Chapter 3. Potentials

3.2 Method of Images

3.2.1 The Classic Image Problem

Suppose a point charge q is held at d **above an infinite grounded conducting plane**.

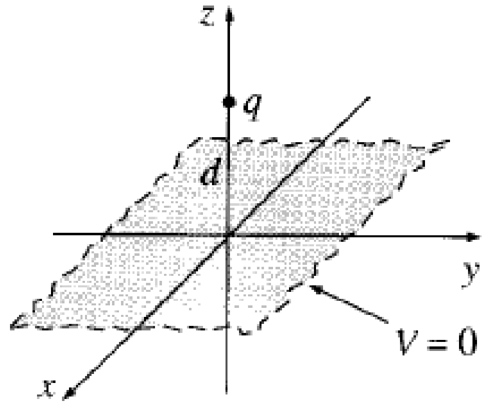
Question: What is the potential in the region above the plane?

→ q will induce a certain amount of negative charge on the nearby surface of the conductor.

→ how can we possibly calculate the potential?

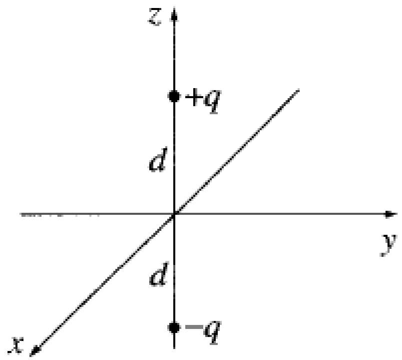
→ we don't know how much charge is induced,

→ or how it is distributed.



This problem is to solve Poisson's equation in the region z > 0, with a single point charge q at (0, 0, *d)*, subject to **the boundary conditions**:

1. $V = 0$ when $z = 0$ (since the conducting plane is grounded),
2. V -+ 0 far from the charge.

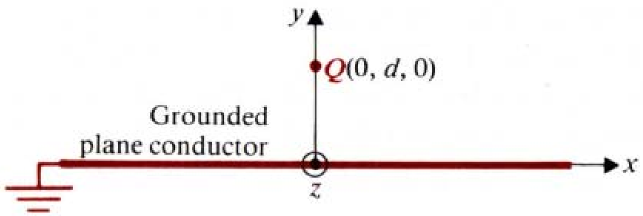


***Trick:* Forget about the actual problem.**

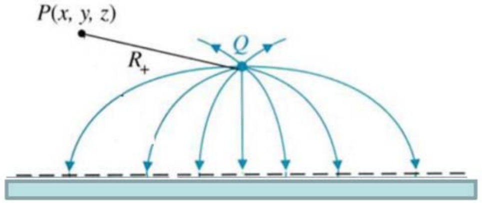
Consider two point charges, +q at (0, 0, d) and -q at (0,0, -d), and no conducting plane.

It produces exactly the same potential as the original configuration, in the "upper" region z > 0.

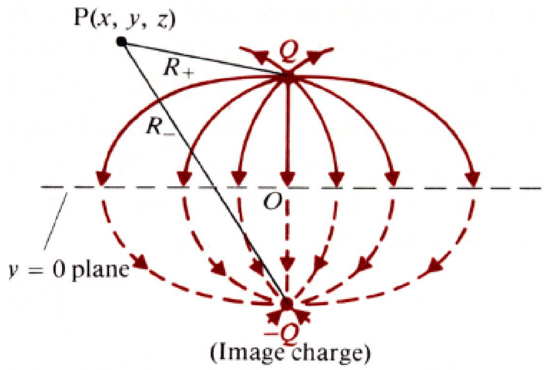
The Classic Image Problem



(a) Physical arrangement.



Induced charge (-q)



(b) Image charge and field lines.

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z+d)^2}}\right]$$

This solution also follows that

1. $V = 0$ when $z = 0$ (since the conducting plane is grounded),
2. V -+ 0 far from the charge.

Notice the **crucial role played by the uniqueness theorem** in this argument:

→ without it, no one would believe this solution,
since it was obtained for a completely different charge distribution.

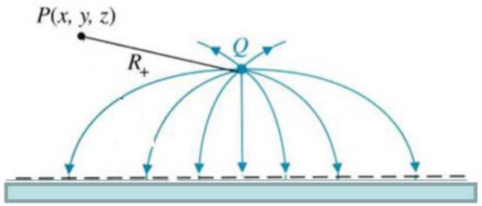
But the uniqueness theorem certifies it:

→ If it satisfies Poisson's equation in the region of interest, and assumes the correct value at the boundaries,

→ **then it must be right!**

3.2.2 Induced Surface Charge

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z+d)^2}}\right]$$



Induced charge (-q)

Boundary condition requires that

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n} \longleftarrow \mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$$

$$\sigma = -\epsilon_0 \left.\frac{\partial V}{\partial z}\right|_{z=0}$$ (in this case)

$$\frac{\partial V}{\partial z} = \frac{1}{4\pi\epsilon_0}\left\{\frac{-q(z-d)}{[x^2 + y^2 + (z-d)^2]^{3/2}} + \frac{q(z+d)}{[x^2 + y^2 + (z+d)^2]^{3/2}}\right\}$$

$$\Rightarrow \quad \sigma(x, y) = \frac{-qd}{2\pi(x^2 + y^2 + d^2)^{3/2}}$$

The total induced charge is:

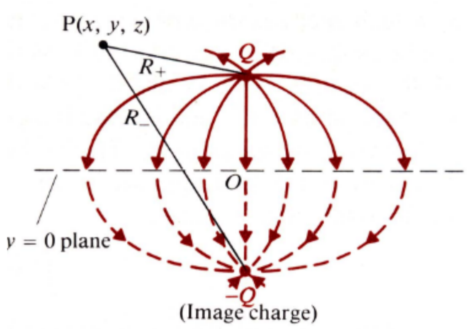
$Q = \int \sigma\, da$ → a little easier to use polar coordinates *(r. ϕ),* $\sigma(r) = \frac{-qd}{2\pi(r^2 + d^2)^{3/2}}$

$$= \int_0^{2\pi}\int_0^{\infty} \frac{-qd}{2\pi(r^2 + d^2)^{3/2}} r\, dr\, d\phi = \left.\frac{qd}{\sqrt{r^2 + d^2}}\right|_0^{\infty} = -q$$

3.2.3 Force and Energy

The charge q is attracted toward the plane, because of the induced charge $-q$.

$$\mathbf{F} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)^2}\hat{\mathbf{z}}$$



(b) Image charge and field lines.

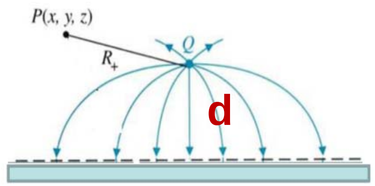
Energy, however, is *not* the same in the two cases.

(1) With the two point charges and no conductor:

$$W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{2d}$$

(2) For a single charge and conducting plane the **energy is *half*** of this:

$$W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d}$$



Induced charge (-q)

(Why?) Think of the energy stored in the fields: $W = \frac{\epsilon_0}{2}\int E^2\, d\tau$

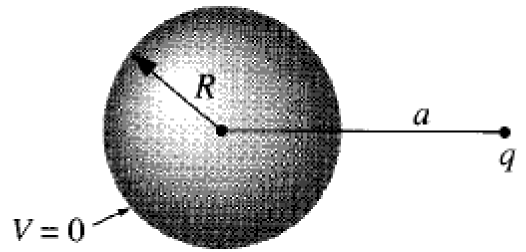
In the first case → both the regions ($z > 0$ and $z < 0$) contribute equally.
But in the second case → only the upper region contains a nonzero field,

(Why?) By calculating the work required to bring q in from infinity:

$$W = \int_{\infty}^{d} \mathbf{F}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\int_{\infty}^{d}\frac{q^2}{4z^2}\,dz = \frac{1}{4\pi\epsilon_0}\left(-\frac{q^2}{4z}\right)\Big|_{\infty}^{d} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d}$$

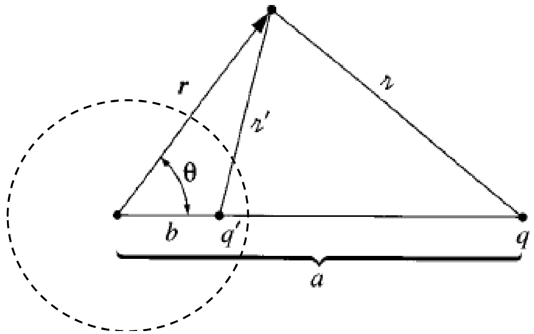
3.2.4 Other Image Problems: Method of images

Example 3.2 Find the potential outside the sphere.



The image charge is $q' = -\frac{R}{a} q$

placed a distance $b = \frac{R^2}{a}$



The potential of this configuration is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right)$$

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{q}{\sqrt{R^2 + (ra/R)^2 - 2ra\cos\theta}}\right]$$