Chapter 3. Potentials

3.1 Laplace's Equation

3.1.1 introduction

The primary task of electrostatics is **to find the electric field** of a given stationary charge distribution:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\boldsymbol{\imath}}}{\imath^2}\rho(\mathbf{r}')\,d\tau'$$

One may get around this by exploiting symmetry and using **Gauss 's law**, but ordinarily the best strategy is **first to calculate the *potential.* V**:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\imath}\rho(\mathbf{r}')\,d\tau'$$

Even *this* integral is often too tough to handle analytically. it is fruitful to recast the problem in **differential form, using Poisson's equation**:

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho$$ together **with appropriate boundary conditions**.

Very often, in fact, we are interested **in finding the potential in a region where $\rho = 0$**.

→ There may be plenty of charge *elsewhere*

→ but we are **confining our attention to places where there is no charge**.

→ **Poisson's equation reduces to Laplace's equation:**

$$\nabla^2 V = 0$$ or, in Cartesian coordinates, $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

➔ ***One might say that electrostatics is the study of Laplace's equation.***

3.1.2 Laplace's Equation in One Dimension

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \quad \text{(Laplacian)}$$

In generalized coordinate,

$$\nabla^2\phi = \nabla\cdot\nabla = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial u_3}\right)\right]$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2}$$

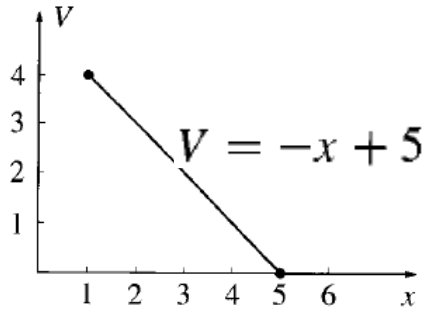
$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

1. $\dfrac{\partial^2 V}{\partial x^2} = 0$
2. $\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial V}{\partial r}\right)$
3. $\dfrac{1}{r^2}\left(\dfrac{\partial^2 V}{\partial \phi^2}\right)$
4. $\dfrac{1}{R^2}\dfrac{\partial}{\partial R}\left(R^2\dfrac{\partial V}{\partial R}\right)$
5. $\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial V}{\partial\theta}\right)$

Laplace's Equation in One Dimension

Suppose *V* depends on only one variable, *x.*

$$\frac{d^2V}{dx^2} = 0$$



The general solution is $\longrightarrow V(x) = mx + b$

- → It contains two undetermined constants *(m* and *b)*
- → *(m* and *b)* are fixed, in any particular case, by the boundary conditions of that problem.

Call attention to two features of this result;

1. V(x) is the average of v(x + a) and v(x - a), for any a:

$$V(x) = \tfrac{1}{2}[V(x+a) + V(x-a)]$$

- **➔ Laplace's equation is a kind of averaging instruction;**
- **➔ Solutions to Laplace's equation are fit the end points properly.**

2. Laplace's equation tolerates no local maxima or minima;

- **➔ Extreme values of *V* must occur at the end points.**

➔ The analogs in two and three dimensions will be powerful.

3.1.3 Laplace's Equation in Two Dimensions

If *V* depends on two variables, $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

This is no longer an *ordinary* differential equation → it is **a *partial* differential equation**

→ The general solution to this equation **doesn't contain just two arbitrary constants** despite the fact that it's a second order equation.

Nevertheless, possible to deduce certain properties common to all solutions.

1. The value of *V* at a point *(x, y)* is the average of those *around* the point.

→ If you draw a circle of any radius *R* about the point *(x, y),* the average value of *V* on the circle is equal to the value at the center:

$$V(x, y) = \frac{1}{2\pi R} \oint_{\text{circle}} V \, dl$$

2. V has no local maxima or minima; all extrema occur at the boundaries.

→ Laplace's equation picks the most **featureless function** possible, consistent with the boundary conditions:

→ Laplace's equation picks no hills, no valleys, **just the smoothest surface available.**

→ Harmonic functions in two dimensions have the same properties.

3.1.4 Laplace's Equation in Three Dimensions

If V depends on three variables, $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = 0$

The same two properties remain true to all solutions.

1. The value of *V* at point *r* is the average of *V* over a spherical surface of radius *R* centered at *r*:

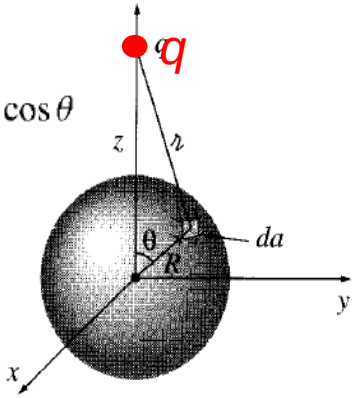
$$V(\mathbf{r}) = \frac{1}{4\pi R^2} \oint_{\text{sphere}} V\, da$$

Proof. Let's begin by calculating the average potential over a spherical surface of radius R due to a *single* point charge q located outside the sphere.

The potential at a point on the surface is $V = \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{\imath}$ $\quad \imath^2 = z^2 + R^2 - 2zR\cos\theta$

$$V_{\text{ave}} = \frac{1}{4\pi R^2}\frac{q}{4\pi\epsilon_0}\int [z^2 + R^2 - 2zR\cos\theta]^{-1/2} R^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}\sqrt{z^2 + R^2 - 2zR\cos\theta}\,\Big|_0^\pi = \frac{1}{4\pi\epsilon_0}\frac{q}{z}$$



➔ **this is precisely the potential due to *q* at the *center* of the sphere!**

➔ **The average potential over the sphere is equal to the net potential at the center**

2. V has no local maxima or minima; all extrema occur at the boundaries.

3.1.5 Boundary Conditions and Uniqueness Theorems

What are appropriate boundary conditions, sufficient to determine the answer and yet not so strong as to generate inconsistencies?

For instance, the value of the function at the two ends,
or the value of the function and its derivative at one end,
or the value at one end and the derivative at the other, and so on.

→ ***It is not so easy to see what would constitute acceptable boundary conditions.***

First uniqueness theorem:

➔ ***The solution to Laplace's equation in some volume V is uniquely determined if V is specified on the boundary surface S.***

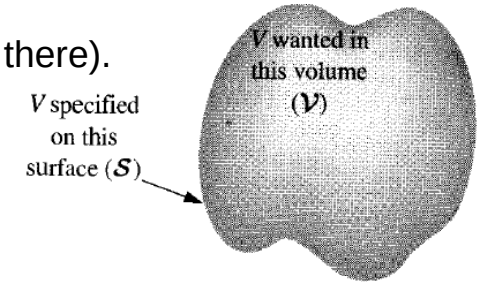
(Corollary) ***The potential in a volume V is uniquely determined if (a) the charge density throughout the region, and (b) the value of V on all boundaries, are specified.***

(Proof) Suppose there were *two* solutions to Laplace's equation: $\nabla^2 V_1 = 0$ and $\nabla^2 V_2 = 0$

Their *difference will* obey Laplace's equation.

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0$$

It takes the value *zero* on all boundaries (since V_1 and V_2 are equal there).
But Laplace's equation allows no local maxima or minima
all extrema occur on the boundaries.
So the maximum and minimum of V_3 are both zero.
Therefore V_3 must be zero everywhere, $\longrightarrow V_1 = V_2$



Boundary Conditions and Uniqueness Theorems

Example 3.1 Show that the potential is *constant* inside an enclosure completely surrounded by conducting material, provided there is no charge within the enclosure.

→ The potential on the cavity wall is some constant, *Vo.*
→ the potential inside is a function that satisfies Laplace's equation and has the constant value *Vo* at the boundary.
→ $V = Vo$ everywhere.
→ The uniqueness theorem guarantees that this is the *only* solution.
→ It follows that the *field* inside an empty cavity is zero.

The first uniqueness theorem also obeys Poisson's equation:

If we assume there is some charge inside the region,

$$\nabla^2 V_1 = -\frac{1}{\epsilon_0}\rho$$

$$\nabla^2 V_2 = -\frac{1}{\epsilon_0}\rho$$

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = -\frac{1}{\epsilon_0}\rho + \frac{1}{\epsilon_0}\rho = 0$$

The *difference* ($V_3 = V_1 - V_2$) satisfies Laplace's equation and is zero on all boundaries, so $V_3 = 0$ → $V_1 = V_2$.

First uniqueness theorem:

→ The solution to Laplace's (Poisson's) equation in some volume V is uniquely determined if V is specified on the boundary surface S.

3.1.6 Conductors and the Second Uniqueness Theorem

First uniqueness theorem ➔ *The solution to Laplace's (Poisson's) equation in some volume V is uniquely determined if V is specified on the boundary surface S.*

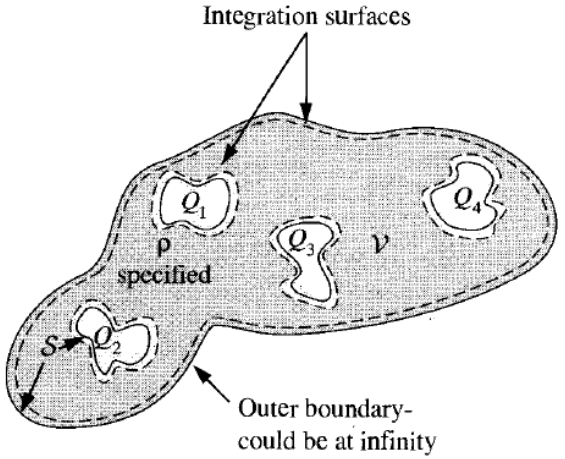
There are other circumstances in which we do not know the *potential* at the boundary, but rather know the *charges* on various conducting surfaces.

Suppose I put charge Q_1 on the first conductor, Q_2 on the second, and so on.

As soon as I put it on, it moves around resulting in some specified charge density ρ in the region between the conductors.

Is the electric field now uniquely determined?

Or are there perhaps a number of different ways the charges could arrange themselves on their respective conductors, each leading to a different field?



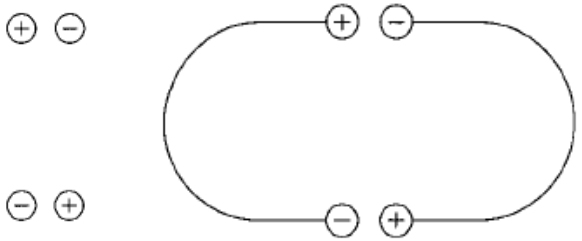
Second uniqueness theorem:

In a volume *V* surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely determined if the *total charge* on each conductor is given.

3.1.6 Conductors and the Second Uniqueness Theorem

First uniqueness theorem ➔ *The solution to Laplace's (Poisson's) equation in some volume V is uniquely determined if V is specified on the boundary surface S.*

Second uniqueness theorem: ➔ *In a volume V surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely determined if the total charge on each conductor is given.*

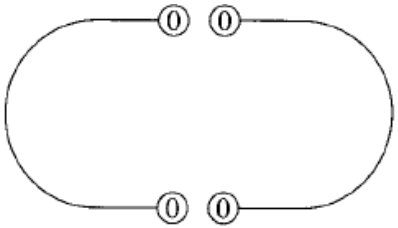


Consider the example of Purcell's:

→ Four conductors with charges ± Q, situated so that the plusses are near the minuses.

→ Now, what happens if we join them in pairs by tiny wires?

nothing will happen? the configuration still looks stable?



➔ ***It's wrong***!

➔ the total charge on each conductor is zero.

➔ One possible way to distribute zero charge over these conductors is to have no accumulation of charge anywhere,

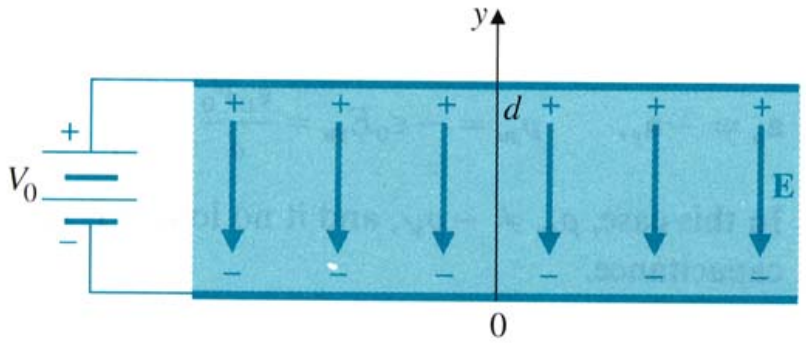
➔ hence zero field everywhere.

➔ ***By the second uniqueness theorem***, this must be **the unique solution**.

Electrostatic Boundary Value Problems $\nabla^2 V = 0$

- Example
  a) The potential at any point between the plates
  b) The surface charge densities on the plates

$$\rho_v(y) = 0$$



$$\frac{d^2V(y)}{dy^2} = 0 \rightarrow \frac{dV}{dy} = C_1$$

$$V(y) = C_1 y + C_2$$

at $y = 0,\ V = 0$

at $y = d,\ V = V_0$

$$V = \frac{V_0}{d} y$$

$$\mathbf{E}(y) = -\mathbf{a}_y \frac{\partial V}{\partial y} = -\mathbf{a}_y \left[\frac{V_0}{d}\right]$$

Surface charge density

$$E_{1n} = \frac{\rho_s}{\varepsilon_0}$$

$$\mathbf{a}_n = \mathbf{a}_y,\ \rho_{sl} = \varepsilon_0 E\big|_{y=0} = -\frac{\varepsilon_0 V_0}{d}$$

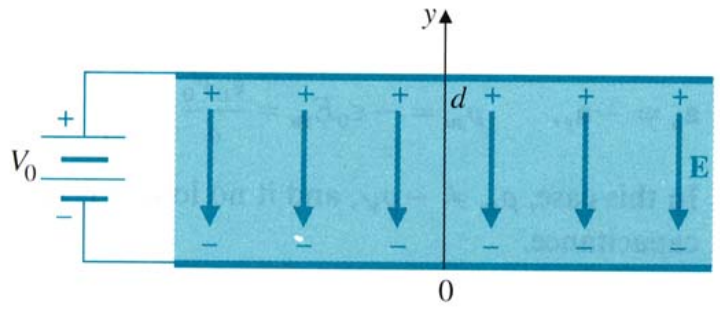
$$\mathbf{a}_n = -\mathbf{a}_y,\ \rho_{su} = -\varepsilon_0 E\big|_{y=d} = \frac{\varepsilon_0 V_0}{d}$$

$\rho_{sl} = -\rho_{su}$ in this case

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho$$

- **Example**
  - The potential at any point between the plates
  - The surface charge densities on the plates

$$\rho_v(y) = -\rho_0 \frac{y}{d}$$



$$\frac{d^2V(y)}{dy^2} = \frac{\rho_0}{\varepsilon_0 d} y$$

$$V(y) = \frac{\rho_0}{6\varepsilon_0 d} y^3 + C_1 y + C_2$$

$$\text{at } y = 0,\ V = 0 = C_2$$

$$\text{at } y = d,\ V = V_0 = \frac{\rho_0 d^2}{6\varepsilon_0} + C_1 d,\quad C_1 = \frac{V_0}{d} - \frac{\rho_0 d^2}{6\varepsilon_0}$$

$$V(y) = \frac{\rho_0}{6\varepsilon_0 d} y^3 + \left(\frac{V_0}{d} - \frac{\rho_0 d^2}{6\varepsilon_0}\right) y$$

$$\mathbf{E}(y) = -\mathbf{a}_y \frac{\partial V}{\partial y} = -\mathbf{a}_y \left[\frac{\rho_0}{2\varepsilon_0 d} y^2 + \left(\frac{V_0}{d} - \frac{\rho_0 d}{6\varepsilon_0}\right)\right]$$

Surface charge density

$$\mathbf{a}_n = \mathbf{a}_y,\quad \rho_{sl} = \varepsilon_0 E\big|_{y=0} = -\frac{\varepsilon_0 V_0}{d} + \frac{\rho_0 d}{6}$$

$$\mathbf{a}_n = -\mathbf{a}_y,\ \rho_{su} = -\varepsilon_0 E\big|_{y=d} = \frac{\varepsilon_0 V_0}{d} + \frac{\rho_0 d}{3}$$

$\rho_{sl} \neq \rho_{su}$ in this case

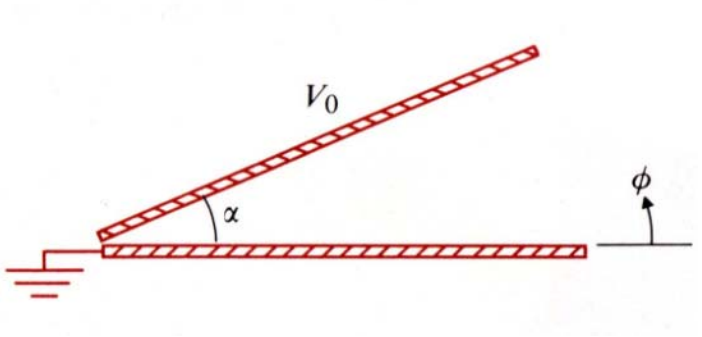
Electrostatic Boundary Value Problems

- Example

Determine the potential distribution fo the regions :

a) $0 < \phi < \alpha$

b) $\alpha < \phi < 2\pi$



For $0 < \phi < \alpha$, $\dfrac{d^2V}{d\phi^2} = 0 \quad \rightarrow \quad V = A\phi + B$

$$\begin{cases} V(0) = 0 \\ V(\alpha) = V_0 \end{cases} \rightarrow \boxed{V(\phi) = \frac{V_0}{\alpha}\phi, \quad 0 < \phi < \alpha}$$

For $\alpha < \phi < 2\pi$,

$V(\alpha) = V_0 = \alpha K_1 + K_2$

$V(2\pi) = 0 = 2\pi K_1 + K_2$

$$K_1 = -\frac{V_0}{2\pi - \alpha}, \; K_2 = \frac{2\pi V_0}{2\pi - \alpha}$$

Finally,

$$\boxed{V(\phi) = \frac{V_0}{2\pi - \alpha}(2\pi - \phi), \; \alpha < \phi < 2\pi}$$

$\mathbf{E} = ?$

$$\mathbf{E} = -\frac{1}{r}\frac{dV}{d\phi}\mathbf{a}_\phi = -\frac{1}{r}\frac{V_0}{\alpha}\mathbf{a}_\phi$$