Clase 22 211121 ①

Lentes gravitacionales (Deflexión de la luz).

Sabemos que fotones moviéndose en el espacio-tiempo de Schwarzschild, las ecs. de movimiento (relevantes)

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - f(r)\frac{L^2}{r^2}$$

$$\left(\frac{d\phi}{d\lambda}\right) = \frac{L}{r^2}$$

$$f(r) = 1 - \frac{r_s}{r} + (\text{otros ingredientes})$$

carga, constante cosmologica, energía oscura, nube de cuerdas, etc)

$$\frac{dr}{d\lambda} = \frac{d\phi}{d\lambda}\frac{dr}{d\phi} = \frac{L}{r^2}\frac{dr}{d\phi} \quad ; \quad u = \frac{1}{r}$$

$$= -L\frac{du}{d\phi}$$

$$\left(\frac{du}{d\phi}\right)^2 = \left(\frac{E}{L}\right)^2 - u^2 f(u) \quad ; \quad f(u) = 1 - r_s u$$

$$\left(\frac{du}{d\phi}\right)^2 = \frac{1}{b^2} - u^2 + r_s u^3$$

$$\frac{du}{\sqrt{\frac{1}{b^2}-u^2+r_s u^3}} = \mp\, d\phi$$

i) Aprox. de Einstein (campo débil)

$$r_s u \ll 1$$

Usando el cambio de variable

$$Y \equiv u\left(1-\frac{r_s}{2}u\right)$$

$$\therefore \quad u = \frac{Y}{1-\frac{r_s}{2}u} \simeq Y\left(1+\frac{r_s}{2}u+\dots\right)$$

$$u \simeq Y\left(1+\frac{r_s}{2}\left(\frac{Y}{1-\frac{r_s}{2}u}\right)+\dots\right)$$

$$u \simeq Y\left(1+\frac{r_s}{2}Y\left(1+\frac{r_s}{2}u+\dots\right)+\dots\right)$$

$$\boxed{u \simeq Y\left(1+\frac{r_s}{2}Y\right)}+\dots$$

$$\boxed{du = (1+r_s Y)\,dY}$$

- $$\frac{1}{b^2}-u^2+r_s u^3 = \frac{1}{b^2}-Y^2\left(1+\frac{r_s}{2}Y\right)^2+r_s\left(1+\frac{r_s}{2}Y\right)^3$$

$$= \frac{1}{b^2}-Y^2\left(1+\frac{r_s}{2}Y^2\right)\left[1-r_s\left(1+\frac{r_s}{2}Y\right)\right]$$

$$\simeq \frac{1}{b^2}-Y^2+\dots$$

CL22

$$\int_0^y \frac{(1+r_s y')\,dy'}{\sqrt{\frac{1}{b^2}-y'^2}} = \int_{\phi_0}^{\phi} d\phi$$

$$\phi = \phi_0 + \int_0^y \frac{dy'}{\sqrt{\frac{1}{b^2}-y'^2}} + r_s \int_0^y \frac{y'\,dy'}{\sqrt{\frac{1}{b^2}-y'^2}}$$

$$* \int_0^y \frac{dy'}{\sqrt{\frac{1}{b^2}-y'^2}} = \quad y' = \frac{1}{b}\sin x \quad ; \; dy' = \frac{1}{b}\cos x\,dx$$

$$\int_0^{\frac{1}{b}\sin x} \frac{1/b \cos x'\,dx'}{\sqrt{\frac{1}{b^2}-\frac{1}{b^2}\sin^2 x'}} = x' \Big|_0^{\frac{1}{b}\sin x} =$$

$$= \text{ArcSin}(yb)$$

$$* \int_0^y \frac{y'\,dy'}{\sqrt{\frac{1}{b^2}-y'^2}} = -\sqrt{\frac{1}{b^2}-y'^2}\,\Big|_0^y =$$

$$= -\sqrt{\frac{1}{b^2}-y^2} + \frac{1}{b}$$

$$\phi = \phi_0 + \frac{r_s}{b} + \text{ArcSin}(yb) - \sqrt{\frac{1}{b^2}-y^2}$$

Término correctivo Einstein.

La ecuación original

$$\phi - \phi_0 = \int_0^u \frac{du}{\sqrt{\frac{1}{b^2} - u^2 + r_s u^3}}$$

Veamos el polinomio

$$r_s u^3 - u^2 + \frac{1}{b^2} = r_s \left(u^3 - \frac{1}{r_s} u^2 + \frac{1}{r_s b^2} \right)$$

$$u = y + \frac{1}{3r_s} \; ; \; u^2 = y^2 + \frac{2}{3r_s} y + \frac{1}{9r_s^2}$$

$$u^3 = y^3 + \cancel{3} y^2 \frac{1}{\cancel{3} r_s} + \cancel{3} y \cdot \frac{1}{\cancel{9}_3 r_s^2} + \frac{1}{27 r_s^3}$$

$$u^3 - \frac{1}{r_s} u^2 = y^3 + \cancel{\frac{y^2}{r_s}} + \frac{y}{3r_s^2} + \frac{1}{27 r_s^3} - \cancel{\frac{1}{r_s} y^2} - \frac{2}{3r_s^2} y - \frac{1}{9 r_s^3}$$

$$= y^3 - \frac{y}{3r_s^2} - \frac{2}{27 r_s^3}$$

$$\therefore \; r_s u^3 - u^2 + \frac{1}{b^2} = r_s \left[y^3 - \frac{y}{3r_s^2} + \frac{1}{r_s b^2} - \frac{2}{27 r_s^3} \right]$$

$$= \frac{r_s}{4} \left[4y^3 - \frac{4}{3r_s^2} y - \frac{4}{r_s} \left(\frac{2}{27 r_s^2} - \frac{1}{b^2} \right) \right]$$

$$\phi - \phi_0 = \frac{2}{\sqrt{r_s}} \int_{-1/3r_s}^{y} \frac{dy}{\sqrt{4y^3 - g_2 y - g_3}}$$

donde $\{g_2, g_3\}$ son los llamados invariantes de Weierstrass, y la integral

$$\int_y^{\infty} \frac{dy}{\sqrt{4y^3 - g_2 y - g_3}} = \wp^{-1}(y; g_2, g_3)$$

~~luego~~ es la integral elíptica de Weierstrass, y $\wp(y; g_2, g_3) \equiv \wp(y)$ es la función elíptica $\wp$-Weierstrass.

$$\frac{\sqrt{r_s}}{2}(\phi - \phi_0) = \int_{-1/3r_s}^{\infty} \frac{dy}{\sqrt{4y^3 - g_2 y - g_3}} - \int_y^{\infty} \frac{dy}{\sqrt{4y^3 - g_2 y - g_3}}$$

$$\frac{\sqrt{r_s}}{2} \Delta\phi = \wp^{-1}\left(-\frac{1}{3r_s}\right) - \wp^{-1}(y)$$

$$\wp^{-1}(y) = \wp^{-1}\left(-\frac{1}{3r_s}\right) - \frac{\sqrt{r_s}}{2}\Delta\phi \quad \Big/ \wp$$

$$y = \wp\Big[\underbrace{\wp^{-1}\left(-\frac{1}{3r_s}\right) - \frac{\sqrt{r_s}}{2}\Delta\phi}_{\varphi}\,; g_2, g_3\Big]$$

$$\frac{1}{r} - \frac{1}{3r_s} = \wp(\varphi) \Rightarrow \frac{1}{r} = \wp(\varphi) + \frac{1}{3r_s} = \frac{1 + 3r_s\wp(\varphi)}{3r_s}$$

$$\Rightarrow \boxed{r(\phi) = \frac{3r_s}{1 + 3r_s \wp(\varphi)}}$$