Section 11.4: Equations of Lines and Planes

Definition: The line containing the point (x_0, y_0, z_0) and parallel to the vector $\vec{v} = \langle A, B, C \rangle$ has **parametric equations**

$$x = x_0 + At, \quad y = y_0 + Bt, \quad z = z_0 + Ct,$$

where $t \in \mathbb{R}$ is a **parameter**. These equations can be expressed in vector form as

$$\vec{R}(t) = \langle x_0 + At, y_0 + Bt, z_0 + Ct \rangle.$$

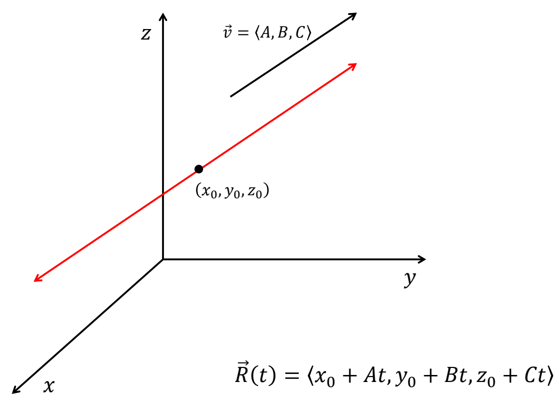

Figure 1: Graph of the line with direction vector $\vec{v}$ passing through (x_0, y_0, z_0).

Note: If A, B, and C are nonzero, then the **symmetric equations** of the line are

$$\frac{x - x_0}{A} = \frac{y - y_0}{B} = \frac{z - z_0}{C}.$$

Example: Find parametric and symmetric equations of the line passing through the points $(1, -1, 2)$ and $(2, 1, 5)$.

The line is defined by the vector

$$\vec{v} = \langle 2 - 1, 1 - (-1), 5 - 2 \rangle = \langle 1, 2, 3 \rangle.$$

Thus, parametric equations of the line are

$$x = 2 + t, \quad y = 1 + 2t, \quad z = 5 + 3t.$$

The symmetric equations of the line are

$$x - 2 = \frac{y-1}{2} = \frac{z-5}{3}.$$

Definition: A vector $\vec{N}$ that is orthogonal to every vector in a plane is called a **normal vector** to the plane.

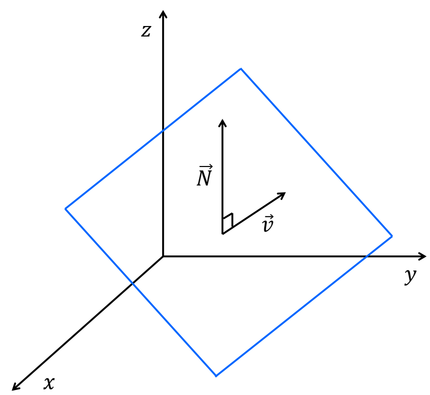

Figure 2: Illustration of a normal vector, $\vec{N}$, to a plane.

Theorem: (Equation of a Plane)
An equation of the plane containing the point (x_0, y_0, z_0) with normal vector $\vec{N} = \langle A, B, C\rangle$ is

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Note: The equation of any plane can be expressed as

$$Ax + By + Cz = D.$$

This is called the **standard form** of the equation of a plane.

Example: Find an equation of the plane passing through the points $P = (-1, 2, 1)$, $Q = (0, -3, 2)$, and $R = (1, 1, -4)$.

Two vectors in the plane are

$$\overrightarrow{PQ} = \langle 1, -5, 1\rangle,$$
$$\overrightarrow{PR} = \langle 2, -1, -5\rangle.$$

A normal vector to the plane is

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -5 & 1 \\ 2 & -1 & -5 \end{vmatrix} = \langle 26, 7, 9 \rangle.$$

Thus, an equation of the plane is

$$\begin{gathered} 26(x+1) + 7(y-2) + 9(z-1) = 0 \\ 26x + 7y + 9z = -3. \end{gathered}$$

<u>Example:</u> Find an equation of the plane passing through the point $P = (1, 6, 4)$ and containing the line defined by $\vec{R}(t) = \langle 1 + 2t, 2 - 3t, 3 - t \rangle$.

The line passes through the point $Q = (1, 2, 3)$ and has direction vector $\vec{v} = \langle 2, -3, -1 \rangle$. Another vector in the plane is

$$\overrightarrow{PQ} = \langle 0, -4, -1 \rangle.$$

A normal vector to the plane is

$$\overrightarrow{PQ} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -4 & -1 \\ 2 & -3 & -1 \end{vmatrix} = \langle 1, -2, 8 \rangle.$$

The equation of the plane is

$$\begin{gathered} (x-1) - 2(y-6) + 8(z-4) = 0 \\ x - 2y + 8z = 21. \end{gathered}$$

<u>Example:</u> Find the point at which the line defined by $\vec{R}(t) = \langle 4 - t, 3 + t, 2t \rangle$ intersects the plane defined by $x - y + 3z = 5$.

Substituting the parametric equations into the equation of the plane gives

$$\begin{aligned} x - y + 3z &= 5 \\ (4-t) - (3+t) + 3(2t) &= 5 \\ 4t + 1 &= 5 \\ t &= 1. \end{aligned}$$

Thus, the point of intersection is $(3, 4, 2)$.

Definition: Two planes are **parallel** if they have the same normal vector (i.e. their normal vectors are parallel).

Note: If two planes are not parallel, then they intersect in a line. The angle between the two planes is the angle between their normal vectors.

Example: Consider the planes defined by $4x - 2y + z = 2$ and $2x + y - 4z = 3$.

(a) Find the angle between the planes.

The normal vectors are $\vec{N_1} = \langle 4, -2, 1 \rangle$ and $\vec{N_2} = \langle 2, 1, -4 \rangle$. If θ is the angle between the planes, then

$$\cos\theta = \frac{\vec{N_1} \cdot \vec{N_2}}{||\vec{N_1}||||\vec{N_2}||} = \frac{2}{\sqrt{21}\sqrt{21}} = \frac{2}{21}.$$

Therefore,

$$\theta = \cos^{-1}\left(\frac{2}{21}\right) \approx 84.5^\circ.$$

(b) Find parametric equations for the line of intersection.

To find the equation of the line of intersection, we need a point on the line and a direction vector. Since the line lies in both planes, it is orthogonal to both $\vec{N_1}$ and $\vec{N_2}$. Thus, a direction vector for the line is

$$\vec{N_1} \times \vec{N_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -2 & 1 \\ 2 & 1 & -4 \end{vmatrix} = \langle 7, 18, 8 \rangle.$$

Since the direction vector is not horizontal, the line must intersect the xy-plane ($z = 0$). Substituting $z = 0$ into the equations for the planes gives

$$\begin{cases} 4x - 2y &= 2 \\ 2x + y &= 3. \end{cases}$$

It follows that $x = 1$ and $y = 1$. Thus, $(1, 1, 0)$ is a point on the line of intersection. The parametric equations are

$$x = 1 + 7t, \quad y = 1 + 18t, \quad z = 8t.$$

Theorem: (Distance from a Point to a Plane)
The distance from (x_0, y_0, z_0) to the plane $Ax + By + Cz + D = 0$ is given by

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

Example: Find the distance between the parallel planes $2x-2y+z = 10$ and $4x-4y+2z = 2$.

The planes are parallel since their normal vectors are parallel. Indeed,

$$2\vec{N}_1 = 2\langle 2, -2, 1\rangle = \langle 4, -4, 2\rangle = \vec{N}_2.$$

Find a point on one plane and then find the distance from this point to the other plane.
Setting $x = y = 0$, the second plane contains $(0, 0, 1)$.
Now the distance between the planes is

$$d = \frac{|2(0) - 2(0) + 1 - 10|}{\sqrt{4 + 4 + 1}} = \frac{9}{3} = 3.$$

Definition: Two lines in $\mathbb{R}^3$ are **skew** if they are not parallel and do not intersect.

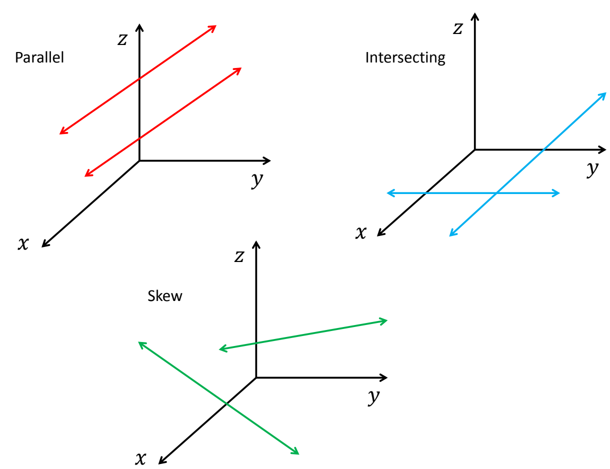

Figure 3: Illustration of parallel (red), intersecting (blue), and skew (green) lines.

Note: Skew lines lie in parallel planes.

Example: Find the distance between the skew lines defined by

$$L_1: \quad \vec{R}_1(t) = \langle 1 + t, 2 + 6t, 2t\rangle,$$
$$L_2: \quad \vec{R}_2(s) = \langle 2 + 2s, 4 + 14s, -3 + 5s\rangle.$$

The lines lie in parallel planes P_1 and P_2. Thus, the distance between the lines is the distance between the planes.
The common normal vector of the planes is orthogonal to the direction vectors $\vec{v_1} = \langle 1, 6, 2\rangle$ and $\vec{v_2} = \langle 2, 14, 5\rangle$. That is,

$$\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 6 & 2 \\ 2 & 14 & 5 \end{vmatrix} = \langle 2, -1, 2\rangle.$$

To define the planes P_1 and P_2, we need points on the planes (on the lines L_1 and L_2). Setting $t = 0$, we find that $(1, 2, 0)$ lies on L_1 and in P_1. Setting $s = 0$, we find that $(2, 4, -3)$ lies on L_2 and in P_2. Thus, the equation of plane P_2 is

$$\begin{gathered} 2(x-2) - (y-4) + 2(z+3) = 0 \\ 2x - y + 2z + 6 = 0. \end{gathered}$$

The distance from the point $(1, 2, 0)$ (in P_1) to the plane P_2 defined by $2x - y + 2z + 6 = 0$ is

$$d = \frac{|2(1) - 2 + 2(0) + 6|}{\sqrt{4+1+4}} = \frac{6}{3} = 2.$$