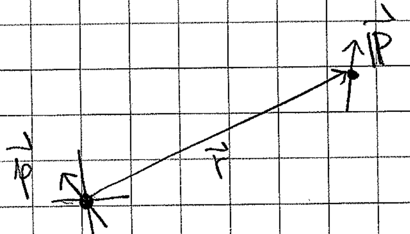
Fuerza entre dipolos

dipolo $\vec{p}$ en el origen

dipolo $\vec{\mathbb{P}}$ en la posición $\vec{r}$



$$\underbrace{\vec{F}}_{\text{fuerza sobre } \vec{\mathbb{P}}} = -\nabla U(\vec{r}) \; ; \; U = -\vec{\mathbb{P}}\cdot\underbrace{\vec{E}(\vec{r})}_{\text{debido a } \vec{p}}$$

$$\therefore \; \vec{F} = \nabla\, \vec{\mathbb{P}}\cdot\vec{E}(\vec{r})$$

$$\vec{F} = \mathbb{P}_j \nabla E_j(\vec{r})$$

$$\Downarrow$$

componente i-ésima de la fuerza

$$F_i = \mathbb{P}_j\, \partial_i E_j(\vec{r})$$

luego $E_j = -\partial_j \phi(\vec{r})$ donde $\phi(\vec{r}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{\vec{p}\cdot\vec{r}}{r^3}$

$$\Downarrow$$

$$E_j = -\frac{1}{4\pi\epsilon_0}\partial_j\left(\frac{p_k x_k}{r^3}\right) = -\frac{p_k}{4\pi\epsilon_0}\partial_j\left(\frac{x_k}{r^3}\right)$$

Luego $\partial_j\left(\frac{X_k}{r^3}\right) = \frac{\partial_j X_k}{r^3} + X_k\, \partial_j (r^{-3})$

$$= \frac{\delta_{jk}}{r^3} + X_k \frac{(-3)}{r^4}\, \partial_j (r)$$

$$= \frac{\delta_{jk}}{r^3} - 3\frac{X_k}{r^4}\, \partial_j (X_\ell X_\ell)^{1/2}$$

$$= \frac{\delta_{jk}}{r^3} - 3\frac{X_k}{r^4}\, \frac{1}{2} \cdot \frac{2 X_\ell \delta_{\ell j}}{r}$$

$$= \frac{\delta_{jk}}{r^3} - 3\frac{X_k}{r^5} X_j$$

$\therefore$

$$E_j = -\frac{p_k}{4\pi\varepsilon_0}\left[\frac{\delta_{jk}}{r^3} - 3\frac{X_j X_k}{r^5}\right]$$

$$= -\frac{1}{4\pi\varepsilon_0}\left[\frac{p_j}{r^3} - 3\frac{\vec{p}\cdot\vec{r}}{r^5} X_j\right]$$

$$= -\frac{1}{4\pi\varepsilon_0}\left[\frac{p_j}{r^3} - 3\frac{p_k X_k X_j}{r^5}\right]$$

$\therefore$

$$F_i = \mathbb{P}_j\, \partial_i \left[\frac{-1}{4\pi\varepsilon_0}\frac{p_j}{r^3} + \frac{1}{4\pi\varepsilon_0}\, 3\frac{p_k X_k X_j}{r^5}\right]$$

$$= -\frac{\mathbb{P}_j p_j}{4\pi\varepsilon_0}\, \partial_i\left(\frac{1}{r^3}\right) + \frac{3}{4\pi\varepsilon_0}\mathbb{P}_j p_k\, \partial_i\left(\frac{X_k X_j}{r^5}\right)$$

luego $\partial_i\left(\frac{1}{r^3}\right) = \partial_i(r^{-3}) = \frac{-3}{r^4}\,\partial_i(r)$

$$= \frac{-3}{r^4}\,\partial_i(x_s x_s)^{1/2}$$

$$= \frac{-3}{r^4}\,\frac{x_s}{r}\,\delta_{is} = \frac{-3}{r^5}\,x_i \quad //$$

y $\partial_i\left(\frac{x_k x_j}{r^5}\right) = \delta_{ik}\frac{x_j}{r^5} + \delta_{ij}\frac{x_k}{r^5} + x_k x_j\,\partial_i(r^{-5})$

$$= \frac{x_j}{r^5}\delta_{ik} + \frac{x_k}{r^5}\delta_{ij} - \frac{5}{r^6}x_k x_j\,\partial_i(r)$$

$$= \frac{x_j}{r^5}\delta_{ik} + \frac{x_k}{r^5}\delta_{ij} - \frac{5}{r^7}x_k x_j x_i \quad //$$

Finalmente

$$F_i = -\frac{P_j p_j}{4\pi\varepsilon_0}\left(-3\frac{x_i}{r^5}\right) + \frac{3}{4\pi\varepsilon_0}P_j p_k\left[x_j\frac{\delta_{ik}}{r^5} + x_k\frac{\delta_{ij}}{r^5} - \frac{5}{r^7}x_k x_j x_i\right]$$

$$= +\frac{3}{4\pi\varepsilon_0}\frac{\vec{P}\cdot\vec{p}}{r^5}x_i + \frac{3}{4\pi\varepsilon_0}\frac{\vec{P}\cdot\vec{r}}{r^5}p_i + \frac{3}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^5}P_i - \frac{15}{4\pi\varepsilon_0}\frac{(\vec{P}\cdot\vec{r})(\vec{p}\cdot\vec{r})}{r^7}x_i$$

$$\Downarrow$$

$$\vec{F} = \frac{3}{4\pi\varepsilon_0}\left[\frac{(\vec{P}\cdot\vec{r})}{r^5}\vec{p} + \frac{(\vec{p}\cdot\vec{r})}{r^5}\vec{P} + \frac{(\vec{P}\cdot\vec{p})}{r^5}\vec{r} - 5\frac{(\vec{P}\cdot\vec{r})(\vec{p}\cdot\vec{r})}{r^7}\vec{r}\right] \quad ///$$