Universidad de Valparaíso
CHILE

Tarea Voluntaria V

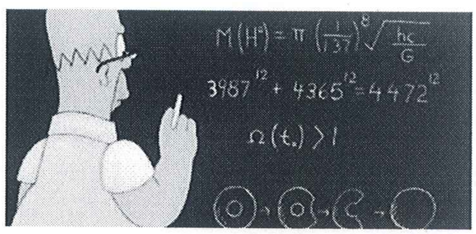
MMF II

Licenciatura en Física - 2020

Utilice MoB para demostrar que:

$$I = \int_0^\infty \frac{J_2(x)\sin(\beta x)}{x}\, dx = \beta\sqrt{1-\beta^2}$$

donde $J_2(x)$ es la función de Bessel de primera especie de orden 2 (en Complemento I se encuentra la representación hipergeométrica de esta función).



Recuerde como aplicar la Regla 2 y Regla 3 descritas en el apunte 5.

$$J = \int_0^{\infty} \frac{J_2(x)\,\mathrm{sen}(\beta x)}{x}\,dx$$

PASO 1: Expansión integrando

$$\mathrm{sen}(\beta x) = \sum_{n\geq 0} (-1)^n \beta^{2n+1} \frac{x^{2n+1}}{(2n+1)!} \cdot \frac{n!}{n!}$$

$$= \beta \sum_{n\geq 0} \phi_n \frac{\Gamma(n+1)}{\Gamma(2n+2)} \beta^{2n} x^{2n+1}$$

$$J_\alpha(x) = 2^{-\alpha} \sum_m \phi_m \frac{1}{4^m \Gamma(1+\alpha+m)} x^{2m+\alpha} \quad ; \text{ con } \alpha = 2$$

PASO 2: Serie de brackets de la integral

$$I = \frac{1}{4} \beta \sum_n \sum_m \phi_{n,m} \frac{\Gamma(n+1)}{\Gamma(2n+2)} \beta^{2n} \frac{1}{4^m \Gamma(3+m)} \underbrace{\int_0^{\infty} x^{2n+2m+3-1}\,dx}_{\langle 2m+2n+3 \rangle}$$

$$\langle 2m+2n+3 \rangle \rightarrow \frac{1}{2} \langle m+n+3/2 \rangle$$

Luego

$$I = \frac{\beta}{8} \sum_n \sum_m \phi_{n,m} \frac{\Gamma(n+1)}{\Gamma(2n+2)} \beta^{2n} \frac{1}{4^m \Gamma(3+m)} \langle m+n+3/2 \rangle$$

PASO 3 : SOLUCIONES

$\boxed{\text{Caso 1}}$: n libre $\Longrightarrow J_n$

$$J_n = \frac{\beta}{8}\sum_{n\geq 0} \phi_n \frac{\Gamma(n+1)}{\Gamma(2n+1)}\, \beta^{2n} \frac{1}{4^m\,\Gamma(3+m)}\,\Gamma(-m)\Bigg|_{m=-n-3/2}$$

$$\Downarrow$$

$$J_n = \frac{\beta}{8}\sum_{n\geq 0} \frac{(-1)^n}{\cancel{n!}} \frac{\cancel{\Gamma(n+1)}\,\beta^{2n}}{\Gamma(2n+2)} \frac{1\ \ \Gamma(3/2+n)}{\Gamma(3/2-n)\,4^{-n}\,4^{-3/2}}$$

$$= \frac{\beta}{\cancel{8}} \frac{\cancel{\Gamma(3/2)}}{\underbrace{\Gamma(2)}_{1}\,\cancel{\Gamma(3/2)}} \sum_{n\geq 0} (-1)^n \frac{\beta^{2n}\,(3/2)_n\,4^n}{(2)_{2n}\,(3/2)_{-n}} \cdot \cancel{8}$$

$$= \beta \sum_{n\geq 0} \frac{\cancel{(-1)^n}\,\cancel{(3/2)_n}\,\cancel{(-1)^n}(-1/2)_n\ \beta^{2n}\,\cancel{4^n}}{\cancel{4^n}\,(1)_n\,\cancel{(3/2)_n}} \quad ;$$

se utilizó las sigtes. identidades:

$$(2)_{2n} = 4^n (1)_n (3/2)_n$$

$$\frac{1}{(3/2)_{-n}} = (-1/2)_n (-1)^n$$

Finalmente:

$$J_n = \beta \sum_{n\geq 0} (-1/2)_n \frac{\beta^{2n}}{(1)_n} = \beta \sum_{n\geq 0} (-1/2)_n \frac{\beta^{2n}}{n!}$$

$$= \beta\, {}_1F_0\left(\begin{matrix} -1/2 \\ - \end{matrix}\,\middle|\, \beta^2\right) = \beta\, \frac{1}{(1-\beta^2)^{-1/2}}$$

$$J_n = \beta\sqrt{1-\beta^2}$$

$\boxed{\text{Caso 2}}$: m libre $\Rightarrow J_m$

$$J_m = \frac{\beta}{8} \sum_{m\geq 0} \phi_m \frac{\Gamma(n+1)}{\Gamma(2n+2)} \beta^{2n} \frac{1}{4^m\,\Gamma(3+m)} \Gamma(-n) \Big|_{n=-m-3/2}$$

$$= \frac{\beta}{8} \sum_{m\geq 0} \frac{(-1)^m}{m!} \frac{\Gamma(-m-1/2)}{\Gamma(-2m-1)} \beta^{-2m-3} \frac{\Gamma(3/2+m)}{4^m\,\Gamma(3+m)}$$

Obs. La fn. Gamma $\Gamma(-2m-1)$ es divergente para todo m, haciendo que cada término de la serie sea nulo, ∴ la serie completa es nula, se descarta.

La solución de la integral es:

$$\int_0^\infty \frac{J_2(x)\,\text{sen}(\beta x)}{x}\,dx = J_n = \beta\sqrt{1-\beta^2}$$

QED.