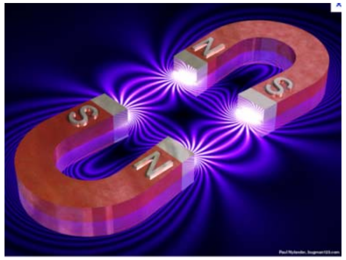
Chapter 5. Magnetostatics



5.4 Magnetic Vector Potential

5.1.1 The Vector Potential

In electrostatics, $\nabla \times \mathbf{E} = 0 \rightarrow \mathbf{E} = -\nabla V$ → Scalar potential *(V)*

In magnetostatics, $\nabla \cdot \mathbf{B} = 0 \rightarrow \boxed{\mathbf{B} = \nabla \times \mathbf{A}}$ → ***Vector potential (A)***

*(Note) The name is "potential", but **A** cannot be interpreted as potential energy per unit charge.*

So far, for a steady current where $\nabla \cdot \mathbf{J} = 0$ the Magnetic field is defined by the Biot-Savart law:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\imath}}}{\imath^2} d\tau'$$

This Magnetic field given by the Biot-Savart law always satisfies the relations of

$\nabla \cdot \mathbf{B} = 0$ (always zero, even for not steady current)

$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ (as long as the current is steady, $\nabla \cdot \mathbf{J} = 0$)

(If the current is not steady, $\nabla \cdot \mathbf{J} = -\partial \rho / \partial t \neq 0$ → $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 (\partial \mathbf{D} / \partial t)$)

The relations of $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$, called Ampere's Law, is very useful, just like the Gauss's Law in E.

The relations of $\nabla \cdot \mathbf{B} = 0$ is always satisfied for any current flow, therefore we may put the **B** field, **either**

$\mathbf{B} = \nabla \times \mathbf{A}$ (for any arbitrary vector **A**)

Or, $\mathbf{B} = \nabla \times (\mathbf{A} + \nabla \lambda)$ (for any arbitrary vector **A** and any scalar λ**,** since $\nabla \times \nabla \lambda = 0$)

$\mathbf{B} = \nabla \times \mathbf{A}$ ***WHY not ?*** $\mathbf{B} = \nabla \times (\mathbf{A} + \nabla \lambda)$

Actually we want to use the Ampere's Law, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ ***, to find out B field.***

(1) Let's use the form of $\mathbf{B} = \nabla \times \mathbf{A}$

$$\longrightarrow \nabla \times \mathbf{B} = \nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$$

If we choose **A** so as to eliminate the divergence of **A**: $\nabla \cdot \mathbf{A} = 0$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \longrightarrow \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \longrightarrow$$ This is a Poisson's equation.

We know how to solve it, just like the electrostatic potential problems.

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \xrightarrow{\text{If } \rho \text{ goes to zero at infinite,}} V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho}{\mathscr{r}} d\tau'$$

Ampere's Law

$$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}}$$

$$\xrightarrow{\text{If } \mathbf{J} \text{ goes to zero at infinite,}} \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}} d\tau'$$

$$\xrightarrow{\text{If } \mathbf{I} \text{ goes to zero at infinite,}} \mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}}{\mathscr{r}} dl'$$

$$\xrightarrow{\text{If } \mathbf{K} \text{ goes to zero at infinite,}} \mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}}{\mathscr{r}} da'$$

(If the current does not go to zero at infinite, we have to find other ways to get A.)

➔ In summary, the definition $\mathbf{B} = \nabla \times \mathbf{A}$ ***under the condition of*** $\nabla \cdot \mathbf{A} = 0$ ***make it possible to transform the Ampere's law into a Poisson's equation on A and J !***

$\mathbf{B} = \nabla \times \mathbf{A}$ *WHY not ?* $\mathbf{B} = \nabla \times (\mathbf{A} + \nabla \lambda)$

(2) Now consider the other form, $\mathbf{B} = \nabla \times (\mathbf{A} + \nabla \lambda)$

Suppose that a vector potential $\mathbf{A_0}$ satisfying $\mathbf{B} = \nabla \times \mathbf{A}_0$, but it is not divergenceless, $\nabla \cdot \mathbf{A}_0 \neq 0$.

If we add to $\mathbf{A_0}$ the gradient of any scalar λ, $\mathbf{A} = \mathbf{A}_0 + \nabla \lambda$, **A** is also satisfied $\mathbf{B} = \nabla \times \mathbf{A}$.

$(\nabla \times (\mathbf{A}_0 + \nabla \lambda) = \nabla \times \mathbf{A} = \mathbf{B}$ since $\nabla \times \nabla \lambda = 0)$

The divergence of **A** is $\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A}_0 + \nabla^2 \lambda$

$\longrightarrow$ We can accommodate the condition of $\nabla \cdot \mathbf{A} = 0$
as long as a scalar function λ can be found that satisfies, $\nabla^2 \lambda = -\nabla \cdot \mathbf{A}_0$

$\longrightarrow$ But this is mathematically identical to Poisson's equation, thus

$$\lambda = \frac{1}{4\pi} \int \frac{\nabla \cdot \mathbf{A}_0}{r} d\tau' \longrightarrow$$ We can always find λ to make $\nabla \cdot \mathbf{A} = 0$

➔ ***Therefore, let's use the simple form of*** $\mathbf{B} = \nabla \times \mathbf{A}$ ***with the divergenceless vector potential,*** $\nabla \cdot \mathbf{A} = 0$

May we use a scalar potential, $\mathbf{B} = -\nabla U$, just like used in $\mathbf{E} = -\nabla V$?

The introduction of Vector Potential (A) may not be as *useful* as *V*,

→ *It is still a vector with many components.*

It would be nice if we could get away with a scalar potential, for instant, $\mathbf{B} = -\nabla U$

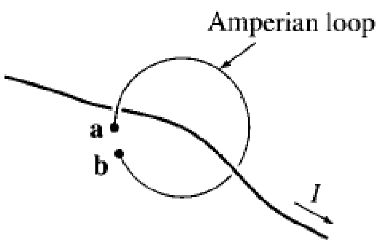
➔ But, this is ***incompatible with Ampere's law***, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$
since the curl of a gradient is always zero → $\nabla \times \mathbf{B} = -\nabla \times \nabla U = 0$

➔ Such a magnetostatic scalar potential *could* be used,
if you stick scrupulously to simply-connected current-free regions (**J = 0**).

Prob. 5-29 Suppose that $\mathbf{B} = -\nabla U$.
Show, by applying Ampere's law to a path that starts at **a** and circles the wire returning to **b**
that the scalar potential cannot be single-valued, $U(\mathbf{a}) \neq U(\mathbf{b})$, even if they are the same point (a = b).

$$\mu_0 I = \oint \mathbf{B} \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \nabla U \cdot d\mathbf{l} = -[U(\mathbf{b}) - U(\mathbf{a})]$$ (by the gradient theorem),

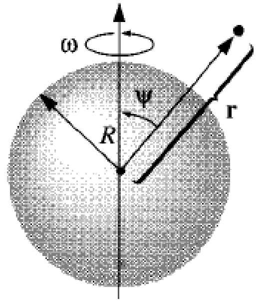
$$\longrightarrow U(\mathbf{b}) \neq U(\mathbf{a})$$



The Vector Potential $\mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{I}}{\imath}dl'$ $\mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{K}}{\imath}da'$ $\mathbf{A}(\mathbf{r})=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')}{\imath}d\tau'$

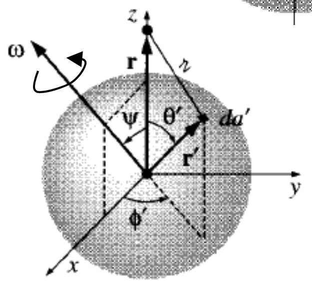
Example 5.11

A spherical shell, of radius *R*, carrying a uniform surface charge σ, is set spinning at angular velocity $\boldsymbol{\omega}$.
Find the vector potential it produces at point ***r***.



For surface current → $\mathbf{A}(\mathbf{r})=\frac{\mu_0}{4\pi}\int\frac{\mathbf{K}(\mathbf{r}')}{\imath}da'$ where $\mathbf{K}=\sigma\mathbf{v}$

In fact the integration is easier if we let ***r*** lie on the *z* axis,
so that ω is tilted at an angle ψ, and orient the x axis so that ω lies in the *xz* plane.



$$\imath=\sqrt{R^2+r^2-2Rr\cos\theta'}\qquad da'=R^2\sin\theta'\,d\theta'\,d\phi'$$

The velocity of a point **r'** in a rotating rigid body is

$$\mathbf{v}=\boldsymbol{\omega}\times\mathbf{r}'=\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \omega\sin\psi & 0 & \omega\cos\psi\\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta'\end{vmatrix}$$

$$=R\omega[-(\cos\psi\sin\theta'\sin\phi')\,\hat{\mathbf{x}}+(\cos\psi\sin\theta'\cos\phi'-\sin\psi\cos\theta')\,\hat{\mathbf{y}}+(\sin\psi\sin\theta'\sin\phi')\,\hat{\mathbf{z}}]$$

Since $\int_0^{2\pi}\sin\phi'\,d\phi'=\int_0^{2\pi}\cos\phi'\,d\phi'=0\longrightarrow\mathbf{A}(\mathbf{r})=-\frac{\mu_0R^3\sigma\omega\sin\psi}{2}\left(\int_0^{\pi}\frac{\cos\theta'\sin\theta'}{\sqrt{R^2+r^2-2Rr\cos\theta'}}d\theta'\right)\hat{\mathbf{y}}$

Letting $u\equiv\cos\theta'$, $\longrightarrow\int_{-1}^{+1}\frac{u}{\sqrt{R^2+r^2-2Rru}}du=-\frac{1}{3R^2r^2}\left[(R^2+r^2+Rr)|R-r|-(R^2+r^2-Rr)(R+r)\right]=\begin{cases}(2r/3R^2) & R>r\\ 2R/3r^2 & R<r\end{cases}$

Noting that $(\boldsymbol{\omega}\times\mathbf{r})=-\omega r\sin\psi\,\hat{\mathbf{y}}$, $\longrightarrow\mathbf{A}(\mathbf{r})=\begin{cases}\frac{\mu_0R\sigma}{3}(\boldsymbol{\omega}\times\mathbf{r}), & R>r\\ \frac{\mu_0R^4\sigma}{3r^3}(\boldsymbol{\omega}\times\mathbf{r}), & R<r\end{cases}$

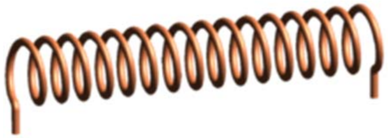
Go back to the original coordinates, in which ω coincides with the z axis and the position ***r*** is at (r,θ,ϕ):

$$\mathbf{A}(r,\theta,\phi)=\begin{cases}\frac{\mu_0R\omega\sigma}{3}r\sin\theta\,\hat{\boldsymbol{\phi}}, & (r\le R),\\ \frac{\mu_0R^4\omega\sigma}{3}\frac{\sin\theta}{r^2}\hat{\boldsymbol{\phi}}, & (r\ge R).\end{cases}\qquad\mathbf{B}=\nabla\times\mathbf{A}=\frac{2\mu_0R\omega\sigma}{3}(\cos\theta\,\hat{\mathbf{r}}-\sin\theta\,\hat{\boldsymbol{\theta}})=\frac{2}{3}\mu_0\sigma R\omega\hat{\mathbf{z}}=\frac{2}{3}\mu_0\sigma R\boldsymbol{\omega}$$

➔ Curiously, the field inside this spherical shell is *uniform!*

The Vector Potential $\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}}{\mathscr{r}}\,dl'$ $\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}}{\mathscr{r}}\,da'$ $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\,d\tau'$

Example 5.12 Find the vector potential of an **infinite solenoid** with *n* turns per unit length, radius *R,* and current *I.*



Since the current itself extends to infinity → *we cannot use such a form of* $\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}}{\mathscr{r}}\,dl'$

We need to use other methods. Here's a cute method that does the job:

Note that $\oint \mathbf{A}\cdot d\mathbf{l} = \int (\nabla\times\mathbf{A})\cdot d\mathbf{a} = \int \mathbf{B}\cdot d\mathbf{a}$ → This means the flux of B through the loop.

To find B, we can use the Ampere's law in integral form,

$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{\text{enc}} = \mu_0 n I$ → Uniform longitudinal magnetic field inside the solenoid and no field outside)

Using a circular "amperian loop" at radius *s* *inside* the solenoid,

$$\oint \mathbf{A}\cdot d\mathbf{l} = A(2\pi s) = \int \mathbf{B}\cdot d\mathbf{a} = \mu_0 n I(\pi s^2) \longrightarrow \mathbf{A} = \frac{\mu_0 n I}{2} s\,\hat{\boldsymbol{\phi}}, \quad \text{for } s < R.$$

For an amperian loop *outside* the solenoid, since no field out to *R,*

$$\oint \mathbf{A}\cdot d\mathbf{l} = A(2\pi s) = \int \mathbf{B}\cdot d\mathbf{a} = \mu_0 n I(\pi R^2) \longrightarrow \mathbf{A} = \frac{\mu_0 n I}{2}\frac{R^2}{s}\,\hat{\boldsymbol{\phi}}, \quad \text{for } s > R$$

Does $\nabla\cdot\mathbf{A} = 0$? $\longrightarrow$ ***If so, we're done.***

5.4.2 Summary; Relations of B – J – A

From just two experimental observations:

(1) the principle of superposition - a broad general rule

(2) Biot-Savert's law - the fundamental law of magnetorostatics.

⇨ ***B – J – A***

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\imath}\, d\tau'$$

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

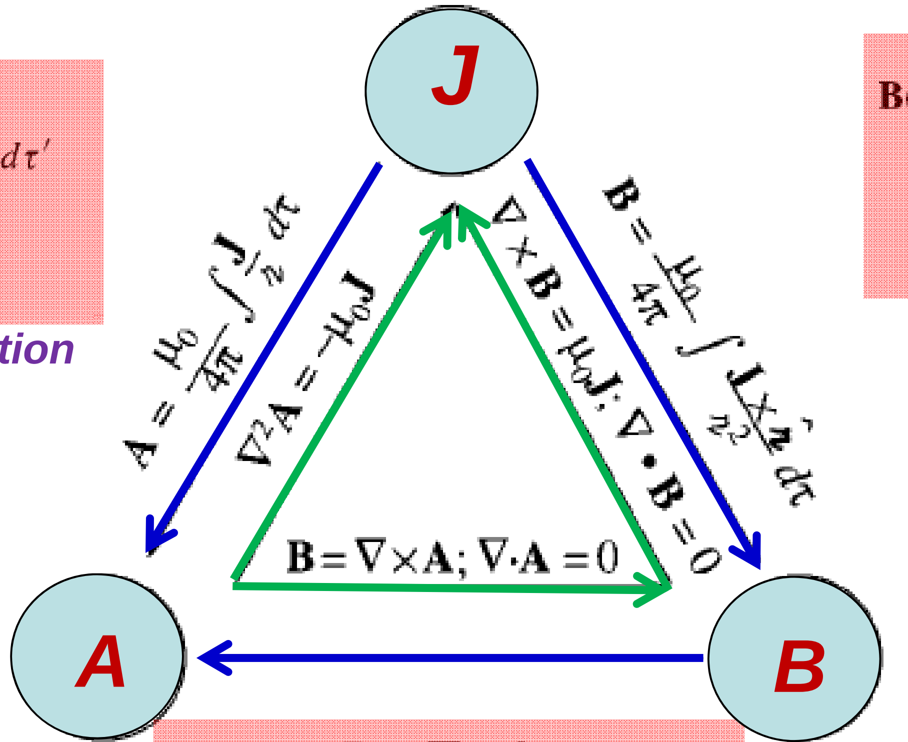
Poisson's equation

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\imath}}}{\imath^2}\, d\tau'$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

Ampere's Law



$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\oint \mathbf{A} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int \mathbf{B} \cdot d\mathbf{a}$$

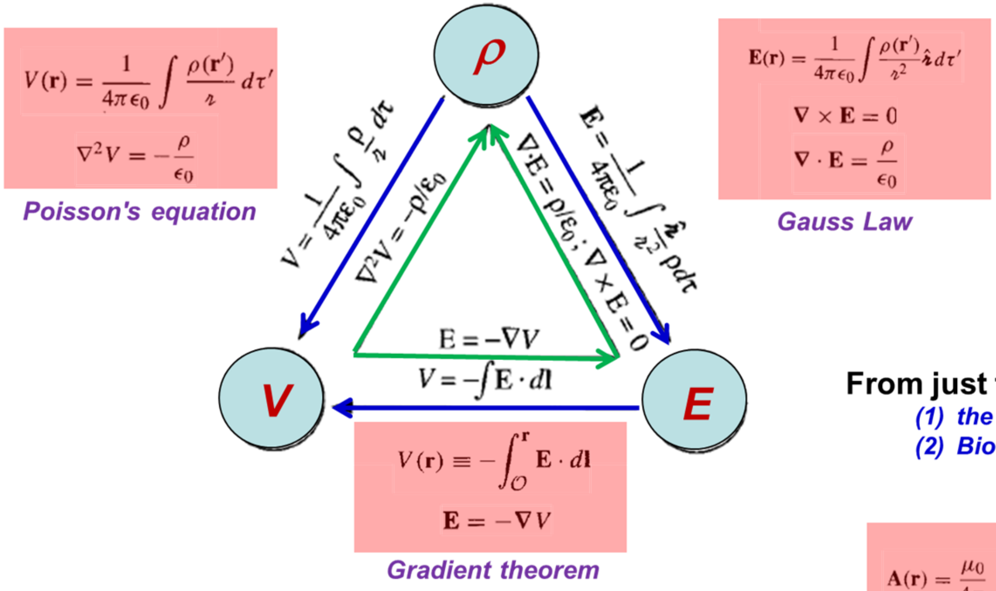
Stoke's theorem

$E - \rho - V$ ⟷ $B - J - A$

From just two experimental observations:

(1) the principle of superposition - a broad general rule
(2) Coulomb's law - the fundamental law of electrostatics.

⇨ $E - \rho - V$



From just two experimental observations:

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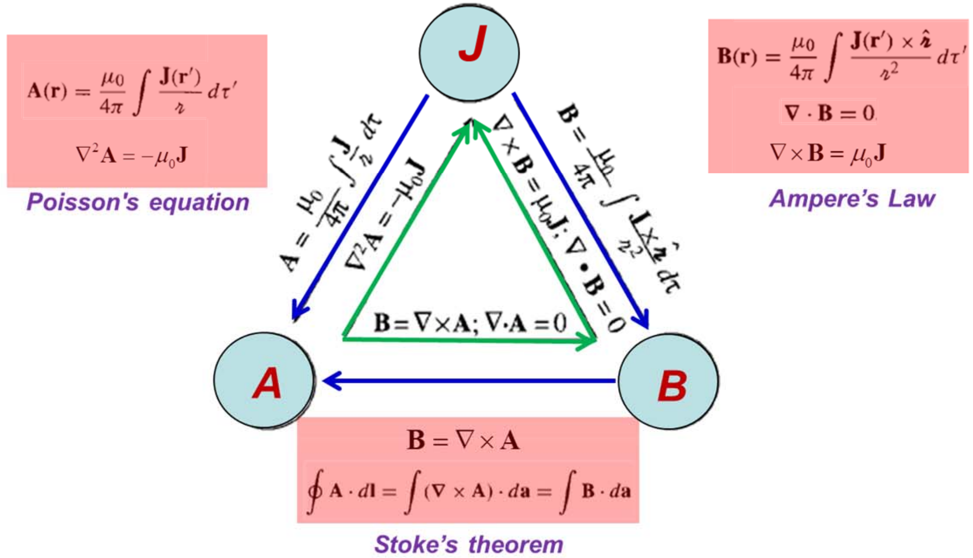
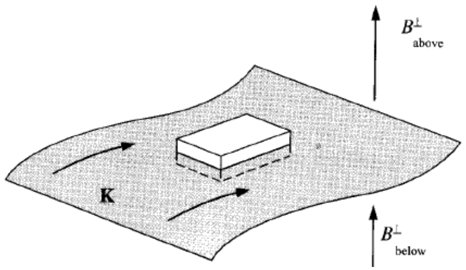
⇨ $B - J - A$



5.4.2 Summary; Magnetostatic Boundary Conditions

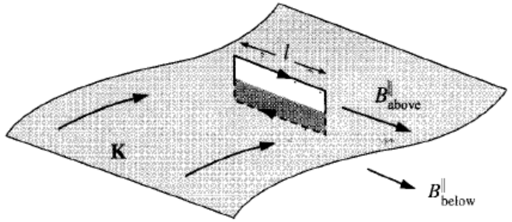
(Normal components)

$$\nabla \cdot \mathbf{B} = 0 \Longrightarrow \oint \mathbf{B} \cdot d\mathbf{a} = 0 \Longrightarrow \boxed{B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}}$$



(Tangential components)

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$



For an amperian loop running perpendicular to the current,

$$\oint \mathbf{B} \cdot d\mathbf{l} = (B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}})l = \mu_0 I_{\text{enc}} = \mu_0 K l \longrightarrow \boxed{B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}} = \mu_0 K}$$

For an amperian loop running parallel to the current,

$$\oint \mathbf{B} \cdot d\mathbf{l} = (B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}})l = \mu_0 I_{\text{enc}} = 0 \longrightarrow \boxed{B^{\parallel}_{\text{above}} = B^{\parallel}_{\text{below}}}$$

Finally, we can summarize in a single formula: $\Longrightarrow \boxed{\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0 (\mathbf{K} \times \hat{\mathbf{n}})}$

The vector potential is continuous across any boundary: $\Longrightarrow \boxed{\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}}}$

(Because, $\nabla \cdot \mathbf{A} = 0$ guarantees the normal continuity; $\nabla \times \mathbf{A} = \mathbf{B}$ the tangential one.)

$$\nabla \times \mathbf{A} = \mathbf{B} \longrightarrow \oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{a} = \Phi = 0$$ (the flux through an amperian loop of vanishing thickness is zero)

Summary of Boundary conditions: E ←→ B

$$\mathbf{E}^{\parallel}_{\text{above}} - \mathbf{E}^{\parallel}_{\text{below}} = 0$$

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{1}{\epsilon_0}\sigma$$

$$V_{above} = V_{below}$$

$$B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}} = \mu_0 K$$

$$B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}$$

$$\mathbf{A}_{above} = \mathbf{A}_{below}$$

$$\boxed{\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}})}$$

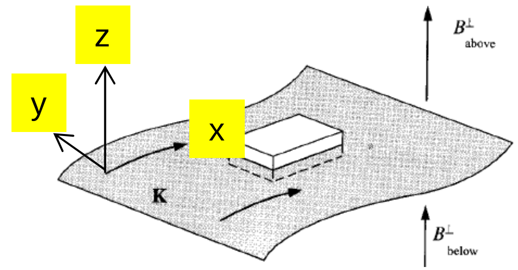
Note that the derivative of vector potential A inherits the discontinuity of B ➔ $\frac{\partial \mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial \mathbf{A}_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K}$

Problem 5.32 (Prove it) Let's take the Cartesian coordinates with z perpendicular to the surface and *x* parallel to the current.

Because $\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}}$ at every point on the surface, it follows that $\frac{\partial \mathbf{A}}{\partial x}$ and $\frac{\partial \mathbf{A}}{\partial y}$ are the same above and below.

Any discontinuity is confined to the normal derivative.

$$\mathrm{B}_{\text{above}} - \mathrm{B}_{\text{below}} = \left(-\frac{\partial A_{y_{\text{above}}}}{\partial z} + \frac{\partial A_{y_{\text{below}}}}{\partial z}\right)\hat{\mathrm{x}} + \left(\frac{\partial A_{x_{\text{above}}}}{\partial z} - \frac{\partial A_{x_{\text{below}}}}{\partial z}\right)\hat{\mathrm{y}}.$$



$$\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}}) \xrightarrow{\mathbf{K} = K\,\hat{x}} \mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0 K(-\hat{\mathbf{y}})$$

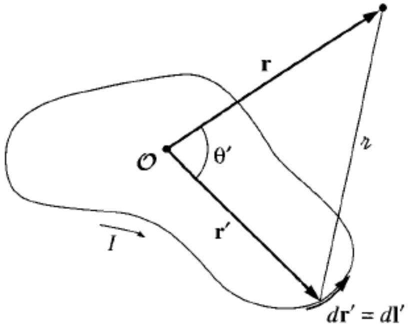
$$\frac{\partial A_{y_{\text{above}}}}{\partial z} = \frac{\partial A_{y_{\text{below}}}}{\partial z} \qquad \frac{\partial A_{x_{\text{above}}}}{\partial z} - \frac{\partial A_{x_{\text{below}}}}{\partial z} = -\mu_0 K$$

$$\boxed{\frac{\partial \mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial \mathbf{A}_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K}}$$

5.4.3 Multipole Expansion of the Vector Potential

If we want an approximate formula for the vector potential of a localized current distribution, valid at distant points, a multipole expansion is required.

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}}{\mathscr{r}} \, dl'$$



Multipole expansion means

➔ ***to write the potential in the form of a power series in 1/r, if r is sufficiently large.***

$$\frac{1}{\mathscr{r}} = \frac{1}{\sqrt{r^2 + (r')^2 - 2rr'\cos\theta'}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta') \quad \text{(Eq. 3.94)}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{1}{\mathscr{r}} \, d\mathbf{l}' = \frac{\mu_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint (r')^n P_n(\cos\theta') \, d\mathbf{l}'$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\oint d\mathbf{l}' + \frac{1}{r^2}\oint r'\cos\theta' \, d\mathbf{l}' + \frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right) d\mathbf{l}' + \cdots\right]$$

monopole ***dipole*** ***quadrupole***

Just for comparison:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(\mathbf{r}')\,d\tau' + \frac{1}{r^2}\int r'\cos\theta'\rho(\mathbf{r}')\,d\tau' + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\mathbf{r}')\,d\tau' + \ldots\right]$$

monopole ***dipole*** ***quadrupole***

Multipole Expansion of the Vector Potential

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\oint d\mathbf{l}' + \frac{1}{r^2}\oint r'\cos\theta'\, d\mathbf{l}' + \frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right) d\mathbf{l}' + \cdots\right]$$

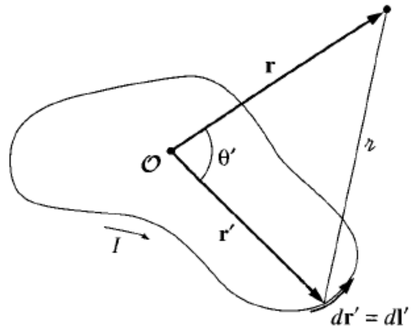
monopole ***dipole*** ***quadrupole***

(Magnetic Monopole Term) ➔ Always zero!

→ For the integral is just the total displacement around a closed path, $\oint d\mathbf{l}' = 0$

→ This reflects the fact that there are (apparently) no magnetic monopoles in nature.

(Magnetic Dipole Term) ➔ It is dominant !



$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi r^2}\oint r'\cos\theta'\, d\mathbf{l}' = \frac{\mu_0 I}{4\pi r^2}\oint (\hat{\mathbf{r}}\cdot\mathbf{r}')\, d\mathbf{l}'$$

$$\oint (\hat{\mathbf{r}}\cdot\mathbf{r}')\, d\mathbf{l}' = -\hat{\mathbf{r}}\times\int d\mathbf{a}'$$

$\boxed{\mathbf{m} \equiv I\int d\mathbf{a} = I\mathbf{a}}$ ➔ **Magnetic dipole moment** $\boxed{\mathbf{p} \equiv \int \mathbf{r}'\rho(\mathbf{r}')\, d\tau'}$

$$\boxed{\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}}$$

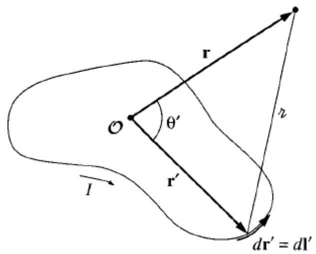
$$\boxed{V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}}$$

Magnetic Dipole field $\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}$ $\mathbf{m} \equiv I\int d\mathbf{a} = I\mathbf{a}$

Magnetic dipole moment is independent of the choice of origin. $\mathbf{m} \equiv I\int d\mathbf{a} = I\mathbf{a}$

(Electric dipole moment was independent of the choice of origin, only when the total charge Q = 0) $\mathbf{p} \equiv \int \mathbf{r}'\rho(\mathbf{r}')\,d\tau'$

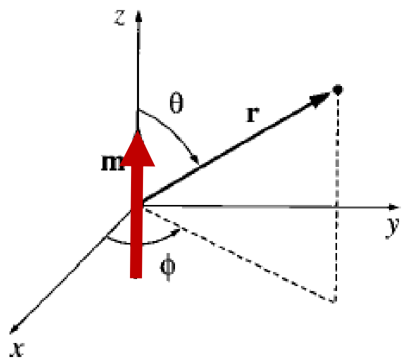
➔ The Independence of origin for magnetic dipole moment is therefore corresponding to no magnetic monopole.

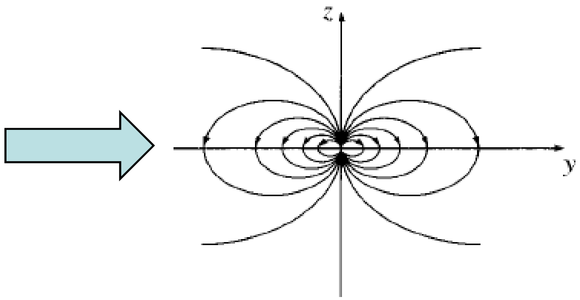


Magnetic field of a (pure) dipole moment placed at the origin.

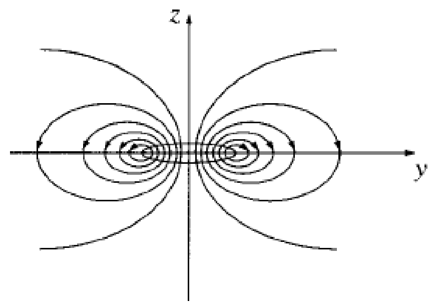
$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} \longrightarrow \mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\boldsymbol{\phi}}$$

$$\longrightarrow \mathbf{B}_{\text{dip}}(\mathbf{r}) = \nabla\times\mathbf{A} = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$





Field B of a "pure" dipole



Field B of a “physical" dipole