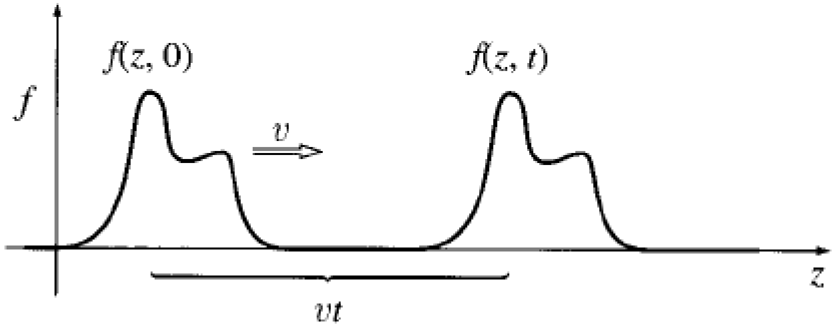
Chapter 9. Electromagnetic waves

9.1.1 The (classical or Mechanical) waves equation

Given the *initial* shape of the string, $g(z) \equiv f(z, 0)$, what is the subsequent form, $f(z, t)$?



The displacement at point *z,* at the later time *t,* is the same as the displacement a distance *vt* to the left (i.e. at *z* - *vt),* back at time *t* = 0:

$$f(z, t) = f(z - vt, 0) = g(z - vt)$$

It represents a wave of fixed shape traveling in the z direction at speed v.

(O) $f_1(z, t) = Ae^{-b(z-vt)^2}$, $f_2(z, t) = A\sin[b(z - vt)]$, $f_3(z, t) = \dfrac{A}{b(z - vt)^2 + 1}$

(X) $f_4(z, t) = Ae^{-b(bz^2+vt)}$, and $f_5(z, t) = A\sin(bz)\cos(bvt)^3$

(Classical) waves equation with a solution of the form: $f(z,t) = g(z \pm vt)$

$$\frac{\partial f}{\partial z} = \frac{dg}{du}\frac{\partial u}{\partial z} = \frac{dg}{du}, \qquad \frac{\partial f}{\partial t} = \frac{dg}{du}\frac{\partial u}{\partial t} = -v\frac{dg}{du}$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{dg}{du}\right) = \frac{d^2 g}{du^2}\frac{\partial u}{\partial z} = \frac{d^2 g}{du^2}, \qquad \frac{\partial^2 f}{\partial t^2} = -v\frac{\partial}{\partial t}\left(\frac{dg}{du}\right) = -v\frac{d^2 g}{du^2}\frac{\partial u}{\partial t} = v^2\frac{d^2 g}{du^2}$$

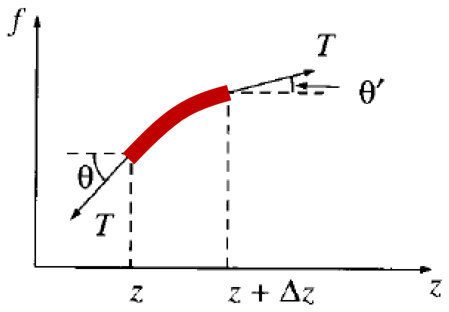
$$\Rightarrow \boxed{\frac{d^2 g}{du^2} = \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}}$$

➔ **Waves equation means a equation of motion governed by Newton's second law!**

(Example) Consider a stretched string which supports wave motion.

The net transverse force on the segment between *z* and (*z* + *Δz*) is

$$\Delta F = T\sin\theta' - T\sin\theta$$



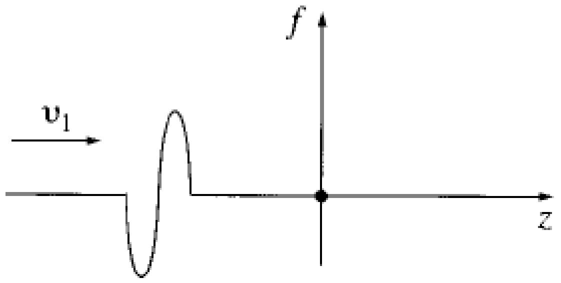
If the distortion of the string is not too great, sinθ ~ tanθ.

$$\Delta F \cong T(\tan\theta' - \tan\theta) = T\left(\left.\frac{\partial f}{\partial z}\right|_{z+\Delta z} - \left.\frac{\partial f}{\partial z}\right|_{z}\right) \cong T\frac{\partial^2 f}{\partial z^2}\Delta z$$

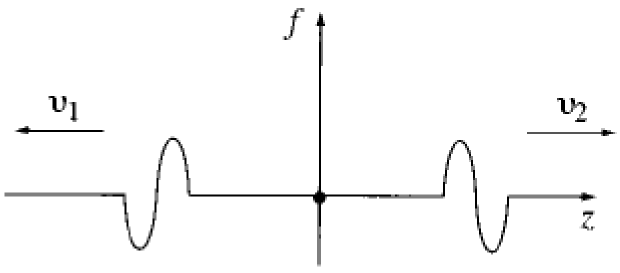
If the mass per unit length is μ, Newton's second law says

$$\Delta F = \mu(\Delta z)\frac{\partial^2 f}{\partial t^2} \Rightarrow \frac{\partial^2 f}{\partial z^2} = \frac{\mu}{T}\frac{\partial^2 f}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \qquad v = \sqrt{\frac{T}{\mu}}$$

9.1.3 Boundary conditions: Reflection and Transmission



(a) Incident pulse



(b) Reflected and transmitted pulses

For a sinusoidal incident wave, then, the net disturbance of the string is:

$$\tilde{f}(z,t) = \begin{cases} \tilde{A}_I e^{i(k_1 z - \omega t)} + \tilde{A}_R e^{i(-k_1 z - \omega t)}, & \text{for } z < 0, \\ \tilde{A}_T e^{i(k_2 z - \omega t)}, & \text{for } z > 0. \end{cases}$$

At the join *(z* = 0), the **displacement** and **slope** just slightly to the left *(z* = 0^-) **must equal** those slightly to the right *(z* = 0^+), or else there would be a break between the two strings.

$$\tilde{f}(0^-,t) = \tilde{f}(0^+,t), \quad \left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^-} = \left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^+} \quad \begin{cases} \tilde{A}_I + \tilde{A}_R = \tilde{A}_T, \quad k_1(\tilde{A}_I - \tilde{A}_R) = k_2 \tilde{A}_T \\ \tilde{A}_R = \left(\dfrac{k_1 - k_2}{k_1 + k_2}\right)\tilde{A}_I, \quad \tilde{A}_T = \left(\dfrac{2k_1}{k_1 + k_2}\right)\tilde{A}_I \\ \tilde{A}_R = \left(\dfrac{v_2 - v_1}{v_2 + v_1}\right)\tilde{A}_I, \quad \tilde{A}_T = \left(\dfrac{2v_2}{v_2 + v_1}\right)\tilde{A}_I \end{cases}$$

9.2 Electromagnetic waves in Vacuum

In Vacuum, $\rho = 0, \quad \mathrm{J} = 0, \quad q = 0, \quad I = 0$ (no free charges and no currents)

$$\boxed{\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}}$$

Let's derive the wave equation for E and B from the curl equations.

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\cancel{\nabla \cdot \mathbf{E}}) - \nabla^2 \mathbf{E} = \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\Rightarrow \quad \nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla(\cancel{\nabla \cdot \mathbf{B}}) - \nabla^2 \mathbf{B} = \nabla \times \left(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) = \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

$$\Rightarrow \quad \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

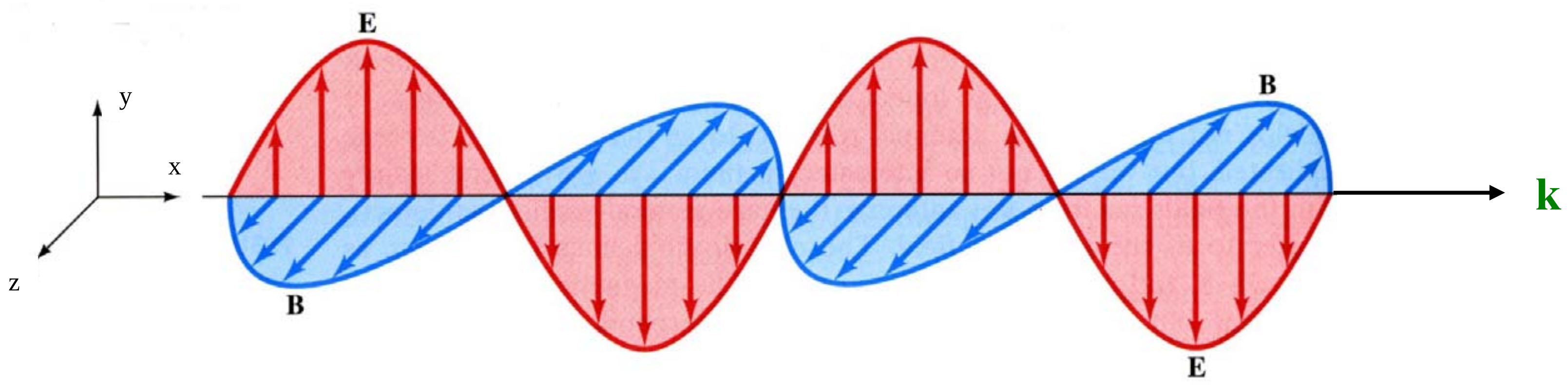
Each Cartesian component of E and B satisfies $\boxed{\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}}$ $\quad v = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3.00 \times 10^8 \ \mathrm{m/s}$

Notice the crucial role played by Maxwell's contribution to Ampere's law $(\mu_0 \epsilon_0 \partial \mathbf{E} / \partial t)$;
without it, the wave equation would not emerge,
and there would be no electromagnetic theory of light.

Algebraic form of Maxwell's Equations in free space

- $i\,\vec{k}\cdot\vec{E}=0$ (i.e. $\vec{E}$ is perpendicular to $\vec{k}$)
- $i\,\vec{k}\cdot\vec{B}=0$ (i.e. $\vec{B}$ is perpendicular to $\vec{k}$)
- $i\,\vec{k}\times\vec{E}=i\omega\,\vec{B}$
- $i\,\vec{k}\times\vec{B}=-i\varepsilon_o\mu_o\,\omega\,\vec{E}$

E and B are mutually perpendicular to each other,
E and B are perpendicular to the direction of propagation of wave.



What is the relation between E and B ?

Or show that E and B are in same phase at any time in space.

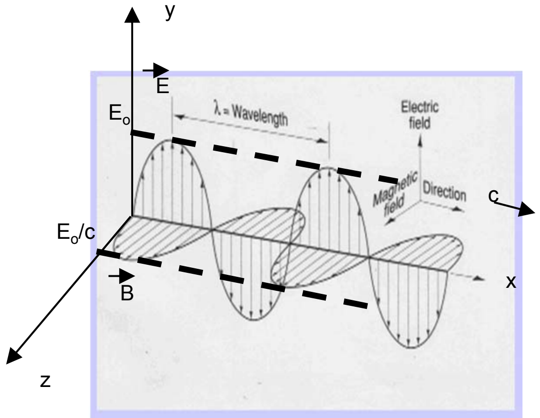
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \Rightarrow \quad \mathbf{k} \times \mathbf{E} = \omega \mathbf{B}$$

when

$$\left.\begin{array}{l} E \rightarrow E_y \\ B \rightarrow B_z \\ k \rightarrow x \end{array}\right\} \Rightarrow \left.\begin{array}{l} k E_y = \omega B_z \\ E_y = \dfrac{\omega}{k} B_z \end{array}\right\} \Rightarrow \begin{array}{l} E_y = c B_z \\ B_z = \dfrac{E_y}{c} \end{array}$$

But

$$\left.\begin{array}{l} E_y = E_o e^{i(kx - wt)} \\ B_z = B_o e^{i(kx - wt)} \end{array}\right\} \Rightarrow \boxed{B_o = \frac{E_o}{c}}$$



Since k/ω is a real number, the electric and magnetic vectors are in phase; thus if at any instant, E is zero then B is also zero, when E attains its maximum value, B also attains its maximum value, etc.

Both E_y and B_z are in same phase.

Summary of Important Properties of Electromagnetic Waves

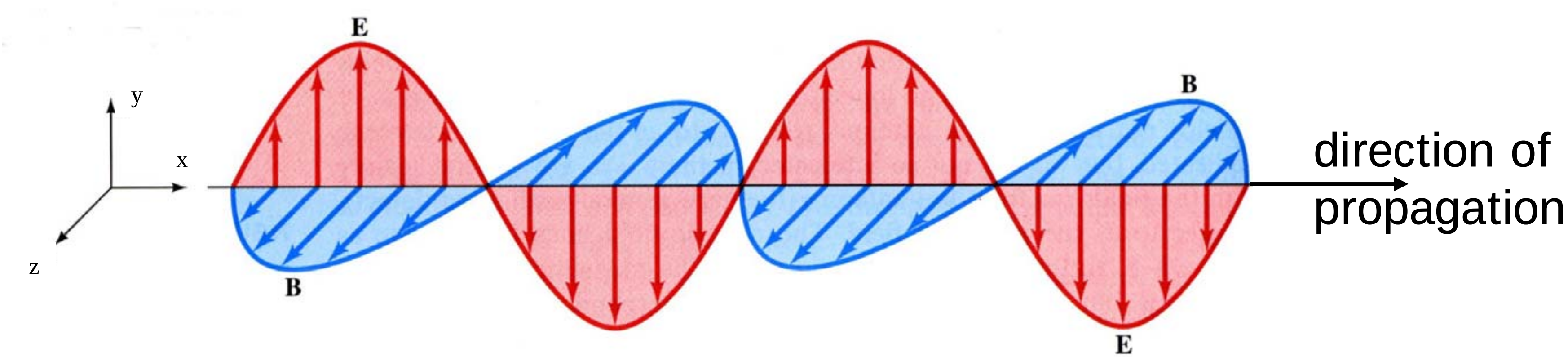
- The solutions (plane wave) of Maxwell's equations are wave-like with both E and B satisfying a wave equation.

$$E_y = E_o \cos(kx - \omega t)$$

$$B_z = B_o \cos(kx - \omega t)$$

- Electromagnetic waves travel through empty space with the speed of light $c = 1/(\mu_0\varepsilon_0)^{1/2}$.

- The plane wave as represented by above is said to be linearly polarized because the electric vector is always along y-axis and, similarly, the magnetic vector is always along z-axis.



- The components of the electric and magnetic fields of plane EM waves are perpendicular to each other and perpendicular to the direction of wave propagation. The latter property says that

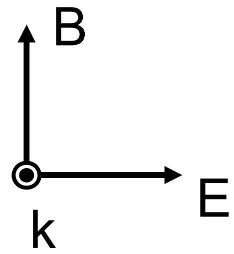
  EM waves are transverse waves.

- The magnitudes of E and B in empty space are related by

  E/B = c.

$$\frac{E_o}{B_o} = \frac{E}{B} = \frac{\omega}{k} = c$$

- Both E and B are at right angles to the direction of propagation. Thus the waves are transverse.



- The electric and magnetic waves are interdependent; neither can exist without the other. Physically, an electric field varying in time produces a magnetic field varying in space and time; this changing magnetic field produces an electric field varying in space and time and so on.

  This mutual generation of electric and magnetic fields result in the propagation of the EM waves.

Numerical example

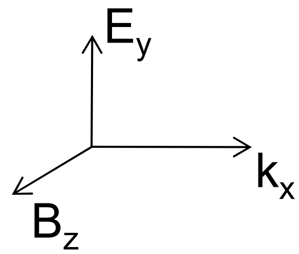
In free space the Electric field is given as

$$\vec{E} = 10\sin(2x - 100t)\,\hat{j}.$$

Determine D, B and H by using Maxwell's equations.

Sol: Wave is propagating along x direction.

(1) $$\vec{D} = \varepsilon_0 \vec{E} = 10\varepsilon_0 \sin(2x - 100t)\,\hat{j}.$$



(2) $$\text{Using } \nabla \times E = -\frac{\partial B}{\partial t},$$

$$20\,Cos(2x - 100t)\,\hat{k} = -\frac{\partial B}{\partial t},$$

$$\frac{1}{5} Sin(2x - 100t)\,\hat{k} = B. \qquad \vec{B} = \frac{1}{5} Sin(2x - 100t)\,\hat{k}$$

(3) $$\vec{H} = \frac{\vec{B}}{\mu_0} = \frac{1}{5\mu_0} Sin(2x - 100t)\,\hat{k}$$

9.3 Electromagnetic waves in Matter

In linear and homogeneous media with no free charge and no free current,

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathrm{B} = \mu\varepsilon \frac{\partial \mathrm{E}}{\partial t}$$

$$\mathbf{D} = \epsilon \mathbf{E}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B}$$

$$v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{n}$$

$$n \equiv \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}$$

$$u = \frac{1}{2}\left(\epsilon E^2 + \frac{1}{\mu}B^2\right)$$

$$\mathbf{S} = \frac{1}{\mu}(\mathbf{E} \times \mathbf{B})$$

$$I = \frac{1}{2}\epsilon v E_0^2 \quad \longleftarrow \quad I = S_{average} = \langle S \rangle$$

Boundary conditions

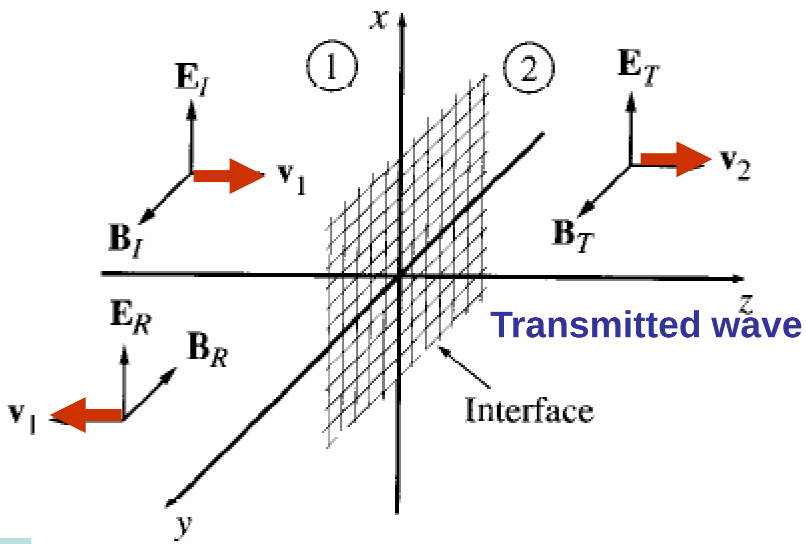
$$\text{(i)}\ \ \epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}, \qquad \text{(iii)}\ \ \mathbf{E}_1^{\parallel} = \mathbf{E}_2^{\parallel},$$

$$\text{(ii)}\ \ B_1^{\perp} = B_2^{\perp}, \qquad \text{(iv)}\ \ \frac{1}{\mu_1}\mathbf{B}_1^{\parallel} = \frac{1}{\mu_2}\mathbf{B}_2^{\parallel}.$$

9.3.2 Reflection and Transmission at Normal Incidence

Suppose *xy* plane forms the boundary between two linear media.

A plane wave of frequency ω is traveling in the z direction (from left), polarized along x direction (TE polarization).



Incident wave

$$\vec{E}_I(z,t) = E_{0I}\exp\big(i(k_1 z - \omega t)\big)\,\hat{x}$$

$$\vec{B}_I(z,t) = \frac{1}{v_1}E_{0I}\exp\big(i(k_1 z - \omega t)\big)\,\hat{y}$$

Reflected wave

$$\vec{E}_R(z,t) = E_{0R}\exp\big(i(-k_1 z - \omega t)\big)\,\hat{x}$$

$$\vec{B}_R(z,t) = -\frac{1}{v_1}E_{0R}\exp\big(i(-k_1 z - \omega t)\big)\,\hat{y}$$

Transmitted wave

$$\vec{E}_T(z,t) = E_{0T}\exp\big(i(k_2 z - \omega t)\big)\,\hat{x}$$

$$\vec{B}_T(z,t) = \frac{1}{v_2}E_{0T}\exp\big(i(k_2 z - \omega t)\big)\,\hat{y}$$

At z = 0,

$$\mathbf{E}_1^{\parallel} = \mathbf{E}_2^{\parallel} \;\longrightarrow\; \tilde{E}_{0_I} + \tilde{E}_{0_R} = \tilde{E}_{0_T}$$

$$\frac{1}{\mu_1}\mathbf{B}_1^{\parallel} = \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} \;\longrightarrow\; \frac{1}{\mu_1}\left(\frac{1}{v_1}\tilde{E}_{0_I} - \frac{1}{v_1}\tilde{E}_{0_R}\right) = \frac{1}{\mu_2}\left(\frac{1}{v_2}\tilde{E}_{0_T}\right)$$

$$\mu_1 = \mu_2 = \mu_0,$$

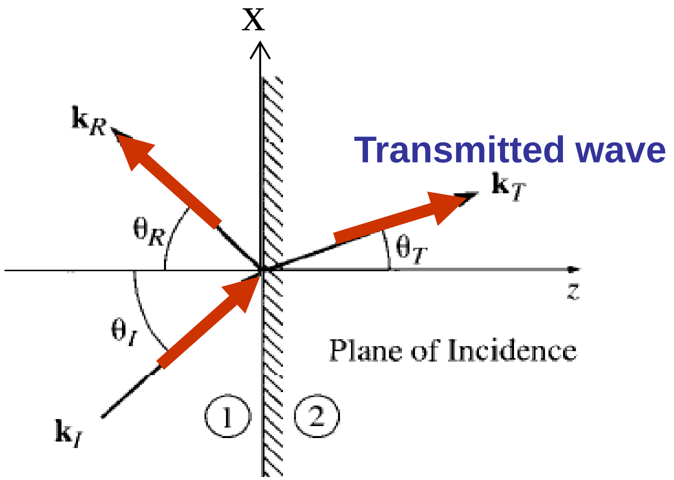
Prove!

$$R \equiv \frac{I_R}{I_I} = \left(\frac{E_{0_R}}{E_{0_I}}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

$$T \equiv \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{0_T}}{E_{0_I}}\right)^2 = \frac{4 n_1 n_2}{(n_1 + n_2)^2}$$

$$\boldsymbol{R + T = 1}$$

9.3.3 Reflection and Transmission at Oblique Incidence



Because the boundary conditions must hold at all points on the plane, and for all times, the exponential factors must be equal at z = 0 plane.

$$\boxed{\mathbf{k}_I \cdot \mathbf{r} = \mathbf{k}_R \cdot \mathbf{r} = \mathbf{k}_T \cdot \mathbf{r}, \quad \text{when } z = 0}$$

$$\rightarrow \quad x(k_I)_x + y(k_I)_y = x(k_R)_x + y(k_R)_y = x(k_T)_x + y(k_T)_y$$

$$\rightarrow \quad (k_I)_x = (k_R)_x = (k_T)_x \qquad \text{if } y = 0$$

$$\rightarrow \quad k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T$$

$$\Rightarrow \quad \theta_I = \theta_R \qquad \textbf{law of reflection}$$

$$\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2} \qquad \textbf{law of refraction}$$

Incident wave

$$\tilde{\mathbf{E}}_I(\mathbf{r}, t) = \tilde{\mathbf{E}}_{0_I} e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega t)}$$

$$\tilde{\mathbf{B}}_I(\mathbf{r}, t) = \frac{1}{v_1}(\hat{\mathbf{k}}_I \times \tilde{\mathbf{E}}_I)$$

Reflected wave

$$\tilde{\mathbf{E}}_R(\mathbf{r}, t) = \tilde{\mathbf{E}}_{0_R} e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)}$$

$$\tilde{\mathbf{B}}_R(\mathbf{r}, t) = \frac{1}{v_1}(\hat{\mathbf{k}}_R \times \tilde{\mathbf{E}}_R)$$

Transmitted wave

$$\tilde{\mathbf{E}}_T(\mathbf{r}, t) = \tilde{\mathbf{E}}_{0_T} e^{i(\mathbf{k}_T \cdot \mathbf{r} - \omega t)}$$

$$\tilde{\mathbf{B}}_T(\mathbf{r}, t) = \frac{1}{v_2}(\hat{\mathbf{k}}_T \times \tilde{\mathbf{E}}_T)$$

Prove!

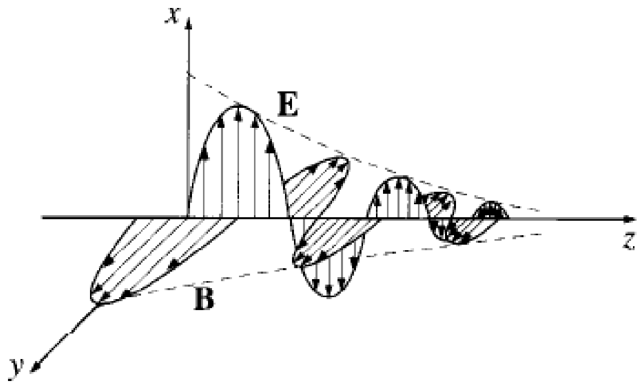
$$R \equiv \frac{I_R}{I_I} = \left(\frac{E_{0_R}}{E_{0_I}}\right)^2 = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2$$

$$T \equiv \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{0_T}}{E_{0_I}}\right)^2 \frac{\cos\theta_T}{\cos\theta_I} = \alpha\beta\left(\frac{2}{\alpha + \beta}\right)^2$$

$$\alpha \equiv \frac{\cos\theta_T}{\cos\theta_I} \qquad \beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$$

9.4 Absorption and Dispersion

9.4.1 Electromagnetic waves in Conductors



According to Ohm's law, the (free) current density is proportional to the electric field: $\mathbf{J}_f = \sigma \mathbf{E}$

Maxwell' s equations for linear media with no free charge assume the form,

$$\left.\begin{array}{ll} \text{(i)} \quad \nabla \cdot \mathbf{E} = 0, & \text{(iii)} \quad \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}, \\ \text{(ii)} \quad \nabla \cdot \mathbf{B} = 0, & \text{(iv)} \quad \nabla \times \mathbf{B} = \mu\epsilon \dfrac{\partial \mathbf{E}}{\partial t} + \mu\sigma\mathbf{E}. \end{array}\right\} \Rightarrow \begin{array}{l} \nabla^2 \mathbf{E} = \mu\epsilon \dfrac{\partial^2 \mathbf{E}}{\partial t^2} + \mu\sigma \dfrac{\partial \mathbf{E}}{\partial t} \\ \nabla^2 \mathbf{B} = \mu\epsilon \dfrac{\partial^2 \mathbf{B}}{\partial t^2} + \mu\sigma \dfrac{\partial \mathbf{B}}{\partial t} \end{array}$$

Plane-wave solutions are $\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(\tilde{k}z - \omega t)}, \quad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i(\tilde{k}z - \omega t)}$

complex wavenumber

$$\tilde{k} = k + i\kappa$$

Prove!

$$\tilde{k} = \omega\sqrt{\mu\varepsilon}\sqrt{1 - i\frac{\sigma}{\omega\varepsilon}}$$

$$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega$$

$$k \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} + 1\right]^{1/2}$$

$$\kappa \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1\right]^{1/2}$$

$$\boxed{\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{-\kappa z} e^{i(kz - \omega t)}, \quad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{-\kappa z} e^{i(kz - \omega t)}}$$

The imaginary part, κ, results in an attenuation of the wave (decreasing amplitude with increasing *z*):

$d \equiv \dfrac{1}{\kappa}$; skin depth

Determine the relative amplitudes, phases, and polarizations of E and B in conductors

For E field polarized along the x direction, $\tilde{\mathbf{E}}(z,t) = \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)}\,\hat{\mathbf{x}}$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \rightarrow \tilde{\mathbf{B}}(z,t) = \frac{\tilde{k}}{\omega}\tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)}\,\hat{\mathbf{y}}$$

Let's express the complex wavenumber in terms of its modulus and phase

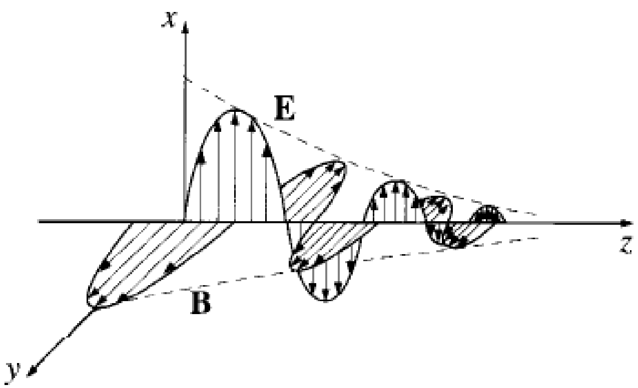
$$\tilde{k} = k + i\kappa = K e^{i\phi} \quad \begin{cases} K \equiv |\tilde{k}| = \sqrt{k^2+\kappa^2} = \omega\sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}} \\ \phi \equiv \tan^{-1}(\kappa/k) \end{cases}$$

$$\tilde{E}_0 = E_0 e^{i\delta_E} \text{ and } \tilde{B}_0 = B_0 e^{i\delta_B} \rightarrow B_0 e^{i\delta_B} = \frac{K e^{i\phi}}{\omega} E_0 e^{i\delta_E}$$

$\rightarrow$ $\delta_B - \delta_E = \phi$; **E and B fields are no longer in phase**

B field *lags behind* E field.

$$\rightarrow \frac{B_0}{E_0} = \frac{K}{\omega} = \sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}}$$



$\rightarrow$ The (real) electric and magnetic fields are, finally,

$$\left.\begin{aligned} \mathbf{E}(z,t) &= E_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E)\,\hat{\mathbf{x}}, \\ \mathbf{B}(z,t) &= B_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E+\phi)\,\hat{\mathbf{y}}. \end{aligned}\right\}$$

@ Helmholtz equation (wave equation in temporal frequency-domain)

$$\nabla^2 \mathbf{E} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla^2 \mathbf{B} = \mu\epsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} + \mu\sigma \frac{\partial \mathbf{B}}{\partial t}$$

$$\boxed{\nabla^2 \psi(r,t) - \mu\sigma \frac{\partial \psi(r,t)}{\partial t} - \mu\varepsilon \frac{\partial^2 \psi(r,t)}{\partial t^2} = 0}$$

Scalar wave equation in space-domain

Let us consider the Fourier transform of the electromagnetic field: $\psi(\mathbf{r}, t) \leftrightarrow \tilde{\psi}(\mathbf{r}, \omega)$

$$\tilde{\psi}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} \psi(\mathbf{r}, t)\, e^{-j\omega t}\, dt \longleftrightarrow \psi(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\psi}(\mathbf{r}, \omega)\, e^{j\omega t}\, d\omega$$

$\tilde{\psi}(\mathbf{r}, \omega)$, *frequency spectrum* of $\psi(\mathbf{r}, t)$

$$\frac{\partial}{\partial t} \psi(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} j\omega \tilde{\psi}(\mathbf{r}, \omega)\, e^{j\omega t}\, d\omega$$

$$\Rightarrow \nabla^2 \psi(r,t) - \mu\sigma \frac{\partial \psi(r,t)}{\partial t} - \mu\varepsilon \frac{\partial^2 \psi(r,t)}{\partial t^2} = \left(\nabla^2 - \mu\sigma \frac{\partial}{\partial t} - \mu\epsilon \frac{\partial^2}{\partial t^2} \right) \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\psi}(\mathbf{r}, \omega)\, e^{j\omega t}\, d\omega = 0$$

$$\Rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\left(\nabla^2 - j\omega\mu\sigma + \omega^2 \mu\epsilon \right) \tilde{\psi}(\mathbf{r}, \omega) \right] e^{j\omega t}\, d\omega = 0$$

Helmholtz equation $\Rightarrow$ $$\boxed{\left(\nabla^2 + \tilde{k}^2 \right) \tilde{\psi}(\mathrm{r}, \omega) = 0}$$ where $\tilde{k} = k + j\kappa = \omega\sqrt{\mu\varepsilon}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}}$

@ Frequency-domain Maxwell equations in a source-free space

Using the temporal inverse Fourier transform,

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\mathbf{E}}(\mathbf{r},\omega)\, e^{j\omega t}\, d\omega \qquad \text{where} \quad \tilde{\mathbf{E}}(\mathbf{r},\omega) = \sum_{i=1}^{3} \hat{\mathbf{i}}_i \tilde{E}_i(\mathbf{r},\omega) = \sum_{i=1}^{3} \hat{\mathbf{i}}_i |\tilde{E}_i(\mathbf{r},\omega)| e^{j\xi_i^E(\mathbf{r},\omega)}$$

$$\nabla\times \mathrm{H} = J + \frac{\partial \mathrm{D}}{\partial t} \;\Rightarrow\; \nabla\times\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\mathbf{H}}(\mathbf{r},\omega)\, e^{j\omega t}\, d\omega = \frac{\partial}{\partial t}\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\mathbf{D}}(\mathbf{r},\omega)\, e^{j\omega t}\, d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\mathbf{J}}(\mathbf{r},\omega)\, e^{j\omega t}\, d\omega$$

$$\Rightarrow\; \frac{1}{2\pi}\int_{-\infty}^{\infty}[\nabla\times\tilde{\mathbf{H}}(\mathbf{r},\omega) - j\omega\tilde{\mathbf{D}}(\mathbf{r},\omega) - \tilde{\mathbf{J}}(\mathbf{r},\omega)]e^{j\omega t}\, d\omega = 0 \;\Rightarrow\; \nabla\times\tilde{\mathbf{H}} = j\omega\tilde{\mathbf{D}} + \tilde{\mathbf{J}}$$

By similar reasoning, finally we have the frequency-domain equations of

$$\nabla\times\tilde{\mathrm{H}}(\mathrm{r},\omega) = \tilde{J}(\mathrm{r},\omega) + j\omega\tilde{\mathrm{D}}(\mathrm{r},\omega)$$

$$\nabla\times\tilde{\mathrm{E}}(\mathrm{r},\omega) = -j\omega\tilde{\mathrm{B}}(\mathrm{r},\omega)$$

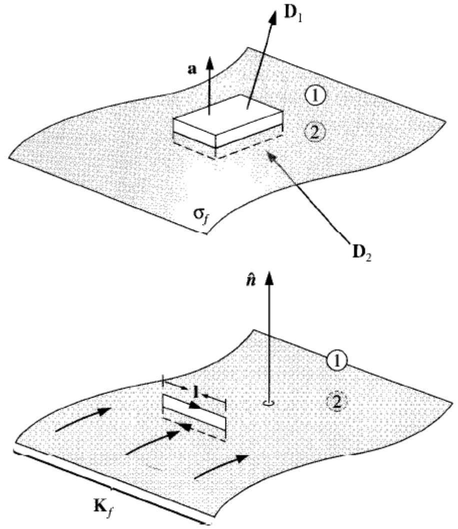
$$\nabla\cdot\tilde{\mathrm{D}}(\mathrm{r},\omega) = \tilde{\rho}(\mathrm{r},\omega)$$

$$\nabla\cdot\tilde{\mathrm{B}}(\mathrm{r},\omega) = 0$$

and $\nabla\cdot\tilde{\mathrm{J}}(\mathrm{r},\omega) = -j\omega\tilde{\rho}(\mathrm{r},\omega)$

The frequency-domain equations involve one fewer derivative
(the time derivative has been replaced by multiplication by $j\omega$), hence may be easier to solve.
However, the inverse transform may be difficult to compute.

9.4.2 Reflection at a conducting surface



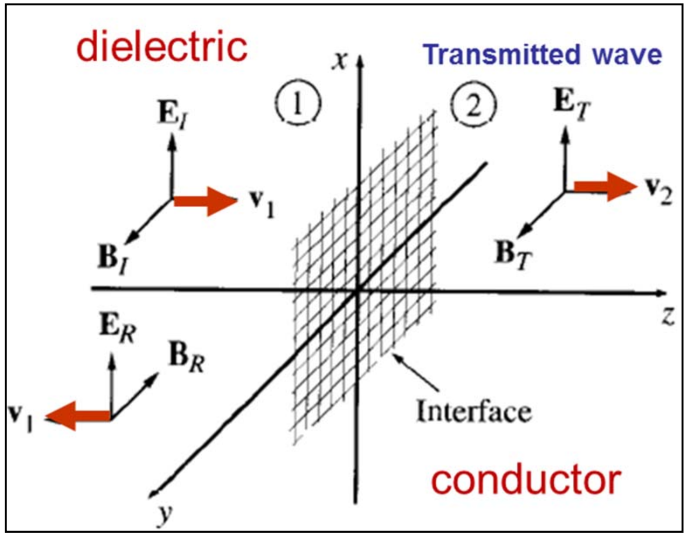
The general boundary conditions for electrodynamics;

$$\text{(i)}\ \ \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f, \quad \text{(iii)}\ \ \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0,$$

$$\text{(ii)}\ \ B_1^{\perp} - B_2^{\perp} = 0, \quad \text{(iv)}\ \ \frac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}},$$

(σ_f : the free surface charge, K_f : the free surface current)

A monochromatic plane wave, traveling in *z,* polarized in *x*, approaches from the left,



$$\tilde{\mathbf{E}}_I(z,t) = \tilde{E}_{0_I} e^{i(k_1 z - \omega t)}\,\hat{\mathbf{x}}, \quad \tilde{\mathbf{B}}_I(z,t) = \frac{1}{v_1}\tilde{E}_{0_I} e^{i(k_1 z - \omega t)}\,\hat{\mathbf{y}}$$

$$\tilde{\mathbf{E}}_R(z,t) = \tilde{E}_{0_R} e^{i(-k_1 z - \omega t)}\,\hat{\mathbf{x}}, \quad \tilde{\mathbf{B}}_R(z,t) = -\frac{1}{v_1}\tilde{E}_{0_R} e^{i(-k_1 z - \omega t)}\,\hat{\mathbf{y}}$$

$$\tilde{\mathbf{E}}_T(z,t) = \tilde{E}_{0_T} e^{i(\tilde{k}_2 z - \omega t)}\,\hat{\mathbf{x}}, \quad \tilde{\mathbf{B}}_T(z,t) = \frac{\tilde{k}_2}{\omega}\tilde{E}_{0_T} e^{i(\tilde{k}_2 z - \omega t)}\,\hat{\mathbf{y}}$$

attenuated as it penetrates into the conductor

Since $E^{\perp} = 0$ on both sides, boundary condition (i) yields $\sigma_f = 0$.

and (iv) (with $\mathbf{K}_f = 0$) says

$$\tilde{E}_{0_I} - \tilde{E}_{0_R} = \tilde{\beta}\tilde{E}_{0_T} \quad \text{where} \quad \tilde{\beta} \equiv \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}_2$$

Prove!

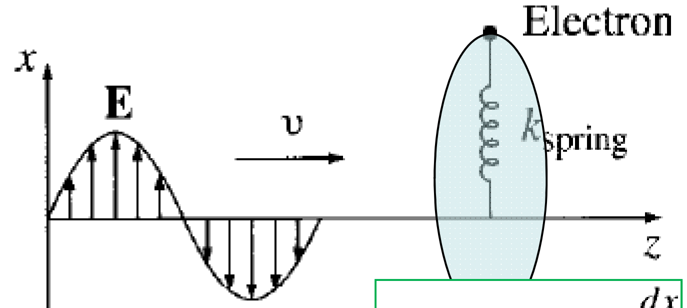
$$\tilde{E}_{0_R} = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\tilde{E}_{0_I}, \quad \tilde{E}_{0_T} = \left(\frac{2}{1+\tilde{\beta}}\right)\tilde{E}_{0_I}$$

9.4.3 Frequency dependence of permittivity in dielectric media (Dispersion)

The electrons in a dielectric are bounded to specific molecules.

$$\boxed{F_{net} = F_{binding} + F_{damping} + F_{driving} = m\frac{d^2\tilde{x}}{dt^2}}$$

$$\boxed{F_{\text{binding}} = -k_{\text{spring}}x = -m\omega_0^2 x}$$



$$\boxed{F_{\text{driving}} = qE = qE_0\cos(\omega t)}$$

$$\boxed{F_{\text{damping}} = -m\gamma\frac{dx}{dt}}$$

$$\frac{d^2\tilde{x}}{dt^2} + \gamma\frac{d\tilde{x}}{dt} + \omega_0^2\tilde{x} = \frac{q}{m}E_0 e^{-i\omega t}$$

$$\tilde{x}(t) = \tilde{x}_0 e^{-i\omega t}$$

$$\tilde{x}_0 = \frac{q/m}{\omega_0^2 - \omega^2 - i\gamma\omega}E_0 \Rightarrow \tilde{p}(t) = q\tilde{x}(t) = \frac{q^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega}E_0 e^{-i\omega t}$$: dipole moment

$$\tilde{\mathbf{P}} = \frac{Nq^2}{m}\left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right)\tilde{\mathbf{E}} \longrightarrow \tilde{\mathbf{P}} = \epsilon_0\tilde{\chi}_e\tilde{\mathbf{E}}$$: polarization vector

$\tilde{\epsilon} = \epsilon_0(1+\tilde{\chi}_e)$: complex permittivity $\longrightarrow \tilde{k} \equiv \sqrt{\tilde{\epsilon}\mu_0}\,\omega = k + i\kappa$

$$\tilde{\epsilon}_r = 1 + \frac{Nq^2}{m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}$$

$\alpha \equiv 2\kappa$ **: absorption coefficient**

$n = \dfrac{ck}{\omega}$ **: refractive index**

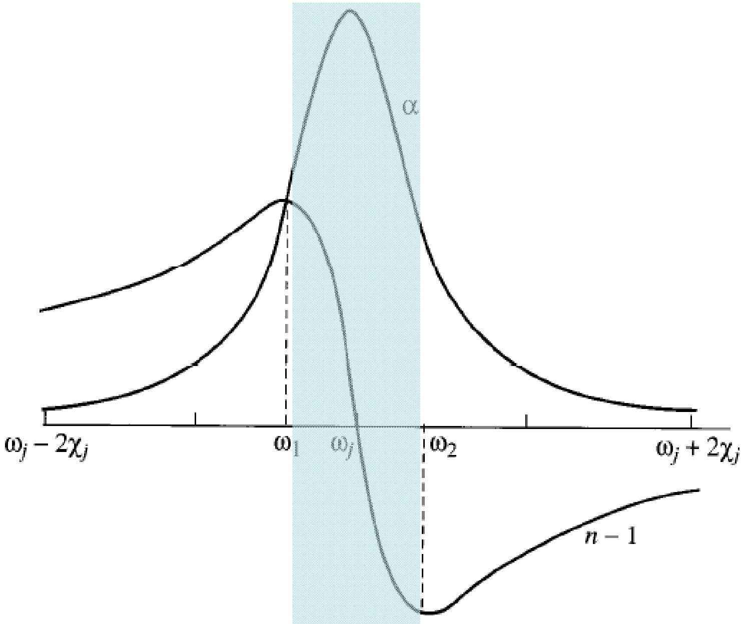
9.4.3 Frequency dependence of permittivity in dielectric media

$$\tilde{\epsilon}_r = 1 + \frac{Nq^2}{m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}$$

$$\tilde{k} = \frac{\omega}{c}\sqrt{\tilde{\epsilon}_r} \cong \frac{\omega}{c}\left[1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right]$$

$$n = \frac{ck}{\omega} \cong 1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2}$$

$$\alpha = 2\kappa \cong \frac{Nq^2\omega^2}{m\epsilon_0 c}\sum_j \frac{f_j\gamma_j}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2}$$



anomalous dispersion

If you agree to stay away from the resonances, the damping can be ignored

$$n = 1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2}$$

$$\frac{1}{\omega_j^2 - \omega^2} = \frac{1}{\omega_j^2}\left(1 - \frac{\omega^2}{\omega_j^2}\right)^{-1} \cong \frac{1}{\omega_j^2}\left(1 + \frac{\omega^2}{\omega_j^2}\right)$$

$$n = 1 + \left(\frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2}\right) + \omega^2\left(\frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^4}\right)$$

$$\boxed{n = 1 + A\left(1 + \frac{B}{\lambda^2}\right)} \quad (\lambda = 2\pi c/\omega)$$

Cauchy's formula