Universidad de Valparaíso
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Prueba Módulo II
Mecánica Cuántica II
Licenciatura en Física - 2022

Problema I

Exercise 5.13
Consider a system which is described by the state

$$\psi(\theta, \varphi) = \sqrt{\frac{3}{8}} Y_{11}(\theta, \varphi) + \sqrt{\frac{1}{8}} Y_{10}(\theta, \varphi) + A Y_{1,-1}(\theta, \varphi),$$

where A is a real constant

(a) Calculate A so that $|\psi\rangle$ is normalized.
(b) Find $\hat{L}_+ \psi(\theta, \varphi)$.
(c) Calculate the expectation values of $\hat{\vec{L}}^2$ in the state $|\psi\rangle$.
(d) Find the probability associated with a measurement that gives zero for the z-component of the angular momentum.
(e) Calculate $\langle \Phi | \hat{L}_z | \psi \rangle$ where :

$$\Phi(\theta, \varphi) = \sqrt{\frac{8}{15}} Y_{21}(\theta, \varphi) + \sqrt{\frac{4}{15}} Y_{10}(\theta, \varphi) + \sqrt{\frac{3}{15}} Y_{2,-1}(\theta, \varphi).$$

Problema II

Exercise 5.34
Find the energy levels of a spin $\frac{5}{2}$ particle whose Hamiltonian is given by

$$\hat{H} = \frac{\varepsilon_0}{\hbar^2}(\hat{S}_x^2 + \hat{S}_y^2) + \frac{\varepsilon_0}{\hbar}\hat{S}_z,$$

where ε_0 is a constant having the dimensions of energy. Are the energy levels degenerate?

Problema III

Exercise 7.29
Let the Hamiltonian of two nonidentical spin $\frac{1}{2}$ particles be

$$\hat{H} = \frac{\varepsilon_1}{\hbar^2}(\vec{\hat{S}}_1 + \vec{\hat{S}}_2) \cdot \vec{\hat{S}}_1 - \frac{\varepsilon_2}{\hbar}(\hat{S}_{1_z} + \hat{S}_{2_z}),$$

where ε_1 and ε_2 are constants having the dimensions of energy. Find the energy levels and their degeneracies.

Probl. I /

$$\psi(\theta,\varphi) = \sqrt{\frac{3}{8}}\, Y_{11}(\theta,\varphi) + \sqrt{\frac{1}{8}}\, Y_{10}(\theta,\varphi) + A\, Y_{1,-1}(\theta,\varphi)$$

$\Downarrow$ VERSIÓN matricial

$$|\psi\rangle = \sqrt{\frac{3}{8}}\, |1,1\rangle + \sqrt{\frac{1}{8}}\, |1,0\rangle + A\, |1,-1\rangle$$

a) Se debe cumplir que:

$$\langle\psi|\psi\rangle = 1$$

$\Downarrow$

$$\frac{3}{8} + \frac{1}{8} + |A|^2 = 1 \implies |A|^2 = \frac{1}{2}$$

sin perder generalidad suponemos $A \in \mathbb{R}$

$\Downarrow$

$|A|^2 = A^2$

$$\therefore \quad A = \sqrt{\frac{1}{2}} = \sqrt{\frac{4}{8}} \quad //$$

b) Si $\hat{L}_+ |l,m\rangle = \hbar\,[l(l+1) - m(m+1)]^{1/2}\, |l,m+1\rangle$

$$\therefore \quad \hat{L}_+ |1,1\rangle = 0 \quad ; \quad \hat{L}_+ |1,0\rangle = \hbar\sqrt{2}\, |1,1\rangle$$

$$\hat{L}_+ |1,-1\rangle = \hbar\sqrt{2}\, |1,0\rangle$$

$\boxed{2}$

$\therefore \hat{L}_+|\psi\rangle = \sqrt{\frac{1}{4}}\hbar|1,1\rangle + \hbar|1,0\rangle$ //

c) En el estado $|\psi\rangle$ el único valor que puede asumir $|\vec{L}|^2$ es

$$|\vec{L}|^2 = L^2 = l(l+1)\hbar^2\Big|_{l=1} = 2\hbar^2 \quad //$$

d)

$$\text{Prob}(\underset{\substack{\uparrow \\ \text{no especifica} \\ l.}}{L_z=0}) = \sum_{l=0}^{\infty} |\langle l,0|\psi\rangle|^2 = \sum_{l=0}^{\infty} \frac{1}{8} |\underbrace{\langle l,0|1,0\rangle}_{\delta_{l1}}|^2$$

$$= \frac{1}{8} \quad //$$

e)

$$\langle\phi|\hat{L}_z|\psi\rangle = \sqrt{\frac{4}{15}}\underbrace{\langle 1,0|\hat{L}_z|1,0\rangle}_{m\hbar \left(\substack{\text{con} \\ m=0}\right)}\sqrt{\frac{1}{8}}$$

$$= 0 \quad //$$

Probl. II /

$$\hat{H} = \frac{\varepsilon_0}{\hbar^2}\left(\hat{S}_x^2 + \hat{S}_y^2\right) + \frac{\varepsilon_0}{\hbar}\hat{S}_z$$

pero $\hat{S}_x^2 + \hat{S}_y^2 = \hat{S}^2 - \hat{S}_z^2$

$$\therefore \quad \hat{H} = \frac{\varepsilon_0}{\hbar^2}\left(\hat{S}^2 - \hat{S}_z^2\right) + \frac{\varepsilon_0}{\hbar}\hat{S}_z$$

La base $\{|s, m_s\rangle\}$ es una base adecuada que diagonaliza el Hamiltoniano, ya:

$$\left.\begin{aligned} \hat{S}^2|s, m_s\rangle &= s(s+1)\hbar^2|s, m_s\rangle \\ \text{y} \quad \hat{S}_z|s, m_s\rangle &= m_s\hbar|s, m_s\rangle \end{aligned}\right\} [\hat{S}^2, \hat{S}_z] = 0$$

$$\therefore \quad \hat{H}|s, m_s\rangle = \frac{\varepsilon_0}{\hbar^2}\left(\hat{S}^2 - \hat{S}_z^2\right)|s, m_s\rangle + \frac{\varepsilon_0 \hat{S}_z}{\hbar}|s, m_s\rangle$$

$$= \left[\frac{\varepsilon_0}{\hbar^2}\left[s(s+1)\hbar^2 - m_s^2\hbar^2\right] + \frac{\varepsilon_0}{\hbar}m_s\hbar\right]|s, m_s\rangle$$

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$\therefore$ el espectro de energía se obtiene a partir de:

$$E = E_{s,m_s} = \varepsilon_0 \left[s(s+1) - m_s^2 + m_s \right]$$

$$= \varepsilon_0 \left[s(s+1) - m_s(m_s - 1) \right]$$

si $s = 5/2 \Longrightarrow$ 6 estados

$\Downarrow$

	Energía
$\vert 5/2, -5/2 \rangle \longrightarrow$	0
$\vert 5/2, -3/2 \rangle \longrightarrow$	$5\varepsilon_0$
$\vert 5/2, -1/2 \rangle \longrightarrow$	$8\varepsilon_0$
$\vert 5/2, 1/2 \rangle \longrightarrow$	$9\varepsilon_0$
$\vert 5/2, 3/2 \rangle \longrightarrow$	$8\varepsilon_0$
$\vert 5/2, 5/2 \rangle \longrightarrow$	$5\varepsilon_0$

$\uparrow$

4 estados degenerados

Probl. 3 /

$$\hat{H} = \frac{\varepsilon_1}{\hbar^2}\left(\vec{\hat{S}}_1 + \vec{\hat{S}}_2\right)\cdot\vec{\hat{S}}_1 - \frac{\varepsilon_2}{\hbar}\left(\hat{S}_{1z} + \hat{S}_{2z}\right)$$

$$= \frac{\varepsilon_1}{\hbar^2}\left(\hat{S}_1^2 + \vec{\hat{S}}_1\cdot\vec{\hat{S}}_2\right) - \frac{\varepsilon_2}{\hbar^2}\left(\hat{S}_{1z} + \hat{S}_{2z}\right)$$

Sea $\vec{J} = \vec{S}_1 + \vec{S}_2 \Longrightarrow J_z = J_{1z} + J_{2z}$

$$\Downarrow \qquad\qquad \Downarrow$$

$$J^2 = S_1^2 + S_2^2 + 2\vec{S}_1\cdot\vec{S}_2 \qquad \hat{J}_z = \hat{J}_{1z} + \hat{J}_{2z}$$

$$\Downarrow$$

$$\hat{J}^2 = \hat{S}_1^2 + \hat{S}_2^2 + 2\vec{\hat{S}}_1\cdot\vec{\hat{S}}_2$$

$$\Downarrow$$

$$\vec{\hat{S}}_1\cdot\vec{\hat{S}}_2 = \frac{1}{2}\left(\hat{J}^2 - \hat{S}_1^2 - \hat{S}_2^2\right)$$

"Buena base" $\Longrightarrow \{|j, m_j, s_1, s_2\rangle\}$

Set de operadores compatibles

$$\begin{cases} \hat{J}^2|\ldots\rangle = j(j+1)\hbar^2|\ldots\rangle \\ \hat{S}_1^2|\ldots\rangle = s_1(s_1+1)\hbar^2|\ldots\rangle \\ \hat{S}_2^2|\ldots\rangle = s_2(s_2+1)\hbar^2|\ldots\rangle \\ \hat{J}_z|\ldots\rangle = m_j\hbar|\ldots\rangle \end{cases}$$

Todos conmutan entre sí.

$\therefore$ reescribimos $\hat{H}$:

$$\hat{H} = \frac{\varepsilon_1}{\hbar^2}\left(\hat{S}_1^2 + \frac{1}{2}\left(\hat{J}^2 - \hat{S}_1^2 - \hat{S}_2^2\right)\right) - \frac{\varepsilon_2}{\hbar}\hat{J}_z$$

$$\hat{H} = \frac{\varepsilon_1}{2\hbar^2}\left(\hat{J}^2 + \hat{S}_1^2 - \hat{S}_2^2\right) - \frac{\varepsilon_2}{\hbar}\hat{J}_z$$

$\rightarrow m_j = -j, \ldots, j$

luego si $s_1 = s_2 = 1/2 \Rightarrow j = 0, 1$

$\Downarrow$

$$E = E_{j,m_j} = \frac{\varepsilon_1}{2} j(j+1) - \varepsilon_2 m_j \;//$$

Estado		Energía
$\|0,0,1/2,1/2\rangle$	$\longrightarrow$	0
$\|1,-1,1/2,1/2\rangle$	$\longrightarrow$	$E_1 + E_2$
$\|1,0,1/2,1/2\rangle$	$\longrightarrow$	E_1
$\|1,1,1/2,1/2\rangle$	$\longrightarrow$	$E_1 - E_2$

No hay degenerancia //