PHY481 - Lecture 27: Dielectric materials
Griffiths: Chapter 4

General Relations

In dielectric materials, polarization leads to a reduction (screening) of applied electric fields so that the electric field inside the material is reduced as compared to vacuum. In that sense metals are perfect dielectrics as they screen out all electric fields. In contrast paraelectric and ferroelectric materials experience enhanced internal electric fields. There is close similarity with magnetic materials where paramagnetic and ferromagnetic materials have enhanced internal fields while diamagnetic materials screen out magnetic fields. Superconductors are the best diamagnets. In our discussion of electric polarization in materials, we shall only discuss the dielectric case, as it is most common.

Dielectric screening is produced by opposing electric fields due to bound charge which in turn originates from the polarization. Bound charge may occur at the surface of a dielectric material in which case it is denoted by σ_b, the surface bound charge density or in the bulk where it is the bulk bound charge density, ρ_b. The relations between these quantities and the polarization in a material are,

$$\sigma_b = \hat{n} \cdot \vec{P}; \quad \rho_b = -\vec{\nabla} \cdot \vec{P} \tag{1}$$

where $\vec{P}$ is the polarization density, i.e. the polarization per unit volume. The bound charge acts as a source of the electric field so we have,

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}(\rho_f + \rho_b) \tag{2}$$

where ρ_f is the so-called free charge and is the excess charge on the material. If we integrate over the whole material, the bound charge sums to zero. The excess charge can be considered to produce a field of its own that we define to be the displacement field so that,

$$\vec{\nabla} \cdot \vec{D} = \rho_f. \tag{3}$$

Using this definition and the definition of the bound charge in terms of the polarization, we have,

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}(\vec{\nabla} \cdot \vec{D} - \vec{\nabla} \cdot \vec{P}); \quad \text{so that} \quad \epsilon_0 \vec{E} = \vec{D} - \vec{P}. \tag{4}$$

The displacement field can be found by using Gauss's law, that is,

$$\vec{\nabla} \cdot \vec{D} = \rho_f; \quad \text{implies} \quad \oint \vec{D} \cdot d\vec{a} = q_f \tag{5}$$

From Faraday's law in the absence of magnetic fields implies that in dielectrics we still have, $\vec{E} = -\vec{\nabla}V$.

In order to solve dielectric problems we need to be given either $\vec{P}$, ρ_b and ρ_f, and/or, we can be given a relation between $\vec{P}$ and $\vec{E}$, i.e. some function or constitutive relation $\vec{P}(\vec{E})$. However usually we don't know $\vec{P}$ and we need more information than that contained in the results above in order to solve EM problems in dielectric media. The key additional information is usually in the form of a constitutive law giving a relation between the polarization and the local electric field, i.e. some function $\vec{P}(\vec{E})$.

In this course we restrict attention to situations where the material is a linear isotropic ielectric, so that $\vec{P} = \epsilon_0 \chi_e \vec{E}$, which is related to the atomic scale relation between polarizeability $\vec{p} = \alpha \vec{E}$ and electric field. The relation between α and χ_e is simple in the gas phase, but more complex in dense media. In some problems we are given the polarization, in which case the system of equations are closed and can be solved (see for example the case of a uniformly polarizared sphere in Lecture 26).

Linear isotropic dielectrics where: $\vec{P} = \epsilon_0 \chi_e \vec{E}$, $\vec{D} = \epsilon\vec{E}$, $\epsilon_r = \epsilon/\epsilon_0$

Most of the problems that we solve will take the simplest, but still very important, case of a linear isotropic dielectric where,

$$\vec{P} = \epsilon_0 \chi_e \vec{E}; \tag{6}$$

Using the general relation, $\epsilon_0 \vec{E} = \vec{D} - \vec{P}$ with the above equation, we find that,

$$\vec{P} = \vec{D} - \epsilon_0 \vec{E} = \epsilon_0 \chi_e \vec{E}; \quad \text{so that} \quad \vec{D} = \epsilon\vec{E} \quad \text{with} \quad \epsilon = \epsilon_0(1 + \chi_e) \tag{7}$$

where χ_e is the dielectric susceptibility and ϵ is the permittivity. Dielectric materials are usually characterized by the ratio $\epsilon_r = \epsilon/\epsilon_0$, the relative permittivity. The relative permittivity of air is close to 1, while that of engineering polymers is around 3. High values of ϵ_r are found in perovskite materials such as Strontium Titanate and Barium Titanate.

Solution strategies for problems involving dielectrics

The goal is again to find the electric field and the electrostatic potential from which all of the physics follows. The following two strategies are generally useful.

- (i) Use Gauss's law for $\vec{D}$, then find $\vec{E} = \vec{D}/\epsilon$.
- (ii) In regions where there are no free charges, and where ϵ is uniform, then,

$$\vec{\nabla} \cdot \vec{D} = 0 = \epsilon \vec{\nabla} \cdot \vec{E} = -\epsilon \nabla^2 V = 0 \tag{8}$$

That is, Laplace's equation holds.

Charges inside dielectrics, use $\vec{\nabla} \cdot \vec{D} = \rho_f, \vec{D} = \epsilon \vec{E}$

Solving problems where there are charges inside dielectric materials proceeds by first solving for $\vec{D}$ using either Poisson's equation or by using Gauss's law. Then find the electric field is using $\vec{D} = \epsilon \vec{E}$. Consider a point charge in a dielectric medium. The field around the point charge is found by using

$$\oint \vec{D} \cdot d\vec{a} = q; \quad \text{so that} \quad \vec{D} = \frac{q}{4\pi r^2}\hat{r}; \quad \text{and hence} \quad \vec{E} = \frac{q}{4\pi \epsilon r^2}\hat{r} \tag{9}$$

The electric field inside the dielectric is *reduced* due to the polarization of the dielectric (dielectric screening). Now consider a point charge at the center of a finite dielectric sphere. The field inside the sphere is given above, but what about the field outside the sphere? Following the same considerations as above, we find that the electric field outside the dielectric sphere is *unaltered* by the dielectric - it is useful to compare this result to a charge at the center of an isolated uncharged metal shell.

Now consider a sheet of charge lying in the x-y plane, with charge density σ, inside a dielectric slab that is centered at the origin and has normal $\hat{k}$. If the thickness of the slab is $2d$, the considerations above indicate that the field inside the slab is $\sigma/(2\epsilon)$ if $d > z > 0$. However outside the slab (e.g. for $z > d$ the field is $\sigma/(2\epsilon_0)$ - i.e. the dielectric medium has no effect on the far field.

Planar capacitor containing a dielectric

Consider a parallel plate capacitor of area A and plate separation d that has dielectric material with dielectric permittivity ϵ between the places. (i) Find the electric field between the plates when a charge Q is placed on the capacitor. Find the voltage across the capacitor and the energy stored in the capacitor. (ii) Now consider connecting a battery to the capacitor, with voltage V on the top plate and the lower plate at ground. Find the charge on the capacitor plates and the energy stored in the capacitor. In both cases compare your results to the case where there is no dielectric between the plates. That is, in the two cases of fixed charge and fixed voltage, does the stored energy go up or down when the dielectric is added? Also, if the dielectric slab is free to move, is it drawn into the space between the capacitor plates or is it pushed out. Find the force in both cases (i) and (ii). (This is a typical electro-mechanical actuator, used for example in doorbells).

Solution to (i) When a charge Q is placed on the capacitor, we can use Gauss's law (or superposition), to find the diplacement field,

$$\vec{D} = -\sigma\hat{k}; \quad \text{so that} \quad \vec{E} = -\frac{\sigma}{\epsilon}\hat{k} \tag{10}$$

The voltage across the plates is given by $V = |E|d = \frac{\sigma d}{\epsilon}$. The capacitance is found from $Q = CV$, and using,

$$V = \frac{\sigma d}{\epsilon} = \frac{Qd}{A\epsilon} \quad \text{so that} \quad C = \frac{\epsilon A}{d} \tag{11}$$

and the energy stored in the capacitor is $U = Q^2/2C$. If there is no dielectric between the plates, the capacitance is $C_0 = \epsilon_0 A/d$, and hence,

$$\frac{U}{U_0} = (\frac{Q^2}{2C})/(\frac{Q^2}{2C_0}) = \frac{C_0}{C} = 1/\epsilon_r \quad \text{for} \quad \textit{isolated capacitor} \tag{12}$$

so that the energy is reduced when the dielectric is between the plates. Note that from this calculation we deduce that the energy density by writing,

$$U = \frac{Q^2}{2C} = \frac{\sigma^2 A^2 d}{2\epsilon A} = uV \quad with \quad u = \frac{1}{2}\epsilon_0\epsilon_r E^2 \tag{13}$$

where u is the energy density, $V = Ad$ is volume and we used $|E| = \sigma/\epsilon$. Now note that,

$$u = \frac{1}{2}\epsilon_0\epsilon_r E^2 = \frac{1}{2}\vec{D}\cdot\vec{E}. \tag{14}$$

The result $u = \frac{1}{2}\vec{D}\cdot\vec{E}$ is a general result for linear isotropic dielectrics.

Solution to (ii) Now consider the case where the voltage across the plates is fixed to V. In that case

$$U/U_0 = \frac{\frac{1}{2}CV^2}{\frac{1}{2}C_0V^2} = C/C_0 = \epsilon_r \quad \text{for} \quad fixed\ voltage\ across\ capacitor \tag{15}$$

The case of a fixed voltage applies to energy storage, as capacitors are usually charged at fixed voltage, e.g. from a generator or solar cell. High energy storage materials require high values of ϵ_r (highly polarizable material) and high V (very good insulator with low leakage).

Note: In either the constant charge or constant voltage situation, the lowest TOTAL energy state is with the dielectric material between the plates. This is obvious in the constnat charge case as the system is isolated. In the constant voltage case however the total energy of the system is energy of the battery plus the energy of the capacitor. When the dielectric is placed between the plates, the energy stored in the capacitor goes up, but the energy stored in the battery goes down by a factor of two more, so the total energy is negative. The amount by which the battery energy is reduced is equal the integral of the power, $\delta U_{bat} = -V\int\frac{dQ}{dt}dt = -VQ$. The energy stored in the capacitor is $QV/2$, so the total change in energy is $-VQ/2$. The energy is thus reduced by maximizing Q and this is achieved by having the dielectric material between the plates.