

Sergio Bravo\*, Ivan Gonzalez, Karen Kohl, and Victor H. Moll

# Integrals of Frullani type and the method of brackets

DOI 10.1515/math-2017-0001

Received September 13, 2016; accepted December 27, 2016.

**Abstract:** The method of brackets is a collection of heuristic rules, some of which have been made rigorous, that provide a flexible, direct method for the evaluation of definite integrals. The present work uses this method to establish classical formulas due to Frullani which provide values of a specific family of integrals. Some generalizations are established.

**Keywords:** Definite integrals, Frullani integrals, Method of brackets

**MSC:** 33C67, 81T18

## 1 Introduction

The integral

$$\int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} dx = \log\left(\frac{b}{a}\right) \quad (1)$$

appears as entry 3.434.2 in [12]. It is one of the simplest examples of the so-called *Frullani integrals*. These are examples of the form

$$S(a, b) = \int_0^{\infty} \frac{f(ax) - f(bx)}{x} dx, \quad (2)$$

and Frullani's theorem states that

$$S(a, b) = [f(0) - f(\infty)] \log\left(\frac{b}{a}\right). \quad (3)$$

The identity (3) holds if, for example,  $f'$  is a continuous function and the integral in (3) exists. Other conditions for the validity of this formula are presented in [3, 13, 16]. The reader will find in [1] a systematic study of the Frullani integrals appearing in [12].

The goal of the present work is to use the *method of brackets*, a new procedure for the evaluation of definite integrals, to compute a variety of integrals similar to those in (1). The method itself is described in Section 2. This is based on a small number of *heuristic rules*, some of which have been rigorously established [2, 8]. The point to be stressed here is that the application of the method of brackets is direct and it reduces the evaluation of a definite integral to the solution of a linear system of equations.

**\*Corresponding Author: Sergio Bravo:** Departamento de Física, Universidad Técnica Federico Santa María, Valparaíso, Chile, E-mail: sergio.bravo.14@sansano.usm.cl

**Ivan Gonzalez:** Departamento de Física y Astronomía, Universidad de Valparaíso, Avda. Gran Bretaña 1111, Valparaíso, Chile, E-mail: ivan.gonzalez@uv.cl

**Karen Kohl:** Department of Mathematics, University of Southern Mississippi, Long Beach, MS 39560, USA, E-mail: karen.kohl@usm.edu

**Victor H. Moll:** Department of Mathematics, Tulane University, New Orleans, LA 70118, USA, E-mail: vhm@math.tulane.edu

## 2 The method of brackets

A method to evaluate integrals over the half-line  $[0, \infty)$ , based on a small number of rules has been developed in [6, 9–11]. This *method of brackets* is described next. The heuristic rules are currently being placed on solid ground [2]. The reader will find in [5, 7, 8] a large collection of evaluations of definite integrals that illustrate the power and flexibility of this method.

For  $a \in \mathbb{R}$ , the symbol

$$\langle a \rangle = \int_0^\infty x^{a-1} dx, \quad (4)$$

is the *bracket* associated to the (divergent) integral on the right. The symbol

$$\phi_n = \frac{(-1)^n}{\Gamma(n+1)}, \quad (5)$$

is called the *indicator* associated to the index  $n$ . The notation  $\phi_{n_1 n_2 \dots n_r}$ , or simply  $\phi_{12 \dots r}$ , denotes the product  $\phi_{n_1} \phi_{n_2} \dots \phi_{n_r}$ .

### Rules for the production of bracket series

**Rule P<sub>1</sub>.** If the function  $f$  is given by the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n + \beta - 1}, \quad (6)$$

with  $\alpha, \beta \in \mathbb{C}$ , then the integral of  $f$  over  $[0, \infty)$  is converted into a *bracket series* by the procedure

$$\int_0^\infty f(x) dx = \sum_n a_n \langle \alpha n + \beta \rangle. \quad (7)$$

**Rule P<sub>2</sub>.** For  $\alpha \in \mathbb{C}$ , the multinomial power  $(a_1 + a_2 + \dots + a_r)^\alpha$  is assigned the  $r$ -dimension bracket series

$$\sum_{n_1} \sum_{n_2} \dots \sum_{n_r} \phi_{n_1 n_2 \dots n_r} a_1^{n_1} \dots a_r^{n_r} \frac{\langle -\alpha + n_1 + \dots + n_r \rangle}{\Gamma(-\alpha)}. \quad (8)$$

### Rules for the evaluation of a bracket series

**Rule E<sub>1</sub>.** The one-dimensional bracket series is assigned the value

$$\sum_n \phi_n f(n) \langle an + b \rangle = \frac{1}{|a|} f(n^*) \Gamma(-n^*), \quad (9)$$

where  $n^*$  is obtained from the vanishing of the bracket; that is,  $n^*$  solves  $an + b = 0$ . This is precisely the Ramanujan's Master Theorem.

The next rule provides a value for multi-dimensional bracket series of index 0, that is, the number of sums is equal to the number of brackets.

**Rule E<sub>2</sub>.** Assume the matrix  $A = (a_{ij})$  is non-singular, then the assignment is

$$\begin{aligned} \sum_{n_1} \dots \sum_{n_r} \phi_{n_1 \dots n_r} f(n_1, \dots, n_r) \langle a_{11}n_1 + \dots + a_{1r}n_r + c_1 \rangle \dots \langle a_{r1}n_1 + \dots + a_{rr}n_r + c_r \rangle \\ = \frac{1}{|\det(A)|} f(n_1^*, \dots, n_r^*) \Gamma(-n_1^*) \dots \Gamma(-n_r^*) \end{aligned}$$

where  $\{n_i^*\}$  is the (unique) solution of the linear system obtained from the vanishing of the brackets.

**Rule E<sub>3</sub>.** The value of a multi-dimensional bracket series of positive index is obtained by computing all the contributions of maximal rank by Rule E<sub>2</sub>. These contributions to the integral appear as series in the free parameters. Series converging in a common region are added and divergent series are discarded.

### 3 The formula in one dimension

The goal of this section is to establish Frullani's evaluation (3) by the method of brackets. The notation  $\phi_k = (-1)^k / \Gamma(k+1)$  is used in the statement of the next theorem.

**Theorem 3.1.** Assume  $f(x)$  admits an expansion of the form

$$f(x) = \sum_{k=0}^{\infty} \phi_k C(k) x^{\alpha k}, \text{ for some } \alpha > 0 \text{ with } C(0) \neq 0 \text{ and } C(0) < \infty. \quad (1)$$

Then,

$$\begin{aligned} S(a, b) &:= \int_0^{\infty} \frac{f(ax) - f(bx)}{x} dx \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{|\alpha|} \Gamma\left(\frac{\varepsilon}{\alpha}\right) C\left(-\frac{\varepsilon}{\alpha}\right) (a^{-\varepsilon} - b^{-\varepsilon}) \\ &= C(0) \log\left(\frac{b}{a}\right), \end{aligned} \quad (2)$$

independently of  $\alpha$ .

*Proof.* Introduce an extra parameter and write

$$S(a, b) = \lim_{\varepsilon \rightarrow 0} \int_0^{\infty} \frac{f(ax) - f(bx)}{x^{1-\varepsilon}} dx. \quad (3)$$

Then,

$$\begin{aligned} S(a, b) &= \lim_{\varepsilon \rightarrow 0} \int_0^{\infty} \sum_{k=0}^{\infty} \phi_k C(k) (a^{\alpha k} - b^{\alpha k}) \int_0^{\infty} x^{\alpha k + \varepsilon - 1} dx \\ &= \lim_{\varepsilon \rightarrow 0} \sum_k \phi_k C(k) (a^{\alpha k} - b^{\alpha k}) (\alpha k + \varepsilon). \end{aligned}$$

The method of brackets gives

$$S(a, b) = \lim_{\varepsilon \rightarrow 0} \frac{1}{|\alpha|} \Gamma\left(\frac{\varepsilon}{\alpha}\right) C\left(-\frac{\varepsilon}{\alpha}\right) (a^{-\varepsilon} - b^{-\varepsilon}). \quad (4)$$

The result follows from the expansions  $\Gamma(\varepsilon/\alpha) = \alpha/\varepsilon - \gamma + O(\varepsilon)$ ,  $C(-\varepsilon/\alpha) = C(0) + O(\varepsilon)$  and  $a^{-\varepsilon} - b^{-\varepsilon} = (\log b - \log a) \varepsilon + O(\varepsilon^2)$ .  $\square$

In the examples given below, observe that  $C(0) = f(0)$  and that  $f(\infty) = 0$  is imposed as a condition on the integrand.

**Example 3.2.** Entry 3.434.2 of [12] states the value

$$\int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} dx = \log \frac{b}{a}. \quad (5)$$

This follows directly from (2).

**Note 3.3.** The method of brackets gives a direct approach to Frullani style problems if the expansion (1) is replaced by the more general one

$$f(x) = \sum_{k=0}^{\infty} \phi_k C(k) x^{\alpha k + \beta}, \quad (6)$$

with  $\beta \neq 0$  and if the function  $f$  does not necessarily have a limit at infinity.

**Example 3.4.** Consider the evaluation of

$$I = \int_0^{\infty} \frac{\sin ax - \sin bx}{x} dx, \quad (7)$$

for  $a, b > 0$ . The integral is evaluated directly as

$$I = \int_0^{\infty} \frac{\sin ax}{x} dx - \int_0^{\infty} \frac{\sin bx}{x} dx, \quad (8)$$

and since  $a, b > 0$ , both integrals are  $\pi/2$ , giving  $I = 0$ . The classical version of Frullani theorem does not apply, since  $f(x)$  does not have a limit as  $x \rightarrow \infty$ . Ostrowski [15] shows that in the case  $f(x)$  is periodic of period  $p$ , the value  $f(\infty)$  might be replaced by

$$\frac{1}{p} \int_0^p f(x) dx. \quad (9)$$

In the present case,  $f(x) = \sin x$  has period  $2\pi$  and mean 0. This yields the vanishing of the integral.

The computation of (7) by the method of brackets begins with the expansion

$$\sin x = x \cdot {}_0F_1 \left( \frac{-}{\frac{3}{2}} \middle| -\frac{1}{4}x^2 \right). \quad (10)$$

Here

$${}_pF_q \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \quad (11)$$

with  $(a)_n = a(a+1) \cdots (a+n-1)$ , is the classical hypergeometric function. The integrand has the series expansion

$$\sum_{n \geq 0} \phi_n \frac{(a^{2n+1} - b^{2n+1})}{(\frac{3}{2})_n 4^n} x^{2n}, \quad (12)$$

that yields

$$I = \sum_n \phi_n \frac{(a^{2n+1} - b^{2n+1})}{(\frac{3}{2})_n 4^n} \langle 2n+1 \rangle. \quad (13)$$

The vanishing of the bracket gives  $n^* = -1/2$  and the bracket series vanishes in view of the factor  $a^{2n+1} - b^{2n+1}$ .

**Example 3.5.** The next example is the evaluation of

$$I = \int_0^{\infty} \frac{\cos ax - \cos bx}{x} dx = \log \left( \frac{b}{a} \right), \quad (14)$$

for  $a, b > 0$ . The expansion

$$\cos x = \sum_{n=0}^{\infty} \phi_n \frac{n!}{(2n)!} x^{2n}, \quad (15)$$

and  $C(n) = \frac{n!}{(2n)!} = \frac{\Gamma(n+1)}{\Gamma(2n+1)}$  in (1). Then  $C(0) = 1$  and the integral is  $I = \log \left( \frac{b}{a} \right)$ , as claimed.

**Example 3.6.** The integral

$$I = \int_0^{\infty} \frac{\tan^{-1}(e^{-ax}) - \tan^{-1}(e^{-bx})}{x} dx, \quad (16)$$

is evaluated next. The expansion of the integrand is

$$\tan^{-1}(e^{-t}) = e^{-t} \cdot {}_2F_1 \left( \frac{1}{2}, 1 \middle| -\frac{3}{2} \right) e^{-2t}$$

$$\begin{aligned}
&= \frac{1}{2} \sum_{n=0}^{\infty} \phi_n \frac{\Gamma(n + \frac{1}{2}) \Gamma(n+1)}{\Gamma(n + \frac{3}{2})} \sum_{k=0}^{\infty} \phi_k (2n+1)^k t^k \\
&= \sum_{k=0}^{\infty} \phi_k \left[ \frac{1}{2} \sum_{n=0}^{\infty} \phi_n \frac{\Gamma(n + \frac{1}{2}) \Gamma(n+1)}{\Gamma(n + \frac{3}{2})} (2n+1)^k \right] t^k.
\end{aligned}$$

Therefore,

$$C(k) = \frac{1}{2} \sum_{n=0}^{\infty} \phi_n \frac{\Gamma(n + \frac{1}{2}) \Gamma(n+1)}{\Gamma(n + \frac{3}{2})} (2n+1)^k, \quad (17)$$

and from here it follows that

$$C(0) = \frac{1}{2} \sum_{n=0}^{\infty} \phi_n \frac{\Gamma(n + \frac{1}{2}) \Gamma(n+1)}{\Gamma(n + \frac{3}{2})} = \tan^{-1}(1) = \frac{\pi}{4}. \quad (18)$$

Thus, the integral is

$$I = C(0) \log\left(\frac{b}{a}\right) = \frac{\pi}{4} \log\left(\frac{b}{a}\right). \quad (19)$$

## 4 A first generalization

This section describes examples of Frullani-type integrals that have an expansion of the form

$$f(x) = \sum_{k \geq 0} \phi_k C(k) x^{\alpha k + \beta}, \quad (20)$$

with  $\beta \neq 0$ .

**Theorem 4.1.** Assume  $f(x)$  admits an expansion of the form (20). Then,

$$\begin{aligned}
S(a, b) &= \int_0^{\infty} \frac{f(ax) - f(bx)}{x} dx \\
&= \lim_{\varepsilon \rightarrow 0} \frac{1}{|\alpha|} \Gamma\left(\frac{\beta}{\alpha} + \frac{\varepsilon}{\alpha}\right) C\left(-\frac{\beta}{\alpha} - \frac{\varepsilon}{\alpha}\right) [a^{-\varepsilon} - b^{-\varepsilon}].
\end{aligned} \quad (21)$$

*Proof.* The method of brackets gives

$$\begin{aligned}
S(a, b; \varepsilon) &= \int_0^{\infty} \frac{f(ax) - f(bx)}{x^{1-\varepsilon}} dx \\
&= \sum_{k \geq 0} \phi_k C(k) [a^{\alpha k + \beta} - b^{\alpha k + \beta}] \int_0^{\infty} x^{\alpha k + \beta + \varepsilon - 1} dx \\
&= \sum_k \phi_k C(k) [a^{\alpha k + \beta} - b^{\alpha k + \beta}] \langle \alpha k + \beta + \varepsilon \rangle \\
&= \frac{1}{|\alpha|} \Gamma(-k) C(k) [a^{\alpha k + \beta} - b^{\alpha k + \beta}]
\end{aligned} \quad (22)$$

with  $k = -(\beta + \varepsilon)/\alpha$  in the last line. The result follows by taking  $\varepsilon \rightarrow 0$ .  $\square$

**Example 4.2.** The integral

$$\int_0^{\infty} \frac{\tan^{-1} ax - \tan^{-1} bx}{x} dx = -\frac{\pi}{2} \log\left(\frac{b}{a}\right) \quad (23)$$

appears as entry 4.536.2 in [12]. It is evaluated directly by the classical Frullani theorem. Its evaluation by the method of brackets comes from the expansion

$$\begin{aligned}\tan^{-1} x &= x \cdot {}_2F_1\left(\frac{1}{2}, 1 \middle| -x^2\right) \\ &= \sum_{k \geq 0} \phi_k \frac{(\frac{1}{2})_k (1)_k}{(\frac{3}{2})_k} x^{2k+1}.\end{aligned}\quad (24)$$

Therefore,  $\alpha = 2$ ,  $\beta = 1$  and

$$C(k) = \frac{\Gamma(\frac{1}{2} + k) \Gamma(1 + k)}{2\Gamma(\frac{3}{2} + k)} = \frac{\Gamma(1 + k)}{2k + 1}.\quad (25)$$

Then

$$\begin{aligned}\int_0^\infty \frac{\tan^{-1} ax - \tan^{-1} bx}{x} dx &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \Gamma\left(\frac{1 + \varepsilon}{2}\right) C\left(-\frac{1 + \varepsilon}{2}\right) [a^{-\varepsilon} - b^{-\varepsilon}] \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \Gamma\left(\frac{1 + \varepsilon}{2}\right) \Gamma\left(\frac{1 - \varepsilon}{2}\right) \frac{[a^{-\varepsilon} - b^{-\varepsilon}]}{-\varepsilon} \\ &= -\frac{\pi}{2} \log\left(\frac{b}{a}\right).\end{aligned}$$

## 5 A second class of Frullani type integrals

Let  $f_1, \dots, f_N$  be a family of functions. This section uses the method of brackets to evaluate

$$I = I(f_1, \dots, f_N) = \int_0^\infty \frac{1}{x} \sum_{k=1}^N f_k(x) dx, \quad (1)$$

subject to the condition  $\sum_{k=1}^N f_k(0) = 0$ , required for convergence.

The functions  $\{f_k(x)\}$  are assumed to admit a series representation of the form

$$f_k(x) = \sum_{n=0}^\infty \phi_n C_k(n) x^{\alpha n}, \quad (2)$$

where  $\alpha > 0$  is independent of  $k$  and  $C_k(0) \neq 0$ . The coefficients  $C_k$  are assumed to admit a meromorphic extension from  $n \in \mathbb{N}$  to  $n \in \mathbb{C}$ .

**Theorem 5.1.** *The integral  $I$  is given by*

$$I = -\frac{1}{|\alpha|} \sum_{k=1}^N C'_k(0), \quad (3)$$

where

$$C'_k(0) = \left. \frac{dC_k(\varepsilon)}{d\varepsilon} \right|_{\varepsilon=0}. \quad (4)$$

*Proof.* The proof begins with the expansion

$$\frac{f_k(x)}{x^{1-\varepsilon}} = \sum_{n=0}^\infty \phi_n C_k(n) x^{\alpha n - 1 + \varepsilon} \quad (5)$$

and the bracket series for the integral is

$$\begin{aligned} I &= \lim_{\varepsilon \rightarrow 0} \sum_n \phi_n \left( \sum_{k=1}^N C_k(n) \right) \langle \alpha n + \varepsilon \rangle \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{|\alpha|} \Gamma \left( -\frac{\varepsilon}{\alpha} \right) \sum_{k=1}^N C_k \left( -\frac{\varepsilon}{\alpha} \right). \end{aligned} \quad (6)$$

The result follows by letting  $\varepsilon \rightarrow 0$ .  $\square$

**Example 5.2.** Entry 3.429 in [12] states that

$$I = \int_0^\infty [e^{-x} - (1+x)^{-\mu}] \frac{dx}{x} = \psi(\mu), \quad (7)$$

where  $\mu > 0$  and  $\psi(x) = \Gamma'(x)/\Gamma(x)$  is the digamma function. This is one of many integral representation for this basic function. The reader will find a classical proof of this identity in [14]. The method of brackets gives a direct proof.

The functions appearing in this example are

$$f_1(x) = e^{-x} = \sum_{n=0}^\infty \phi_n x^n, \quad (8)$$

and

$$f_2(x) = -(1+x)^{-\mu} = -\sum_{n=0}^\infty \phi_n(\mu) x^n, \quad (9)$$

where  $(\mu)_n = \mu(\mu+1)\cdots(\mu+n-1)$  is the Pochhammer symbol (this comes directly from the binomial theorem). The condition  $f_1(0) + f_2(0) = 0$  is satisfied and the coefficients are identified as

$$C_1(n) = 1 \text{ and } C_2(n) = -(\mu)_n = -\frac{\Gamma(\mu+n)}{\Gamma(\mu)}. \quad (10)$$

Then,  $C'_1(0) = 0$  and  $C'_2(0) = -\frac{\Gamma'(\mu)}{\Gamma(\mu)}$ . This gives the evaluation.

**Example 5.3.** The elliptic integrals  $\mathbf{K}(x)$  and  $\mathbf{E}(x)$  may be expressed in hypergeometric form as

$$\mathbf{K}(x) = \frac{\pi}{2} {}_2F_1 \left( \begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix} \middle| x^2 \right) \text{ and } \mathbf{E}(x) = \frac{\pi}{2} {}_2F_1 \left( \begin{matrix} -\frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix} \middle| x^2 \right) \quad (11)$$

The reader will find information about these integrals in [4, 17].

Theorem 5.1 is now used to establish the value

$$\int_0^\infty \frac{\pi e^{-ax^2} - \mathbf{K}(bx) - \mathbf{E}(cx)}{x} dx = \frac{\pi}{2} \left[ \log \left( \frac{bc}{a} \right) - \gamma - 4 \log 2 + 1 \right]. \quad (12)$$

Here  $\gamma = -\Gamma'(1)$  is Euler's constant.

The first step is to compute series expansions of each of the terms in the integrand. The exponential term is easy:

$$\pi e^{-ax^2} = \pi \sum_{n_1=0}^\infty \frac{(-ax^2)^{n_1}}{n_1!} = \sum_{n_1} \phi_{n_1} a^{n_1} x^{2n_1}, \quad (13)$$

and this gives  $C_1(n) = a^n$ . For the first elliptic integral,

$$\mathbf{K}(bx) = \frac{\pi}{2} {}_2F_1 \left( \begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix} \middle| b^2 x^2 \right)$$

$$\begin{aligned}
&= \frac{\pi}{2} \sum_{n_2=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{n_2} \left(\frac{1}{2}\right)_{n_2}}{(1)_{n_2} n_2!} b^{2n_2} x^{2n_2} \\
&= \sum_{n_2} \phi_{n_2} \frac{\pi}{2} \left( \frac{(-1)^{n_2} b^{2n_2}}{n_2!} \left(\frac{1}{2}\right)_{n_2}^2 \right) x^{2n_2}.
\end{aligned}$$

Therefore,

$$C_2(n) = \frac{\pi}{2} \frac{\cos(\pi n) \Gamma^2(n + \frac{1}{2})}{\Gamma(n+1)} b^{2n}, \quad (14)$$

where the term  $(-1)^n$  has been replaced by  $\cos(\pi n)$ . A similar calculation gives

$$C_3(n) = \frac{\pi}{4} \frac{\cos(\pi n) \Gamma(n - \frac{1}{2}) \Gamma(n + \frac{1}{2})}{\Gamma(n+1)} c^{2n}. \quad (15)$$

A direct calculation gives

$$C'_1(0) = \log a, \quad C'_2(0) = -\frac{\gamma}{2} - \log b - \psi\left(\frac{1}{2}\right) \text{ and } C'_3(0) = -\frac{\gamma}{2} - \log c - \psi\left(-\frac{1}{2}\right).$$

The result now comes from the values

$$\psi\left(\frac{1}{2}\right) = -2 \log 2 - \gamma \text{ and } \psi\left(-\frac{1}{2}\right) = -2 \log 2 - \gamma + 2. \quad (16)$$

**Example 5.4.** Let  $a, b \in \mathbb{R}$  with  $a > 0$ . Then

$$\int_0^{\infty} \frac{\exp(-ax^2) - \cos bx}{x} dx = \frac{\gamma - \log a + 2 \log b}{2}. \quad (17)$$

To apply Theorem 5.1 start with the series

$$f_1(x) = e^{-ax^2} = \sum_n \phi_n a^n x^{2n} \quad (18)$$

and

$$f_2(x) = \cos bx = \sum_n \phi_n \left[ \frac{\Gamma(n+1)}{\Gamma(2n+1)} b^{2n} \right] x^{2n}. \quad (19)$$

In both expansions  $\alpha = 2$  and the coefficients are given by

$$C_1(n) = a^n \text{ and } C_2(n) = \frac{\Gamma(n+1)}{\Gamma(2n+1)} b^{2n}. \quad (20)$$

Then,  $C'_1(0) = \log a$  and  $C'_2(n) = \frac{b^{2n} \Gamma(n+1)}{\Gamma(2n+1)} [2 \log b + \psi(n+1) - \psi(2n+1)]$  yield  $C'_2(0) = 2 \log b - \psi(1) = 2 \log b + \gamma$ . The value (17) follows from here.

**Example 5.5.** The next example in this section involves the Bessel function of order 0

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!^2} \left(\frac{x}{2}\right)^{2n} \quad (21)$$

and Theorem 5.1 is used to evaluate

$$\int_0^{\infty} \frac{J_0(x) - \cos ax}{x} dx = \log 2a. \quad (22)$$

This appears as entry 6.693.8 in [12]. The expansions

$$J_0(x) = \sum_{n=0}^{\infty} \phi_n \frac{1}{n! 2^{2n}} x^{2n} \text{ and } \cos ax = \sum_{n=0}^{\infty} \phi_n \frac{n!}{(2n)!} a^{2n} x^{2n}, \quad (23)$$



show  $\alpha = 2$  and

$$C_1(n) = \frac{1}{\Gamma(n+1)2^{2n}} \text{ and } C_2(n) = -\frac{\Gamma(n+1)}{\Gamma(2n+1)}a^{2n}. \quad (24)$$

Differentiation gives

$$C'_1(n) = -\frac{2\ln 2 + \psi(n+1)}{2^{2n}\Gamma(n+1)}, \quad (25)$$

and

$$C'_2(n) = -\frac{a^{2n}\Gamma(n+1)(2\log a + \psi(n+1) - 2\psi(2n+1))}{\Gamma(2n+1)}. \quad (26)$$

Then,

$$C'_1(0) = \gamma - 2\log 2 \text{ and } C'_2(0) = -(\gamma + 2\log a), \quad (27)$$

and the result now follows from Theorem 5.1. The reader is invited to use the representation

$$J_0^2(x) = {}_1F_2\left(\frac{1}{2} \middle| -x^2\right), \quad (28)$$

to verify the identity

$$\int_0^\infty \frac{J_0^2(x) - \cos x}{x} dx = \log 2. \quad (29)$$

**Example 5.6.** The final example in this section is

$$I = \int_0^\infty \frac{J_0^2(x) - e^{-x^2} \cos x}{x} dx. \quad (30)$$

The evaluation begins with the expansions

$$J_0(x) = \sum_{k=0}^\infty \phi_k \frac{x^{2k}}{4^k \Gamma(k+1)} \text{ and } \cos x = \sum_{k=0}^\infty \phi_k \frac{\sqrt{\pi}}{4^k \Gamma(k+\frac{1}{2})}. \quad (31)$$

Then,

$$J_0^2(x) = \sum_{k,n} \phi_{k,n} \frac{1}{4^{k+n} \Gamma(k+1) \Gamma(n+1)} x^{2k+2n}, \quad (32)$$

and

$$e^{-x^2} \cos x = \sum_{k,n} \phi_{k,n} \frac{\sqrt{\pi}}{4^k \Gamma(k+\frac{1}{2})} x^{2k+2n}. \quad (33)$$

Integration yields

$$\begin{aligned} I &= \int_0^\infty \frac{J_0^2(x) - e^{-x^2} \cos x}{x^{1-\varepsilon}} dx \\ &= \sum_{k,n} \phi_{k,n} \left[ \frac{1}{4^{k+n} \Gamma(k+1) \Gamma(n+1)} - \frac{\sqrt{\pi}}{4^k \Gamma(k+\frac{1}{2})} \right] \int_0^\infty x^{2k+2n+\varepsilon-1} dx \\ &= \sum_{k,n} \phi_{k,n} \left[ \frac{1}{4^{k+n} \Gamma(k+1) \Gamma(n+1)} - \frac{\sqrt{\pi}}{4^k \Gamma(k+\frac{1}{2})} \right] \langle 2k+2n+\varepsilon \rangle. \end{aligned}$$

The method of brackets now gives

$$I = \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \sum_{k=0}^\infty \frac{(-1)^k \Gamma(k+\frac{\varepsilon}{2})}{k!} \left[ \frac{1}{2^{-\varepsilon} \Gamma(k+1) \Gamma(1-k-\varepsilon/2)} - \frac{\sqrt{\pi}}{2^{2k} \Gamma(k+\frac{1}{2})} \right].$$



uses the expansion

$$f(s, t) = e^{-st^2} \cos(st) = \sum_{n_1} \sum_{n_2} \phi_{1,2} \frac{\sqrt{\pi}}{\Gamma(n_2 + \frac{1}{2}) 4^{n_2}} s^{n_1+2n_2} t^{2n_1+2n_2}, \quad (5)$$

with parameters  $\rho_1 = -\frac{1}{2}$ ,  $\rho_2 = -1$ ,  $b_1 = a^2/\mu$ ,  $b_2 = \mu/a$ ,  $a_1 = b^2/\mu$ ,  $a_2 = \mu/b$ . The solution to the linear system is  $n_1^* = -\frac{1}{2}$  and  $n_2^* = -\frac{\varepsilon}{2}$  and  $|\det A| = 2$ . Then

$$\begin{aligned} I &= \frac{1}{2} \lim_{\varepsilon \rightarrow 0} \left[ \left( \frac{a^2}{\mu} \right)^{-1/2-\varepsilon} \left( \frac{\mu}{a} \right)^{-1-\varepsilon} - \left( \frac{b^2}{\mu} \right)^{-1/2-\varepsilon} \left( \frac{\mu}{b} \right)^{-1-\varepsilon} \right] \times \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{\varepsilon}{2}\right) \frac{\sqrt{\pi}}{\Gamma\left(\frac{1-\varepsilon}{2}\right) 4^{-\varepsilon/2}} \\ &= \sqrt{\frac{\pi}{\mu}} \lim_{\varepsilon \rightarrow 0} \left[ \frac{b^\varepsilon - a^\varepsilon}{\varepsilon} \right] \times \frac{\Gamma(1+\varepsilon) \cos\left(\frac{\pi\varepsilon}{2}\right)}{(ab)^\varepsilon} \\ &= \sqrt{\frac{\pi}{\mu}} \log\left(\frac{b}{a}\right). \end{aligned}$$

The double integral (4) has been evaluated.

**Example 6.3.** The method is now used to evaluate

$$\int_0^\infty \int_0^\infty \frac{\sin(\mu xy^2) \cos(axy) - \sin(\mu xy^2) \cos(bxy)}{xy} = \frac{\pi}{2} \log \frac{b}{a}. \quad (6)$$

The evaluation begins with the expansion

$$\begin{aligned} f(x, y) &= \sin(xy^2) \cos(xy) \\ &= \left( xy^2 \sum_{n_1 \geq 0} \phi_{n_1} \frac{\Gamma\left(\frac{3}{2}\right) (xy^2)^{2n_1}}{\Gamma\left(n_1 + \frac{3}{2}\right) 4^{n_1}} \right) \left( \sum_{n_2 \geq 0} \phi_{n_2} \frac{\Gamma\left(\frac{1}{2}\right) (xy)^{2n_2}}{\Gamma\left(n_2 + \frac{1}{2}\right) 4^{n_2}} \right) \\ &= \sum_{n_1} \sum_{n_2} \phi_{n_1} \phi_{n_2} \frac{\pi}{2\Gamma\left(n_1 + \frac{3}{2}\right) \Gamma\left(n_2 + \frac{1}{2}\right) 4^{n_1+n_2}} x^{2n_1+2n_2+1} y^{4n_1+2n_2}. \end{aligned}$$

The parameters are  $b_1 = a^2/\mu$ ,  $b_2 = \mu/a$ ,  $a_1 = b^2/\mu$ ,  $a_2 = \mu/b$  and  $\rho_1 = \rho_2 = 0$ . The solution to the linear system is  $n_1^* = -\frac{1}{2}$  and  $n_2^* = -\frac{\varepsilon}{2}$  and  $|\det A| = 4$ . Then,

$$\begin{aligned} I &= \lim_{\varepsilon \rightarrow 0} \frac{a^{-\varepsilon} - b^{-\varepsilon}}{4} \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{\varepsilon}{2}\right) \frac{\pi}{2\Gamma(1)\Gamma\left(\frac{1-\varepsilon}{2}\right) 4^{-\varepsilon-1/2}} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\pi^{3/2} 4^{\varepsilon/2}}{4} \frac{b^\varepsilon - a^\varepsilon}{(ab)^\varepsilon} \frac{2^{1-2\varepsilon} \sqrt{\pi} \Gamma(\varepsilon)}{\pi \csc\left(\frac{1}{2} + \frac{\varepsilon}{2}\right)} \\ &= \frac{\pi}{2} \log\left(\frac{b}{a}\right), \end{aligned}$$

as claimed.

## 7 Conclusions

The method of brackets consists of a small number of heuristic rules that reduce the evaluation of a definite integral to the solution of a linear system of equations. The method has been used to establish a classical theorem of Frullani and to evaluate, in an algorithmic manner, a variety of integrals of *Frullani type*. The flexibility of the method yields a direct and simple solution to these evaluations.

**Acknowledgement:** The second author wishes to thank the partial support of the Centro de Astrofísica de Valparaíso (CAV). The last author acknowledges the partial support of NSF-DMS 1112656.

## References

- [1] M. Albano, T. Amdeberhan, E. Beyerstedt, and V. Moll. The integrals in Gradshteyn and Ryzhik. Part 15: Frullani integrals. *Scientia*, 19:113–119, 2010.
- [2] T. Amdeberhan, O. Espinosa, I. Gonzalez, M. Harrison, V. Moll, and A. Straub. Ramanujan Master Theorem. *The Ramanujan Journal*, 29:103–120, 2012.
- [3] J. Arias-de Reyna. On the theorem of Frullani. *Proc. Amer. Math. Soc.*, 109:165–175, 1990.
- [4] J. M. Borwein and P. B. Borwein. *Pi and the AGM- A study in analytic number theory and computational complexity*. Wiley, New York, 1st edition, 1987.
- [5] I. Gonzalez, K. Kohl, and V. Moll. Evaluation of entries in Gradshteyn and Ryzhik employing the method of brackets. *Scientia*, 25:65–84, 2014.
- [6] I. Gonzalez and M. Loewe. Feynman diagrams and a combination of the Integration by Parts (IBP) and the Integration by Fractional Expansion (IBFE) Techniques. *Physical Review D*, 81:026003, 2010.
- [7] I. Gonzalez and V. Moll. Definite integrals by the method of brackets. Part 1. *Adv. Appl. Math.*, 45:50–73, 2010.
- [8] I. Gonzalez, V. Moll, and A. Straub. The method of brackets. Part 2: Examples and applications. In T. Amdeberhan, L. Medina, and Victor H. Moll, editors, *Gems in Experimental Mathematics*, volume 517 of *Contemporary Mathematics*, pages 157–172. American Mathematical Society, 2010.
- [9] I. Gonzalez and I. Schmidt. Optimized negative dimensional integration method (NDIM) and multiloop Feynman diagram calculation. *Nuclear Physics B*, 769:124–173, 2007.
- [10] I. Gonzalez and I. Schmidt. Modular application of an integration by fractional expansion (IBFE) method to multiloop Feynman diagrams. *Phys. Rev. D*, 78:086003, 2008.
- [11] I. Gonzalez and I. Schmidt. Modular application of an integration by fractional expansion (IBFE) method to multiloop Feynman diagrams II. *Phys. Rev. D*, 79:126014, 2009.
- [12] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products*. Edited by D. Zwillinger and V. Moll. Academic Press, New York, 8th edition, 2015.
- [13] G. H. Hardy. On the Frullanian integral  $\int_0^\infty ([\varphi(ax^m) - \varphi(bx^n)]/x)(\log x)^p dx$ . *Quart. J. Math.*, 33:113–144, 1902.
- [14] L. Medina and V. Moll. The integrals in Gradshteyn and Ryzhik. Part 10: The digamma function. *Scientia*, 17:45–66, 2009.
- [15] A. M. Ostrowski. On some generalizations of the Cauchy-Frullani integral. *Proc. Nat. Acad. Sci. U.S.A*, 35:612–616, 1949.
- [16] A. M. Ostrowski. On Cauchy-Frullani integrals. *Comment. Math. Helvetici*, 51:57–91, 1976.
- [17] E. T. Whittaker and G. N. Watson. *Modern Analysis*. Cambridge University Press, 1962.