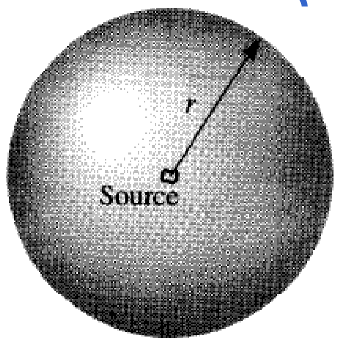
Chapter 11. Radiation

How accelerating charges and changing currents produce electromagnetic waves, how they radiate.

11.1.1 What is Radiation?



Assume a radiation source is localized near the origin.

Total power passing out through a spherical shell is the integral of the Poynting vector:

$$P(r) = \oint \mathbf{S} \cdot d\mathbf{a} = \frac{1}{\mu_0} \oint (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{a}$$

The total power *radiated from the source* is the limit of this quantity as *r* goes to infinity:

$$P_{\text{rad}} \equiv \lim_{r \to \infty} P(r)$$

Since the area of the sphere is $4\pi r^2$, so for radiation to occur (for P_{rad} not to be zero), the Poynting vector must decrease (at large *r*) no faster than $1/r^2$.

But, according to Coulomb's law and Bio-Savart law, S ~ $1/r^4$ for static configurations.

➔ **Static sources do not radiate!**

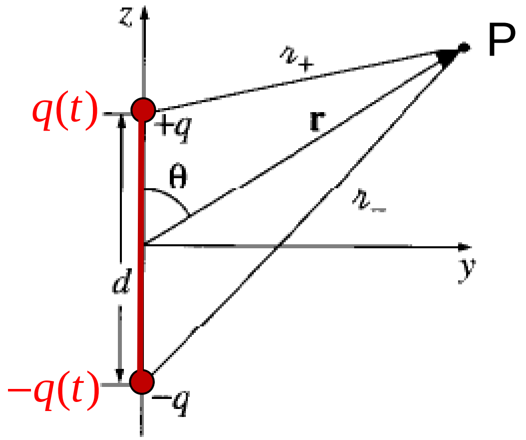
➔ **Jefimenko's Equations** indicate that *time dependent* fields include terms that go like 1/ *r;* $(\dot{\rho}$ and $\dot{J})$ it is *these* terms that are responsible for electromagnetic radiation.

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \left[\frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}} + \frac{\dot{\rho}(\mathbf{r}', t_r)}{c\mathscr{r}} \hat{\boldsymbol{\mathscr{r}}} - \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{c^2 \mathscr{r}} \right] d\tau' \qquad \mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \left[\frac{\mathbf{J}(\mathbf{r}', t_r)}{\mathscr{r}^2} + \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{c\mathscr{r}} \right] \times \hat{\boldsymbol{\mathscr{r}}}\, d\tau'$$

The study of radiation, then, involves picking out the parts of E and B that go like 1/ *r* at large distances from the source, constructing from them the $1/r^2$ term in S, integrating over a large spherical surface, and taking the limit as *r* ➔ infinite.

11.1.2 Electric Dipole Radiation

Suppose the charge back and forth through the wire, from one end to the other, at an angular frequency ω:



Dipole charge: $q(t) = q_0 \cos(\omega t)$

Current: $\mathbf{I}(t) = \dfrac{dq}{dt}\,\hat{\mathbf{z}} = -q_0\omega \sin(\omega t)\,\hat{\mathbf{z}}$

Electric dipole: $\mathbf{p}(t) = p_0 \cos(\omega t)\,\hat{\mathbf{z}},\quad$ where $\quad p_0 \equiv q_0 d$

From the retarded potentials of a point charge given by (Eq. 10.19),

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\frac{1}{\mathscr{r}}\int \rho(\mathbf{r}', t_r)\,d\tau' = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}} \qquad q = \int \rho(\mathbf{r}', t_r)\,d\tau'$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t_r)}{\mathscr{r}}\,d\tau' \qquad \boxed{t_r \equiv t - \frac{\mathscr{r}}{c}}$$

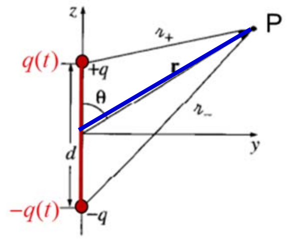
the retarded scalar and vector potentials at P are

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q_0\cos[\omega(t - \mathscr{r}_+/c)]}{\mathscr{r}_+} - \frac{q_0\cos[\omega(t - \mathscr{r}_-/c)]}{\mathscr{r}_-}\right\} \qquad \mathscr{r}_\pm = \sqrt{r^2 \mp rd\cos\theta + (d/2)^2}.$$

$$\mathbf{A}(s, t) = \frac{\mu_0}{4\pi}\,\hat{\mathbf{z}}\int_{-d/2}^{d/2}\frac{I(t_r)}{\mathscr{r}}\,dz$$

Retarded scalar potential: $V(\mathbf{r},t)=\dfrac{1}{4\pi\epsilon_0}\left\{\dfrac{q_0\cos[\omega(t-\mathscr{r}_+/c)]}{\mathscr{r}_+}-\dfrac{q_0\cos[\omega(t-\mathscr{r}_-/c)]}{\mathscr{r}_-}\right\}$

Approximation 1 : *d « r* ➔ To make a perfect dipole, assume ***d*** to be extremely small



$$\mathscr{r}_\pm=\sqrt{r^2\mp rd\cos\theta+(d/2)^2}.\longrightarrow\frac{1}{\mathscr{r}_\pm}\cong\frac{1}{r}\left(1\pm\frac{d}{2r}\cos\theta\right)$$

$$\cos[\omega(t-\mathscr{r}_\pm/c)]\cong\cos\left[\omega(t-r/c)\pm\frac{\omega d}{2c}\cos\theta\right]$$

$$=\cos[\omega(t-r/c)]\cos\left(\frac{\omega d}{2c}\cos\theta\right)\mp\sin[\omega(t-r/c)]\sin\left(\frac{\omega d}{2c}\cos\theta\right)$$

Approximation 2 : *d « λ = 2πc/ω* ➔ Assume ***d*** to be extremely smaller than wavelength

$$\cos[\omega(t-\mathscr{r}_\pm/c)]\cong\cos[\omega(t-r/c)]\mp\frac{\omega d}{2c}\cos\theta\sin[\omega(t-r/c)]$$

$$V(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\left\{\frac{q_0\cos[\omega(t-\mathscr{r}_+/c)]}{\mathscr{r}_+}-\frac{q_0\cos[\omega(t-\mathscr{r}_-/c)]}{\mathscr{r}_-}\right\}$$

$$\Rightarrow V(r,\theta,t)=\frac{p_0\cos\theta}{4\pi\epsilon_0 r}\left\{-\frac{\omega}{c}\sin[\omega(t-r/c)]+\frac{1}{r}\cos[\omega(t-r/c)]\right\}$$

In the static limit ($\omega\to 0$),

$$V=\frac{p_0\cos\theta}{4\pi\epsilon_0 r^2}$$

Approximation 3 : *r » λ = 2πc/ω* ➔ Assume ***r*** to be larger than wavelength (far-field radiation)

$$\Rightarrow \boxed{V(r,\theta,t)=-\frac{p_0\omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin[\omega(t-r/c)]}$$

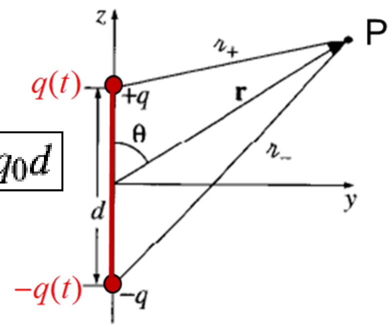
Retarded vector potential: $\mathbf{A}(s,t) = \frac{\mu_0}{4\pi}\,\hat{\mathbf{z}} \int_{-d/2}^{d/2} \frac{I(t_r)}{\mathscr{r}}\,dz$

$$\mathbf{I}(t) = \frac{dq}{dt}\,\hat{\mathbf{z}} = -q_0\omega\sin(\omega t)\,\hat{\mathbf{z}}$$

$$\longrightarrow \quad \mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int_{-d/2}^{d/2} \frac{-q_0\omega\sin[\omega(t-\mathscr{r}/c)]\,\hat{\mathbf{z}}}{\mathscr{r}}\,dz$$

$\boxed{p_0 \equiv q_0 d}$

$d \ll \lambda \ll r \quad \longrightarrow \quad \boxed{\mathbf{A}(r,\theta,t) = -\frac{\mu_0 p_0\omega}{4\pi r}\sin[\omega(t-r/c)]\,\hat{\mathbf{z}}}$



Retarded potentials:

$$V(r,\theta,t) = -\frac{p_0\omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin[\omega(t-r/c)]$$

$$\Rightarrow \quad \nabla V = \frac{\partial V}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial\theta}\,\hat{\boldsymbol{\theta}}$$

$$= -\frac{p_0\omega}{4\pi\epsilon_0 c}\left\{\cos\theta\left(-\frac{1}{r^2}\sin[\omega(t-r/c)] - \frac{\omega}{rc}\cos[\omega(t-r/c)]\right)\hat{\mathbf{r}} - \frac{\sin\theta}{r^2}\sin[\omega(t-r/c)]\,\hat{\boldsymbol{\theta}}\right\}$$

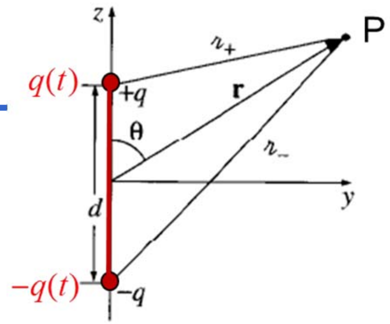
$$\cong \frac{p_0\omega^2}{4\pi\epsilon_0 c^2}\left(\frac{\cos\theta}{r}\right)\cos[\omega(t-r/c)]\,\hat{\mathbf{r}}$$

$$\mathbf{A}(r,\theta,t) = -\frac{\mu_0 p_0\omega}{4\pi r}\sin[\omega(t-r/c)]\,\hat{\mathbf{z}}$$

$$\Rightarrow \quad \frac{\partial\mathbf{A}}{\partial t} = -\frac{\mu_0 p_0\omega^2}{4\pi r}\cos[\omega(t-r/c)](\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}})$$

$$\Rightarrow \quad \nabla\times\mathbf{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} = -\frac{\mu_0 p_0\omega}{4\pi r}\left\{\frac{\omega}{c}\sin\theta\cos[\omega(t-r/c)] + \frac{\sin\theta}{r}\sin[\omega(t-r/c)]\right\}\hat{\boldsymbol{\phi}}$$

➔ Electric dipole radiation



$$\nabla V = \frac{p_0\omega^2}{4\pi\epsilon_0 c^2}\left(\frac{\cos\theta}{r}\right)\cos[\omega(t-r/c)]\,\hat{\mathbf{r}}.$$

$$\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi r}\cos[\omega(t-r/c)](\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}})$$

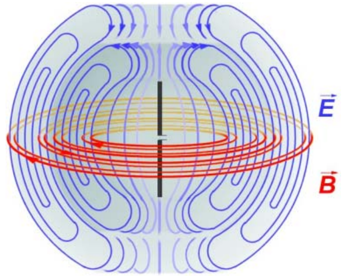
$$\nabla\times\mathbf{A} = -\frac{\mu_0 p_0 \omega}{4\pi r}\left\{\frac{\omega}{c}\sin\theta\cos[\omega(t-r/c)] + \frac{\sin\theta}{r}\sin[\omega(t-r/c)]\right\}\hat{\boldsymbol{\phi}}$$

Far-field radiation

$d \ll \lambda \ll r$

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\,\hat{\boldsymbol{\theta}}.$$

$$\mathbf{B} = \nabla\times\mathbf{A} = -\frac{\mu_0 p_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\,\hat{\boldsymbol{\phi}}.$$

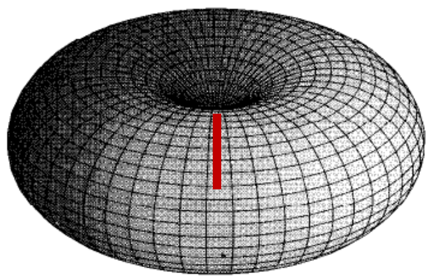


E and B are in phase, mutually perpendicular, and transverse; the ratio of their amplitudes is E_0/B_0 = c.
These are actually spherical waves, not plane waves, and their amplitude decreases like 1/r.

The energy radiated by an oscillating electric dipole is determined by the Poynting vector:

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\right\}^2\hat{\mathbf{r}}$$

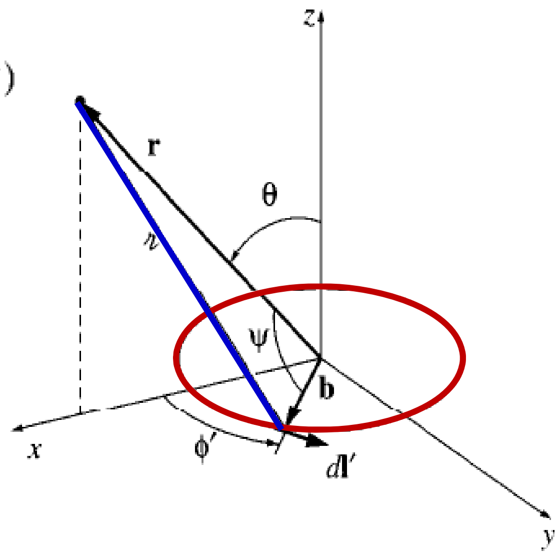
$$\langle\mathbf{S}\rangle = \left(\frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\,\hat{\mathbf{r}}$$ **: Intensity obtained by averaging**



$$\langle P\rangle = \int\langle\mathbf{S}\rangle\cdot d\mathbf{a} = \frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\int\frac{\sin^2\theta}{r^2}r^2\sin\theta\,d\theta\,d\phi = \frac{\mu_0 p_0^2\omega^4}{12\pi c}$$ **: total power radiated**

11.1.3 Magnetic Dipole Radiation

Magnetic dipole moment of an oscillating loop current : $I(t) = I_0 \cos(\omega t)$



$$\mathbf{m}(t) = \pi b^2 I(t)\,\hat{\mathbf{z}} = m_0 \cos(\omega t)\,\hat{\mathbf{z}} \qquad \text{where} \quad m_0 \equiv \pi b^2 I_0$$

➔ The loop is uncharged, so the scalar potential is zero.

➔ The retarded vector potential is

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{I_0 \cos[\omega(t - \mathscr{r}/c)]}{\mathscr{r}}\, d\mathbf{l}'$$

For a point ***r*** directly above the *x* axis, **A** must aim in the *y* direction,
since the *x* components from symmetrically placed points on either side of the *x* axis will cancel.

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 I_0 b}{4\pi}\,\hat{\mathbf{y}} \int_0^{2\pi} \frac{\cos[\omega(t - \mathscr{r}/c)]}{\mathscr{r}} \cos\phi'\, d\phi'$$

(cos ϕ' serves to pick out the y-component of *dl'*).

$$\mathscr{r} = \sqrt{r^2 + b^2 - 2rb\cos\psi}$$

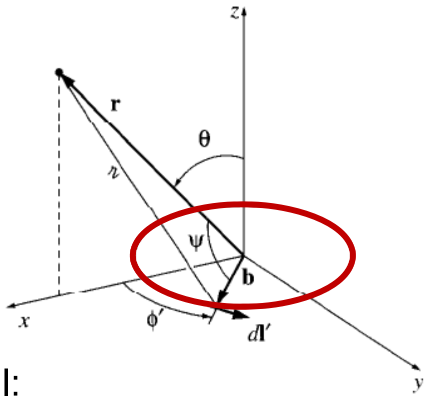
$$\mathbf{r} = r\sin\theta\,\hat{\mathbf{x}} + r\cos\theta\,\hat{\mathbf{z}}, \quad \mathbf{b} = b\cos\phi'\,\hat{\mathbf{x}} + b\sin\phi'\,\hat{\mathbf{y}} \longrightarrow rb\cos\psi = \mathbf{r}\cdot\mathbf{b} = rb\sin\theta\cos\phi'$$

$$\mathscr{r} = \sqrt{r^2 + b^2 - 2rb\sin\theta\cos\phi'}$$

Magnetic Dipole Radiation

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0 I_0 b}{4\pi}\,\hat{\mathbf{y}} \int_0^{2\pi} \frac{\cos[\omega(t - \mathscr{r}/c)]}{\mathscr{r}} \cos\phi'\, d\phi'$$

$$\mathscr{r} = \sqrt{r^2 + b^2 - 2rb\sin\theta\cos\phi'}$$



Approximation 1 : *b « r* For a "perfect" dipole, the loop must be extremely small:

$$\mathscr{r} \cong r\left(1 - \frac{b}{r}\sin\theta\cos\phi'\right) \longrightarrow \frac{1}{\mathscr{r}} \cong \frac{1}{r}\left(1 + \frac{b}{r}\sin\theta\cos\phi'\right)$$

$$\cos[\omega(t - \mathscr{r}/c)] \cong \cos\left[\omega(t - r/c) + \frac{\omega b}{c}\sin\theta\cos\phi'\right]$$

$$= \cos[\omega(t - r/c)]\cos\left(\frac{\omega b}{c}\sin\theta\cos\phi'\right) - \sin[\omega(t - r/c)]\sin\left(\frac{\omega b}{c}\sin\theta\cos\phi'\right)$$

Approximation 2 : *b « λ = 2πc/ω* ➔ Assume ***b*** to be extremely smaller than wavelength

$$\cos[\omega(t - \mathscr{r}/c)] \cong \cos[\omega(t - r/c)] - \frac{\omega b}{c}\sin\theta\cos\phi'\sin[\omega(t - r/c)]$$

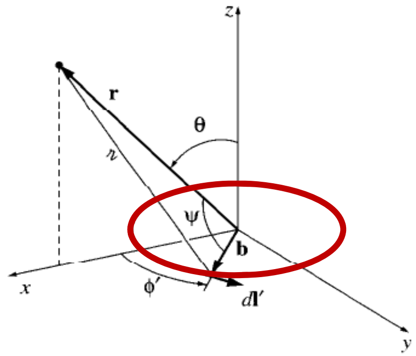
$$\mathbf{A}(\mathbf{r},t) \cong \frac{\mu_0 I_0 b}{4\pi r}\,\hat{\mathbf{y}} \int_0^{2\pi} \left\{\cos[\omega(t - r/c)] + b\sin\theta\cos\phi'\left(\frac{1}{r}\cos[\omega(t - r/c)] - \frac{\omega}{c}\sin[\omega(t - r/c)]\right)\right\}\cos\phi'\, d\phi'$$

Magnetic Dipole Radiation

$$\mathbf{A}(\mathbf{r},t) \cong \frac{\mu_0 I_0 b}{4\pi r}\hat{\mathbf{y}} \int_0^{2\pi} \left\{ \cos[\omega(t-r/c)] + b\sin\theta\cos\phi' \left(\frac{1}{r}\cos[\omega(t-r/c)] - \frac{\omega}{c}\sin[\omega(t-r/c)] \right) \right\} \cos\phi'\, d\phi'$$

$$\int_0^{2\pi} \cos\phi'\, d\phi' = 0 \qquad \int_0^{2\pi} \cos^2\phi'\, d\phi' = \pi$$

And, noting that in general **A** points in the ϕ-direction.



$$\mathbf{A}(r,\theta,t) = \frac{\mu_0 m_0}{4\pi}\left(\frac{\sin\theta}{r}\right)\left\{ \frac{1}{r}\cos[\omega(t-r/c)] - \frac{\omega}{c}\sin[\omega(t-r/c)] \right\} \hat{\boldsymbol{\phi}}$$

In the static limit (ω = 0), ➔ $\mathbf{A}(r,\theta) = \dfrac{\mu_0}{4\pi}\dfrac{m_0\sin\theta}{r^2}\hat{\boldsymbol{\phi}}$

Approximation 3 : *r » λ = 2πc/ω* ➔ Assume ***r*** to be larger than wavelength (far-field radiation)

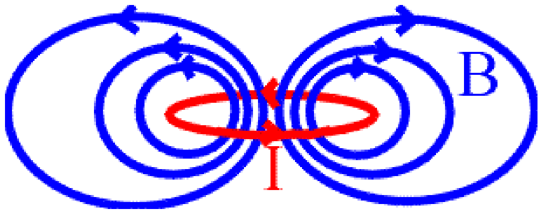
$$\mathbf{A}(r,\theta,t) = -\frac{\mu_0 m_0 \omega}{4\pi c}\left(\frac{\sin\theta}{r}\right)\sin[\omega(t-r/c)]\hat{\boldsymbol{\phi}}$$

Far-field radiation

d « λ « *r*

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = \frac{\mu_0 m_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\hat{\boldsymbol{\phi}}$$

$$\mathbf{B} = \nabla\times\mathbf{A} = -\frac{\mu_0 m_0 \omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\hat{\boldsymbol{\theta}}$$



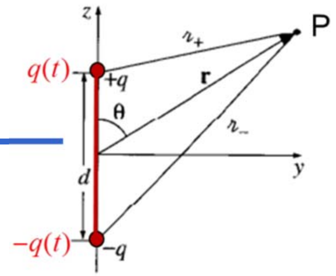
These fields are in phase, mutually perpendicular, and transverse to the direction of propagation (r) and the ratio of their amplitudes is E_o/B_o = c, all of which is as expected for electromagnetic waves.

$d \ll \lambda \ll r$

Electric dipole radiation

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\boldsymbol{\theta}}.$$

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\mu_0 p_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\boldsymbol{\phi}}$$



Energy flux: $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\right\}^2 \hat{\mathbf{r}}$ $\qquad \langle \mathbf{S}\rangle = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\,\hat{\mathbf{r}}$

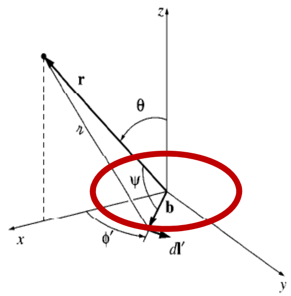
Total power radiated: $\langle P\rangle = \int \langle \mathbf{S}\rangle \cdot d\mathbf{a} = \frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\int \frac{\sin^2\theta}{r^2} r^2 \sin\theta\, d\theta\, d\phi = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$

$b \ll \lambda \ll r$

Magnetic dipole radiation

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = \frac{\mu_0 m_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\boldsymbol{\phi}}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\mu_0 m_0 \omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\boldsymbol{\theta}}$$



Energy flux: $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{\mu_0}{c}\left\{\frac{m_0\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\right\}^2 \hat{\mathbf{r}}$ $\qquad \langle \mathbf{S}\rangle = \left(\frac{\mu_0 m_0^2 \omega^4}{32\pi^2 c^3}\right)\frac{\sin^2\theta}{r^2}\,\hat{\mathbf{r}}$

Total power radiated: $\langle P\rangle = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3}$

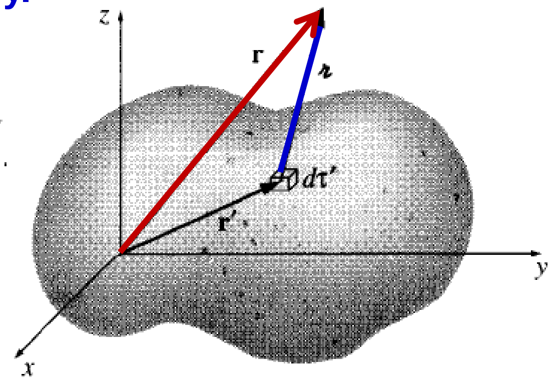
➔ **There is a remarkable similarity between the fields of oscillating *electric and magnetic* dipoles.**

One important difference between electric and magnetic dipole radiation is that for configurations with comparable dimensions, the power radiated electrically is enormously greater.

$$\frac{P_{\text{magnetic}}}{P_{\text{electric}}} = \left(\frac{m_0}{p_0 c}\right)^2 \xrightarrow[\text{Setting } d = \pi b]{m_0 = \pi b^2 I_0,\ \text{and}\ p_0 = q_0 d \quad I_0 = q_0\omega} \frac{P_{\text{magnetic}}}{P_{\text{electric}}} = \left(\frac{\omega b}{c}\right)^2 \xrightarrow{\text{Approximation 2: } b \ll \lambda = 2\pi c/\omega} \ll 1$$

11.1.4 Radiation from an Arbitrary Source

Consider a configuration of charge and current that is entirely arbitrary.



The retarded scalar potential is

$$V(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}',t-\mathscr{r}/c)}{\mathscr{r}}\,d\tau' \qquad \mathscr{r}=\sqrt{r^2+r'^2-2\mathbf{r}\cdot\mathbf{r}'}.$$

Approximation 1 : *r' « r (far field)*

$$\mathscr{r}\cong r\left(1-\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right)\longrightarrow\frac{1}{\mathscr{r}}\cong\frac{1}{r}\left(1+\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right)$$

$$\longrightarrow\rho(\mathbf{r}',t-\mathscr{r}/c)\cong\rho\left(\mathbf{r}',t-\frac{r}{c}+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right)$$

Expanding ρ as a Taylor series in t about the retarded time at the origin, $t_0\equiv t-\dfrac{r}{c}$,

$$\rho(\mathbf{r}',t-\mathscr{r}/c)\cong\rho(\mathbf{r}',t_0)+\dot{\rho}(\mathbf{r}',t_0)\left(\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right)+\left\{\frac{1}{2}\ddot{\rho}\left(\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right)^2,\quad\frac{1}{3!}\dddot{\rho}\left(\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right)^3,\ \ldots\right\}$$

Approximation 2 : *r' « λ = 2πc/ω*

$$\longrightarrow r'\ll\frac{c}{|\ddot{\rho}/\dot{\rho}|},\ \frac{c}{|\dddot{\rho}/\dot{\rho}|^{1/2}},\ \frac{c}{|\ddddot{\rho}/\dot{\rho}|^{1/3}},\ \ldots$$

r' « λ « r

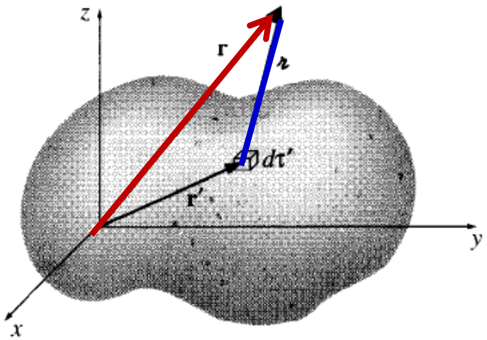
$$V(\mathbf{r},t)\cong\frac{1}{4\pi\epsilon_0 r}\left[\int\rho(\mathbf{r}',t_0)\,d\tau'+\frac{\hat{\mathbf{r}}}{r}\cdot\int\mathbf{r}'\rho(\mathbf{r}',t_0)\,d\tau'+\frac{\hat{\mathbf{r}}}{c}\cdot\frac{d}{dt}\int\mathbf{r}'\rho(\mathbf{r}',t_0)\,d\tau'\right]$$

$$=\frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r}+\frac{\hat{\mathbf{r}}\cdot\mathbf{p}(t_0)}{r^2}+\frac{\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}(t_0)}{rc}\right]\qquad\text{(Q: total charge)}$$

→ In the static case, the first two terms are the monopole and dipole contributions

Radiation from an Arbitrary Source

Now, consider the vector potential:



$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t - \mathscr{r}/c)}{\mathscr{r}}\, d\tau'$$

To first order in r' it suffices to replace $\mathscr{r}$ by r in the integrand:

$$\mathscr{r} \cong r \longrightarrow t_0 \cong t - \frac{r}{c}$$ *(Ignore the effect of magnetic* dipole moment)

$$\mathbf{A}(\mathbf{r}, t) \cong \frac{\mu_0}{4\pi r}\int \mathbf{J}(\mathbf{r}', t_0)\, d\tau' \cong \frac{\mu_0}{4\pi}\frac{\dot{\mathbf{p}}(t_0)}{r}$$ (according to Prob. 5.7)

Approximation 3 : *r » λ = 2πc/ω* (discard *1/r²* terms in E and B)

$$V(\mathbf{r}, t) \cong \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\hat{\mathbf{r}}\cdot\mathbf{p}(t_0)}{r^2} + \frac{\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}(t_0)}{rc}\right]$$

$$\nabla V \cong \nabla\left[\frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}(t_0)}{rc}\right] \cong \frac{1}{4\pi\epsilon_0}\left[\frac{\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}}(t_0)}{rc}\right]\nabla t_0 = -\frac{1}{4\pi\epsilon_0 c^2}\frac{[\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}}(t_0)]}{r}\hat{\mathbf{r}} \longleftarrow \nabla t_0 = -\frac{1}{c}\nabla r = -\frac{1}{c}\hat{\mathbf{r}}$$

$$\nabla\times\mathbf{A} \cong \frac{\mu_0}{4\pi r}[\nabla\times\dot{\mathbf{p}}(t_0)] = \frac{\mu_0}{4\pi r}[(\nabla t_0)\times\ddot{\mathbf{p}}(t_0)] = -\frac{\mu_0}{4\pi r c}[\hat{\mathbf{r}}\times\ddot{\mathbf{p}}(t_0)]$$

$$\frac{\partial\mathbf{A}}{\partial t} \cong \frac{\mu_0}{4\pi}\frac{\ddot{\mathbf{p}}(t_0)}{r}$$

$$\boxed{\mathbf{E}(\mathbf{r}, t) \cong \frac{\mu_0}{4\pi r}[(\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}})\hat{\mathbf{r}} - \ddot{\mathbf{p}}] = \frac{\mu_0}{4\pi r}[\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\ddot{\mathbf{p}})] \qquad \mathbf{B}(\mathbf{r}, t) \cong -\frac{\mu_0}{4\pi r c}[\hat{\mathbf{r}}\times\ddot{\mathbf{p}}]}$$

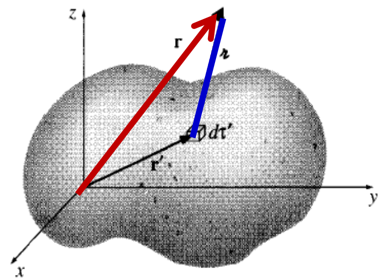
Radiation from an Arbitrary Source

$$\mathbf{E}(\mathbf{r}, t) \cong \frac{\mu_0}{4\pi r}[(\hat{\mathbf{r}} \cdot \ddot{\mathbf{p}})\hat{\mathbf{r}} - \ddot{\mathbf{p}}] = \frac{\mu_0}{4\pi r}[\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \ddot{\mathbf{p}})] \qquad \mathbf{B}(\mathbf{r}, t) \cong -\frac{\mu_0}{4\pi r c}[\hat{\mathbf{r}} \times \ddot{\mathbf{p}}]$$

In particular, if we use spherical polar coordinates, with the z axis in the direction of $\ddot{\mathbf{p}}(t_0)$,

$$\mathbf{E}(r, \theta, t) \cong \frac{\mu_0 \ddot{p}(t_0)}{4\pi}\left(\frac{\sin\theta}{r}\right)\hat{\boldsymbol{\theta}}$$

$$\mathbf{B}(r, \theta, t) \cong \frac{\mu_0 \ddot{p}(t_0)}{4\pi c}\left(\frac{\sin\theta}{r}\right)\hat{\boldsymbol{\phi}}$$



➔ Notice that E and B are mutually perpendicular, transverse to the direction of propagation (r) and in the ratio *E/B* = c, as always for radiation fields.

Poynting vector: $\mathbf{S} \cong \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) = \frac{\mu_0}{16\pi^2 c}[\ddot{p}(t_0)]^2\left(\frac{\sin^2\theta}{r^2}\right)\hat{\mathbf{r}}$

Total radiated power: $P \cong \int \mathbf{S} \cdot d\mathbf{a} = \frac{\mu_0 \ddot{p}^2}{6\pi c}$

$$\mathscr{r} \cong r\left(1 - \frac{\mathbf{r} \cdot \mathbf{r}'}{r^2}\right)$$

➔ If the electric dipole moment should happen to vanish (or, at any rate, if its second time derivative is zero), then there is no electric dipole radiation, and one must look to the next term: the one of *second* order in *r'.*

➔As it happens, this term can be separated into two parts, one of which is related to the *magnetic* dipole moment of the source, the other to its electric *quadrupole* moment (The former is a generalization of the magnetic dipole radiation).

➔ If the magnetic dipole and electric quadrupole contributions vanish, the *(r')3* term must be considered.

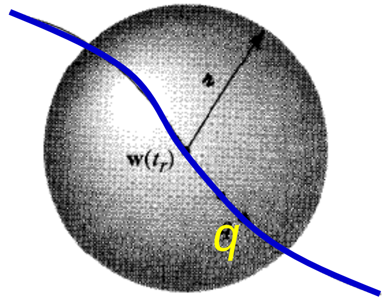
➔ This yields magnetic quadrupole and electric octopole radiation ... and so it goes.

11.2. Point charges: Power Radiated by a Moving Point Charge

The fields of a point charge *q* in arbitrary motion is (Eq. 10.65)

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}(\mathbf{r}, t)$$

The magnetic field of a point charge is always perpendicular to the electric field.



$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3}[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})] \quad \text{where } \mathbf{u} = c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}$$

velocity field (nonradiation field?)

acceleration field (radiation field?)

The Poynting vector is ➔ $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) = \frac{1}{\mu_0 c}[\mathbf{E} \times (\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E})] = \frac{1}{\mu_0 c}[E^2\hat{\boldsymbol{\mathscr{r}}} - (\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{E})\mathbf{E}]$

Consider a huge sphere of radius $\mathscr{r}$, the area of the sphere is proportional to $\mathscr{r}^2$

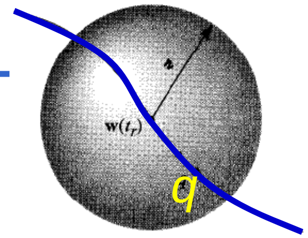
So any term in S that goes like $1/\mathscr{r}^2$ will yield a finite answer, but terms like $1/\mathscr{r}^3$ or $1/\mathscr{r}^4$ will contribute nothing in the limit $\mathscr{r} \to \infty$.

The velocity fields carry energy as the charge moves this energy is dragged along, **but it's not *radiation*.**

Only the *acceleration* fields represent true radiation (hence their other name, radiation fields):

$$\mathbf{E}_{\text{rad}} = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3}[\boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})] \qquad \mathbf{S}_{\text{rad}} = \frac{1}{\mu_0 c}E_{\text{rad}}^2\,\hat{\boldsymbol{\mathscr{r}}}$$

Power Radiated by a Point Charge



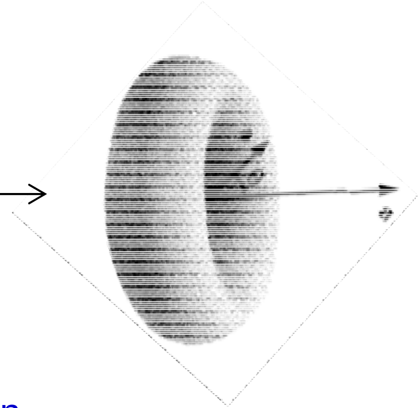
$$\mathbf{E}_{\text{rad}} = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}[\boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})] \qquad \mathbf{B}(\mathbf{r},t) = \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{E}(\mathbf{r},t)$$

$$\mathbf{S}_{\text{rad}} = \frac{1}{\mu_0 c}E_{\text{rad}}^2\,\hat{\boldsymbol{\mathscr{r}}} \qquad \text{where } \mathbf{u} = c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}$$

If the charge is instantaneously at *rest* (at time t_r), then $\mathbf{u} = c\hat{\boldsymbol{\mathscr{r}}}$, ***(It is good approximation as long as v « c.)***

$$\mathbf{E}_{\text{rad}} = \frac{q}{4\pi\epsilon_0 c^2\mathscr{r}}[\hat{\boldsymbol{\mathscr{r}}}\times(\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{a})] = \frac{\mu_0 q}{4\pi\mathscr{r}}[(\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{a})\hat{\boldsymbol{\mathscr{r}}} - \mathbf{a}]$$

$$\mathbf{S}_{\text{rad}} = \frac{1}{\mu_0 c}\left(\frac{\mu_0 q}{4\pi\mathscr{r}}\right)^2[a^2 - (\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{a})^2]\hat{\boldsymbol{\mathscr{r}}} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\frac{\sin^2\theta}{\mathscr{r}^2}\right)\hat{\boldsymbol{\mathscr{r}}}$$



where θ is the angle between $\hat{\boldsymbol{\mathscr{r}}}$ and $\mathbf{a}$.

No power is radiated in the forward or backward direction-rather,
it is emitted in a donut about the direction of instantaneous acceleration.

The total power radiated is

$$P = \oint \mathbf{S}_{\text{rad}}\cdot d\mathbf{a} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\int\frac{\sin^2\theta}{\mathscr{r}^2}\mathscr{r}^2\sin\theta\,d\theta\,d\phi \Rightarrow \boxed{P = \frac{\mu_0 q^2 a^2}{6\pi c}}$$

Larmor formula

An exact treatment of the case $v \neq 0$ is more difficult. Let's simply quote the result:

$$P = \frac{\mu_0 q^2\gamma^6}{6\pi c}\left(a^2 - \left|\frac{\mathbf{v}\times\mathbf{a}}{c}\right|^2\right) \quad \text{where } \gamma \equiv 1/\sqrt{1 - v^2/c^2}.$$

Liénard's generalization of the Larmor formula

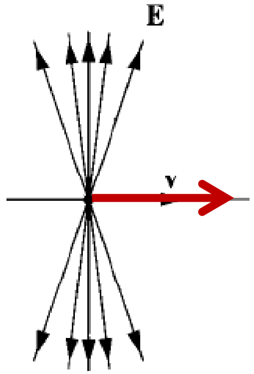
➔ ***The factor γ^6 means that the radiated power increases enormously as the velocity approaches the speed of light.***

Comparison: Non-radiated fields and radiated fields from a Point Charge

$$\mathbf{E}(\mathbf{r},t)=\frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}[(c^2-v^2)\mathbf{u}+\boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})] \quad \text{where } \mathbf{u}=c\hat{\boldsymbol{\mathscr{r}}}-\mathbf{v}$$

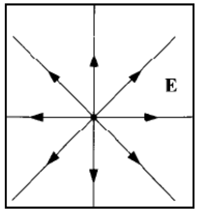
$\mathbf{E}_{non\text{-}rad}$ velocity field (non-radiation field) acceleration field (radiation field) $\mathbf{E}_{rad}$

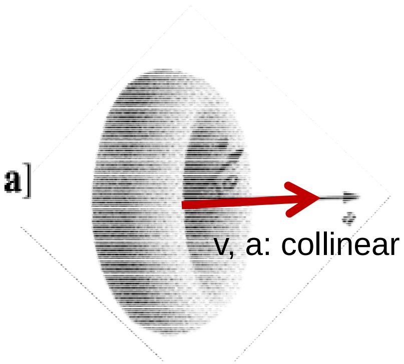
Note that the velocity fields also do carry energy; they just don't transport it out to infinity.



$\boxed{a=0, \upsilon\neq 0}$ $\quad E_{rad}=0 \quad E_{non-rad}=\frac{q}{4\pi\epsilon_0}\frac{(c^2-v^2)\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\mathbf{u}=\frac{q}{4\pi\epsilon_0}\frac{1-v^2/c^2}{\left(1-v^2\sin^2\theta/c^2\right)^{3/2}}\frac{\hat{\mathbf{R}}}{R^2}$

$\boxed{a=0, \upsilon=0}$ $\quad E_{rad}=0 \quad E_{non-rad}=\frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}$

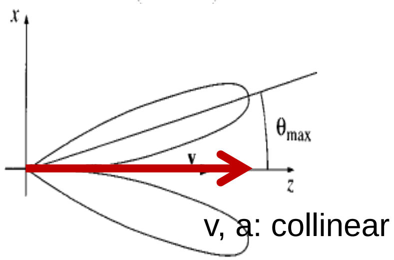




$\boxed{a\neq 0, \upsilon\neq 0;\ \upsilon\ll c}$ $\quad E_{non-rad}\neq 0 \quad E_{rad}=\frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}[\boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})]=\frac{\mu_0 q}{4\pi\mathscr{r}}[(\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{a})\hat{\boldsymbol{\mathscr{r}}}-\mathbf{a}]$

$$\mathbf{S}_{\text{rad}}=\frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\frac{\sin^2\theta}{\mathscr{r}^2}\right)\hat{\boldsymbol{\mathscr{r}}} \qquad P=\oint \mathbf{S}_{\text{rad}}\cdot d\mathbf{a}=\frac{\mu_0 q^2 a^2}{6\pi c}$$

$\boxed{a\neq 0, \upsilon\neq 0; \upsilon\sim c}$ $\quad E_{non-rad}\neq 0 \quad E_{rad}=\frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}[\boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})]$



$$P=\frac{\mu_0 q^2 a^2\gamma^6}{6\pi c} \qquad \frac{dP}{d\Omega}=\frac{\mu_0 q^2 a^2}{16\pi^2 c}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \qquad \beta\equiv v/c \qquad \gamma\equiv 1/\sqrt{1-v^2/c^2}$$

11.2.2 Radiation Reaction

Radiation from an accelerating charge carries off energy → resulting in reduction of the particle's kinetic energy.
→ Under a given force, therefore, a charged particle accelerates *less* than a neutral one of the same mass.
→ The radiation evidently exerts a force (**F**$_{\text{rad}}$) back on the charge – *recoil (or, radiation reaction) force*.

For a nonrelativistic particle *(v « c)* the total power radiated is given by the Larmor formula (Eq. 11.70):

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} \longrightarrow \mathbf{F}_{\text{rad}} \cdot \mathbf{v} = -\frac{\mu_0 q^2 a^2}{6\pi c} \qquad (11.77)$$

Conservation of energy suggests that this is also the rate at which the particle loses energy, under the influence of the radiation reaction force **F**$_{\text{rad}}$:

The energy lost by the particle in any given time interval: $\displaystyle\int_{t_1}^{t_2} \mathbf{F}_{\text{rad}} \cdot \mathbf{v}\, dt = -\frac{\mu_0 q^2}{6\pi c}\int_{t_1}^{t_2} a^2\, dt$

$$\int_{t_1}^{t_2} a^2\, dt = \int_{t_1}^{t_2} \left(\frac{d\mathbf{v}}{dt}\right) \cdot \left(\frac{d\mathbf{v}}{dt}\right) dt = \left(\mathbf{v} \cdot \frac{d\mathbf{v}}{dt}\right)\Bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d^2\mathbf{v}}{dt^2} \cdot \mathbf{v}\, dt$$

0

If the motion is periodic-the velocities and accelerations are identical at t_1 and t_2,
or if $v \cdot a = 0$ at t_1 and t_2,

$$\int_{t_1}^{t_2} \left(\mathbf{F}_{\text{rad}} - \frac{\mu_0 q^2}{6\pi c}\dot{\mathbf{a}}\right) \cdot \mathbf{v}\, dt = 0 \longrightarrow \mathbf{F}_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c}\dot{\mathbf{a}}$$

Abraham-Lorentz formula
for the radiation reaction force

Radiation Reaction

$$\int_{t_1}^{t_2}\left(\mathbf{F}_{\text{rad}}-\frac{\mu_0 q^2}{6\pi c}\dot{\mathbf{a}}\right)\cdot\mathbf{v}\,dt=0 \qquad \mathbf{F}_{\text{rad}}=\frac{\mu_0 q^2}{6\pi c}\dot{\mathbf{a}}$$

Abraham-Lorentz formula
for the radiation reaction force

For suppose a particle is subject to no *external* forces (F = 0); then Newton's second law says

$$F_{\text{rad}}=\frac{\mu_0 q^2}{6\pi c}\dot{a}=ma \longrightarrow a(t)=a_0 e^{t/\tau}, \quad \text{where} \quad \tau\equiv\frac{\mu_0 q^2}{6\pi mc}$$

➔ In the case of the electron, $\tau = 6 \times 10^{-24}$ s. → only the time taken for light to travel ~ 10^{-15} m

➔ The acceleration spontaneously *increases* exponentially with time!
→ "runaway" under no external force!

If you *do* apply an external force,

$$ma = F_{rad} + F, \quad F_{rad} = \tau\dot{a} \quad \Rightarrow \quad a = \tau\dot{a} + \frac{F}{m}$$

: Abraham-Lorentz equation of motion

If an external force is applied to the particle for times t > 0, the equation of motion predicts "preaccelaeration" before the force is actually applied. ➔ It starts to respond *before the force acts!*
→ *"preacceleration" acausalty!*

(Problem 11.19) Assume that a particle is subjected to a constant force *F*, beginning at time *t* = 0 and lasting until time *T*. Show that you can *either* eliminate the runaway in region (iii) *or* avoid preacceleration in region (i), *but not both.*

Radiation Reaction

Problem 11.19 If you apply an external force, **F**, acting on the particle, Newton's second law for a charged particle becomes

$$\boxed{a = \tau \dot{a} + \frac{F}{m}} \qquad \tau \equiv \frac{\mu_0 q^2}{6\pi mc}$$

(b) A particle is subjected to a constant force *F,* beginning at time *t* = 0 and lasting until time *T.*
Find the most general solution *a (t)* to the equation of motion in each of the three periods: (i) *t* < 0; (ii) 0 < *t* < *T;* (iii) *t* > *T.*

$$\text{(i) } a = \tau\dot{a} = \tau\frac{da}{dt} \Rightarrow \frac{da}{a} = \frac{1}{\tau}dt \Rightarrow \int\frac{da}{a} = \frac{1}{\tau}\int dt \Rightarrow \ln a = \frac{t}{\tau} + \text{constant} \Rightarrow \boxed{a(t) = Ae^{t/\tau},}$$

$$\text{(ii) } a = \tau\dot{a} + \frac{F}{m} \Rightarrow \tau\frac{da}{dt} = a - \frac{F}{m} \Rightarrow \frac{da}{a - F/m} = \frac{1}{\tau}dt \Rightarrow \ln(a - F/m) = \frac{t}{\tau} + \text{constant} \Rightarrow a - \frac{F}{m} = Be^{t/\tau} \Rightarrow \boxed{a(t) = \frac{F}{m} + Be^{t/\tau},}$$

$$\text{(iii) Same as (i): } \boxed{a(t) = Ce^{t/\tau},}$$

(c) Impose the continuity condition (a) at *t* = 0 and *t* = *T.*
Show that you can *either* eliminate the runaway in region (iii) *or* avoid preacceleration in region (i), *but not both.*

At $t = 0$, $A = F/m + B$; at $t = T$, $F/m + Be^{T/\tau} = Ce^{T/\tau} \Rightarrow C = (F/m)e^{-T/\tau} + B$.

$$a(t) = \begin{cases} [(F/m) + B]\, e^{t/\tau}, & t \le 0; \\ \left[(F/m) + Be^{t/\tau}\right], & 0 \le t \le T; \\ \left[(F/m)e^{-T/\tau} + B\right] e^{t/\tau}, & t \ge T. \end{cases}$$

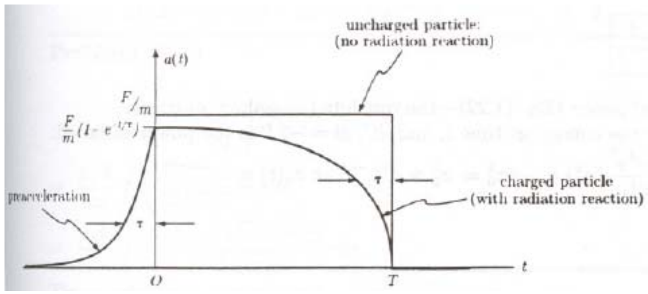
To eliminate the runaway in region (iii), we'd need $B = -(F/m)e^{-T/\tau}$;

to avoid preacceleration in region (i), we'd need $B = -(F/m)$.

Obviously, we cannot do both at once.

(d) If you choose to eliminate the runaway,

$$a(t) = \begin{cases} (F/m)\left[1 - e^{-T/\tau}\right] e^{t/\tau}, & t \le 0; \\ (F/m)\left[1 - e^{(t-T)/\tau}\right], & 0 \le t \le T; \\ 0, & t \ge T. \end{cases}$$



Radiation Reaction (J. Jacksons, p.780)

To estimate the range of parameters where radiative effects on reaction are important or not, consider the radiative energy of a charge ***e*** under an external force to have acceleration ***a*** for a period of time T:

For a particle at rest initially a typical energy is its kinetic energy after the period of acceleration:

$$E_0 \sim m(aT)^2$$

On the other hand, From the Larmor formula the energy radiated is of the order of

$$E_{rad} \sim P \cdot T \sim \frac{\mu_0 e^2}{6\pi c} a^2 T \quad \longleftarrow \quad P = \frac{\mu_0 q^2 a^2}{6\pi c}$$

The criterion for the regime where radiative effects are not important can thus be expressed by

$$E_{rad} \ll E_0 \longrightarrow \frac{\mu_0 e^2}{6\pi c} a^2 T \ll m a^2 T^2 \longrightarrow T \gg \frac{\mu_0 e^2}{6\pi m c} = \tau$$

➔ In the case of the electron, $\tau = 6 \times 10^{-24}$ s. → only the time taken for light to travel ~ 10^{-15} m

➔ *Only for phenomena involving such distances or times will we expect radiative effects to play a crucial role.*

(ex) If the motion is quasi-periodic with a typical amplitude ***d*** and characteristic frequency ω_0: $E_0 \sim m\omega_0^2 d^2$

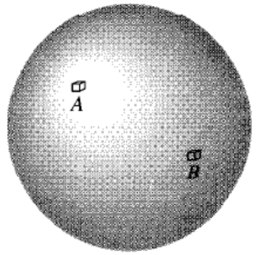
The acceleration are typically a ~ ω_0^2 d, and the time interval T ~ $(1/\omega_0)$:

$$E_{rad} \ll E_0 \longrightarrow \frac{\mu_0 e^2}{6\pi c} \frac{\left(\omega_0^2 d\right)^2}{\omega_0} \ll m\omega_0^2 d^2 \longrightarrow \frac{1}{\omega_0} \sim T \gg \tau$$

➔ *If the mechanical time interval is much longer than τ, radiative reaction effects will be unimportant.*

11.2.3 The Physical Basis of the Radiation Reaction

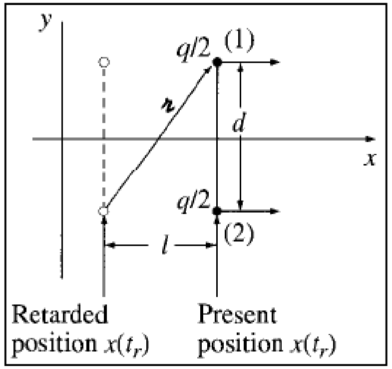
Conclusion:* “*The radiation reaction is due to the force of the charge on itself (“**self-force**”). Or, more elaborately, the net force exerted by the fields generated by different parts of the charge distribution acting on one another.”



Consider a moving charge with an *extended* charge distribution:
In general, the electromagnetic force of one part (A) on another part (B) is not equal and opposite to the force of B on A.

Let’s simplify the situation into a “bumble” : the total charge **q** is divided into two halves separated by a fixed **d**:

➔ In the point limit *(d* → 0), it must yield the Abraham-Lorentz formula.



The electric field at (1) due to (2) is

$$\mathbf{E}_1 = \frac{(q/2)}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}[(c^2+\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{u}-(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\mathbf{a}] \qquad \mathbf{u} = c\,\hat{\boldsymbol{\mathscr{r}}} \text{ and } \boldsymbol{\mathscr{r}} = l\,\hat{\mathbf{x}} + d\,\hat{\mathbf{y}}$$

$$u_x = \frac{cl}{\mathscr{r}} \longrightarrow E_{1_x} = \frac{q}{8\pi\epsilon_0 c^2}\frac{(lc^2-ad^2)}{(l^2+d^2)^{3/2}} \quad \text{By symmetry, } E_{2_x} = E_{1_x},$$

$$\mathbf{F}_{\text{self}} = \frac{q}{2}(\mathbf{E}_1+\mathbf{E}_2) = \frac{q^2}{8\pi\epsilon_0 c^2}\frac{(lc^2-ad^2)}{(l^2+d^2)^{3/2}}\,\hat{\mathbf{x}} = \frac{q^2}{4\pi\epsilon_0}\left[-\frac{a(t)}{4c^2d}+\frac{\dot{a}(t)}{3c^3}+(\ )d+\cdots\right]\hat{\mathbf{x}}$$

The first term ~ E_0

The second term survives in the "point dumbbell" limit *d* → 0: $F_{\text{rad}}^{\text{int}} = \frac{\mu_0 q^2 \dot{a}}{12\pi c}$

This term (x 2) is equal to the radiation reaction force given by the Abraham-Lorentz formula!

➔ ***In conclusion,* “*the radiation reaction is due to the force of the charge on itself (“**self-force**”).***