PROBLEMA GUÍA 2 | #5

$\phi_1 = \dfrac{(\vec{P}\cdot\vec{r})^2}{r^2}$, luego la componente i-ésima de $\nabla\phi_1$

está dada por

$$(\nabla\phi_1)_i = \partial_i \phi_1 = \partial_i\left[\frac{(\vec{P}\cdot\vec{r})^2}{r^2}\right]$$

$$= (\partial_i r^{-2})(\vec{P}\cdot\vec{r})^2 + \frac{1}{r^2}(\partial_i(\vec{P}\cdot\vec{r})^2)$$

$$= -\frac{2}{r^3}(\partial_i r)(\vec{P}\cdot\vec{r})^2 + \frac{1}{r^2}2(\vec{P}\cdot\vec{r})(\partial_i(\vec{P}\cdot\vec{r}))$$

$$= -2\frac{(\vec{P}\cdot\vec{r})^2}{r^3}\frac{x_i}{r} + 2\frac{(\vec{P}\cdot\vec{r})}{r^2}P_\ell(\partial_i x_\ell)$$

$$= -2\frac{(\vec{P}\cdot\vec{r})^2}{r^4}x_i + 2\frac{(\vec{P}\cdot\vec{r})}{r^2}P_i$$

$$\therefore \quad \nabla\phi_1 = 2\frac{(\vec{P}\cdot\vec{r})}{r^2}\left[\vec{P} - \frac{(\vec{P}\cdot\vec{r})}{r^2}\vec{r}\right]$$

Por otro lado $\phi_2 = \dfrac{r^2}{(\vec{P}\cdot\vec{r})^2}$ y

$$(\nabla\phi_2)_i = \partial_i\left[\frac{r^2}{(\vec{P}\cdot\vec{r})^2}\right] = (\partial_i r^2)\frac{1}{(\vec{P}\cdot\vec{r})^2} + r^2(\partial_i(\vec{P}\cdot\vec{r})^{-2})$$

$$= 2r(\partial_i r)\frac{1}{(\vec{p}\cdot\vec{r})^2} + r^2\frac{(-2)}{(\vec{p}\cdot\vec{r})^3}(\partial_i(\vec{p}\cdot\vec{r}))$$

$$= \frac{2}{(\vec{p}\cdot\vec{r})^2}x_i - 2\frac{r^2}{(\vec{p}\cdot\vec{r})^3}p_\ell(\partial_i x_\ell)$$

$$= \frac{2}{(\vec{p}\cdot\vec{r})^2}x_i - 2\frac{r^2}{(\vec{p}\cdot\vec{r})^3}p_i$$

$$\therefore \nabla\phi_2 = \frac{2}{(\vec{p}\cdot\vec{r})^2}\vec{r} - \frac{2r^2}{(\vec{p}\cdot\vec{r})^3}\vec{p}$$

$$= \frac{2}{(\vec{p}\cdot\vec{r})^2}\left[\vec{r} - \frac{r^2\vec{p}}{(\vec{p}\cdot\vec{r})}\right]$$

Finalmente:

$$\nabla\phi_1 \times \nabla\phi_2 = \frac{4}{r^2(\vec{p}\cdot\vec{r})}\left(\vec{p} - \frac{(\vec{p}\cdot\vec{r})\vec{r}}{r^2}\right) \times \left(\vec{r} - \frac{r^2\vec{p}}{(\vec{p}\cdot\vec{r})}\right)$$

$$= \frac{4}{r^2(\vec{p}\cdot\vec{r})}\left(\vec{p}\times\vec{r} + \underbrace{\vec{r}\times\vec{p}}_{-\vec{p}\times\vec{r}}\right)$$

$$= \vec{0}$$