

Assignment 5

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a) Step 1.1 is already done. Therefore, let's carry out steps 1.2 to 1.5.

$$\begin{aligned}
 \underline{\omega}^{ca} &= \underline{\omega}^{cb} + \underline{\omega}^{ba} \\
 &= \underline{\mathcal{F}}_c^T \mathbf{1}_2 \dot{\theta} + \underline{\mathcal{F}}_b^T \mathbf{1}_3 \dot{\phi} \\
 &= \underline{\mathcal{F}}_c^T \left(\mathbf{1}_2 \dot{\theta} + \mathbf{C}_{cb} \mathbf{1}_3 \dot{\phi} \right) \\
 &= \underline{\mathcal{F}}_c^T \begin{bmatrix} -\sin(\theta) \dot{\phi} \\ \dot{\theta} \\ \cos(\theta) \dot{\phi} \end{bmatrix}, \\
 \boldsymbol{\omega}_c^{ca \times} &= \begin{bmatrix} 0 & -\cos(\theta) \dot{\phi} & \dot{\theta} \\ \cos(\theta) \dot{\phi} & 0 & \sin(\theta) \dot{\phi} \\ -\dot{\theta} & -\sin(\theta) \dot{\phi} & 0 \end{bmatrix}, \\
 \underline{r}^{pw} &= \underline{\mathcal{F}}_c^T \begin{bmatrix} 0 \\ 0 \\ l \end{bmatrix}, \\
 \underline{v}^{pw/a} &= \underline{r}^{pw \bullet a} \\
 &= \underbrace{\underline{r}^{pw \bullet c}}_{\underline{0}} + \underline{\omega}^{ca} \times \underline{r}^{pw} \\
 &= \underline{\mathcal{F}}_c^T \boldsymbol{\omega}_c^{ca \times} \mathbf{r}_c^{pw} \\
 &= \underline{\mathcal{F}}_c^T \begin{bmatrix} 0 & -\cos(\theta) \dot{\phi} & \dot{\theta} \\ \cos(\theta) \dot{\phi} & 0 & \sin(\theta) \dot{\phi} \\ -\dot{\theta} & -\sin(\theta) \dot{\phi} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ l \end{bmatrix} \\
 &= \underline{\mathcal{F}}_c^T \begin{bmatrix} l \dot{\theta} \\ l \sin(\theta) \dot{\phi} \\ 0 \end{bmatrix}.
 \end{aligned}$$

Since we don't have constraints, we can now compute the different energies, i.e. steps 2.1 to 2.3:

$$\begin{aligned}
T_{pw/a} &= \frac{1}{2}m \underline{v}^{pw/a} \cdot \underline{v}^{pw/a} \\
&= \frac{1}{2}m \underline{\mathbf{v}}_c^{pw/a \top} \underline{\mathbf{v}}_c^{pw/a} \\
&= \frac{1}{2}m \begin{bmatrix} l\dot{\theta} & l\sin(\theta)\dot{\phi} & 0 \end{bmatrix} \begin{bmatrix} l\dot{\theta} \\ l\sin(\theta)\dot{\phi} \\ 0 \end{bmatrix} \\
&= \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2). \\
U_{pw} &= -m \underline{g} \cdot \underline{r}^{pw} \\
&= -m \begin{bmatrix} 0 & 0 & -g \end{bmatrix} \begin{bmatrix} l\cos(\phi)\sin(\theta) \\ l\sin(\phi)\sin(\theta) \\ l\cos(\theta) \end{bmatrix} \\
&= mgl\cos(\theta). \\
L_{pw/a} &= T_{pw/a} - U_{pw} \\
&= \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2) - mgl\cos(\theta).
\end{aligned}$$

Since no external force is applied, we have

$$\mathbf{f} \triangleq \begin{bmatrix} \mathbf{f}_\phi \\ \mathbf{f}_\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (1)$$

Applying steps 4.1 and 4.2, we get:

$$\begin{aligned}
\frac{d}{dt} \left(\frac{\partial L_{pw/a}}{\partial \dot{\mathbf{q}}} \right) &= \frac{d}{dt} \left(\begin{bmatrix} \frac{\partial L_{pw/a}}{\partial \dot{\phi}} & \frac{\partial L_{pw/a}}{\partial \dot{\theta}} \end{bmatrix} \right) \\
&= \frac{d}{dt} \left(\begin{bmatrix} ml^2 \sin^2(\theta)\dot{\phi} & ml^2\dot{\theta} \end{bmatrix} \right) \\
&= \begin{bmatrix} ml^2 \left\{ 2\sin(\theta)\cos(\theta)\dot{\theta}\dot{\phi} + \sin^2(\theta)\ddot{\phi} \right\} & ml^2\ddot{\theta} \end{bmatrix} \\
&= \begin{bmatrix} ml^2 \left\{ \sin(2\theta)\dot{\theta}\dot{\phi} + \sin^2(\theta)\ddot{\phi} \right\} & ml^2\ddot{\theta} \end{bmatrix}, \quad (2)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L_{pw/a}}{\partial \mathbf{q}} &= \begin{bmatrix} \frac{\partial L_{pw/a}}{\partial \phi} & \frac{\partial L_{pw/a}}{\partial \theta} \end{bmatrix} \\
&= \begin{bmatrix} 0 & ml^2\dot{\phi}^2 \sin(\theta)\cos(\theta) + mgl\sin(\theta) \end{bmatrix} \\
&= \begin{bmatrix} 0 & \frac{1}{2}ml^2\dot{\phi}^2 \sin(2\theta) + mgl\sin(\theta) \end{bmatrix}. \quad (3)
\end{aligned}$$

Finally, since our generalized coordinates are unconstrained, we can apply the Lagrange's Equation for an Unconstrained Particle, i.e.

$$\frac{d}{dt} \left(\frac{\partial L_{pw/a}}{\partial \dot{\mathbf{q}}} \right)^\top - \left(\frac{\partial L_{pw/a}}{\partial \mathbf{q}} \right)^\top = \mathbf{f}. \quad (4)$$

Substituting (1), (2) and (3) into (4), we obtain the following set of differential equations:

$$\begin{aligned} ml^2 \left\{ \sin(2\theta) \dot{\theta} \dot{\phi} + \sin^2(\theta) \ddot{\phi} \right\} &= 0, \\ ml^2 \ddot{\theta} - \frac{1}{2} ml^2 \dot{\phi}^2 \sin(2\theta) - mgl \sin(\theta) &= 0, \end{aligned}$$

which can be further simplified as follows:

$$\begin{aligned} \sin^2(\theta) \ddot{\phi} + \sin(2\theta) \dot{\theta} \dot{\phi} &= 0, \\ \ddot{\theta} - \frac{1}{2} \dot{\phi}^2 \sin(2\theta) - \frac{g}{l} \sin(\theta) &= 0. \end{aligned}$$

To verify partly this result, we can see that if we set $\dot{\phi} = \ddot{\phi} = 0$ and if we use $\alpha = \pi - \theta$, we obtain the following differential equation:

$$\ddot{\alpha} + \frac{g}{l} \sin(\alpha) = 0,$$

which is the equation describing the motion of a simple gravity pendulum of length l .

b) Following the same procedure as in a):

$$\begin{aligned} \vec{r}^{pw} &= \mathcal{F}_a^T \begin{bmatrix} x_a \\ y_a \\ \sqrt{\ell^2 - x_a^2 - y_a^2} \end{bmatrix}, \\ \vec{v}^{pw/a} &= \mathcal{F}_a^T \begin{bmatrix} \dot{x}_a \\ \dot{y}_a \\ -\frac{x_a \dot{x}_a + y_a \dot{y}_a}{\sqrt{\ell^2 - x_a^2 - y_a^2}} \end{bmatrix}, \\ T_{pw/a} &= \frac{1}{2} m \vec{v}^{pw/a} \cdot \vec{v}^{pw/a} \\ &= \frac{1}{2} m \left(\dot{x}_a^2 + \dot{y}_a^2 + \frac{(x_a \dot{x}_a + y_a \dot{y}_a)^2}{\ell^2 - x_a^2 - y_a^2} \right), \\ U_{pw} &= -m \vec{g} \cdot \vec{r}^{pw} \\ &= -m \begin{bmatrix} 0 & 0 & -g \end{bmatrix} \begin{bmatrix} x_a \\ y_a \\ \sqrt{\ell^2 - x_a^2 - y_a^2} \end{bmatrix} \\ &= mg \sqrt{\ell^2 - x_a^2 - y_a^2}, \\ L_{pw/a} &= T_{pw/a} - U_{pw} \\ &= \frac{1}{2} m \left(\dot{x}_a^2 + \dot{y}_a^2 + \frac{(x_a \dot{x}_a + y_a \dot{y}_a)^2}{\ell^2 - x_a^2 - y_a^2} \right) - mg \sqrt{\ell^2 - x_a^2 - y_a^2}, \\ \mathbf{f} &\triangleq \begin{bmatrix} f_{x_a} \\ f_{y_a} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} \left(\frac{\partial L_{pw/a}}{\partial \dot{\mathbf{q}}} \right) &= \frac{d}{dt} \left(\begin{bmatrix} \frac{\partial L_{pw/a}}{\partial \dot{x}_a} & \frac{\partial L_{pw/a}}{\partial \dot{y}_a} \end{bmatrix} \right) \\
&= \frac{d}{dt} \left(\begin{bmatrix} \frac{1}{2}m \left\{ 2\dot{x}_a + 2x_a \frac{x_a\dot{x}_a + y_a\dot{y}_a}{\ell^2 - x_a^2 - y_a^2} \right\} & \frac{1}{2}m \left\{ 2\dot{y}_a + 2y_a \frac{x_a\dot{x}_a + y_a\dot{y}_a}{\ell^2 - x_a^2 - y_a^2} \right\} \end{bmatrix} \right) \\
&= m \begin{bmatrix} \ddot{x}_a + \frac{2x_a\dot{x}_a^2 + \dot{x}_a\dot{y}_ay_a + x_a^2\ddot{x}_a + x_a\dot{y}_a^2 + x_a y_a\ddot{y}_a}{\ell^2 - x_a^2 - y_a^2} + 2x_a \frac{(x_a\dot{x}_a + y_a\dot{y}_a)^2}{(\ell^2 - x_a^2 - y_a^2)^2} \\ \ddot{y}_a + \frac{2y_a\dot{y}_a^2 + \dot{y}_a\dot{x}_ax_a + y_a^2\ddot{y}_a + y_a\dot{x}_a^2 + y_a x_a\ddot{x}_a}{\ell^2 - x_a^2 - y_a^2} + 2y_a \frac{(x_a\dot{x}_a + y_a\dot{y}_a)^2}{(\ell^2 - x_a^2 - y_a^2)^2} \end{bmatrix}^T, \\
\frac{\partial L_{pw/a}}{\partial \mathbf{q}} &= \begin{bmatrix} \frac{\partial L_{pw/a}}{\partial x_a} & \frac{\partial L_{pw/a}}{\partial y_a} \end{bmatrix} \\
&= m \begin{bmatrix} \frac{gx_a}{\sqrt{\ell^2 - x_a^2 - y_a^2}} + \dot{x}_a \frac{x_a\dot{x}_a + y_a\dot{y}_a}{\ell^2 - x_a^2 - y_a^2} + x_a \frac{(x_a\dot{x}_a + y_a\dot{y}_a)^2}{(\ell^2 - x_a^2 - y_a^2)^2} \\ \frac{gy_a}{\sqrt{\ell^2 - x_a^2 - y_a^2}} + \dot{y}_a \frac{x_a\dot{x}_a + y_a\dot{y}_a}{\ell^2 - x_a^2 - y_a^2} + y_a \frac{(x_a\dot{x}_a + y_a\dot{y}_a)^2}{(\ell^2 - x_a^2 - y_a^2)^2} \end{bmatrix}^T.
\end{aligned}$$

Finally, substituting everything into (4) and rearranging:

$$\begin{aligned}
\ddot{x}_a + \frac{x_a\dot{x}_a^2 + x_a^2\ddot{x}_a + x_a\dot{y}_a^2 + x_a y_a\ddot{y}_a}{\ell^2 - x_a^2 - y_a^2} + x_a \frac{(x_a\dot{x}_a + y_a\dot{y}_a)^2}{(\ell^2 - x_a^2 - y_a^2)^2} - \frac{gx_a}{\sqrt{\ell^2 - y_a^2 - y_a^2}} &= 0, \\
\ddot{y}_a + \frac{y_a\dot{y}_a^2 + y_a^2\ddot{y}_a + y_a\dot{x}_a^2 + y_a x_a\ddot{x}_a}{\ell^2 - x_a^2 - y_a^2} + y_a \frac{(x_a\dot{x}_a + y_a\dot{y}_a)^2}{(\ell^2 - x_a^2 - y_a^2)^2} - \frac{gy_a}{\sqrt{\ell^2 - y_a^2 - y_a^2}} &= 0.
\end{aligned}$$

c) As in a) and b):

$$\begin{aligned}
\vec{r}^{pw} &= \vec{\mathcal{F}}_a^T \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix}, \\
\vec{v}^{pw/a} &= \vec{\mathcal{F}}_a^T \begin{bmatrix} \dot{x}_a \\ \dot{y}_a \\ \dot{z}_a \end{bmatrix}, \\
T_{pw/a} &= \frac{1}{2}m \vec{v}^{pw/a} \cdot \vec{v}^{pw/a} = \frac{1}{2}m (\dot{x}_a^2 + \dot{y}_a^2 + \dot{z}_a^2), \\
U_{pw} &= -m \vec{g} \cdot \vec{r}^{pw} = -m \begin{bmatrix} 0 & 0 & -g \end{bmatrix} \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} = mgz_a, \\
L_{pw/a} &= T_{pw/a} - U_{pw} = \frac{1}{2}m (\dot{x}_a^2 + \dot{y}_a^2 + \dot{z}_a^2) - mgz_a, \\
\mathbf{f} &\triangleq \begin{bmatrix} \mathbf{f}_{x_a} \\ \mathbf{f}_{y_a} \\ \mathbf{f}_{z_a} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \\
\frac{d}{dt} \left(\frac{\partial L_{pw/a}}{\partial \dot{\mathbf{q}}} \right) &= \frac{d}{dt} \left(\begin{bmatrix} \frac{\partial L_{pw/a}}{\partial \dot{x}_a} & \frac{\partial L_{pw/a}}{\partial \dot{y}_a} & \frac{\partial L_{pw/a}}{\partial \dot{z}_a} \end{bmatrix} \right) = \begin{bmatrix} m\ddot{x}_a & m\ddot{y}_a & m\ddot{z}_a \end{bmatrix}, \\
\frac{\partial L_{pw/a}}{\partial \mathbf{q}} &= \begin{bmatrix} \frac{\partial L_{pw/a}}{\partial x_a} & \frac{\partial L_{pw/a}}{\partial y_a} & \frac{\partial L_{pw/a}}{\partial z_a} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -mg \end{bmatrix}.
\end{aligned}$$

However, we know have a holonomic constraint:

$$\phi(\mathbf{q}) = \mathbf{q}^T \mathbf{q} - \ell^2 = x_a^2 + y_a^2 + z_a^2 - \ell^2 = 0.$$

Defining the quantities

$$\begin{aligned} \Xi &\triangleq \begin{bmatrix} \frac{\partial \phi(\mathbf{q})}{\partial x_a} & \frac{\partial \phi(\mathbf{q})}{\partial y_a} & \frac{\partial \phi(\mathbf{q})}{\partial z_a} \end{bmatrix} = \begin{bmatrix} 2x_a & 2y_a & 2z_a \end{bmatrix}, \\ \Xi_t &\triangleq \frac{\partial \phi(\mathbf{q})}{\partial t} = 0, \end{aligned}$$

we can write the rate form of the constraint as follows:

$$\Xi \dot{\mathbf{q}} + \Xi_t = 2x_a \dot{x}_a + 2y_a \dot{y}_a + 2z_a \dot{z}_a = 0. \quad (5)$$

This gives us the first differential equation of motion. To get the others, we apply the Lagrange's Equation for a Constraint System of Particles (λ is a scalar since we only have 1 constraint):

$$\frac{d}{dt} \left(\frac{\partial L_{pw/a}}{\partial \dot{\mathbf{q}}} \right)^T - \left(\frac{\partial L_{pw/a}}{\partial \mathbf{q}} \right)^T = \mathbf{f} + \lambda \Xi^T. \quad (6)$$

After substitution, we obtain three additional differential equations:

$$m\ddot{x}_a = 2\lambda x_a, \quad (7)$$

$$m\ddot{y}_a = 2\lambda y_a, \quad (8)$$

$$m\ddot{z}_a + mg = 2\lambda z_a. \quad (9)$$

Together, (7), (8), (9) and (5) describe the motion of p .

(i)

Equations (7), (8) and (9) can be rewritten in matrix form:

$$\mathbf{M} \ddot{\mathbf{q}} - \lambda \Xi^T = \mathbf{f}_{\text{non}}, \quad (10)$$

where

$$\mathbf{M} = m\mathbf{1}, \quad \Xi^T = \begin{bmatrix} 2x_a \\ 2y_a \\ 2z_a \end{bmatrix}, \quad \mathbf{f}_{\text{non}} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix}.$$

Moreover, taking the time derivative of (5),

$$\Xi \ddot{\mathbf{q}} + \dot{\Xi} \mathbf{q} + \dot{\Xi}_t = \mathbf{0}. \quad (11)$$

We can combine (10) and (11) in matrix form to obtain

$$\begin{bmatrix} \mathbf{M} & -\Xi^T \\ -\Xi & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{\text{non}} \\ \dot{\Xi} \mathbf{q} + \dot{\Xi}_t \end{bmatrix}.$$

I can be shown that the left-hand side matrix is nonsingular, and therefore the equation can be solved for $\begin{bmatrix} \ddot{\mathbf{q}} & \lambda \end{bmatrix}^T$:

$$\begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{M} & -\Xi^T \\ -\Xi & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{f}_{\text{non}} \\ \dot{\Xi}\mathbf{q} + \dot{\Xi}_t \end{bmatrix} \triangleq \begin{bmatrix} \mathbf{f}_{\ddot{\mathbf{q}}}(\mathbf{q}, \dot{\mathbf{q}}) \\ \mathbf{f}_{\lambda}(\mathbf{q}, \dot{\mathbf{q}}) \end{bmatrix}.$$

This gives us a direct equation for the Lagrange multiplier λ . Plus, in order to solve for \mathbf{q} versus time, we can numerically integrate the equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),$$

where

$$\mathbf{x} \triangleq \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix}, \quad \mathbf{f}(\mathbf{x}) \triangleq \begin{bmatrix} \dot{\mathbf{q}} \\ \mathbf{f}_{\ddot{\mathbf{q}}}(\mathbf{q}, \dot{\mathbf{q}}) \end{bmatrix}.$$

(ii)

Writing

$$\begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} = \begin{bmatrix} l \cos(\phi) \sin(\theta) \\ l \sin(\phi) \sin(\theta) \\ l \cos(\theta) \end{bmatrix}$$

and taking the time derivative on both sides, we obtain the following:

$$\begin{aligned} \dot{\mathbf{q}} &= \begin{bmatrix} \dot{x}_a \\ \dot{y}_a \\ \dot{z}_a \end{bmatrix} \\ &= \begin{bmatrix} l\{\cos(\phi)\cos(\theta)\dot{\theta} - \sin(\phi)\sin(\theta)\dot{\phi}\} \\ l\{\sin(\phi)\cos(\theta)\dot{\theta} + \cos(\phi)\sin(\theta)\dot{\phi}\} \\ -l\sin(\theta)\dot{\theta} \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} -l\sin(\phi)\sin(\theta) & l\cos(\phi)\cos(\theta) \\ l\cos(\phi)\sin(\theta) & l\sin(\phi)\cos(\theta) \\ 0 & -l\sin(\theta) \end{bmatrix}}_{\mathbf{\Upsilon}} \underbrace{\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \end{bmatrix}}_{\dot{\mathbf{q}}}. \end{aligned} \quad (12)$$

Using this definition, we have

$$\dot{\mathbf{\Upsilon}} = \begin{bmatrix} -l\{\cos(\phi)\sin(\theta)\dot{\phi} + \cos(\theta)\sin(\phi)\dot{\theta}\} & -l\{\cos(\theta)\sin(\phi)\dot{\phi} + \cos(\phi)\sin(\theta)\dot{\theta}\} \\ l\{-\sin(\phi)\sin(\theta)\dot{\phi} + \cos(\phi)\cos(\theta)\dot{\theta}\} & l\{\cos(\phi)\cos(\theta)\dot{\phi} - \sin(\phi)\sin(\theta)\dot{\theta}\} \\ 0 & -l\cos(\theta)\dot{\theta} \end{bmatrix}.$$

Substituting (12) into (10) and premultiplying by $\mathbf{\Upsilon}^T$, it yields

$$\mathbf{\Upsilon}^T \mathbf{M} \mathbf{\Upsilon} \ddot{\mathbf{q}} + \mathbf{\Upsilon}^T \mathbf{M} \dot{\mathbf{\Upsilon}} \dot{\mathbf{q}} = \mathbf{\Upsilon}^T \mathbf{f}_{\text{non}}, \quad (13)$$

where we used the fact that $\mathbf{\Upsilon}^T \mathbf{\Xi}^T = \mathbf{0}$. Developing (13) and simplifying, we get

$$\begin{bmatrix} ml^2 \sin^2(\theta) & 0 \\ 0 & ml^2 \end{bmatrix} \begin{bmatrix} \ddot{\phi} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} ml^2 \sin(\theta) \cos(\theta) \dot{\theta} & ml^2 \sin(\theta) \cos(\theta) \dot{\phi} \\ -ml^2 \sin(\theta) \cos(\theta) \dot{\phi} & 0 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ mgl \sin(\theta) \end{bmatrix}.$$

Using the trigonometric identity $\sin(x) \cos(x) = \frac{1}{2} \sin(2x)$ and dividing on both side by ml^2 , we obtain the same two differential equations as in a):

$$\begin{aligned} \sin^2(\theta) \ddot{\phi} + \sin(2\theta) \dot{\theta} \dot{\phi} &= 0, \\ \ddot{\theta} - \frac{1}{2} \dot{\phi}^2 \sin(2\theta) - \frac{g}{l} \sin(\theta) &= 0. \end{aligned}$$

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a) First, the DCM and the angular velocity between \mathcal{F}_a and \mathcal{F}_b are given by:

$$\begin{aligned} \mathbf{C}_{ba} &= \mathbf{C}_{bq} \mathbf{C}_{qa} = \mathbf{C}_2(\theta) \mathbf{C}_3(\gamma) = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\gamma) & \sin(\gamma) & 0 \\ -\sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta) \cos(\gamma) & \cos(\theta) \sin(\gamma) & -\sin(\theta) \\ -\sin(\gamma) & \cos(\gamma) & 0 \\ \sin(\theta) \cos(\gamma) & \sin(\theta) \sin(\gamma) & \cos(\theta) \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \underline{\omega}^{ba} &= \underline{\omega}^{bq} + \underline{\omega}^{qa} = \underline{\mathcal{F}}_b \mathbf{1}_2 \dot{\theta} + \underline{\mathcal{F}}_a \mathbf{1}_3 \dot{\gamma} \\ &= \underline{\mathcal{F}}_b \left(\mathbf{1}_2 \dot{\theta} + \mathbf{C}_{ba} \mathbf{1}_3 \dot{\gamma} \right) \\ &= \underline{\mathcal{F}}_b \begin{bmatrix} -\dot{\gamma} \sin(\theta) \\ \dot{\theta} \\ \dot{\gamma} \cos(\theta) \end{bmatrix}. \end{aligned}$$

Next,

$$\begin{aligned} \underline{r}^{pw} &= \underline{r}^{pc} + \underline{r}^{cw} \\ &= \underline{\mathcal{F}}_b \mathbf{1}_3 r + \underline{\mathcal{F}}_a \mathbf{r}_a^{cw}, \\ \underline{v}^{pw/a} &= \underline{r}^{pw \cdot a} = \underline{r}^{pc \cdot a} + \underbrace{\underline{r}^{cw \cdot a}}_{\underline{0}} = \underbrace{\underline{r}^{pc \cdot b}}_{\underline{0}} + \underline{\omega}^{ba} \times \underline{r}^{pc} \\ &= \underline{\mathcal{F}}_b \left(\boldsymbol{\omega}_b^{ba \times} \mathbf{r}_b^{pc} \right) \\ &= \underline{\mathcal{F}}_b \begin{bmatrix} \dot{\theta} r \\ r \dot{\gamma} \sin(\theta) \\ 0 \end{bmatrix}. \end{aligned}$$

The kinetic and potential energy become

$$\begin{aligned}
 T_{pw/a} &= \frac{1}{2} m \underline{v}^{pw/a} \cdot \underline{v}^{pw/a} \\
 &= \frac{1}{2} m \underline{\mathbf{v}}_b^{pw/a \top} \underline{\mathbf{v}}_b^{pw/a} \\
 &= \frac{1}{2} m \begin{bmatrix} \dot{\theta} r & r \dot{\gamma} \sin(\theta) & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} r \\ r \dot{\gamma} \sin(\theta) \\ 0 \end{bmatrix} \\
 &= \frac{1}{2} m \left(\dot{\theta}^2 r^2 + r^2 \dot{\gamma}^2 \sin^2(\theta) \right) \\
 &= \frac{1}{2} \underbrace{\begin{bmatrix} \dot{\theta} & \dot{\gamma} \end{bmatrix}}_{\dot{\mathbf{q}}^\top} \underbrace{\begin{bmatrix} mr^2 & 0 \\ 0 & mr^2 \sin^2(\theta) \end{bmatrix}}_{\mathbf{M}(\mathbf{q})} \underbrace{\begin{bmatrix} \dot{\theta} \\ \dot{\gamma} \end{bmatrix}}_{\dot{\mathbf{q}}}.
 \end{aligned}$$

$$\begin{aligned}
 U_{pw} &= -m \underline{g} \cdot \underline{r}^{pw} \\
 &= mg (\mathbf{C}_{ba} \mathbf{1}_3)^\top (\mathbf{r}_b^{pc} + \mathbf{C}_{ba} \mathbf{r}_a^{cw}) \\
 &= mg \left(\mathbf{1}_3^\top \mathbf{C}_{ba}^\top \mathbf{r}_b^{pc} + \mathbf{1}_3^\top \mathbf{C}_{ba}^\top \mathbf{C}_{ba} \mathbf{r}_a^{cw} \right) \\
 &= mg \left(r \mathbf{1}_3^\top \mathbf{C}_{ba}^\top \mathbf{1}_3 + \mathbf{1}_3^\top \mathbf{r}_a^{cw} \right) \\
 &= mgr \cos(\theta) + mg \mathbf{1}_3^\top \mathbf{r}_a^{cw}.
 \end{aligned}$$

And finally,

$$L_{pw/a} = T_{pw/a} - U_{pw} = \frac{1}{2} m \left(\dot{\theta}^2 r^2 + r^2 \dot{\gamma}^2 \sin^2(\theta) \right) - mgr \cos(\theta) - mg \mathbf{1}_3^\top \mathbf{r}_a^{cw}.$$

b)

$$\begin{aligned}
 \frac{d}{dt} \left(\frac{\partial L_{pw/a}}{\partial \dot{\mathbf{q}}} \right) &= \frac{d}{dt} \left(\frac{\partial T_{pw/a}}{\partial \dot{\mathbf{q}}} \right) \\
 &= \frac{d}{dt} \left(\dot{\mathbf{q}}^\top \mathbf{M}(\mathbf{q}) \right) \\
 &= \frac{d}{dt} \left(\begin{bmatrix} mr^2 \dot{\theta} & mr^2 \dot{\gamma} \sin^2(\theta) \end{bmatrix} \right) \\
 &= \begin{bmatrix} mr^2 \ddot{\theta} & mr^2 (\ddot{\gamma} \sin^2(\theta) + 2 \dot{\gamma} \sin(\theta) \cos(\theta) \dot{\theta}) \end{bmatrix}, \tag{14}
 \end{aligned}$$

$$\frac{\partial L_{pw/a}}{\partial \mathbf{q}} = \begin{bmatrix} mr^2 \dot{\gamma}^2 \sin(\theta) \cos(\theta) + mgr \sin(\theta) & 0 \end{bmatrix}. \tag{15}$$

Since the generalized coordinates are not constrained and no external forces is applied on the particle, the Lagrange's Equation can be written as follows:

$$\frac{d}{dt} \left(\frac{\partial L_{pw/a}}{\partial \dot{\mathbf{q}}} \right)^\top - \frac{\partial L_{pw/a}}{\partial \mathbf{q}}^\top = \mathbf{0}. \tag{16}$$

Substituting (14) and (15) into (16), we obtain

$$\begin{bmatrix} mr^2\ddot{\theta} - mr^2\dot{\gamma}^2 \sin(\theta) \cos(\theta) - mgr \sin(\theta) \\ mr^2(\ddot{\gamma} \sin^2(\theta) + 2\dot{\gamma} \sin(\theta) \cos(\theta)\dot{\theta}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which can be rewritten as follows

$$\underbrace{\begin{bmatrix} mr^2 & 0 \\ 0 & mr^2 \sin^2(\theta) \end{bmatrix}}_{\mathbf{M}(\mathbf{q})} \underbrace{\begin{bmatrix} \ddot{\theta} \\ \ddot{\gamma} \end{bmatrix}}_{\ddot{\mathbf{q}}} = \underbrace{\begin{bmatrix} \frac{1}{2}mr^2\dot{\gamma}^2 \sin(2\theta) + mgr \sin(\theta) \\ -mr^2\dot{\gamma}\dot{\theta} \sin(2\theta) \end{bmatrix}}_{\mathbf{f}_{non}(\mathbf{q}, \dot{\mathbf{q}})}$$

c) Let's define \mathbf{x} as follows:

$$\mathbf{x} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix}, \quad \dot{\mathbf{x}} = \begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix}$$

so that we can describe the particle's motion with the following first order DE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} \dot{\mathbf{q}} \\ \mathbf{M}^{-1}(\mathbf{q})\mathbf{f}_{non}(\mathbf{q}, \dot{\mathbf{q}}) \end{bmatrix}$$

See the following MATLAB code for the numerical integration and the verification. The mechanical energy of the system should remain constant (neglecting the numerical errors).

Problem 2 c)

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In this code, we verify that the mechanical energy of the system remains constant through time.

Parameters

```
close all
clear all
clc

m=0.5;           %kg
r=1;             %m
g=9.81;          %m/s^2

% Initial conditions (arbitrary)
q0=[pi/3; 0];    %rad
dq0=[0; pi];     %rad/s
x0=[q0;dq0];

% Integration limits
tspan=[0 10];    %s
```

Useful functions

```
Mass matrix

M=@(q) m*r^2*diag([1,sin(q(1))^2]);
invM=@(q) 1/(m*r^2)*diag([1,1/(sin(q(1))^2)]);

% Non linear forces
fnon=@(q,dq) [0.5*m*r^2*(dq(2)).^2*sin(2*q(1))+m*g*r*sin(q(1)) ;...
              -m*r^2*dq(2).*dq(1)*sin(2*q(1))];

% Function to integrate
f=@(t,x) [x([3 4]) ; invM(x([1 2]))*fnon(x([1 2]),x([3 4]))];
```

Integration

```
options = odeset('RelTol',1e-10);
[t,x] = ode45(f,tspan,x0,options);
```

Energies

```

Tpw_a=@(x) 0.5*x([3 4])'*M(x([1 2]))*x([3 4]);
Upw=@(x) m*g*r*cos(x(1));
Epw_a=@(x) Tpw_a(x)+Upw(x);

Emech=NaN(length(t),1);
T=NaN(length(t),1);
U=NaN(length(t),1);
for i=1:length(t)
    Emech(i)=Epw_a(x(i,:))';
    T(i)=Tpw_a(x(i,:))';
    U(i)=Upw(x(i,:))';
end

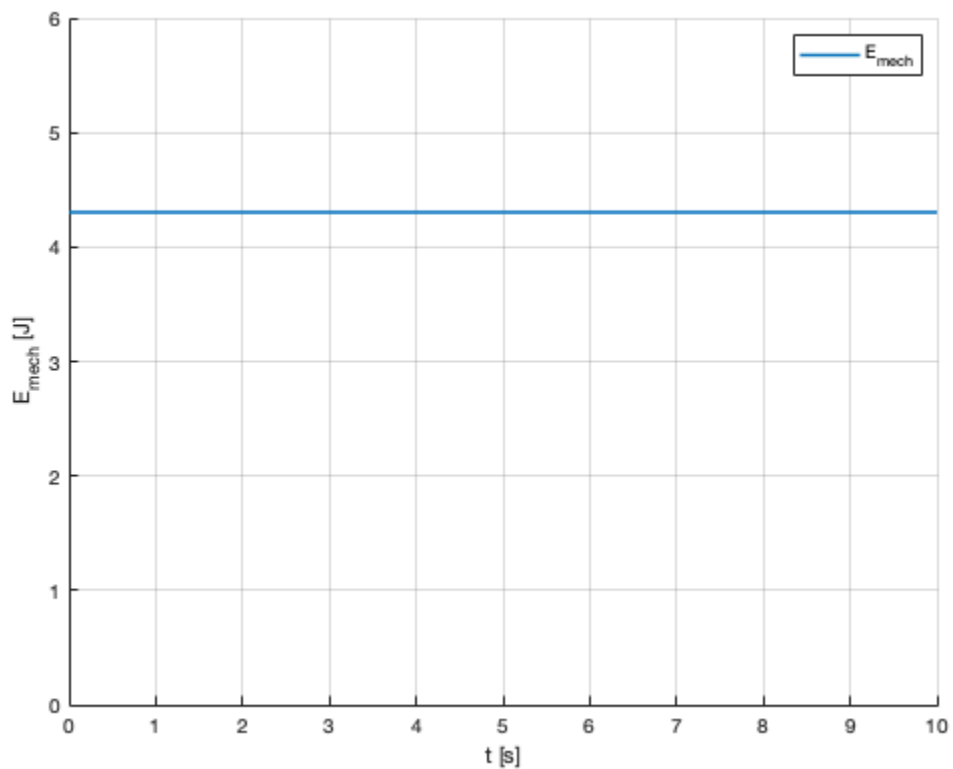
```

Plot

```

figure('Numbertitle','off','Name','Mechanical Energy');
grid on
hold on
plot(t,Emech,'LineWidth',1.5,'DisplayName','E_{mech}');
ylim([0 6]);
xlabel('t [s]');ylabel('E_{mech} [J]');
legend show

```



Comments

As expected, the mechanical energy is constant through time. This is a good indicator of a correct dynamics analysis.

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