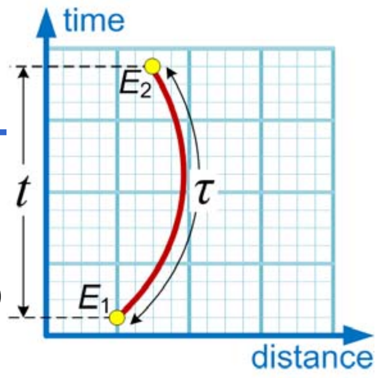
Chapter 12.
Electrodynamics and Relativity

Does the principle of relativity apply to the laws of electrodynamics?

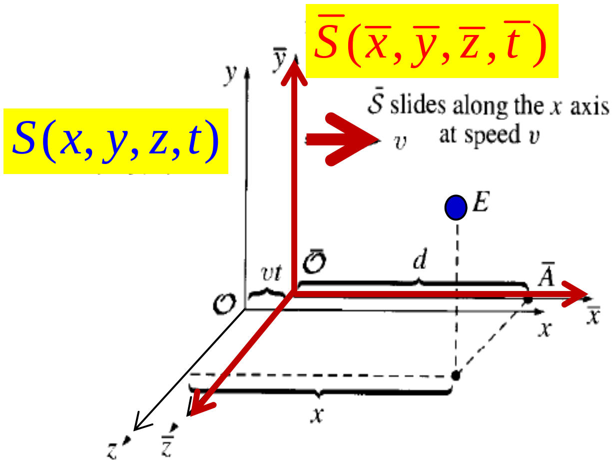
12.2 Relativistic Mechanics



12.2.1 Proper Time and Proper Velocity

proper time (τ): (The word suggests a mistranslation of the French *propre,* meaning "own.")

→ The elapsed time between two events as measured by a clock that passes through both events.
→ The proper time depends not only on the events but also on the motion of the clock between the events.



$$\left.\begin{array}{ll}\text{(i)} & \bar{x} = \gamma(x - vt), \\ \text{(ii)} & \bar{y} = y, \\ \text{(iii)} & \bar{z} = z, \\ \text{(iv)} & \bar{t} = \gamma\left(t - \frac{v}{c^2}x\right)\end{array}\right\} \qquad \left.\begin{array}{ll}\text{(i}'\text{)} & x = \gamma(\bar{x} + v\bar{t}), \\ \text{(ii}'\text{)} & y = \bar{y}, \\ \text{(iii}'\text{)} & z = \bar{z}, \\ \text{(iv}'\text{)} & t = \gamma\left(\bar{t} + \frac{v}{c^2}\bar{x}\right).\end{array}\right\}$$

$$d\bar{t} = \gamma\left(1 - \frac{u}{c^2}\frac{dx}{dt}\right)dt = \gamma\left(1 - \frac{u^2}{c^2}\right)dt = \frac{1}{\gamma}dt$$

$$dt = \gamma\left(1 - \frac{u}{c^2}\frac{d\bar{x}}{d\bar{t}}\right)d\bar{t} = \gamma\left(1 - \frac{u^2}{c^2}\right)d\bar{t} = \frac{1}{\gamma}d\bar{t}$$

$$d\tau = \sqrt{1 - u^2/c^2}\,dt$$

→ *In some cases τ may be a more relevant or useful quantity than t.*
→ *For one thing, proper time is invariant,*
whereas "ordinary" time depends on the particular reference frame you have in mind.

Proper Time and Proper Velocity

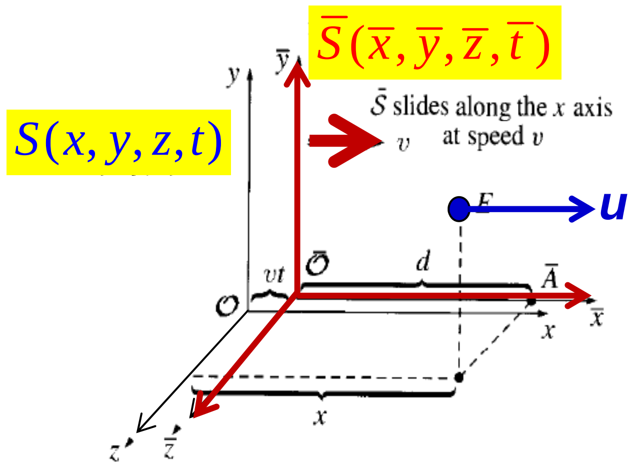
Ordinary velocity (u) ➔ $\mathbf{u} = \dfrac{d\mathbf{l}}{dt}$: distance measured on the ground, over time measured in the ground

Proper velocity (η) ➔ $\boldsymbol{\eta} \equiv \dfrac{d\mathbf{l}}{d\tau}$: distance ($\boldsymbol{l}$) measured on S, over the proper time (hybrid combination)

$$d\tau = \sqrt{1 - u^2/c^2}\, dt$$

$$\boldsymbol{\eta} = \frac{1}{\sqrt{1 - u^2/c^2}}\mathbf{u}$$ **: proper velocity 4-vector (4-velocity)**

$$\rightarrow \quad \eta^{\mu} \equiv \frac{dx^{\mu}}{d\tau}$$



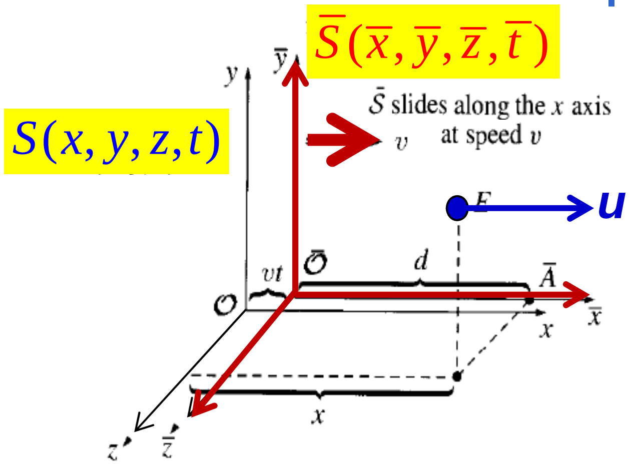
whose zeroth component is $\eta^0 = \dfrac{dx^0}{d\tau} = c\dfrac{dt}{d\tau} = \dfrac{c}{\sqrt{1 - u^2/c^2}}$

Note: the particle velocities under Lorentz transformations

υ: $\bar{S}$ frame velocity with respect to S frame

$u = \dfrac{dx}{dt}$: velocity of the *particle* with respect to S frame

$\bar{u} = \dfrac{d\bar{x}}{d\bar{t}}$: velocity of the *particle* with respect to $\bar{S}$ frame



$$
\left.\begin{aligned}
\bar{u}_x &= \frac{d\bar{x}}{d\bar{t}} = \frac{u_x - v}{(1 - vu_x/c^2)},\\
\bar{u}_y &= \frac{d\bar{y}}{d\bar{t}} = \frac{u_y}{\gamma(1 - vu_x/c^2)},\\
\bar{u}_z &= \frac{d\bar{z}}{d\bar{t}} = \frac{u_z}{\gamma(1 - vu_x/c^2)}.
\end{aligned}\right\}
$$

→ The transformation rule for *ordinary* velocities
→ It is extremely cumbersome!

$\eta = \dfrac{dl}{d\tau}$ **: proper velocity of the *particle*:** $\boldsymbol{\eta} = \dfrac{1}{\sqrt{1 - u^2/c^2}}\mathbf{u}$ $\quad \eta = \dfrac{dx}{d\bar{t}} \quad \bar{\eta} = \dfrac{d\bar{x}}{dt}$

$$
\left.\begin{aligned}
\bar{\eta}^0 &= \gamma(\eta^0 - \beta\eta^1),\\
\bar{\eta}^1 &= \gamma(\eta^1 - \beta\eta^0),\\
\bar{\eta}^2 &= \eta^2,\\
\bar{\eta}^3 &= \eta^3.
\end{aligned}\right\}
$$

→ Proper velocity has all enormous advantage over ordinary velocity:

→ it transforms simply, when you go from one inertial system to another.

12.2.2 Relativistic Energy and Momentum

$$\boldsymbol{\eta} = \frac{1}{\sqrt{1-u^2/c^2}}\mathbf{u}$$

Relativistic Momentum: $\boxed{\mathbf{p} \equiv m\boldsymbol{\eta} = \frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}}}$ $\longrightarrow$ $p^\mu \equiv m\eta^\mu$ (*m*: rest mass)

$$m_{\text{rel}} \equiv \frac{m}{\sqrt{1-u^2/c^2}}$$: Einstein called **relativistic mass**

➔ **Relativistic momentum is conserved!**

(Prove: Prob. 12.29)
The conservation law of momentum would be inconsistent with the principle of relativity if we were to define momentum as ***mu***.

Problem 12.2: In inertial frame *S*, particle A (mass m_A, velocity u_A) hits particle *B* (mass m_B, velocity u_B).
In the course of the collision some mass rubs off *A* and onto *B*,
and we are left with particles C (mass m_C, velocity u_C) and *D* (mass m_D, velocity u_D).

(a) Prove that momentum is also conserved in inertial frame *S-bar,* which moves with velocity v relative to *S.*

If we use Galileo's velocity addition rule of $v_{AC} = v_{AB} + v_{BC}$

Assuming mass is conserved, $(m_A + m_B) = (m_C + m_D)$, it follows that momentum is conserved.

$$m_A\mathbf{u}_A + m_B\mathbf{u}_B = m_C\mathbf{u}_C + m_D\mathbf{u}_D; \qquad \mathbf{u}_i = \bar{\mathbf{u}}_i + \mathbf{v}.$$

$$m_A(\bar{\mathbf{u}}_A + \mathbf{v}) + m_B(\bar{\mathbf{u}}_B + \mathbf{v}) = m_C(\bar{\mathbf{u}}_C + \mathbf{v}) + m_D(\bar{\mathbf{u}}_D + \mathbf{v})$$

$$m_A\bar{\mathbf{u}}_A + m_B\bar{\mathbf{u}}_B + (m_A + m_B)\mathbf{v} = m_C\bar{\mathbf{u}}_C + m_D\bar{\mathbf{u}}_D + (m_C + m_D)\mathbf{v}$$

Assuming *mass* is conserved, $(m_A + m_B) = (m_C + m_D)$,

$m_A\bar{\mathbf{u}}_A + m_B\bar{\mathbf{u}}_B = m_C\bar{\mathbf{u}}_C + m_D\bar{\mathbf{u}}_D$, so momentum is conserved in $\bar{S}$.

Relativistic Momentum conservation

$$\mathbf{p} \equiv m\boldsymbol{\eta} = \frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}}$$

(Prob. 12.29)

(a) The conservation law of momentum would be inconsistent with the principle of relativity if we were to define momentum as p= *m**u***, but with the (correct) Einstein velocity addition rule.

$$v_{AC} = \frac{v_{AB} + v_{BC}}{1 + (v_{AB}v_{BC}/c^2)}$$

(a) $m_A u_A + m_B u_B = m_C u_C + m_D u_D$; $\quad u_i = \dfrac{\bar{u}_i + v}{1 + (\bar{u}_i v/c^2)}$.

$$m_A \frac{\bar{u}_A + v}{1 + (\bar{u}_A v/c^2)} + m_B \frac{\bar{u}_B + v}{1 + (\bar{u}_B v/c^2)} = m_C \frac{\bar{u}_C + v}{1 + (\bar{u}_C v/c^2)} + m_D \frac{\bar{u}_D + v}{1 + (\bar{u}_D v/c^2)}.$$

This time, because the denominators are all different, we *cannot* conclude that $m_A\bar{u}_A + m_B\bar{u}_B = m_C\bar{u}_C + m_D\bar{u}_D$.

(b) Now do the same using the correct definition, p = $m\eta$.

(b) $m_A\eta_A + m_B\eta_B = m_C\eta_C + m_D\eta_D$; $\quad \eta_i = \gamma(\bar{\eta}_i + \beta\bar{\eta}_i^0)$. (The inverse Lorentz transformation.)

$$m_A\gamma(\bar{\eta}_A + \beta\bar{\eta}_A^0) + m_B\gamma(\bar{\eta}_B + \beta\bar{\eta}_B^0) = m_C\gamma(\bar{\eta}_C + \beta\bar{\eta}_C^0) + m_D\gamma(\bar{\eta}_D + \beta\bar{\eta}_D^0).$$

The gamma's cancel:

$$m_A\bar{\eta}_A + m_B\bar{\eta}_B + \beta(m_A\bar{\eta}_A^0 + m_B\bar{\eta}_B^0) = m_C\bar{\eta}_C + m_D\bar{\eta}_D + \beta(m_C\bar{\eta}_C^0 + m_D\bar{\eta}_D^0).$$

But $m_i\eta_i^0 = p_i^0 = E_i/c$, so if $\boxed{\textit{energy}\text{ is conserved}}$ in $\bar{S}$ $(\bar{E}_A + \bar{E}_B = \bar{E}_C + \bar{E}_D)$,

$$m_A\bar{\eta}_A + m_B\bar{\eta}_B = m_C\bar{\eta}_C + m_D\bar{\eta}_D.$$

Relativistic Energy

Relativistic energy: $$\boxed{E \equiv \frac{mc^2}{\sqrt{1-u^2/c^2}}}$$

➔ **Relativistic energy is conserved!**

$E_{\text{rest}} \equiv mc^2$ ➔ ***Rest energy (u = 0)***

➔ The relativistic energy is nonzero *even when the object is stationary!*

$$E_{\text{kin}} \equiv E - mc^2 = mc^2\left(\frac{1}{\sqrt{1-u^2/c^2}} - 1\right) = \frac{1}{2}mu^2 + \frac{3}{8}\frac{mu^4}{c^2} + \cdots$$ **: kinetic energy**

Note: $p^0 = m\eta^0 = \frac{mc}{\sqrt{1-u^2/c^2}} = \frac{E}{c}$

The scalar product of p^μ with itself: $p^\mu p_\mu = -(p^0)^2 + (\mathbf{p}\cdot\mathbf{p}) = -m^2c^2$

$$\Rightarrow \boxed{E^2 - p^2c^2 = m^2c^4}$$

This result is extremely useful, for it enables you to calculate *E* (if you know *p),* or *p* (knowing *E),* without ever having to determine the velocity.

Total relativistic energy and momentum are conserved.

Relativistic Momentum: $\mathbf{p} \equiv m\boldsymbol{\eta} = \dfrac{m\mathbf{u}}{\sqrt{1-u^2/c^2}} = m_{rel}\mathbf{u}$

Relativistic energy: $E \equiv \dfrac{mc^2}{\sqrt{1-u^2/c^2}} = m_{rel}c^2$

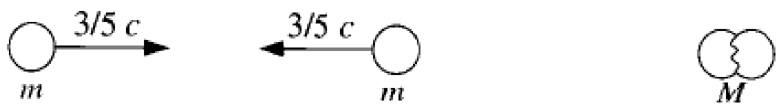
$$E^2 - p^2c^2 = m^2c^4$$

Note the distinction between an **invariant** quantity (same value in all inertial systems) and a **conserved** quantity (same value before and after some process).

- → **rest mass** is invariant, but not conserved;
- → **energy** is conserved but not invariant;
- → **electric charge** is both conserved *and* invariant;
- → **velocity** is neither conserved *nor* invariant.

(Example 12.7)

Two lumps of clay, each of (rest) mass *m,* collide head-on at (3/5)c. They stick together. What is the mass *(M)* of the composite lump?



The energy of each lump: $\dfrac{mc^2}{\sqrt{1-(3/5)^2}} = \frac{5}{4}mc^2$

Conservation energy: $\frac{5}{4}mc^2 + \frac{5}{4}mc^2 = Mc^2.$

$$M = \frac{5}{2}m.$$

M is greater than the sum of the initial masses!
Mass was not conserved.
Kinetic energy was converted into rest energy,
→ the mass increased.

12.2.3 Relativistic Kinematics

$$\mathbf{p} \equiv m\boldsymbol{\eta} = \frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}} \qquad E \equiv \frac{mc^2}{\sqrt{1-u^2/c^2}} \qquad \boxed{E^2 - p^2c^2 = m^2c^4}$$

For a massless particle (m = 0): photon

➔ If ***u* = c**,

→ then (zero) over (zero), leaving p and *E* indeterminate.

→ therefore, that a massless particle could carry energy and momentum at the speed of light.

→ ***E = pc***

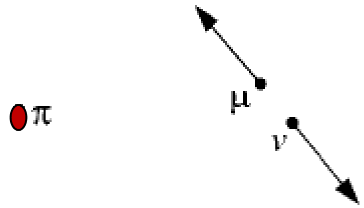
➔ What distinguishes a photon with a *lot* of energy from one with very little?

"they just have the same mass (zero) and the same speed (c)!"

→ Relativity offers no answer to this question.

→ Quantum mechanics *does,* according to the Planck formula, $E = h\nu$

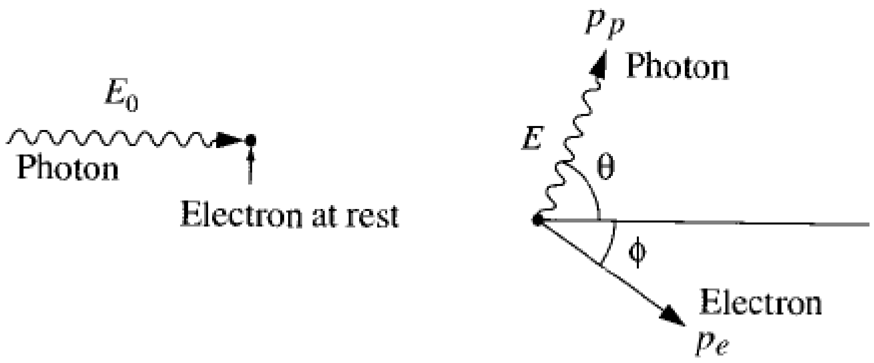
(Example 12.8) A pion at rest decays into a muon and a neutrino. Find the energy of the outgoing muon, in terms of the two masses, m_π and m_μ (assume $m_\nu = 0$).



$$\begin{aligned} E_{\text{before}} &= m_\pi c^2 \\ E_{\text{after}} &= E_\mu + E_\nu \end{aligned} \qquad E_\mu + E_\nu = m_\pi c^2 \qquad \begin{aligned} \mathbf{p}_{\text{before}} &= 0 \\ \mathbf{p}_{\text{after}} &= \mathbf{p}_\mu + \mathbf{p}_\nu \end{aligned} \qquad \mathbf{p}_\nu = -\mathbf{p}_\mu$$

$$\begin{aligned} \mathbf{p}_\nu &= -\mathbf{p}_\mu \\ E_\nu &= |\mathbf{p}_\nu| c \\ |\mathbf{p}_\mu| &= \sqrt{E_\mu^2 - m_\mu^2 c^4}\,/c \end{aligned} \qquad E_\mu + \sqrt{E_\mu^2 - m_\mu^2 c^4} = m_\pi c^2 \qquad \boxed{E_\mu = \frac{(m_\pi^2 + m_\mu^2)c^2}{2m_\pi}}$$

(Example 12.9) Compton scattering: $\lambda = \lambda_0 + \frac{h}{mc}(1-\cos\theta)$



Conservation of momentum:

$\boxed{\text{vertical}}$ $\quad p_e \sin\phi = p_p \sin\theta \qquad p_p = E/c \longrightarrow \sin\phi = \frac{E}{p_e c}\sin\theta$

$\boxed{\text{horizontal}}$ $\quad \frac{E_0}{c} = p_p\cos\theta + p_e\cos\phi = \frac{E}{c}\cos\theta + p_e\sqrt{1-\left(\frac{E}{p_e c}\sin\theta\right)^2}$

$$\longrightarrow p_e^2c^2 = (E_0 - E\cos\theta)^2 + E^2\sin^2\theta = E_0^2 - 2E_0E\cos\theta + E^2$$

Conservation of energy:

$$E_0 + mc^2 = E + E_e = E + \sqrt{m^2c^4 + p_e^2c^2} = E + \sqrt{m^2c^4 + E_0^2 - 2E_0E\cos\theta + E^2}$$

$$\Rightarrow E = \frac{1}{(1-\cos\theta)/mc^2 + (1/E_0)} \longrightarrow E = h\nu = \frac{hc}{\lambda} \Rightarrow \boxed{\lambda = \lambda_0 + \frac{h}{mc}(1-\cos\theta)}$$

(h/mc) is called the **Compton wavelength** of the electron

12.2.4 Relativistic Dynamics

***Newton's first* law** **is built into the principle of relativity.**

***Newton's second* law**: $\boxed{\mathbf{F} = \frac{d\mathbf{p}}{dt}} \longleftarrow \mathbf{p} \equiv m\boldsymbol{\eta} = \frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}}$

***Newton's third* law** **does *not,* in general, extend to the relativistic domain.**

→ the third law is incompatible with the relativity of simultaneity:

→ If the two objects in question are separated in space, a moving observer will report the reaction force at different time, therefore, the third law is *violated.*

work-energy theorem: the net work done on a particle equals the increase in its kinetic energy

$$W \equiv \int \mathbf{F}\cdot d\mathbf{l} = \int \frac{d\mathbf{p}}{dt}\cdot d\mathbf{l} = \int \frac{d\mathbf{p}}{dt}\cdot\frac{d\mathbf{l}}{dt}\,dt = \int \frac{d\mathbf{p}}{dt}\cdot\mathbf{u}\,dt$$

$$\frac{d\mathbf{p}}{dt}\cdot\mathbf{u} = \frac{d}{dt}\left(\frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}}\right)\cdot\mathbf{u} = \frac{m\mathbf{u}}{(1-u^2/c^2)^{3/2}}\cdot\frac{d\mathbf{u}}{dt} = \frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-u^2/c^2}}\right) = \frac{dE}{dt}$$

$$\Rightarrow\quad W = \int \frac{dE}{dt}\,dt = E_{\text{final}} - E_{\text{initial}}$$

Relativistic Dynamics: *Newton's second law* $\boxed{\mathbf{F} = \frac{d\mathbf{p}}{dt}}$ $\quad \mathbf{p} \equiv m\boldsymbol{\eta} = \frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}}$

Example 12.10 Motion under a constant force. A particle of mass *m* is subject to a constant force *F.* If it starts from rest at the origin, at time *t* = 0, find its position x(t), as a function of time.

Classically, it is a parabola function $x(t) = (F/2m)t^2$.

In relativistic,

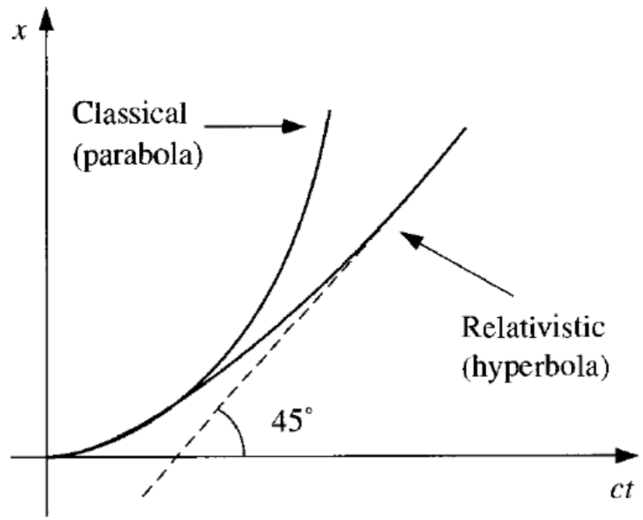
$$\frac{dp}{dt} = F \quad \Rightarrow \quad p = Ft + \text{constant.}$$

but since $p = 0$ at $t = 0$,

$$p = \frac{mu}{\sqrt{1-u^2/c^2}} = Ft$$

$$u = \frac{(F/m)t}{\sqrt{1+(Ft/mc)^2}}$$

$$\begin{aligned} x(t) &= \frac{F}{m}\int_0^t \frac{t'}{\sqrt{1+(Ft'/mc)^2}}\,dt' \\ &= \frac{mc^2}{F}\sqrt{1+(Ft'/mc)^2}\,\Big|_0^t \\ &= \frac{mc^2}{F}\left[\sqrt{1+(Ft/mc)^2}-1\right] \end{aligned}$$



→ hyperbolic motion

Relativistic Dynamics: ***Newton's Third*** law

$$\boxed{\mathbf{F} = \frac{d\mathbf{p}}{dt}} \qquad \mathbf{p} \equiv m\boldsymbol{\eta} = \frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}}$$

The third law is incompatible with the relativity of simultaneity:

→ If the two objects in question are separated in space, a moving observer will report the reaction force at different time, therefore, the third law is *violated.*

→ Only in the case of contact interactions, where the two forces are applied at the *same physical point* (and in the trivial case where the forces are *constant),* can the third law be retained.

Consider the transformation of force F between S and S-bar frames with velocity v in x:

Because **F** is the derivative of (relativistic) momentum with respect to *ordinary* time,

$$\bar{F}_y = \frac{d\bar{p}_y}{d\bar{t}} \longrightarrow = \frac{dp_y}{\gamma\, dt - \frac{\gamma\beta}{c}\, dx} = \frac{dp_y/dt}{\gamma\left(1 - \frac{\beta}{c}\frac{dx}{dt}\right)} \longrightarrow \bar{F}_y = \frac{F_y}{\gamma(1-\beta u_x/c)}$$

$$\left.\begin{aligned} \bar{\eta}^0 &= \gamma(\eta^0 - \beta\eta^1), \\ \bar{\eta}^1 &= \gamma(\eta^1 - \beta\eta^0), \\ \bar{\eta}^2 &= \eta^2, \\ \bar{\eta}^3 &= \eta^3. \end{aligned}\right\} \quad \left\{\begin{aligned} &\text{(i)}\ \ \bar{x} = \gamma(x - vt), \\ &\text{(ii)}\ \ \bar{y} = y, \\ &\text{(iii)}\ \ \bar{z} = z, \\ &\text{(iv)}\ \ \bar{t} = \gamma\left(t - \frac{v}{c^2}x\right) \end{aligned}\right.$$

Similarly for the ***Z*** component: $\longrightarrow$ $$\bar{F}_z = \frac{F_z}{\gamma(1-\beta u_x/c)}$$

The *x* component is even worse:

$$\bar{F}_x = \frac{d\bar{p}_x}{d\bar{t}} = \frac{\gamma\, dp_x - \gamma\beta\, dp^0}{\gamma\, dt - \frac{\gamma\beta}{c}\, dx} = \frac{\frac{dp_x}{dt} - \beta\frac{dp^0}{dt}}{1 - \frac{\beta}{c}\frac{dx}{dt}} = \frac{F_x - \frac{\beta}{c}\left(\frac{dE}{dt}\right)}{1-\beta u_x/c} \longrightarrow \bar{F}_x = \frac{F_x - \beta(\mathbf{u}\cdot\mathbf{F})/c}{1-\beta u_x/c}$$

Relativistic Dynamics: ***Newton's Third*** law

$$\boxed{\mathbf{F} = \frac{d\mathbf{p}}{dt}} \qquad \mathbf{p} \equiv m\boldsymbol{\eta} = \frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}}$$

The third law is incompatible with the relativity of simultaneity:

$$\left.\begin{aligned}\bar{\eta}^0 &= \gamma(\eta^0 - \beta\eta^1),\\ \bar{\eta}^1 &= \gamma(\eta^1 - \beta\eta^0),\\ \bar{\eta}^2 &= \eta^2,\\ \bar{\eta}^3 &= \eta^3.\end{aligned}\right\} \left|\begin{aligned}&\text{(i)}\ \bar{x} = \gamma(x - vt),\\ &\text{(ii)}\ \bar{y} = y,\\ &\text{(iii)}\ \bar{z} = z,\\ &\text{(iv)}\ \bar{t} = \gamma\left(t - \frac{v}{c^2}x\right)\end{aligned}\right\}$$

$$\bar{F}_x = \frac{F_x - \beta(\mathbf{u}\cdot\mathbf{F})/c}{1 - \beta u_x/c} \qquad \bar{F}_y = \frac{F_y}{\gamma(1-\beta u_x/c)} \qquad \bar{F}_z = \frac{F_z}{\gamma(1-\beta u_x/c)}$$

Only in one special case are these equations reasonably tractable:

If the particle is (instantaneously) at rest in S, for example, so that if ***u*** = 0,

$$\bar{\mathbf{F}}_\perp = \frac{1}{\gamma}\mathbf{F}_\perp, \qquad \bar{F}_\parallel = F_\parallel$$

➔ the component of *F parallel* to the motion *of S* is unchanged,
➔ Perpendicular components are divided by *γ.*

We could avoid the bad transformation behavior of **F** by introducing **a "proper" force**,
analogous to proper velocity, which would be **the derivative of momentum with respect to *proper* time**:

➔ **Minkowski force:** the derivative of momentum with respect to *proper* time:

$$\boxed{K^\mu \equiv \frac{dp^\mu}{d\tau}}$$

The spatial components → $\mathbf{K} = \left(\frac{dt}{d\tau}\right)\frac{d\mathbf{p}}{dt} = \frac{1}{\sqrt{1-u^2/c^2}}\mathbf{F}$

The zeroth component → $K^0 = \frac{dp^0}{d\tau} = \frac{1}{c}\frac{dE}{d\tau}$

Minkowski force $\boxed{K^{\mu} \equiv \frac{dp^{\mu}}{d\tau}}$

When we wish to generalize some classical force law, such as Lorentz's force, to the relativistic domain, the question arises: "Does the classical formula correspond to the *ordinary* force or to the Minkowski force?"

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})\ ? \qquad \mathbf{F} = \frac{d\mathbf{p}}{dt} \quad \mathbf{p} \equiv m\boldsymbol{\eta} = \frac{m\mathbf{u}}{\sqrt{1 - u^2/c^2}}$$

$$\mathbf{K} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})? \qquad \mathbf{K} = \left(\frac{dt}{d\tau}\right)\frac{d\mathbf{p}}{dt} = \frac{1}{\sqrt{1 - u^2/c^2}}\mathbf{F}$$

By defining the Field tensor: $F^{\mu\nu} = \left\{\begin{array}{cccc} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{array}\right\}$

$$K^{\mu} = q\eta_{\nu}F^{\mu\nu}$$

$$\longrightarrow \mathbf{K} = \frac{q}{\sqrt{1 - u^2/c^2}}[\mathbf{E} + (\mathbf{u} \times \mathbf{B})]$$

$$\longrightarrow \mathbf{F} = q[\mathbf{E} + (\mathbf{u} \times \mathbf{B})]$$