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The Equation of Time

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Summary. An equation is developed which gives the Equation of Time as a function of Universal Time. This enables it to be calculated for any epoch within 30 centuries of the present day, to a precision of about 3 s of time. We also give several expressions for the Equation of Ephemeris Time which ignores the distinction between the time-scale of the Ephemeris and Universal Time, and so may be compared with expressions given in old text books.

1 Introduction

The Equation of Time is the difference between solar time and mean time. Solar time at any instant is the hour angle of the Sun at that instant. The difference is calculated by considering two Suns, the actual Sun (known usually as the apparent Sun) which moves along the celestial ecliptic at a varying rate, this variation being due to the eccentricity of the Earth's orbit around the Sun; and the mean Sun (a fictitious object) that moves along the celestial equator at a uniform rate. Both orbit the Earth in exactly one tropical year. So mean time is indicated by the mean Sun, and apparent solar time is indicated by the actual Sun and is, for example, measured by sundials.

The precise definition of the Equation of Time is

$$E = GHA \text{ (apparent Sun)} - GHA \text{ (mean Sun)} \tag{1}$$

where GHA is the Greenwich hour angle. An alternative formula often quoted in text books is

$$E = \text{right ascension of the fictitious mean Sun} - \text{right ascension of the apparent (actual) Sun.} \tag{2}$$

In older text books the terms on the right-hand side of equation (2) are usually reversed. Rewriting equation (1) in terms that are calculable gives

$$E^\circ = (ST^\circ - \alpha^\circ) - (15\,UT^{\mathrm{h}} - 180^\circ) \tag{3}$$

where ST° is the Greenwich apparent sidereal time in degrees, which is a function of UT, α° is the apparent right ascension of the Sun in degrees and UT^{h} is the Universal Time in hours.

Since 1960 most text books have taken into account the difference between Universal Time (UT) and Ephemeris Time (ET). The precise meaning of these terms is important as they both occur as arguments in the Equation of Time. Definitions of these time-scales taken from the Glossary of *The Astronomical Almanac* are repeated in this section for clarification.

Universal Time (UT) conforms closely with the mean motion of the Sun and is the basis of all civil timekeeping. Since the Earth is slowing down it is not a uniform time-scale. UT is formally defined by a mathematical formula as a function of sidereal time. UT is determined from observations of the diurnal motions of the stars. The time-scale that is determined directly from such observations is designated UT0 and is slightly dependent on the place of observation. When UT0 is corrected for the shift in longitude of the observing station caused by polar motion, the time-scale UT1 is obtained. Whenever UT is used in this paper UT1 is implied. The time-scale available from broadcast signals is called Coordinated Universal Time (UTC). It is based on the International Atomic Time Scale TAI (which is uniform) and is maintained within $\pm 0^{\rm s}.9$ of UT1.

Ephemeris Time (ET) was the uniform time-scale used before 1984 as the independent variable in gravitational theories of the solar system. In 1984, ET was replace by Dynamical Time (DT). The time-scale of ephemerides for observations from the surface of the Earth is called Terrestial Dynamical Time (TDT); it is called Barycentric Dynamical Time (TDB) when it is referred to the barycentre of the Solar System. In the context of the Equation of Time it is not necessary to distinguish between them because the difference, due to variations in the gravitational potential around the Earth's orbit, does not exceed $0^{\rm s}.002$.

Older text books (e.g. Ferguson 1803; Hymers 1890; Godfray 1906; Ball 1908; Hosmer 1921) did not distinguish between UT and ET when discussing the Equation of Time. Woolard & Clemence (1966) suggested that replacing the term Equation of Time by the Equation of Ephemeris Time was more appropriate in such cases. In this paper we have adopted this suggestion so that the term Equation of Ephemeris Time is used when the difference between UT and ET has been ignored. On the other hand the term the Equation of Time is used when it is expressed as a function of UT, which is more useful in practical cases.

The use of the Equation of Time is complicated by two other factors. First, the exact form of the Equation varies slightly throughout the leap year cycle. Secondly, the Equation varies over longer time periods due to changes with the three parameters that it is dependent upon, these being the position of the perihelion of the Earth's orbit with respect to the First Point of Aries, the obliquity of the ecliptic and the eccentricity of the Earth's orbit. The main reason for the variation of the Equation of Ephemeris Time as a function of time over long periods of time is the precession of the line of apsides of the Earth's orbit. In essence the longitude of perigee is increasing by about $1^\circ.72$ per century. So around the year 4000 BC at the time of the building of the Great Pyramid in Egypt, perihelion passage occurred near the autumnal equinox, whereas now it occurs around January 2.

In Section 2 of this paper we give an algorithm for calculating the Equation of Time, valid over 60 centuries, to an accuracy of 3 s of time. The formulation of the Equation of Ephemeris Time in Section 3 allows us to look at the form of the function through its parameters. Section 4 gives several expressions for the Equation of Ephemeris Time which can be compared with versions in old text books.

2 Algorithm for calculating the Equation of Time

This algorithm calculates the Equation of Time for any specified calendar date and Universal Time accurate to 3 s of time over a period of within 30 centuries of the present day.

The arguments of the solar orbit are based on the epoch of J2000.0 and on the formulae published in *The Astronomical Almanac*, while the expression for sidereal time in terms of Universal Time is based on the 1976 IAU system of astronomical constants, the 1980 IAU theory of nutation and the equinox of the FK5 catalogue. An approximation made in this algorithm is to ignore the effect of nutation, which is small, and replace apparent quantities by mean quantities. There are new improved theories for the motion of the Earth based on Bretagnon (1982) which should be used if greater accuracy were desired.

Step A

The Julian date JD at $0^{\rm h}$ UT (i.e. midnight) is calculated (see Hatcher 1984) from the calendar date $Y-M-D$ where Y is the year, M is the month and D is the day of the month as follows:

(1) If $M>2$ set $y=Y$ and $m=M-3$, otherwise set $y=Y-1$ and $m=M+9$.

(2) Calculate

$$J=[365.25(y+4712)]+[30.6m+0.5]+59+D-0.5$$

where $[x]$ means 'take the integer part of x'.

(3) If the calendar date is Gregorian, set

$$G_{\rm n}=38-[3[49+y/100]/4].$$

For the Julian calendar, $G_{\rm n}=0$.

(4) Calculate the Julian date from

$$JD=J+G_{\rm n}.$$

Step B

Calculate the time arguments, t, the interval from 2000 January 1 at $12^{\rm h}$ UT to the date (JD) and time (UT) required in Julian centuries of 36525 d, and T the corresponding time interval in Dynamical Time:

$$t=(JD+UT^{\rm h}/24-2451545.0)/36525$$

$$T=t+\Delta T(t),$$

where $\Delta T(t)=0$ between AD 1650 and AD 1900

$\Delta T(t)=[-3.36+1.35(t+2.33)^2]\times 10^{-8}$ elsewhere.

This expression for ΔT is based on the work by Muller (1975). More recent estimates are given by Stephenson & Morrison (1984).

Step C

Calculate the Greenwich mean sidereal time from

$$ST^{\circ}=100.4606-36000.77005t+0.000388\,t^2-3\times 10^{-8}\,t^3$$

Step D

Calculate the right ascension of the apparent Sun using the Dynamical Time interval T.

(1) Calculate the solar arguments; the geometric mean ecliptic longitude of date (L), the mean anomaly (G), the mean obliquity of the ecliptic (ε), and the equation of the centre (C). Thus

$$L^{\circ}=280^{\circ}\!.46607+36000^{\circ}\!.76980\,T+0^{\circ}\!.000\,3025\,T^2$$

$$G^{\circ}=357^{\circ}\!.528+35999^{\circ}\!.0503\,T$$

$$\varepsilon^{\circ}=23^{\circ}\!.4393-0^{\circ}\!.01300\,T-0^{\circ}\!.000\,0002\,T^2+0^{\circ}\!.000\,0005\,T^3$$

and

$$C^{\circ}=(1^{\circ}\!.9146-0^{\circ}\!.00484\,T-0^{\circ}\!.000014\,T^2)\sin G+(0^{\circ}\!.01999-0^{\circ}\!.00008\,T)\sin 2G.$$

(2) Calculate the ecliptic longitude of date, ($L_\odot$), from the mean longitude by applying aberration and the correction to centre. This gives

$L^\circ_\odot = L^\circ + \check{C}^\circ - 0^\circ.0057.$

(3) Calculate the right ascension from

$\alpha^\circ = L^\circ_\odot - yf \sin 2L_\odot + 0.5\, y^2 f \sin 4L_\odot,$

where $y = \tan^2 \varepsilon/2$ and $f = 180/\pi$.

Step E

As stated in equation (3) the Equation of Time, E, in degrees is given by

$E = (ST^\circ - \alpha^\circ) - (15\,UT^{\mathrm{h}} - 180^\circ)$

If $E > 10^\circ$ then $E = E - 360^\circ$

This last line ensures that the discontinuities at 360° are taken into account.

4 The Equation of Ephemeris Time

In the past many text books quote the Equation of Ephemeris Time formula in the simplified form

$$E = L - \alpha, \tag{4}$$

where L is the longitude of the mean Sun and α is the right ascension of the apparent Sun. This formula makes the approximation that $ST^\circ = L^\circ - 180$ and that Dynamical Time = UT. The approximation that sidereal time equals the Sun's mean longitude shifted by 180° ignores small T and T^2 terms which nevertheless introduces significant errors after about a thousand years from the present epoch. These terms appear as the first two terms of equation (9) where the correct formula for sidereal time has been used to form a long-term expression for calculating the Equation of Ephemeris Time.

An approximate expression for α is given in Step D(3) in Section 2; a more precise expansion is

$$\alpha = L_\odot - \tan^2\frac{\varepsilon}{2}\sin 2L_\odot + \frac{1}{2}\tan^4\frac{\varepsilon}{2}\sin 4L_\odot - \frac{1}{3}\tan^6\frac{\varepsilon}{2}\sin 6L_\odot + \frac{1}{4}\tan^8\frac{\varepsilon}{2}\sin 8L_\odot + \ldots, \tag{5}$$

where

$$L_\odot = L + \left(2e - \frac{e^3}{4}\right)\sin(L-\omega) + \frac{5}{4}e^2\sin(2L-2\omega) + \frac{13}{12}e^3\sin(3L-3\omega) + \ldots \tag{6}$$

and ε is the obliquity of the ecliptic, $L_\odot$ is the longitude of the apparent Sun and is obtained by expressing the true anomaly of the Sun in terms of the mean anomaly, $\omega = L - G$ is the longitude of perigee of the Sun's orbit around the Earth and e is the eccentricity of this orbit. As e is approximately 0.017, terms beyond the cube can be ignored here. Note that equations (5) and (6) for $L_\odot$ and α are in radians.

Using this formulation it can be seen that the Equation of Time has two major components. The first (marked E in Fig. 1) is due to the ellipticity of the Earth's orbit. By substituting into equation (6) it can be seen that the amplitude of the 'component E' curve in Fig. 1 is less than 8 minutes of time. The E curve crosses the zero abscissa at perihelion and aphelion passage time. At perihelion (January 2, 0500^{h} in 1986) the apparent Sun and the mean Sun are coincident. The apparent Sun, however, has its greatest velocity at this time and will shoot ahead of the mean Sun, the interval between them continuing to increase as long as the angular velocity of

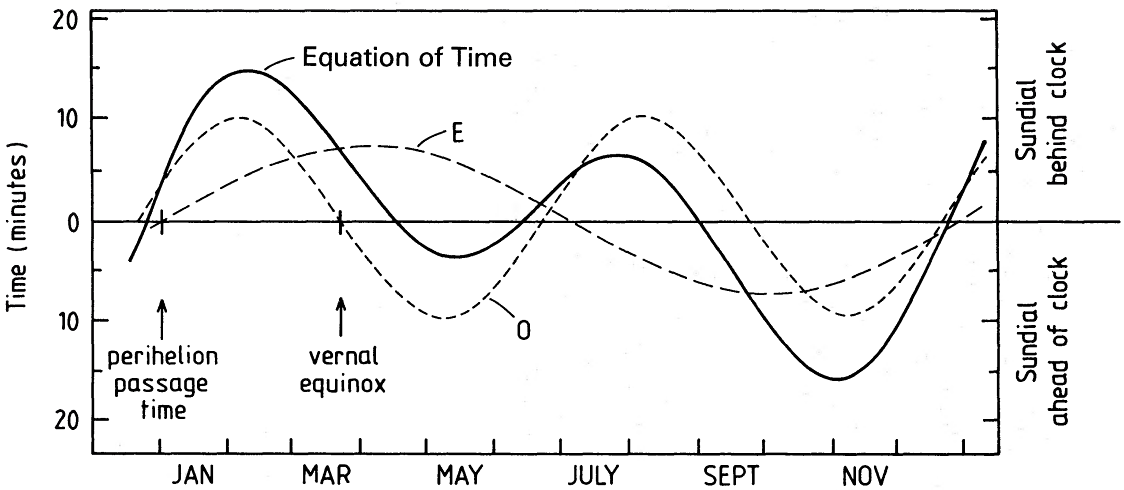

Figure 1. The present day Equation of Time is plotted as a function of time throughout the year. When the curve is above the abscissa the apparent solar time is less than the mean solar time, i.e. the sundial is slow and the Equation of Time has to be added to its reading to give mean-time. The Equation of Time is made up of two components both shown as dashed curves. The one marked 'E' is due to the ellipticity of the Sun's orbit. It has a period of one year and crosses the abscissa at perihelion passage time and aphelion passage time. The one marked 'O' is due to the apparent Sun moving along the ecliptic whereas the mean Sun moves along the celestial equator. This crosses the abscissa at the equinoxes and solstices. These are not separated by 90° of longitude. At the present time, for example, winter, spring, summer and autumn occupy 87°.71, 91°.42, 92°.32 and 88°.55 respectively.

the apparent Sun exceeds its mean value. Just beyond a solar longitude of 90° from perihelion, the two angular velocities become equal and after that time the interval between the two diminishes until it becomes zero at aphelion. Between perihelion and aphelion a specific earth meridian overtakes the mean Sun before the apparent Sun. So mean noon occurs before true noon and the component of the Equation of Time due to this cause is additive. Similar reasoning shows that the aphelion and perihelion movements results in a negative quantity.

The second component (marked O in Fig. 1) is due to the mean Sun moving along the celestial equator whereas the apparent Sun moves along the ecliptic. This curve has an amplitude of about 9.9 minutes of time and crosses the zero abscissa at the equinoxes and solstices. The curve is only quasi-sinusoidal because the lengths of the seasons are not constant (for example, at the present time, summer solstice to autumnal equinox is ≈ 93.6 d whereas autumnal equinoxes to winter solstice is ≈ 89.8 d, see Hughes 1989).

The journey beyond the complexities of equations (5) and (6) is not helped by the fact that the Earth's orbit is only an ellipse to a first order approximation. The gravitational influence of the Moon and the other planets gives the Earth a somewhat irregular path.

The calculation of, for example, the perihelion passage time is not straightforward. The simplest approach is to look each year for the minima in dr/dt, the time differential of the Earth–Sun distance. As this is a very slowly varying function, this minima is usually only given to the nearest hour (see for example, Roberts & Boksenberg, 1985). Prior to 1975 it was only given to the nearest day. It should also be noted that the time between perihelion and aphelion passage is only approximately half a year and easily can vary by a day or two around this value. An alternative approach to the problem is to produce a long-term mean orbit for the Earth/Moon barycentre and obtain the perihelion passage time for this orbit.

4 Approximate expressions for the Equation of Ephemeris Time

A much simplified expression for the Equation of Time can be obtained by substituting equations (5) and (6) into equation (4) and then neglecting the minor terms. Thus

$$E = -2e\sin(L-\omega) + \tan^2\frac{\varepsilon}{2}\sin 2L \tag{7}$$

where E is expressed in radians, and does not deviate, for the present epoch, by more than 18 s of time from those obtained by using the more complete approach.

Yallop (1978) quoted a formula for the Equation of Ephemeris Time based on one by Smart (1971), precise to 1 arcmin. His expression for E in degrees is

$$\begin{aligned} E = & -(0.388+0.0593T-0.00006T^2)\sin L-(1.802-0.0155T-0.00086T^2)\cos L \\ & +(2.487-0.0034T-0.00004T^2)\sin 2L-(0.006+0.0012T)\cos 2L \\ & +(0.016+0.0025T)\sin 3L+(0.081-0.0009T-0.00004T^2)\cos 3L \\ & -(0.053-0.0001T)\sin 4L, \end{aligned} \tag{8}$$

where T is measured in Julian centuries from 1900 January 0.5 ET (i.e. Julian Day 2415020.0). For epoch AD 2000, the difference between Yallop's value and one derived from a precise formulation never exceeds 4 s of time (1 arcmin). For epoch 0 the difference between the two calculated values never exceeds 13 s of time.

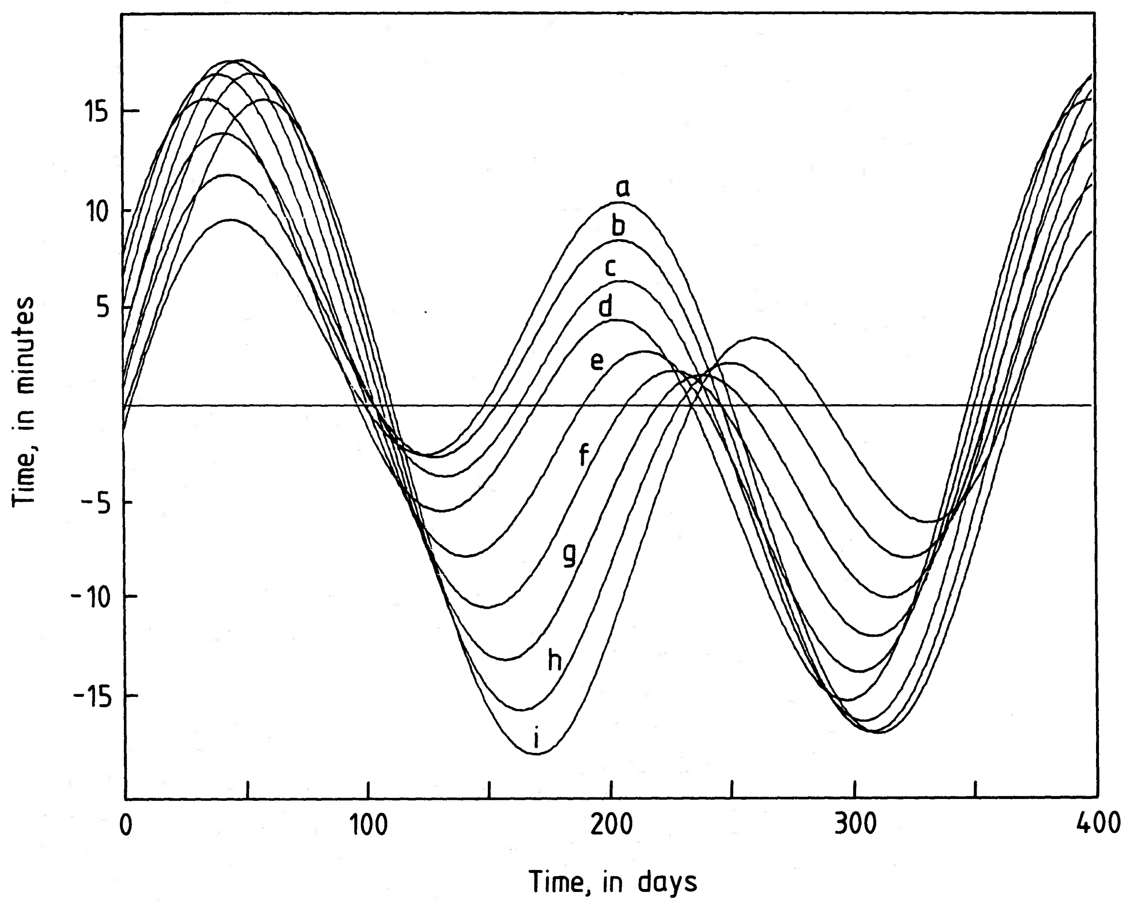

Figure 2. The Equation of Ephemeris Time is plotted as a function of time throughout the year. This equation has been calculated for nine different epochs. Anno Domini 4000(a), 3000(b), 2000(c), 1000(d), 0(e), −1000(f), −2000(g), −3000(h) and −4000(i). In each case the zero point on the abscissa corresponds to January 0, at noon, of the year in question. These zeros are equivalent to Julian Days 3182030.5, 2816788.0, 2451545.0, 2086308.0, 1721058.0, 1355808.0, 990558.0, 625308.0 and 260058.0, respectively. The Equation has been plotted for the following 400 d in each case. Dates after, and including, AD 2000 are according to the Gregorian calendar, dates before AD 2000 are Julian.

A short, medium precision expression for the Equation of Ephemeris Time in radians, adequate for use over a time period of 60 centuries may be derived from the method in Section 2 by setting $\Delta T = 0$, i.e. $T = t$. Thus

$$E = 4.47\times 10^{-6}\, T + 1.49\times 10^{-6}\, T^2 - 2e\sin G - \tfrac{5}{4}\, e^2 \sin 2G + y\sin 2L - \tfrac{1}{2}\, y^2 \sin 4L$$

$$+ 4ey\sin G\cos 2L + \tfrac{5}{2}\, e^2\, y\sin 2G\cos 2L - 4ey^2 \sin G\cos 4L - 8e^2 y\sin^2 G\sin 2L, \qquad (9)$$

where $e = 0.016708 - 0.000423T - 0.00000013T^2$, is the eccentricity of the Earth's orbit, and the equations for L, G and ε are given in Step D(1). The two most dominant periodic terms in this expression are those given in equation (7) above. Over a range of 60 centuries the errors produced by using this formula are within $\pm 0\overset{\prime}{.}8$ or about ± 3.2 s.

Fig. 2 shows the variation of the Equation of Ephemeris Time, using this expression, every thousand years between 4000 BC and AD 4000.

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