

Assignment 4 - Problem 2 & 3

Frédéric Berdoz
260867318

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a) Let θ be the angle between \underline{b}^2 and \underline{a}^2 . Therefore

$$\mathbf{C}_{ba} = \mathbf{C}_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Moreover,

$$\underline{\omega}^{ba} = \underline{\mathcal{F}}_b^\top \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{aligned} \underline{r}^{\kappa w} &= \underline{r}^{\kappa c} + \underline{r}^{cw} \\ &= \underline{\mathcal{F}}_b^\top \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} + \underline{\mathcal{F}}_b^\top \begin{bmatrix} 0 \\ 0 \\ -d \end{bmatrix} \\ &= \underline{\mathcal{F}}_b^\top \begin{bmatrix} 0 \\ y \\ -d \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \underline{r}^{dmw} &= \underline{r}^{dmc} + \underline{r}^{cw} \\ &= \underline{\mathcal{F}}_b^\top \begin{bmatrix} \rho_{b1} \\ \rho_{b2} \\ \rho_{b3} \end{bmatrix} + \underline{\mathcal{F}}_b^\top \begin{bmatrix} 0 \\ 0 \\ -d \end{bmatrix} \\ &= \underline{\mathcal{F}}_b^\top \begin{bmatrix} \rho_{b1} \\ \rho_{b2} \\ \rho_{b3} - d \end{bmatrix}, \end{aligned}$$

$$\begin{aligned}
 \underline{r}^{\kappa w \bullet a} &= \underline{r}^{\kappa w \bullet b} + \underline{\omega}^{ba} \times \underline{r}^{\kappa w} \\
 &= \underline{\mathcal{F}}_b^T \left(\dot{\mathbf{r}}_b^{\kappa w} + \underline{\omega}_b^{ba \times} \mathbf{r}_b^{\kappa w} \right) \\
 &= \underline{\mathcal{F}}_b^T \left(\begin{bmatrix} 0 \\ \dot{y} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta} \\ 0 & \dot{\theta} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ y \\ -d \end{bmatrix} \right) \\
 &= \underline{\mathcal{F}}_b^T \begin{bmatrix} 0 \\ \dot{y} + d\dot{\theta} \\ y\dot{\theta} \end{bmatrix},
 \end{aligned}$$

$$\begin{aligned}
 \underline{r}^{dmw \bullet a} &= \underline{r}^{dmw \bullet b} + \underline{\omega}^{ba} \times \underline{r}^{dmw} \\
 &= \underline{\mathcal{F}}_b^T \left(\underbrace{\dot{\mathbf{r}}_b^{dmw}}_{=0} + \underline{\omega}_b^{ba \times} \mathbf{r}_b^{dmw} \right) \\
 &= \underline{\mathcal{F}}_b^T \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta} \\ 0 & \dot{\theta} & 0 \end{bmatrix} \begin{bmatrix} \rho_{b1} \\ \rho_{b2} \\ \rho_{b3} - d \end{bmatrix} \right) \\
 &= \underline{\mathcal{F}}_b^T \begin{bmatrix} 0 \\ (d - \rho_{b3})\dot{\theta} \\ \rho_{b2}\dot{\theta} \end{bmatrix}.
 \end{aligned}$$

b) Figures 1 and 2 show the Free Body Diagrams of the particle κ and the body \mathcal{B} . Let \underline{f}^{r1} and \underline{f}^{r1} be the reaction forces applied by the massless bars on the body \mathcal{B} .

The norm of $\underline{r}^{\kappa w}$ is

$$\|\underline{r}^{\kappa w}\|_2 = \mathbf{r}_b^{\kappa w T} \mathbf{r}_b^{\kappa w} = y^2 + d^2.$$

Therefore, the total force applied on κ is given by:

$$\begin{aligned}
 \underline{f}^{\kappa} &= \underline{f}^{\kappa \mathcal{B}} + \underline{f}^{\kappa g} + \underline{f}^{\kappa s} \\
 &= \underline{\mathcal{F}}_b^T \left(\begin{bmatrix} f_{b1}^{\kappa \mathcal{B}} \\ 0 \\ f_{b3}^{\kappa \mathcal{B}} \end{bmatrix} + \underline{\mathcal{F}}_b^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -m_{\kappa}g \end{bmatrix} - k \frac{l_s - \bar{l}_s}{y^2 + d^2} \begin{bmatrix} 0 \\ y \\ -d \end{bmatrix} \right) \\
 &= \underline{\mathcal{F}}_b^T \begin{bmatrix} f_{b1}^{\kappa \mathcal{B}} \\ -m_{\kappa}g \sin(\theta) - ky \frac{l_s - \bar{l}_s}{y^2 + d^2} \\ f_{b3}^{\kappa \mathcal{B}} - m_{\kappa}g \cos(\theta) + kd \frac{l_s - \bar{l}_s}{y^2 + d^2} \end{bmatrix}.
 \end{aligned}$$

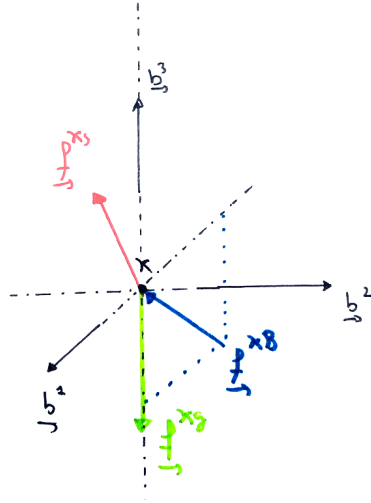


Figure 1: FBD of κ .

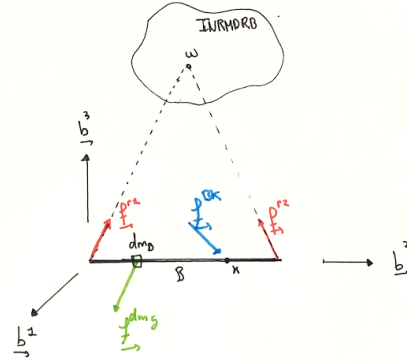


Figure 2: FBD of \mathcal{B} .

The total force acting on \mathcal{B} is given by:

$$\begin{aligned} \vec{f}^{\mathcal{B}} &= \vec{f}^{r1} + \vec{f}^{r2} + \vec{f}^{\mathcal{B}\kappa} + \int_{\mathcal{B}} d\vec{f}^{dmg} \\ &= \mathcal{F}_{\vec{b}}^T \left(\begin{bmatrix} f_{b1}^{r1} \\ f_{b2}^{r1} \\ f_{b3}^{r1} \end{bmatrix} + \begin{bmatrix} f_{b1}^{r2} \\ f_{b2}^{r2} \\ f_{b3}^{r2} \end{bmatrix} - \begin{bmatrix} f_{b1}^{\kappa\mathcal{B}} \\ 0 \\ f_{b3}^{\kappa\mathcal{B}} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix} \int_{\mathcal{B}} \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} dm \right) \\ &= \mathcal{F}_{\vec{b}}^T \begin{bmatrix} f_{b1}^{r1} + f_{b1}^{r2} - f_{b1}^{\kappa\mathcal{B}} \\ f_{b2}^{r1} + f_{b2}^{r2} - m_{\mathcal{B}}g \sin(\theta) \\ f_{b3}^{r1} + f_{b3}^{r2} - f_{b3}^{\kappa\mathcal{B}} - m_{\mathcal{B}}g \cos(\theta) \end{bmatrix}. \end{aligned}$$

Finally, the total moment applied on \mathcal{B} relative to w is given by:

$$\begin{aligned} \vec{m}^{\mathcal{B}w} &= \underbrace{\vec{r}^{r1w} \times \vec{f}^{r1}}_{=0} + \underbrace{\vec{r}^{r2w} \times \vec{f}^{r2}}_{=0} + \vec{r}^{\kappa w} \times \vec{f}^{\mathcal{B}\kappa} + \int_{\mathcal{B}} \vec{r}^{dmw} \times d\vec{f}^{dmg} \\ &= \vec{r}^{\kappa w} \times \vec{f}^{\mathcal{B}\kappa} + \underbrace{\left(\int_{\mathcal{B}} \vec{r}^{dmw} dm \right)}_{=m_{\mathcal{B}} \vec{r}_{\vec{w}}^{cw}} \times \vec{g} \\ &= \mathcal{F}_{\vec{b}}^T \left(\begin{bmatrix} 0 \\ y \\ -d \end{bmatrix} \times \begin{bmatrix} -f_{b1}^{\kappa\mathcal{B}} \\ 0 \\ -f_{b3}^{\kappa\mathcal{B}} \end{bmatrix} + m_{\mathcal{B}} \begin{bmatrix} 0 \\ 0 \\ -d \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} \right) \\ &= \mathcal{F}_{\vec{b}}^T \begin{bmatrix} -yf_{b3}^{\kappa\mathcal{B}} - dgm_{\mathcal{B}} \sin(\theta) \\ df_{b1}^{\kappa\mathcal{B}} \\ yf_{b1}^{\kappa\mathcal{B}} \end{bmatrix} \end{aligned}$$

c)

$$\underline{p}_{\rightarrow}^{\kappa w/a} = m_{\kappa} \underline{r}_{\rightarrow}^{\kappa w/a} \cdot^a = m_{\kappa} \underline{\mathcal{F}}_b^{\top} \begin{bmatrix} 0 \\ \dot{y} + d\dot{\theta} \\ y\dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ m_{\kappa}(\dot{y} + d\dot{\theta}) \\ m_{\kappa}y\dot{\theta} \end{bmatrix}. \quad \square$$

$$\begin{aligned} \underline{h}_{\rightarrow}^{Bw/a} &= \int_B \underline{r}_{\rightarrow}^{dmw} \times \underline{r}_{\rightarrow}^{dmw} \cdot^a dm \\ &= \underline{\mathcal{F}}_b^{\top} \left(\int_B \begin{bmatrix} \rho_{b1} \\ \rho_{b2} \\ \rho_{b3} - d \end{bmatrix} \times \begin{bmatrix} 0 \\ (d - \rho_{b3})\dot{\theta} \\ \rho_{b2}\dot{\theta} \end{bmatrix} dm \right) \\ &= \underline{\mathcal{F}}_b^{\top} \left(\int_B \begin{bmatrix} 0 & -(\rho_{b3} - d) & \rho_{b2} \\ (\rho_{b3} - d) & 0 & -\rho_{b1} \\ -\rho_{b2} & \rho_{b1} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ (d - \rho_{b3})\dot{\theta} \\ \rho_{b2}\dot{\theta} \end{bmatrix} dm \right) \\ &= \underline{\mathcal{F}}_b^{\top} \left(\dot{\theta} \int_B \begin{bmatrix} (d - \rho_{b3})^2 + \rho_{b2}^2 \\ -\rho_{b1}\rho_{b2} \\ \rho_{b1}(d - \rho_{b3}) \end{bmatrix} dm \right) \\ &= \underline{\mathcal{F}}_b^{\top} \left(\dot{\theta} \int_{V_B} \begin{bmatrix} (d - \rho_{b3})^2 + \rho_{b2}^2 \\ -\rho_{b1}\rho_{b2} \\ \rho_{b1}(d - \rho_{b3}) \end{bmatrix} \left(\frac{m_B}{lth} \right) dV \right) \\ &= \underline{\mathcal{F}}_b^{\top} \left(\frac{m_B \dot{\theta}}{lth} \begin{bmatrix} lt \left[\int_{-h/2}^{h/2} (d - \rho_{b3})^2 d\rho_{b3} \right] + th \left[\int_{-l/2}^{l/2} \rho_{b2}^2 d\rho_{b2} \right] \\ -lth^2 \left[\underbrace{\int_{-t/2}^{t/2} \rho_{b1} d\rho_{b1}}_0 \right] \left[\underbrace{\int_{-l/2}^{l/2} \rho_{b2} d\rho_{b2}}_0 \right] \\ l^2 ht \left[\underbrace{\int_{-t/2}^{t/2} \rho_{b1} d\rho_{b1}}_0 \right] \left[\int_{-h/2}^{h/2} (d - \rho_{b3}) d\rho_{b3} \right] \end{bmatrix} \right) \\ &= \underline{\mathcal{F}}_b^{\top} \left(\frac{m_B \dot{\theta}}{lth} \begin{bmatrix} -\frac{lt}{3} [(d - \rho_{b3})^3]_{-h/2}^{h/2} + \frac{th}{3} [\rho_{b2}^3]_{-l/2}^{l/2} \\ 0 \\ 0 \end{bmatrix} \right) \\ &= \underline{\mathcal{F}}_b^{\top} \left(\frac{m_B \dot{\theta}}{lth} \begin{bmatrix} -\frac{lt}{3} [-3d^2h - \frac{h^3}{4}] + \frac{th}{3} \frac{l^3}{4} \\ 0 \\ 0 \end{bmatrix} \right) \\ &= \underline{\mathcal{F}}_b^{\top} \left(m_B \dot{\theta} \begin{bmatrix} d^2 + \frac{1}{12}(h^2 + l^2) \\ 0 \\ 0 \end{bmatrix} \right). \quad \square \end{aligned}$$

d) First, from N2L,

$$\underline{p}_{\rightarrow}^{\kappa w/a \cdot a} = \underline{f}_{\rightarrow}^{\kappa}.$$

Using the Transport Theorem and developing:

$$\begin{aligned} \underline{p}_{\rightarrow}^{\kappa w/a \cdot a} &= \underline{p}_{\rightarrow}^{\kappa w/a \cdot b} + \underline{\omega}^{ba} \times \underline{p}_{\rightarrow}^{\kappa w/a} \\ &= \underline{\mathcal{F}}_{\rightarrow b}^T \left(\begin{bmatrix} 0 \\ m_{\kappa}(\ddot{y} + d\ddot{\theta}) \\ m_{\kappa}(\dot{y}\dot{\theta} + y\ddot{\theta}) \end{bmatrix} + \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ m_{\kappa}(\dot{y} + d\dot{\theta}) \\ m_{\kappa}y\dot{\theta} \end{bmatrix} \right) \\ &= \underline{\mathcal{F}}_{\rightarrow b}^T \begin{bmatrix} 0 \\ m_{\kappa}(\ddot{y} + d\ddot{\theta}) - m_{\kappa}y\dot{\theta}^2 \\ m_{\kappa}(\dot{y}\dot{\theta} + y\ddot{\theta}) + \dot{\theta}m_{\kappa}(\dot{y} + d\dot{\theta}) \end{bmatrix} \\ &= \underline{\mathcal{F}}_{\rightarrow b}^T \begin{bmatrix} 0 \\ m_{\kappa}(\ddot{y} + d\ddot{\theta}) - m_{\kappa}y\dot{\theta}^2 \\ 2m_{\kappa}\dot{y}\dot{\theta} + m_{\kappa}y\ddot{\theta} + m_{\kappa}d\dot{\theta}^2 \end{bmatrix} \end{aligned}$$

Therefore,

$$\begin{bmatrix} 0 \\ m_{\kappa}(\ddot{y} + d\ddot{\theta}) - m_{\kappa}y\dot{\theta}^2 \\ 2m_{\kappa}\dot{y}\dot{\theta} + m_{\kappa}y\ddot{\theta} + m_{\kappa}d\dot{\theta}^2 \end{bmatrix} = \begin{bmatrix} f_{b1}^{\kappa B} \\ -m_{\kappa}g \sin(\theta) - ky \frac{l_s - \bar{l}_s}{y^2 + d^2} \\ f_{b3}^{\kappa B} - m_{\kappa}g \cos(\theta) + kd \frac{l_s - \bar{l}_s}{y^2 + d^2} \end{bmatrix}. \quad (1)$$

We can see straight away that

$$f_{b1}^{\kappa B} = 0.$$

Secondly,

$$\begin{aligned} \underline{h}_{\rightarrow}^{\mathcal{B}w/a \cdot a} &= \underline{h}_{\rightarrow}^{\mathcal{B}w/a \cdot b} + \underline{\omega}^{ba} \times \underline{h}_{\rightarrow}^{\mathcal{B}w/a} \\ &= \underline{\mathcal{F}}_{\rightarrow b}^T \left(\begin{bmatrix} m_{\mathcal{B}}\ddot{\theta}[d^2 + \frac{1}{12}(h^2 + l^2)] \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} m_{\mathcal{B}}\dot{\theta}[d^2 + \frac{1}{12}(h^2 + l^2)] \\ 0 \\ 0 \end{bmatrix} \right) \\ &= \underline{\mathcal{F}}_{\rightarrow b}^T \begin{bmatrix} m_{\mathcal{B}}\ddot{\theta}[d^2 + \frac{1}{12}(h^2 + l^2)] \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

From N2LR,

$$\underline{h}_{\rightarrow}^{\mathcal{B}w/a \cdot a} = \underline{m}_{\rightarrow}^{\mathcal{B}w}.$$

Therefore,

$$\underline{\mathcal{F}}_{\rightarrow b}^T \begin{bmatrix} m_{\mathcal{B}}\ddot{\theta}[d^2 + \frac{1}{12}(h^2 + l^2)] \\ 0 \\ 0 \end{bmatrix} = \underline{\mathcal{F}}_{\rightarrow b}^T \begin{bmatrix} -yf_{b3}^{\kappa B} - dgm_{\mathcal{B}}\sin(\theta) \\ df_{b1}^{\kappa B} \\ yf_{b1}^{\kappa B} \end{bmatrix}$$

In particular,

$$f_{b3}^{\kappa B} = -\frac{m_B}{y} \left(\ddot{\theta} [d^2 + \frac{1}{12}(h^2 + l^2)] + dg \sin(\theta) \right) \quad (2)$$

Finally, substituting (2) into (1),

$$m_\kappa(\ddot{y} + d\ddot{\theta}) - m_\kappa y \dot{\theta}^2 = -m_\kappa g \sin(\theta) - ky \frac{l_s - \bar{l}_s}{y^2 + d^2} \quad (3)$$

$$\begin{aligned} 2m_\kappa \dot{y} \dot{\theta} + m_\kappa y \ddot{\theta} + m_\kappa d \dot{\theta}^2 &= -\frac{m_B}{y} \left(\ddot{\theta} [d^2 + \frac{1}{12}(h^2 + l^2)] + dg \sin(\theta) \right) \\ &\quad - m_\kappa g \cos(\theta) + kd \frac{l_s - \bar{l}_s}{y^2 + d^2} \end{aligned} \quad (4)$$

e) The total moment applied on \mathcal{S} relative to w is given by:

$$\begin{aligned} \underline{M}_{\rightarrow}^{Sw} &= m_B \underline{r}_{\rightarrow}^{cw} \times \underline{g}_{\rightarrow} + m_\kappa \underline{r}_{\rightarrow}^{\kappa w} \times \underline{g}_{\rightarrow} \\ &= \underline{\mathcal{F}}_{\rightarrow b}^T \left(m_B \begin{bmatrix} 0 \\ 0 \\ -d \end{bmatrix} \times \begin{bmatrix} 0 \\ -\sin(\theta)g \\ -\cos(\theta)g \end{bmatrix} + m_\kappa \begin{bmatrix} 0 \\ y \\ -d \end{bmatrix} \times \begin{bmatrix} 0 \\ -\sin(\theta)g \\ -\cos(\theta)g \end{bmatrix} \right) \\ &= \underline{\mathcal{F}}_{\rightarrow b}^T \begin{bmatrix} -m_B dg \sin(\theta) + m_\kappa [-yg \cos(\theta) - dg \sin(\theta)] \\ 0 \\ 0 \end{bmatrix}. \quad \square \end{aligned}$$

Secondly,

$$\begin{aligned} \underline{h}_{\rightarrow}^{Sw/a} &= \underline{h}_{\rightarrow}^{Bw/a} + \underline{r}_{\rightarrow}^{\kappa w} \times \underline{p}_{\rightarrow}^{\kappa w/a} \\ &= \underline{\mathcal{F}}_{\rightarrow b}^T \left(\begin{bmatrix} m_B \dot{\theta} [d^2 + \frac{1}{12}(h^2 + l^2)] \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ -d \end{bmatrix} \times \begin{bmatrix} 0 \\ m_\kappa(\dot{y} + d\dot{\theta}) \\ m_\kappa y \dot{\theta} \end{bmatrix} \right) \\ &= \underline{\mathcal{F}}_{\rightarrow b}^T \begin{bmatrix} m_B \dot{\theta} [d^2 + \frac{1}{12}(h^2 + l^2)] + m_\kappa [y^2 \dot{\theta} + d(\dot{y} + d\dot{\theta})] \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} \underline{h}_{\rightarrow}^{Sw/a \cdot a} &= \underline{h}_{\rightarrow}^{Sw/a \cdot b} + \underbrace{\underline{\omega}^{ba} \times \underline{h}_{\rightarrow}^{Sw/a}}_{=0} \\ &= \underline{\mathcal{F}}_{\rightarrow b}^T \begin{bmatrix} m_B \ddot{\theta} [d^2 + \frac{1}{12}(h^2 + l^2)] + m_\kappa [2y \dot{y} \dot{\theta} + y^2 \ddot{\theta} + d(\ddot{y} + d\ddot{\theta})] \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

And finally, applying N2LR, i.e. $\underline{h}^{Sw/a \cdot a} = \underline{m}^{Sw}$:

$$\begin{aligned} m_B \ddot{\theta} [d^2 + \frac{1}{12}(h^2 + l^2)] + m_\kappa [2y\dot{y}\dot{\theta} + y^2\ddot{\theta} + d(\ddot{y} + d\ddot{\theta})] \\ = -m_B dg \sin(\theta) + m_\kappa [-yg \cos(\theta) - dg \sin(\theta)] \end{aligned} \quad (5)$$

We can easily check that (5) is nothing else than $d \cdot (3) + y \cdot (4)$, and thus the two sets of DEs describe the same motion.

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a)

$$\begin{aligned} \underline{\omega}^{sa} &= \underline{\omega}^{sb} + \underline{\omega}^{bq} + \underline{\omega}^{qa} \\ &= \underline{\mathcal{F}}_s^T \underline{\omega}_s^{sb} + \underline{\mathcal{F}}_b^T \underline{\omega}_b^{bq} + \underline{\mathcal{F}}_q^T \underline{\omega}_q^{qa} \\ &= \underline{\mathcal{F}}_s^T \underline{\omega}_s^{sb} + \underline{\mathcal{F}}_s^T \mathbf{C}_{sb} \underline{\omega}_b^{bq} + \underline{\mathcal{F}}_s^T \mathbf{C}_{sq} \underline{\omega}_q^{qa} \\ &= \underline{\mathcal{F}}_s^T \underline{\omega}_s^{sb} + \underline{\mathcal{F}}_s^T \mathbf{C}_3(\gamma) \underline{\omega}_b^{bq} + \underline{\mathcal{F}}_s^T \mathbf{C}_3(\gamma) \mathbf{C}_2(\phi) \underline{\omega}_q^{qa} \\ &= \underline{\mathcal{F}}_s^T \left(\mathbf{1}_3 \dot{\gamma} + \mathbf{C}_3(\gamma) \mathbf{1}_2 \dot{\phi} + \mathbf{C}_3(\gamma) \mathbf{C}_2(\phi) \mathbf{1}_3 \dot{\theta} \right) \\ &= \underline{\mathcal{F}}_s^T \underbrace{\begin{bmatrix} \mathbf{C}_3(\gamma) \mathbf{C}_2(\phi) \mathbf{1}_3 & \mathbf{C}_3(\gamma) \mathbf{1}_2 & \mathbf{1}_3 \end{bmatrix}}_{\underline{\omega}_s^{sa}} \begin{bmatrix} \dot{\theta} \\ \dot{\phi} \\ \dot{\gamma} \end{bmatrix}. \quad \square \end{aligned}$$

b)

$$\begin{aligned} \underline{r}^{dmw \cdot a} &= \left(\underline{r}^{dmc} + \underline{r}^{cw} \right)^{\cdot a} \\ &= \left(\underline{r}^{dmc} + \underline{r}^{cw} \right)^{\cdot s} + \underline{\omega}^{sa} \times \left(\underline{r}^{dmc} + \underline{r}^{cw} \right) \\ &= \underline{\mathcal{F}}_s^T \left(\underbrace{\underline{\mathbf{r}}_s^{dmc}}_{=\mathbf{0}} + \underbrace{\underline{\mathbf{r}}_s^{cw}}_{=\mathbf{0}} \right) + \underline{\omega}^{sa} \times \left(\underline{r}^{dmc} + \underline{r}^{cw} \right) \\ &= \underline{\omega}^{sa} \times \left(\underline{r}^{dmc} + \underline{r}^{cw} \right). \quad \square \end{aligned}$$

Where we've used the facts that the material element dm is fixed in the body frame \mathcal{F}_s and that $\underline{\mathbf{r}}_s^{cw} = \underline{l} \mathbf{1}_3 = \mathbf{0}$.

c)

$$\begin{aligned}
\vec{h}^{\mathcal{B}w/a} &= \int_{\mathcal{B}} \vec{r}^{dmw} \times \vec{r}^{dmw \cdot a} dm \\
&= \int_{\mathcal{B}} \vec{r}^{dmw} \times \left[\vec{\omega}^{sa} \times \left(\vec{r}^{dmc} + \vec{r}^{cw} \right) \right] dm \\
&= \int_{\mathcal{B}} \left(\vec{r}^{dmc} + \vec{r}^{cw} \right) \times \left[\vec{\omega}^{sa} \times \vec{r}^{dmc} + \vec{\omega}^{sa} \times \vec{r}^{cw} \right] dm \\
&= \int_{\mathcal{B}} \left(\vec{r}^{dmc} + \vec{r}^{cw} \right) \times \left[-\vec{r}^{dmc} \times \vec{\omega}^{sa} - \vec{r}^{cw} \times \vec{\omega}^{sa} \right] dm \\
&= \int_{\mathcal{B}} -\vec{r}^{dmc} \times (\vec{r}^{dmc} \times \vec{\omega}^{sa}) - \vec{r}^{dmc} \times (\vec{r}^{cw} \times \vec{\omega}^{sa}) \\
&\quad - \vec{r}^{cw} \times (\vec{r}^{dmc} \times \vec{\omega}^{sa}) - \vec{r}^{cw} \times (\vec{r}^{cw} \times \vec{\omega}^{sa}) dm \\
&= \int_{\mathcal{B}} -\vec{r}^{dmc} \times (\vec{r}^{dmc} \times \vec{\omega}^{sa}) + (\vec{r}^{cw} \times \vec{\omega}^{sa}) \times \vec{r}^{dmc} \\
&\quad + \vec{r}^{cw} \times (\vec{\omega}^{sa} \times \vec{r}^{dmc}) - \vec{r}^{cw} \times (\vec{r}^{cw} \times \vec{\omega}^{sa}) dm \\
&= \underline{\mathcal{F}}_s^T \left\{ \int_{\mathcal{B}} -\mathbf{r}_s^{dmc \times} \mathbf{r}_s^{dmc \times} \omega_s^{sa} + (\mathbf{r}_s^{cw \times} \omega_s^{sa}) \times \mathbf{r}_s^{dmc} + \mathbf{r}_s^{cw \times} \omega_s^{sa} \times \mathbf{r}_s^{dmc} - \mathbf{r}_s^{cw \times} \mathbf{r}_s^{cw \times} \omega_s^{sa} dm \right\} \\
&= \underline{\mathcal{F}}_s^T \left\{ \underbrace{\left[\int_{\mathcal{B}} -\mathbf{r}_s^{dmc \times} \mathbf{r}_s^{dmc \times} dm \right]}_{\mathbf{J}_s^{\mathcal{B}c}} \omega_s^{sa} + [(\mathbf{r}_s^{cw \times} \omega_s^{sa}) \times + \mathbf{r}_s^{cw \times} \omega_s^{sa} \times] \underbrace{\int_{\mathcal{B}} \mathbf{r}_s^{dmc} dm}_0 \right. \\
&\quad \left. + \underbrace{\int_{\mathcal{B}} dm [-\mathbf{r}_s^{cw \times} \mathbf{r}_s^{cw \times}]}_{\mathbf{J}_s^{m\mathcal{B}w}} \omega_s^{sa} \right\} \\
&= \underline{\mathcal{F}}_s^T \{ (\mathbf{J}_s^{\mathcal{B}c} + \mathbf{J}_s^{m\mathcal{B}w}) \omega_s^{sa} \} \\
&= (\underline{J}^{\mathcal{B}c} + \underline{J}^{m\mathcal{B}w}) \cdot \underline{\omega}^{sa} \\
&= \underline{J}^{\mathcal{B}w} \cdot \underline{\omega}^{sa}. \quad \square
\end{aligned}$$

e) From ERL,

$$\vec{h}^{\mathcal{B}w/a \cdot a} = \underline{m}^{\mathcal{B}w}. \quad (6)$$

Developing on both sides:

$$\begin{aligned}
 \underline{h}^{\mathcal{B}w/a} \dot{\mathbf{a}} &= \underline{h}^{\mathcal{B}w/a} \dot{\mathbf{s}} + \underline{\omega}^{sa} \times \underline{h}^{\mathcal{B}w/a} \\
 &= \left\{ \underline{\mathcal{F}}_s^T (\mathbf{J}_s^{\mathcal{B}c} + \mathbf{J}_s^{m\mathcal{B}w}) \underline{\omega}_s^{sa} \right\}^{\cdot s} + \underline{\mathcal{F}}_b^T \underline{\omega}_s^{sa \times} (\mathbf{J}_s^{\mathcal{B}c} + \mathbf{J}_s^{m\mathcal{B}w}) \underline{\omega}_s^{sa} \\
 &= \underline{\mathcal{F}}_s^T \left\{ \underbrace{(\mathbf{j}_s^{\mathcal{B}c} + \mathbf{j}_s^{m\mathcal{B}w})}_{=\mathbf{0} \text{ (Body frame)}} \underline{\omega}_s^{sa} + (\mathbf{J}_s^{\mathcal{B}c} + \mathbf{J}_s^{m\mathcal{B}w}) \dot{\underline{\omega}}_s^{sa} + \underline{\omega}_s^{sa \times} (\mathbf{J}_s^{\mathcal{B}c} + \mathbf{J}_s^{m\mathcal{B}w}) \underline{\omega}_s^{sa} \right\} \\
 &= \underline{\mathcal{F}}_s^T \left\{ (\mathbf{J}_s^{\mathcal{B}c} + \mathbf{J}_s^{m\mathcal{B}w}) \dot{\underline{\omega}}_s^{sa} + \underline{\omega}_s^{sa \times} (\mathbf{J}_s^{\mathcal{B}c} + \mathbf{J}_s^{m\mathcal{B}w}) \underline{\omega}_s^{sa} \right\} \stackrel{(6)}{=} \underline{\mathcal{F}}_s^T \mathbf{m}_s^{\mathcal{B}w}. \quad \square
 \end{aligned}$$

f) We have previously shown the following:

$$\begin{aligned}
 \underline{\omega}_s^{sa} &= \mathbf{S}_s^{sa} \dot{\underline{\theta}}^{sa}, \\
 (\mathbf{J}_s^{\mathcal{B}c} + \mathbf{J}_s^{m\mathcal{B}w}) \dot{\underline{\omega}}_s^{sa} + \underline{\omega}_s^{sa \times} (\mathbf{J}_s^{\mathcal{B}c} + \mathbf{J}_s^{m\mathcal{B}w}) \underline{\omega}_s^{sa} &= \mathbf{m}_s^{\mathcal{B}w}.
 \end{aligned}$$

Rearranging these equations:

$$\begin{aligned}
 \dot{\underline{\theta}}^{sa} &= (\mathbf{S}_s^{sa})^{-1} \underline{\omega}_s^{sa}, \\
 \dot{\underline{\omega}}_s^{sa} &= (\mathbf{J}_s^{\mathcal{B}c} + \mathbf{J}_s^{m\mathcal{B}w})^{-1} (\mathbf{m}_s^{\mathcal{B}w} - \underline{\omega}_s^{sa \times} (\mathbf{J}_s^{\mathcal{B}c} + \mathbf{J}_s^{m\mathcal{B}w}) \underline{\omega}_s^{sa}).
 \end{aligned}$$

Using the following notation,

$$\mathbf{x} = \begin{bmatrix} \underline{\theta}^{sa} \\ \underline{\omega}_s^{sa} \end{bmatrix}, \quad \dot{\mathbf{x}} = \begin{bmatrix} \dot{\underline{\theta}}^{sa} \\ \dot{\underline{\omega}}_s^{sa} \end{bmatrix},$$

we can write a first-order state-space system of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}).$$

Using MATLAB, we obtain the following plots (the code is given at the end of the assignment):

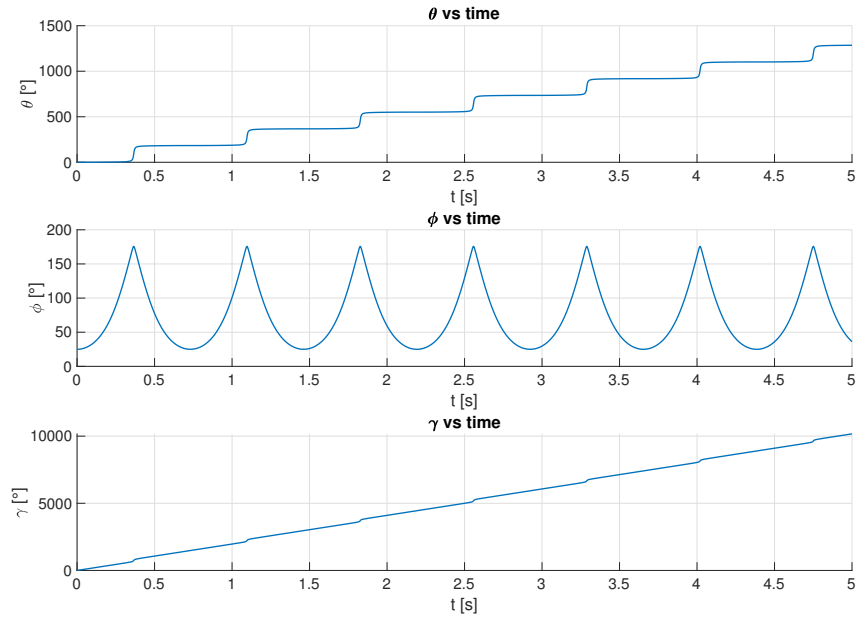


Figure 3: θ , ϕ and γ vs. time.

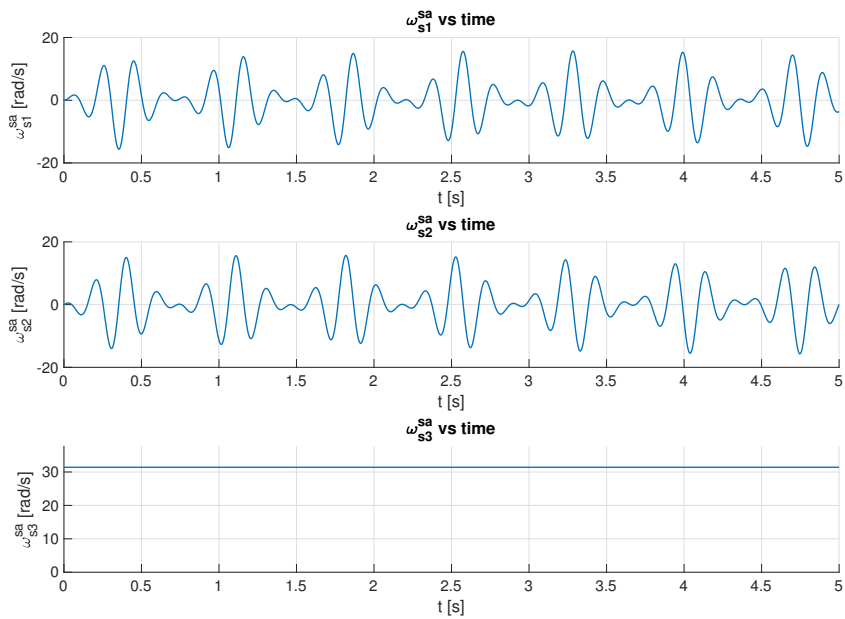


Figure 4: ω_{s1}^{sa} , ω_{s2}^{sa} and ω_{s3}^{sa} vs. time.

g) First,

$$\begin{aligned}
 T_{Bw/a} &= \frac{1}{2} \boldsymbol{\omega}_s^{sa \top} \mathbf{J}_s^{Bw} \boldsymbol{\omega}_s^{sa}, \\
 U_{Bw} &= - \int_B \vec{g} \cdot \vec{r}^{dmw} dm \\
 &= - \vec{g} \cdot \underbrace{\left(\int_B \vec{r}^{dmw} dm \right)}_{\vec{r}^{cw} m_B} \\
 &= (\mathbf{C}_3(\gamma) \mathbf{C}_2(\phi) \mathbf{C}_3(\theta) \mathbf{1}_3 g)^\top \underline{\mathcal{F}}_s \cdot \underline{\mathcal{F}}_s^\top \mathbf{r}_s^{cw} m_B \\
 &= m_B g l (\mathbf{C}_3(\gamma) \mathbf{C}_2(\phi) \mathbf{C}_3(\theta) \mathbf{1}_3)^\top \mathbf{1}_3.
 \end{aligned}$$

We can now use MATLAB to obtain the plot shown in Figure 5. As expected, $E_{Bw/a}$ is constant with time, which is a good indicator of an accurate simulation.

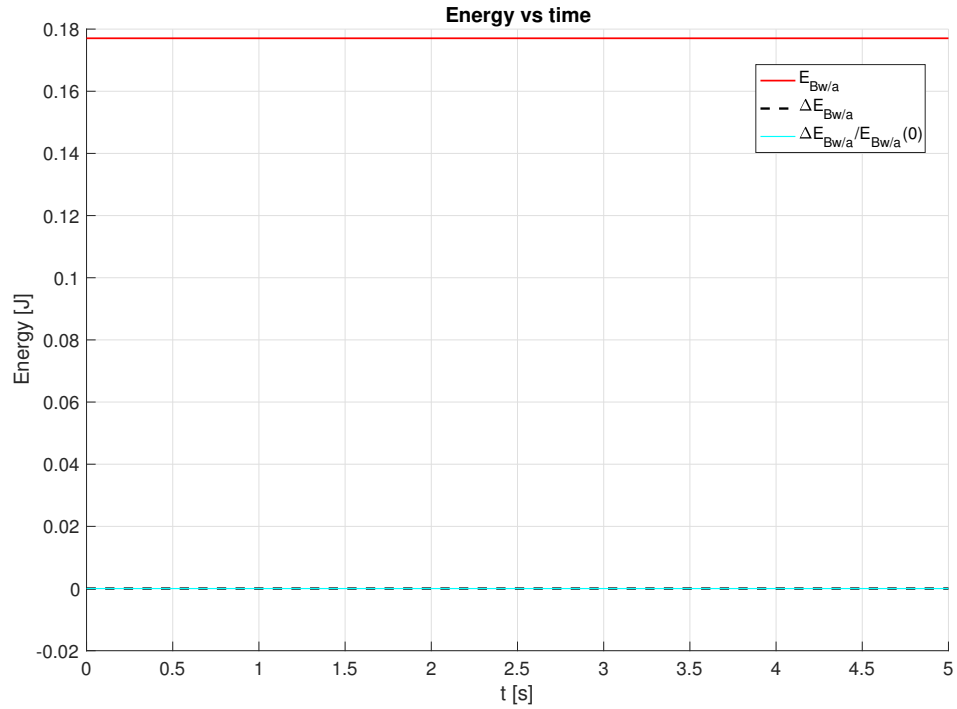


Figure 5: Energy vs. time.

Problem 3

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DCM

```
clear all; close all; clc;
C3=@(a) [cos(a) sin(a) 0; -sin(a) cos(a) 0; 0 0 1];
C2=@(a) [cos(a) 0 -sin(a); 0 1 0; sin(a) 0 cos(a)];
```

Parameters

```
O_ba0= [0; 25* pi/180; 0]; % rad
ws_ba0= [0; 0; 10*pi]; % rad/s
x0=[O_ba0;ws_ba0];
tspan=[0 5]; % s
g=9.81; % m/s^2
l=0.15; % m
s=8050; % kg/m^3
D=0.06; % m
h=0.005; % m
mB=pi*(D/2)^2*h*s; % kg
Js_Bc=diag([1/12*mB*(3*(D/2)^2 + h^2), 1/12*mB*(3*(D/2)^2 + h^2),
1/2*mB*(D/2)^2]); % kg*m^2
rs_cw=[0;0;l]; % m
rs_cw_x=[0 -l 0;l 0 0; 0 0 0];
Js_mBw=-mB*rs_cw_x*rs_cw_x; % kg*m^2

ms_Bw=@(theta,phi,gamma)
cross(rs_cw,C3(gamma)*C2(phi)*C3(theta)*[0;0;-g*mB]); % Nm
Ss_sa=@(gamma,phi) [C3(gamma)*C2(phi)*[0;0;1], C3(gamma)*[0;1;0],
[0;0;1]];
```

State-space simulation

```
options = odeset('RelTol',1e-10);
odef=@(t,x) [Ss_sa(x(3),x(2))\x(4:6); (Js_Bc
+Js_mBw)\(ms_Bw(x(1),x(2),x(3))-cross(x(4:6),(Js_Bc+Js_mBw)*x(4:6)))];
[t, x]=ode45(odef,tspan,x0,options);
```

Plots for f)

```
figure('Name','Angle
      plot','NumberTitle','off','DefaultAxesFontSize',16)

subplot(3,1,1)
grid on; hold on;
plot(t,x(:,1)*180/pi,'LineWidth',1.3,'DisplayName','\theta');
title('\theta vs time'); xlabel('t [s]'); ylabel('\theta [°]');

subplot(3,1,2)
grid on; hold on;
plot(t,x(:,2)*180/pi,'LineWidth',1.3,'DisplayName','\phi');
title('\phi vs time'); xlabel('t [s]'); ylabel('\phi [°]');

subplot(3,1,3)
grid on; hold on;
plot(t,x(:,3)*180/pi,'LineWidth',1.3,'DisplayName','\gamma');
title('\gamma vs time'); xlabel('t [s]'); ylabel('\gamma [°]');

figure('Name','Angular Velocity
      plot','NumberTitle','off','DefaultAxesFontSize',16)

subplot(3,1,1)
grid on; hold on;
plot(t,x(:,4),'LineWidth',1.3,'DisplayName','\omega_{s1}^{sa}');
title('\omega_{s1}^{sa} vs time'); xlabel('t [s]');
      ylabel('\omega_{s1}^{sa} [rad/s]');

subplot(3,1,2)
grid on; hold on;
plot(t,x(:,5),'LineWidth',1.3,'DisplayName','\omega_{s2}^{sa}');
title('\omega_{s2}^{sa} vs time'); xlabel('t [s]');
      ylabel('\omega_{s2}^{sa} [rad/s]');

subplot(3,1,3)
grid on; hold on;
plot(t,x(:,6),'LineWidth',1.3,'DisplayName','\omega_{s3}^{sa}');
title('\omega_{s3}^{sa} vs time'); xlabel('t [s]');
      ylabel('\omega_{s3}^{sa} [rad/s]');
ylim([0 12*pi]);
```

Part g)

```
TBwa=diag(1/2*x(:,4:6)*(Js_Bc+Js_mBw)*(x(:,4:6))');

UBw=[];
for i=1:size(t,1);
    UBw=[UBw;
        mB*g*l*(C3(x(i,3))*C2(x(i,2))*C3(x(i,1))*[0;0;1])'*[0;0;1]];
end
```

```
EBwa=TBwa+UBw;  
DEBwa=EBwa-ones(size(t,1),1)*EBwa(1);
```

Energy Plots

```
figure('Name','Energy','NumberTitle','off','DefaultAxesFontSize',16)  
grid on; hold on;  
title('Energy vs time'); xlabel('t [s]'); ylabel('Energy [J]');  
plot(t,EBwa,'r','LineWidth',1.5,'DisplayName','E_{Bw/a}');  
plot(t,DEBwa,'k--','LineWidth',2,'DisplayName','\Delta E_{Bw/a}');  
plot(t,DEBwa/EBwa(1),'c-','LineWidth',1,'DisplayName','\Delta E_{Bw/a}/  
E_{Bw/a}(0)');  
legend show;
```

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