PHY481 - Lecture 23: Inductance and mutual inductance
Griffiths: Chapter 7

Microscopic origin of conductivity

Consider that we have a number density (number per unit volume), n_q, of charged carriers which each have charge q. Suppose that these charge carriers have net "drift" velocity $\vec{v}_d$ in a specified direction. The current density is then given by,

$$\vec{j} = n_q q \vec{v}_d \tag{1}$$

The physical origin of the drift velocity is most easily seen using the so-called free electron model. In this model, metals are composed of a sea of free electrons which floats in between positive nuclei whose positions are essentially fixed (there are only some small vibrations about these fixed positions). The electrons move at high speed and in random directions. They scatter off each other, off the positive nuclei and from impurities. The time between scattering events is τ the scattering time of the material. This varies from metal to metal. Now consider applying an electric field in a given direction. The electric field leads to an acceleration $ma = -eE$ of each electron. Over a time τ, the velocity of the electron thus increases by $eE\tau/m$ in a direction opposite to the direction of the applied field. This is the drift velocity,

$$v_d = \frac{-eE\tau}{m} \tag{2}$$

Note that after each scattering event, the velocity of the electron is considered to be random. Using $\vec{j} = n_q q \vec{v}_d$, we find,

$$\vec{j} = -\frac{n_e e^2 \tau}{m} E \tag{3}$$

Within the free electron model we then have the elegant formula,

$$\sigma_r = \frac{n_e e^2 \tau}{m} \tag{4}$$

which provides a connection between the continuum variables σ_r, ρ_r and the behavior of electrons at the atomic scale. These ideas form the basis for understanding most types of conduction processes.

Charging a capacitor

Consider a capacitor that is originally uncharged and it is in series with a battery with DC voltage V_0, a resistor R, and a switch that is initially open. If we close the switch at $t = 0$ the battery charges until it reaches charge $Q_0 = CV_0$ and current ceases to flow. The charging of the capacitor is described by,

$$V_0 - iR - \frac{Q}{C} = 0 \quad \text{or} \quad V_0 - R\frac{dQ}{dt} - \frac{Q}{C} = 0 \tag{5}$$

This first order linear differential equation has solution $Q(t) = Q_0(1 - e^{\frac{-t}{RC}})$ Energy is transfered from the battery to the capacitor. The energy is stored in the electric field. What happens with magnetic fields?

Self Inductance

Faraday's law relates the induced emf to the change in magnetic flux, however the magnetic flux in circuits is often generated by currents, so we would like to relate flux to the current and to write Faraday's law in terms of the current. This is easy to do as the flux comes from the magnetic field and the magnetic field is proportional to the current, as is seen from the Biot-Savart law or from Ampere's law, therefore $\phi_B \propto i$. In the case of one loop, the flux through the loop is defined to be $\phi_B = Li$, where L is the self-inductance, that is usually abbreviated to simply the "inductance". If there are N loops, then the "flux linkage" $N\phi_B = Li$. Taking a time derivative of this equation and using Faraday's law we have,

$$\epsilon = -N\frac{d\phi_B}{dt} = -L\frac{di}{dt} \tag{6}$$

The self-inductance, L, depends on the geometry and on the materials used but not on the current or the voltage supplied to the circuit.

Calculation of L is important as every circuit has a self inductance and its influence can be dominant in some frequency ranges. In general inductance is important at high frequency as there the current is changing rapidly so that the flux changes rapidly leading to high voltage through Faraday's law. In most cases calculation of L has to be carried out numerically or at least is quite complicated. However there are a few cases where it can be done analytically, and these cases are tabulated. We now go through three simple cases.

Solenoid

We find L by considering a solenoid with N turns and length l, and cross-sectional area A in air. A current i passes through the solenoid, and from Faraday's law we have $N\phi_B = Li$ (setting the constant of integration to zero). Since the magnetic field in a solenoid is constant and takes the value $B = ni\mu_0$ (where $n = N/l$), the flux is $\phi_B = NiA\mu_0/l$, so that,

$$N\phi_B = Li \quad \text{implies} \quad L = N^2 A\mu_0/l \tag{7}$$

Coaxial

Consider a co-axial consisting of a thin conducting cylinder of length l and of radius a that is concentric with a thin conducting cylinder of radius $b > a$. The flux through a loop that passes along the inner cylinder and then returns along the outer cylinder is given by,

$$\phi_B = \int \vec{B} \cdot d\vec{A} = l \int_a^b \frac{\mu_0 i}{2\pi r} dr = \frac{\mu_0 il}{2\pi} ln(b/a) \tag{8}$$

so the inductance is $L = \mu_0 l ln(b/a)/(2\pi)$.

Rectangular loop

It is harder to do a square loop as it is not trivial to find the magnetic field everywhere (we only found it on the z-axis). However if we take a rectangular loop with sides l and a and $l/a \to \infty$, then we can use the result for a field near a wire (multiplied by two), ie,

$$\phi_B = 2l \int_w^a \frac{\mu_0 i}{2\pi r} dr = 2\frac{\mu_0 ln(a/w)li}{2\pi} \tag{9}$$

The factor of two is due to the presence of two wires. The inductance is then, $L = \mu_0 ln(a/w)l/\pi$. In this expression, the wire is assumed to have radius w and current is assumed to be flowing at the surface of the wire.

Energizing an inductor

Consider an inductor, L, that is initially in a zero current condition. It is connected in series with an applied DC voltage V_0, a resistor R, and a switch that is initially open. The switch is closed at time $t = 0$. Assuming that the inductor has no internal resistance, the final current in the cicuit is $i_0 = V_0/R$. The current as a function of time is given by,

$$V_0 = iR + L\frac{di}{dt} \tag{10}$$

Solving this equation, we find that $i(t) = i_0(1 - e^{-Rt/L})$. Energy is transferred from the battery to energize the inductor. Energy is stored in the magnetic field.

Mutual inductance (transformers, wireless energy transfer, em communications)

Consider a drive coil carrying current i_1 and a pickup coil through which flux ϕ_2 passes. This flux is generated by the drive coil. We write,

$$N_2\phi_2 = M_{12}i_1 \quad \text{so that} \quad V_2 = -N_2\frac{d\phi_2}{dt} = -M_{12}\frac{di_1}{dt} \tag{11}$$

This coupling is the basis of antennas, transformers and a variety of other essential elements of technology. Note that we still have $N_1\phi_1 = Li_1$ so the mutual inductance is in addition to the self-inductance. Calculations of mutual inductance are similar to that of self-inductance. Here are two examples.

A rectangular loop near a wire carrying current i_1, and $N_2 = N_1 = 1$

We take the loop to have its long sides, of length l_2, parallel to a long straight wire carrying current i_1. The side which is closest to the wire is distance a from the wire. The short sides have length b_1, and the normal to the loop

is in the $\hat{\phi}$ direction with respect to the axis of the wire. In that case, we need to find $\phi_2 = M_{12}i_1$, so we have a by now quite familiar calculation,

$$\phi_2 = \int_0^l dy \int_a^{a+b} \frac{\mu_0 i}{2\pi r} = \frac{\mu_0 i l}{2\pi} ln((a+b)/a), \quad \text{so that} \quad M_{12} = \frac{\mu_0 l}{2\pi} ln((a+b)/a) \tag{12}$$

Two solenoids - one inside the other

The two solenoids are characterized by cross-sectional areas A_1, A_2 turns N_1, N_2 and lengths $l_1 = l_2 = l$, with the former having $A_1 > A_2$. We also take the lengths of both solenoids to be much larger than their radii, so the formula $B = ni\mu_0$ applies, with $n = N/l$.

If a current i_1 flows in the larger solenoid, the flux in the smaller solenoid is then $\phi_2 = N_1 i_1 A_2 \mu_0/l$ and so,

$$N_2\phi_2 = M_{12}i_1 \quad \text{implies that} \quad M_{12} = N_1 N_2 \mu_0 A_2/l \tag{13}$$

Now lets consider the reverse situation where a current i_2 flows in the smaller solenoid. In that case there is no magnetic field outside the smaller solenoid, so the flux in the larger solenoid is, $\phi_1 = N_2 i_2 \mu_0/l$, then using

$$N_1\phi_1 = M_{12}i_2 \quad \text{implies that} \quad M_{21} = N_1 N_2 \mu_0 A_2/l \tag{14}$$