The Method of Brackets and the Bernoulli Symbol

An Abstract
Submitted On The Second Day Of March, 2016
To The Department of Mathematics
Of The School Of Science And Engineering Of
Tulane University
In Partial Fulfillment Of The Requirements
For The Degree Of
Doctor Of Philosophy
By

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Abstract

Symbolic computation has been widely applied to Combinatorics, Number Theory, and also other fields. Many reliable and fast algorithms with corresponding implementations now have been established and developed. Using the tool of Experimental Mathematics, especially with the help of mathematical software, in particularly `Mathematica`, we could visualize the data, manipulate algorithms and implementations. The work presented here, based on symbolic computation, involves the following two parts.

The first part introduces a systematic integration method, called the Method of Brackets. It only consists of a small number of simple and direct rules coming from the Schwinger parametrization of Feynman diagrams. Verification of each rule makes this method rigorous. Then it follows a necessary theorem that different series representations of the integrand, though lead to different processes of computations, do not affect the result. Examples of application lead to further discussions on analytic continuation, especially on Pochhammer symbol, divergent series and connection to Mellin transform of the Method of Brackets. In the end, comparison with other integration methods and a `Mathematica` package manual are presented.

The second part provides a symbolic approach on the study of Bernoulli numbers and its generalizations. The Bernoulli symbol $\mathcal{B}$ originally comes from Umbral Calculus, as a formal approach to Sheffer sequences. Recently, a rigorous footing by probabilistic proof makes it also a random variable with its density function a shifted

hyperbolic trigonometric function. Such an approach together with general method on random variables gives a variety of results on generalized Bernoulli polynomials, multiple zeta functions, and also other related topics.

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Contents

Chapter 1

Overview

The work presented here in this dissertation corresponds to papers that have been finished during my study in Tulane. Some of them either have been published or have been accepted for publication, while some are just submitted or in preparation.

The work presented in Part I, namely Chapters 2-9 studies an integration method which originally comes from the evaluation of integrals of Feynman diagrams. The idea of the method is to expression the integral as a (bracket) series and evaluate to either get a single expression or a collection of series. We overview of these chapters as follows.

- **Chapter 2** introduces the concrete rules of this method of brackets and verifies these rules, including a collection of previous results in the literature. This is part of the paper [38]:

  On the Method of Brackets: Independence on Factorization of Integrand.

  submitted for publication.

- **Chapter 3** provides some examples, either from the literature of previous work or also from [38].

- **Chapter 4** provides the theorem that the method of brackets does not depend on the factorization of the integrand, which is necessary for an integration method, but was missing from the literature. This is also the main result in [38].

- **Chapter 5** discusses the divergent series obtained from the method of brackets. By old rules, these divergent series are discarded. Critical examples show that their analytic continuation should be considered. Further discussion shows the use of these divergent series in further computations.

- **Chapter 6** begins with an evaluation of an integral, from which the value of the Pochhammer symbol for both base and index being negative integers is inspired to study. It is the result of [24]:
  Pochhammer Symbol with Negative Indices. A New Rule for the Method of Brackets.
  to appear in Open Mathematics.

- **Chapter 7** studies an inverse application of the method of brackets. It provides a way for a function to obtain its coefficients of series expansion through the Mellin Transform. A paper on this study together with Ivan Gonzalez, Karen Kohl and Victor H. Moll is in preparation.

- **Chapter 8** is a survey that compares the method of brackets with other two integration methods: negative dimensional integration method (NDIM) and integrating by differentiating (IBD).

- **Chapter 9** is the user manual for a `Mathematica` package that applies the method of brackets. Commands are introduced in details through concrete examples.

The work in Part II, i.e., Chapters 10-14 presents results related to Bernoulli numbers and Euler numbers. The technique here are all related to random variables in Probability Theory, especially in Chapters 10-13 where the Bernoulli symbol is introduced and applied. These chapters are organized as follows.

- **Chapter 10** introduces the basic properties of the Bernoulli symbol, especially its probabilistic interpretation, and some examples. At the end of this chapter, a result on Nörlund numbers are given, which is part of the paper [4]:
  Recursion Rules for the Hypergeometric Zeta Functions.
  (with Alyssa Byrnes, Victor H. Moll and Christophe Vignat)
  published in International Journal of Number Theory. Vol. 10, No. 7, 1761-1782, 2014.

- **Chapter 11** presents the application of the Bernoulli symbol on Bernoulli-Barnes polynomials, which is the content of the paper [40]:
  A Symbolic Approach to Some Identities for Bernoulli-Barnes Polynomials.
  (with Victor H. Moll and Christophe Vignat.)
  to appear in International Journal of Number Theory.

- **Chapter 12** shows the use the symbolic approach that generalizes the Bernoulli symbol to study the multiple zeta values at negative integers. This presents the results of paper [39]:
  A Symbolic Approach to Multiple Zeta Values at the Negative Integers.
  (with Victor H. Moll and Christophe Vignat)
  submitted for Publication.

- **Chapter 13** studies the hypergeometric Bernoulli numbers and hypergeometric zeta function that involves Kummer-${}_1F_1$ function, which are the main results in [4].

- **Chapter 14** presents the work on generalized Euler polynomials, by an approach involving random variables and Chebyshev polynomials. All results are in the paper [41]:

Identities for Generalized Euler Polynomials.

(with Victor H. Moll and Christophe Vignat)

published in Integral Transforms and Special Functions. Vol. 25, No. 10, 777-789, 2014.

Part I

The Method of Brackets

Chapter 2

Introduction

2.1 Introduction

Many problems in different areas of mathematics and physics involve integrations that require closed-form expressions and accurate results. For this purpose, many tables of integrals have been compiled. The classic table by I. S. Gradshteyn and I. M. Ryzhik [31] is currently in its 8th edition. Many methods, human and symbolic, have been developed to evaluate integrals. The method discussed here, called *the method of brackets* that was developed by I. Gonzalez [28, 29], has its origin on the evaluation of definite integrals arising from the Schwinger parametrization of Feynman diagrams. Besides examples of its use appearing in [5, 23, 26, 27], implementation has been produced in [45] using `Sage` with internal use of `Mathematical`. This method has also been used in [25] for the evaluation of entries in [31], which forms part of a project, initiated in [49], to produce proofs and context of all the entries in this table.

The key idea of the method of brackets is the Ramanujan's Master Theorem, which refers to the formal identity that

$$\int_0^\infty x^{s-1}\left\{f\left(0\right)-\frac{x}{1!}f\left(1\right)+\frac{x^2}{2!}f\left(2\right)-\cdots\right\}dx=f\left(-s\right)\Gamma\left(s\right). \tag{2.1.1}$$

Here, in a neighborhood of $x=0$, it is required that $f\left(0\right)\neq 0$. When $s>0$, the integral above is convergent. Details on restrictions of the parameter s are given in [5]. A proof of Ramanujan's Master Theorem and its precise conditions appear in [32].

2.2 Rules of the Method of Brackets

The method of brackets evaluates definite integrals over the positive half real line $[0, \infty)$ of a function $f(x)$, i.e., it evaluates

$$\int_0^\infty f(x)\, dx. \tag{2.2.1}$$

Before describing the rules, some necessary definitions of symbols are introduced first.

Definition 2.2.1. For $a \in \mathbb{R}$, the symbol

$$\langle a \rangle := \int_0^\infty x^{a-1} dx \tag{2.2.2}$$

is the *bracket* associated to the (divergent) integral on the right.

Definition 2.2.2. The *indicator* of n is defined as

$$\phi_n := \frac{(-1)^n}{n!} = \frac{(-1)^n}{\Gamma(n+1)} \tag{2.2.3}$$

and moreover, the product of multiple indicators is denoted by

$$\phi_{1,\ldots,r} = \phi_{n_1,\ldots,n_r} := \prod_{i=1}^{r} \phi_{n_i}. \tag{2.2.4}$$

The method of brackets consists of the following operational rules.

Rules for the production of the bracket series.

Rule P_1: If the function f is given by the formal power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n + \beta - 1}, \tag{2.2.5}$$

then the improper integral of f over the positive real line is formally written as the *bracket series*

$$\int_0^\infty f(x)\,dx = \sum_n a_n \langle \alpha n + \beta \rangle . \tag{2.2.6}$$

Rule P_2: The expansion of a multinomial as bracket series is given by, $\forall \alpha \notin \mathbb{N}$,

$$(a_1 + \cdots + a_r)^\alpha = \sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} a_1^{n_1} \cdots a_r^{n_r} \frac{\langle -\alpha + n_1 + \cdots + n_r \rangle}{\Gamma(-\alpha)}. \tag{2.2.7}$$

Rule P_3: Each representation of an integral by the bracket series has associated an index of the representation, defined by

$$\text{index} = \text{number of sums} - \text{number of brackets}. \tag{2.2.8}$$

Rules for the evaluation of the bracket series.

Rule E_1: The one-dimensional bracket series is assigned the value

$$\sum_n \phi_n f(n) \langle an + b \rangle = \frac{1}{|a|} f(n^*)\, \Gamma(-n^*), \tag{2.2.9}$$

where n^* solves $an + b = 0$.

Rule E_2: For the multi-dimensional bracket series with index 0, the value is given by the following rule. Assume that the matrix $A = (a_{ij})_{r\times r}$ is non-singular, then

$$\sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} f(n_1,\ldots,n_r) \prod_{i=1}^{r} \langle a_{i1} n_1 + \cdots + a_{ir} n_r + c_i \rangle = \frac{f(n_1^*,\ldots,n_r^*) \prod_{i=1}^{r} \Gamma(-n_i^*)}{|\det A|}, \tag{2.2.10}$$

where $(n_1^*,\ldots,n_r^*)$ is the unique solution to the linear system

$$A(n_1,\ldots n_r)^T + (c_1,\ldots,c_r)^T = \vec{0}^{\,T}. \tag{2.2.11}$$

Rule E_3: The value of a multi-dimensional bracket series of positive index is obtained by computing all the contributions of maximal rank by Rule E_2. These contributions to the integral appear as series in the free parameters. Series converging in a common region are added and divergent series are discarded. Any series producing a non-real contribution is also discarded. There is no assignment to a bracket series of negative index.

Modified Rule $\tilde{E}_3$: The value of a multi-dimensional bracket series of positive index is obtained by computing all the contributions of maximal rank by Rule E_2. These contributions to the integral appear as series in the free parameters. Series converging in a common region are added. *Divergent series are evaluated by their analytic continuations, if such continuations exist. Divergent series having the same analytic continuation are combined and treated as the common analytic continuation on its domain.* Any series producing a non-real contribution is also discarded. There is no assignment to a bracket of negative index.

Added Rule E_4: Let $k \in \mathbb{N}$ be fixed. $\forall m \in \mathbb{N}$, in the evaluation of series, the rule

$$(-km)_{-m} = \frac{k}{k+1} \cdot \frac{(-1)^m\left((km)!\right)}{\left((k+1)\,m\right)!} \tag{2.2.12}$$

must be used to eliminate Pochhammer symbols with negative index and negative integer base.

Further discussions on $\tilde{E}_3$ and E_4 will appear in Chapters 5 and 6

2.3 Verification for the Rules

This section verifies basic rules P_1, P_2, E_1 and E_2 of the method of brackets. Some have already been proven and appear here as a collection. Also see [5].

2.3.1 Rule P_1

Being the first rule, P_1 has been left unproven for a long time in the literature. A formal interpretation involving Laplace Transform is first represented here.

Definition 2.3.1. Define the *ε-bracket* by

$$\langle a \rangle_\varepsilon := \int_0^\infty x^{a-1} e^{-\varepsilon x} dx = \frac{\Gamma(a)}{\varepsilon^a} \tag{2.3.1}$$

and formally

$$\langle a\rangle = \lim_{\varepsilon\to 0} \langle a\rangle_{\varepsilon}\,. \tag{2.3.2}$$

Fact 2.3.2. *The rule P_1 is the (formal) limit case of the ϵ-bracket series. In concrete, assume the function f admits an expansion of the form*

$$f\left(x\right) = \sum_{n=0}^{\infty} a_n x^{\alpha n+\beta-1}, \tag{2.3.3}$$

then formally

$$\int_0^{\infty} f\left(x\right) dx = \lim_{\varepsilon\to 0} \sum_{n=0}^{\infty} a_n \left\langle \alpha n+\beta\right\rangle_{\varepsilon} = \sum_{n=0}^{\infty} a_n \left\langle \alpha n+\beta\right\rangle. \tag{2.3.4}$$

Proof. Associate to $f\left(x\right)$ the formal power series

$$\tilde{f}\left(x\right) := \sum_{n=0}^{\infty} a_n x^{\alpha n+\beta-1}\Gamma\left(\alpha n+\beta\right), \tag{2.3.5}$$

and denote the Laplace transform of f by

$$F\left(s\right) := \mathfrak{L}\left(f\right)\left(s\right) := \int_0^{\infty} f\left(x\right) e^{-sx} dx. \tag{2.3.6}$$

Then, by (2.3.1),

$$\sum_{n=0}^{\infty} a_n \left\langle \alpha n+\beta\right\rangle_{\varepsilon} = \sum_{n=0}^{\infty} a_n \frac{\Gamma\left(\alpha n+\beta\right)}{\varepsilon^{\alpha n+\beta}} = \frac{1}{\varepsilon}\tilde{f}\left(\frac{1}{\varepsilon}\right). \tag{2.3.7}$$

On the other hand,

$$\begin{aligned} \tilde{f}\left(x\right) &:= \sum_{n=0}^{\infty} a_n x^{\alpha n+\beta-1}\Gamma\left(\alpha n+\beta\right) \\ &= \sum_{n=0}^{\infty} a_n x^{\alpha n+\beta-1} \int_0^{\infty} t^{\alpha n+\beta-1} e^{-t} dt \\ &= \int_0^{\infty} e^{-t} \sum_{n=0}^{\infty} a_n \left(xt\right)^{\alpha n+\beta-1} dt \\ &= \int_0^{\infty} e^{-t} f\left(xt\right) dt \\ &= \frac{1}{x} F\left(\frac{1}{x}\right). \end{aligned} \tag{2.3.8}$$

Thus,

$$\sum_{n=0}^{\infty} a_n \langle \alpha n + \beta \rangle_{\varepsilon} = \frac{1}{\varepsilon} \tilde{f}\left(\frac{1}{\varepsilon}\right) = F(\varepsilon), \tag{2.3.9}$$

from which it follows that

$$\sum_n a_n \langle \alpha n + \beta \rangle = \lim_{\varepsilon \to 0} \sum_{n=0}^{\infty} a_n \langle \alpha n + \beta \rangle_{\varepsilon} = \lim_{\varepsilon \to 0} \int_0^{\infty} e^{-\varepsilon x} f(x)\, dx = \int_0^{\infty} f(x)\, dx. \tag{2.3.10}$$

□

It is obvious that the discussion above is formal, since, besides formally taking the limit of $\varepsilon \to 0$, it also interchanges the sum of series and the definite integral. In fact, if allowing interchanging the sum and integral, rule P_1 becomes trivial:

$$\int_0^{\infty} f(x)\, dx = \int_0^{\infty} \sum_{n=0}^{\infty} a_n x^{\alpha n + \beta - 1} dx = \sum_{n=0}^{\infty} a_n \int_0^{\infty} x^{\alpha n + \beta - 1} dx = \sum_n a_n \langle \alpha n + \beta \rangle. \tag{2.3.11}$$

And introducing the ε-brackets, which at the very beginning aimed at regularization of the interchanging above, fails to be rigorous as the limit and also seems to be redundant. However, this is not completely useless. Further formal connection with another integration method based on the discussion above will appear in Section 8.2.

Realizing that later applying evaluations rules E_1 and E_2 does not require to interchange the integral and the sum of the series, we could actually interpret the rule P_1 as a notation instead of a computation.

Definition 2.3.3. Rule P_1 is just another expression of the integral of a series integrand in terms that

$$\int_0^{\infty} \sum_{n=0}^{\infty} a_n x^{\alpha n + \beta - 1} dx = \sum_n a_n \langle \alpha n + \beta \rangle. \tag{2.3.12}$$

Here, we simplify the expression by the following two concrete ways:

(1) omitting the definite integrals sign $\int_0^{\infty}$ and also the range of series sum that n is from 0 to ∞;

(2) rewrite the power of the integration variable $x^{\alpha n+\beta-1}$ by a bracket of the power adding one, i.e. $\langle \alpha n+\beta\rangle$, which provides the linear system that needs to be solved;

Now, after reconsidering P_1 just as a rule of notation, one can easily read either side of (2.3.12) and convert it to the other side. And, more importantly, it works for multi-dimensional integrals. The following two integrals will appear in the next chapter of examples.

Example 2.3.4. (1)

$$\int_0^\infty \sum_{n=0}^{\infty} (-1)^n x^{2n} dx = \sum_n (-1)^n \langle 2n+1\rangle . \tag{2.3.13}$$

(2)

$$\begin{aligned} &\sum_{n_1,n_2,n_3} \phi_{n_1,n_2,n_3} \frac{y^{2n_3} a^{2n_1} \left\langle n_1+n_2+\frac{1}{2}\right\rangle \langle 2n_2+2n_3+2\rangle}{\Gamma(n_3+1)\Gamma\left(\frac{1}{2}\right) 2^{2n_3}} \\ = &\int_0^\infty \int_0^\infty \sum_{n_1,n_2,n_3=0}^{\infty} \frac{\phi_{n_1,n_2,n_3} y^{2n_3} a^{2n_1}}{\Gamma(n_3+1)\Gamma\left(\frac{1}{2}\right) 2^{2n_3}} x_1^{n_1+n_2-\frac{1}{2}} x_2^{2n_2+2n_3+1} dx_1 dx_2 \end{aligned} \tag{2.3.14}$$

2.3.2 Rule P_2 and Mellin Transform

Definition 2.3.5. Mellin Transform of a function f is defined by

$$\mathcal{M}(f)(s) = \varphi(s) = \int_0^\infty x^{s-1} f(x)\, dx, \tag{2.3.15}$$

and its inverse transform by

$$\mathcal{M}^{-1}(\varphi)(x) = f(x) = \frac{1}{2\pi i} \int_{c-\iota\infty}^{c+\iota\infty} x^{s-1} \varphi(s)\, ds. \tag{2.3.16}$$

Remark 2.3.6. It is not hard to see that the method of brackets can be applied to compute the Mellin Transform of a given function $f(x)$.

Theorem 2.3.7. *Rule P_2 is a restatement of the fact that the Mellin transform of e^{-x} is $\Gamma(s)$.*

Proof. Considering the change of variables that

$$x = \frac{t}{a_1 + \cdots + a_r}, \tag{2.3.17}$$

the following sequence of steps is direct

$$\begin{aligned} \Gamma(-\alpha)(a_1 + \cdots + a_r)^{\alpha} &= (a_1 + \cdots + a_r)^{\alpha} \int_0^{\infty} t^{-\alpha-1} e^{-t} dt \\ &= \int_0^{\infty} x^{-\alpha-1} e^{-(a_1+\cdots+a_r)x} dx \\ &= \int_0^{\infty} x^{-\alpha-1} \prod_{i=1}^{r} e^{-a_i x} dx \\ &= \int_0^{\infty} x^{-\alpha-1} \prod_{i=1}^{r} \sum_{n_i=0}^{\infty} \phi_{n_i} (a_i x)^{n_i} dx \\ &= \sum_{n_1,\dots,n_r} \phi_{1,\dots,r} a_1^{n_1} \cdots a_r^{n_r} \langle -\alpha + n_1 + \cdots + n_r \rangle . \end{aligned} \tag{2.3.18}$$

□

2.3.3 Rules E_1, E_2 and Ramanujan's Master Theorem

This section collects detailed proofs of rules E_1 and E_2 in the literature.

We first restate the Ramanujan's Master theorem.

Theorem 2.3.8. *[Ramanujan's Master Theorem] Assume $f(x)$ has an expression of the form*

$$f(x) = \sum_{k=0}^{\infty} \phi_k c(k) x^k, \tag{2.3.19}$$

then the Mellin Transform of $f(x)$ is given by

$$\mathcal{M}(f)(s) = \int_0^{\infty} x^{s-1} f(x) dx = \Gamma(s) c(-s). \tag{2.3.20}$$

Now, the rule E_1 becomes obvious.

Theorem 2.3.9. *The rule E_1 is precisely the Ramanujan's Master Theorem.*

Proof. Using the change of variable $t = x^{\alpha}$ and also applying (2.3.8), it follows that

$$\begin{aligned}\sum_n \phi_n f(n) \langle \alpha n + \beta \rangle &= \int_0^\infty \sum_{n=0}^\infty \phi_n f(n) x^{\alpha n+\beta-1} dx \\ &= \int_0^\infty x^{\beta-1} \sum_{n=0}^\infty \phi_n f(n) (x^\alpha)^n dx \\ &= \int_0^\infty t^{\frac{\beta-1}{\alpha}} \sum_{n=0}^\infty \phi_n f(n) t^n \left(\frac{1}{|\alpha|} t^{\frac{1}{\alpha}-1}\right) dt \\ &= \frac{1}{|\alpha|} \int_0^\infty t^{\frac{\beta}{\alpha}-1} \left(\sum_{n=0}^\infty \phi_n f(n)\right) dt \\ &= \frac{1}{|\alpha|} f\left(-\frac{\beta}{\alpha}\right) \Gamma\left(\frac{\beta}{\alpha}\right).\end{aligned} \tag{2.3.21}$$

□

Theorem 2.3.10. *The rule E_2 follows upon iterating the one-dimensional case.*

Proof. It suffices to only consider the two-dimensional case, since higher cases are similar by induction.

The change of variables

$$\begin{cases} u = x^{a_{11}} y^{a_{21}}, \\ v = x^{a_{12}} y^{a_{22}}, \end{cases} \tag{2.3.22}$$

gives

$$\begin{cases} du = a_{11} x^{a_{11}-1} y^{a_{21}} dx + a_{21} x^{a_{11}} y^{a_{21}-1} dy, \\ dv = a_{12} x^{a_{12}-1} y^{a_{22}} dx + a_{22} x^{a_{12}} y^{a_{22}-1} dy, \end{cases} \tag{2.3.23}$$

and

$$\frac{dudv}{uv} = \frac{(a_{11}a_{22} - a_{12}a_{21})\, dxdy}{xy}, \tag{2.3.24}$$

by noting

$$\begin{cases} dx \cdot dx = dy \cdot dy = 0, \\ dx \cdot dy = -dy \cdot dx. \end{cases} \tag{2.3.25}$$

Thus, with $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$,

$$\begin{aligned}
&\sum_{n_1,n_2} \phi_{n_1,n_2} f(n_1,n_2)\langle a_{11}n_1+a_{12}n_2+c_1\rangle \langle a_{21}n_1+a_{22}n_2+c_2\rangle \qquad (2.3.26)\\
&= \int_0^\infty \int_0^\infty \sum_{n_1,n_2=0}^{\infty} \phi_{n_1,n_2} f(n_1,n_2)\, x^{a_{11}n_1+a_{12}n_2+c_1} y^{a_{21}n_1+a_{22}n_2+c_2} \frac{dxdy}{xy}\\
&= \frac{1}{|\det A|}\int_0^\infty \int_0^\infty \sum_{n_1,n_2=0}^{\infty} \phi_{n_1,n_2} f(n_1,n_2)\, u^{n_1-n_1^*-1} v^{n_2-n_2^*-1} dudv,
\end{aligned}$$

where (n_1^*,n_2^*) satisfies the linear system

$$A\begin{bmatrix} n_1^* \\ n_2^* \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad (2.3.27)$$

i.e.,

$$\begin{cases} a_{11}n_1+a_{12}n_2+c_1 = 0 = a_{11}(n_1-n_1^*)+a_{12}(n_2-n_2^*), \\ a_{21}n_1+a_{22}n_2+c_2 = 0 = a_{21}(n_1-n_1^*)+a_{22}(n_2-n_2^*). \end{cases} \qquad (2.3.28)$$

Therefore, by applying rule E_1 twice,

$$\begin{aligned}
&\sum_{n_1,n_2} \phi_{n_1,n_2} f(n_1,n_2)\langle a_{11}n_1+a_{12}n_2+c_1\rangle \langle a_{21}n_1+a_{22}n_2+c_2\rangle \qquad (2.3.29)\\
&= \frac{1}{|\det A|}\int_0^\infty \int_0^\infty \sum_{n_1,n_2} \phi_{n_1,n_2} f(n_1,n_2)\, u^{n_1-n_1^*-1} v^{n_2-n_2^*-1} dudv\\
&= \frac{1}{|\det A|}\int_0^\infty \sum_{n_2} \phi_{n_2} v^{n_2-n_2^*-1}\left[\int_0^\infty u^{-n_1^*-1}\sum_{n_1}\phi_{n_1} f(n_1,n_2)\, u^{n_1} du\right] dv\\
&= \frac{1}{|\det A|}\Gamma(-n_1^*)\int_0^\infty v^{-n_2^*-1}\sum_{n_1}\phi_{n_1} f(n_1^*,n_2)\, v^{n_2} dv\\
&= \frac{1}{|\det A|} f(n_1^*,n_2^*)\,\Gamma(-n_1^*)\,\Gamma(-n_2^*).
\end{aligned}$$

$\square$

Chapter 3

Examples

3.1 Examples of Index 0

Example 3.1.1. Entry **3.310** in [31] states the simple result

$$I = \int_0^\infty e^{-x} dx = -e^{-x}\Big|_{x=0}^{x=\infty} = 1. \tag{3.1.1}$$

The method of brackets evaluates this entry in elementary manner. Since the integrand is

$$f(x) = e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \phi_n x^n, \tag{3.1.2}$$

it follows that

$$I = \int_0^\infty \sum_{n=0}^{\infty} \phi_n x^n dx \overset{P_1}{=} \sum_n \phi_n \langle n+1 \rangle \overset{E_1}{=} 1 \cdot \Gamma(1) = 1. \tag{3.1.3}$$

Example 3.1.2. The special case of entry **3.037.1** in [31] is

$$I = \int_0^\infty \frac{1}{1+x^2} dx = \tan^{-1} x\Big|_{x=0}^{x=\infty} = \frac{\pi}{2}. \tag{3.1.4}$$

Similarly as the previous example, one can see that

$$f(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} \phi_n \Gamma(n+1) x^{2n}, \tag{3.1.5}$$

and

$$I=\int_0^{\infty} f\left(x\right)dx \overset{P_1}{=} \sum_n \phi_n \Gamma\left(n+1\right)\left\langle 2n+1\right\rangle \overset{E_1}{=} \frac{1}{|2|}\Gamma\left(n^*+1\right)\Gamma\left(-n^*\right), \tag{3.1.6}$$

where n^* solves $2n+1=0$, i.e., $n^*=-\frac{1}{2}$. Thus,

$$I=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)^2=\frac{\pi}{2}. \tag{3.1.7}$$

Alternatively, note that the integrand also has the form

$$f\left(x\right)=\left(1+x^2\right)^{-1}, \tag{3.1.8}$$

which allows us to apply the rule P_2 to get

$$f\left(x\right) \overset{P_2}{=} \sum_{n_1,n_2} \phi_{n_1,n_2} 1^{n_1} x^{2n_2} \frac{\left\langle 1+n_1+n_2\right\rangle}{\Gamma\left(1\right)} = \sum_{n_1,n_2} \phi_{n_1,n_2} x^{2n_2}\left\langle n_1+n_2+1\right\rangle, \tag{3.1.9}$$

and therefore

$$I \overset{P_1}{=} \sum_{n_1,n_2} \phi_{n_1,n_2}\left\langle n_1+n_2+1\right\rangle\left\langle 2n_2+1\right\rangle. \tag{3.1.10}$$

The corresponding linear system

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}+\begin{bmatrix} 1 \\ 1 \end{bmatrix}=0 \tag{3.1.11}$$

has the unique solution

$$\begin{bmatrix} n_1^* \\ n_2^* \end{bmatrix}=\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}. \tag{3.1.12}$$

Hence,

$$I=\frac{1}{\left|\det\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\right|}\Gamma\left(\frac{1}{2}\right)^2=\frac{\pi}{2}. \tag{3.1.13}$$

3.2 Example of Positive Index

Example 3.2.1. Entry **6.554.1** in [31] reads

$$I = \int_0^\infty xJ_0(xy)\,\frac{dx}{(a^2+x^2)^{\frac{1}{2}}} = y^{-1}e^{-ay}.\ [y>0,\ \mathrm{Re}\,(a)>0] \tag{3.2.1}$$

Here, $J_0(x)$ is *the Bessel function of the first kind of order* 0, whose power series expansion is given by

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!m!}\left(\frac{x}{2}\right)^{2m} = \sum_{m=0}^{\infty} \frac{\phi_m x^{2m}}{\Gamma(m+1)\,2^{2m}}. \tag{3.2.2}$$

First of all, we apply Rule P_2 for the factor $(a^2+x^2)^{-\frac{1}{2}}$ of the integrand:

$$\left(a^2+x^2\right)^{-\frac{1}{2}} = \sum_{n_1,n_2} \phi_{n_1,n_2} a^{2n_1} x^{2n_2} \frac{\left\langle \frac{1}{2}+n_1+n_2\right\rangle}{\Gamma\left(\frac{1}{2}\right)}. \tag{3.2.3}$$

Then, a bracket series of index 1 is obtained as follows:

$$\begin{aligned} I &= \int_0^\infty \frac{x}{\Gamma\left(\frac{1}{2}\right)}\left[\sum_{n_3}\frac{\phi_{n_3} y^{2n_3} x^{2n_3}}{\Gamma(n_3+1)\,2^{2n_3}}\right]\left[\sum_{n_1,n_2}\phi_{n_1,n_2}a^{2n_1}x^{2n_2}\left\langle\frac{1}{2}+n_1+n_2\right\rangle\right]dx \\ &= \sum_{n_1,n_2,n_3}\phi_{1,2,3}\frac{y^{2n_3}a^{2n_1}}{\Gamma(n_3+1)\,\Gamma\left(\frac{1}{2}\right)2^{2n_3}}\left\langle n_1+n_2+\frac{1}{2}\right\rangle\langle 2n_2+2n_3+2\rangle. \end{aligned} \tag{3.2.4}$$

i> If we choose n_1 as the free parameter, then letting $A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$, implying $\det A = 2$, the linear system for the brackets is given by

$$A\begin{pmatrix} n_2 \\ n_3 \end{pmatrix} + \begin{pmatrix} n_1+\frac{1}{2} \\ 2 \end{pmatrix} = 0, \tag{3.2.5}$$

with solution

$$\begin{cases} n_2^* = -\frac{1}{2} - n_1, \\ n_3^* = -\frac{1}{2} + n_1. \end{cases} \tag{3.2.6}$$

Thus,

$$\begin{aligned} I &= \frac{1}{|2|}\sum_{n_1=0}^{\infty}\phi_{n_1}\frac{a^{2n_1}y^{2n_1-1}\Gamma\left(n_1+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-n_1\right)}{\Gamma\left(n_1+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)2^{2n_1-1}} \\ &= \frac{1}{y}\sum_{n_1=0}^{\infty}\frac{(-1)^{n_1}}{n_1!}\left(\frac{ay}{2}\right)^{2n_1}\frac{\Gamma\left(\frac{1}{2}-n_1\right)}{\Gamma\left(\frac{1}{2}\right)}. \end{aligned} \tag{3.2.7}$$

Recall that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\Gamma\left(\frac{1}{2} - n_1\right) = \frac{(-4)^{n_1} n_1!}{(2n_1)!}\sqrt{\pi}$, so it follows that

$$I = \frac{1}{y} \sum_{n_1=0}^{\infty} \frac{(ay)^{2n_1}}{(2n_1)!} = \frac{1}{y} \cosh(ay), \tag{3.2.8}$$

which is a convergent series with radius of convergence $+\infty$.

ii> If choose n_2 as the free parameter, similar computation shows

$$I = \frac{1}{\sqrt{\pi} y} \sum_{n_2=0}^{\infty} \frac{\Gamma\left(n_2 + \frac{1}{2}\right)}{\Gamma(-n_2)} \left(\frac{2}{ay}\right)^{2n_2+1}, \tag{3.2.9}$$

which always equals to 0 since the Gamma function has poles at non-positive integers.

iii> When choosing n_3 as the free parameter, by similar computations, we get that

$$I = -\frac{1}{y} \sum_{n_3=0}^{\infty} \frac{1}{(2n_3+1)!} \left(\frac{ay}{2}\right)^{2n_3+1} = -\frac{\sinh(ay)}{y}, \tag{3.2.10}$$

whose radius of convergence is also $+\infty$, the same as that in i>.

Now, since all three series converge in the same region, they should be added to get

$$I = \frac{\cosh(ay) + 0 - \sinh(ay)}{y} = \frac{1}{y} e^{-ay}. \tag{3.2.11}$$

The formula is verified.

3.3 Example of Higher Dimensions

Example 3.3.1. The following double integral

$$I = \int_0^{\infty} \int_0^{\infty} \frac{1}{(1 + x^2 + y^2)^2} dx dy = \frac{\pi}{4} \tag{3.3.1}$$

can be easily verified through polar coordinate that

$$\begin{cases} r = \sqrt{x^2 + y^2}, & 0 \le r < \infty \\ \theta = \arctan \frac{y}{x}, & 0 \le \theta \le \frac{\pi}{2} \end{cases}, \tag{3.3.2}$$

which yields

$$I = \int_0^{\frac{\pi}{2}} \int_0^{\infty} \frac{r}{(1 + r^2)^2} dr d\theta = \frac{\pi}{2} \cdot \frac{1}{2} \cdot \left(-\frac{1}{1 + r^2}\right) \Bigg|_{r=0}^{r=\infty} = \frac{\pi}{4}. \tag{3.3.3}$$

Now, application of the method of brackets begins with rule P_2 that

$$f\left(x,y\right)=\left(1+x^2+y^2\right)^{-2}=\sum_{n_1,n_2,n_3}\phi_{n_1,n_2,n_3}1^{n_1}x^{2n_2}y^{2n_3}\frac{\langle n_1+n_2+n_3+2\rangle}{\Gamma\left(2\right)}. \tag{3.3.4}$$

Then,

$$I=\sum_{n_1,n_2,n_3}\phi_{1,2,3}\left\langle n_1+n_2+n_3+2\right\rangle\left\langle 2n_2+1\right\rangle\left\langle 2n_3+1\right\rangle. \tag{3.3.5}$$

The linear system

$$\begin{bmatrix}1&1&1\\0&2&0\\0&0&2\end{bmatrix}\begin{bmatrix}n_1\\n_2\\n_3\end{bmatrix}+\begin{bmatrix}2\\1\\1\end{bmatrix}=0 \tag{3.3.6}$$

gives

$$\det\begin{bmatrix}1&1&1\\0&2&0\\0&0&2\end{bmatrix}=4 \text{ and } \begin{bmatrix}n_1^*\\n_2^*\\n_3^*\end{bmatrix}=\begin{bmatrix}-1\\-\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}. \tag{3.3.7}$$

Therefore,

$$I=\frac{1}{|4|}\Gamma\left(1\right)\Gamma\left(\frac{1}{2}\right)^2=\frac{\pi}{4}. \tag{3.3.8}$$

Chapter 4

Independence on Series Representations of Integrand

The rules of the method of brackets rely on the representation of the integrand as a power series. In many instances, it is possible to decompose the integrand as a product of several factors, each with its corresponding series. The most elementary example $e^{ax} = e^{bx} \times e^{(a-b)x}$ produces a variety of such representations. The goal of this section is to prove that the evaluation of integrals by the method of brackets is *independent* of these decompositions.

The discussion begins with the simple example that has been mentioned in last chapter.

4.1 Leading Example

Example 4.1.1. As mentioned above, Entry **3.310** in [31] can be verified by

$$I = \int_0^\infty e^{-x} dx = \int_0^\infty \sum_{n=0}^\infty \phi_n x^n dx = \sum_n \phi_n \langle n+1 \rangle = 1 \cdot \Gamma(1) = 1. \tag{4.1.1}$$

On the other hand, if the integrand is rewritten as $e^{-x} = e^{-\frac{x}{3}} \times e^{-\frac{2x}{3}}$, expanding both factors gives

$$e^{-x} = \left(\sum_{n_1=0}^{\infty} \frac{\left(-\frac{x}{3}\right)^{n_1}}{n_1!} \right) \left(\sum_{n_2=0}^{\infty} \frac{\left(-\frac{2x}{3}\right)^{n_2}}{n_2!} \right) = \sum_{n_1,n_2} \phi_{1,2} \frac{2^{n_2} x^{n_1+n_2}}{3^{n_1+n_2}}. \tag{4.1.2}$$

Integration produces a bracket series of index 1,

$$I = \sum_{n_1,n_2} \phi_{1,2} \frac{2^{n_2}}{3^{n_1+n_2}} \langle n_1 + n_2 + 1 \rangle . \tag{4.1.3}$$

The evaluation of I is obtained by choosing free parameters, either n_1 or n_2.

i> If we choose n_1 as the free parameter, then solving $n_1 + n_2^* + 1 = 0$ gives $n_2^* = -1 - n_1$. Thus,

$$I = \sum_{n_1=0}^{\infty} \phi_{n_1} \frac{2^{-1-n_1}}{3^{-1}} \Gamma(n_1 + 1) = \frac{3}{2} \sum_{n_1=0}^{\infty} \left(-\frac{1}{2} \right)^{n_1} = 1. \tag{4.1.4}$$

ii> If we choose n_2 as the free parameter, then by symmetry, $n_1^* = -1 - n_2$ and

$$I = \sum_{n_2} \phi_{n_2} \frac{2^{n_2}}{3^{-1}} \Gamma(n_2 + 1) = 3 \sum_{n_2=0}^{\infty} (-2)^{n_2} , \tag{4.1.5}$$

which is a divergent series. Rule E_3 states that this choice should be discarded. The value of I has been confirmed.

A natural question raised by this example is whether different series representations of the integrand lead to the same answer. In other words, how does factorization of the integrand affect the result? It is necessary for an integration method to be independent on such different expressions. Main theorems in the next section prove this result.

4.2 Main Theorems

Theorem 4.2.1. *Suppose $f(x) = g(x) h(x)$, where all f, g, and h have power series expressions. Then, the method of bracket gives the same value for the two integrals*

$$I_1 = \int_0^{\infty} f(x)\, dx \text{ and } I_2 = \int_0^{\infty} g(x)\, h(x)\, dx. \tag{4.2.1}$$

Proof. Suppose that

$$\begin{cases} f(x) = \sum\limits_{n=0}^{\infty} \phi_n a(n)\, x^{\alpha n+\beta-1} & = x^{\beta-1} \sum\limits_{n=0}^{\infty} a(n) \frac{(-x^{\alpha})^n}{n!}, \\ g(x) = \sum\limits_{n_1=0}^{\infty} \phi_{n_1} b(n_1)\, x^{\alpha_1 n_1+\beta_1-1} & = x^{\beta_1-1} \sum\limits_{n_1=0}^{\infty} b(n_1) \frac{(-x^{\alpha_1})^{n_1}}{n_1!}, \\ h(x) = \sum\limits_{n_2=0}^{\infty} \phi_{n_2} c(n_2)\, x^{\alpha_2 n_2+\beta_2-1} & = x^{\beta_2-1} \sum\limits_{n_2=0}^{\infty} c(n_2) \frac{(-x^{\alpha_2})^{n_2}}{n_2!}. \end{cases} \tag{4.2.2}$$

The rules of the method of brackets yield

$$I_1 = \int_0^{\infty} f(x)\, dx \overset{P_1}{=} \sum_n \phi_n a(n) \langle \alpha n + \beta \rangle \overset{E_1}{=} a\left(-\frac{\beta}{\alpha}\right) \Gamma\left(\frac{\beta}{\alpha}\right), \tag{4.2.3}$$

and for the second integral,

$$\begin{aligned} I_2 &= \int_0^{\infty} g(x)\, h(x)\, dx \qquad (4.2.4) \\ &= \int_0^{\infty} \sum_{n_1,n_2} \phi_{1,2} b(n_1)\, c(n_2)\, x^{\alpha_1 n_1+\beta_1+\alpha_2 n_2+\beta_2-2} \\ &= \sum_{n_1,n_2} \phi_{1,2} b(n_1)\, c(n_2) \langle \alpha_1 n_1 + \beta_1 + \alpha_2 n_2 + \beta_2 - 1 \rangle, \end{aligned}$$

which is a bracket series of index 1. Different choices of free parameters produce two series for the integral I_2, denoted by $I_{2,1}$ and $I_{2,2}$, respectively.

i>Choosing n_1 as the free parameter yields

$$I_{2,1} = \frac{1}{|\alpha_2|} \sum_{n_1=0}^{\infty} \phi_{n_1} b(n_1)\, c\left(\frac{1-\alpha_1 n_1 - \beta_1 - \beta_2}{\alpha_2}\right) \Gamma\left(\frac{\alpha_1 n_1 + \beta_1 + \beta_2 - 1}{\alpha_2}\right). \tag{4.2.5}$$

ii>Similarly, the choice of n_2 as the free parameter gives, by symmetry,

$$I_{2,2} = \frac{1}{|\alpha_1|} \sum_{n_2=0}^{\infty} \phi_{n_2} c(n_2)\, b\left(\frac{1-\alpha_2 n_2 - \beta_1 - \beta_2}{\alpha_1}\right) \Gamma\left(\frac{\alpha_2 n_2 + \beta_1 + \beta_2 - 1}{\alpha_1}\right). \tag{4.2.6}$$

We shall show that both $I_{2,1}$ and $I_{2,2}$ agree with I_1, which implies that $I_1 = I_{2,1} = I_{2,2}$.

The identity $f(x) = g(x)\, h(x)$ gives

$$x^{\beta-1} \sum_{n=0}^{\infty} a(n)\, \phi_n x^{\alpha n} = x^{\beta_1+\beta_2-2} \sum_{n=0}^{\infty} \phi_n \left[\sum_{k=0}^{n} \binom{n}{k} b(k)\, c(n-k)\, x^{\alpha_1 k + \alpha_2 (n-k)} \right]. \tag{4.2.7}$$

Matching powers of x produces $\alpha = \alpha_1 = \alpha_2$, $\beta = \beta_1 + \beta_2 - 1$ and

$$a(n) = \sum_{k=0}^{n} \binom{n}{k} b(k)\, c(n-k). \tag{4.2.8}$$

Thus, with $u = \beta/\alpha$

$$\begin{aligned} I_{2,1} &= \frac{1}{|\alpha|} \sum_{n_1=0}^{\infty} \phi_{n_1} b(n_1)\, c(-u-n_1)\, \Gamma(u+n_1) \qquad (4.2.9) \\ &= \frac{1}{|\alpha|} \sum_{n_1=0}^{\infty} \frac{(-1)^{n_1} \Gamma(u+n_1)}{n_1!} b(n_1)\, c(-u-n_1) \\ &= \frac{\Gamma(u)}{|\alpha|} \sum_{n_1=0}^{\infty} \frac{(-1)^{n_1} (u)_{n_1}}{n_1!} b(n_1)\, c(-u-n_1), \end{aligned}$$

where we use the Pochhammer symbol

$$(u)_m := \frac{\Gamma(u+m)}{\Gamma(u)} = u(u+1)\cdots(u+m-1). \tag{4.2.10}$$

In order to have $I_1 = I_{2,1}$, it suffices to show that

$$a(-u) = \sum_{k=0}^{\infty} \frac{(-1)^k (u)_k}{k!} b(k)\, c(-u-k). \tag{4.2.11}$$

This follows by the analytic continuation of the representation

$$a(n) = \sum_{k=0}^{n} \binom{n}{k} b(k)\, c(n-k) = \sum_{k=0}^{\infty} \binom{n}{k} b(k)\, c(n-k), \tag{4.2.12}$$

since

$$\binom{-u}{k} = \frac{(-u)(-u-1)\cdots(-u-k+1)}{k!} = \frac{(-1)^k (u)_k}{k!}. \tag{4.2.13}$$

This completes the argument. Similar computation shows $I_1 = I_{2,2}$. □

The result is now extended to the case of a finite number of factors.

Theorem 4.2.2. *Assume f admits a representation of the form $f(x) = \prod_{i=1}^{r} f_i(x)$. Then the value of the integral, obtained by method of bracket, is the same for both series representations.*

Proof. Assume for all $i = 1, \ldots, r$

$$f_i(x) = \sum_{n_i=0}^{\infty} \phi_{n_i} a_i(n_i) x^{\alpha_i n_i + \beta_i - 1} = x^{\beta_i - 1} \sum_{n_i=0}^{\infty} a_i(n_i) \frac{(-x^{\alpha_i})^{n_i}}{n_i!}. \tag{4.2.14}$$

Then,

$$\begin{aligned} \int_0^{\infty} \prod_{i=1}^{r} f_i(x)\, dx &= \int_0^{\infty} \sum_{n_1,\ldots,n_r=0}^{\infty} \phi_{1,\ldots,r} \prod_{i=1}^{r} a_i(n_i)\, x^{\sum\limits_{i=1}^{r}(\alpha_i n_i + \beta_i - 1)} \qquad (4.2.15) \\ &= \sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} \left[\prod_{i=1}^{r} a_i(n_i)\right] \left\langle \sum_{i=1}^{r} (\alpha_i n_i + \beta_i - 1) + 1 \right\rangle. \end{aligned}$$

From $f(x) = \prod\limits_{i=1}^{r} f_i(x)$, it follows that

$$\alpha = \alpha_1 = \cdots = \alpha_r, \text{ and } \beta = \left(\sum_{i=1}^{r} \beta_i\right) - (r-1). \tag{4.2.16}$$

Also, we have

$$a(n) = \sum_{k_1,\ldots,k_r} \binom{n}{k_1, \ldots, k_r} \prod_{i=1}^{k} a_i(k_i), \tag{4.2.17}$$

where the multinomial coefficients are defined by

$$\binom{n}{k_1, \ldots, k_r} := \begin{cases} \frac{n!}{k_1! \cdots k_r!}, & \text{if } k_1 + \cdots + k_r = n, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2.18}$$

Without loss of generality, we let $n_1, \ldots, n_{r-1}$ be free parameters, and let $u = \frac{\beta}{\alpha}$, then

$$\begin{aligned} \int_0^{\infty} \prod_{i=1}^{r} f_i(x)\, dx &= \sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} \left[\prod_{i=1}^{r} a_i(n_i)\right] \left\langle \alpha \sum_{i=1}^{r} n_i + \beta \right\rangle \qquad (4.2.19) \\ &= \frac{1}{|\alpha|} \sum_{n_1,\ldots,n_{r-1}} \phi_{1,\ldots,r} \left[\prod_{i=1}^{r-1} a_i(n_i)\right] a_r \left(-u - \sum_{i=1}^{r-1} n_i\right) \Gamma\left(u + \sum_{i=1}^{r-1} n_i\right) \\ &= \frac{\Gamma(u)}{|\alpha|} \sum_{n_1,\ldots,n_{r-1}} \phi_{1,\ldots,r} \left[\prod_{i=1}^{r-1} a_i(n_i)\right] a_r \left(-u - \sum_{i=1}^{r-1} n_i\right) (u)_{\sum\limits_{i=1}^{r-1} n_i}. \end{aligned}$$

Now, denoting

$$n_r := -u - \sum_{i=1}^{r} n_i, \tag{4.2.20}$$

it follows that

$$\begin{aligned}
\int_0^\infty \prod_{i=1}^{r} f_i(x)\,dx &= \frac{\Gamma(u)}{|\alpha|} \sum_{n_1,\ldots,n_r} \frac{(-1)^{-u-n_r} a_r(n_r)(u)_{-u-n_r}}{n_1! \cdots n_{r-1}!} \left[\prod_{i=1}^{r-1} a_i(n_i)\right] \\
&= \frac{\Gamma(u)}{|\alpha|} \sum_{n_1,\ldots,n_r} \frac{\Gamma(-u+1)}{n_1! \cdots n_{r-1}! n_r!} \left[\prod_{i=1}^{r} a_i(n_i)\right] \\
&= \frac{\Gamma(u)}{|\alpha|} \sum_{n_1,\ldots,n_r} \binom{-u}{n_1,\ldots,n_r} \left[\prod_{i=1}^{r} a_i(n_i)\right] \\
&= \frac{1}{|\alpha|}\Gamma(u)\, a(-u) \\
&= \int_0^\infty f(x)\,dx. \qquad (4.2.21)
\end{aligned}$$

This completes the proof. □

Chapter 5

Discussion on Divergent Series, Analytic Continuation and Modification for Rule E_3

This section first suggests, via an example, a possible modification of Rule E_3.

5.1 Leading Example

Example 5.1.1. Reconsider the example provided that

$$I = \int_0^\infty e^{-x} dx = 1. \tag{5.1.1}$$

When factoring $e^{-x} = e^{-\frac{x}{3}} e^{-\frac{2x}{3}}$, one of the choices of parameters yields the series (4.1.5)

$$H = \sum_{n_2} \phi_{n_2} \frac{2^{n_2}}{3^{-1}} \Gamma(n_2+1) = 3 \sum_{n_2=0}^{\infty} (-2)^{n_2}, \tag{5.1.2}$$

which was discarded by rule E_3 since it is divergent. This divergence is interpreted as the value at $x = 2$ of the series

$$H(x) = 3 \sum_{n=0}^{\infty} (-x)^n. \tag{5.1.3}$$

Although this series converges only for $|x| < 1$, it can be analytically continued to $\mathbb{C} \setminus \{1\}$. Such a continuation is given by

$$H(x) = \frac{3}{1+x} \text{ and } H(2) = 1. \tag{5.1.4}$$

This phenomenon is also observed in the general factorization of

$$e^{-x} = e^{-ax}e^{-bx}, \text{ where } a+b=1, \text{ and } a,b>0. \tag{5.1.5}$$

Note that the result is actually related to the following formal series expansions:

$$1 = \frac{1}{a+b} = \begin{cases} a \cdot \frac{1}{1+\frac{b}{a}} = \sum\limits_{n=0}^{\infty} \frac{b^n}{a^{n+1}} & \text{if } a > b > 0, \\ b \cdot \frac{1}{1+\frac{a}{b}} = \sum\limits_{n=0}^{\infty} \frac{a^n}{b^{n+1}} & \text{if } b > a > 0. \end{cases} \tag{5.1.6}$$

Now, consider the critical case $a = b = \frac{1}{2}$ in (5.1.5). Similar computation as in Example 4.1.1 shows

$$I = \int_0^\infty \left(e^{-\frac{x}{2}}\right)^2 dx = \sum_{n_1,n_2} \phi_{n_1,n_2} 2^{-(n_1+n_2)} \langle n_1 + n_2 + 1 \rangle . \tag{5.1.7}$$

Both choices of free parameter, n_1 or n_2, lead to the same divergent series but correct answer by analytic continuation:

$$I = 2\sum_{n=0}^{\infty} (-1)^n = 2 \cdot \sum_{n=0}^{\infty} x^n \Big|_{x=-1} = 2 \cdot \frac{1}{1-x}\Big|_{x=-1} = 1. \tag{5.1.8}$$

If applying the old rule E_3, since both series diverge, we discard them and conclude that the integral diverges or the method of brackets cannot compute it. On the other hand, Theorem 4.2.2 shows that the answer should be the same without the factorization of the integrand. Such contradiction can be solved by considering analytic continuation, which was already applied in the proof of both theorems in the previous chapter, and also could make divergent series meaningful.

5.2 Conclusion on New Rule E_3.

It suggests to modify the rule E_3 as follows:

Rule $\tilde{E}_3$: The value of a multi-dimensional bracket series of positive index is obtained by computing all the contributions of maximal rank by Rule E_2. These contributions to the integral appear as series in the free parameters. Series converging in a common region are added. *Divergent series are evaluated by their analytic continuations, if such continuations exist. Divergent series having the same analytic continuation are combined and treated as the common analytic continuation on its domain.* Any series producing a non-real contribution is also discarded. There is no assignment to a bracket of negative index.

5.3 Further Discussion on Divergent Series

It happens sometimes that applying the method of brackets will not result in convergent series. Namely, all possible series obtained are either divergent or having all term vanish, which we call *null series*.

Example 5.3.1. Entry **3.754.2** in [31] is

$$\int_0^\infty \frac{\cos(ax)\,dx}{\sqrt{\beta^2+x^2}} = K_0(a\beta), \tag{5.3.1}$$

where $K_0(x)$ is the *modified Bessel function of the second kind of order* 0, defined by

$$K_0(x) := \lim_{t\to 0}\frac{\pi}{2}\cdot\frac{I_{-t}(x)-I_t(x)}{\sin(t\pi)}, \tag{5.3.2}$$

and

$$I_0(x) := J_0(\iota x) = \sum_{m=0}^{\infty}\frac{x^{2m}}{m!m!2^{2m}} \tag{5.3.3}$$

is the *modified Bessel function of the first kind of order* 0. Take $\beta = 1$ to have

$$K_0(x) = \int_0^\infty \frac{\cos(tx)\,dt}{\sqrt{1+t^2}}. \tag{5.3.4}$$

From the rules, we have

$$\begin{cases} \cos(tx) = \sum\limits_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}t^{2n} = \sum\limits_{n=0}^{\infty}\phi_n\frac{\Gamma(n+1)x^{2n}}{\Gamma(2n+1)}t^{2n}, \\ (1+t^2)^{-\frac{1}{2}} = \sum\limits_{m,k=0}^{\infty}\phi_{m,k}t^{2k}\frac{\langle m+k+\frac{1}{2}\rangle}{\Gamma(\frac{1}{2})}. \end{cases} \tag{5.3.5}$$

Thus,

$$K_0(x) = \frac{1}{\sqrt{\pi}} \sum_{m,n,k} \phi_{m,n,k} \frac{\Gamma(n+1)\, x^{2n}}{\Gamma(2n+1)} \left\langle m + k + \frac{1}{2} \right\rangle \langle 2k + 2n + 1 \rangle . \tag{5.3.6}$$

The index of this bracket series is $3 - 2 = 1$. Choosing different free parameters among m, n and k leads to the following results:

Numbers	Free Parameter	Solution	Result	Type
(1)	m	$n^* = m$ $k^* = -\frac{1}{2} - m$	$K_0(x) = \frac{1}{2} \sum_m \phi_m \Gamma(-m) \frac{x^{2m}}{4^m}$	DS
(2)	n	$m^* = n$ $k^* = -\frac{1}{2} - n$	$K_0(x) = \frac{1}{2} \sum_n \phi_n \Gamma(-n) \frac{x^{2n}}{4^n}$	
(3)	k	$m^* = -\frac{1}{2} - k$ $n^* = -\frac{1}{2} - k$	$K_0(x) = \sum_k \phi_k \frac{\Gamma\left(k+\frac{1}{2}\right)^2}{\Gamma(-k)} \cdot \frac{4^k}{x^{2k+1}}$	NS

Here, "DS" stands for "Divergent Series" and "NS" for "Null Series". Both of them are due to poles of Gamma function at non-positive integers. Also, we remark the following.

Remark 5.3.2. (i) By symmetry of m and n, (1) and (2) are the same.

(ii) The duplication formula that

$$\Gamma(z)\,\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z} \sqrt{\pi}\,\Gamma(2z) \tag{5.3.7}$$

is applied in (1) and (2) in the case $z = m + \frac{1}{2}$ and in (3) for $z = -m$.

(iii) It is not quite astonishing not to obtain convergent series since $K_0(x)$ does not admit a series representation.

In fact, the two results that

$$K_0(x) = \frac{1}{2} \sum_n \phi_n \Gamma(-n) \frac{x^{2n}}{4^n} \tag{5.3.8}$$

and

$$K_0(x) = \sum_n \phi_n \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(-n)} \cdot \frac{4^n}{x^{2n+1}} \tag{5.3.9}$$

can be further applied when series representations for $K_0(x)$ is required.

Example 5.3.3. The Mellin Transform of $K_0(x)$ is given by (a special case of) entry **6.561.16** in [31]:

$$\mathcal{M}(K_0)(s) = \int_0^\infty x^{s-1} K_0(x)\, dx = 2^{s-2}\Gamma\left(\frac{s}{2}\right)^2. \tag{5.3.10}$$

We could apply the method of brackets based on the result of previous example to see that

$$\begin{aligned}
\mathcal{M}(K_0)(s) &= \int_0^\infty x^{s-1} K_0(x)\, dx \qquad (5.3.11)\\
&= \int_0^\infty x^{s-1} \sum_n \phi_n \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(-n)} \cdot \frac{4^n}{x^{2n+1}} dx \\
&= \sum_n \phi_n \frac{\Gamma\left(n+\frac{1}{2}\right)^2 4^n}{\Gamma(-n)} \langle s-2n-1 \rangle \\
&= \frac{1}{|-2|} \cdot \frac{\Gamma\left(\frac{s}{2}\right)^2 4^{\frac{s-1}{2}}}{\Gamma\left(-\frac{s-1}{2}\right)} \cdot \Gamma\left(-\frac{s-1}{2}\right) \\
&= 2^{s-2}\Gamma\left(\frac{s}{2}\right)^2,
\end{aligned}$$

and also

$$\begin{aligned}
\mathcal{M}(K_0)(s) &= \int_0^\infty x^{s-1} K_0(x)\, dx \qquad (5.3.12)\\
&= \int_0^\infty \frac{x^{s-1}}{2} \sum_n \phi_n \Gamma(-n) \frac{x^{2n}}{4^n} dx \\
&= \frac{1}{2} \sum_n \phi_n \frac{\Gamma(-n)}{4^n} \langle 2n+s \rangle \\
&= \frac{1}{2} \cdot \frac{1}{|2|} \cdot \frac{\Gamma\left(\frac{s}{2}\right)}{4^{-\frac{s}{2}}} \cdot \Gamma\left(\frac{s}{2}\right) \\
&= 2^{s-2}\Gamma\left(\frac{s}{2}\right)^2.
\end{aligned}$$

Chapter 6

Discussion on Pochhamer Symbol at Negative Indices

6.1 Quick Introduction on Pochhammer Symbol

Rule E_1 of the method of brackets requires the evaluation of $f(n^*)$. In many instances, this involves the evaluation of the Pochhammer symbol $(x)_m$ defined in (4.2.10). Two important identities

$$(-x)_n = (-1)^n (x-n+1)_n \tag{6.1.1}$$

and

$$(x)_{-n} = \frac{(-1)^n}{(1-x)_n} \tag{6.1.2}$$

are usually used to compute its analytic continuation, i.e., for the case that x or n being negative, especially negative integers. This section discusses the case when both x and n are negative integers. It is important to discuss this if noticing `Mathematica`, which is one of the main computer language being widely uses, provides different results in the following example.

Example 6.1.1. In `Mathematica`, it is not hard to check that

$$(-2)_{-2} = \frac{1}{12} \text{ and } \lim_{n \to -2} (n)_n = \frac{1}{24}, \tag{6.1.3}$$

by the commands:

```
In[1]:= Pochhammer[-2, -2]

Out[1]= 1/12

In[2]:= Limit[Pochhammer[x, x], x -> -2]

Out[2]= 1/24
```

In the coming section, we present an important example that draws our attention to such difference when applying the method of brackets.

6.2 Leading Example

Example 6.2.1. Entry **6.671.7** in [31] reads:

$$I := \int_0^\infty J_0(ax)\sin(bx)dx = \begin{cases} 0, & \text{if } 0 < b < a, \\ 1/\sqrt{b^2 - a^2}, & \text{if } 0 < a < b, \end{cases} \tag{6.2.1}$$

whose proof appears in [6]. The evaluation uses the series

$$J_0(ax) = \sum_{m=0}^{\infty} \phi_m \frac{a^{2m}}{\Gamma(m+1)2^{2m}} x^{2m} \tag{6.2.2}$$

and

$$\sin(bx) = \sum_{n=0}^{\infty} \phi_n \frac{\Gamma(n+1)}{\Gamma(2n+2)} b^{2n+1} x^{2n+1}. \tag{6.2.3}$$

Therefore the integral is given by

$$I = \sum_{m,n} \phi_{m,n} \frac{a^{2m} b^{2n+1} \Gamma(n+1)}{2^{2m}\Gamma(m+1)\Gamma(2n+2)} \langle 2m + 2n + 2 \rangle. \tag{6.2.4}$$

The duplication formula (5.3.7) transforms this expression to

$$I = \frac{\sqrt{\pi}}{2} \sum_{m,n} \phi_{m,n} \frac{a^{2m} b^{2n+1}}{2^{2m+2n}\Gamma(m+1)\Gamma(n+3/2)} \langle 2m + 2n + 2 \rangle. \tag{6.2.5}$$

(i) Eliminating the parameter n using Rule E_1 gives $n^* = -m-1$ and produces

$$\begin{aligned} I &= \frac{\sqrt{\pi}}{b} \sum_m \phi_m \frac{1}{\Gamma\left(-m+\frac{1}{2}\right)} \left(\frac{a}{b}\right)^{2m} && (6.2.6) \\ &= \frac{1}{b} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\left(\frac{a^2}{b^2}\right)^m}{\left(\frac{1}{2}\right)_{-m}} \\ &= \frac{1}{b} \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)_m \frac{1}{m!} \left(\frac{a^2}{b^2}\right)^m \\ &= \frac{1}{\sqrt{b^2-a^2}}. \end{aligned}$$

The condition $|b| > |a|$ is imposed to guarantee the convergence of the series on the third line of the previous argument.

(ii) The series obtained by eliminating the parameter m by $m^* = -n-1$ vanishes because of the factor $\Gamma(m+1)$ in the denominator.

Example 6.2.2. A second evaluation of **6.671.7** begins with

$$\int_0^{\infty} J_0(ax)\sin(bx)dx = \sum_{m,n} \phi_{m,n} \frac{a^{2m}b^{2n+1}}{2^{2m}m!(2n+1)!} \langle 2m+2n+2 \rangle. \tag{6.2.7}$$

(i) Choose n as the free parameter. Then $m^* = -n-1$ and the contribution to the integral is

$$I_1 = \frac{1}{2} \sum_{n=0}^{\infty} \phi_n \left(\frac{a}{2}\right)^{-2n-2} b^{2n+1} \frac{1}{\Gamma(-n)} \frac{n!}{(2n+1)!} \Gamma(n+1). \tag{6.2.8}$$

Each term in the sum vanishes.

(ii) Choose m as a free parameter. Then $n^* = -m-1$ and

$$I_2 = \frac{1}{2} \sum_m \phi_m \left(\frac{a}{2}\right)^{2m} b^{-2m-1} \frac{1}{m!} \cdot \frac{n!}{(2n+1)!}\bigg|_{n=-m-1} \Gamma(m+1). \tag{6.2.9}$$

Now write

$$\frac{n!}{(2n+1)!} = \frac{1}{(n+1)_{n+1}} \tag{6.2.10}$$

and (6.2.9) becomes

$$I_2 = \frac{1}{2b} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{a}{2b}\right)^{2m} \frac{1}{(-m)_{-m}}. \tag{6.2.11}$$

Transforming the term $(-m)_{-m}$ by using (6.1.2) gives

$$(-m)_{-m} = \frac{(-1)^m}{(1+m)_m}. \tag{6.2.12}$$

Replacing in (6.2.11) produces

$$I_2 = \frac{1}{2b} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)_k \frac{1}{k!} \left(\frac{a^2}{b^2}\right)^k = \frac{1}{2} \frac{1}{\sqrt{b^2 - a^2}}. \tag{6.2.13}$$

Now, it is astonishing that the method of brackets produces only *half of the expected answer*. Admittedly, it is possible that the entry in [31] is erroneous (which happens in the book once in a while). However, some numerical computations and especially the evaluation in Example 6.2.1 should be convincible that this is not the case.

Now, two different evaluations above are obtained, based on different series representations of $\sin(bx)$, which should not affect the result as proven before. In the coming section, we will see that the source of the error is the use of (6.1.2) for the evaluation of the term $(-m)_{-m}$, which appears in Example 6.2.2 but is avoided in Example 6.2.1.

6.3 Pochhamer Symbol at Negative Indices

As discussed above, the question of the value

$$(-m)_{-m} \text{ for } m \in \mathbb{N} \tag{6.3.1}$$

is at the core of the missing factor of 2 in the example shown above.

The first extension of $(x)_m$ to negative values of n comes from the identity

$$(x)_{-m} = \frac{(-1)^m}{(1-x)_m}. \tag{6.3.2}$$

This is obtained from

$$(x)_{-m} = \frac{\Gamma(x-m)}{\Gamma(x)} = \frac{\Gamma(x-m)}{(x-1)(x-2)\cdots(x-m)\Gamma(x-m)} \tag{6.3.3}$$

and then changing the signs of each of the factors. It is valid as long as x is not a negative integer. The limiting value of the right-hand side in (6.3.2) as $x \mapsto -km$, with $k \in \mathbb{N}$, is

$$(-km)_{-m} = \frac{(-1)^m\,(km)!}{((k+1)m)!}. \tag{6.3.4}$$

On the other hand, the limiting value of the left-hand side is

$$\begin{aligned}
\lim_{\varepsilon\to 0}(-k(m+\varepsilon))_{-(m+\varepsilon)} &= \lim_{\varepsilon\to 0}\frac{\Gamma(-(k+1)m-(k+1)\varepsilon)}{\Gamma(-km-k\varepsilon)} \\
&= \lim_{\varepsilon\to 0}\frac{\Gamma(-(k+1)\varepsilon)(-(k+1)\varepsilon)_{-(k+1)m}}{\Gamma(-k\varepsilon)(-k\varepsilon)_{-km}} \\
&= \lim_{\varepsilon\to 0}\frac{\Gamma(-(k+1)\varepsilon)}{\Gamma(-k\varepsilon)}\frac{(-1)^{(k+1)m}}{(1+(k+1)\varepsilon)_{(k+1)m}}\cdot\frac{(1+k\varepsilon)_{km}}{(-1)^{km}} \\
&= \frac{(-1)^m(km)!}{((k+1)m)!}\cdot\frac{k}{k+1}.
\end{aligned} \tag{6.3.5}$$

Therefore the function $(x)_{-m}$ is discontinuous at $x = -km$, with

$$\frac{\text{Direct }(-km)_{-m}}{\text{Limiting }(-km)_{-m}} = \frac{k+1}{k}. \tag{6.3.6}$$

For $k = 1$, this ratio becomes 2, which explains the missing $\frac{1}{2}$ in the calculation in Example 6.2.2. Therefore it is the discontinuity of Pochhamer symbol at negative integer values of the variables, that is responsible for the error in the evaluation.

Thus, it suggests that the rules of the method of brackets should be supplemented with an additional one:

Rule E_4. Let $k,\ m \in \mathbb{N}$ be fixed. In the evaluation of series, the rule

$$(-km)_{-m} = \frac{k}{k+1}\cdot\frac{(-1)^m(km)!}{((k+1)m)!} \tag{6.3.7}$$

must be used to eliminate Pochhammer symbols with negative integer index and negative integer base.

Chapter 7

An Inverse Application of the Method of Brackets on Mellin Transform

7.1 Series Representation from Mellin Transform

Given a function $f(x)$ and its Mellin transform $\mathcal{M}(f)(s)$. We could assume f admits a series representation that

$$f(x) = \sum_{n} \phi_n C(n) x^{\alpha n+\beta}, \tag{7.1.1}$$

for some $\alpha \neq 0$ and β. Applying the method of brackets yields

$$\begin{aligned} \mathcal{M}(f)(s) &= \int_0^\infty x^{s-1} f(x)\, dx \\ &\overset{P_1}{=} \sum_n \phi(n) C(n) \langle \alpha n + \beta + s \rangle \\ &\overset{E_1}{=} \frac{1}{|\alpha|} C\left(-\frac{\beta+s}{\alpha}\right) \Gamma\left(\frac{\beta+s}{\alpha}\right), \end{aligned} \tag{7.1.2}$$

which implies

$$C\left(-\frac{\beta+s}{\alpha}\right) = \frac{|\alpha| \mathcal{M}(f)(s)}{\Gamma\left(\frac{\beta+s}{\alpha}\right)}, \tag{7.1.3}$$

and therefore

$$C(n) = \frac{|\alpha| \mathcal{M}(f)(-\alpha n - \beta)}{\Gamma(-n)} \tag{7.1.4}$$

gives the coefficients for the series representation of $f(x)$.

Example 7.1.1. Recall the Mellin Transform of Bessel-K function that

$$\mathcal{M}(K_0)(s) = \int_0^\infty x^{s-1} K_0(x)\, dx = 2^{s-2}\Gamma\left(\frac{s}{2}\right)^2. \tag{7.1.5}$$

Then (7.1.4) shows that

$$K_0(x) = \sum_n \phi_n \frac{|\alpha|\,\Gamma\left(\frac{-\alpha n-\beta}{2}\right)^2}{2^{\alpha n+\beta+2}\Gamma(-n)} x^{\alpha n+\beta}. \tag{7.1.6}$$

(i) Choosing $\alpha = 2$ and $\beta = 0$ to cancel the $\Gamma(-n)$ gives the desired divergent series

$$K_0(x) = \frac{1}{2}\sum_n \phi_n \Gamma(-n)\frac{x^{2n}}{4^n}. \tag{7.1.7}$$

(ii) Choosing $\alpha = -2$ and $\beta = -1$, one can obtain the null series

$$\sum_n \phi_n \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(-n)} \cdot \frac{4^n}{x^{2n+1}}. \tag{7.1.8}$$

Other choices of α and β give a variety of results. It verifies the results in Chapter 5.

Consider an elementary function $f(x) = \sin(bx)$, whose Mellin transform is the entry **3.761.4** in [31] that

$$\mathcal{M}(f)(s) = \int_0^\infty x^{s-1}\sin(bx)\, dx = \frac{\pi\Gamma(s)}{b^s\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}. \tag{7.1.9}$$

Thus, by (7.1.4), we have

$$\sin(bx) = \sum_n \phi_n \frac{|\alpha|\, b^{\alpha n+\beta}\pi\Gamma(-\alpha n-\beta)}{\Gamma\left(\frac{-\alpha n-\beta}{2}\right)\Gamma\left(1+\frac{\alpha n+\beta}{2}\right)\Gamma(-n)} x^{\alpha n+\beta}. \tag{7.1.10}$$

As we know, from the Taylor expansion of the sine function, $\alpha = 2$ and $\beta = 1$. Then,

$$\sin(bx) = \sum_n \phi_n \frac{2b^{2n+1}\pi\Gamma(-2n-1)}{\Gamma\left(-n-\frac{1}{2}\right)\Gamma\left(n+\frac{3}{2}\right)\Gamma(-n)} x^{2n+1}. \tag{7.1.11}$$

Duplication formula (5.3.7) in the cases $z = -n-\frac{1}{2}$ and $z = n+1$ gives both

$$\frac{\Gamma(-2n-1)}{\Gamma\left(-n-\frac{1}{2}\right)\Gamma(-n)} = \frac{1}{2^{2n+2}\sqrt{\pi}}, \tag{7.1.12}$$

and

$$\frac{1}{\Gamma(n+1)\,\Gamma\left(n+\frac{3}{2}\right)}=\frac{2^{2n+1}}{\Gamma(2n+2)}=\frac{2^{2n+1}}{(2n+1)!}. \tag{7.1.13}$$

Therefore, we recover the desired result

$$\sin(bx)=\sum_{n=0}^{\infty}\frac{(-1)^n(bx)^{2n+1}}{(2n+1)!}. \tag{7.1.14}$$

In fact, without knowing the expected result, one can, in order to balance the $\Gamma(-n)$ on the denominator, choose $\alpha=1$ and $\beta=0$ to obtain

$$\sin(bx)=\sum_n\phi_n\frac{b^n\pi}{\Gamma\left(-\frac{n}{2}\right)\Gamma\left(\frac{n}{2}+1\right)}x^n=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\sin\left(\frac{n\pi}{2}\right)}{n!}(bx)^n, \tag{7.1.15}$$

which is the same and correct result.

7.2 Further Application on Product of Functions

Assume that in the process of evaluation of the integral

$$I=\int_0 f_1(x)\,f_2(x)\,dx, \tag{7.2.1}$$

one knows an expansion of $f_1(x)$ in the form

$$f_1(x)=\sum_{k=0}^{\infty}\phi_k A(k)\,x^{\alpha_1 k+\beta_1}, \tag{7.2.2}$$

and the Mellin transform of the function $f_2(x)$

$$\mathcal{M}(s)=\mathcal{M}(f_2)(s)=\int_0^{\infty}x^{s-1}f_2(x)\,dx. \tag{7.2.3}$$

From (7.1.4),

$$f_2(x)=\sum_{n=0}^{\infty}\phi_n\frac{|\alpha_2|\,\mathcal{M}(-\alpha_2 n-\beta_2)}{\Gamma(-n)}x^{\alpha_2 n+\beta_2}, \tag{7.2.4}$$

which further leads to

$$I=\sum_{k,n}\phi_{k,n}\frac{|\alpha_2|\,A(k)\,\mathcal{M}(-\alpha_2 n-\beta_2)}{\Gamma(-n)}\langle\alpha_1 k+\alpha_2 n+\beta_1+\beta_2+1\rangle. \tag{7.2.5}$$

Choosing k or n as the free parameter provides to the following theorem.

Theorem 7.2.1. *Given that*

$$f_1(x) = \sum_{k=0}^{\infty} \phi_k A(k)\, x^{\alpha_1 k + \beta_1}, \tag{7.2.6}$$

and

$$\mathcal{M}(s) = \mathcal{M}(f_2)(s) = \int_0^\infty x^{s-1} f_2(x)\, dx, \tag{7.2.7}$$

we have

$$I := \int_0^\infty f_1(x) f_2(x)\, dx = \sum_k \phi_k A(k)\, \mathcal{M}(\alpha_1 k + \beta_1 + 1), \tag{7.2.8}$$

and

$$I = \left|\frac{\alpha_2}{\alpha_1}\right| \sum_n \frac{\phi_n A\left(-\frac{\alpha_2 n + \beta_1 + \beta_2 + 1}{\alpha_1}\right) \mathcal{M}(-\alpha_2 n - \beta_2)\, \Gamma\left(\frac{\alpha_2 n + \beta_1 + \beta_2 + 1}{\alpha_1}\right)}{\Gamma(-n)}. \tag{7.2.9}$$

Example 7.2.2. Entry **3.893.1** in [31] is

$$I = \int_0^\infty e^{-px} \sin(qx + \lambda)\, dx = \frac{1}{p^2 + q^2} (q \cos\lambda + p \sin\lambda) \text{ with } \mathrm{Re} p > 0. \tag{7.2.10}$$

Taking $\lambda = 0$, $q = b$ and $p = a > 0$ to get an easy version:

$$I = \int_0^\infty e^{-ax} \sin(bx)\, dx = \frac{b}{a^2 + b^2}. \tag{7.2.11}$$

Note that (7.1.9) implies

$$\begin{cases} A(k) = a^k, \\ \alpha_1 = 1, \\ \beta_1 = 0, \end{cases} \tag{7.2.12}$$

and we also have (3.1.2). By Theorem 7.2.1,

$$I = \sum_k \phi_k \frac{a^k \pi \Gamma(k+1)}{b^{k+1} \Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{1-k}{2}\right)} = \frac{1}{b} \sum_k \frac{\pi}{\Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{1-k}{2}\right)} \left(-\frac{a}{b}\right)^k. \tag{7.2.13}$$

Since terms for odd k vanish due to $\Gamma\left(\frac{1-k}{2}\right)$, we assume $k = 2j$. Note that

$$\Gamma\left(\frac{2j+1}{2}\right) \Gamma\left(\frac{1-2j}{2}\right) = \frac{\pi}{\cos(\pi j)} = (-1)^j \pi. \tag{7.2.14}$$

Thus,

$$I = \frac{1}{b} \sum_j \left(-\frac{a^2}{b^2} \right)^j = \frac{b}{a^2 + b^2}, \tag{7.2.15}$$

provided that $|a| < |b|$. On the other hand,

$$I = |\alpha_2| \sum_n \frac{\phi_n a^{-(\alpha_2 n + \beta_2 + 1)} \mathcal{M}\left(-\alpha_2 n - \beta_2\right) \Gamma\left(\alpha_2 n + \beta_2 + 1\right)}{\Gamma\left(-n\right)}, \tag{7.2.16}$$

with

$$\mathcal{M}\left(-\alpha_2 n - \beta_2\right) = \frac{\sqrt{\pi}}{2} \left(\frac{b}{2} \right)^{\alpha_2 n + \beta_2} \frac{\Gamma\left(\frac{-\alpha_2 n - \beta_2 + 1}{2}\right)}{\Gamma\left(1 + \frac{\alpha_2 n + \beta_2}{2}\right)}. \tag{7.2.17}$$

To balance the pole of $\Gamma\left(-n\right)$, one could choose $\alpha_2 = 2$ and $\beta_1 = 1$, which also satisfies the power series of sine function. Then,

$$I = 2 \sum_n \frac{\phi_n a^{-2n-2} \Gamma\left(2n+2\right)}{\Gamma\left(n + \frac{3}{2}\right)} \left(\frac{b}{2} \right)^{2n+1} \frac{\sqrt{\pi}}{2}. \tag{7.2.18}$$

Note that

$$\Gamma\left(n + \frac{3}{2}\right) = \frac{\Gamma\left(2n+3\right)}{2^{2n+2}\Gamma\left(n+2\right)} \sqrt{\pi}. \tag{7.2.19}$$

Therefore,

$$\begin{aligned} I &= \frac{b}{a^2} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma\left(2n+2\right) 2^{2n+2} \Gamma\left(n+2\right)}{\Gamma\left(n+1\right) \Gamma\left(2n+3\right) 2^{2n+1}} \left(\frac{b}{a} \right)^{2n} \\ &= \frac{b}{a^2} \sum_{n=0}^{\infty} \left(-\frac{b^2}{a^2} \right)^n \\ &= \frac{b}{a^2 + b^2}, \end{aligned} \tag{7.2.20}$$

provided that $|a| > |b|$. So the formula is confirmed.

Example 7.2.3. Entry **6.512.1** in [31]

$$I = \int_0^\infty J_\mu\left(ax\right) J_\nu\left(bx\right) dx \tag{7.2.21}$$

involves the Bessel function $J_\beta\left(x\right)$ with series expansion

$$J_\beta\left(x\right) = \sum_{k=0}^{\infty} \frac{\phi_k}{\Gamma\left(k + \beta + 1\right) 2^{2k+\beta}} x^{2k+\beta}. \tag{7.2.22}$$

Then,

$$f_1(x) = J_\mu(ax) = \sum_{k=0}^{\infty} \frac{\phi_k a^{2k+\mu}}{\Gamma(k+\mu+1)\,2^{2k+\mu}} x^{2k+\mu}, \tag{7.2.23}$$

so that $\alpha_1 = 2$, $\beta_1 = \mu$ and $A(k) = \frac{a^{2k+\mu}}{\Gamma(k+\mu+1)2^{2k+\mu}}$. The second factor is $f_2(x) = J_\nu(bx)$ and its Mellin transform is given as entry **6.561.14** in [31] that

$$\mathcal{M}(x) = \mathcal{M}(J_\nu(bx))(s) = \int_0^\infty x^{s-1} J_\nu(x)\,dx = \frac{2^{s-1}\Gamma\left(\frac{\nu+s}{2}\right)}{b^s\Gamma\left(1+\frac{\nu-s}{2}\right)}. \tag{7.2.24}$$

Theorem 7.2.1 gives

$$\begin{aligned} I &= \sum_{k=0}^{\infty} \phi_k A(k)\,\mathcal{M}(\alpha_1 k + \beta + 1) \\ &= \sum_{k=0}^{\infty} \phi_k \frac{a^{2k+\mu}}{\Gamma(k+\mu+1)\,2^{2k+\mu}} \cdot \frac{2^{2k+\mu}\Gamma\left(k+\frac{\mu+\nu+1}{2}\right)}{b^{2k+\mu+1}\Gamma\left(-k+\frac{\nu-\mu+1}{2}\right)} \\ &= a^\mu b^{-\mu-1} \sum_{k=0}^{\infty} \frac{\phi_k \Gamma\left(k+\frac{\mu+\nu+1}{2}\right)}{\Gamma(k+\mu+1)\,\Gamma\left(-k+\frac{\nu-\mu+1}{2}\right)} \cdot \left(\frac{a^2}{b^2}\right)^k. \end{aligned} \tag{7.2.25}$$

Note that

$$\begin{cases} \Gamma\left(k+\frac{\mu+\nu+1}{2}\right) = \Gamma\left(\frac{\mu+\nu+1}{2}\right) \cdot \left(\frac{\mu+\nu+1}{2}\right)_k, \\ \Gamma(k+\mu+1) = \Gamma(\mu+1) \cdot (\mu+1)_k, \end{cases} \tag{7.2.26}$$

and especially, by using (4.2.10) and (6.1.2)

$$\Gamma\left(-k+\frac{\nu-\mu+1}{2}\right) = \frac{\Gamma\left(\frac{\nu-\mu+1}{2}\right)(-1)^k}{\left(\frac{\mu-\nu+1}{2}\right)_k}. \tag{7.2.27}$$

Therefore,

$$\begin{aligned} I &= a^\mu b^{-\mu-1} \sum_{k=0}^{\infty} \frac{\phi_k \Gamma\left(k+\frac{\mu+\nu+1}{2}\right)}{\Gamma(k+\mu+1)\,\Gamma\left(-k+\frac{\nu-\mu+1}{2}\right)} \cdot \left(\frac{a^2}{b^2}\right)^k \\ &= a^\mu b^{-\mu-1} \frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma(\mu+1)\,\Gamma\left(\frac{\nu-\mu+1}{2}\right)} \sum_{k=0}^{\infty} \frac{\left(\frac{\mu+\nu+1}{2}\right)_k \cdot \left(\frac{\mu-\nu+1}{2}\right)_k}{(\mu+1)_k \cdot k!} \left(\frac{a^2}{b^2}\right)^k \\ &= a^\mu b^{-\mu-1} \frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma(\mu+1)\,\Gamma\left(\frac{\nu-\mu+1}{2}\right)}. \end{aligned} \tag{7.2.28}$$

This is convergent for $b > a$. On the other hand,

$$I = \frac{|\alpha_2|}{2} \sum_{n=0}^{\infty} \frac{\phi_n}{\Gamma(-n)} A\left(-\frac{\alpha_2 n + \mu + \beta_2 + 1}{2}\right) \mathcal{M}(-\alpha_2 n - \beta_2)\,\Gamma\left(\frac{\alpha_2 n + \mu + \beta_2 + 1}{2}\right). \tag{7.2.29}$$

Either to balance the singular term $\Gamma(-n)$ or to recover the series expansion of Bessel function, we choose $\alpha_2 = 2$ and $\beta_2 = \nu$. Then

$$\begin{aligned} I &= \sum_{n=0}^{\infty} \frac{\phi_n}{\Gamma(-n)} A\left(-n - \frac{\mu+\nu+1}{2}\right) \mathcal{M}(-2n-\nu)\,\Gamma\left(n + \frac{\mu+\nu+1}{2}\right) \qquad (7.2.30)\\ &= \sum_{n=0}^{\infty} \frac{\phi_n a^{-2n-\nu-1} 2^{-2n-\nu-1}\Gamma(-n)\,\Gamma\left(n+\frac{\mu+\nu+1}{2}\right)}{\Gamma(-n)\,\Gamma\left(-n+\frac{\mu-\nu+1}{2}\right) 2^{-2n-\nu-1} b^{-2n-\nu}\Gamma(n+\nu+1)} \\ &= b^{\nu} a^{-\nu-1} \sum_{n=0}^{\infty} \frac{\phi_n \Gamma\left(n+\frac{\mu+\nu+1}{2}\right)}{\Gamma(n+\nu+1)\,\Gamma\left(-n+\frac{\mu-\nu+1}{2}\right)} \cdot \left(\frac{b^2}{a^2}\right)^n \\ &= b^{\nu} a^{-\nu-1} \frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma(\nu+1)\,\Gamma\left(\frac{\mu-\nu+1}{2}\right)}\ {}_2F_1\left(\left.\begin{matrix}\frac{\mu+\nu+1}{2}, \frac{\nu-\mu+1}{2} \\ \nu+1\end{matrix}\right| \frac{b^2}{a^2}\right), \end{aligned}$$

which is convergent if $a > b$.

Chapter 8

Comparison with Other Integration Methods

8.1 Negative Dimensional Integration Method (NDIM)

The Negative Dimensional Integration Method (NDIM) comes from evaluating D-dimensional Feynman loop integrals. It was first devised by Halliday et al. [19, 30] aiming at Feynman loop integrals in quantum field theory (QFT) and has been developed by A. T. Suzuki [58, 59, 60, 61, 62]. Therefore, NDIM specifically deals with integrals of a certain type. Similarly as Ramanujan's master theorem being the core of the method of brackets, NDIM is based on the Gaussian integral:

$$\int_{\mathbb{R}^D} e^{-\alpha \boldsymbol{x}^2} d^D \boldsymbol{x} = \left(\frac{\pi}{\alpha}\right)^{\frac{D}{2}}, \tag{8.1.1}$$

where we denote

$$\begin{cases} \mathbb{R}^D & = \{x_1, x_2, \ldots, x_D\}, \\ \boldsymbol{x}^2 & = x_1^2 + x_2^2 + \cdots + x_D^2, \\ d^D \boldsymbol{x} & = dx_1 dx_2 \cdots dx_D. \end{cases} \tag{8.1.2}$$

Basic requirement of analytic continuation (AC) is applied to perform

$$\int \frac{d^D\boldsymbol{x}}{A \cdot B \cdot C} \xrightarrow{AC} \int d^D\boldsymbol{x}\,(A \cdot B \cdot C)\,, \tag{8.1.3}$$

which one can see easily from the following examples.

Example 8.1.1. Recall

$$\int_0^\infty \frac{1}{1+x^2}dx = \frac{\pi}{2}. \tag{8.1.4}$$

First of all, we notice that

$$\int_0^\infty \frac{1}{1+x^2}dx = \frac{1}{2}\int_{\mathbb{R}} \frac{1}{1+x^2}dx. \tag{8.1.5}$$

Now, consider the following integral

$$I = \int_{\mathbb{R}} e^{-\alpha\left(1+x^2\right)}dx. \tag{8.1.6}$$

On one hand, applying (8.1.1) for $D = 1$ to get

$$I = e^{-\alpha}\int_{\mathbb{R}} e^{-\alpha x^2}dx = \sqrt{\frac{\pi}{\alpha}}e^{-\alpha} = \sqrt{\pi}\sum_{m=0}^{\infty}\phi_m\alpha^{m-\frac{1}{2}}. \tag{8.1.7}$$

On the other hand, expanding the integrand directly yields

$$I = \int_{\mathbb{R}}\sum_{n=0}^{\infty}\phi_n\alpha^n\left(1+x^2\right)^n dx = \sum_{n=0}^{\infty}\phi_n\alpha^n\int_{\mathbb{R}}\left(1+x^2\right)^n dx. \tag{8.1.8}$$

Denote that

$$I_n := \int_{\mathbb{R}}\left(1+x^2\right)^n dx, \tag{8.1.9}$$

we equate the two expressions to get

$$\sqrt{\pi}\sum_{m=0}^{\infty}\phi_m\alpha^{m-\frac{1}{2}} = \sum_{n=0}^{\infty}\phi_n I_n\alpha^n. \tag{8.1.10}$$

Comparing the coefficients of α to obtain that $m - \frac{1}{2} = n$ and

$$I_n = \sqrt{\pi}\frac{\phi_{n+\frac{1}{2}}}{\phi_n} = \sqrt{\pi}\,(-1)^{\frac{1}{2}}\,\frac{\Gamma\left(n+1\right)}{\Gamma\left(n+\frac{3}{2}\right)}. \tag{8.1.11}$$

Notice that eventually, we shall evaluate I_n for $n = -1$. Therefore, analytic continuation is required to transform the result into that is valid for negative integers n. Noting that

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)} = \frac{1}{(n+1)_{\frac{1}{2}}} = \frac{(-n)_{-\frac{1}{2}}}{(-1)^{-\frac{1}{2}}}, \tag{8.1.12}$$

one can see, by applying (6.1.2),

$$I_n = \sqrt{\pi}\,(-n)_{-\frac{1}{2}} = \sqrt{\pi}\frac{\Gamma\left(-\frac{1}{2}-n\right)}{\Gamma(-n)}. \tag{8.1.13}$$

Let $n = -1$ to get

$$\int_0^\infty \frac{1}{1+x^2}dx = \frac{1}{2}I_{-1} = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \frac{\pi}{2}. \tag{8.1.14}$$

In fact, NDIM provides the general formula that

$$\int_0^\infty \frac{1}{(1+x^2)^n}dx = \frac{1}{2}I_{-n} = \frac{\sqrt{\pi}\Gamma\left(-\frac{1}{2}+n\right)}{2\Gamma(n)}. \tag{8.1.15}$$

For instance, the cases for $n = 2$ and $n = 3$ are

$$\begin{cases} \int_0^\infty \frac{1}{(1+x^2)^2}dx & = \frac{\sqrt{\pi}\Gamma\left(\frac{3}{2}\right)}{2\Gamma(2)} = \frac{\pi}{4}, \\ \int_0^\infty \frac{1}{(1+x^2)^3}dx & = \frac{\sqrt{\pi}\Gamma\left(\frac{5}{2}\right)}{2\Gamma(3)} = \frac{3\pi}{16}. \end{cases} \tag{8.1.16}$$

Remark 8.1.2. The method of brackets can also handle these integrals easily through

$$\begin{aligned} \int_0^\infty \frac{1}{(1+x^2)^n}dx &\overset{P_2}{=} \int_0^\infty \sum_{k,l}\phi_{k,l}x^{2l}\frac{\langle n+k+l\rangle}{\Gamma(n)}dx \\ &\overset{P_1}{=} \frac{1}{\Gamma(n)}\sum_{k,l}\phi_{k,l}\langle n+k+l\rangle\langle 2l+1\rangle \\ &\overset{E_2}{=} \frac{1}{\Gamma(n)}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right) \\ &= \frac{\sqrt{\pi}\Gamma\left(-\frac{1}{2}+n\right)}{2\Gamma(n)}, \end{aligned} \tag{8.1.17}$$

which is exactly the same result by using NDIM.

The next example contains more than one factor.

Example 8.1.3. We consider another integral that is also mentioned in [62]:

$$\int_0^\infty \frac{x^2 dx}{\left(1+x^2\right)^3} = \frac{\pi}{16}. \tag{8.1.18}$$

Consider

$$I = \int_{\mathbb{R}} e^{-\alpha x^2 - \beta\left(1+x^2\right)} dx, \tag{8.1.19}$$

Two ways of expanding this integral result in both

$$\begin{aligned} I &= e^{-\beta} \int_{\mathbb{R}} e^{-(\alpha+\beta)x^2} dx \\ &= \left(\sum_a \phi_a \beta^a\right) \sqrt{\frac{\pi}{\alpha+\beta}} \\ &= \sqrt{\pi} \sum_a \phi_a \beta^a \sum_{b+c=-\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(b+1)\Gamma(c+1)} \alpha^b \beta^c \\ &= \pi \sum_{a,b} \frac{\phi_a}{\Gamma(b+1)\Gamma\left(\frac{1}{2}-b\right)} \alpha^b \beta^{a-\frac{1}{2}-b}, \end{aligned} \tag{8.1.20}$$

and

$$I = \sum_{m,n} \phi_{m,n} \alpha^m \beta^n \int_{\mathbb{R}} x^{2m} \left(1+x^2\right)^n dx. \tag{8.1.21}$$

Comparing coefficients of the pair (α, β) leads to the following equation and solution

$$(m,n) = \left(b, a - \frac{1}{2} - b\right) \Rightarrow (a,b) = \left(m+n+\frac{1}{2}, m\right) \tag{8.1.22}$$

Therefore, we obtain again the general formula

$$\begin{aligned} I(m,n) &:= \int_{\mathbb{R}} x^{2m} \left(1+x^2\right)^n dx \\ &= \frac{\pi \phi_{n+m+\frac{1}{2}}}{\phi_{m,n} \Gamma(m+1) \Gamma\left(\frac{1}{2}-m\right)} \\ &= \frac{\pi (-1)^{\frac{1}{2}} \Gamma(n+1)}{\Gamma\left(\frac{1}{2}-m\right) \Gamma\left(n+m+\frac{3}{2}\right)} \\ &= \frac{\pi (-1)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-m\right) (n+1)_{m+\frac{1}{2}}} \\ &= \frac{\pi (-1)^m (-n)_{-m-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-m\right)}. \end{aligned} \tag{8.1.23}$$

This allows us to plug in $(m,n)=(2,-3)$ and to obtain

$$\int_0^\infty \frac{x^2dx}{\left(1+x^2\right)^3} = \frac{1}{2}I\left(1,-3\right) = \pi\frac{\left(-1\right)\left(3\right)_{-\frac{5}{2}}}{2\Gamma\left(-\frac{1}{2}\right)} = -\frac{\pi\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(-\frac{1}{2}\right)\Gamma\left(3\right)} = \frac{\pi}{16}. \tag{8.1.24}$$

Remark 8.1.4. (1) Examples containing more factors are also presented in [62].

(2) The method of brackets can also directly handle the example above as follows.

$$\begin{aligned}
\int_0^\infty \frac{x^2dx}{\left(1+x^2\right)^3} &= \sum_{k,l}\phi_{k,l}x^{2l+2}\frac{\langle 3+k+l\rangle}{\Gamma\left(3\right)}dx \qquad (8.1.25)\\
&= \frac{1}{2}\sum_{k,l}\phi_{k,l}\left\langle k+l+3\right\rangle\left\langle 2l+3\right\rangle\\
&= \frac{1}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{3}{2}\right)^2 = \frac{\pi}{16}.
\end{aligned}$$

8.2 Integrate by Differentiating (IBD)

In [42, 43], A. Kempf et al. study Dirac-delta function to obtain the following formulas:

$$\begin{aligned}
\int_a^b f\left(x\right)dx &= \lim_{\varepsilon\to 0} f\left(\partial_\varepsilon\right)\frac{e^{\varepsilon b}-e^{\varepsilon a}}{\varepsilon}, &(8.2.1)\\
\int_{-\infty}^\infty f\left(x\right)dx &= \lim_{\varepsilon\to 0} 2\pi f\left(-\iota\partial_\varepsilon\right)\delta\left(\varepsilon\right) = 2\pi\delta\left(\iota\partial_\varepsilon\right)f\left(\varepsilon\right), &(8.2.2)\\
\int_0^\infty f\left(x\right)dx &= \lim_{\varepsilon\to 0} f\left(-\partial_\varepsilon\right)\frac{1}{\varepsilon}, &(8.2.3)\\
\int_{-\infty}^0 f\left(x\right)dx &= \lim_{\varepsilon\to 0} f\left(\partial_\varepsilon\right)\frac{1}{\varepsilon}, &(8.2.4)\\
\int_{-\infty}^\infty f\left(x\right)dx &= \lim_{\varepsilon\to 0} \left[f\left(-\partial_\varepsilon\right)+f\left(\partial_\varepsilon\right)\right]\frac{1}{\varepsilon}, &(8.2.5)
\end{aligned}$$

where ∂_ε denotes the derivative with respect to ε. Among these formulas above, (8.2.3) deals with the same integral as the method of brackets and definitely draws our attention. The following two examples directly come from [43].

Example 8.2.1. Entry **3.721** of [31] provides, with a special case,

$$I = \int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}. \tag{8.2.6}$$

Express that

$$f(x) = \frac{1}{x} \cdot \frac{1}{2\iota} \left(e^{\iota x} - e^{-\iota x} \right), \tag{8.2.7}$$

to see

$$I = \lim_{\varepsilon \to 0} f(-\partial_\varepsilon) \frac{1}{\varepsilon} = \frac{1}{2\iota} \lim_{\varepsilon \to 0} \left(e^{-\iota \partial_\varepsilon} - e^{\iota \partial_\varepsilon} \right) \frac{1}{\partial_\varepsilon} \circ \frac{1}{\epsilon}. \tag{8.2.8}$$

Note that $1/\partial_\varepsilon$ is the inverse operation of derivative, i.e., integration. One can see

$$I = \frac{1}{2\iota} \lim_{\varepsilon \to 0} \left(e^{-\iota \partial_\varepsilon} - e^{\iota \partial_\varepsilon} \right) \circ (\ln \varepsilon + c), \tag{8.2.9}$$

for some constant c. Recall that for the derivative operator ∂_ε, it is also connected to the forward translation operator via

$$e^{a \partial_\varepsilon} \circ f(\varepsilon) = f(\varepsilon + a). \tag{8.2.10}$$

Thus, we have

$$\begin{aligned} I &= \frac{1}{2\iota} \lim_{\varepsilon \to 0} \left[(\ln(\varepsilon - \iota) + c) - (\ln(\varepsilon + \iota) + c) \right] \\ &= \frac{1}{2\iota} \lim_{\varepsilon \to 0} \left[\ln(\varepsilon - \iota) - \ln(\varepsilon + \iota) \right] \\ &= \frac{1}{2\iota} \left(\frac{-\iota \pi}{2} - \frac{\iota \pi}{2} \right) \\ &= \frac{\pi}{2}. \end{aligned} \tag{8.2.11}$$

Remark 8.2.2. The method of brackets works as well.

$$\begin{aligned} I &= \int_0^\infty \frac{1}{x} \sum_n \phi_n \frac{\Gamma(n+1)}{\Gamma(2n+2)} x^{2n+1} dx \\ &= \sum_n \phi_n \frac{\Gamma(n+1)}{\Gamma(2n+2)} \langle 2n+1 \rangle \\ &= \frac{1}{2} \cdot \frac{\Gamma\left(-\frac{1}{2}+1\right)}{\Gamma\left(2\left(-\frac{1}{2}\right)+2\right)} \Gamma\left(\frac{1}{2}\right) \\ &= \frac{\pi}{2}. \end{aligned} \tag{8.2.12}$$

The next examples shows that when dealing with non-entire function, IBD encounters some trouble, even if the integrand is simple.

Example 8.2.3. Recall the elementary integral that

$$I=\int_0^{\infty}\frac{1}{1+x^2}dx=\frac{\pi}{2}. \tag{8.2.13}$$

The integrand is not entire on $\mathbb{C}$ due to poles at $\{\pm\iota\}$. We have to split the integral as

$$I=\int_0^{\infty}\frac{1}{1+x^2}dx=\int_0^{1}\frac{1}{1+x^2}dx+\int_1^{\infty}\frac{1}{1+x^2}dx. \tag{8.2.14}$$

Note that, by the change of variables

$$t=\frac{1}{x}, \tag{8.2.15}$$

the two parts are actually the same since

$$\int_1^{\infty}\frac{1}{1+x^2}dx=\int_1^{0}\frac{1}{1+\frac{1}{t^2}}\left(-\frac{1}{t^2}\right)dt=\int_0^{1}\frac{1}{1+t^2}dt. \tag{8.2.16}$$

Therefore, we have

$$I=2\int_0^{1}\frac{1}{1+x^2}dx. \tag{8.2.17}$$

Thus, we have to apply the general formula (8.2.1) to get

$$\begin{aligned}
I &= 2\lim_{\varepsilon\to 0}\frac{1}{1+\partial_\varepsilon^2}\circ\frac{e^{\varepsilon}-1}{\varepsilon} \qquad (8.2.18)\\
&= 2\lim_{\varepsilon\to 0}\sum_{n=0}^{\infty}(-1)^n\left(\partial_\varepsilon^{2n}\right)\circ\sum_{m=0}^{\infty}\frac{\varepsilon^m}{(m+1)!}\\
&= 2\lim_{\varepsilon\to 0}\sum_{n=0}^{\infty}(-1)^n\left(\partial_\varepsilon^{2n}\right)\circ\sum_{m=2n}^{\infty}\frac{\varepsilon^m}{(m+1)!}\\
&= 2\lim_{\varepsilon\to 0}\sum_{n=0}^{\infty}(-1)^n\sum_{m=0}^{\infty}\frac{\varepsilon^{m-2n}}{(m+1)(m-2n)!}\\
&= 2\sum_{n=0}^{\infty}(-1)^n\lim_{\varepsilon\to 0}(-x)^{-2n-1}\gamma(2n+1,0,-x)\\
&= 2\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\\
&= 2\tan^{-1}(1)\\
&= \frac{\pi}{2}.
\end{aligned}$$

To our surprise, considering the following formal proof, i.e., allowing interchanging limits and ignoring convergence, IBD does have connection to the method of brackets.

Recall that a formal proof of rule P_1 shows that if

$$f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n + \beta - 1}, \tag{8.2.19}$$

then,

$$\sum_n a_n \langle \alpha n + \beta \rangle = \lim_{\varepsilon \to 0} \sum_{n=0}^{\infty} a_n \langle \alpha n + \beta \rangle_{\varepsilon} = \lim_{\varepsilon \to 0} \int_0^{\infty} e^{-\varepsilon x} f(x)\, dx. \tag{8.2.20}$$

On the other hand,

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_0^{\infty} e^{-\varepsilon x} f(x)\, dx &= \lim_{\varepsilon \to 0} \int_0^{\infty} e^{-\varepsilon x} \left(\sum_{n=0}^{\infty} a_n x^{\alpha n + \beta - 1} \right) dx \\ &= \lim_{\varepsilon \to 0} \sum_{n=0}^{\infty} a_n \int_0^{\infty} e^{-\varepsilon x} x^{\alpha n + \beta - 1} dx \\ &= \lim_{\varepsilon \to 0} \sum_{n=0}^{\infty} a_n \int_0^{\infty} \left((-\partial_{\varepsilon})^{\alpha n + \beta - 1} \circ e^{-\varepsilon x} \right) dx \\ &= \lim_{\varepsilon \to 0} \sum_{n=0}^{\infty} a_n (-\partial_{\varepsilon})^{\alpha n + \beta - 1} \circ \int_0^{\infty} e^{-\varepsilon x} dx \\ &= \lim_{\varepsilon \to 0} f(-\partial_{\varepsilon}) \circ \frac{1}{\varepsilon}. \end{aligned} \tag{8.2.21}$$

8.3 Conclusions

Based on the two sections above, we could conclude the following.

With Negative Dimensional Integration Method (NDIM):

(1) NDIM relies on the Gaussian integral (8.1.1) instead of Ramanujan's Master Theorem, so it has completely different core to the method of brackets.

(2) NDIM only deals with certain type of integrals which can also be handled by the method of brackets.

(3) NDIM interchanges the sum and integral always, which makes it completely formal.

(4) It is very important that NDIM requires analytic continuation in almost every computation of integrals, which cannot be completely fulfilled by computer algebra while the method of brackets does not always requires unless special divergent series is encountered.

With Integration by Differentiating (IBD):

(1) Formal statement shows that Laplace transform could connect both IBD and the method of brackets.

(2) IBD fails or is not convenient to handle non-entire functions.

(3) IBD has obvious connection with operator algebra, which deserves further discussion, especially for the case when the interval for integration is $[a, b]$.

Chapter 9

Mathematical Package on the Method of Brackets

9.1 Introduction

As we have seen from Section 2.2, the rules of the method of brackets are simple, direct and suitable for computer program implementation. In fact, K. Kohl [45] has already established her packages using `Sage` with internal use of `Mathematica`. For these year, it have been developing dramatically for free, open-sourced computer languages, especially for `Sage` whose core is `Python`. However, only using `Sage` is not enough now. As we have seen from examples above, evaluations of the bracket series usually result in series. Simplification for those series in terms of special functions requires not only valid algorithms but also huge database. Being a commercial computer software for years, `Mathematica` definitely has the advantage on this aspect, which explains why Kohl's implementation calls `Mathematica` when evaluating bracket series.

The following example shows the advantage of `Mathematica` on handling series expressions clearly.

Example 9.1.1. In [7], we study the following computation

$$\sum_{n=0}^{\infty} \frac{z^n}{C_n} = {}_2F_1 \left(\left. \begin{matrix} 1, 2 \\ \frac{1}{2} \end{matrix} \right| \frac{z}{4} \right), \tag{9.1.1}$$

which can be obtained easily from `Mathematica`:

```
In[3]:= Sum[z^n/CatalanNumber[n],{n,0,Infinity}]

Out[3]= Hypergeometric[1,2,1/2,z/4]
```

While, on the other hand, `Sage` originally provides the answer that

$$\begin{aligned} \frac{1}{16\left[x^2\left(-\frac{1}{4}x+1\right)^{\frac{5}{2}} - 8x\left(-\frac{1}{4}x+1\right)^{\frac{5}{2}} + 16\left(-\frac{1}{4}x+1\right)^{\frac{5}{2}}\right]} &\times \Bigg\{3x^{\frac{7}{2}}\sin^{-1}\left(\frac{\sqrt{x}}{2}\right) - 3\Bigg[x^{\frac{3}{2}}\sin^{-1}\left(\frac{\sqrt{x}}{2}\right) \\ +12\sqrt{x}\sin^{-1}\left(\frac{\sqrt{x}}{2}\right)\Bigg]x^2 - 256\left(-\frac{1}{4}x+1\right)^{\frac{5}{2}} &- 48x^{\frac{3}{2}}\sin^{-1}\left(\frac{\sqrt{x}}{2}\right) - 8\Bigg(4\left(-\frac{1}{4}x+1\right)^{\frac{5}{2}} \\ -x^{\frac{3}{2}}\sin^{-1}\left(\frac{\sqrt{x}}{2}\right) - 18\sqrt{x}\sin^{-1}\left(\frac{\sqrt{x}}{2}\right)\Bigg)x &- 192\sqrt{x}\sin^{-1}\left(\frac{\sqrt{x}}{2}\right)\Bigg\}, \end{aligned}$$

and it cannot be simplified by `Sage` itself.

In early versions of `Sage` (version 5) and `Mathematica` (version 8), the interface to `Mathematica` in `Sage` can be established. However, such an interface between them fail for higher versions of `Mathematica`, for instance 9 and 10. Realizing the necessity of applying `Mathematica` for current version of `Sage` and impracticability for forcing users to use early versions, a pure `Mathematica` package becomes the solution to this problem.

9.2 Examples as Manual of Package

The `Mathematica` package mainly contains three commands:

- **MakeTheBracket**: By applying rules P_1 and P_2, it expresses the integral as the bracket series. It also provides the linear system of brackets.

- **EvaluateTheBracket**: From the bracket series that contains the linear system and index, it applies E_1 and E_2 for all possible free parameter(s).

- **TheMethodofBrackets**: This basically combines the first two command. In addition, all important information during the process of computation such as linear system, all brackets, index, number of sums, etc. is also provided as part of the output.

The examples presented here are all from [26].

9.2.1 Examples of Index 0

Example 9.2.1. The trivial example would be the definition of gamma function that

$$\Gamma(a) := I_a = \int_0^\infty x^{a-1} e^{-x} dx. \tag{9.2.1}$$

Step 1. Production of the bracket series

```
In[4]:= MakeTheBracket[{x^(a-1)Exp[-x]}]
```

Out[4]= $\{\frac{(-1)^{n[1]}}{\Gamma(n[1]+1)}\}, \{a+n[1]\}, 1, 0, \{\{1\}\}, \{\{a\}\}\}$

This shows:

(i) the function in the bracket series

$$f(n_1) = \frac{(-1)^{n_1}}{\Gamma(n_1+1)} = \phi_{n_1}; \tag{9.2.2}$$

(ii) the single bracket is

$$\langle n_1 + a \rangle; \tag{9.2.3}$$

(iii) the number of sum is 1 and the index is 0;

(iv) the linear system is

$$1 \cdot n_1 + a = 0. \tag{9.2.4}$$

Easily, we can have verify it through the computation:

$$I_a = \int_0^\infty x^{a-1} \sum_{n_1=0}^{\infty} \phi_{n_1} x^{n_1} dx = \sum_{n_1} \phi_{n_1} \langle n_1 + a \rangle. \tag{9.2.5}$$

Step 2. Evaluation of the bracket series

```
In[5]:=EvaluateTheBracket[(-1)^n[1]/Gamma[1+n[1]],
      {a+n[1]},1,0,{{1}},{{a}}]
```

Out[5]= $\Gamma(a)$

In fact, if one only wants the answer, we could use:

```
In[6]:= TheMethodofBrackets[{x^(a-1)Exp[-x]}]
```

Out[6]= $\Gamma(a)$

Evaluating the bracket series is straight forward:

$$I_a = \sum_{n_1} \phi_{n_1} \langle n_1 + a \rangle = \Gamma(a). \tag{9.2.6}$$

Example 9.2.2. Consider

$$J_{2,m} := \int_0^\infty \frac{dx}{(1+x^2)^{m+1}} = \frac{\pi}{2^{2m+1}}\binom{2m}{m}. \tag{9.2.7}$$

Step 1. Production of the bracket series

```
In[7]:= MakeTheBracket[{1/(1+x^2)^(m+1)}]
```

Out[7]= $\{\frac{(-1)^{n[1]+n[2]}}{\Gamma(m+1)\Gamma(n[1]+1)\Gamma(n[2]+1)}\}, \{1+m+n[1]+n[2],$
$1+2n[2]\}, 2, 0, \{\{1,1\},\{0,2\},\{\{1+m\},\{1\}\}\}.$

The result shows that

(i) the function of the bracket series is

$$f(n_1,n_2) = \frac{(-1)^{n_1+n_2}\langle n_1+n_2+m+1\rangle \langle 2n_2+1\rangle}{\Gamma(m+1)\Gamma(n_1+1)\Gamma(n_2+1)} = \frac{\phi_{1,2}}{\Gamma(m+1)}; \tag{9.2.8}$$

(ii) the two brackets are

$$\langle n_1+n_2+m+1\rangle \langle 2n_2+1\rangle, \tag{9.2.9}$$

(iii) the number of sum is 2 and the index is 0;

(iv) the linear system from the brackets is

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} + \begin{pmatrix} m+1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{9.2.10}$$

Easily, we can verify it through the computation:

$$
\begin{aligned}
J_{2,m} &\overset{P_2}{=} \int_0^\infty \left(\sum_{n_1,n_2} \phi_{1,2} 1^{n_1} x^{2n_2} \frac{\langle m+1+n_1+n_2 \rangle}{\Gamma(m+1)} \right) dx \\
&= \int_0^\infty \sum_{n_1,n_2} \phi_{1,2} \frac{\langle m+1+n_1+n_2 \rangle}{\Gamma(m+1)} x^{2n_2} dx \\
&\overset{P_1}{=} \sum_{n_1,n_2} \phi_{1,2} \frac{\langle n_1+n_2+m+1 \rangle \langle 2n_2+1 \rangle}{\Gamma(m+1)}.
\end{aligned} \tag{9.2.11}
$$

Step 2. Evaluation of the bracket series

```
In[8]:=EvaluateTheBracket[{(-1)^(n[1]+n[2])/(Gamma[1+m]
       Gamma[1+n[1]]Gamma[1+n[2]]),{1+m+n[1]+n[2],1+2n[2]},
       2,0,{{1,1},{0,2}},{{1+m},{1}}}]
```

Out[8]= $\frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}(1+2m)\right)}{2\Gamma(m+1)}$

Or, simply use

```
In[9]:= TheMethodofBrackets[{1/(1+x^2)^(m+1)}]
```

Out[9]= $\frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}(1+2m)\right)}{2\Gamma(m+1)}$

Evaluating the bracket series is straight forward:

$$
\begin{aligned}
J_{2,m} &= \sum_{n_1,n_2} \phi_{1,2} \frac{\langle n_1+n_2+m+1 \rangle \langle 2n_2+1 \rangle}{\Gamma(m+1)} \\
&= \frac{1}{\left| \det \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \right|} \frac{1}{\Gamma(m+1)} \Gamma\left(m+\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right) \\
&= \frac{\sqrt{\pi}\Gamma\left(m+\frac{1}{2}\right)}{2\Gamma(m+1)}.
\end{aligned} \tag{9.2.12}
$$

The following identity guarantees the evaluation is valid

$$
\Gamma\left(\frac{1}{2}+m\right) = \frac{(2m)!}{2^{2m} m!}\sqrt{\pi}. \tag{9.2.13}
$$

9.2.2 Example of Positive Index

Example 9.2.3. The following integral

$$I = \int_0^\infty \frac{1}{1+e^{x^2}} dx = \frac{\sqrt{2}-1}{2}\zeta\left(\frac{1}{2}\right) \tag{9.2.14}$$

can be computed directly by `Mathematica`:

```
In[10]:= Integrate[{1/(1+Exp[x^2]),{x,0,Infinity}]
```

Out[10]= $-\frac{1}{2}(-1+\sqrt{2})\sqrt{\pi}\zeta[\frac{1}{2}]$

On the other hand, using the method of brackets, it gives

```
In[11]:= TheMethodofBrackets[{1/(1 + Exp[x^2])}]
```

Out[11]= $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\frac{(-1)^{n_1+n_2} n_2^{n_3}}{\Gamma(n_1+1)\Gamma(n_2+1)\Gamma(n_3+1)}$

{Choose n_1 as free parameter(s).}

$\left\{\frac{\iota(-1)^{n_1}\sqrt{\pi}}{2\sqrt{-n_1-1}}, \{\{n_1, 0, \infty\}\}\right\}$

Direct Computation

$-\frac{1}{2}(-1+\sqrt{2})\sqrt{\pi}\zeta[\frac{1}{2}]$

Using bracket.m

{$-\frac{1}{2}(-1+\sqrt{2})\sqrt{\pi}\zeta[\frac{1}{2}], -\frac{1}{2}(-1+\sqrt{2})\sqrt{\pi}\zeta[\frac{1}{2}]$, True, $\{n_1\}$}

{Choose n_2 as free parameter(s).}

$\left\{\frac{\iota(-1)^{n_2}\sqrt{\pi}}{2\sqrt{n_2}}, \{\{n_2, 0, \infty\}\}\right\}$

Direct Computation

$\sum_{n_2=0}^{\infty} \frac{\iota(-1)^{n_2}\sqrt{\pi}}{2\sqrt{n_2}}$

Using bracket.m

{Infinity, Infinity, False, $\{n_2\}$}

{Choose n_3 as free parameter(s).}

```
LinearSolve::nosol:Linear equation
encountered that has no solution.>>
{ComplexInfinity, {{n3, 0, ∞}}}
Direct Computation
Sum::div:  Sum does not converge.  >>
 ∞
 ∑ ComplexInfinity
n3=0
Using bracket.m
{Infinity, Infinity, False, {n3}}
```

There are a lot of output since the command provides almost all details of the process. From all the outputs above, we first could read that

$$
\begin{aligned}
I &= \sum_{n_1,n_2,n_3} \frac{(-1)^{n_1+n_2}\, n_2^{n_3} \left\langle \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle}{\Gamma(n_1+1)\,\Gamma(n_2+1)\,\Gamma(n_3+1)} \\
&= \sum_{n_1,n_2,n_3} \phi_{1,2,3} (-1)^{n_3} \langle n_1+n_2+1 \rangle \langle 2n_3+1 \rangle \\
&= \begin{cases} \sum_{n_1=0}^{\infty} \frac{\iota(-1)^{n_1}\sqrt{\pi}}{2\sqrt{-n_1-1}} & \text{if } n_1 \text{ is free,} \\ \sum_{n_2=0}^{\infty} \frac{\iota(-1)^{n_1}\sqrt{\pi}}{2\sqrt{n_2}} & \text{if } n_2 \text{ is free,} \\ \text{Linear system has no solution} & \text{if } n_3 \text{ is free,} \end{cases} \\
&= \begin{cases} \frac{\sqrt{2}-1}{2}\zeta\left(\frac{1}{2}\right) & \text{if } n_1 \text{ is free,} \\ \text{Divergent} & \text{if } n_2 \text{ is free,} \\ \text{Linear system has no solution} & \text{if } n_3 \text{ is free,} \end{cases} \\
&\overset{E_3}{=} \frac{\sqrt{2}-1}{2}\zeta\left(\frac{1}{2}\right).
\end{aligned} \tag{9.2.15}
$$

The integral is verified.

Remark 9.2.4. When calculating series, two ways are provided here, namely either by direct computation, or through a computation using the file 'bracket.m'. This file/package is established by K. Kohl [45], which is the part that `Sage` uses from `Mathematica` to deal with massive series representations. The direct computation is provided since in early versions of `Mathematica`, for example version 8, using 'bracket.m' leads to wrong answer that

$$\sum_{n_1=0}^{\infty} \frac{\iota\,(-1)^{n_1}\sqrt{\pi}}{2\sqrt{-n_1-1}} = \begin{cases} \frac{\sqrt{2}-1}{2}\zeta\left(\frac{1}{2}\right) & \text{by direct computation,} \\ \text{Infinity} & \text{by using bracket.m.} \end{cases} \tag{9.2.16}$$

Meanwhile, the advantage of 'bracket.m' is that it provides not only the result, but also the region of convergence, which one can see from the next example.

Example 9.2.5. Entry **6.574.1** shows for $0 < \alpha < \beta$,

$$I = \int_0^\infty x^{-\lambda} J_\nu(\alpha x)\, J_\mu(\beta x)\, dx = \frac{\alpha^\nu \Gamma\left(\frac{\nu+\mu-\lambda+1}{2}\right) {}_2F_1\left(\left.\begin{matrix} \frac{\nu+\mu-\lambda+1}{2}, \frac{-\nu+\mu+\lambda+1}{2} \\ \nu+1 \end{matrix}\right| \frac{\alpha^2}{\beta^2}\right)}{2^\lambda \beta^{\nu-\lambda+1}\Gamma\left(\frac{-\nu+\mu+\lambda+1}{2}\right)\Gamma(\nu+1)}. \tag{9.2.17}$$

And the case for $0 < \beta < \alpha$ can be easily seen by symmetry.

In[12]:= `TheMethodofBrackets[{x^(-`λ`)besselj[`ν`,`α`x]` `besselj[`μ`,`β`x]}]`

Out[12]= $\begin{pmatrix} 2 & 2 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} + (1-\lambda+\mu+\nu)$

$$\frac{(-1)^{n_1+n_2} 2^{-\mu-\nu-2n_1-2n_2} \alpha^{\nu+2n_2} \beta^{\mu+2n_1}}{\Gamma(n_1+1)\Gamma(n_1+\mu+1)\Gamma(n_2+1)\Gamma(n_2+\nu+1)}$$

`{Choose` n_1 `as free parameter(s).}`

$$\left\{ \frac{(-1)^{n_1} 2^{-1-\mu-\nu-2\left(-\frac{1}{2}+\frac{\lambda}{2}-\frac{\mu}{2}-\frac{\nu}{2}-n_1\right)-2n_1} \alpha^{\nu+2\left(-\frac{1}{2}+\frac{\lambda}{2}-\frac{\mu}{2}-\frac{\nu}{2}-n_1\right)} \beta^{\mu+2n_1}}{\Gamma\left(\frac{1}{2}+\frac{\lambda}{2}-\frac{\mu}{2}+\frac{\nu}{2}-n_1\right)\Gamma(n_1+1)\Gamma(n_1+\mu+1)} \times\Gamma\left(\frac{1}{2}-\frac{\lambda}{2}+\frac{\mu}{2}+\frac{\nu}{2}+n_1\right), \{\{n_1, 0, \infty\}\} \right\}$$

`Direct Computation`

$$\frac{2^{-\lambda}\alpha^{-1+\lambda-\mu}\beta^\mu \Gamma\left(\frac{\nu+\mu-\lambda+1}{2}\right) {}_2F_1\left(\left.\begin{matrix} \frac{1}{2}-\frac{\lambda}{2}+\frac{\mu}{2}-\frac{\nu}{2}, \frac{1}{2}-\frac{\lambda}{2}+\frac{\mu}{2}+\frac{\nu}{2} \\ 1+\mu \end{matrix}\right| \frac{\beta^2}{\alpha^2}\right)}{\Gamma\left(\frac{-\nu+\mu+\lambda+1}{2}\right)\Gamma(\nu+1)}$$

`Using bracket.m`

$$\left\{ \frac{2^{-\lambda}\alpha^{-1+\lambda-\mu}\beta^\mu \Gamma\left(\frac{\nu+\mu-\lambda+1}{2}\right) {}_2F_1\left(\left.\begin{matrix} \frac{1-\lambda+\mu-\nu}{2}, \frac{1-\lambda+\mu+\nu}{2} \\ 1+\mu \end{matrix}\right| \frac{\beta^2}{\alpha^2}\right)}{\Gamma\left(\frac{-\nu+\mu+\lambda+1}{2}\right)\Gamma(\nu+1)},,\ |\beta|<|\alpha|,\ \{n_1\} \right\}$$

```
{Choose n2 as free parameter(s).}
```

$$\left\{\frac{(-1)^{n_2}2^{-1-\mu-\nu-2\left(-\frac{1}{2}+\frac{\lambda}{2}-\frac{\mu}{2}-\frac{\nu}{2}-n_2\right)-2n_2}\beta^{\mu+2\left(-\frac{1}{2}+\frac{\lambda}{2}-\frac{\mu}{2}-\frac{\nu}{2}-n_1\right)}\alpha^{\nu+2n_1}}{\Gamma\left(\frac{1}{2}+\frac{\lambda}{2}+\frac{\mu}{2}-\frac{\nu}{2}-n_2\right)\Gamma(n_2+1)\Gamma(n_2+\nu+1)} \times\Gamma\left(\frac{1}{2}-\frac{\lambda}{2}+\frac{\mu}{2}+\frac{\nu}{2}+n_2\right),\{\{n_2,0,\infty\}\}\right\}$$

```
Direct Computation
```

$$\frac{2^{-\lambda}\alpha^{\lambda}\beta^{-1+\lambda-\nu}\Gamma\left(\frac{\nu+\mu-\lambda+1}{2}\right){}_2F_1\left(\left.\begin{matrix}\frac{1}{2}-\frac{\lambda}{2}+\frac{\mu}{2}-\frac{\nu}{2},\frac{1}{2}-\frac{\lambda}{2}+\frac{\mu}{2}+\frac{\nu}{2}\\1+\mu\end{matrix}\right|\frac{\alpha^2}{\beta^2}\right)}{\Gamma\left(\frac{\nu-\mu+\lambda+1}{2}\right)\Gamma(\mu+1)}$$

```
Using bracket.m
```

$$\left\{\frac{2^{-\lambda}\alpha^{\nu}\beta^{-1+\lambda-\nu}\Gamma\left(\frac{\nu+\mu-\lambda+1}{2}\right){}_2F_1\left(\left.\begin{matrix}\frac{1-\lambda-\mu+\nu}{2},\frac{1-\lambda+\mu+\nu}{2}\\1+\nu\end{matrix}\right|\frac{\alpha^2}{\beta^2}\right)}{\Gamma\left(\frac{\nu-\mu+\lambda+1}{2}\right)\Gamma(\mu+1)},,\ |\alpha|<|\beta|,\ \{n_2\}\right\}$$

It is easy to write the result as

$$\begin{aligned}
I &= \int_0^\infty x^{-\lambda}J_\nu(\alpha x)J_\mu(\beta x)\,dx \qquad (9.2.18)\\
&= \sum_{n_1,n_2}\phi_{1,2}\frac{2^{-\mu-\nu-2n_1-2n_2}\alpha^{\nu+2n_2}\beta^{\mu+2n_1}}{\Gamma(n_1+\mu+1)\Gamma(n_2+\nu+1)}\langle 2n_1+2n_2+1-\lambda+\mu+\nu\rangle\\
&= \begin{cases}\displaystyle\sum_{n_1=0}^{\infty}\frac{\phi_1 2\alpha^{\nu+2\left(-\frac{1}{2}+\frac{\lambda}{2}-\frac{\mu}{2}-\frac{\nu}{2}-n_1\right)}\beta^{\mu+2n_1}\Gamma\left(\frac{1}{2}-\frac{\lambda}{2}+\frac{\mu}{2}+\frac{\nu}{2}+n_1\right)}{2^{1+\mu+\nu+2\left(-\frac{1}{2}+\frac{\lambda}{2}-\frac{\mu}{2}-\frac{\nu}{2}-n_1\right)+2n_1}\Gamma\left(\frac{1}{2}+\frac{\lambda}{2}-\frac{\mu}{2}+\frac{\nu}{2}-n_1\right)\Gamma(n_1+\mu+1)} & \text{, if } n_1 \text{ is free}\\[2ex]
\displaystyle\sum_{n_2=0}^{\infty}\frac{\phi_2\beta^{\mu+2\left(-\frac{1}{2}+\frac{\lambda}{2}-\frac{\mu}{2}-\frac{\nu}{2}-n_1\right)}\alpha^{\nu+2n_1}\Gamma\left(\frac{1}{2}-\frac{\lambda}{2}+\frac{\mu}{2}+\frac{\nu}{2}+n_2\right)}{2^{1+\mu+\nu+2\left(-\frac{1}{2}+\frac{\lambda}{2}-\frac{\mu}{2}-\frac{\nu}{2}-n_2\right)+2n_2}\Gamma\left(\frac{1}{2}+\frac{\lambda}{2}+\frac{\mu}{2}-\frac{\nu}{2}-n_2\right)\Gamma(n_2+\nu+1)} & \text{, if } n_2 \text{ is free}\end{cases}\\
&= \begin{cases}\dfrac{2^{-\lambda}\alpha^{-1+\lambda-\mu}\beta^{\mu}\Gamma\left(\frac{\nu+\mu-\lambda+1}{2}\right){}_2F_1\left(\left.\begin{matrix}\frac{1-\lambda+\mu-\nu}{2},\frac{1-\lambda+\mu+\nu}{2}\\1+\mu\end{matrix}\right|\frac{\beta^2}{\alpha^2}\right)}{\Gamma\left(\frac{-\nu+\mu+\lambda+1}{2}\right)\Gamma(\nu+1)} & \text{, if } |\beta|<|\alpha|\\[2ex]
\dfrac{2^{-\lambda}\alpha^{\nu}\beta^{-1+\lambda-\nu}\Gamma\left(\frac{\nu+\mu-\lambda+1}{2}\right){}_2F_1\left(\left.\begin{matrix}\frac{1-\lambda-\mu+\nu}{2},\frac{1-\lambda+\mu+\nu}{2}\\1+\nu\end{matrix}\right|\frac{\alpha^2}{\beta^2}\right)}{\Gamma\left(\frac{\nu-\mu+\lambda+1}{2}\right)\Gamma(\mu+1)} & \text{, if } |\alpha|<|\beta|\end{cases}
\end{aligned}$$

from which one can see that the condition for convergence (i.e., $|\beta| < |\alpha|$ and $|\alpha| < |\beta|$) can also be found in the output when using bracket.m.

9.2.3 Factorization of the Integrand

One, who knows `Mathematica` well, can see that the input of *MakeTheBracket* and *TheMethodofBrackets* both require the input in the form of a list, i.e., with a pair of braces. Concretely, examples above are all of the form:

`MakeTheBracket[{f}]` and `TheMethodofBrackets[{f}]`.

In the study of independence on factorization of integrand, this list helps us check different possible cases.

Example 9.2.6. Recall the example we mentioned that

$$I = \int_0^\infty e^{-x} dx = 1. \tag{9.2.19}$$

```
In[13]:= TheMethodofBrackets[{Exp[-x]}]
```

Out[13]= $(1)\,(n_1) + (1)$

$\frac{(-1)^{n_1}}{\Gamma(n_1+1)}$

1

Namely, the computation is

$$I = \sum_{n_1} \phi_1 \langle n_1 + 1 \rangle = 1. \tag{9.2.20}$$

On the other hand, we could rewrite the integrand as $e^{-x} = e^{-\frac{2}{3}x} e^{-\frac{1}{3}x}$, and the result does not change, though through different process:

```
In[12]:= TheMethodofBrackets[{x^(-λ)besselj[ν,αx]
besselj[μ,βx]}]
```

Out[12]= $\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} + (1)$

$\frac{(-2)^{n_1}(-1)^{n_2} 3^{-n_1-n_2}}{\Gamma(n_1+1)\Gamma(n_2+1)}$

{Choose n_1 as free parameter(s).}

$\{3\,(-2)^{n_1}, \{\{n_1, 0, \infty\}\}\}$

Direct Computation

Sum::div: Sum does not converge. >>

$\sum_{n_1=0}^{\infty} 3\,(-2)^{n_1}$

Using bracket.m

{Infinity, Infinity, False, {n_1}}

```
{Choose n_2 as free parameter(s).}
```

$$\left\{3\left(-2\right)^{-1-n_2}\left(-1\right)^{1+2n_2},\{\{n_2,0,\infty\}\}\right\}$$

```
Direct Computation
```

1

```
Using bracket.m

{Infinity, Infinity, False, {n_2}}
```

Note that in this example, using bracket.m shows the second series obtained by using n_2 as the free parameter is divergent, which is not correct. Fortunately, direct computation provides the correct answer.

9.3 Conclusion

To keep modifying the package is definitely part of the future work. As one can see that, expressing the integral as the bracket series is straightforward. While evaluation of series needs large database on special functions to find the compact expression. Currently, `Sage` seems to be too young to handle it, which is the reason that `Mathematica` has to been involved. Besides the requirement of such database for `Sage`, effective, universal algorithms for finding compact form of series and finding analytic continuation of divergent series would also be a solution.

Part II

The Bernoulli Symbol $\mathcal{B}$ and Other Results in Number Theory Related to Bernoulli and Euler Numbers

Chapter 10

Introduction to the Bernoulli Symbol

10.1 Introduction

The Bernoulli symbol $\mathcal{B}$ originally comes from Umbral Calculus, which provides a symbolic approach to the study of Sheffer sequences, and especially of Bernoulli numbers and Bernoulli polynomials. One could find more details in [52] and applications in [22].

Bernoulli numbers are defined by the following exponential generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \tag{10.1.1}$$

and Bernoulli polynomials by

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n\left(x\right) \frac{t^n}{n!}, \tag{10.1.2}$$

The Bernoulli symbol $\mathcal{B}$ has a very easy evaluation rule that

$$eval\left(\mathcal{B}^n\right) = B_n,\ \text{the } n^{\text{th}} \text{ Bernoulli number.} \tag{10.1.3}$$

In future, "*eval*" will be omitted, leaving only

$$\mathcal{B}^n = B_n. \tag{10.1.4}$$

It is also not surprising that Bernoulli polynomials are given by

$$B_n(x) = (\mathcal{B} + x)^n \tag{10.1.5}$$

since they satisfy

$$B_n(x) = \sum_{k=0}^{n} B_k x^{n-k}. \tag{10.1.6}$$

Before we introducing basic results and examples, an important property of the Bernoulli symbol comes first.

Theorem 10.1.1. *The Bernoulli symbol satisfies that*

$$-\mathcal{B} = \mathcal{B} + 1. \tag{10.1.7}$$

Proof. The required identity is equivalent to that

$$(-1)^n B_n = (-\mathcal{B})^n = (\mathcal{B} + 1)^n = B_n(1), \tag{10.1.8}$$

which can be easily verified by comparing the generating functions of $\{(-1)^n B_n\}$ and $\{B_n(1)\}$ as the following computation:

$$\sum_{n=0}^{\infty} (-1)^n B_n \frac{t^n}{n!} = \sum_{n=0}^{\infty} B_n \frac{(-t)^n}{n!} = \frac{-t}{e^{-t} - 1} = \frac{t}{e^t - 1} e^{1 \cdot t} = \sum_{n=0}^{\infty} B_n(1) \frac{t^n}{n!}. \tag{10.1.9}$$

□

10.2 Results and Examples from Umbral Calculus

The benefit of applying the Bernoulli symbol $\mathcal{B}$ involves the following.

10.2.1 Simplify Formulas

Example 10.2.1. It is easy to rewrite the generating function by

$$e^{\mathcal{B}t} = \sum_{n=0}^{\infty} \frac{\mathcal{B}^n t^n}{n!} = \sum_{n=0}^{\infty} \frac{B_n t^n}{n!} = \frac{t}{e^t - 1}. \tag{10.2.1}$$

Also (10.1.7) can be expressed (and proven) by

$$e^{-\mathcal{B}t} = e^{\mathcal{B}(-t)} = \frac{-t}{e^{-t}-1} = \frac{t}{e^{t}-1}e^{1\cdot t} = e^{\mathcal{B}t}\cdot e^{t} = e^{(\mathcal{B}+1)t}. \tag{10.2.2}$$

Example 10.2.2. In the literature, Bernoulli numbers can also be introduced, besides the generating function, by the following partial sum

$$S_m(n) = \sum_{k=1}^{n} k^m = \frac{1}{m+1}\sum_{l=0}^{m}\binom{m+1}{l}B_l n^{m+1-l}, \tag{10.2.3}$$

i.e., B_m is the coefficient of n in the polynomial $S_m(n)$. Then, by (10.1.4),

$$S_m(n) = \frac{1}{m+1}\sum_{l=0}^{m}\binom{m+1}{l}\mathcal{B}^l n^{m+1-l} = \frac{1}{m+1}\left[(\mathfrak{B}+n)^{m+1} - \mathfrak{B}^{m+1}\right], \tag{10.2.4}$$

which provides a closed form rather than a finite sum. Recall that $e^{-a\partial_x}$ is a forward translation operator that

$$e^{-a\partial_x}\circ f(x) = f(x+a). \tag{10.2.5}$$

So we could further express

$$S_m(n) = \left(e^{-a\partial_x}\circ\int\right)\circ x^m\Big|_{x=\mathcal{B}}, \tag{10.2.6}$$

which is a much simpler expression.

10.2.2 Visualize Properties and Formulas

Example 10.2.3. The symbolic expression of Bernoulli polynomial can also be verified by

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}(\mathcal{B}+x)^n\frac{t^n}{n!} = e^{(\mathcal{B}+x)t} = e^{\mathcal{B}t}\cdot e^{xt} = \frac{te^{xt}}{e^t-1}. \tag{10.2.7}$$

Example 10.2.4. From the definition of Bernoulli polynomials that

$$\frac{te^{tx}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{10.2.8}$$

we could differentiate both sides with respect to x to obtain

$$\frac{t^2e^{tx}}{e^t-1} = \sum_{n=0}^{\infty} B_n'(x)\frac{t^n}{n!}. \tag{10.2.9}$$

Note that the left hand side is

$$t \cdot \frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^{n+1}}{n!} = \sum_{n=1}^{\infty} nB_{n-1}(x) \frac{t^n}{n!}, \tag{10.2.10}$$

which implies, by comparing coefficients of two sides,

$$B_n'(x) = nB_{n-1}(x). \tag{10.2.11}$$

On the other hand, through the Bernoulli symbol, the property becomes obvious in basic calculus manner

$$B_n'(x) = [(\mathcal{B} + x)^n]' = n(\mathcal{B} + x)^{n-1} = nB_{n-1}(x). \tag{10.2.12}$$

10.3 Probabilistic Interpretation

10.3.1 Interpretation In Terms of Random Variables

Umbral Calculus works on formal power series, which is not enough to become a rigorous method. In [18], authors manage to interpret the Bernoulli symbol in terms of a random variable and succeed to provide its density function.

Theorem 10.3.1. *(A. Dixit, V. H. Moll and C. Vignat) $\mathcal{B} \sim \iota L_B - \frac{1}{2}$, where L_B is a random variable with density function*

$$p_{L_B}(x) = \frac{\pi}{2\cosh^2(\pi x)} = \frac{\pi}{2}\operatorname{sech}^2(x) \quad on\ \mathbb{R}. \tag{10.3.1}$$

Based on this setup, the evaluation of powers of the Bernoulli symbol is identical to computing expectation, namely

$$eval = \mathbb{E}, \tag{10.3.2}$$

since one could check the following computations

$$eval(\mathcal{B}^n) = \mathbb{E}(\mathcal{B}^n) = \mathbb{E}\left[\left(\iota L_B - \frac{1}{2}\right)^n\right] = \frac{\pi}{2}\int_{\mathbb{R}}\left(\iota t - \frac{1}{2}\right)^n \operatorname{sech}^2(t)\,dt = B_n, \tag{10.3.3}$$

and

$$eval\left[(\mathcal{B}+x)^n\right] = \mathbb{E}\left[(\mathcal{B}+x)^n\right] = \frac{\pi}{2}\int_{\mathbb{R}}\left(x+\imath t-\frac{1}{2}\right)^n \operatorname{sech}^2(t)\,dt = B_n(x)\,. \quad (10.3.4)$$

Remark 10.3.2. The last evaluation first appears in [64], to the best knowledge.

10.3.2 The Intimate Uniform Symbol

Beside providing a rigorous background in analysis, the theorem above also suggests further probabilistic approaches on Bernoulli symbols. This results in the following new symbol $\mathcal{U}$, called the *uniform symbol*.

Definition 10.3.3. The Uniform Symbol $\mathcal{U}$ is the uniform random variable on $[0,1]$, i.e., $\mathcal{U}$ has density function

$$p_{\mathcal{U}}(u) = \begin{cases} 1 & u \in [0,1]\,, \\ 0 & \text{otherwise.} \end{cases} \quad (10.3.5)$$

Now, recall the moment generating function, for random variable X with density function $p_X(x)$, is defined by

$$M_X(t) = \mathbb{E}\left(e^{tX}\right) = \int e^{tx} p(x)\,dx. \quad (10.3.6)$$

It plays an important role in probability theory. Given another random variable Y, independent of X, with its density function $p_Y(y)$ so that $M_Y(t) = \mathbb{E}\left(e^{tY}\right)$, the sum of X and Y has its moment generating function

$$M_{X+Y}(t) = M_X(t)\,M_Y(t)\,. \quad (10.3.7)$$

Now it is trivial to compute that

$$M_{\mathcal{B}}(t) = \mathbb{E}\left(e^{t\mathcal{B}}\right) = e^{t\mathcal{B}} = \frac{t}{e^t-1}, \quad (10.3.8)$$

and

$$M_{\mathcal{U}}(t) = \mathbb{E}\left(e^{t\mathcal{U}}\right) = \int_0^1 e^{tu}du = \frac{e^t-1}{t}, \quad (10.3.9)$$

which implies

$$M_{\mathcal{B}+\mathcal{U}}(t) = M_{\mathcal{B}}(t)\, M_{\mathcal{U}}(t) = 1. \tag{10.3.10}$$

Therefore, we are led to the following fact.

Theorem 10.3.4.

$$(\mathcal{B}+\mathcal{U})^n = \mathbb{E}\left[(\mathcal{B}+\mathcal{U})^n\right] = n^{th} \textit{ moment of } \mathcal{B}+\mathcal{U} = \delta_{0,n} = \begin{cases} 1 & n=0, \\ 0 & \textit{otherwise.} \end{cases} \tag{10.3.11}$$

Moreover, for an analytic function $f(x)$, one has

$$f(x+\mathcal{B}+\mathcal{U}) = f(x). \tag{10.3.12}$$

Proof. Only (10.3.12) needs to be verified. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then the following computation is direct:

$$\begin{aligned} f(x+\mathcal{B}+\mathcal{U}) &= \sum_{n=0}^{\infty} a_n (x+\mathcal{B}+\mathcal{U})^n \\ &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} x^k (\mathcal{B}+\mathcal{U})^{n-k} \\ &= \sum_{n=0}^{\infty} a_n \left[\binom{n}{n} x^n (\mathcal{B}+\mathcal{U})^0\right] \\ &= \sum_{n=0}^{\infty} a_n x^n = f(x). \end{aligned} \tag{10.3.13}$$

□

Remark 10.3.5. It does not mean that two random variables are cancelling each other, but means that, expect for the 0^{th} moment, all moments of the random variable $\mathcal{B}+\mathcal{U}$ vanish. If fact, for any random variable X,

$$M_X(0) = \int e^0 p_X(x)\, dx = \int x^0 p_X(x)\, dx = \int p_X(x)\, dx = 1, \tag{10.3.14}$$

namely 0^{th} moment is always 1. We could simply state that the random variable $\mathcal{B}+\mathcal{U}$ has zero moments. Therefore, we give a generalized definition for such two random variables as follows.

Definition 10.3.6. The independent random variables X and Y are called *conjugate* if

$$\mathbb{E}\left[(X+Y)^n\right] = \delta_{n,0}. \tag{10.3.15}$$

Example 10.3.7. To prove the identity

$$B_n(x+1) - B_n(x) = nx^{n-1}, \tag{10.3.16}$$

classical method refers to the generating functions

$$\begin{cases} \frac{te^{tx}}{e^t-1} = \sum\limits_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \\ \frac{te^{t(x+1)}}{e^t-1} = \sum\limits_{n=0}^{\infty} B_n(x+1) \frac{t^n}{n!}, \end{cases}$$

and their difference

$$\sum_{n=0}^{\infty} \left[B_n(x+1) - B_n(x)\right] \frac{t^n}{n!} = \frac{te^{t(x+1)}}{e^t-1} - \frac{te^{tx}}{e^t-1} = te^{tx} = \sum_{n=0}^{\infty} \frac{t^{n+1}x^n}{n!}. \tag{10.3.17}$$

With the help of Bernoulli and uniform symbols, we could consider a general case. Let $f(x)$ be analytic,

$$f'(x+\mathcal{U}) = \int_0^1 f'(x+u)\,du = f(x+1) - f(x). \tag{10.3.18}$$

Now replace x by $x+\mathcal{B}$ to get

$$f'(x) = f'(x+\mathcal{B}+\mathcal{U}) = f(x+\mathcal{B}+1) - f(x+\mathcal{B}). \tag{10.3.19}$$

The special case that $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$ gives the required identity.

10.3.3 Further Applications on Nörlund Polynomials

Definition 10.3.8. The Nörlund polynomials $B_n^{(p)}(x)$ are defined by the exponential generating function

$$\left(\frac{t}{e^t-1}\right)^p e^{tx} = \sum_{n=0}^{\infty} B_n^{(p)}(x) \frac{t^n}{n!}. \tag{10.3.20}$$

When $x = 0$, we obtain the Nörlund numbers $B_n^{(p)}$:

$$\left(\frac{t}{e^t - 1}\right)^p = \sum_{n=0}^{\infty} B_n^{(p)} \frac{t^n}{n!}. \tag{10.3.21}$$

In terms of Bernoulli symbol, we consider $\{\mathcal{B}_i\}_{i=1}^p$, a sequence of independent and identically distributed (i.i.d.) random variables such that $\mathcal{B}_i \sim \mathcal{B}$. Then,

$$M_{\left(x+\sum\limits_{i=1}^{p}\mathcal{B}_i\right)}(t) = e^{tx}\prod_{i=1}^{p} e^{t\mathcal{B}} = \left(\frac{t}{e^t - 1}\right)^p, \tag{10.3.22}$$

which yields

$$B_n^{(p)}(x) = (\mathcal{B}_1 + \cdots + \mathcal{B}_p + x)^n. \tag{10.3.23}$$

An essential result for Nörlund polynomial is the Lucas formula on its recurrence.

Theorem 10.3.9. *[Lucas Formula]*

$$B_n^{(p+1)} = \left(1 - \frac{n}{p}\right) B_n^{(p)} - nB_{n-1}^{(p)}. \tag{10.3.24}$$

If we further define another symbol β by

$$eval\left(\beta^n\right) = \beta^n = \frac{B_n}{n}, \tag{10.3.25}$$

and its Pochhammer

$$(\beta)_p = \beta\left(\beta + 1\right)\cdots\left(\beta + p - 1\right), \tag{10.3.26}$$

then, we could rewrite

$$B_n^{(p+1)} = (-1)^p p \binom{n}{p} \beta^{n-p}. \tag{10.3.27}$$

And the Lucas formula becomes

$$B_n^{(p+1)}(x) = \left(1 - \frac{n}{p}\right) B_n^{(p)}(x) - n\left(1 - \frac{x}{p}\right) B_{n-1}^{(p)}(x). \tag{10.3.28}$$

Proof. [Symbolic Proof.] It suffices to only prove (10.3.28) since (10.3.24) is the special case for $x = 0$, from which (10.3.27) is guaranteed by induction.

First of all, we recall (10.3.18) and let

$$f(x) = xB_n(x) \Rightarrow f'(x) = B_n(x) + nxB_{n-1}(x) \tag{10.3.29}$$

to get

$$
\begin{aligned}
B_n(x+\mathcal{U}) + n(x+\mathcal{U}) B_{n-1}(x+\mathcal{U}) &= (x+1) B_n(x+1) - x B_n(x) \quad (10.3.30)\\
&= x\left[B_n(x+1) - B_n\right] + B_n(x+1)\\
&= n x^n + n x^{n-1} + B_n(x).\\
&= n x^n + n x^{n-1} + (x+\mathcal{B})^n.
\end{aligned}
$$

Substitution $x \mapsto x + \mathcal{B}'$ gives

$$
\begin{aligned}
B_n(x) + n x B_{n-1}(x) &= n(x+\mathcal{B}')^n + n(x+\mathcal{B}')^{n-1} + (x+\mathcal{B}'+\mathcal{B})^n\\
&= n B_n(x) + n B_{n-1}(x) + B_n^{(2)}(x), \qquad (10.3.31)
\end{aligned}
$$

namely,

$$
B_n^{(2)}(x) = \left(1 - \frac{n}{1}\right) B_n(x) + n\left(1 - \frac{x}{1}\right) B_{n-1}(x). \qquad (10.3.32)
$$

For inductive step, we begin with

$$
B_n^{(p+1)}(x) = \left(1 - \frac{n}{p}\right) B_n^{(p)}(x) - n\left(1 - \frac{x}{p}\right) B_{n-1}^{(p)}(x), \qquad (10.3.33)
$$

and replace x by $x + \mathcal{B}$ to get

$$
B_n^{(p+2)}(x) = \left(1 - \frac{n}{p}\right) B_n^{(p+1)}(x) - n\left(1 - \frac{x+\mathcal{B}}{p}\right) B_{n-1}^{(p+1)}(x), \qquad (10.3.34)
$$

by noting that $\forall k \in \mathbb{N}$,

$$
B_n^{(k)}(x+\mathcal{B}) = (x + \mathcal{B}_1 + \cdots + \mathcal{B}_k + \mathcal{B}) = B_n^{(k+1)}(x). \qquad (10.3.35)
$$

Therefore,

$$
B_n^{(p+2)}(x) = \left(1 - \frac{n}{p}\right) B_n^{(p+1)}(x) - n\left(1 - \frac{x}{p}\right) B_{n-1}^{(p+1)}(x) + \frac{n}{p}\mathcal{B} B_{n-1}^{(p+1)}(x). \quad (10.3.36)
$$

To evaluate the last term, we consider the symmetry among $\{\mathcal{B}_1, \ldots, \mathcal{B}_p, \mathcal{B}\}$ to obtain that

$$\begin{aligned}
n\mathcal{B}B_{n-1}^{(p+1)}(x) &= n\mathcal{B}\left(x+\mathcal{B}_1+\cdots+\mathcal{B}_p+\mathcal{B}\right)^{n-1} && (10.3.37)\\
\text{[By Symmetry]} &= \frac{n}{p+1}\left(\mathcal{B}_1+\cdots+\mathcal{B}_p+\mathcal{B}\right)\left(x+\mathcal{B}_1+\cdots+\mathcal{B}_p+\mathcal{B}\right)^{n-1}\\
&= \frac{n}{p+1}\sum_{k=0}^{n-1}\binom{n-1}{k}\left(\mathfrak{B}_1+\cdots+\mathfrak{B}_p+\mathfrak{B}\right)^{k+1}x^{n-1-k}\\
[l=k-1] &= \frac{n}{p+1}\sum_{l=1}^{n}\binom{n-1}{l-1}\left(\mathfrak{B}_1+\cdots+\mathfrak{B}_p+\mathfrak{B}\right)^{l}x^{n-l}\\
\left[\frac{n}{l}\binom{n-1}{l-1}=\binom{n}{l}\right] &= \frac{1}{p+1}\sum_{l=1}^{n}\binom{n}{l}l\left(\mathfrak{B}_1+\cdots+\mathfrak{B}_p+\mathfrak{B}\right)^{l}x^{n-l}\\
&= \frac{1}{p+1}\sum_{l=0}^{n}\binom{n}{l}(l-n)\left(\mathfrak{B}_1+\cdots+\mathfrak{B}_p+\mathfrak{B}\right)^{l}x^{n-l}\\
&\quad+\frac{n}{p+1}\sum_{l=0}^{n}\binom{n}{l}\left(\mathfrak{B}_1+\cdots+\mathfrak{B}_p+\mathfrak{B}\right)x^{n-l}\\
&= \frac{1}{p+1}\left[-x\partial_x\circ B_n^{(p+1)}(x)+nB_n^{(p+1)}(x)\right]\\
&= \frac{1}{p+1}\left[-nxB_{n-1}^{(p+1)}(x)+nB_n^{(p+1)}(x)\right],
\end{aligned}$$

which gives the desired inductive formula that

$$B_n^{(p+2)}(x)=\left(1-\frac{n}{p+1}\right)B_n^{(p+1)}(x)-n\left(1-\frac{x}{p+1}\right)B_{n-1}^{(p+1)}(x). \qquad (10.3.38)$$

□

Another classical result that connecting Nörlund polynomials and hypergeometric functions, but not involving the Bernoulli symbol, is presented here, as the end of this chapter.

Theorem 10.3.10. $\forall 0\le r\le p-1$,

$$\frac{B_r^{(p+1)}(p)}{r!}=\sum_{k=1}^{r+1}\frac{1}{k}\cdot\frac{B_{r+1-k}^{(p+1-k)}(p-k)}{(r+1-k)!}. \qquad (10.3.39)$$

Proof. We first begins with entry **7.3.1.136** in [51] that

$$ {}_2F_1\left(\begin{matrix}1,1\\ p+2\end{matrix}\middle| z\right) = \frac{p+1}{z}\left[\sum_{l=0}^{p-1}\frac{1}{p-l}\left(\frac{z-1}{z}\right)^l - \left(\frac{z-1}{z}\right)^p \log\left(1-z\right)\right]. \quad (10.3.40) $$

Substitution $z \mapsto 1-e^z$ leads to

$$ {}_2F_1\left(\begin{matrix}1,1\\ p+2\end{matrix}\middle| 1-e^z\right) = \left(p+1\right)\left[\frac{ze^{pz}}{\left(e^z-1\right)^{p+1}} - \sum_{l=0}^{p-1}\frac{e^{lz}}{\left(p-l\right)\left(e^z-1\right)^{l+1}}\right]. \quad (10.3.41) $$

Note that

$$ \frac{ze^{pz}}{\left(e^z-1\right)^{p+1}} = z^{-p}e^{pz}\left(\frac{z}{e^z-1}\right)^{p+1} = z^{-p}\sum_{j=0}^{\infty}B_j^{(p+1)}\left(p\right)\frac{z^j}{j!} = \sum_{j=-p}\frac{B_{j+p}^{(p+1)}\left(p\right)}{\left(j+p\right)!}z^j, \quad (10.3.42) $$

and similarly

$$ \sum_{l=0}^{p-1}\frac{e^{lz}}{\left(p-l\right)\left(e^z-1\right)^{l+1}} = \sum_{l=0}^{p-1}\frac{1}{p-l}\sum_{j=-l-1}^{\infty}\frac{B_{j+l+1}^{(l+1)}\left(l\right)}{\left(j+l+1\right)!}z^j. \quad (10.3.43) $$

Since hypergeometric function is analytic at $z=0$, coefficients for negative powers must all vanish, which gives

$$ \frac{B_{j+p}^{(p+1)}\left(p\right)}{\left(j+p\right)!} = \sum_{l=-j-1}^{p-1}\frac{1}{p-l}\cdot\frac{B_{j+l+1}^{(l+1)}\left(l\right)}{\left(j+l+1\right)!}. \quad (10.3.44) $$

The theorem is verified by shifting sum index j by $j+p \mapsto r$. $\square$

Chapter 11

A Symbolic Approach to Identities for Bernoulli-Barnes Polynomials

11.1 Introduction

11.1.1 The Bernoulli-Barnes Polynomials and Bernoulli-Barnes Numbers

The Bernoulli-Barnes polynomials are defined by the exponential generating function

$$e^{xt}\prod_{j=1}^{n}\frac{t}{e^{a_j t}-1}=\sum_{k=0}^{\infty}B_k\left(x;\boldsymbol{a}\right)\frac{t^k}{k!}, \tag{11.1.1}$$

depending on a multi-dimensional parameter

$$\boldsymbol{a}=(a_1,\ldots,a_n)\in\mathbb{C}\text{ and }a_j\neq 0,\ j=1,\ldots,n. \tag{11.1.2}$$

Being another generalization of Bernoulli number/polynomial, the special case of Bernoulli-Barnes polynomial with $\boldsymbol{a}=\mathbf{1}=(1,\ldots,1)\in\mathbb{C}^n$ gives the Nörlund polynomials:

$$\sum_{k=0}^{\infty}B_k\left(x;\mathbf{1}\right)\frac{t^k}{k!}=e^{xt}\left(\frac{t}{e^t-1}\right)^n=\sum_{k=0}^{\infty}B_k^{(n)}\left(x\right)\frac{t^k}{k!}. \tag{11.1.3}$$

Remark 11.1.1. In the section, we use "k" instead of "n" as the degree of polynomials, since now "n" is used for the dimension of the parameter $\boldsymbol{a} \in \mathbb{C}^n$.

Similarly as one can see, Bernoulli-Barnes numbers $\{B_k(\boldsymbol{a})\}$ are defined in the case $x = 0$, or equivalently from the exponential generating function

$$\prod_{j=1}^{n} \frac{t}{e^{a_j t} - 1} = \sum_{k=0}^{\infty} B_k(a) \frac{t^k}{k!}, \tag{11.1.4}$$

and are connected to Bernoulli-Barnes polynomials also by

$$B_k(x; \boldsymbol{a}) = \sum_{l=1}^{k} \binom{k}{l} B_{k-l}(\boldsymbol{a}) x^l. \tag{11.1.5}$$

In particular, the Bernoulli-Barnes Numbers can be expressed in terms of Bernoulli numbers B_k by the multiple sum

$$B_k(\boldsymbol{a}) = \sum_{m_1+\cdots+m_n=k} \binom{k}{m_1, \ldots, m_n} a_1^{m_1-1} \cdots a_n^{m_n-1} B_{m_1} \cdots B_{m_n}, \tag{11.1.6}$$

from which one can tell that $a_1 \cdots a_n B_k(\boldsymbol{a})$ is also a polynomial in $\boldsymbol{a}$. This explains a share of nomenclature that some parts of the literature refer to $a_1 \cdots a_n B_k(\boldsymbol{a})$ as the Bernoulli-Barnes polynomials (in $\boldsymbol{a}$).

11.1.2 The Symbolic Expression

Given an i.i.d sequences of (random variables) Bernoulli symbols $\{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$, we further define

$$\boldsymbol{\mathcal{B}} := (\mathcal{B}_1, \ldots, \mathcal{B}_n) \tag{11.1.7}$$

and

$$|\boldsymbol{a}| = \prod_{j=1}^{n} a_j \neq 0. \tag{11.1.8}$$

Then, our main symbolic result follows.

Theorem 11.1.2. *Bernoulli-Barnes polynomials are given by*

$$B_k(x; \boldsymbol{a}) = \frac{(x + \boldsymbol{a} \cdot \boldsymbol{\mathcal{B}})^k}{|\boldsymbol{a}|}, \tag{11.1.9}$$

and Bernoulli-Barnes numbers by

$$B_k(x;\boldsymbol{a}) = \frac{(\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})^k}{|\boldsymbol{a}|}, \tag{11.1.10}$$

where

$$\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}} = \sum_{j=1}^{n} a_j\mathcal{B}_j \tag{11.1.11}$$

is the inner product.

Proof. It is obvious to see that $\forall a \in \mathbb{C}\setminus\{0\}$

$$e^{at\mathcal{B}} = \mathbb{E}\left[e^{at\mathcal{B}}\right] = M_{\mathcal{B}}(at) = \frac{at}{e^{at}-1}, \tag{11.1.12}$$

which is sufficient to prove the result since each $a_j\mathcal{B}_j$ contribute to a factor of $\frac{a_j t}{e^{a_j t}-1}$ due to independence among all $\mathcal{B}_j'$s. □

11.2 A Difference Formula

We first introduce a subset notation for convenience. For any $L \subset \{1,\dots,n\}$, say $L = \{i_1,\dots,i_r\}$, i.e., $|L| = r$, we define

$$\boldsymbol{a}_L := (a_{i_1},\dots,a_{i_r}) \text{ and } |\boldsymbol{a}|_L = \prod_{j=1}^{r} a_{i_j}. \tag{11.2.1}$$

For instance, $\boldsymbol{a}_{\{2,5\}} = (a_2, a_5)$ and $|\boldsymbol{a}|_{\{2,5\}} = a_2a_5$.

In [18], authors establish a difference formula as Theorem 5.1 as follows.

Theorem 11.2.1. *For $\boldsymbol{a} = (a_1,\dots,a_n) \in \mathbb{C}^n$ with $|\boldsymbol{a}| \neq 0$ and $A = \sum_{j=1}^{n} a_j$, we have the difference formula*

$$(-1)^k B_k(-x;\boldsymbol{a}) - B_k(x;\boldsymbol{a}) = k!\sum_{l=0}^{n-1}\sum_{|L|=l}\frac{B_{k-n+l}(x;\boldsymbol{a}_L)}{(k-n+l)!}, \tag{11.2.2}$$

with $B_k(x;\boldsymbol{a}_L) = x^m$ if $L = \emptyset$, i.e., $|L| = 0$. Furthermore,

$$B_k(x+A;\boldsymbol{a}) = (-1)^k B_k(-x;\boldsymbol{a}). \tag{11.2.3}$$

The next theorem established, being a generalization of Theorem 11.2.1, requires conditions of function f that certain orders of derivatives exist. We shall call functions satisfying such conditions *reasonable*. Apparently, analytic functions are reasonable, and in particular, polynomials are reasonable.

Theorem 11.2.2. *Let $f^{(j)}(x)$ denote the j^{th} derivative of a reasonable function f. Then,*

$$f(x-\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})=\sum_{l=0}^{n}\sum_{|L|=l}|\boldsymbol{a}|_{L^*}f^{(n-l)}(x+(\boldsymbol{a}_L\cdot\boldsymbol{\mathcal{B}}_L)), \tag{11.2.4}$$

where $L^*=\{1,\ldots,n\}\setminus L$ is the complement of L. Moreover,

$$f(x+A+\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})=f(x-\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}}). \tag{11.2.5}$$

Example 11.2.3. For $n=2$, the theorem gives

$$\begin{aligned} f(x-a_1\mathcal{B}_1-a_2\mathcal{B}_2) &= f(x+a_1\mathcal{B}_1+a_2\mathcal{B}_2)+a_1f'(x+a_2\mathcal{B}_2) \\ &\quad +a_2f'(x+a_1\mathcal{B}_1)+a_1a_2f''(x). \end{aligned} \tag{11.2.6}$$

Remark 11.2.4. From the classical differentiation formula that

$$\left[\frac{B_k(x)}{k!}\right]^{(j)}=\frac{B_{k-j}(x)}{(k-j)!}, \tag{11.2.7}$$

it is obvious that Theorem 11.2.1 is a special case in Theorem 11.2.2 for $f(x)=\frac{x^k}{k!}$, since the left hand side is

$$LHS=\frac{(x-\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})^k}{k!}=\frac{(-1)^k(-x+\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})^k}{k!}=|\boldsymbol{a}|\frac{(-1)^kB_k(x;\boldsymbol{a})}{k!}, \tag{11.2.8}$$

while the right hand side

$$RHS=\sum_{l=0}^{n}\sum_{|L|=l}|\boldsymbol{a}|_{L^*}\frac{(x+(\boldsymbol{a}_L\cdot\boldsymbol{\mathcal{B}}_L))^{k-n+l}}{(k-n+l)!}=|\boldsymbol{a}|_L\sum_{l=0}^{n}\sum_{|L|=l}\frac{B_{k-n+l}(x;\boldsymbol{a}_L)}{(k-n+l)!}. \tag{11.2.9}$$

Multiplying both sides by $k!/|\boldsymbol{a}|$ and move the term for $l=n$ from the right hand side to the left hand side gives the result.

In the next three subsection, we shall provide three different proofs for Theorem 11.2.2.

11.2.1 Direct Symbolic Proof

We first establish a necessary lemma.

Lemma 11.2.5. *Let g be a reasonable function. Then,*

$$g(-\mathcal{B}) = g(\mathcal{B}+1) = g(\mathcal{B}) + g'(0). \tag{11.2.10}$$

Proof. It suffices to show the case $g(y) = y^k$, the general case follows by linearity. Also noting that $-\mathcal{B} = \mathcal{B}+1$ has already been proven, it only remains to show that

$$(\mathcal{B}+1)^k = \mathcal{B}^k + \delta_{k-1,0}, \tag{11.2.11}$$

since

$$g(y) = y^k \Rightarrow g'(0) = \delta_{k-1,0}. \tag{11.2.12}$$

This follows from (10.3.18) for the case $f(y) = g(y) = y^k$ that

$$(y+1)^k = y^k + g'(y+\mathcal{U}). \tag{11.2.13}$$

Replace $y \mapsto y + \mathcal{B}$ to obtain

$$(y+\mathcal{B}+1)^k = (y+\mathcal{B})^k + g'(y), \tag{11.2.14}$$

which therefore implies the required identity, by making $y = 0$. □

Proof. [Proof of Theorem 11.2.2]. We shall prove it by induction on the dimension n.

i> When $n = 1$, previous lemma gives, by considering $g(y) = f(x+ay)$, directly the result that

$$f(x - a\mathcal{B}) = g(-\mathcal{B}) = g(\mathcal{B}) + g'(0) = f(x + a\mathcal{B}) + af'(x). \tag{11.2.15}$$

ii> Suppose the identity holds for n that

$$f(x - \boldsymbol{a}\cdot\boldsymbol{\mathcal{B}}) = \sum_{l=0}^{n}\sum_{|L|=l} |\boldsymbol{a}|_{L^*} f^{(n-l)}(x + (\boldsymbol{a}_L \cdot \boldsymbol{\mathcal{B}}_L)). \tag{11.2.16}$$

For the case $n+1$, i.e., new component a_{n+1} and $\mathcal{B}_{n+1}$ being added to have

$$\bar{\boldsymbol{a}} = (\boldsymbol{a}, a_{n+1}) \text{ and } \bar{\boldsymbol{\mathcal{B}}} = (\boldsymbol{\mathcal{B}}, \mathcal{B}_{n+1}), \tag{11.2.17}$$

we define

$$g(y) = f(x - \boldsymbol{a} \cdot \boldsymbol{\mathcal{B}} + a_{n+1} y). \tag{11.2.18}$$

Apply the previous lemma again to get the inductive result. □

11.2.2 Symbolic Proof Fully Involving the Uniform Symbol

Proof. [Proof of Theorem 11.2.2]. For the case $n = 1$, we need to show that

$$f(x - a\mathcal{B}) = f(x + a\mathcal{B}) + af'(x). \tag{11.2.19}$$

Replacing x by $x + a\mathcal{U}$, one can see that

$$\begin{cases} LHS = f(x + a\mathcal{U} - a\mathcal{B}) = f(x + a\mathcal{U} + a(\mathcal{B} + 1)) = f(x + a), \\ RHS = f(x) + af'(x + a\mathcal{U}) = f(x) + a\int_0^1 f'(x + au)\,du, \end{cases} \tag{11.2.20}$$

which means the identity for $n = 2$ is equivalent to

$$f(x + a) - f(x) = a\int_0^1 f'(x + au)\,du = \int_0^a f(x + t)\,dt, \tag{11.2.21}$$

the fundamental theorem of calculus. We introduce Δ_a, the forward difference operator of step a, to rewrite

$$\Delta_a f(x) = f(x + a) - f(a) = af'(x + a\mathcal{U}). \tag{11.2.22}$$

Then, the general case for arbitrary n is equivalent to the elementary identity

$$\prod_{j=1}^{n} \Delta_{a_i} f(x) = |\boldsymbol{a}|\, f^{(n)}(x + a_1\mathcal{U}_1 + \cdots + a_n\mathcal{U}_n). \tag{11.2.23}$$

Replacement $x \mapsto x + \boldsymbol{a} \cdot \boldsymbol{\mathcal{B}}$ gives the desired result. □

11.2.3 An Operational Calculus Proof

Proof. [Proof of Theorem 11.2.2] At the end of the previous proof, the forward difference operator inspires us to consider the problem via operator. Now, we define an operator T_a that acts on reasonable function $f(x)$ by

$$T_a[f(x)] = f(x - a\mathcal{B}). \tag{11.2.24}$$

It is also obvious to see that

$$T_{a_1} \circ T_{a_2}[f(x)] = f(x - a_1\mathcal{B}_1 - a_2\mathcal{B}_2) = T_{a_2} \circ T_{a_1}[f(x)], \tag{11.2.25}$$

implying operators $\left\{T_{a_j}\right\}_{j=1}^{n}$ are mutually commutative. Recall that

$$T_a[f(x)] = f(x - a\mathcal{B}) = f(x + a\mathcal{B}) + af'(x). \tag{11.2.26}$$

Thus, we could express T_a formally by

$$T_a = e^{a\mathcal{B}\partial_x} + a\partial_x, \tag{11.2.27}$$

namely a forward translation of step $a\mathcal{B}$ plus a scaled derivative. Composition can be easily computed that

$$T_{a_1} \circ T_{a_2} = e^{(a_1\mathcal{B}_1 + a_2\mathcal{B}_2)\partial_x} + a_1\partial_x e^{a_2\mathcal{B}_2\partial_x} + a_2\partial_x e^{a_1\mathcal{B}_1\partial_x} + a_1a_2(\partial_x)^2, \tag{11.2.28}$$

which exactly gives the result for $n = 2$:

$$\begin{aligned} f(x - a_1\mathcal{B}_1 - a_2\mathcal{B}_2) &= T_{a_1} \circ T_{a_2}[f(x)] \\ &= f(x + a_1\mathcal{B}_1 + a_2\mathcal{B}_2) + a_1 f'(x + a_2\mathcal{B}_2) \\ &\quad + a_2 f'(x + a_1\mathcal{B}_1) + a_1a_2 f''(x). \end{aligned} \tag{11.2.29}$$

The general case follows from the direct computation

$$T_{a_1} \circ \cdots \circ T_{a_n} = \prod_{j=1}^{n}\left(e^{a_j\mathcal{B}_j\partial_x} + a_j\partial_x\right) = \sum_{l=0}^{n}\sum_{|L|=l}|\boldsymbol{a}|_{L^*}(\partial_x)^{n-j} e^{\boldsymbol{a}_L \cdot \boldsymbol{\mathcal{B}}_L\partial_x}. \tag{11.2.30}$$

$\square$

11.3 Self-dual Property

Definition 11.3.1. Given a sequence $\{a_k\}$, its dual, denoted by $\{a_k^*\}$, is defined by

$$a_k^* := \sum_{l=0}^{k} \binom{k}{l} (-1)^l a_l. \tag{11.3.1}$$

The inversion formula, for instance in [51],

$$a_k^* := \sum_{l=0}^{k} \binom{k}{l} (-1)^l a_l \tag{11.3.2}$$

shows that dual sequences are reciprocal. A self-dual sequence is that whose dual coincides with the sequence itself. Examples of self-dual sequences have been discussed in the literature, for example in [56] and [57].

Example 11.3.2. Define $a_k := (-1)^k B_k$, then $\{a_k\}$ is self-dual. Namely,

$$a_k = (-1)^k B_k = \sum_{l=0}^{k} \binom{k}{l} B_l = \sum_{l=0}^{k} \binom{k}{l} (-1)^l a_l. \tag{11.3.3}$$

Symbolic proof of the identity is trivial since

$$(-1)^k B_k = (-\mathcal{B})^k = (\mathcal{B}+1)^k = \sum_{l=0}^{k} \binom{k}{l} \mathcal{B}^l = \sum_{l=0}^{k} \binom{k}{l} B_l. \tag{11.3.4}$$

In [8], the author prove the next two theorems but not directly, and therefore also ask for direct proofs, which can be obtained through symbolic approach.

Theorem 11.3.3. *Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ with $|\boldsymbol{a}| \neq 0$ and $A = a_1 + \cdots + a_n \neq 0$. Then the sequence*

$$p_k := (-1)^k A^{-k} B_n(\boldsymbol{a}) \tag{11.3.5}$$

is self-dual.

Proof. The following computation is direct:

$$\begin{aligned}
p_k^* &:= \sum_{l=0}^{k} \binom{k}{l} (-1)^k p_k && (11.3.6)\\
&= \sum_{l=0}^{k} \binom{k}{l} A^{-k} \frac{(\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})^k}{|\boldsymbol{a}|}\\
&= \frac{1}{|\boldsymbol{a}|}\left(1+\frac{\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}}}{A}\right)^k\\
&= \frac{1}{|\boldsymbol{a}|\,A^k}\left(A+\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}}\right)^k\\
&= A^{-k}\frac{1}{|\boldsymbol{a}|}\left[\sum_{j=1}^{n} a_j\left(\mathcal{B}_j+1\right)\right]^k\\
&= A^{-k}\frac{1}{|\boldsymbol{a}|}\left(-\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}}\right)^k\\
&= (-1)^k A^{-k} B_k(\boldsymbol{a})\\
&= p_n.
\end{aligned}$$

□

Theorem 11.3.4. *Let* $\boldsymbol{a} = (a_1, \ldots, a_n)$ *with* $|\boldsymbol{a}| \neq 0$ *and* $A = a_1 + \cdots + a_n \neq 0$. *Then,* $\forall l, m \in \mathbb{N} \cup \{0\}$,

$$(-1)^m \sum_{k=0}^{m} \binom{m}{k} A^{m-k} B_{l+k}(x;\boldsymbol{a}) = (-1)^l \sum_{k=0}^{l} \binom{l}{k} A^{l-k} B_{m+k}(-x;\boldsymbol{a}), \qquad (11.3.7)$$

and

$$\begin{aligned}
&(-1)^{m+1} B_{l+m+1}(x;\boldsymbol{a}) + (-1)^{l+1} B_{l+m+1}(-x;\boldsymbol{a}) = && (11.3.8)\\
&\qquad \frac{(-1)^m}{m+l+2}\sum_{k=0}^{m}\binom{m+1}{k}(l+k+1) \quad A^{m-1-k} B_{l+k}(x;\boldsymbol{a})\\
&\qquad +\frac{(-1)^m}{m+l+2}\sum_{k=0}^{l}\binom{l+1}{k}(m+k+1) \quad A^{l+1-k} B_{m+k}(x;\boldsymbol{a}).
\end{aligned}$$

Proof. It only suffices to prove (11.3.7) since (11.3.8) comes directly from substitutions $m \mapsto m+1$, $l \mapsto l+1$ and differentiation with respect to x. Note that for the left hand side

$$\begin{aligned} LHS &= (-1)^m \sum_{k=0}^{m} \binom{m}{k} A^{m-k} \frac{(x+\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})^{l+k}}{|\boldsymbol{a}|} \\ &= \frac{1}{|\boldsymbol{a}|}(-1)^m (x+\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})^l (x+\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}}+A)^m \\ &= \frac{1}{|\boldsymbol{a}|}(-1)^m (x-A-\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})^l (x-\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})^m, \end{aligned} \tag{11.3.9}$$

and similarly by interchanging m and l and also replacing x by $-x$, for the right hand side

$$\begin{aligned} RHS &= \frac{1}{|\boldsymbol{a}|}(-1)^l (-x+\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})^m (-x+\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}}+A)^l \\ &= \frac{1}{|\boldsymbol{a}|}(-1)^m (x-A-\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})^l (x-\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})^m. \end{aligned} \tag{11.3.10}$$

This completes the proof. □

11.4 Linear Identities for Bernoulli-Barnes Numbers

In this section, by providing symbolical proofs, we recover two theorems in [8], of which, we correct a typo for the first one and establish a generalization for the second.

Theorem 11.4.1. *Let $m \in \mathbb{N}$, then*

$$B_{2m+1}(\boldsymbol{a}) = -\frac{1}{2(m+1)} \sum_{k=0}^{m} \binom{m+1}{k} (m+k+1) A^{m+1-k} B_{m+k}(\boldsymbol{a}), \tag{11.4.1}$$

and

$$\begin{aligned} B_{2m}(\boldsymbol{a}) &= -\frac{1}{(m+1)(2m+1)} \sum_{k=0}^{m-1} \binom{m+1}{k} (m+k+1) A^{m-k} B_{m+k}(\boldsymbol{a}) \\ &\quad + \frac{(2m)!}{A} \sum_{k=0}^{n-1} \sum_{|K|=k} \frac{B_{2m+1-n+k}(\boldsymbol{a}_I)}{(2m+1-m+k)!}. \end{aligned} \tag{11.4.2}$$

Proof. We begin with an elementary binomial identity that

$$\begin{aligned} f(x,y) &= \sum_{k=0}^{m}\binom{m+1}{k}(m+k+1)\,x^{m+1-k}y^{m+k} \\ &= y^m\left[\sum_{k=0}^{m+1}\binom{m+1}{k}(m+k+1)\,x^{m+1-k}y^{k}-(2m+2)\,y^{m+1}\right] \\ &= y^m\left[(m+1)(x+y)^{m+1}-2(m+1)\,y^{m+1}+y\partial_y(x+y)^{m+1}\right] \\ &= y^m\left[(m+1)(x+y)^{m+1}-2(m+1)\,y^{m+1}+(m+1)(x+y)^m y\right] \\ &= y^m\left[(m+1)(x+y)^m(x+2y)-2(m+1)\,y^{m+1}\right] \\ &= -(m+1)\,y^m\left[2y^{m+1}-(x+y)^m(x+2y)\right]. \end{aligned} \tag{11.4.3}$$

Let $x=A=a_1+\cdots+a_n$ and $y=\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}}$ to obtain

$$f(A,\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})=-(m+1)(\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})^m\left[2(\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})^{m+1}-(A+\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})^m(A+2\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})\right], \tag{11.4.4}$$

and also

$$\begin{aligned} f(A,\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}}) &= \sum_{k=0}^{m}\binom{m+1}{k}(m+k+1)\,A^{m+1-k}(\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})^{m+k} \\ &= |\boldsymbol{a}|\sum_{k=0}^{m}\binom{m+1}{k}(m+k+1)\,A^{m+1-k}B_{m+k}(\boldsymbol{a}). \end{aligned} \tag{11.4.5}$$

Note that

$$(\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})^m(A+\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})^{m+1}=(-A-\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})^m(-\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})^{m+1}=(\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})^{m+1}(A+\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})^m, \tag{11.4.6}$$

which leads to

$$(\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})^m(A+\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})^m(A+2\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})=0. \tag{11.4.7}$$

Thus,

$$f(A,\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})=-2(m+1)(\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}})^{2m+1}=-2|\boldsymbol{a}|(m+1)\,B_{2m+1}(\boldsymbol{a}), \tag{11.4.8}$$

which gives (11.4.1). For (11.4.2), use Theorem 11.2.1 with $x=0$ and $m\mapsto 2m+1$ to obtain

$$f(A,\boldsymbol{a}\cdot\boldsymbol{\mathcal{B}}) = |\boldsymbol{a}|(m+1)(2m+1)!\sum_{k=0}^{n-1}\sum_{|K|=k}\frac{B_{2m+1-n+k}(\boldsymbol{a}_K)}{(2m+1-n+k)!}. \tag{11.4.9}$$

Matching with (11.4.1), we see

$$\frac{(2m)!}{A}\sum_{k=0}^{n-1}\sum_{|K|=k}\frac{B_{2m+1-n+k}\left(\boldsymbol{a}_K\right)}{(2m+1-n+k)!} = \tag{11.4.10}$$

$$\frac{1}{(m+1)(2m+1)}\sum_{k=0}^{m}\binom{m+1}{k}\quad(m+k+1)\,A^{m+1-k}B_{m+k}\left(\boldsymbol{a}\right),$$

where $B_{2m}\left(\boldsymbol{a}\right)$ lies on the right hand side for $k=m$. Solving for it gives (11.4.2). $\square$

Theorem 11.4.2. *For $n \geq 3$, $m \geq 1$ both odd and $a_i \in \mathbb{R}\backslash\{0\}$,*

$$\sum_{j=n-m}^{n}\binom{n+j-4}{j-2}\frac{\sum_{|J|=j} B_{m-n+j}\left(\boldsymbol{a}_J\right)}{(m-n+j)!} = \begin{cases} \frac{1}{2} & \text{if } n=m=3, \\ 0 & \text{otherwise.} \end{cases} \tag{11.4.11}$$

We next establish a generalization which not only shows identity above is part of a general class, but also explains the appearance of the puzzling $\binom{n+j-4}{j-2}$.

Theorem 11.4.3. *Let $\left\{\alpha_j^{(n)} : 1 \leq j \leq n\right\}$ be a sequence of numbers satisfying the palindromic condition $\alpha_{n-j}^{(n)} = \alpha_j^{(n)}$, where n is odd, and let f be an odd function. Then,*

$$\sum_{j=0}^{n}\alpha_j^{(n)}\sum_{|J|=j} f\left(\boldsymbol{a}_J\cdot\boldsymbol{\mathcal{B}}_J - \boldsymbol{a}_{J^*}\cdot\boldsymbol{\mathcal{B}}_{J^*}\right) = 0, \tag{11.4.12}$$

where $J^* = \{1,\ldots,n\}\backslash J$.

Proof. Note that $\forall j \in \{0,1,\ldots,n\}$, let $|J|=j$, then we could denote $J'=J^*$ to see, since f is odd,

$$\begin{aligned}
\alpha_{n-j}^{(n)} f\left(\boldsymbol{a}_{J'}\cdot\boldsymbol{\mathcal{B}}_{J'} - \boldsymbol{a}_{J'^*}\cdot\boldsymbol{\mathcal{B}}_{J'^*}\right) &= \alpha_{n-j}^{(n)} f\left(\boldsymbol{a}_{J^*}\cdot\boldsymbol{\mathcal{B}}_{J^*} - \boldsymbol{a}_J\cdot\boldsymbol{\mathcal{B}}_J\right) \\
&= \alpha_{n-j}^{(n)} f\left[-\left(\boldsymbol{a}_J\cdot\boldsymbol{\mathcal{B}}_J - \boldsymbol{a}_{J^*}\cdot\boldsymbol{\mathcal{B}}_{J^*}\right)\right] \\
&= -\alpha_{n-j}^{(n)} f\left(\boldsymbol{a}_J\cdot\boldsymbol{\mathcal{B}}_J - \boldsymbol{a}_{J^*}\cdot\boldsymbol{\mathcal{B}}_{J^*}\right),
\end{aligned} \tag{11.4.13}$$

which leads to

$$\alpha_j^{(n)} f\left(\boldsymbol{a}_{J'}\cdot\boldsymbol{\mathcal{B}}_{J'} - \boldsymbol{a}_{J'^*}\cdot\boldsymbol{\mathcal{B}}_{J'^*}\right) + \alpha_{n-j}^{(n)} f\left(\boldsymbol{a}_{J'}\cdot\boldsymbol{\mathcal{B}}_{J'} - \boldsymbol{a}_{J'^*}\cdot\boldsymbol{\mathcal{B}}_{J'^*}\right) = 0, \tag{11.4.14}$$

due to the palindromic condition. Finally, since n is odd, such pair (J,J') always exists, which makes the sum vanish. $\square$

Example 11.4.4. We consider the special choice of

$$\alpha_j^{(n)} = \begin{cases} \binom{n-4}{j-2} & \text{if } 2 \leq j \leq n-2. \\ 0 & \text{otherwise.} \end{cases} \tag{11.4.15}$$

In Theorem 11.2.2, we let $j \mapsto n-j$, which interchanges J and J^* in the formula, and restrict the set from $\{1, \ldots, n\}$ down to K^*, which makes $J^* \mapsto K^* \backslash J$. Then, we obtain

$$f\left(x - \boldsymbol{a}_{K^*} \cdot \boldsymbol{\mathcal{B}}_{K^*}\right) = \sum_{j=0}^{|K^*|} \sum_{|J|=j} |\boldsymbol{a}_J| \, f^{(j)}\left(x + \boldsymbol{a}_{K^*\backslash J} \cdot \boldsymbol{\mathcal{B}}_{K^*\backslash J^*}\right). \tag{11.4.16}$$

Further let $x = \boldsymbol{a}_K \cdot \boldsymbol{\mathcal{B}}_K$, which yields

$$x + \boldsymbol{a}_{K^*\backslash J} \cdot \boldsymbol{\mathcal{B}}_{K^*\backslash J^*} = \boldsymbol{a}_K \cdot \boldsymbol{\mathcal{B}}_K + \boldsymbol{a}_{K^*\backslash J} \cdot \boldsymbol{\mathcal{B}}_{K^*\backslash J^*} = \boldsymbol{a}_{J^*} \cdot \boldsymbol{\mathcal{B}}_{J^*}, \tag{11.4.17}$$

and therefore

$$f\left(\boldsymbol{a}_K \cdot \boldsymbol{\mathcal{B}}_K - \boldsymbol{a}_{K^*} \cdot \boldsymbol{\mathcal{B}}_{K^*}\right) = \sum_{j=0}^{|K^*|} \sum_{|J|=j} |\boldsymbol{a}_J| \, f^{(j)}\left(\boldsymbol{a}_{J^*} \cdot \boldsymbol{\mathcal{B}}_{J^*}\right). \tag{11.4.18}$$

Now consider the sum to get

$$\sum_{k=2}^{n-2} \sum_{|K|=k} \alpha_k^{(n)} f\left(\boldsymbol{a}_K \cdot \boldsymbol{\mathcal{B}}_K - \boldsymbol{a}_{K^*} \cdot \boldsymbol{\mathcal{B}}_{K^*}\right) = \sum_{k=2}^{n-2} \sum_{|K|=k} \alpha_k^{(n)} \sum_{j=0}^{|K^*|} \sum_{|J|=j} |\boldsymbol{a}_J| \, f^{(j)}\left(\boldsymbol{a}_{J^*} \cdot \boldsymbol{\mathcal{B}}_{J^*}\right). \tag{11.4.19}$$

Since there are $\binom{n-j}{k}$ subsets of K in $\{1, \ldots, n\}$ that overlap with J for the right hand side, we know

$$\sum_{|K|=k} \binom{n-j}{k} = 1. \tag{11.4.20}$$

Hence,

$$\sum_{k=2}^{n-2} \sum_{|K|=k} \alpha_k^{(n)} = \sum_{k=2}^{n-2} \binom{n-4}{k-2} \binom{n-j}{n-j-k} = \binom{2n-j-4}{n-j-2}, \tag{11.4.21}$$

by the Chu-Vandermonde identity. Now, we have

$$\sum_{k=2}^{n-2} \sum_{|K|=k} \alpha_k^{(n)} f\left(\boldsymbol{a}_K \cdot \boldsymbol{\mathcal{B}}_K - \boldsymbol{a}_{K^*} \cdot \boldsymbol{\mathcal{B}}_{K^*}\right) = \sum_{j=0}^{n} \binom{2n-j-4}{n-j-4} \sum_{|J|=j} |\boldsymbol{a}_J| \, f^{(j)}\left(\boldsymbol{a}_{J^*} \cdot \boldsymbol{\mathcal{B}}_{J^*}\right). \tag{11.4.22}$$

The change of summation index $j \mapsto n - j$ leads to

$$\binom{2n-j-4}{n-j-4} \mapsto \binom{n+j-4}{j-2} \text{ and } J \mapsto J^{*}. \tag{11.4.23}$$

Eventually, apply it to odd function $f(x) = \frac{x^m}{m!}$ to obtain Theorem 11.4.2.

Chapter 12

A Symbolic Approach to Multiple Zeta Values at Negative Integers

12.1 Introduction

The multiple zeta functions, first introduced by Euler and generalized by D. Zagier [65], appear in diverse areas such as quantum field theory [11] and knot theory [63]. These are defined by

$$\zeta_r(n_1, \ldots, n_r) = \sum_{0<k_1<\cdots<k_r} \frac{1}{k_1^{n_1} \ldots k_r^{n_r}}, \tag{12.1.1}$$

where $\{n_i\}$ are complex values, and (12.1.1) converges when the constraints

$$\mathrm{Re}(n_r) \geq 1, \text{ and } \sum_{j=1}^{k} \mathrm{Re}(n_{r+1-j}) \geq k, \; 2 \leq k \leq r, \tag{12.1.2}$$

are satisfied (see [66]). Their values at integer points $\mathbf{n} = (n_1, \ldots, n_r)$ satisfying (12.1.2) are called *multiple zeta values*. The sum of the exponents $n_1 + \cdots + n_r$ is called the *weight* of the zeta value, and the number r of these exponents is called its *depth*.

An equivalent definition of these values is

$$\zeta_r(n_1, \ldots, n_r) = \sum_{k_1>0,\ldots,k_r>0} \frac{1}{k_1^{n_1}(k_1+k_2)^{n_2} \ldots (k_1+\cdots+k_r)^{n_r}}. \tag{12.1.3}$$

And the multiple Hurwitz-zeta function is given by

$$Z(\mathbf{n},\mathbf{z}) = \sum_{k_1,\ldots,k_r=1}^{\infty} \frac{1}{(k_1+z_1)^{n_1}(k_1+z_1+k_2+z_2)^{n_2}\ldots(k_1+z_1+\cdots+k_r+z_r)^{n_r}}. \tag{12.1.4}$$

Following the result by Zhao [66], that the multiple zeta function has an analytic continuation to the whole space $\mathbb{C}^r$, several authors have recently proposed different analytic continuations based on a variety of approaches: Akiyama et al. [3] used the Euler-Maclaurin summation formula and Matsumoto [48] the Mellin-Barnes integral formula.

B. Sadaoui [53] provided recently such analytic continuation based on Raabe's identity, which links the multiple integral

$$Y_{\mathbf{a}}(\mathbf{n}) = \int_{[1,+\infty)^r} \frac{d\mathbf{x}}{(x_1+a_1)^{n_1}(x_1+a_1+x_2+a_2)^{n_2}\ldots(x_1+a_1+\cdots+x_r+a_r)^{n_r}} \tag{12.1.5}$$

to the multiple zeta function $Z(\mathbf{n},\mathbf{z})$ by

$$Y_{\mathbf{0}}(\mathbf{n}) = \int_{[0,1]^r} Z(\mathbf{n},\mathbf{z})\,d\mathbf{z}. \tag{12.1.6}$$

B. Sadaoui uses a classical inversion argument to obtain an analytic continuation of the multiple zeta function defined at negative integer arguments $-\mathbf{n} = (-n_1,\ldots,-n_r)$. The argument uses the following three steps:

(I) the integral $Y_{\mathbf{a}}(\mathbf{n})$ is computed for values of $n_1,\ldots,n_r$ that satisfy the convergence conditions (12.1.2);

(II) the values $\mathbf{n}$ are replaced by $-\mathbf{n}$ in this result: it is then shown that $Y_{\mathbf{a}}(-\mathbf{n})$ is a polynomial in the variable $\mathbf{a}$;

(III) the variables $\mathbf{a} = (a_1,\ldots,a_r)$ are replaced by $(\mathcal{B}_1,\ldots,\mathcal{B}_r)$, where $\{B_i\}_{i=1}^r$ is a sequence of i.i.d. Bernoulli symbols/variables.

Example 12.1.1. A multiple zeta value of depth 2, appearing in [53], is now computed using the rules above. The integral $Y_{\mathbf{a}}(n_1,n_2)$ is explicitly computed and, replacing (n_1,n_2) by $(-n_1,-n_2)$ gives

$$Y_{a_1,a_2}\left(-n_1,-n_2\right)=\frac{1}{n_2+1}\sum_{k_2=0}^{n_2+1}\sum_{l_1=0}^{n_1+n_2+2-k_2}\sum_{l_2=0}^{k_2}\frac{\binom{n_2+1}{k_2}\binom{n_1+n_2+2-k_2}{l_1}\binom{k_2}{l_2}}{n_1+n_2+2-k_2}a_1^{l_1}a_2^{l_2}. \quad (12.1.7)$$

Then substituting the variables a_1 and a_2 by the Bernoulli symbols $\mathcal{B}_1$ and $\mathcal{B}_2$ gives

$$\zeta_2\left(-n_1,-n_2\right)=\frac{1}{n_2+1}\sum_{k_2=0}^{n_2+1}\sum_{l_1=0}^{n_1+n_2+2-k_2}\sum_{l_2=0}^{k_2}\frac{\binom{n_2+1}{k_2}\binom{n_1+n_2+2-k_2}{l_1}\binom{k_2}{l_2}}{n_1+n_2+2-k_2}\mathcal{B}^{l_1}\mathcal{B}^{l_2}. \quad (12.1.8)$$

Evaluating the Bernoulli symbols to get the multiple zeta value of depth 2 at $(-n_1,-n_2)$:

$$\zeta_2\left(-n_1,-n_2\right)=\frac{1}{n_2+1}\sum_{k_2=0}^{n_2+1}\sum_{l_1=0}^{n_1+n_2+2-k_2}\sum_{l_2=0}^{k_2}\frac{\binom{n_2+1}{k_2}\binom{n_1+n_2+2-k_2}{l_1}\binom{k_2}{l_2}}{n_1+n_2+2-k_2}B_{l_1}B_{l_2}. \quad (12.1.9)$$

The general case is given in [53, eq. (4.10)] as the $(2r-1)$-fold sum (which corrects a typo)

$$\begin{aligned}\zeta_r\left(-n_1,\dots,-n_r\right) &= (-1)^r\sum_{k_2,\dots,k_r}\frac{1}{\left(\bar{n}+r-\bar{k}\right)} \quad (12.1.10)\\ &\times\prod_{j=2}^{r}\frac{\binom{\sum_{i=j}^{r}n_i+r-j+1-\sum_{i=j+1}^{n}k_i}{k_j}}{\left(\sum_{i=j}^{r}n_i+r-j+1-\sum_{i=j+1}^{n}k_i\right)}\\ &\times\sum_{l_1,\dots,l_r}\binom{\bar{n}+r-\bar{k}}{l_1}\binom{k_2}{l_2}\cdots\binom{k_r}{l_r}B_{l_1}\dots B_{l_r}\end{aligned}$$

where $k_2,\dots,k_r\geq 0$, $l_j\leq k_j$ for $2\leq j\leq r$ and $l_1\leq\bar{n}+r+\bar{k}$ and

$$\bar{n}=\sum_{j=1}^{r}n_j,\ \ \bar{k}=\sum_{j=2}^{r}k_j. \quad (12.1.11)$$

It is presented next the symbolic approach to derive some specific zeta values at negative integers, contiguity identities for the multiple zeta functions, recursions on their depth and generating functions.

12.2 Main Results

12.2.1 On Multiple Zeta Values at Negative Integers

Introduce first the symbols $\mathcal{C}_{1,2,\ldots,k}$ defined recursively in terms of the Bernoulli symbols $\mathcal{B}_1, \ldots, \mathcal{B}_r$ as

$$\mathcal{C}_1^n = \frac{\mathcal{B}_1^n}{n},\ \mathcal{C}_{1,2}^n = \frac{(\mathcal{C}_1 + \mathcal{B}_2)^n}{n}, \ldots \text{ and } \mathcal{C}_{1,2,\ldots,k+1}^n = \frac{(\mathcal{C}_{1,2,\ldots,k} + \mathcal{B}_{k+1})^n}{n} \tag{12.2.1}$$

with the $\mathcal{C}-$**symbols rule** that all symbols $\mathcal{C}_{1,2,\ldots,k}$ are expanded using the above identities to express them only in terms of $\mathcal{B}_k$.

Example 12.2.1. The rules above are illustrated by

$$\begin{aligned} \mathcal{C}_1^{n_1}\mathcal{C}_{1,2}^{n_2} &= \mathcal{C}_1^{n_1}\frac{(\mathcal{C}_1 + \mathcal{B}_2)^{n_2}}{n_2} \\ &= \sum_{k=0}^{n_2}\binom{n_2}{k}\frac{\mathcal{C}_1^{n_1+k}\mathcal{B}_2^{n_2-k}}{n_2} \\ &= \sum_{k=0}^{n_2}\binom{n_2}{k}\frac{\mathcal{B}_1^{n_1+k}\mathcal{B}_2^{n_2-k}}{(n_1+k)\,n_2} \\ &= \frac{1}{n_2}\sum_{k=0}^{n_2}\binom{n_2}{k}\frac{B_{n_1+k}}{n_1+k}B_{n_2-k}. \end{aligned} \tag{12.2.2}$$

The next result is given in terms of this notation.

Theorem 12.2.2. *The multiple zeta values (12.1.10) at the negative integers* $(-n_1, \ldots, -n_r)$ *are given by*

$$\zeta_r(-n_1, \ldots, -n_r) = \prod_{k=1}^{r}(-1)^{n_k}\,\mathcal{C}_{1,\ldots,k}^{n_k+1}. \tag{12.2.3}$$

Proof. The inner sum in (12.1.10), in its Bernoulli symbols version,

$$\sum_{l_1,\ldots,l_r}\binom{\bar{n}+r-\bar{k}}{l_1}\binom{k_2}{l_2}\ldots\binom{k_r}{l_r}\mathcal{B}^{l_1}\ldots\mathcal{B}^{l_r}, \tag{12.2.4}$$

can be summed to

$$(1+\mathcal{B}_1)^{\bar{n}+r-\bar{k}}(1+\mathcal{B}_2)^{k_2}\ldots(1+\mathcal{B}_r)^{k_r}. \tag{12.2.5}$$

The identity (10.1.7) that $\mathcal{B}+1=-\mathcal{B}$, with $\bar{n}$ defined in (12.1.11), reduces this to

$$(-1)^{\bar{n}+1}\,\mathcal{B}_1^{\bar{n}+r-\bar{k}}\mathcal{B}_2^{k_2}\ldots\mathcal{B}_r^{k_r}. \tag{12.2.6}$$

It follows that

$$\begin{aligned}\zeta_r(-\mathbf{n}) &= \frac{(-1)^{\bar{n}}}{(n_r+1)}\sum_{k_2,\ldots,k_r}\mathcal{C}_1^{\bar{n}+r-\bar{k}}\mathcal{B}_2^{k_2}\ldots\mathcal{B}_r^{k_r} \\ &\quad\times\prod_{j=2}^{r}\frac{\binom{\sum_{i=j}^{r}n_i+r-j+1-\sum_{i=j+1}^{n}k_i}{k_j}}{\left(\sum_{i=j}^{r}n_i+r-j+1-\sum_{i=j+1}^{n}k_i\right)}.\end{aligned} \tag{12.2.7}$$

Summing first over k_2 gives

$$\begin{aligned}\zeta_r(-\mathbf{n}) &= \frac{(-1)^{\bar{n}}}{(n_r+1)}\sum_{k_3,\ldots,k_r}\mathcal{C}_1^{n_1+1}\mathcal{C}_2^{n_2+\cdots+n_r+r-1}\mathcal{B}_3^{k_3}\ldots\mathcal{B}_r^{k_r} \\ &\quad\times\prod_{j=3}^{r}\frac{\binom{\sum_{i=j}^{r}n_i+r-j+1-\sum_{i=j+1}^{n}k_i}{k_j}}{\left(\sum_{i=j}^{r}n_i+r-j+1-\sum_{i=j+1}^{n}k_i\right)}.\end{aligned} \tag{12.2.8}$$

The result now follows by summing, in order, over the remaining indices. □

Observe that the reduction (12.2.6) performed in the proof allows to restate a simpler version of Sadaoui's formula (12.1.10) as the more tractable $(r-1)$−fold sum

$$\begin{aligned}\zeta_r(-n_1,\ldots,-n_r) &= (-1)^{\bar{n}}\sum_{k_2,\ldots,k_r}\frac{1}{(\bar{n}+r-\bar{k})} \\ &\quad\times\prod_{j=2}^{r}\frac{\binom{\sum_{i=j}^{r}n_i+r-j+1-\sum_{i=j+1}^{n}k_i}{k_j}B_{l_1}\ldots B_{l_r}}{\left(\sum_{i=j}^{r}n_i+r-j+1-\sum_{i=j+1}^{n}k_i\right)}.\end{aligned} \tag{12.2.9}$$

Moreover, the derivation of (12.2.3) is unchanged if the symbols $\mathcal{B}_1,\ldots,\mathcal{B}_r$ are replaced by a generalization of the Bernoulli symbol $\mathcal{B}$, namely the polynomial Bernoulli symbol $\mathcal{B}+z$. Similar proof as above yields the next statement.

Theorem 12.2.3. *The analytic continuation of the zeta function, as given in [53], is written as*

$$\zeta_r\left(-n_1,\ldots,-n_r,z_1,\ldots,z_r\right)=\prod_{i=1}^{r}\mathcal{C}_{1,\ldots,i}^{n_i+1}\left(z_1,\ldots,z_i\right) \tag{12.2.10}$$

with

$$\mathcal{C}_1^n\left(z_1\right)=\frac{\left(z_1+\mathcal{B}_1\right)^n}{n}=\frac{B_n\left(z_1\right)}{n},\ \mathcal{C}_{1,2}^n\left(z_1,z_2\right)=\frac{\left(\mathcal{C}_1\left(z_1\right)+\mathcal{B}_2+z_2\right)^n}{n},\ldots \tag{12.2.11}$$

and

$$\mathcal{C}_{1,2,\ldots,k+1}^n\left(z_1,\ldots,z_{k+1}\right)=\frac{\left(\mathcal{C}_{1,2,\ldots,k}\left(z_1,\ldots,z_k\right)+\mathcal{B}_{k+1}+z_{k+1}\right)^n}{n}. \tag{12.2.12}$$

12.2.2 Recursion Formula on the Depth

The methods above are now used to produce a general recursion formula on the depth of the multiple zeta function.

Theorem 12.2.4. *The multiple zeta functions satisfy the recursion rule that*

$$\zeta_r\left(-\boldsymbol{n};\boldsymbol{z}\right)=\frac{(-1)^{n_r}}{n_r+1}\sum_{k=0}^{n_r+1}\binom{n_r+1}{k}(-1)^k\,\zeta_r\left(-n_1,\ldots,-n_{r-1}-k;\boldsymbol{z}\right)\left(\mathcal{B}+z_r\right)^{n_r+1-k}. \tag{12.2.13}$$

If we further introduce the new zeta symbol $\mathcal{Z}_r$ by

$$\mathcal{Z}_r^k=\zeta_r\left(-n_1,\ldots,-n_{r-1},-n_r-k;\mathbf{z}\right), \tag{12.2.14}$$

(also noting $\mathcal{Z}_r^0\neq 1$) this recursion rule can be written symbolically as

$$\zeta_r\left(-\mathbf{n};\mathbf{z}\right)=(-1)^{n_r}\frac{\left(\mathcal{B}-\mathcal{Z}_{r-1}\right)^{n_r+1}}{n_r+1}=\zeta_1\left(-n_r;-\mathcal{Z}_{r-1}\right). \tag{12.2.15}$$

Proof. Start from (12.2.10) and expand the last term

$$\mathcal{C}_{1,\ldots,r}^{n_r+1}\left(z_1,\ldots,z_r\right)=\frac{\left[\mathcal{C}_{1,\ldots,r-1}^{n_{r-1}+1}\left(z_1,\ldots,z_{r-1}\right)+\mathcal{B}_r\left(z_r\right)\right]^{n_r+1}}{n_r+1} \tag{12.2.16}$$

using the binomial theorem to expand and produce

$$\begin{aligned}\zeta_r\left(-n_1,\ldots,-n_r,z_1,\ldots,z_r\right) &= \frac{(-1)^{n_r}}{n_{r+1}}\sum_{k=0}^{n_r+1}\binom{n_r+1}{k}\prod_{i=1}^{r-2}\mathcal{C}_{1,\ldots,i}^{n_i+1}\left(z_1,\ldots,z_i\right) \\ &\times\ \mathcal{C}_{1,\ldots,r-1}^{n_r+1+k}\left(z_1,\ldots,z_{r-1}\right)\mathcal{B}_r^{n_r+1-k}\left(z_r\right). \qquad (12.2.17)\end{aligned}$$

Then identify

$$\left(\prod_{i=1}^{r-2}\mathcal{C}_{1,\ldots,i}^{n_i+1}\left(z_1,\ldots,z_i\right)\right)\mathcal{C}_{1,\ldots,r-1}^{n_r+1+k}\left(z_1,\ldots,z_{r-1}\right) \qquad (12.2.18)$$

as

$$(-1)^{n_1+\cdots+n_{r-2}+n_{r-1}+k}\,\zeta_{r-1}\left(-n_1,\ldots,-n_{r-2},-n_{r-1}-k;\mathbf{z}\right) \qquad (12.2.19)$$

to obtain the desired result.

Now, by using the symbol $\mathcal{Z}$, this identity is written as

$$\zeta_r\left(-n_1,\ldots,-n_r,z_1,\ldots,z_r\right) = \frac{(-1)^{n_r}}{n_{r+1}}\left(\mathcal{B}-\mathcal{Z}_{r-1}\right)^{n_r+1} \qquad (12.2.20)$$

and the initial value

$$\zeta_1\left(-n;z\right) = (-1)^n\,\frac{\left(z+\mathcal{B}\right)^{n+1}}{n+1} \qquad (12.2.21)$$

provides the stated recursion. □

12.2.3 Contiguity Identities

The multiple zeta function at negative integer values satisfies contiguity identities in the $\boldsymbol{z}$ variables. Two of them are presented here.

Theorem 12.2.5. *The zeta function satisfies the contiguity identity*

$$\begin{aligned}\zeta_r\left(-n_1,\ldots,-n_r;z_1,\ldots,z_{r-1},z_r+1\right) &= \zeta_r\left(-n_1,\ldots,-n_r;z_1,\ldots,z_{r-1},z_r\right) \\ &+(-1)^{n_r}\left(z_r-\mathcal{Z}_{r-1}\right)^{n_r}. \qquad (12.2.22)\end{aligned}$$

Proof. Expand

$$\begin{aligned} &\zeta_r\left(-n_1,\ldots,-n_r;z_1,\ldots,z_{r-1},z_r+1\right) \\ = \quad &\frac{(-1)^{\bar{n}}}{n_r+1}\mathcal{C}_1^{n_1+1}\left(z_1\right)\ldots\mathcal{C}_{1,\ldots,r-2}^{n_{r-2}+1}\left(z_1,\ldots,z_{r-2}\right) \\ &\times\sum_{k=0}^{n_r+1}\binom{n_r+1}{k}\mathcal{C}_{1,\ldots,r-1}^{n_{r-1}+1+k}\left(z_1,\ldots,z_{r-1}\right)B_{n_r+1-k}\left(z_r+1\right) \end{aligned} \tag{12.2.23}$$

and use the identity for Bernoulli polynomials

$$B_{n_r+1-k}\left(z_r+1\right)=B_{n_r+1-k}\left(z_r\right)+\left(n_r-k+1\right)z_r^{n_r-k} \tag{12.2.24}$$

to produce the result. □

Example 12.2.6. In the case of the zeta function of depth 2,

$$\zeta_2\left(-n_1,-n_2,z_1,z_2+1\right)=\zeta_2\left(-n_1,-n_2,z_1,z_2\right)+(-1)^{n_1+1}\left(z_2-\mathcal{Z}_1\right)^{n_2}, \tag{12.2.25}$$

where the second term is expanded as

$$(-1)^{n_1+1}\sum_{k=0}^{n_2}\binom{n_2}{k}z_2^{n_2-k}(-1)^k\zeta_1\left(-n_1-k;z_1\right). \tag{12.2.26}$$

The corresponding result for a shift in the first variable admits a similar proof.

Theorem 12.2.7. *The depth-2 zeta function satisfies the contiguity identities*

$$\zeta_2\left(-n_1,-n_2,z_1+1,z_2\right)=\zeta_2\left(-n_1,-n_2,z_1,z_2\right)+\frac{(-1)^{n_1+n_2}}{n_2+1}z_1^{n_1}B_{n_2+1}\left(z_1+z_2\right). \tag{12.2.27}$$

12.2.4 A Generating Function

The generating function of the zeta values at negative integers is defined by

$$F_r\left(w_1,\ldots,w_r\right)=\sum_{n_1,\ldots,n_r\geq 0}\frac{w_1^{n_1}\ldots w_r^{n_r}}{n_1!\ldots n_r!}\zeta_r\left(-n_1,\ldots,-n_r\right). \tag{12.2.28}$$

A recurrence for F_r is presented below. The initial condition is given in terms of the generating function for Bernoulli numbers

$$F_B(w) = \sum_{n=0}^{\infty} \frac{B_n}{n!} w^n = e^{w\mathcal{B}} = \frac{w}{e^w - 1}. \tag{12.2.29}$$

Theorem 12.2.8. *The generating function of the zeta values at negative integers satisfies the recurrence*

$$F_r(w_1, \ldots, w_r) = \frac{F_{r-1}(w_1, \ldots, w_{r-1}) - F_B(-w_r) F_{r-1}(w_1, \ldots, w_{r-2}, w_{r-1} + w_r)}{w_r}, \tag{12.2.30}$$

with the initial value

$$F_1(w_1) = -\frac{1}{w_1}\left[e^{-w_1\mathcal{B}_1} - 1\right] = \frac{1 - F_B(-w_1)}{w_1}. \tag{12.2.31}$$

Moreover, recall the representation of the translation operator as mentioned before that

$$\exp\left(a\frac{\partial}{\partial w}\right) \circ f(w) = f(w + a) \tag{12.2.32}$$

and

$$F_1(w, z) = -\frac{1}{w}\left[e^{-w(\mathcal{B}+z)} - 1\right], \tag{12.2.33}$$

one could express the recursion symbolically as

$$F_r(w_1, \ldots, w_r) = F_1\left(w_r, -\frac{\partial}{\partial w_{r-1}}\right) \circ F_{r-1}(w_1, \ldots, w_{r-1}), \tag{12.2.34}$$

so that

$$\begin{aligned} F_r(w_1, \ldots, w_r) &= F_1\left(w_r, -\frac{\partial}{\partial w_{r-1}}\right) \circ F_1\left(w_{r-1}, -\frac{\partial}{\partial w_{r-2}}\right) \\ &\quad \circ \cdots \circ F_1\left(w_2, -\frac{\partial}{\partial w_1}\right) \circ F_1(w_1). \end{aligned} \tag{12.2.35}$$

Proof. Start from

$$\begin{aligned} F_r(w_1, \ldots, w_r) &= \sum_{n_1, \ldots n_r} \frac{w_1^{n_1} \cdots w_r^{n_r}}{n_1! \cdots n_r!} (-1)^{n_1 + \cdots + n_r} \mathcal{C}_1^{n_1+1} \cdots \mathcal{C}_{1,\ldots,r}^{n_r+1} \\ &= \prod_{j=1}^{r} \mathcal{C}_{1,\ldots,j} e^{-w_j \mathcal{C}_{1,\ldots,j}}, \end{aligned} \tag{12.2.36}$$

and expand

$$\begin{aligned}\mathcal{C}_{1,\ldots,r}e^{-w_r\mathcal{C}_{1,\ldots,r}} &= \sum_{n=0}^{\infty}\frac{(-w_r)^n}{n!}\cdot\frac{(-1)^{n+1}}{n+1}\left(\mathcal{C}_{1,\ldots,r-1}+\mathcal{B}_r\right)^{n+1} \\ &= -\frac{1}{w_r}\left(e^{-w_r(\mathcal{C}_{1,\ldots,r-1}+\mathcal{B}_r)}-1\right),\end{aligned} \tag{12.2.37}$$

to deduce

$$\begin{aligned}& F_r\left(w_1,\ldots,w_r\right) \\ = \; & \frac{1}{w_r}\left(\prod_{j=1}^{r-1}\mathcal{C}_{1,\ldots,j}e^{-w_j\mathcal{C}_{1,\ldots,j}}\right) \\ = \; & -\frac{1}{w_r}\left(\prod_{j=1}^{r-2}\mathcal{C}_{1,\ldots,j}e^{-w_j\mathcal{C}_{1,\ldots,j}}\right)e^{-w_r\mathcal{B}_r}\mathcal{C}_{1,\ldots,r-1}e^{-(w_{r-1}+w_r)\mathcal{C}_{1,\ldots,r-1}} \\ = \; & \frac{1}{w_r}F_{r-1}\left(w_1,\ldots,w_{r-1}\right)-\frac{1}{w_r}F_B\left(-w_r\right)F_{r-1}\left(w_1,\ldots,w_{r-2},w_{r-1}+w_r\right).\end{aligned} \tag{12.2.38}$$

This completes the proof. □

12.2.5 Shuffle Identity

Multiple zeta values at positive integers satisfy *shuffle identities*, such as

$$\zeta_2\left(n_1,n_2\right)+\zeta_2\left(n_2,n_1\right)+\zeta_1\left(n_1+n_2\right)=\zeta_1\left(n_1\right)\zeta_1\left(n_2\right). \tag{12.2.39}$$

The analytic continuation technique used in [53] does not preserve this identity at negative integers, while others do (for example, see [47]). The following theorem gives the correction terms.

Theorem 12.2.9. *The zeta values at negative integers satisfy the identity*

$$\begin{aligned}& \zeta_2\left(-n_1,-n_2\right)+\zeta_2\left(-n_2,-n_1\right)+\zeta_1\left(-n_1-n_2\right)-\zeta_1\left(-n_1\right)\zeta_1\left(-n_2\right) \\ = \; & \frac{(-1)^{n_1+1}n_1!n_2!}{(n_1+n_2+2)!}B_{n_1+n_2+2}.\end{aligned} \tag{12.2.40}$$

Remark 12.2.10. When n_1+n_2 is odd, $B_{n_1+n_2+2}=0$ so that the shuffle identity (12.2.40) holds for $\zeta_2\left(-n_1,-n_2\right)$ as expected, since the depth-2 zeta function is holomorphic at these points.

Proof. Define that

$$\delta\left(w_1,w_2\right)=F_2\left(w_1,w_2\right)+F_2\left(w_2,w_1\right)+F_1\left(w_1+w_2\right)-F_1\left(w_1\right)F_1\left(w_2\right). \quad (12.2.41)$$

An elementary calculation gives

$$\delta\left(w_1,w_2\right)=\frac{\frac{1}{w_1}+\frac{1}{w_2}-\frac{1}{2}\coth\left(\frac{w_1}{2}\right)-\frac{1}{2}\coth\left(\frac{w_2}{2}\right)}{w_1+w_2}. \quad (12.2.42)$$

Then, expansions

$$\frac{1}{w_1}-\frac{1}{2}\coth\left(\frac{w_1}{2}\right)=-\sum_{k=0}^{\infty}\frac{w_1^{2k+1}}{(2k+2)!}B_{2k+2} \quad (12.2.43)$$

and

$$\frac{1}{w_1+w_2}=\frac{1}{w_2}\sum_{l=0}^{\infty}\left(-\frac{w_1}{w_2}\right)^l \quad (12.2.44)$$

now produce

$$\delta\left(w_1,w_2\right)=-\sum_{k,l=0}^{\infty}\left(-1\right)^l\frac{B_{2k+2}}{(2k+2)!}\left(w_1^{2k+l+1}w_2^{-l-1}+w_1^lw_2^{2k-l}\right). \quad (12.2.45)$$

Identifying the coefficient of $w_1^{n_1}w_2^{n_2}$ in this series expansion gives the result. □

12.2.6 An Empirical Derivation of Raabe's Identity and Its Analytic Continuation

Define the new symbol $\mathcal{V}$ by

$$f\left(z+\mathcal{V}\right)=\int_1^{+\infty}f\left(z+v\right)dv. \quad (12.2.46)$$

Observe that

$$\begin{aligned} f\left(z+\mathcal{V}\right) &= \int_1^{+\infty}f\left(z+v\right)dv \quad (12.2.47)\\ &= \sum_{k\geq1}\int_k^{k+1}f\left(z+v\right)dv\\ &= \sum_{k\geq1}\int_0^1 f\left(z+v+k\right)dv\\ &= \sum_{k\geq1}f\left(z+k+\mathcal{U}\right) \end{aligned}$$

gives

$$f(z+\mathcal{V})=\sum_{k\geq 1}f(z+k+\mathcal{U}) \tag{12.2.48}$$

or equivalently, replacing z by $z+\mathcal{B}$,

$$\sum_{k\geq 1}f(z+k)=f(z+\mathcal{V}+\mathcal{B}). \tag{12.2.49}$$

Repeating this operation produces

$$\begin{aligned}&\sum_{k_1,k_2\geq 1}f_1(z_1+k_1)\,f_2(z_1+k_1+x_2+k_2) \\ =\ & f_1(z_1+\mathcal{V}_1+\mathcal{B}_1)\,f_2(z_1+\mathcal{V}_1+\mathcal{B}_1+z_2+\mathcal{V}_2+\mathcal{B}_2),\end{aligned} \tag{12.2.50}$$

which is the symbolic expression of Raabe's identity for $r=2$.

The mechanism behind the analytic continuation of this identity can be better understood by considering the $r=1$ case. Recall the Hurwitz-zeta function

$$\zeta(z;n)=\sum_{k=1}^{\infty}\frac{1}{(z+k)^n}, \tag{12.2.51}$$

and compute

$$\zeta(z+\mathcal{U};n)=\sum_{k=1}^{\infty}\frac{1}{(z+\mathcal{U}+k)^n}=\frac{1}{n-1}(z+1)^{1-n}. \tag{12.2.52}$$

Thus

$$\zeta(z;n)=\zeta(z+\mathcal{U}+\mathcal{B};n)=\frac{1}{n-1}(z+\mathcal{B}+1)^{1-n}. \tag{12.2.53}$$

Replacing now n by $-2n$ gives the analytic continuation of the Hurwitz-zeta function

$$\zeta(z;-2n)=\frac{-1}{2n+1}B_{2n+1}(z+1). \tag{12.2.54}$$

This is also the mechanism used in Sadaoui's paper.

Remark 12.2.11. Let $z=0$ in (12.2.53), $n\mapsto -n$, and also note (10.1.7), we have

$$\zeta(-n)=\frac{1}{-n-1}(\mathcal{B}+1)^{n+1}=\frac{(-1)^n}{n+1}\mathcal{B}^{n+1}=\frac{(-1)^n}{n+1}B_{n+1}. \tag{12.2.55}$$

Case 1. When n is odd, then

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}, \tag{12.2.56}$$

satisfying the analytic continuation at negative integers.

Case 2. When $n > 0$ and is even, then automatically,

$$\zeta(-n) = 0, \tag{12.2.57}$$

which also satisfies the analytic continuation.

Case 3. When $n = 0$,

$$\zeta(0) = \frac{(-1)^0}{0+1} B_{0+1} = -\frac{1}{2}. \tag{12.2.58}$$

Therefore, it suggests that to uniformly rewrite the analytic continuation of Riemann-zata function at non-positive integers as

$$\zeta(-n) = \frac{(-1)^n}{n+1} \mathcal{B}^{n+1} = \frac{(-1)^n}{n+1} B_{n+1}. \tag{12.2.59}$$

12.3 Examples on Specific Multiple Zeta Values

This final section gives some examples of the evaluation at negative integers of the zeta function, obtained from (12.1.10) and (12.2.15).

Example 12.3.1. For depth $r = 2$, we have

$$\zeta_2(-n, 0) = (-1)^n \left[\frac{B_{n+2}}{n+2} - \frac{1}{2} \frac{B_{n+1}}{n+1} \right], \tag{12.3.1}$$

and

$$\zeta_2(0, -n) = \frac{(-1)^{n+1}}{n+1} \left[B_{n+1} + B_{n+2} \right]. \tag{12.3.2}$$

Example 12.3.2. For depth $r = 3$, one can compute

$$\zeta_3(-n, 0, 0) = \frac{(-1)^n}{2} \left[\frac{B_{n+3}}{n+3} - 2 \frac{B_{n+2}}{n+2} + \frac{2}{3} \frac{B_{n+1}}{n+1} \right], \tag{12.3.3}$$

and

$$\zeta_3(0,-n,0)=\frac{(-1)^{n+1}}{2}\left[\frac{n}{(n+1)(n+2)}B_{n+2}-\frac{B_{n+1}}{n+1}+2\frac{B_{n+3}}{n+2}\right]. \tag{12.3.4}$$

Apparently, $\zeta_3(0,0,-n)$ can be obtained by rule (12.2.13). For instance, let $n=2$, we have

$$\begin{aligned}\zeta_3(0,0,-2) &= \frac{(\mathcal{B}-\mathcal{Z}_2)^3}{3}=\frac{1}{3}\left(\mathcal{B}^3\mathcal{Z}_2^0-3\mathcal{B}^2\mathcal{Z}_2^1+3\mathcal{B}\mathcal{Z}_2^2-\mathcal{Z}_2^3\right) \\ &= \frac{1}{3}\left(B_3\zeta_2(0,0)-3B_2\zeta_2(0,-1)+3B_1\zeta_2(0,-2)-\zeta_2(0,-3)\right) \\ &= -\frac{1}{60}.\end{aligned} \tag{12.3.5}$$

Chapter 13

Hypergeometric Bernoulli Numbers and Zeta Functions

13.1 Introduction and Preliminaries

13.1.1 Definitions and Examples of Zeta Functions

The zeta function attached to a collection of non-zero complex numbers $\mathbb{A} = \{a_n \neq 0 : n \in \mathbb{N}\}$ is defined by

$$\zeta_{\mathbb{A}}(s) = \sum_{n=1}^{\infty} \frac{1}{a_n^s}, \text{ for } \operatorname{Re}(s) > c. \tag{13.1.1}$$

The most common choice of sequences $\mathbb{A}$ includes those coming from the zeros of a given function f:

$$\mathbb{A}(f) = \{z \in \mathbb{C} : f(z) = 0\} = \{z_n \in \mathbb{C} : f(z_n) = 0,\ n \in \mathbb{N}\}, \tag{13.1.2}$$

to produce the associated zeta function

$$\zeta_f(s) = \sum_{n=1}^{\infty} \frac{1}{z_n^s}. \tag{13.1.3}$$

The prototypical example is the classical Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{13.1.4}$$

coming from half of the zeros $\mathbb{A} = \{z_n = n > 0\}$ of the function $f(z) = \dfrac{\sin \pi z}{\pi z}$. The literature contains a variety of zeta functions $\zeta_{\mathbb{A}}$ and their study is concentrated in reproducing the basic properties of (13.1.4).

Carlitz introduced in [12] coefficients β_n by

$$\frac{x^2}{e^x - 1 - x} = \sum_{n=0}^{\infty} \beta_n \frac{x^n}{n!}, \tag{13.1.5}$$

and stated that *nothing is known about them.* Howard [36] used the notation $A_s = \frac{1}{2}\beta_s$, and introduced in [37] the generalization $A_{k,r}$ by

$$\frac{x^k}{k!}\left(e^x - \sum_{s=0}^{k-1} \frac{x^s}{s!}\right)^{-1} = \sum_{r=0}^{\infty} A_{k,r} \frac{x^r}{r!}. \tag{13.1.6}$$

These numbers satisfy the recurrence

$$\sum_{r=0}^{n} \binom{n+k}{r} A_{k,r} = 0, \text{ for } n > 0 \tag{13.1.7}$$

with $A_{k,0} = 1$. It follows that $A_{k,n}$ is a rational number.

Definition 13.1.1. The work in this chapter considers a zeta function constructed in terms of the *Kummer function*

$$M(a, b; z) = {}_1F_1\left(\left.\begin{matrix} a \\ b \end{matrix}\right| z\right) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}. \tag{13.1.8}$$

For simplicity, introduce the notation that for $a, b \in \mathbb{R}_+$

$$\Phi_{a,b}(z) := {}_1F_1\left(\left.\begin{matrix} a \\ a+b \end{matrix}\right| z\right) = M(a, a+b; z). \tag{13.1.9}$$

The *hypergeometric zeta function* is defined by

$$\zeta^H_{a,b}(s) = \sum_{k=1}^{\infty} \frac{1}{z^s_{k;a,b}} \text{ for } \operatorname{Re}(s) > 1, \tag{13.1.10}$$

where $z_{k;a,b}$ is the sequence of complex zeros of the function $\Phi_{a,b}(z)$.

Remark 13.1.2. The special case $\Phi_{1,1}$:

$$\frac{1}{\Phi_{1,1}(z)} = \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \tag{13.1.11}$$

is the reciprocal of the exponential generating function for the Bernoulli numbers (and therefore the moment generating function of the uniform symbol).

Definition 13.1.3. Also, we define coefficients $B_n^{(b)}$ by the exponential generating function as follows:

$$\frac{1}{\Phi_{1,b}(z)} = \sum_{n=0}^{\infty} B_n^{(b)} \frac{z^n}{n!}. \tag{13.1.12}$$

In the case $b \in \mathbb{N}$, these numbers are the coefficients $A_{k,r}$ defined by Howard in (13.1.6) (with $k = b$ and $r = n$). They are discussed in Section 13.2.2. The function $\Phi_{1,2}(z)$ also appeared in [14] in the asymptotic expansion of $n!$. Indeed, it can be shown that the coefficients a_k in the expansion

$$n! \sim \frac{n^n \sqrt{2\pi n}}{e^n} \sum_{k=0}^{\infty} \frac{a_k}{n^k} \text{ as } n \to \infty, \tag{13.1.13}$$

are given by

$$a_k = \frac{1}{2^k k!} \left(\frac{d}{dz} \right)^{2k} \Phi_{1,2}^{-(k+1/2)} \bigg|_{z=0}. \tag{13.1.14}$$

K. Dilcher [15, 16] considered the zeta function $\zeta_{a,b}^H$. In particular, he established an expression for $\zeta_{a,b}^H(m)$, for a, b, $m \in \mathbb{N}$, in terms of the hypergeometric Bernoulli numbers $B_{a,b}^n$, which are introduced in Section 13.2.4.

Example 13.1.4. Many examples of zeta functions discussed in the literature include the *Bessel zeta function*

$$\zeta_{Bes,a}(s) = \sum_{n=1}^{\infty} \frac{1}{j_{a,n}^s}, \tag{13.1.15}$$

where $\{j_{a,n}\}$ are the zeros of $J_a(z)/z^a$, including the Bessel function of the first kind. Papers considering $\zeta_{Bes,a}$ include [2, 20, 35, 55].

Example 13.1.5. A second example is the *Airy-zeta function*, defined by

$$\zeta_{Ai}(s) = \sum_{n=1}^{\infty} \frac{1}{a_n^s}, \tag{13.1.16}$$

where $\{a_n\}$ are the zeros of the Airy function

$$Ai(x) = \frac{1}{\pi} \int_0^{\infty} \cos\left(\tfrac{1}{3}t^3 + xt\right)\, dt. \tag{13.1.17}$$

This is considered by R. Crandall [13] in the so-called *quantum bouncer*. Special values include the remarkable

$$\zeta_{Ai}(2) = \frac{3^{5/3}}{4\pi^2}\Gamma^4\left(\frac{2}{3}\right). \tag{13.1.18}$$

Example 13.1.6. A third example is the zeta function studied by A. Hassen and H. Nguyen [33, 34]. This is defined by the integral

$$\zeta_{HN,b}(s) = \frac{1}{\Gamma(s+b-1)} \int_0^{\infty} \frac{x^{s+b-2}\, dx}{e^x - 1 - z - z^2/2! - \cdots - z^{b-1}/(b-1)!}. \tag{13.1.19}$$

Notation. It is an unfortunate fact that many of the terms used in the present work are denoted by the letter B. The list below shows the symbols employed here.

Notation	**Name**	**Definition**
B_n	Bernoulli number	(10.1.1)
$M(a,b;z)$	Kummer function	(13.1.8)
$\Phi_{a,b}(z)$	Kummer function	(13.1.9)
$\zeta_{a,b}^{H}(s)$	hypergeometric zeta function	(13.1.10)
$B_n^{(b)}$	hypergeometric Bernoulli number	(13.1.12)
$B(a,b)$	the beta function	(13.2.6)
$\mathfrak{B}_{a,b}$	a beta distributed random variable	(13.2.28)
$\mathfrak{Z}_{a,b}$	a complex random variable	(13.2.34)
$B_n^{(a,b)}(x)$	hypergeometric Bernoulli polynomial	(13.2.54)
$B_n^{(a,b)}$	hypergeometric Bernoulli number	(13.2.56)

13.1.2 Properties of the Kummer Function $\Phi_{a,b}(z)$.

The function

$$\Phi_{a,b}(z) ={}_1 F_1 \left(\begin{matrix} a \\ a+b \end{matrix} \Bigg| z \right) = M(a, a+b; z), \text{ for } a, b \in \mathbb{R} \tag{13.1.20}$$

defined in terms of the Kummer function $M(a, b; z)$ is the main object considered in the present work. The function $M(a, b; z)$ satisfies the differential equation

$$z\frac{d^2M}{dz^2} + (b - z)\frac{dM}{dz} - aM = 0, \tag{13.1.21}$$

obtained from the standard hypergeometric equation

$$z\frac{dw^2}{dz^2} + [c - (a + b + 1)z]\frac{dw}{dz} - abw = 0 \tag{13.1.22}$$

by scaling $z \mapsto z/b$, letting $b \to \infty$ and replacing the parameter c by b.

The first result shows that the special case $a = 1$ gives the function considered by Howard [37].

Theorem 13.1.7. *For $b \in \mathbb{N}$, the function $\Phi_{1,b}(z)$ is given by*

$$\Phi_{1,b}(z) = \frac{b!}{z^b}\left(e^z - \sum_{k=0}^{b-1} \frac{z^k}{k!}\right). \tag{13.1.23}$$

Proof. This follows directly from the expansion

$$\Phi_{1,b}(z) = {}_1F_1 \left(\begin{matrix} 1 \\ 1+b \end{matrix} \Bigg| z \right) = \sum_{k=0}^{\infty} \frac{(1)_k z^k}{(1+b)_k \, k!} = \sum_{k=0}^{\infty} \frac{b!}{(b+k)!} z^k = \frac{b!}{z^b}\sum_{k=b}^{\infty} \frac{z^k}{k!}. \tag{13.1.24}$$

□

Corollary 13.1.8. *The zeta function $\zeta_{HN,b}(s)$ in (13.1.19) is given by*

$$\zeta_{HN,b}(s) = \frac{b!}{\Gamma(s+b-1)} \int_0^\infty \frac{x^{s-2}\, dx}{\Phi_{1,b}(x)}. \tag{13.1.25}$$

The next property of $\Phi_{a,b}(z)$ is a representation as an infinite product. The result comes from the classical Hadamard factorization theorem for entire functions. A preliminary lemma is given first.

Lemma 13.1.9. *The Kummer function* $\Phi_{a,b}(z)$ *satisfies*

$$\frac{d}{dz}\Phi_{a,b}(z) = \frac{a}{a+b}\Phi_{a+1,b}(z). \tag{13.1.26}$$

Proof. This comes directly from formula

$$\frac{d}{dz} = {}_1F_1\left(\begin{matrix} a \\ b \end{matrix}\middle| z\right) = \frac{a}{b} = {}_1F_1\left(\begin{matrix} a+1 \\ b+1 \end{matrix}\middle| z\right), \tag{13.1.27}$$

which is entry **13.3.15** on page 325 of [46]. □

Definition 13.1.10. The *order* of an entire function $h(z)$ is defined as the infimum of $\alpha \in \mathbb{R}$ for which there exists a radius $r_0 > 0$ such that

$$|h(z)| < e^{|z|^\alpha} \text{ for } |z| > r_0. \tag{13.1.28}$$

The order of $h(z)$ is denoted by $\rho(h)$. See [9] for further information.

The main result used here is Hadamard's theorem stated below.

Theorem 13.1.11. *[Hadamard's Theorem] For* $p \in \mathbb{N}$, *define the elementary factors*

$$E_p(z) = \begin{cases} 1 - z & \textit{if } p = 0, \\ (1-z)exp\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^p}{p}\right) & \textit{otherwise.} \end{cases} \tag{13.1.29}$$

Assume h *is an entire function of finite order* $\rho = \rho(h)$. *Let* $\{a_n\}$ *be the collection of zeros of* h *repeated according to multiplicity. Then* h *admits the factorization*

$$h(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right). \tag{13.1.30}$$

where $g(z)$ *is a polynomial of degree* $q \leq \rho$, $p = \lfloor \rho \rfloor$ *and* $m \geq 0$ *is the order of the zero of* h *at the origin.*

Theorem 13.1.12. *Let* a, $b > 0$. *Then* $\Phi_{a,b}$ *is an entire function of order* 1.

Proof. The ratio test shows that the function $\Phi_{a,b}(z)$ is entire. Moreover,

$$\frac{(a)_\ell}{(a+b)_\ell} = \prod_{k=0}^{\ell-1} \frac{a+k}{a+b+k} < 1. \tag{13.1.31}$$

Therefore

$$|\Phi_{a,b}(z)| \leq \sum_{\ell=0}^{\infty} \frac{(a)_\ell}{(a+b)_\ell} \left| \frac{z^\ell}{\ell!} \right| \leq \sum_{\ell=0}^{\infty} \frac{|z|^\ell}{\ell!} = e^{|z|}. \tag{13.1.32}$$

This proves $\rho(h) \leq 1$.

To establish the opposite inequality, use the asymptotic behavior

$$M(a, b; z) \sim \frac{e^{-z} z^{a-b}}{\Gamma(a)}, \text{ as } z \to \infty \tag{13.1.33}$$

(see [46], page 323) to see that, for every $0 \leq \varepsilon < 1$ and $z \in \mathbb{R}$,

$$\lim_{z\to\infty} \frac{\Phi_{a,b}(z)}{\exp(z^\varepsilon)} = +\infty. \tag{13.1.34}$$

Hence, for any given $r_0 > 0$, there is $r > r_0$ such that

$$|\Phi_{a,b}(r)| = \Phi_{a,b}(r) > \exp\left(r^\varepsilon\right) = \exp\left(|r|^\varepsilon\right). \tag{13.1.35}$$

This proves $\rho(h) \geq 1$ and the proof is completed. □

The factorization of $\Phi_{a,b}(z)$ in terms of its zeros $\{z_{a,b;k}\}$ is discussed next. Section 13.9 of [46] states that if a and $b \neq 0, -1, -2, \cdots$, then $\Phi_{a,b}(z)$ has infinitely many complex zeros. Moreover, if a, $b \geq 0$, then there are no real zeros. The growth of the large zeros of $M(a, a+b; z)$ is given by

$$z_{a,b;n} = \pm(2n+a)\pi\imath + \ln\left(-\frac{\Gamma(a)}{\Gamma(b)}(\pm 2n\pi\imath)^{b-a}\right) + O(n^{-1}\ln n), \tag{13.1.36}$$

where n is a large positive integer, and the logarithm takes its principal value.

Theorem 13.1.13. *The function $\Phi_{a,b}(z)$ admits the factorization*

$$\Phi_{a,b}(z) = e^{az/(a+b)} \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_{a,b;k}}\right) e^{z/z_{a,b;k}}. \tag{13.1.37}$$

Proof. Hadamard's theorem shows the existence of two constants A, B such that

$$\Phi_{a,b}(z) = e^{Az+B} \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_{a,b;k}} \right) e^{z/z_{a,b;k}}. \tag{13.1.38}$$

Evaluating at $z = 0$, using $\Phi_{a,b}(0) = 1$, gives $B = 0$. To obtain the value of the parameter A, take the logarithmic derivative of (13.1.38) and use Lemma 13.1.9 to produce

$$\frac{a}{a+b} \cdot \frac{{}_1F_1\left(\left. \begin{matrix} a+1 \\ a+b+1 \end{matrix} \right| z \right)}{{}_1F_1\left(\left. \begin{matrix} a \\ a+b \end{matrix} \right| z \right)} = A + \sum_{k=0}^{\infty} \left[\frac{1}{z_{a,b;k}} - \frac{1}{z_{a,b;k} - z} \right]. \tag{13.1.39}$$

Expanding both sides near $z = 0$ gives

$$\frac{a}{a+b} + \frac{ab}{(a+b)^2(1+a+b)} z + \mathcal{O}(z^2) = A - \sum_{k=1}^{\infty} \frac{z}{z_{a,b;k}^2} + \mathcal{O}(z^2). \tag{13.1.40}$$

This gives $A = a/(a+b)$, completing the proof. $\square$

Corollary 13.1.14. *The hypergeometric zeta function has the special value*

$$\zeta_{a,b}^{H}(2) = -\frac{ab}{(a+b)^2(1+a+b)}. \tag{13.1.41}$$

Proof. Compare the coefficients of z in (13.1.40). $\square$

The next statement presents additional properties of the Kummer function which will be useful in the next section. It appears as entries **13.4.12** and **13.4.13** in [1].

Lemma 13.1.15. *The Kummer function satisfies*

$$\frac{d}{dz}\Phi_{a,b}(z) = -\frac{b}{a+b}\Phi_{a,b+1}(z) + \Phi_{a,b}(z) \tag{13.1.42}$$

and

$$\frac{d}{dz}\Phi_{a,b+1}(z) = \frac{a+b}{z}\left(\Phi_{a,b}(z) - \Phi_{a,b+1}(z)\right). \tag{13.1.43}$$

13.2 Main Results

13.2.1 Recurrence for the Hypergeometric Zeta Function

This section describes some recurrences for the values $\zeta_{a,b}^H(k)$. The proofs are based on a relation between the Kummer function $\Phi_{a,b}(z)$ and these values. It is the analog of standard result for the usual zeta function

$$\sum_{k=1}^{\infty} \zeta(k+1) z^k = -\gamma - \psi(1-z), \tag{13.2.1}$$

where $\psi(z) = \frac{d}{dz} \log \Gamma(z)$ is the digamma function and γ is the Euler constant. This relation is obtained directly from the product representation of $\Gamma(z)$. See entry **6.3.14** in [1].

Proposition 13.2.1. *The Kummer function $\Phi_{a,b}(z)$ and the hypergeometric zeta function are related by*

$$\frac{\Phi_{a,b+1}(z)}{\Phi_{a,b}(z)} = 1 + \frac{a+b}{b} \sum_{k=1}^{\infty} \zeta_{a,b}^H(k+1) z^k. \tag{13.2.2}$$

Proof. The relation (13.1.42) gives

$$\frac{\Phi_{a,b+1}(z)}{\Phi_{a,b}(z)} = -\frac{a+b}{b} \left[\frac{\Phi'_{a,b}(z)}{\Phi_{a,b}(z)} - 1 \right]. \tag{13.2.3}$$

The fraction on the right-hand side is the logarithmic derivative of the product in Theorem 13.1.13. This yields

$$\frac{\Phi_{a,b+1}(z)}{\Phi_{a,b}(z)} = -\frac{a+b}{b} \left(\frac{a}{a+b} + \sum_{k=0}^{\infty} \left[\frac{1}{z_{a,b;k}} - \frac{1}{z_{a,b;k} - z} \right] - 1 \right). \tag{13.2.4}$$

To establish the result, use the expansion

$$\frac{1}{z_{a,b;k} - z} = \frac{1}{z_{a,b;k}} \frac{1}{1 - z/z_{a,b;k}}, \tag{13.2.5}$$

and expand the last term as the sum of a geometric series. Since $\min_k \{|z_{a,b;k}|\} \neq 0$, this series has a positive radius of convergence. □

The next result gives a linear recurrence for the hypergeometric zeta function. This involves the beta function

$$B(u,v) = \int_0^1 x^{u-1}(1-x)^{v-1}dx = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \tag{13.2.6}$$

with values

$$B(u,v) = \frac{u+v}{uv}\binom{u+v}{u}^{-1} \text{ for } u,\, v \in \mathbb{N}. \tag{13.2.7}$$

Theorem 13.2.2. *The hypergeometric zeta function satisfies the linear recurrence*

$$\sum_{\ell=1}^{p} B(a+p-\ell,b)\frac{p!}{(p-\ell)!}\zeta_{a,b}^{H}(\ell+1) = -\frac{bp}{(a+b)(a+b+p)}B(a+p,b). \tag{13.2.8}$$

Proof. Proposition 13.2.1 gives

$$\Phi_{a,b+1}(z) = \Phi_{a,b}(z) + \frac{a+b}{b}\Phi_{a,b}(z)\sum_{k=1}^{\infty}\zeta_{a,b}^{H}(k+1)z.^{k} \tag{13.2.9}$$

Matching the coefficient of z^p gives the identity

$$\frac{(a)_p}{(a+b+1)_p} = \frac{(a)_p}{(a+b)_p} + \frac{a+b}{b}\sum_{\ell=1}^{p}\frac{(a)_{p-\ell}}{(a+b)_{p-\ell}}\frac{p!}{(p-\ell)!}\zeta_{a,b}^{H}(\ell+1). \tag{13.2.10}$$

This simplifies to produce the result. □

Example 13.2.3. The recurrence (13.2.8) can be written as

$$\zeta_{a,b}^{H}(p) = -\frac{bB(a+p-1,b)}{(a+b)(a+b+p-1)(p-2)!B(a,b)} - \sum_{r=1}^{p-2}\frac{B(a+r,b)}{B(a,b)r!}\zeta_{a,b}^{H}(p-r). \tag{13.2.11}$$

The initial condition given in (13.1.41) shows that, for $p \in \mathbb{N}$, the value $\zeta_{a,b}^{H}(p)$ is a rational function of a, b. The first few values are

$$\zeta_{a,b}^{H}(2) = -\frac{ab}{(a+b)^2(1+a+b)} \tag{13.2.12}$$

$$\zeta_{a,b}^{H}(3) = \frac{ab(a-b)}{(a+b)^3(a+b+1)(a+b+2)} \tag{13.2.13}$$

$$\zeta_{a,b}^{H}(4) = -\frac{abP_4(a,b)}{(a+b)^4(a+b+1)^2(a+b+2)(a+b+3)} \tag{13.2.14}$$

where

$$P_4(a,b) = a^2 + a^3 - 4ab - 2a^2b + b^2 - 2ab^2 + b^3. \qquad (13.2.15)$$

The patterns in $\zeta_{a,b}^H(p)$ are still now left as unknown.

The next result presents a different type of recurrence for $\zeta_{a,b}^H(s)$. It is the analogue of the classical identity

$$\left(n+\tfrac{1}{2}\right)\zeta(2n) = \sum_{k=1}^{n-1} \zeta(2k)\zeta(2n-2k), \text{ for } n \geq 2. \qquad (13.2.16)$$

See **25.6.16** in [46].

Theorem 13.2.4. *The hypergeometric zeta function satisfies the quadratic recurrence*

$$\sum_{k=1}^{p} \zeta_{a,b}^H(k+1)\zeta_{a,b}^H(p-k+1) = (a+b+p+1)\zeta_{a,b}^H(p+2) + \left(\frac{a-b}{a+b}\right)\zeta_{a,b}^H(p+1). \qquad (13.2.17)$$

Proof. The function $f(z) = \Phi_{a,b+1}(z)/\Phi_{a,b}(z)$ satisfies the differential equation

$$f'(z) = \frac{a+b}{z}(1-f(z)) - f(z) + \frac{b}{a+b}f^2(z). \qquad (13.2.18)$$

This can be verified directly using the results of Lemma 13.1.15. Now use Theorem 13.2.1 to match the coefficients of z^p and produce

$$\begin{aligned} \frac{a+b}{b}(p+1)\zeta_{a,b}^H(p+2) &= -\frac{(a+b)^2}{b}\zeta_{a,b}^H(p+2) - \frac{a+b}{b}\zeta_{a,b}^H(p+1) \qquad (13.2.19) \\ &\quad + \frac{a+b}{b}\sum_{k=1}^{p}\zeta_{a,b}^H(k+1)\zeta_{a,b}^H(p-k+1) + 2\zeta_{a,b}^H(p+1), \end{aligned}$$

which, after simplification, yields the result. $\square$

Remark 13.2.5. Matching the constant terms recovers the value of $\zeta_{a,b}^H(2)$ in (13.1.41).

13.2.2 On the Hypergeometric Bernoulli Numbers

Definition 13.2.6. The *hypergeometric Bernoulli numbers* $B_n^{(b)}$ are defined by the relation

$$\frac{1}{\Phi_{1,b}(z)} = \sum_{n=0}^{\infty} B_n^{(b)} \frac{z^n}{n!}. \qquad (13.2.20)$$

These are precisely the numbers $A_{b,n}$ studied by Howard [37], following from Theorem 13.1.7 and (13.1.6). The special case $b=1$ corresponds to the Bernoulli numbers.

The next result appears in [37] in the case $b=2$.

Theorem 13.2.7. *Let $b \in \mathbb{N}$. The hypergeometric Bernoulli numbers $B_n^{(b)}$ are expressed in terms of the hypergeometric zeta function as*

$$B_n^{(b)} = \begin{cases} 1 & \text{for } n=0, \\ -1/(1+b) & \text{for } n=1, \\ -n!\zeta_{1,b}^H(n)/b & \text{for } n \geq 2. \end{cases} \tag{13.2.21}$$

Proof. The product representation of $\Phi_{1,b}(z)$ given in Theorem 13.1.13 is

$$\Phi_{1,b}(z) = e^{z/(1+b)} \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_{1,b;k}}\right) e^{z/z_{1,b;k}}. \tag{13.2.22}$$

Logarithmic differentiation yields

$$\begin{aligned} \frac{\Phi'_{1,b}(z)}{\Phi_{1,b}(z)} &= \frac{1}{1+b} + \sum_{k=1}^{\infty} \left[\frac{1}{z_{1,b;k}} - \frac{1}{z_{1,b;k}-z}\right] && (13.2.23) \\ &= \frac{1}{1+b} + \sum_{k=1}^{\infty} \frac{1}{z_{1,b;k}} \left(1 - \sum_{\ell=0}^{\infty} \left(\frac{z}{z_{1,b;k}}\right)^{\ell}\right) \\ &= \frac{1}{1+b} - \sum_{\ell=1}^{\infty} z^{\ell} \zeta_{1,b}^H(\ell+1). \end{aligned}$$

On the other hand, Theorem 13.1.7 gives

$$\Phi_{1,b}(z) = \frac{b!}{z^b} \left(e^z - \sum_{j=0}^{b-1} \frac{z^j}{j!}\right), \tag{13.2.24}$$

and logarithmic differentiation produces

$$\frac{\Phi'_{1,b}(z)}{\Phi_{1,b}(z)} = 1 + \frac{b}{z} \cdot \frac{1}{\Phi_{1,b}(z)} - \frac{b}{z}. \tag{13.2.25}$$

Therefore,

$$\frac{1}{\Phi_{1,b}(z)} = 1 - \frac{z}{1+b} - \frac{1}{b} \sum_{\ell=2}^{\infty} z^{\ell} \zeta_{1,b}^H(\ell) = \sum_{n=0}^{\infty} B_n^{(b)} \frac{z^n}{n!}. \tag{13.2.26}$$

The conclusion follows by comparing coefficients of powers of z. $\square$

Remark 13.2.8. The previous theorem suggests the definition

$$\zeta_{1,b}^{H}(1) = \frac{b}{1+b}. \tag{13.2.27}$$

13.2.3 A Probabilistic Approach

This section presents an interpretation of the Kummer function $\Phi_{a,b}(z)$ as the expectation of a complex random variable.

Definition 13.2.9. Let $a,\ b > 0$. The random variable $\mathfrak{B}_{a,b}$ is called *beta distributed*, or simply *a beta random variable*, if its distribution function is given by

$$f_{\mathfrak{B}_{a,b}}(x) = \begin{cases} \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{13.2.28}$$

Here $B(a,b)$ is the beta function (13.2.6).

The integral representation [1, 13.2.1]

$$\Phi_{a,b}(z) = {}_1F_1\left(\begin{matrix} a \\ a+b \end{matrix} \middle| z \right) = \frac{1}{B(a,b)} \int_0^1 e^{tz} t^{a-1}(1-t)^{b-1}\, dt, \tag{13.2.29}$$

shows that $\Phi_{a,b}(z)$ is the moment generating function of a beta random variable $\mathfrak{B}_{a,b}$:

$$\mathbb{E}\left(e^{z\mathfrak{B}_{a,b}}\right) = \Phi_{a,b}(z). \tag{13.2.30}$$

Definition 13.2.10. A random variable Γ is said to be *exponentially distributed* if its distribution function is given by

$$f_{\Gamma}(x) = \begin{cases} e^{-x} & \text{if } x \geq 0, \\ 0 & \text{elsewhere.} \end{cases} \tag{13.2.31}$$

The moment generating function of an exponentially distributed random variable Γ is

$$\mathbb{E}\left[e^{z\Gamma}\right] = \frac{1}{1-z}, \text{ for } |z| < 1. \tag{13.2.32}$$

Remark 13.2.11. Similarly as the Bernoulli random variable, we make the choice for clear reason: if Γ is exponentially distributed, then

$$\mathbb{E}\left[\Gamma^{\alpha-1}\right] = \Gamma(\alpha). \tag{13.2.33}$$

Consider a sequence $\{\Gamma_k\}_{k\geq 1}$ of i.i.d. random variables, each with the same exponential distribution (13.2.31).

Definition 13.2.12. Let $\{z_{a,b;k} : k \in \mathbb{N}\}$ be the collection of zeros of the Kummer function $\Phi_{a,b}(z)$. The complex-valued random variable $\mathfrak{Z}_{a,b}$ is defined by

$$\mathfrak{Z}_{a,b} = -\frac{a}{a+b} + \sum_{k=1}^{\infty} \frac{\Gamma_k - 1}{z_{a,b;k}}. \tag{13.2.34}$$

Some properties of $\mathfrak{Z}_{a,b}$ are given below. The main relation between $\mathfrak{Z}_{a,b}$ and the Kummer function $\Phi_{a,b}(z)$ is stated first.

Theorem 13.2.13. *The complex-valued random variable $\mathfrak{Z}_{a,b}$ satisfies*

$$\mathbb{E}e^{z\mathfrak{Z}_{a,b}} = \frac{1}{\Phi_{a,b}(z)}. \tag{13.2.35}$$

This theorem makes $\mathfrak{Z}_{a,b}$ and $\mathfrak{B}_{a,b}$ conjugate to each other, and also guarantees for analytic function f

$$\mathbb{E}f\left(z + \mathfrak{B}_{a,b} + \mathfrak{Z}_{a,b}\right) = f(z) \tag{13.2.36}$$

holds, provided that the expectation is finite.

Proof. The independence of the family $\{\Gamma_k\}$ and Theorem 13.1.13 give

$$\mathbb{E}e^{z\mathfrak{Z}_{a,b}} = e^{-az/(a+b)} \prod_{k=1}^{\infty} \frac{1}{1 - z/z_{a,b;k}} e^{-z/z_{a,b;k}}. \tag{13.2.37}$$

which is the stated result. $\square$

Lemma 13.2.14. *The symmetry property*

$$\mathfrak{Z}_{a,b} = -1 - \mathfrak{Z}_{b,a}, \tag{13.2.38}$$

holds in the sense of distribution.

Proof. The symmetry property follows from Kummer transformation

$$ {}_1F_1\left(\begin{matrix} a \\ b \end{matrix}\middle| z\right) = e^z {}_1F_1\left(\begin{matrix} b-a \\ b \end{matrix}\middle| -z\right). \tag{13.2.39}$$

□

Remark 13.2.15. Hassen and Nguyen [33] show that

$$\int_0^1 x^{a-1}(1-x)^{b-1}B_n^{(a,b)}(x)\,dx = \begin{cases} B(a,b) & \text{if } n=0, \\ 0 & \text{if } n \neq 0. \end{cases} \tag{13.2.40}$$

This follows directly from (13.2.36) by taking $f(z) = z^n$ and then $z = 0$.

The next item in this section gives a probabilistic point of view of the linear recurrence in Theorem 13.2.2. Some preliminary background is discussed first.

Definition 13.2.16. The sequence of *cumulants* $\kappa_X(n)$ is defined by

$$\log \varphi_X(z) = \sum_{m=1}^{\infty} \kappa_X(m)\frac{z^m}{m!}. \tag{13.2.41}$$

Example 13.2.17. The cumulant generating function for the $\mathfrak{B}_{a,b}$ distribution is

$$\log \Phi_{a,b}(z) = \frac{a}{a+b}z - \sum_{p=2}^{\infty} \frac{z^p}{p}\zeta_{a,b}^H(p), \tag{13.2.42}$$

so the cumulants are

$$\kappa_{\mathfrak{B}_{a,b}}(1) = \frac{a}{a+b} \text{ and } \kappa_{\mathfrak{B}_{a,b}}(p) = -(p-1)!\zeta_{a,b}^H(p) \text{ for } p \geq 2. \tag{13.2.43}$$

For a general random variable X, the moments $\mathbb{E}X^n$ and its cumulants $\kappa_X(n)$ are related by

$$\kappa_X(n) = \mathbb{E}X^n - \sum_{j=1}^{n-1}\binom{n-1}{j-1}\kappa_X(j)\mathbb{E}X^{n-j}. \tag{13.2.44}$$

See [54] for details. In the special case of a random variable with a beta distribution,

$$(n-1)!\sum_{j=2}^{n}\frac{B(a+n-j,b)}{(n-j)!}\zeta_{a,b}^H(j) = \frac{a}{a+b}B(a+n-1,b) - B(a+n,b) \tag{13.2.45}$$

that is equivalent to the linear recurrence identity in Theorem 13.2.2.

13.2.4 The Generalized Bernoulli Polynomials

K. Dilcher introduced in [15] the generalized Bernoulli polynomials by

$$\frac{e^{xz}}{\Phi_{1,b}(z)} = \sum_{k=0}^{\infty} B_k^{(b)}(x)\frac{z^k}{k!}. \tag{13.2.46}$$

These polynomials are now interpreted as moments.

Theorem 13.2.18. *The generalized Bernoulli polynomials are given by*

$$B_k^{(b)}(x) = \mathbb{E}(x + \mathfrak{Z}_{1,b})^k. \tag{13.2.47}$$

Proof. This follows directly from Theorem 13.2.13. □

Dilcher [15] used generating functions to provide the following recursion for these polynomials.

Theorem 13.2.19. *The generalized Bernoulli polynomials $B_k^{(b)}(x)$ satisfy*

$$B_k^{(b)}(x+1) = \sum_{p=0}^{b-1} \binom{k}{p} B_{k-p}^{(b)}(x) + \binom{k}{b} x^{k-b}. \tag{13.2.48}$$

A probabilistic proof is presented next. A preliminary result is stated first.

Lemma 13.2.20. *Let $\mathfrak{B}_{a,b}$ be a random variable with a beta distribution and $g \in C^1[0,1]$. Then, for $a, b > 1$,*

$$\mathbb{E}g'(x + \mathfrak{B}_{a,b}) = (a+b-1)\left[\mathbb{E}g(x+\mathfrak{B}_{a,b-1}) - \mathbb{E}g(x+\mathfrak{B}_{a-1,b})\right]. \tag{13.2.49}$$

For $a = 1$ and $b > 1$

$$\mathbb{E}g'(x+\mathfrak{B}_{1,b}) = -bg(x) + b\mathbb{E}g(x+\mathfrak{B}_{1,b-1}). \tag{13.2.50}$$

Proof. A direct calculation shows

$$\begin{aligned} B(a,b)\mathbb{E}g'(x+\mathfrak{B}_{a,b}) &= \int_0^1 g'(x+t)t^{a-1}(1-t)^{b-1}dt \qquad (13.2.51)\\ &= g(x+t)t^{a-1}(1-t)^{b-1}\Big|_0^1 \\ &\quad -(a-1)\int_0^1 g(x+t)t^{a-2}(1-t)^{b-1}dt \\ &\quad +(b-1)\int_0^1 g(x+t)t^{a-1}(1-t)^{b-2}dt. \end{aligned}$$

The result follows by simplification. The case $a = 1$ is straightforward. $\square$

The formula (13.2.50) can be extended directly to higher order derivatives.

Lemma 13.2.21. *Let $g \in C^k[0,1]$ and $\mathfrak{B}_{1,b}$ as before. Then, provided $b \geq k$,*

$$\mathbb{E}g^{(k)}(x + \mathfrak{B}_{1,b}) = -\sum_{\ell=1}^{k} \frac{b!}{(b-\ell)!} g^{(k-\ell)}(x) + \frac{b!}{(b-k)!}\mathbb{E}g(x + \mathfrak{B}_{1,b-k}). \tag{13.2.52}$$

To prove Dilcher's theorem, replace x by $x + \mathfrak{B}_{1,b}$ and take the expectation with respect to $\mathfrak{B}_{1,b}$, then it follows that Theorem 13.2.19 is equivalent to the identity

$$(x+1)^k = \sum_{p=0}^{b-1} \binom{k}{p} x^{k-p} + \mathbb{E}(x + \mathfrak{B}_{1,b-k})^k. \tag{13.2.53}$$

This is precisely the statement of Lemma 13.2.21 for $g(x) = x^k$.

The expression for the polynomials $B_k^{(b)}(x)$ given in Theorem 13.2.18 provides a natural way to extend them to a two-parameter family.

Definition 13.2.22. The *hypergeometric Bernoulli polynomials* are, symbolically defined by

$$B_n^{(a,b)}(x) = \mathbb{E}(x + \mathfrak{Z}_{a,b})^n, \tag{13.2.54}$$

whose exponential generating function is given by

$$\sum_{n=0}^{\infty} B_n^{(a,b)}(x)\frac{z^n}{n!} = \frac{e^{xz}}{\Phi_{a,b}(z)}. \tag{13.2.55}$$

In particular, the *hypergeometric Bernoulli numbers* by

$$B_n^{(a,b)} = B_n^{(a,b)}(0) = \mathbb{E}(\mathfrak{Z}_{a,b})^n. \tag{13.2.56}$$

Remark 13.2.23. The case considered by Howard is $B_n^{(b)} = B_n^{(1,b)}$.

The next result appears, in the special case $x = 0$, as Proposition 2.1 in [15]. The result gives a change of basis formula from $\left\{B_n^{(a,b)}(x) : n = 0,\, 1,\, 2,\, \cdots\right\}$ to $\{x^n : n = 0,\, 1,\, 2,\, \cdots\}$.

Theorem 13.2.24. *The polynomial $B_n^{(a,b)}(x)$ satisfy $B_0^{(a,b)}(x) = 1$ and, for $n \geq 1$,*

$$\sum_{k=0}^{n} \binom{a+b+n-1}{k} \binom{a-1+n-k}{a-1} B_k^{(a,b)}(x) = (a+b)_n \frac{x^n}{n!}. \tag{13.2.57}$$

Proof. Theorem 13.2.36, in the special case $f(x) = x^n$, gives

$$\mathbb{E}(x + \mathfrak{B}_{a,b} + \mathfrak{Z}_{a,b})^n = x^n. \tag{13.2.58}$$

The binomial theorem now gives

$$\sum_{k=0}^{n} \binom{n}{k} \mathbb{E}(x + \mathfrak{Z}_{a,b})^k \mathbb{E}\mathfrak{B}_{a,b}^{n-k} = x^n. \tag{13.2.59}$$

The moments of the beta random variable $\mathfrak{B}_{a,b}$ are

$$\begin{aligned} \mathbb{E}\mathfrak{B}_{a,b}^p &= \frac{1}{B(a,b)} \int_0^1 x^p \cdot x^{a-1}(1-x)^{b-1} dx \\ &= \frac{B(a+p,b)}{B(a,b)} = \frac{\Gamma(a+p)}{\Gamma(a+b+p)} \cdot \frac{\Gamma(a+b)}{\Gamma(a)}. \end{aligned} \tag{13.2.60}$$

The expression (13.2.59) is now expressed as

$$\frac{\Gamma(a+b)}{\Gamma(a)} \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(a+n-k)}{\Gamma(a+b+n-k)} B_k^{(a,b)}(x) = x^n, \tag{13.2.61}$$

which is equivalent to the stated result. $\square$

The probabilistic approach presented here, provides a direct proof of a symmetry property established in [15]. It extends the classical relation

$$B_n(1-x) = (-1)^n B_n(x) \tag{13.2.62}$$

of the Bernoulli polynomials.

Theorem 13.2.25. *The polynomials $B_n^{(a,b)}(x)$ satisfy the symmetry*

$$B_n^{(a,b)}(1-x) = (-1)^n B_n^{(b,a)}(x). \tag{13.2.63}$$

Proof. The moment representation

$$B_n^{(a,b)}(x) = \mathbb{E}(x + \mathfrak{Z}_{a,b})^n \tag{13.2.64}$$

and using (13.2.38) yields

$$\begin{aligned} B_n^{(a,b)}(1-x) &= \mathbb{E}(1 - x + \mathfrak{Z}_{a,b})^n \\ &= \mathbb{E}(-x - \mathfrak{Z}_{b,a})^n \\ &= (-1)^n \mathbb{E}(x + \mathfrak{Z}_{b,a})^n. \end{aligned} \tag{13.2.65}$$

□

The next result presents a linear recurrence for the polynomials $B_n^{(a,b)}(x)$.

Theorem 13.2.26. *Let X and Y be conjugate random variables defined by Definition 10.3.6. Define the polynomials*

$$P_n(z) = \mathbb{E}(z + X)^n \text{ and } Q_n(z) = \mathbb{E}(z + Y)^n. \tag{13.2.66}$$

Then P_n and Q_n satisfy the recurrences

$$P_{n+1}(z) - zP_n(z) = \sum_{j=0}^{n} \binom{n}{j} \kappa_X(j+1) P_{n-j}(z), \tag{13.2.67}$$

and

$$Q_{n+1}(z) - zQ_n(z) = -\sum_{j=0}^{n} \binom{n}{j} \kappa_X(j+1) Q_{n-j}(z). \tag{13.2.68}$$

Proof. Let X_1 and X_2 be two independent random variables distributed as X. Then,

$$\begin{aligned} f(z) &= \mathbb{E}\left[X_1(X_1 + Y + z + X_2)^n - X_1(z + X_2)^n\right] \\ &= \sum_{j=0}^{n} \binom{n}{j} \mathbb{E}\left[X_1(X_1+Y)^j (z + X_2)^{n-j}\right] - \mathbb{E}X_1(z+X_2)^n \\ &= \sum_{j=1}^{n} \binom{n}{j} \mathbb{E}\left[X_1(X_1+Y)^j\right] \mathbb{E}(z+X_2)^{n-j}. \end{aligned} \tag{13.2.69}$$

Theorem 3.3 in [17] shows that the cumulants satisfy

$$\kappa_X(p) = \mathbb{E}\left[X(X+Y)^{p-1}\right], \text{ for } p \geq 1. \tag{13.2.70}$$

Therefore

$$f(z) = \sum_{j=1}^{n} \binom{n}{j} \kappa_X(j+1) P_{n-j}(z). \tag{13.2.71}$$

The function $f(z)$ may also be expressed as

$$f(z) = \sum_{j=0}^{n} \binom{n}{j} \mathbb{E}\left[X_1(X_1+z)^{n-j}(Y+X_2)^j\right] - \mathbb{E}X_1\,\mathbb{E}(z+X_2)^n. \tag{13.2.72}$$

The relation $\mathbb{E}(Y+X_2)^j = \delta_j$ holds since X_2 and Y are conjugate random variables. This shows

$$f(z) = \mathbb{E}X_1(X_1+z)^n - \mathbb{E}X_1\,P_n(z), \tag{13.2.73}$$

which can be simplified using

$$\mathbb{E}X_1(X_1+z)^n = \mathbb{E}(X_1+z)^{n+1} - z\mathbb{E}(X_1+z)^n = P_{n+1}(z) - zP_n(z). \tag{13.2.74}$$

So, the function f has been expressed as, by using $\mathbb{E}(X) = \kappa_X(1)$,

$$f(z) = P_{n+1}(z) - (z + \kappa_X(1))P_n(z). \tag{13.2.75}$$

The recurrence for P_n comes by comparing (13.2.71) and (13.2.75).

The second identity is obtained by replacing X and Y and remarking that $\kappa_X(p) = -\kappa_Y(p)$ and $\mathbb{E}(X+Y)=0$, since X and Y are conjugate random variables. $\square$

Theorem 13.2.27. *The hypergeometric Bernoulli polynomials $B_n^{(a,b)}(z)$ and the companion family $C_n^{(a,b)}(z)$ defined by*

$$B_n^{(a,b)}(z) = \mathbb{E}(z+\mathfrak{Z}_{a,b})^n \text{ and } C_n^{(a,b)}(z) = \mathbb{E}(z+\mathfrak{B}_{a,b})^n \tag{13.2.76}$$

satisfy the recurrences

$$B_{n+1}^{(a,b)}(z) - zB_n^{(a,b)}(z) = \sum_{j=0}^{n} \frac{n!}{(n-j)!} \zeta_{a,b}^H(j+1) B_{n-j}^{(a,b)}(z), \tag{13.2.77}$$

and

$$C_{n+1}^{(a,b)}(z) - zC_n^{(a,b)}(z) = -\sum_{j=0}^{n} \frac{n!}{(n-j)!} \zeta_{a,b}^H(j+1) C_{n-j}^{(a,b)}(z). \tag{13.2.78}$$

Proof. The result now follows from Theorem 13.2.26 and the cumulants for the beta distribution given in Example 13.2.17. □

Our last result provides a probabilistic approach to the linear recurrences for the hypergeometric zeta function.

For a random variable X, the moments $\mathbb{E}X^n$ and its cumulants $\kappa_X(n)$ satisfy the relation (13.2.44). This is now used to produce a linear recurrence for the hypergeometric zeta function.

Theorem 13.2.28. *The hypergeometric zeta function $\zeta_{a,b}^H$ satisfies*

$$(n-1)!\sum_{j=2}^{n}\frac{B_{n-j}^{(a,b)}}{(n-j)!}\zeta_{a,b}^H(j) = \frac{a}{a+b}B_{n-1}^{(a,b)} + B_n^{(a,b)}. \tag{13.2.79}$$

Proof. Use the identity (13.2.44) to the random variable $\mathfrak{Z}_{a,b}$. Its moments are the hypergeometric Bernoulli numbers

$$\mathbb{E}\mathfrak{Z}_{a,b}^p = B_p^{(a,b)} \tag{13.2.80}$$

and its cumulants are

$$\kappa_{\mathfrak{Z}_{a,b}}(n) = \begin{cases} (n-1)!\zeta_{a,b}^H(n), & \text{for } n \geq 2, \\ -\frac{a}{a+b} & \text{for } n = 1, \end{cases} \tag{13.2.81}$$

since $\mathfrak{Z}_{a,b}$ and $\mathfrak{B}_{a,b}$ are conjugate random variables. A second proof is obtained by letting $z = 0$ in (13.2.77). □

Remark 13.2.29. Surprisingly, the two linear recurrences for $\zeta_{a,b}^H$, given in Theorem 13.2.2 and in Theorem 13.2.28 are different. For example, choosing $a = 5$, $b = 3$ these produce for $n = 3$ the relations

$$\begin{cases} 2\zeta_{5,3}^H(3) + \frac{5}{4}\zeta_{5,3}^H(2) + \frac{1}{32} & = 0, \\ 2\zeta_{5,3}^H(3) - \frac{5}{4}\zeta_{5,3}^H(2) - \frac{13}{384} & = 0. \end{cases} \tag{13.2.82}$$

Chapter 14

Identities for Generalized Euler Polynomials

14.1 Introduction

14.1.1 Definitions

Definition 14.1.1. The *Euler numbers* E_n, defined by the exponential generating function

$$\frac{1}{\cosh z} = \sum_{n=0}^{\infty} E_n \frac{z^n}{n!} \tag{14.1.1}$$

and the *Euler polynomials* $E_n(x)$ by

$$\sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} = \frac{2e^{xz}}{e^z + 1} \tag{14.1.2}$$

(see [31, 9.630, 9.651]) are examples of basic special functions. It follows directly from the definition that $E_n = 0$ for n odd. Moreover, the relation

$$E_n = 2^n E_n\left(\tfrac{1}{2}\right) \tag{14.1.3}$$

follows by setting $x = \frac{1}{2}$ in (14.1.2), replacing z by $2z$ and comparing with (14.1.1).

Moreover, the identity

$$\frac{2e^{xz}}{e^z+1} = \frac{2e^{(x-1/2)z}}{e^{z/2}+e^{-z/2}} \tag{14.1.4}$$

produces

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} \left(x - \tfrac{1}{2}\right)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} E_k\left(\tfrac{1}{2}\right)\left(x - \tfrac{1}{2}\right)^{n-k}, \tag{14.1.5}$$

that gives $E_n(x)$ in terms of the Euler numbers (see [31, 9.650]).

Definition 14.1.2. The *generalized Euler polynomials* $E_n^{(p)}(z)$, defined by the generating function

$$\sum_{n=0}^{\infty} E_n^{(p)}(x)\frac{z^n}{n!} = \left(\frac{2}{1+e^z}\right)^p e^{xz}, \quad \text{for } p \in \mathbb{N} \tag{14.1.6}$$

are polynomials extending $E_n(x)$, the case $p=1$. These appear in [50, Section 24.16]. The definition leads directly to the expression

$$E_n^{(p)}(x) = \sum_{k=0}^{n} \binom{n}{k} x^k E_{n-k}^{(p)}(0), \tag{14.1.7}$$

where the *generalized Euler numbers* $E_n^{(p)}(0)$ are defined recursively by

$$E_n^{(p)}(0) = \sum_{k=0}^{n} \binom{n}{k} E_k^{(p-1)}(0) E_{n-k}(0), \tag{14.1.8}$$

for $p > 1$ and initial condition $E_n^{(1)}(0) = E_n(0)$.

14.1.2 A Probabilistic Representation of Euler Polynomials And Their Generalizations

This section discusses probabilistic representations of the Euler polynomials and their generalizations.

Theorem 14.1.3. *Let L be a random variable with hyperbolic secant density, i.e., for $x \in \mathbb{R}$,*

$$f_L(x) = \operatorname{sech}(\pi x). \tag{14.1.9}$$

Then the Euler polynomial is given by

$$E_n(x) = \mathbb{E}\left(x + \imath L - \tfrac{1}{2}\right)^n. \tag{14.1.10}$$

Proof. The right hand-side of (14.1.10) is

$$\begin{aligned} \mathbb{E}\left(x + \imath L - \tfrac{1}{2}\right)^n &= \int_{-\infty}^{\infty} \left(x - \tfrac{1}{2} + \imath t\right)^n \operatorname{sech}(\pi t)\, dt \qquad (14.1.11) \\ &= \sum_{j=0}^{n} \binom{n}{j} \left(x - \tfrac{1}{2}\right)^{n-j} \imath^j \int_{-\infty}^{\infty} t^j \operatorname{sech}(\pi t)\, dt \end{aligned}$$

The identity

$$\int_{-\infty}^{\infty} t^k \operatorname{sech}(\pi t)\, dt = \frac{|E_k|}{2^k} \tag{14.1.12}$$

holds for k odd, since both sides vanish and for k even, it appears as entry **3.523.4** in [31]. A proof of this entry may be found in [10]. Then, using $|E_{2n}| = (-1)^n E_{2n}$ (entry **9.633** in [31])

$$\mathbb{E}(x + \imath L - \tfrac{1}{2})^n = \sum_{j=0}^{n} \binom{n}{j} (x - \tfrac{1}{2})^{n-j} \frac{E_j}{2^j} = E_n(x). \tag{14.1.13}$$

□

There is a natural extension to the case of $E_n^{(p)}(x)$. The proof is similar to the previous case, so it is omitted.

Theorem 14.1.4. *Let $p \in \mathbb{N}$ and L_j, $1 \leq j \leq p$ a collection of i.i.d. random variables with hyperbolic secant distribution. Then*

$$E_n^{(p)}(x) = \mathbb{E}\left[x + \sum_{j=1}^{p} \left(\imath L_j - \tfrac{1}{2}\right)\right]^n. \tag{14.1.14}$$

L. B. Klebanov et al., in their paper [44], considered random sums of independent random variables of the form

$$\frac{1}{N}\sum_{j=1}^{\mu_N} L_j, \tag{14.1.15}$$

where μ_N is independent of the L_j's and described below.

Definition 14.1.5. Let $N \in \mathbb{N}$ and $T_N(z)$ be the Chebyshev polynomial of the first kind. The random variable μ_N taking values in $\mathbb{N}$, is defined by its generating function

$$\mathbb{E}z^{\mu_N} = \frac{1}{T_N(1/z)}. \tag{14.1.16}$$

Detailed information about the Chebyshev polynomials appears in [31] and [50]. One way to define the Chebyshev polynomial of the first kind is through the identity

$$T_n\left(\cos\theta\right) = \cos\left(n\theta\right). \tag{14.1.17}$$

Example 14.1.6. Take $N = 2$, then $T_2(z) = 2z^2 - 1$ gives

$$\mathbb{E}z^{\mu_2} = \frac{1}{T_2(1/z)} = \frac{z^2}{2 - z^2} = \sum_{\ell=1}^{\infty} \frac{z^{2\ell}}{2^\ell}. \tag{14.1.18}$$

Therefore μ_2 takes the value 2ℓ, with $\ell \in \mathbb{N}$, with probability

$$\Pr(\mu_2 = 2\ell) = 2^{-\ell}. \tag{14.1.19}$$

In [44], Klebanov et al. also prove the following result, which is the key to the result in next section.

Theorem 14.1.7. *Assume $\{L_j\}$ is a sequence of i.i.d. random variables with hyperbolic secant distribution. Then, for all $N \geq 2$ and μ_N defined in (14.1.16), the following random variable has the same hyperbolic secant distribution:*

$$L := \frac{1}{N}\sum_{j=1}^{\mu_N} L_j. \tag{14.1.20}$$

14.2 Main Results

14.2.1 The Euler polynomials in terms of the generalized ones

The identifies (14.1.7) and (14.1.8) can be used to express the generalized Euler polynomial $E_n^{(p)}(x)$ in terms of the standard Euler polynomials $E_n(x)$. However, to the best of our knowledge, there is no formula that allows to express $E_n(x)$ in terms of $E_n^{(p)}(x)$. This section presents such a formula.

Definition 14.2.1. Let $N \in \mathbb{N}$. The sequence $\left\{ p_\ell^{(N)} \right\}_{l=0}^{\infty}$ is defined by

$$\frac{1}{T_N(1/z)} = \sum_{\ell=0}^{\infty} p_\ell^{(N)} z^\ell. \tag{14.2.1}$$

Definition 14.1.5 shows that for $\ell \in \mathbb{N}$,

$$p_\ell^{(N)} = \Pr(\mu_N = \ell). \tag{14.2.2}$$

The numbers $p_\ell^{(N)}$ will be referred as the *probability numbers.*

Example 14.2.2. For $N = 2$, Example 14.1.6 gives

$$p_\ell^{(2)} = \begin{cases} 0 & \text{if } \ell \text{ is odd,} \\ 2^{-\ell/2} & \text{if } \ell \text{ is even, } \ell \neq 0. \end{cases} \tag{14.2.3}$$

Now, $p_\ell^{(N)}$ are used to produce expansions of $E_n(x)$, in terms of $E_n^{(p)}(x)$.

Theorem 14.2.3. *The Euler polynomials satisfy, for all $N \in \mathbb{N}$,*

$$E_n(x) = \frac{1}{N^n} \mathbb{E}\left[E_n^{(\mu_N)}\left(\tfrac{1}{2}\mu_N + N(x - \tfrac{1}{2}) \right) \right]. \tag{14.2.4}$$

Proof. From (14.1.10) and (14.1.20)

$$E_n\left(\tfrac{1}{2}\right) = \mathbb{E}(\imath L)^n = \frac{1}{N^n} \mathbb{E}\left[\imath \sum_{j=1}^{\mu_N} L_j \right]^n, \tag{14.2.5}$$

with Theorem 14.1.4, this yields

$$\mathbb{E}\left[E_n^{(\mu_N)}\left(\frac{\mu_N}{2}\right)\right] = \mathbb{E}\left[\imath\sum_{j=1}^{\mu_N} L_j\right]^n = N^n E_n\left(\tfrac{1}{2}\right). \tag{14.2.6}$$

Using identity (14.1.5), it follows that

$$\begin{aligned}
E_n(x) &= \sum_{k=0}^{n}\binom{n}{k}E_k\left(\tfrac{1}{2}\right)\left(x-\tfrac{1}{2}\right)^{n-k} \qquad (14.2.7)\\
&= \mathbb{E}\left[\sum_{k=0}^{n}\binom{n}{k}N^{-k}E_k^{(\mu_N)}\left(\tfrac{1}{2}\mu_N\right)\left(x-\tfrac{1}{2}\right)^{n-k}\right]\\
&= \mathbb{E}\left[\sum_{k=0}^{n}\binom{n}{k}N^{-k}(\imath L_1+\cdots+\imath L_{\mu_N})^k\left(x-\tfrac{1}{2}\right)^{n-k}\right]\\
&= \mathbb{E}\left[\frac{1}{N^n}\sum_{k=0}^{n}\binom{n}{k}(\imath L_1+\cdots+\imath L_{\mu_N})^k\left(N(x-\tfrac{1}{2})\right)^{n-k}\right]\\
&= \mathbb{E}\left[\frac{1}{N^n}\left(\imath L_1+\cdots+\imath L_{\mu_N}+N(x-\tfrac{1}{2})\right)^n\right]\\
&= \mathbb{E}\left[\frac{1}{N^n}\left(\imath L_1+\cdots+\imath L_{\mu_N}+z-\tfrac{1}{2}\mu_N\right)^n\right]\\
&= \frac{1}{N^n}\mathbb{E}\left[E_n^{(\mu_N)}(z)\right],
\end{aligned}$$

where $z = \frac{1}{2}\mu_N + N\left(x-\frac{1}{2}\right)$. This completes the proof. □

The next result is established using the fact that the expectation operator $\mathbb{E}$ satisfies

$$\mathbb{E}[h(\mu_N)] = \sum_{k=0}^{\infty} p_k^{(N)} h(k), \tag{14.2.8}$$

for any function h such that the right-hand side exists.

Corollary 14.2.4. *The Euler polynomials satisfy*

$$E_n(x) = \frac{1}{N^n}\sum_{k=N}^{\infty} p_k^{(N)} E_n^{(k)}\left(\tfrac{1}{2}k + N\left(x-\tfrac{1}{2}\right)\right). \tag{14.2.9}$$

Remark 14.2.5. Corollary 14.2.4 gives an infinite family of expressions for $E_n(x)$ in terms of the generalized Euler polynomials $E_n^{(k)}(x)$, one for each value of $N \geq 2$.

Example 14.2.6. The expansion (14.2.9) with $N = 2$ gives

$$E_n(x) = \frac{1}{2^n} \sum_{\ell=1}^{\infty} \frac{1}{2^\ell} E_n^{(2\ell)}(\ell + 2x - 1). \tag{14.2.10}$$

For instance, when $n = 1$,

$$E_1(x) = \frac{1}{2} \sum_{\ell=1}^{\infty} \frac{1}{2^\ell} E_1^{(2\ell)}(\ell + 2x - 1), \tag{14.2.11}$$

and the value $E_1^{(\ell)}(x) = x - \frac{\ell}{2}$ gives

$$E_1(x) = \frac{1}{2} \sum_{\ell=1}^{\infty} \frac{1}{2^\ell}(\ell + 2x - 1 - \ell) = x - \tfrac{1}{2}, \tag{14.2.12}$$

as expected.

14.2.2 The Probability Numbers

For fixed $N \in \mathbb{N}$, the random variable μ_N has been defined by its moment generating function

$$\mathbb{E}z^{\mu_N} = \frac{1}{T_N(1/z)} = \sum_{\ell=0}^{\infty} p_\ell^{(N)} z^\ell. \tag{14.2.13}$$

For small N, the coefficients $p_\ell^{(N)}$ can be computed directly by expanding the rational function $1/T_N(1/z)$ in partial fractions. Example 14.1.6 gave the case $N = 2$. The cases $N = 3$ and $N = 4$ are presented below.

Example 14.2.7. For $N = 3$, the Chebyshev polynomial is

$$T_3(z) = 4z^3 - 3z = 4z(z - \alpha)(z + \alpha), \tag{14.2.14}$$

with $\alpha = \sqrt{3}/2$. This yields

$$\frac{1}{T_3(1/z)} = \frac{z^3}{4(1 - \alpha z)(1 + \alpha z)} = \sum_{k=0}^{\infty} \frac{3^k}{2^{2k+2}} z^{2k+3}. \tag{14.2.15}$$

It follows that $p_\ell^{(3)} = 0$ unless $\ell = 2k + 3$ and

$$p_{2k+3}^{(3)} = \frac{3^k}{2^{2k+2}}. \tag{14.2.16}$$

Corollary 14.2.4 now gives a companion to (14.2.10):

$$E_n(x) = \frac{1}{3^n} \sum_{k=0}^{\infty} \frac{3^k}{2^{2k+2}} E_n^{(2k+3)}(3x+k). \tag{14.2.17}$$

Example 14.2.8. The probability numbers for $N = 4$ are computed from the expression

$$\frac{1}{T_4(1/z)} = \frac{z^4}{z^4 - 8z^2 + 8}. \tag{14.2.18}$$

The factorization

$$z^4 - 8z^2 + 8 = (z^2 - \beta)(z^2 - \gamma) \tag{14.2.19}$$

with $\beta = 2(2+\sqrt{2})$ and $\gamma = 2(2-\sqrt{2})$ and the partial fraction decomposition

$$\frac{z^4}{z^4 - 8z^2 + 8} = \frac{\beta}{\beta - \gamma} \frac{1}{1 - \beta/z^2} - \frac{\gamma}{\beta - \gamma} \frac{1}{1 - \gamma/z^2} \tag{14.2.20}$$

show that $p_\ell^{(4)} = 0$ for ℓ odd or $\ell = 2$ and, for $\ell \geq 2$,

$$p_{2\ell}^{(4)} = \frac{\sqrt{2}}{2^{2\ell+1}} \left[(2+\sqrt{2})^{\ell-1} - (2-\sqrt{2})^{\ell-1} \right]. \tag{14.2.21}$$

Corollary 14.2.4 now gives

$$E_n(x) = \sqrt{2} \sum_{\ell=2}^{\infty} \frac{\left[(2+\sqrt{2})^{\ell-1} - (2-\sqrt{2})^{\ell-1} \right]}{2^{2\ell+1}} E_n^{(2\ell)}(4x + \ell - 2). \tag{14.2.22}$$

Some elementary properties of the probability numbers are presented next.

Proposition 14.2.9. *The probability numbers $p_\ell^{(N)}$ vanish if $\ell < N$.*

Proof. The Chebyshev polynomial $T_N(z)$ has the form $2^{N-1}z^N +$ lower order terms. Then the expansion of $1/T_N(1/z)$ has a zero of order N at $z = 0$. This proves the statement. □

Proposition 14.2.10. *The probability numbers $p_\ell^{(N)}$ vanish if $\ell \not\equiv N$ (mod 2).*

Proof. The polynomial $T_N(z)$ has the same parity as N. The same holds for the rational function $1/T_N(1/z)$. □

An expression for the probability numbers is given next.

Theorem 14.2.11. *Let $N \in \mathbb{N}$ be fixed and define*

$$\theta_k^{(N)} = \frac{(2k-1)\pi}{2N}. \tag{14.2.23}$$

Then

$$p_\ell^{(N)} = \frac{1}{N}\sum_{k=1}^{N}(-1)^{k+1}\sin\theta_k^{(N)}\cos^{\ell-1}\theta_k^{(N)}. \tag{14.2.24}$$

Proof. The Chebyshev polynomial is defined by $T_N(\cos\theta) = \cos(N\theta)$, so its roots are $z_k^{(N)} = \cos\theta_k^{(N)}$, with $\theta_k^{(N)}$ as above. The leading coefficient of $T_N(z)$ is 2^{N-1}. Thus

$$\frac{1}{T_N(z)} = \frac{2^{1-N}}{\prod\limits_{k=1}^{N}(z-z_k)}. \tag{14.2.25}$$

In the remainder of the proof, the superscript N has been dropped from $z_k^{(N)}$ and $\theta_k^{(N)}$, for clarity. Define

$$Q(z) = \prod_{k=1}^{N}(z-z_k). \tag{14.2.26}$$

The roots z_k of Q are distinct, therefore

$$\frac{1}{Q(z)} = \sum_{k=1}^{N}\frac{1}{Q'(z_k)}\frac{1}{z-z_k}. \tag{14.2.27}$$

The identity $T_N'(z) = NU_{N-1}(z)$ gives

$$Q'(z_k) = N2^{1-N}U_{N-1}(z_k), \tag{14.2.28}$$

where $U_j(z)$ is the Chebyshev polynomial of the second kind defined by

$$U_j(\cos\theta) = \frac{\sin(j+1)\theta}{\sin\theta}. \tag{14.2.29}$$

Then

$$U_{N-1}(z_k) = U_{N-1}(\cos\theta_k) = \frac{\sin N\theta_k}{\sin\theta_k}. \tag{14.2.30}$$

and the value $\sin(N\theta_k) = (-1)^{k+1}$ yields

$$Q'(z_k) = \frac{(-1)^{k+1}}{\sin\theta_k} N 2^{1-N}. \tag{14.2.31}$$

Therefore (14.2.27) now gives

$$\frac{1}{Q(z)} = \frac{2^{N-1}}{N} \sum_{k=1}^{N} \frac{(-1)^{k+1}\sin\theta_k}{z-\cos\theta_k}. \tag{14.2.32}$$

It follows that

$$\begin{aligned} \frac{1}{T_N(1/z)} = \frac{2^{1-N}}{Q(1/z)} &= \frac{1}{N}\sum_{k=1}^{N}(-1)^{k+1}\frac{z\sin\theta_k}{1-z\cos\theta_k} \\ &= \frac{1}{N}\sum_{k=1}^{N}(-1)^{k+1}\sin\theta_k\sum_{\ell=0}^{\infty} z^{\ell+1}\cos^{\ell}\theta_k \\ &= \frac{1}{N}\sum_{\ell=0}^{\infty} z^{\ell+1}\sum_{k=1}^{N}(-1)^{k+1}\sin\theta_k\cos^{\ell}\theta_k. \end{aligned} \tag{14.2.33}$$

The proof is complete. $\square$

The next result provides another explicit formula for the probability numbers. The coefficients $A(n,k)$ appear in The On-Line Encyclopedia of Integer Sequences (OEIS) entry **A008315**, as entries of the Catalan triangle.

Theorem 14.2.12. *Let $A(n,k) = \binom{n}{k} - \binom{n}{k-1}$ and assume N satisfy $N \equiv \ell \bmod 2$. If ℓ is not an odd multiple of N, then*

$$p_{\ell}^{(N)} = \frac{1}{2^{\ell}} \sum_{t=\lfloor \frac{1}{2}\left(\frac{2-\ell}{N}-1\right)\rfloor}^{\lfloor \frac{1}{2}\left(\frac{\ell}{N}-1\right)\rfloor} (-1)^t A(\ell-1, \tfrac{1}{2}(\ell-(2t+1)N)), \tag{14.2.34}$$

otherwise, with $k = \frac{1}{2}(\ell/N-1)$

$$p_{\ell}^{(N)} = \frac{1}{2^{\ell}}\left[\sum_{s=1}^{\lfloor \ell/N-1\rfloor} (-1)^{k-s} A(\ell-1, sN)\right] + \frac{(-1)^k}{2^{\ell-1}}. \tag{14.2.35}$$

The proof begins with a preliminary result.

Lemma 14.2.13. *Let $N \in \mathbb{N}$ and $\theta_k = \frac{\pi}{2}\frac{(2k-1)}{N}$. Then*

$$f_N(z) = \sum_{k=1}^{N} (-1)^{k+1} e^{\imath\theta_k z} \tag{14.2.36}$$

is given by

$$f_N(z) = \frac{1-(-1)^N e^{\pi\imath z}}{2\cos\left(\frac{\pi z}{2N}\right)} \qquad \text{if } z \neq (2t+1)N \text{ with } t \in \mathbb{Z}, \tag{14.2.37}$$

and

$$f_N(z) = (-1)^t N\imath \qquad \text{if } z = (2t+1)N \text{ for some } t \in \mathbb{Z}. \tag{14.2.38}$$

In particular

$$f_N(k) = \begin{cases} (-1)^{(k/N-1)/2} N\imath & \text{if } \frac{k}{N} \text{ is an odd integer,} \\ \frac{1-(-1)^{N+k}}{2\cos\left(\frac{\pi k}{2N}\right)} & \text{otherwise.} \end{cases} \tag{14.2.39}$$

Proof. The function f_N is the sum of a geometric progression. The formula (14.2.38) comes from (14.2.37) by passing to the limit. □

The proof of Theorem 14.2.12 is given now.

Proof. The expression for $p_\ell^{(N)}$ given in Theorem 14.2.11 yields

$$\begin{aligned} p_\ell^{(N)} &= \frac{1}{N}\sum_{k=1}^{N}(-1)^{k+1}\frac{(e^{\imath\theta_k}-e^{-\imath\theta_k})}{2i}\left(\frac{e^{\imath\theta_k}+e^{-\imath\theta_k}}{2}\right)^{\ell-1} \\ &= \frac{1}{2^\ell N\imath}\sum_{k=1}^{N}(-1)^{k+1}\sum_{r=0}^{\ell-1}\binom{\ell-1}{r}\left[e^{\imath(\ell-2r)\theta_k}-e^{\imath(\ell-2r-2)\theta_k}\right] \\ &= \frac{1}{2^\ell N\imath}\sum_{r=0}^{\ell-1}\binom{\ell-1}{r}\left[f_N(l-2r)-f_N(l-2r-2)\right] \\ &= \frac{1}{2^\ell N\imath}\left[\sum_{r=1}^{\ell-1}A(\ell-1,r)f_N(\ell-2r)+f_N(\ell)-f_N(-\ell)\right]. \end{aligned} \tag{14.2.40}$$

Now $f_N(\ell) = f_N(-\ell) = 0$ if ℓ/N is not an odd integer. On the other hand, if $\ell = (2t+1)N$, with $t \in \mathbb{Z}$, then

$$f_N(\ell) = (-1)^t N\imath \text{ and } f_N(-\ell) = -(-1)^t N\imath. \tag{14.2.41}$$

Thus,

$$f_N(\ell) - f_N(-\ell) = \begin{cases} 2N\imath(-1)^{(\ell/N-1)/2} & \text{if } \ell \text{ is an odd multiple of } N, \\ 0 & \text{otherwise.} \end{cases} \tag{14.2.42}$$

The simplification of the previous expression for $p_\ell^{(N)}$ is divided in two cases, according to whether ℓ is an odd multiple of N or not.

Case 1. Assume ℓ is not an odd multiple of N. Then

$$p_\ell^{(N)} = \frac{1}{2^\ell N\imath} \sum_{r=0}^{\ell-1} A(\ell-1, r) f_N(\ell - 2r). \tag{14.2.43}$$

Moreover,

$$f_N(\ell - 2r) = \begin{cases} (-1)^t N\imath & \text{if } \frac{l-2r}{N} = 2t+1 \\ 0 & \text{otherwise.} \end{cases} \tag{14.2.44}$$

Therefore

$$p_\ell^{(N)} = \frac{1}{2^\ell} \sum_{\substack{t=\frac{1}{2}\left(\frac{2-\ell}{N}-1\right) \\ \ell-2r=(2t+1)N}}^{\frac{1}{2}\left(\frac{\ell}{N}-1\right)} (-1)^t A(\ell-1, r). \tag{14.2.45}$$

Observe that $\ell-(2t+1)N$ is always an even integer, thus the index r may be eliminated from the previous expression to obtain

$$p_\ell^{(N)} = \frac{1}{2^\ell} \sum_{t=\lfloor\frac{1}{2}\left(\frac{2-\ell}{N}-1\right)\rfloor}^{\lfloor\frac{1}{2}\left(\frac{\ell}{N}-1\right)\rfloor} (-1)^t A(\ell-1, \tfrac{1}{2}(\ell-(2t+1)N)). \tag{14.2.46}$$

Case 2. Assume ℓ is an odd multiple of N, say $\ell = (2k+1)N$. Then

$$\begin{aligned} p_\ell^{(N)} &= \frac{1}{2^\ell N\imath} \left[\sum_{r=0}^{\ell-1} A(\ell-1, r) f_N(\ell-2r) + 2Ni(-1)^k \right] \qquad (14.2.47) \\ &= \frac{1}{2^\ell N\imath} \left[\sum_{r=0}^{\ell-1} A(\ell-1, r) f_N(\ell-2r) \right] + \frac{(-1)^k}{2^{\ell-1}}. \end{aligned}$$

The term $f_N(\ell - 2r)$ vanishes unless $\ell - 2r$ is an odd multiple of N. Given that $\ell = (2k+1)N$, the term is non-zero provided $2r$ is an even multiple of N; say

$r = sN$ for $s \in \mathbb{N}$. The range of s is $1 \leq s \leq \frac{\ell-1}{N} = 2k+1-\frac{1}{N}$. This implies $1 \leq s \leq 2k = \ell/N - 1$, and it follows that

$$p_\ell^{(N)} = \frac{1}{2^\ell}\left[\sum_{s=1}^{\ell/N-1}(-1)^{k-s}A(\ell-1, sN)\right] + \frac{(-1)^k}{2^{\ell-1}}, \text{ with } k = \tfrac{1}{2}\left(\ell/N - 1\right). \quad (14.2.48)$$

The proof is complete. □

Remark 14.2.14. The expression in Theorem 14.2.12 shows that $p_\ell^{(N)}$ is a rational number with a denominator a power of 2 of exponent at most ℓ.

14.2.3 An Asymptotic Expansion

The final result deals with the asymptotic behavior of $p_\ell^{(N)}$.

Theorem 14.2.15. *Let $\varphi_N(z) = \mathbb{E}\left[z^{\mu_N}\right]$. Then, for fixed z in the unit disk $|z| < 1$,*

$$\varphi_N(z) \sim \left(\frac{z}{1+\sqrt{1-z^2}}\right)^N, \text{ as } N \to \infty. \quad (14.2.49)$$

Proof. The generating function satisfies

$$\varphi_N(z) = 1/T_N(1/z) = \frac{z^N}{2^{N-1}}\prod_{k=1}^{N}\left(1 - z\,\cos\theta_k^{(N)}\right)^{-1}, \quad (14.2.50)$$

with $\theta_k^{(N)} = (2k-1)\pi/2N$ as before. Then

$$\log\varphi_N(z) = \log 2 + N\log\frac{z}{2} - \sum_{k=1}^{N}\log\left(1 - z\cos\theta_k^{(N)}\right). \quad (14.2.51)$$

The last sum is approximated by a Riemann integral

$$\frac{1}{N}\sum_{k=1}^{N}\log\left(1 - z\cos\theta_k^{(N)}\right) \sim \frac{1}{\pi}\int_0^\pi \log(1 - z\cos\theta)\,d\theta = \log\left(\frac{1+\sqrt{1-z^2}}{2}\right). \quad (14.2.52)$$

The last evaluation is elementary. It appears as entry **4.224.9** in [31]. It follows that

$$\log\varphi_N(z) \sim \log 2 + N\log\left(\frac{z}{2}\right) - N\log\left(\frac{1+\sqrt{1-z^2}}{2}\right), \quad (14.2.53)$$

which is equivalent to the result. □

The function

$$A(z) = \frac{2}{1+\sqrt{1-4z}} = \sum_{n=0}^{\infty} C_n z^n \tag{14.2.54}$$

is the generating function for the Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{14.2.55}$$

The final result follows directly from the Binet's formula for Chebyshev polynomial

$$T_N(z) = \frac{(z-\sqrt{z^2-1})^N + (z+\sqrt{z^2-1})^N}{2}. \tag{14.2.56}$$

Some standard notation is recalled.

Given two sequences $\mathbf{a} = \{a_n\}$, $\mathbf{b} = \{b_n\}$, their convolution $\mathbf{c} = \mathbf{a} * \mathbf{b}$ is the sequence $\mathbf{c} = \{c_n\}$, with

$$c_n = \sum_{j=0}^{n} a_j b_{n-j}. \tag{14.2.57}$$

The *convolution power* $\mathbf{c}^{(*N)}$ is the convolution of $\mathbf{c}$ with itself, N times.

Theorem 14.2.16. *For $N \in \mathbb{N}$ fixed, the first N nonzero terms of the sequence $q_\ell^{(N)} = 2^{\ell-1} p_\ell^{(N)}$ agree with the first N terms of the N-th convolution power $C_n^{(*N)}$ of the Catalan sequence:*

$$q_N^{(N)} = C_0^{(*N)},\ q_{N+2}^{(*N)} = C_1^{(*N)},\ \cdots,\ q_{N+2k}^{(N)} = C_k^{(*N)}, \cdots, q_{3N-2}^{(N)} = C_{N-1}^{(*N)}. \tag{14.2.58}$$

In terms of generating functions, this is equivalent to

$$\left(\sum_{n=0}^{\infty} C_n z^{2n+1}\right)^N - \sum_{\ell=0}^{\infty} q_\ell^{(N)} z^\ell \sim 2^N z^{3N}. \tag{14.2.59}$$

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Biography

Lin Jiu got both his Bachelor and Master degrees of Sciences in Beijing Institute of Technology under the supervision of Professor Huafei Sun. He then became a graduate student of Tulane in August 2011, under the supervision of Professor Victor H. Moll. During his study in Tulane, he got an opportunity to visit Research Institute for Symbolic Computation in Austria from September 2013 to February 2014 as an exchange student, working with Professor Carsten Schneider. Right now, he is offered by Professor Carsten Schneider again as a postdoc starting from May 2016.