# Project Kinematics - ADR Spacecraft

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## I. Geometrical parametrization

In order to perform an accurate kinematic analysis on the spacecraft and the debris, it is necessary to describe fully and accurately their geometry with parameters. Let $o$ be a fixed point on earth. The spacecraft is modeled as a closed end hollow cylinder with a wall thickness $t_s$, an outer radius $\rho_o$ and a length $l_s$. Its main propeller P1 is attached to the main body at the point $p$, at the center of its base. On the top, the two auxiliary propellers P2 and P3 sit diametrically opposite each other, at a distance $\rho_p$ from $A_s$ (the spacecraft axis). The two reaction wheels W1 and W2 are modeled as full cylinders of radius and length $\rho_1$ and $l_1$ (respectively $\rho_2$ and $l_2$). W1 is oriented radially and W2 axially, in such a way that their center of gravity $g_1$ and $g_2$ are both on $A_s$. Moreover, W1 is aligned with P2 and P3. The length of point $g_1$ (and $g_2$) relative to point $p$ is $z_1$ (respectively $z_2$). The centre of mass of the wall $g_s$ coincide with the geometrical centre of the spacecraft. With this symmetrical geometry, the spacecraft centre of gravity $g$ must lie on the main axis $A_s$. Let $z_g$ be the length of $g$ relative to $p$ and let $m_s$, $m_1$ and $m_2$ be the mass of the spacecraft wall, of W1 and of W2, respectively. Neglecting the mass of all the other parts, it yields

$$z_g = \frac{\frac{l_s}{2}m_s + z_1 m_1 + z_2 m_2}{m_w + m_1 + m_2}. \tag{1}$$

To simplify, the debris is modeled as a full cube of size $l_d$. Finally, let the point $w$ be one of its corner and $g_d$ its center of gravity.

## II. Reference frames

(From this point, the notation used is from [1].) In this project, 6 different reference frames will be used:

- $\mathcal{F}_e$ : An inertial reference frame attached to the earth.
- $\mathcal{F}_s$ : The reference frame attached to the body of the spacecraft. $\underrightarrow{s}^2$ is aligned with W1, $\underrightarrow{s}^3$ is aligned with $A_s$ and oriented towards the front of the spacecraft. Finally, $\underrightarrow{s}^1 = \underrightarrow{s}^2 \times \underrightarrow{s}^3$.
- $\mathcal{F}_a$ : The reference frame attached to W1, obtained by rotating $\mathcal{F}_s$ about $\underrightarrow{s}^2 = \underrightarrow{a}^2$. The angle of this rotation is refered to as $\alpha$.
- $\mathcal{F}_b$ : The reference frame attached to W2, obtained by rotating $\mathcal{F}_s$ about $\underrightarrow{s}^3 = \underrightarrow{b}^3$. The angle of this rotation is refered to as $\beta$.
- $\mathcal{F}_d$ : The reference frame attached to the body of the debris and aligned with its edges.
- $\mathcal{F}_u$ : The inertial reference frame obtained by rotating $\mathcal{F}_e$ in such a way that $\underrightarrow{e}^3$ goes to $\underrightarrow{u}^3$, which is itself aligned with $\underrightarrow{v}^{g_d o/e}$. (This frame is not really useful for the dynamic/kinematic part but might help if some control is added at the end of the project.)

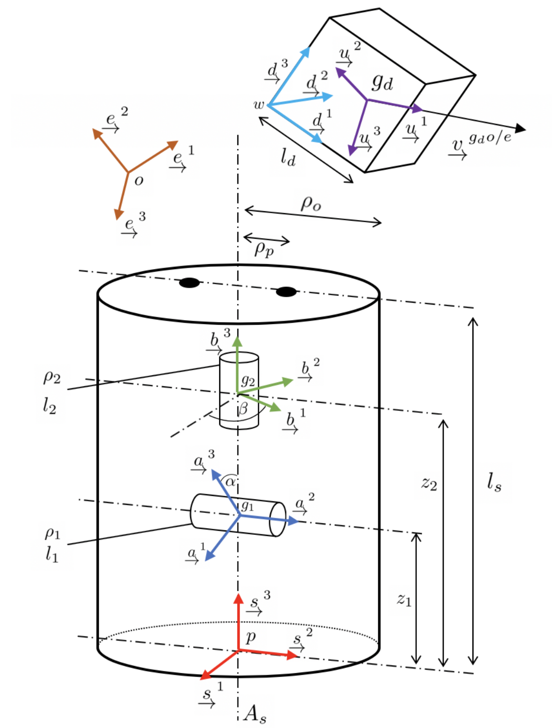


Fig. 1. Geometrical parametrization and reference frames definition.

## III. Attitude description

Since this problem is highly three dimensional, all the mutual configurations of the different reference frames are likely to happen. Using attitude parametrization that are not global or have kinematic singularities is risky and might in the end complicate the problem even more. This is why Direction Cosine Matrices will mostly be used. However, one DCM ($\mathbf{C}_{ue}$) will be parametrized using the Angle/Axis (Rodrigues's formula), but it will be constant in time and kinematic singularities will not be a problem.

## IV. Measurables

This problem has 14 degrees of freedom (6 for the shell, 1 for each reaction wheel and 6 for the debris). Therefore, in order to have a well defined and observable problem, 28 independent measurables are needed (14 positions and 14 velocities). The following assumptions will be made:

- The earth has a radar system that can determine $\mathbf{r}_e^{g_d o}$, $\mathbf{v}_e^{g_d o/e}$, $\mathbf{r}_e^{po}$ and $\mathbf{v}_e^{po/e}$. Moreover, it communicates with the spacecraft sensors in order to determine $\mathbf{s}_e^1, \mathbf{s}_e^2$ and $\mathbf{s}_e^3$. Lastly, it can compute $\boldsymbol{\omega}_e^{se}$.
- The spacecraft system knows $\alpha$, $\beta$, $\dot{\alpha}$ and $\dot{\beta}$.

- The spacecraft has sensors that can observe the debris' behavior, i.e. $\mathbf{d}_s^1, \mathbf{d}_s^2$, $\mathbf{d}_s^3$ and $\boldsymbol{\omega}_s^{ds}$ (It is assumed that the radars on earth are too far to observe directly the orientation and the spin of the debris).

All the other states can be derived from these 28 variables.

## V. Direction Cosine Matrices

*A priori*, 15 DCM are needed in order to be able to jump freely between 6 reference frames. However, all of them can be expressed as a combination of 5 elementary DCM (or their transpose). Here is a possible set that can be used to derived all the other DCM:

$$\mathbf{C}_{es} = \underrightarrow{\mathcal{F}}_e \cdot \underrightarrow{\mathcal{F}}_s^{\mathsf{T}} = \begin{bmatrix} \mathbf{s}_e^1 & \mathbf{s}_e^2 & \mathbf{s}_e^3 \end{bmatrix}, \tag{2}$$

$$\mathbf{C}_{as} = \mathbf{C}_2(\alpha) = \begin{bmatrix} \cos(\alpha) & 0 & -\sin(\alpha) \\ 0 & 1 & 0 \\ \sin(\alpha) & 0 & \cos(\alpha) \end{bmatrix}, \tag{3}$$

$$\mathbf{C}_{bs} = \mathbf{C}_3(\beta) = \begin{bmatrix} \cos(\beta) & \sin(\beta) & 0 \\ -\sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4}$$

$$\mathbf{C}_{sd} = \underrightarrow{\mathcal{F}}_s \cdot \underrightarrow{\mathcal{F}}_d^{\mathsf{T}} = \begin{bmatrix} \mathbf{d}_s^1 & \mathbf{d}_s^2 & \mathbf{d}_s^3 \end{bmatrix}. \tag{5}$$

Lastly, the way $\mathcal{F}_u$ was previously defined does not determine uniquely $\mathbf{C}_{ue}$. Therefore, the following conventions will be used:

$$\mathbf{C}_{ue} = \cos(\phi_v)\mathbf{1} + (1 - \cos(\phi_v))\mathbf{a}_v\mathbf{a}_v^{\mathsf{T}} - \sin(\phi_v)\mathbf{a}_v^{\times} \tag{6}$$

where, using $v_d = ||\mathbf{v}_e^{g_d o/e}||_2$,

$$\phi_v = \begin{cases} 0 & \text{for} \quad v_d = 0 \\ \arccos\left(\frac{\mathbf{1}_3^{\mathsf{T}}\mathbf{v}_e^{g_d o/e}}{v_d}\right) & \text{for} \quad v_d \neq 0\,, \end{cases}$$

$$\mathbf{a}_v = \begin{cases} \mathbf{0} & \text{for} \quad \phi_v \in \{0, \pi\} \\ \frac{\mathbf{1}_3^{\times}\mathbf{v}_e^{g_d o/e}}{v_d \sin(\phi_v)} & \text{for} \quad \phi_v \notin \{0, \pi\}\,. \end{cases}$$

As discussed in the Proposal, $\underrightarrow{v}^{g_d o/e}$ is assumed to be constant in time. Therefore, $\mathbf{C}_{ue}$ is also constant in time, i.e. $\underrightarrow{\omega}^{ue} = \underrightarrow{0}$.

## VI. Angular Velocities

As for the DCM, a set of 5 angular velocity physical vectors is sufficient to describe every possible kinematic transformation. In particular,

$$\underrightarrow{\omega}^{es} = \underrightarrow{\mathcal{F}}_e^{\mathsf{T}}\boldsymbol{\omega}_e^{es} = -\underrightarrow{\mathcal{F}}_e^{\mathsf{T}}\boldsymbol{\omega}_e^{se}, \tag{7}$$

$$\underrightarrow{\omega}^{as} = \underrightarrow{\mathcal{F}}_s\mathbf{1}_2\dot{\alpha} = \underrightarrow{\mathcal{F}}_a\mathbf{1}_2\dot{\alpha}, \tag{8}$$

$$\underrightarrow{\omega}^{bs} = \underrightarrow{\mathcal{F}}_s\mathbf{1}_3\dot{\beta} = \underrightarrow{\mathcal{F}}_b\mathbf{1}_3\dot{\beta}, \tag{9}$$

$$\underrightarrow{\omega}^{sd} = \underrightarrow{\mathcal{F}}_s^{\mathsf{T}}\boldsymbol{\omega}_s^{sd} = -\underrightarrow{\mathcal{F}}_s^{\mathsf{T}}\boldsymbol{\omega}_s^{ds}, \tag{10}$$

$$\underrightarrow{\omega}^{ue} = \underrightarrow{0}. \tag{11}$$

## VII. Components Parametrization

Due to the cylindrical symmetry of the spacecraft and the reaction wheels, the components of the physical vectors resolved in frame $\mathcal{F}_s$, $\mathcal{F}_a$ and $\mathcal{F}_b$ will be parametrized using the cylindrical coordinates. Let $dm_s$, $dm_1$ and $dm_2$ be material elements of the spacecraft wall, of W1 and of W2, respectively. The parametrization of the components of $\mathbf{r}_s^{dm_s p}$, $\mathbf{r}_a^{dm_1 g_1}$ and $\mathbf{r}_b^{dm_2 g_2}$ will be done accordingly to Figure 2.

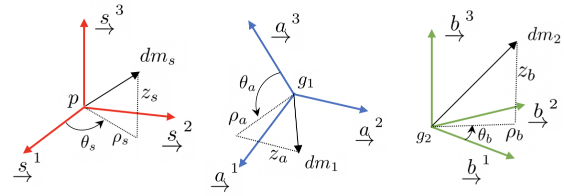

Fig. 2. Component parametrization in $\mathcal{F}_s$, $\mathcal{F}_a$ and $\mathcal{F}_b$.

As for the debris, the position of its material elements $dm_d$ relative to $g_d$ resolved in $\mathcal{F}_d$ will be described using a cartesian coordinate system. More precisely, $x_d$, $y_d$ and $z_d$ for the components along $\underrightarrow{d}^1$, $\underrightarrow{d}^2$ and $\underrightarrow{d}^3$, respectively. In matrix form:

$$\mathbf{r}_s^{dm_s p} = \begin{bmatrix} \rho_s\cos(\theta_s) \\ \rho_s\sin(\theta_s) \\ z_s \end{bmatrix}, \quad \mathbf{r}_a^{dm_1 g_1} = \begin{bmatrix} \rho_a\cos(\theta_a) \\ z_a \\ \rho_a\sin(\theta_a) \end{bmatrix},$$

$$\mathbf{r}_b^{dm_2 g_2} = \begin{bmatrix} \rho_b\cos(\theta_b) \\ \rho_b\sin(\theta_b) \\ z_b \end{bmatrix}, \quad \mathbf{r}_d^{dm_d g_d} = \begin{bmatrix} x_d \\ y_d \\ z_d \end{bmatrix}.$$

All the mathematical tools are now derived in order to express the position, velocity and acceleration of any material element in the inertial reference frame $\mathcal{F}_e$ and in function of the measurables.

## VIII. Position

$$\underrightarrow{r}^{dm_s o} = \underrightarrow{\mathcal{F}}_e^{\mathsf{T}}\left(\mathbf{r}_e^{po} + \mathbf{C}_{es}\mathbf{r}_s^{dm_s p}\right), \tag{12}$$

$$\underrightarrow{r}^{dm_1 o} = \underrightarrow{\mathcal{F}}_e^{\mathsf{T}}\left(\mathbf{r}_e^{po} + \mathbf{C}_{es}\mathbf{1}_3 z_1 + \mathbf{C}_{es}\mathbf{C}_{as}^{\mathsf{T}}\mathbf{r}_a^{dm_1 g_1}\right), \tag{13}$$

$$\underrightarrow{r}^{dm_2 o} = \underrightarrow{\mathcal{F}}_e^{\mathsf{T}}\left(\mathbf{r}_e^{po} + \mathbf{C}_{es}\mathbf{1}_3 z_2 + \mathbf{C}_{es}\mathbf{C}_{bs}^{\mathsf{T}}\mathbf{r}_b^{dm_2 g_2}\right), \tag{14}$$

$$\underrightarrow{r}^{dm_d o} = \underrightarrow{\mathcal{F}}_e^{\mathsf{T}}\left(\mathbf{r}_e^{g_d o} + \mathbf{C}_{es}\mathbf{C}_{sd}\mathbf{r}_d^{dm_d g_d}\right). \tag{15}$$

## IX. Velocity

$$\underrightarrow{v}^{dm_s o/e} = \underrightarrow{\mathcal{F}}_e^{\mathsf{T}}\left(\mathbf{v}_e^{po/e} + \boldsymbol{\omega}_e^{se\times}\mathbf{C}_{es}\mathbf{r}_s^{dm_s p}\right), \tag{16}$$

$$\underrightarrow{v}^{dm_1 o/e} = \underrightarrow{\mathcal{F}}_e^{\mathsf{T}}\Big([\mathbf{C}_{es}\mathbf{1}_2\dot{\alpha} + \boldsymbol{\omega}_e^{se}]^{\times}\,\mathbf{C}_{es}\mathbf{C}_{as}^{\mathsf{T}}\mathbf{r}_a^{dm_1 g_1} + \boldsymbol{\omega}_e^{se\times}\mathbf{C}_{es}\mathbf{1}_3 z_1 + \mathbf{v}_e^{po/e}\Big), \tag{17}$$

$$\underrightarrow{v}^{dm_2 o/e} = \underrightarrow{\mathcal{F}}_e^{\mathsf{T}}\left(\left[\mathbf{C}_{es}\mathbf{1}_3\dot{\beta} + \boldsymbol{\omega}_e^{se}\right]^{\times}\mathbf{C}_{es}\mathbf{C}_{bs}^{\mathsf{T}}\mathbf{r}_b^{dm_2 g_2} + \boldsymbol{\omega}_e^{se\times}\mathbf{C}_{es}\mathbf{1}_3 z_2 + \mathbf{v}_e^{po/e}\right), \tag{18}$$

$$\underrightarrow{v}^{dm_d o/e} = \underrightarrow{\mathcal{F}}_e^{\mathsf{T}}\left(\left[\mathbf{C}_{es}\boldsymbol{\omega}_s^{ds} + \boldsymbol{\omega}_e^{se}\right]^{\times}\mathbf{C}_{es}\mathbf{C}_{sd}\mathbf{r}_d^{dm_d g_d} + \mathbf{v}_e^{g_d o/e}\right). \tag{19}$$

## X. ACCELERATION

$$
\begin{aligned}
\underset{\rightarrow}{a}^{dm_s o/e} &= \underset{\rightarrow}{\mathcal{F}}_e^{\mathsf{T}} \Big( \dot{\mathbf{v}}_e^{po/e} + \dot{\boldsymbol{\omega}}_e^{se\,\times} \mathbf{C}_{es}\mathbf{r}_s^{dm_s p} \\
&\quad + \boldsymbol{\omega}_e^{se\times}\boldsymbol{\omega}_e^{se\times} \mathbf{C}_{es}\mathbf{r}_s^{dm_s p} \Big), && (20) \\
\underset{\rightarrow}{a}^{dm_1 o/e} &= \underset{\rightarrow}{\mathcal{F}}_e^{\mathsf{T}} \Big( \left[ \mathbf{C}_{es}\mathbf{1}_2\ddot{\alpha} + \dot{\boldsymbol{\omega}}_e^{se} - \mathbf{C}_{es}(\mathbf{1}_2\dot{\alpha})^{\times}\boldsymbol{\omega}_e^{se} \right]^{\times} \dots \\
&\quad \mathbf{C}_{es}\mathbf{C}_{as}^{\mathsf{T}}\mathbf{r}_a^{dm_1 g_1} + \left[ \mathbf{C}_{es}\mathbf{1}_2\dot{\alpha} + \boldsymbol{\omega}_e^{se} \right]^{\times} \dots \\
&\quad \left[ \mathbf{C}_{es}\mathbf{1}_2\dot{\alpha} + \boldsymbol{\omega}_e^{se} \right]^{\times} \mathbf{C}_{es}\mathbf{C}_{as}^{\mathsf{T}}\mathbf{r}_a^{dm_1 g_1} \\
&\quad + \left[ \dot{\boldsymbol{\omega}}_e^{se\,\times} + \boldsymbol{\omega}_e^{se\times}\boldsymbol{\omega}_e^{se\times} \right] \mathbf{C}_{es}\mathbf{1}_3 z_1 \\
&\quad + \dot{\mathbf{v}}_e^{po/e} \Big), && (21) \\
\underset{\rightarrow}{a}^{dm_2 o/e} &= \underset{\rightarrow}{\mathcal{F}}_e^{\mathsf{T}} \Bigg( \Big[ \mathbf{C}_{es}\mathbf{1}_3\ddot{\beta} + \dot{\boldsymbol{\omega}}_e^{se} - \mathbf{C}_{es}(\mathbf{1}_3\dot{\beta})^{\times}\boldsymbol{\omega}_e^{se} \Big]^{\times} \dots \\
&\quad \mathbf{C}_{es}\mathbf{C}_{bs}^{\mathsf{T}}\mathbf{r}_b^{dm_2 g_2} + \Big[ \mathbf{C}_{es}\mathbf{1}_3\dot{\beta} + \boldsymbol{\omega}_e^{se} \Big]^{\times} \dots \\
&\quad \Big[ \mathbf{C}_{es}\mathbf{1}_3\dot{\beta} + \boldsymbol{\omega}_e^{se} \Big]^{\times} \mathbf{C}_{es}\mathbf{C}_{bs}^{\mathsf{T}}\mathbf{r}_b^{dm_2 g_2} \\
&\quad + \left[ \dot{\boldsymbol{\omega}}_e^{se\,\times} + \boldsymbol{\omega}_e^{se\times}\boldsymbol{\omega}_e^{se\times} \right] \mathbf{C}_{es}\mathbf{1}_3 z_2 \\
&\quad + \dot{\mathbf{v}}_e^{po/e} \Bigg), && (22) \\
\underset{\rightarrow}{a}^{dm_d o/e} &= \underset{\rightarrow}{\mathcal{F}}_e^{\mathsf{T}} \Bigg( \Big[ \mathbf{C}_{es}\dot{\boldsymbol{\omega}}_s^{ds} + \dot{\boldsymbol{\omega}}_e^{se} - \left\{ \mathbf{C}_{es}\boldsymbol{\omega}_s^{ds} \right\}^{\times} \boldsymbol{\omega}_e^{se} \Big]^{\times} \dots \\
&\quad \mathbf{C}_{es}\mathbf{C}_{sd}\mathbf{r}_d^{dm_d g_d} \\
&\quad + \left[ \mathbf{C}_{es}\boldsymbol{\omega}_s^{ds} + \boldsymbol{\omega}_e^{se} \right]^{\times} \left[ \mathbf{C}_{es}\boldsymbol{\omega}_s^{ds} + \boldsymbol{\omega}_e^{se} \right]^{\times} \dots \\
&\quad \mathbf{C}_{es}\mathbf{C}_{sd}\mathbf{r}_d^{dm_d g_d} + \dot{\mathbf{v}}_e^{g_d o/e} \Bigg). && (23)
\end{aligned}
$$

## XI. DERIVATION

I didn't write explicitly the derivations of these expressions due to their lengths. However, it is quite straight forward to check them using *Poisson's Equation* [1, Eq. 3.12, p. 61], the product rule for derivatives and the fact that

$$
\dot{\mathbf{r}}_s^{dm_s p} = \dot{\mathbf{r}}_a^{dm_1 g_1} = \dot{\mathbf{r}}_b^{dm_2 g_2} = \dot{\mathbf{r}}_d^{dm_d g_d} = \dot{\mathbf{r}}_s^{g_1 p} = \dot{\mathbf{r}}_s^{g_2 p} = \mathbf{0}.
$$

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## REFERENCES

[1] J. Forbes, "A Systematic Approach to Dynamics", Course Notes for MECH 642, 2019.