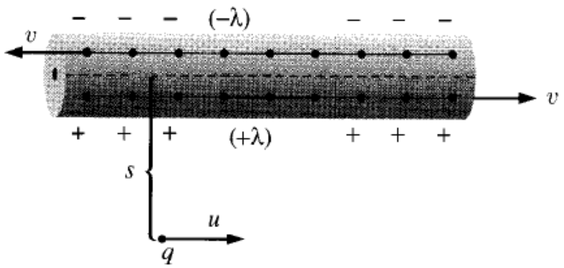
Chapter 12.
Electrodynamics and Relativity

Does the principle of relativity apply to the laws of electrodynamics?

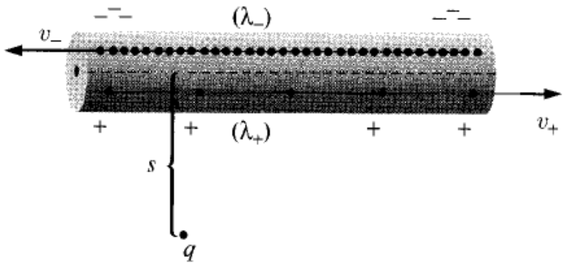
12.3 Relativistic Electrodynamics

12.3.1 Magnetism as a Relativistic Phenomenon

net current: $I = 2\lambda v$



A point charge ***q*** traveling to the right at speed $u < v$



In the reference frame where q is at rest, system $\bar{S}$,

by the Einstein velocity addition rule, the velocities of the positive and negative lines are

$$v_\pm = \frac{v \mp u}{1 \mp vu/c^2}$$

Because $v_- > v_+$, the Lorentz contraction of the spacing between negative charges is more severe;

→ ***the wire carries a net negative charge!***

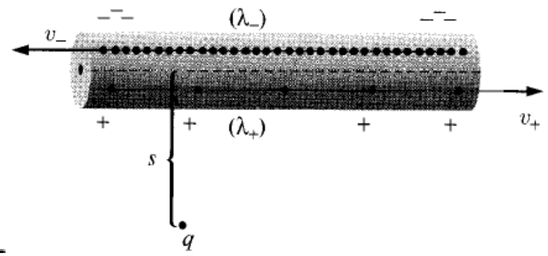
$$\lambda_\pm = \pm(\gamma_\pm)\lambda_0 \longrightarrow \lambda_{\text{tot}} = \lambda_+ + \lambda_- = \lambda_0(\gamma_+ - \gamma_-) = \frac{-2\lambda uv}{c^2\sqrt{1-u^2/c^2}}$$

$$\text{where} \quad \gamma_\pm = \frac{1}{\sqrt{1-v_\pm^2/c^2}} = \gamma\frac{1 \mp uv/c^2}{\sqrt{1-u^2/c^2}}$$

Conclusion: As a result of unequal Lorentz contraction of the positive and negative lines, a current-carrying wire that is electrically neutral in one inertial system will be charged in another.

Magnetism as a Relativistic Phenomenon

In the reference frame where *q* is at rest, system $\bar{S}$,



$$\lambda_{\text{tot}} = \lambda_+ + \lambda_- = \lambda_0(\gamma_+ - \gamma_-) = \frac{-2\lambda u v}{c^2\sqrt{1-u^2/c^2}}$$

The line charge sets up an *electric* field: $$E = \frac{\lambda_{\text{tot}}}{2\pi\epsilon_0 s}$$

so *there is an electrical force on q in* $\bar{S}$, $$\bar{F} = qE = -\frac{\lambda v}{\pi\epsilon_0 c^2 s}\frac{qu}{\sqrt{1-u^2/c^2}}$$

➔ **In $\bar{S}$ system, the wire is attracted toward the charge by a purely electrical force.**

The force $\bar{F}$ can be transformed into F in S (wire at rest) by (Eq. 12.68)

$$F = \frac{1}{\gamma}\bar{F} = \sqrt{1-u^2/c^2}\,\bar{F} = -\frac{\lambda v}{\pi\epsilon_0 c^2}\frac{qu}{s}$$

But, in the wire frame (S) the total charge is neutral !

→ **what does the force F imply?**

→ Electrostatics and relativity imply the existence of another force in view point of S frame.

➔ **magnetic force**.

In fact, by using $c^2 = (\epsilon_0\mu_0)^{-1}$ and $I = 2\lambda v$

$$F = -\frac{\lambda v}{\pi\epsilon_0 c^2}\frac{qu}{s} = -qu\left(\frac{\mu_0 I}{2\pi s}\right)$$, magnetic field, B = $\left(\frac{\mu_0 I}{2\pi s}\right)$

➔ **One observer's electric field is another's magnetic field!**

➔ **Therefore, the relativistic force F is the Lorentz force in system S, not Minkowski!**

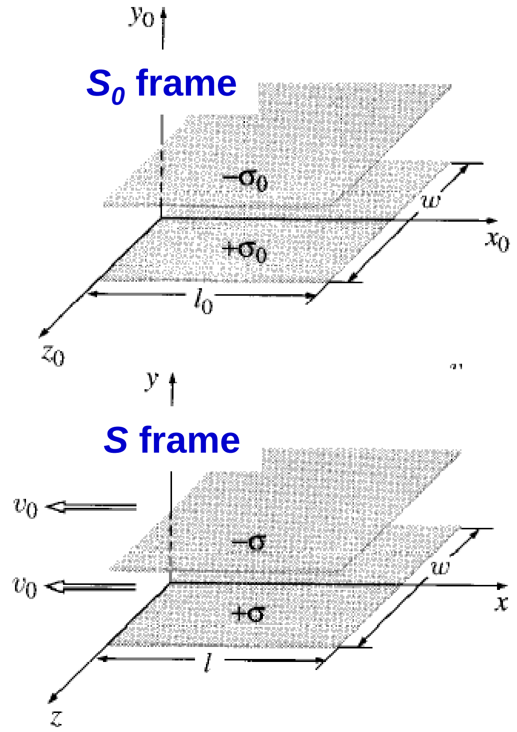
12.3.2 How the Fields Transform

Let's find the general transformation rules for electromagnetic fields:

➔ Given the fields in a frame ($\mathcal{S}$), what are the fields in another frame ($\bar{\mathcal{S}}$)?

consider the *simplest possible* electric field in a large parallel-plate capacitor in ***S_0*** frame.

$$\mathbf{E}_0 = \frac{\sigma_0}{\epsilon_0}\,\hat{\mathbf{y}}$$



In the system ***S***, moving to the right at speed v_0,
the plates are moving to the left with the different surface charge σ:

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\,\hat{\mathbf{y}}$$

The total charge on each plate is invariant, and the *width* (*w)* is unchanged, but the *length (l)* is Lorentz-contracted by a factor

$$\frac{1}{\gamma_0} = \sqrt{1 - v_0^2/c^2} \longrightarrow \sigma = \gamma_0\sigma_0 \longrightarrow \boxed{\mathbf{E}^{\perp} = \gamma_0 \mathbf{E}_0{}^{\perp}}$$

→ This rule pertains to components of E that are *perpendicular* to the direction of motion of *S.*

How the Fields Transform

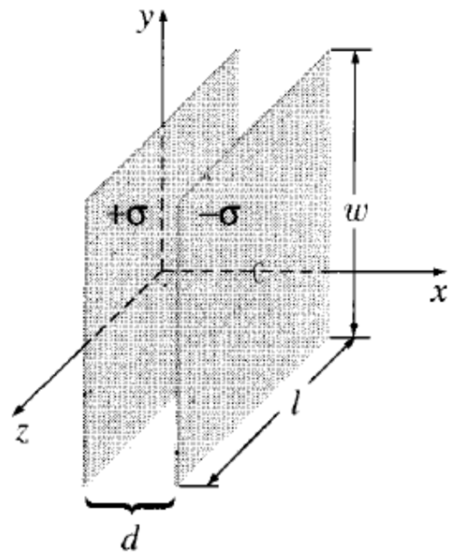
Let's find the general transformation rules for electromagnetic fields:

➔ Given the fields in a frame ($\mathcal{S}$), what are the fields in another frame ($\bar{\mathcal{S}}$)?

For *parallel* components, consider the capacitor lined up with the *y z* plane.

→ the plate separation *(d)* that is Lorentz-contracted,

→ whereas *l* and *w* (and hence also σ) are the same in both frames.



$$E^{\parallel} = E_0^{\parallel}$$

$$E^{\parallel} = E_0^{\parallel} \qquad \mathbf{E}^{\perp} = \gamma_0 \mathbf{E}_0^{\perp}$$

This case is not the most general case:
we began with a system S_o in which the charges were at rest
and where, consequently, there was no magnetic field.

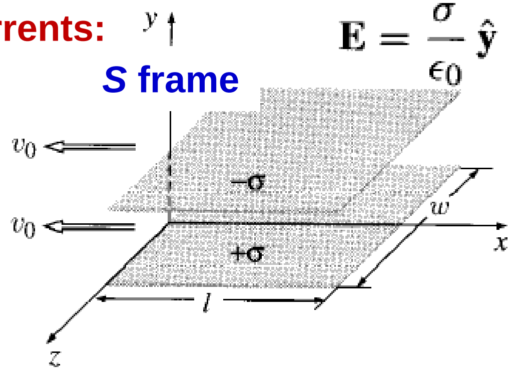
To derive the *general* rule we must start out in a system with both electric and magnetic fields.

How the Fields Transform

To derive the *general* rule we must start out in a system with both electric and magnetic fields.

Consider the S system, there is also a *magnetic* field due to the surface currents:

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\,\hat{\mathbf{y}}$$



$$\mathbf{K}_{\pm} = \mp\sigma v_0\,\hat{\mathbf{x}} \quad \left(\upsilon_0 :\ \text{velocity of } S \text{ relative to } S_0\right)$$

By the right-hand rule, this field points in the negative *z* direction;

$$B_z = -\mu_0 \sigma v_0 \quad \text{by Ampère's law}$$

What we need to derive the *general* rule is an introduction of another frame $\bar{S}$,

then, derivation of the transformation of $\left(E, B\right)$ fields in S system into $\left(\bar{E}, \bar{B}\right)$ fields in $\bar{S}$ system.

In a *third* system, $\bar{S}$, traveling to the right with speed $\left(\upsilon :\ \text{velocity of } \bar{S} \text{ relative to } S\right)$

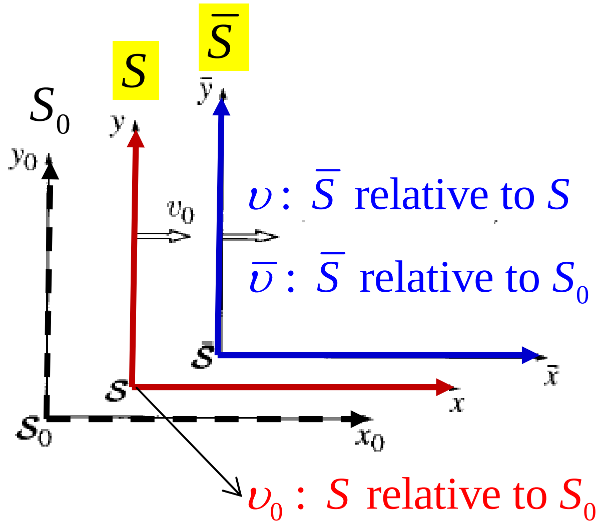
$$\bar{E}_y = \frac{\bar{\sigma}}{\epsilon_0}, \quad \bar{B}_z = -\mu_0 \bar{\sigma}\,\bar{v}$$

$$\bar{v} = \frac{v + v_0}{1 + v v_0/c^2} \quad \left(\bar{\upsilon} :\ \text{velocity of } \bar{S} \text{ relative to } S_0\right)$$

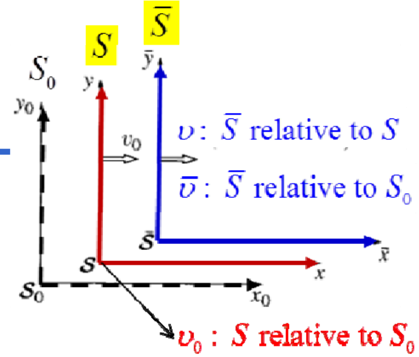
$$\bar{\sigma} = \bar{\gamma}\sigma_0 \quad \bar{\gamma} = \frac{1}{\sqrt{1 - \bar{v}^2/c^2}}$$

also, since $\sigma = \gamma_0 \sigma_0 \quad \frac{1}{\gamma_0} = \sqrt{1 - v_0^2/c^2}$

$$\bar{E}_y = \left(\frac{\bar{\gamma}}{\gamma_0}\right)\frac{\sigma}{\epsilon_0}, \quad \bar{B}_z = -\left(\frac{\bar{\gamma}}{\gamma_0}\right)\mu_0 \sigma\,\bar{v}$$



How the Fields Transform



$$\bar{E}_y = \left(\frac{\bar{\gamma}}{\gamma_0}\right)\frac{\sigma}{\epsilon_0}, \quad \bar{B}_z = -\left(\frac{\bar{\gamma}}{\gamma_0}\right)\mu_0\sigma\bar{v}$$

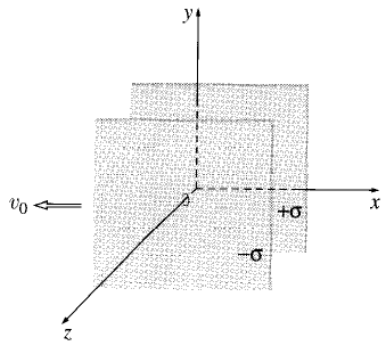
$$\frac{\bar{\gamma}}{\gamma_0} = \frac{\sqrt{1-v_0^2/c^2}}{\sqrt{1-\bar{v}^2/c^2}} = \frac{1+vv_0/c^2}{\sqrt{1-v^2/c^2}} = \gamma\left(1+\frac{vv_0}{c^2}\right) \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

$$\bar{E}_y = \gamma\left(1+\frac{vv_0}{c^2}\right)\frac{\sigma}{\epsilon_0} = \gamma\left(E_y - \frac{v}{c^2\epsilon_0\mu_0}B_z\right) \longleftarrow B_z = -\mu_0\sigma v_0$$

$$\bar{B}_z = -\gamma\left(1+\frac{vv_0}{c^2}\right)\mu_0\sigma\left(\frac{v+v_0}{1+vv_0/c^2}\right) = \gamma(B_z - \mu_0\epsilon_0 v E_y) \longleftarrow \mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{y}}$$

since $\mu_0\epsilon_0 = 1/c^2$,

$$\left.\begin{aligned} \bar{E}_y &= \gamma(E_y - vB_z), \\ \bar{B}_z &= \gamma\left(B_z - \frac{v}{c^2}E_y\right). \end{aligned}\right\}$$

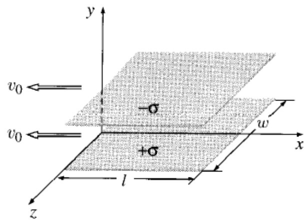


Similarly, to do E_z and B_y simply align the same capacitor parallel to *xy* plane instead of *xz* plane

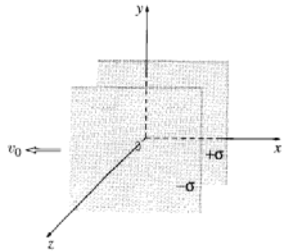
$$\left.\begin{aligned} \bar{E}_z &= \gamma(E_z + vB_y), \\ \bar{B}_y &= \gamma\left(B_y + \frac{v}{c^2}E_z\right). \end{aligned}\right\}$$

How the Fields Transform

$$\begin{aligned}\bar{E}_y &= \gamma(E_y - vB_z),\\ \bar{B}_z &= \gamma\left(B_z - \frac{v}{c^2}E_y\right).\end{aligned}$$



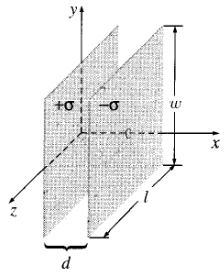
$$\begin{aligned}\bar{E}_z &= \gamma(E_z + vB_y),\\ \bar{B}_y &= \gamma\left(B_y + \frac{v}{c^2}E_z\right).\end{aligned}$$

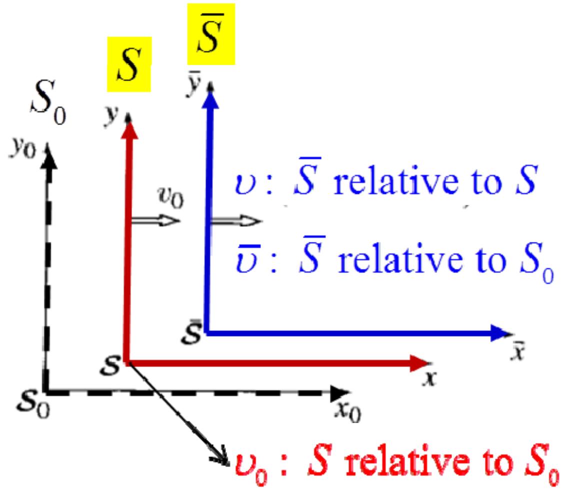


$\bar{E}_x = E_x$

$\bar{B}_x = B_x$

the field component s *parallel* to the motion is unchanged.





$$\boxed{\begin{aligned}&\bar{E}_x = E_x, \quad &\bar{E}_y = \gamma(E_y - vB_z), \quad &\bar{E}_z = \gamma(E_z + vB_y),\\ &\bar{B}_x = B_x, \quad &\bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), \quad &\bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)\end{aligned}}$$

where $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$

$(v: \bar{S}$ relative to $S)$

How the Fields Transform

$$\boxed{\begin{aligned}&\bar{E}_x = E_x, \quad \bar{E}_y = \gamma(E_y - vB_z), \quad \bar{E}_z = \gamma(E_z + vB_y),\\ &\bar{B}_x = B_x, \quad \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), \quad \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)\end{aligned}}$$

where $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$

$\left(v:\ \bar{S} \text{ relative to } S\right)$

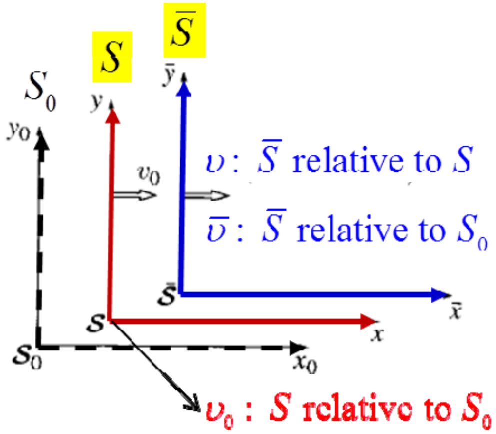
Two special cases:

(1) *If* $B = 0$ *in* S *frame*, $(E \neq 0)$;

$$\bar{\mathbf{B}} = \gamma \frac{v}{c^2}(E_z \hat{\mathbf{y}} - E_y \hat{\mathbf{z}})$$

or, since $\mathbf{E}^{\perp} = \gamma_0 \mathbf{E}_0^{\perp}$ $\longrightarrow$ $\bar{\mathbf{B}} = \frac{v}{c^2}(\bar{E}_z \hat{\mathbf{y}} - \bar{E}_y \hat{\mathbf{z}})$

or, since $\mathbf{v} = v\,\hat{\mathbf{x}}$, $\longrightarrow$ $\bar{\mathbf{B}} = -\frac{1}{c^2}(\mathbf{v} \times \bar{\mathbf{E}})$



(2) *If* $E = 0$ *in* S *frame*, $(B \neq 0)$;

$$\bar{\mathbf{E}} = -\gamma v(B_z \hat{\mathbf{y}} - B_y \hat{\mathbf{z}}) = -v(\bar{B}_z \hat{\mathbf{y}} - \bar{B}_y \hat{\mathbf{z}}) \longrightarrow \bar{\mathbf{E}} = \mathbf{v} \times \bar{\mathbf{B}}$$

➔ If either E or B is zero (at a particular point) in *one* system,
then in any other system the fields (at that point) are very simply related.

12.3.3 The Field Tensor $F^{\mu\nu}$

$$\bar{E}_x = E_x, \quad \bar{E}_y = \gamma(E_y - vB_z), \quad \bar{E}_z = \gamma(E_z + vB_y),$$

$$\bar{B}_x = B_x, \quad \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), \quad \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

The components of **E** and **B** are stirred together when you go from one inertial system to another.
→ What sort of an object is this, which has six components and transforms according to the above relations?
→ It's an ***antisymmetric, second-rank tensor***.

Lorentz transformation matrix

Remember that a 4-vector transforms by the rule → $\bar{a}^\mu = \Lambda^\mu_\nu a^\nu$

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A (second-rank) tensor is an object with *two* indices, which transform with *two* factors of Λ (one for each index):

$$\bar{t}^{\mu\nu} = \Lambda^\mu_\lambda \Lambda^\nu_\sigma t^{\lambda\sigma}$$

A tensor (in 4 dimensions) has 4 x 4 = 16 components, which we can display in a 4 x 4 array:

$$t^{\mu\nu} = \begin{Bmatrix} t^{00} & t^{01} & t^{02} & t^{03} \\ t^{10} & t^{11} & t^{12} & t^{13} \\ t^{20} & t^{21} & t^{22} & t^{23} \\ t^{30} & t^{31} & t^{32} & t^{33} \end{Bmatrix}$$

However, the 16 elements need not all be different.

The Field Tensor $F^{\mu\nu}$

$$\bar{t}^{\mu\nu} = \Lambda^\mu_\lambda \Lambda^\nu_\sigma t^{\lambda\sigma} \qquad t^{\mu\nu} = \begin{Bmatrix} t^{00} & t^{01} & t^{02} & t^{03} \\ t^{10} & t^{11} & t^{12} & t^{13} \\ t^{20} & t^{21} & t^{22} & t^{23} \\ t^{30} & t^{31} & t^{32} & t^{33} \end{Bmatrix}$$

$t^{\mu\nu} = t^{\nu\mu}$ **(symmetric tensor)** → 10 distinct elements

$t^{\mu\nu} = -t^{\nu\mu}$ **(antisymmetric tensor)** → 6 distinct elements, and four are zero $(t^{00},\ t^{11}, t^{22},\ \text{and}\ t^{33})$

Thus, the general **antisymmetric tensor** has the form

$$t^{\mu\nu} = \begin{Bmatrix} 0 & t^{01} & t^{02} & t^{03} \\ -t^{01} & 0 & t^{12} & t^{13} \\ -t^{02} & -t^{12} & 0 & t^{23} \\ -t^{03} & -t^{13} & -t^{23} & 0 \end{Bmatrix} \qquad \bar{t}^{\mu\nu} = \Lambda^\mu_\lambda \Lambda^\nu_\sigma t^{\lambda\sigma}$$

Let's see how the transformation rule works, for the six distinct components of an antisymmetric tensor.

$$\bar{t}^{01} = \Lambda^0_\lambda \Lambda^1_\sigma t^{\lambda\sigma}$$

$$\Lambda^0_\lambda = 0 \text{ unless } \lambda = 0 \text{ or } 1, \text{ and } \Lambda^1_\sigma = 0 \text{ unless } \sigma = 0 \text{ or } 1.$$

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\bar{t}^{01} = \Lambda^0_0 \Lambda^1_0 t^{00} + \Lambda^0_0 \Lambda^1_1 t^{01} + \Lambda^0_1 \Lambda^1_0 t^{10} + \Lambda^0_1 \Lambda^1_1 t^{11}$$

$$t^{00} = t^{11} = 0, \text{ while } t^{01} = -t^{10},$$

$$\bar{t}^{01} = (\Lambda^0_0 \Lambda^1_1 - \Lambda^0_1 \Lambda^1_0) t^{01} = (\gamma^2 - (\gamma\beta)^2) t^{01} = t^{01}$$

The Field Tensor $F^{\mu\nu}$

Lorentz transformation of an antisymmetric tensor: $\bar{t}^{\mu\nu} = \Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma}t^{\lambda\sigma}$

$$t^{\mu\nu} = \left\{ \begin{array}{cccc} 0 & t^{01} & t^{02} & t^{03} \\ -t^{01} & 0 & t^{12} & t^{13} \\ -t^{02} & -t^{12} & 0 & t^{23} \\ -t^{03} & -t^{13} & -t^{23} & 0 \end{array} \right\} \qquad \Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The complete set of transformation rules is

$$\boxed{\begin{array}{lll} \bar{t}^{01} = t^{01}, & \bar{t}^{02} = \gamma(t^{02} - \beta t^{12}), & \bar{t}^{03} = \gamma(t^{03} + \beta t^{31}), \\ \bar{t}^{23} = t^{23}, & \bar{t}^{31} = \gamma(t^{31} + \beta t^{03}), & \bar{t}^{12} = \gamma(t^{12} - \beta t^{02}). \end{array}\Bigg\}}$$

Now we can construct the **field tensor $F_{\mu\nu}$** by direct comparison:

$$\boxed{\begin{array}{lll} \bar{E}_x = E_x, & \bar{E}_y = \gamma(E_y - vB_z), & \bar{E}_z = \gamma(E_z + vB_y), \\ \bar{B}_x = B_x, & \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), & \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right) \end{array}}$$

$$F^{01} \equiv \frac{E_x}{c}, \quad F^{02} \equiv \frac{E_y}{c}, \quad F^{03} \equiv \frac{E_z}{c}, \quad F^{12} \equiv B_z, \quad F^{31} \equiv B_y, \quad F^{23} \equiv B_x$$

$$\boxed{F^{\mu\nu} = \left\{ \begin{array}{cccc} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{array} \right\}}$$

The Field Tensor $F^{\mu\nu}$

$$F^{\mu\nu} = \begin{Bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{Bmatrix} \xrightarrow[\mathbf{B} \to -\mathbf{E}/c]{\mathbf{E}/c \to \mathbf{B}} G^{\mu\nu} = \begin{Bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{Bmatrix}$$

Dual tensor

Properties

Antisymmetry: $F^{\mu\nu} = -F^{\nu\mu}$

Six independent components: In Cartesian coordinates, the three spatial components of (E_x, E_y, E_z) and (B_x, B_y, B_z).

Inner product: If one forms an inner product of the field strength tensor a Lorentz invariant is formed

$$F_{\mu\nu}F^{\mu\nu} = 2\left(B^2 - \frac{E^2}{c^2}\right)$$

→ meaning this number does not change from one frame of reference to another.

Pseudoscalar invariant: The product of the tensor $(F^{\mu\nu})$ with its **dual tensor** $(G^{\mu\nu})$ gives the Lorentz invariant:

$$G_{\gamma\delta}F^{\gamma\delta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}F^{\alpha\beta}F^{\gamma\delta} = -\frac{4}{c}(\mathbf{B}\cdot\mathbf{E})$$

Determinant: $\det(F) = \frac{1}{c^2}(\mathbf{B}\cdot\mathbf{E})^2$

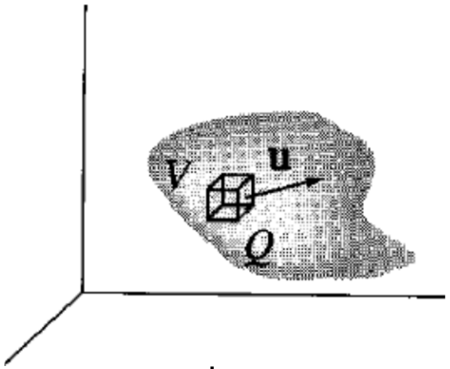
12.3.4 Electrodynamics in Tensor Notation $F^{\mu\nu}$

$$F^{\mu\nu} = \begin{Bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{Bmatrix}$$

$$\boxed{\bar{F}^{\mu\nu} = \Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma}F^{\lambda\sigma}}$$

To begin with, we must determine how the *sources* of the fields, ρ and **J,** transform.

Imagine a cloud of charge drifting by, we concentrate on an infinitesimal volume V, which contains charge Q moving at velocity **u**.



charge density → $\rho = \dfrac{Q}{V}$ current density → $\mathbf{J} = \rho\mathbf{u}$

The charge density in the rest system of the charge: $\rho_0 = \dfrac{Q}{V_0}$

Because one dimension (the one along the direction of motion) is Lorentz-contracted,

$$V = \sqrt{1 - u^2/c^2}\, V_0 \longrightarrow \rho = \rho_0 \frac{1}{\sqrt{1 - u^2/c^2}} \qquad \mathbf{J} = \rho_0 \frac{\mathbf{u}}{\sqrt{1 - u^2/c^2}}$$

$$\boldsymbol{\eta} = \frac{1}{\sqrt{1 - u^2/c^2}}\mathbf{u}$$

$$\eta^0 = \frac{c}{\sqrt{1 - u^2/c^2}}$$

$$\boxed{J^{\mu} = \rho_0 \eta^{\mu}}$$

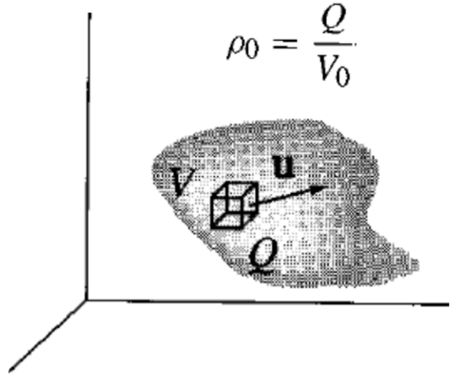
$$J^{\mu} = (c\rho,\ J_x,\ J_y,\ J_z,)$$ → **current density 4-vector.**

Continuity equation in Tensor Notation

Transformation of the **charge density** and **current density**

$$J^{\mu} = \rho_0 \eta^{\mu}$$

$$J^{\mu} = (c\rho, J_x, J_y, J_z,)$$ → **current density 4-vector.**



The **continuity equation** in terms of J^{μ}

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \longrightarrow \boxed{\frac{\partial J^{\mu}}{\partial x^{\mu}} = 0}$$

➔ The current density 4-vector is divergenceless.

$$\nabla \cdot \mathbf{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = \sum_{i=1}^{3} \frac{\partial J^i}{\partial x^i}$$

$$\frac{\partial \rho}{\partial t} = \frac{1}{c}\frac{\partial J^0}{\partial t} = \frac{\partial J^0}{\partial x^0}$$

Current density 4-vector (charge and current densities) $J^{\mu} = \rho_0 \eta^{\mu} = (c\rho, J_x, J_y, J_z,)$

Continuity equation $\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \longrightarrow \frac{\partial J^{\mu}}{\partial x^{\mu}} = 0$

Maxwell's Equations in Tensor Notation:

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} \qquad G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

$$\boxed{\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu, \quad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0} \longrightarrow \textbf{4 Maxwell's Equations}$$

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu \Rightarrow$$

If $\mu = 0$, Gauss's law: $\dfrac{\partial F^{0\nu}}{\partial x^\nu} = \mu_0 J^0 \longrightarrow \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho$

$$\frac{\partial F^{0\nu}}{\partial x^\nu} = \frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{01}}{\partial x^1} + \frac{\partial F^{02}}{\partial x^2} + \frac{\partial F^{03}}{\partial x^3} = \frac{1}{c}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) = \frac{1}{c}(\nabla \cdot \mathbf{E})$$

$$\mu_0 J^0 = \mu_0 c\rho$$

If $\mu = 1, 2$, and 3, Ampere's law with Maxwell's correction: $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}$

$$\frac{\partial F^{1\nu}}{\partial x^\nu} = \frac{\partial F^{10}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{12}}{\partial x^2} + \frac{\partial F^{13}}{\partial x^3} = -\frac{1}{c^2}\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \left(-\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B}\right)_x$$

$\mu_0 J^1 = \mu_0 J_x$ Combine this with the corresponding results for $\mu = 2$ and 3.

Maxwell's Equations in Tensor Notation:

$$F^{\mu\nu} = \begin{Bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{Bmatrix} \qquad G^{\mu\nu} = \begin{Bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{Bmatrix}$$

$$\boxed{\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu, \quad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0} \longrightarrow$$ **4 Maxwell's Equations**

$$\frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0 \Rightarrow$$

If $\mu = 0$, $\longrightarrow \dfrac{\partial G^{0\nu}}{\partial x^\nu} = 0 \longrightarrow \nabla \cdot \mathbf{B} = 0$

$$\frac{\partial G^{0\nu}}{\partial x^\nu} = \frac{\partial G^{00}}{\partial x^0} + \frac{\partial G^{01}}{\partial x^1} + \frac{\partial G^{02}}{\partial x^2} + \frac{\partial G^{03}}{\partial x^3} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \nabla \cdot \mathbf{B} = 0$$

If $\mu = 1, 2,$ and 3, **Faraday's law:** $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$

$$\frac{\partial G^{1\nu}}{\partial x^\nu} = \frac{\partial G^{10}}{\partial x^0} + \frac{\partial G^{11}}{\partial x^1} + \frac{\partial G^{12}}{\partial x^2} + \frac{\partial G^{13}}{\partial x^3}$$

$$= -\frac{1}{c}\frac{\partial B_x}{\partial t} - \frac{1}{c}\frac{\partial E_z}{\partial y} + \frac{1}{c}\frac{\partial E_y}{\partial z} = -\frac{1}{c}\left(\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E}\right)_x = 0$$

Combine this with the corresponding results for $\mu = 2$ and 3.

Minkowski force in Tensor Notation

$$F^{\mu\nu} = \begin{Bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{Bmatrix} \qquad G^{\mu\nu} = \begin{Bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{Bmatrix}$$

$\boxed{K^\mu = q\eta_\nu F^{\mu\nu}}$ **: Minkowski force (Lorentz force in relativistic notation)**

If $\mu = 1$,

$$\begin{aligned} K^1 &= q\eta_\nu F^{1\nu} = q(-\eta^0 F^{10} + \eta^1 F^{11} + \eta^2 F^{12} + \eta^3 F^{13}) \\ &= q\left[\frac{-c}{\sqrt{1-u^2/c^2}}\left(\frac{-E_x}{c}\right) + \frac{u_y}{\sqrt{1-u^2/c^2}}(B_z) + \frac{u_z}{\sqrt{1-u^2/c^2}}(-B_y)\right] \\ &= \frac{q}{\sqrt{1-u^2/c^2}}[\mathbf{E} + (\mathbf{u} \times \mathbf{B})]_x. \end{aligned}$$

With a similar formula for $\mu = 2$, and 3,

$$K^\mu = q\eta_\nu F^{\mu\nu} \longrightarrow \mathbf{K} = \frac{q}{\sqrt{1-u^2/c^2}}[\mathbf{E} + (\mathbf{u} \times \mathbf{B})]$$

➔ **Lorentz force law in relativistic notation**

12.3.5 Relativistic Potentials

$$F^{\mu\nu} = \begin{Bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{Bmatrix} \qquad G^{\mu\nu} = \begin{Bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{Bmatrix}$$

$$\boxed{F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}} \longrightarrow \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

$A^\mu = (V/c, A_x, A_y, A_z)$ **: 4-vector potential**

For $\mu = 0$, $\nu = 1$ (2,3): $\longrightarrow \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$

$$F^{01} = \frac{\partial A^1}{\partial x_0} - \frac{\partial A^0}{\partial x_1} = -\frac{\partial A_x}{\partial (ct)} - \frac{1}{c}\frac{\partial V}{\partial x} = -\frac{1}{c}\left(\frac{\partial \mathbf{A}}{\partial t} + \nabla V\right)_x = \frac{E_x}{c}$$

For $\mu = 1$, $\nu = 2$ ($\mu = 1$, $\nu = 2$) ($\mu = 2$, $\nu = 3$): $\longrightarrow \mathbf{B} = \nabla \times \mathbf{A}$

$$F^{12} = \frac{\partial A^2}{\partial x_1} - \frac{\partial A^1}{\partial x_2} = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = (\nabla \times \mathbf{A})_z = B_z$$

Relativistic Potentials

$$F^{\mu\nu} = \begin{Bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{Bmatrix} \qquad G^{\mu\nu} = \begin{Bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{Bmatrix}$$

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} \longrightarrow \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

$A^\mu = (V/c, A_x, A_y, A_z)$ **: 4-vector potential**

Maxwell's Equations

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu \longrightarrow \frac{\partial}{\partial x_\mu}\left(\frac{\partial A^\nu}{\partial x^\nu}\right) - \frac{\partial}{\partial x_\nu}\left(\frac{\partial A^\mu}{\partial x^\nu}\right) = \mu_0 J^\mu$$

The Lorentz gauge condition in relativistic notation,

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t} \longrightarrow \frac{\partial A^\nu}{\partial x^\nu} = 0.$$

In the Lorentz gauge, Maxwell's Equations reduces to,

$$\frac{\partial}{\partial x_\nu}\left(\frac{\partial A^\mu}{\partial x^\nu}\right) = -\mu_0 J^\mu \longrightarrow \boxed{\Box^2 A^\mu = -\mu_0 J^\mu}$$

(d' Alembertian) $\Box^2 \equiv \frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x^\nu} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$

➔ **The most elegant (and the simplest) formulation of Maxwell's equations**

Introduction to Electrodynamics, David J. Griffiths

1. Vector analysis
2. Electrostatics
3. Special techniques
4. Electric fields in mater
5. Magnetostatics

6. Magnetic fields in matter
7. Electrodynamics
8. Conservation laws
9. Electromagnetic waves

10. Potentials and fields
11. Radiation
12. Electrodynamics and relativity

$$\boxed{\Box^2 A^\mu = -\mu_0 J^\mu}$$

$$\Box^2 \equiv \frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x^\nu} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

$$A^\mu = (V/c, A_x, A_y, A_z)$$

4-vector potential

$$J^\mu = (c\rho, J_x, J_y, J_z,)$$

4-vector density