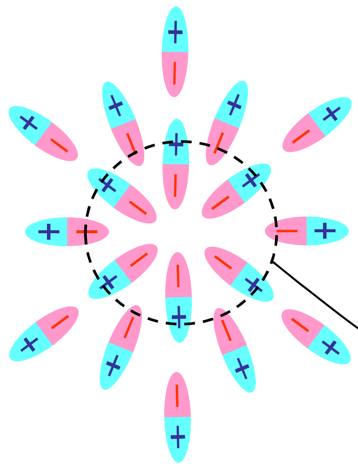
# Chapter 4. Electric Fields in Matter

# 4.2 The Field of a Polarized Object

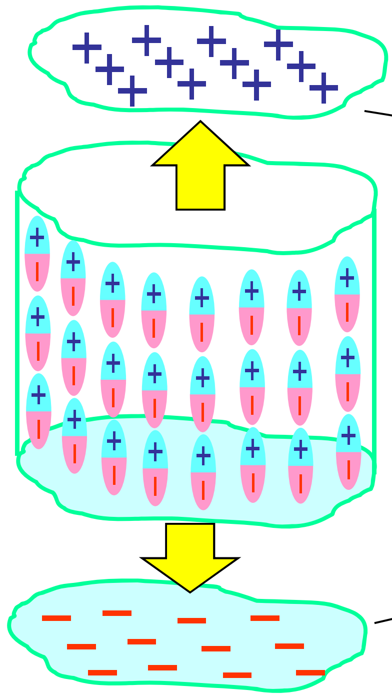
***What is the field produced by P in a polarized material?***
***(not the field that may have caused the polarization, but the field the polarization itself causes)***

## 4.2.1 Bound Charges

The polarization **P** can produces ***bound volume and surface charges***.



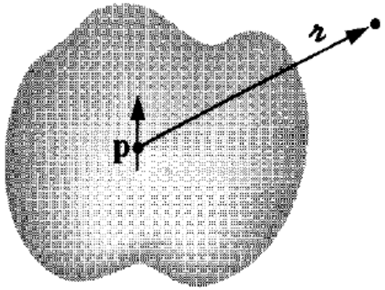
$$\rho_b = -\nabla \cdot \mathbf{P}$$



$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$$

## 4.2.1 Bound Charges

For a single dipole **p**, the potential is, $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{p}}{\mathscr{r}^2}$




A dipole moment in each volume element dτ': $\mathbf{p} = \mathbf{P}\,d\tau'$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{P}(\mathbf{r}')}{\mathscr{r}^2}\,d\tau'$$

$$\nabla'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$ ← the differentiation is with respect to the *source* coordinates (r')

$$V = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \mathbf{P}\cdot\nabla'\left(\frac{1}{\mathscr{r}}\right)d\tau'$$

Integrating by parts , $$V = \frac{1}{4\pi\epsilon_0}\left[\int_{\mathcal{V}} \nabla'\cdot\left(\frac{\mathbf{P}}{\mathscr{r}}\right)d\tau' - \int_{\mathcal{V}} \frac{1}{\mathscr{r}}(\nabla'\cdot\mathbf{P})\,d\tau'\right]$$

Using the divergence theorem, $$V = \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{1}{\mathscr{r}}\mathbf{P}\cdot d\mathbf{a}' - \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{1}{\mathscr{r}}(\nabla'\cdot\mathbf{P})\,d\tau'$$

$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{\sigma_b}{\mathscr{r}}\,da' + \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\rho_b}{\mathscr{r}}\,d\tau'}$$

$\rho_b \equiv -\nabla\cdot\mathbf{P}$ **: Volume charge density**

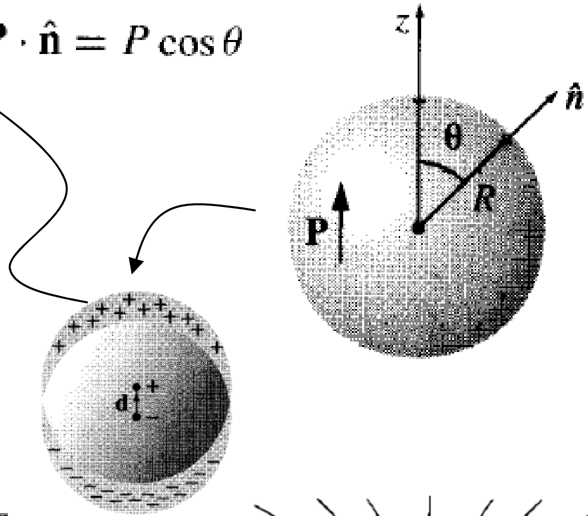
$\sigma_b \equiv \mathbf{P}\cdot\hat{\mathbf{n}}$ **: Surface charge density**

# Potential induced by polarization

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{\sigma_b}{\mathscr{r}}\,da' + \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\rho_b}{\mathscr{r}}\,d\tau' \qquad \rho_b \equiv -\nabla\cdot\mathbf{P} \qquad \sigma_b \equiv \mathbf{P}\cdot\hat{\mathbf{n}}$$

**Example 4.2** Find the electric field produced by **a uniformly polarized sphere** of radius *R*.

$\rho_b = 0$, since **P** is uniform. $\quad \sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = P\cos\theta$

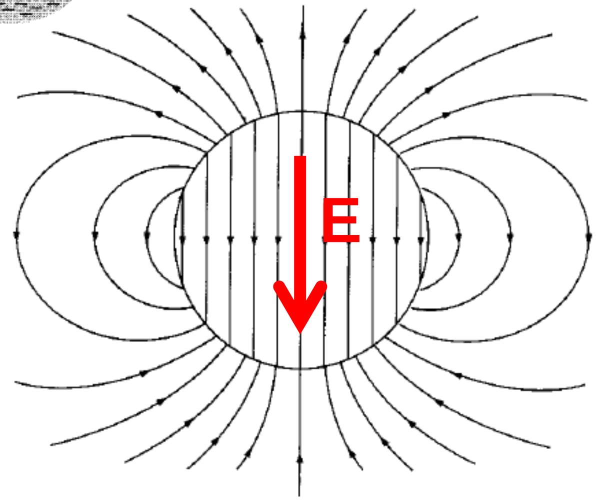



From Eqs. (3.85), (3.86), and (3.87) in Example 3.9,

$$V(r,\theta) = \begin{cases} \dfrac{P}{3\epsilon_0} r\cos\theta, & \text{for } r \le R, \\[2ex] \dfrac{P}{3\epsilon_0}\dfrac{R^3}{r^2}\cos\theta, & \text{for } r \ge R. \end{cases}$$

The *field* inside the sphere is uniform, since $r\cos\theta = z$,

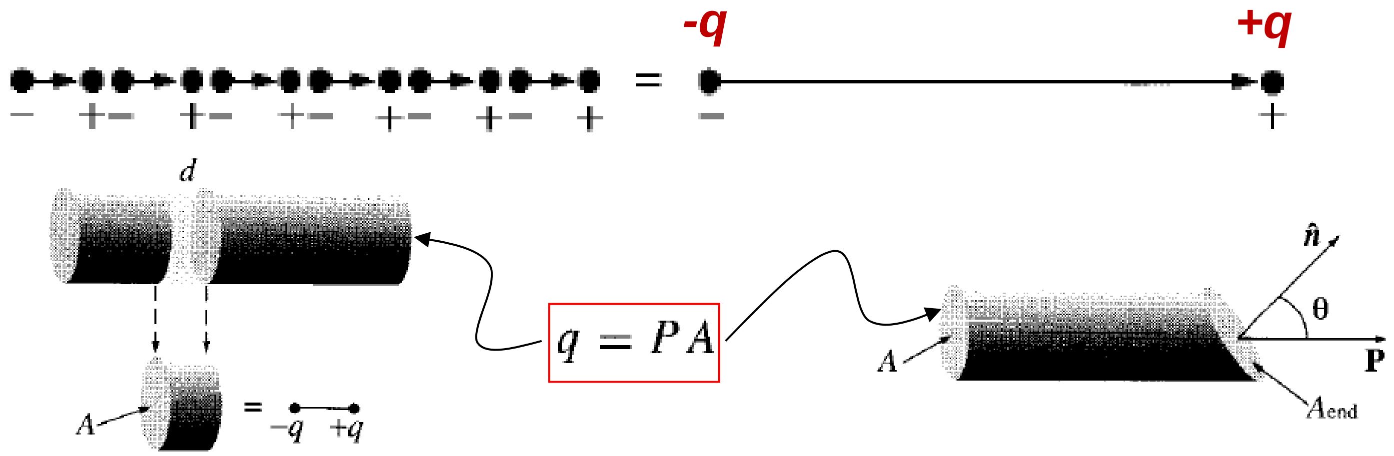
$$\mathbf{E} = -\nabla V = -\frac{P}{3\epsilon_0}\hat{\mathbf{z}} = -\frac{1}{3\epsilon_0}\mathbf{P}, \quad \text{for } r < R.$$




Outside the sphere the potential is identical to that of a perfect dipole at the origin,

$$V = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}, \quad \text{for } r \ge R, \qquad \mathbf{p} = \tfrac{4}{3}\pi R^3\mathbf{P}$$

# 4.2.2 Physical Interpretation of Bound Charges




If the ends have been sliced off perpendicularly,

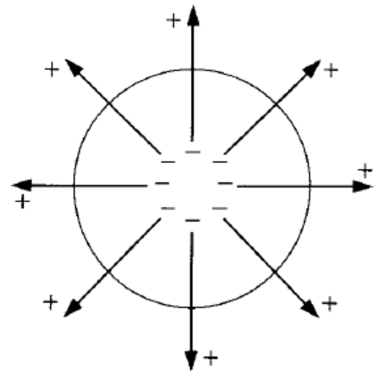
$$\sigma_b = \frac{q}{A} = P$$

For an oblique cut, $A = A_{\text{end}} \cos\theta$,

$$\sigma_b = \frac{q}{A_{\text{end}}} = P\cos\theta = \mathbf{P}\cdot\hat{\mathbf{n}}$$

If the polarization is nonuniform, we get accumulations of bound charge *within* the material.

The net bound charge $\int \rho_b \, d\tau$ in a given volume is equal and opposite to the amount that has been pushed out through the surface:



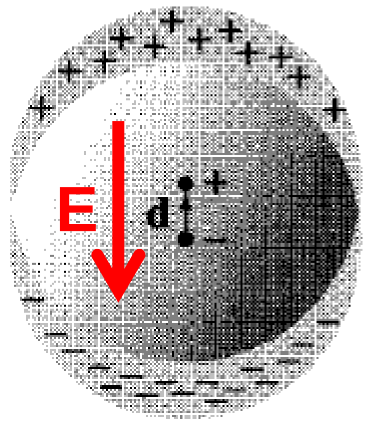
$$\int_{\mathcal{V}} \rho_b \, d\tau = -\oint_{\mathcal{S}} \mathbf{P}\cdot d\mathbf{a} = -\int_{\mathcal{V}} (\nabla\cdot\mathbf{P})\, d\tau$$

# Electric field produced by a uniformly polarized sphere

**Example 4.3** Another way of analyzing the uniformly polarized sphere.

(Example 4.2) **Electric field produced by a uniformly polarized sphere.**

*Consider two* spheres of charge:
A positive sphere and a negative sphere.



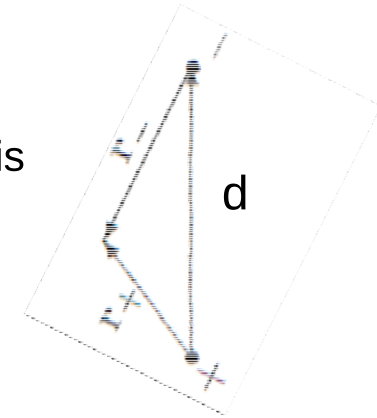

$$\mathbf{E} = -\frac{1}{3\epsilon_0}\mathbf{P}$$

The field inside a uniformly charged sphere is

$$\oint \mathbf{E}\cdot d\mathbf{a} = E \cdot 4\pi r^2 = \frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{1}{\epsilon_0}\frac{4}{3}\pi r^3 \rho.$$

$$\longrightarrow \quad \mathbf{E} = \frac{1}{3\epsilon_0}\rho r \hat{\mathbf{r}}$$

The field in the region of overlap between two uniformly charged spheres is




$$\mathbf{E} = \frac{\rho}{3\epsilon_0}(\mathbf{r}_+ - \mathbf{r}_-) = \frac{\rho}{3\epsilon_0}\mathbf{d} \quad \longrightarrow \quad \mathbf{E} = -\frac{1}{4\pi\epsilon_0}\frac{q\mathbf{d}}{R^3}$$

$$\mathbf{p} = q\mathbf{d} = \left(\frac{4}{3}\pi R^3\right)\mathbf{P} \quad \longrightarrow \quad \boxed{\mathbf{E} = -\frac{1}{3\epsilon_0}\mathbf{P}}$$

# 4.2.3 The Field Inside a Dielectric

***The electric field inside matter must be very complicated, on the microscopic level***

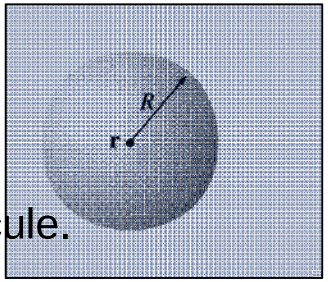
Moreover, an instant later, as the atoms move about, the field will have altered entirely.

→ This true microscopic field would be utterly impossible to calculate!

Therefore, consider the **macroscopic field** defined as the *average* field over regions large enough to contain many thousands of atoms.

➔ ***The field inside matter means the macroscopic field in usual.***

***Suppose the macroscopic field at some point r within a dielectric.***




For a small sphere about **r,** of radius, say, a thousand times the size of a molecule.

The macroscopic field at **r,** then, consists of two parts:

$$\mathbf{E} = \mathbf{E}_{\text{out}} + \mathbf{E}_{\text{in}}$$

→ the average field over the sphere due to all charges *outside,*

→ plus the average due to all charges *inside:*

Inside the sphere, we know already the field: $\mathbf{E}_{\text{in}} = -\frac{1}{3\epsilon_0}\mathbf{P}$

Outside the sphere, the potential is $V_{\text{out}} = \frac{1}{4\pi\epsilon_0}\int_{\text{outside}} \frac{\hat{\mathbf{\imath}} \cdot \mathbf{P}(\mathbf{r}')}{\imath^2}\, d\tau' \longrightarrow \mathbf{E}_{\text{out}} = -\frac{1}{3\epsilon_0}\mathbf{P}$

by assumption the sphere is small enough that P does not vary significantly over volume,

➔ **The macroscopic field, then, is given by the potential:** $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\mathbf{\imath}} \cdot \mathbf{P}(\mathbf{r}')}{\imath^2}\, d\tau'$

# The Field Inside a Dielectric

***The field inside matter means the macroscopic field in usual.***

➔ **The macroscopic field, then, is given by the potential:**

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\boldsymbol{\imath}}\cdot\mathbf{P}(\mathbf{r}')}{\imath^2}\,d\tau'$$ where the integral runs over the *entire* volume.

➔ The macroscopic field is certainly independent of the geometry of the averaging region.

➔ Therefore, the field inside a dielectric can be determined from Eq. (4.13):

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\oint_{\mathcal{S}} \frac{\sigma_b}{\imath}\,da' + \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\rho_b}{\imath}\,d\tau'$$

**Problem 4.10** A sphere of radius *R* carries a polarization $\mathbf{P}(\mathbf{r}) = k\mathbf{r}$

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \boxed{kR;} \qquad \rho_b = -\nabla\cdot\mathbf{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 kr) = -\frac{1}{r^2}3kr^2 = \boxed{-3k.}$$

For $r < R$, $\mathbf{E} = \frac{1}{3\epsilon_0}\rho r\,\hat{\mathbf{r}}$, so $\mathbf{E} = \boxed{-(k/\epsilon_0)\,\mathbf{r}.}$

For $r > R$, $Q_{\text{tot}} = (kR)(4\pi R^2) + (-3k)(\frac{4}{3}\pi R^3) = 0$, so $\boxed{\mathbf{E} = \mathbf{0}.}$