Problem 6.19.

An ultrarelativistic gas with $N \gg 1$ particles is in a volume V. The total energy is

$$E = c \sum_{i=1}^{N} p_i$$

where $\boldsymbol{p}_i$ is the momentum of the $i-$th particle with $p_i = |\boldsymbol{p}_i|$ its absolute value, and where c is the speed of light. The total energy is fixed and the particles are indistinguishable. Give an estimate for the entropy and write down the equation of state. Finally, determine the specific heat at constant pressure C_P. If D is the region of the phase space such that $\sum_{i=1}^{N} |\boldsymbol{p}_i| \leq E/c$, the following integral

$$\mathscr{I}_N(E) = \int_D \prod_{i=1}^{N} p_i^2 dp_i = \frac{2^N}{(3N)!}\left(\frac{E}{c}\right)^{3N}$$

may be useful.

$$\Omega[E, V, N] = \frac{1}{h^{3N}} \int d^{3N}q_i \, d^{3N}p_i$$

$$p_i \to (p, \phi, \phi)$$

$$d^3p_i = p^2 \sin\theta \, dp \, d\theta \, d\phi$$

$$\int \prod^N d^3q_i \, d^3p_i = V^N \left(\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \right)^N \int_D \prod_j^N p_j^2 \, dp_j$$

$$\Omega = \frac{V^N (4\pi)^N}{h^{3N}} \cdot \mathscr{I}_N[E]$$

microestados distinguibles.

indistinguible

$$\Omega = \frac{1}{N!} \left(\frac{V 4\pi}{h^3} \right)^N \cdot \frac{2^N}{(3N)!}\left(\frac{E}{c}\right)^{3N} = \frac{1}{N! \, (3N)!} \left(\frac{8\pi V E^3}{h^3 c^3} \right)^N$$

$$hf = E \quad / \lambda$$

$$h\lambda f = \lambda E$$

$$hc = \lambda E$$

$$\left(\frac{E}{hc}\right)^3 = \frac{1}{\lambda^3}$$

$$\Omega = \frac{1}{N!\,(3N)!}\left(\frac{8\pi V}{\lambda^3}\right)^N$$

longitud de onda termal
(gas ideal
ultra relativista)

$$P = \frac{h}{\lambda}$$

$$\frac{P}{h} = \frac{1}{\lambda}$$

$$E = c\sum_{i=1}^{N} p_i \quad / \frac{1}{ch}$$

$$\frac{E}{ch} = \sum_{i=1}^{N} \frac{p_i}{h}$$

$$\boxed{\frac{1}{\lambda} = \sum_{i=1}^{N} \frac{1}{\lambda_i}}$$

$$S = k \ln \Omega = k \ln\left(\frac{1}{N!\,(3N)!}\left(\frac{8\pi V}{\lambda^3}\right)^N\right)$$

$$= k\left\{N \ln\left(\frac{8\pi V}{\lambda^3}\right) - \ln N! - \ln(3N)!\right\}$$

$$= k\left\{N \ln\left(\frac{8\pi V}{\lambda^3}\right) - N\ln N + N - 3N\ln(3N) + 3N\right\}$$

$$= k\left\{N \ln\left(\frac{8\pi V}{\lambda^3}\right) - N\left(\ln N + 3\ln(3N)\right) + 4N\right\}$$

$$S = k\left\{N \ln\left(\frac{8\pi V E^3}{h^3 c^3}\right) - N\left(\ln N + 3\ln(3N)\right) + 4N\right\}$$

$$-N\left(\ln N + \ln(27N^3)\right)$$

$$= k\{N\ln(\quad.\quad) - N(\ln[27N^4]) + 4N\}$$

$$S = k\left\{N\ln\left[\frac{8\pi E^3 V}{27N^4 h^3 c^3}\right] + 4N\right\} \quad E, N, V$$

$E = 3kT$

$\lambda = \left(\frac{h^2}{2\pi mkT}\right)^{1/2}$ $\quad \lambda = (8\pi)^{1/3}\frac{hc}{kT}$ en este caso

$$\frac{8\pi}{27h^3c^3}\frac{E^3V}{N^4} = \eta\frac{E^3V}{N^4}$$

$$S = Nk\ln\left[\eta\frac{E^3V}{N^4}\right] + 4Nk$$

$$dE = TdS - PdV + \mu dN$$

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{N,V} = \frac{Nk}{\cancel{\eta}\frac{E^3\cancel{V}}{\cancel{N^4}}}\cdot\frac{\cancel{\eta V}}{\cancel{N^4}}3E^2 = \frac{3Nk}{E} \rightarrow \boxed{T = \frac{E}{3Nk}}$$

$$P = T\left(\frac{\partial S}{\partial V}\right)_{E,N} = T\frac{Nk}{V}$$

la ecuación de estado $\quad PV = NkT \quad ; \quad E = 3NkT$

$$\frac{TNk}{V}V = NkT \quad \checkmark\checkmark$$

$$y \quad C_p = T\left(\frac{\partial S}{\partial T}\right)_{P,N} \quad \text{con } S[P,T,N]$$