— Problemas I —

Ej. $$I = \int_0^\infty \int_0^\infty x^\alpha y^\beta \operatorname{sen}(x^2 y)\, e^{-\frac{x}{y}}\, dx\, dy$$

donde

$$\operatorname{sen}(x^2 y) = \sum_n \frac{(-1)^n}{\Gamma(2n+2)} x^{4n+2} y^{2n+1}$$

$$= \sum_n \phi_n \frac{\Gamma(n+1)}{\Gamma(2n+2)} x^{4n+2} y^{2n+1}$$

$$e^{-\frac{x}{y}} = \sum_m \phi_m x^m y^{-m}$$

$$\therefore \quad I = \sum_n \sum_m \phi_{n,m} \frac{\Gamma(n+1)}{\Gamma(2n+2)} \underbrace{\int_0^\infty x^{4n+3+m+\alpha-1} dx}_{\langle 4n+3+m+\alpha \rangle} \underbrace{\int_0^\infty y^{2n+2-m+\beta-1} dy}_{\langle 2n+2-m+\beta \rangle}$$

(Notación $\phi_n \phi_m$ corta)

Luego la integral se transforma en serie de brackets:

$$I = \sum_n \sum_m \phi_{n,m} \frac{\Gamma(n+1)}{\Gamma(2n+2)} \langle 4n+3+m+\alpha \rangle \langle 2n+2-m+\beta \rangle$$

La anulación de los brackets produce el siguiente sistema de ecs. lineales:

$$\underbrace{\begin{pmatrix} 4 & 1 \\ 2 & -1 \end{pmatrix}}_{A} \begin{pmatrix} n \\ m \end{pmatrix} = \begin{pmatrix} -3-\alpha \\ -2-\beta \end{pmatrix}$$

con $\det(A) = -6$ y $n^* = -\frac{\alpha}{6} - \frac{\beta}{6} - \frac{5}{6}$

$m^* = -\frac{\alpha}{3} + \frac{2}{3}\beta + 1/3$

La solución final es entonces:

$$I = \frac{1}{|\det(A)|} \Gamma(-m)\Gamma(-n) \frac{\Gamma(n+1)}{\Gamma(2n+2)} \Bigg|_{\substack{n=n^* \\ m=m^*}}$$

$$\Downarrow$$

$$I = \frac{1}{6} \frac{\Gamma\left(\frac{\alpha}{3} - \frac{2}{3}\beta - \frac{1}{3}\right)\Gamma\left(\frac{\alpha}{6} + \frac{\beta}{6} + \frac{5}{6}\right)\Gamma\left(\frac{1}{6} - \frac{\beta}{6} - \frac{\alpha}{6}\right)}{\Gamma\left(\frac{1}{3} - \frac{\beta}{3} - \frac{\alpha}{3}\right)}$$