Chapter 1. Vector Analysis

1.1 Vector Algebra

1.1.1 Vector Operations

(i) Addition of two vectors.

Addition is *commutative*: A + B = B + A

Addition is *associative*: (A + B) + C = A + (B + C)

To *subtract* is to add its opposite: A - B = A + (-B)

(ii) Multiplication by a scalar. $a(\mathbf{A}+\mathbf{B}) = a\mathbf{A} + a\mathbf{B}$

(iii) Dot product of two vectors. $\mathbf{A}\cdot\mathbf{B} \equiv AB\cos\theta$

Dot product (= scalar product) is *commutative*: A · B = B · A

Dot product (= scalar product) is *distributive*: A · (B + C) = A · B + A · C

(iv) Cross product of two vectors. $\mathbf{A}\times\mathbf{B} \equiv AB\sin\theta\,\hat{\mathbf{n}}$

Cross product (= vector product) is *not commutative*: B x A = A x B

Dot product (= vector product) is *distributive*: A x (B + C) = A x B + A x C

Example 1.1

Let $\mathbf{C} = \mathbf{A} - \mathbf{B}$ (Fig. 1.7), and calculate the dot product of $\mathbf{C}$ with itself.

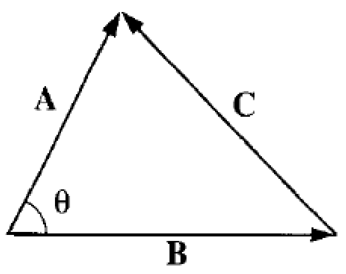
Solution:

$$\mathbf{C}\cdot\mathbf{C} = (\mathbf{A}-\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) = \mathbf{A}\cdot\mathbf{A} - \mathbf{A}\cdot\mathbf{B} - \mathbf{B}\cdot\mathbf{A} + \mathbf{B}\cdot\mathbf{B},$$

or

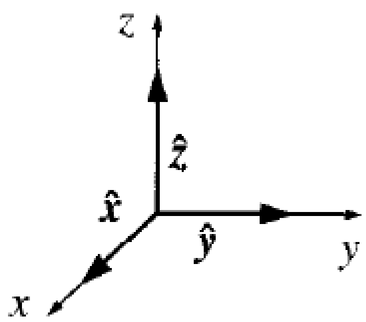
$$C^2 = A^2 + B^2 - 2AB\cos\theta.$$

This is the **law of cosines**.

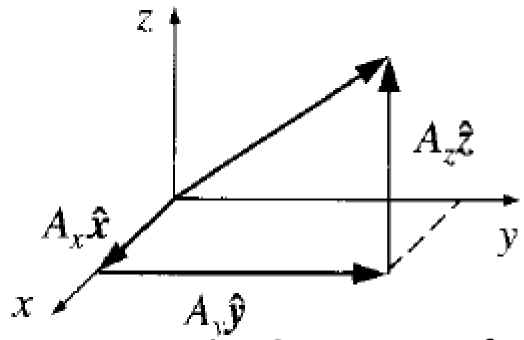


1.1.2 Vector Algebra: Component form

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}.$$



Unit vectors



Component form

(i) Rule: *To add vectors, add like components.*

$$\begin{aligned}\mathbf{A} + \mathbf{B} &= (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) + (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}) \\ &= (A_x + B_x)\hat{\mathbf{x}} + (A_y + B_y)\hat{\mathbf{y}} + (A_z + B_z)\hat{\mathbf{z}}.\end{aligned}$$

(ii) Rule: *To multiply by a scalar, multiply each component.*

$$a\mathbf{A} = (aA_x)\hat{\mathbf{x}} + (aA_y)\hat{\mathbf{y}} + (aA_z)\hat{\mathbf{z}}.$$

(iii) Rule: *To calculate the dot product, multiply like components, and add.*

$$\mathbf{A}\cdot\mathbf{B} = (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}})\cdot(B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}) = A_xB_x + A_yB_y + A_zB_z$$

$$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1; \quad \hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{x}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = 0.$$

1.1.2 Vector Algebra: Component form

(iv) Rule: *To calculate the cross product, form the determinant whose first row is* $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$, *whose second row is* $\mathbf{A}$ *(in component form), and whose third row is* $\mathbf{B}$.

$$
\begin{aligned}
\mathbf{A} \times \mathbf{B} &= (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \times (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}) \\
&= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\
&= (A_yB_z - A_zB_y)\hat{\mathbf{x}} + (A_zB_x - A_xB_z)\hat{\mathbf{y}} + (A_xB_y - A_yB_x)\hat{\mathbf{z}}
\end{aligned}
$$

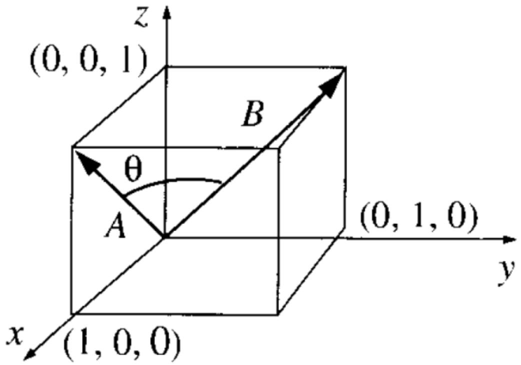
$$
\begin{aligned}
\hat{\mathbf{x}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{y}} &= \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0, \\
\hat{\mathbf{x}} \times \hat{\mathbf{y}} = -\hat{\mathbf{y}} \times \hat{\mathbf{x}} &= \hat{\mathbf{z}}, \\
\hat{\mathbf{y}} \times \hat{\mathbf{z}} = -\hat{\mathbf{z}} \times \hat{\mathbf{y}} &= \hat{\mathbf{x}}, \\
\hat{\mathbf{z}} \times \hat{\mathbf{x}} = -\hat{\mathbf{x}} \times \hat{\mathbf{z}} &= \hat{\mathbf{y}}.
\end{aligned}
$$

Example 1.2 Find the angle between the face diagonals of a cube.

$$\mathbf{A} = 1\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + 1\,\hat{\mathbf{z}}; \qquad \mathbf{B} = 0\,\hat{\mathbf{x}} + 1\,\hat{\mathbf{y}} + 1\,\hat{\mathbf{z}}.$$

$$
\begin{aligned}
\mathbf{A} \cdot \mathbf{B} &= 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 1 = 1 \\
&= AB\cos\theta = \sqrt{2}\sqrt{2}\cos\theta = 2\cos\theta.
\end{aligned}
$$

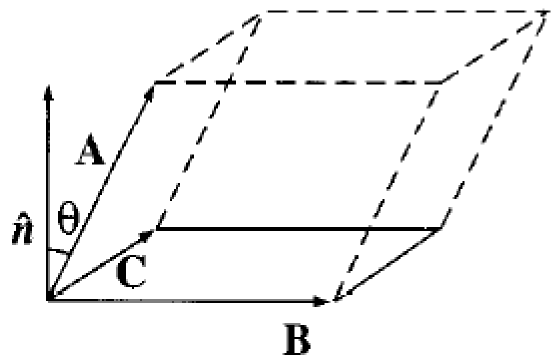
Therefore, $\cos\theta = 1/2$, or $\theta = 60°$.



1.1.3 Triple Products

(i) Scalar triple product: $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$

$|\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})|$ is the volume of the parallelepiped generated by **A**, **B**, and **C**, since $|\mathbf{B} \times \mathbf{C}|$ is the area of the base, and $|\mathbf{A}\cos\theta|$ is the altitude



$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$ Note that "alphabetical" order is preserved.

In component form, $$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Note that the dot and cross can be interchanged: $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$

1.1.3 Triple Products

(ii) Vector triple product: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$ ➔ **BAC-CAB** rule

Notice that $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ is an entirely different vector.

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = -\mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = -\mathbf{A}(\mathbf{B} \cdot \mathbf{C}) + \mathbf{B}(\mathbf{A} \cdot \mathbf{C})$$

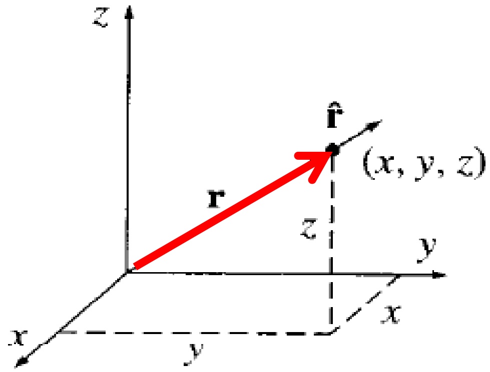
all *higher* vector products can be similarly reduced,
so it is never necessary for an expression
to contain more than one cross product in any term.

$$\begin{aligned} (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}); \\ \mathbf{A} \times (\mathbf{B} \times (\mathbf{C} \times \mathbf{D})) &= \mathbf{B}(\mathbf{A} \cdot (\mathbf{C} \times \mathbf{D})) - (\mathbf{A} \cdot \mathbf{B})(\mathbf{C} \times \mathbf{D}). \end{aligned}$$

$$[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})] + [\mathbf{B} \times (\mathbf{C} \times \mathbf{A})] + [\mathbf{C} \times (\mathbf{A} \times \mathbf{B})] = 0.$$

1.1.4 Position, Displacement, and Separation Vectors

Position vector: $\mathbf{r} \equiv x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$.



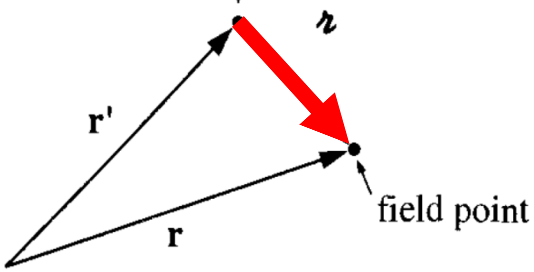
a unit vector pointing radially outward is

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}}$$

Infinitesimal displacement vector: $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$

Separation vector from source point to field point:

$$\boldsymbol{\mathscr{r}} \equiv \mathbf{r} - \mathbf{r}' \qquad \boldsymbol{\mathscr{r}} = (x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}$$



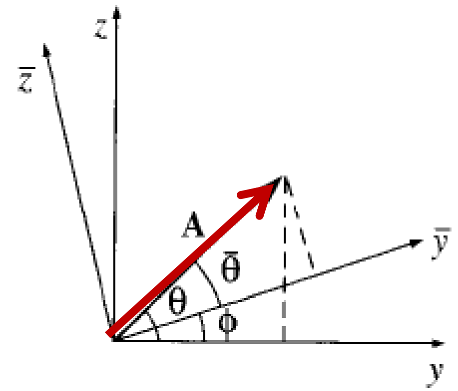
Its magnitude is

$$\mathscr{r} = |\mathbf{r} - \mathbf{r}'| \qquad \mathscr{r} = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

a unit vector in the direction from $\mathbf{r}'$ to $\mathbf{r}$ is

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \qquad \hat{\boldsymbol{\mathscr{r}}} = \frac{(x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}$$

1.1.5 How Vectors transform



Suppose, for instance, the $\bar{x}$, $\bar{y}$, $\bar{z}$ system is rotated by angle ϕ, relative to x, y, z, about the common $x = \bar{x}$ axes.

$$A_y = A\cos\theta, \qquad A_z = A\sin\theta,$$

$$\bar{A}_y = A\cos\bar{\theta} = A\cos(\theta-\phi) = A(\cos\theta\cos\phi + \sin\theta\sin\phi) = \cos\phi A_y + \sin\phi A_z$$

$$\bar{A}_z = A\sin\bar{\theta} = A\sin(\theta-\phi) = A(\sin\theta\cos\phi - \cos\theta\sin\phi) = -\sin\phi A_y + \cos\phi A_z$$

$$\Rightarrow \begin{pmatrix} \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_y \\ A_z \end{pmatrix}$$

More generally, for rotation about an *arbitrary* axis in three dimensions,

$$\begin{pmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \Rightarrow \bar{A}_i = \sum_{j=1}^{3} R_{ij} A_j$$

a vector is a tensor of rank 1

1.2 Differential Calculus

1.2.1 "Ordinary" Derivatives

$$df = \boxed{\left(\frac{df}{dx}\right)} dx$$

how rapidly the function $f(x)$ varies when we change the argument x

the *slope* of the graph of f versus x

1.2.2 Gradient

Suppose, now, that we have a function of *three* variables— $T(x, y, z)$

how T changes when we alter all three variables by the infinitesimal amounts dx, dy, dz.

$$\begin{aligned} dT &= \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz \\ &= \left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right) \cdot (dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}) \\ &= (\nabla T) \cdot (d\mathbf{l}) \end{aligned}$$

$$\nabla T \equiv \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

➔ **Gradient** of T

$$\nabla T \equiv \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

→ **What's the physical meaning of the Gradient:**

Geometrical Interpretation of the Gradient:

$$dT = \nabla T \cdot d\mathbf{l} = |\nabla T||d\mathbf{l}|\cos\theta$$ where θ is the angle between ∇T and $d\mathbf{l}$.

the *maximum* change in T evidentally occurs when $\theta = 0$ (for then $\cos\theta = 1$)

That is, for a fixed distance $|d\mathbf{l}|$, dT is greatest when I move in the *same direction* as ∇T.

➔ **Gradient** is a vector that points in the direction of maximum increase of a function. Its magnitude gives the slope (rate of increase) along this maximal direction.

➔ Gradient represents both the magnitude and the direction of the maximum rate of increase of a scalar function.

Example 1.3

Find the gradient of $r = \sqrt{x^2 + y^2 + z^2}$ (the magnitude of the position vector).

$$\nabla r = \frac{\partial r}{\partial x}\hat{\mathbf{x}} + \frac{\partial r}{\partial y}\hat{\mathbf{y}} + \frac{\partial r}{\partial z}\hat{\mathbf{z}} = \frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}} = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}.$$

it says that the distance from the origin increases most rapidly in the radial direction, its *rate* of increase in that direction is 1.

1.2.3 The Del Operator: ∇

$$\nabla T \equiv \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)T$$

$$\Rightarrow \boxed{\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}}$$

: a **vector operator**, **not a vector.**

there are three ways the operator ∇ can act:

1. On a scalar function T : ∇T **(gradient)**

   ➔ **Gradient** represents both the magnitude and the direction of the maximum rate of increase of a scalar function.

2. On a vector function $\mathbf{v}$, via the dot product: $\nabla \cdot \mathbf{v}$ **(divergence)**

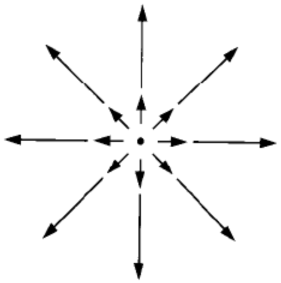
3. On a vector function $\mathbf{v}$, via the cross product: $\nabla \times \mathbf{v}$ **(curl)**

1.2.4 The Divergence $div\ \mathbf{A} = \nabla \bullet \mathbf{A}$

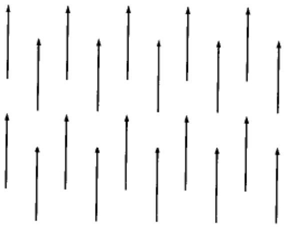
$$\nabla \cdot \mathbf{v} = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot (v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\boxed{\nabla \bullet \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

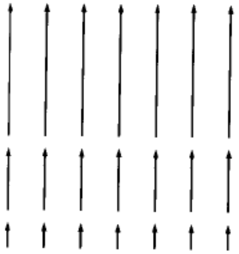
: **scalar, a** measure of how much the vector **A** spread out (diverges) from the point in question



: positive (negative if the arrows pointed in) divergence



: zero divergence



: positive divergence

Example 1.4

Suppose the functions in Fig. 1.18 are $\mathbf{v}_a = \mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$, $\mathbf{v}_b = \hat{\mathbf{z}}$, and $\mathbf{v}_c = z\,\hat{\mathbf{z}}$. Calculate their divergences.

$$\nabla \cdot \mathbf{v}_a = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$$

$$\nabla \cdot \mathbf{v}_b = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(1) = 0 + 0 + 0 = 0$$

$$\nabla \cdot \mathbf{v}_c = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(z) = 0 + 0 + 1 = 1$$

1.2.5 The Curl

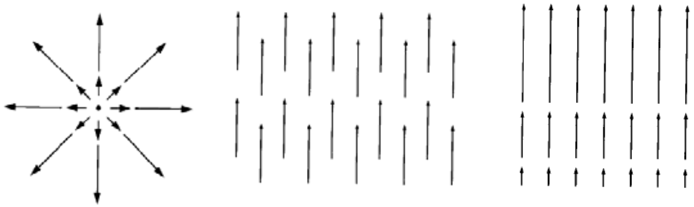
$curl\ \mathbf{A} = rot\ \mathbf{A} = \nabla \times \mathbf{A}$

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix} = \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

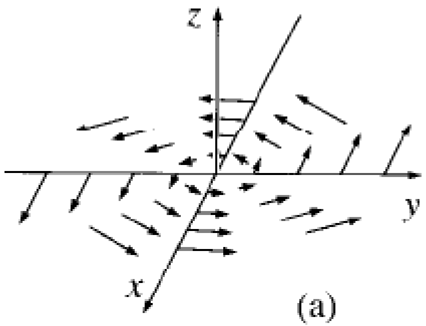
$$\nabla \times \boldsymbol{A} = \begin{pmatrix} \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix}$$

: a **vector,** a measure of how much the vector **A** curl (rotate) around the point in question.

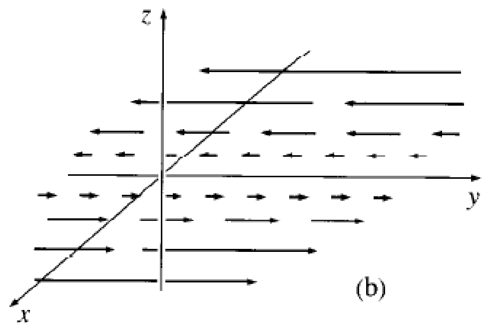
Zero curl :



Non-zero curl :



(a)



(b)

$$\mathbf{v}_a = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$$

$$\nabla \times \mathbf{v}_a = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = 2\hat{\mathbf{z}}$$

$$\mathbf{v}_b = x\hat{\mathbf{y}}$$

$$\nabla \times \mathbf{v}_b = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & x & 0 \end{vmatrix} = \hat{\mathbf{z}}$$

1.2.6 Product Rules (six rules)

two for gradients:

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

two for divergences:

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

two for curls:

$$\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$$

Prove al the six rules!

$$\begin{aligned}
\nabla\cdot(f\mathbf{A}) &= \frac{\partial}{\partial x}(fA_x) + \frac{\partial}{\partial y}(fA_y) + \frac{\partial}{\partial z}(fA_z) \\
&= \left(\frac{\partial f}{\partial x}A_x + f\frac{\partial A_x}{\partial x}\right) + \left(\frac{\partial f}{\partial y}A_y + f\frac{\partial A_y}{\partial y}\right) + \left(\frac{\partial f}{\partial z}A_z + f\frac{\partial A_z}{\partial z}\right) \\
&= (\nabla f)\cdot\mathbf{A} + f(\nabla\cdot\mathbf{A}).
\end{aligned}$$

1.2.7 Second Derivatives

∇T is a *vector*,	(1) Divergence of gradient: $\nabla \cdot (\nabla T)$. (2) Curl of gradient: $\nabla \times (\nabla T)$.
$\nabla \cdot \mathbf{v}$ is a *scalar*.	(3) Gradient of divergence: $\nabla(\nabla \cdot \mathbf{v})$
$\nabla \times \mathbf{v}$ is a *vector*,	(4) Divergence of curl: $\nabla \cdot (\nabla \times \mathbf{v})$. (5) Curl of curl: $\nabla \times (\nabla \times \mathbf{v})$.

(1) Divergence of gradient: $\nabla \cdot (\nabla T) \longrightarrow \boxed{\text{Laplacian: } \Delta = \nabla \cdot \nabla = \nabla^2}$

$$\nabla \cdot (\nabla T) = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

(2) The curl of a gradient is always *zero:* $\longrightarrow \nabla \times (\nabla T) = 0$

➔ ***The curl of the gradient of any scalar field is identically zero!***

(3) $\nabla(\nabla \cdot \mathbf{v})$ for some reason seldom occurs in physical applications $\quad \nabla^2 \mathbf{v} = (\nabla \cdot \nabla)\mathbf{v} \neq \nabla(\nabla \cdot \mathbf{v})$

(4) The divergence of a curl, is *always zero:* $\longrightarrow \nabla \cdot (\nabla \times \mathbf{v}) = 0$

➔ **The divergence of the curl of any vector field is identically zero.**

(5) Curl of curl: $\longrightarrow \quad \nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$

(Note) Two Null Identities of second derivatives

(I) The curl of the gradient of any scalar field is identically zero.

$$\boxed{\nabla \times (\nabla V) = 0}$$

(ex) If a vector is curl-free, then it can be expressed as the gradient of a scalar field.

$$\nabla \times \mathbf{E} = 0 \longrightarrow \mathbf{E} = -\nabla V$$

(II) The divergence of the curl of any vector field is identically zero.

$$\boxed{\nabla \cdot (\nabla \times \mathbf{A}) = 0}$$

(ex) If a vector is divergenceless, then it can be expressed as the curl of another vector field.

$$\nabla \cdot \mathbf{B} = 0 \longrightarrow \mathbf{B} = \nabla \times \mathbf{A}$$

Summary of the useful vector formulas

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$ ***(BAC-CAB rule)***

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

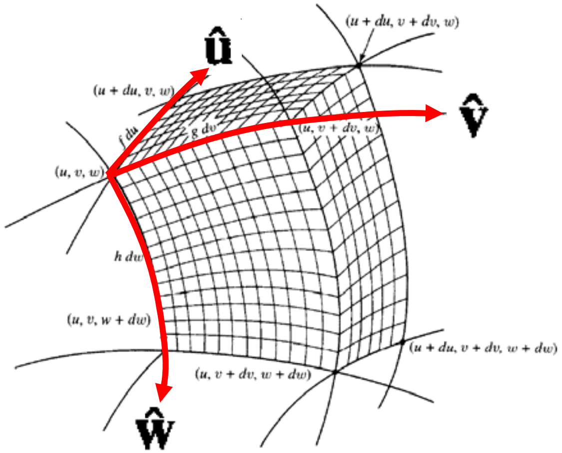
Appendix A: Vector Calculus in Curvilinear Coordinates

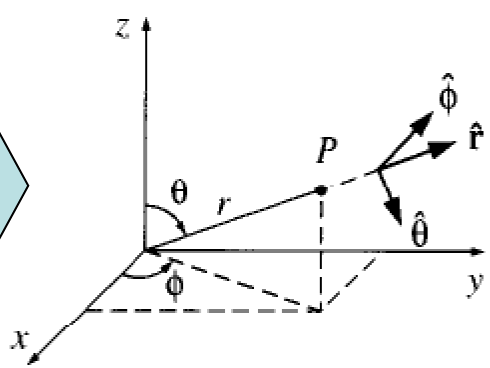
A.1 (orthogonal) Curvilinear Coordinates: (u, v, w)

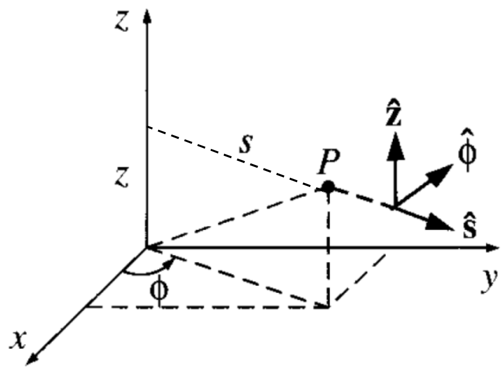
in the Cartesian system, (x, y, z);

in the spherical system, (r, θ, ϕ); $\quad x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$

in the cylindrical system, (s, ϕ, z) $\quad x = s\cos\phi, \quad y = s\sin\phi, \quad z = z.$







A.2 Notation

the infinitesimal displacement vector from (u, v, w) to $(u + du, v + dv, w + dw)$

$$d\mathbf{l} = f\,du\,\hat{\mathbf{u}} + g\,dv\,\hat{\mathbf{v}} + h\,dw\,\hat{\mathbf{w}}$$

in Cartesian coordinates $f = g = h = 1$;

in spherical coordinates $f = 1,\ g = r,\ h = r\sin\theta$;

in cylindrical coordinates $f = h = 1,\ g = s$.

System	u	v	w	f	g	h
Cartesian	x	y	z	1	1	1
Spherical	r	θ	ϕ	1	r	$r\sin\theta$
Cylindrical	s	ϕ	z	1	s	1

A.3 Gradient in Curvilinear Coordinates:

a scalar function $t(u, v, w)$ changes by an amount

$$dt = \frac{\partial t}{\partial u}\, du + \frac{\partial t}{\partial v}\, dv + \frac{\partial t}{\partial w}\, dw$$

We can write it as a dot product,

$$dt = \nabla t \cdot d\mathbf{l} = (\nabla t)_u\, f\, du + (\nabla t)_v\, g\, dv + (\nabla t)_w\, h\, dw \longleftarrow d\mathbf{l} = f\, du\, \hat{\mathbf{u}} + g\, dv\, \hat{\mathbf{v}} + h\, dw\, \hat{\mathbf{w}}$$

$$(\nabla t)_u \equiv \frac{1}{f}\frac{\partial t}{\partial u}, \quad (\nabla t)_v \equiv \frac{1}{g}\frac{\partial t}{\partial v}, \quad (\nabla t)_w \equiv \frac{1}{h}\frac{\partial t}{\partial w}$$

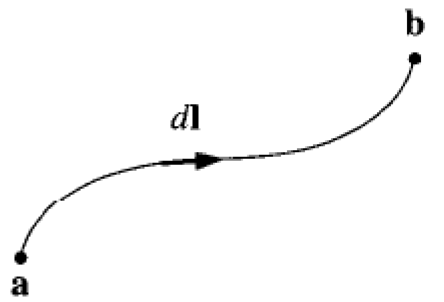
$$\Rightarrow \nabla t \equiv \frac{1}{f}\frac{\partial t}{\partial u}\hat{\mathbf{u}} + \frac{1}{g}\frac{\partial t}{\partial v}\hat{\mathbf{v}} + \frac{1}{h}\frac{\partial t}{\partial w}\hat{\mathbf{w}}$$

System	u	v	w	f	g	h
Cartesian	x	y	z	1	1	1
Spherical	r	θ	ϕ	1	r	$r\sin\theta$
Cylindrical	s	ϕ	z	1	s	1

→ **Gradient** of t in arbitrary curvilinear coordinates.

the *total* change in t, as you go from point $\mathbf{a}$ to point $\mathbf{b}$

$$t(\mathbf{b}) - t(\mathbf{a}) = \int_{\mathbf{a}}^{\mathbf{b}} dt = \int_{\mathbf{a}}^{\mathbf{b}} (\nabla t) \cdot d\mathbf{l}$$



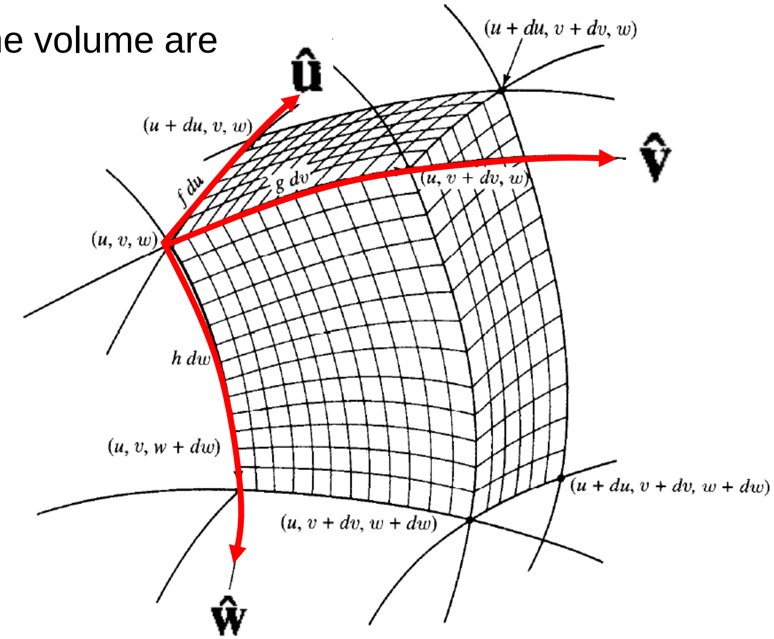
➔ ***Fundamental theorem for gradients***

A.4 Divergence in Curvilinear Coordinates:

Suppose that we have a *vector* function,

$$\mathbf{A}(u, v, w) = A_u\,\hat{\mathbf{u}} + A_v\,\hat{\mathbf{v}} + A_w\,\hat{\mathbf{w}}$$

we wish to evaluate the integral $\oint \mathbf{A} \cdot d\mathbf{a}$ over the surface of the infinitesimal volume



Because $d\mathbf{l} = f\,du\,\hat{\mathbf{u}} + g\,dv\,\hat{\mathbf{v}} + h\,dw\,\hat{\mathbf{w}}$, the side lengths of the volume are

$$dl_u = f\,du, \quad dl_v = g\,dv, \text{ and } dl_w = h\,dw$$

Therefore the volume of the infinitesimal volume is

$$d\tau = dl_u\,dl_v\,dl_w = (fgh)\,du\,dv\,dw$$

For the *front* surface: the area is $d\mathbf{a} = -(gh)\,dv\,dw\,\hat{\mathbf{u}}$

$$\mathbf{A} \cdot d\mathbf{a} = -(ghA_u)\,dv\,dw$$

For the *back* surface at (u + du): $d\mathbf{a} = (gh)\,dv\,dw\,\hat{\mathbf{u}}$

$$\mathbf{A} \cdot d\mathbf{a} = (ghA_u)\,dv\,dw$$

Since for any (differentiable) function $F(u)$, $F(u + du) - F(u) = \dfrac{dF}{du}\,du$

$\mathbf{A} \cdot d\mathbf{a}$ at (u + du) - $\mathbf{A} \cdot d\mathbf{a}$ at (u) ➔ $\left[\dfrac{\partial}{\partial u}(ghA_u)\right] du\,dv\,dw = \dfrac{1}{fgh}\dfrac{\partial}{\partial u}(ghA_u)\,d\tau$

Divergence in Curvilinear Coordinates:

we wish to evaluate the integral $\oint \mathbf{A} \cdot d\mathbf{a}$ over the surface of the infinitesimal volume

The front and back sides yield, $\quad \mathbf{A} \cdot d\mathbf{a} \longrightarrow \dfrac{1}{fgh} \dfrac{\partial}{\partial u}(ghA_u)\, d\tau.$

By the same token, the right and left sides yield $\longrightarrow \dfrac{1}{fgh} \dfrac{\partial}{\partial v}(fhA_v)\, d\tau$

the top and bottom give $\longrightarrow \dfrac{1}{fgh} \dfrac{\partial}{\partial w}(fgA_w)\, d\tau.$

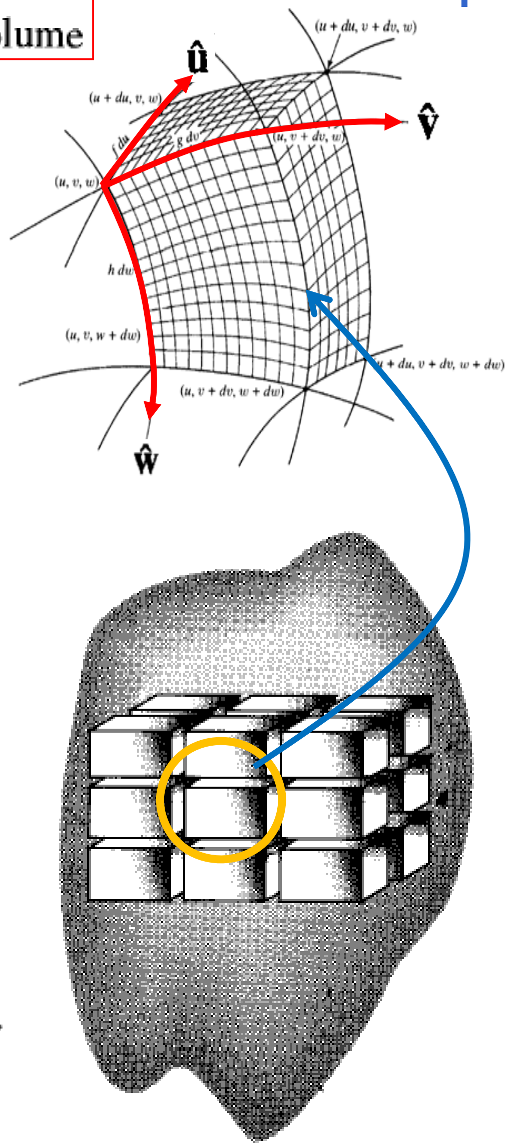
$$\Rightarrow \oint \mathbf{A} \cdot d\mathbf{a} = \frac{1}{fgh}\left[\frac{\partial}{\partial u}(ghA_u) + \frac{\partial}{\partial v}(fhA_v) + \frac{\partial}{\partial w}(fgA_w)\right] d\tau.$$

The **divergence of A in curvilinear coordinates** is defined by

$$\nabla \cdot \mathbf{A} \equiv \frac{1}{fgh}\left[\frac{\partial}{\partial u}(ghA_u) + \frac{\partial}{\partial v}(fhA_v) + \frac{\partial}{\partial w}(fgA_w)\right]$$

$$\Rightarrow \oint \mathbf{A} \cdot d\mathbf{a} = (\nabla \cdot \mathbf{A})\, d\tau.$$ it pertains only to *infinitesimal* volumes.

since $d\mathbf{a}$ always points *outward*,
$\mathbf{A} \cdot d\mathbf{a}$ has the opposite sign for the two members of each pair
only those at the outer boundary survive when everything is added up.



$$\Rightarrow \boxed{\oint \mathbf{A} \cdot d\mathbf{a} = \int (\nabla \cdot \mathbf{A})\, d\tau}$$ ➔ **Divergence theorem**

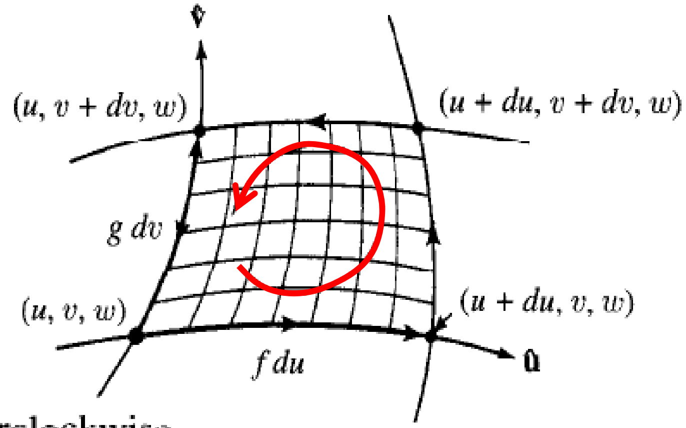
➔ It converts a volume integral to a closed surface integral, and vice versa.

A.5 Curl in Curvilinear Coordinates:

To obtain the curl in curvilinear coordinates, we calculate the line integral, $\boxed{\oint \mathbf{A}\cdot d\mathbf{l}}$,

around the infinitesimal loop generated by starting at (u, v, w) of length $dl_u = f\,du$, width $dl_v = g\,dv$

The area is $\quad d\mathbf{a} = (fg)du\,dv\,\hat{\mathbf{w}} \longleftarrow$ $\hat{\mathbf{w}}$ points out of the page



Along the bottom segment,

$$d\mathbf{l} = f\,du\,\hat{\mathbf{u}} \quad \longrightarrow \quad \mathbf{A}\cdot d\mathbf{l} = (fA_u)\,du$$

Assuming the coordinate system is right-handed,

obliged by the right-hand rule to run the line integral counterclockwise

Along the top leg, the sign is reversed, $\longrightarrow \mathbf{A}\cdot d\mathbf{l} = -(fA_u)\big|_{v+dv}$

Taken together, these two edges give

$$\mathbf{A}\cdot d\mathbf{l}: \longrightarrow \left[-(fA_u)\big|_{v+dv} + (fA_u)\big|_v\right] du = -\left[\frac{\partial}{\partial v}(fA_u)\right] du\,dv$$

Similarly, the right and left sides yield $\quad \mathbf{A}\cdot d\mathbf{l}: \longrightarrow \left[\frac{\partial}{\partial u}(gA_v)\right] du\,dv$

so the total is

$$\oint \mathbf{A}\cdot d\mathbf{l} = \left[\frac{\partial}{\partial u}(gA_v) - \frac{\partial}{\partial v}(fA_u)\right] du\,dv = \frac{1}{fg}\left[\frac{\partial}{\partial u}(gA_v) - \frac{\partial}{\partial v}(fA_u)\right]\hat{\mathbf{w}}\cdot d\mathbf{a}$$

Curl in Curvilinear Coordinates:

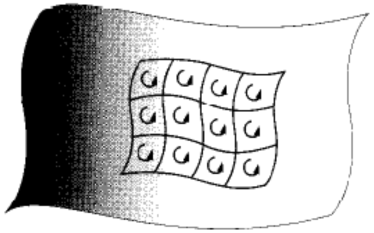
$$\oint \mathbf{A} \cdot d\mathbf{l} = \left[\frac{\partial}{\partial u}(g A_v) - \frac{\partial}{\partial v}(f A_u) \right] du\, dv = \frac{1}{fg} \left[\frac{\partial}{\partial u}(g A_v) - \frac{\partial}{\partial v}(f A_u) \right] \hat{\mathbf{w}} \cdot d\mathbf{a}$$

The **curl of A in curvilinear coordinates** is defined by

$$\nabla \times \mathbf{A} \equiv \frac{1}{gh} \left[\frac{\partial}{\partial v}(h A_w) - \frac{\partial}{\partial w}(g A_v) \right] \hat{\mathbf{u}} + \frac{1}{fh} \left[\frac{\partial}{\partial w}(f A_u) - \frac{\partial}{\partial u}(h A_w) \right] \hat{\mathbf{v}} + \frac{1}{fg} \left[\frac{\partial}{\partial u}(g A_v) - \frac{\partial}{\partial v}(f A_u) \right] \hat{\mathbf{w}}$$

Now we generalize the line integral to the u, v, and w components,

$$\oint \mathbf{A} \cdot d\mathbf{l} = (\nabla \times \mathbf{A}) \cdot d\mathbf{a}.$$



Fortunately, as before, the internal contributions cancel in pairs, because every internal line is the edge of *two* adjacent loops running in opposite directions.

Therefore, we can extend it to finite surface:

$$\boxed{\oint \mathbf{A} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a}}$$

➔ **Stokes' theorem**

➔ ***It converts a volume integral to a closed surface integral, and vice versa.***

A.6 Laplacian in Curvilinear Coordinates:

Laplacian = "the divergence of the gradient of " $\nabla^2 = \nabla \bullet \nabla$

$$\nabla t \equiv \frac{1}{f}\frac{\partial t}{\partial u}\hat{\mathbf{u}} + \frac{1}{g}\frac{\partial t}{\partial v}\hat{\mathbf{v}} + \frac{1}{h}\frac{\partial t}{\partial w}\hat{\mathbf{w}}$$ ← **Gradient** of T

$$\nabla \cdot \mathbf{A} \equiv \frac{1}{fgh}\left[\frac{\partial}{\partial u}(ghA_u) + \frac{\partial}{\partial v}(fhA_v) + \frac{\partial}{\partial w}(fgA_w)\right]$$ ← **Divergence** of **A**

$$\boxed{\nabla^2 t \equiv \frac{1}{fgh}\left[\frac{\partial}{\partial u}\left(\frac{gh}{f}\frac{\partial t}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{fh}{g}\frac{\partial t}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{fg}{h}\frac{\partial t}{\partial w}\right)\right]}$$

System	u	v	w	f	g	h
Cartesian	x	y	z	1	1	1
Spherical	r	θ	ϕ	1	r	$r\sin\theta$
Cylindrical	s	ϕ	z	1	s	1

(Ex) *Laplace equation:* $\nabla^2 V = 0$

Poisson equation: $\nabla^2 V = -\dfrac{\rho}{\varepsilon_0}$