

Chapter 3. Special Techniques for Calculating Potentials

Given a stationary charge distribution $\rho(\vec{r})$ we can, in principle, calculate the electric field:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\Delta\hat{r}}{(\Delta r)^2} \rho(\vec{r}') d\tau'$$

where $\Delta\vec{r} = \vec{r}' - \vec{r}$. This integral involves a vector as an integrand and is, in general, difficult to calculate. In most cases it is easier to evaluate first the electrostatic potential V which is defined as

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{\Delta r} \rho(\vec{r}') d\tau'$$

since the integrand of the integral is a scalar. The corresponding electric field \vec{E} can then be obtained from the gradient of V since

$$\vec{E} = -\vec{\nabla} V$$

The electrostatic potential V can only be evaluated analytically for the simplest charge configurations. In addition, in many electrostatic problems, conductors are involved and the charge distribution ρ is not known in advance (only the total charge on each conductor is known).

A better approach to determine the electrostatic potential is to start with **Poisson's equation**

$$\vec{\nabla}^2 V = -\frac{\rho}{\epsilon_0}$$

Very often we only want to determine the potential in a region where $\rho = 0$. In this region Poisson's equation reduces to **Laplace's equation**

$$\vec{\nabla}^2 V = 0$$

There are an infinite number of functions that satisfy Laplace's equation and the appropriate solution is selected by specifying the appropriate **boundary conditions**. This Chapter will concentrate on the various techniques that can be used to calculate the solutions of Laplace's equation and on the boundary conditions required to uniquely determine a solution.

3.1. Solutions of Laplace's Equation in One-, Two, and Three Dimensions

3.1.1. Laplace's Equation in One Dimension

In one dimension the electrostatic potential V depends on only one variable x . The electrostatic potential $V(x)$ is a solution of the one-dimensional Laplace equation

$$\frac{d^2V}{dx^2} = 0$$

The general solution of this equation is

$$V(x) = sx + b$$

where s and b are arbitrary constants. These constants are fixed when the value of the potential is specified at two different positions.

Example

Consider a one-dimensional world with two point conductors located at $x = 0$ m and at $x = 10$ m. The conductor at $x = 0$ m is grounded ($V = 0$ V) and the conductor at $x = 10$ m is kept at a constant potential of 200 V. Determine V .

The boundary conditions for V are

$$V(0) = b = 0V$$

and

$$V(10) = 10s + b = 200V$$

The first boundary condition shows that $b = 0$ V. The second boundary condition shows that $s = 20$ V/m. The electrostatic potential for this system of conductors is thus

$$V(x) = 20x$$

The corresponding electric field can be obtained from the gradient of V

$$E(x) = -\frac{dV(x)}{dx} = -20 \text{ V / m}$$

The boundary conditions used here, can be used to specify the electrostatic potential between $x = 0$ m and $x = 10$ m but not in the region $x < 0$ m and $x > 10$ m. If the solution obtained here was the general solution for all x , then V would approach infinity when x approaches infinity and V would approach minus infinity when x approaches minus infinity. The boundary conditions therefore provide the information necessary to uniquely define a solution to Laplace's equation, but they also define the boundary of the region where this solution is valid (in this example $0 \text{ m} < x < 10 \text{ m}$).

The following properties are true for any solution of the one-dimensional Laplace equation:

Property 1:

$V(x)$ is the average of $V(x + R)$ and $V(x - R)$ for any R as long as $x + R$ and $x - R$ are located in the region between the boundary points. This property is easy to proof:

$$\frac{V(x + r) + V(x - R)}{2} = \frac{s(x + R) + b + s(x - R) + b}{2} = sx + b = V(x)$$

This property immediately suggests a powerful analytical method to determine the solution of Laplace's equation. If the boundary values of V are

$$V(x = a) = V_a$$

and

$$V(x = b) = V_b$$

then property 1 can be used to determine the value of the potential at $(a + b)/2$:

$$V\left(x = \frac{a + b}{2}\right) = \frac{1}{2}[V_a + V_b]$$

Next we can determine the value of the potential at $x = (3a + b)/4$ and at $x = (a + 3b)/4$:

$$V\left(x = \frac{3a + b}{2}\right) = \frac{1}{2}\left[V(x = a) + V\left(x = \frac{a + b}{2}\right)\right] = \frac{1}{2}\left[\frac{3}{2}V_a + \frac{1}{2}V_b\right]$$

$$V\left(x = \frac{a + 3b}{2}\right) = \frac{1}{2}\left[V\left(x = \frac{a + b}{2}\right) + V(x = b)\right] = \frac{1}{2}\left[\frac{1}{2}V_a + \frac{3}{2}V_b\right]$$

This process can be repeated and V can be calculated in this manner at any point between $x = a$ and $x = b$ (but not in the region $x > b$ and $x < a$).

Property 2:

The solution of Laplace's equation can not have local maxima or minima. Extreme values must occur at the end points (the boundaries). This is a direct consequence of property 1.

Property 2 has an important consequence: a charged particle can not be held in stable equilibrium by electrostatic forces alone (**Earnshaw's Theorem**). A particle is in a stable equilibrium if it is located at a position where the potential has a minimum value. A small displacement away from the equilibrium position will increase the electrostatic potential of the particle, and a restoring force will try to move the particle back to its equilibrium position. However, since there can be no local maxima or minima in the electrostatic potential, the particle can not be held in stable equilibrium by just electrostatic forces.

3.1.2. Laplace's Equation in Two Dimensions

In two dimensions the electrostatic potential depends on two variables x and y . Laplace's equation now becomes

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

This equation does not have a simple analytical solution as the one-dimensional Laplace equation does. However, the properties of solutions of the one-dimensional Laplace equation are also valid for solutions of the two-dimensional Laplace equation:

Property 1:

The value of V at a point (x, y) is equal to the average value of V around this point

$$V(x, y) = \frac{1}{2\pi R} \oint_{\text{Circle}} V R d\phi$$

where the path integral is along a circle of arbitrary radius, centered at (x, y) and with radius R .

Property 2:

V has no local maxima or minima; all extremes occur at the boundaries.

3.1.3. Laplace's Equation in Three Dimensions

In three dimensions the electrostatic potential depends on three variables x , y , and z . Laplace's equation now becomes

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

This equation does not have a simple analytical solution as the one-dimensional Laplace equation does. However, the properties of solutions of the one-dimensional Laplace equation are also valid for solutions of the three-dimensional Laplace equation:

Property 1:

The value of V at a point (x, y, z) is equal to the average value of V around this point

$$V(x, y, z) = \frac{1}{4\pi R^2} \oint_{\text{Sphere}} V R^2 \sin\theta \, d\theta \, d\phi$$

where the surface integral is across the surface of a sphere of arbitrary radius, centered at (x, y, z) and with radius R .

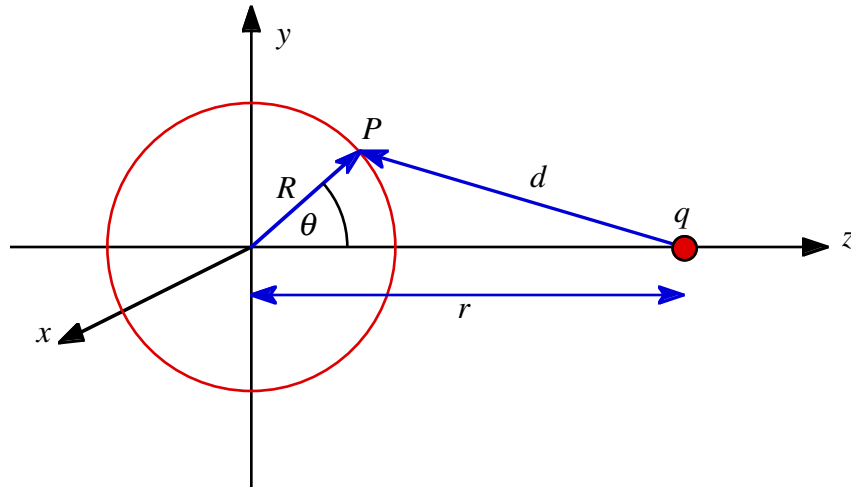


Figure 3.1. Proof of property 1.

To proof this property of V consider the electrostatic potential generated by a point charge q located on the z axis, a distance r away from the center of a sphere of radius R (see Figure 3.1). The potential at P , generated by charge q , is equal to

$$V_P = \frac{1}{4\pi\epsilon_0} \frac{q}{d}$$

where d is the distance between P and q . Using the cosine rule we can express d in terms of r , R and θ

$$d^2 = r^2 + R^2 - 2rR \cos \theta$$

The potential at P due to charge q is therefore equal to

$$V_P = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{r^2 + R^2 - 2rR \cos \theta}}$$

The average potential on the surface of the sphere can be obtained by integrating V_P across the surface of the sphere. The average potential is equal to

$$\begin{aligned} V_{average} &= \frac{1}{4\pi R^2} \int_{\text{Sphere}} V_P R^2 \sin \theta d\theta d\phi = \frac{1}{4\pi} \frac{1}{4\pi\epsilon_0} \int \frac{q}{\sqrt{r^2 + R^2 - 2rR \cos \theta}} 2\pi \sin \theta d\theta = \\ &= \frac{q}{8\pi\epsilon_0} \frac{\sqrt{r^2 + R^2 - 2rR \cos \theta}}{rR} \Bigg|_0^\pi = \frac{q}{8\pi\epsilon_0} \left(\frac{r+R}{rR} - \frac{r-R}{rR} \right) = \frac{1}{4\pi\epsilon_0} \frac{q}{r} \end{aligned}$$

which is equal to the potential due to q at the center of the sphere. Applying the principle of superposition it is easy to show that the average potential generated by a collection of point charges is equal to the net potential they produce at the center of the sphere.

Property 2:

The electrostatic potential V has no local maxima or minima; all extremes occur at the boundaries.

Example: Problem 3.3

Find the general solution to Laplace's equation in spherical coordinates, for the case where V depends only on r . Then do the same for cylindrical coordinates.

Laplace's equation in spherical coordinates is given by

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0$$

If V is only a function of r then

$$\frac{\partial V}{\partial \theta} = 0$$

and

$$\frac{\partial V}{\partial \phi} = 0$$

Therefore, Laplace's equation can be rewritten as

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) = 0$$

The solution V of this second-order differential equation must satisfy the following first-order differential equation:

$$r^2 \frac{\partial V}{\partial r} = a = \text{constant}$$

This differential equation can be rewritten as

$$\frac{\partial V}{\partial r} = \frac{a}{r^2}$$

The general solution of this first-order differential equation is

$$V(r) = -\frac{a}{r} + b$$

where b is a constant. If $V = 0$ at infinity then b must be equal to zero, and consequently

$$V(r) = -\frac{a}{r}$$

Laplace's equation in cylindrical coordinates is

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

If V is only a function of r then

$$\frac{\partial V}{\partial \phi} = 0$$

and

$$\frac{\partial V}{\partial z} = 0$$

Therefore, Laplace's equation can be rewritten as

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) = 0$$

The solution V of this second-order differential equation must satisfy the following first-order differential equation:

$$r \frac{\partial V}{\partial r} = a = \text{constant}$$

This differential equation can be rewritten as

$$\frac{\partial V}{\partial r} = \frac{a}{r}$$

The general solution of this first-order differential equation is

$$V(r) = a \ln(r) + b$$

where b is a constant. The constants a and b are determined by the boundary conditions.

3.1.4. Uniqueness Theorems

Consider a volume (see Figure 3.2) within which the charge density is equal to zero. Suppose that the value of the electrostatic potential is specified at every point on the surface of this volume. The **first uniqueness theorem** states that in this case the solution of Laplace's equation is uniquely defined.

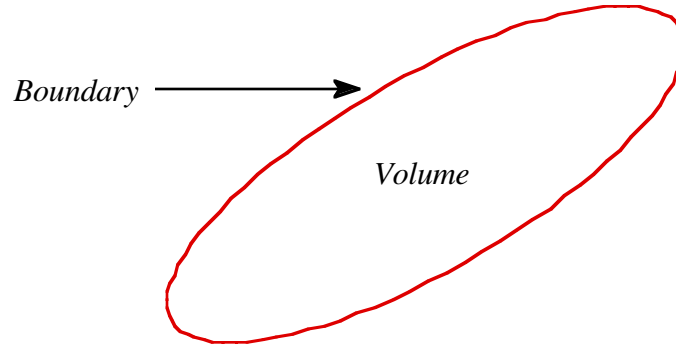


Figure 3.2. First Uniqueness Theorem

To prove the first uniqueness theorem we will consider what happens when there are two solutions V_1 and V_2 of Laplace's equation in the volume shown in Figure 3.2. Since V_1 and V_2 are solutions of Laplace's equation we know that

$$\nabla^2 V_1 = 0$$

and

$$\nabla^2 V_2 = 0$$

Since both V_1 and V_2 are solutions, they must have the same value on the boundary. Thus $V_1 = V_2$ on the boundary of the volume. Now consider a third function V_3 , which is the difference between V_1 and V_2

$$V_3 = V_1 - V_2$$

The function V_3 is also a solution of Laplace's equation. This can be demonstrated easily:

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0$$

The value of the function V_3 is equal to zero on the boundary of the volume since $V_1 = V_2$ there. However, property 2 of any solution of Laplace's equation states that it can have no local maxima or minima and that the extreme values of the solution must occur at the boundaries. Since V_3 is a solution of Laplace's equation and its value is zero everywhere on the boundary of the volume, the maximum and minimum value of V_3 must be equal to zero. Therefore, V_3 must be equal to zero everywhere. This immediately implies that

$$V_1 = V_2$$

everywhere. This proves that there can be no two different functions V_1 and V_2 that are solutions of Laplace's equation and satisfy the same boundary conditions. Therefore, the solution of Laplace's equation is uniquely determined if its value is a specified function on all boundaries of the region. This also indicates that it does not matter how you come by your solution: if (a) it is a solution of Laplace's equation, and (b) it has the correct value on the boundaries, then it is the right and only solution.

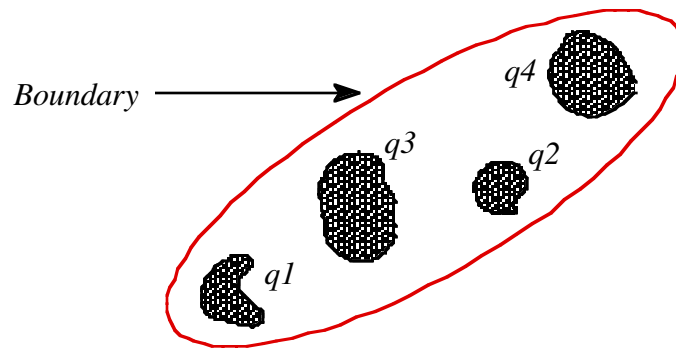


Figure 3.3. System with conductors.

The first uniqueness theorem can only be applied in those regions that are free of charge and surrounded by a boundary with a known potential (not necessarily constant). In the laboratory the boundaries are usually conductors connected to batteries to keep them at a fixed potential. In many other electrostatic problems we do not know the potential at the boundaries of the system. Instead we might know the total charge on the various conductors that make up the system (note: knowing the total charge on a conductor does not imply a knowledge of the charge distribution ρ since it is influenced by the presence of the other conductors). In addition to the conductors that make up the system, there might be a charge distribution ρ filling the regions between the conductors (see Figure 3.3). For this type of system the first uniqueness theorem does not apply. The **second uniqueness theorem** states that the electric field is uniquely determined if the total charge on each conductor is given and the charge distribution in the regions between the conductors is known.

The proof of the second uniqueness theorem is similar to the proof of the first uniqueness theorem. Suppose that there are two fields \bar{E}_1 and \bar{E}_2 that are solutions of Poisson's equation in the region between the conductors. Thus

$$\bar{\nabla} \cdot \bar{E}_1 = \frac{\rho}{\epsilon_0}$$

and

$$\bar{\nabla} \cdot \bar{E}_2 = \frac{\rho}{\epsilon_0}$$

where ρ is the charge density at the point where the electric field is evaluated. The surface integrals of \bar{E}_1 and \bar{E}_2 , evaluated using a surface that is just outside one of the conductors with charge Q_i , are equal to Q_i / ϵ_0 . Thus

$$\int_{\text{Surface conductor } i} \bar{E}_1 \cdot d\bar{a} = \frac{Q_i}{\epsilon_0}$$

$$\int_{\text{Surface conductor } i} \bar{E}_2 \cdot d\bar{a} = \frac{Q_i}{\epsilon_0}$$

The difference between \bar{E}_1 and \bar{E}_2 , $\bar{E}_3 = \bar{E}_1 - \bar{E}_2$, satisfies the following equations:

$$\bar{\nabla} \cdot \bar{E}_3 = \bar{\nabla} \cdot \bar{E}_1 - \bar{\nabla} \cdot \bar{E}_2 = \frac{\rho}{\epsilon_0} - \frac{\rho}{\epsilon_0} = 0$$

$$\int_{\text{Surface conductor } i} \bar{E}_3 \cdot d\bar{a} = \int_{\text{Surface conductor } i} \bar{E}_1 \cdot d\bar{a} - \int_{\text{Surface conductor } i} \bar{E}_2 \cdot d\bar{a} = \frac{Q_i}{\epsilon_0} - \frac{Q_i}{\epsilon_0} = 0$$

Consider the surface integral of \bar{E}_3 , integrated over all surfaces (the surface of all conductors and the outer surface). Since the potential on the surface of any conductor is constant, the electrostatic potential associated with \bar{E}_1 and \bar{E}_2 must also be constant on the surface of each conductor. Therefore, $V_3 = V_1 - V_2$ will also be constant on the surface of each conductor. The surface integral of $V_3 \bar{E}_3$ over the surface of conductor i can be written as

$$\int_{\text{Surface conductor } i} V_3 \bar{E}_3 \cdot d\bar{a} = V_3 \int_{\text{Surface conductor } i} \bar{E}_3 \cdot d\bar{a} = 0$$

Since the surface integral of $V_3 \bar{E}_3$ over the surface of conductor i is equal to zero, the surface integral of $V_3 \bar{E}_3$ over all conductor surfaces will also be equal to zero. The surface integral of $V_3 \bar{E}_3$ over the outer surface will also be equal to zero since $V_3 = 0$ on this surface. Thus

$$\int_{\text{All surfaces}} V_3 \bar{E}_3 \cdot d\bar{a} = 0$$

The surface integral of $V_3 \bar{E}_3$ can be rewritten using Green's identity as

$$\begin{aligned} 0 &= \int_{\text{All surfaces}} V_3 \bar{E}_3 \cdot d\bar{a} = - \int_{\text{All surfaces}} V_3 \bar{\nabla} V_3 \cdot d\bar{a} = - \int_{\text{Volume between conductors}} \left(V_3 \bar{\nabla}^2 V_3 + (\bar{\nabla} V_3) \cdot (\bar{\nabla} V_3) \right) d\tau = \\ &= - \int_{\text{Volume between conductors}} \left(-V_3 (\bar{\nabla} \cdot \bar{E}_3) + \bar{E}_3 \cdot \bar{E}_3 \right) d\tau = - \int_{\text{Volume between conductors}} E_3^2 d\tau = 0 \end{aligned}$$

where the volume integration is over all space between the conductors and the outer surface. Since E_3^2 is always positive, the volume integral of E_3^2 can only be equal to zero if $E_3^2 = 0$ everywhere. This implies immediately that $E_1 = E_2$ everywhere, and proves the second uniqueness theorem.

3.2. Method of Images

Consider a point charge q held at a distance d above an infinite grounded conducting plane (see Figure 3.4). The electrostatic potential of this system must satisfy the following two boundary conditions:

$$\begin{aligned} V(x, y, 0) &= 0 \\ V(x, y, z) &\rightarrow 0 \text{ when } \begin{cases} x \rightarrow \infty \\ y \rightarrow \infty \\ z \rightarrow \infty \end{cases} \end{aligned}$$

A direct calculation of the electrostatic potential can not be carried out since the charge distribution on the grounded conductor is unknown. **Note:** the charge distribution on the surface of a grounded conductor does not need to be zero.

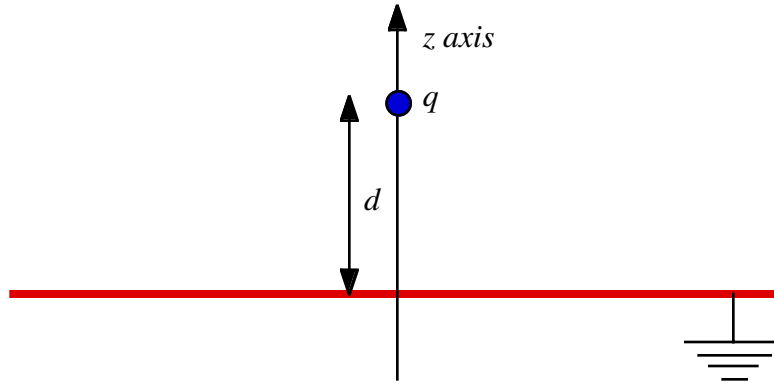


Figure 3.4. Method of images.

Consider a second system, consisting of two point charges with charges $+q$ and $-q$, located at $z = d$ and $z = -d$, respectively (see Figure 3.5). The electrostatic potential generated by these two charges can be calculated directly at any point in space. At a point $P = (x, y, 0)$ on the xy plane the electrostatic potential is equal to

$$V(x, y, 0) = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\sqrt{x^2 + y^2 + d^2}} + \frac{-q}{\sqrt{x^2 + y^2 + d^2}} \right] = 0$$

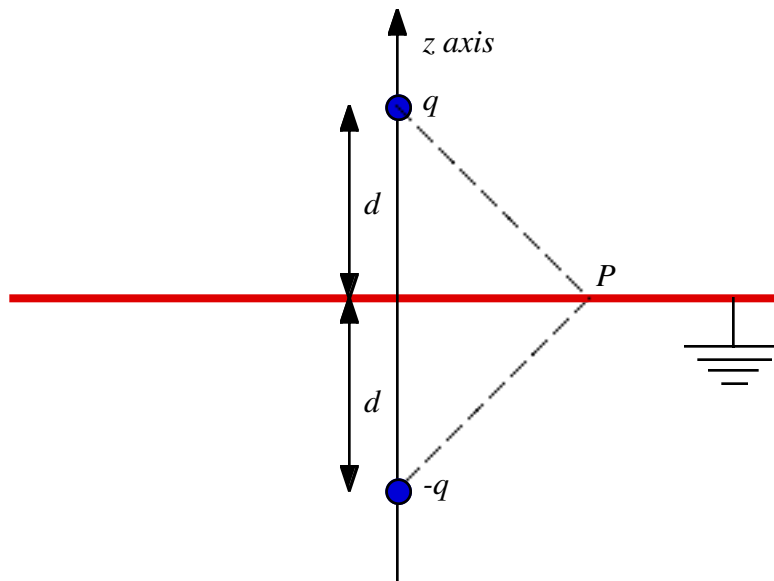


Figure 3.5. Charge and image charge.

The potential of this system at infinity will approach zero since the potential generated by each charge will decrease as $1/r$ with increasing distance r . Therefore, the electrostatic potential

generated by the two charges shown in Figure 3.5 satisfies the same boundary conditions as the system shown in Figure 3.4. Since the charge distribution in the region $z > 0$ (bounded by the xy plane boundary and the boundary at infinity) for the two systems is identical, the corollary of the first uniqueness theorem states that the electrostatic potential in this region is uniquely defined. Therefore, if we find any function that satisfies the boundary conditions and Poisson's equation, it will be the right answer. Consider a point (x, y, z) with $z > 0$. The electrostatic potential at this point can be calculated easily for the charge distribution shown in Figure 3.5. It is equal to

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} + \frac{-q}{\sqrt{x^2 + y^2 + (z+d)^2}} \right]$$

Since this solution satisfies the boundary conditions, it must be the correct solution in the region $z > 0$ for the system shown in Figure 3.4. This technique of using image charges to obtain the electrostatic potential in some region of space is called the **method of images**.

The electrostatic potential can be used to calculate the charge distribution on the grounded conductor. Since the electric field inside the conductor is equal to zero, the boundary condition for \vec{E} (see Chapter 2) shows that the electric field right outside the conductor is equal to

$$\vec{E}_{Outside} = \frac{\sigma}{\epsilon_0} \hat{n} = \frac{\sigma}{\epsilon_0} \hat{k}$$

where σ is the surface charge density and \hat{n} is the unit vector normal to the surface of the conductor. Expressing the electric field in terms of the electrostatic potential V we can rewrite this equation as

$$\sigma = \epsilon_0 E_z = -\epsilon_0 \left. \frac{\partial V}{\partial z} \right|_{z=0}$$

Substituting the solution for V in this equation we find

$$\sigma = -\frac{q}{4\pi} \left[\frac{-(z-d)}{(x^2 + y^2 + (z-d)^2)^{3/2}} + \frac{(z+d)}{(x^2 + y^2 + (z+d)^2)^{3/2}} \right]_{z=0} = -\frac{q}{2\pi} \frac{d}{(x^2 + y^2 + d^2)^{3/2}}$$

Only in the last step of this calculation have we substituted $z = 0$. The induced charge distribution is negative and the charge density is greatest at $(x = 0, y = 0, z = 0)$. The total charge on the conductor can be calculated by surface integrating of σ .

$$Q_{total} = \int_{Surface} \sigma da = \int_0^{2\pi} \int_0^\infty \sigma(r) r dr d\theta$$

where $r^2 = x^2 + y^2$. Substituting the expression for σ in the integral we obtain

$$Q_{total} = -qd \int_0^\infty \frac{1}{(r^2 + d^2)^{3/2}} r dr = \frac{qd}{\sqrt{r^2 + d^2}} \Big|_0^\infty = qd \left[0 - \frac{1}{d} \right] = -q$$

As a result of the induced surface charge on the conductor, the point charge q will be attracted towards the conductor. Since the electrostatic potential generated by the charge image-charge system is the same as the charge-conductor system in the region where $z > 0$, the associated electric field (and consequently the force on point charge q) will also be the same. The force exerted on point charge q can be obtained immediately by calculating the force exerted on the point charge by the image charge. This force is equal to

$$F = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{(2d)^2} \hat{k}$$

There is however one important difference between the image-charge system and the real system. This difference is the total electrostatic energy of the system. The electric field in the image-charge system is present everywhere, and the magnitude of the electric field at (x, y, z) will be the same as the magnitude of the electric field at $(x, y, -z)$. On the other hand, in the real system the electric field will only be non zero in the region with $z > 0$. Since the electrostatic energy of a system is proportional to the volume integral of E^2 the electrostatic energy of the real system will be 1/2 of the electrostatic energy of the image-charge system (only 1/2 of the total volume has a non-zero electric field in the real system). The electrostatic energy of the image-charge system is equal to

$$W_{image} = \frac{1}{4\pi\epsilon_0} \frac{q^2}{2d}$$

The electrostatic energy of the real system is therefore equal to

$$W_{real} = \frac{1}{2} W_{image} = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{4d}$$

The electrostatic energy of the real system can also be obtained by calculating the work required to be done to assemble the system. In order to move the charge q to its final position we will

have to exert a force opposite to the force exerted on it by the grounded conductor. The work done to move the charge from infinity along the z axis to $z = d$ is equal to

$$W_{real} = \frac{1}{4\pi\epsilon_0} \int_{\infty}^d \frac{q^2}{4z^2} dz = \frac{1}{4\pi\epsilon_0} \left. \frac{-q^2}{4z} \right|_{\infty}^d = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{4d}$$

which is identical to the result obtained using the electrostatic potential energy of the image-charge system.

Example: Example 3.2 + Problem 3.7

A point charge q is situated a distance s from the center of a grounded conducting sphere of radius R (see Figure 3.6).

- Find the potential everywhere.
- Find the induced surface charge on the sphere, as function of q . Integrate this to get the total induced charge.
- Calculate the electrostatic energy of this configuration.

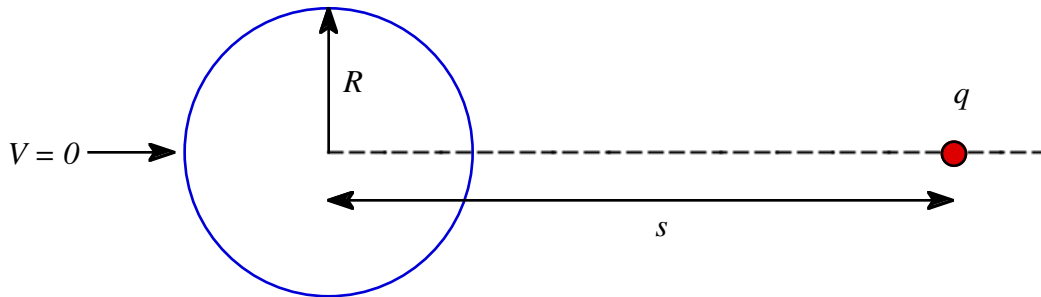


Figure 3.6. Example 3.2 + Problem 3.7.

a) Consider a system consisting of two charges q and q' , located on the z axis at $z = s$ and $z = z'$, respectively. If the potential produced by this system is identical everywhere to the potential produced by the system shown in Figure 3.6 then the position of point charge q' must be chosen such that the potential on the surface of a sphere of radius R , centered at the origin, is equal to zero (in this case the boundary conditions for the potential generated by both systems are identical).

We will start with determining the correct position of point charge q' . The electrostatic potential at P (see Figure 3.7) is equal to

The position of the image charge is equal to

$$z' = \frac{R^2}{s}$$

The value of the image charge is equal to

$$q' = -\frac{q}{s+R}(R+z') = -\frac{q}{s+R}\left(R + \frac{R^2}{s}\right) = -q\frac{R}{s}$$

Now consider an arbitrary point P' on the circle. The distance between P' and charge q is d and the distance between P' and charge q' is equal to d' . Using the cosine rule (see Figure 3.7) we can express d and d' in terms of R , s , and θ :

$$d = \sqrt{R^2 + s^2 - 2Rs\cos\theta}$$

$$d' = \sqrt{R^2 + z'^2 - 2Rz'\cos\theta} = \sqrt{R^2 + \frac{R^4}{s^2} - 2R\frac{R^2}{s}\cos\theta}$$

The electrostatic potential at P' is equal to

$$V_{P'} = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{d} + \frac{q'}{d'} \right] = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\sqrt{R^2 + s^2 - 2Rs\cos\theta}} + \frac{-q\frac{R}{s}}{\sqrt{R^2 + \frac{R^4}{s^2} - 2R\frac{R^2}{s}\cos\theta}} \right] =$$

$$= \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\sqrt{R^2 + s^2 - 2Rs\cos\theta}} - \frac{q}{\sqrt{R^2 + s^2 - 2Rs\cos\theta}} \right] = 0$$

Thus we conclude that the configuration of charge and image charge produces an electrostatic potential that is zero at any point on a sphere with radius R and centered at the origin. Therefore, this charge configuration produces an electrostatic potential that satisfies exactly the same boundary conditions as the potential produced by the charge-sphere system. In the region outside the sphere, the electrostatic potential is therefore equal to the electrostatic potential produced by the charge and image charge. Consider an arbitrary point (r, θ, ϕ) . The distance between this point and charge q is d and the distance between this point and charge q' is equal to d' . These distances can be expressed in terms of r , s , and θ using the cosine rule:

$$d = \sqrt{r^2 + s^2 - 2rs\cos\theta}$$

$$d' = \sqrt{r^2 + z'^2 - 2rz' \cos \theta} = \sqrt{r^2 + \frac{R^4}{s^2} - 2r \frac{R^2}{s} \cos \theta}$$

The electrostatic potential at (r, θ, ϕ) will therefore be equal to

$$\begin{aligned} V(r, \theta, \phi) &= \frac{1}{4\pi\epsilon_0} \left[\frac{q}{d} + \frac{q'}{d'} \right] = \frac{1}{4\pi\epsilon_0} \left\{ \frac{q}{\sqrt{r^2 + s^2 - 2rs \cos \theta}} + \frac{-q \frac{R}{s}}{\sqrt{r^2 + \frac{R^4}{s^2} - 2r \frac{R^2}{s} \cos \theta}} \right\} = \\ &= \frac{1}{4\pi\epsilon_0} \left\{ \frac{q}{\sqrt{r^2 + s^2 - 2rs \cos \theta}} - \frac{q}{\sqrt{\left(\frac{rs}{R}\right)^2 + R^2 - 2rs \cos \theta}} \right\} \end{aligned}$$

b) The surface charge density σ on the sphere can be obtained from the boundary conditions of \vec{E}

$$\vec{E}_{outside} - \vec{E}_{inside} = \vec{E}_{outside} = \frac{\sigma}{\epsilon_0} \hat{n} = \frac{\sigma}{\epsilon_0} \hat{r}$$

where we have used the fact that the electric field inside the sphere is zero. This equation can be rewritten as

$$\sigma = \epsilon_0 E_r = -\epsilon_0 \frac{\partial V}{\partial r}$$

Substituting the general expression for V into this equation we obtain

$$\begin{aligned} \sigma &= -\frac{q}{4\pi} \left\{ \frac{-r + s \cos \theta}{(r^2 + s^2 - 2rs \cos \theta)^{3/2}} - \frac{-\frac{rs^2}{R^2} + s \cos \theta}{\left(\left(\frac{rs}{R} \right)^2 + R^2 - 2rs \cos \theta \right)^{3/2}} \right\} \bigg|_{r=R} = \\ &= -\frac{q}{4\pi} \left\{ \frac{-R + s \cos \theta}{(R^2 + s^2 - 2Rs \cos \theta)^{3/2}} - \frac{-\frac{s^2}{R} + s \cos \theta}{(s^2 + R^2 - 2Rs \cos \theta)^{3/2}} \right\} = \\ &= -\frac{q}{4\pi R} \frac{s^2 - R^2}{(R^2 + s^2 - 2Rs \cos \theta)^{3/2}} \end{aligned}$$

The total charge on the sphere can be obtained by integrating σ over the surface of the sphere. The result is

$$\begin{aligned}
 Q &= \int \sigma R^2 \sin \theta d\theta d\phi = -\frac{q}{2} R (s^2 - R^2) \int_0^\pi \frac{\sin \theta d\theta}{(R^2 + s^2 - 2Rs \cos \theta)^{3/2}} = \\
 &= -\frac{q}{2} R (s^2 - R^2) \left[\frac{-\frac{1}{sR}}{\sqrt{R^2 + s^2 - 2Rs \cos \theta}} \right]_0^\pi = \frac{q}{2} \frac{s^2 - R^2}{s} \left[\frac{1}{|R + s|} - \frac{1}{|R - s|} \right] = -q \frac{R}{s}
 \end{aligned}$$

c) To obtain the electrostatic energy of the system we can determine the work it takes to assemble the system by calculating the path integral of the force that we need to exert in charge q in order to move it from infinity to its final position ($z = s$). Charge q will feel an attractive force exerted by the induced charge on the sphere. The strength of this force is equal to the force on charge q exerted by the image charge q' . This force is equal to

$$\bar{F}_{qq'} = \frac{1}{4\pi\epsilon_0} \frac{qq'}{(s - z')^2} \hat{k} = \frac{1}{4\pi\epsilon_0} \frac{q\left(-\frac{R}{s}q\right)}{\left(s - \frac{R^2}{s}\right)^2} \hat{k} = -\frac{1}{4\pi\epsilon_0} \frac{sRq^2}{(s^2 - R^2)^2} \hat{k}$$

The force that we must exert on q to move it from infinity to its current position is opposite to $\bar{F}_{qq'}$. The total work required to move the charge is therefore equal to

$$W = \int_\infty^s -\bar{F}_{qq'} \cdot d\vec{l} = \frac{1}{4\pi\epsilon_0} \int_\infty^s \frac{zRq^2}{(z^2 - R^2)^2} dz = \frac{1}{4\pi\epsilon_0} \frac{-Rq^2}{2(z^2 - R^2)} \Big|_\infty^s = -\frac{1}{8\pi\epsilon_0} \frac{Rq^2}{s^2 - R^2}$$

Example: Problem 3.10

Two semi-infinite grounded conducting planes meet at right angles. In the region between them, there is a point charge q , situated as shown in Figure 3.8. Set up the image configuration, and calculate the potential in this region. What charges do you need, and where should they be located? What is the force on q ? How much work did it take to bring q in from infinity?

Consider the system of four charges shown in Figure 3.9. The electrostatic potential generated by this charge distribution is zero at every point on the yz plane and at every point on the xz plane. Therefore, the electrostatic potential generated by this image charge distribution satisfies the same boundary conditions as the electrostatic potential of the original system. The

potential generated by the image charge distribution in the region where $x > 0$ and $y > 0$ will be identical to the potential of the original system. The potential at a point $P = (x, y, z)$ is equal to

$$V_p = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{(x-a)^2 + (y-b)^2 + z^2}} + \frac{1}{4\pi\epsilon_0} \frac{-q}{\sqrt{(x-a)^2 + (y+b)^2 + z^2}} +$$

$$+ \frac{1}{4\pi\epsilon_0} \frac{-q}{\sqrt{(x+a)^2 + (y-b)^2 + z^2}} + \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{(x+a)^2 + (y+b)^2 + z^2}}$$

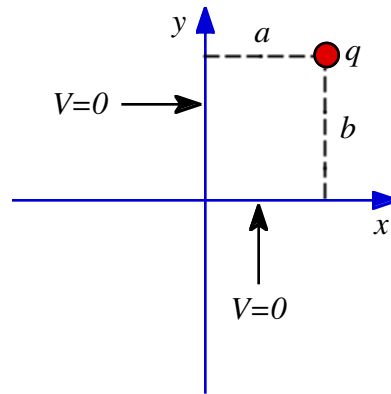


Figure 3.8. Problem 3.10.

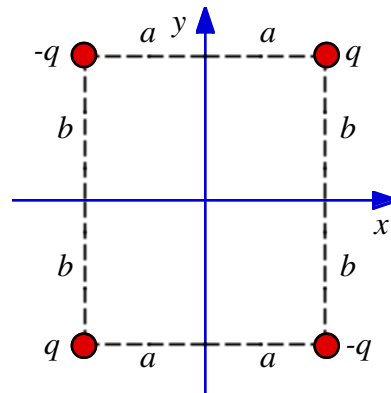


Figure 3.9. Image charges for problem 3.10.

The force exerted on q can be obtained by calculating the force exerted on q by the image charges. The total force is equal to the vector sum of the forces exerted by each of the three

image charges. The force exerted by the image charge located at $(-a, b, 0)$ is directed along the negative x axis and is equal to

$$\bar{F}_1 = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{4a^2} \hat{i}$$

The force exerted by the image charge located at $(a, -b, 0)$ is directed along the negative y axis and is equal to

$$\bar{F}_2 = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{4b^2} \hat{j}$$

The force exerted by the image charge located at $(-a, -b, 0)$ is directed along the vector connecting $(-a, -b, 0)$ and $(a, b, 0)$ and is equal to

$$\bar{F}_3 = \frac{1}{4\pi\epsilon_0} \frac{q^2}{4a^2 + 4b^2} \frac{a\hat{i} + b\hat{j}}{\sqrt{a^2 + b^2}} = \frac{1}{16\pi\epsilon_0} \frac{q^2}{(a^2 + b^2)^{3/2}} (a\hat{i} + b\hat{j})$$

The total force on charge q is the vector sum of \bar{F}_1 , \bar{F}_2 and \bar{F}_3 :

$$\bar{F}_{tot} = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 = -\frac{q^2}{16\pi\epsilon_0} \left(\left(\frac{1}{a^2} - \frac{a}{(a^2 + b^2)^{3/2}} \right) \hat{i} + \left(\frac{1}{b^2} - \frac{b}{(a^2 + b^2)^{3/2}} \right) \hat{j} \right)$$

The electrostatic potential energy of the system can, in principle, be obtained by calculating the path integral of $-\bar{F}_{tot}$ between infinity and $(a, b, 0)$. However, this is not trivial since the force $-\bar{F}_{tot}$ is a rather complex function of a and b . An easier technique is to calculate the electrostatic potential energy of the system with charge and image charges. The potential energy of this system is equal to

$$W_{image} = \frac{1}{4\pi\epsilon_0} \left(\frac{-q^2}{2a} + \frac{-q^2}{2b} + \frac{q^2}{\sqrt{4a^2 + 4b^2}} \right) = \frac{q^2}{8\pi\epsilon_0} \left(\frac{1}{\sqrt{a^2 + b^2}} - \frac{1}{a} - \frac{1}{b} \right)$$

However, in the real system the electric field is only non-zero in the region where $x > 0$ and $y > 0$. Therefore, the total electrostatic potential energy of the real system is only 1/4 of the total electrostatic potential energy of the image charge system. Thus

$$W_{real} = \frac{1}{4} W_{image} = \frac{q^2}{32\pi\epsilon_0} \left(\frac{1}{\sqrt{a^2 + b^2}} - \frac{1}{a} - \frac{1}{b} \right)$$

3.3. Separation of Variables

3.3.1. Separation of variables: Cartesian coordinates

A powerful technique very frequently used to solve partial differential equations is **separation of variables**. In this section we will demonstrate the power of this technique by discussing several examples.

Example: Example 3.3 (Griffiths)

Two infinite, grounded, metal plates lie parallel to the xz plane, one at $y = 0$, the other at $y = \pi$ (see Figure 3.10). The left end, at $x = 0$, is closed off with an infinite strip insulated from the two plates and maintained at a specified potential $V_0(y)$. Find the potential inside this "slot".

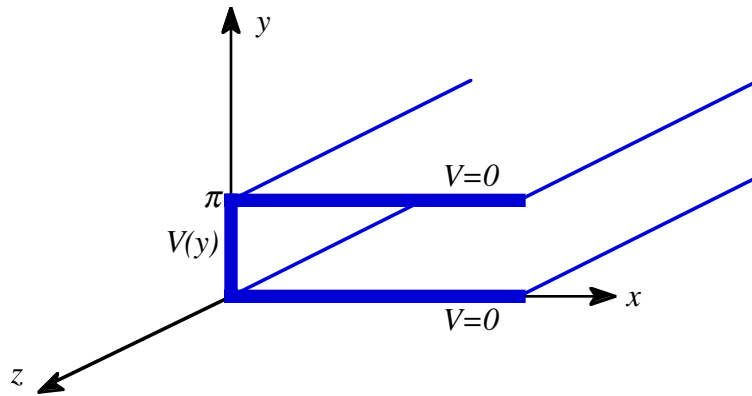


Figure 3.10. Example 3.3 (Griffiths).

The electrostatic potential in the slot must satisfy the three-dimensional Laplace equation. However, since V does not have a z dependence, the three-dimensional Laplace equation reduces to the two-dimensional Laplace equation:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

The boundary conditions for the solution of Laplace's equation are:

1. $V(x, y = 0) = 0$ (grounded bottom plate).

2. $V(x, y = \pi) = 0$ (grounded top plate).

3. $V(x = 0, y) = V_0(y)$ (plate at $x = 0$).

4. $V \rightarrow 0$ when $x \rightarrow \infty$.

These four boundary conditions specify the value of the potential on **all** boundaries surrounding the slot and are therefore sufficient to uniquely determine the solution of Laplace's equation inside the slot. Therefore, if we find one solution of Laplace's equation satisfying these boundary conditions than it must be the correct one. Consider solutions of the following form:

$$V(x, y) = X(x)Y(y)$$

If this is a solution of the two-dimensional Laplace equation than we must require that

$$\frac{\partial^2}{\partial x^2} [X(x)Y(y)] + \frac{\partial^2}{\partial y^2} [X(x)Y(y)] = Y(y) \frac{\partial^2 X(x)}{\partial x^2} + X(x) \frac{\partial^2 Y(y)}{\partial y^2} = 0$$

This equation can be rewritten as

$$\frac{1}{X(x)} \frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{Y(y)} \frac{\partial^2 Y(y)}{\partial y^2} = 0$$

The first term of the left-hand side of this equation depends only on x while the second term depends only on y . Therefore, if this equation must hold for all x and y in the slot we must require that

$$\frac{1}{X(x)} \frac{\partial^2 X(x)}{\partial x^2} = C_1 = \text{constant}$$

and

$$\frac{1}{Y(y)} \frac{\partial^2 Y(y)}{\partial y^2} = -C_1$$

The differential equation for X can be rewritten as

$$\frac{\partial^2 X(x)}{\partial x^2} - C_1 X(x) = 0$$

If C_1 is a negative number then this equation can be rewritten as

$$\frac{\partial^2 X(x)}{\partial x^2} + k^2 X(x) = 0$$

where $k^2 = -C_1$. The most general solution of this equation is

$$X(x) = A \cos(kx) + B \sin(kx)$$

However, this function is an oscillatory function and does not satisfy boundary condition # 4, which requires that V approaches zero when x approaches infinity. We therefore conclude that C_1 **can not** be a negative number. If C_1 is a positive number then the differential equation for X can be written as

$$\frac{\partial^2 X(x)}{\partial x^2} - k^2 X(x) = 0$$

The most general solution of this equation is

$$X(x) = A e^{kx} + B e^{-kx}$$

This solution will approach zero when x approaches infinity if $A = 0$. Thus

$$X(x) = B e^{-kx}$$

The solution for Y can be obtained by solving the following differential equation:

$$\frac{\partial^2 Y(y)}{\partial y^2} + k^2 Y(y) = 0$$

The most general solution of this equation is

$$Y(y) = C \cos(ky) + D \sin(ky)$$

Therefore, the general solution for the electrostatic potential $V(x,y)$ is equal to

$$V(x,y) = e^{-kx} (C \cos(ky) + D \sin(ky))$$

where we have absorbed the constant B into the constants C and D . The constants C and D must be chosen such that the remaining three boundary conditions (1, 2, and 3) are satisfied. The first boundary condition requires that $V(x, y = 0) = 0$:

$$V(x, y = 0) = e^{-kx} (C \cos(0) + D \sin(0)) = Ce^{-kx} = 0$$

which requires that $C = 0$. The second boundary condition requires that $V(x, y = \pi) = 0$:

$$V(x, y = \pi) = De^{-kx} \sin(k\pi) = 0$$

which requires that $\sin(k\pi) = 0$. This condition limits the possible values of k to positive integers:

$$k = 1, 2, 3, 4, \dots$$

Note: negative values of k are not allowed since $\exp(-kx)$ approaches zero at infinity only if $k > 0$. To satisfy boundary condition # 3 we must require that

$$V(x = 0, y) = D \sin(ky) = V_0(y)$$

This last expression suggests that the only time at which we can find a solution of Laplace's equation that satisfies all four boundary conditions has the form $\exp(-kx) \sin(ky)$ is when $V_0(y)$ happens to have the form $\sin(ky)$. However, since k can take on an infinite number of values, there will be an infinite number of solutions of Laplace's equation satisfying boundary conditions # 1, # 2 and # 4. The most general form of the solution of Laplace's equation will be a linear superposition of all possible solutions. Thus

$$V(x, y) = \sum_{k=1}^{\infty} D_k e^{-kx} \sin(ky)$$

Boundary condition # 3 can now be written as

$$V(x = 0, y) = \sum_{k=1}^{\infty} D_k \sin(ky) = V_0(y)$$

Multiplying both sides by $\sin(ny)$ and integrating each side between $y = 0$ and $y = \pi$ we obtain

$$\sum_{k=1}^{\infty} D_k \int_0^{\pi} \sin(ny) \sin(ky) dy = \int_0^{\pi} \sin(ny) V_0(y) dy$$

The integral on the left-hand side of this equation is equal to zero for all values of k except $k = n$. Thus

$$\sum_{k=1}^{\infty} D_k \int_0^{\pi} \sin(ny) \sin(ky) dy = \sum_{k=1}^{\infty} D_k \frac{\pi}{2} \delta_{kn} = \frac{\pi}{2} D_n$$

The coefficients D_k can thus be calculated easily:

$$D_k = \frac{2}{\pi} \int_0^{\pi} \sin(ky) V_0(y) dy$$

The coefficients D_k are called the **Fourier coefficients** of $V_0(y)$. The solution of Laplace's equation in the slot is therefore equal to

$$V(x, y) = \sum_{k=1}^{\infty} D_k e^{-kx} \sin(ky)$$

where

$$D_k = \frac{2}{\pi} \int_0^{\pi} \sin(ky) V_0(y) dy$$

Now consider the special case in which $V_0(y) = \text{constant} = V_0$. In this case the coefficients D_k are equal to

$$D_k = \frac{2}{\pi} \int_0^{\pi} \sin(ky) V_0(y) dy = -\frac{2}{\pi} V_0 \left. \frac{\cos(ky)}{k} \right|_0^{\pi} = \frac{2}{\pi} \frac{V_0}{k} (1 - \cos(k\pi)) = \begin{cases} 0 & \text{if } k \text{ is even} \\ \frac{4}{\pi} \frac{V_0}{k} & \text{if } k \text{ is odd} \end{cases}$$

The solution of Laplace's equation is thus equal to

$$V(x, y) = \frac{4V_0}{\pi} \sum_{k=1,3,5,\dots}^{\infty} \frac{1}{k} e^{-kx} \sin(ky)$$

Example: Problem 3.12

Find the potential in the infinite slot of Example 3.3 (Griffiths) if the boundary at $x = 0$ consists to two metal stripes: one, from $y = 0$ to $y = \pi/2$, is held at constant potential V_0 , and the other, from $y = \pi/2$ to $y = \pi$ is at potential $-V_0$.

The boundary condition at $x = 0$ is

$$V(0,y) = \begin{cases} V_0 & \text{for } 0 < y < \pi/2 \\ -V_0 & \text{for } \pi/2 < y < \pi \end{cases}$$

The Fourier coefficients of the function $V_0(y)$ are equal to

$$\begin{aligned} D_k &= \frac{2}{\pi} \int_0^\pi \sin(ky) V_0(y) dy = \frac{2}{\pi} V_0 \int_0^{\pi/2} \sin(ky) dy - \frac{2}{\pi} V_0 \int_{\pi/2}^\pi \sin(ky) dy = \\ &= \frac{2}{\pi} \frac{V_0}{k} \left(1 + \cos(k\pi) - 2\cos\left(\frac{1}{2}k\pi\right) \right) = \frac{2}{\pi} \frac{V_0}{k} C_k \end{aligned}$$

The values for the first four C coefficients are

$$C_1 = 0 \quad C_3 = 0$$

$$C_2 = 4 \quad C_4 = 0$$

It is easy to see that $C_{k+4} = C_k$ and therefore we conclude that

$$C_k = \begin{cases} 4 & \text{for } k = 2, 6, 10, \dots \\ 0 & \text{otherwise} \end{cases}$$

The Fourier coefficients C_k are thus equal to

$$D_k = \begin{cases} \frac{8V_0}{k\pi} & \text{for } k = 2, 6, 10, \dots \\ 0 & \text{otherwise} \end{cases}$$

The electrostatic potential is thus equal to

$$V(x,y) = \frac{8V_0}{\pi} \sum_{k=2,6,10,\dots}^{\infty} \frac{1}{k} e^{-kx} \sin(ky)$$

Example: Problem 3.13

For the infinite slot (Example 3.3 Griffiths) determine the charge density $\sigma(y)$ on the strip at $x=0$, assuming it is a conductor at constant potential V_0 .

The electrostatic potential in the slot is equal to

$$V(x,y) = \frac{4V_0}{\pi} \sum_{k=1,3,5,\dots}^{\infty} \frac{1}{k} e^{-kx} \sin(ky)$$

The charge density at the plate at $x = 0$ can be obtained using the boundary condition for the electric field at a boundary:

$$\bar{E}_{x=+0} - \bar{E}_{x=-0} = \bar{E}_{x=+0} = \frac{\sigma}{\epsilon_0} \hat{n}$$

where \hat{n} is directed along the positive x axis. Since $\bar{E} = -\bar{\nabla} V$ this boundary condition can be rewritten as

$$\left. \frac{\partial V}{\partial x} \right|_{x=+0} = -\frac{\sigma}{\epsilon_0}$$

Differentiating $V(x,y)$ with respect to x we obtain

$$\frac{\partial V}{\partial x} = -\frac{4V_0}{\pi} \sum_{k=1,3,5,\dots}^{\infty} e^{-kx} \sin(ky)$$

At the $x = 0$ boundary we obtain

$$\left. \frac{\partial V}{\partial x} \right|_{x=+0} = -\frac{4V_0}{\pi} \sum_{k=1,3,5,\dots}^{\infty} \sin(ky)$$

The charge density σ on the $x = 0$ strip is therefore equal to

$$\sigma = -\epsilon_0 \left. \frac{\partial V}{\partial x} \right|_{x=+0} = \frac{4V_0 \epsilon_0}{\pi} \sum_{k=1,3,5,\dots}^{\infty} \sin(ky)$$

Example: Double infinite slots

The slot of example 3.3 in Griffiths and its mirror image at negative x are separated by an insulating strip at $x = 0$. If the charge density $\sigma(y)$ on the dividing strip is given, determine the potential in the slot.

The boundary condition at $x = 0$ requires that

$$\bar{E}_{x=+0} - \bar{E}_{x=-0} = 2\bar{E}_{x=+0} = \frac{\sigma}{\epsilon_0} \hat{n}$$

where \hat{n} is directed along the positive x axis. Here we have used the symmetry of the configuration which requires that the electric field in the region $x < 0$ is the mirror image of the field in the region $x > 0$. Since $\bar{E} = -\bar{\nabla} V$ this boundary condition can be rewritten as

$$\left. \frac{\partial V}{\partial x} \right|_{x=+0} = -\frac{1}{2} \frac{\sigma(y)}{\epsilon_0}$$

We will first determine the potential in the $x > 0$ region. Following the same procedure as in Example 3 we obtain for the electrostatic potential

$$V(x, y) = \sum_{k=1}^{\infty} D_k e^{-kx} \sin(ky)$$

where the constants D_k must be chosen such that the boundary condition at $x = 0$ is satisfied. This requires that

$$\left. \frac{\partial V}{\partial x} \right|_{x=+0} = -\sum_{k=1}^{\infty} k D_k e^{-kx} \sin(ky) \Big|_{x=+0} = -\sum_{k=1}^{\infty} k D_k \sin(ky) = -\frac{1}{2} \frac{\sigma(y)}{\epsilon_0}$$

Thus

$$\sum_{k=1}^{\infty} k D_k \sin(ky) = \frac{1}{2} \frac{\sigma(y)}{\epsilon_0}$$

The constants D_k can be determined by multiplying both sides of this equation with $\sin(my)$ and integrating both sides with respect to y between $y = 0$ and $y = \pi$. The result is

$$\frac{1}{2\epsilon_0} \int_0^{\pi} \sigma(y) \sin(my) dy = \sum_{k=1}^{\infty} k D_k \int_0^{\pi} \sin(ky) \sin(my) dy = m D_m \frac{\pi}{2}$$

The constants C_k are thus equal to

$$D_k = \frac{1}{\pi \epsilon_0 k} \int_0^{\pi} \sigma(y) \sin(ky) dy$$

The electrostatic potential is thus equal to

$$V(x,y) = \frac{1}{\pi \epsilon_0} \sum_{k=1}^{\infty} \left[\frac{1}{k} \left\{ \int_0^{\pi} \sigma(y) \sin(ky) dy \right\} e^{-kx} \sin(ky) \right]$$

3.3.2. Separation of variables: spherical coordinates

Consider a spherical symmetric system. If we want to solve Laplace's equation it is natural to use spherical coordinates. Assuming that the system has azimuthal symmetry ($\partial V / \partial \phi = 0$) Laplace's equation reads

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) = 0$$

Multiplying both sides by r^2 we obtain

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) = 0$$

Consider the possibility that the general solution of this equation is the product of a function $R(r)$, which depends only on the distance r , and a function $\alpha(\theta)$, which depends only on the angle θ :

$$V(r, \theta) = R(r) \alpha(\theta)$$

Substituting this "solution" into Laplace's equation we obtain

$$\alpha(\theta) \frac{\partial}{\partial r} \left(r^2 \frac{\partial R(r)}{\partial r} \right) + \frac{R(r)}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \alpha(\theta)}{\partial \theta} \right) = 0$$

Dividing each term of this equation by $R(r) \alpha(\theta)$ we obtain

$$\frac{1}{R(r)} \frac{\partial}{\partial r} \left(r^2 \frac{\partial R(r)}{\partial r} \right) + \frac{1}{\alpha(\theta)} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \alpha(\theta)}{\partial \theta} \right) = 0$$

The first term in this expression depends only on the distance r while the second term depends only on the angle θ . This equation can only be true for all r and θ if

$$\frac{1}{R(r)} \frac{\partial}{\partial r} \left(r^2 \frac{\partial R(r)}{\partial r} \right) = m(m+1) = \text{constant}$$

and

$$\frac{1}{\alpha(\theta)} \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial\alpha(\theta)}{\partial\theta} \right) = -m(m+1)$$

Consider a solution for R of the following form:

$$R(r) = Ar^k$$

where A and k are arbitrary constants. Substituting this expression in the differential equation for $R(r)$ we obtain

$$\frac{1}{Ar^k} \frac{\partial}{\partial r} (kr^2 Ar^{k-1}) = \frac{k}{r^k} (k+1)r^k = k(k+1) = m(m+1)$$

Therefore, the constant k must satisfy the following relation:

$$k(k+1) = k^2 + k = m(m+1)$$

This equation gives us the following expression for k

$$k = \frac{-1 \pm \sqrt{1 + 4m(m+1)}}{2} = \frac{-1 \pm 2\left(m + \frac{1}{2}\right)}{2} = \begin{cases} m \\ \text{or} \\ -(m+1) \end{cases}$$

The general solution for $R(r)$ is thus given by

$$R(r) = Ar^m + \frac{B}{r^{m+1}}$$

where A and B are arbitrary constants.

The angle dependent part of the solution of Laplace's equation must satisfy the following equation

$$\frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial\alpha(\theta)}{\partial\theta} \right) + m(m+1)\alpha(\theta)\sin\theta = 0$$

The solutions of this equation are known as the **Legendre polynomial** $P_m(\cos\theta)$. The Legendre polynomials have the following properties:

1. if m is even: $P_m(x) = P_m(-x)$
2. if m is odd: $P_m(x) = -P_m(-x)$

$$3. P_m(1) = 1 \text{ for all } m$$

$$4. \int_{-1}^1 P_n(x)P_m(x)dx = \frac{2}{2m+1}\delta_{nm} \text{ or } \int_{-1}^1 P_n(\cos\theta)P_m(\cos\theta)\sin\theta d\theta = \frac{2}{2m+1}\delta_{nm}$$

Combining the solutions for $R(r)$ and $\alpha(\theta)$ we obtain the most general solution of Laplace's equation in a spherical symmetric system with azimuthal symmetry:

$$V(r, \theta) = \sum_{m=0}^{\infty} \left(A_m r^m + \frac{B_m}{r^{m+1}} \right) P_m(\cos\theta)$$

Example: Problem 3.18

The potential at the surface of a sphere is given by

$$V_0(\theta) = k \cos(3\theta)$$

where k is some constant. Find the potential inside and outside the sphere, as well as the surface charge density $\sigma(\theta)$ on the sphere. (Assume that there is no charge inside or outside of the sphere.)

The most general solution of Laplace's equation in spherical coordinates is

$$V(r, \theta) = \sum_{m=0}^{\infty} \left(A_m r^m + \frac{B_m}{r^{m+1}} \right) P_m(\cos\theta)$$

First consider the region inside the sphere ($r < R$). In this region $B_m = 0$ since otherwise $V(r, \theta)$ would blow up at $r = 0$. Thus

$$V(r, \theta) = \sum_{m=0}^{\infty} A_m r^m P_m(\cos\theta)$$

The potential at $r = R$ is therefore equal to

$$V(R, \theta) = \sum_{m=0}^{\infty} A_m R^m P_m(\cos\theta) = k \cos(3\theta)$$

Using trigonometric relations we can rewrite $\cos(3\theta)$ as

$$\cos(3\theta) = 4 \cos^3 \theta - 3 \cos \theta = \frac{8}{5} P_3(\cos\theta) - \frac{3}{5} P_1(\cos\theta)$$

Substituting this expression in the equation for $V(R, \theta)$ we obtain

$$V(R, \theta) = \sum_{m=0}^{\infty} A_m R^m P_m(\cos \theta) = \frac{8k}{5} P_3(\cos \theta) - \frac{3k}{5} P_1(\cos \theta)$$

This equation immediately shows that $A_m = 0$ unless $m = 1$ or $m = 3$. If $m = 1$ or $m = 3$ then

$$A_1 = -\frac{3}{5} \frac{k}{R}$$

$$A_3 = \frac{8}{5} \frac{k}{R^3}$$

The electrostatic potential inside the sphere is therefore equal to

$$V(r, \theta) = \frac{8k}{5} \frac{r^3}{R^3} P_3(\cos \theta) - \frac{3k}{5} \frac{r}{R} P_1(\cos \theta)$$

Now consider the region outside the sphere ($r > R$). In this region $A_m = 0$ since otherwise $V(r, \theta)$ would blow up at infinity. The solution of Laplace's equation in this region is therefore equal to

$$V(r, \theta) = \sum_{m=0}^{\infty} \frac{B_m}{r^{m+1}} P_m(\cos \theta)$$

The potential at $r = R$ is therefore equal to

$$V(R, \theta) = \sum_{m=0}^{\infty} \frac{B_m}{R^{m+1}} P_m(\cos \theta) = \frac{8k}{5} P_3(\cos \theta) - \frac{3k}{5} P_1(\cos \theta)$$

The equation immediately shows that $B_m = 0$ except when $m = 1$ or $m = 3$. If $m = 1$ or $m = 3$ then

$$B_1 = -\frac{3}{5} k R^2$$

$$B_3 = \frac{8}{5} k R^4$$

The electrostatic potential outside the sphere is thus equal to

$$V(r, \theta) = -\frac{3k}{5} \frac{R^2}{r^2} P_1(\cos \theta) + \frac{8k}{5} \frac{R^4}{r^4} P_3(\cos \theta)$$

The charge density on the sphere can be obtained using the boundary conditions for the electric field at a boundary:

$$\bar{E}_{r=R+} - \bar{E}_{r=R-} = \frac{\sigma(\theta)}{\epsilon_0} \hat{r}$$

Since $\bar{E} = -\bar{\nabla} V$ this boundary condition can be rewritten as

$$\left. \frac{\partial V}{\partial r} \right|_{r=R+} - \left. \frac{\partial V}{\partial r} \right|_{r=R-} = -\frac{\sigma(\theta)}{\epsilon_0}$$

The first term on the left-hand side of this equation can be calculated using the electrostatic potential just obtained:

$$\left. \frac{\partial V}{\partial r} \right|_{r=R+} = \left(\frac{6k}{5} \frac{R^2}{r^3} P_1(\cos\theta) - \frac{32k}{5} \frac{R^4}{r^5} P_3(\cos\theta) \right) \bigg|_{r=R+} = \frac{k}{5R} (6P_1(\cos\theta) - 32P_3(\cos\theta))$$

In the same manner we obtain

$$\left. \frac{\partial V}{\partial r} \right|_{r=R-} = \left(-\frac{3k}{5} \frac{1}{R} P_1(\cos\theta) + \frac{24k}{5} \frac{r^2}{R^3} P_3(\cos\theta) \right) \bigg|_{r=R-} = \frac{k}{5R} (-3P_1(\cos\theta) + 24P_3(\cos\theta))$$

Therefore,

$$\left. \frac{\partial V}{\partial r} \right|_{r=R+} - \left. \frac{\partial V}{\partial r} \right|_{r=R-} = \frac{k}{5R} (9P_1(\cos\theta) - 56P_3(\cos\theta)) = -\frac{\sigma(\theta)}{\epsilon_0}$$

The charge density on the sphere is thus equal to

$$\sigma(\theta) = \frac{k\epsilon_0}{5R} (-9P_1(\cos\theta) + 56P_3(\cos\theta))$$

Example: Problem 3.19

Suppose the potential $V_0(\theta)$ at the surface of a sphere is specified, and there is no charge inside or outside the sphere. Show that the charge density on the sphere is given by

$$\sigma(\theta) = \frac{\epsilon_0}{2R} \sum_{m=0}^{\infty} (2m+1)^2 C_m P_m(\cos\theta)$$

where

$$C_m = \int_0^\pi V_0(\theta) P_m(\cos\theta) \sin\theta d\theta$$

Most of the solution of this problem is very similar to the solution of Problem 3.18. First consider the electrostatic potential inside the sphere. The electrostatic potential in this region is given by

$$V(r, \theta) = \sum_{m=0}^{\infty} A_m r^m P_m(\cos\theta)$$

and the boundary condition is

$$V(R, \theta) = \sum_{m=0}^{\infty} A_m R^m P_m(\cos\theta) = V_0(\theta)$$

The coefficients A_m can be determined by multiplying both sides of this equation by $P_n(\cos\theta) \sin\theta$ and integrating with respect to θ between $\theta = 0$ and $\theta = \pi$:

$$\int_0^\pi V_0(\theta) P_n(\cos\theta) \sin\theta d\theta = \sum_{m=0}^{\infty} A_m R^m \int_0^\pi P_m(\cos\theta) P_n(\cos\theta) \sin\theta d\theta = \frac{2}{2n+1} A_n R^n$$

Thus

$$A_m = \frac{2m+1}{2} \frac{1}{R^m} \int_0^\pi V_0(\theta) P_m(\cos\theta) \sin\theta d\theta$$

In the region outside the sphere the electrostatic potential is given by

$$V(r, \theta) = \sum_{m=0}^{\infty} \frac{B_m}{r^{m+1}} P_m(\cos\theta)$$

and the boundary condition is

$$V(R, \theta) = \sum_{m=0}^{\infty} \frac{B_m}{R^{m+1}} P_m(\cos\theta) = V_0(\theta)$$

The coefficients B_m are given by

$$B_m = \frac{2m+1}{2} R^{m+1} \int_0^\pi V_0(\theta) P_m(\cos\theta) \sin\theta d\theta$$

The charge density $\sigma(\theta)$ on the surface of the sphere is equal to

$$\sigma(\theta) = -\epsilon_0 \left\{ \left. \frac{\partial V}{\partial r} \right|_{r=R+} - \left. \frac{\partial V}{\partial r} \right|_{r=R-} \right\}$$

Differentiating $V(r, \theta)$ with respect to r in the region $r > R$ we obtain

$$\left. \frac{\partial V}{\partial r} \right|_{r=R+} = -\sum_{m=0}^{\infty} (m+1) \frac{B_m}{R^{m+2}} P_m(\cos \theta)$$

Differentiating $V(r, \theta)$ with respect to r in the region $r < R$ we obtain

$$\left. \frac{\partial V}{\partial r} \right|_{r=R-} = \sum_{m=0}^{\infty} m A_m R^{m-1} P_m(\cos \theta)$$

The charge density is therefore equal to

$$\begin{aligned} \sigma(\theta) &= -\epsilon_0 \left\{ -\sum_{m=0}^{\infty} \left[(m+1) \frac{B_m}{R^{m+2}} P_m(\cos \theta) \right] - \sum_{m=0}^{\infty} \left[m A_m R^{m-1} P_m(\cos \theta) \right] \right\} = \\ &= \epsilon_0 \left\{ \sum_{m=0}^{\infty} \left[\left((m+1) \frac{B_m}{R^{m+2}} + m A_m R^{m-1} \right) P_m(\cos \theta) \right] \right\} \end{aligned}$$

Substituting the expressions for A_m and B_m into this equation we obtain

$$\begin{aligned} \sigma(\theta) &= \epsilon_0 \left\{ \sum_{m=0}^{\infty} \left[\left((m+1) \frac{(2m+1)R^{m+1}}{2R^{m+2}} + m \frac{(2m+1)}{2R^m} R^{m-1} \right) C_m P_m(\cos \theta) \right] \right\} = \\ &= \frac{\epsilon_0}{2R} \left\{ \sum_{m=0}^{\infty} \left[(2m+1)^2 C_m P_m(\cos \theta) \right] \right\} \end{aligned}$$

where

$$C_m = \int_0^\pi V_0(\theta) P_m(\cos \theta) \sin \theta d\theta$$

Example: Problem 3.23

Solve Laplace's equation by separation of variables in *cylindrical* coordinates, assuming there is no dependence on z (cylindrical symmetry). Make sure that you find *all* solutions to the radial equation. Does your result accommodate the case of an infinite line charge?

For a system with cylindrical symmetry the electrostatic potential does not depend on z . This immediately implies that $\partial V / \partial z = 0$. Under this assumption Laplace's equation reads

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \phi^2} = 0$$

Consider as a possible solution of V :

$$V(r, \phi) = R(r)\alpha(\phi)$$

Substituting this solution into Laplace's equation we obtain

$$\frac{\alpha(\phi)}{r} \frac{\partial}{\partial r} \left(r \frac{\partial R(r)}{\partial r} \right) + \frac{R(r)}{r^2} \frac{\partial^2 \alpha(\phi)}{\partial \phi^2} = 0$$

Multiplying each term in this equation by r^2 and dividing by $R(r)\alpha(\phi)$ we obtain

$$\frac{r}{R(r)} \frac{\partial}{\partial r} \left(r \frac{\partial R(r)}{\partial r} \right) + \frac{1}{\alpha(\phi)} \frac{\partial^2 \alpha(\phi)}{\partial \phi^2} = 0$$

The first term in this equation depends only on r while the second term in this equation depends only on ϕ . This equation can therefore be only valid for every r and every ϕ if each term is equal to a constant. Thus we require that

$$\frac{1}{R(r)} \frac{\partial}{\partial r} \left(r \frac{\partial R(r)}{\partial r} \right) = \text{constant} = \gamma$$

and

$$\frac{1}{\alpha(\phi)} \frac{\partial^2 \alpha(\phi)}{\partial \phi^2} = -\gamma$$

First consider the case in which $\gamma = -m^2 < 0$. The differential equation for $\alpha(\phi)$ can be rewritten as

$$\frac{\partial^2 \alpha(\phi)}{\partial \phi^2} + m^2 \alpha(\phi) = 0$$

The most general solution of this differential solution is

$$\alpha_m(\phi) = C_m e^{m\phi} + D_m e^{-m\phi}$$

However, in cylindrical coordinates we require that any solution for a given ϕ is equal to the solution for $\phi + 2\pi$. Obviously this condition is not satisfied for this solution, and we conclude that $\gamma = m^2 \geq 0$. The differential equation for $\alpha(\phi)$ can be rewritten as

$$\frac{\partial^2 \alpha(\phi)}{\partial \phi^2} + m^2 \alpha(\phi) = 0$$

The most general solution of this differential equation is

$$\alpha_m(\phi) = C_m \cos(m\phi) + D_m \sin(m\phi)$$

The condition that $\alpha_m(\phi) = \alpha_m(\phi + 2\pi)$ requires that m is an integer. Now consider the radial function $R(r)$. We will first consider the case in which $\gamma = m^2 > 0$. Consider the following solution for $R(r)$:

$$R(r) = Ar^k$$

Substituting this solution into the previous differential equation we obtain

$$\frac{r}{Ar^k} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} (Ar^k) \right) = \frac{1}{Ar^{k-1}} \frac{\partial}{\partial r} (r(kAr^{k-1})) = \frac{1}{Ar^{k-1}} \frac{\partial}{\partial r} (kAr^k) = \frac{1}{Ar^{k-1}} k^2 Ar^{k-1} = k^2 = m^2$$

Therefore, the constant k can take on the following two values:

$$k_+ = m$$

$$k_- = -m$$

The most general solution for $R(r)$ under the assumption that $m^2 > 0$ is therefore

$$R(r) = A_m r^m + \frac{B_m}{r^m}$$

Now consider the solutions for $R(r)$ when $m^2 = 0$. In this case we require that

$$\frac{\partial}{\partial r} \left(r \frac{\partial R(r)}{\partial r} \right) = 0$$

or

$$r \frac{\partial R(r)}{\partial r} = \text{constant} = a_0$$

This equation can be rewritten as

$$\frac{\partial R(r)}{\partial r} = \frac{a_0}{r}$$

If $a_0 = 0$ then the solution of this differential equation is

$$R(r) = b_0 = \text{constant}$$

If $a_0 \neq 0$ then the solution of this differential equation is

$$R(r) = a_0 \ln(r) + b_0$$

Combining the solutions obtained for $m^2 = 0$ with the solutions obtained for $m^2 > 0$ we conclude that the most general solution for $R(r)$ is given by

$$R(r) = a_0 \ln(r) + b_0 + \sum_{m=1}^{\infty} \left[A_m r^m + \frac{B_m}{r^m} \right]$$

Therefore, the most general solution of Laplace's equation for a system with cylindrical symmetry is

$$V(r, \phi) = a_0 \ln(r) + b_0 + \sum_{m=1}^{\infty} \left[\left(A_m r^m + \frac{B_m}{r^m} \right) (C_m \cos(m\phi) + D_m \sin(m\phi)) \right]$$

Example: Problem 3.25

A charge density

$$\sigma = a \sin(5\phi)$$

is glued over the surface of an infinite cylinder of radius R . Find the potential inside and outside the cylinder.

The electrostatic potential can be obtained using the general solution of Laplace's equation for a system with cylindrical symmetry obtained in Problem 3.24. In the region inside the cylinder the coefficient B_m must be equal to zero since otherwise $V(r, \phi)$ would blow up at $r = 0$. For the same reason $a_0 = 0$. Thus

$$V(r, \phi) = b_{in,0} + \sum_{m=1}^{\infty} \left[r^m (C_{in,m} \cos(m\phi) + D_{in,m} \sin(m\phi)) \right]$$

In the region outside the cylinder the coefficients A_m must be equal to zero since otherwise $V(r, \phi)$ would blow up at infinity. For the same reason $a_0 = 0$. Thus

$$V(r, \phi) = b_{out,0} + \sum_{m=1}^{\infty} \left[\frac{1}{r^m} (C_{out,m} \cos(m\phi) + D_{out,m} \sin(m\phi)) \right]$$

Since $V(r, \phi)$ must approach 0 when r approaches infinity, we must also require that $b_{out,0}$ is equal to 0. The charge density on the surface of the cylinder is equal to

$$\sigma(\phi) = -\epsilon_0 \left[\frac{\partial V}{\partial r} \Big|_{r=R+} - \frac{\partial V}{\partial r} \Big|_{r=R-} \right]$$

Differentiating $V(r, \phi)$ in the region $r > R$ and setting $r = R$ we obtain

$$\frac{\partial V}{\partial r} \Big|_{r=R+} = \sum_{m=1}^{\infty} \left[-\frac{m}{R^{m+1}} (C_{out,m} \cos(m\phi) + D_{out,m} \sin(m\phi)) \right]$$

Differentiating $V(r, \phi)$ in the region $r < R$ and setting $r = R$ we obtain

$$\frac{\partial V}{\partial r} \Big|_{r=R-} = \sum_{m=1}^{\infty} \left[mR^{m-1} (C_{in,m} \cos(m\phi) + D_{in,m} \sin(m\phi)) \right]$$

The charge density on the surface of the cylinder is therefore equal to

$$\sigma(\phi) = \epsilon_0 \sum_{m=1}^{\infty} \left[\left(mR^{m-1} C_{in,m} + \frac{m}{R^{m+1}} C_{out,m} \right) \cos(m\phi) + \left(mR^{m-1} D_{in,m} + \frac{m}{R^{m+1}} D_{out,m} \right) \sin(m\phi) \right]$$

Since the charge density is proportional to $\sin(5\phi)$ we can conclude immediately that $C_{in,m} = C_{out,m} = 0$ for all m and that $D_{in,m} = D_{out,m} = 0$ for all m except $m = 5$. Therefore

$$\sigma(\phi) = \epsilon_0 \left(5R^4 D_{in,5} + \frac{5}{R^6} D_{out,5} \right) \sin(5\phi) = a \sin(5\phi)$$

This requires that

$$5R^4 D_{in,5} + \frac{5}{R^6} D_{out,5} = \frac{a}{\epsilon_0}$$

A second relation between $D_{in,5}$ and $D_{out,5}$ can be obtained using the condition that the electrostatic potential is continuous at any boundary. This requires that

$$V_{in}(R, \phi) = b_{in,0} + R^5 D_{in,5} \sin(5\phi) = V_{out}(R, \phi) = \frac{D_{out,5}}{R^5} \sin(5\phi)$$

Thus

$$b_{in,0} = 0$$

and

$$D_{out,5} = R^{10} D_{in,5}$$

We now have two equations with two unknown, $D_{in,5}$ and $D_{out,5}$, which can be solved with the following result:

$$D_{in,5} = \frac{a}{10\epsilon_0} \frac{1}{R^4}$$

and

$$D_{out,5} = \frac{a}{10\epsilon_0} R^6$$

The electrostatic potential inside the cylinder is thus equal to

$$V_{in}(r, \phi) = r^5 D_{in,5} \sin(5\phi) = \frac{a}{10\epsilon_0} \frac{r^5}{R^4} \sin(5\phi)$$

The electrostatic potential outside the cylinder is thus equal to

$$V_{out}(r, \phi) = \frac{D_{out,5}}{r^5} \sin(5\phi) = \frac{a}{10\epsilon_0} \frac{R^6}{r^5} \sin(5\phi)$$

Example: Problem 3.37

A conducting sphere of radius a , at potential V_0 , is surrounded by a thin concentric spherical shell of radius b , over which someone has glued a surface charge

$$\sigma(\theta) = \sigma_0 \cos\theta$$

where σ_0 is a constant.

a) Find the electrostatic potential in each region:

i) $r > b$

ii) $a < r < b$

b) Find the induced surface charge $\sigma(\theta)$ on the conductor.

c) What is the total charge of the system? Check that your answer is consistent with the behavior of V at large r .

a) The system has spherical symmetry and we can therefore use the most general solution of Laplace's equation in spherical coordinates:

$$V(r, \theta) = \sum_{m=0}^{\infty} \left(A_m r^m + \frac{B_m}{r^{m+1}} \right) P_m(\cos \theta)$$

In the region inside the sphere $B_m = 0$ since otherwise $V(r, \theta)$ would blow up at $r = 0$. Therefore

$$V(r, \theta) = \sum_{m=0}^{\infty} A_m r^m P_m(\cos \theta)$$

The boundary condition for $V(r, \theta)$ is that it is equal to V_0 at $r = a$:

$$V(a, \theta) = \sum_{m=0}^{\infty} A_m a^m P_m(\cos \theta) = V_0 = V_0 P_0(\cos \theta)$$

This immediately shows that $A_m = 0$ for all m except $m = 0$:

$$A_0 = V_0$$

The electrostatic potential inside the sphere is thus given by

$$V_{r < a}(r, \theta) = V_0$$

which should not come as a surprise.

In the region outside the shell $A_m = 0$ since otherwise $V(r, \theta)$ would blow up at infinity. Thus

$$V_{r > b}(r, \theta) = \sum_{m=0}^{\infty} \frac{B_{out, m}}{r^{m+1}} P_m(\cos \theta)$$

In the region between the sphere and the shell the most general solution for $V(r, \theta)$ is given by

$$V_{a < r < b}(r, \theta) = \sum_{m=0}^{\infty} \left(A_{in,m} r^m + \frac{B_{in,m}}{r^{m+1}} \right) P_m(\cos \theta)$$

The boundary condition for $V_{a < r < b}(r, \theta)$ at $r = a$ is

$$V_{a < r < b}(a, \theta) = \sum_{m=0}^{\infty} \left(A_{in,m} a^m + \frac{B_{in,m}}{a^{m+1}} \right) P_m(\cos \theta) = V_0 = V_0 P_0(\cos \theta)$$

This equation can only be satisfied if

$$A_{in,m} a^m + \frac{B_{in,m}}{a^{m+1}} = 0 \quad \text{if } m > 0$$

$$A_{in,0} + \frac{B_{in,0}}{a} = V_0 \quad \text{if } m = 0$$

The requirement that the electrostatic potential is continuous at $r = b$ requires that

$$\sum_{m=0}^{\infty} \left(A_{in,m} b^m + \frac{B_{in,m}}{b^{m+1}} \right) P_m(\cos \theta) = \sum_{m=0}^{\infty} \frac{B_{out,m}}{b^{m+1}} P_m(\cos \theta)$$

or

$$A_{in,m} b^m + \frac{B_{in,m}}{b^{m+1}} = \frac{B_{out,m}}{b^{m+1}}$$

This condition can be rewritten as

$$B_{out,m} - B_{in,m} = A_{in,m} b^{2m+1}$$

The other boundary condition for the electrostatic potential at $r = b$ is that it must produce the charge distribution given in the problem. This requires that

$$\begin{aligned} \sigma(\phi) &= -\epsilon_0 \left[\frac{\partial V}{\partial r} \Big|_{r=b+} - \frac{\partial V}{\partial r} \Big|_{r=b-} \right] = \epsilon_0 \sum_{m=0}^{\infty} \left(\frac{m+1}{b^{m+2}} (B_{out,m} - B_{in,m}) + m A_{in,m} b^{m-1} \right) P_m(\cos \theta) = \\ &= \sigma_0 \cos \theta = \sigma_0 P_1(\cos \theta) \end{aligned}$$

This condition is satisfied if

$$\frac{m+1}{b^{m+2}} (B_{out,m} - B_{in,m}) + m A_{in,m} b^{m-1} = 0 \quad \text{if } m \neq 1$$

$$\frac{2}{b^3}(B_{out,1} - B_{in,1}) + A_{in,1} = \frac{\sigma_0}{\epsilon_0} \quad \text{if } m=1$$

Substituting the relation between the various coefficients obtained by applying the continuity condition we obtain

$$\frac{m+1}{b^{m+2}} A_{in,m} b^{2m+1} + m A_{in,m} b^{m-1} = (2m+1) A_{in,m} b^{m-1} = 0 \quad \text{if } m \neq 1$$

$$\frac{2}{b^3} A_{in,1} b^3 + A_{in,1} = 3 A_{in,1} = \frac{\sigma_0}{\epsilon_0} \quad \text{if } m=1$$

These equations show that

$$A_{in,m} = 0 \quad \text{if } m \neq 1$$

$$A_{in,1} = \frac{\sigma_0}{3\epsilon_0} \quad \text{if } m=1$$

Using these values for $A_{in,m}$ we can show that

$$B_{out,m} - B_{in,m} = 0 \quad \text{if } m \neq 1$$

$$B_{out,1} - B_{in,1} = \frac{\sigma_0}{3\epsilon_0} b^3 \quad \text{if } m=1$$

The boundary condition for V at $r = a$ shows that

$$B_{in,m} = -A_{in,m} a^{2m+1} = 0 \quad \text{if } m \geq 2$$

$$B_{in,1} = -A_{in,1} a^3 = -\frac{\sigma_0}{3\epsilon_0} a^3 \quad \text{if } m=1$$

$$B_{in,0} = a(V_0 - A_{in,0}) = aV_0 \quad \text{if } m=0$$

These values for $B_{in,m}$ immediately fix the values for $B_{out,m}$:

$$B_{out,m} = B_{in,m} = 0 \quad \text{if } m \geq 2$$

$$B_{out,1} = \frac{\sigma_0}{3\epsilon_0} b^3 + B_{in,1} = \frac{\sigma_0}{3\epsilon_0} (b^3 - a^3) \quad \text{if } m=1$$

$$B_{out,0} = B_{in,0} = aV_0 \quad \text{if } m=0$$

The potential in the region outside the shell is therefore equal to

$$V_{r>b}(r, \theta) = \frac{aV_0}{r} P_0(\cos\theta) + \frac{\sigma_0}{3\epsilon_0} \frac{1}{r^2} (b^3 - a^3) P_1(\cos\theta)$$

The potential in the region between the sphere and the shell is equal to

$$V_{a<r<b}(r, \theta) = \frac{aV_0}{r} P_0(\cos\theta) + \frac{\sigma_0}{3\epsilon_0} \left(r - \frac{a^3}{r^2} \right) P_1(\cos\theta)$$

b) The charge density on the surface of the sphere can be found by calculating the slope of the electrostatic potential at this surface:

$$\sigma(\theta) = -\epsilon_0 \left[\frac{\partial V}{\partial r} \Big|_{r=a+} - \frac{\partial V}{\partial r} \Big|_{r=a-} \right] = -\epsilon_0 \left[-\frac{V_0}{a} + \frac{\sigma_0}{\epsilon_0} \cos\theta \right] = \frac{\epsilon_0 V_0}{a} - \sigma_0 \cos\theta$$

c) The total charge on the sphere is equal to

$$Q_a = \int_0^\pi \int_0^{2\pi} \sigma(\theta) a^2 \sin\theta d\theta d\phi = 2\pi a^2 \left\{ 2 \frac{\epsilon_0 V_0}{a} - \sigma_0 \int_0^\pi \cos\theta \sin\theta d\theta \right\} = 4\pi a \epsilon_0 V_0$$

The total charge on the shell is equal to zero. Therefore the total charge of the system is equal to

$$Q_{total} = 4\pi a \epsilon_0 V_0$$

The electrostatic potential at large distances will therefore be approximately equal to

$$V = \frac{1}{4\pi\epsilon_0} \frac{Q_{total}}{r} = \frac{1}{4\pi\epsilon_0} \frac{4\pi a \epsilon_0 V_0}{r} = \frac{aV_0}{r}$$

This is equal to limit of the exact electrostatic potential when $r \rightarrow \infty$.

3.4. Multipole Expansions

Consider a given charge distribution ρ . The potential at a point P (see Figure 3.11) is equal to

$$V(P) = \frac{1}{4\pi\epsilon_0} \int_{Volume} \frac{\rho}{d} d\tau$$

where d is the distance between P and a infinitesimal segment of the charge distribution. Figure 3.11 shows that d can be written as a function of r, r' and θ :

$$d^2 = r^2 + r'^2 - 2rr' \cos \theta = r^2 \left(1 + \left(\frac{r'}{r} \right)^2 - 2 \left(\frac{r'}{r} \right) \cos \theta \right)$$

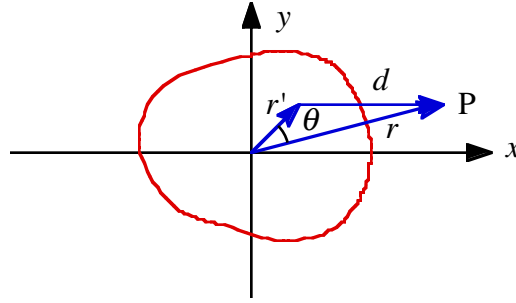


Figure 3.11. Charge distribution ρ .

This equation can be rewritten as

$$\frac{1}{d} = \frac{1}{r} \frac{1}{\sqrt{\left(1 + \left(\frac{r'}{r} \right)^2 - 2 \left(\frac{r'}{r} \right) \cos \theta \right)}}$$

At large distances from the charge distribution $r \gg r'$ and consequently $r'/r \ll 1$. Using the following expansion for $1/\sqrt{1+x}$:

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \dots$$

we can rewrite $1/d$ as

$$\begin{aligned} \frac{1}{d} &\approx \frac{1}{r} \left\{ 1 - \frac{1}{2} \left(\frac{r'}{r} \right) \left(\left(\frac{r'}{r} \right) - 2 \cos \theta \right) + \frac{3}{8} \left(\frac{r'}{r} \right)^2 \left(\left(\frac{r'}{r} \right) - 2 \cos \theta \right)^2 - \dots \right\} = \\ &= \frac{1}{r} \left\{ 1 + \left(\frac{r'}{r} \right) \cos \theta + \left(\frac{r'}{r} \right)^2 \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) + \dots \right\} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r} \right)^n P_n(\cos \theta) \end{aligned}$$

Using this expansion of $1/d$ we can rewrite the electrostatic potential at P as

$$V(P) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int_{Volume} \rho(\vec{r}') r'^n P_n(\cos\theta) d\tau$$

This expression is valid for all r (not only $r \gg r'$). However, if $r \gg r'$ then the potential at P will be dominated by the first non-zero term in this expansion. This expansion is known as the **multipole expansion**. In the limit of $r \gg r'$ only the first terms in the expansion need to be considered:

$$V(P) = \frac{1}{4\pi\epsilon_0} \left\{ \frac{1}{r} \int_{Volume} \rho d\tau + \frac{1}{r^2} \int_{Volume} \rho r' \cos\theta d\tau + \frac{1}{r^3} \int_{Volume} \rho r'^2 \left(\frac{3}{2} \cos^2\theta - \frac{1}{2} \right) d\tau + \dots \right\}$$

The first term in this expression, proportional to $1/r$, is called the **monopole term**. The second term in this expression, proportional to $1/r^2$, is called the **dipole term**. The third term in this expression, proportional to $1/r^3$, is called the **quadrupole term**.

3.4.1. The monopole term.

If the total charge of the system is non zero then the electrostatic potential at large distances is dominated by the monopole term:

$$V(P) = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \int_{Volume} \rho d\tau = \frac{1}{4\pi\epsilon_0} \frac{Q}{r}$$

where Q is the total charge of the charge distribution.

The electric field associated with the monopole term can be obtained by calculating the gradient of $V(P)$:

$$\vec{E}(P) = -\vec{\nabla} V(P) = -\frac{Q}{4\pi\epsilon_0} \vec{\nabla} \left(\frac{1}{r} \right) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r}$$

3.4.2. The dipole term.

If the total charge of the charge distribution is equal to zero ($Q = 0$) then the monopole term in the multipole expansion will be equal to zero. In this case the dipole term will dominate the electrostatic potential at large distances

$$V(P) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \int_{Volume} \rho r' \cos\theta d\tau$$

Since θ is the angle between \vec{r} and \vec{r}' we can rewrite $r' \cos \theta$ as

$$r' \cos \theta = \hat{r} \bullet \vec{r}'$$

The electrostatic potential at P can therefore be rewritten as

$$V(P) = \frac{1}{4\pi\epsilon_0} \frac{\hat{r}}{r^2} \bullet \int_{Volume} \rho \vec{r}' d\tau = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \bullet \hat{r}}{r^2}$$

In this expression \vec{p} is the **dipole moment** of the charge distribution which is defined as

$$\vec{p} = \int_{Volume} \rho \vec{r}' d\tau$$

The electric field associated with the dipole term can be obtained by calculating the gradient of $V(P)$:

$$E_r(P) = -\frac{\partial V(P)}{\partial r} = \frac{2p \cos \theta}{4\pi\epsilon_0} \frac{1}{r^3}$$

$$E_\theta(P) = -\frac{1}{r} \frac{\partial V(P)}{\partial \theta} = \frac{p \sin \theta}{4\pi\epsilon_0} \frac{1}{r^3}$$

$$E_\phi(P) = -\frac{1}{r \sin \theta} \frac{\partial V(P)}{\partial \phi} = 0$$

Example

Consider a system of two point charges shown in Figure 3.12. The total charge of this system is zero, and therefore the monopole term is equal to zero. The dipole moment of this system is equal to

$$\vec{p} = (-q)\vec{r}_- + (+q)\vec{r}_+ = q(\vec{r}_+ - \vec{r}_-) = q\vec{s}$$

where \vec{s} is the vector pointing from $-q$ to $+q$.

The dipole moment of a charge distribution depends on the origin of the coordinate system chosen. Consider a coordinate system S and a charge distribution ρ . The dipole moment of this charge distribution is equal to

$$\bar{p}_S = \int_{Volume} \rho \bar{r}_S d\tau$$

A second coordinate system S' is displaced by \bar{d} with respect to S :

$$\bar{r}_{S'} = \bar{r}_S + \bar{d}$$

The dipole moment of the charge distribution in S' is equal to

$$\bar{p}_{S'} = \int_{Volume} \rho \bar{r}_{S'} d\tau = \int_{Volume} \rho \bar{r}_S d\tau + \bar{d} \int_{Volume} \rho d\tau = \bar{p}_S + \bar{d}Q$$

This equation shows that if the total charge of the system is zero ($Q = 0$) then the dipole moment of the charge distribution is independent of the choice of the origin of the coordinate system.

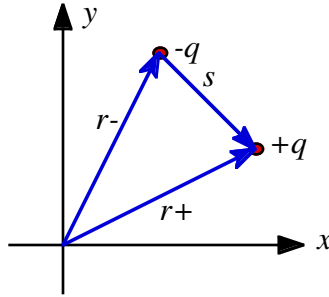


Figure 3.12. Electric dipole moment.

Example: Problem 3.40

A thin insulating rod, running from $z = -a$ to $z = +a$, carries the following line charges:

- a) $\lambda = \lambda_0 \cos\left(\frac{\pi z}{2a}\right)$
- b) $\lambda = \lambda_0 \sin\left(\frac{\pi z}{a}\right)$
- c) $\lambda = \lambda_0 \cos\left(\frac{\pi z}{a}\right)$

In each case, find the leading term in the multipole expansion of the potential.

- a) The total charge on the rod is equal to

$$Q_{tot} = \int_{-a}^{+a} \lambda dz = \int_{-a}^{+a} \lambda_0 \cos\left(\frac{\pi z}{2a}\right) dz = \frac{4a}{\pi} \lambda_0$$

Since $Q_{tot} \neq 0$, the monopole term will dominate the electrostatic potential at large distances. Thus

$$V_P = \frac{1}{4\pi\epsilon_0} \frac{4a}{\pi} \lambda_0 \frac{1}{r}$$

b) The total charge on the rod is equal to zero. Therefore, the electrostatic potential at large distances will be dominated by the dipole term (if non-zero). The dipole moment of the rod is equal to

$$p = \int_{-a}^{+a} z \lambda dz = \int_{-a}^{+a} z \lambda_0 \sin\left(\frac{\pi z}{a}\right) dz = \frac{2a^2}{\pi} \lambda_0$$

Since the dipole moment of the rod is not equal to zero, the dipole term will dominate the electrostatic potential at large distances. Therefore

$$V_P = \frac{1}{4\pi\epsilon_0} \frac{2a^2}{\pi} \lambda_0 \frac{1}{r^2} \cos\theta$$

c) For this charge distribution the total charge is equal to zero and the dipole moment is equal to zero. The electrostatic potential of this charge distribution is dominated by the quadrupole term.

$$I_2 = \int_{-a}^{+a} z^2 \lambda dz = \int_{-a}^{+a} z^2 \lambda_0 \cos\left(\frac{\pi z}{a}\right) dz = \frac{4a^3}{\pi^2} \lambda_0$$

The electrostatic potential at large distance from the rod will be equal to

$$V_P = \frac{1}{4\pi\epsilon_0} \left(-\frac{4a^3}{\pi^2} \lambda_0 \right) \frac{1}{r^3} \frac{1}{2} (3 \cos^2 \theta - 1)$$

Example: Problem 3.27

Four particles (one of charge q , one of charge $3q$, and two of charge $-2q$) are placed as shown in Figure 3.12, each a distance d from the origin. Find a simple approximate formula for the electrostatic potential, valid at a point P far from the origin.

The total charge of the system is equal to zero and therefore the monopole term in the multipole expansion is equal to zero. The dipole moment of this charge distribution is equal to

$$\vec{p} = \sum_i q_i \vec{r}_i = (-2q)d\hat{j} + (q)(-d)\hat{k} + (-2q)(-d)\hat{j} + (3q)d\hat{k} = 2qd\hat{k}$$

The Cartesian coordinates of P are

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

The scalar product between \vec{p} and \hat{r} is therefore

$$\vec{p} \bullet \hat{r} = 2qd \cos\theta$$

The electrostatic potential at P is therefore equal to

$$V_P = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \bullet \hat{r}}{r^2} = \frac{1}{4\pi\epsilon_0} \frac{2qd \cos\theta}{r^2}$$

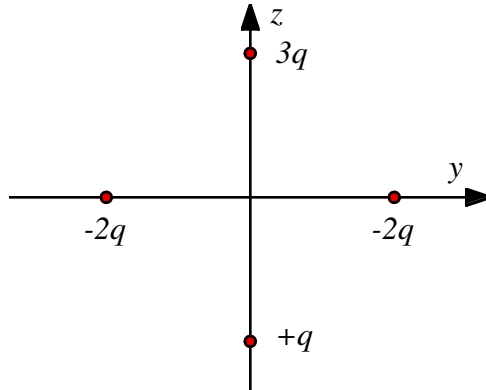


Figure 3.13. Problem 3.27.

Example: Problem 3.38

A charge Q is distributed uniformly along the z axis from $z = -a$ to $z = a$. Show that the electric potential at a point (r, θ) is given by

$$V(r, \theta) = \frac{Q}{4\pi\epsilon_0} \frac{1}{r} \left(1 + \frac{1}{3} \left(\frac{a}{r} \right)^2 P_2(\cos\theta) + \frac{1}{5} \left(\frac{a}{r} \right)^4 P_4(\cos\theta) + \dots \right)$$

for $r > a$.

The charge density along this segment of the z axis is equal to

$$\rho = \frac{Q}{2a}$$

Therefore, the n th moment of the charge distribution is equal to

$$I_n = \int_{-a}^a z^n \rho dz = \frac{Q}{2a} \int_{-a}^a z^n dz = \frac{Q}{2a} \frac{z^{n+1}}{n+1} \Big|_{-a}^a = \frac{Q}{2a} \frac{a^{n+1}}{n+1} \{1 - (-1)^{n+1}\} = \frac{Q}{2} \frac{a^n}{n+1} \{1 - (-1)^{n+1}\}$$

This equation immediately shows that

$$I_n = \frac{a^n}{n+1} Q \quad \text{if } n \text{ is even}$$

$$I_n = 0 \quad \text{if } n \text{ is odd}$$

The electrostatic potential at P is therefore equal to

$$V(r, \theta) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} I_n P_n(\cos\theta) = \frac{Q}{4\pi\epsilon_0} \frac{1}{r} \left(1 + \frac{1}{3} \left(\frac{a}{r} \right)^2 P_2(\cos\theta) + \frac{1}{5} \left(\frac{a}{r} \right)^4 P_4(\cos\theta) + \dots \right)$$