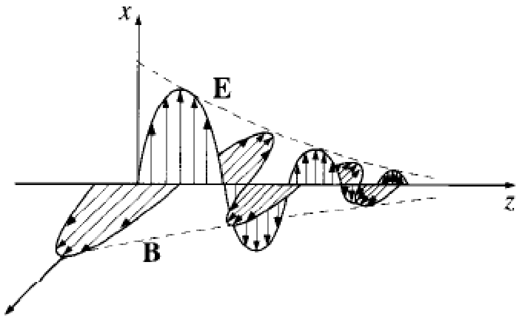
Chapter 9. Electromagnetic Waves

9.4 Absorption and Dispersion



9.4.1 Electromagnetic waves in Conductors

According to Ohm's law, the (free) current density is proportional to the electric field: $\mathbf{J}_f = \sigma \mathbf{E}$

Maxwell' s equations for linear media with no free charge assume the form,

$$\left.\begin{array}{ll}\text{(i)}\ \nabla\cdot\mathbf{E}=0, & \text{(iii)}\ \nabla\times\mathbf{E}=-\dfrac{\partial\mathbf{B}}{\partial t},\\[2ex] \text{(ii)}\ \nabla\cdot\mathbf{B}=0, & \text{(iv)}\ \nabla\times\mathbf{B}=\mu\epsilon\dfrac{\partial\mathbf{E}}{\partial t}+\mu\sigma\mathbf{E}.\end{array}\right\} \Rightarrow \begin{array}{l}\nabla^2\mathbf{E}=\mu\epsilon\dfrac{\partial^2\mathbf{E}}{\partial t^2}+\mu\sigma\dfrac{\partial\mathbf{E}}{\partial t}\\[2ex] \nabla^2\mathbf{B}=\mu\epsilon\dfrac{\partial^2\mathbf{B}}{\partial t^2}+\mu\sigma\dfrac{\partial\mathbf{B}}{\partial t}\end{array}$$

Plane-wave solutions are $\tilde{\mathbf{E}}(z,t)=\tilde{\mathbf{E}}_0 e^{i(\tilde{k}z-\omega t)}, \quad \tilde{\mathbf{B}}(z,t)=\tilde{\mathbf{B}}_0 e^{i(\tilde{k}z-\omega t)}$

complex wave number

$$\tilde{k}^2=\mu\epsilon\omega^2+i\mu\sigma\omega \xrightarrow{\tilde{k}=k+i\kappa} \begin{array}{l} k\equiv\omega\sqrt{\dfrac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\dfrac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{1/2}\\[2ex] \kappa\equiv\omega\sqrt{\dfrac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\dfrac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{1/2}\end{array}$$

$$\boxed{\tilde{\mathbf{E}}(z,t)=\tilde{\mathbf{E}}_0 e^{-\kappa z}e^{i(kz-\omega t)}, \quad \tilde{\mathbf{B}}(z,t)=\tilde{\mathbf{B}}_0 e^{-\kappa z}e^{i(kz-\omega t)}}$$

The imaginary part, κ, results in an attenuation of the wave (decreasing amplitude with increasing *z*):

$d\equiv\dfrac{1}{\kappa}$; **→ skin depth**

Determine the relative amplitudes, phases, and E and B in conductors

For E field polarized along the x direction, $\tilde{\mathbf{E}}(z,t) = \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)}\,\hat{\mathbf{x}}$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \;\longrightarrow\; \tilde{\mathbf{B}}(z,t) = \frac{\tilde{k}}{\omega}\tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)}\,\hat{\mathbf{y}}$$

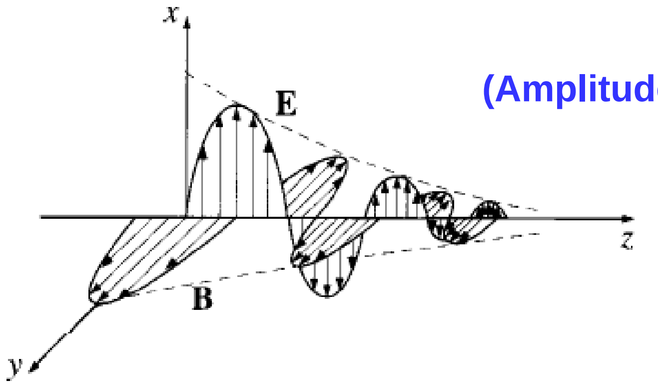
Let's express the complex wave number in terms of its modulus and phase

$$\tilde{k} = k + i\kappa = K e^{i\phi} \quad \begin{cases} K \equiv |\tilde{k}| = \sqrt{k^2+\kappa^2} = \omega\sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}} \\ \phi \equiv \tan^{-1}(\kappa/k) \end{cases}$$

$$\tilde{E}_0 = E_0 e^{i\delta_E} \text{ and } \tilde{B}_0 = B_0 e^{i\delta_B} \;\longrightarrow\; B_0 e^{i\delta_B} = \frac{K e^{i\phi}}{\omega} E_0 e^{i\delta_E}$$

(Phase) $\longrightarrow$ $\delta_B - \delta_E = \phi;$ **E and B fields are no longer in phase**

→ *B field lags behind E field.*



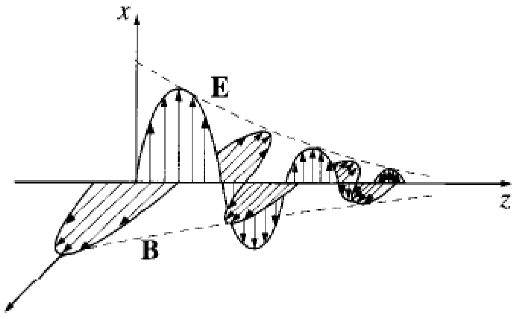
(Amplitude) $\longrightarrow$ $$\frac{B_0}{E_0} = \frac{K}{\omega} = \sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}}$$

$\longrightarrow$ The (real) electric and magnetic fields are, finally,

$$\left.\begin{aligned} \mathbf{E}(z,t) &= E_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E)\,\hat{\mathbf{x}}, \\ \mathbf{B}(z,t) &= B_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E+\phi)\,\hat{\mathbf{y}}. \end{aligned}\right\}$$

Energy density and intensity in conductors

$$\left.\begin{aligned}\mathbf{E}(z,t) &= E_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E)\,\hat{\mathbf{x}},\\ \mathbf{B}(z,t) &= B_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E+\phi)\,\hat{\mathbf{y}}.\end{aligned}\right\}$$



Problem 9.21 (a) Calculate the (time averaged) energy density of an electromagnetic plane wave in a conducting medium. Show that the magnetic contribution always dominates.

(b) Show that the intensity is $(k/2\mu\omega)E_0^2 e^{-2\kappa z}$

$$\text{(a)}\ u = \frac{1}{2}\left(\epsilon E^2 + \frac{1}{\mu}B^2\right) = \frac{1}{2}e^{-2\kappa z}\left[\epsilon E_0^2\cos^2(kz-\omega t+\delta_E) + \frac{1}{\mu}B_0^2\cos^2(kz-\omega t+\delta_E+\phi)\right]$$

$$\langle u\rangle = \frac{1}{2}e^{-2\kappa z}\left[\frac{\epsilon}{2}E_0^2 + \frac{1}{2\mu}B_0^2\right] = \frac{1}{4}e^{-2\kappa z}\left[\epsilon E_0^2 + \frac{1}{\mu}E_0^2\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}\right] = \frac{1}{4}e^{-2\kappa z}\epsilon E_0^2\left[1+\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}\right]$$

$$k \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{1/2} \longrightarrow 1+\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2} = \frac{2}{\epsilon\mu}\frac{k^2}{\omega^2} \longrightarrow$$

The ratio of the magnetic contribution to the electric contribution is $\dfrac{\langle u_{\text{mag}}\rangle}{\langle u_{\text{elec}}\rangle} = \dfrac{B_0^2/\mu}{E_0^2\epsilon} = \dfrac{1}{\mu\epsilon}\mu\epsilon\sqrt{1+\left(\dfrac{\sigma}{\epsilon\omega}\right)^2} = \sqrt{1+\left(\dfrac{\sigma}{\epsilon\omega}\right)^2} > 1$

$$\text{(b)}\ \mathbf{S} = \frac{1}{\mu}(\mathbf{E}\times\mathbf{B}) = \frac{1}{\mu}E_0B_0e^{-2\kappa z}\cos(kz-\omega t+\delta_E)\cos(kz-\omega t+\delta_E+\phi)\,\hat{\mathbf{z}} \longrightarrow$$

The average of the product of the cosines is $(1/2\pi)\int_0^{2\pi}\cos\theta\cos(\theta+\phi)\,d\theta = (1/2)\cos\phi$

$$I = \frac{1}{2\mu}E_0B_0e^{-2\kappa z}\cos\phi = \frac{1}{2\mu}E_0^2e^{-2\kappa z}\left(\frac{K}{\omega}\cos\phi\right) \xrightarrow[K\cos\phi = k]{\tilde{k} = k+i\kappa = Ke^{i\phi}}$$

Helmholtz Equation (Wave Equation in Frequency Domain)

$$\nabla^2 \mathbf{E} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{E}}{\partial t}$$
$$\nabla^2 \mathbf{B} = \mu\epsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla^2 \psi(r,t) - \mu\sigma \frac{\partial \psi(r,t)}{\partial t} - \mu\varepsilon \frac{\partial^2 \psi(r,t)}{\partial t^2} = 0$$

→ **Wave equation in space-domain**

Let us consider the Fourier transform of the electromagnetic field: $\psi(\mathbf{r}, t) \leftrightarrow \tilde{\psi}(\mathbf{r}, \omega)$

$$\tilde{\psi}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} \psi(\mathbf{r}, t)\, e^{-j\omega t}\, dt \longleftrightarrow \psi(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\psi}(\mathbf{r}, \omega)\, e^{j\omega t}\, d\omega$$

$\tilde{\psi}(\mathbf{r}, \omega)$, *frequency spectrum* of $\psi(\mathbf{r}, t)$

$$\frac{\partial}{\partial t}\psi(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} j\omega\tilde{\psi}(\mathbf{r}, \omega)\, e^{j\omega t}\, d\omega$$

$$\nabla^2 \psi(r,t) - \mu\sigma \frac{\partial \psi(r,t)}{\partial t} - \mu\varepsilon \frac{\partial^2 \psi(r,t)}{\partial t^2} = \left(\nabla^2 - \mu\sigma\frac{\partial}{\partial t} - \mu\epsilon\frac{\partial^2}{\partial t^2}\right) \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\psi}(\mathbf{r}, \omega)\, e^{j\omega t}\, d\omega = 0$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\left(\nabla^2 - j\omega\mu\sigma + \omega^2\mu\epsilon\right) \tilde{\psi}(\mathbf{r}, \omega)\right] e^{j\omega t}\, d\omega = 0$$

Helmholtz equation

$$\left(\nabla^2 + \tilde{k}^2\right)\tilde{\psi}(\mathrm{r},\omega) = 0$$

where $\tilde{k} = k + j\kappa = \omega\sqrt{\mu\varepsilon}\sqrt{1 - j\dfrac{\sigma}{\omega\varepsilon}}$

→ **Wave equation in frequency domain**

Frequency-domain Maxwell equations in a source-free space

Using the temporal inverse Fourier transform,

$$\mathbf{E}(\mathbf{r},t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\mathbf{E}}(\mathbf{r},\omega)\,e^{j\omega t}\,d\omega \qquad \text{where} \quad \tilde{\mathbf{E}}(\mathbf{r},\omega)=\sum_{i=1}^{3}\hat{\mathbf{i}}_i\,\tilde{E}_i(\mathbf{r},\omega)=\sum_{i=1}^{3}\hat{\mathbf{i}}_i\,|\tilde{E}_i(\mathbf{r},\omega)|e^{j\xi_i^E(\mathbf{r},\omega)}$$

$$\nabla\times\mathrm{H}=J+\frac{\partial \mathrm{D}}{\partial t} \Rightarrow \nabla\times\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\mathbf{H}}(\mathbf{r},\omega)\,e^{j\omega t}\,d\omega=\frac{\partial}{\partial t}\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\mathbf{D}}(\mathbf{r},\omega)\,e^{j\omega t}\,d\omega+\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\mathbf{J}}(\mathbf{r},\omega)\,e^{j\omega t}\,d\omega$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty}[\nabla\times\tilde{\mathbf{H}}(\mathbf{r},\omega)-j\omega\tilde{\mathbf{D}}(\mathbf{r},\omega)-\tilde{\mathbf{J}}(\mathbf{r},\omega)]e^{j\omega t}\,d\omega=0 \Rightarrow \nabla\times\tilde{\mathbf{H}}=j\omega\tilde{\mathbf{D}}+\tilde{\mathbf{J}}$$

By similar reasoning, finally we can have the Maxwell's equations in frequency domain:

$$\nabla\times\tilde{\mathrm{H}}(\mathrm{r},\omega)=\tilde{J}(\mathrm{r},\omega)+j\omega\tilde{\mathrm{D}}(\mathrm{r},\omega)$$
$$\nabla\times\tilde{\mathrm{E}}(\mathrm{r},\omega)=-j\omega\tilde{\mathrm{B}}(\mathrm{r},\omega)$$
$$\nabla\cdot\tilde{\mathrm{D}}(\mathrm{r},\omega)=\tilde{\rho}(\mathrm{r},\omega)$$
$$\nabla\cdot\tilde{\mathrm{B}}(\mathrm{r},\omega)=0$$

and $\nabla\cdot\tilde{\mathrm{J}}(\mathrm{r},\omega)=-j\omega\tilde{\rho}(\mathrm{r},\omega)$

→ *The frequency-domain equations involve one fewer derivative (the time derivative has been replaced by multiplication by jω), hence may be easier to solve.*
→ *However, the inverse transform may be difficult to compute.*

9.4.2 Reflection at a Conducting Surface

The general boundary conditions for electrodynamics;

$$\text{(i)}\ \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f, \qquad \text{(iii)}\ \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0,$$

$$\text{(ii)}\ B_1^{\perp} - B_2^{\perp} = 0, \qquad \text{(iv)}\ \frac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}},$$

(σ_f : the free surface charge)
(K_f *:* the free surface current)

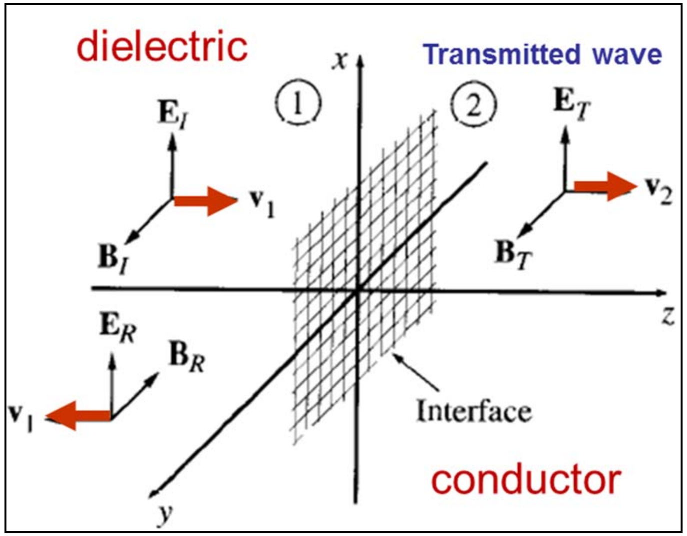
Consider a monochromatic plane wave, traveling in *z*, polarized in x (TM), approaches from the left,

$$\tilde{\mathbf{E}}_I(z,t) = \tilde{E}_{0_I} e^{i(k_1 z - \omega t)}\,\hat{\mathbf{x}}, \quad \tilde{\mathbf{B}}_I(z,t) = \frac{1}{v_1}\tilde{E}_{0_I} e^{i(k_1 z - \omega t)}\,\hat{\mathbf{y}}$$

$$\tilde{\mathbf{E}}_R(z,t) = \tilde{E}_{0_R} e^{i(-k_1 z - \omega t)}\,\hat{\mathbf{x}}, \quad \tilde{\mathbf{B}}_R(z,t) = -\frac{1}{v_1}\tilde{E}_{0_R} e^{i(-k_1 z - \omega t)}\,\hat{\mathbf{y}}$$

$$\tilde{\mathbf{E}}_T(z,t) = \tilde{E}_{0_T} e^{i(\tilde{k}_2 z - \omega t)}\,\hat{\mathbf{x}}, \quad \tilde{\mathbf{B}}_T(z,t) = \frac{\tilde{k}_2}{\omega}\tilde{E}_{0_T} e^{i(\tilde{k}_2 z - \omega t)}\,\hat{\mathbf{y}}$$

attenuated as it penetrates into the conductor



Since $E^{\perp} = 0$ on both sides, boundary condition (i) yields $\sigma_f = 0$.

(iii) gives $\tilde{E}_{0_I} + \tilde{E}_{0_R} = \tilde{E}_{0_T}$

and (iv) (with $\mathbf{K}_f = 0$) says $\tilde{E}_{0_I} - \tilde{E}_{0_R} = \tilde{\beta}\tilde{E}_{0_T}$ where $\tilde{\beta} \equiv \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}_2$

$$\boxed{\tilde{E}_{0_R} = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\tilde{E}_{0_I}, \quad \tilde{E}_{0_T} = \left(\frac{2}{1+\tilde{\beta}}\right)\tilde{E}_{0_I}}$$

For a *perfect* conductor $(\sigma = \infty)$, $k_2 = \infty \longrightarrow \tilde{\beta} = \infty \longrightarrow \tilde{E}_{0_R} = -\tilde{E}_{0_I}, \quad \tilde{E}_{0_T} = 0$

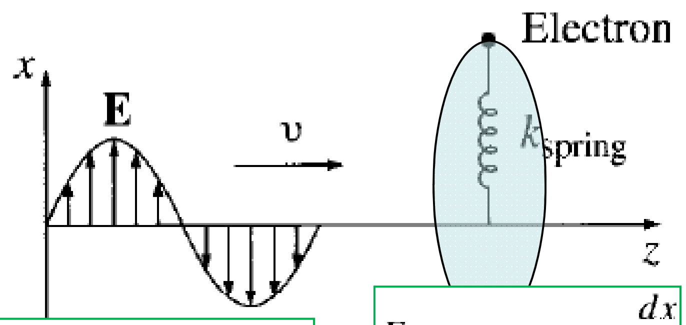
→ The wave is totally reflected, with a 180º phase shift.

9.4.3 Frequency dependence of permittivity in dielectric media

The electrons in a dielectric are bounded to specific molecules.

$$F_{net} = F_{binding} + F_{damping} + F_{driving} = m\frac{d^2\tilde{x}}{dt^2}$$

$$F_{\text{binding}} = -k_{\text{spring}}x = -m\omega_0^2 x$$



$$F_{\text{driving}} = qE = qE_0\cos(\omega t)$$

$$F_{\text{damping}} = -m\gamma\frac{dx}{dt}$$

$$\frac{d^2\tilde{x}}{dt^2} + \gamma\frac{d\tilde{x}}{dt} + \omega_0^2\tilde{x} = \frac{q}{m}E_0 e^{-i\omega t}$$

$$\tilde{x}(t) = \tilde{x}_0 e^{-i\omega t}$$

$$\tilde{x}_0 = \frac{q/m}{\omega_0^2 - \omega^2 - i\gamma\omega}E_0$$

Dipole moment ➔ $\tilde{p}(t) = q\tilde{x}(t) = \frac{q^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega}E_0 e^{-i\omega t}$

If there are *N* molecules per unit volume

$$\tilde{\mathbf{P}} = \frac{Nq^2}{m}\left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right)\tilde{\mathbf{E}} \longrightarrow \tilde{\mathbf{P}} = \epsilon_0\tilde{\chi}_e\tilde{\mathbf{E}}$$

$\tilde{\epsilon} = \epsilon_0(1+\tilde{\chi}_e)$ **: complex permittivity** $\longrightarrow$ $\tilde{k} \equiv \sqrt{\tilde{\epsilon}\mu_0}\,\omega = k + i\kappa$

$$\tilde{\epsilon}_r = 1 + \frac{Nq^2}{m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}$$

$\alpha \equiv 2\kappa$ **: Absorption coefficient**

$n = \frac{ck}{\omega}$ **: Refractive index**

Frequency dependence of permittivity (Dispersion)

$\tilde{\epsilon} = \epsilon_0(1+\tilde{\chi}_e)$ **: complex permittivity** $\longrightarrow$ $\tilde{\epsilon}_r = 1 + \frac{Nq^2}{m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}$

$\tilde{k} \equiv \sqrt{\tilde{\epsilon}\mu_0}\,\omega = k + i\kappa$ $\longrightarrow$

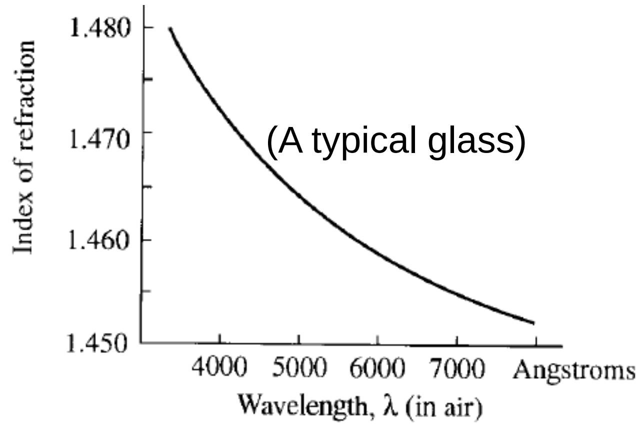
$\alpha \equiv 2\kappa$ **: Absorption coefficient**

$n = \frac{ck}{\omega}$ **: Refractive index**

➔ n is a function of frequency (wavelength)

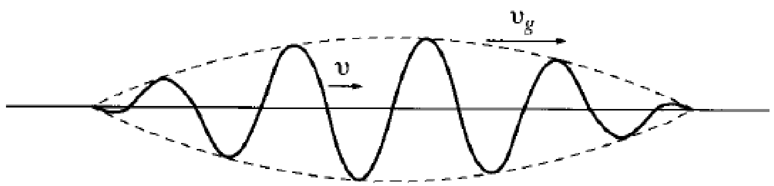
A prism spreads white light out into a rainbow of colors.
→ This phenomenon is called **dispersion**.

The speed of a wave depends on its frequency,
→ The supporting medium is called **dispersive**.



Wave velocity (Phase velocity) → $v = \frac{\omega}{k}$

Group velocity → $v_g = \frac{d\omega}{dk}$

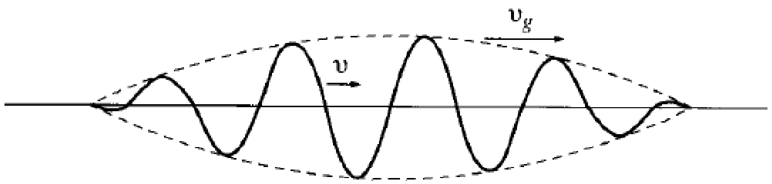


The energy carried by a wave packet in a dispersive medium ordinarily travels at the group velocity, not the phase velocity.

Phase Velocity and Group Velocity

Phase velocity → $v = \dfrac{\omega}{k}$

Group velocity → $v_g = \dfrac{d\omega}{dk}$



Problem 9.23 In quantum mechanics, a free particle of mass *m* traveling in the *x* direction is described by the wave function

$$\Psi(x,t) = Ae^{i(px-Et)/\hbar} \qquad E = p^2/2m$$

Calculate the group velocity and the phase velocity.
Which one corresponds to the classical speed of the particle?

$$\frac{i(px-Et)}{\hbar} = i(kx-\omega t) \longrightarrow k = \frac{p}{\hbar} \quad \omega = \frac{E}{\hbar} \longrightarrow$$

$$v_g = \frac{d\omega}{dk} \longrightarrow$$

$$v = \frac{1}{2}v_g$$ → Note that the phase (wave) velocity is *half* the group velocity.

Since $p = mv_c$ (where v_c is the classical speed of the particle),

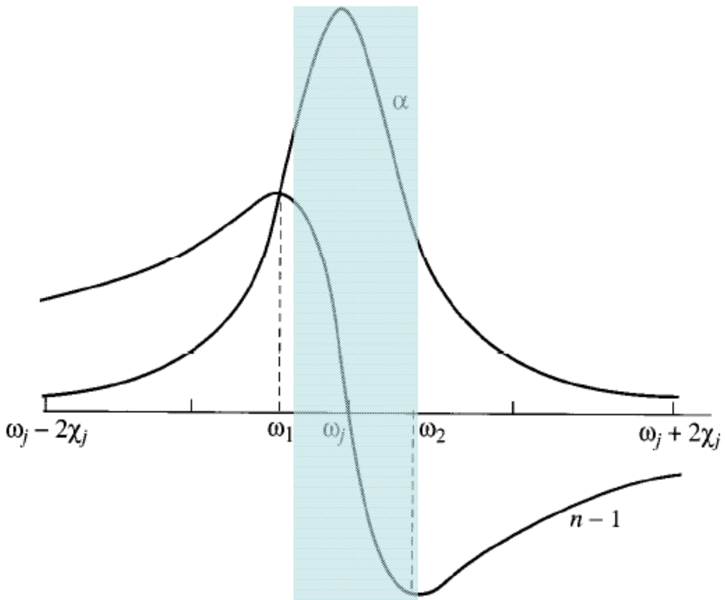
$\longrightarrow$ v_g (not v) corresponds to the classical veloctity.

9.4.3 Frequency dependence of permittivity in dielectric media

$$\tilde{\epsilon}_r = 1 + \frac{Nq^2}{m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega} \longrightarrow \tilde{k} = \frac{\omega}{c}\sqrt{\tilde{\epsilon}_r} \cong \frac{\omega}{c}\left[1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right]$$

$$n = \frac{ck}{\omega} \cong 1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2}$$

$$\alpha = 2\kappa \cong \frac{Nq^2\omega^2}{m\epsilon_0 c}\sum_j \frac{f_j\gamma_j}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2}$$



anomalous dispersion

If you agree to stay away from the resonances, the damping γ can be ignored.

$$n = 1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2}$$

$$\frac{1}{\omega_j^2 - \omega^2} = \frac{1}{\omega_j^2}\left(1 - \frac{\omega^2}{\omega_j^2}\right)^{-1} \cong \frac{1}{\omega_j^2}\left(1 + \frac{\omega^2}{\omega_j^2}\right)$$

$$n = 1 + \left(\frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2}\right) + \omega^2\left(\frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^4}\right)$$

$$\boxed{n = 1 + A\left(1 + \frac{B}{\lambda^2}\right)} \quad (\lambda = 2\pi c/\omega)$$

➔ **Cauchy's formula**

Anomalous Dispersion

Problem 9.25 Find the width of the anomalous dispersion region for the case of a single resonance at frequency ω_0

Assume $\gamma \ll \omega_0$

$$n = \frac{ck}{\omega} \cong 1 + \frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \xrightarrow[\text{resonance at frequency } \omega_0]{\text{for the case of a single}} n = 1 + \frac{Nq^2}{2m\epsilon_0} \frac{(\omega_0^2 - \omega^2)}{[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2]}$$

Let the denominator $\equiv D$.

At the extreme frequencies ω_1 and ω_2,

$$\frac{dn}{d\omega} = \frac{Nq^2}{2m\epsilon_0}\left\{\frac{-2\omega}{D} - \frac{(\omega_0^2-\omega^2)}{D^2}\left[2(\omega_0^2-\omega^2)(-2\omega)+\gamma^2 2\omega\right]\right\} = 0$$

$$2\omega D = (\omega_0^2 - \omega^2)\left[2(\omega_0^2-\omega^2) - \gamma^2\right] 2\omega$$

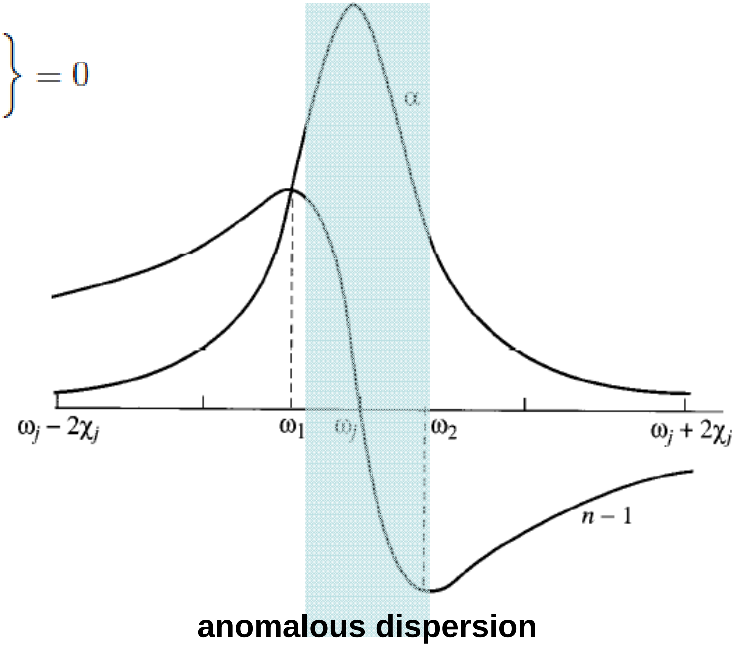
$$(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2 = 2(\omega_0^2-\omega^2)^2 - \gamma^2(\omega_0^2-\omega^2)$$

$$(\omega_0^2-\omega^2)^2 = \gamma^2(\omega^2+\omega_0^2-\omega^2) = \gamma^2\omega_0^2$$

$$(\omega_0^2-\omega^2) = \pm\omega_0\gamma \longrightarrow \omega^2 = \omega_0^2 \mp \omega_0\gamma$$

$$\omega = \omega_0\sqrt{1 \mp \gamma/\omega_0} \cong \omega_0\left(1 \mp \gamma/2\omega_0\right) = \omega_0 \mp \gamma/2$$

$$\omega_1 = \omega_0 - \gamma/2 \qquad \omega_2 = \omega_0 + \gamma/2$$



anomalous dispersion

➔ The width of the anomalous region:

➔ *The index of refraction assumes its maximum and minimum values at points where the absorption coefficient is at half-maximum.*

➔ *The full-width at half maximum (FWHM) of the absorption coefficient is γ* ➔ $\Delta\omega = \gamma$.