Assignment 4 - Problem 1

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a) First,

$$\mathbf{C}_{ba} = \mathbf{C}_3(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The position of p relative to w is given by:

$$\underset{\rightarrow}{r}^{pw} = \underset{\rightarrow}{\mathcal{F}}_b^\mathsf{T} \mathbf{r}_b^{pw} = \underset{\rightarrow}{\mathcal{F}}_a^\mathsf{T} \mathbf{C}_{ba}^\mathsf{T} \mathbf{r}_b^{pw} = \underset{\rightarrow}{\mathcal{F}}_a^\mathsf{T} \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}^\mathsf{T} \begin{bmatrix} x_b \\ 0 \\ 0 \end{bmatrix} = \underset{\rightarrow}{\mathcal{F}}_a^\mathsf{T} \underbrace{\begin{bmatrix} x_b \cos(\theta) \\ x_b \sin(\theta) \\ 0 \end{bmatrix}}_{\mathbf{r}_a^{pw}}$$

Thus, the velocity of p relative to w w.r.t. $\mathcal{F}_a$ is:

$$\underset{\rightarrow}{v}^{pw/a} = \underset{\rightarrow}{r}^{pw \bullet a} = \left(\underset{\rightarrow}{\mathcal{F}}_a^\mathsf{T} \mathbf{r}_a^{pw}\right)^{\bullet a} = \underset{\rightarrow}{\mathcal{F}}_a^\mathsf{T} \dot{\mathbf{r}}_a^{pw} = \underset{\rightarrow}{\mathcal{F}}_a^\mathsf{T} \underbrace{\begin{bmatrix} \dot{x}_b \cos(\theta) - x_b \sin(\theta)\dot{\theta} \\ \dot{x}_b \sin(\theta) + x_b \cos(\theta)\dot{\theta} \\ 0 \end{bmatrix}}_{\mathbf{v}_a^{pw/a}}$$

Finally, the acceleration of p relative to w w.r.t. $\mathcal{F}_a$ is:

$$\begin{aligned} \underset{\rightarrow}{a}^{pw/a} &= \underset{\rightarrow}{v}^{pw/a \bullet a} = \left(\underset{\rightarrow}{\mathcal{F}}_a^\mathsf{T} \mathbf{v}_a^{pw/a}\right)^{\bullet a} = \underset{\rightarrow}{\mathcal{F}}_a^\mathsf{T} \dot{\mathbf{v}}_a^{pw/a} \\ &= \underset{\rightarrow}{\mathcal{F}}_a^\mathsf{T} \underbrace{\begin{bmatrix} \ddot{x}_b \cos(\theta) - 2\dot{x}_b \sin(\theta)\dot{\theta} - x_b \cos(\theta)\dot{\theta}^2 - x_b \sin(\theta)\ddot{\theta} \\ \ddot{x}_b \sin(\theta) + 2\dot{x}_b \cos(\theta)\dot{\theta} - x_b \sin(\theta)\dot{\theta}^2 + x_b \cos(\theta)\ddot{\theta} \\ 0 \end{bmatrix}}_{\mathbf{a}_a^{pw/a}} \\ &= \underset{\rightarrow}{\mathcal{F}}_b^\mathsf{T} \mathbf{C}_{ba} \mathbf{a}_a^{pw/a} = \underset{\rightarrow}{\mathcal{F}}_b^\mathsf{T} \underbrace{\begin{bmatrix} \ddot{x}_b - x_b\dot{\theta}^2 \\ 2\dot{x}_b\dot{\theta} + x_b\ddot{\theta} \\ 0 \end{bmatrix}}_{\mathbf{a}_b^{pw/a}} \end{aligned}$$

Let $\underset{\rightarrow}{f}^{pt} = \begin{bmatrix} f_{b1}^{pt} & f_{b2}^{pt} & f_{b3}^{pt} \end{bmatrix} \underset{\rightarrow}{\mathcal{F}}_b$ be the reaction force of the table, $\underset{\rightarrow}{f}^{pg} = \begin{bmatrix} 0 & 0 & -mg \end{bmatrix} \underset{\rightarrow}{\mathcal{F}}_b$ the gravitational force applied on p and $\underset{\rightarrow}{f}^{ps} = \begin{bmatrix} -kx_b & 0 & 0 \end{bmatrix} \underset{\rightarrow}{\mathcal{F}}_b$ the spring force. Using the fact that the cut in the table is frictionless, i.e. $f_{b1}^{pt} = 0$, the total force acting on p becomes:

$$\underset{\rightarrow}{f}^{p} = \underset{\rightarrow}{f}^{pt} + \underset{\rightarrow}{f}^{pg} + \underset{\rightarrow}{f}^{ps} = \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \underbrace{\begin{bmatrix} -kx_b \\ f_{b2}^{pt} \\ f_{b3}^{pt} - mg \end{bmatrix}}_{\mathbf{f}_b^p}$$

Since $\mathcal{F}_a$ is an inertial frame and w is unforced, we can use Newton's Second Law to derive the differential equation that describes the motion of p:

$$m\,\underset{\rightarrow}{a}^{pw/a} = \underset{\rightarrow}{f}^{p}$$

$$m\underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}}\mathbf{a}_b^{pw/a} = \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}}\mathbf{f}_b^p$$

$$m\mathbf{a}_b^{pw/a} = \mathbf{f}_b^p$$

$$m\begin{bmatrix} \ddot{x}_b - x_b\dot{\theta}^2 \\ 2\dot{x}_b\dot{\theta} + x_b\ddot{\theta} \\ 0 \end{bmatrix} = \begin{bmatrix} -kx_b \\ f_{b2}^{pt} \\ f_{b3}^{pt} - mg \end{bmatrix} \tag{1}$$

We can see straight away that $f_{b3}^{pt} = mg$.

b) Using the fact that $\dot{\theta}$ is constant (i.e. $\ddot{\theta} = 0$) and rearranging, (1) can be rewritten as follows:

$$\ddot{x}_b + (\frac{k}{m} - \dot{\theta}^2)x_b = 0 \tag{2}$$

$$2\dot{x}_b\dot{\theta} - \frac{f_{b2}^{pt}}{m} = 0 \tag{3}$$

The general solution to (2), knowing that $\frac{k}{m} > \dot{\theta}^2$, is given by:

$$x_b(t) = A\cos(\omega t) + B\sin(\omega t), \quad A, B \in \mathbb{R}, \quad \omega = \sqrt{\frac{k}{m} - \dot{\theta}^2} \tag{4}$$

c) First, using the initial condition $x_b(0) = 0.1$, it yields:

$$x_b(0) = A = 0.1$$

. Secondly, using $\dot{x}_b(0) = 0$:

$$\dot{x}_b(0) = [-A\omega\sin(\omega t) + B\omega\cos(\omega t)]_{t=0} = B\omega = 0.$$

Therefore:

$$B = 0$$

and (4) can be rewritten as follows:

$$x_b(t) = 0.1\cos(\omega t)\ [\text{m}], \quad \omega = \sqrt{\frac{k}{m} - \dot{\theta}^2}. \tag{5}$$

Lastly,

$$\dot{x}_b = -0.1\omega\sin(\omega t)\ [\text{m/s}], \tag{6}$$

$$\ddot{x}_b = -0.1\omega^2\cos(\omega t)\ [\text{m/s}^2]. \tag{7}$$

Using MATLAB, we obtain the following plots:

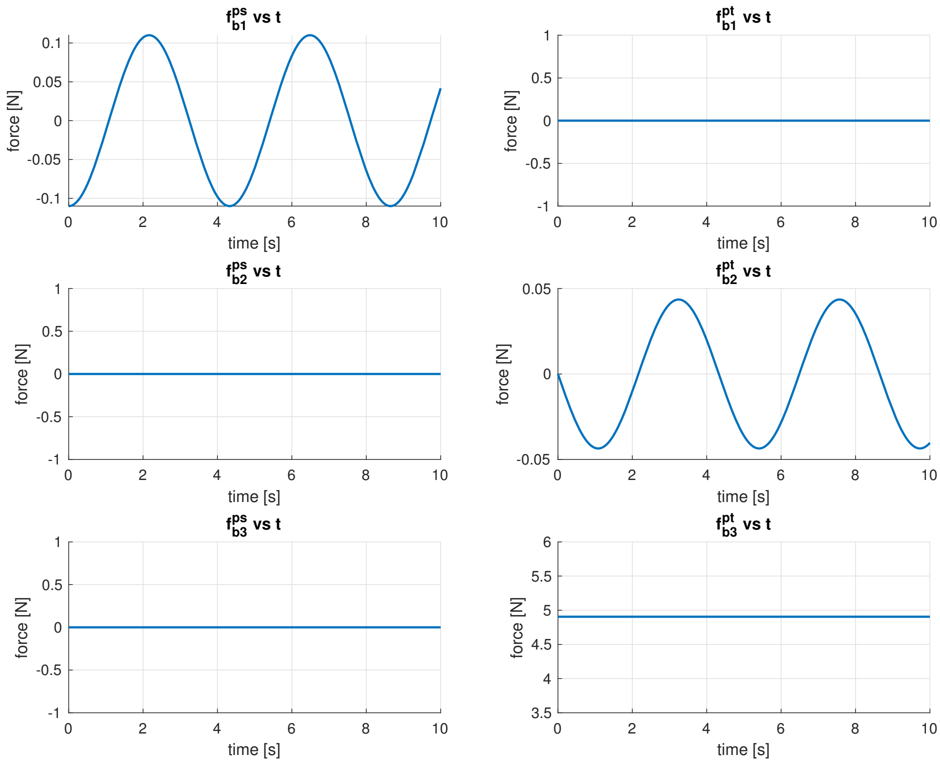

Figure 1: $\mathbf{f}_b^{pt}$ and $\mathbf{f}_b^{ps}$ vs time.