EJEMPLO

Halle la solución y sus distintas representaciones en serie de la integral:

$$J = \int_0^{\infty}\int_0^{\infty} \frac{e^{-x}\, e^{\frac{xy}{x+y}A}}{(x+y)^{\alpha}}\, dx\, dy$$

PASO 1: Expansión del integrando

$$e^{-x} = \sum_n \phi_n\, x^n$$

$$e^{\frac{xy}{x+y}A} = e^{\frac{g}{g}\frac{xy}{x+y}A}$$; Hacemos $g=-1$ en el numerador al principio y $g=-1$ en el denominador al final cuando ya no existan brackets.

esto es:

$$e^{-\frac{xy}{x+y}\frac{A}{g}} = \sum_m \phi_m \frac{x^m y^m}{(x+y)^m} \frac{A^m}{g^m}$$

$\therefore$ el integrando hasta ahora queda expandido como sigue:

$$\frac{e^{-x} e^{-\frac{xy}{x+y}\frac{A}{g}}}{(x+y)^{\alpha}} = \sum_n \sum_m \phi_{n,m} \frac{x^{n+m}\, y^m}{(x+y)^{\alpha+m}} A^m g^{-m}$$

Ahora expandimos el binomio:

$$\frac{1}{(x+y)^{\alpha+m}} = \sum_l \sum_j \phi_{l,j}\, x^l y^j \frac{\langle \alpha+m+l+j \rangle}{\Gamma(\alpha+m)}$$

Finalmente:

$$\frac{e^{-x} e^{-\frac{xy}{x+y}\frac{A}{g}}}{(x+y)^{\alpha}} = \sum_{n}\sum_{m}\sum_{\ell}\sum_{j} \phi_{n,m,\ell,j}\, A^{m} g^{-m}\, x^{m+n+\ell}\, y^{m+j}$$

$$\times \frac{\langle \alpha+m+\ell+j \rangle}{\Gamma(\alpha+m)}$$

PASO 2: serie de brackets de la integral.

$$J \equiv \sum_{n}\sum_{m}\sum_{\ell}\sum_{j} \phi_{n,m,\ell,j}\, A^{m} g^{-m} \frac{\langle \alpha+m+\ell+j \rangle}{\Gamma(\alpha+m)} \underbrace{\int_0^{\infty} x^{m+n+\ell}\, dx}_{\langle m+n+\ell+1 \rangle} \underbrace{\int_0^{\infty} y^{m+j}\, dy}_{\langle m+j+1 \rangle}$$

$$\Downarrow$$

$$J = \sum_{n}\sum_{m}\sum_{\ell}\sum_{j} \phi_{n,m,\ell,j}\, A^{m} g^{-m} \frac{\langle \alpha+m+\ell+j \rangle}{\Gamma(\alpha+m)} \langle m+n+\ell+1 \rangle \langle m+j+1 \rangle$$

PASO 3: Soluciones

$\boxed{\text{caso 1}}$: n Libre $\Rightarrow J_n$

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{M_n} \begin{pmatrix} m \\ \ell \\ j \end{pmatrix} = \begin{pmatrix} -\alpha \\ -1-n \\ -1 \end{pmatrix}$$

luego $\det M_n = -1$ y $\begin{pmatrix} m^* \\ l^* \\ j^* \end{pmatrix} = \begin{pmatrix} \alpha-2-n \\ 1-\alpha \\ n-\alpha+1 \end{pmatrix}$

entonces

$$J_n = \frac{1}{\underbrace{|\cancel{\det} M_n|}_{1}} \sum_{n\geq 0} \phi_n A^m g^{-m} \frac{\Gamma(-m)\Gamma(-l)\Gamma(-j)}{\Gamma(\alpha+m)} \Bigg|_{\substack{m=m^* \\ l=l^* \\ j=j^*}}$$

$$= \sum_{n\geq 0} \frac{(-1)^n}{n!} A^{\alpha-2-n} g^{-(\alpha-2-n)} \frac{\Gamma(2-\alpha+n)\Gamma(\alpha-1)\Gamma(\alpha-1-n)}{\Gamma(2\alpha-2-n)}$$

$$= A^{\alpha-2}\Gamma(\alpha-1) g^{-\alpha-2} \sum_{n\geq 0} \frac{(-1)^n}{n!} A^{-n} g^{n} \frac{\Gamma(2-\alpha)(2-\alpha)_n \Gamma(\alpha-1)(\alpha-1)_{-n}}{\Gamma(2\alpha-2)(2\alpha-2)_n}$$

haciendo ahora $g=-1$.

$$J_n = A^{\alpha-2}(-1)^{-\alpha} \frac{\Gamma(2-\alpha)\Gamma(\alpha-1)^2}{\Gamma(2\alpha-2)} \sum_{n\geq 0} \frac{(2-\alpha)_n (\alpha-1)_{-n}}{(2\alpha-2)_{-n}} \frac{A^{-n}}{n!}$$

$$= A^{\alpha-2}(-1)^{-\alpha} \frac{\Gamma(2-\alpha)\Gamma(\alpha-1)^2}{\Gamma(2\alpha-2)} \sum_{n\geq 0} \frac{(2-\alpha)_n (3-2\alpha)_n}{(2-\alpha)_n} \frac{(1/A)^n}{n!}$$

$$J_n = A^{\alpha-2}(-1)^{-\alpha} \frac{\Gamma(2-\alpha)\Gamma(\alpha-1)^2}{\Gamma(2\alpha-2)} \, {}_2F_1\left(\begin{matrix} 2-\alpha, 3-2\alpha \\ 2-\alpha \end{matrix} \middle| \frac{1}{A} \right)$$

Sin embargo J_n puede ser simplificado:

$$J_n = A^{\alpha-2}(-1)^{-\alpha}\,\frac{\Gamma(2-\alpha)\Gamma(\alpha-1)^2}{\Gamma(2\alpha-2)}\;\underbrace{{}_1F_0\left(\begin{matrix}3-2\alpha\\ -\end{matrix}\,\middle|\,\frac{1}{A}\right)}_{\dfrac{1}{\left(1-\frac{1}{A}\right)^{3-2\alpha}}}$$

$$J_n = A^{\alpha-2}(-1)^{-\alpha}\,\frac{\Gamma(2-\alpha)\Gamma(\alpha-1)^2}{\Gamma(2\alpha-2)}\;\frac{A^{3-2\alpha}}{(A-1)^{3-2\alpha}}$$

$\boxed{\text{caso 2}}$: m libre $\Rightarrow J_m$

$$\underbrace{\begin{pmatrix}0 & 1 & 1\\ 1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}}_{M_m}\begin{pmatrix}n\\ l\\ j\end{pmatrix} = \begin{pmatrix}-\alpha-m\\ -1-m\\ -1-m\end{pmatrix}$$

Π $\det M_m = -1$ y $\begin{pmatrix}n^*\\ l^*\\ j^*\end{pmatrix} = \begin{pmatrix}-m+\alpha-2\\ 1-\alpha\\ -m-1\end{pmatrix}$

luego

$$J_m = \frac{1}{\underbrace{|\cancel{\det M_m}|}_{1}}\sum_{m\geq 0}\phi_m\,A^m g^{-m}\,\frac{\Gamma(-n)\Gamma(-l)\Gamma(-j)}{\Gamma(\alpha+m)}\Bigg|_{\substack{m=m^*\\ l=l^*\\ j=j^*}}$$

Luego

$$J_m = \sum_{m\geq 0} \frac{\cancel{(-1)^m}}{m!} A^m \cancel{(-1)^m} \frac{\Gamma(m-\alpha+2)\Gamma(\alpha-1)\Gamma(m+1)}{\Gamma(\alpha+m)}$$

$$= \Gamma(\alpha-1) \sum_{m\geq 0} \frac{\Gamma(2-\alpha)(2-\alpha)_m (1)_m}{\Gamma(\alpha)(\alpha)_m} \frac{A^m}{m!}$$

$$= \frac{\Gamma(\alpha-1)\Gamma(2-\alpha)}{\Gamma(\alpha)} \sum_{m\geq 0} \frac{(2-\alpha)_m (1)_m}{(\alpha)_m} \frac{A^m}{m!}$$

$$J_m = \frac{\Gamma(\alpha-1)\Gamma(2-\alpha)}{\Gamma(\alpha)} \, {}_2F_1\left(\begin{matrix} 2-\alpha \, ; \, 1 \\ \alpha \end{matrix} \middle| A\right)$$

$\boxed{\text{caso 3}}$: l libre $\Rightarrow J_l$

$$\underbrace{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}}_{M_l} \begin{pmatrix} n \\ m \\ j \end{pmatrix} = \begin{pmatrix} -\alpha-l \\ -1-l \\ -1 \end{pmatrix}$$

Obs. $\det M_l = 0$

$\Downarrow$

Esta combinación $\Sigma / \langle\rangle$ no genera un término

$\boxed{\text{Caso 4}}$: j Libre $\Longrightarrow J_j$

$$\underbrace{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{M_j} \begin{pmatrix} n \\ m \\ l \end{pmatrix} = \begin{pmatrix} -\alpha-j \\ -1 \\ -1-j \end{pmatrix}$$

luego

$$\det M_j = 1 \quad y \quad \begin{pmatrix} n^* \\ m^* \\ l^* \end{pmatrix} = \begin{pmatrix} j+\alpha-1 \\ -j-1 \\ -\alpha+1 \end{pmatrix}$$

Luego

$$J_j = \frac{1}{\cancel{|\det M_j|}_{1}} \sum_{j\geq 0} \frac{(-1)^j}{j!} A^m g^{-m} \frac{\Gamma(-n)\Gamma(-m)\Gamma(-l)}{\Gamma(\alpha+m)} \Bigg|_{\substack{m=m^* \\ n=n^* \\ l=l^*}}$$

$$J_j = \sum_{j\geq 0} \frac{(-1)^j}{j!} A^{-j-1} g^{j+1} \frac{\Gamma(1-j-\alpha)\Gamma(1+j)\Gamma(\alpha-1)}{\Gamma(\alpha-j-1)} \quad , g=-1$$

$$= \frac{-\Gamma(\alpha-1)}{A} \sum_{j\geq 0} \frac{(1/A)^j}{j!} \frac{\Gamma(1-\alpha)(1-\alpha)_{-j}(1)_j}{\Gamma(\alpha-1)(\alpha-1)_{-j}}$$

luego

$$J_g = -\frac{\Gamma(1-\alpha)}{A} \sum_{j\geq 0} \frac{(1/A)^j}{j!} \frac{(1)_j (2-\alpha)_j}{(\alpha)_j}$$

$$J_g = -\frac{\Gamma(1-\alpha)}{A} \, {}_2F_1\left(\begin{matrix} 1, 2-\alpha \\ \alpha \end{matrix} \middle| \frac{1}{A}\right)$$

$$\therefore \quad J = \begin{cases} J_m & \text{(Serie en potencias de } A) \\ \text{ó} & \\ J_n + J_g & \text{(Serie en potencias de } \frac{1}{A}) \end{cases}$$