## COMPLEMENTO III

$\alpha_1$

Si $L \equiv L(\{q\},\{\dot{q}\},t)$ y $F \equiv F(\{q\},t)$ demuestre que $L' = L + \frac{dF}{dt}$ da origen a las mismas ecuaciones de movimiento que para el lagrangiano $L$.

* Se conoce que para $L$ se cumplen las siguientes ecuaciones de movimiento:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

Por otro lado se tiene que:

$$dF = \sum_{j=1}^{s} \frac{\partial F}{\partial q_j} dq_j + \frac{\partial F}{\partial t} dt$$

$$\Downarrow$$

$$\dot{F} = \sum_{j=1}^{s} \frac{\partial F}{\partial q_j} \dot{q}_j + \frac{\partial F}{\partial t}$$

* Para $L'$ las ecs. de movimiento son:

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) - \frac{\partial L'}{\partial q_j} = 0$$

$\alpha_2$

entonces

$$\frac{\partial L'}{\partial \dot{q}_{t_j}} = \frac{\partial L}{\partial \dot{q}_{t_j}} + \frac{\partial}{\partial \dot{q}_{t_j}} \dot{F} = \frac{\partial L}{\partial \dot{q}_{t_j}} + \frac{\partial}{\partial \dot{q}_{t_j}} \left( \sum_l \frac{\partial F}{\partial q_{t_l}} \dot{q}_{t_l} + \frac{\partial F}{\partial t} \right)$$

$$\Downarrow$$

$$\frac{\partial L'}{\partial \dot{q}_{t_j}} = \frac{\partial L}{\partial \dot{q}_{t_j}} + \sum_l \left( \frac{\partial}{\partial \dot{q}_{t_j}} \frac{\partial F}{\partial q_{t_l}} \dot{q}_{t_l} + \frac{\partial F}{\partial q_{t_l}} \frac{\partial \dot{q}_{t_l}}{\partial \dot{q}_{t_j}} \right) + \frac{\partial}{\partial \dot{q}_{t_j}} \frac{\partial F}{\partial t}$$

Obs! $\frac{\partial}{\partial \dot{q}_{t_j}} \frac{\partial F}{\partial q_{t_l}} = \frac{\partial}{\partial q_{t_l}} \cancelto{0}{\frac{\partial F}{\partial \dot{q}_{t_j}}} = 0$

Obs. $\frac{\partial F}{\partial q_{t_l}} \frac{\partial \dot{q}_{t_l}}{\partial \dot{q}_{t_j}} = \frac{\partial F}{\partial q_{t_l}} \delta_{lj} = \frac{\partial F}{\partial q_{t_j}} \delta_{lj}$

Obs. $\frac{\partial}{\partial \dot{q}_{t_j}} \frac{\partial F}{\partial t} = \frac{\partial}{\partial t} \cancelto{0}{\frac{\partial F}{\partial \dot{q}_{t_j}}} = 0$

$\therefore$ $$\frac{\partial L'}{\partial \dot{q}_{t_j}} = \frac{\partial L}{\partial \dot{q}_{t_j}} + \frac{\partial F}{\partial q_{t_j}} \quad \Big| \ \frac{d}{dt}$$

(i) $$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_{t_j}}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_{t_j}}\right) + \frac{d}{dt}\left(\frac{\partial F}{\partial q_{t_j}}\right)$$

Por otro lado se cumple que:

$$(ii)\ \frac{\partial L'}{\partial q_j} = \frac{\partial L}{\partial q_j} + \frac{\partial \dot{F}}{\partial q_j} \quad ; \quad \underline{\text{Obs.}}\ \frac{\partial \dot{F}}{\partial q_j} = \frac{d}{dt}\frac{\partial F}{\partial q_j}.$$

Finalmente de (i) y (ii) se cumple que:

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) - \frac{\partial L'}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} \qquad \text{QED.}$$