m} - \dot{\theta}^2}. \tag{5}$$

Lastly,

$$\dot{x}_b = -0.1\omega\sin(\omega t)\ [\mathrm{m/s}], \tag{6}$$
$$\ddot{x}_b = -0.1\omega^2\cos(\omega t)\ [\mathrm{m/s^2}]. \tag{7}$$

Using MATLAB, we obtain the following plots:

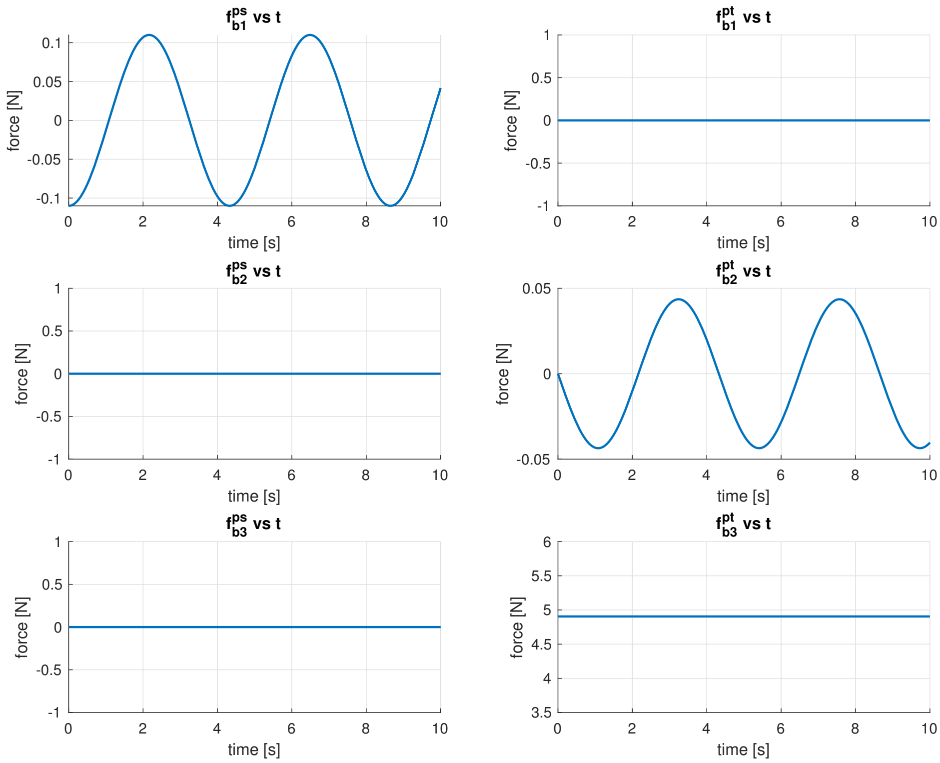

Figure 1: $\mathbf{f}_b^{pt}$ and $\mathbf{f}_b^{ps}$ vs time.

2

a) Let θ be the angle between $\underrightarrow{b}^2$ and $\underrightarrow{a}^2$. Therefore

$$\mathbf{C}_{ba} = \mathbf{C}_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Moreover,

$$\underrightarrow{\omega}^{ba} = \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{aligned}
\underrightarrow{r}^{\kappa w} &= \underrightarrow{r}^{\kappa c} + \underrightarrow{r}^{cw} \\
&= \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} + \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} 0 \\ 0 \\ -d \end{bmatrix} \\
&= \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} 0 \\ y \\ -d \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\underrightarrow{r}^{dmw} &= \underrightarrow{r}^{dmc} + \underrightarrow{r}^{cw} \\
&= \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} \rho_{b1} \\ \rho_{b2} \\ \rho_{b3} \end{bmatrix} + \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} 0 \\ 0 \\ -d \end{bmatrix} \\
&= \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} \rho_{b1} \\ \rho_{b2} \\ \rho_{b3} - d \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\underrightarrow{r}^{\kappa w \bullet a} &= \underrightarrow{r}^{\kappa w \bullet b} + \underrightarrow{\omega}^{ba} \times \underrightarrow{r}^{\kappa w} \\
&= \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \left(\dot{\mathbf{r}}_b^{\kappa w} + \boldsymbol{\omega}_b^{ba\times} \mathbf{r}_b^{\kappa w} \right) \\
&= \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \left(\begin{bmatrix} 0 \\ \dot{y} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta} \\ 0 & \dot{\theta} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ y \\ -d \end{bmatrix} \right) \\
&= \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} 0 \\ \dot{y} + d\dot{\theta} \\ y\dot{\theta} \end{bmatrix},
\end{aligned}$$

$$
\begin{aligned}
{\underrightarrow{r}^{dmw}}^{\bullet a} &= {\underrightarrow{r}^{dmw}}^{\bullet b} + \underrightarrow{\omega}^{ba} \times \underrightarrow{r}^{dmw} \\
&= \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \left(\underbrace{\dot{\mathbf{r}}_b^{dmw}}_{=\mathbf{0}} + \boldsymbol{\omega}_b^{ba\times} \mathbf{r}_b^{dmw} \right) \\
&= \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta} \\ 0 & \dot{\theta} & 0 \end{bmatrix} \begin{bmatrix} \rho_{b1} \\ \rho_{b2} \\ \rho_{b3} - d \end{bmatrix} \right) \\
&= \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} 0 \\ (d - \rho_{b3})\dot{\theta} \\ \rho_{b2}\dot{\theta} \end{bmatrix}.
\end{aligned}
$$

b) Figures 2 and 3 show the Free Body Diagrams of the particle κ and the body $\mathcal{B}$. Let $\underrightarrow{f}^{r1}$ and $\underrightarrow{f}^{r1}$ be the reaction forces applied by the massless bars on the body $\mathcal{B}$.

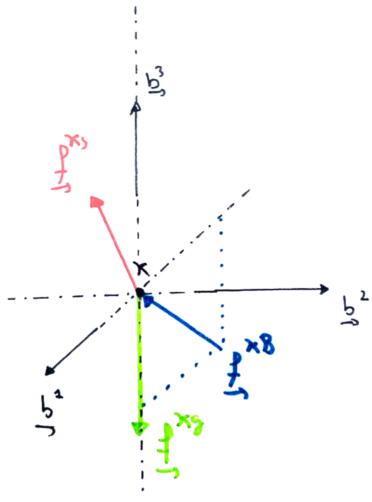

Figure 2: FBD of κ.

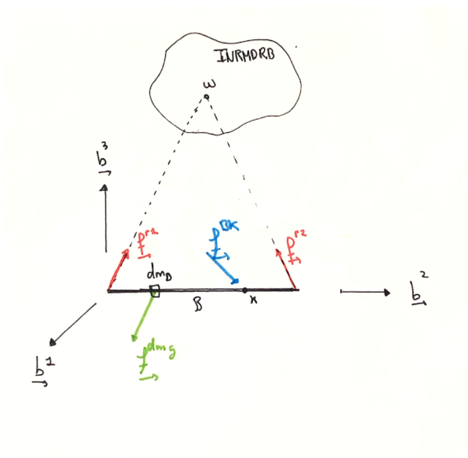

Figure 3: FBD of $\mathcal{B}$.

The norm of $\underrightarrow{r}^{\kappa w}$ is

$$||\underrightarrow{r}^{\kappa w}||_2 = \mathbf{r}_b^{\kappa w\mathsf{T}} \mathbf{r}_b^{\kappa w} = y^2 + d^2.$$

Therefore, the total force applied on κ is given by:

$$
\begin{aligned}
\underset{\rightarrow}{f}^{\kappa} &= \underset{\rightarrow}{f}^{\kappa\mathcal{B}} + \underset{\rightarrow}{f}^{\kappa g} + \underset{\rightarrow}{f}^{\kappa s} \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \left(\begin{bmatrix} f_{b1}^{\kappa\mathcal{B}} \\ 0 \\ f_{b3}^{\kappa\mathcal{B}} \end{bmatrix} + \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -m_\kappa g \end{bmatrix} - k\frac{l_s - \bar{l}_s}{y^2+d^2} \begin{bmatrix} 0 \\ y \\ -d \end{bmatrix} \right) \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} f_{b1}^{\kappa\mathcal{B}} \\ -m_\kappa g \sin(\theta) - ky\frac{l_s-\bar{l}_s}{y^2+d^2} \\ f_{b3}^{\kappa\mathcal{B}} - m_\kappa g\cos(\theta) + kd\frac{l_s-\bar{l}_s}{y^2+d^2} \end{bmatrix}.
\end{aligned}
$$

The total force acting on $\mathcal{B}$ is given by:

$$
\begin{aligned}
\underset{\rightarrow}{f}^{\mathcal{B}} &= \underset{\rightarrow}{f}^{r1} + \underset{\rightarrow}{f}^{r2} + \underset{\rightarrow}{f}^{\mathcal{B}\kappa} + \int_{\mathcal{B}} d\underset{\rightarrow}{f}^{dmg} \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \left(\begin{bmatrix} f_{b1}^{r1} \\ f_{b2}^{r1} \\ f_{b3}^{r1} \end{bmatrix} + \begin{bmatrix} f_{b1}^{r2} \\ f_{b2}^{r2} \\ f_{b3}^{r3} \end{bmatrix} - \begin{bmatrix} f_{b1}^{\kappa\mathcal{B}} \\ 0 \\ f_{b3}^{\kappa\mathcal{B}} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix} \int_{\mathcal{B}} \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} dm \right) \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} f_{b1}^{r1} + f_{b1}^{r2} - f_{b1}^{\kappa\mathcal{B}} \\ f_{b2}^{r1} + f_{b2}^{r2} - m_{\mathcal{B}} g\sin(\theta) \\ f_{b3}^{r1} + f_{b3}^{r2} - f_{b3}^{\kappa\mathcal{B}} - m_{\mathcal{B}} g\cos(\theta) \end{bmatrix}.
\end{aligned}
$$

Finally, the total moment applied on $\mathcal{B}$ relative to w is given by:

$$
\begin{aligned}
\underset{\rightarrow}{m}^{\mathcal{B}w} &= \underbrace{\underset{\rightarrow}{r}^{r1w} \times \underset{\rightarrow}{f}^{r1}}_{=\underset{\rightarrow}{0}} + \underbrace{\underset{\rightarrow}{r}^{r2w} \times \underset{\rightarrow}{f}^{r2}}_{=\underset{\rightarrow}{0}} + \underset{\rightarrow}{r}^{\kappa w} \times \underset{\rightarrow}{f}^{\mathcal{B}\kappa} + \int_{\mathcal{B}} \underset{\rightarrow}{r}^{dmw} \times d\underset{\rightarrow}{f}^{dmg} \\
&= \underset{\rightarrow}{r}^{\kappa w} \times \underset{\rightarrow}{f}^{\mathcal{B}\kappa} + \underbrace{\left(\int_{\mathcal{B}} \underset{\rightarrow}{r}^{dmw} dm \right)}_{=m_{\mathcal{B}} \underset{\rightarrow}{r}^{cw}} \times \underset{\rightarrow}{g} \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \left(\begin{bmatrix} 0 \\ y \\ -d \end{bmatrix}^{\times} \begin{bmatrix} -f_{b1}^{\kappa\mathcal{B}} \\ 0 \\ -f_{b3}^{\kappa\mathcal{B}} \end{bmatrix} + m_{\mathcal{B}} \begin{bmatrix} 0 \\ 0 \\ -d \end{bmatrix}^{\times} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} \right) \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} -y f_{b3}^{\kappa\mathcal{B}} - dgm_{\mathcal{B}} \sin(\theta) \\ d f_{b1}^{\kappa\mathcal{B}} \\ y f_{b1}^{\kappa\mathcal{B}} \end{bmatrix}
\end{aligned}
$$

c)

$$
\underset{\rightarrow}{p}^{\kappa w/a} = m_\kappa \underset{\rightarrow}{r}^{\kappa w/a^{\bullet a}} = m_\kappa \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} 0 \\ \dot{y} + d\dot{\theta} \\ y\dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ m_\kappa(\dot{y} + d\dot{\theta}) \\ m_\kappa y\dot{\theta} \end{bmatrix}. \quad \square
$$

$$
\begin{aligned}
\underset{\rightarrow}{h}^{\mathcal{B}w/a} &= \int_{\mathcal{B}} \underset{\rightarrow}{r}^{dmw} \times \underset{\rightarrow}{r}^{dmw^{\bullet a}} dm \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \left(\int_{\mathcal{B}} \begin{bmatrix} \rho_{b1} \\ \rho_{b2} \\ \rho_{b3} - d \end{bmatrix}^{\times} \begin{bmatrix} 0 \\ (d-\rho_{b3})\dot{\theta} \\ \rho_{b2}\dot{\theta} \end{bmatrix} dm \right) \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \left(\int_{\mathcal{B}} \begin{bmatrix} 0 & -(\rho_{b3}-d) & \rho_{b2} \\ (\rho_{b3}-d) & 0 & -\rho_{b1} \\ -\rho_{b2} & \rho_{b1} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ (d-\rho_{b3})\dot{\theta} \\ \rho_{b2}\dot{\theta} \end{bmatrix} dm \right) \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \left(\dot{\theta} \int_{\mathcal{B}} \begin{bmatrix} (d-\rho_{b3})^2 + \rho_{b2}^2 \\ -\rho_{b1}\rho_{b2} \\ \rho_{b1}(d-\rho_{b3}) \end{bmatrix} dm \right) \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \left(\dot{\theta} \int_{V_{\mathcal{B}}} \begin{bmatrix} (d-\rho_{b3})^2 + \rho_{b2}^2 \\ -\rho_{b1}\rho_{b2} \\ \rho_{b1}(d-\rho_{b3}) \end{bmatrix} \left(\frac{m_{\mathcal{B}}}{lth}\right) dV \right) \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \left(\frac{m_{\mathcal{B}}\dot{\theta}}{lth} \begin{bmatrix} lt[\int_{-h/2}^{h/2} (d-\rho_{b3})^2 d\rho_{b3}] + th[\int_{-l/2}^{l/2} \rho_{b2}^2 d\rho_{b2}] \\ -lth^2 \underbrace{[\int_{-t/2}^{t/2} \rho_{b1} d\rho_{b1}]}_{0} \underbrace{[\int_{-l/2}^{l/2} \rho_{b2} d\rho_{b2}]}_{0} \\ l^2ht \underbrace{[\int_{-t/2}^{t/2} \rho_{b1} d\rho_{b1}]}_{0} [\int_{-h/2}^{h/2} (d-\rho_{b3}) d\rho_{b3}] \end{bmatrix} \right) \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \left(\frac{m_{\mathcal{B}}\dot{\theta}}{lth} \begin{bmatrix} -\frac{lt}{3}[(d-\rho_{b3})^3]_{-h/2}^{h/2} + \frac{th}{3}[\rho_{b2}^3]_{-l/2}^{l/2} \\ 0 \\ 0 \end{bmatrix} \right) \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \left(\frac{m_{\mathcal{B}}\dot{\theta}}{lth} \begin{bmatrix} -\frac{lt}{3}[-3d^2h - \frac{h^3}{4}] + \frac{th}{3}\frac{l^3}{4} \\ 0 \\ 0 \end{bmatrix} \right) \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \left(m_{\mathcal{B}}\dot{\theta} \begin{bmatrix} d^2 + \frac{1}{12}(h^2+l^2) \\ 0 \\ 0 \end{bmatrix} \right). \quad \square
\end{aligned}
$$

d) First, from N2L,

$$
\underset{\rightarrow}{p}^{\kappa w/a^{\bullet a}} = \underset{\rightarrow}{f}^{\kappa}.
$$

Using the Transport Theorem and developing:

$$\begin{aligned}
\underset{\rightarrow}{p}^{\kappa w/a \bullet a} &= \underset{\rightarrow}{p}^{\kappa w/a \bullet b} + \underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{p}^{\kappa w/a} \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \left(\begin{bmatrix} 0 \\ m_\kappa(\ddot{y} + d\ddot{\theta}) \\ m_\kappa(\dot{y}\dot{\theta} + y\ddot{\theta}) \end{bmatrix} + \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix}^{\times} \begin{bmatrix} 0 \\ m_\kappa(\dot{y} + d\dot{\theta}) \\ m_\kappa y\dot{\theta} \end{bmatrix} \right) \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} 0 \\ m_\kappa(\ddot{y} + d\ddot{\theta}) - m_\kappa y\dot{\theta}^2 \\ m_\kappa(\dot{y}\dot{\theta} + y\ddot{\theta}) + \dot{\theta}m_\kappa(\dot{y} + d\dot{\theta}) \end{bmatrix} \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} 0 \\ m_\kappa(\ddot{y} + d\ddot{\theta}) - m_\kappa y\dot{\theta}^2 \\ 2m_\kappa\dot{y}\dot{\theta} + m_\kappa y\ddot{\theta} + m_\kappa d\dot{\theta}^2) \end{bmatrix}
\end{aligned}$$

Therefore,

$$\begin{bmatrix} 0 \\ m_\kappa(\ddot{y} + d\ddot{\theta}) - m_\kappa y\dot{\theta}^2 \\ 2m_\kappa\dot{y}\dot{\theta} + m_\kappa y\ddot{\theta} + m_\kappa d\dot{\theta}^2) \end{bmatrix} = \begin{bmatrix} f_{b1}^{\kappa\mathcal{B}} \\ -m_\kappa g\sin(\theta) - ky\frac{l_s - \bar{l}_s}{y^2+d^2} \\ f_{b3}^{\kappa\mathcal{B}} - m_\kappa g\cos(\theta) + kd\frac{l_s - \bar{l}_s}{y^2+d^2} \end{bmatrix}. \tag{8}$$

We can see straight away that

$$f_{b1}^{\kappa\mathcal{B}} = 0.$$

Secondly,

$$\begin{aligned}
\underset{\rightarrow}{h}^{\mathcal{B}w/a \bullet a} &= \underset{\rightarrow}{h}^{\mathcal{B}w/a \bullet b} + \underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{h}^{\mathcal{B}w/a} \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \left(\begin{bmatrix} m_\mathcal{B}\ddot{\theta}[d^2 + \frac{1}{12}(h^2 + l^2)] \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix}^{\times} \begin{bmatrix} m_\mathcal{B}\dot{\theta}[d^2 + \frac{1}{12}(h^2 + l^2)] \\ 0 \\ 0 \end{bmatrix} \right) \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} m_\mathcal{B}\ddot{\theta}[d^2 + \frac{1}{12}(h^2 + l^2)] \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

From N2LR,

$$\underset{\rightarrow}{h}^{\mathcal{B}w/a \bullet a} = \underset{\rightarrow}{m}^{\mathcal{B}w}.$$

Therefore,

$$\underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} m_\mathcal{B}\ddot{\theta}[d^2 + \frac{1}{12}(h^2 + l^2)] \\ 0 \\ 0 \end{bmatrix} = \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} -yf_{b3}^{\kappa\mathcal{B}} - dgm_\mathcal{B}\sin(\theta) \\ df_{b1}^{\kappa\mathcal{B}} \\ yf_{b1}^{\kappa\mathcal{B}} \end{bmatrix}$$

In particular,

$$f_{b3}^{\kappa\mathcal{B}} = -\frac{m_\mathcal{B}}{y}\left(\ddot{\theta}[d^2 + \frac{1}{12}(h^2 + l^2)] + dg\sin(\theta)\right) \tag{9}$$

Finally, substituting (9) into (8),

$$m_\kappa(\ddot{y} + d\ddot{\theta}) - m_\kappa y\dot{\theta}^2 = -m_\kappa g\sin(\theta) - ky\frac{l_s - \bar{l}_s}{y^2 + d^2} \tag{10}$$

$$\begin{aligned} 2m_\kappa\dot{y}\dot{\theta} + m_\kappa y\ddot{\theta} + m_\kappa d\dot{\theta}^2 = &-\frac{m_\mathcal{B}}{y}\left(\ddot{\theta}[d^2 + \frac{1}{12}(h^2 + l^2)] + dg\sin(\theta)\right) \\ &- m_\kappa g\cos(\theta) + kd\frac{l_s - \bar{l}_s}{y^2 + d^2} \end{aligned} \tag{11}$$

e) The total moment applied on $\mathcal{S}$ relative to w is given by:

$$\begin{aligned} \underset{\rightarrow}{m}^{\mathcal{S}w} =& m_\mathcal{B}\underset{\rightarrow}{r}^{cw} \times \underset{\rightarrow}{g} + m_\kappa \underset{\rightarrow}{r}^{\kappa w} \times \underset{\rightarrow}{g} \\ =& \underset{\rightarrow}{\mathcal{F}}_b^\mathsf{T}\left(m_\mathcal{B}\begin{bmatrix} 0 \\ 0 \\ -d \end{bmatrix}^\times \begin{bmatrix} 0 \\ -\sin(\theta)g \\ -\cos(\theta)g \end{bmatrix} + m_\kappa \begin{bmatrix} 0 \\ y \\ -d \end{bmatrix}^\times \begin{bmatrix} 0 \\ -\sin(\theta)g \\ -\cos(\theta)g \end{bmatrix}\right) \\ =& \underset{\rightarrow}{\mathcal{F}}_b^\mathsf{T}\begin{bmatrix} -m_\mathcal{B}dg\sin(\theta) + m_\kappa[-yg\cos(\theta) - dg\sin(\theta)] \\ 0 \\ 0 \end{bmatrix}. \quad \square \end{aligned}$$

Secondly,

$$\begin{aligned} \underset{\rightarrow}{h}^{\mathcal{S}w/a} &= \underset{\rightarrow}{h}^{\mathcal{B}w/a} + \underset{\rightarrow}{r}^{\kappa w} \times \underset{\rightarrow}{p}^{\kappa w/a} \\ &= \underset{\rightarrow}{\mathcal{F}}_b^\mathsf{T}\left(\begin{bmatrix} m_\mathcal{B}\dot{\theta}[d^2 + \frac{1}{12}(h^2 + l^2)] \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ -d \end{bmatrix}^\times \begin{bmatrix} 0 \\ m_\kappa(\dot{y} + d\dot{\theta}) \\ m_\kappa y\dot{\theta} \end{bmatrix}\right) \\ &= \underset{\rightarrow}{\mathcal{F}}_b^\mathsf{T}\begin{bmatrix} m_\mathcal{B}\dot{\theta}[d^2 + \frac{1}{12}(h^2 + l^2)] + m_\kappa[y^2\dot{\theta} + d(\dot{y} + d\dot{\theta})] \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} \underset{\rightarrow}{h}^{\mathcal{S}w/a\,\bullet a} &= \underset{\rightarrow}{h}^{\mathcal{S}w/a\,\bullet b} + \underbrace{\underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{h}^{\mathcal{S}w/a}}_{=\underset{\rightarrow}{0}} \\ &= \underset{\rightarrow}{\mathcal{F}}_b^\mathsf{T}\begin{bmatrix} m_\mathcal{B}\ddot{\theta}[d^2 + \frac{1}{12}(h^2 + l^2)] + m_\kappa[2y\dot{y}\dot{\theta} + y^2\ddot{\theta} + d(\ddot{y} + d\ddot{\theta})] \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

And finally, applying N2LR, i.e. $\underset{\rightarrow}{h}^{\mathcal{S}w/a^{\bullet a}} = \underset{\rightarrow}{m}^{\mathcal{S}w}$:

$$m_{\mathcal{B}}\ddot{\theta}[d^2 + \frac{1}{12}(h^2 + l^2)] + m_\kappa[2y\dot{y}\dot{\theta} + y^2\ddot{\theta} + d(\ddot{y} + d\ddot{\theta})]$$
$$= -m_{\mathcal{B}}dg\sin(\theta) + m_\kappa[-yg\cos(\theta) - dg\sin(\theta)] \tag{12}$$

We can easily check that (12) is nothing else than $d\cdot(10) + y\cdot(11)$, and thus the two sets of DEs describe the same motion.