Universidad de Valparaíso
CHILE

Prueba I
Métodos Matemáticos de la Física II
Licenciatura en Física - 2018
IPGG

(I) (30%)

Sea:

$$R_{ij} = \delta_{ij} \cos(\theta) + n_i n_j \left[1 - \cos(\theta)\right] - \sin(\theta)\, \epsilon_{ijk} n_k$$

donde n_k son las componentes de un vector unitario. Halle la traza de la matriz $\mathbf{C}$ cuyas componentes están dadas por la ecuación indicial $C_{ij} = R_{ik} R_{jk}$

Obs.: $i, j, k \in \{1, 2, 3\}$.

(II) (30%)

Si r es la magnitud del vector posición $\mathbf{r}$, demuestre lo siguiente:

1. $\nabla f(r) = \dfrac{\mathbf{r}}{r} \dfrac{df(r)}{dr}$

2. $\nabla \cdot \mathbf{F}(r) = \dfrac{\mathbf{r}}{r} \cdot \dfrac{d\mathbf{F}(r)}{dr}$

3. $\nabla f(\boldsymbol{\xi}) = \mathbf{A} \dfrac{df(\xi)}{d\xi}$, donde $\xi = \mathbf{A} \cdot \mathbf{r}$, siendo $\mathbf{A}$ un vector constante.

Obs. : Todas las funciones consideradas aquí son arbitrarias.

(III) (40%)

Sean los campos escalares $\phi_1(r) = \dfrac{(\mathbf{P} \cdot \mathbf{r})^2}{r^2}$ y $\phi_2(r) = \dfrac{r^2}{(\mathbf{P} \cdot \mathbf{r})^2}$, donde $\mathbf{P}$ es un vector constante. Determine el producto $\nabla \phi_1(r) \times \nabla \phi_2(r)$.

---

1) $R_{ij} = \delta_{ij}\cos\theta + n_i n_j(1-\cos\theta) - \sin\theta\, \epsilon_{ijk} n_k$

$$C_{ii} = R_{il}R_{il}$$

$$= \left[\delta_{il}\cos\theta + n_i n_l(1-\cos\theta) - \sin\theta\, \epsilon_{ils} n_s\right]$$

$$\times \left[\delta_{il}\cos\theta + n_i n_l(1-\cos\theta) - \sin\theta\, \epsilon_{ilk} n_k\right]$$

$$= -\left[\delta_{il}\cos\theta + (1-\cos\theta) n_i n_l\right] \sin\theta\, \epsilon_{ilk} n_k$$

$$- \left[\delta_{il}\cos\theta + (1-\cos\theta) n_i n_l\right] \sin\theta\, \epsilon_{ils} n_s$$

$$+ \left[\delta_{il}\cos\theta + n_i n_l(1-\cos\theta)\right]\left[\delta_{il}\cos\theta + n_i n_l(1-\cos\theta)\right]$$

$$+ \sin^2\theta\, \epsilon_{ils}\epsilon_{ilk} n_s n_k$$

Obs. Los 2 primeros términos son nulos $\Rightarrow \delta_{il}\epsilon_{ilk} = 0$

y $\epsilon_{ilk} n_k n_i n_l = 0$.

$\therefore$

$$C_{ii} = \overset{3}{\cancel{\delta_{il}\delta_{il}}} \cos^2\theta + 2\,\delta_{il}\, n_i n_l \cos\theta(1-\cos\theta) + n_i n_l n_i n_l (1-\cos\theta)^2$$

$$+ \sin^2\theta \; \epsilon_{ils}\epsilon_{ilk} n_s n_k$$

$$= 3\cos^2\theta + \overset{1\,(\text{vector unitario})}{\cancel{n_i n_i}} (1-\cos\theta)\cos\theta + \overset{1}{\cancel{n_i n_i}}\, \overset{1}{\cancel{n_l n_l}} (1-\cos\theta)^2$$

$$+ \sin^2\theta \left(\delta_{ll}\delta_{ks} - \delta_{lk}\delta_{sl}\right) n_s n_k$$

$$= 3\cos^2\theta + (1-\cos\theta)\cos\theta + (1-\cos\theta)^2 + 3\sin^2\theta\, \overset{1}{\cancel{n_k n_k}}$$

$$- \sin^2\theta\, \overset{1}{\cancel{n_l n_l}}$$

$$= 3\cos^2\theta + \cos\theta - \cancel{\cos^2\theta} + 1 - 2\cos\theta + \cancel{\cos^2\theta} + 3\sin^2\theta$$

$$- \sin^2\theta$$

$$= 3 + 1 - \cos\theta + \sin^2\theta = 4 + \sin^2\theta - \cos\theta$$

2) a) La componente i-ésima está dada por:

Regla de la cadena.

$$[\nabla f(r)]_i = \partial_i f(r) = \frac{\partial}{\partial x_i} f(r) = \frac{\partial f(r)}{\partial r}\frac{\partial r}{\partial x_i}$$

luego $$\frac{\partial r}{\partial x_i} = \partial_i r = \partial_i (x_\ell x_\ell)^{1/2} = \frac{1}{2}(x_\ell x_\ell)^{-1/2}\underbrace{\partial_i(x_\ell x_\ell)}_{2x_i}$$

$$= \frac{x_i}{r}$$

Se transforma en derivada total dado que f solo depende de $|\vec{r}|=r$.

$$\therefore \; [\nabla f(r)]_i = \frac{x_i}{r}\frac{df(r)}{dr}$$

$$\Downarrow$$

$$\nabla f(r) = \frac{\vec{r}}{r}\frac{df(r)}{dr} \; //$$

Regla de la cadena.

b) $$\nabla\cdot\vec{F}(r) = \partial_i F_i(r) = \frac{\partial}{\partial x_i}F_i(r) = \frac{dF_i(r)}{dr}\underbrace{\frac{\partial r}{\partial x_i}}_{\frac{x_i}{r}}$$

$$= \frac{x_i}{r}\frac{dF_i(r)}{dr} = \frac{\vec{r}}{r}\cdot\frac{d\vec{F}(r)}{dr} \; //$$

c) $\left[\nabla f(\xi)\right]_i = \partial_i f(\xi)$

$$= \frac{\partial \xi}{\partial x_i} \frac{\partial f(\xi)}{\partial \xi} = \frac{\partial \xi}{dx_i} \frac{df(\xi)}{d\xi}$$

por otro lado: $\frac{\partial \xi}{\partial x_i} = \frac{\partial}{\partial x_i}(A_\ell x_\ell) = A_\ell \frac{\partial x_\ell}{\partial x_i} = A_\ell \delta_{\ell i}$

$$= A_i$$

$\therefore \left[\nabla f(\xi)\right]_i = A_i \frac{df(\xi)}{d\xi}$

$$\Downarrow$$

$$\nabla f(\xi) = \vec{A} \frac{df(\xi)}{d\xi} \quad //$$

3) $\phi_1 = \dfrac{(\vec{P}\cdot\vec{r})^2}{r^2} = \dfrac{P_\ell P_j X_\ell X_j}{r^2}$

entonces:

$[\nabla \phi_1]_i = \partial_i \left[P_\ell P_j X_\ell X_j (X_s X_s)^{-1} \right]$

$= P_\ell P_j \left[\partial_i (X_\ell X_j (X_s X_s)^{-1} \right]$

$= P_\ell P_j \left[\delta_{i\ell} \dfrac{X_j}{r^2} + \delta_{ij} \dfrac{X_\ell}{r^2} - X_\ell X_j (X_s X_s)^{-2} \times 2 X_i \right]$

$= P_i \dfrac{(\vec{P}\cdot\vec{r})}{r^2} + P_i \dfrac{(\vec{P}\cdot\vec{r})}{r^2} - 2 \dfrac{(\vec{P}\cdot\vec{r})^2}{r^4} X_i$

$\therefore$

$\Downarrow$

$\nabla \phi_1 = 2 \left[\dfrac{(\vec{P}\cdot\vec{r})}{r^2} \vec{P} - \dfrac{(\vec{P}\cdot\vec{r})^2}{r^4} \vec{r} \right]$

Por otro lado:

$\phi_2 = \dfrac{r^2}{(\vec{P}\cdot\vec{r})^2}$

$[\nabla \phi_2]_i = \partial_i \left[X_s X_s (P_\ell X_\ell)^{-2} \right]$

$$= \frac{\partial i (X_s X_s)}{(\vec{p}\cdot\vec{r})^2} + r^2 \partial i (P_e X_e)^{-2}$$

$$= \frac{2 X i}{(\vec{p}\cdot\vec{r})^2} - 2 r^2 (P_e X_e)^{-3} \partial i (P_e X_e)$$

$$= 2 \frac{X_i}{(\vec{p}\cdot\vec{r})^2} - 2 \frac{r^2}{(\vec{p}\cdot\vec{r})^3} P_i$$

$$= 2 \left[\frac{\vec{r}}{(\vec{p}\cdot\vec{r})^2} - \frac{r^2}{(\vec{p}\cdot\vec{r})^3} \vec{p} \right]$$

$$\therefore \nabla\phi_1 \times \nabla\phi_2 = 4 \left[\frac{(\vec{p}\cdot\vec{r})}{r^2} \vec{p} - \frac{(\vec{p}\cdot\vec{r})^2}{r^4} \vec{r} \right] \times \left[\frac{\vec{r}}{(\vec{p}\cdot\vec{r})^2} - \frac{r^2 \vec{p}}{(\vec{p}\cdot\vec{r})^3} \right]$$

solo sobreviven los siguientes términos!!

$$= 4 \left(\frac{1}{r^2 (\vec{p}\cdot\vec{r})} \vec{p}\times\vec{r} + \frac{1}{r^2 (\vec{p}\cdot\vec{r})} \vec{r}\times\vec{p} \right) = \vec{0}$$