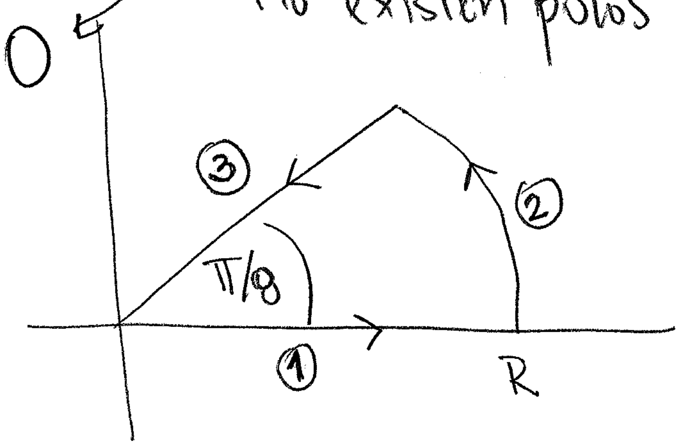
< Pauta Prueba III >

1) $\oint e^{-z^2}dz = 0$ No existen polos



— Para 1 —

$z = t \; ; \; t : 0 \to R$

$$\int_{\gamma_1} e^{-z^2}dz = \int_0^R e^{-t^2}dt \xrightarrow{R\to\infty} \int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}$$

— Para 2 —

$z = Re^{it} \; ; \; t = 0 \to \pi/8$

$$\int_{\gamma_2} e^{-z^2}dz = \int_0^{\pi/8} e^{-R^2e^{2it}} iRe^{it}dt$$

donde $Re^{-R^2e^{2it}} = Re^{-R^2(\cos 2t + i\,\mathrm{sen}\,2t)}$

$= Re^{-R^2\cos 2t}e^{-iR^2\mathrm{sen}\,2t}.$

dado que $\cos 2t > 0$ en $[0, \pi/8]$ entonces el factor

$Re^{-R^2\cos 2t} \to 0$ cuando $R \to \infty \; \therefore \; \int_{\gamma_2} e^{-z^2}dz = 0$

— Paso 3 —

$\mathrm{tg}(\pi/8)$

$z = x + iy \rightarrow y = \mathrm{tg}(\pi/8)\, x = mx \wedge x = t$

$\therefore\ z = t + imt = t(1+im)\ ;$ con $t: R \rightarrow 0$

$$\therefore \int_{\gamma_3} e^{-z^2} dz = \int_R^0 e^{-t^2(1+im)^2}(1+im)\,dt$$

$$= \int_R^0 e^{-t^2(1+2im-m^2)}(1+im)\,dt$$

$$= \int_R^0 e^{-t^2(1-m^2)} e^{-t^2 2im}(1+im)\,dt$$

Obs. $(1-m^2)^2 = (1-\mathrm{tg}^2(\tfrac{\pi}{8}))^2 = (1-(\sqrt{2}-1)^2)$

$= (1-(2-2\sqrt{2}+1))$

$= (-2+2\sqrt{2})^2 = (2(\sqrt{2}-1))$

$= (2\,\mathrm{tg}(\pi/8)) = 2m$

luego

$$= \int_R^0 e^{-t^2 2m} e^{-it^2 2m}(1+im)\,dt$$

$$= -\int_0^R e^{-t^2 2m}\left(\cos(2mt^2) - i\,\mathrm{sen}(2mt^2)\right)(1+im)\,dt$$

en el límite $R \to \infty$

$$\int_{\gamma_3} e^{-z^2}\, dz = -\int_0^\infty e^{-2mt^2}\left(\cos(2mt^2) - i\,\mathrm{sen}(2mt^2)\right)(1+im)\, dt.$$

haciendo $x^2 = 2mt^2$

$x = \sqrt{2m}\, t$.

$dt = \frac{1}{\sqrt{2m}}\, dx$.

$$= -\int_0^\infty e^{-x^2}\left[\cos(x^2) - i\,\mathrm{sen}(x^2)\right]\frac{(1+im)}{\sqrt{2m}}\, dx$$

Finalmente:

$$\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} = 0$$

$$\frac{\sqrt{\pi}}{2} - \int_0^\infty e^{-x^2}\left(\cos(x^2) + m\,\mathrm{sen}(x^2)\right)\frac{dx}{\sqrt{2m}} - i\int_0^\infty e^{-x^2}\left(m\cos(x^2) - \mathrm{sen}(x^2)\right)\frac{dx}{\sqrt{2m}} = 0$$

Igualando componentes:

$$* \quad \frac{\sqrt{\pi}}{2}\sqrt{2m} = \int_0^\infty e^{-x^2}\cos(x^2)\,dx + m\int_0^\infty e^{-x^2}\,\mathrm{sen}(x^2)\,dx$$

$$** \quad m\int_0^\infty e^{-x^2}\cos(x^2)\,dx = \int_0^\infty e^{-x^2}\,\mathrm{sen}(x^2)\,dx.$$

$\overline{4}$

$$\frac{\sqrt{\pi}}{2}\sqrt{2m} = (1+m^2)\int_0^\infty e^{-x^2}\cos(x^2)\,dx.$$

$$1+m^2 = 1+(\sqrt{2}-1)^2 = 1+(2-2\sqrt{2}+1)$$

$$= 2+2(1-\sqrt{2})$$

$$= 2-2m = 2(1-m)$$

$\therefore$

$$\frac{\sqrt{\pi}}{2}\,\frac{\sqrt{2m}}{2(1-m)} = \int_0^\infty e^{-x^2}\cos(x^2)\,dx$$

$$\frac{\sqrt{\pi}}{4}\,\frac{\sqrt{2\sqrt{2}-2}}{2-\sqrt{2}} = \int_0^\infty e^{-x^2}\cos(x^2)\,dx$$

$$\frac{\sqrt{\pi}}{4}\,\frac{\sqrt{2\sqrt{2}-2}\,.\,(2+\sqrt{2})}{4-2} = \int_0^\infty e^{-x^2}\cos(x^2)\,dx$$

$$\frac{\sqrt{\pi}}{4}\,\frac{\sqrt{(2\sqrt{2}-2)(2+\sqrt{2})^2}}{2} = \int_0^\infty e^{-x^2}\cos(x^2)\,dx$$

$$\frac{\sqrt{\pi}}{4}\,\frac{\sqrt{4\sqrt{2}+4}}{2} = \int_0^\infty e^{-x^2}\cos(x^2)\,dx$$

Finally

$$\int_0^\infty e^{-x^2}\cos(x^2)\,dx = \frac{\sqrt{\pi}}{4}\sqrt{\sqrt{2}+1}$$

2) $I = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+2x+2)^2}$

Se evalúa

$$\oint \frac{z^2}{(z^2+2z+2)^2} = -\frac{d}{da}\oint \frac{z^2}{(z^2+2z+a)}\Big|_{a=2}$$

$$= -\frac{d}{da}\oint \frac{z^2}{(z-(\sqrt{1-a}-1))(z+(\sqrt{1-a}-1))}dz$$

ahora $a>1$.

$$= -\frac{d}{da}\oint \frac{z^2}{[z+1-i\sqrt{a-1}][z+1+i\sqrt{a-1}]}$$

Este factor (polo) contribuye a la integral.

Evaluación del residuo → límite:

$$= -\frac{d}{da}\left[2\pi i \frac{z^2}{[z+1+i\sqrt{a-1}]}\right]_{z=i\sqrt{a-1}-1}$$

$$= -\frac{d}{da}\left[2\pi i \frac{[i\sqrt{a-1}-1]^2}{2i\sqrt{a-1}}\right] = -\pi\frac{d}{da}\left[\frac{-(a-1)-2i\sqrt{a-1}+1}{\sqrt{a-1}}\right]$$

$$= -\pi\frac{d}{da}\left[-\sqrt{a-1}-2i+\frac{1}{\sqrt{a-1}}\right]$$

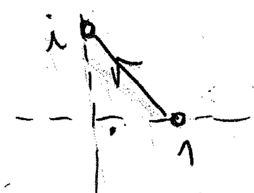
$$= -\pi\left[-\frac{1}{2\sqrt{a-1}} - \frac{1}{2}\frac{1}{(a-1)^{3/2}}\right] \quad \text{haciendo } a=2.$$

$$= -\pi\left[-\frac{1}{2} - \frac{1}{2}\right] = \pi$$

Por otro lado $\displaystyle\oint \frac{z^2}{(z^2+2z+2)^2} = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+2x+2)^2}dx + \cancel{\int_{C_1} \frac{z^2}{(z^2+2z+2)^2}dz}$

Finalmente:

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+2x+2)^2}dx = \pi$$



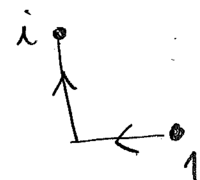
3) a) $\displaystyle\int_\gamma xy\, dz$; $z = x+iy$; $y = -x+1$.

$$\int_\gamma xy\,(dx+i\,dy) = \int_1^0 xy\,dx + i\int_0^1 xy\,dy$$

$$= \int_1^0 x(1-x)dx + i\int_0^1 y(1-y)\,dy$$

$$= -\frac{1}{6} + i\frac{1}{6} = -\frac{1}{6}(1-i)$$

b) $\displaystyle\int_\gamma xy\,dz = \int_{\substack{\text{Eje}\\ \text{horiz.}\\ (y=0)}} \cancelto{0}{xy}\,dz + \int_{\substack{\text{Eje}\\ \text{vertical}\\ (x=0)}} \cancelto{0}{x}y\,dz = 0.$



c) $$\int_{\gamma} XY\,dz = \int_{|z|=1} XY\,dz$$

luego
$z = e^{it}$; $t: 0 \to \pi/2$
$= \underbrace{\cos t}_{X} + i\,\underbrace{\operatorname{sen} t}_{Y}$

$dz = i\,e^{it}dt$

$$= i \int_0^{\pi/2} \cos t \operatorname{sen} t\, e^{it}\, dt$$

$$= i\left[\frac{1}{3}(1+i)\right] = \frac{1}{3}(i-1)$$

4) $$\int_{|z|=1} \frac{e^{2z}\cot(z)}{(2-z)(9-z^2)}\,dz$$

<u>Obs.</u> en la circunferencia (interior) solo hay un polo ($z=0$) debido al $\cot(z) = \dfrac{\cos(z)}{\operatorname{sen}(z)}$

Usando la serie de Laurent ...

* $z\cot(z) = 1 - \frac{1}{3}z^2 + \dots \Rightarrow \cot(z) = \frac{1}{z} - \frac{1}{3}z + \dots$

* $e^{2z} = 1 + 2z + \dots$

* $\dfrac{1}{(2-z)} = \dfrac{1}{2}\,\dfrac{1}{1-\frac{z}{2}} = \dfrac{1}{2}\left(1 + \dfrac{z}{2} + \dots\right)$

Por serie geométrica.

$$* \quad \frac{1}{(9-z^2)} = \frac{1}{9}\,\frac{1}{\left(1-\frac{z^2}{9}\right)} = \frac{1}{9}\left(1+\frac{z^2}{9}+\dots\right)$$

Finalmente:

$$\frac{e^z \cot(z)}{(2-z)(9-z^2)} = \frac{1}{2}\cdot\frac{1}{9}\left(\frac{1}{z}-\frac{1}{3}z+\dots\right)\left(1+2z+\dots\right)\left(1+\frac{z}{2}+\dots\right)\left(1+\frac{z^2}{9}+\dots\right)$$

$$= \frac{1}{18}\left(\frac{1}{z}+O(z^0)\right)$$

luego $C_{-1} = \dfrac{1}{18}$.

$$\therefore \quad \int_{|z|=1} \frac{e^{2z}\cot(z)}{(2-z)(9-z^2)}\,dz = 2\pi i\, C_{-1} = \frac{\pi i}{9}$$