Assignment 4 - Problem 2 & 3

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a) Let θ be the angle between $\underrightarrow{b}^2$ and $\underrightarrow{a}^2$. Therefore

$$\mathbf{C}_{ba} = \mathbf{C}_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Moreover,

$$\underrightarrow{\omega}^{ba} = \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{aligned}
\underrightarrow{r}^{\kappa w} &= \underrightarrow{r}^{\kappa c} + \underrightarrow{r}^{cw} \\
&= \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} + \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} 0 \\ 0 \\ -d \end{bmatrix} \\
&= \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} 0 \\ y \\ -d \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\underrightarrow{r}^{dmw} &= \underrightarrow{r}^{dmc} + \underrightarrow{r}^{cw} \\
&= \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} \rho_{b1} \\ \rho_{b2} \\ \rho_{b3} \end{bmatrix} + \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} 0 \\ 0 \\ -d \end{bmatrix} \\
&= \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} \rho_{b1} \\ \rho_{b2} \\ \rho_{b3} - d \end{bmatrix},
\end{aligned}$$

$$
\begin{aligned}
\underset{\rightarrow}{r}^{\kappa w \bullet a} &= \underset{\rightarrow}{r}^{\kappa w \bullet b} + \underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{r}^{\kappa w} \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \left(\dot{\mathbf{r}}_b^{\kappa w} + \boldsymbol{\omega}_b^{ba\times} \mathbf{r}_b^{\kappa w} \right) \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \left(\begin{bmatrix} 0 \\ \dot{y} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta} \\ 0 & \dot{\theta} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ y \\ -d \end{bmatrix} \right) \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} 0 \\ \dot{y} + d\dot{\theta} \\ y\dot{\theta} \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\underset{\rightarrow}{r}^{dmw \bullet a} &= \underset{\rightarrow}{r}^{dmw \bullet b} + \underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{r}^{dmw} \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \left(\underbrace{\dot{\mathbf{r}}_b^{dmw}}_{=\mathbf{0}} + \boldsymbol{\omega}_b^{ba\times} \mathbf{r}_b^{dmw} \right) \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta} \\ 0 & \dot{\theta} & 0 \end{bmatrix} \begin{bmatrix} \rho_{b1} \\ \rho_{b2} \\ \rho_{b3} - d \end{bmatrix} \right) \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} 0 \\ (d - \rho_{b3})\dot{\theta} \\ \rho_{b2}\dot{\theta} \end{bmatrix}.
\end{aligned}
$$

b) Figures 1 and 2 show the Free Body Diagrams of the particle κ and the body $\mathcal{B}$. Let $\underset{\rightarrow}{f}^{r1}$ and $\underset{\rightarrow}{f}^{r1}$ be the reaction forces applied by the massless bars on the body $\mathcal{B}$.

The norm of $\underset{\rightarrow}{r}^{\kappa w}$ is

$$||\underset{\rightarrow}{r}^{\kappa w}||_2 = \mathbf{r}_b^{\kappa w \mathsf{T}} \mathbf{r}_b^{\kappa w} = y^2 + d^2.$$

Therefore, the total force applied on κ is given by:

$$
\begin{aligned}
\underset{\rightarrow}{f}^{\kappa} &= \underset{\rightarrow}{f}^{\kappa\mathcal{B}} + \underset{\rightarrow}{f}^{\kappa g} + \underset{\rightarrow}{f}^{\kappa s} \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \left(\begin{bmatrix} f_{b1}^{\kappa\mathcal{B}} \\ 0 \\ f_{b3}^{\kappa\mathcal{B}} \end{bmatrix} + \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -m_\kappa g \end{bmatrix} - k \frac{l_s - \bar{l}_s}{y^2 + d^2} \begin{bmatrix} 0 \\ y \\ -d \end{bmatrix} \right) \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} f_{b1}^{\kappa\mathcal{B}} \\ -m_\kappa g \sin(\theta) - ky\frac{l_s - \bar{l}_s}{y^2+d^2} \\ f_{b3}^{\kappa\mathcal{B}} - m_\kappa g \cos(\theta) + kd\frac{l_s - \bar{l}_s}{y^2+d^2} \end{bmatrix}.
\end{aligned}
$$

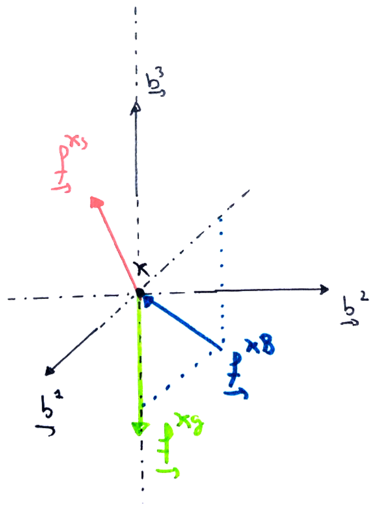

Figure 1: FBD of κ.

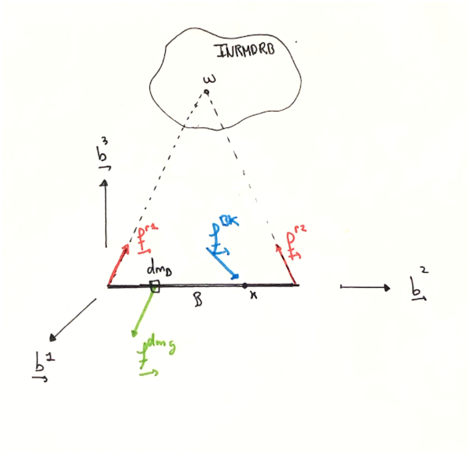

Figure 2: FBD of $\mathcal{B}$.

The total force acting on $\mathcal{B}$ is given by:

$$
\begin{aligned}
\underrightarrow{f}^{\mathcal{B}} &= \underrightarrow{f}^{r1} + \underrightarrow{f}^{r2} + \underrightarrow{f}^{\mathcal{B}\kappa} + \int_{\mathcal{B}} d\,\underrightarrow{f}^{dmg} \\
&= \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \left(\begin{bmatrix} f_{b1}^{r1} \\ f_{b2}^{r1} \\ f_{b3}^{r1} \end{bmatrix} + \begin{bmatrix} f_{b1}^{r2} \\ f_{b2}^{r2} \\ f_{b3}^{r3} \end{bmatrix} - \begin{bmatrix} f_{b1}^{\kappa\mathcal{B}} \\ 0 \\ f_{b3}^{\kappa\mathcal{B}} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix} \int_{\mathcal{B}} \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} dm \right) \\
&= \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} f_{b1}^{r1} + f_{b1}^{r2} - f_{b1}^{\kappa\mathcal{B}} \\ f_{b2}^{r1} + f_{b2}^{r2} - m_{\mathcal{B}} g \sin(\theta) \\ f_{b3}^{r1} + f_{b3}^{r2} - f_{b3}^{\kappa\mathcal{B}} - m_{\mathcal{B}} g \cos(\theta) \end{bmatrix}.
\end{aligned}
$$

Finally, the total moment applied on $\mathcal{B}$ relative to w is given by:

$$
\begin{aligned}
\underrightarrow{m}^{\mathcal{B}w} &= \underbrace{\underrightarrow{r}^{r1w} \times \underrightarrow{f}^{r1}}_{=\underrightarrow{0}} + \underbrace{\underrightarrow{r}^{r2w} \times \underrightarrow{f}^{r2}}_{=\underrightarrow{0}} + \underrightarrow{r}^{\kappa w} \times \underrightarrow{f}^{\mathcal{B}\kappa} + \int_{\mathcal{B}} \underrightarrow{r}^{dmw} \times d\,\underrightarrow{f}^{dmg} \\
&= \underrightarrow{r}^{\kappa w} \times \underrightarrow{f}^{\mathcal{B}\kappa} + \underbrace{\left(\int_{\mathcal{B}} \underrightarrow{r}^{dmw} dm \right)}_{= m_{\mathcal{B}} \underrightarrow{r}^{cw}} \times \underrightarrow{g} \\
&= \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \left(\begin{bmatrix} 0 \\ y \\ -d \end{bmatrix}^{\times} \begin{bmatrix} -f_{b1}^{\kappa\mathcal{B}} \\ 0 \\ -f_{b3}^{\kappa\mathcal{B}} \end{bmatrix} + m_{\mathcal{B}} \begin{bmatrix} 0 \\ 0 \\ -d \end{bmatrix}^{\times} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} \right) \\
&= \underrightarrow{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} -y f_{b3}^{\kappa\mathcal{B}} - d g m_{\mathcal{B}} \sin(\theta) \\ d f_{b1}^{\kappa\mathcal{B}} \\ y f_{b1}^{\kappa\mathcal{B}} \end{bmatrix}
\end{aligned}
$$

c)

$$\underset{\rightarrow}{p}^{\kappa w/a} = m_\kappa \underset{\rightarrow}{r}^{\kappa w/a^{\bullet a}} = m_\kappa \underset{\rightarrow}{\mathcal{F}}_b^\mathsf{T} \begin{bmatrix} 0 \\ \dot{y} + d\dot{\theta} \\ y\dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ m_\kappa(\dot{y} + d\dot{\theta}) \\ m_\kappa y\dot{\theta} \end{bmatrix}. \quad \square$$

$$\begin{aligned}
\underset{\rightarrow}{h}^{\mathcal{B}w/a} &= \int_\mathcal{B} \underset{\rightarrow}{r}^{dmw} \times \underset{\rightarrow}{r}^{dmw^{\bullet a}} dm \\
&= \underset{\rightarrow}{\mathcal{F}}_b^\mathsf{T} \left(\int_\mathcal{B} \begin{bmatrix} \rho_{b1} \\ \rho_{b2} \\ \rho_{b3} - d \end{bmatrix}^\times \begin{bmatrix} 0 \\ (d-\rho_{b3})\dot{\theta} \\ \rho_{b2}\dot{\theta} \end{bmatrix} dm \right) \\
&= \underset{\rightarrow}{\mathcal{F}}_b^\mathsf{T} \left(\int_\mathcal{B} \begin{bmatrix} 0 & -(\rho_{b3}-d) & \rho_{b2} \\ (\rho_{b3}-d) & 0 & -\rho_{b1} \\ -\rho_{b2} & \rho_{b1} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ (d-\rho_{b3})\dot{\theta} \\ \rho_{b2}\dot{\theta} \end{bmatrix} dm \right) \\
&= \underset{\rightarrow}{\mathcal{F}}_b^\mathsf{T} \left(\dot{\theta} \int_\mathcal{B} \begin{bmatrix} (d-\rho_{b3})^2 + \rho_{b2}^2 \\ -\rho_{b1}\rho_{b2} \\ \rho_{b1}(d-\rho_{b3}) \end{bmatrix} dm \right) \\
&= \underset{\rightarrow}{\mathcal{F}}_b^\mathsf{T} \left(\dot{\theta} \int_{V_\mathcal{B}} \begin{bmatrix} (d-\rho_{b3})^2 + \rho_{b2}^2 \\ -\rho_{b1}\rho_{b2} \\ \rho_{b1}(d-\rho_{b3}) \end{bmatrix} \left(\frac{m_\mathcal{B}}{lth}\right) dV \right) \\
&= \underset{\rightarrow}{\mathcal{F}}_b^\mathsf{T} \left(\frac{m_\mathcal{B}\dot{\theta}}{lth} \begin{bmatrix} lt[\int_{-h/2}^{h/2}(d-\rho_{b3})^2 d\rho_{b3}] + th[\int_{-l/2}^{l/2}\rho_{b2}^2 d\rho_{b2}] \\ -lth^2 [\underbrace{\int_{-t/2}^{t/2}\rho_{b1} d\rho_{b1}}_{0}][\underbrace{\int_{-l/2}^{l/2}\rho_{b2} d\rho_{b2}}_{0}] \\ l^2ht[\underbrace{\int_{-t/2}^{t/2}\rho_{b1} d\rho_{b1}}_{0}][\int_{-h/2}^{h/2}(d-\rho_{b3}) d\rho_{b3}] \end{bmatrix} \right) \\
&= \underset{\rightarrow}{\mathcal{F}}_b^\mathsf{T} \left(\frac{m_\mathcal{B}\dot{\theta}}{lth} \begin{bmatrix} -\frac{lt}{3}[(d-\rho_{b3})^3]_{-h/2}^{h/2} + \frac{th}{3}[\rho_{b2}^3]_{-l/2}^{l/2} \\ 0 \\ 0 \end{bmatrix} \right) \\
&= \underset{\rightarrow}{\mathcal{F}}_b^\mathsf{T} \left(\frac{m_\mathcal{B}\dot{\theta}}{lth} \begin{bmatrix} -\frac{lt}{3}[-3d^2h - \frac{h^3}{4}] + \frac{th}{3}\frac{l^3}{4} \\ 0 \\ 0 \end{bmatrix} \right) \\
&= \underset{\rightarrow}{\mathcal{F}}_b^\mathsf{T} \left(m_\mathcal{B}\dot{\theta} \begin{bmatrix} d^2 + \frac{1}{12}(h^2 + l^2) \\ 0 \\ 0 \end{bmatrix} \right). \quad \square
\end{aligned}$$

d) First, from N2L,

$$\underset{\rightarrow}{p}^{\kappa w/a^{\bullet a}} = \underset{\rightarrow}{f}^{\kappa}.$$

Using the Transport Theorem and developing:

$$\begin{aligned}
\underset{\rightarrow}{p}^{\kappa w/a^{\bullet a}} &= \underset{\rightarrow}{p}^{\kappa w/a^{\bullet b}} + \underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{p}^{\kappa w/a} \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \left(\begin{bmatrix} 0 \\ m_\kappa(\ddot{y} + d\ddot{\theta}) \\ m_\kappa(\dot{y}\dot{\theta} + y\ddot{\theta}) \end{bmatrix} + \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix}^{\times} \begin{bmatrix} 0 \\ m_\kappa(\dot{y} + d\dot{\theta}) \\ m_\kappa y\dot{\theta} \end{bmatrix} \right) \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} 0 \\ m_\kappa(\ddot{y} + d\ddot{\theta}) - m_\kappa y\dot{\theta}^2 \\ m_\kappa(\dot{y}\dot{\theta} + y\ddot{\theta}) + \dot{\theta} m_\kappa(\dot{y} + d\dot{\theta}) \end{bmatrix} \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} 0 \\ m_\kappa(\ddot{y} + d\ddot{\theta}) - m_\kappa y\dot{\theta}^2 \\ 2m_\kappa\dot{y}\dot{\theta} + m_\kappa y\ddot{\theta} + m_\kappa d\dot{\theta}^2) \end{bmatrix}
\end{aligned}$$

Therefore,

$$\begin{bmatrix} 0 \\ m_\kappa(\ddot{y} + d\ddot{\theta}) - m_\kappa y\dot{\theta}^2 \\ 2m_\kappa\dot{y}\dot{\theta} + m_\kappa y\ddot{\theta} + m_\kappa d\dot{\theta}^2) \end{bmatrix} = \begin{bmatrix} f_{b1}^{\kappa\mathcal{B}} \\ -m_\kappa g\sin(\theta) - ky\frac{l_s - \bar{l}_s}{y^2+d^2} \\ f_{b3}^{\kappa\mathcal{B}} - m_\kappa g\cos(\theta) + kd\frac{l_s - \bar{l}_s}{y^2+d^2} \end{bmatrix}. \tag{1}$$

We can see straight away that

$$f_{b1}^{\kappa\mathcal{B}} = 0.$$

Secondly,

$$\begin{aligned}
\underset{\rightarrow}{h}^{\mathcal{B}w/a^{\bullet a}} &= \underset{\rightarrow}{h}^{\mathcal{B}w/a^{\bullet b}} + \underset{\rightarrow}{\omega}^{ba} \times \underset{\rightarrow}{h}^{\mathcal{B}w/a} \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \left(\begin{bmatrix} m_\mathcal{B}\ddot{\theta}[d^2 + \frac{1}{12}(h^2 + l^2)] \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix}^{\times} \begin{bmatrix} m_\mathcal{B}\dot{\theta}[d^2 + \frac{1}{12}(h^2 + l^2)] \\ 0 \\ 0 \end{bmatrix} \right) \\
&= \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} m_\mathcal{B}\ddot{\theta}[d^2 + \frac{1}{12}(h^2 + l^2)] \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

From N2LR,

$$\underset{\rightarrow}{h}^{\mathcal{B}w/a^{\bullet a}} = \underset{\rightarrow}{m}^{\mathcal{B}w}.$$

Therefore,

$$\underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} m_\mathcal{B}\ddot{\theta}[d^2 + \frac{1}{12}(h^2 + l^2)] \\ 0 \\ 0 \end{bmatrix} = \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \begin{bmatrix} -yf_{b3}^{\kappa\mathcal{B}} - dgm_\mathcal{B}\sin(\theta) \\ df_{b1}^{\kappa\mathcal{B}} \\ yf_{b1}^{\kappa\mathcal{B}} \end{bmatrix}$$

In particular,

$$f_{b3}^{\kappa\mathcal{B}} = -\frac{m_{\mathcal{B}}}{y}\left(\ddot{\theta}[d^2 + \frac{1}{12}(h^2 + l^2)] + dg\sin(\theta)\right) \tag{2}$$

Finally, substituting (2) into (1),

$$m_\kappa(\ddot{y} + d\ddot{\theta}) - m_\kappa y\dot{\theta}^2 = -m_\kappa g\sin(\theta) - ky\frac{l_s - \bar{l}_s}{y^2 + d^2} \tag{3}$$

$$\begin{aligned} 2m_\kappa\dot{y}\dot{\theta} + m_\kappa y\ddot{\theta} + m_\kappa d\dot{\theta}^2 = &-\frac{m_{\mathcal{B}}}{y}\left(\ddot{\theta}[d^2 + \frac{1}{12}(h^2 + l^2)] + dg\sin(\theta)\right) \\ &- m_\kappa g\cos(\theta) + kd\frac{l_s - \bar{l}_s}{y^2 + d^2} \end{aligned} \tag{4}$$

e) The total moment applied on $\mathcal{S}$ relative to w is given by:

$$\begin{aligned} \underrightarrow{m}^{\mathcal{S}w} =& m_{\mathcal{B}}\underrightarrow{r}^{cw} \times \underrightarrow{g} + m_\kappa\underrightarrow{r}^{\kappa w} \times \underrightarrow{g} \\ =& \underrightarrow{\mathcal{F}}_b^{\mathsf{T}}\left(m_{\mathcal{B}}\begin{bmatrix} 0 \\ 0 \\ -d \end{bmatrix}^{\times}\begin{bmatrix} 0 \\ -\sin(\theta)g \\ -\cos(\theta)g \end{bmatrix} + m_\kappa\begin{bmatrix} 0 \\ y \\ -d \end{bmatrix}^{\times}\begin{bmatrix} 0 \\ -\sin(\theta)g \\ -\cos(\theta)g \end{bmatrix}\right) \\ =& \underrightarrow{\mathcal{F}}_b^{\mathsf{T}}\begin{bmatrix} -m_{\mathcal{B}}dg\sin(\theta) + m_\kappa[-yg\cos(\theta) - dg\sin(\theta)] \\ 0 \\ 0 \end{bmatrix}. \quad \square \end{aligned}$$

Secondly,

$$\begin{aligned} \underrightarrow{h}^{\mathcal{S}w/a} &= \underrightarrow{h}^{\mathcal{B}w/a} + \underrightarrow{r}^{\kappa w} \times \underrightarrow{p}^{\kappa w/a} \\ &= \underrightarrow{\mathcal{F}}_b^{\mathsf{T}}\left(\begin{bmatrix} m_{\mathcal{B}}\dot{\theta}[d^2 + \frac{1}{12}(h^2 + l^2)] \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ -d \end{bmatrix}^{\times}\begin{bmatrix} 0 \\ m_\kappa(\dot{y} + d\dot{\theta}) \\ m_\kappa y\dot{\theta} \end{bmatrix}\right) \\ &= \underrightarrow{\mathcal{F}}_b^{\mathsf{T}}\begin{bmatrix} m_{\mathcal{B}}\dot{\theta}[d^2 + \frac{1}{12}(h^2 + l^2)] + m_\kappa[y^2\dot{\theta} + d(\dot{y} + d\dot{\theta})] \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} \underrightarrow{h}^{\mathcal{S}w/a\,\bullet a} &= \underrightarrow{h}^{\mathcal{S}w/a\,\bullet b} + \underbrace{\underrightarrow{\omega}^{ba} \times \underrightarrow{h}^{\mathcal{S}w/a}}_{=\underrightarrow{0}} \\ &= \underrightarrow{\mathcal{F}}_b^{\mathsf{T}}\begin{bmatrix} m_{\mathcal{B}}\ddot{\theta}[d^2 + \frac{1}{12}(h^2 + l^2)] + m_\kappa[2y\dot{y}\dot{\theta} + y^2\ddot{\theta} + d(\ddot{y} + d\ddot{\theta})] \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

And finally, applying N2LR, i.e. $\underset{\rightarrow}{h}^{\mathcal{S}w/a^{\bullet a}} = \underset{\rightarrow}{m}^{\mathcal{S}w}$:

$$m_{\mathcal{B}}\ddot{\theta}[d^2 + \frac{1}{12}(h^2 + l^2)] + m_\kappa[2y\dot{y}\dot{\theta} + y^2\ddot{\theta} + d(\ddot{y} + d\ddot{\theta})] \\ = -m_{\mathcal{B}}dg\sin(\theta) + m_\kappa[-yg\cos(\theta) - dg\sin(\theta)] \qquad (5)$$

We can easily check that (5) is nothing else than $d{\cdot}(3)$ + $y{\cdot}(4)$, and thus the two sets of DEs describe the same motion.

3

a)

$$\begin{aligned}
\underset{\rightarrow}{\omega}^{sa} &= \underset{\rightarrow}{\omega}^{sb} + \underset{\rightarrow}{\omega}^{bq} + \underset{\rightarrow}{\omega}^{qa} \\
&= \underset{\rightarrow}{\mathcal{F}}_s^{\mathsf{T}}\boldsymbol{\omega}_s^{sb} + \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}}\boldsymbol{\omega}_b^{bq} + \underset{\rightarrow}{\mathcal{F}}_q^{\mathsf{T}}\boldsymbol{\omega}_q^{qa} \\
&= \underset{\rightarrow}{\mathcal{F}}_s^{\mathsf{T}}\boldsymbol{\omega}_s^{sb} + \underset{\rightarrow}{\mathcal{F}}_s^{\mathsf{T}}\mathbf{C}_{sb}\boldsymbol{\omega}_b^{bq} + \underset{\rightarrow}{\mathcal{F}}_s^{\mathsf{T}}\mathbf{C}_{sq}\boldsymbol{\omega}_q^{qa} \\
&= \underset{\rightarrow}{\mathcal{F}}_s^{\mathsf{T}}\boldsymbol{\omega}_s^{sb} + \underset{\rightarrow}{\mathcal{F}}_s^{\mathsf{T}}\mathbf{C}_3(\gamma)\boldsymbol{\omega}_b^{bq} + \underset{\rightarrow}{\mathcal{F}}_s^{\mathsf{T}}\mathbf{C}_3(\gamma)\mathbf{C}_2(\phi)\boldsymbol{\omega}_q^{qa} \\
&= \underset{\rightarrow}{\mathcal{F}}_s^{\mathsf{T}}\left(\mathbf{1}_3\dot{\gamma} + \mathbf{C}_3(\gamma)\mathbf{1}_2\dot{\phi} + \mathbf{C}_3(\gamma)\mathbf{C}_2(\phi)\mathbf{1}_3\dot{\theta}\right) \\
&= \underset{\rightarrow}{\mathcal{F}}_s^{\mathsf{T}}\underbrace{\begin{bmatrix} \mathbf{C}_3(\gamma)\mathbf{C}_2(\phi)\mathbf{1}_3 & \mathbf{C}_3(\gamma)\mathbf{1}_2 & \mathbf{1}_3 \end{bmatrix}\begin{bmatrix} \dot{\theta} \\ \dot{\phi} \\ \dot{\gamma} \end{bmatrix}}_{\boldsymbol{\omega}_s^{sa}}. \quad \square
\end{aligned}$$

b)

$$\begin{aligned}
\underset{\rightarrow}{r}^{\mathrm{d}mw^{\bullet a}} &= \left(\underset{\rightarrow}{r}^{\mathrm{d}mc} + \underset{\rightarrow}{r}^{cw}\right)^{\bullet a} \\
&= \left(\underset{\rightarrow}{r}^{\mathrm{d}mc} + \underset{\rightarrow}{r}^{cw}\right)^{\bullet s} + \underset{\rightarrow}{\omega}^{sa} \times \left(\underset{\rightarrow}{r}^{\mathrm{d}mc} + \underset{\rightarrow}{r}^{cw}\right) \\
&= \underset{\rightarrow}{\mathcal{F}}_s^{\mathsf{T}}\left(\underbrace{\dot{\mathbf{r}}_s^{\mathrm{d}mc}}_{=\mathbf{0}} + \underbrace{\dot{\mathbf{r}}_s^{cw}}_{=\mathbf{0}}\right) + \underset{\rightarrow}{\omega}^{sa} \times \left(\underset{\rightarrow}{r}^{\mathrm{d}mc} + \underset{\rightarrow}{r}^{cw}\right) \\
&= \underset{\rightarrow}{\omega}^{sa} \times \left(\underset{\rightarrow}{r}^{\mathrm{d}mc} + \underset{\rightarrow}{r}^{cw}\right). \quad \square
\end{aligned}$$

Where we've used the facts that the material element $\mathrm{d}m$ is fixed in the body frame $\mathcal{F}_s$ and that $\dot{\mathbf{r}}_s^{cw} = \dot{l}\mathbf{1}_3 = \mathbf{0}$.

c)

$$
\begin{aligned}
\underset{\rightarrow}{h}^{\mathcal{B}w/a} &= \int_{\mathcal{B}} \underset{\rightarrow}{r}^{\mathrm{d}mw} \times \underset{\rightarrow}{r}^{\mathrm{d}mw^{\bullet a}} \,\mathrm{d}m \\
&= \int_{\mathcal{B}} \underset{\rightarrow}{r}^{\mathrm{d}mw} \times \left[\underset{\rightarrow}{\omega}^{sa} \times \left(\underset{\rightarrow}{r}^{\mathrm{d}mc} + \underset{\rightarrow}{r}^{cw}\right)\right] \mathrm{d}m \\
&= \int_{\mathcal{B}} \left(\underset{\rightarrow}{r}^{\mathrm{d}mc} + \underset{\rightarrow}{r}^{cw}\right) \times \left[\underset{\rightarrow}{\omega}^{sa} \times \underset{\rightarrow}{r}^{\mathrm{d}mc} + \underset{\rightarrow}{\omega}^{sa} \times \underset{\rightarrow}{r}^{cw}\right] \mathrm{d}m \\
&= \int_{\mathcal{B}} \left(\underset{\rightarrow}{r}^{\mathrm{d}mc} + \underset{\rightarrow}{r}^{cw}\right) \times \left[-\underset{\rightarrow}{r}^{\mathrm{d}mc} \times \underset{\rightarrow}{\omega}^{sa} - \underset{\rightarrow}{r}^{cw} \times \underset{\rightarrow}{\omega}^{sa}\right] \mathrm{d}m \\
&= \int_{\mathcal{B}} -\underset{\rightarrow}{r}^{\mathrm{d}mc} \times (\underset{\rightarrow}{r}^{\mathrm{d}mc} \times \underset{\rightarrow}{\omega}^{sa}) - \underset{\rightarrow}{r}^{\mathrm{d}mc} \times (\underset{\rightarrow}{r}^{cw} \times \underset{\rightarrow}{\omega}^{sa}) \\
&\quad - \underset{\rightarrow}{r}^{cw} \times (\underset{\rightarrow}{r}^{\mathrm{d}mc} \times \underset{\rightarrow}{\omega}^{sa}) - \underset{\rightarrow}{r}^{cw} \times (\underset{\rightarrow}{r}^{cw} \times \underset{\rightarrow}{\omega}^{sa}) \,\mathrm{d}m \\
&= \int_{\mathcal{B}} -\underset{\rightarrow}{r}^{\mathrm{d}mc} \times (\underset{\rightarrow}{r}^{\mathrm{d}mc} \times \underset{\rightarrow}{\omega}^{sa}) + (\underset{\rightarrow}{r}^{cw} \times \underset{\rightarrow}{\omega}^{sa}) \times \underset{\rightarrow}{r}^{\mathrm{d}mc} \\
&\quad + \underset{\rightarrow}{r}^{cw} \times (\underset{\rightarrow}{\omega}^{sa} \times \underset{\rightarrow}{r}^{\mathrm{d}mc}) - \underset{\rightarrow}{r}^{cw} \times (\underset{\rightarrow}{r}^{cw} \times \underset{\rightarrow}{\omega}^{sa}) \,\mathrm{d}m \\
&= \underset{\rightarrow}{\mathcal{F}}_s^{\mathsf{T}} \left\{ \int_{\mathcal{B}} -\mathbf{r}_s^{\mathrm{d}mc\times} \mathbf{r}_s^{\mathrm{d}mc\times} \boldsymbol{\omega}_s^{sa} + (\mathbf{r}_s^{cw\times} \boldsymbol{\omega}_s^{sa})^{\times} \mathbf{r}_s^{\mathrm{d}mc} + \mathbf{r}_s^{cw\times} \boldsymbol{\omega}_s^{sa\times} \mathbf{r}_s^{\mathrm{d}mc} - \mathbf{r}_s^{cw\times} \mathbf{r}_s^{cw\times} \boldsymbol{\omega}_s^{sa} \,\mathrm{d}m \right\} \\
&= \underset{\rightarrow}{\mathcal{F}}_s^{\mathsf{T}} \left\{ \underbrace{\left[\int_{\mathcal{B}} -\mathbf{r}_s^{\mathrm{d}mc\times} \mathbf{r}_s^{\mathrm{d}mc\times} \,\mathrm{d}m\right]}_{\mathbf{J}_s^{\mathcal{B}c}} \boldsymbol{\omega}_s^{sa} + \left[(\mathbf{r}_s^{cw\times} \boldsymbol{\omega}_s^{sa})^{\times} + \mathbf{r}_s^{cw\times} \boldsymbol{\omega}_s^{sa\times}\right] \underbrace{\int_{\mathcal{B}} \mathbf{r}_s^{\mathrm{d}mc} \,\mathrm{d}m}_{\mathbf{0}} \right. \\
&\quad \left. + \underbrace{\overbrace{\int_{\mathcal{B}} \mathrm{d}m}^{m_{\mathcal{B}}} \left[-\mathbf{r}_s^{cw\times} \mathbf{r}_s^{cw\times}\right]}_{\mathbf{J}_s^{m_{\mathcal{B}}w}} \boldsymbol{\omega}_s^{sa} \right\} \\
&= \underset{\rightarrow}{\mathcal{F}}_s^{\mathsf{T}} \left\{ (\mathbf{J}_s^{\mathcal{B}c} + \mathbf{J}_s^{m_{\mathcal{B}}w}) \boldsymbol{\omega}_s^{sa} \right\} \\
&= (\underset{\rightarrow}{J}^{\mathcal{B}c} + \underset{\rightarrow}{J}^{m_{\mathcal{B}}w}) \cdot \underset{\rightarrow}{\omega}^{sa} \\
&= \underset{\rightarrow}{J}^{\mathcal{B}w} \cdot \underset{\rightarrow}{\omega}^{sa}. \quad \square
\end{aligned}
$$

e) From ERL,

$$
\underset{\rightarrow}{h}^{\mathcal{B}w/a^{\bullet a}} = \underset{\rightarrow}{m}^{\mathcal{B}w}. \tag{6}
$$

Developing on both sides:

$$\begin{aligned}
\underset{\rightarrow}{h}^{\mathcal{B}w/a\ \bullet a} &= \underset{\rightarrow}{h}^{\mathcal{B}w/a\ \bullet s} + \underset{\rightarrow}{\omega}^{sa} \times \underset{\rightarrow}{h}^{\mathcal{B}w/a} \\
&= \left\{ \underset{\rightarrow}{\mathcal{F}}_s^{\mathsf{T}} (\mathbf{J}_s^{\mathcal{B}c} + \mathbf{J}_s^{m_{\mathcal{B}w}}) \boldsymbol{\omega}_s^{sa} \right\}^{\bullet s} + \underset{\rightarrow}{\mathcal{F}}_b^{\mathsf{T}} \boldsymbol{\omega}_s^{sa\times} (\mathbf{J}_s^{\mathcal{B}c} + \mathbf{J}_s^{m_{\mathcal{B}w}}) \boldsymbol{\omega}_s^{sa} \\
&= \underset{\rightarrow}{\mathcal{F}}_s^{\mathsf{T}} \left\{ \underbrace{(\dot{\mathbf{J}}_s^{\mathcal{B}c} + \dot{\mathbf{J}}_s^{m_{\mathcal{B}w}})}_{=\mathbf{0} \text{ (Body frame)}} \boldsymbol{\omega}_s^{sa} + (\mathbf{J}_s^{\mathcal{B}c} + \mathbf{J}_s^{m_{\mathcal{B}w}}) \dot{\boldsymbol{\omega}}_s^{sa} + \boldsymbol{\omega}_s^{sa\times} (\mathbf{J}_s^{\mathcal{B}c} + \mathbf{J}_s^{m_{\mathcal{B}w}}) \boldsymbol{\omega}_s^{sa} \right\} \\
&= \underset{\rightarrow}{\mathcal{F}}_s^{\mathsf{T}} \left\{ (\mathbf{J}_s^{\mathcal{B}c} + \mathbf{J}_s^{m_{\mathcal{B}w}}) \dot{\boldsymbol{\omega}}_s^{sa} + \boldsymbol{\omega}_s^{sa\times} (\mathbf{J}_s^{\mathcal{B}c} + \mathbf{J}_s^{m_{\mathcal{B}w}}) \boldsymbol{\omega}_s^{sa} \right\} \overset{(6)}{=} \underset{\rightarrow}{\mathcal{F}}_s^{\mathsf{T}} \mathbf{m}_s^{\mathcal{B}w}. \quad \square
\end{aligned}$$

f) We have previously shown the following:

$$\begin{aligned}
\boldsymbol{\omega}_s^{sa} &= \mathbf{S}_s^{sa} \dot{\boldsymbol{\theta}}^{sa}, \\
(\mathbf{J}_s^{\mathcal{B}c} + \mathbf{J}_s^{m_{\mathcal{B}w}}) \dot{\boldsymbol{\omega}}_s^{sa} + \boldsymbol{\omega}_s^{sa\times} (\mathbf{J}_s^{\mathcal{B}c} + \mathbf{J}_s^{m_{\mathcal{B}w}}) \boldsymbol{\omega}_s^{sa} &= \mathbf{m}_s^{\mathcal{B}w}.
\end{aligned}$$

Rearranging these equations:

$$\begin{aligned}
\dot{\boldsymbol{\theta}}^{sa} &= (\mathbf{S}_s^{sa})^{-1} \boldsymbol{\omega}_s^{sa}, \\
\dot{\boldsymbol{\omega}}_s^{sa} &= (\mathbf{J}_s^{\mathcal{B}c} + \mathbf{J}_s^{m_{\mathcal{B}w}})^{-1} \left(\mathbf{m}_s^{\mathcal{B}w} - \boldsymbol{\omega}_s^{sa\times} (\mathbf{J}_s^{\mathcal{B}c} + \mathbf{J}_s^{m_{\mathcal{B}w}}) \boldsymbol{\omega}_s^{sa} \right).
\end{aligned}$$

Using the following notation,

$$\mathbf{x} = \begin{bmatrix} \boldsymbol{\theta}^{sa} \\ \boldsymbol{\omega}_s^{sa} \end{bmatrix}, \qquad \dot{\mathbf{x}} = \begin{bmatrix} \dot{\boldsymbol{\theta}}^{sa} \\ \dot{\boldsymbol{\omega}}_s^{sa} \end{bmatrix},$$

we can write a first-order state-space system of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}).$$

Using MATLAB, we obtain the following plots (the code is given at the end of the assignment):

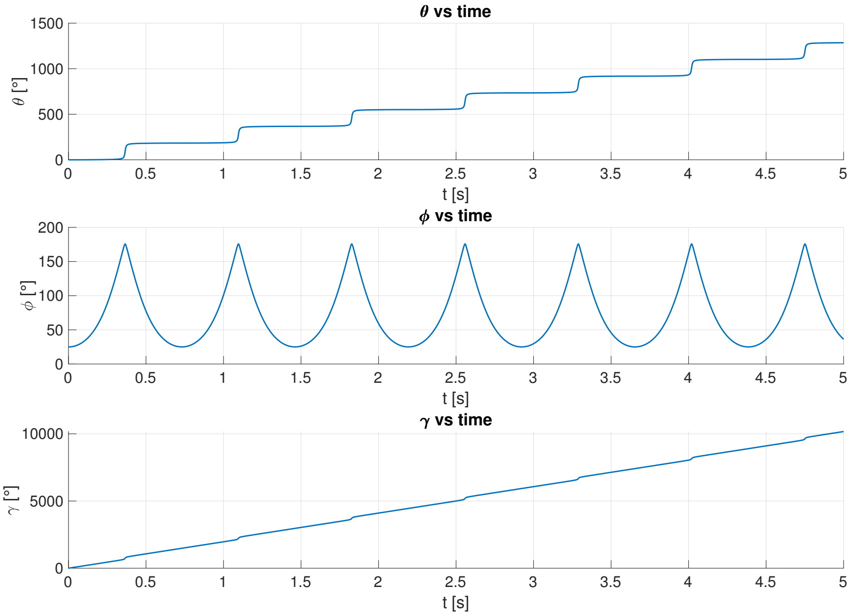

Figure 3: θ, ϕ and γ vs. time.

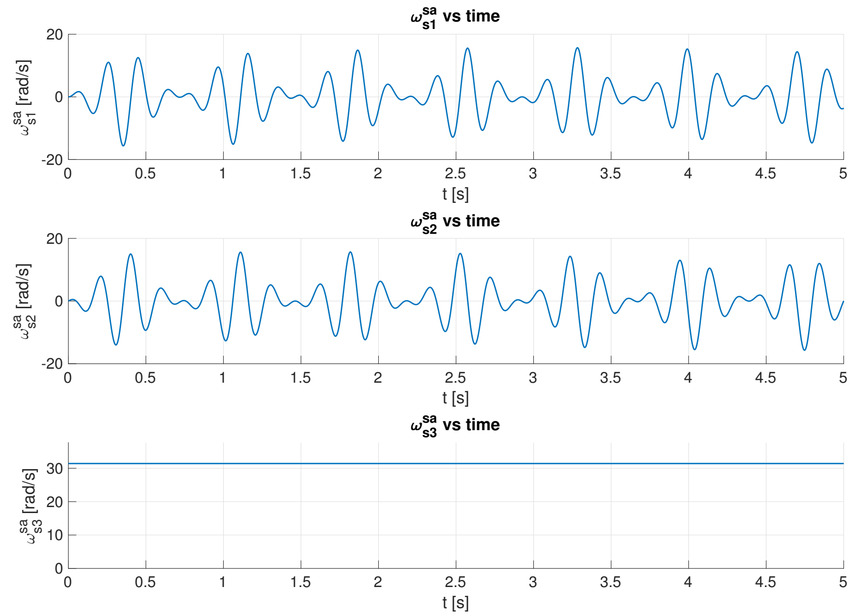

Figure 4: ω_{s1}^{sa}, ω_{s2}^{sa} and ω_{s3}^{sa} vs. time.

g) First,

$$
\begin{aligned}
T_{\mathcal{B}w/a} &= \tfrac{1}{2}\boldsymbol{\omega}_s^{sa\mathsf{T}}\mathbf{J}_s^{\mathcal{B}w}\boldsymbol{\omega}_s^{sa},\\
U_{\mathcal{B}w} &= -\int_{\mathcal{B}} \underset{\rightarrow}{g}\cdot\underset{\rightarrow}{r}^{\,\mathrm{d}mw}\,\mathrm{d}m\\
&= -\underset{\rightarrow}{g}\cdot\underbrace{\left(\int_{\mathcal{B}}\underset{\rightarrow}{r}^{\,\mathrm{d}mw}\,\mathrm{d}m\right)}_{\underset{\rightarrow}{r}^{\,cw}m_{\mathcal{B}}}\\
&= \left(\mathbf{C}_3(\gamma)\mathbf{C}_2(\phi)\mathbf{C}_3(\theta)\mathbf{1}_3 g\right)^{\mathsf{T}}\underset{\rightarrow}{\mathcal{F}}_s\cdot\underset{\rightarrow}{\mathcal{F}}_s^{\mathsf{T}}\mathbf{r}_s^{cw}m_{\mathcal{B}}\\
&= m_{\mathcal{B}}gl\left(\mathbf{C}_3(\gamma)\mathbf{C}_2(\phi)\mathbf{C}_3(\theta)\mathbf{1}_3\right)^{\mathsf{T}}\mathbf{1}_3.
\end{aligned}
$$

We can now use MATLAB to obtain the plot shown in Figure 5. As expected, $E_{\mathcal{B}w/a}$ is constant with time, which is a good indicator of an accurate simulation.

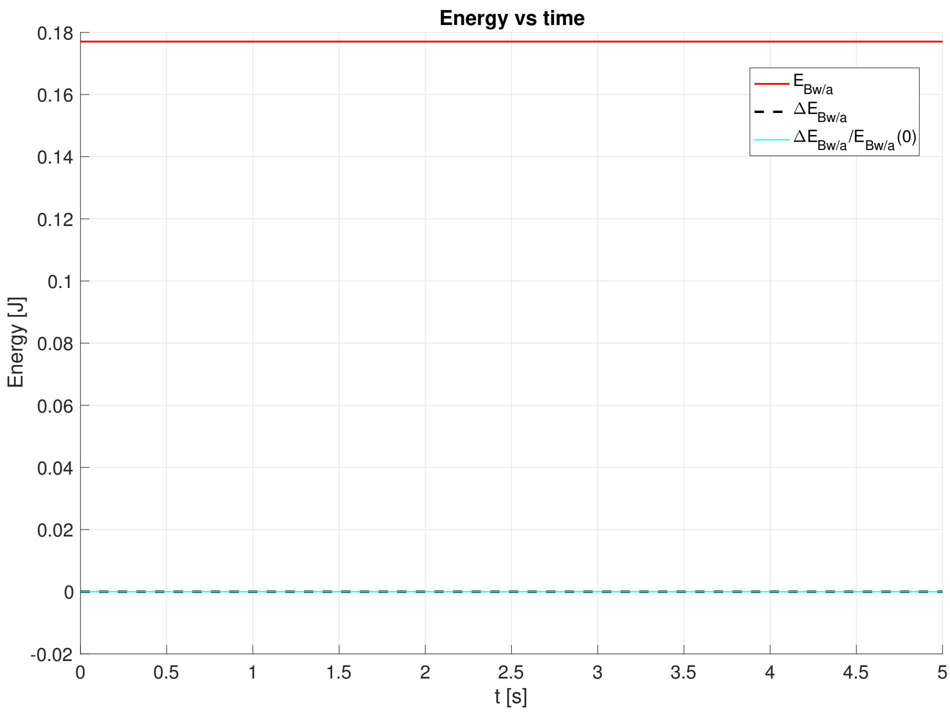

Figure 5: Energy vs. time.