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Ramanujan's Master Theorem

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Abstract S. Ramanujan introduced a technique, known as Ramanujan's Master Theorem, which provides an explicit expression for the Mellin transform of a function in terms of the analytic continuation of its Taylor coefficients. The history and proof of this result are reviewed, and a variety of applications is presented. Finally, a multi-dimensional extension of Ramanujan's Master Theorem is discussed.

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O. Espinosa is deceased.

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1 Introduction

Ramanujan's Master Theorem refers to the formal identity

$$\int_0^\infty x^{s-1}\left\{\lambda(0) - \frac{x}{1!}\lambda(1) + \frac{x^2}{2!}\lambda(2) - \cdots\right\} dx = \Gamma(s)\lambda(-s) \tag{1.1}$$

stated by S. Ramanujan's in his *Quarterly Reports* [2, p. 298]. It was widely used by him as a tool in computing definite integrals and infinite series. In fact, as G.H. Hardy puts it in [14], he "was particularly fond of them [(1.1) and (2.4)], and used them as one of his commonest tools."

The goal of this semi-expository paper is to discuss the history of (1.1) and to describe a selection of applications of this technique. Section 2 discusses evidence that (1.1) was nearly discovered as early as 1874 by J.W.L. Glaisher and J. O'Kinealy. Section 3 briefly outlines Hardy's proof of Ramanujan's Master Theorem. The critical issue is the extension of the function λ from $\mathbb{N}$ to $\mathbb{C}$. Section 4 presents the evaluation of a collection of definite integrals with most of the examples coming from the classical table [13]. Further examples of definite integrals are given in Sect. 8 which collects integrals derived from classical polynomials.

Section 5 is a recollection on the evaluation of the quartic integral

$$N_{0,4}(a;m) = \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}. \tag{1.2}$$

This section provides a personal historical context: it was the evaluation of (1.2) that lead one of the authors to (1.1).

Sections 6 and 9 outline the use of Ramanujan's Master Theorem to ongoing research projects: Sect. 6 deals with an integral related to the distance traveled by a uniform random walk in a fixed number of steps; finally, Sect. 9 presents a multidimensional version of the main theorem that has appeared in the context of Feynman diagrams.

The use of Ramanujan's Master Theorem has been restricted here mostly to the evaluation of definite integrals. Many other applications appear in the literature. For instance, Ramanujan himself employed it to derive various expansions: the two examples given in [14, 11.9] are the expansion of e^{-ax} in powers of xe^{bx} as well as an expansion of the powers x^r of a root of $aqx^p + x^q = 1$ in terms of powers of a.

2 History

The first integral theorem in the spirit of Ramanujan's Master Theorem appears to have been given by Glaisher in 1874, [10]:

$$\int_0^\infty \left(a_0 - a_1x^2 + a_2x^4 - \cdots\right) dx = \frac{\pi}{2}a_{-\frac{1}{2}}. \tag{2.1}$$

Glaisher writes, "of course, a_n being only defined for n a positive integer, $a_{-\frac{1}{2}}$ is without meaning. But in cases where a_n involves factorials, there is a strong presumption, derived from experience in similar questions, that the formula will give correct results if the continuity of the terms is preserved by the substitution of gamma functions for the factorials. This I have found to be true in every case to which I have applied (2.1)."

Glaisher in [10] formally obtained (2.1) by integrating term-by-term the identity

$$a_0 - a_1x^2 + a_2x^4 - \cdots = \frac{a_0}{1+x^2} - \Delta a_0 \frac{x^2}{(1+x^2)^2} + \Delta^2 a_0 \frac{x^4}{(1+x^2)^3} - \cdots. \tag{2.2}$$

Here Δ is the forward-difference operator defined by $\Delta a_n = a_{n+1} - a_n$.

Glaisher's argument, published in July 1874, was picked up in October of the same year by O'Kinealy who critically simplified it in [15]. Employing the forward-shift operator E defined by $E \cdot \lambda(n) = \lambda(n+1)$, O'Kinealy writes the left-hand side of (2.2) as $\frac{1}{1+x^2E} \cdot a_0$ which he then integrates treating E as a number to obtain

$$\frac{\pi}{2} E^{-1/2} \cdot a_0 = \frac{\pi}{2} a_{-\frac{1}{2}},$$

thus arriving at the identity (2.1). O'Kinealy, [15], remarks that "it is evident that there are numerous theorems of the same kind". As an example, he proposes integrating $\cos(xE) \cdot a_0$ and $\sin(xE) \cdot a_0$.

O'Kinealy's improvements are emphatically received by Glaisher in a short letter [9] to the editors in which he remarks that he had examined O'Kinealy's work and that, "after developing the method so far as to include these formulae and several others, I communicated it, with the examples, to Professor Cayley, in a letter on the 22nd or 23rd of July, which gave rise to a short correspondence between us on the matter at the end of July. My only reason for wishing to mention this at once is that otherwise, as I hope soon to be able to return to the subject and somewhat develop the principle, which is to a certain extent novel, it might be thought at some future time that I had availed myself of Mr. O'Kinealy's idea without proper acknowledgement."

Unfortunately, no further work seems to have appeared along these lines so that one can only speculate as to what Glaisher and Cayley have figured out. It is not unreasonable to guess that they might very well have developed an idea somewhat similar to Ramanujan's Master Theorem (1.1). In fact, just slightly generalizing O'Kinealy's argument is enough to formally obtain (1.1). This is shown next.

Formal proof of (1.1)

$$\begin{aligned}
\int_0^\infty x^{s-1} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \lambda(n) x^n \, dx &= \int_0^\infty x^{s-1} \sum_{n=0}^\infty \frac{(-1)^n}{n!} E^n x^n \, dx \cdot \lambda(0) \\
&= \int_0^\infty x^{s-1} e^{-Ex} \, dx \cdot \lambda(0) \\
&= \frac{\Gamma(s)}{E^s} \cdot \lambda(0) \\
&= \Gamma(s) \lambda(-s)
\end{aligned}$$

where in the penultimate step the integral representation

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx \tag{2.3}$$

of the gamma function was employed and the operator E treated as a number. It is this step which renders the proof formal: clearly the coefficient function $\lambda(n)$ needs to satisfy certain conditions for the result to be valid. This will be discussed in Sect. 3. □

The identity

$$\int_0^\infty x^{s-1}\{\varphi(0) - x\varphi(1) + x^2\varphi(2) - \cdots\}\, dx = \frac{\pi}{\sin s\pi}\varphi(-s), \tag{2.4}$$

is given by Ramanujan alongside (1.1) (see [2]). The formulations are equivalent: the relation $\varphi(n) = \lambda(n)/\Gamma(n+1)$ converts (2.4) into (1.1).

The integral theorem (2.1) also appears in the text [8] as Exercise 7 on Chap. XXVI. It is attributed there to Glaisher. The exercise asks to show (2.1) and to "apply this theorem to find $\int_0^\infty \frac{\sin ax}{x}\, dx$."

The argument that Ramanujan gives for (1.1) appears in Hardy [14] where the author demonstrates that, while the argument can be made rigorous in certain cases, it usually leads to false intermediate formulas which "excludes practically all of Ramanujan's examples".

A rigorous proof of (1.1) and its special case (2.1) was given in Chap. XI of [14]. This text is based on a series of lectures on Ramanujan's work given in the Fall semester of 1936 at Harvard University.

3 Rigorous treatment of the Master Theorem

The proof of Ramanujan's Master Theorem provided by Hardy in [14] employs Cauchy's residue theorem as well as the well-known Mellin inversion formula which is recalled next followed by an outline of the proof.

Theorem 3.1 (Mellin inversion formula) *Assume that $F(s)$ is analytic in the strip $a < \text{Re}\, s < b$ and define f by*

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty} F(s) x^{-s}\, ds.$$

If this integral converges absolutely and uniformly for $c \in (a,b)$ then

$$F(s) = \int_0^\infty x^{s-1} f(x)\, dx.$$

Theorem 3.2 (Ramanujan's Master Theorem) *Let $\varphi(z)$ be an analytic (single-valued) function, defined on a half-plane*

$$H(\delta) = \{z \in \mathbb{C} : \text{Re}\, z \geq -\delta\} \tag{3.1}$$

for some $0 < \delta < 1$. *Suppose that, for some* $A < \pi$, φ *satisfies the growth condition*

$$\left|\varphi(v + iw)\right| < Ce^{Pv + A|w|} \tag{3.2}$$

for all $z = v + iw \in H(\delta)$. *Then* (2.4) *holds for all* $0 < \operatorname{Re} s < \delta$, *that is*,

$$\int_0^\infty x^{s-1}\left\{\varphi(0) - x\varphi(1) + x^2\varphi(2) - \cdots\right\} dx = \frac{\pi}{\sin s\pi}\varphi(-s). \tag{3.3}$$

Proof Let $0 < x < e^{-P}$. The growth conditions show that the series

$$\Phi(x) = \varphi(0) - x\varphi(1) + x^2\varphi(2) - \cdots$$

converges. The residue theorem yields

$$\Phi(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty} \frac{\pi}{\sin \pi s}\varphi(-s)x^{-s}\, ds \tag{3.4}$$

for any $0 < c < \delta$. Observe that $\pi/\sin(\pi s)$ has poles at $s = -n$ for $n = 0, 1, 2, \dots$ with residue $(-1)^n$. The integral in (3.4) converges absolutely and uniformly for $c \in (a, b)$ for any $0 < a < b < \delta$. The claim now follows from Theorem 3.1. □

Remark 3.3 The conversion $\varphi(u) = \lambda(u)/\Gamma(u + 1)$ establishes Ramanujan's Master Theorem in the form (1.1). The condition $\delta < 1$ ensures convergence of the integral in (3.3). Analytic continuation may be employed to validate (3.3) to a larger strip in which the integral converges. See also Sect. 7.

4 A collection of elementary examples

This section contains a collection of definite integrals that can be evaluated directly from Ramanujan's Master Theorem 3.2. For the convenience of the reader, the main theorem in the form (1.1) is reproduced below. Its hypotheses are described in Sect. 3.

Theorem 4.1 *Assume* f *admits an expansion of the form*

$$f(x) = \sum_{k=0}^\infty \frac{\lambda(k)}{k!}(-x)^k. \tag{4.1}$$

Then, the Mellin transform of f *is given by*

$$\int_0^\infty x^{s-1} f(x)\, dx = \Gamma(s)\lambda(-s). \tag{4.2}$$

Example 4.2 Instances of series expansions involving factorials are particularly well-suited for the application of Ramanujan's Master Theorem. To illustrate this fact, use the binomial theorem for $a > 0$ in the form

$$(1 + x)^{-a} = \sum_{k=0}^\infty \binom{k + a - 1}{k}(-x)^k = \sum_{k=0}^\infty \frac{\Gamma(k + a)}{\Gamma(a)}\frac{(-x)^k}{k!}. \tag{4.3}$$

Ramanujan's Master Theorem (1.1), with $\lambda(k) = \Gamma(a+k)/\Gamma(a)$, then yields

$$\int_0^\infty \frac{x^{s-1}\,dx}{(1+x)^a} = \frac{\Gamma(s)\Gamma(a-s)}{\Gamma(a)} = B(s, a-s) \tag{4.4}$$

where B is the beta integral.

Example 4.3 Several of the functions appearing in this paper are special cases of the hypergeometric function

$${}_pF_q(\mathbf{c}; \mathbf{d}; -x) = \sum_{k=0}^\infty \frac{(c_1)_k\,(c_2)_k \cdots (c_p)_k}{(d_1)_k\,(d_2)_k \cdots (d_q)_k} \frac{(-x)^k}{k!} \tag{4.5}$$

where $\mathbf{c} = (c_1, \dots, c_p)$, $\mathbf{d} = (d_1, \dots, d_q)$, and $(a)_k = a(a+1)\cdots(a+k-1)$ denotes the rising factorial. To apply Ramanujan's Master Theorem, write $(a)_k = \Gamma(a+k)/\Gamma(a)$. The result is the standard evaluation

$$\int_0^\infty x^{s-1}\,{}_pF_q(\mathbf{c}; \mathbf{d}; -x)\,dx = \Gamma(s)\frac{\Gamma(c_1-s)\,\cdots, \Gamma(c_p-s)\Gamma(d_1)\cdots\Gamma(d_q)}{\Gamma(c_1)\,\cdots\Gamma(c_p)\Gamma(d_1-s)\cdots\Gamma(d_q-s)}, \tag{4.6}$$

which appears as Entry 7.511 in [13].

Example 4.4 The Bessel function $J_\nu(x)$ admits the hypergeometric representation

$$J_\nu(x) = \frac{1}{\Gamma(\nu+1)} \frac{x^\nu}{2^\nu}\,{}_0F_1\left(-; \nu+1; -\frac{x^2}{4}\right). \tag{4.7}$$

Its Mellin transform is therefore obtained from (4.6) as

$$\int_0^\infty x^{s-1} J_\nu(x)\,dx = \frac{2^{s-1}\Gamma(\frac{s+\nu}{2})}{\Gamma(\frac{\nu-s}{2}+1)}. \tag{4.8}$$

This formula appears as 6.561.14 in [13].

Example 4.5 The expansion

$$\frac{\cos(t\tan^{-1}\sqrt{x})}{(1+x)^{t/2}} = \sum_{k=0}^\infty \frac{\Gamma(t+2k)\,\Gamma(k+1)}{\Gamma(t)\,\Gamma(2k+1)} \frac{(-x)^k}{k!}.$$

was established in [4] in the process of evaluating of a class of definite integrals (alternatively, as pointed out by the referee, the expansion may be deduced hypergeometrically; in fact, the conversion is done automatically by `Mathematica` 7 upon expressing the series as a hypergeometric function). A direct application of Ramanujan's Master Theorem yields

$$\int_0^\infty x^{\nu-1} \frac{\cos(2t\tan^{-1}\sqrt{x})}{(1+x)^t}\,dx = \frac{\Gamma(2t-2\nu)\,\Gamma(1-\nu)\,\Gamma(\nu)}{\Gamma(2t)\,\Gamma(1-2\nu)},$$

and $x = \tan^2\theta$ gives

$$\int_0^{\pi/2} \sin^{\mu}\theta \cos^{2t-\mu}\theta \cos(2t\theta)\, d\theta = \frac{\pi\,\Gamma(2t-\mu-1)}{2\sin(\pi\mu/2)\,\Gamma(2t)\Gamma(-\mu)}. \tag{4.9}$$

Similarly, the expansion

$$\frac{\sin(2t\tan^{-1}\sqrt{x})}{\sqrt{x}(1+x)^t} = \sum_{k=0}^{\infty} \frac{\Gamma(2t+2k+1)\,\Gamma(k+1)}{\Gamma(2t)\Gamma(2k+2)} \frac{(-x)^k}{k!}$$

produces

$$\int_0^{\pi/2} \sin^{\mu-1}\theta \cos^{2t-\mu}\theta\, \sin(2t\theta)\, d\theta = \frac{\pi\,\Gamma(2t-\mu)}{2\sin(\pi\mu/2)\Gamma(2t)\Gamma(1-\mu)}. \tag{4.10}$$

Example 4.6 The Mellin transform of the function $\log(1+x)/(1+x)$ is obtained from the expansion

$$\frac{\log(1+x)}{1+x} = -\sum_{k=1}^{\infty} H_k(-x)^k, \tag{4.11}$$

where $H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$ is the kth harmonic number. The analytic continuation of the harmonic numbers, required for an application of Ramanujan's Master Theorem, is achieved by the relation

$$H_k = \gamma + \psi(k+1), \tag{4.12}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function and $\gamma = -\Gamma'(1)$ is the Euler constant. The expansion (4.11) and Ramanujan's Master Theorem now give

$$\int_0^{\infty} \frac{x^{\nu-1}}{1+x} \log(1+x)\, dx = -\frac{\pi}{\sin\pi\nu}\big(\gamma + \psi(1-\nu)\big). \tag{4.13}$$

The special case $\nu = \frac{1}{2}$ produces the logarithmic integral

$$\int_0^{\infty} \frac{\log(1+t^2)}{1+t^2}\, dt = \pi \log 2 \tag{4.14}$$

which is equivalent to the classic evaluation

$$\int_0^{\pi/2} \log\sin x\, dx = -\frac{\pi}{2}\log 2 \tag{4.15}$$

given by Euler.

Example 4.7 The infinite product representation of the gamma function

$$\Gamma(x) = \frac{e^{-\gamma x}}{x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right)^{-1} e^{x/n} \tag{4.16}$$

is equivalent to the expansion

$$\log \Gamma(1+x) = -\gamma x + \sum_{k=2}^{\infty} \frac{\zeta(k)}{k}(-x)^k. \tag{4.17}$$

Hence Ramanujan's Master Theorem implies

$$\int_0^\infty x^{\nu-1} \frac{\gamma x + \log \Gamma(1+x)}{x^2}\, dx = \frac{\pi}{\sin \pi \nu} \frac{\zeta(2-\nu)}{2-\nu}, \tag{4.18}$$

valid for $0 < \nu < 1$.

5 A quartic integral

The authors' first encounter with Ramanujan's Master Theorem occurred while evaluating the quartic integral

$$N_{0,4}(a;m) = \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}. \tag{5.1}$$

The goal was to provide a proof of the experimental observation that

$$N_{0,4}(a;m) = \frac{\pi}{2^{m+3/2}\,(a+1)^{m+1/2}} P_m(a), \tag{5.2}$$

where

$$P_m(a) = 2^{-2m} \sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}(a+1)^k. \tag{5.3}$$

The reader will find in [1] a variety of proofs of this identity, but it was Ramanujan's Master Theorem that was key to the first proof of (5.2). This proof is outlined next.

The initial observation is that the double square root function $\sqrt{a+\sqrt{1+c}}$ satisfies the unexpected relation

$$\frac{d}{dc}\sqrt{a+\sqrt{1+c}} = \frac{1}{\pi\sqrt{2}} \int_0^\infty \frac{dx}{x^4+2ax^2+1+c}. \tag{5.4}$$

This leads naturally to the Taylor series expansion

$$\sqrt{a+\sqrt{1+c}} = \sqrt{a+1} + \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} N_{0,4}(a;k-1)c^k. \tag{5.5}$$

Thus, in terms of

$$\lambda(k) = -\frac{(k-1)!}{\pi\sqrt{2}} N_{0,4}(a;k-1), \tag{5.6}$$

Ramanujan's Master Theorem implies that

$$\Gamma(s)\lambda(-s) = \int_0^\infty c^{s-1}\sqrt{a+\sqrt{1+c}}\,dc. \tag{5.7}$$

The next ingredient emerges from a direct differentiation of the integral $N_{0,4}$:

$$\left(\frac{d}{da}\right)^j N_{0,4}(a;k-1) = \frac{(-1)^j 2^j (k+j-1)!}{(k-1)!}\int_0^\infty \frac{x^{4k+2j-2}\,dx}{(x^4+2ax^2+1)^{k+j}}.$$

Note that the integral on the right-hand side can be expressed in terms of $N_{0,4}$ if $j = 1-2k$. In this case, the formal relation

$$\left(\frac{d}{da}\right)^{1-2k}\lambda(k) = (-2)^{1-2k}\lambda(1-k) \tag{5.8}$$

is obtained. This may be rewritten as

$$\lambda(m+1) = \left(-\frac{1}{2}\frac{d}{da}\right)^{2m+1}\lambda(-m) \tag{5.9}$$

and relates the quartic integral $N_{0,4}(a;m)$, as a function in m, to its analytic continuation appearing in (5.7). Combining (5.9) and (5.7) one arrives at

$$\begin{aligned} N_{0,4}(a;m) &= \frac{\pi\sqrt{2}}{2^{2m+1}(m-1)!m!}\left(\frac{d}{da}\right)^{2m+1}\int_0^\infty c^{m-1}\sqrt{a+\sqrt{1+c}}\,dc \\ &= \frac{m\pi\sqrt{2}}{2^{6m+2}}\binom{4m}{2m}\binom{2m}{m}\int_0^\infty \frac{c^{m-1}\,dc}{(a+\sqrt{1+c})^{2m+1/2}}. \end{aligned} \tag{5.10}$$

The substitution $u = \sqrt{1+c}$ shows that

$$N_{0,4}(a;m) = \frac{m\pi\sqrt{2}}{2^{6m+1}}\binom{4m}{2m}\binom{2m}{m}\int_1^\infty f_m(u)(a+u)^{-(2m+1/2)}\,du, \tag{5.11}$$

with $f_m(u) = u(u^2-1)^{m-1}$. This final integral can now be evaluated to give the desired expression (5.2) for $N_{0,4}$. To this end one integrates by parts and uses the fact that the derivatives of f_m at $u=1$ have a closed-form evaluation. Further details can be found in [3].

6 Random Walk Integrals

In this section, the n-dimensional integral

$$W_n(s) := \int_{[0,1]^n}\left|\sum_{k=1}^n e^{2\pi i x_k}\right|^s dx_1\,dx_2\cdots dx_n \tag{6.1}$$

is considered which has recently been studied in [5] and [6]. This integral is connected to planar random walks. In detail, such a walk is said to be *uniform* if it starts at the origin and at each step takes a unit-step in a random direction. As such, (6.1) expresses the s-th moment of the distance to the origin after n steps. The study of these walks originated with K. Pearson more than a century ago [16].

For s an even integer, the moments $W_n(s)$ take integer values. In fact, for integers $k \geq 0$, the explicit formula

$$W_n(2k) = \sum_{a_1+\cdots+a_n=k} \binom{k}{a_1, \ldots, a_n}^2 \tag{6.2}$$

has been established in [5]. The evaluation of $W_n(s)$ for values of $s \neq 2k$ is more challenging. In particular, the definition (6.1) is not well-suited for high-precision numerical evaluations, and other representations are needed.

In the remainder of this section, it is indicated how Ramanujan's Master Theorem may be applied to find a one-dimensional integral representation for $W_n(s)$. While (6.1) may be used to justify a priori that Ramanujan's Master Theorem 3.2 applies, it should be noted that one may proceed formally with only the sequence (6.2) given. This is the approach taken below in the proof of Theorem 6.1. Ramanujan's Master Theorem produces a formal candidate for an analytic extension of the sequence $W_n(2k)$. This argument yields the following Bessel integral representation of (6.1), previously obtained by D. Broadhurst [7].

Theorem 6.1 *Let $s \in \mathbb{C}$ with $2k > \operatorname{Re} s > \max(-2, -\frac{n}{2})$. Then*

$$W_n(s) = 2^{s+1-k} \frac{\Gamma(1+\frac{s}{2})}{\Gamma(k-\frac{s}{2})} \int_0^\infty x^{2k-s-1} \left(-\frac{1}{x}\frac{d}{dx}\right)^k J_0^n(x)\, dx. \tag{6.3}$$

Proof The evaluation (6.2) yields the generating function for the even moments:

$$\sum_{k\geq 0} W_n(2k) \frac{(-x)^k}{(k!)^2} = \left(\sum_{k\geq 0} \frac{(-x)^k}{(k!)^2}\right)^n = J_0\big(2\sqrt{x}\big)^n, \tag{6.4}$$

with $J_0(z)$ the Bessel function of the first kind as in (4.7). Applying Ramanujan's Master Theorem (1.1) to $\lambda(k) = W_n(2k)/k!$ produces

$$\Gamma(\nu)\lambda(-\nu) = \int_0^\infty x^{\nu-1} J_0^n\big(2\sqrt{x}\big)\, dx. \tag{6.5}$$

A change of variables and setting $s = 2\nu$ gives

$$W_n(-s) = 2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \int_0^\infty x^{s-1} J_0^n(x)\, dx. \tag{6.6}$$

The claim now follows from the fact that if $F(s)$ is the Mellin transform of $f(x)$ then $(s-2)(s-4)\cdots(s-2k)F(s-2k)$ is the corresponding transform of $(-\frac{1}{x}\frac{d}{dx})^k f(x)$. The latter is a consequence of Ramanujan's Master Theorem. $\square$

7 Extending the domain of validity

The region of validity of the identity given by Ramanujan's Master Theorem is restricted by the region of convergence of the integral. For example, the integral representation of the gamma function given in (2.3) holds for $\text{Re}\, s > 0$. In this section it is shown that analytic continuations of such representations are readily available by dropping the first few terms of the Taylor series of the defining integrand. This provides an alternative to the method used at the end of the proof of Theorem 6.1.

Theorem 7.1 *Suppose φ satisfies the conditions of Theorem 3.2 so that for all* $0 < \text{Re}\, s < \delta$

$$\int_0^\infty x^{s-1} \sum_{k=0}^\infty \varphi(k)(-x)^k \, dx = \frac{\pi}{\sin s\pi} \varphi(-s).$$

Then, for any positive integer N and $-N < \text{Re}\, s < -N+1$,

$$\int_0^\infty x^{s-1} \sum_{k=N}^\infty \varphi(k)(-x)^k \, dx = \frac{\pi}{\sin s\pi} \varphi(-s). \tag{7.1}$$

Proof Applying Theorem 3.2 to the function $\varphi(\cdot + N)$ shows that

$$\int_0^\infty x^{s-1} \sum_{k=0}^\infty \varphi(k+N)(-x)^k \, dx = \frac{\pi}{\sin s\pi} \varphi(-s+N).$$

Now shift s to obtain (7.1). □

Example 7.2 Apply the result (7.1) with $N = 1$ to obtain

$$\Gamma(s) = \int_0^\infty x^{s-1} \left(e^{-x} - 1 \right) dx. \tag{7.2}$$

This integral representation now gives an analytic continuation of (2.3) to $-1 < \text{Re}\, s < 0$.

8 Some classical polynomials

In this section the explicit formulas for the generating functions of classical polynomials are employed to derive some definite integrals.

8.1 The Bernoulli polynomials

The generating function for the Bernoulli polynomials $B_m(q)$ is given by

$$\frac{te^{qt}}{e^t - 1} = \sum_{m=0}^\infty B_m(q) \frac{t^m}{m!}. \tag{8.1}$$

These polynomials relate to the Hurwitz zeta function

$$\zeta(z,q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^z} \tag{8.2}$$

via $B_m(q) = -m\zeta(1-m,q)$ for $m \geq 1$. Then (8.1) yields

$$\frac{e^{-qt}}{1-e^{-t}} - \frac{1}{t} = \sum_{m=0}^{\infty} \zeta(-m,q)\frac{(-t)^m}{m!}. \tag{8.3}$$

Ramanujan's Master Theorem now provides the integral representation

$$\int_0^{\infty} t^{\nu-1}\left(\frac{e^{-qt}}{1-e^{-t}} - \frac{1}{t}\right) dt = \Gamma(\nu)\zeta(\nu,q), \tag{8.4}$$

valid in the range $0 < \operatorname{Re} \nu < 1$.

8.2 The Hermite polynomials

The generating function for the Hermite polynomials $H_m(x)$ is

$$e^{2xt-t^2} = \sum_{m=0}^{\infty} H_m(x)\frac{t^m}{m!}. \tag{8.5}$$

Their analytic continuation, as a function in the index m, is given by

$$H_m(x) = 2^m U\left(-\frac{m}{2}, \frac{1}{2}, x^2\right) \tag{8.6}$$

where U is Whittaker's confluent hypergeometric function. Ramanujan's Master Theorem now provides the integral evaluation

$$\int_0^{\infty} t^{s-1} e^{-2xt-t^2}\, dt = \frac{\Gamma(s)}{2^s}\, U\left(\frac{s}{2}, \frac{1}{2}, x^2\right). \tag{8.7}$$

An equivalent form of this evaluation appears as Entry 3.462.1 in [13].

8.3 The Laguerre polynomials

The Laguerre polynomials $L_n(x)$ given by

$$\frac{1}{1-t}\exp\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n(x)t^n \tag{8.8}$$

can be expressed also as $L_n(x) = M(-n,1;x)$, where

$$M(a,c;x) = {}_1F_1\left(\left.\begin{matrix} a \\ c \end{matrix}\right| x\right) = \sum_{j=0}^{\infty} \frac{(a)_j}{(c)_j}\frac{x^j}{j!} \tag{8.9}$$

is the confluent hypergeometric or Kummer function. Ramanujan's Master Theorem yields the evaluation

$$\int_0^\infty \frac{t^{\nu-1}}{1+t} \exp\left(\frac{xt}{1+t}\right) dt = \Gamma(\nu)\Gamma(1-\nu)M(\nu, 1; x). \tag{8.10}$$

The change of variables $r = t/(1+t)$ then gives

$$M(\nu, 1; x) = \frac{1}{\Gamma(\nu)\Gamma(1-\nu)} \int_0^1 r^{\nu-1}(1-r)^{-\nu} e^{rx}\, dr, \tag{8.11}$$

which is Entry 9.211.2 in [13].

8.4 The Jacobi polynomials

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are defined by the generating function

$$\sum_{n=0}^\infty P_n^{(\alpha,\beta)}(x)t^n = \frac{2^{\alpha+\beta}}{R^*(x,t)}\big(1-t+R^*(x,t)\big)^{-\alpha}\big(1+t+R^*(x,t)\big)^{-\beta}, \tag{8.12}$$

where $R^*(x,t) = \sqrt{1-2xt+t^2}$. These polynomials admit the hypergeometric representation

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n+1+\alpha)}{n!\,\Gamma(1+\alpha)} {}_2F_1\left(n+\alpha+\beta+1, -n; 1+\alpha; \frac{1-x}{2}\right). \tag{8.13}$$

Now write $R(x,t) = R^*(x,-t)$, so that $R(x,t) = \sqrt{1+2xt+t^2}$, to obtain

$$2^{\alpha+\beta} R^{-1}(1+t+R)^{-\alpha}(1-t+R)^{-\beta} = \sum_{k=0}^\infty \lambda(k)\frac{(-t)^k}{k!} \tag{8.14}$$

where

$$\lambda(k) = \frac{\Gamma(k+1+\alpha)}{\Gamma(1+\alpha)} {}_2F_1\left(k+\alpha+\beta+1, -k; 1+\alpha; \frac{1-x}{2}\right). \tag{8.15}$$

Ramanujan's Master Theorem produces

$$\int_0^\infty \frac{t^{\nu-1}\,dt}{R(1+t+R)^\alpha(1-t+R)^\beta} = \frac{B(\nu, 1+\alpha-\nu)}{2^{\alpha+\beta}} {}_2F_1\left(\begin{matrix} 1+\alpha+\beta-\nu, \nu \\ 1+\alpha \end{matrix} \,\middle|\, \frac{1-x}{2}\right).$$

8.5 The Chebyshev polynomials of the second kind

These polynomials are defined by

$$U_n(a) = \frac{\sin((n+1)x)}{\sin x}, \quad \text{where } \cos x = a, \tag{8.16}$$

and have the generating function

$$\sum_{k=0}^{\infty} U_k(a)x^k = \frac{1}{1-2ax+x^2}. \tag{8.17}$$

The usual application of Ramanujan's Master Theorem yields

$$\int_0^\infty \frac{x^{\nu-1}\,dx}{1+2ax+x^2} = \frac{\pi}{\sin \pi\nu}\,\frac{\sin[(1-\nu)\cos^{-1} a]}{\sqrt{1-a^2}}. \tag{8.18}$$

This result appears as Entry 3.252.12 in [13].

9 The method of brackets

The focus of this final section will be on a multi-dimensional extension of Ramanujan's Master Theorem. This has been called the *method of brackets* and it was originally presented in [12] in the context of integrals arising from Feynman diagrams. A complete description of the operational rules of the method, together with a variety of examples, was first discussed in [11]. The basic idea is the assignment of a formal symbol $\langle a\rangle$ to the divergent integral

$$\int_0^\infty x^{a-1}\,dx. \tag{9.1}$$

The rules for operating with brackets are described below. These rules employ the symbol

$$\phi_n = \frac{(-1)^n}{\Gamma(n+1)}, \tag{9.2}$$

called the *indicator of n*.

Rule 1 *The bracket expansion*

$$\frac{1}{(a_1+a_2+\cdots+a_r)^\alpha} = \sum_{m_1,\ldots,m_r} \phi_{m_1,\ldots,m_r} a_1^{m_1}\cdots a_r^{m_r} \frac{\langle \alpha+m_1+\cdots+m_r\rangle}{\Gamma(\alpha)}$$

holds. Here $\phi_{m_1,\ldots,m_r}$ is a shorthand notation for the product $\phi_{m_1}\cdots\phi_{m_r}$. Where there is no possibility of confusion this will be further abridged as $\phi_{\{m\}}$. The notation $\sum_{\{m\}}$ is to be understood likewise.

Rule 2 *A series of brackets*

$$\sum_{\{n\}} \phi_{\{n\}} f(n_1,\ldots,n_r)\langle a_{11}n_1+\cdots+a_{1r}n_r+c_1\rangle\cdots\langle a_{r1}n_1+\cdots+a_{rr}n_r+c_r\rangle$$

is assigned the value

$$\frac{1}{|\det(A)|} f\left(n_1^*, n_r^*\right)\Gamma\left(-n_1^*\right)\cdots\Gamma\left(-n_r^*\right),$$

where A is the matrix of coefficients (a_{ij}) and (n_i^*) is the solution of the linear system obtained by the vanishing of the brackets. No value is assigned if the matrix A is singular.

Rule 3 *In the case where a higher dimensional series has more summation indices than brackets, the appropriate number of free variables is chosen among the indices. For each such choice, Rule 2 yields a series. Those converging in a common region are added to evaluate the desired integral.*

Example 9.1 Apply the method of brackets to

$$\int_0^\infty x^{\nu-1} F(x)\, dx \tag{9.3}$$

where F has the series representation

$$F(x) = \sum_{k=0}^{\infty} \phi_k \lambda(k) x^k.$$

Then (9.3) can be written as the bracket series

$$\int_0^\infty x^{\nu-1} F(x)\, dx = \int_0^\infty \sum_{k=0}^{\infty} \phi_k \lambda(k) x^{k+\nu-1}\, dx = \sum_k \phi_k \lambda(k) \langle k+\nu \rangle.$$

Rule 2 assigns the value

$$\sum_k \phi_k \lambda(k) \langle k+\nu \rangle = \lambda\big(k^*\big) \Gamma\big(-k^*\big) \tag{9.4}$$

where k^* is the solution of $k+\nu=0$. Thus one obtains

$$\int_0^\infty x^{\nu-1} F(x)\, dx = \lambda(-\nu)\Gamma(\nu). \tag{9.5}$$

This is precisely Ramanujan's Master Theorem as given by Theorem 3.2.

Rule 1 is a restatement of the fact that the Mellin transform of e^{-x} is $\Gamma(s)$:

$$\begin{aligned}
\frac{\Gamma(s)}{(a_1+\cdots+a_r)^s} &= \int_0^\infty x^{s-1} e^{-(a_1+\cdots+a_r)x}\, dx \\
&= \int_0^\infty x^{s-1} \prod_{i=1}^{r} \sum_{m_i} \phi_{m_i} (a_i x)^{m_i}\, dx \\
&= \sum_{\{m\}} \phi_{\{m\}} a_1^{m_1} \cdots a_r^{m_r} \langle s+m_1+\cdots+m_r \rangle.
\end{aligned}$$

Example 9.1 has shown that the 1-dimensional version of Rule 2 is Ramanujan's Master Theorem. A formal argument is now presented to show that the multi-dimensional version of Rule 2 follows upon iterating the one-dimensional result. The exposition is restricted to the 2-dimensional case. Consider the bracket series

$$\sum_{n_1,n_2} \phi_{n_1}\phi_{n_2} f(n_1,n_2)\langle a_{11}n_1 + a_{12}n_2 + c_1\rangle\langle a_{21}n_1 + a_{22}n_2 + c_2\rangle \tag{9.6}$$

which encodes the integral

$$\int_0^\infty \int_0^\infty \sum_{n_1,n_2} \phi_{n_1}\phi_{n_2} f(n_1,n_2) x^{a_{11}n_1+a_{12}n_2+c_1-1} y^{a_{21}n_1+a_{22}n_2+c_2-1}\, dx\, dy.$$

Substituting $(u,v) = (x^{a_{11}}y^{a_{21}}, x^{a_{12}}y^{a_{22}})$ yields $\frac{dx\,dy}{xy} = \frac{1}{|a_{11}a_{22}-a_{12}a_{21}|}\frac{du\,dv}{uv}$, and hence the above integral simplifies to

$$\frac{1}{|a_{11}a_{22}-a_{12}a_{21}|}\int_0^\infty \int_0^\infty \sum_{n_1,n_2} \phi_{n_1}\phi_{n_2} f(n_1,n_2) u^{n_1-n_1^*-1} v^{n_2-n_2^*-1}\, du\, dv.$$

Here (n_1^*, n_2^*) is the solution to $a_{11}n_1^* + a_{12}n_2^* + c_1 = 0$, $a_{21}n_1^* + a_{22}n_2^* + c_2 = 0$. Ramanujan's Master Theorem gives

$$\int_0^\infty \sum_{n_1} \phi_{n_1} f(n_1,n_2) u^{n_1-n_1^*-1}\, du = f\left(n_1^*, n_2\right)\Gamma\left(-n_1^*\right).$$

A second application of Ramanujan's Master Theorem shows that the bracket series (9.6) evaluates to

$$\frac{1}{|a_{11}a_{22}-a_{12}a_{21}|} f\left(n_1^*, n_2^*\right)\Gamma\left(-n_1^*\right)\Gamma\left(-n_2^*\right).$$

This is Rule 2.

9.1 A gamma-like higher dimensional integral

The next example illustrates the power and ease of the method of brackets for the treatment of certain multi-dimensional integrals such as

$$\int_0^\infty \cdots \int_0^\infty \exp\left(-(x_1 + \cdots + x_n)^\alpha\right) \prod_{i=1}^n x_i^{s_i-1}\, dx_i. \tag{9.7}$$

It should be pointed out that this class of integrals is beyond the scope of current computer algebra systems including `Mathematica` 7 and `Maple` 12.

For simplicity of exposition, take $n = 2$ in (9.7). The n-dimensional case presents no additional difficulties.

$$\begin{aligned}
&\int_0^\infty \int_0^\infty x^{s-1} y^{t-1} \exp\bigl(-(x+y)^\alpha\bigr)\, dx\, dy \\
&\quad = \sum_j \phi_j \int_0^\infty \int_0^\infty x^{s-1} y^{t-1} (x+y)^{\alpha j}\, dx\, dy \\
&\quad = \sum_j \phi_j \int_0^\infty \int_0^\infty x^{s-1} y^{t-1} \sum_{n,m} \phi_{n,m} x^n y^m \frac{\langle n+m-\alpha j\rangle}{\Gamma(-\alpha j)}\, dx\, dy \\
&\quad = \sum_{j,n,m} \phi_{j,n,m} \frac{1}{\Gamma(-\alpha j)} \langle n+m-\alpha j\rangle \langle n+s\rangle \langle m+t\rangle.
\end{aligned}$$

Solving the linear equations for the vanishing of the brackets gives $n^* = -s$, $m^* = -t$, and $j^* = -\frac{s+t}{\alpha}$. The determinant of the system is α, therefore the integral is

$$\frac{1}{\alpha} \frac{1}{\Gamma(-\alpha j^*)} \Gamma\bigl(-n^*\bigr) \Gamma\bigl(-m^*\bigr) \Gamma\bigl(-j^*\bigr) = \frac{1}{\alpha} \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \Gamma\left(\frac{s+t}{\alpha}\right).$$

The full statement of this result is presented as the next theorem.

Theorem 9.2

$$\begin{aligned}
&\int_0^\infty \cdots \int_0^\infty \exp\bigl(-(x_1+\cdots+x_n)^\alpha\bigr) \prod_{i=1}^n x_i^{s_i-1}\, dx_i \\
&\quad = \frac{1}{\alpha} \frac{\Gamma(s_1)\Gamma(s_2)\cdots\Gamma(s_n)}{\Gamma(s_1+\cdots+s_n)} \Gamma\left(\frac{s_1+\cdots+s_n}{\alpha}\right).
\end{aligned}$$

Remark 9.3 The correct interpretation of Rule 3 is work in-progress. The next example illustrates the subtleties associated with this question. The evaluation

$$\int_0^\infty x^{s-1} e^{-2x}\, dx = \frac{\Gamma(s)}{2^s} \tag{9.8}$$

follows directly from the bracket expansion

$$\int_0^\infty x^{s-1} e^{-2x}\, dx = \sum_n \phi_n 2^n \langle n+s\rangle$$

and Rule 2. On the other hand, rewriting the integrand as $e^{-2x} = e^{-x}e^{-x}$ and expanding it in a bracket series produces

$$\int_0^\infty x^{s-1} e^{-x} e^{-x}\, dx = \sum_{n,m} \phi_{n,m} \langle n+m+s\rangle.$$

The resulting bracket series has more summation indices than brackets. The choice of n as a free variable, gives $m^* = -n - s$ and Rule 2 produces the convergent series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Gamma(n+s) = \Gamma(s) {}_1F_0\left(\begin{matrix} s \\ - \end{matrix} \middle| -1 \right) = \frac{\Gamma(s)}{2^s}. \tag{9.9}$$

Symmetry dictates that the choice of m as a free variable leads to the same result. Rule 3, as stated currently, would yield the correct evaluation (9.8), twice.

The trouble has its origin in that the series in (9.9) has been evaluated at the boundary of its region of convergence. Rule 3 should be modified by introducing extra parameters to distinguish different regions of convergence. This remains to be clarified. For instance,

$$\int_0^{\infty} x^{s-1} e^{-Ax} e^{-Bx} \, dx = \sum_{n,m} \phi_{n,m} A^n B^m \langle n+m+s \rangle \tag{9.10}$$

which, upon choosing n and m as free variables, yields the two series

$$\frac{\Gamma(s)}{B^s} {}_1F_0\left(\begin{matrix} s \\ - \end{matrix} \middle| -\frac{A}{B} \right), \qquad \frac{\Gamma(s)}{A^s} {}_1F_0\left(\begin{matrix} s \\ - \end{matrix} \middle| -\frac{B}{A} \right),$$

respectively. Both series evaluate to $\Gamma(s)/(A+B)^s$, but it is now apparent that their regions of convergence are different. Accordingly, they should not be added in order to obtain the value (9.10). The original integral (9.8) appears as the limit $A, B \to 1$.

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