

Thoughts on measures of central tendency

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Introduction

Students of high school statistics learn three basic measures of central tendency: the mean, median, and mode. All of them have their strengths and weaknesses. The mean has a disadvantage of being easily affected by extreme values. The median, on the other hand, while being a true measure of the center, does not give sufficient information about the data because it is unaffected by the actual values of the points. The mode tends to be too unstable.

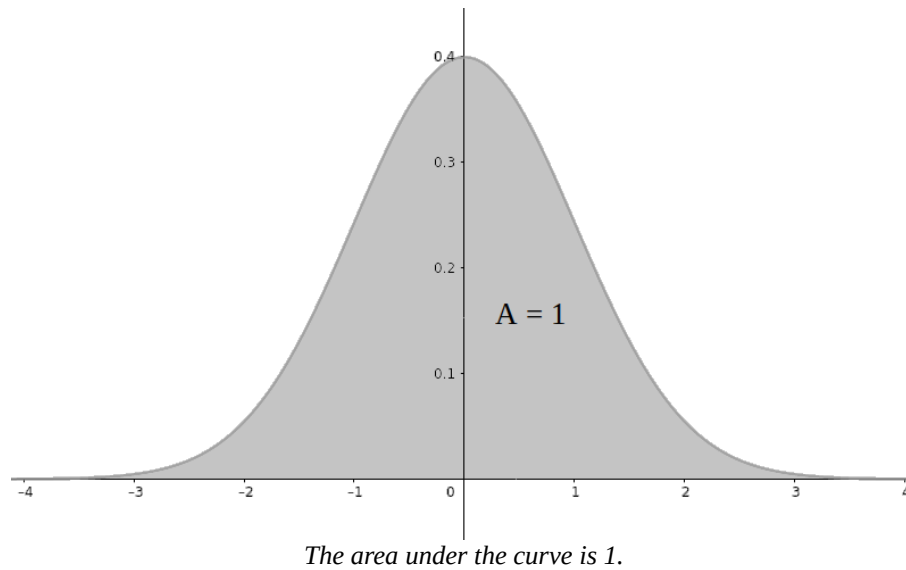
To work around these weaknesses, analysts would sometimes discard the highest and lowest values and take the mean of the remaining, or would discard the first and fourth quartiles and get the mean of the middle 50%, or some other variation of this. The disadvantage of this approach is that it involves some arbitrariness in the selection of how far from the center we consider significant enough to include. These methods build on the idea that values that are farther from the center should be considered less significant than those closer to the center, but do not provide a quantitative way to determine the actual significance of each data point.

Central Tendency

Thus I thought to look for an approach to measure central tendency such that the significance of each data point is quantified based on how far away they are from the center, so that extreme values would count less and those closer to the middle would count more. I could then get a weighted average according to this significance. In fact, I thought, the mean is just an average where all the values have equal weight, and the median is one where all but the middle one (or two) have zero weight. In my mind we should not be limited to these two extremes—there must be a middle ground, a better way to mathematically assign appropriate weights to each data point.

I considered several ways to do this, some more mathematical (a function of their distance from the mean or median), some based on physics (an inverse square relationship like how bodies that are farther away from the center of a force field have less effect on the system), but decided to work on an approach that is purely statistical.

While thinking about how the means of random subsets of a population would still form a normal distribution even though the population is non-normal (Central Limit Theorem), it randomly occurred to me that a normal curve can be used to assign a weight as a function of distance to a central value, even for distributions that are non-normal. The bell curve peaks at the center and diminishes rapidly at both ends, and we often set the area under the entire curve to be 1, thus the curve is a good candidate for assigning weights.



The problem with this approach is that the distribution assumes an uncountably infinite number of data points. Thus we need a way to apply this idea to discrete values.

Binomial Distribution

The coefficients of a power of a binomial can be expressed in terms of Pascal's Triangle, as seen in the following diagram we have learned in school. Each row represents a sequence of coefficients for a given power n of a binomial, where rows and columns are counted from 0.

			1			
		1		1		
	1		2		1	
	1	3		3	1	
	1	4	6	4	1	
	1	5	10	10	5	1
1	6	15	20	15	6	1

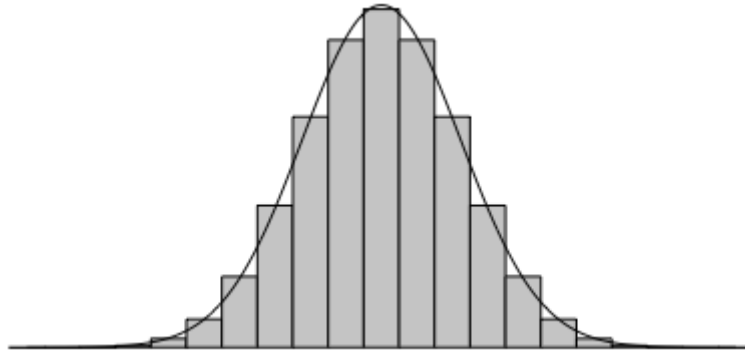
Pascal's Triangle

The numbers form a binomial distribution, and can be written in terms of combinations, e.g., the n^{th} row will have the following entries:

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n-1}, \binom{n}{n}$$

where each term $\binom{n}{k} = {}_nC_k = \frac{n!}{k!(n-k)!}$ is the number of combinations of n taken k at a time.

If we examine each row in Pascal's Triangle, we see that the number at the center is the largest, and tapers to 1 at both ends. The beauty of this sequence is that when n is very large, the shape of its graph looks that of the bell curve, as illustrated below:



The sequence of binomial coefficients form a bell curve.

If we get the sum of each row, we see that the entries of the n^{th} row just add up to 2^n . Thus we can convert it to a sequence whose sum is 1 by dividing each term by 2^n , as follows,

$$\frac{\binom{n}{0}}{2^n}, \frac{\binom{n}{1}}{2^n}, \frac{\binom{n}{2}}{2^n}, \dots, \frac{\binom{n}{n-1}}{2^n}, \frac{\binom{n}{n}}{2^n}$$

so that the area under the curve becomes 1. We can then write this as a function of k where for a given n the k^{th} term is defined by

$$f(k, n) = \frac{{}_nC_k}{2^n}$$

where n and k are non-negative integers and $k < n$. This is a discrete function that approaches a normal distribution as $n \rightarrow \infty$, and is called the *probability mass function* $f(k, n, p)$ for $p = 0.5$.

Applying Weights

I realized that this can be used as a way to apply weights to discrete data points so that values farther away from the center become less significant.

We first arrange the set of values in increasing order, as we would when getting the median, but instead of finding only the central value, we assign the centermost terms the largest weights, waning as it goes farther away from the median in the list order.

Let us take for example a set of seven numbers $N_1, N_2, N_3, N_4, N_5, N_6, N_7$ arranged in increasing order such that N_1 is the smallest and N_7 is the largest. We get the 7th row in Pascal's Triangle ($n = 6$, since we start counting at 0). We see that the values are 1, 6, 15, 20, 15, 6, 1. We then apply these as weights, as follows:

$$1N_1 + 6N_2 + 15N_3 + 20N_4 + 15N_5 + 6N_6 + 1N_7$$

The sum of the coefficients is $1 + 6 + 15 + 20 + 15 + 6 + 1 = 64$, which we can also see from the formula $2^n = 2^6 = 64$.

When getting an average in the usual sense, we divide the sum by 7 which is the number of terms. But since this is a *weighted* average, we divide it by 64 which is the sum of the weights:

$$\frac{1 N_1 + 6 N_2 + 15 N_3 + 20 N_4 + 15 N_5 + 6 N_6 + 1 N_7}{64}$$

This gives us a weighted average where the median N_4 has *20 times* more significance than the endpoints N_1 and N_7 , but where each endpoint still contributes one-64th of the value. Unlike the median, every term contributes some weight. But unlike the mean, extreme values are less likely to influence our measure. The more data points there are, the significance of the endpoints quickly vanish.

Take note that the assigned weights only depend on the order which they appear on the sequence, and is not affected at all by their actual values. This is on purpose, in order to keep the idea more in line with the median. By doing this, we are able to calculate a central value that is more representative of the data than either the mean or the median.

The formula

I started with an example to show how this works. But now I attempt to write down a general formula that can be used to calculate it in a spreadsheet or a computer algorithm.

Let $N_1, N_2, N_3, \dots, N_{n-1}, N_n$ be a set of n data points, arranged in increasing order such that N_1 is the smallest and N_n is the largest. The formula for my modified measure of central tendency can therefore be written as:

$$\frac{\sum_{k=0}^n \binom{n}{k} N_k}{2^n}$$

I do not have a name for it yet, nor a symbol for it. I intentionally did not think of one because I am unaware if such a thing already exists, or what it is called. But I decided to write this down in order to help those for whom this idea might be useful.

Whatever the case, this formula represents a measure of central tendency that, in my opinion, could be more representative of a central value for a set of data points than either the mean, median, or mode, and should be useful in many situations.