Method of Images for Magnetostatics

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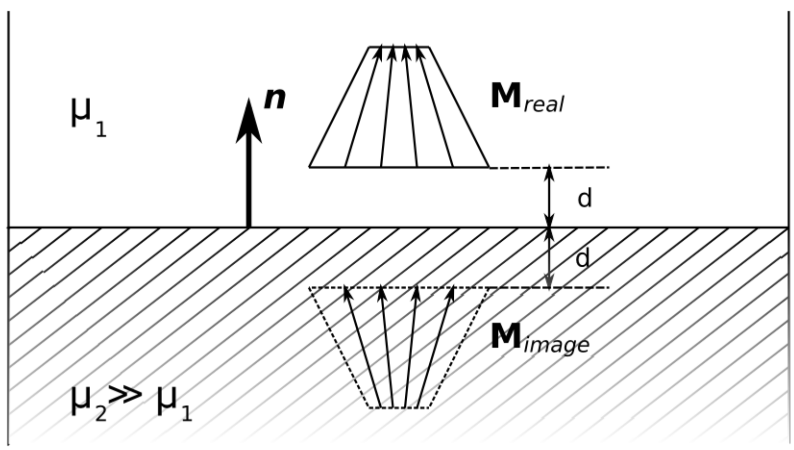

Fig. 1:
Cross section of a truncated cone with magnetization $\vec{M}_{real}$ embeded in a region of low permeability μ_1. The cone is a distance d from a region of highly permeable material μ_2. Both regions are separated by a plane defined by the normal vector $\vec{n}$. The associated image magnetization $\vec{M}_{image}$ is also displayed. (Note the pseudovector nature of $\vec{M}$.)

Introduction

In electrostatics you may have encountered a method for solving field equations known as the method of images. In electrostatics this method is often exploited due to a physical property of conducting surfaces. A conducting surface is necessarily an equipotential of an electric potential field. Because of this property a conducting surface can be replaced with a boundary having properties inherited from the equipotential that coincides with it. This equipotential (and hence the boundary) is well defined enough to be used as a boundary condition for electric field equations.

In electrostatics this method involves creating a mirror image of the electric charge distribution about a symmetry plane and then reversing the polarity of this image. The virtual field created by the virtual charge distribution is actually the solution to the field generated by conducting surface when restricted to the problem domain. Thus the full electric field solution can be found by adding the virtual field within the problem domain and the real field.

This method can be used in magnetostatics despite some difficulties to overcome. First there is no such thing as an isolated "magnetic charge." Also there is typically not any actual case in magnetostatics where an exact equipotential surface coincides with a material boundary. However, in the case of highly permeable materials the material boundaries can *approximate* equipotentials. This allows the method of images to be used in some cases as a good approximation.

Before we can conduct such an analysis we first have to characterize the nature of the potential field implied by stating the surface of a highly permeable material is approximately an "equipotential." After establishing the potential field we must investigate the properties of the potential field to ensure that it is useful in the method of images.

Method of Images

Magnetizing Field Potential

The potential field we will investigate is the potential field associated with the magnetizing field, $\vec{H}$ as it is often expressed in Maxwell's equations. We do not consider the magnetic field (B-field) as a candidate for our potential field as in general it does not have a scalar potential field.

The magnetizing field is defined in terms of the magnetic field $\vec{B}$ and magnetization field $\vec{M}$ as,

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M} \tag{1}$$

In the case of magnetostatics, Maxwell's equation,

$$\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} \tag{2}$$

becomes, in the absence of free currents,

$$\nabla \times \vec{H} = 0 \tag{3}$$

This implies from the Helmholtz decomposition theorem,

$$\vec{F} = -\nabla\varphi + \nabla \times \vec{A} \tag{4}$$

that the magnetizing field $\vec{H}$ is *irrotational*, thus we can define it entirely in terms of the gradient of some scalar potential as,

$$\vec{H} = -\nabla\varphi \tag{5}$$

We'll define this potential φ_H for $\vec{H}$ as the "magnetizing scalar potential,"

$$-\nabla\varphi_H = \vec{H} \tag{6}$$

we'll regard it as a mathematical tool and not consider its physical interpretation.

Magnetizing Field Boundary Conditions

As stated earlier our objective is to replace the material boundary with an equipotential that we can use as a symmetry plane and boundary surface. To do this we will construct a model for an idealized, highly permeable and linear ferromagnetic medium boundary. The space outside of the material boundary has a permeability μ_1. The permeable material has a permeability μ_2 such that $\mu_2 \gg \mu_1$ (See figure 1). In linear media the magnetic field and magnetizing field are related as,

$$\vec{B} = \mu\vec{H} \tag{7}$$

We can determine the boundary condition at the equipotential surface that will replace the idealized material boundary by considering the limiting behavior of the boundary conditions as $\mu_2 \to \infty$.

The magnetizing field boundary conditions for magnetostatics are,

$$\mu_1\vec{H}_1 \cdot \vec{n} = \mu_2\vec{H}_2 \cdot \vec{n} \tag{8}$$

$$\vec{H}_1 \times \vec{n} = \vec{H}_2 \times \vec{n} \tag{9}$$

Where $\vec{n}$ is a vector normal to the boundary between the two materials, directed from μ_2 to μ_1. We want to know the angles between the normal and the magnetizing field on each side of the boundary. To do this we will first expand the vector products in eq. 8 and eq. 9 to their trigonometric forms,

$$\mu_1 H_1 \cos\theta_1 = \mu_2 H_2 \cos\theta_2$$

$$H_1 \sin\theta_1 = H_2 \sin\theta_2$$

Dividing these equations gives,

$$\mu_1 \cot\theta_1 = \mu_2 \cot\theta_2$$

We now only need to examine the limiting behavior of this expression. In the limiting case μ_1 is a very small, non-vanishing number and $\mu_2 \to \infty$. This implies,

$$\cot\theta_1 \to \infty$$

which is true when $\theta_1 \to 0$ (the angle between $\vec{H}_1$ and $\vec{n}$ is 0). This implies that in our idealized model of an interface between a highly permeable, very low permeability linear materials; it is a good approximation to assume that $\vec{H}_1$ is *everywhere normal to the boundary* such that,

$$\vec{H}_1 \cdot \vec{n} = H_1 \tag{10}$$

Magnetizing Field Boundary as an Equipotential

With the property of of the magnetizing field at the boundary expressed by eq. 10 we can show that the boundary is an equipotential. By eq. 3 it follows that we have an irrotational and conservative field. Thus we can use the gradient theorem,

$$\varphi(\vec{b}) - \varphi(\vec{a}) = \int_L \nabla\varphi \cdot d\vec{r} \tag{11}$$

to investigate the difference in potential between arbitrary points on the material boundary.

We can express the gradient theorem for our particular boundary model by first introducing two arbitrary coordinates restricted to the boundary plane; $\vec{q}_1$ and $\vec{q}_2$. The path $\vec{r}(t)$ is a path restricted to the boundary plane over which the line integral in eq. 11 is integrated.

$$\varphi_H(\vec{q}_2) - \varphi_H(\vec{q}_1) = \int_L -\vec{H} \cdot d\vec{r}$$

considering the property expressed in eq. 10, this becomes

$$\varphi_H(\vec{q}_2) - \varphi_H(\vec{q}_1) = \int_L -H\vec{n} \cdot d\vec{r}$$

The symmetry plane normal $\vec{n}$ is everywhere orthogonal to $\vec{r}$ so the line integral goes to zero and we are left with,

$$\varphi_H(\vec{q}_2) = \varphi_H(\vec{q}_1)$$

it follows that the entire boundary is an equipotential. To use this equipotential as a boundary for the problem domain we need an exact value for this equipotential. If assign a potential of zero to points at infinity then all points on the symmetry plane boundary must also be zero since this plane extends to infinity as well.

Uniqueness of the Magnetizing Potential Field

The uniqueness theorem for the Poisson equation allows us to establish that a particular solution is unique if a specific set of boundary conditions are satisfied. In other words the uniqueness theorem allows us to determine how much information we need to know about the boundary conditions of a problem domain to know for certain a field solution has only one solution.

Before determining the uniqueness conditions for our field we need a Poisson field equation in terms of quantities that are known. Specifically we would like to know the magnetizing scalar potential as it relates to a given magnetization distribution.

Poisson's equation is a partial differential equation of the form,

$$\nabla^2 \varphi = f$$

where φ and f are real or complex valued functions. We will begin by deriving the form of the Poisson equation for the magnetizing scalar potential. We begin by taking the divergence of eq. 1 and noting that,

$$\nabla \cdot \vec{B} = 0$$

We are then left with the relation,

$$\nabla \cdot \vec{H} = -\nabla \cdot \vec{M} \tag{12}$$

When we substitute the definition for the magnetizing field potential into eq. 12 we obtain,

$$\begin{aligned} \nabla \cdot (-\nabla \varphi_H) &= -\nabla \cdot \vec{M} \\ \nabla^2 \varphi_H &= \nabla \cdot \vec{M} \end{aligned} \tag{13}$$

Eq. 13 is the Poisson equation we must find uniqueness conditions for.

If we assume that a solution to eq. 13 is not unique then there must be at least two solutions, φ_1 and φ_2. If there is no difference between these solutions then we have established conditions for uniqueness. We begin by defining a potential that is the difference between solutions,

$$\phi = \varphi_2 - \varphi_1$$

We then apply the Laplacian operator to this potential,

$$\nabla^2 \phi = \nabla^2 \varphi_2 - \nabla^2 \varphi_1$$

We can write two magnetizing Poisson equations for our candidate solutions. It is important to note that although the solutions are assumed to be unique they are both associated with the same magnetization distribution. Therefore these equations can be expressed as,

$$\nabla^2 \varphi_1 = \nabla \cdot \vec{M}$$

$$\nabla^2 \varphi_2 = \nabla \cdot \vec{M}$$

Thus,

$$\nabla^2 \phi = \nabla \cdot \vec{M} - \nabla \cdot \vec{M}$$

Giving us a field equation for the solution difference potential,

$$\nabla^2 \phi = 0 \tag{14}$$

We now will derive another useful identity that can be used in the divergence theorem. We begin with the following product rule,

$$\nabla \cdot (f\vec{A}) = f(\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla f)$$

then substitute $f = \phi$ and $\vec{A} = \nabla\phi$. Expanding we have

$$\nabla \cdot (\phi\nabla\phi) = \phi(\nabla \cdot \nabla\phi) + \nabla\phi \cdot \nabla\phi$$

$$\nabla \cdot (\phi\nabla\phi) = \phi(\nabla^2\phi) + (\nabla\phi)^2$$

Note that from eq. 14 the term $\phi(\nabla^2\phi) \to 0$ thus this expression simplifies to

$$\nabla \cdot (\phi\nabla\phi) = (\nabla\phi)^2 \tag{15}$$

Note that the term $\nabla \cdot (\phi\nabla\phi)$ is in the form that the divergence theorem

$$\int_V (\nabla \cdot \vec{A})dV = \int_{\partial V} \vec{A} \cdot d\vec{a}$$

The divergence theorem can be applied with $\vec{A} = \phi\nabla\phi$, V is the problem domain volume (the area above the symmetry plane in figure 1), $d\vec{a}$ is an element of the boundary surface enclosing the domain, ∂V.

$$\int_V (\nabla \cdot (\phi\nabla\phi))dV = \int_{\partial V} (\phi\nabla\phi) \cdot d\vec{a}$$

From eq. 15 the left term simplifies this expression to,

$$\int_V (\nabla\phi)^2 dV = \int_{\partial V} (\phi\nabla\phi) \cdot d\vec{a}$$

Our first condition is that ϕ exist, be well defined and real valued everywhere. It follows that the the integrand of the volume integral must have the property,

$$(\nabla\phi)^2 \geq 0$$

If the volume integral is non-vanishing then we do not have conditions for a unique solution. We are only interested situations where the volume integral *is* vanishing. Thus our boundary counstraints are characterized by,

$$\int_{\partial V} (\phi\nabla\phi) \cdot d\vec{a} = 0 \tag{16}$$

If $\phi = 0$ then the boundary constraint integral in eq. 16 is satisfied. This implies

$$\varphi_2 - \varphi_1 = 0$$

$$\varphi_2 = \varphi_1$$

It follows that the solution to the magnetizing field equation is unique if the values of the field are defined on the entire boundary of the field. This is known as a *Dirichlet* boundary condition.

Furthermore it is also clear by inspection that if $\nabla\phi = 0$ in eq. 16 then the boundary constraint integral again vanishes. It follows that

$$\nabla\varphi_2 - \nabla\varphi_1 = 0$$

$$\nabla\varphi_2 = \nabla\varphi_1$$

It follows that the solution is unique if the first derivative of the field is known everywhere on the boundary. This is known as a *Neumann* boundary condition.

In a mixed boundary condition both Dirichlet and Neumann boundary conditions can be specified. They will both cause the boundary constraint integral to vanish and thus produce a unique solution.

We now have a sufficient battery of tests to verify whether a proposed image solution is unique.

Uniqueness of the Image Solution

The proposed image solution is to take the magnetizing potential and apply a reflection operator. We will see that the magnetizing potential is, in fact, a pseudo-scalar thus it will require a polarity change after undergoing a reflection operation (a form of improper rotation where at least one coordinate axis inversion occurs).

To begin we will define some fields.

Let $\varphi_{real}(\vec{r})$ be the magnetizing potential field associated with the real magnetization distribution $\vec{M}_{real}(\vec{r})$.

Let $\varphi_{image}(\vec{r})$ be the magnetizing potential field associated with the image magnetization distribution $\vec{M}_{image}(\vec{r})$

The magnetizing field $\Phi_H = \varphi_{real} + \varphi_{image}$ is the solution for the magnetizing field within our problem domain. It is the solution to test for uniqueness.

The real and image magnetizing potential fields both will go to zero at infinity so if we sum them this property will hold, this satisfies one boundary condition for Φ_H. The second and final boundary condition is that $\Phi_H = 0$ everywhere on the symmetry plane. To satisfy this we can define the image magnetizing potential as,

$$\varphi_{image} = -\varphi_{real}(\boldsymbol{R}\vec{r})$$

where $\boldsymbol{R}$ is a mirror transformation operator that mirrors a vector about the symmetry plane defined by $\vec{n}$. We have to also change the polarity of the field when we reflect it. We can see why this is necessary by checking the solution on the boundary by first defining a vector constrained to the symmetry plane $\vec{s}$ and noting that,

$$\varphi_{real}(\boldsymbol{R}\vec{s}) = \varphi_{real}(\vec{s})$$

In other words coordinates *on* the symmetry plane are not transformed under a reflection. We then have,

$$\Phi_H(\vec{s}) = \varphi_{real}(\vec{s}) + [-\varphi_{real}(\vec{s})]$$
$$\Phi_H(\vec{s}) = 0 \tag{17}$$

With all boundary conditions satisfied we have confirmed that the solution in the problem domain is,

$$\Phi_H(\vec{r}) = \varphi_{real}(\vec{r}) - \varphi_{real}(\boldsymbol{R}\vec{r})$$

Taking the gradient of this solution,

$$\nabla\Phi_H(\vec{r}) = \nabla\varphi_{real}(\vec{r}) - \nabla\varphi_{real}(\boldsymbol{R}\vec{r})$$

recalling that the chain rule still applies to a gradient,

$$\nabla\varphi_{real}(\boldsymbol{R}\vec{r}) = \boldsymbol{R}\nabla\varphi_{real}(\vec{u})$$

with $\vec{u} = \boldsymbol{R}\vec{r}$. Also substituting the definition for the magnetizing field gradient in eq. 6 we get,

$$\vec{H}_{solution} = \vec{H}_{real}(\vec{r}) - \boldsymbol{R}\vec{H}_{real}(\boldsymbol{R}\vec{r}) \tag{18}$$

Because of the uniqueness theorem this solution is identical to the field solution if we had *not* used the image method and instead left the highly permeable material where it was. This also implies that the field "felt" by the real magnetization distribution from a highly permeable material arises from a magnetization distribution,

$$\vec{M}_{image} = -\boldsymbol{R}\vec{M}_{real}(\boldsymbol{R}\vec{r}) \tag{19}$$

Eq. 19 would prove most useful in calculating the force exerted on a permanent magnet by a large ferrous plate of high permeability. The plate can be replaced by the image of the magnetization distribution of the magnet.

Sample Problem

A bar magnet of radius $R = 0.03m$ and length $L = 0.15m$ has a magnetization of $1 \times 10^6 A/m$. One of its flat ends is placed in contact with an infinitely permeable plate of iron of side $A \gg L$. Find the force needed to separate the magnet from the sheet.

Bound Current Approach

With the image method in mind; one approach to solving this problem is to determine the magnetic field resulting from the bound currents in the magnet. Once those are determined we can create a virtual magnetization distribution according to eq. 19. We then calculate the total force on these dipoles due to the magnetic field created by the bound current distribution. The total force on these virtual dipoles will be the same as the force on the metal plate by the magnet.

We'll begin by defining the magnetization of the real magnet as a function of the position using cylindrical coordinates. In fact this entire problem will be done in cylindrical coordinates because this will exploit the symmetry of the problem.

$$\vec{M} = C_M \mathcal{H}(R-r)[\mathcal{H}(-z) - \mathcal{H}(L-z)]\hat{z} \tag{20}$$

Applying the image transformation in eq. 19 with the plate surface normal aligned to the z-axis gives the magnetization distribution image,

$$\vec{M}_i = C_M \mathcal{H}(R-r)[\mathcal{H}(-z_i) - \mathcal{H}(-z_i-L)]\hat{z} \tag{21}$$

Where $\mathcal{H}$ is the Heaviside step distribution,

$$\mathcal{H}(x) = \begin{cases} 0 & x < 0 \\ 1 & otherwise \end{cases}$$

whose derivative is the Dirac Delta distribution

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & otherwise \end{cases} \tag{22}$$

which has the property,

$$\int_a^b f(x)\delta(x-c)\,dx = f(c)\, where\, a \leq c \leq b\,. \tag{23}$$

Bound Current

We can determine the bound current by starting with the definition of the magnetic field in terms of the magnetizing field and magnetization field,

$$\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M} \tag{24}$$

We take the curl of eq. 24,

$$\nabla \times \vec{B} = \mu_0 \nabla \times \vec{H} + \mu_0 \nabla \times \vec{M} \tag{25}$$

Because this is a magnetostatics problem there are no free currents so the magnetizing field has no curl,

$$\nabla \times \vec{H} = 0\,.$$

Substituting into eq. 25 gives the curl of the magnetic field inside of the domain of the magnet in terms of the magnetization distribution,

$$\nabla \times \vec{B} = \mu_0 \nabla \times \vec{M} \tag{26}$$

With the following Maxwell equation,

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

in the case of magnetostatics, eq. 26 becomes,

$$\mu_0 \vec{J} = \mu_0 \nabla \times \vec{M}$$

Note that the total current can be separated into two terms as follows,

$$\vec{J} = \vec{J_b} + \vec{J_f}$$

There is no free current so the definition of the bound current density is,

$$\vec{J_b} = \nabla \times \vec{M} \tag{27}$$

Using the definition for curl in cylindrical coordinates,

$$\nabla \times \vec{v} = \left[\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right]\hat{r} + \left[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{z}$$

eq. 27 becomes,

$$\vec{J_b} = -\frac{\partial}{\partial r}\left[C_M \mathcal{H}(R-r)[\mathcal{H}(-z) - \mathcal{H}(L-z)]\right]\hat{\theta}$$

$$\vec{J_b} = C_M \delta(R-r)[\mathcal{H}(-z) - \mathcal{H}(L-z)]\hat{\theta}\,. \tag{28}$$

Magnetic Field

With the bound current density known we can use it to calculate the magnetic field from the Biot-Savart Law in MKS units.

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{(\vec{J}dV) \times \vec{\mathfrak{r}}}{|\mathfrak{r}|^3} \tag{29}$$

Where $\vec{\mathfrak{r}}$ is the displacement vector from a volume element dV to $\vec{x}$. In this specific case we will define $\vec{s}$ as the coordinate of dV which is being integrated over. We will define $\vec{s}_i$ as the coordinate in the image region we wish to know the magnetic field at. With this definition

$$\vec{\mathfrak{r}} = \vec{s}_i - \vec{s}$$

where

$$\vec{s}_i = \left\langle \begin{array}{c} r_i \cos\theta_i \\ r_i \sin\theta_i \\ z_i \end{array} \right\rangle,$$

$$\vec{s} = \left\langle \begin{array}{c} r \cos\theta \\ r \sin\theta \\ z \end{array} \right\rangle$$

are cartesian vectors in terms of their polar coordinate parameters.

Since we are trying to find the force on a volume of dipoles we use the dipole force law,

$$\vec{F} = \nabla(\vec{m} \cdot \vec{B}) \,. \tag{30}$$

We substitute an arbitrary, infinitesimal dipole element in the image region,

$$d\vec{m}_i = \vec{M}_i dV_i$$

into eq. 30 and get,

$$d\vec{F} = \nabla(\vec{M}_i dV_i) \cdot \vec{B}$$

We substitute eq. 29 and only determine the z component of $d\vec{F}$ since we are only interested in the force between the plate and the magnet which will be aligned to the z-axis.

$$d\vec{F} = \nabla(\vec{M} dV_i) \cdot \int \frac{\mu_0}{4\pi} \frac{(\vec{J}dV) \times \vec{\mathfrak{r}}}{|\mathfrak{r}|^3}$$

$$dF_z = \frac{\partial}{\partial z_i}(M_{iz} dV_i) \frac{\mu_0}{4\pi} \int \frac{[\vec{J}dV \times \vec{\mathfrak{r}}]_z}{|\mathfrak{r}|^3}$$

$$[\vec{J}dV \times \vec{\mathfrak{r}}]_z = (J_x \mathfrak{r}_y - J_y \mathfrak{r}_x) dV$$

$$dF_z = \frac{\partial}{\partial z_i}(M_{iz} r_i dr_i d\theta_i dz_i)\frac{\mu_0}{4\pi}\int \frac{(J_x \mathfrak{r}_y - J_y \mathfrak{r}_x) r dr d\theta dz}{|\mathfrak{r}|^3}$$

$$\hat{\theta} = \left\langle \begin{array}{c} -\sin\theta \\ \cos\theta \\ 0 \end{array} \right\rangle$$

$$dF_z = \frac{\partial}{\partial z_i}(M_{iz} r_i dr_i d\theta_i dz_i)\frac{\mu_0}{4\pi}\int \frac{J[-\sin\theta(r_i\sin\theta_i - r\sin\theta) - \cos\theta(r_i\cos\theta_i - r\cos\theta)] r dr d\theta dz}{[(r_i\cos\theta_i - r\cos\theta)^2 + (r_i\sin\theta - r\sin\theta)^2 + (z_i - z)^2]^{\frac{3}{2}}}$$

Applying trigonometric identities this greatly simplifies to,

$$dF_z = \frac{\partial}{\partial z_i}(M_{iz} r_i dr_i d\theta_i dz_i)\frac{\mu_0}{4\pi}\int \frac{J[r - r_i\cos(\theta - \theta_i)] r dr d\theta dz}{[r^2 + r_i^2 - 2rr_i\cos(\theta - \theta_i) + (z_i - z)^2]^{\frac{3}{2}}}$$

Expanding J and M_{iz},

$$dF_z = \frac{\partial}{\partial z_i} C_M^2[\mathcal{H}(-z_i) - \mathcal{H}(-z_i - L)] dr_i d\theta_i dz_i \ldots$$
$$\frac{\mu_0}{4\pi}\int \frac{\delta(R - r)\mathcal{H}(R - r)[\mathcal{H}(-z) - \mathcal{H}(L - z)][r - r_i\cos(\theta - \theta_i)] r_i r dr d\theta dz}{[r^2 + r_i^2 - 2rr_i\cos(\theta - \theta_i) + (z_i - z)^2]^{\frac{3}{2}}}$$

factoring the delta function out and integrating over r while applying the definition of the delta distribution.

$$dF_z = \frac{\partial}{\partial z_i} C_M^2 dr_i d\theta_i dz_i \frac{\mu_0}{4\pi}\int\int_0^R \delta(R - r) u(r) dr d\theta dz$$

$$dF_z = \frac{\partial}{\partial z_i} C_M^2 dr_i d\theta_i dz_i \frac{\mu_0}{4\pi}\int u(R) d\theta dz$$

Expanding u

$$dF_z = \frac{\partial}{\partial z_i} C_M^2 dr_i d\theta_i dz_i \frac{\mu_0}{4\pi}\int \frac{\mathcal{H}(R - R)[\mathcal{H}(-z) - \mathcal{H}(L - z)][R - r_i\cos(\theta - \theta_i)] r_i R d\theta dz}{[R^2 + r_i^2 - 2Rr_i\cos(\theta - \theta_i) + (z_i - z)^2]^{\frac{3}{2}}}$$

Placing the appropriate limits on the integration allows us to assume the heaviside terms are constant within the range of integration and thus reduce to 1. Also note by definition $\mathcal{H}(R - R) = 1$. I am pairing the differentials with their associated definite integral to make it clear which variable is associated with which range of integration.

$$dF_z = \frac{\partial}{\partial z_i} C_M^2 dr_i d\theta_i dz_i \frac{\mu_0}{4\pi}\int_0^{2\pi} d\theta \int_0^L dz \frac{[R - r_i\cos(\theta - \theta_i)] r_i R}{[R^2 + r_i^2 - 2Rr_i\cos(\theta - \theta_i) + (z_i - z)^2]^{\frac{3}{2}}}$$

Magnetic Force

So far we have only focused on the magnetic field integral. Of course we actually want the force integral, that's why we've been dragging the differentials along so far. We have put this off because the partial derivative operator arising from the gradient operator in the magnetic dipole force law requires some special consideration. It is nothing complex; but, the partial derivative does not simply "cancel" the associated differential out. First we will construct the full force integral and isolate z_i,

$$F_z = \frac{C_M^2\mu_0}{4\pi}\int_0^{2\pi} d\theta_i \int_0^R dr_i \int_0^{2\pi} d\theta \int_0^L dz \left[\int_0^{-L} dz_i \frac{\partial}{\partial z_i}\frac{[R - r_i\cos(\theta-\theta_i)]r_i R}{[R^2 + r_i^2 - 2Rr_i\cos(\theta-\theta_i) + (z_i - z)^2]^{\frac{3}{2}}}\right]$$

Evaluating the brackets involves nothing less than the fundamental theorem of calculus,

$$\int_a^b \frac{d}{dx} f(x)dx = f(b) - f(a)$$

$$F_z = \frac{C_M^2\mu_0}{4\pi}\left[\int_0^{2\pi} d\theta_i \int_0^R dr_i \int_0^{2\pi} d\theta \int_0^L dz \frac{[R - r_i\cos(\theta-\theta_i)]r_i R}{[R^2 + r_i^2 - 2Rr_i\cos(\theta-\theta_i) + (z_i - z)^2]^{\frac{3}{2}}}\right]_{z_i=0}^{z_i=-L}$$

To further simplify this integral it is possible to set $\theta_i = 0$ from the cos arguments. As a result it simplifies to a three variable integral that must be evaluated numerically.

$$F_z = \frac{C_M^2\mu_0}{2}\left[\int_0^R dr_i \int_0^{2\pi} d\theta \int_0^L dz \frac{[R - r_i\cos\theta]r_i R}{[R^2 + r_i^2 - 2Rr_i\cos\theta + (z_i - z)^2]^{\frac{3}{2}}}\right]_{z_i=0}^{z_i=-L}$$

Evaluating this integral with

$$C_M = 1\times 10^6 A/m$$

$$\mu_0 = 4\pi\times 10^{-7} N/A^2$$

$$L = 0.15m$$

$$R = 0.03m$$

Gives an attraction force between the magnet and the plate of,

$$\boxed{F_z = 1718\, N}$$

Depending on the method of integration this triple integral may take considerable time to evaluate.

Pole Strength Approach

The method of using pole strength involves treating a magnetostatics problem as though it consists of magnetic charges. This isn't a physically accurate model as magnetic fields are created by current loops and thus magnetic poles cannot be isolated. However, it is possible to use this model without violating Maxwell's equations globally and obtain a good approximation for magnetostatics problems. Since I do not wish to imply the actual existence of isolated magnetic charges I will refer to a region that has a non-zero divergence of the magnetization distribution as a "pole" the magnitude of a pole will be referred to as "pole strength."

The method of images can easily be applied to the pole strength approach. Since we are using an analog to electric charges it is reasonable to assume that applying the usual image transformation will work. Throughout this problem we will refer to the image magnet as though it is a real magnet. For all practical purposes an image problem reduces to calculating the force between two magnets.

Magnetic Poles

We can define pole density distribution by observing that eq. 13 is very similar to the Poisson equation for electrostatics. By analog we can express this as,

$$\nabla^2 \varphi_H = \rho_g$$

where,

$$\rho_g = \nabla \cdot \vec{M}$$

Additionally since,

$$\nabla \cdot \vec{H} = -\nabla \cdot \vec{M}$$

we can express the divergence of the magnetizing field in terms of the pole density distribution as,

$$\nabla \cdot \vec{H} = -\rho_g$$

Force Law

First we'll begin by introducing a force law by drawing an analogy between magnetic and electric dipoles. We will use the Gilbert model of treating a magnetic dipole as two poles of magnitude g separated by a distance d. We will now use empirically verified torque laws to infer a force law that involves the magnetic field and pole strength,

$$\vec{\tau}_m = \vec{d} \times \vec{F}_m$$

$$\begin{aligned}\vec{\tau}_m &= \vec{m} \times \vec{B} \\ \vec{p} = q\vec{d} &\implies \vec{m} = g\vec{d} \\ \vec{m} \times \vec{B} &= \vec{d} \times \vec{F}_m \\ g\vec{d} \times \vec{B} &= \vec{d} \times \vec{F}_m\end{aligned}$$

The cross product is a linear operator and thus the former statement is equivalent to

$$\vec{d} \times g\vec{B} = \vec{d} \times \vec{F}_m \implies \vec{F}_m = g\vec{B} \,.$$

Pole Distribution

Before we can apply our force law we need to know the distribution of poles both real and imaged. The real pole distribution will be the divergence of eq. 20. Using the definition for the divergence in cylindrical coordinates,

$$\nabla \cdot \vec{v} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

The magnetization distribution,

$$\vec{M} = C_M \mathcal{H}(R-r)[\mathcal{H}(-z) - \mathcal{H}(L-z)]\hat{z}$$

becomes the pole density distribution,

$$\rho_{real} = C_M \mathcal{H}(R-r)[-\delta(-z) + \delta(L-z)] \tag{31}$$

The image pole density can be found with eq. 19,

$$\rho_{image}(\vec{r}) = \nabla \cdot (-\boldsymbol{R}\vec{M}(\boldsymbol{R}\vec{r}))$$

In this case $\boldsymbol{R}$ serves to invert the z-axis.

$$\rho_{image}(\vec{r}) = \nabla \cdot (C_M \mathcal{H}(R-r)[\mathcal{H}(z) - \mathcal{H}(L+z)]\hat{z})$$

$$\rho_{image}(\vec{r}) = C_M \mathcal{H}(R-r)[\delta(z) - \delta(L+z)] \tag{32}$$

We should stop to examine how both of our pole density distributions create "poles." Note that the distributional derivative of the magnetization distribution results in two delta functions. Those both completely localize the pole density strength to two separate locations. Each of the poles could actually be regarded as entirely separate disks at both ends of the magnet. However, it really isn't even necessary to worry about this due to the property of the delta functions in eq. 23. The delta distribution gives us a convenient way of expressing a discontinuous distribution in a way that we can simply evaluate volume integrals over the distributions and get the same solutions as if we had explicitly treated them as some kind of surface object.

With the distribution of poles known we can now calculate the forces between the poles and determine the net force between the real magnet and virtual magnet and hence the force between the magnet and the iron plate.

Symmetry Plane Poles

The poles near the symmetry plane can be regarded as being a very small distance ϵ from the symmetry plane. This small gap represents the real physical gap that would exist between two magnets in physical contact. Additionally if the poles are regarded as coincident then there is no free space available for a field to exist between them that can exert any force. Because the distance between these two poles is so small we can ignore effects of fringing.

We calculate the magnetic field in the air gap as follows,

$$\nabla \cdot \vec{H} = -\nabla \cdot \vec{M}$$

$$\int_V -\nabla \cdot \vec{M} dV = \int_{\partial V} \vec{H} \cdot d\vec{a}$$

In the following boundary integral we use a very thin, axis aligned cylinder of radius R,

$$\int -C_M \mathcal{H}(R-r)[\delta(z) - \delta(z-L)]dV = \int_{\partial V} \vec{H} \cdot d\vec{a}$$

$$dV = r dr d\theta dz$$

$$\int_0^R dr \int_0^{2\pi} d\theta \int_{-\epsilon}^{\epsilon} dz(-C_M \mathcal{H}(R-r)[\delta(z) - \delta(z-L)]r)$$

$$-C_M \int_0^R r dr \int_0^{2\pi} d\theta \int_{-\epsilon}^{\epsilon} \delta(z) - \delta(L-z)dz = H \oint da$$

$$-C_M \frac{1}{2} R^2 2\pi = H 2\pi R^2$$

$$H = -\frac{1}{2} C_M$$

$$B = -\frac{\mu_0}{2} C_M$$

With the magnetic field and the force law we calculate the force between these poles as follows,

$$dg = \rho_{image} dV = \rho_{image} r dr d\theta dz$$

$$dF = dgB$$

$$dF = C_M \mathcal{H}(R-r)[\delta(z) - \delta(L+z)] \left(-\frac{\mu_0}{2} C_M\right) r dr d\theta dz$$

$$F = \int_0^R dr \int_0^{2\pi} d\theta \int_{-\epsilon}^{\epsilon} dz C_M [\delta(z) - \delta(L+z)] \left(-\frac{\mu_0}{2} C_M\right) r$$

$$F = -\frac{1}{2} \mu_0 R^2 C_M^2 \pi \tag{33}$$

Distant Poles

The rest of the poles are far enough apart that we cannot disregard the distance between them. We will have to compute the force between the poles a distance $2L$ apart and the force from the poles a distance L apart. Because all the poles are the same shape we need only construct one integral and evaluate it for the two distances we are interested in. We will be somewhat less rigorous than the bound current approach in our evaluation of the integral as the finer details of using Heaviside and delta distributions are covered sufficiently above.

We are going to be integrating the force between pole elements so we need an equation that gives us the magnetic field due to a point pole. We will use

$$\vec{B} = \mu_0 \frac{g}{4\pi r^2}\hat{r} \tag{34}$$

Substituting this into the force on a pole we have the force between two point poles,

$$\vec{F} = \mu_0 \frac{g_1 g_2}{4\pi r^2}\hat{r}$$

$$d\vec{F} = \mu_0 \frac{dg_1 dg_2 \vec{\varrho}}{4\pi |\varrho|^3}$$

The displacement vector between two pole elements is,

$$\vec{\varrho} = \vec{r}_2 - \vec{r}_1$$

The pole elements will be restricted to the pole surfaces separated by a distance L,

$$\vec{r}_2 = \left\langle \begin{array}{c} r_2 \cos\theta_2 \\ r_2 \sin\theta_2 \\ L \end{array} \right\rangle$$

$$\vec{r}_1 = \left\langle \begin{array}{c} r_1 \cos\theta_1 \\ r_1 \sin\theta_1 \\ 0 \end{array} \right\rangle$$

We only need the force along the z-axis so we express the differential force as,

$$dF_z = \mu_0 \frac{C_M^2 L r_1 dr_1 d\theta_1 r_2 dr_2 d\theta_2}{4\pi \left[r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2) + L^2\right]^{3/2}}$$

Letting $\theta_2 = 0$, changing L to a parameter h and integrating,

$$F_z(h) = \frac{1}{2}\mu_0 C_M^2 h \int_0^R dr_1 \int_0^R dr_2 \int_0^{2\pi} d\theta_1 \frac{r_1 r_2}{\left[r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1) + h^2\right]^{3/2}}$$

This integral must be evaluated numerically. The total force will be,

$$F = -\frac{1}{2}\mu_0 R^2 C_M^2 \pi + F(L) - F(2L)$$

Evaluating this with the parameters defined in the problem gives,

$$\boxed{-1698.6N}$$

Where the negative sign merely indicates this is the force of the plate on the magnet and not vice versa as is done in the bound current approach earlier. The choice of symmetry is really irrelevant as long as the polarity of the charges and dipoles are kept straight in both methods. Both methods are in close agreement.