Chapter 6. Magnetic Fields in Matter

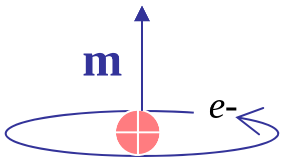
6.1 Magnetization . . . .
6.1.1 Diamagnets, Paramagnets, Ferromagnets . . . .
6.1.2 Torques and Forces on Magnetic Dipoles . . . .
6.1.3 Effect of a Magnetic Field on Atomic Orbits . . . .
6.1.4 Magnetization . . . .
6.2 The Field of a Magnetized Object . . . .
6.2.1 Bound Currents . . . .
6.2.2 Physical Interpretation of Bound Currents . . . .
6.2.3 The Magnetic Field Inside Matter . . . .
6.3 The Auxiliary Field H . . . .
6.3.1 Ampère's law in Magnetized Materials . . . .
6.3.2 A Deceptive Parallel . . . .
6.3.3 Boundary Conditions . . . .
6.4 Linear and Nonlinear Media . . . .
6.4.1 Magnetic Susceptibility and Permeability . . . .
6.4.2 Ferromagnetism . . . .

6.1 Magnetization

- All matters are composed of atoms, each with a positive charged nucleus and a number of orbiting electrons.

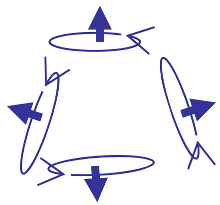
- In addition, both electrons and the nucleus of an atom rotate (spin) on their own axes with certain magnetic dipole moments.

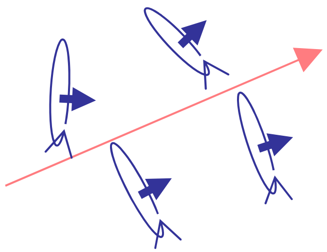
$$m_{nucleus} << m_{electron} \text{ because } M_{nucleus} >> M_{electron} \text{ and } \omega_{\text{nucleus}} << \omega_{electron}$$



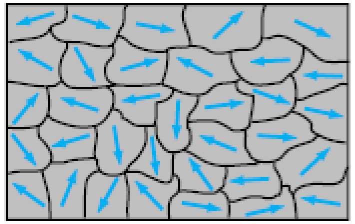
- In the absence of an external magnetic field, the magnetic dipoles of the atoms of most materials (excepts permanent magnets) have random orientations, resulting in no net magnetic moment.

- The application of an external magnetic fields cause

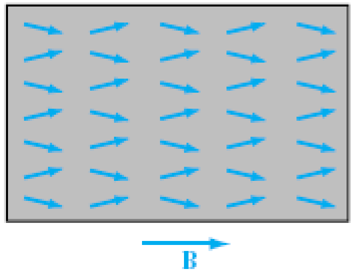




6.1.1 Magnetic Materials: Diamagnets, Paramagnets, Ferromagnets



(a) Unmagnetized domains



(b) Magnetized domains

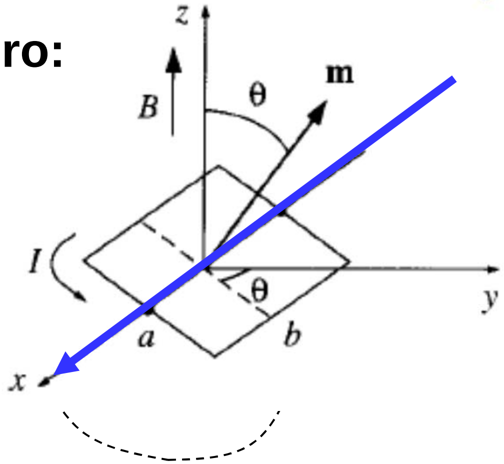
- Diamagnetic, if $\mu_r \leq 1$ → magnetization *opposite* to B
  - $\chi_m \sim -10^{-5}$
  - the orbital motion of the electrons
  - Copper, germanium, silver, gold
- Paramagnetic, if $\mu_r \geq 1$ → magnetization *parallel* to B
  - $\chi_m \sim 10^{-5}$
  - Magnetic dipole moments of the spinning electrons
  - Aluminum, magnesium, titanium and tungsten
- Ferromagnetic, if $\mu_r >> 1$ → retain magnetization even after B removed
  - $\chi_m >> 1$ (100~ 100,000)
  - Magnetized domains (strong coupling forces between the magnetic dipole moments of the atoms
  - Nickel, cobalt, iron (pure), mumetal

Magnetic Materials

	Diamagnetism	Paramagnetism	Ferromagnetism
Permanent magnetic dipole moment	No	Yes, but weak	Yes, and strong
Primary magnetization mechanism	Electron orbital magnetic moment	Electron spin magnetic moment	Magnetized domains
Direction of induced magnetic field (relative to external field)	Opposite	Same	Hysteresis [see Fig. 4-22]
Common substances	Bismuth, copper, diamond, gold, lead, mercury, silver, silicon	Aluminum, calcium, chromium, magnesium, niobium, platinum, tungsten	Iron, nickel, cobalt
Typical value of χ_m **Typical value of μ_r**	$\approx -10^{-5}$ ≈ 1	$\approx 10^{-5}$ ≈ 1	$\lvert\chi_m\rvert \gg 1$ and hysteretic $\lvert\mu_r\rvert \gg 1$ and hysteretic

6.1.2 Torques and Forces on Magnetic Dipoles

In a uniform field B, the net force on a current loop is zero:



$$\mathbf{F} = I\oint (d\mathbf{l} \times \mathbf{B}) = I\left(\oint d\mathbf{l}\right) \times \mathbf{B} = 0$$

But, Torque on a rectangular current loop is not zero:

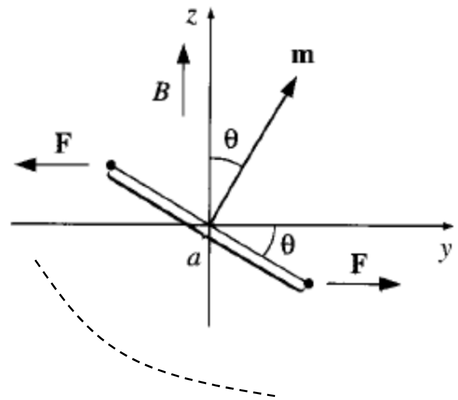
The forces on the "horizontal" sides are likewise equal and opposite (so the net *force* on the loop is zero), but they do generate a torque:

$$\mathbf{N} = aF\sin\theta\,\hat{\mathbf{x}} \qquad F = IbB$$

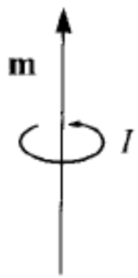
$$\mathbf{N} = IabB\sin\theta\,\hat{\mathbf{x}} = mB\sin\theta\,\hat{\mathbf{x}}$$

$$\mathbf{N} = \mathbf{m} \times \mathbf{B}$$

where $m = Iab$ is the magnetic dipole moment of the loop.



Magnetism is not due to magnetic monopoles.
but rather to moving electric charges;
→ magnetic dipoles are tiny current loops.



(Ampere model)

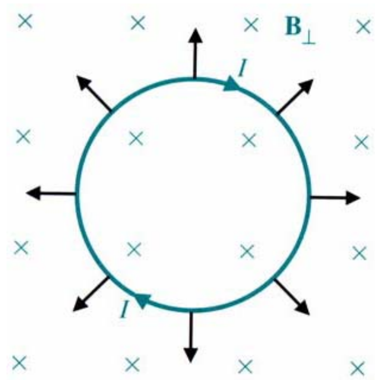
Magnetic Torques

- Small circular loop of radius *b* and carrying a current *I* in uniform **B**.

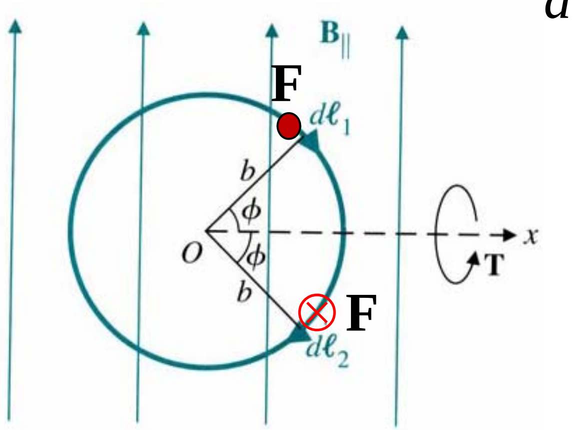
$$dl \propto \mathbf{a}_\theta, \mathbf{B} = B\mathbf{a}_z$$
$$\rightarrow \mathbf{F} \propto dl \times \mathbf{B}_\perp \propto \mathbf{a}_r, \text{ no net force}$$

$$\mathbf{B} = \mathbf{B}_\parallel + \mathbf{B}_\perp$$

$$\mathbf{F} \propto dl \times \left(\mathbf{B}_\parallel + \mathbf{B}_\perp\right) = dl \times \mathbf{B}_\parallel + \boxed{dl \times \mathbf{B}_\perp}$$



$$d\mathrm{N}\left(\text{torque}\right) = \mathbf{r} \times d\mathbf{F}, \quad \underline{d\mathbf{F} = I dl \times \mathbf{B}}$$



$$d\mathrm{N}_1 = \mathbf{a}_x\left(dF_1\right)\left(b\sin\phi\right) = \mathbf{a}_x\left(Idl_1 B_\parallel \sin\phi\right) b\sin\phi$$

$$dF = \left|d\mathbf{F}_1\right| = \left|d\mathbf{F}_2\right|, \; dl = \left|dl_1\right| = \left|dl_2\right| = b\,d\phi$$

$$d\mathrm{N} = 2\times\left(\mathbf{a}_x I b^2 B_\parallel \sin^2\phi\, d\phi\right)$$

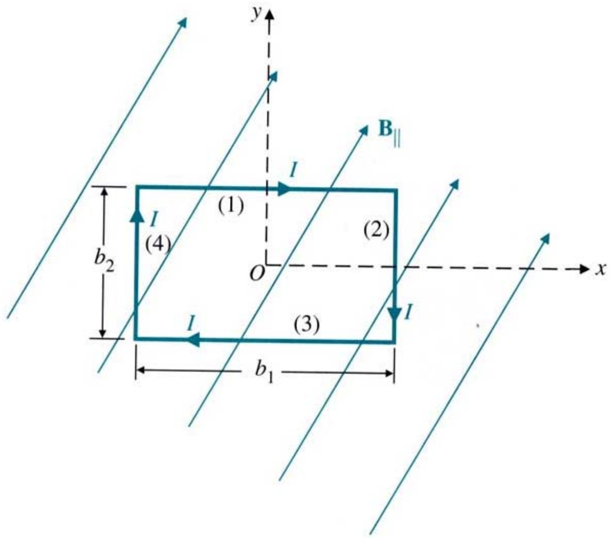
The total torque acting on the loop

$$\mathrm{N} = \int d\mathrm{N} = \mathbf{a}_x 2Ib^2 B_\parallel \int_0^\pi \sin^2\phi\, d\phi = \mathbf{a}_x I\left(\pi b^2\right) B_\parallel$$

Magnetic moment: $\longrightarrow$ $\boxed{\mathbf{m} = \mathbf{a}_n I\left(\pi b^2\right)}$ $\Rightarrow$ $\boxed{\mathrm{N} = \mathbf{a}_x I\left(\pi b^2\right) B_\parallel = \mathbf{m} \times \mathbf{B}}$

Magnetic Torques

- Example: The force and torque on the loop



$$\mathbf{B}_{\perp} = \mathbf{a}_z B_z,$$
$$\mathbf{B}_{\|} = \mathbf{a}_x B_x + \mathbf{a}_y B_y$$

$\mathbf{F} = dl \times \mathbf{B}_{\perp} \rightarrow$ sum of (1)~(4) forces, all directed toward center is zero. $\rightarrow$ no torgue is produced.

$$\mathbf{F}_1 = Ib_1 \mathbf{a}_x \times \left(\mathbf{a}_x B_x + \mathbf{a}_y B_y\right) = \mathbf{a}_z I b_1 B_y = -\mathbf{F}_3$$
$$\mathbf{F}_2 = Ib_2 \left(-\mathbf{a}_y\right) \times \left(\mathbf{a}_x B_x + \mathbf{a}_y B_y\right) = \mathbf{a}_z I b_2 B_x = -\mathbf{F}_4$$

Net force on the loop, $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 = 0,$ however, they result in a net torque.

$$\mathrm{N}_{13}\left(\mathbf{F}_1, \mathbf{F}_3\right) = \mathbf{a}_x 2 \times \frac{b_2}{2} I b_1 B_y = \mathbf{a}_x I b_1 b_2 B_y$$
$$\mathrm{N}_{24}\left(\mathbf{F}_2, \mathbf{F}_4\right) = \mathbf{a}_y I b_1 b_2 B_x$$

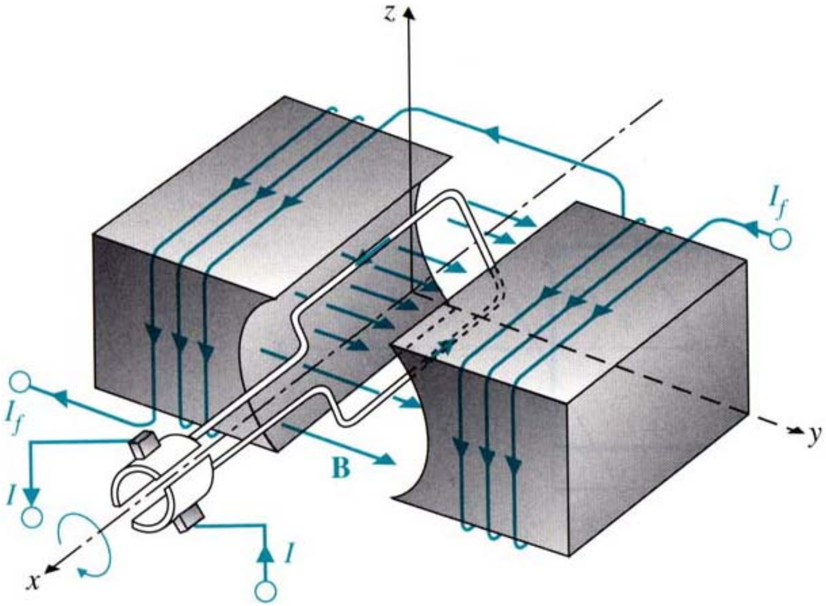
The total torque on the rectangular loop

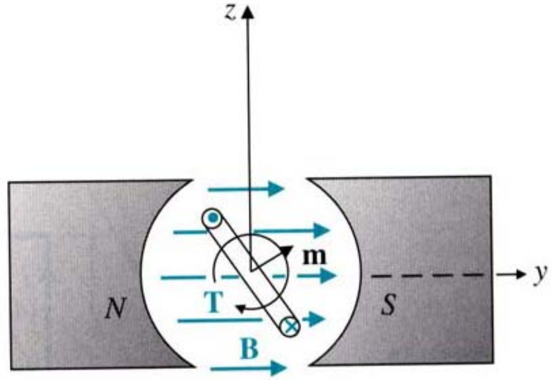
$$\mathrm{N} = \mathrm{N}_{13} + \mathrm{N}_{24} = I b_1 b_2 \left(\mathbf{a}_x B_y - \mathbf{a}_y B_x\right) \quad (\mathrm{N} \bullet \mathrm{m}) \quad \boxed{\leftarrow \mathbf{N} = \mathbf{m} \times \left(\mathbf{a}_x B_x + \mathbf{a}_y B_y\right)}$$

Magnetic Torques

- Direct current motors

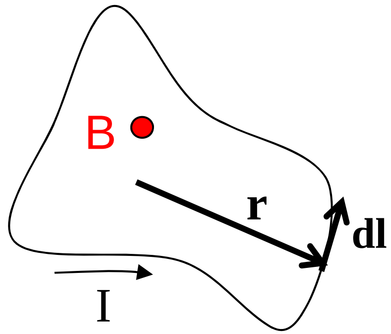
$$\mathbf{N} = \mathbf{m} \times \mathbf{B}$$





Torques and Forces on Magnetic Dipoles

Problem 6.2 Starting from the Lorentz force law, show that the torque on *any* steady current distribution (not just a square loop) **in a uniform field B is *m x B***.



$$\mathbf{F} = I\oint (d\mathbf{l}\times\mathbf{B})$$: Lorentz force law

$$d\mathbf{F} = I\, d\mathbf{l}\times\mathbf{B}$$

$$d\mathbf{N} = \mathbf{r}\times d\mathbf{F} = I\,\mathbf{r}\times(d\mathbf{l}\times\mathbf{B})$$

$$\mathbf{r}\times(d\mathbf{l}\times\mathbf{B}) + d\mathbf{l}\times(\mathbf{B}\times\mathbf{r}) + \mathbf{B}\times(\mathbf{r}\times d\mathbf{l}) = \mathbf{0} \longleftarrow \mathbf{A}\times(\mathbf{B}\times\mathbf{C})+\mathbf{B}\times(\mathbf{C}\times\mathbf{A})+\mathbf{C}\times(\mathbf{A}\times\mathbf{B})=0$$

$$d[\mathbf{r}\times(\mathbf{r}\times\mathbf{B})] = d\mathbf{r}\times(\mathbf{r}\times\mathbf{B}) + \mathbf{r}\times(d\mathbf{r}\times\mathbf{B})$$

$$d\mathbf{r} = d\mathbf{l}$$

$$d\mathbf{l}\times(\mathbf{B}\times\mathbf{r}) = \mathbf{r}\times(d\mathbf{l}\times\mathbf{B}) - d[\mathbf{r}\times(\mathbf{r}\times\mathbf{B})]$$

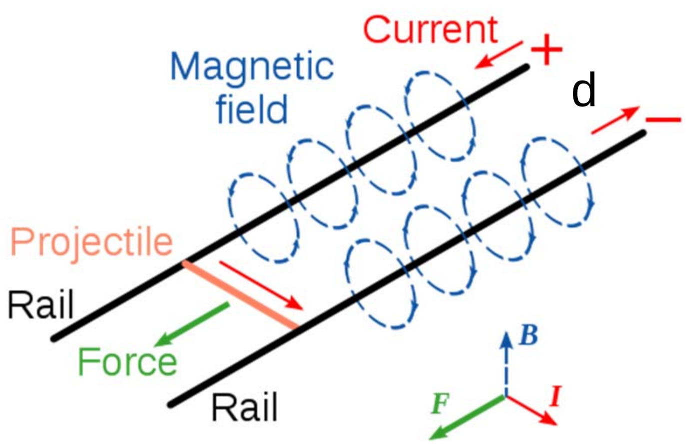
$$2\mathbf{r}\times(d\mathbf{l}\times\mathbf{B}) = d[\mathbf{r}\times(\mathbf{r}\times\mathbf{B})] - \mathbf{B}\times(\mathbf{r}\times d\mathbf{l})$$

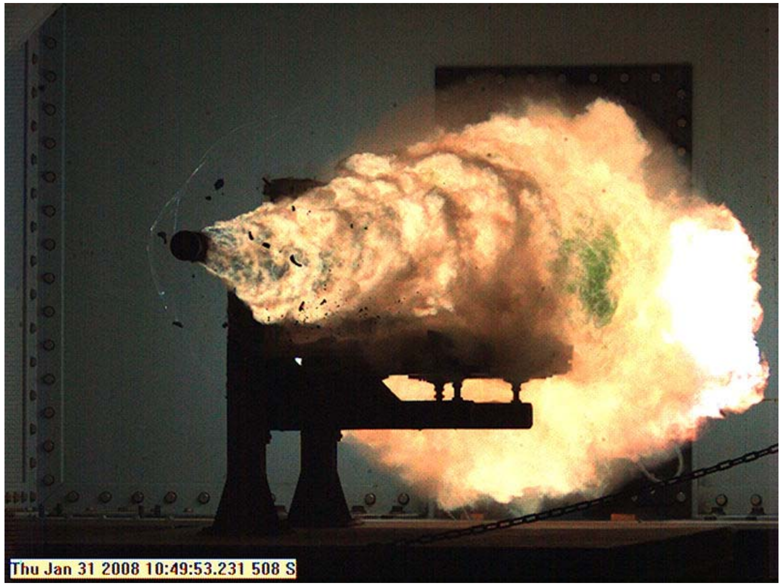
$$d\mathbf{N} = \tfrac{1}{2}I\,\{d[\mathbf{r}\times(\mathbf{r}\times\mathbf{B})] - \mathbf{B}\times(\mathbf{r}\times d\mathbf{l})\}$$

$$\mathbf{N} = \tfrac{1}{2}I\,\{\underbrace{\oint d[\mathbf{r}\times(\mathbf{r}\times\mathbf{B})]}_{0} - \mathbf{B}\times \underbrace{\oint(\mathbf{r}\times d\mathbf{l})}_{2\mathbf{a}}\}$$

$$\mathbf{N} = -I(\mathbf{B}\times\mathbf{a}) = \mathbf{m}\times\mathbf{B}.$$

Magnetic Forces: Rail Gun





$$\mathbf{B}(s) = \frac{1}{2}\frac{\mu_0 I}{2\pi s}\hat{\phi}$$

So, in the space between two semi-infinite wires separated by a distance, d, the magnitude of the field is:

$$B(s) = \frac{\mu_0 I}{4\pi}\left(\frac{1}{s} + \frac{1}{d-s}\right)$$

To obtain the average magnetic field in the space between the two wires, we assume that r is small compared with d and compute the following integral:

$$B_{\text{avg}} = \frac{1}{d}\int_r^{d-r} B(s)\mathrm{d}s = \frac{\mu_0 I}{4\pi d}\int_r^{d-r}\left(\frac{1}{s} + \frac{1}{d-s}\right)\mathrm{d}s = \frac{\mu_0 I}{2\pi d}$$

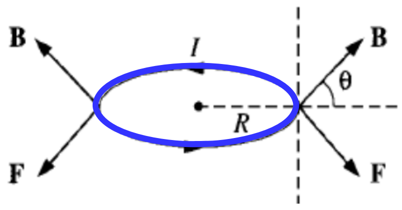
By the Lorentz force law, the magnetic force on a current-carrying wire is given by IdB, so since the width of the conductive projectile is d, we have

$$F = IdB_{\text{avg}} = \frac{\mu_0 I^2}{2\pi}$$

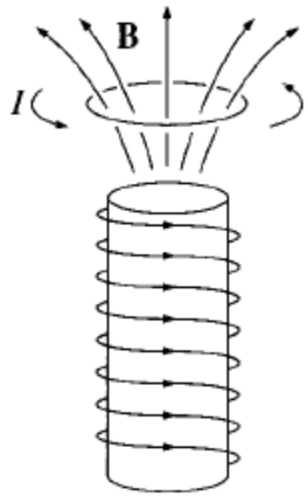
Torques and Forces on Magnetic Dipoles

In a uniform field B, the net force on a current loop is zero:

$$\mathbf{F} = I\oint (d\mathbf{l}\times\mathbf{B}) = I\left(\oint d\mathbf{l}\right)\times\mathbf{B} = 0$$



But, in a *nonuniform* field B, F is not zero:



For a circular wire of radius R ➔ $\mathbf{F} = I\oint (d\mathbf{l}\times\mathbf{B}) = 2\pi I R B\cos\theta.$

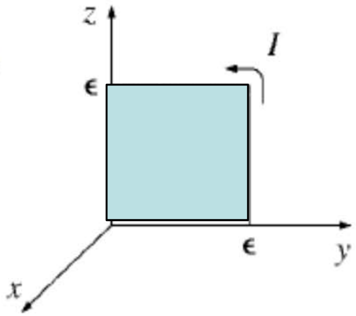
For an infinitesimal loop, with dipole moment **m,** in a field **B** ➔ $\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$

Problem 6.4

$$d\mathbf{F} = I\{(dy\,\hat{\mathbf{y}})\times\mathbf{B}(0,y,0) + (dz\,\hat{\mathbf{z}})\times\mathbf{B}(0,\epsilon,z) - (dy\,\hat{\mathbf{y}})\times\mathbf{B}(0,y,\epsilon) - (dz\,\hat{\mathbf{z}})\times\mathbf{B}(0,0,z)\}$$

$$= I\Big\{-(dy\,\hat{\mathbf{y}})\times\underbrace{[\mathbf{B}(0,y,\epsilon)-\mathbf{B}(0,y,0)]}_{\approx\,\epsilon\frac{\partial\mathbf{B}}{\partial z}} + (dz\,\hat{\mathbf{z}})\times\underbrace{[\mathbf{B}(0,\epsilon,z)-\mathbf{B}(0,0,z)]}_{\approx\,\epsilon\frac{\partial\mathbf{B}}{\partial y}}\Big\}$$

$$\Rightarrow I\epsilon^2\left\{\hat{\mathbf{z}}\times\frac{\partial\mathbf{B}}{\partial y} - \hat{\mathbf{y}}\times\frac{\partial\mathbf{B}}{\partial z}\right\}.\quad\left[\text{Note that } \int dy\,\frac{\partial\mathbf{B}}{\partial z}\Big|_{0,y,0}\approx\epsilon\,\frac{\partial\mathbf{B}}{\partial z}\Big|_{0,0,0}\text{ and } \int dz\,\frac{\partial\mathbf{B}}{\partial y}\Big|_{0,0,z}\approx\epsilon\,\frac{\partial\mathbf{B}}{\partial y}\Big|_{0,0,0}\right]$$



$$\mathbf{F} = m\left\{\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ 0 & 0 & 1\\ \frac{\partial B_x}{\partial y} & \frac{\partial B_y}{\partial y} & \frac{\partial B_z}{\partial y}\end{vmatrix} - \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ 0 & 1 & 0\\ \frac{\partial B_x}{\partial z} & \frac{\partial B_y}{\partial z} & \frac{\partial B_z}{\partial z}\end{vmatrix}\right\} = m\left\{\hat{\mathbf{y}}\frac{\partial B_x}{\partial y} - \hat{\mathbf{x}}\frac{\partial B_y}{\partial y} - \hat{\mathbf{x}}\frac{\partial B_z}{\partial z} + \hat{\mathbf{z}}\frac{\partial B_x}{\partial z}\right\}$$

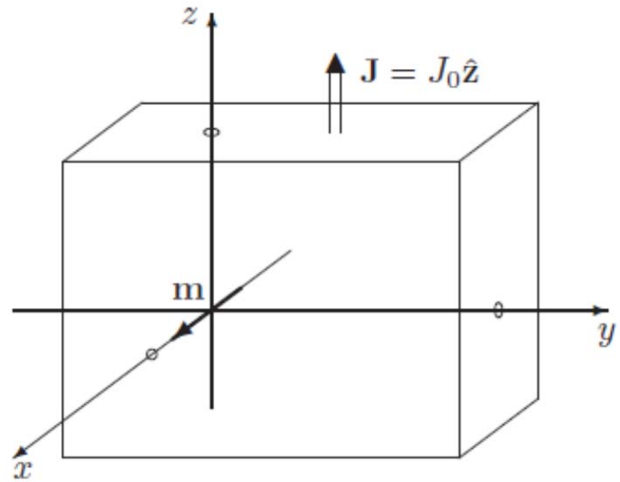
$$= m\left[\hat{\mathbf{x}}\frac{\partial B_x}{\partial x} + \hat{\mathbf{y}}\frac{\partial B_x}{\partial y} + \hat{\mathbf{z}}\frac{\partial B_x}{\partial z}\right]\quad\left(\text{using } \nabla\cdot\mathbf{B}=0 \text{ to write } \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = -\frac{\partial B_x}{\partial x}\right).$$

But $\mathbf{m}\cdot\mathbf{B} = mB_x$ (since $\mathbf{m} = m\hat{\mathbf{x}}$, here), so $\nabla(\mathbf{m}\cdot\mathbf{B}) = m\nabla(B_x) = m\left(\frac{\partial B_x}{\partial x}\hat{\mathbf{x}} + \frac{\partial B_x}{\partial y}\hat{\mathbf{y}} + \frac{\partial B_x}{\partial z}\hat{\mathbf{z}}\right)$.

⇨ $\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$.

Torques and Forces on Magnetic Dipoles

Problem 6.5 A uniform current density $\mathbf{J} = J_0$ z fills a slab straddling the *yz* plane, from $x = -a$ to $x = +a$. A magnetic dipole $\mathbf{m} = m_0$ x is situated at the origin.



(a) Find the force on the dipole

$$\oint \mathbf{B} \cdot d\mathbf{l} = Bl = \mu_0 I_{\text{enc}} \quad \longrightarrow \quad \mathbf{B} = \mu_0 J_0 x \hat{\mathbf{y}}$$ at a point x

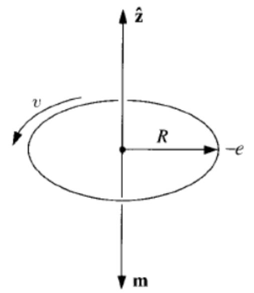
$$\mathbf{F} = \boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B}). \xrightarrow{\mathbf{m}\cdot\mathbf{B} = 0,} \mathbf{F} = \mathbf{0}.$$

(b) Do the same for a dipole pointing in the *y* direction: $\mathbf{m} = m_0$ y .

$$\mathbf{m}\cdot\mathbf{B} = m_0 \mu_0 J_0 x, \longrightarrow \mathbf{F} = m_0 \mu_0 J_0 \hat{\mathbf{x}}.$$

6.1.3 Effect of a Magnetic Field on Atomic Orbits

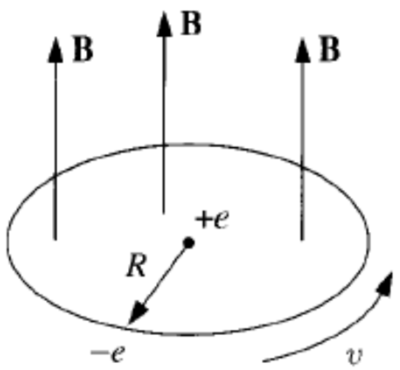
Electrons not only spin; they also revolve around the nucleus.
For simplicity, let's assume the orbit is a circle of radius *R.*



The orbital motion with the period $T = 2\pi R / v$ → look like a steady current

$$I = \frac{e}{T} = \frac{ev}{2\pi R}$$

The orbital dipole moment: $\mathbf{m} = I\pi R^2 = -\frac{1}{2} e v R\,\hat{\mathbf{z}}.$



When the atom is placed in a magnetic field,

→ The magnetic dipole is subject to a torque (m x B).
But it's a lot harder to tilt the entire orbit than it is the spin,
→ So the orbital contribution to paramagnetism is small.

➔ A more significant effect on the orbital motion is
The electron speeds up or slows down, depending on the orientation of B.

$$\frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} = m_e \frac{v^2}{R} \longrightarrow \frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} + e\bar{v}B = m_e\frac{\bar{v}^2}{R}$$

$$e\bar{v}B = \frac{m_e}{R}(\bar{v}^2 - v^2) = \frac{m_e}{R}(\bar{v}+v)(\bar{v}-v) \xrightarrow{\text{assuming the change } \Delta v = \bar{v} - v \text{ is small,}} \Delta v = \frac{eRB}{2m_e}.$$

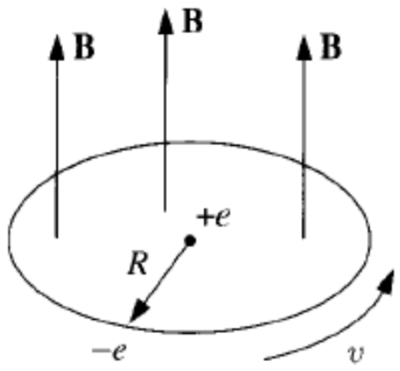
(An electron circling the other way would be slowed down.)

Effect of a Magnetic Field on Atomic Orbits

When the atom is placed in a magnetic field, the electron speeds up → $\Delta v = \frac{eRB}{2m_e}$.

➔ A change in orbital speed means a change in the dipole moment

$$\Delta \mathbf{m} = -\frac{1}{2}e(\Delta v)R\,\hat{\mathbf{z}} = -\frac{e^2R^2}{4m_e}\mathbf{B}$$



➔ Notice that *the change in* **m** is *opposite to the direction of* ***B***.

➔ this increments is ***antiparallel*** to the field.

➔ ***This is the mechanism responsible for diamagnetism.***

➔ It is a universal phenomenon, affecting all atoms.

6.1.4 Magnetization

In the presence of a magnetic field, matter becomes *magnetized,*

(l) Paramagnetism ➔ the dipoles associated with the spins of unpaired electrons experience a torque tending to line them up parallel to the field)

(2) Diamagnetism ➔ the orbital speed of the electrons is altered in such a way as to change the orbital dipole moment in a direction opposite to the field).

Problem 6.6 Of the following materials, which would you expect to be paramagnetic and which diamagnetic? Aluminum, copper, copper chloride (CuCl2), carbon, lead, nitrogen (N2), salt (NaCl), sodium, sulfur, water.

→ Aluminum, copper, copper chloride, and sodium → an odd number of electrons → paramagnetic.
→ The rest (having an even number) should be diamagnetic.

The state of magnetic polarization ➔ Magnetization, M

M = magnetic dipole moment per unit volume

$$\boxed{\mathbf{M} = \lim_{\Delta\upsilon \to 0} \frac{1}{\Delta\upsilon} \sum_k \mathbf{m}_k \quad (\mathrm{A/m})}$$

➔ Analogous to the polarization **P** in electrostatics

Chapter 6. Magnetic Fields in Matter

6.1 Magnetization . . . .

6.1.1 Diamagnets, Paramagnets, Ferromagnets . . . .

6.1.2 Torques and Forces on Magnetic Dipoles . . . .

6.1.3 Effect of a Magnetic Field on Atomic Orbits . . . .

6.1.4 Magnetization . . . .

6.2 The Field of a Magnetized Object . . . .

6.2.1 Bound Currents . . . .

6.2.2 Physical Interpretation of Bound Currents . . . .

6.2.3 The Magnetic Field Inside Matter . . . .

6.3 The Auxiliary Field H . . . .

6.3.1 Ampère's law in Magnetized Materials . . . .

6.3.2 A Deceptive Parallel . . . .

6.3.3 Boundary Conditions . . . .

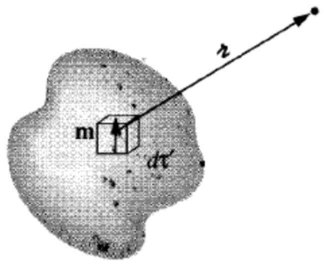
6.4 Linear and Nonlinear Media . . . .

6.4.1 Magnetic Susceptibility and Permeability . . . .

6.4.2 Ferromagnetism . . . .

6.2 The Field of a Magnetized Object

6.2.1 Bound Currents



Vector potential of a single dipole **m** was $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$

Total vector potential in a magnetized object is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{M}(\mathbf{r}')\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'$$

$$\nabla'\frac{1}{\mathscr{r}} = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\left[\mathbf{M}(\mathbf{r}')\times\left(\nabla'\frac{1}{\mathscr{r}}\right)\right]d\tau'$$

$$= \frac{\mu_0}{4\pi}\left\{\int\frac{1}{\mathscr{r}}[\nabla'\times\mathbf{M}(\mathbf{r}')]\,d\tau' - \int\nabla'\times\left[\frac{\mathbf{M}(\mathbf{r}')}{\mathscr{r}}\right]d\tau'\right\}$$ ← integrating by parts

$$= \frac{\mu_0}{4\pi}\int\frac{1}{\mathscr{r}}[\nabla'\times\mathbf{M}(\mathbf{r}')]\,d\tau' + \frac{\mu_0}{4\pi}\oint\frac{1}{\mathscr{r}}[\mathbf{M}(\mathbf{r}')\times d\mathbf{a}']$$

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}}\frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}}\,d\tau' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}}\frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}}\,da'}$$

$\mathbf{J}_b = \nabla\times\mathbf{M}$ → **volume current density**

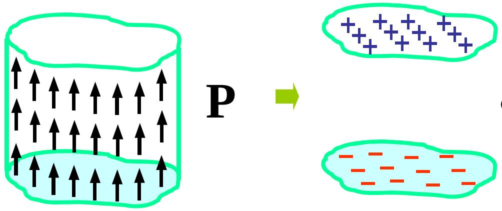
➔ Magnetization current density

$\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}}$ → **Surface current density**

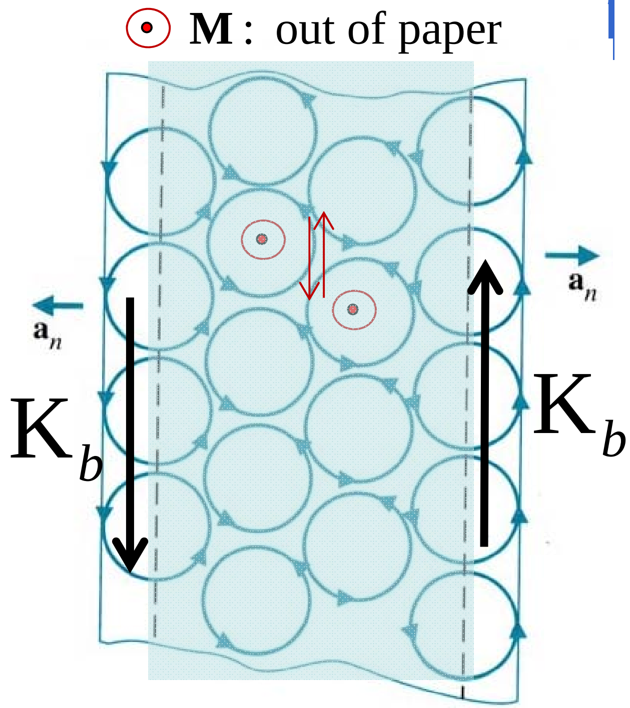
Magnetization current density

- Magnetization surface current density

$$\boxed{\mathbf{K}_b = \mathbf{M} \times \mathbf{n} \quad (\text{A/m})}$$



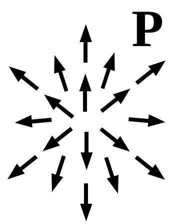
$\sigma_b = \mathbf{P} \bullet \mathbf{n}$



- Magnetization volume current density

$$\boxed{\mathbf{J}_b = \nabla \times \mathbf{M} \quad (\text{A/m}^2)}$$

$\rho_\upsilon = -\nabla \bullet \mathbf{P}$

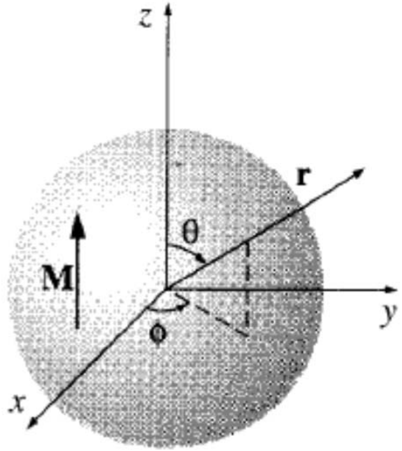


If $\mathbf{M}$ is uniform (space invariant) inside, the current of the neighboring atomic dipoles will cancelled averywhere.

On the surface of the material, $K_b = \mathbf{M} \times \text{n} \neq 0$.
No net current in the interior ($\mathbf{J}_b = 0$),
or, $\nabla \times \mathbf{M} = \mathbf{J}_b = 0$ if $\mathbf{M}$ is space invariant.

Bound Currents

Example 6.1 Find the magnetic field of a uniformly magnetized sphere.



Choosing the ***z* axis** along the direction of **M**,

$$\mathbf{J}_b = \nabla \times \mathbf{M} = 0, \quad \mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = M \sin\theta\, \hat{\boldsymbol{\phi}}.$$

The surface current density corresponds to a rotating spherical shell, of uniform surface charge σ,

$$\mathbf{K} = \sigma \mathbf{v} = \sigma \omega R \sin\theta\, \hat{\boldsymbol{\phi}}. \quad \longrightarrow \quad \sigma R \boldsymbol{\omega} \to \mathbf{M}$$

(Example 5.11) $$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{r}')}{\mathscr{r}}\, da'$$

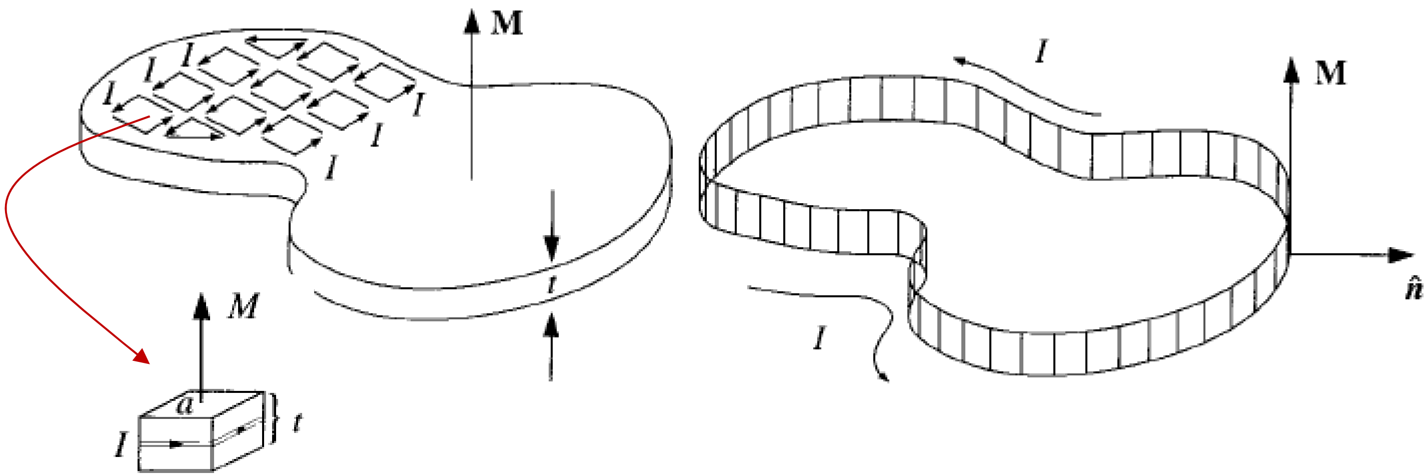
$$\mathbf{A}(r, \theta, \phi) = \begin{cases} \dfrac{\mu_0 R \omega \sigma}{3} r \sin\theta\, \hat{\boldsymbol{\phi}}, & (r \leq R), \\[2ex] \dfrac{\mu_0 R^4 \omega \sigma}{3} \dfrac{\sin\theta}{r^2}\, \hat{\boldsymbol{\phi}}, & (r \geq R). \end{cases}$$

Inside the sphere, $\mathbf{B} = \nabla \times \mathbf{A} = \frac{2}{3} \mu_0 \sigma R \boldsymbol{\omega}$

$$\sigma R \boldsymbol{\omega} \to \mathbf{M} \quad \longrightarrow \quad \mathbf{B} = \frac{2}{3} \mu_0 \mathbf{M},$$

➔ *the internal field is uniform.*

6.2.2 Physical Interpretation of Bound Currents



For uniformly magnetized material, all the internal currents cancel.

$$J_b = \nabla \times M = 0$$

However, at the edge there is a current.

$$m = \mathbf{M}(at) \ \& \ m = Ia$$

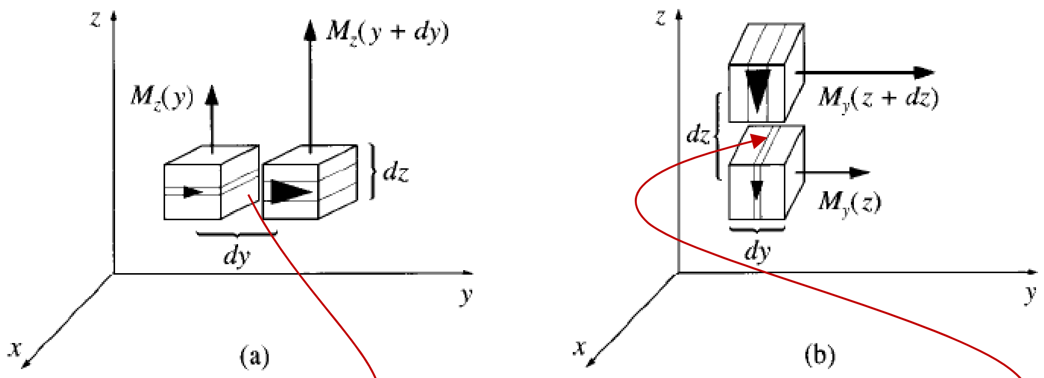
$$\rightarrow I = Mt$$

$$\Rightarrow \text{ surface current: } K_b = I / t = M$$

$$\boxed{K_b = \mathbf{M} \times \mathrm{n}}$$

Physical Interpretation of Bound Currents

For a ***nonuniformly magnetized material***, the internal currents no longer cancel.



(a) (b)

$$I = Mt$$

$$\rightarrow I_x = \left[M_z(y+dy) - M_z(y)\right]dz$$

$$\rightarrow I_x = \frac{\partial M_z}{\partial y}dydz$$

$$\Rightarrow (J_b)_x = \frac{\partial M_z}{\partial y}$$

$$I = Mt$$

$$\rightarrow I_x = \left[-M_y(z+dz) + M_y(z)\right]dy$$

$$\rightarrow I_x = -\frac{\partial M_y}{\partial z}dzdy$$

$$\Rightarrow (J_b)_x = -\frac{\partial M_y}{\partial z}$$

$$\Rightarrow (J_b)_x = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z} \Rightarrow \boxed{\mathbf{J}_b = \nabla \times \mathbf{M} \quad \left(\mathrm{A/m^2}\right)}$$

NOTE: $\mathbf{J}_b$ should obey the conservation law: $\boxed{\nabla \cdot \mathbf{J}_b = 0}$

6.2.3 The Magnetic Field Inside Matter

The effect of magnetization is to establish bound currents $\mathbf{J}_b = \nabla \times \mathbf{M}$ and $\mathbf{K}_b = \mathbf{M} \times \mathbf{n}$

→ The field due to magnetization of the medium is just the field produced by these bound currents.

Now, we can use the **Ampere's law** to find the Magnetic field from **(bound currents) + (free currents).**

$$\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f.$$

$$\frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \mathbf{J} = \mathbf{J}_f + \mathbf{J}_b = \mathbf{J}_f + (\nabla \times \mathbf{M})$$