Algorithmic Methods for Definite Integration

An Abstract
Submitted On The First Day Of August, 2011
To The Department Of Mathematics
Of The Graduate School Of
Tulane University
In Partial Fulfillment Of The Requirements
For The Degree Of
Doctor Of Philosophy
By

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Abstract

Definite integrals of many special functions such as Bessel functions and orthogonal polynomials are of use to physicists and engineers. Without closed forms for the indefinite integrals of such functions, separate algorithms for definite integration must be developed. This thesis presents the analysis and implementation of two algorithms for symbolic definite integration. The methods presented here can be used to evaluate or verify a variety of table entries of definite integrals and give conditions on the parameters required for convergence of the integral.

The first method presented in this thesis, based on the Mellin transform, is a combination of algorithmic and analytic techniques. This algorithmic Mellin Transform Method has been applied to definite integrals involving elementary functions as well as special functions such as Bessel functions and a variety of orthogonal polynomials. After a definite integral is rewritten as a Mellin-Barnes complex contour integral involving hypergeometric terms, Wegschaider's algorithm is used to compute recurrences for the integrand as well as the integral itself.

The second method in the thesis, called the Method of Brackets, is a heuristic approach developed in the evaluation of Feynman integrals for particle physics. This method is based on Ramanujan's Master Theorem and is applicable to a large class of single or multiple integrals involving elementary and special functions. The rules act upon the series representations of the integrand and require solving a system of linear equations. The implementation of this algorithm in Sage with calls to Mathematica allows for testing and modification of the original rules.

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Contents

Chapter 1

Introduction

1.1 The Problem

The goal of this thesis is the analysis and implementation of algorithms for symbolic definite integration. Integrals involving special functions such as orthogonal polynomials, Bessel functions, and other hypergeometric functions are of particular interest in Mathematical Physics and other applications. The indefinite integral of such functions often cannot be written in closed form in terms of elementary functions.

The Risch algorithm [15] for indefinite integration is first and foremost a decision algorithm, deciding whether a function has an elementary function as its indefinite integral, and, if so, finding it. However, many integrals of interest to physicists and engineers are definite integrals. For this reason, the validity of entries in tables of integrals is a concern, as they are used in lookups within computer algebra software systems.

The methods presented here can be used to verify a variety of table entries of definite integrals, including conditions on the parameters required for convergence of the integral. The use of these methods have shown several errors in the Gradshteyn-Ryzhik table [7]. Several integral identities have been corrected by application of classical techniques in Chapter 2. Others errors have been discovered through testing of the second method; these are presented in Section 4.9.

The first method presented in this thesis, based on the Mellin transform method, is a combination of algorithmic and analytic techniques. Wegschaider's algorithm [19] (described in Section 1.2.3) determines recurrences for hypergeometric summands in multi-sums. Here it is applied to find recurrences for integrands in Mellin-Barnes complex contour integrals involving hypergeometric terms. These recurrences may be used to evaluate or verify entries in the table [7].

The second method in the thesis, called the method of brackets [5, 6], is a heuristic approach developed in the evaluation of Feynman integrals for particle physics. This method is based on Ramanujan's Master Theorem [1] and is applicable to a large class of single or multiple integrals involving elementary and special functions.

1.2 Background

This section presents introductory definitions of special functions and algorithmic methods for summation that will be applied in this study of algorithms for definite integration. See Appendix A for alternate definitions of special functions.

1.2.1 Gamma and related functions

The Gamma function

The gamma function $\Gamma(s)$ is defined by the integral

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt \quad [\text{Re } (s) > 0]. \tag{1.1}$$

Its functional equation

$$\Gamma(s+1) = s\Gamma(s) \tag{1.2}$$

is easily verified through integration by parts. The base case $\Gamma(1) = 1$ together with the functional equation show that the gamma function is an extension of the factorial function: $\Gamma(n) = (n-1)!$ for n a positive integer. The functional equation (1.2) allows extension of the gamma function to all complex numbers except the nonnegative integers. The gamma function has simple poles at the nonnegative integers $z = -n = 0, -1, -2, -3, \ldots$, with residues

$$\mathrm{Res}(\Gamma, -n) = \frac{(-1)^n}{n!}$$

Useful properties of the gamma function include the reflection formula (1.3), the duplication formula (1.3), and the multiplication formula (1.5):

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}, \quad [s \notin \mathbb{Z}] \tag{1.3}$$

$$\Gamma(s)\Gamma(s+\frac{1}{2}) = 2^{1-2s}\sqrt{\pi}\Gamma(2s), \tag{1.4}$$

$$\Gamma(s)\Gamma(s+\frac{1}{m})\Gamma(s+\frac{2}{m})\cdots\Gamma(s+\frac{m-1}{m}) = (2\pi)^{(m-1)/2}m^{1/2-ms}\Gamma(ms) \tag{1.5}$$

As $|s| \to \infty$ in the region where $|\arg s| \le \pi - \delta$ and $|\arg(s+a)| \le \pi - \delta$ for $\delta > 0$, the gamma function has the asymptotic behavior ([14], Theorem 13)

$$\log\Gamma(s+a) = (s+a-\frac{1}{2})\mathrm{Log}(s) - s + O(1). \tag{1.6}$$

The Beta function

The beta function $B(p,q)$ is defined by an integral and is closely related to the gamma function as follows

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx \quad [\mathrm{Re}(p)>0, \mathrm{Re}(q)>0] \tag{1.7}$$
$$= \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{1.8}$$

Special functions related to the beta function include the incomplete beta function $B(x;p,q)$ and the regularized incomplete beta function $I_x(a,b)$:

$$B(x;p,q) = \int_0^x t^{p-1}(1-t)^{q-1}\,dt \tag{1.9}$$
$$I_x(a,b) = \frac{B(x;a,b)}{B(a,b)}. \tag{1.10}$$

Rising Factorials

The *Pochhammer symbol* $(a)_k$ denotes the *rising factorial* defined for $a \in \mathbb{C}$ and $n \in \mathbb{Z}$ as

$$(a)_n := \begin{cases} a(a+1)\cdots(a+n-1) & \text{if } n>0 \\ 1 & \text{if } n=0 \\ \frac{1}{(a-1)(a-2)\cdots(a+n)} & \text{if } n<0 \text{ and } a \notin \{1,2,\ldots,-n\}. \end{cases}$$

For $n > 0$, this last case may be written as

$$(a)_{-n} = \frac{(-1)^n}{(1-a)_n}. \tag{1.11}$$

Rising factorials satisfy the useful property

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \text{ if } a \in \mathbb{C} \text{ and } a+n \notin \{0,-1,-2,\ldots\}.$$

This property, along with the definition and the gamma duplication formula (1.4), produces the useful identity ([2], 1.5.1):

$$(a)_{2n} = 2^{2n}\left(\frac{a}{2}\right)_n\left(\frac{a+1}{2}\right)_n \tag{1.12}$$

1.2.2 Hypergeometric Series

A series $\sum_{k=0}^{\infty} c_k$ is called *hypergeometric* if the ratio of two consecutive terms is a rational function of k, i.e., if there exist polynomials $p(k)$ and $q(k)$ such that

$$\frac{c_{k+1}}{c_k} = \frac{p(k)}{q(k)}.$$

By factoring the polynomials p and q, this ratio may be written in the form

$$\frac{p(k)}{q(k)} = \frac{(k+a_1)(k+a_2)\cdots(k+a_p)x}{(k+b_1)(k+b_2)\cdots(k+b_q)(k+1)},$$

where the constant x is the leading coefficient of $p(k)$ if it is not monic. Even if the factor $(k+1)$ does not appear in the factorization of $q(k)$, this factor can be inserted to the numerator to obtain this traditional form. Now the series can be written explicitly:

$$\sum_{k=0}^{\infty} c_k = c_0 \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{x^k}{k!}$$

This last generalized form is represented in the notation

$$ {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; x\right) = {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{x^k}{k!}. $$

If any of the numerator parameters $a_1, \ldots, a_p$ is a negative integer, the series terminates and convergence is not an issue. The hypergeometric series converges absolutely for all x if $p \leq q$ and for $|x| < 1$ if $p = q + 1$. If $p > q + 1$ and the series is not terminating, then the series diverges for $x \neq 0$.

The hypergeometric function with $p = 2$ and $q = 1$ is known as the *Gauss hypergeometric function* or the ordinary hypergeometric function. The hypergeometric function with $p = q = 1$ is called the *confluent hypergeometric function*.

Many elementary functions can be represented in terms of hypergeometric functions:

$$ e^x = {}_0F_0(-; -; x) \tag{1.13} $$

$$ \sin(x) = x\, {}_0F_1\left(-; \frac{3}{2}; -\frac{x^2}{4}\right) \tag{1.14} $$

$$ \cos(x) = {}_0F_1\left(-; \frac{1}{2}; -\frac{x^2}{4}\right) \tag{1.15} $$

$$ \log(1 + x) = x\, {}_2F_1\left(1, 1; 2; -x\right) \tag{1.16} $$

$$ (1 + z)^{2b} - (1 - z)^{2b} = 4bz\, {}_2F_1(1 - b, 1/2 - b; 3/2; z^2) \tag{1.17} $$

$$ \arctan(z) = z\, {}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; -z^2\right) \tag{1.18} $$

The binomial theorem can also be expressed in hypergeometric form:

$$ (1 - x)^{-a} = {}_1F_0\left(a; -; x\right) \tag{1.19} $$

Special functions such as the incomplete beta function, the error function, Bessel functions, and orthogonal polynomials may also be represented through hypergeometric functions. For alternate definitions of these functions, see A.

$$I_x(a,b) = \frac{x^a(1-x)^b}{aB(a,b)}\, {}_2F_1\left(1, a+b; a+1; x\right) \tag{1.20}$$

$$\Phi(z) = \frac{2z}{\sqrt{\pi}}\, {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -z^2\right) \tag{1.21}$$

$$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}\, {}_0F_1\left(-; \nu+1; -\frac{z^2}{4}\right) \tag{1.22}$$

$$I_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}\, {}_0F_1\left(-; \nu+1; \frac{z^2}{4}\right) \tag{1.23}$$

$$P_n(z) = {}_2F_1\left(-n, n+1; 1; \frac{1-z}{2}\right) \tag{1.24}$$

$$T_n(x) = {}_2F_1\left(-n, n; \frac{1}{2}; \frac{1-x}{2}\right) \tag{1.25}$$

$$U_n(x) = (n+1)\, {}_2F_1\left(-n, n+2; \frac{3}{2}; \frac{1-x}{2}\right) \tag{1.26}$$

$$L_n^\lambda(z) = \frac{(\lambda+1)_n}{n!}\, {}_1F_1(-n; \lambda+1; z) \tag{1.27}$$

$$H_n(z) = 2^n\sqrt{\pi}\left[\frac{1}{\Gamma\left(\frac{1-n}{2}\right)}\, {}_1F_1\left(-\frac{n}{2}; \frac{1}{2}; z^2\right) - \frac{2z}{\Gamma(-n/2)}\, {}_1F_1\left(\frac{1-n}{2}; \frac{3}{2}; z^2\right)\right] \tag{1.28}$$

Specific values of hypergeometric functions

Many hypergeometric identities exist for special values on the boundary of convergence in the case $p = q+1$. The most important was proved by Gauss in 1812 ([2], Theorem 2.2.2):

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{for } \mathrm{Re}(c-a-b) > 0 \tag{1.29}$$

Hypergeometric Transformation Identities

Transformation identities relate two hypergeometric functions at differing values of the argument x and provide analytic continuations of ${}_{p+1}F_p$ outside the unit disc. For Gauss hypergeometric functions, Pfaff's and Euler's transformations are useful:

$$
\begin{aligned}
{}_2F_1(a,b;c;x) &= (1-x)^{-a}\,{}_2F_1(a,c-b;c;x/(x-1)) && \text{(Pfaff)} \qquad (1.30)\\
{}_2F_1(a,b;c;x) &= (1-x)^{-b}\,{}_2F_1(c-a,b;c;x/(x-1)) && \text{(Pfaff alternate)} \qquad (1.31)\\
{}_2F_1(a,b;c;x) &= (1-x)^{c-a-b}\,{}_2F_1\left(c-a,c-b;c;x\right) && \text{(Euler)} \qquad (1.32)
\end{aligned}
$$

Linear combinations of solutions to the hypergeometric differential equation ([2], 2.3.5) also create identities of the Gauss hypergeometric function.

At 1, the identity is the following for $|\arg(1-z)| < \pi$ ([18], 5.10):

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix};z\right) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,{}_2F_1\left(\begin{matrix} a,b \\ a+b-c+1 \end{matrix};1-z\right)\\
&\quad + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}\,{}_2F_1\left(\begin{matrix} c-a,c-b \\ c-a-b+1 \end{matrix};1-z\right) \qquad (1.33)
\end{aligned}
$$

At ∞, the identity is the following for $|\arg(-z)| < \pi$ ([18], 5.11):

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix};z\right) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}\,{}_2F_1\left(\begin{matrix} a,1-c+a \\ 1-b+a \end{matrix};\frac{1}{z}\right)\\
&\quad + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}\,{}_2F_1\left(\begin{matrix} b,1-c+b \\ 1-a+b \end{matrix};\frac{1}{z}\right) \qquad (1.34)
\end{aligned}
$$

A useful identity for the confluent hypergeometric function ${}_1F_1$ is Kummer's identity, which will be used in classical proofs of integral identities presented in Section 2.2.2. For b not zero or a negative integer,

$$ {}_1F_1(a;b;z) = e^z \, {}_1F_1(b-a;b;-z). \tag{1.35} $$

1.2.3 An Introduction to Hypergeometric Summation

Sister Celine's Method

The first automatic proof method of summation identities was developed by Sister Celine Fasenmyer in her 1945 doctoral thesis [4]. This method solves summation problems of the form

$$ \sum_k F(n,k) = f(n) \tag{1.36} $$

where the summation is over all values of k and where the summand $F(n,k)$ is hypergeometric in both n and k.

The method proceeds as follows:

1. Find a recurrence for the summand $F(n,k)$:

   (a) Make an ansatz that summand $F(n,k)$ satisfies a "k-free" recurrence of the form

   $$ \sum_{(i,j)\in S} a_{i,j}(n) F(n-i,k-j) = 0. \tag{1.37} $$

   where S is a *structure set* of integer tuples and the coefficients $a_{i,j}(n)$ do not depend on k.

   (b) Divide (1.37) by $F(n,k)$ and simplify each term $F(n-i,k-j)/F(n,k)$ so that only rational functions in n and k remain.

   (c) Clear the denominators. Collect by powers of k.

(d) Equate the coefficients of each power of k to zero. Solve the resulting system of linear equations for the unknown $a_{i,j}(n)$ values. If the system has no solution, try again with a larger structure set S.

2. Sum this k-free recurrence over all values of k to produce a recurrence for the sum $f(n)$.

3. Solve this recurrence if possible. If no solution is possible but a conjectured value of $f(n)$ is available, check that $f(n)$ satisfies the recurrence and has the same initial conditions.

With an extension of a single k to $\mathbf{k} = (k_1, k_2, \ldots, k_r)$, Sister Celine's algorithm can be used to solve multiple sums $\sum_{\mathbf{k}} F(n, \mathbf{k})$. Examples of summation identities that can be proved by Sister Celine's method include the following:

1. $\displaystyle\sum_k k\binom{n}{k} = n2^{n-1}$

2. $\displaystyle\sum_k \binom{n}{k}\binom{2k}{k}(-2)^{n-k} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \dbinom{n}{n/2} & \text{if } n \text{ is even} \end{cases}$

3. $\displaystyle\sum_i \sum_j \binom{n}{j}\binom{j}{i} x^i y^{j-i} z^{n-j} = (x+y+z)^n.$

An Introduction to WZ summation

The use of *creative telescoping* in Wilf-Zeilberger summation methods [23] produces an improvement over Sister Celine's method [4]. Like Sister Celine's method, these methods are also used to prove summation identities of the form (1.36) where, again, $F(n, k)$ is hypergeometric in n and in k.

If the right side $r(n)$ is nonzero, divide (1.36) through by this right side to obtain the equivalent identity to be proved. Then, without loss of generality, identities to be proved are of the form

$$\sum_k F(n,k) = \mathit{constant}. \tag{1.38}$$

If the sum is denoted by $f(n)$, then the goal is to prove that $f(n) = \mathit{constant}$. Determining a recurrence $f(n+1) - f(n) = 0$ that is true for all n would be one way to prove that a function $f(n)$ is constant.

A nice way to certify that $f(n+1) - f(n) = 0$ for all n is to find a function $G(n,k)$ such that

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k). \tag{1.39}$$

If such a G can be found, then summing over all k in (1.39) producess the desired recurrence $f(n+1) - f(n) = 0$. In the class of identities involving summands $F(n,k)$ that are hypergeometric in both n and k, it will be the case that

$$G(n,k) = R(n,k)F(n,k),$$

for some function $R(n,k)$ which is rational in both n and k.

The pair of functions (F,G) satisfying (1.39) is called a *WZ-pair*, and the rational function $R(n,k)$ is called the *WZ proof certificate.* Zeilberger's algorithm determines the proof certificate $R(n,k)$ by "creative telescoping" techniques.

To prove an identity (1.36) from its WZ proof certificate $R(n,k)$, one must verify that equation (1.39) is true and check the identity for a single value of n.

Proper Hypergeometric terms

The summation algorithms described here will produce k-free recurrence relations if the summands are of a certain form called *proper hypergeometric.*

Definition 1. *[19] Let $r \in \mathbb{N}$, $l \in \mathbb{N}_0$, and let $V = \{n, k_1, \ldots, k_r, \alpha_1, \ldots, \alpha_l\}$ be a set of variables. Also let*

1. *$pp \in \mathbb{N}_0$ and $qq \in \mathbb{N}_0$, and*

2. *for every $p \in [1, \ldots, pp]$, let $a_p \in \mathbb{Z}^r$, $\mathbf{b}_p \in \mathbb{Z}$ and $c_p \in \mathbb{C}[\alpha]$, and*

3. *for every $q \in [1, \ldots, aa]$, let $u_q \in \mathbb{Z}^r$, $\mathbf{v}_q \in \mathbb{Z}$ and $w_{\in}\mathbb{C}[\alpha]$, and*

4. *let $P(n, k) \in \mathbb{C}[V]$ be a polynomial, and*

5. *let $x_0, x_1, \ldots, x_r \in \mathbb{C}[\alpha]$ be polynomials.*

Then

$$t = P(n,k)\frac{\prod_{p=1}^{pp}\Gamma(a_p n + \mathbf{b}_p \cdot \mathbf{k} + c_p)}{\prod_{q=1}^{qq}\Gamma(u_q n + \mathbf{v}_q \cdot \mathbf{k} + w_q)} x_0^n x_1^{k_1} \cdots x_r^{k_r}. \tag{1.40}$$

is a proper hypergeometric term *with hypergeometric variables n and $\mathbf{k}$ and additional parameters α. The linear forms $a_p n + \mathbf{b}_p \cdot \mathbf{k} + c_p$ are called the* numerator factorial expressions *of t; the linear forms $u_q n + \mathbf{v}_q \cdot \mathbf{k} + w_q$ are called the* denominator factorial expressions of t.

Wegschaider's Algorithm

Wegschaider's algorithm [19] is an extension of Fasenmyer/WZ summation methods [4, 21] and can be used to compute recurrences for sums of the form

$$Sum(\mu) = \sum_{\kappa_1 \in R_1} \cdots \sum_{\kappa_r \in R_r} \mathcal{F}(\mu, \kappa_1, \ldots, \kappa_r) \tag{1.41}$$

where the summand $\mathcal{F}(\mu, \kappa_1, \ldots, \kappa_r)$ is hypergeometric in the summation variables $\mu_i \in \mu = (\mu_1, \ldots, \mu_p)$ and $\kappa = (\kappa_1, \ldots, \kappa_r)$. Note that if $p = 1$ and $r = 1$, this is Zeilberger's algorithm [23].

Wegschaider's algorithm [19] can be applied if the summands $\mathcal{F}(\mu, \kappa)$ are proper hypergeometric in all integer variables μ_i from $\mu = (\mu_1, \ldots, \mu_p)$ and in all summation variables κ_j from $\kappa = (\kappa_1, \ldots, \kappa_r) \in \mathcal{R}$ where $\mathcal{R} := \mathcal{R}_1 \times \cdots \times \mathcal{R}_r \subseteq \mathbb{Z}^r$ is the summation range. The structure set is a finite nonempty set $\mathbb{S} \subseteq \mathbb{Z}^{p+r}$.

The algorithm first finds a recurrence for the summand $\mathcal{F}(\mu, \kappa)$, called the *certificate recurrence*, of the form

$$\sum_{m \in \mathbb{S}} a_m(\mu)\mathcal{F}(\mu + m, \kappa) = \sum_{j=1}^{r} \Delta_{\kappa_j} \left(\sum_{(m,k) \in \mathbb{S}_j} b_{m,k}(\mu, \kappa)\mathcal{F}(\mu + m, \kappa + k) \right), \qquad (1.42)$$

where the polynomials $a_m(\mu)$, not all zero, $b_{m,k}(\mu, \kappa)$ and the sets $\mathbb{S}_j \subset \mathbb{Z}^{p+r}$ are determined algorithmically and the forward shift operators Δ_{κ_j} are defined as

$$\Delta_{\kappa_j}\mathcal{F}(\mu, \kappa) := \mathcal{F}(\mu, \kappa_1, \ldots, \kappa_j + 1, \ldots, \kappa_r) - \mathcal{F}(\mu, \kappa).$$

Because the summand $\mathcal{F}(\mu, \kappa)$ is hypergeometric, the right hand side of (1.42) can always be rewritten as

$$\sum_{j=1}^{r} \Delta_{\kappa_j} \left(\sum_{(m,k) \in \mathbb{S}_j} b_{m,k}(\mu, \kappa)\mathcal{F}(\mu + m, \kappa + k) \right) = \sum_{j=1}^{r} \Delta_{\kappa_j}(r_j(\mu, \kappa)(F(\mu, \kappa)),$$

where r_j are rational functions of all variables from $\mu = (\mu_1, \ldots, \mu_p)$ and $\kappa = (\kappa_1, \ldots, \kappa_r)$.

In the certificate recurrence (1.42), the coefficients $a_m(\mu)$ are polynomials free of the summation variables κ_j and the coefficients $b_{m,k}(\mu, \kappa)$ of the Δ-parts are polynomials in all the variables from μ and κ.

A recurrence for the multisum (1.41) is obtained by summing the certificate recurrence (1.42) over all variables from κ in the given summation range $\mathcal{R}$. Since it can be easily checked whether the summand $\mathcal{F}(\mu, \kappa)$ does, in fact, satisfy the certificate recurrence (1.42), the certificate recurrence also provides a proof of the recurrence for the multisum $Sum(\mu)$ in (1.41).

Wegschaider's algorithm determines certificate recurrences after fixing the structure set $\mathbb{S}$ by solving large system of linear equations over a field of rational functions. Since computations can be rather time consuming, the procedure `FindStructureSet` included in the package `MultiSum` finds small structure sets using an algorithm based on modular computation. To use this procedure and the summation algorithm [19], a user loads the package `MultiSum` within a Mathematica session:

In[1]:= **<< MultiSum.m**
MultiSum Package by Kurt Wegschaider (enhanced by Axel Riese and Burkhard Zimmermann) – © RISC Linz – V2.02β (02/21/05)

Wegschaider's algorithm [19] terminates successfully for a large enough structure set if the input class is restricted to *proper hypergeometric* summands. In many applications, the function $\mathcal{F}(\mu, \kappa)$ has finite support. In these cases, if the certificate recurrence (1.42) is summed over a domain that is larger than the support of the function, the Δ-parts on the right hand side telescope and the values that are not in the support vanish. The result is that in such situations, one can obtain a homogeneous recurrence of the form

$$\sum_{m \in \mathbb{S}} a_m(\mu) Sum(\mu + m) = 0. \tag{1.43}$$

It is not the case in general that the recurrence is homogeneous. In specific situations, inspection may be necessary to pass from the certificate recurrence (1.42) to a homogeneous or inhomogeneous recurrence for the sum (1.41). See [21] for further details.

1.3 The Methods

1.3.1 Classical approaches

As a preface to algorithmic approaches, Chapter 2 presents proofs of integral identities by classical techniques. These proof involve integral representations of hypergeometric functions, the techniques in the proofs of such representations, and frequent changes of variables. These proofs illustrate difficulties with relying solely on these techniques for proofs of table entries.

1.3.2 Mellin Transform approach

In Chapter 3, the classical Mellin transform method is given more power by an algorithmic extension. This method has been applied to definite integrals involving elementary functions, Bessel functions, and various orthogonal polynomials. Rewriting an integrand factor using its inverse Mellin transform representation and reversing the order of integration will, in many cases, turn the problem into a complex contour Mellin-Barnes integral over hypergeometric terms. The algorithmic procedure is to use Wegschaider's algorithm [19] to compute recurrences for the integrand as well as the integral itself.

This algorithmic aspect of the method must be combined with analytic analysis of the Mellin-Barnes integrand, such as the checking of growth conditions on the integrand and initial conditions of the recurrences.

In the paper [8], this algorithmic Mellin transform method was used to generate proofs of definite integral identities in the Gradshteyn-Ryzhik table [7]. A variety of examples will be presented in Chapter 3.

1.3.3 Method of Brackets approach

Chapter 4 presents the method of brackets, a heuristic method originating in the evaluation of Feynmann integrals. Currently, parts of this method are still only heuristic. In spite of this, its short list of rules can be easily automated and have been implemented in the open-source computer algebra software system Sage. The implementation allows for testing and modification of the current rules. Still under development, this method and its implementation are being tested against entries from classical tables of integrals such as [7]. The code is given in the appendices.

In the method of brackets, the integrand is replaced by a (multi-)series representation. Next, the integral is replaced with a bracket series. The final step is to apply rules to evaluate the bracket series; these rules involve solving a system of linear equations and often return series which must be evaluated and analyzed for convergence.

The equivalent representations of the integrand as well as the order of series expansions produce various bracket series. The production of all these will be computationally intense. Among these, the most appropriate and efficient one must be selected. The form of the bracket series and the linear system will also determine much of the computational complexity of the solution.

The final product of this method will be an implementation which evaluates integrals on $[0, \infty)$. The method will be implemented in the open-source software system Sage with calls to Mathematica for evaluation of infinite sums.

Modifications to the original set of rules are proposed based on the testing the method against table entries. An algorithm to determine the best representation of the integrand and the most appropriate order of application of rules will also be created and implemented.

1.4 Examples

The following integral identities may be proved by one or both of the algorithmic methods presented in this thesis.

The integral ([7], 6.565.2) involving a single Bessel function will be proved by the classical Mellin transform method in Section 3.2 and by the method of brackets in Section 4.8.2.

$$\int_0^\infty x^{\nu+1}(x^2+a^2)^{-\nu-\frac{1}{2}} J_\nu(bx)\,dx = \frac{\sqrt{\pi}b^{\nu-1}}{2^\nu e^{ab}\Gamma(\nu+\frac{1}{2})}.$$

The integral ([7], 6.512.3) involves a product of Bessel functions. Both algorithmic methods generate proofs, one in Section 3.5.1 and one in Section 4.8.2.

$$\int_0^\infty J_\nu(\alpha x)J_{\nu-1}(\beta x)dx = \frac{\beta^{\nu-1}}{\alpha^\nu} \quad [\beta < \alpha]\,.$$

Each of these algorithms handles certain general-degree orthogonal polynomials. The algorithmic extension of the Mellin transform method handles integrals of polynomials that can be written as terminating ${}_2F_1$'s. Proofs of identities such as ([7], 7.231.2) involving one or more Legendre or Chebyshev polynomials are shown in Section 3.5.2:

$$\int_0^1 x^\lambda P_{2m+1}(x)\,dx = \frac{(-1)^m\Gamma\left(m+\frac{1}{2}-\frac{1}{2}\lambda\right)\Gamma\left(1+\frac{1}{2}\lambda\right)}{2\Gamma\left(\frac{1}{2}-\frac{1}{2}\lambda\right)\Gamma\left(m+2+\frac{1}{2}\lambda\right)}.$$

Identity ([7], 7.345.3) contains two arbitrary degree Chebyshev polynomials and is also proved in Section 3.5.2:

$$\int_{-1}^1 (1-x)^{1/2}(1+x)^{m+n+3/2}U_m(x)U_n(x)\,dx = \frac{\pi(2m+2n+2)!}{2^{m+n+2}(2m+1)!(2n+1)!}$$

The method of brackets evaluates ${}_1F_1$ polynomials such as Hermite and Laguerre polynomials that are integrated over $[0,\infty)$. The identity ([7], 7.376.2) involving a Hermite polynomial is evaluated by the method of brackets in Section 4.8.2:

$$\int_0^\infty e^{-2\alpha x^2} x^\nu H_{2n}(x)\,dx = (-1)^n 2^{2n-\frac{3}{2}-\frac{1}{2}\nu} \frac{\Gamma(\frac{\nu+1}{2})\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\alpha^{\frac{1}{2}(\nu+1)}} \, {}_2F_1\left(-n, \frac{\nu+1}{2}; \frac{1}{2}; \frac{1}{2\alpha}\right).$$

Identity ([7], 7.414.9) with a product of Laguerre polynomials is evaluated by the method of brackets in Section 4.8.3:

$$\int_0^\infty e^{-x} x^{\alpha+\beta} L_m^\alpha(x) L_n^\beta(x) dx = (-1)^{m+n}(\alpha+\beta)! \binom{\alpha+m}{n}\binom{\beta+n}{m}.$$

Chapter 2

Classical Proofs

This chapter presents proofs of integrals from the table [7] involving classical techniques. Specifically, the integrals presented here are closely related to hypergeometric functions ${}_pF_q$, which have integral representations as well as series representations.

2.1 Integrals involving the Hypergeometric Function ${}_2F_1$

Gauss' hypergeometric function is defined for $|z| < 1$ by the series

$$ {}_2F_1\left(\begin{matrix} a, b, \\ c \end{matrix} ; z\right) = \sum_{k=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \tag{2.1} $$

except for $c = 0, -1, -2, \ldots$.

2.1.1 Integral Represetation of ${}_2F_1$

In addition to the series representation, the ordinary hypergeometric function ${}_2F_1$ also has an integral representation. This was proved by Euler in 1748 by expanding the $(1 - tx)^{-a}$ factor via the binomial theorem (1.19) and integrating termwise.

$$
{}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix};x\right) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}dt
$$

$$
[\text{Re } c > \text{Re } b > 0, \arg(1-z) < \pi]. \quad (2.2)
$$

This integral representation of the ${}_2F_1$ function appears in [7] as identity 3.197.3:

$$
\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}(1-\beta x)^{-\nu}\,dx = B(\lambda,\mu)\,{}_2F_1(\nu,\lambda;\lambda+\mu;\beta) \quad (2.3)
$$

2.1.2 Examples

Identity ([7], 3.197.1)

$$
\int_0^\infty x^{\nu-1}(\beta+x)^{-\mu}(x+\gamma)^{-\varrho}\,dx = \beta^{-\mu}\gamma^{\nu-\varrho}B(\nu,\mu-\nu+\varrho)\,{}_2F_1(\mu,\nu;\mu+\varrho;1-\frac{\gamma}{\beta}) \quad (2.4)
$$

for $|\arg\beta| < \pi, |\arg\gamma| < \pi, \text{Re } \nu > 0, \text{Re } \mu > \text{Re}(\nu-\varrho)$. The proof of 3.197.1 requires a change of variables from identity 3.197.3. With different parameter notation, the details appear in [9].

Identity ([7], 3.227.2)

There are typos in the $7th$ edition of [7]. The original entry of ([7], 3.227.2) is

$$
\int_0^\infty \frac{x^{-\varrho}(\beta-x)^{-\sigma}}{\gamma+x}\,dx = \pi\gamma^{-\varrho}(\beta-\gamma)^{-\sigma}\csc(\varrho\pi)I_{1-\gamma/\beta}(\sigma,\varrho)
$$

$$
[|\arg\beta| < \pi, |\arg\gamma| < \pi, -\text{Re}\,\sigma < \text{Re}\,\varrho < 1]
$$

The corrected entry is

$$\int_0^\infty \frac{x^{-\rho}(\beta-x)^{-\sigma}}{\gamma+x}\,dx = \frac{\pi}{\gamma^\rho(\beta+\gamma)^\sigma\sin(\pi\rho)} I_{1+\frac{\gamma}{\beta}}(\sigma,\rho). \tag{2.5}$$

where $I_x(a,b)$ represents the regularized incomplete beta function (1.10).

Application of identity 3.197.1 (2.4) transforms this identity into

$$\int_0^\infty \frac{x^{-\rho}(\beta-x)^{-\sigma}}{\gamma+x}\,dx = (-1)^{-\sigma}\frac{(-\beta)^{1-\rho-\sigma}}{\gamma} B(1-\rho,\rho+\sigma)\,{}_2F_1\left(1, 1-\rho; 1+\sigma; 1+\frac{\beta}{\gamma}\right).$$

By the Pfaff's identity (1.30), the hypergeometric representation of $I_x(a,b)$ (1.20), and the reflection formula (1.3), the identity is proved.

This correction will also be verified by the method of brackets in Section 4.8.2.

Identity ([7], 3.236)

There is a typo in the *7th* edition of [7]. The original version of ([7], 3.236) is

$$\int_0^1 x^{\mu/2}(1-x)^{-(\mu+1)/2}(1-a^2x)^{-(\mu+1)/2}\,dx = \frac{\Gamma\left(\frac{\mu}{2}+1\right)\Gamma\left(\frac{1-\mu}{2}\right)}{2a\mu\sqrt{\pi}}\left[(1-a)^{-\mu}-(1+a)^{-\mu}\right]$$

The corrected version is shown here without the factor of 2 in the denominator:

$$\int_0^1 x^{\mu/2}(1-x)^{-(\mu+1)/2}(1-a^2x)^{-(\mu+1)/2}\,dx = \frac{\Gamma\left(\frac{\mu}{2}+1\right)\Gamma\left(\frac{1-\mu}{2}\right)}{a\mu\sqrt{\pi}}\left[(1-a)^{-\mu}-(1+a)^{-\mu}\right] \tag{2.6}$$

By identity 3.197.3 (2.3), the identity can be written as

$$\int_0^1 x^{\mu/2}(1-x)^{-(\mu+1)/2}(1-a^2x)^{-(\mu+1)/2}\,dx = B\left(\frac{\mu}{2}+1, \frac{1-\mu}{2}\right)\,{}_2F_1\left(\frac{\mu+1}{2}, \frac{\mu}{2}+1; \frac{3}{2}; a^2\right).$$

The symmetry of hypergeometric series with respect to numerator parameters and the identity (1.17) are used to complete the proof.

Identity ([7], 3.315.1)

$$\int_{-\infty}^{\infty} \frac{e^{-\mu x}\,dx}{(e^{\beta}+e^{-x})^{\nu}(e^{\gamma}+e^{-x})^{\rho}} = e^{\gamma(\mu-\rho)-\beta\nu} B(\mu, \nu+\rho-\mu)\,{}_2F_1\left(\nu, \mu; \nu+\rho; 1-e^{\nu-\beta}\right)$$

$$[|\operatorname{Im}\beta| < \pi, |\operatorname{Im}\gamma| < \pi, 0 < \operatorname{Re}\mu < \operatorname{Re}(\nu+\rho)] \quad (2.7)$$

The proof of this identity requires only the change of variables $y = e^{-x}$ and identity 3.197.1 (2.4).

Identity ([7], 3.228.3)

$$\int_a^b (x-a)^{\nu-1}(b-x)^{\mu-1}(x-c)^{-1}\,dx$$

$$= \begin{cases} (b-c)^{-1}(b-a)^{\mu+\nu-1} B(\mu,\nu)\,{}_2F_1\left(1, \mu; \mu+\nu; \frac{a-b}{c-b}\right) \\ \qquad \text{for } c < a \text{ or } c > b; \\ \pi(c-a)^{\nu-1}(b-c)^{\mu-1}\cot(\mu\pi) \\ \quad -(b-a)^{\mu+\nu-2} B(\mu-1,\nu)\,{}_2F_1\left(\begin{matrix} 2-\mu-\nu, 1 \\ 2-\mu \end{matrix}; \frac{b-c}{b-a}\right) \\ \qquad \text{for } a < c < b \end{cases}$$

$$[\operatorname{Re}\mu > 0,\ \operatorname{Re}\nu > 0, \mu+\nu \neq 1, \mu \neq 1, 2, \ldots] \quad (2.8)$$

For the first case, use the change of variables $t = x - a$, ([7], 3.197.8), and Pfaff's identity (1.30). The other case is proved by application of the identity (1.34) given in ([18], 5.11).

Identity ([7], 3.228.5)

$$\int_0^\infty \frac{x^{\nu-1}(x+a)^{1-\mu}}{x-c}\,dx$$
$$= \begin{cases} a^{1-\mu}(-c)^{\nu-1}B(\mu-\nu,\nu)\,{}_2F_1\left(\mu-1,\nu;\mu;1+\frac{c}{a}\right) \\ \qquad \text{for } c<0; \\ \pi c^{\nu-1}(a+c)^{1-\mu}\cot[(\mu-\nu)\pi] - \frac{a^{1-\mu-\nu}}{a+c}B(\mu-\nu-1,\nu)\,{}_2F_1\left(\begin{matrix} 2-\mu,1 \\ \\ 2-\mu+\nu \end{matrix};\frac{a}{a+c}\right) \\ \qquad \text{for } c>0 \end{cases}$$

$$[a>0,\ 0<\operatorname{Re}\nu<\operatorname{Re}\mu] \quad (2.9)$$

For the first case, identity 3.197.1 (2.4) is used. The second case is proved by application of the identity (1.34) on the first case.

2.2 Integrals involving the Generalized Hypergeometric Function ${}_pF_q$

2.2.1 Integral Representation of ${}_pF_q$

The generalized hypergeometric function ${}_pF_q$ also has an integral representation similar to (2.2) for Re $c >$ Re $d > 0$:

$$
{}_{p+1}F_{q+1}\left(\begin{matrix} a_1, a_2, \ldots, a_p, c \\ b_1, b_2 \ldots, b_q, d \end{matrix}; z\right)
$$

$$
= \frac{1}{B(c, d-c)} \int_0^1 t^{c-1}(1-t)^{d-c-1} {}_pF_q\left(\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}; zt\right) dt. \quad (2.10)
$$

This identity appears in [7] as 7.512.12:

$$
\int_0^1 (1-x)^{\mu-1} x^{\nu-1} {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; ax\right) dx
$$

$$
= \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} {}_{p+q}F_{q+1}\left(\begin{matrix} \nu, a_1, \ldots, a_p \\ \mu+\nu, b_1, \ldots, b_q \end{matrix}; a\right). \quad (2.11)
$$

This identity is proved by writing the ${}_pF_q$ in its series notation, interchanging integration and summation, and recognizing the integral as a beta function.

Integral Representation of ${}_1F_1$

A useful special case of (2.10) is the integral representation for the ${}_1F_1$ confluent hypergeometric function:

$$
{}_1F_1(a; b; z) = \frac{1}{B(a, b-a)} \int_0^1 e^{zt} t^{a-1}(1-t)^{b-a-1} dt \quad (2.12)
$$

This identity is proved by (2.10) with the hypergeometric ${}_0F_0$ representation (1.13)of the exponential function. This representation is given in ([18], 7.7).

Identity ([7], 7.522.5)

Another useful integral representation is given as identity 7.522.5 in [7]:

$$\int_0^\infty e^{-x}x^{s-1}\,{}_pF_q\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q\end{matrix};ax\right)dx=\Gamma(s)\,{}_{p+1}F_q\left(\begin{matrix} s,a_1,\ldots,a_p \\ b_1,\ldots,b_q\end{matrix};a\right). \tag{2.13}$$

This identity is proved by writing the ${}_pF_q$ in its series notation, interchanging integration and summation, and recognizing the integral as a gamma function.

2.2.2 Examples

Identity ([7], 3.254.1)

$$\int_0^u x^{\lambda-1}(u-x)^{\mu-1}(x^2+\beta^2)^\nu\,dx=u^{\mu+\lambda-1}\beta^{2\nu}B(\mu,\lambda)\,{}_3F_2\left(\begin{matrix} -\nu,\frac{\lambda}{2},\frac{\lambda+1}{2} \\ \frac{\lambda+\mu}{2},\frac{\lambda+\mu+1}{2}\end{matrix};-\frac{u^2}{\beta^2}\right). \tag{2.14}$$

This identity is proved by applying the change of variables $t=x/u$ and following the techniques in the proof of Euler's integral formula (2.2) with an application of the duplication formula (1.4)

Identity ([7], 3.254.2)

The identity 3.254.2 contains a typo in the *7th* edition of [7]. The original identity is

$$\int_u^\infty (x^{-\lambda}(x-u)^{\mu-1}(x^2+\beta^2))^\nu\,dx$$
$$=u^{\mu-\lambda+2\nu}B(\mu,\lambda-\mu-2\nu)\,{}_3F_2\left(\begin{matrix} -\nu,\frac{\lambda-\mu}{2}-\nu,\frac{1+\lambda-\mu}{2}-\nu \\ \frac{\lambda}{2}-\nu,\frac{1+\lambda}{2}-\nu\end{matrix};-\frac{\beta^2}{u^2}\right)$$

The corrected version is

$$\int_u^\infty x^{-\lambda}(x-u)^{\mu-1}(x^2+\beta^2)^\nu\,dx$$
$$= u^{\mu-\lambda+2\nu}B(\mu,\lambda-\mu-2\nu)\,{}_3F_2\left(\begin{matrix} -\nu,\,\frac{\lambda-\mu}{2}-\nu,\,\frac{1+\lambda-\mu}{2}-\nu \\ \frac{\lambda}{2}-\nu,\,\frac{1+\lambda}{2}-\nu \end{matrix};-\frac{\beta^2}{u^2}\right) \quad (2.15)$$

This identity is proved by using the change of variables $y = u^2/x$ and identity 3.254.1 (2.14).

Identity ([7], 3.259.2)

$$\int_0^u x^{\nu-1}(u-x)^{\mu-1}(x^m+\beta^m)^\lambda\,dx$$
$$= \beta^{m\lambda}u^{\mu+\nu-1}B(\mu,\nu)\,{}_{m+1}F_m\left(\begin{matrix} -\lambda,\,\frac{\nu}{m},\,\frac{\nu+1}{m},\ldots,\frac{\nu+m-1}{m} \\ \frac{\mu+\nu}{m},\,\frac{\mu+\nu+1}{m},\ldots,\frac{\mu+\nu+m-1}{m} \end{matrix};-\frac{u^m}{\beta^m}\right) \quad (2.16)$$

Following the techniques used in the proof of the integral representation of ${}_2F_1$ (2.2), the change of variables $t = x/u$ is made, the hypergeometric form of the binomial theorem (1.19) is used, summation and integration are intergchanged, the integral is recognized as a beta function, and the multiplication formula (1.5) for the gamma function is applied twice. See [9] for details.

Identity ([7], 7.347.2)

The printed version of identity 7.347.2 in the *7th* edition of [7] contains a typo. The original identity is

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}U_n(x)\,dx$$
$$= \frac{2^{\alpha+\beta+2n+2}\left[(n+1)!\right]^2 B(\alpha+1,\beta+1)}{(2n+2)!}\,{}_3F_2\left(\begin{matrix} -n, n+2, \alpha+1 \\ \frac{3}{2}, \alpha+\beta+2 \end{matrix} ;1\right)$$

The corrected form is

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}U_n(x)\,dx = 2^{\alpha+\beta+1}(n+1)B(\alpha+1,\beta+1)\,{}_3F_2\left(\begin{matrix} -n, n+2, \alpha+1 \\ \frac{3}{2}, \alpha+\beta+2 \end{matrix} ;1\right) \tag{2.17}$$

The proof of this corrected identity requires the hypergeometric representation of the Chebyshev polynomial of the second kind (1.26), the change of variables $y = (1-x)/2$, and the identity (2.10). This corrected identity will be used in the proof of ([7], 7.345.3) presented in Section 3.5.2.

Identity ([7], 3.383.1)

$$\int_0^u x^{\nu-1}(u-x)^{\mu-1}e^{\beta x}\,dx = B(\mu,\nu)u^{\mu+\nu-1}\,{}_1F_1(\nu;\mu+\nu;\beta u) \tag{2.18}$$

The proof of this identity requires the change of variables $t = x/u$ and the following integral representation for the confluent hypergeometric function (2.12).

Identity ([7], 3.478.3)

$$\int_0^u x^{\nu-1}(u-x)^{\mu-1}e^{\beta x^n}\,dx = B(\mu,\nu)u^{\mu+\nu-1}\,{}_nF_n\left(\begin{matrix} \frac{\nu}{n},\ldots,\frac{\nu+n-1}{n} \\ \frac{\nu+\mu}{n},\ldots,\frac{\nu+\mu+n-1}{n}\end{matrix};\beta u^n\right) \tag{2.19}$$

The change of variables $t = x/u$ is applied, the hypergeometric form of the exponential function is used, integration and summation are interchanged, the inner definite integral is recognized as a beta function, and the multiplication formula for the gamma function (1.5) is applied.

Identity ([7], 3.952.7)

$$\int_0^\infty x^{\mu-1}e^{-\beta x^2}\sin(\gamma x)\,dx = \frac{\gamma}{2\beta^{(\mu+1)/2}}\Gamma\left(\frac{\mu+1}{2}\right)e^{-\gamma^2/4\beta}\,{}_1F_1\left(1-\frac{\mu}{2};\frac{3}{2};\frac{\gamma^2}{4\beta}\right) \tag{2.20}$$

This identity is proved by the change of variables $t = \beta x^2$, the representation of the sine function in its hypergeometric form, the integration of a ${}_0F_1$ to a ${}_1F_1$ by 7.522.5 (2.13), and Kummer's identity (1.35) for ${}_1F_1$.

Identity ([7], 3.952.8)

$$\int_0^\infty x^{\mu-1}e^{-\beta x^2}\cos(ax)\,dx = \frac{1}{2}\beta^{-\mu/2}\Gamma\left(\frac{\mu}{2}\right)e^{-a^2/4\beta}\,{}_1F_1\left(\frac{1}{2}-\frac{\mu}{2};\frac{1}{2};\frac{a^2}{4\beta}\right) \tag{2.21}$$

Just as with identity 3.952.7, this identity may be proved using the change of variables $t = \beta x^2$, the expression of the cosine in its ${}_0F_1$ hypergeometric form, the identity 7.522.5 (2.13), and Kummer's identity (1.35) for ${}_1F_1$.

Identity ([7], 3.389.1)

$$\int_0^u x^{2\nu-1}(u^2-x^2)^{\rho-1}e^{\mu x}\,dx$$
$$=\frac{u^{2\rho+2\nu-2}B(\rho,\nu)}{2}\,{}_1F_2(\nu;\nu+\rho,\frac{1}{2};\frac{\mu^2u^2}{4})$$
$$+\frac{u^{2\rho+2\nu-1}\mu B(\rho,\nu+1/2)}{2}\,{}_1F_2(\nu+1;\nu+\rho+1/2,\frac{3}{2};\frac{\mu^2u^2}{4})$$

The proof of this identity requires the change of variables $y = x^2/u^2$, the hypergeometric representation of the exponential function, the interchange of integration and summation, recognition of the inner definite integral as a beta function, a splitting of the resulting series into two series according to the parity of summation index, and the gamma duplication formula (1.4). Details appear in [9].

Identity ([7], 3.771.3)

$$\int_0^u x^{2\nu-1}(u^2-x^2)^{\mu-1}\sin(ax)\,dx$$
$$=\frac{a}{2}u^{2\nu+2\mu-1}B(\nu+1/2,\mu)\,{}_1F_2\left(\nu+\frac{1}{2};\frac{3}{2},\mu+\nu+\frac{1}{2};-\frac{a^2u^2}{4}\right) \quad (2.22)$$

The identity ([7], 3.771.3) is proved by the change of variables $x = u\sqrt{t}$, the expression of the sine function in its hypergeometric form (1.14), and the identity (2.11).

Identity ([7], 3.771.4)

$$\int_0^u x^{2\nu-1}(u^2-x^2)^{\mu-1}\cos(ax)\,dx = \frac{1}{2}u^{2\nu+2\mu-2}B(\nu,\mu)\,{}_1F_2\left(\nu;\frac{1}{2},\mu+\nu;-\frac{a^2u^2}{4}\right) \tag{2.23}$$

The identity ([7], 3.771.4) is proved by the change of variables $x = u\sqrt{t}$, the expression of the cosine function in its hypergeometric form (1.15), and the identity (2.11).

2.3 Possibilities of Automation

The proofs shown in this chapter illustrate why classical proof techniques could be automated only to a certain extent. There exists a large number of hypergeometric transformations, a few of which were listed in Section 1.2.2. The transformations for ${}_2F_1$'s include Euler's and Pfaff's transformations (1.32 and 1.30) plus quadratic transformations and contiguous relations. Similar transformations exist for generalized ${}_pF_q$. No database of transformations would ever be comprehensive since a new transformation may be derived through a sequence of previous transformations.

Chapter 3

The Algorithmic Mellin Transform Method

This chapter presents an algorithmic extension of the classic Mellin transform method for definite integration. In the classic Mellin transform method for definite integration, an integral is transformed to a contour integral of Mellin-Barnes type via the inverse Mellin transform. In this algorithmic approach, Wegschaider's algorithm [19] is applied to determine recurrences for multiple contour integrals over hypergeometric integrands. If these recurrences cannot be solved, they may at least be checked against a known value from a table such as [7]. This method can be used to prove identities involving such special functions as Bessel functions and orthogonal polynomials.

3.1 The Mellin Transform and Its Inverse

Definition 2. *The* Mellin transform *of a locally integrable function* $f : (0, \infty) \to \mathbb{C}$ *is defined by*

$$M[f; z] = \tilde{f}(z) = \int_0^\infty t^{z-1} f(t) dt \tag{3.1}$$

whenever the integral converges.

In general, the integral converges absolutely and is analytic only for values of z in an infinite *strip of analyticity* generally of the form $\alpha < \mathrm{Re}(z) < \beta$.

For example, with $c > 0$, the Mellin transform of $f(t) = e^{-ct}$ is

$$M[f; z] = \int_0^\infty t^{z-1} e^{-ct}\, dt = c^{-z}\Gamma(z).$$

Since the gamma function $\Gamma(z)$ is analytic for $\mathrm{Re}(z) > 0$, the strip of analyticity is a half-plane in this case.

3.1.1 Inverse Mellin Transform

A function $f(t)$ may be recovered from its transform via the *inversion formula*:

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-z}\tilde{f}(z)dz. \tag{3.2}$$

The original function $f(t)$ is determined uniquely from $\tilde{f}(z)$ at all points $t \geq 0$ where $f(t)$ is continuous. The contour of integration is a vertical line in the z-plane which must lie in the strip of analyticity.

With $c = 1$ in the earlier example, the function e^{-t} (for $t > 0$) is the inverse Mellin transform of the $\Gamma(z)$ for $z > 0$. By the inversion formula (3.2), e^{-t} has the contour integral representation

$$e^{-t} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} t^{-z}\Gamma(z)\, dz, \qquad [a > 0] \tag{3.3}$$

Since the gamma function can be analytically continued in the left half-plane except at the poles located at $0, -1, -2, \ldots$, the inverse Mellin transform of the gamma function for different strips of analyticity can be found by shifting the contour in (3.3). Shifting the contour to the left will cause the integral to pick up the values of the residues at each pole. If $a > 0$ and $-N < a' < -N + 1$ for N a positive integer, then

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} t^{-z}\Gamma(z)\,dz = \sum_{n=0}^{N-1}\frac{(-1)^n}{n!}t^n + \frac{1}{2\pi i}\int_{a'-i\infty}^{a'+i\infty} t^{-z}\Gamma(z)\,dz.$$

The inversion formula of the gamma function in the strip $-N < \mathrm{Re}(z) < -N+1$ gives the result:

$$\frac{1}{2\pi i}\int_{a'-i\infty}^{a'+i\infty} t^{-z}\Gamma(z)\,dz = e^{-t} - \sum_{n=0}^{N-1}\frac{(-1)^n}{n!}t^n,$$

This representation shows that the integral term represents the remainder in the Taylor expansion of e^{-t} and can be shown by Stirling's formula to vanish as $N \to \infty$.

3.1.2 Mellin convolution

The Mellin transform method to proving identities is motivated by the proof of the main property of the *Mellin convolution*, i.e.,

$$\int_0^\infty g(xt)h(t)dt = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} x^{-z}\tilde{g}(z)\tilde{h}(1-z)dz \tag{3.4}$$

where $g, h : (0,\infty) \to \mathbb{C}$ are defined so that the integral on the left exists and the Mellin transforms $\tilde{g}(z)$ and $\tilde{h}(1-z)$ have a common domain of analyticity with δ lying in this common domain. The special case $x = 1$ of (3.4) is caled the *Parseval formula* for the Mellin transform ([12], Section 3.1).

The proof of (3.4) begins by using the inversion formula to insert $\tilde{g}(z)$. The order of integration is then reversed by the absolute convergence of the double integral and the application of Fubini's theorem. The result is

$$\int_0^\infty g(xt)h(t)dt = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} x^{-z}\tilde{g}(z)\left(\int_0^\infty t^{-z}h(t)dt\right)dz.$$

The recognition of the inner definite integral as $\tilde{h}(1-z)$ completes the proof.

3.2 The Mellin Transform Method

The *Mellin transform method* used in the proof above may be used for integration problems, obtaining equivalent Mellin-Barnes integral representations which can be rewritten as sums of residues at certain poles of the integrands. A definite integral is rewritten by inserting a Mellin-Barnes integral representation of type (3.2) for one or more factors of the integrand. The hope is that after interchanging the order of integration, the inner definite integral can be recognized as an easily computable definite integral so that only a contour integral of Barnes' type over a hypergeometric integrand remains.

The identity ([7], 6.565.2) can be proved using the classic Mellin transform method:

$$\int_0^\infty x^{\nu+1}(x^2+a^2)^{-\nu-\frac{1}{2}} J_\nu(bx)dx = \frac{\sqrt{\pi}b^{\nu-1}}{2^\nu e^{ab}\Gamma(\nu+\frac{1}{2})} \quad [\operatorname{Re} a > 0,\, b > 0,\, \operatorname{Re}\nu > -1/2] \tag{3.5}$$

The Bessel function has the Mellin-Barnes representation obtained through the inversion formula (3.2) with the Mellin transform of $J_\nu(bx)$ given in ([11], 1.10.1):

$$J_\nu(bx) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} x^{-z}\frac{\Gamma\left(\frac{\nu+z}{2}\right)}{2\Gamma\left(1+\frac{\nu-z}{2}\right)}\left(\frac{b}{x}\right)^{-z} dz, \qquad [-\nu < \delta < 3/2].$$

After replacing $J_\nu(bx)$ with the above representation and reversing the order of integration, the integral becomes

$$\int_0^\infty x^{\nu+1}(x^2+a^2)^{-\nu-\frac{1}{2}} J_\nu(bx)dx$$
$$= \frac{1}{4\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{\Gamma\left(\frac{\nu+z}{2}\right)}{\Gamma\left(1+\frac{\nu-z}{2}\right)}\left(\frac{b}{2}\right)^{-z}\left(\int_0^\infty x^{1+\nu-z}(x^2+a^2)^{-\nu-\frac{1}{2}}dx\right)dz$$

Through changes of variables, the inner definite integral is recognized as

$$\int_0^\infty x^{1+\nu-z}(x^2+a^2)^{-\nu-\frac{1}{2}}dx = \frac{a^{-\nu-z+1}\Gamma\left(\frac{2+\nu-z}{2}\right)\Gamma\left(\frac{-1+\nu+z}{2}\right)}{2\Gamma\left(\frac{1}{2}+\nu\right)}.$$

By the duplication formula (1.4), the identity (3.5) can now been written as

$$\frac{a^{1-\nu}\sqrt{\pi}}{2\pi i\Gamma\left(\nu+\frac{1}{2}\right)2^\nu}\int_{\delta-i\infty}^{\delta+i\infty}(ab)^{-z}\Gamma(-1+\nu+z)dz = \frac{\sqrt{\pi}b^{\nu-1}}{2^\nu e^{ab}\Gamma(\nu+\frac{1}{2})}. \tag{3.6}$$

The identity is proved with recognition of integral as the Mellin-Barnes integral representation for the right side.

3.3 Summation to Integration

This section shows how Wegschaider's algorithm [19] can be used to determine recurrences for multiple contour integrals of Barnes' type

$$Int\,(\mu) = \int_{\mathcal{C}_{\kappa_1}}\dots\int_{\mathcal{C}_{\kappa_r}} \mathcal{F}\,(\mu,\kappa_1,\dots,\kappa_r)\,d\kappa_1\dots d\kappa_r, \tag{3.7}$$

where the integrands $\mathcal{F}\,(\mu,\kappa)$ are hypergeometric in all integer variables μ_i from $\mu = (\mu_1,\dots,\mu_p)$ and in all integration variables κ_j from $\kappa = (\kappa_1,\dots,\kappa_r) \in \mathbb{C}^r$. This idea was used in [16] to prove recurrences for a class of Ising integrals.

As in the summation problem (1.41), the fundamental theorem of hypergeometric summation stated by Wilf and Zeilberger in [21] proves the existence of non-trivial certificate recurrences of the form (1.42) for the function $\mathcal{F}(\mu, \kappa)$. Using WZ summation methods, Wegschaider's algorithm [19] delivers recurrences of the form (1.42) for the hypergeometric integrand from (3.7) with coefficients free of all integration variables $\kappa = (\kappa_1, \ldots, \kappa_r)$.

Remark: Although discrete functions are especially of interest, one can evaluate the function $\mathcal{F}(\mu, \kappa)$ also for complex values of the variables μ_i and κ_j for all $1 \leq i \leq p$ and $1 \leq j \leq r$ except at certain poles. The singularities of the numerator gamma functions need to be excluded from the evaluation domain. The function $\mathcal{F}(\mu, \kappa)$ is then continuous on its evaluation domain, and by taking limits it can be shown that the computed recurrences hold in $\mathbb{C}^{p+r}$.

After successively integrating over the Barnes paths of integration $\mathcal{C}_{\kappa_j}$ for $1 \leq j \leq r$, (1.42) leads, in some cases, to a homogeneous recurrence for the integration problem (3.7), i.e.,

$$\sum_{m \in \mathbb{S}} a_m(\mu) \, Int(\mu + m) = 0. \tag{3.8}$$

However, after integrating over the contours of integration $\mathcal{C}_{\kappa_j}$ for $1 \leq j \leq r$, it is not clear in general that the recurrence will be homogeneous as in (3.8). Consequently, one needs to analyze the behavior of the contour integrals over the left hand side of (1.42).

For this purpose, consider the following integration problems:

$$I_j := \int_{\mathcal{C}_{\kappa_j}} \Delta_{\kappa_j} \mathcal{F}(\mu, \kappa) \, d\kappa_j = \int_{\mathcal{C}'_{\kappa_j}} \mathcal{F}(\mu, \kappa) \, d\kappa_j - \int_{\mathcal{C}_{\kappa_j}} \mathcal{F}(\mu, \kappa) \, d\kappa_j, \tag{3.9}$$

where the Barnes path $\mathcal{C}_{\kappa_j}$ runs vertically over $(c_j - i\infty, c_j + i\infty)$ while $\mathcal{C}'_{\kappa_j}$ denotes the shifted path $(1 + c_j - i\infty, 1 + c_j + i\infty)$ for all $1 \leq j \leq r$.

For any $1 \leq j \leq r$, consider now the contour integral I_j^N over a rectangle with vertices at the points $c_j - iN$, $c_j + iN$, $1 + c_j + iN$ and $1 + c_j - iN$ with $N \in \mathbb{N}$; i.e.,

$$I_j^N = \int\limits_{c_j - iN}^{c_j + iN} \mathcal{F}(\mu, \kappa)\, d\kappa_j + \int\limits_{c_j + iN}^{1 + c_j + iN} \mathcal{F}(\mu, \kappa)\, d\kappa_j \tag{3.10}$$
$$- \int\limits_{1 + c_j - iN}^{1 + c_j + iN} \mathcal{F}(\mu, \kappa)\, d\kappa_j + \int\limits_{1 + c_j - iN}^{c_j - iN} \mathcal{F}(\mu, \kappa)\, d\kappa_j.$$

If in any such rectangular region of integration, the integrand has the asymptotic behavior

$$\mathcal{F}(\mu, \kappa) = \mathcal{O}\left(\frac{1}{|\kappa_j|^d} e^{-c|\kappa_j|} \right) \quad \text{as} \quad |\kappa_j| \to \infty \quad \text{with} \quad c \geq 0, \quad d \geq 0 \tag{3.11}$$

then $I_j^N \to I_j$ as $N \to \infty$. When the function $\mathcal{F}(\mu, \kappa)$ is dominated by an exponential with negative exponent, it suffices to to analyze the integrals (3.9) instead of the integrals over the right hand side of (1.2.3).

The integrals (3.10) may also be calculated by considering the residues of the function $\mathcal{F}(\mu, \kappa)$ at the poles lying inside the closed rectangular contours. If for all $1 \leq j \leq r$, the Barnes paths of integration $\mathcal{C}_{\kappa_j}$ can be chosen such that the function $\mathcal{F}(\mu, \kappa)$ has no poles inside these rectangular regions, then the integrals (3.9) will be zero.

Under these restrictions, the recurrence obtained from the certificate recurrence (1.42) will be homogeneous as in (3.8) for the multiple Barnes' type integral (3.7). Note that a different choice of the integration contours will lead to inhomogeneous recurrences for multiple Barnes' integrals which satisfy the asymptotic condition (3.11).

3.4 Analytic Continuation of Mellin Transforms

In the case of a polynomial of order $n \in \mathbb{N}$ the strip of analyticity will require $\alpha = 0$ and $\beta = -n$. Hence, the Mellin transform does not exist as defined in (3.1).

A constructive approach to this problem is presented in ([3], 4.3). The function $f(x)$ is first decomposed into two functions defined on disjoint intervals:

$$f_1(x) = \begin{cases} f(x), & x \in [0,1) \\ 0, & x \in [1,\infty) \end{cases} \quad , \quad f_2(x) = \begin{cases} 0, & x \in [0,1) \\ f(x), & x \in [1,\infty) \end{cases} .$$

Then, by analytic continuation of their Mellin transforms, the Mellin transform of the function f is obtained as a meromorphic function defined by

$$\tilde{f}(z) = \tilde{f}_1(z) + \tilde{f}_2(z)$$

on the entire z-plane.

For $f(x) = (1-x)^n$ with $Re(n) > 0$,

$$\tilde{f}(z) = \Gamma(n+1)\left[\frac{\Gamma(z)}{\Gamma(n+z+1)} + (-1)^n \frac{\Gamma(-n-z)}{\Gamma(1-z)}\right], \tag{3.12}$$

for all $z \in \mathbb{C}$ except at its simple poles. The asymptotic behavior of these generalized Mellin transforms and the Parseval formula are considered in section 4.5 of [3].

From the algorithmic point of view, the Mellin transform (3.12) is particularly interesting as it is the sum of two proper hypergeometric terms which are shadows [20] of each other through the reflection formula (1.3). Therefore, the same certificate recurrence is found for both terms and is also satisfied by their sum.

In more general situations with differing recurrences, in order to compute the recurrence for the sum from those of the terms, the command `REPlus` from the package `GeneratingFunctions` [10] can be applied since the recurrences are holonomic recurrences [22].

From (3.12) and Euler's integral representation (2.2), the Barnes' type integral form of the terminating ${}_2F_1$ can be found as follows:

$$
{}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix} ; x\right) = \frac{\Gamma(c)\Gamma(n+1)}{2\pi i \Gamma(b)} \left[\int_{\delta-i\infty}^{\delta+i\infty} \frac{\Gamma(z)}{\Gamma(n+z+1)} \frac{\Gamma(b-z)}{\Gamma(c-z)} x^{-z} dz \right.
$$
$$
\left. +(-1)^n \int_{\eta-i\infty}^{\eta+i\infty} \frac{\Gamma(-n-z)}{\Gamma(1-z)} \frac{\Gamma(b-z)}{\Gamma(c-z)} x^{-z} dz \right] \quad (3.13)
$$

where $Re(c) > Re(b) > 0$, $Re(b) > \delta > 0$ and $\eta < -Re(n)$.

With this Barnes' type representation for a terminating ${}_2F_1$, integral identities involving ${}_2F_1$ orthogonal polynomials can now proved.

3.5 Examples

This section presents proofs of integral identities in the table [7]. These integrals involve Bessel functions and orthogonal polynomials of general degree.

3.5.1 Integrals involving Bessel Functions

Identity ([7], 6.512.3)

An integral involving a product Bessel functions is ([7], **6.512.3**):

$$
\int_0^\infty J_\nu(\alpha x) J_{\nu-1}(\beta x) dx = \frac{\beta^{\nu-1}}{\alpha^\nu}, \quad [\beta < \alpha] \quad (3.14)
$$

where J_ν denotes the Bessel function of the first kind of order ν; see for instance ([2], 4.5.2). The Mellin-Barnes integral representation of J_ν is given by ([11], 10.1) for $-\nu < \delta < \frac{3}{2}$.

$$J_\nu(\alpha x) = \frac{1}{4\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{\Gamma\left(\frac{\nu+z}{2}\right)}{\Gamma\left(1+\frac{\nu-z}{2}\right)}\left(\frac{\alpha x}{2}\right)^{-z}. \tag{3.15}$$

Using the Mellin transform method presented in section 3.2, a Mellin-Barnes integral representation is obtained for the left hand side of (3.14):

$$\int_0^\infty J_\nu(\alpha x)J_{\nu-1}(\beta x)dx = \frac{1}{2\pi i\beta}\int_{\delta-i\infty}^{\delta+i\infty}\left(\frac{\beta}{\alpha}\right)^z \frac{dz}{\nu - z} \tag{3.16}$$

The integral on the right hand side of (3.16) is denoted by

$$Int[\nu] = \int F[\nu,z]dz \tag{3.17}$$

and it is observed that $F[\nu,z] = \mathcal{O}\left(|z|^{-1}e^{|z|\,log\,\frac{\beta}{\alpha}}\right)$, satisfying condition (3.11) if $\beta < \alpha$.

In Mathematica, the integrand is defined

In[2]:= $\boldsymbol{F[\nu\_, z\_] := \frac{1}{2\pi i\beta(\nu - z)}\left(\frac{\beta}{\alpha}\right)^z}$

and a certificate recurrence is computed integer parameter ν by the command

In[3]:= **FindRecurrence** $[\boldsymbol{F[\nu, z], \nu, z, 1}]$;

from the package `MultiSum` and shift this recurrence:

In[4]:= **ShiftRecurrence** $[\%[[\mathbf{1}]], \{\boldsymbol{\nu}, \mathbf{1}\}, \{\boldsymbol{z}, \mathbf{1}\}]$

Out[4]= $\beta F[\nu,z] - \alpha F[\nu+1,z] = \Delta_z\left[\alpha F[\nu+1,z]\right].$

Since δ can be chosen such that the rectangular regions described above do not contain the pole of the function $F[\nu+1,z]$, the homogeneous recurrence satisfied by the left hand side of (3.14) is found as the output of the following command:

In[5]:= $\boldsymbol{rec1}$ **= SumCertificate** $[\%]$ **/.SUM → INT**

Out[5]= $\beta\mathrm{INT}[\nu] - \alpha\mathrm{INT}[\nu+1] = 0.$

In this simple case, the solution may be read off from the recurrence relation. In general situations, solving might be done using the package `Hyper` [13]. In this case since the right side of the identity (3.14) is given in the table, one can simply check that it also satisfies the recurrence above:

In[6]:= $\mathbf{RHS[\nu\_] := \dfrac{\beta^{\nu-1}}{\alpha^{\nu}}}$

In[7]:= $\mathbf{CheckRecurrence\,[\mathit{rec}1, RHS[\nu]]}$

Out[7]= True.

The initial value that must be checked is a known property of the Bessel function. A similar approach works for the other two cases given in the table for this identity.

Identity ([7], 6.671.9)

Now consider an integral involving a Bessel function and a trigonometric function:

$$\int_0^\infty J_{2n+1}(ax)\sin(bx)\,dx = \frac{(-1)^n}{\sqrt{a^2-b^2}}T_{2n+1}\left(\frac{b}{a}\right), \quad [b<a] \tag{3.18}$$

where $T_n(x)$ represents the n^{th} Chebyshev polynomial of the first kind on $[-1,1]$ ([18], 6.21). See (1.25) for its hypergeometric representation or Appendix A for its Rodrigues' formula.

Here the Mellin transform of $\sin(bx)$ ([11], 5.1) is used to rewrite

$$\sin(bx) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} x^{-z}b^{-z}\Gamma(z)\sin\left(\frac{\pi z}{2}\right)\,dz, \quad [-1<\delta<1]$$

and the reflection formula (1.3) is applied to rewrite (3.18) as follows:

$$\int_{\delta-i\infty}^{\delta+i\infty} \frac{\Gamma(z)}{2ib^z\Gamma\left(\frac{z}{2}\right)\Gamma\left(1-\frac{z}{2}\right)}\int_0^\infty x^{-z}J_{2n+1}(ax)\,dx\,dz = \frac{(-1)^n}{\sqrt{a^2-b^2}}T_{2n+1}\left(\frac{b}{a}\right), \quad [b<a]. \tag{3.19}$$

The inner definite integral may be simplified by ([7], 6.561.14) so that the identity to be proved, after the change of variables $z = 2s$, is

$$\int_{\delta-i\infty}^{\delta+i\infty} \frac{a^{2s-1}\Gamma(2s)\Gamma(1+n-s)}{i2^{2s}b^{2s}\Gamma(s)\Gamma(1-s)\Gamma(1+n+s)}\,ds = \frac{(-1)^n}{\sqrt{a^2-b^2}}T_{2n+1}\left(\frac{b}{a}\right), \quad [b<a] \tag{3.20}$$

The integral on the right hand side of (3.20) is denoted by

$$Int[n] = \int F[n,s]\,ds \tag{3.21}$$

and it can be found via application of (1.6) that $F[n,s] = \mathcal{O}(|s|^{2n+2\operatorname{Re}s+1/2}e^{-2\operatorname{Im}s\arg(s)})$. In both cases ($\operatorname{Im}s > 0$ or $\operatorname{Im}s < 0$), the exponential decay factor will dominate and condition (3.11) is met. The certificate recurrence is computed as follows:

In[8]:= $\boldsymbol{F[n\_, s\_] := \dfrac{\Gamma[2s]a^{2s-1}\Gamma[1+n-s]}{i2^{2s}b^{2s}\Gamma[s]\Gamma[1-s]\Gamma[1+n+s]}}$

In[9]:= **FindRecurrence** $\boldsymbol{[F[n,s], n, s, 1]\,[[1]];}$

Out[9]= $a^2(1+n)F[-1+n,-1+s] - 2(a^2-2b^2)(1+n)F[n,-1+s] + a^2(1+n)F[1+n,-1+s] =$
$\Delta_s[-a^2(1+n-s)F[-1+n,-1+s] + (-a^2-4b^2+a^2n-4b^2n+a^2s)F[n,-1+s]]$

The homogeneous recurrence satisfied by the left hand side of (3.18) is the output of the following command:

In[10]:= **SumCertificate[%]/.SUM → INT**

Out[10]= $a^2\mathrm{INT}[-1+n] - 2(a^2-2b^2)\mathrm{INT}[n] + a^2\mathrm{INT}[1+n] = 0$

It can be verified that the right side of the identity does satisfy the recurrence since the Chebyshev polynomials satisfy the recurrence relation $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$.

The initial conditions to be checked are $n = 0$ and $n = 1$. For $n = 0$, the identity is $\int_0^\infty J_1(ax)\sin(bx)\,dx = \dfrac{b}{a\sqrt{a^2-b^2}}$, which can be proved using the fact that $J_1(x) = -J_0'(x)$, integration by parts, and the identity ([7], 6.671.8). For $n = 1$, $\int_0^\infty J_3(ax)\sin(bx)\,dx = -\dfrac{b(4b^2-3a^2)}{a^3\sqrt{a^2-b^2}}$. With these initial conditions checked, the proof is now complete.

Note that the command `CheckRecurrence` cannot be used with the function `ChebyshevT` in Mathematica since that Mathematica representation cannot be checked to be proper hypergeometric.

3.5.2 Integrals involving Orthogonal Polynomials

Identity ([7], 7.231.2)

A simple integral involving Legendre polynomials is ([7], **7.231.2**):

$$\int_0^1 x^\lambda P_{2m+1}(x)\,dx = \frac{(-1)^m \Gamma\left(m+\frac{1}{2}-\frac{1}{2}\lambda\right)\Gamma\left(1+\frac{1}{2}\lambda\right)}{2\Gamma\left(\frac{1}{2}-\frac{1}{2}\lambda\right)\Gamma\left(m+2+\frac{1}{2}\lambda\right)}, \qquad [\operatorname{Re}\lambda > -2] \tag{3.22}$$

where P_n denotes the Legendre polynomial of order n; see (1.24) or Appendix A.

The Legendre polynomial can be written using an alternate hypergeometric representation ([18], Section 8.2):

$$P_\nu(z) = {}_2F_1\left(\begin{matrix} -\frac{\nu}{2}, \frac{\nu+1}{2} \\ 1 \end{matrix}; 1-z^2\right). \tag{3.23}$$

Using the Mellin-Barnes integral representation of ${}_2F_1$ in ([14], Theorem 35), a Mellin-Barnes integral representation for the left hand side of (3.22) is

$$\int_0^1 x^\lambda P_{2m+1}(x)\,dx$$
$$= \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{(-1)^s\Gamma\left(-m-\frac{1}{2}+s\right)\Gamma\left(m+1+s\right)\Gamma\left(-s\right)\Gamma\left(\frac{1+\lambda}{2}\right)}{4\pi i\Gamma\left(-m-\frac{1}{2}\right)\Gamma\left(m+1\right)\Gamma\left(\frac{3}{2}+\frac{\lambda}{2}+s\right)}\,ds \quad (3.24)$$

where the path of integration is curved to place the poles of $\Gamma(-m-1/2+s)$ and $\Gamma(m+1+s)$ to the left of the path and the poles of $\Gamma(-s)$ to the right of the path.

The integral on the right hand side of (3.24) is denoted by

$$Int[m] = \int F[m,s]ds. \quad (3.25)$$

After the integrand is defined, Wegschaider's algorithm [19] is used to compute a certificate recurrence in the integer parameter m:

In[11]:= $\boldsymbol{F[m\_, s\_] := \frac{\Gamma[-m-1/2+s]\Gamma[m+1+s]\Gamma[-s](-1)^s\Gamma[(1+\lambda)/2]}{4\pi i\Gamma[-m-1/2]\Gamma[m+1]\Gamma[3/2+\lambda/2+s]}}$

In[12]:= **FindRecurrence** $[\boldsymbol{F[\nu, z], \nu, z, 1}]$ **;**

from the package `MultiSum` and shift this recurrence:

In[13]:= **ShiftRecurrence** $[\%[[\mathbf{1}]], \{\boldsymbol{\nu}, \mathbf{1}\}, \{\boldsymbol{z}, \mathbf{1}\}]$

Out[13]= $-(1+m)(3+2m)(1-\lambda+2m)(9+4m)F[m,s]-(7+4m)(-6+11\lambda-7m+14\lambda m-2m^2+4\lambda m^2)F[1+m,s]+(2+m)(5+2m)(6+\lambda+2m)(5+4m)F[2+m,s]=\Delta_s[(1+m)(3+2m)(1-\lambda+2m)(9+4m)F[m,s]+(7+4m)(39+11\lambda+49m+14\lambda m+14m^2+4\lambda m^2+45s+56ms+16m^2s)F[1+m,s]-(2+m)(5+2m)(6+\lambda+2m)(5+4m)F[2+m,s]]$

The homogeneous recurrence satisfied by the left hand side of (3.22) is the output of the following command:

In[14]:= $\boldsymbol{rec1}$ **= SumCertificate** [%] **/.SUM → INT**

Out[14]= $-(1+m)(3+2m)(1-\lambda+2m)(9+4m)\mathrm{INT}[m]-(7+4m)(-6+11\lambda-7m+14\lambda m-2m^2+4\lambda m^2)\mathrm{INT}[1+m]+(2+m)(5+2m)(6+\lambda+2m)(5+4m)\mathrm{INT}[2+m]=0$

Since the right side of the identity (3.22) is given, one can simply check that it also satisfies the recurrence above:

In[15]:= $\mathbf{RHS}[\boldsymbol{m}\_] := \dfrac{(-1)^{m}\Gamma\left(m+\frac{1}{2}-\frac{1}{2}\lambda\right)\Gamma\left(1+\frac{1}{2}\lambda\right)}{2\Gamma\left(\frac{1}{2}-\frac{1}{2}\lambda\right)\Gamma\left(m+2+\frac{1}{2}\lambda\right)}$

In[16]:= **CheckRecurrence** [$\boldsymbol{rec1}$, **RHS**[$\boldsymbol{\nu}$]]

Out[16]= True.

With the initial values of $m = 0$ and $m = 1$, the initial conditions are easily verified with the lookup of $P_1(x) = x$ and $P_3(x) = \frac{1}{2}(5x^3 - 3x)$.

Identity ([7], 7.226.2)

$$\int_{-1}^{1} x(1-x^2)^{-1/2} P_{2m+1}(x)\, dx = \frac{\Gamma\left(\frac{1}{2}+m\right)\Gamma\left(\frac{3}{2}+m\right)}{m!(m+1)!} \tag{3.26}$$

After the change of variables $y := x + 1$, the identity (3.26) becomes

$$\int_{0}^{2} (y-1)(2-y)^{-1/2} y^{-1/2} P_{2m+1}(y-1)\, dy = \frac{\Gamma\left(\frac{1}{2}+m\right)\Gamma\left(\frac{3}{2}+m\right)}{m!(m+1)!} \tag{3.27}$$

Using the Mellin transform of P_{2m+1} found in ([11], 8.42) and reversing the order of integration, the equivalent contour integral identity is

$$\int_{\delta-i\infty}^{\delta+i\infty} \frac{-s\Gamma(s)\Gamma(s)\Gamma\left(\frac{1}{2}-s\right)}{2\sqrt{\pi} i\Gamma(s-2m-1)\Gamma(2m+2+s)\Gamma(2-s)}\, ds = \frac{\Gamma\left(\frac{1}{2}+m\right)\Gamma\left(\frac{3}{2}+m\right)}{m!(m+1)!}. \tag{3.28}$$

The integrand is input and a certificate recurrence is computed for the integrand:

In[17]:= $\boldsymbol{F}[\boldsymbol{m}\_, \boldsymbol{s}\_] := \dfrac{-s\Gamma[s]\Gamma[s]\Gamma[1/2-s]}{2\sqrt{\pi} i\Gamma[s-2m-1]\Gamma[2m+s+2]\Gamma[2-s]}$

In[18]:= $\boldsymbol{rec0}$ = **FindRecurrence**[$\boldsymbol{F}[\boldsymbol{m}, \boldsymbol{s}], \boldsymbol{m}, \boldsymbol{s}, \mathbf{1}$]

In[19]:= $\boldsymbol{rec1}$ = **ShiftRecurrence**[$\boldsymbol{rec0}$[[1]], {$\boldsymbol{s}$, 1}]

Out[19]= $(-1+2m)(1+2m)F[-1+m,s]-4m(1+m)F[m,s] = \Delta_s[-(1+4m)(2m-s)F[-1+m,s]+(1+4m)(1+2m+s)F[m,s]]$

Integrating over the integrand recurrence produces a homogeneous recurrence for the integral:

In[20]:= $\boldsymbol{rec2} = \textbf{SumCertificate}[\boldsymbol{rec1}]/.\textbf{SUM} \to \textbf{INT}$

Out[20]= $(-1+2m)(1+2m)\mathrm{INT}[-1+m]-4m(1+m)\mathrm{INT}[m]=0$

The right hand side satisfies the same recurrence:

In[21]:= $\boldsymbol{RHS}[\boldsymbol{m\_}] := \dfrac{\boldsymbol{\Gamma[1/2+m]\Gamma[3/2+m]}}{\boldsymbol{m!(m+1)!}}$

In[22]:= $\textbf{CheckRecurrence}[\boldsymbol{rec2}, \boldsymbol{RHS}[\boldsymbol{m}]]$

Out[22]= $True$

The initial condition corresponding to $m = 0$ must be checked. With $P_1(x) = x$, the original left side of (3.26) can be transformed into a beta integral.

Identity ([7], 7.126.1)

The identity 7.126.1 will be verified for nonnegative integer values of ν:

$$\int_0^1 P_\nu(x)x^\sigma dx = \frac{\sqrt{\pi}2^{-\sigma-1}\Gamma(1+\sigma)}{\Gamma\left(1+\frac{1}{2}\sigma-\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}\sigma+\frac{1}{2}\nu+\frac{3}{2}\right)}, \quad [\operatorname{Re}\sigma > -1] \tag{3.29}$$

With the representation (1.24) for the Legendre polynomial and the representation (3.13) for a terminating ${}_2F_1$, the identity (3.29) becomes

$$\begin{aligned}\frac{1}{2\pi i}\Bigg[\int_{\delta-i\infty}^{\delta+i\infty} \frac{2^s\Gamma(s)\Gamma(n+1-s)\Gamma(\sigma+1)}{\Gamma n+s+1\Gamma(\sigma-s+2)}\,ds \\ +(-1)^n\int_{\eta-i\infty}^{\eta+i\infty}\frac{2^s\Gamma(-n-s)\Gamma(n+1-s)\Gamma(\sigma+1)}{\Gamma 1-s\Gamma(\sigma-s+2)}\,ds\Bigg] \\ = \frac{\sqrt{\pi}2^{-\sigma-1}\Gamma(1+\sigma)}{\Gamma\left(1+\frac{1}{2}\sigma-\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}\sigma+\frac{1}{2}\nu+\frac{3}{2}\right)}\end{aligned} \tag{3.30}$$

where the reversing the order of integration produced inner definite integrals which could both be recognized as beta functions $B(\sigma+1, 1-s)$ for $\operatorname{Re}\sigma > -1$.

Both integrands are expected to satisfy the same recurrence because they are shadows of each other. This is verified by finding the recurrence for the first integral:

In[23]:= $\boldsymbol{F1[n\_, s\_] := \frac{2^s\Gamma[s]\Gamma[n+1-s]\Gamma[\sigma+1]}{2\pi i\Gamma[n+s+1]\Gamma[\sigma-s+2]}}$

In[24]:= $\boldsymbol{rec1a = \mathrm{FindRecurrence}[F1[n,s], n, s, 1]}$

In[25]:= $\boldsymbol{rec1b = \mathrm{ShiftRecurrence}[rec1a[[1]], \{n,-1\}, \{s,1\}]}$

Out[25]= $n(-2+n-\sigma)F[-2+n,s]+n(1+n+sig)F[n,s] = \Delta_s[-(-1+n-s)(-2+n-\sigma)F[-2+n,s]+(2-2n-n^2-2s+ns+\sigma-n\sigma-s\sigma)F[-1+n,s]]$

In[26]:= $\boldsymbol{rec1c = \mathrm{SumCertificate}[rec1b]/.\mathrm{SUM} \to \mathrm{INT}}$

Out[26]= $(-2+n-\sigma)\mathrm{INT}[-2+n]+(1+n+\sigma)\mathrm{INT}[n] = 0$

For the second integral, the recurrence is found similarly:

In[27]:= $\boldsymbol{F2[n\_, s\_] := \frac{2^s\Gamma[-n-s]\Gamma[n+1-s]\Gamma[\sigma+1]}{2\pi i\Gamma[1-s]\Gamma[\sigma-s+2]}}$

In[28]:= $\boldsymbol{rec2a = \mathrm{FindRecurrence}[F2[n,s], n, s, 1]}$

In[29]:= $\boldsymbol{rec2b = \mathrm{ShiftRecurrence}[rec2a[[1]], \{n,-1\}, \{s,1\}]}$

Out[29]= $n(-2+n-\sigma)F[-2+n,s]+n(1+n+\sigma)F[n,s] = \Delta_s[-(-1+n-s)(-2+n-\sigma)F[-2+n,s]+(-2+2n+n^2+2s-ns-\sigma+n\sigma+s\sigma)F[-1+n,s]]$

In[30]:= $\boldsymbol{rec2c = \mathrm{SumCertificate}[rec2b]/.\mathrm{SUM} \to \mathrm{INT}}$

Out[30]= $(-2+n-\sigma)\mathrm{INT}[-2+n]+(1+n+\sigma)\mathrm{INT}[n] = 0$

Each integral does indeed satisfy the same recurrence.

Since the right side of (3.29) is not in proper hypergeometric form, the command `CheckRecurrence` cannot be used for verification. But this recurrence is easily checked for the right side by hand. Since the recurrence skips by two, two initial conditions must be checked. For the even case, the initial value is $n = 0$ and $P_0(x) = 1$, and the duplication formula is used to simplify the right side. For the odd case, the initial value $n = 1$ and $P_1(x) = x$ and the simplifications are easily performed with the duplication formula.

Identity ([7], 7.222.5)

Using the same techniques, the identity 7.222.5 can be proved with a multiple Barnes-type integral:

$$\int_0^1 x^2 P_{n+1}(x) P_{n-1}(x) dx = \frac{n(n+1)}{(2n-1)(2n+1)(2n+3)}. \tag{3.31}$$

By the representation (3.13) for a terminating ${}_2F_1$, the Legendre polynomials are represented as

$$\begin{aligned} P_m &= {}_2F_1\left(\begin{matrix} -m, m+1 \\ 1 \end{matrix}; \frac{1-x}{2}\right) \\ &= \frac{1}{2\pi i}\left[\int_{\delta-i\infty}^{\delta+i\infty} \tilde{f}_{1,m}(s)(1-x)^{-s}dx + \int_{\eta-i\infty}^{\eta+i\infty} \tilde{f}_{2,m}(s)(1-x)^{-s}dx\right], \end{aligned}$$

where the following notations have been introduced for simplicity:

$$\begin{aligned} \tilde{f}_{1,m}(s) &= \frac{\Gamma(s)\Gamma(m+1-s)2^s}{\Gamma(m+s+1)\Gamma(1-s)} \\ \tilde{f}_{2,m}(s) &= \frac{\Gamma(-m-s)\Gamma(m+1-s)2^s}{\Gamma(1-s)\Gamma(1-s)} \end{aligned}$$

After expanding and then reversing the order of integration, the inner definite integrals will recognized as beta integrals of the form

$$g(v) = \int_0^1 x^2(1-x)^{-v}dx = B(3, 1-v), \quad [\operatorname{Re} v < 1],$$

and the identity (3.31) can be written as

$$\frac{1}{2\pi i} \sum_{i,j\in\{1,2\}} \int_{C_i} \int_{C_j} \tilde{f}_{i,n+1}(s)\tilde{f}_{j,n-1}(t) g(s+t)\, ds\, dt = \frac{n(n+1)}{(2n-1)(2n+1)(2n+3)}, \tag{3.32}$$

where the contours of integration are of the form $C_1 = (\delta - i\infty, \delta + i\infty)$ and $C_2 = (\eta - i\infty, \eta + i\infty)$.

Expansion will produce a sum of four integrals, each of which satisfies the same recurrence since the integrands are shadows of one another. A generic one of these four integrals may be denoted by

$$INT[n] = \int\int F[n,s,t]\,ds\,dt.$$

Wegschaider's algorithm delivers a certificate recurrence in the integer parameter n for the integrand (using the first integrand, $F1$, here):

In[31]:= $\boldsymbol{f1[m\_, s\_] := \dfrac{\Gamma[s]\Gamma[m+1-s]2^s}{\Gamma[m+s+1]\Gamma[1-s]}}$

In[32]:= $\boldsymbol{f2[m\_, s\_] := (-1)^m \dfrac{\Gamma[-m-s]\Gamma[m+1-s]2^s}{\Gamma[1-s]\Gamma[1-s]}}$

In[33]:= $\boldsymbol{g[s\_] := \dfrac{\Gamma[3]\Gamma[1-s]}{\Gamma[4-s]}}$

In[34]:= $\boldsymbol{F1[n\_, s\_, t\_] := f1[n+1, s] f1[n-1, t] g[s+t]}$

In[35]:= $\boldsymbol{rec1a = \text{FindRecurrence}[F1[n,s,t], n, \{s,t\}, 1];}$

In[36]:= $\boldsymbol{rec1b = ShiftRecurrence[rec1a[[1]], \{s,1\}, \{t,1\}]}$

Out[36]= $(-2+n)^2n^2(1+n)(-5+2n)(-7+3n)F[-2+n,s,t] + 3(-2+n)(1+n)(-1+2n)(-7+3n)(-1-n+n^2)F[-1+n,s,t] - (-2+n)(-1+n)^2(1+n)^2(3+2n)(-7+3n)F[n,s,t] = \Delta_s[-(-2+n)^3n(1+n)s(-3+n+t)F[-2+n,s,t]-(-2+n)^2n(1+n)(10n-17n^2+5n^3-4s+9ns-3n^2s-n^2t+2st-nst)F[-2+n,s,1+t]-(1+n)(7-13n+2n^2+n^3+4n^4-2n^5+s+16ns-20n^2s+6n^3s+5t-13nt-26n^2t+54n^3t-31n^4t+6n^5t-9st-10nst+19n^2st-6n^3st)F[-1+n,s,t]+(-2+n)^2n(-3+2n)(-3n-n^2+2n^3-s+2ns+st+2nst)F[-1+n,s,1+t]-(-2+n)^2(1+n)(-n-n^2+s-ns-3n^2s+2n^3s-3nt-n^2t+6n^3t+st)F[-1+n,1+s,t]-(-1+n)^2(1+n)(-11-5n+6n^2+5s-6ns+n^3s-11t-5nt+6n^2t-15st+10nst-n^2st)F[n,s,t]+(-2+n)(-7n+3n^2+8n^3-5n^4-2n^5+n^6-7ns+6n^2s+n^4s-n^5s+3t-nt-12n^2t+3n^3t+8n^4t-3n^5t+3st-8nst-6n^2st+15n^3st-5n^4st)F[n,s,1+t]]+\Delta_t[-(-2+n)^2n(1+n)(35n-29n^2+6n^3-6s+5ns-n^2s+2st-nst)F[-2+n,s,t]+(1+n)(-35+68n+80n^2-191n^3+109n^4-20n^5+s+16ns-20n^2s+6n^3s+5t-13nt-26n^2t+54n^3t-31n^4t+6n^5t-9st-10nst+19n^2st-6n^3st)F[-1+n,s,t]-(-2+n)(1+n)(-5+2n)(2-3n+n^2-2s+13ns-17n^2s+6n^3s+t-nt-3n^2t+2n^3t-st)F[-1+n,s,1+t]+(-1+n)^2(1+n)(31+26n-22n^2-11n^3+6n^4+5s-6ns+n^3s-11t-5nt+6n^2t-15st+10nst-n^2st)F[n,s,t]]$

Integrating over this certificate recurrence produces a recurrence for the sum of integrals from (3.32):

In[37]:= $\boldsymbol{rec1c = \text{SumCertificate}[rec1b] /. \text{SUM} \to \text{INT}}$

Out[37]= $(-2+n)n^2(-5+2n)\mathrm{INT}[-2+n]+3(-1+2n)(-1-n+n^2)\mathrm{INT}[-1+n]-(-1+n)^2(1+n)(3+2n)\mathrm{INT}[n]=0$

In[38]:= $\boldsymbol{RHS[n\_] := n(n+1)/(2n-1)/(2n+1)/(2n+3)}$

In[39]:= **CheckRecurrence**$[\boldsymbol{rec1c}, \boldsymbol{RHS[n]}]$

Out[39]= $True$

Lastly, two initial conditions must be checked. For $n = 1, 2$, the necessary Legendre polynomials are $P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = \frac{1}{2}(3x^2 - 1)$, and the calculations are trivial.

Identity ([7],7.345.3)

Now consider an integral involving two Chebyshev polynomials of the second kind, each of arbitrary order:

$$\int_{-1}^{1}(1-x)^{1/2}(1+x)^{m+n+3/2}U_m(x)U_n(x)\,dx = \frac{\pi(2m+2n+2)!}{2^{m+n+2}(2m+1)!(2n+1)!} \tag{3.33}$$

The hypergeometric representation of the Chebyshev polynomials of the second kind is given by

$$U_n(x) = (n+1)\,{}_2F_1\left(\begin{matrix}-n, n+2\\ \frac{3}{2}\end{matrix}\,;\frac{1-x}{2}\right). \tag{3.34}$$

With the change of variables $y = \frac{1-x}{2}$, the representation (3.13) for a terminating ${}_2F_1$, and the reversal the order of integration, the identity (3.33) can be written as

$$h_{m,n}\sum_{i,j\in\{1,2\}}\int_{C_i}\int_{C_j} f_{i,m}(z)f_{j,n}(s)g(z,s)\,ds\,dz = \frac{\pi(2m+2n+2)!}{2^{m+n+2}(2m+1)!(2n+1)!}, \tag{3.35}$$

where, for simplicity of notation, the following notations are introduced:

$$
\begin{aligned}
h_{m,n} &= -\frac{2^{n+m-1}}{\pi} \\
f_{1,m}(z) &= \frac{\Gamma(z)\Gamma(m+2-z)}{\Gamma(m+z+1)\Gamma\left(\frac{3}{2}-z\right)} \\
f_{2,m}(z) &= (-1)^m \frac{\Gamma(-m-z)\Gamma(m+2-z)}{\Gamma(1-z)\Gamma\left(\frac{3}{2}-z\right)} \\
g(z,s) &= \int_0^1 y^{1/2-z-s}(1-y)^{m+n+3/2}\,dy = B\left(\frac{3}{2}-z-s, m+n+\frac{5}{2}\right)
\end{aligned}
$$

and the contours of integration are of the form $C_1 = (\delta - i\infty, \delta + i\infty)$ and $C_2 = (\eta - i\infty, \eta + i\infty)$.

Since all four integrands on the left hand side of (3.35) are shadows of one another, the integrands and integrals themselves will all satisfy the same certificate recurrence:

In[40]:= $\boldsymbol{rec1a = \mathbf{FindRecurrence}[F1[m,n,z,s],m,n,z,s,1]}$

In[41]:= $\boldsymbol{rec1b = \mathbf{ShiftRecurrence}[rec1a[[1]],\{n,-1\},\{m,-1\}]}$

In[42]:= $\boldsymbol{rec1c = \mathbf{SumCertificate}[rec1b]/.SUM \to INT}$

Out[42]= $-(-1+2m+2n)(1+2m+2n)(45+3m-74m^2-16m^3+16m^4+69n-33mn-186m^2n-56m^3n+64m^4n-330n^2-234mn^2+152m^2n^2+32m^3n^2+312n^3-72mn^3-192m^2n^3+96n^4-384mn^4-192n^5)INT[-2+m,-1+n]+6(-1+n)n(1+2m+2n)(1+m+4m^2+10n+2mn+16n^2+8mn^2+16n^3)INT[-2+m,n]+2(-1+2m+2n)(1+2m+2n)(33-13m-70m^2-8m^3+16m^4+85n-49mn-186m^2n-24m^3n+64m^4n-114n^2-82mn^2+88m^2n^2+32m^3n^2+152n^3-232mn^3-192m^2n^3+96n^4-384mn^4-192n^5)INT[-1+m,-2+n]+4(1+2m+2n)(11-12m-61m^2-62m^3-8m^4+16m^5-21n-20mn-17m^2n-146m^3n-40m^4n+64m^5n-142n^2+70mn^2+344m^2n^2+112m^3n^2-32m^4n^2+216n^3+56mn^3-624m^2n^3-224m^3n^3+56n^4+224mn^4-192m^2n^4+48n^5+192mn^5+192n^6)INT[-1+m,-1+n]+96(-1+n)n(m+n)(1+m-m^2+5n+6mn+8n^2+4mn^2+4n^3)INT[-1+m,n]-(-1+2m+2n)(1+2m+2n)(-3+11m-34m^2-32m^3+16m^4-27n+63mn-58m^2n-120m^3n+64m^4n+6n^2-26mn^2+24m^2n^2+32m^3n^2+120n^3-264mn^3-192m^2n^3+96n^4-384mn^4-192n^5)INT[m,-3+n]-2(1+2m+2n)(-6+36m-98m^2-188m^3-16m^4+32m^5-45n+125mn-142m^2n-484m^3n-80m^4n+128m^5n+93n^2-327mn^2-116m^2n^2+480m^3n^2-64m^4n^2+222n^3-962mn^3-1440m^2n^3-448m^3n^3-192n^4-936mn^4-384m^2n^4-816n^5+384mn^5+384n^6)INT[m,-2+n]-12(-1+n)(m+n)(1+m-4m^2-4m^3+12n+4mn-32m^2n-16m^3n+36n^2+12mn^2-32m^2n^2+48n^3+16mn^3+32n^4)INT[m,-1+n] =$

0

In[43]:= $\boldsymbol{RHS[m\_, n\_] := \dfrac{\pi\Gamma[2m+2n+3]}{2^{m+n+2}(2m+1)!(2n+1)!}}$

In[44]:= $\boldsymbol{CheckRecurrence[rec1c, RHS[m, n]]}$

Out[44]= $True$

After solving for $INT[m, n-1]$ in terms of $INT[m, n-2]$, $INT[m, n-3]$, $INT[m-1, n]$, $INT[m-1, n-1]$, $INT[m-1, n-2]$, $INT[m-2, n]$, and $INT[m-2, n-1]$, it can be seen that it is sufficient to check the identity (3.33) for $m = 0$, $m = 1$, $n = 0$, and $n = 1$. By the symmetry of m and n in the identity, only $m = 0$ and $m = 1$ must be checked.

For the initial condition $m = 0$, $U_0(x) = 1$ and this case is proved

$$\int_{-1}^{1} (1-x)^{1/2}(1+x)^{n+3/2}U_n(x)\,dx = \frac{\pi(n+1)}{2^{n+1}} \tag{3.36}$$

by using the corrected solution (2.17) for ([7], 7.347.2), the formula (1.29) for a ${}_2F_1$ at $x = 1$, and the duplication formula.

For the initial condition $m = 1$, $U_1(x) = 2x$, and the hypergeometric representation of $U_n(x)$ along with the change of variables $y = \frac{1-x}{2}$ transforms the identity into

$$(n+1)2^{n+5}\int_0^1 (1-2y)y^{1/2}(1-y)^{n+5/2}\,{}_2F_1\left(\begin{matrix} -n, n+2 \\ \frac{3}{2} \end{matrix}; y\right)dy = \frac{\pi(2n+4)!}{2^{n+3}3!(2n+1)!}.$$

Splitting the $(1-2y)$ factor produces two integrals which can be evaluated by (2.10):

$$
\begin{aligned}
I_1 =&(n+1)2^{n+5}\int_0^1 y^{1/2}(1-y)^{n+5/2}\,{}_2F_1\left(\begin{matrix}-n, n+2 \\ \frac{3}{2}\end{matrix};y\right)dy \\
=&(n+1)2^{n+5}\,{}_3F_2\left(\begin{matrix}-n, n+2, \frac{3}{2} \\ \frac{3}{2}, n+5\end{matrix};1\right)\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(n+\frac{7}{2}\right)}{\Gamma(n+5)} \\
=&\frac{\pi(n+1)(2n+5)}{2^{n+2}} \\
I_2 =&-(n+1)2^{n+6}\int_0^1 y^{3/2}(1-y)^{n+5/2}\,{}_2F_1\left(\begin{matrix}-n, n+2 \\ \frac{3}{2}\end{matrix};y\right)dy \\
=&-(n+1)2^{n+6}\,{}_3F_2\left(\begin{matrix}-n, n+2, \frac{5}{2} \\ \frac{3}{2}, n+6\end{matrix};1\right)\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(n+\frac{7}{2}\right)}{\Gamma(n+6)} \\
=&\frac{\pi(n+1)(2n^2+n-9)}{3\cdot 2^{2+n}}
\end{aligned}
$$

Summing and simplifying $I_1 + I_2$ produces the desired result.

3.6 Conclusions

This algorithmic approach extends the classical Mellin transform method by applying Wegschaider's algorithm [19] to multiple nested Mellin-Barnes integrals. Weg-schaider's algorithm computes recurrences for multi-sums as well as for nested Barnes type integrals over hypergeometric terms. Once the Mellin transform is used to put definite integrals into a suitable Barnes type integral form, entries from [7] can be proved by algorithmically finding a recurrence satisfied by both sides of the identity. This application of Wegschaider's algorithm extends domain of applicability of the Mellin transform method.

When this method was first attempted for integrals involving ${}_2F_1$ and ${}_1F_1$

orthogonal polynomials, the Mellin-Barnes integral representation for ${}_pF_q$ was used without regard to the existence of the strip of analyticity. For the ${}_2F_1$ polynomials, the recurrences produced were the same as those produced using the representation (3.13). Without a similar Mellin-Barnes representation for ${}_1F_1$ polynomials, integral identities involving Hermite or Laguerre polynomials have not been proved using this method.

Chapter 4

The Method of Brackets

The *method of brackets* [5, 6], a heuristic process appearing in the evaluation of Feynman diagrams, can be used to evaluate symbolically a large class of single or multiple integrals on $[0, \infty)$. With only a short list of rules, the method can be implemented and tested relatively easily.

The method of brackets handles a much larger variety of integrals than the algorithmic Mellin transform method and is able to do so with less necessary user analysis or knowledge.

4.1 The Method

In the method of brackets, the integrand is replaced by a series representation via a sequence of ad-hoc rules. *Brackets* appear in these rules. The preliminary setup converts the integral into a *bracket series*. Rules for the evaluation of the bracket series involve solving a system of linear equations. These rules also determine conditions required on parameters for the convergence of the integral.

4.1.1 Definitions

1. A *bracket* $\langle a \rangle$ is a symbol associated to the divergent integral $\displaystyle\int_0^\infty x^{a-1}\, dx$.

2. The symbol $\phi_n := \dfrac{(-1)^n}{\Gamma(n+1)}$ is called the *indicator of* n.

3. The symbol $\phi_{1,2,\ldots,k}$ is shorthand notation for $\phi_{n_1}\phi_{n_2}\cdots\phi_{n_k}$.

4.1.2 Rules for production of bracket series

The rules P_1, P_2, and P_3 control production of a bracket series for a given integral. Empirical evidence has shown that Rule P_3 should be explicitly added to the original set of rules in [5, 6].

P_1. Integrals involving power series in the integrand are converted into *bracket series*. For $f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n+\beta-1}$, the bracket series associated with the integral $\int_0^\infty f(x)\, dx$ is $\sum_n a_n \langle \alpha n + \beta \rangle$.

P_2. For complex α, the expression $(a_1 + a_2 + \cdots + a_r)^\alpha$ is assigned the bracket series $\sum_{n_1,\cdots,n_r} \phi_1\phi_2\cdots\phi_r a_1^{n_1}\cdots a_r^{m_r} \frac{\langle -\alpha + n_1 + \cdots + n_r\rangle}{\Gamma(-\alpha)}$.

P_3. Each bracket series associated with an integral has an *Index* given by

$$\text{Index} = \text{number of sums} - \text{number of brackets}$$

The index is attached to a specific representation of the integrand and order of application of rules. Experimentation has shown that among all bracket series associated with an integral, the one with the *minimal index* should be chosen.

4.1.3 Rules for evaluation of bracket series

Given a bracket series associated with an integral, the Rules E_1, E_2, and E_3 determine the value assigned to the integral. Based on empirical evidence, Rule E_3 has been modified from its original form in [5, 6] so that imaginary series and zeros are discarded.

E_1. The one-dimensional bracket series $\sum_n \phi_n f(n) \langle an+b \rangle$ is assigned the value $\frac{1}{|a|} f(n^*)\Gamma(-n^*)$, where n^* solves $an+b=0$.

E_2. The r-dimensional bracket series

$$\sum_{n_1} \cdots \sum_{n_r} \phi_{n_1} \cdots \phi_{n_r} f(n_1, \cdots, n_r)$$
$$\times \langle a_{11}n_1 + \cdots + a_{1r}n_r + c_1 \rangle \cdots \langle a_{r1}n_1 + \cdots + a_{rr}n_r + c_r \rangle$$

is assigned the value $\frac{1}{|\det(A)|} f(n_1^*, \cdots, n_r^*)\Gamma(n_1^*) \cdots \Gamma(n_r^*)$ where A is the matrix of coefficients (a_{ij}) and $\{n_i^*\}$ is the solution of the linear system obtained by the vanishing of the brackets.

E_3. The value of a multi-dimensional bracket series of positive index is obtained by computing all the contributions of maximal rank by Rule E_2. These contributions to the integral appear as series in the free parameters. Series converging in a common region are added and divergent series are discarded. Any series producing a non-real contribution is also discarded. Contributions of zeros are also discarded.

4.2 Theoretical Background

4.2.1 Ramanujan's Master Theorem

The method of brackets is an extension of *Ramanujan's Master Theorem* [1]:

Theorem 3. *(Ramanujan's Master Theorem as stated in [1]) Let $\varphi(z)$ be an analytic* *function, defined on a half-plane*

$$H(\delta) = \{z \in \mathbb{C} : Re\, z \geq -\delta\}$$

for some $0 < \delta < 1$. *Suppose that, for some* $A < \pi$, φ *satisfies the growth condition*

$$|\varphi(v + iw)| < Ce^{Pv + A|w|} \tag{4.1}$$

for all $z = v + iw \in H(\delta)$. *Then, for all* $0 < Re\, s < \delta$,

$$\int_0^\infty x^{s-1} \left(\varphi(0) - x\varphi(1) + x^2\varphi(2) - \ldots \right) \, dx = \frac{\pi}{\sin(s\pi)} \varphi(-s). \tag{4.2}$$

The proof presented in [1] makes use of the Mellin inversion formula (3.2).

Rule E_1 is a direct application of Ramanujan's Master Theorem. See [1] for some examples of direct application of the theorem to evaluate integrals on $[0, \infty)$. Rule E_2 is a multi-dimensional extension of Rule E_1. See [1] for justification in the two-dimensional case.

4.2.2 Theoretical Foundation for P_2

The motivation shown for Rule P_2 in [5] is reproduced here. In the identity

$$\frac{1}{A^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-Ax} \, dx,$$

substitute $A = a_1 + \cdots + a_r$ to produce

$$(a_1 + \cdots + a_r)^\alpha = \frac{1}{\Gamma(-\alpha)} \int_0^\infty x^{-\alpha-1} e^{-a_1 x} \cdots e^{-a_r x} \, dx.$$

Expanding the exponentials gives

$$(a_1 + \cdots + a_r)^\alpha = \frac{1}{\Gamma(-\alpha)} \sum_{n_1} \cdots \sum_{n_r} \phi_{1,\ldots,r} a_1^{n_1} \cdots a_r^{n_r} \int_0^\infty x^{-\alpha - n_1 - \cdots - n_r - 1}\, dx$$

or

$$(a_1 + \cdots + a_r)^\alpha = \sum_{n_1} \cdots \sum_{n_r} \phi_{1,\ldots,r} a_1^{n_1} \cdots a_r^{n_r} \frac{\langle -\alpha + n_1 + \cdots + n_r \rangle}{\Gamma(-\alpha)}.$$

4.3 Elementary Examples

The examples in this section require only the original set of rules described in the papers [5, 6].

Identity ([7], 3.310)

$$\int_0^\infty e^{-px}\, dx = \frac{1}{p} \qquad [\operatorname{Re} p > 0] \tag{4.3}$$

The integral in 3.310 is evaluated by first writing the exponential function e^{-px} in its series representation $\sum_{n_1=0}^{\infty} \frac{(-px)^{n_1}}{n_1!}$. Then the integral is associated with the bracket series

$$\sum_{n_1=0}^{\infty} \frac{(-p)^{n_1}}{n_1!} \langle n_1 + 1 \rangle = \sum_{n_1=0}^{\infty} \phi_{n_1} p^{n_1} \langle n_1 + 1 \rangle, \tag{4.4}$$

with $f(n_1) = p^{n_1}$ in Rule E_1. The vanishing of the bracket gives $n_1^* = -1$ and the integral is assigned the expected value

$$p^{n_1^*} \Gamma(-n_1^*) = \frac{1}{p} \Gamma(1) = \frac{1}{p}.$$

The condition $\operatorname{Re} p > 0$ is needed in order for the integrand to satisfy the growth condition (4.1) of Ramanujan's Master Theorem.

Identity ([7], 6.511.1)

An integral of a single Bessel function can also be found through only the original rules.

$$\int_0^\infty J_\nu(bx)\,dx = \frac{1}{b} \qquad [\mathrm{Re}\ \nu > -1, b > 0] \tag{4.5}$$

To evaluate the integral in 6.511.1, the series representation (1.22) of the order ν Bessel function is used so that the integral is associated with the bracket series

$$\sum \phi_n \frac{\left(\frac{b}{2}\right)^\nu \left(\frac{b^2}{2^2}\right)^n}{\Gamma(n+\nu+1)} \langle 2n+\nu+1 \rangle\,. \tag{4.6}$$

Application of Rule E_1 first requires the solution of the equation $2n+\nu+1=0$ for $n^* = \frac{1}{2}(-1-\nu)$. Then Rule E_1 assigns to the bracket series the value

$$\frac{1}{2}\left(\frac{\left(\frac{b}{2}\right)^\nu \left(\frac{b^2}{2^2}\right)^{n^*}}{\Gamma(n^*+\nu+1)}\right)\Gamma(-n^*) = \frac{1}{2}\frac{\left(\frac{b}{2}\right)^\nu \left(\frac{b^2}{2^2}\right)^{(-1-\nu)/2}}{\Gamma(\frac{\nu+1}{2})}\Gamma(\frac{\nu+1}{2}) = \frac{1}{b}. \tag{4.7}$$

Identity ([7],3.194.3)

Entry 3.194.3 illustrates the 2-dimensional version of Rule E_2.

$$\int_0^\infty \frac{x^{\mu-1}\,dx}{(\beta x+1)^\nu} = \beta^{-\mu}B(\mu,\nu-\mu)$$

Expanding the denominator by Rule P_2 produces a bracket series of Index 0:

$$\sum_{n_1\geq 0}\sum_{n_2\geq 0} \phi_{1,2}\frac{\beta^{n_1}}{\Gamma(\nu)} \langle n_1+\mu \rangle \langle n_1+n_2+\nu \rangle$$

The vanishing of the brackets gives the solution to the system of linear equations:

$$n_1^* = -\mu \quad \text{and} \quad n_2^* = \mu - \nu.$$

The matrix of coefficients has determinant 1. By Rule E_2, the integral is

$$\frac{1}{|1|}\frac{\beta^{n_1^*}}{\Gamma(\nu)}\Gamma(-n_1^*)\Gamma(-n_2^*) = \beta^{-\mu}\frac{\Gamma(-\mu+\nu)\Gamma(\mu)}{\Gamma(\nu)} = \beta^{-\mu}B(-\mu+\nu,\mu).$$

Identity ([7],3.311.1)

The integral (3.311.1) illustrates Rule P_2 as well as the portion of Rule E_3 that discards divergent series.

$$\int_0^\infty \frac{1}{e^{px}+1}\,dx = \frac{\ln 2}{p}$$

The first step is to apply Rule P_2 to write the integrand as

$$\sum_{n_1\geq 0}\sum_{n_2\geq 0}\phi_{1,2}(e^{px})^{n_1}1^{n_2}\frac{\langle 1+n_1+n_2\rangle}{\Gamma(1)} = \sum_{n_1\geq 0}\sum_{n_2\geq 0}\phi_{1,2}e^{n_1px}\left\langle 1+n_1+n_2\right\rangle.$$

Then e^{n_1px} is written in its series notation so that the integrand is now represented as

$$\sum_{n_1\geq 0}\sum_{n_2\geq 0}\phi_{1,2}\left(\sum_{n_3\geq 0}\frac{(pxn_1)^{n_3}}{\Gamma(n_3+1)}\right)\left\langle 1+n_1+n_2\right\rangle$$
$$= \sum_{n_1,n_2,n_3\geq 0}\phi_{1,2,3}(-n_1)^{n_3}p^{n_3}x^{n_3}\left\langle 1+n_1+n_2\right\rangle.$$

From this form, the integral can now be associated with the bracket series of Index 1

$$\sum_{n_1\geq 0}\sum_{n_2\geq 0}\sum_{n_3\geq 0}\phi_{1,2,3}(-n_1)^{n_3}p^{n_3}\left\langle n_3+1\right\rangle\left\langle 1+n_1+n_2\right\rangle.$$

With three sums and two brackets, there will be one free summation index. By the bracket $\langle n_3+1\rangle$, n_3 must be fixed so that $n_3^*=-1$. There are then only two choices for free variables.

Case 1: With n_1 free and n_2 fixed, $n_2^*=-n_1-1$ and the determinant is -1. The resulting series is divergent due to its infinite first term and is discarded:

$$\sum_{n_1\geq 0}\phi_1\frac{1}{|-1|}(-n_1)^{n_3^*}p^{n_3^*}\Gamma(-n_2^*)\Gamma(-n_3^*)=\sum_{n_1\geq 0}\frac{(-1)^{n_1+1}}{pn_1}.$$

Case 2: With n_2 free and n_1 fixed, $n_1^*=-n_2-1$ and the determinant is -1. The resulting series give the value shown in the table entry.

$$\sum_{n_2\geq 0}\phi_2\frac{1}{|-1|}(-n_1^*)^{n_3^*}p^{n_3^*}\Gamma(-n_1^*)\Gamma(-n_3)^*=\sum_{n_2\geq 0}\frac{-(-1)^{n_2+1}}{p(n_2+1)}=\frac{\ln 2}{p}$$

Identity ([7], 6.554.4)

$$\int_0^\infty xJ_0(xy)\frac{dx}{(a^2+x^2)^{3/2}}=\frac{e^{-ay}}{a}\quad[y>0,\operatorname{Re}a>0]$$

The Bessel function $J_0(xy)$ is expanded with respect to x through its hypergeometric series representation (1.22), and the denominator is expanded by Rule P_2. The bracket series is

$$\sum_{n_1}\sum_{n_2}\sum_{n_3}\phi_{1,2,3}\frac{y^{2n_3}a^{2n_1}}{4^{n_3}\Gamma(1+n_3)\Gamma(3/2)}\left\langle 3/2+n_1+n_2\right\rangle\left\langle 2n_2+2n_3+2\right\rangle.$$

With a bracket series of Index 1, there are several choices of free variables.

Case 1: If n_1 is free, $|\det(A)| = 2$, $n_3^* = n_1 + 1/2$, and $n_2^* = -n_1 - 3/2$. The resulting series is

$$\begin{aligned} S_1 =& \sum_{n_1} \frac{(-1)^{n_1} 2^{-2n_1-1} a^{2n_1} y^{2n_1+1} \Gamma(-n_1 - 1/2)}{\sqrt{\pi}\Gamma(n_1+1)} \\ =& -\frac{\sinh(ay)}{a} \end{aligned}$$

Case 2: If n_2 is free, $|\det(A)| = 2$, $n_3^* = -n_2 - 1$, and $n_1^* = -n_2 - 3/2$. The resulting series is divergent and therefore does not contribute to the value of the integral:

$$S_2 = \sum_{n_2} \frac{(-1)^{n_2} 2^{2n_2+2} a^{-2n_2-3} y^{-2n_2-2} \Gamma(n_2 + 3/2)}{\sqrt{\pi}\Gamma(-n_2)}$$

Case 3: If n_3 free, $|\det(A)| = 2$, $n_2^* = -n_3 - 1$, and $n_1^* = n_3 - 1/2$. The resulting series is

$$\begin{aligned} S_3 =& \sum_{n_3} \frac{(-1)^{n_3} a^{2n_3-1} y^{2n_3} \Gamma(-n_3 + 1/2)}{\sqrt{\pi} 2^{2n_3} \Gamma(n_3+1)} \\ =& \frac{\cosh(ay)}{a} \end{aligned}$$

Summing these two contributions produces the value of the integral:

$$\begin{aligned} \int_0^\infty x J_0(xy) \frac{dx}{(a^2+x^2)^{3/2}} =& S_1 + S_3 \\ =& -\frac{\sinh(ay)}{a} + \frac{\cosh(ay)}{a} \\ =& \frac{e^{-ay}}{a} \end{aligned}$$

4.4 Discussion of the Method

The method of brackets reduces integration problems to expanding factors into infinite series, solving systems of linear equations, and analyzing infinite series. These few rules can be automated relatively easily. This feature has allowed for experimentation with the set of rules and testing the results of the method against known integral identities in tables of integrals such as [7].

In the solution of the system of linear equations, a fixed n_i^* may not be integer-valued so factorials are extended by the use of gamma functions.

The method of brackets has been applied successfully to a wide variety of single or multiple integrals on $[0, \infty)$ involving elementary functions as well as special functions such as Bessel functions and orthogonal polynomials. Many examples can be found in the original papers [5, 6, 1]. Additional integrals from the Gradshteyn-Ryzhik table [7] will be presented in Section 4.8 here.

It is easier to list which classes of integrals cannot be handled by the method of brackets.

1. The method of brackets should not be applied to integrals involving functions that violate the growth condition (4.1). In particular, the method of brackets is not capable of determining convergence of integrals. Without checking the growth condition, the method would proceed as usual and may return a finite value. For example, proceeding with the application of the rules, the method of brackets would produce values for the following integrals:

    - $\displaystyle\int_0^\infty e^x \, dx = -1$
    - $\displaystyle\int_0^\infty \sin x \, dx = 1$
    - $\displaystyle\int_0^\infty \cos x \, dx = 0$

2. The method of brackets cannot be used to evaluate any integral containing a function of x that cannot be written as a series from $n = 0$ to $n = \infty$. Such functions include $\Gamma(x)$, $\Gamma(s, x)$, and $\mathrm{Ei}(x)$.

3. Rule P_2 applies easily only in certain circumstances.

   - Rule P_2 may not be applied to an expression with a exponent involving a variable of integration. If this were to happen, that variable of integration will appear inside the bracket and in the argument of the gamma function in the denominator. A bracket series should not contain any variable of integration, but the application of Rule P_2 in this case will produce these two instances of the variable of integration in the bracket series. See integral 3.484 in [7].

   - If Rule P_2 is applied to an expression with a positive integer exponent m, then Rule P_2 requires division by $\Gamma(-m)$. A possible solution to this problem is to replace m with $m + \varepsilon$ and take the limit as $\varepsilon \to 0$ in the result.

   - If Rule P_2 applies, but there is a term which should next be converted to a series, then there will be $(series)^{n_i}$, which cannot be evaluated, unless the original term was exponential. A possible solution of changing n_i to $n_i + \varepsilon$ as above should be studied.

4. The method of brackets cannot handle integrals that involve a composition of functions because $(inner\_series)^{n_i}$ cannot be evaluated for general n_i (unless the inner function is an exponential function). Many examples of this composition exist in [7], including 3.893.3.

5. Certain functions have series expansions that depend on sequences of numbers that have not been extended. When solving the system of linear equations

for n_i^*, the solutions may no longer be integer-valued so an extension from the sequence to a function valid at other rational/real points is needed to solve integrals involving such functions as the tangent and secant functions. For example, $\tan x$ requires Alternating (Up/Down) numbers. Examples involving $\tan x$ should be studied to determine whether the relation to Bernoulli numbers and the Riemann zeta function can be applied.

6. Integrals involving division by a series cannot be handled. These will likely will run into issue of extending coefficients mentioned above.

4.5 Modifications to the Method of Brackets

Rule P_3 has been explicitly added to the original set of rules presented in [5, 6, 1]. Rule E_3 is modified so that imaginary results and zeros are discarded.

4.5.1 Experimental Motivation for Rule P_3

Although P_3 is not fully understood, experimental evidence presented in this section experimental suggests its necessity. Through experimentation, it has been found that the bracket series (and therefore the final solution) produced via rules P_1, P_2, and P_3 depend on the representation of the integrand and on the order of application of these rules.

Representation

It appears that as much grouping as possible should be performed to obtain satisfactory representations of the integrand. Experimentation with integrals in [6] illustrates that grouping should be performed even when factoring is not possible. For example, all five integrands below are equivalent and should integrate to $\frac{1}{2}$. However, only the first two representations produce the $\frac{1}{2}$. These both produce bracket series with the same minimal difference of zero. Other representations, such as the last three below without both $(x+y)$ groupings, produce bracket series with one more index than brackets. These sums are all divergent and the method returns no solution for these representations.

1. $\displaystyle\int_0^\infty\int_0^\infty \frac{xy\,dxdy}{(xy(x+y)+(x+y))^2} = \frac{1}{2}$ is of Index 0
2. $\displaystyle\int_0^\infty\int_0^\infty \frac{xy\,dxdy}{(xy+1)^2(x+y)^2} = \frac{1}{2}$ is of Index 0
3. $\displaystyle\int_0^\infty\int_0^\infty \frac{dxdy}{xy(x+y+1/x+1/y)^2}$ $\rightarrow$ produces no solution and is of Index 1
4. $\displaystyle\int_0^\infty\int_0^\infty \frac{xy\,dxdy}{(x^2y+xy^2+x+y)^2}$ $\rightarrow$ produces no solution and is of Index 1
5. $\displaystyle\int_0^\infty\int_0^\infty \frac{xy\,dxdy}{(xy(x+y)+x+y)^2}$ $\rightarrow$ produces no solution and is of Index 1

Order of Series Expansions

Another motivation for Rule P_3 is that the order of series expansions may also affect the Index.

The ordering of series expansions of integrand factors also can affect the Index. For example, in the integral ([7], 3.452.4)

$$\int_0^\infty \frac{xe^{-x}\,dx}{\sqrt{e^{2x}-1}} = 1-\ln 2, \tag{4.8}$$

expanding the square root or the exponential first will produce the differing results.

Case 1: Expanding $(e^{2x}-1)^{-1/2}$ first generates a bracket series of Index 1, from which the correct result is produced.

By applying Rule P_2 first, the integrand has the series representation

$$xe^{-x}\sum_{n_1}\sum_{n_2}\phi_{1,2}e^{2n_1x}(-1)^{n_2}\frac{\langle 1/2+n_1+n_2\rangle}{\Gamma(1/2)}$$
$$=x\sum_{n_1}\sum_{n_2}\phi_{1,2}\left(\sum_{n_3}\phi_{n_3}(1-2n_1)^{n_3}x^{n_3}\right)(-1)^{n_2}\frac{\langle 1/2+n_1+n_2\rangle}{\Gamma(1/2)}.$$

Then the bracket series is of Index 1:

$$\sum_{n_1}\sum_{n_2}\sum_{n_3}\phi_{1,2,3}\frac{(1-2n_1)^{n_3}(-1)^{n_2}}{\Gamma(1/2)}\langle n_3+2\rangle\langle 1/2+n_1+n_2\rangle.$$

The vanishing of the brackets makes $n_3^*=-2$, and n_1 or n_2 will be free:

- n_1 free gives $n_2^*=-n_1-1/2$, $n_3^*=-2$, and $|\det(A)|=1$. The resulting series is purely imaginary and therefore discarded:

$$\sum_{n_1}\frac{-i\Gamma(n_1+1/2)}{(2n_1-1)^2\sqrt{\pi}\Gamma(n_1+1)}=-\frac{i\pi}{2}$$

- n_2 free gives $n_1^*=-n_2-1/2$, $n_3^*=-2$, and $|\det(A)|=1$. The resulting series is the desired result:

$$\sum_{n_2}\frac{\Gamma(n_2+1/2)}{4(n_2+1)^2\sqrt{\pi}\Gamma(n_2+1)}=1-\ln 2$$

Case 2: Expanding the numerator exponential first generates a bracket series of Index 2 and a value that is *twice* the correct value. Introducing additional parameters allows for insight into this doubling issue. Consider instead the integral

$$\int_0^\infty\frac{xe^{-Ax}}{\sqrt{e^{2Bx}-1}}\,dx$$

By expanding e^{-Ax} first, the integrand has the series representation

$$
\begin{aligned}
&x\left(\sum_{n_1}\phi_{n_1}(Ax)^{n_1}\right)\left(\sum_{n_2}\sum_{n_3}\phi_{2,3}e^{2Bn_2x}(-1)^{n_3}\frac{\langle 1/2+n_2+n_3\rangle}{\Gamma(1/2)}\right)\\
&=x\sum_{n_1}\sum_{n_2}\sum_{n_3}\phi_{1,2,3}\frac{(-1)^{n_3}A^{n_1}x^{n_1}}{\Gamma(1/2)}\left(\sum_{n_4}\phi_{n_4}(-1)^{n_4}(2Bn_2x)^{n_4}\right)\langle 1/2+n_2+n_3\rangle
\end{aligned}
$$

Then the bracket series is of Index 2:

$$\sum_{n_1}\sum_{n_2}\sum_{n_3}\sum_{n_4}\phi_{1,2,3,4}\frac{(-1)^{n_3+n_4}A^{n_1}(2Bn_2)^{n_4}}{\sqrt{\pi}}\langle 1/2+n_2+n_3\rangle\langle n_1+n_4+2\rangle$$

- With n_3 and n_4 free, the inner series converges for $|A|>|B+2Bn_3|$:

$$\sum_{n_3}\sum_{n_4}\frac{(-n_3-1/2)^{n_4}(2B)^{n_4}\Gamma(n_4+2)\Gamma(n_3+1/2)}{\sqrt{\pi}\Gamma(n_4+1)\Gamma(n_3+1)A^{n_4+2}}=$$

- With n_2 and n_4 free, the resulting series is purely imaginary and therefore discarded:

$$\sum_{n_2}\sum_{n_4}\frac{-i2^{n_4}(Bn_2)^{n_4}\Gamma(n_4+2)\Gamma(n_2+1/2)}{\sqrt{\pi}\Gamma(n_4+1)\Gamma(n_2+1)A^{n_4-2}}$$

- With n_1 and n_3 free, the inner series converges for $|A|<|B+2Bn_3|$:

$$\sum_{n_3}\sum_{n_1}\frac{(-n_3-1/2)^{-n_1-2}(2B)^{-n_1-2}A^{n_1}\Gamma(n_3+1/2)\Gamma(n_1+2)}{\sqrt{\pi}\Gamma(n_3+1)\Gamma(n_1+1)}$$

- With n_1 and n_2 free, the resulting series is purely imaginary and is discarded:

$$\sum_{n_1}\sum_{n_2}\frac{-i2^{-n_1-2}A^{n_1}(Bn_2)^{-n_1-2}\Gamma(n_2+1/2)\Gamma(n_1+2)}{\sqrt{\pi}\Gamma(n_2+1)\Gamma(n_1+1)}$$

Letting $A, B \to 1$ returns this integral to the form (4.8). In those cases, the real sums would both simplify to $1 - \ln 2$, and summing would produce a value *twice* as much as expected. With the convergence conditions on the inner integrals made explicit with the introduction of A and B, it can be seen that the evaluation of the resulting series would on the boundary of the regions of convergence, rather than within two separate regions. This explains the doubling effect.

In the original integral without the inserted A and B parameters, expanding the square root first by Rule P_2 produces a bracket series of minimal index because the exponentials appear inside and outside the square root and can be combined after the expansion of the square root. Since this issue only appears in integrals involving composite functions, a solution is possible. If there is a "fuzzy match" of the interior function of a composition with another external factor of the integrand, then the exterior function should be expanded into its series representation first. This solution has been implemented.

Efficiency

Experimentation has also shown that several representations may minimize the Index. For efficiency, the one with the fewest summations should be selected among these.

In the following example, both representations below produce the correct answer. The second representation is preferred by Sage, but it requires three summations whereas the first representation requires only two summations. The application of Rule P_2 in the second representation accounts for the additional summation index and bracket.

1. $\displaystyle\int_0^\infty x^{s-1} e^{-\beta x^2} e^{-\gamma x}\, dx$ has 2 summation indices and 1 bracket
2. $\displaystyle\int_0^\infty x^{s-1} e^{-\beta x^2 - \gamma x}\, dx$ has 3 summation indices and 2 brackets

This case of the exponential function of a sum can be handled by an explicit rewrite rule to form the first representation from the second. However, issues of composition for minimizing the Index must also be considered since these issues affect correctness.

4.5.2 Experimental Motivation for Rule E_3

Rule E_3 has been modified so that imaginary series and zeros are discarded. The examples presented here illustrate the necessity of this modification.

Identity ([7], 3.452.1)

One example showing the need to discard imaginary series that result from a choice of free/fixed summation indices is identity 3.452.1. Other examples is Section 3.452 also illustrate this necessity.

$$\int_0^\infty \frac{x}{\sqrt{e^x - 1}}\, dx = 2\pi \ln 2.$$

The bracket series is

$$\sum_{n_1}\sum_{n_2}\sum_{n_3} \phi_{1,2,3} \frac{(-1)^{n_2+n_3} n_1^{n_3}}{\sqrt{\pi}} \langle n_1 + n_2 + 1/2 \rangle \langle n_3 + 2 \rangle$$

By the second bracket, n_3 must be fixed with $n_3^* = -2$. The free variable must be chosen among n_1 and n_2.

Case 1: With n_2 free, the resulting series is

$$\sum_{n_2} \frac{4\Gamma(n_2 + 1/2)}{(2n_2 + 1)^2 \sqrt{\pi}\Gamma(n_2 + 1)},$$

which Mathematica 7 simplifies to $\pi \ln 4$, the result given in the table entry.

Case 2: With n_1 free, the resulting series is

$$\sum_{n_1} \frac{(-1)^{2n_1+5/2}\Gamma(n_1+1/2)}{\sqrt{\pi}n_1^2\Gamma(n_1+1)},$$

which is purely imaginary due to the $(-1)^{5/2}$ factor.

Identity ([7], 3.723.9)

Another example showing the need to discard imaginary series is ([7], 3.723.9):

$$\int_0^\infty \frac{\cos(ax)}{b^2-x^2}\,dx = \frac{\pi}{2b}\sin(ab) \quad [a>0, b>0]$$

The bracket series is of Index 1:

$$\sum_{n_1,n_2,n_3} \phi_{1,2,3} \frac{(-1)^{n_3}\sqrt{\pi}a^{2n_1}b^{2n_2}}{2^{2n_1}\Gamma(n_1+1/2)} \langle 2n_1+2n_3+1\rangle \langle n_2+n_3+1\rangle$$

Case 1: With n_1 free, $n_3^* = -n_1 - 1/2$, $n_2^* = n_1 - 1/2$, $|\det(A)| = 2$, and the series produced is purely imaginary:

$$\sum_{n_1} \frac{-i\sqrt{\pi}2^{-2n_1-1}a^{2n_1}b^{2n_1-1}\Gamma(-n_1+1/2)}{\Gamma(n_1+1)}$$

Case 2: With n_3 free, $n_1^* = -n_3 - 1/2$, $n_2^* = -n_3 - 1$, $|\det(A)| = 2$, and the series is zero due to the $\Gamma(-n_3)$ in the denominator:

$$\sum_{n_3} \frac{\sqrt{\pi}2^{2n_3}a^{-2n_3-1}b^{-2n_3-2}\Gamma(n_3+1/2)}{\Gamma(-n_3)}$$

Case 3: With n_2 free, $n_1^* = n_2 + 1/2$, $n_3^* = -n_2 - 1$, $|\det(A)| = 2$, and the result is

$$\sum_{n_2} \frac{-\sqrt{\pi}2^{-2n_2-2}a^{2n_2+1}|b|^{2n_2}\Gamma(-n_2-1/2)}{\Gamma(n_2+1)}$$

Mathematica 7 simplifies this third series to $\pi \sin(a|b|)/(2|b|)$, agreeing with the table entry assuming $a > 0$ and $b > 0$.

Identity ([7], 7.522.2)

Identities 7.522.2-4 show illustrate that it is necessary to discard series of zeros. Without this discarding, a solution of zero would be returned, and this solution is incorrect. If these zeros are discarded, then the method of brackets will return no solution. Identity([7], 7.522.2) is presented here:

$$\int_0^\infty e^{-bx} x^{a-1} \, {}_2F_1\left(\frac{1}{2}+\nu, \frac{1}{2}-\nu; a; -\frac{x}{2}\right) dx = 2^a e^b \frac{1}{\sqrt{\pi}} \Gamma(a)(2b)^{\frac{1}{2}-a} K_\nu(b) \quad [\operatorname{Re} a > 0, \operatorname{Re} b > 0]$$

The bracket-series is

$$\sum_{n_1} \sum_{n_2} \frac{(-1)^{n_1+n_2} b^{n_1} \Gamma(n_2-\nu+1/2)\Gamma(n_2+\nu+1/2)\Gamma(a)}{2^{n_2}\Gamma(-\nu+1/2)\Gamma(\nu+1/2)\Gamma(n_2+1)\Gamma(n_1+1)\Gamma(a+n_2)} \langle a+n_1+n_2 \rangle$$

Case 1: With n_1 free, the resulting series is zero:

$$\sum_{n_1} \frac{(-1)^{n_1} 2^{a+n_1} b^{n_1} \Gamma(a+n_1)\Gamma(-a-n_1-\nu+1/2)\Gamma(-a-n_1+\nu+1/2)\Gamma(a)}{\Gamma(-\nu+1/2)\Gamma(\nu+1/2)\Gamma(n_1+1)\Gamma(-n_1)}$$

Case 2: With n_2 free, the resulting series is divergent:

$$\sum_{n_2} \frac{(-1)^{n_2} b^{-a-n_2} \Gamma(n_2-\nu+1/2)\Gamma(n_2+\nu+1/2)\Gamma(a)}{2^{n_2}\Gamma(-\nu+1/2)\Gamma(\nu+1/2)\Gamma(n_2+1)}$$

The divergent series must be discarded. If the other series is not discarded, the method of brackets will return a value of zero, clearly incorrect. If both series is discarded, there will be no solution returned by the method of brackets. The discarding of zeros is not an ideal solution to this issue, but it is preferable to an incorrect solution.

Identities 7.522.3 and 7.522.4 produce the same results of one divergent series and one series of zeros. These three integrands are all combinations of exponentials with ${}_2F_1$'s. Identities 7.531.1 and 7.531.2 return two series which are both sums of zeros and therefore must be discarded also.

However, the combination of an exponential and a ${}_2F_1$ in 7.524.1 does not show this problem, but the method of brackets appears to be missing one term of this solution, as discussed in Section 4.10.

4.6 Challenges

Difficulties encountered in applying the method include the representation of the integrand and the order of application of the rules producing a bracket series.

Experimentation has shown that satisfactory representations of the integrand are only among those that minimize the difference between the number of summations and the number of brackets.

For integrals involving the modified Bessel function of the second kind, $K_\nu(x)$, there is another representation issue. The integral representation produces expected results, but the representation involving the sum of two series produces a result which is twice the correct value.

A comprehensive algorithm is needed to determine the "best" representation and correct ordering of the expansions so that the Index is minimized in Rule P_3. Producing all bracket series from various representations and expansion ordering would be very computationally expensive so the representation must be chosen before creation of bracket series. The implemented solution to this is described in the next section. However, this solution does not capture the $K_\nu(x)$ issue described above.

There may be many series produced as a result of the choices of free/fixed indices. These series are sent to Mathematica 7 for an attempt at simplification. When these are multi-sums, all ordering of summation must be attempted until a simplification is found.

4.7 Implementation

The method of brackets has been implemented in the open-source computer algebra system Sage with some calls to Mathematica. The code appears in the appendices. By testing the method against a large number of integrals in the table [7], necessary modifications of rules were discovered. Those include the introduction of Rule P_3 and the discarding of imaginary series in Rule E_3.

4.7.1 Procedure

The Sage implementation is according to the following procedure.

1. Assign a bracket series to the integral:

   (a) Expand the integrand into a series representation recursively:

      i. All terms in polynomial factors should be placed over a common denominator. Enumerate all representations involving all possible groupings of polynomial terms. Choose the representation which minimizes the difference between the number of terms in polynomial factors and the number of such polynomial factors.

      ii. Determine the first series expansion to be performed by considering compositions, and expand that factor into its series by a series lookup or by application of Rule P_2.

      iii. Push all other factors inside this sum, simplify, and recursively expand this summand by step 1(a).

(b) By Definition 1, convert each integral and its corresponding variable of integration into a bracket. Assume each index n_i is a nonnegative integer to assist in simplification.

2. If there are as many brackets as summations, the system of one or more linear equations is solved in the application of Rule E_1 or Rule E_2 to find the value of the integral. Note that the assumptions on the n_i indices must be forgotten before solving this system.

3. In the case with more summations than brackets, the procedure is as follows:

   (a) Assemble a matrix from the list of brackets. Using its reduced row echelon form, determine all possibilities of free and fixed indices so that the matrix of coefficients of fixed indices has maximal rank.

   (b) For each of the above choices, solve a system of linear equations corresponding to the fixed indices, producing a series over the free summation indices. The nonnegative integer assumptions on the n_i must be kept on the free summation indices but forgotten for the fixed indices.

   (c) For each series produced, attempt simplification and determination of convergence conditions in Mathematica. If the series is a multi-sum, then various orderings of the summation must be attempted.

   (d) Use the regions of convergence determined above by Mathematica and sum the simplified series that converge within each region.

4.7.2 I/O

Input

The user calls the function `method_of_brackets_zero_inf` with two arguments. The first argument is the expression of the integrand. The second is a list of the variable(s) of integration.

Before this call is made, all variables should be declared and any assumptions on them should be made.

Output

1. If the $Index = 0$, the result is returned as the only element of a list.
2. If the $Index > 0$, simplification of the resulting sums is attempted in Mathematica. Where possible, Mathematica also returns convergence conditions. These results are returned in a list from Mathematica.

4.7.3 Algorithm Runtime

The main factors affecting the runtime of the algorithm are the number of polynomial factors and the number of terms in each, the Index of the bracket series as well as the number of summation indices in the bracket series. Several aspects of the procedure may be very time-consuming.

When grouping is performed on polynomial factors, the set of terms of each polynomial is partitioned into non-empty subsets using the `SetPartitions` command in Sage. For a set of n terms, the number of such set partitions is given by the Bell number $B(n)$ [17]. For $n = 8$ terms as in the example in [6] concerning I_4, there are $B(8) = 4140$ such partitions.

When the Index is positive, the system of equations must be solved for each choice of free summation indices, and there may be many choices for free/fixed indices. The Index (which counts the number of free summation indices) along with the number of variables determines how many linear systems of equations must be solved. When Index is at least 1, the series produced must each be simplified in Mathematica. When number of free variables is at least 2, the series is a multi-sum, and it may be necessary to attempt more than one ordering of summation to simplify the multi-sum. Simplifying (multi-)sums in Mathematica may be very time-consuming in certain cases. For multi-sums, it may be necessary to attempt all orderings of the indices before Mathematica finds a solution to the multi-sum. Even though simplification each multi-sum is attempted in parallel, the simplification with various orderings of the summation indices is attempted sequentially.

Also worth noting is that the `assume` and `forget` functions of Sage (which calls Maxima) often take upwards of 15 seconds each. These commands are used frequently with the n_i variables. The `assume` function is used on the n_i variables to aide simplification of the bracket series or the final series since the n_i will be nonnegative integers. The `forget` function is applied to the n_i variables when it is time to solve the system of linear equations since the n_i^* solutions are not always nonnegative integers.

4.8 Working Examples

The examples presented in this section are taken from the table [7] so that the results may be verified.

4.8.1 Examples of Index 0

Identity ([7], 3.721.1)

The classical techniques of proving identity (3.721.1) involve complex contour integration.

$$\int_0^\infty \frac{\sin(ax)}{x}\,dx = \frac{\pi}{2}\text{sign } a$$

The $\sin(ax)$ factor is written in its hypergeometric form (1.14), and the resulting bracket series is of Index 0:

$$\sum_{n_1} \phi_{n_1} \frac{(1/2)\sqrt{\pi}(a^2)^{n_1} a}{4^{n_1}\Gamma(n_1+3/2)} \langle 2n_1 + 1 \rangle .$$

The vanishing of the brackets produces $n_1^* = -1/2$. By Rule E_1, the solution is as expected:

$$\frac{1}{|2|}\frac{(1/2)\sqrt{\pi}(a^2)^{n_1^*} a}{4^{n_1^*}\Gamma(n_1^*+3/2)}\Gamma(-n_1^*) = \frac{\pi a}{2|a|}.$$

Identity ([7], 6.561.16)

$$\int_0^\infty x^\mu K_\nu(ax) = 2^{\mu-1}a^{-\mu-1}\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\mu-\nu}{2}\right)$$

$$[\mathrm{Re}\,(\mu+1\pm\nu) > 0, \mathrm{Re}\,a > 0] \quad (4.9)$$

The integral representation of the modified Bessel function of the second kind

$$K_\nu(x) = \frac{2^\nu}{\sqrt{\pi}x^\nu}\Gamma\left(\nu+\frac{1}{2}\right)\int_0^\infty \frac{\cos(xt)\,dt}{(t^2+1)^{\nu+1/2}}, \qquad [\mathrm{Re}(\nu) > -1/2, x > 0] \qquad (4.10)$$

transforms the identity into a double integral identity:

$$\int_0^\infty \int_0^\infty \frac{2^\nu x^{\mu-\nu}\Gamma(\nu+1/2)}{\sqrt{\pi}a^\nu}\frac{\cos(axt)}{(t^2+1)^{\nu+1/2}}\,dt\,dx$$
$$= 2^{\mu-1}a^{-\mu-1}\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\mu-\nu}{2}\right)$$

The expansion of $\cos(axt)$ by (1.15) and of the denominator by Rule P_2 produces the representation of the integrand:

$$\frac{2^\nu x^{\mu-\nu}\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}a^\nu}\left(\sum_{n_1}\phi_{n_1}\frac{\Gamma(1/2)}{\Gamma(n_1+1/2)}\left(\frac{(axt)^2}{4}\right)^{n_1}\right) \times \left(\sum_{n_2}\sum_{n_3}\phi_{n_2,n_3}t^{2n_2}\frac{\langle\nu+1/2+n_2+n_3\rangle}{\Gamma(\nu+1/2)}\right)$$

Then the bracket series has Index 0:

$$\sum_{n_1}\sum_{n_2}\sum_{n_3}\phi_{1,2,3}\frac{2^{\nu-2n_1}a^{2n_1-\nu}}{\Gamma(n_1+1/2)}\langle\mu-\nu+2n_1+1\rangle\langle 2n_1+2n_2+1\rangle\left\langle n_2+n_3+\nu+\frac{1}{2}\right\rangle.$$

The coefficients of n_i in the bracketed expressions form the matrix

$$A=\begin{bmatrix}2 & 0 & 0\\ 2 & 2 & 0\\ 0 & 1 & 1\end{bmatrix}$$

with $|det(A)|=4$. The system created by the vanishing of the brackets is solved by

$$\{n_1^*=\frac{\nu-\mu-1}{2};\quad n_2^*=\frac{\mu-\nu}{2};\quad n_3^*=-\frac{\nu+\mu+1}{2}\}$$

By the multidimensional Rule E_2, the bracket series is assigned the value

$$\frac{1}{4}\left(\frac{2^{\nu-2n_1^*}a^{2n_1^*-\nu}}{\Gamma(n_1^*+1/2)}\right)\Gamma(-n_1^*)\Gamma(-n_2^*)\Gamma(-n_3^*)=2^{\mu-1}a^{-\mu-1}\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\mu-\nu}{2}\right)$$

Note that the method of brackets produces the correct solution with the use of the integral representation of $K_\nu(x)$. However with the representation involving the sum of two series given in Appendix A, the values is doubled. This issue will be examined in Section 4.10.

4.8.2 Examples of Index 1

Identity ([7], 3.197.1)

We first consider one integral representation of the Gauss hypergeometric function that was discussed in Chapter 2:

$$\int_0^{\infty} x^{\nu-1}(\beta+x)^{-\mu}(x+\gamma)^{-\varrho}\,dx = \beta^{-\mu}\gamma^{\nu-\varrho}B(\nu,\mu-\nu+\varrho)\,{}_2F_1\left(\mu,\nu;\mu+\varrho;1-\frac{\gamma}{\beta}\right)$$
$$[|\arg\beta|<\pi, |\arg\gamma|<\pi, \mathrm{Re}\nu>0, \mathrm{Re}\mu>\mathrm{Re}\,(\nu-\varrho)] \quad (4.11)$$

The bracket series is of Index 1:

$$\sum_{n_1,n_2,n_3,n_4}\phi_{1,2,3,4}\frac{\beta^{n_1}\gamma^{n_3}}{\Gamma(\varrho)\Gamma(\mu)}\langle n_2+n_4+\nu\rangle\langle n_3+n_4+\varrho\rangle\langle n_1+n_2+\mu\rangle$$

Case 1: With n_4 free, $n_2^*=-n_4-\nu$, $n_1^*=n_4-\mu+\nu$, $n_3^*=-n_4-\varrho$, and $|\det(A)|=1$. The resulting series is

$$\sum_{n_4}\frac{(-1)^{n_4}\beta^{n_4-\mu+\nu}c^{-n_4-\varrho}\Gamma(n_4+\nu)\Gamma(n_4+\varrho)\Gamma(-n_4+\mu-\nu)}{\Gamma(n_4+1)\Gamma(\varrho)\Gamma(\mu)}$$
$$=\frac{\beta^{-\mu+\nu}\Gamma(\mu-\nu)\Gamma(\nu)}{\gamma^{\varrho}\Gamma(\mu)}\,{}_2F_1\left(\varrho,\nu;1-\mu+\nu;\frac{\beta}{\gamma}\right) \qquad |\beta|<|\gamma|.$$

Case 2: With n_1 free, $n_2^*=-n_1-\mu$, $n_4^*=n_1+\mu-\nu$, $n_3^*=-n_1-\varrho-\mu+\nu$, and

$|\det(A)| = 1$. The resulting series is

$$\sum_{n_1} \frac{(-1)^{n_1}\beta^{n_1}\gamma^{-n_1-\varrho-\mu+\nu}\Gamma(n_1+\mu)\Gamma(-n_1-\mu+\nu)\Gamma(n_1+\varrho+\mu-\nu)}{\Gamma(n_1+1)\Gamma(\varrho)\Gamma(\mu)}$$
$$= \frac{\gamma^{-\varrho-\mu+\nu}\Gamma(\varrho+\mu-\nu)\Gamma(-\mu+\nu)}{\Gamma(\varrho)}\,{}_2F_1\left(\mu,\varrho+\mu-\nu;1+\mu-\nu;\frac{\beta}{\gamma}\right) \quad |\beta|<|\gamma|.$$

Case 3: With n_3 free, $n_2^* = n_3+\varrho-\nu$, $n_1^* = -n_3-\varrho-\mu+\nu$, $n_4^* = -n_3-\varrho$, and $|\det(A)| = 1$. The resulting series is

$$\sum_{n_3} \frac{(-1)^{n_3}\beta^{-n_3-\varrho-\mu+\nu}\gamma^{n_3}\Gamma(n_3+\varrho)\Gamma(-n_3-\varrho+\nu)\Gamma(n_3+\varrho+\mu-\nu)}{\Gamma(n_3+1)\Gamma(\varrho)\Gamma(\mu)}$$
$$= \frac{\beta^{-\varrho-\mu+\nu}\Gamma\varrho+\mu-\nu\Gamma-\varrho+\nu}{\Gamma(\mu)}\,{}_2F_1\left(\varrho,\varrho+\mu-\nu;1+\varrho-\nu;\frac{\gamma}{\beta}\right), \quad |\beta|>|\gamma|.$$

Case 4: With n_2 free, $n_1^* = -n_2-\mu$, $n_4^* = -n_2-\nu$, $n_3^* = n_2-\varrho+\nu$, and $|\det(A)| = 1$. The resulting series is

$$\sum_{n_2} \frac{(-1)^{n_2}\beta^{-n_2-\mu}\gamma^{n_2-\varrho+\nu}\Gamma(n_2+\nu)\Gamma(n_2+\mu)\Gamma(-n_2+\varrho-\nu)}{\Gamma(n_2+1)\Gamma(\varrho)\Gamma(\mu)}$$
$$= \frac{\gamma^{-\varrho+\nu}\Gamma(\varrho-\nu)\Gamma(\nu)}{\beta^{\mu}\Gamma(\varrho)}\,{}_2F_1\left(\mu,\nu;1-\varrho+\nu;\frac{\gamma}{\beta}\right) \quad |\beta|>|\gamma|.$$

In order to write convert these to the form in (4.11), identity (1.33) with $z = 1-\gamma/\beta$ is needed in both regions. Additionally, the alternate Pfaff identity (1.31) is needed for the $|\beta/\gamma|<1$ case.

Identity ([7], 3.223.1)

$$\int_0^\infty \frac{x^{\mu-1}\,dx}{(\beta+x)(\gamma+x)} = \frac{\pi}{\gamma-\beta}(\beta^{\mu-1}-\gamma^{\mu-1})\csc(\mu\pi)$$

$$[|\arg\beta|<\pi, |\arg\gamma|<\pi, 0<\text{Re}\mu<2]$$

The method of brackets gives the bracket series

$$\sum_{n_1}\sum_{n_2}\sum_{n_3}\sum_{n_4}\phi_{1,2,3,4}\beta^{n_3}\gamma^{n_1}\langle n_3+n_4+1\rangle\langle n_1+n_2+1\rangle\langle n_2+n_4+\mu\rangle \tag{4.12}$$

There are four ways to choose a free summation index, and these sums all simplify to hypergeometric ${}_1F_0$ functions, which restate the Binomial Theorem (1.19):

With n_1 or n_4 free, the region of convergence is $|\gamma/\beta|<1$. With n_1 free, the contribution is $\beta^{\mu-2}\Gamma(-\mu+2)\Gamma(\mu-1)\,{}_1F_0(1,\gamma/\beta)=\beta^\mu\Gamma(-\mu+2)\Gamma(\mu-1)/(\beta^2-\beta\gamma)$. With n_4 free, the result is $\gamma^{\mu-1}\Gamma(-\mu+1)\Gamma(\mu)\,{}_1F_0(1,\gamma/\beta)/\beta=\gamma^\mu\Gamma(-\mu+1)\Gamma(\mu)/(\beta\gamma-\gamma^2)$. Therefore in the region $|\gamma/\beta|<1$, the value of the integral is

$$\frac{\gamma^\mu\Gamma(-\mu+1)\Gamma(\mu)}{\beta\gamma-\gamma^2}+\frac{\beta^\mu\Gamma(-\mu+2)\Gamma(\mu-1)}{\beta^2-\beta\gamma}$$

With n_2 or n_3 free, the region of convergence is $|\beta/\gamma|<1$. With n_2 free, the contribution is $\beta^{\mu-1}\Gamma(-\mu+1)\Gamma(\mu)\,{}_1F_0(1,\beta/\gamma)/\gamma=-\beta^\mu\Gamma(-\mu+1)\Gamma(\mu)/(\beta^2-\beta\gamma)$. With n_3 free, the contribution is $\gamma^{\mu-2}\Gamma(-\mu+2)\Gamma(\mu-1)\,{}_1F_0(1,\beta/\gamma)=-\gamma^\mu\Gamma(-\mu+2)\Gamma(\mu-1)/(\beta\gamma-\gamma^2)$.

In the region where $|\beta/\gamma|<1$, the total is

$$-\frac{\gamma^\mu\Gamma(-\mu+2)\Gamma(\mu-1)}{\beta\gamma-\gamma^2}-\frac{\beta^\mu\Gamma(-\mu+1)\Gamma(\mu)}{\beta^2-\beta\gamma}$$

The gamma reflection formula (1.3) is required to write these in the form shown in [7]. The two cases found above agree so the solution is valid everywhere, even on the boundary $|\gamma/\beta| = 1$.

Identity ([7], 3.311.4)

$$\int_0^\infty \frac{e^{-qx}\,dx}{1 - ae^{-px}} = \sum_{k=0}^{\infty} \frac{a^k}{q + kp} \quad [0 < a < 1]$$

In this identity, the denominator must be expanded into its series representation by Rule P_2 first so that the Index will be minimized after combining the exponentials before expansion. The bracket series of Index 1 is given by

$$\sum_{n_1}\sum_{n_2}\sum_{n_3}\sum_{n_4} \frac{-(-p)^{n_3}(-q)^{n_4}(-1)^{n_1+2n_2}a^{n_1}n_1^{n_3}}{\Gamma(n_4+1)\Gamma(n_3+1)\Gamma(n_2+1)\Gamma(n_1+1)} \langle n_3 + n_4 + 1\rangle \langle n_1 + n_2 + 1\rangle$$

Case 1: n_2 free produces $n_1^* = -n_2 - 1$ and $n_3^* = -1$. The reulting series converges for $|a| < 1$. This corresponds to the case presented in [7].

$$\sum_{n_2} \frac{a^{n_2}}{n_2 p + q}$$

Case 2: n_1 free produces $n_2^* = -n_1 - 1$ and $n_3^* = -1$. The resulting series converges for $|a| > 1$.

$$\sum_{n_1} \frac{a^{-n_1-1}}{(n_1+1)p - q}$$

Identity ([7], 3.451.2)

$$\int_0^\infty xe^{-x}\sqrt{1 - e^{-2x}}\,dx = \frac{\pi}{4}\left(\frac{1}{2} + \ln 2\right)$$

Expanding the radical first will produce the minimal index, as required by Rule P_3. The bracket series is

$$\sum_{n_1}\sum_{n_2}\sum_{n_3}\phi_{1,2,3}\frac{(-(1-n1)^{n_3}(-1)^{n_2}2^{n_3-1}}{\sqrt{\pi}}\langle n_3+2\rangle\langle n_1+n_2-1/2\rangle$$

Case 1: n_2 free produces $n_3^*=-2$, $n_1^*=-n_2+1/2$, $|det(A)|=1$, and the resulting series matches the table entry:

$$\sum_{n_2}\frac{-\Gamma(n_2-1/2)}{2(2n_2+1)^2\sqrt{\pi}\Gamma(n_2+1)}=\frac{\pi}{8}(1+2\ln 2)$$

Case 1: n_1 free produces $n_3^*=-2$, $n_2^*=-n_1+1/2$, $|det(A)|=1$, and the resulting series is purely imaginary and therefore discarded.

$$\sum_{n_1}\frac{-i\Gamma(n_1-1/2)}{8(n_1-1)^2\sqrt{\pi}\Gamma(n_1+1)}$$

Identity ([7], 3.471.13)

$$\int_0^\infty\frac{x^{\nu-1}e^{-\beta/x}}{x+\gamma}\,dx=\gamma^{\nu-1}e^{\beta/\gamma}\Gamma(1-\nu)\Gamma(\nu,\beta/\gamma),\quad[|\arg(\gamma)|<\pi,\mathrm{Re}\beta>0,\mathrm{Re}\gamma<1] \tag{4.13}$$

First the exponential factor is expressed in a series in n_1. Next Rule P_2 applies to expand the denominator. Therefore the bracket series is

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\phi_{1,2,3}\beta^{n_1}\gamma^{n_2}\langle 1+n_2+n_3\rangle\langle -n_1+n_3+\nu\rangle$$

With three indices and two brackets, there will be one free variable, and any of the summation indices could be free.

Case 1: With n_1 free, the result is

$$\begin{aligned} S_1 = &\sum_{n_1=0}^{\infty} \frac{(-1)^{n_1}\beta^{n_1}\gamma^{-n_1+\nu-1}\Gamma(-n_1+\nu)\Gamma(n_1-\nu+1)}{\Gamma(n_1+1)} \\ =&\gamma^{-1+\nu}e^{\beta/\gamma}\pi\csc(\pi\nu). \end{aligned}$$

Case 2: With n_2 free, the series diverges and is discarded:

$$S_2 = \sum_{n_2=0}^{\infty}(-\beta)^{-n_2+\nu-1}(-1)^{2n_2-\nu+1}\gamma^{n_2}\Gamma(n_2-\nu+1)$$

Case 3: With n_3 free, the resulting series is

$$\begin{aligned} S_3 = &\sum_{n_3=0}^{\infty}((-\beta)^{n_3+\nu}\gamma^{-n_3-1}\Gamma(-n_3-\nu)/(-1)^{\nu} \\ =&\frac{(-\beta)^{\nu}e^{\beta/\gamma}\Gamma(-\nu)}{(-1)^{\nu}(\beta/\gamma)^{\nu}\gamma}\left[\Gamma(1+\nu)-\nu\Gamma(\nu,\beta/\gamma)\right]. \end{aligned}$$

The first and third do sum to produce the correct solution.

Identity ([7], 3.479.1)

$$\int_0^{\infty}\frac{x^{\nu-1}\exp(-\beta\sqrt{x+1})}{\sqrt{x+1}}\,dx = \frac{2}{\sqrt{\pi}}\left(\frac{\beta}{2}\right)^{\frac{1}{2}-\nu}\Gamma(\nu)K_{\frac{1}{2}-\nu}(\beta) \quad [\mathrm{Re}\beta > 0, \mathrm{Re}\nu > 0] \tag{4.14}$$

In order to minimize the Index, the exponential function must be expanded into its series first. Then the $\sqrt{x+1}$ factors may be combined before expansion by Rule P_2. The bracket series is of Index 1:

$$\sum_{n_1}\sum_{n_2}\sum_{n_3}\phi_{1,2,3}\frac{\beta^{n_1}}{\gamma(-n_1/2+1/2)}\langle n_2+\nu\rangle\langle -n_1/2+n_2+n_3+1/2\rangle$$

By the vanishing of the brackets, it must be that $n_2^* = -\nu$ and either n_1 or n_3 will be free.

Case 1: n_3 free: $n_1^* = 2n_3 - 2\nu + 1$, $|\det(A)| = 1/2$

$$\sum_{n_3} \frac{(2(-1)^{n_3}\beta^{2n_3-2\nu+1}\Gamma(-2n_3+2\nu-1)\Gamma(\nu)}{\Gamma(n_3+1)\Gamma(-n_3+\nu)} = 2\beta^{1-2\nu}\Gamma(-1+2\nu)\,{}_0F_1(-;3/2-\nu;\beta^2/4)$$

Case 2: n_1 free: $n_3^* = 1/2 * n1 + v - 1/2$, $|\det(A)| = 1$

$$\sum_{n_1} \frac{(-1)^{n_1}\beta^{n_1}\Gamma(-n_1/2-\nu+1/2)\Gamma(\nu)}{\Gamma(-n_1/2+1/2)\Gamma(n_1+1)} = \frac{\sqrt{\pi}\Gamma(\nu)\cos(\pi\nu)}{\Gamma(1/2+\nu)}\,{}_0F_1(-;1/2+\nu,\beta^2/4)$$

By the hypergeometric form (1.23) of the modified Bessel function $I_\nu(x)$ and the relation ([18], 9.29)

$$K_\nu(z) = \frac{\pi}{2\sin(\nu\pi)}\left[I_{-\nu}(z) - I_\nu(z)\right], \tag{4.15}$$

the sum of the two contributions above can be verified to match the table entry.

Identity ([7], 3.764.2)

$$\int_0^\infty x^p \cos(ax+b)\,dx = -\frac{1}{a^{p+1}}\Gamma(1+p)\sin\left(b+\frac{\pi p}{2}\right) \quad [a>0, -1<p<0]$$

The bracket-series is

$$\sum_{n_1=0}^\infty\sum_{n_2=0}^\infty\sum_{n_3=0}^\infty \phi_{1,2,3}\frac{\Gamma(\frac{1}{2})a^{n_2}b^{n_3}}{4^{n_1}\Gamma(n_1+\frac{1}{2})\Gamma(-2n_1)}\langle -2n_1+n_2+n_3\rangle\langle n_2+p+1\rangle.$$

By the vanishing of the brackets, n_2 must be fixed with $n_2^* = -p-1$. Then either n_1 or n_3 will be free:

Case 1: With n_1 free, $n_3^* = 2n_1 + p + 1$, $|det(A)| = 1$, and the resulting series has no contribution:

$$\sum_{n_1=0}^{\infty} \phi_{n_1} \frac{\Gamma(\frac{1}{2}) a^{-p-1} b^{2n_1+p+1}}{4^{n_1}\Gamma(n_1+\frac{1}{2})\Gamma(-2n_1)} \Gamma(p+1)\Gamma(-2n_1-p-1) = 0$$

Case 2: With n_3 free, $n_1^* = n_3/2 - p/2 - 1/2$, $|det(A)| = 2$, and the resulting series produces the desired result:

$$\sum_{n_3=0}^{\infty} \phi_{n_3} \frac{\Gamma(\frac{1}{2}) a^{-p-1} b^{n_3} \Gamma\left(-\frac{n_3}{2} + \frac{p}{2} + \frac{1}{2}\right)\Gamma(p+1)}{2^{n_3-p}\Gamma(\frac{n_3}{2} - \frac{p}{2})\Gamma(-n_3+p+1)} = -a^{-p-1}\Gamma(1+p)\sin\left(b + \frac{p\pi}{2}\right)$$

Identity ([7], 4.296.3)

$$\int_0^{\infty} \ln(1 + 2x\cos t + x^2) x^{\mu-1} dx = \frac{2\pi}{\mu} \frac{\cos \mu t}{\sin \mu\pi} \quad [|t| < \pi, -1 < \operatorname{Re}\mu < 0]$$

The hypergeometric representation (1.16) is used to rewrite the integrand as

$$\begin{aligned} & x^{\mu-1}\log(1 + 2x\cos t + x^2) \\ & \qquad = x^{\mu-1} \sum_{n_1} \phi_{n_1} \frac{\Gamma(1+n_1)\Gamma(1+n_1)}{\Gamma(2+n_1)} (2x\cos t + x^2)^{n_1+1}. \end{aligned}$$

The factor $(2x\cos t + x^2)^{n_1+1}$ is expanded via Rule P_2 so that the integrand is

$$\begin{aligned} & x^{\mu-1} \sum_{n_1} \phi_{n_1} \frac{\Gamma(1+n_1)\Gamma(1+n_1)}{\Gamma(2+n_1)} (2x\cos t + x^2)^{n_1+1} \\ & = x^{\mu-1} \sum_{n_1} \phi_{n_1} \frac{\Gamma(1+n_1)\Gamma(1+n_1)}{\Gamma(2+n_1)} \sum_{n_2}\sum_{n_3} \phi_{n_1,n_2} (2x\cos t)^{n_2} x^{2n_3} \frac{\langle -n_1 - 1 + n_2 + n_3 \rangle}{\Gamma(-n_1-1)}, \end{aligned}$$

producing a bracket series of Index 1:

$$\sum_{n_1}\sum_{n_2}\sum_{n_3}\phi_{1,2,3}\frac{2^{n_2}(\Gamma(1+n_1))^2(\cos(t))^{n_2}}{\Gamma(2+n_1)\Gamma(-n_1-1)}\langle -n_1-1+n_2+n_3\rangle\langle \mu+n_2+2n_3\rangle .$$

Case 1: With n_1 free, the resulting series makes no contribution:

$$\begin{aligned}S_1 =& \sum_{n_1}\frac{(-1)^{n_1}2^{2n_1+\mu+2}(\cos t)^{2n_1+\mu+2}\Gamma(n_1+1)\Gamma(-2*n_1-\mu-2)\Gamma(n_1+\mu+1)}{\Gamma(-n_1-1)\Gamma(n_1+2)}\\ =&0\end{aligned}$$

Case 2: With n_2 free, the resulting series matches the value in the table:

$$\begin{aligned}S_2 =& \sum_{n_2}\frac{(-1)^{n_2}2^{n_2-1}(\cos t)^{n_2}\Gamma(-n_2/2+\mu/2+1)\Gamma(n_2/2-\mu/2)^2\Gamma(n_2/2+\mu/2)}{\Gamma(n_2+1)\Gamma(-n_2/2+\mu/2)\Gamma(n_2/2-\mu/2+1)}\\ =&\frac{2\pi\cos(\mu t)}{\mu\sin(\pi\mu)}, \quad [|\cos t|<1]\end{aligned}$$

Case 3: With n_3 free, the resulting series diverges since the condition for convergence will not be satisfied:

$$\begin{aligned}S_3 =& \sum_{n_3}\frac{(-1)^{n_3}2^{-2n_3-\mu}(\cos t)^{-2n_3-\mu}\Gamma(-n_3-\mu)^2\Gamma(n_3+\mu+1)\Gamma(2n_3+\mu)}{\Gamma(n_3+1)\Gamma(-n_3-\mu+1)\Gamma(n_3+\mu)}\\ =&\frac{\pi}{\sin(\pi\mu)\mu(\cos t)^{\mu}(1+\sqrt{-\tan^2 t})^{\mu}}, \quad [|\sec t|<1]\end{aligned}$$

Identity ([7], 6.512.3)

The identity 6.512.3 was proved for the case $\beta < \alpha$ by the algorithmic Mellin Transform method in Chapter 3. The method of brackets finds the result in all cases.

$$\int_0^\infty J_\nu(\alpha x)J_{\nu-1}(\beta x)dx = \begin{cases} \frac{\beta^{\nu-1}}{\alpha^\nu} & [\beta < \alpha] \\ \frac{1}{2\beta} & [\beta = \alpha] \\ 0 & [\beta > \alpha] \end{cases}$$

Expanding both Bessel functions via their hypergeometric representations (1.22) produces the bracket series of Index 1

$$\sum_{n_1}\sum_{n_2} \phi_{1,2} \frac{2^{-2\nu+1-2n_1-2n_2}\alpha^{2n_1+\nu}\beta^{2n_2+\nu-1}}{\Gamma(n_2+\nu)\Gamma(n_1+\nu+1)} \langle 2n_1 + 2n_2 + 2\nu \rangle .$$

Case 1: With n_1 free, $n_2^* = -n_1 - \nu$, and $|\det(A)| = 2$, the resulting series is zero for $|\alpha| < |\beta|$

$$\sum_{n_1} \frac{(-1)^{n_1}\alpha^{2n_1+\nu}\beta^{-\nu-1-2n_1}\Gamma(n_1+\nu)}{\Gamma(n_1+1)\Gamma(n_1+\nu+1)\Gamma(-n_1)} = 0$$

Case 2: With n_2 free, $n_1^* = -n_2 - \nu$, and $|\det(A)| = 2$, the resulting series terminates with only 1 term:

$$\sum_{n_2} \frac{(-1)^{n_2}\alpha^{-\nu-2n_2}\beta^{2n_2+\nu-1}}{\Gamma(-n_2+1)\Gamma(n_2+1)} = \frac{\beta^{\nu-1}}{\alpha^\nu}$$

With $\alpha = \beta$, the method of brackets also finds the third outcome. In Case 2 above, the result will converge and simplify to $1/\alpha$.

Identity ([7], 6.565.2)

$$\int_0^\infty x^{\nu+1}(x^2+a^2)^{-\nu-\frac{1}{2}} J_\nu(bx)\, dx = \frac{\sqrt{\pi} b^{\nu-1}}{2^\nu e^{ab}\Gamma(\nu+\frac{1}{2})}$$

$$[\mathrm{Re}\, a > 0,\ b > 0,\ \mathrm{Re}\nu > -1/2] \quad (4.16)$$

The bracket series is

$$\sum_{n_1}\sum_{n_2}\sum_{n_3} \phi_{1,2,3} \frac{b^{2n_1+\nu} a^{2n_2}}{2^{\nu+2n_1}\Gamma(\nu+1/2)\Gamma(n_1+\nu+1)} \times \langle 2n_1+2n_3+2\nu+2\rangle \langle n_2+n_3+\nu+1/2\rangle$$

Case 1: With n_1 free, $n_2^* = n_1 + 1/2$, $n_3 = -n_1 - \nu - 1$ and $|\det(A)| = 2$, and the resulting series is

$$S_1 = \sum_{n_1} \frac{(-1)^{n_1} 2^{-2n_1-\nu-1} b^{2n_1+\nu} a^{2n_1+1}\Gamma(-n_1-1/2)}{\Gamma(\nu+1/2)\Gamma(n_1+1)} = \frac{b^{-1+\nu}\sqrt{\pi}\cosh(ab)}{2^\nu\Gamma(1/2+\nu)}$$

Case 2: With n_2 free, $n_1^* = n_2 - 1/2$, $n_3^* = -n_2 - \nu - 1/2$, and $|\det(A)| = 2$, and the resulting series is

$$S_2 = \sum_{n_2} \frac{(-1)^{n_2} 2^{-2n_2-\nu} b^{2n_2+\nu-1} a^{2n_2}\Gamma(-n_2+1/2)}{\Gamma(\nu+1/2)\Gamma(n_2+1)} = -\frac{b^{\nu-1}\sqrt{\pi}\sinh(ab)}{2^\nu\Gamma(1/2+\nu)}$$

Case 3: With n_3 free, $n_1^* = -n_3 - \nu - 1$, $n_2^* = -n_3 - \nu - 1/2$, and $|\det(A)| = 2$. The resulting series is discarded.

$$S_3 = \sum_{n_3} \frac{(-1)^{n_3} 2^{2n_3+\nu+1} b^{-2n_3-\nu-2} a^{-2n_3-2\nu-1}\Gamma(n_3+\nu+1/2)\Gamma(n_3+\nu+1)}{\Gamma(\nu+1/2)\Gamma(n_3+1)\Gamma(-n_3)}$$

Summing S_1 and S_2 produces the desired result.

Identity ([7], 6.566.2)

$$\int_0^\infty x^{\nu+1} J_\nu(ax) \frac{dx}{x^2+b^2} = b^\nu K_\nu(ab) \quad [a>0, \operatorname{Re} b > 0, -1 < \operatorname{Re} \nu < \frac{3}{2}]$$

The bracket series is of Index 1:

$$\sum_{n_1} \sum_{n_2} \sum_{n_3} \phi_{1,2,3} \frac{(a^2)^{n_1} a^v abs(b)^{2n_2}}{2^v 4^{n_1} \gamma(n_1+v+1)} \langle 2n_1 + 2n_3 + 2v + 2 \rangle \langle n_2 + n_3 + 1 \rangle$$

Case 1: With n_1 free, the result is

$$\sum_{n_1} \frac{(-1)^{n_1} 2^{-2n_1-v-1} a^{2n_1+v} b^{2n_1+2v} \Gamma(-n_1-v)}{\Gamma(n_1+1)} = -\frac{1}{2} \pi b^\nu I_\nu(ab) \csc(\pi\nu).$$

Case 2: With n_2 free, the result is

$$\sum_{n_2} \frac{(-1)^{n_2} 2^{-2n_2+v-1} a^{-v+2n_2} b^{2n_2} \Gamma(-n_2+v)}{\Gamma(n_2+1)} = \frac{a^v \pi b^v I_{-v}(ab) \csc(\pi\nu)}{2a^v}.$$

Case 3: With n_3 free, the series will be discarded:

$$\sum_{n_3} \frac{(-1)^{n_3} 2^{2n_3+v+1} a^{-2n_3-v-2} b^{-2n_3-2} \Gamma(n_3+v+1)}{\Gamma(-n_3)}$$

With the discarding of S_3, the result will be the sum of S_1 and S_2. By identity (4.15), the identity holds.

Identity ([7], 6.671.9)

This identity was proved by applying the classical Mellin transform method in Section 3.5.1.

$$\int_0^\infty J_{2n+1}(ax)\sin(bx)\,dx = \frac{(-1)^n}{\sqrt{a^2-b^2}}T_{2n+1}\left(\frac{b}{a}\right), \quad [b<a]$$

$$\sum_{n_1}\sum_{n_2}\phi_{1,2}\frac{\sqrt{\pi}2^{-2n-2-2n_1-2n_2}a^{2n_1+2n+1}b^{2n_2+1}}{\Gamma(n_2+3/2)\Gamma(2n+n_1+2)}\langle 2n+2n_1+2n_2+3\rangle$$

Case 1: With n_1 free, $n_2^* = -n-n_1-3/2$, and $|\det(A)| = 2$, and the resulting series converges for $|a|<|b|$

$$\sum_{n_1}\frac{\sqrt{\pi}(-1)^{n_1}a^{2n+2n_1+1}b^{-2n-2n_1-2}\Gamma(n+n_1+3/2)}{\Gamma(n_1+1)\Gamma(-n-n_1)\Gamma(2n+n_1+2)}$$
$$= \frac{-2^{-1-2n}a^{1+2n}\sin(n\pi)}{b^{2(1+n)}}\,{}_2F_1\left(1+n,3/2+n;2+2n;a^2/b^2\right)$$

Case 2: With n_2 free, $n_1^* = -n-n_2-3/2$ and $|\det(A)| = 2$, and the resulting series converges for $|a|>|b|$

$$\sum_{n_2}\frac{\sqrt{\pi}(-1)^{n_2}b^{2*n_2+1}a^{-2*n_2-2}\Gamma(n+n_2+3/2)}{\Gamma(n_2+1)\Gamma(n_2+3/2)\Gamma(n-n_2+1/2)} = \frac{\sin((2n+1)\arcsin(b/a))}{\sqrt{a^2-b^2}}$$

Using the fact that $T_m(x) = \cos(m\arccos x)$ and the trigonometric identity $\arcsin x = \frac{\pi}{2} - \arccos x$, Case 2 agrees with the entry in [7].

Identity ([7], 7.376.2)

$$\int_0^\infty e^{-2\alpha x^2} x^\nu H_{2n}(x)\, dx = (-1)^n 2^{2n-\frac{3}{2}-\frac{1}{2}\nu} \frac{\Gamma(\frac{\nu+1}{2})\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\alpha^{\frac{1}{2}(\nu+1)}} {}_2F_1\left(-n, \frac{\nu+1}{2}; \frac{1}{2}; \frac{1}{2\alpha}\right)$$

$$[\mathrm{Re}\alpha > 0, \mathrm{Re}\nu > -1] \quad (4.17)$$

The integrand is expanded by the hypergeometric representation of the Hermite polynomial (1.28). Since this integral involves even degree Hermite polynomials, only the first term in the expansion will remain. The bracket series is

$$\sum_{n_1}\sum_{n_2} \phi_{1,2} \frac{\sqrt{\pi} a^{n_1} (-1)^{n+n_2} 2^{n_1} \Gamma(2n+1)\Gamma(-n+n_2)}{\Gamma(n_2+1/2)\Gamma(n+1)\Gamma(-n)} \langle 2n_1 + 2n_2 + v + 1 \rangle .$$

Case 1: With n_1 free, the resulting series is imaginary and therefore discarded.

Case 2: With n_2 free, the resulting series converges for $2|a| > 1$:

$$\sum_{n_2} \frac{\sqrt{\pi}(-1)^n 2^{-n_2-\nu/2-3/2}\Gamma(2n+1)\Gamma(n_2+\nu/2+1/2)\Gamma(-n+n_2)}{a^{n_2+\nu/2+1/2}\Gamma(n_2+1/2)\Gamma(n_2+1)\Gamma(n+1)\Gamma(-n)}$$
$$= \frac{(-1)^n\Gamma(1+2n)\Gamma((1+v)/2)}{2^{3/2+\nu/2}a^{1/2+\nu/2}\Gamma(n+1)} {}_2F_1\left(-n, \frac{1+\nu}{2}; 1/2; \frac{1}{2a}\right)$$

The duplication formula (1.4) puts this into the form shown in (4.17).

Identity ([7], 7.383.1)

A similar integral involving Hermite polynomials of odd degree is 7.383.1:

$$\int_0^\infty e^{-xp} H_{2n+1}(\sqrt{x})\, dx = (-1)^n 2^n (2n+1)!! \pi^{1/2} (p-1)^n p^{-n-\frac{3}{2}} \quad [\mathrm{Re}p > 0]. \quad (4.18)$$

$$\sum_{n_1}\sum_{n_2} \frac{\sqrt{\pi}(-p)^{n_1}(-1)^n\Gamma(2n+2)\Gamma(-n+n_2)}{\Gamma(n_2+1)\Gamma(n_2+3/2)\Gamma(n_1+1)\Gamma(n+1)\Gamma(-n)} \langle n_1+n_2+3/2\rangle$$

Case 1: With n_1 free, the resulting series has no contribution:

$$\sum_{n_1} \frac{\sqrt{\pi}(-1)^{n+2n_1+3/2}p^{n_1}\Gamma(n_1+3/2)\Gamma(2n+2)\Gamma(-n-n_1-3/2)}{\Gamma(n_1+1)\Gamma(n+1)\Gamma(-n)\Gamma(-n_1)} = 0 \quad [|p|<1]$$

Case 2: With n_2 free, the series converges for $|p| > 1$ and matches the right side of (4.18) once the double factorial is written in terms of the gamma function.

$$\sum_{n_2} \frac{\sqrt{\pi}p^{-n_2-3/2}(-1)^n\Gamma(2n+2)\Gamma(-n+n_2)}{\Gamma(n_2+1)\Gamma(n+1)\Gamma(-n)} = (-1)^n 2^{1+2n}(p-1)^n p^{-3/2-n}\Gamma(3/2+n)$$

4.8.3 Examples of Index 2

Identity ([7], 7.414.9)

$$\int_0^\infty e^{-x}x^{a+b}L_m^a(x)L_n^b(x)\,dx = (-1)^{m+n}(a+b)!\binom{a+m}{n}\binom{b+n}{m} \quad [\mathrm{Re}(a+b)>-1] \tag{4.19}$$

With the representation (1.27) for each of the Associated Laguerre polynomials, the bracket series is of Index 2:

$$\sum_{n_1}\sum_{n_2}\sum_{n_3} \phi_{1,2,3}\frac{(-1)^{n_2+n_3}\Gamma(-n+n_3)\Gamma(-m+n_2)\Gamma(b+n+1)\Gamma(a+m+1)}{\Gamma(n+1)\Gamma(m+1)\Gamma(b+n_3+1)\Gamma(a+n_2+1)\Gamma(-m)\Gamma(-n)}$$
$$\times \langle a+b+n_1+n_2+n_3+1\rangle$$

There are three choices of free/fixed variables in the solution of the linear equation from the vanishing of the bracket:

Case 1: With n_1 and n_2 free, the resulting series is zero and makes no contribution:

$$\sum_{n_1,n_2} \frac{(-1)^{a+b+2n_1+n_2+1}\Gamma(-m+n_2)\Gamma(b+n+1)\Gamma(a+m+1)}{\Gamma(n_2+1)\Gamma(n_1+1)\Gamma(n+1)\Gamma(m+1)\Gamma(a+n_2+1}$$
$$\times \frac{\Gamma(a+b+n_1+n_2+1)\Gamma(-a-b-n-n_1-n_2-1)}{)\Gamma(-a-n_1-n_2)\Gamma(-m)\Gamma(-n)}$$
$$= \frac{(-1)^{a+b}\Gamma(1+a+b)\Gamma(-m)\Gamma(1+a+m)\Gamma(1+b+n)\Gamma(-a-m+n)}{\pi^2 \sin((a+b+n)\pi)\Gamma(1+n)\Gamma(1+b-m+n)}$$
$$\times \sin(a\pi)\sin(m\pi)\sin(n\pi)$$

Case 2: With n_1 and n_3 free, the resulting series is zero and makes no contribution

$$\sum_{n_1,n_3} \frac{(-1)^{a+b+2n_1+n_3+1}\Gamma(-n+n_3)\Gamma(b+n+1)\Gamma(a+m+1)}{\Gamma(n_3+1)\Gamma(n_1+1)\Gamma(n+1)\Gamma(m+1)\Gamma(b+n_3+1)}$$
$$\times \frac{\Gamma(a+b+n_1+n_3+1)\Gamma(-a-b-m-n_1-n_3-1)}{\Gamma(-b-n_1-n_3)\Gamma(-m)\Gamma(-n)}$$
$$= \frac{(-1)^{a+b}\Gamma(1+a+b)\Gamma(1+a+m)\Gamma(-b+m-n)\Gamma(-n)\Gamma(1+b+n)}{\pi^2 \sin((a+b+m)\pi)\Gamma(1+m)\Gamma(1+a+m-n)}$$
$$\times \sin(b\pi)\sin(m\pi)\sin(n\pi)$$

Case 3: With n_2 and n_3 free, the resulting series does match the value in the table:

$$\sum_{n_2,n_3} \frac{\Gamma(-n+n_3)\Gamma(-m+n_2)\Gamma(b+n+1)\Gamma(a+m+1)\Gamma(a+b+n_2+n_3+1)}{\Gamma(n_3+1)\Gamma(n_2+1)\Gamma(n+1)\Gamma(m+1)\Gamma(b+n_3+1)\Gamma(a+n_2+1)\Gamma(-m)\Gamma(-n)}$$
$$= \frac{\Gamma(1+a+b)\Gamma(1+b+n)\Gamma(-a-m+n)\sin(b\pi)}{\sin((b-m)\pi)\Gamma(-a-m)\Gamma(1+m)\Gamma(1+n)\Gamma(1+b-m+n)}$$
$$= (-1)^{m+n}\Gamma(a+b+1)\frac{\Gamma(a+m+1)}{\Gamma(n+1)\Gamma(a+m-n+1)}\frac{\Gamma(b+n+1)}{\Gamma(m+1)\Gamma(b+n-m+1)}$$

4.9 Examples suggesting corrections to the table [7]

The results of testing the method of brackets against the examples in this section suggest corrections to the table [7].

Identity ([7], 3.227.2)

This identity was corrected in Section 2.1.2 through classical proof techniques. The same correction is motivated here by the method of brackets.

$$\int_0^\infty \frac{x^{-\rho}(\beta - x)^{-\sigma}}{\gamma + x}\, dx = \frac{\pi}{\gamma^\rho(\beta+\gamma)^\sigma \sin(\pi\rho)} I_{1+\frac{\gamma}{\beta}}(\sigma, \rho). \tag{4.20}$$

$$\sum_{n_1}\sum_{n_2}\sum_{n_3}\sum_{n_4} \phi_{1,2,3,4} \frac{(-1)^{n_2}\beta^{n_1}\gamma^{n_3}}{\Gamma(s)} \langle n_2 + n_4 - \rho + 1\rangle \langle n_3 + n_4 + 1\rangle \langle n_1 + n_2 + s\rangle$$

Case 1: With n_1 free, $n_2^* = -n_1 - \sigma$, $n_4^* = n_1 + \rho + \sigma - 1$, $n_3^* = -n_1 - \rho - \sigma$, and $|\det(A)| = 1$. The resulting series converges for $|\gamma| < |\beta|$.

$$\begin{aligned} S_1 =& \sum_{n_1} \frac{\beta^{n_1}\gamma^{-n_1-\rho-\sigma}\Gamma(n_1+\sigma)\Gamma(-n_1-\rho-\sigma+1)\Gamma(n_1+\rho+\sigma)}{(-1)^\sigma\Gamma(n_1+1)\Gamma(\sigma)} \\ =& \frac{\pi\csc(\rho\pi)}{\gamma^\rho(\beta+\gamma)^\sigma} \end{aligned}$$

Case 2: With n_2 free, $n_1^* = -n_2 - \sigma$, $n_4^* = -n_2 + \rho - 1$, $n_3^* = n_2 - \rho$, and $|\det(A)| = 1$.

The resulting series converges for $|\gamma| < |\beta|$.

$$\begin{aligned} S_2 =& \sum_{n_2} \frac{\beta^{-n_2-\sigma}\gamma^{n_2-\rho}\Gamma(-n_2+\rho)\Gamma(n_2+\sigma)\Gamma(n_2-\rho+1)}{\Gamma(n_2+1)\Gamma(\sigma)} \\ =& \frac{(-1)^{\rho}\beta^{-\rho-\sigma}\Gamma(-\rho)\Gamma(\rho+\sigma)}{\Gamma(\sigma)} \, {}_2F_1\left(1, \rho+\sigma, 1+\rho, -\frac{\gamma}{\beta}\right) \end{aligned}$$

Case 3: With n_3 free, $n_2^* = n_3+\rho$, $n_1^* = -n_3-\rho-\sigma$, $n_4^* = -n_3-1$, and $|\det(A)| = 1$. The resulting series converges for $|\beta| < |\gamma|$.

$$\begin{aligned} S_3 =& \sum_{n_2} \frac{(-1)^{2n_3+\rho}\beta^{-n_3-\rho-\sigma}\gamma^{n_3}\Gamma(-n_3-\rho)\Gamma(n_3+\rho+\sigma)}{\Gamma(\sigma)} \\ =& \frac{\pi\csc(\pi(\rho+\sigma))}{(-1)^{\sigma}\gamma^{\rho}(\beta+\gamma)^{\sigma}} \end{aligned}$$

Case 4: With n_4 free, $n_2^* = -n_4+\rho-1$, $n_1^* = n_4-\rho-\sigma+1$, $n_3^* = -n_4-1$, and $|\det(A)| = 1$. The resulting series converges for $|\beta| < |\gamma|$.

$$\begin{aligned} S_4 =& \sum_{n_4} \frac{(-1)^{\rho-1}\beta^{n_4-\rho-\sigma+1}\gamma^{-n_4-1}\Gamma(n_4-\rho+1)\Gamma(-n_4+\rho+\sigma-1)}{\Gamma(\sigma)} \\ =& \frac{(-1)^{\rho}\beta^{1-\rho-\sigma}\pi\csc(\pi(\rho+\sigma))\Gamma(1-\rho)}{\gamma\Gamma(\sigma)\Gamma(2-\rho-\sigma)} \, {}_2F_1\left(1, 1-\rho, 2-\rho-\sigma, -\frac{\beta}{\gamma}\right) \end{aligned}$$

The contributions of S_1 and S_2 are added for the region $|\gamma| < |\beta|$. By the reflection formula (1.3), the representation (1.20), and the identity $I_x(a,b) = 1 - I_{1-x}(b,a)$, the result matches the corrected case given in (4.20).

4.9.1 Identity ([7], 6.512.1)

Entry **6.512.1** gives the value for the integral

$$\int_0^{\infty} J_{\mu}(ax)J_{\nu}(bx)\,dx. \tag{4.21}$$

The answer in [7] is separated into cases according to conditions on the parameters. This separation is unclear, since both cases consider the case $a > b$, clearly a typo.

The integrand has the series expansion

$$J_\mu(ax)J_\nu(bx) = \left(\frac{\left(\frac{ax}{2}\right)^\mu}{\Gamma(\mu+1)} \sum_{n_1=0}^{\infty} \frac{\left(-\frac{(ax)^2}{4}\right)^{n_1}}{(\mu+1)_{n_1}\, n_1!} \right) \left(\frac{\left(\frac{bx}{2}\right)^\nu}{\Gamma(\nu+1)} \sum_{n_2=0}^{\infty} \frac{\left(-\frac{(bx)^2}{4}\right)^{n_2}}{(\nu+1)_{n_2}\, n_2!} \right)$$

Then the bracket series is of Index 1:

$$\frac{a^\mu b^\nu}{2^{\mu+\nu}\Gamma(\mu+1)\Gamma(\nu+1)} \sum_{n_1} \sum_{n_2} \phi_{12} \frac{a^{2n_1}\, b^{2n_2}}{2^{2n_1+2n_2}\, (\mu+1)_{n_1}(\nu+1)_{n_2}} \langle 2n_1 + 2n_2 + \mu + \nu + 1 \rangle.$$

Case 1: With n_2 free, $n_1^* = -\frac{1}{2}(2n_2 + \mu + \nu + 1)$ and the contribution to the integral is

$$S_1 = \frac{b^\nu a^{-\nu-1}\, \Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma(\nu+1)\Gamma\left(\frac{\mu+1-\nu}{2}\right)} {}_2F_1\left(\frac{\mu+\nu+1}{2}, \frac{\nu-\mu+1}{2}; \nu+1; \frac{b^2}{a^2} \right).$$

The series converges for $|b| < |a|$.

Case 2: With n_2 free, the calculation is done as in Case 1. The result is equivalent to the formula above with μ and ν interchanged and a and b interchanged.

4.9.2 Identity ([7], 6.287.1)

In [7], identity 6.287.1 appears as

$$\int_0^\infty \Phi(\beta x) e^{-\mu x^2} x\, dx = \frac{\beta}{2\mu\sqrt{\mu+\beta^2}} \quad [\mathrm{Re}\mu > -\mathrm{Re}\beta^2, \mathrm{Re}\mu > 0].$$

The hypergeometric representation of the error function (1.21) is used in the series expansion of the integrand to produce the bracket series of Index 1:

$$\sum_{n_1}\sum_{n_2}\phi_{n_1}\phi_{n_2}\frac{\beta^{2n_2+1}(\mu)^{n_1}\Gamma(n_2+1/2)}{\sqrt{\pi}\Gamma(n_2+3/2)}\langle 2n_1+2n_2+2\rangle\,.$$

Case 1: With n_1 free, the resulting series converges for $|\mu|<|\beta|^2$:

$$\sum_{n_1}\frac{(-1)^{n_1}\mu^{n_1}\beta^{-2n_1-1}\Gamma(-n_1-1/2)}{2\sqrt{\pi}\Gamma(-n_1+1/2)}=-\frac{\arctan(\sqrt{\mu}/\beta)}{\sqrt{\pi}\sqrt{\mu}}$$

Case 2: With n_2 free, the result converges for $|\beta|<\sqrt{|\mu|}$:

$$\sum_{n_2}\frac{(-1)^{n_2}\beta^{2n_2+1}\mu^{-n_2-1}\Gamma(n_2+1/2)}{2\sqrt{\pi}\Gamma(n_2+3/2)}=\frac{\arctan(\beta/\sqrt{\mu})}{\sqrt{\pi}\sqrt{\mu}}$$

Case 2 agrees with Mathematica 7 output, which requires $\mathrm{Re}(\beta^2+\mu)>0$.

4.10 Examples deserving further investigation

4.10.1 Incorrect Results

Identity ([7], 3.311.2)

$$\int_0^\infty \frac{e^{-\mu x}}{1+e^{-x}}\,dx=\beta(\mu)\quad[\mathrm{Re}\,\mu>0]$$

where a series representation of $\beta(x)$ is given in ([7], 8.372.1):

$$\beta(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{x+k}\qquad[-x\notin\mathbb{N}]$$

Expanding the denominator first in order to minimize the Index produces the bracket series of Index 1:

$$\sum_{n_1,n_2,n_3} \phi_{1,2,3}(n_1-\mu+1)^{n_3}(-1)^{n_3}\langle n_3+1\rangle\langle n_1+n_2+1\rangle$$

The vanishing of the brackets requires $n_3^*=-1$ and either n_1 or n_2 is free.

Case 1: The choice of n_2 free gives $n_1^*=-n_2-1$ and $n_3^*=-1$ with $|det(A)|=1$. The resulting series is

$$\sum_{n_2}\frac{-(-1)^{n_2+1}}{n_2+\mu}=\beta(\mu)$$

Case 2: The choice of n_1 free gives $n_3^*=-1$ and $n_2^*=-n_1-1$ with $|\det(A)|=1$. The resulting series is

$$\sum_{n_1}\frac{(-1)^{n_1+1}}{n_1-\mu+1}=-\beta(1-\mu)$$

With no conditions on convergence, Rule E_3 would return the sum $\beta(\mu)-\beta(1-\mu)$, but the second term should not exist.

Identity ([7], 3.331.2)

As given in [7], identity 3.331.2 is

$$\int_0^\infty \exp(-\beta e^x-\mu x)\,dx=\beta^\mu\Gamma(-\mu,\beta)\quad [\mathrm{Re}\beta>0]$$

The most efficient representation of the integrand is $e^{-\beta e^x}e^{-\mu x}$. Expanding $e^{-\beta e^x}$ as a series in n_1, the integrand has the series representation:

$$\sum_{n_1}\phi_{n_1}\beta^{n_1}e^{-(\mu-n_1)x}=\sum_{n_1}\sum_{n_2}\phi_{1,2}\beta^{n_1}(\mu-n_1)^{n_2}x^{n_2}$$

Then the bracket series is

$$\sum_{n_1}\sum_{n_2}\phi_{1,2}\beta^{n_1}(\mu-n_1)^{n_2}\langle n_2+1\rangle$$

With Index 1, there will be one free summation index, and that must be n_1 since n_1 does not appear within the bracket. Then $n_2^*=-1$.

$$\sum_{n_1}\phi_{n_1}\beta^{n_1}(\mu-n_1)^{-1}=E_{1+\mu}(\beta)-\beta^{\mu}\Gamma(-\mu)$$

where E represents the exponential integral.

Identity ([7], 3.952.10)

$$\begin{aligned}\int_0^\infty x^{2n+1}e^{-\beta^2x^2}\sin(ax)\,dx &= (-1)^n\frac{\sqrt{\pi}}{2^{n+\frac{3}{2}}\beta^{2n+2}}\exp\left(-\frac{a^2}{8\beta^2}\right)D_{2n+1}\left(\frac{a}{\beta\sqrt{2}}\right)\\ &= (-1)^n\frac{\sqrt{\pi}}{(2\beta)^{2n+2}}\exp\left(-\frac{a^2}{4\beta^2}\right)H_{2n+1}\left(\frac{a}{2\beta}\right)\\ &\qquad\left[|\arg\beta|<\frac{\pi}{4},a>0\right]\end{aligned}$$

The bracket series is

$$\sum_{n_1}\sum_{n_2}\phi_{1,2}\frac{\sqrt{\pi}a^{2n_2+1}b^{2n_1}}{2^{1+2n_2}\Gamma(n_2+3/2)}\langle 2n+2n_1+2n_2+2\rangle$$

Case 1: With n_2 free, $n_1^*=-n-n_2-1$, $|\det(A)|=2$, and the resulting series is

$$\sum_{n_2}\frac{\sqrt{\pi}(-1)^{n_2}2^{-2n_2-2}a^{2n_2+1}b^{-2n-2n_2-2}\Gamma(n+n_2+1)}{\Gamma(n_2+1)\Gamma(n_2+3/2)}$$
$$=\frac{a\Gamma(1+n)}{2b^{2n+2}}\,{}_1F_1\left(1+n;3/2;-\frac{a^2}{4b^2}\right)$$

Case 2: With n_1 free, $n_2^* = -n - n_1 - 1$, $|\det(A)| = 2$, but the series is divergent

$$\sum_{n_1} \frac{\sqrt{\pi}(-1)^{n_1} a 2^{2n+2n_1} b^{2n_1} a^{-2n-2n_1-2}\Gamma(n+n_1+1)}{\Gamma(n_1+1)\Gamma(-n-n_1+1/2)}$$

Mathematica output agrees with the table entry. The result from the method of brackets does not check out numerically and needs to be studied further.

Identities involving $K_\nu(x)$

Many integrals of the modified Bessel function of the second kind ($K_\nu(x)$) can be evaluated by the method of brackets. For example, identity 6.561.16 (4.9) was solved in section 4.8.1 using the integral representation (4.10). However, if the representation below (4.22) is utilized, the result will be *twice* the correct value.

$$K_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(-n+\nu)}{2n!} \left(\frac{z}{2}\right)^{2n-\nu} + \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(-n-\nu)}{2n!} \left(\frac{z}{2}\right)^{2n+\nu} \tag{4.22}$$

With this representation, the integral in identity 6.561.16 can be written as a sum of integrals I_1 and I_2:

$$\begin{aligned}
I_1 &= \int_0^\infty x^\mu \sum_m \frac{(-1)^m \Gamma(-m+\nu)}{2m!} \left(\frac{ax}{2}\right)^{2m-\nu} \\
&= \sum_m \phi_m \frac{\Gamma(-m+\nu) a^{2m-\nu}}{2^{2m-\nu+1}} \langle 2m-\nu+\mu+1 \rangle \\
&= \{m^* = (\nu-\mu-1)/2\} \\
&= 2^{\mu-1} a^{-\mu-1} \Gamma\left(\frac{1+\mu+\nu}{2}\right) \Gamma\left(\frac{1+\mu-\nu}{2}\right)
\end{aligned}$$

$$\begin{aligned}
I_2 =& \int_0^\infty x^\mu \sum_n \frac{(-1)^n \Gamma(-n-\nu)}{2n!} \left(\frac{ax}{2}\right)^{2n+\nu} \\
=& \sum_n \phi_n \frac{\Gamma(-n-\nu) a^{2n+\nu}}{2^{2n+\nu+1}} \langle 2n+\nu+\mu+1 \rangle \\
=& \{n^* = (-\nu-\mu-1)/2\} \\
=& 2^{\mu-1} a^{-\mu-1} \Gamma\left(\frac{1+\mu+\nu}{2}\right) \Gamma\left(\frac{1+\mu-\nu}{2}\right)
\end{aligned}$$

Each of the above values matches the expected solution, but Rule E_3 adds these values, producing an incorrectly doubled solution. There is no concern about convergence since the bracket series were of Index 0. Other identities exhibiting this issue include 6.511.12, 6.512.9, and 6.514.8. This issue has been observed only in integrals of $K_\nu(x)$. Until it is observed elsewhere, no solution will be proposed.

Identity ([7], 7.524.1)

$$\int_0^\infty e^{-\lambda x} {}_2F_1\left(a, b; \frac{1}{2}; -x^2\right) dx = \lambda^{a+b-1} S_{1-a-b, a-b}(\lambda) \quad [\mathrm{Re}\lambda > 0]$$

The right side requires the Lommel function $S_{\mu,\nu}(x)$ defined in Appendix A. The bracket series is

$$\sum_{n_1} \sum_{n_2} \frac{\sqrt{\pi}(-1)^{n_1+n_2} \lambda^{n_1} \Gamma(b+n_2)\Gamma(a+n_2)}{\Gamma(n_2+1/2)\Gamma(n_2+1)\Gamma(n_1+1)\Gamma(a)\Gamma(b)} \langle n_1 + 2n_2 + 1 \rangle$$

Case 1: With n_1 free, the resulting series involves the other Lommel function, $s_{\mu,\nu}$:

$$\begin{aligned} S_1 =& \sum_{n_1} \frac{\sqrt{\pi}(-1)^{n_1}\lambda^{n_1}\Gamma(n_1/2+1/2)\Gamma(b-n_1/2-1/2)\Gamma(a-n_1/2-1/2)}{2\Gamma(n_1+1)\Gamma(a)\Gamma(b)\Gamma(-n_1/2)} \\ =& \frac{\lambda}{4(1-a)(1-b)}\,{}_1F_2\left(1;2-a,2-b,-\frac{\lambda^2}{4}\right) \\ =& \lambda^{a+b+1}s_{1-a-b,a-b}(\lambda) \end{aligned}$$

Case 2: With n_2 free, the resulting series diverges and is discarded:

$$S_2 = \sum_{n_2} \frac{\sqrt{\pi}(-1)^{n_2}\lambda^{-2n_2-1}\Gamma(2n_2+1)\Gamma(b+n_2)\Gamma(a+n_2)}{\Gamma(n_2+1/2)\Gamma(n_2+1)\Gamma(a)\Gamma(b)}$$

However, the entry in the table uses the other Lommel function $S_{\mu,\nu}$ rather than $s_{\mu,\nu}$. Since $S_{\mu,\nu}$ can be expressed in terms of $s_{\mu,\nu}$ plus another term, it appears that the method of brackets is missing this other term. Mathematica also returns this additional term.

4.10.2 No result

In the following examples, the method of brackets produces no solution due to the discarding of all divergent and imaginary series in the application of Rule E_3.

Identity ([7], 3.327)

$$\int_0^\infty \exp(-ae^{nx})\,dx = -\frac{1}{n}\mathrm{Ei}(-a) \quad [n>1, \mathrm{Re}\,a>0, a\neq 0]$$

Without any choice of in order of expansion, the integrand has the following series representation:

$$\sum_{n_1=0}^{\infty}\frac{(-ae^{nx})^{n_1}}{n_1!} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{(-a^{n_1})(nxn_1)^{n_2}}{n_2!n_1!}$$

The bracket series is of Index 1:

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\phi_{1,2}a^{n_1}(-nn_1)^{n_2}\langle n_2+1\rangle$$

By the vanishing of the bracket, n_2 must be fixed at $n_2^*=-1$ and n_1 is free. By Rule E_1, the value assigned to the integral is

$$\sum_{n_1=0}^{\infty}\phi_{n_1}a^{n_1}(-nn_1)^{-1}\Gamma(1)=-\frac{1}{n}\sum_{n_1=0}^{\infty}\frac{(-a)^{n_1}}{n_1!n_1}$$

With an infinite term corresponding to $n_1=0$, this result is discarded and the method of brackets produces no solution.

However, the value given in [7] is

$$-\frac{1}{n}\mathrm{Ei}(-a)=-\frac{1}{n}\left[\gamma+\ln x+\sum_{k=1}^{\infty}\frac{(-a)^k}{kk!}\right].$$

With so much similarity in the series, this example deserves further investigation.

Identity ([7], 3.352.4)

$$\int_0^{\infty}\frac{e^{-\mu x}\,dx}{x+\beta}=-e^{\beta\mu}\mathrm{Ei}(-\mu\beta)\quad[|\arg\beta|<\pi,\mathrm{Re}\mu>0]$$

The bracket series is

$$\sum_{n_1,n_2,n_3\geq 0}\frac{(-1)^{n_1+n_2+n_3}b^{n_1}\mu^{n_3}}{\Gamma(n_3+1)\Gamma(n_2+1)\Gamma(n_1+1)}\langle n_1+n_2+1\rangle\langle n_2+n_3+1\rangle$$

There are three choices of free summation indices, but none of the resulting series converge. No solution is possible by the method of brackets.

Identities ([7], 7.522.2-4)

$$\int_0^\infty e^{-bx}x^{a-1}F(\frac{1}{2}+\nu,\frac{1}{2}-\nu;a;-\frac{x}{2})\,dx = 2^a e^b \frac{1}{\sqrt{\pi}}\Gamma(a)(2b)^{\frac{1}{2}-a}K_\nu(b) \quad [\operatorname{Re} a > 0, \operatorname{Re} b > 0]$$

As mentioned in Section 4.5.2, the method of brackets produces only series that diverge or are zero for identities 7.522.2-4. These examples suggested that zeros should be discarded as well. Otherwise a solution of zero would be returned. However, now that all resulting series are discarded, no solution is given by the method of brackets.

4.10.3 Ramanujan's Master Theorem not applicable

The examples in this section should not be attempted because Ramanujan's Master Theorem 3 is not applicable due to violations of the growth condition. However, the method of brackets produces solutions that are similar to the expected result.

Identity ([7], 3.511.1)

$$\int_0^\infty \frac{dx}{\cosh ax} = \frac{\pi}{2a} \quad [a > 0]$$

The method of brackets should not have been attempted due to singularities at $i\pi/2+\pi ki$. However, the correct value does appear from one of the resulting series. The integrand is rewritten as

$$\begin{aligned}\frac{2}{e^{ax}+e^{-ax}} =&2\sum_{n_1}\sum_{n_2}\phi_{1,2}e^{-ax(n_2-n_1)}\langle 1+n_1+n_2\rangle \\ =&2\sum_{n_1}\sum_{n_2}\phi_{1,2}\left(\sum_{n_3}\phi_{n_3}a^{n_3}x^{n_3}(n_2-n_1)^{n_3}\right)\langle 1+n_1+n_2\rangle =\end{aligned}$$

The bracket series is

$$\sum_{n_1}\sum_{n_2}\sum_{n_3}\phi_{1,2,3}2a^{n_3}(n_2-n_1)^{n_3}\langle n_3+1\rangle\langle 1+n_1+n_2\rangle$$

Case 1: With n_2 free, $n_1^*=-n_2-1, n_3^*=-1$ and $|\det(A)|=1$

$$\sum_{n_2}\frac{2(-1)^{n_2}}{(2n_2+1)a}=\frac{\pi}{2a}$$

Case 2: With n_1 free, $n_3^*=-1$, $n_2^*=-n_1-1$, and $|\det(A)|=1$

$$\sum_{n_1}\frac{-2(-1)^{n_1}}{(2n_1+1)a}=-\frac{\pi}{2a}$$

Identity ([7], 4.531.8)

$$\int_0^\infty \frac{x\arctan x}{1+x^4}\,dx=\frac{\pi^2}{16}$$

The method of brackets should not have been attempted due to the $\pm i$ branch points. However, the correct value does appear from one of the resulting series.

The bracket series is of Index 1:

$$\sum_{n_1}\sum_{n_2}\sum_{n_3} \frac{(-1)^{n_1+n_2+n_3}\Gamma(n_1+3/2)^{n_2+n_3}\Gamma(n_1+1/2)}{2}\Gamma(n_3+1)\Gamma(n_2+1)$$
$$\times \langle n_2+n_3+1\rangle \langle 2n_1+4n_2+3\rangle$$

The resulting series are

- $\displaystyle\sum_{n_1} \frac{(-1)^{n_1}\Gamma(-n_1/2+1/4)\Gamma(n_1/2+3/4)\Gamma(n_1+1/2)}{8\Gamma(n_1+3/2)} = \frac{\pi^2}{8}$
- $\displaystyle\sum_{n_3} \frac{(-1)^{n_3}\Gamma(-2n_3-1/2)\Gamma(2n_3+1)\Gamma(2n_3+3/2)}{4\Gamma(2n_3+2)} = -\frac{\pi^2}{16}$
- $\displaystyle\sum_{n_2} \frac{(-1)^{n_2}\Gamma(-2n_2-1)\Gamma(-2n_2-1/2)\Gamma(2n_2+3/2)}{4\Gamma(-2n_2)} = \frac{\pi^2}{16}$

4.11 Non-summable Examples

The examples in this section produce sums (usually multi-sums) which cannot be simplified by Mathematica 7. With series representations returned, these examples cannot generally be checked against table entries.

However, if the summand is hypergeometric in the summation indices, Wegschaider's algorithm [7] may be able to compute recurrences for the summand and the sum, which could be used for check since the right side of identities are available in the table [7].

4.11.1 Identity ([7], 3.331.1)

$$\int_0^\infty \exp(-\beta e^{-x} - \mu x)\, dx = \beta^{-\mu}\gamma(\mu,\beta)[\mathrm{Re}\mu > 0]$$

The bracket series is of Index 2:

$$\sum_{n_1}\sum_{n_2}\sum_{n_3}\sum_{n_4}\phi_{1,2,3,4}\frac{(n_1-n_2)^{n_4}b^{n_3}(-\mu)^{n_2}}{\Gamma(-n_1)}\langle n_2+n_4+1\rangle\langle -n_1+n_2+n_3\rangle$$

The result is 5 double sums which Mathematica 7 could not evaluate.

4.11.2 Identity ([7], 6.522.4)

$$\int_0^\infty xJ_0(ax)K_0(bx)J_0(cx)\,dx = (a^4+b^4+c^4-2a^2c^2+2a^2b^2+2b^2c^2)^{-1/2}$$

$$[\operatorname{Re} b > |\operatorname{Im} a|, c > 0]$$

With the integral representation (4.10) of the modified Bessel function, the integral identity is now written as a double integral:

$$\int_0^\infty\int_0^\infty \frac{\cos(bxt)}{\sqrt{t^2+1}}xJ_0(ax)J_0(cx)\,dt\,dx = (a^4+b^4+c^4-2a^2c^2+2a^2b^2+2b^2c^2)^{-1/2}$$

$$[\operatorname{Re} b > |\operatorname{Im} a|, c > 0]$$

The bracket series is of Index 2:

$$\sum_{n_1}\sum_{n_2}\sum_{n_3}\sum_{n_4}\sum_{n_5}\phi_{1,2,3,4,5}\frac{4^{-n_1-n_2-n_3}(a^2)^{n_2}(b^2)^{n_1}c^{2n_3}}{\Gamma(n_3+1)\Gamma(n_2+1)\Gamma(n_1+1/2)}$$
$$\times\langle 2n_1+2n_4+1\rangle\langle 2n_1+2n_2+2n_3+2\rangle\langle n_4+n_5+1/2\rangle$$

The seven double sums produced by various choices of free/fixed variables could not all be simplified by Mathematica 7.

4.11.3 Identity ([7], 6.697.2)

$$\int_0^\infty \frac{\sin(x+t)}{x+t} J_0(t)\,dt = \frac{\pi}{2} J_0(x) \quad [x>0]$$

The bracket series is of Index 2:

$$\sum_{n_1}\sum_{n_2}\sum_{n_3}\sum_{n_4} \phi_{1,2,3,4} \frac{1/2\sqrt{\pi}4^{-n_1-n_2}x^{n_4}}{\Gamma(n_2+3/2)\Gamma(n_1+1)\Gamma(-2n_2)} \langle 2n_1+n_3+1\rangle \langle -2n_2+n_3+n_4\rangle$$

The result is 5 double sums that could not all be evaluated by Mathematica 7.

4.12 Open Questions

The examples in Section 4.10 should be studied further. Also, identities 7.522.2-4 should be should be investigates further since the method of brackets produced no solution for these.

Another open question is whether any result will include complex series. So far, the only observed non-real series have been purely imaginary. If complex series are found, they will be studied to determine whether the whole series or just the imaginary part should be discarded.

As discussed in Section 4.10, there is an issue of doubling when the series representation is used for the modified Bessel function $K_\nu(x)$ that does not occur when the integral representation is used. Before a solution can be proposed, other similar examples must be found.

The other doubling issue occurred when the Index was not minimized by the choice of ordering of series expansion and was found to be the result of evaluating series at points on boundary of the region of convergence. Grouping and factoring also affected the Index, but more investigation is needed to understand why no solution was produced for representations that were not fully grouped or factored.

4.13 Future Work

In order for this implementation of the method of brackets to be complete for a user without knowledge of the method, it will need a number of enhancements. First, the growth condition in Ramanujan's Master Theorem must be checked automatically to determine applicability of the method. So that the method may apply to general definite integrals, transfomations from a generic interval $[a, b]$ to an integral over $[0, \infty)$ must be investigated with careful consideration of singularities.

The implementation must be extended to utilize integral representations of functions such as $K_\nu(x)$ automatically whenever appropriate without user analysis.

As shown in [5, 6], it is often useful to introduce parameters integrand so that Rule P_2 may apply and/or so that the resulting sums will converge. At the end, the limit should be taken as these parameters tend to 0 or 1 in most cases. These parameters should be inserted automatically when the method produces no solution. Some common insertions include a factor of x^{s-1} (as in the Mellin transform) with $s \to 0$. In applications of Rule P_2, general coefficients $A, B, \ldots$ may be inserted where the coefficient is otherwise just 1. If the exponent in Rule P_2 is a non-negative integer α, change it to $\alpha + \varepsilon$ and let $\varepsilon \to 0$ as the final step.

When the method produces multi-sums that were not summable in Mathematica, the application of Wegschaider's algorithm [19] should be considered. Where possible, the `MultiSum` software will determine recurrences for the summands as well as the sums themselves. In situations with differing recurrences, in order to compute the recurrence for the sum from those of the terms, the command `REPlus` from the package `GeneratingFunctions` [10] can be applied since the recurrences are holonomic recurrences [22]. Since the recurrences are not easily solved in general, this technique would be most applicable for verifying table entries where the right side of an identity could be checked to satisfy the recurrence.

Appendix A

Index of Special Functions

1. Gamma function:

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t}\, dt \quad [\operatorname{Re} z > 0]$$

2. Incomplete Gamma function

$$\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t}\, dt$$

3. Beta function

$$B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1}\, dx \quad [\operatorname{Re} a > 0,\ \operatorname{Re} b > 0]$$

4. Incomplete Beta function

$$B_x(a, b) = \int_0^x t^{a-1} (1-t)^{b-1}\, dt$$

5. Regularized Incomplete Beta function

$$I_x(a, b) = \frac{B_x(a, b)}{B(a, b)}$$

6. Error function

$$\text{erf } (z) = \Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt$$

7. Exponential integral

$$\text{Ei}(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t} dt \quad [x < 0]$$

8. Bessel function of the first kind

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+\nu+1)k!} \left(\frac{z}{2}\right)^{2k+\nu}$$

9. Modified Bessel function of the first kind

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+\nu+1)k!} \left(\frac{z}{2}\right)^{2k+\nu}$$

10. Modified Bessel function of the second kind

$$K_\nu(z) = \frac{\pi}{2\sin(\pi\nu)} \left[I_{-\nu}(z) - I_\nu(z)\right]$$

11. Legendre polynomial

$$P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n(1-x^2)^n}{d\,x^n}$$

12. Chebyshev polynomial of the first kind

$$T_n(x) = \frac{(-1)^n \sqrt{\pi}\sqrt{1-x^2}}{2^n(n-1/2)!} \frac{d^n}{d\,x^n} \left[(1-x^2)^{n-1/2}\right]$$

13. Chebyshev polynomial of the second kind

$$U_n(x) = \frac{(-1)^n (n+1)\sqrt{\pi}}{2^{n+1}(n+1/2)!\sqrt{1-x^2}} \frac{d^n}{d\,x^n}\left[(1-x^2)^{n+1/2}\right]$$

14. Associated Laguerre polynomial

$$L_n^{\lambda}(x) = \frac{e^x x^{-\lambda}}{n!} \frac{d^n}{d\,x^n}\left(e^{-x} x^{n+\lambda}\right)$$

15. Hermite polynomial

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{d\,x^n} e^{-x^2}$$

16. Lommel functions

$$\begin{aligned} s_{\mu,\nu}(z) =& \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\,{}_1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; -\frac{z^2}{4}\right) \\ S_{\mu,\nu}(z) =& s_{\mu,\nu}(z) + \frac{2^{\mu-1}\Gamma\left(\frac{\mu-\nu+1}{2}\right)\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\sin(\nu\pi)} \\ & \times \left(\cos\left[\frac{\pi}{2}(\mu-\nu)\right] J_{-\nu}(z) - \cos\left[\frac{\pi}{2}(\mu+\nu)\right] J_{\nu}(z)\right) \end{aligned}$$

Appendix B

User Manual

B.1 Setup

Since this code takes advantage of parallel processing in Mathematica, the user should set Mathematica preferences so that parallel kernels are launched at startup. The file `PSMS.m` should be located within Mathematica's path.

In Sage, the user must first load the code. Attaching (or loading) brackets.sage will also attach the other necessary files.

```
sage:  attach brackets.sage
```

Many common parameters such as a, b, m, n are declared symbolically in the file `brackets.sage`. If the user wishes to define other parameters as symbolic variables, the `var` command in Sage must be used.

```
sage:  var('eta')
eta
```

B.2 Commands

Input:

The main call is `method_of_brackets` with the following input

- `integrand`
- `intvars` as a list

```
sage: method_of_brackets(INTEGRAND,INTVARS)
```

Output:

1. Intermediate Output:
   (a) the bracket-series,
   (b) the matrix A and its row echelon form,
   (c) in cases where the bracket series has $Index \geq 1$, the solution for the n_i^* values and the sum over each choice of free summation indices.
2. Final Output:
   (a) If the $Index = 0$, the result appears alone in a list.
   (b) If the $Index \geq 1$, the list of sums and their simplifications from Mathematica are returned, but these are not added in matching convergence regions.

If a bracket series has been computed by hand, the user may use the commands `eval_bracket_series` and `make_sum` once any summation indices are defined. The function `make_sum` expects two arguments: the summand and a list of summation indices. The command `eval_bracket_series` will expects a SUM, the output of `make_sum`.

```
sage: var('n1')
n1
sage:  eval_bracket_series(make_sum((-1)^n1*bracket(2*n1+1)/gamma(n1↩
    +1),[n1]))
```

B.3 Examples

B.3.1 Single Integral of Index 0

If the user wishes to evaluate the integral

$$\int_0^\infty e^{-x^2}\,dx,$$

the user makes the call

```
sage: method_of_brackets_zero_inf(e^(-x^2),[x])
```

The output is

```
bracket-series:  SUM((-1)^n1*bracket(2*n1 + 1)/gamma(n1 + 1), n1)
Index= 0
brackets = [2*n1 + 1]
A =
[2]
det(A) =  2
only one equation [2*n1 + 1 == 0] ; var =  n1
solution:   {n1: -1/2}
result =  1/2*sqrt(pi)
[1/2*sqrt(pi)]
```

Since the Index=0, the last line output is the solution ($\sqrt{\pi}/2$).

B.3.2 Double Integral of Index 0

The integral I_2 from [6] is evaluated here with the best representation chosen automatically:

$$\int_0^\infty \int_0^\infty \frac{dx\,dy}{xy(x+1/x+y+1/y)^2} = \frac{1}{2}$$

```
sage: method_of_brackets_zero_inf(1/(x*y*(x+1/x+y+1/y)^2),[x,y])
```

```
bracket-series:  SUM(SUM(SUM(SUM((-1)^(n1 + n2 + n3 + n4)*bracket(n1 +↩
    n4 + 2)*bracket(n1 + n3 + 2)*bracket(n3 + n4 + 2)*bracket(n1 + n2 ↩
   + 2)/(gamma(n4 + 1)*gamma(n3 + 1)*gamma(n2 + 1)*gamma(n1 + 1)), n4)↩
    , n3), n2), n1)
4 indices,  4 brackets
Index= 0
brackets = [n1 + n4 + 2, n1 + n3 + 2, n3 + n4 + 2, n1 + n2 + 2]
A =
[1 0 0 1]
[1 0 1 0]
[0 0 1 1]
[1 1 0 0]
det(A) =  2
solution:   {n4: -1, n1: -1, n2: -1, n3: -1}
result =  1/2
[1/2]
```

Since the Index=0, the last line output is the solution (1/2).

B.3.3 Single Integral of Index 1

Identity ([7], 6.554.4) analyzed in Section 4.3 will be presented here as one example of a integral with Index 1.

$$\int_0^\infty xJ_0(xy)\frac{dx}{(a^2+x^2)^{3/2}} = \frac{e^{-ay}}{a} \quad [y>0, \operatorname{Re} a > 0]$$

Since code in the `brackets.sage` file contains a hypergeometric definition for the function `besselJ`, the input is

```
sage: method_of_brackets_zero_inf(x*besselJ(0,x*y)*(a^2+x^2)^(-3/2),[x↩
    ])
```

The output produced is

```
bracket-series:  SUM(SUM(SUM(2*(-1)^(n1 + n2 + n3)*(y^2)^n1*abs(a)^(2*↩
    n2)*bracket(2*n1 + 2*n3 + 2)*bracket(n2 + n3 + 3/2)/(sqrt(pi)*4^n1*↩
    gamma(n3 + 1)*gamma(n2 + 1)*gamma(n1 + 1)^2), n3), n2), n1)
Index= 1
brackets = [2*n1 + 2*n3 + 2, n2 + n3 + 3/2]
A =
[2 0 2]
[0 1 1]
A's echelon_form =
[1 0 1]
[0 1 1]


fixed vars =  [n1, n2]  and free vars =  [n3]
det(A) =  2
solution:   {n1: -n3 - 1, n2: -n3 - 3/2}
result =  SUM((-1)^n3*2^(2*n3 + 2)*abs(a)^(-2*n3 - 3)*abs(y)^(-2*n3 - ↩
    2)*gamma(n3 + 3/2)/(sqrt(pi)*gamma(-n3)), n3)


fixed vars =  [n1, n3]  and free vars =  [n2]
det(A) =  2
solution:   {n1: n2 + 1/2, n3: -n2 - 3/2}
result =  SUM((-1)^n2*2^(-2*n2 - 1)*abs(a)^(2*n2)*abs(y)^(2*n2 + 1)*↩
    gamma(-n2 - 1/2)/(sqrt(pi)*gamma(n2 + 1)), n2)


fixed vars =  [n3, n2]  and free vars =  [n1]
det(A) =  2
solution:   {n3: -n1 - 1, n2: n1 - 1/2}
result =  SUM((-1)^n1*y^(2*n1)*abs(a)^(2*n1 - 1)*gamma(-n1 + 1/2)/(↩
    sqrt(pi)*2^(2*n1)*gamma(n1 + 1)), n1)


Answer list ( 3 ):   (convergence NOT yet determined)
[SUM((-1)^n3*2^(2*n3 + 2)*abs(a)^(-2*n3 - 3)*abs(y)^(-2*n3 - 2)*gamma(↩
    n3 + 3/2)/(sqrt(pi)*gamma(-n3)), n3), SUM((-1)^n2*2^(-2*n2 - 1)*abs↩
```

```
    (a)^(2*n2)*abs(y)^(2*n2 + 1)*gamma(-n2 - 1/2)/(sqrt(pi)*gamma(n2 + ↩
    1)), n2), SUM((-1)^n1*y^(2*n1)*abs(a)^(2*n1 - 1)*gamma(-n1 + 1/2)/(↩
    sqrt(pi)*2^(2*n1)*gamma(n1 + 1)), n1)]

MMA(anslist) =
{{Infinity, False}, {-(Sinh[Abs[a*y]]/Abs[a]), True}, {Cosh[y*Abs[a]]/↩
    Abs[a], True}}
```

Since the Index=1, the result is the sum of results with the same convergence conditions. The first series is divergent and discarded. The other two have no convergence conditions and may be summed and simplified to find the expected result.

Appendix C

Sage Code

C.1 brackets.sage

```
import itertools
from sage.symbolic.function_factory import function_factory
from sage.symbolic.relation import solve

attach make_bracket_series.sage
attach eval_bracket_series.sage


# set up variables for common parameters:
var('a b c d g k l m n p q r s t u v w y z A B C D E F nu mu eps dummy↩
    ')
var('a1 a2 a3 b1 b2 b3')

# generally have n and m parameters as positive integers
assume(n,'integer')
assume(n >= 0)
assume(m,'integer')
assume(m >= 0)

# function names (without evaluation)
SUM = function_factory('SUM',2)
INT = function_factory('INT',2)
ruleP2 = function_factory('ruleP2',2)
pochhammer = function_factory('pochhammer',2)
bracket = function_factory('bracket',1)
fresnelc=function('fresnelc',x)
fresnels=function('fresnels',x)
hypergeometric0F0=function('hypergeometric0F0',x)
hypergeometric0F1=function('hypergeometric0F1',a,x)
```

```
hypergeometric0F2=function('hypergeometric0F2',a,b,x)
hypergeometric0F3=function('hypergeometric0F3',a,b,c,x)
hypergeometric1F1=function('hypergeometric1F1',a,c,x)
hypergeometric1F0=function('hypergeometric1F0',a,x)
hypergeometric1F2=function('hypergeometric1F2',a,b,c,x)
hypergeometric2F0=function('hypergeometric2F0',a1,a2,x)
hypergeometric2F1=function('hypergeometric2F1',a,b,c,x)
hypergeometric2F2=function('hypergeometric2F2',a,b,c,d,x)
hypergeometric3F2=function('hypergeometric3F2',a1,a2,a3,b1,b2,x)
hypergeometric3F0=function('hypergeometric3F0',a1,a2,a3,x)


# definitions of common special functions in terms of other functions:
# Bessel J function
besselJ(v,x) = (x/2)^v/(gamma(v+1))*hypergeometric0F1(v+1,-(x/2)^2)
# Associated Legendre Fn of 2nd kind :
assocLegQ(l,mu,x) =  sqrt(pi)*gamma(l+mu+1)*(1-x^2)^(mu/2)/(2^(l+1)*↩
    gamma(l+3/2)*z^(l+mu+1))*hypergeometric2F1((l+mu+1)/2,(l+mu+2)/2,l↩
    +3/2,1/x^2)
# lower incomplete gamma function:
lower_inc_gamma(svar,zvar) = zvar^svar*hypergeometric1F1(svar,svar+1,-↩
    zvar)/svar
#complete elliptic integral of the first kind:
elliptic_K(k) = (pi/2)*hypergeometric2F1(1/2,1/2,1,k^2)
#complete elliptic integral of the second kind:
elliptic_E(k) = (pi/2)*hypergeometric2F1(1/2,-1/2,1,k^2)
beta(u,v) = gamma(u)*gamma(v)/gamma(u+v)
# Hermite polynomial:  odd or even or either
hermite_odd(n,x) = (-1)^n*gamma(2*n+2)*2*x*hypergeometric1F1(-n,3/2,x↩
    ^2)/gamma(n+1)
hermite_even(n,x) = (-1)^n*gamma(2*n+1)*hypergeometric1F1(-n,1/2,x^2)/↩
    gamma(n+1)
hermite(n,x) = 2^n*sqrt(pi)*(hypergeometric1F1(-n/2,1/2,x^2)/gamma((1-↩
    n)/2)-2*x*hypergeometric1F1((1-n)/2,3/2,x^2)/gamma(-n/2))
# Laguerre polynomial:
laguerre_l(n,x) = hypergeometric1F1(-n,1,x)
assoc_laguerre(n,a,x) = gamma(n+a+1)*hypergeometric1F1(-n,a+1,x)/gamma↩
    (a+1)/gamma(n+1)


# Definition of indicator(n)=phi_n in Method of Brackets
indicator(n) = (-1)^n/gamma(n+1)


def method_of_brackets_zero_inf(integrand,xvarlist):
    # MAIN PROCEDURE
    # INPUT:   integrand expression
    #          xvarlist—list of variables of integration
    # OUTPUT:  If Index = 0, return the solution
    #          If Index > 0, return a Mathematica-simplified list
    #                          with convergence conditions
    nviter = iter(range(100))
```

```
    nviter.next()
    # assume each var of integration is positive
    #  (for simplification)
    #  since integration is over positive half of real line.
    for xvar in xvarlist:
        assume(xvar > 0)
    #  convert all factors to series representations
    final_sumd = write_as_series(integrand,xvarlist,nviter)[1]
    nvars = get_nvars(final_sumd)
    summand = factor(innermost_summand(final_sumd))
    # make/find brackets:
    #   make from definition--by vars of integration
    blist1 = [get_brackets(summand,xvar) for xvar in xvarlist]
    #   find brackets created by Rule P2:
    blist2 = [f2.operands()[0] for f2 in summand.operands() if str(f2.↩
       operator())=='bracket']
    bseries_sumd = summand*prod([bracket(bexp) for bexp in blist1])/↩
       prod([(bx[1])^(bx[0]-1) for bx in zip(blist1,xvarlist)])
    bracket_series = make_sum(bseries_sumd,nvars)
    print "\nbracket-series: ", bracket_series
    if any([bseries_sumd.has(xvar) for xvar in xvarlist]):
        print bracket_series, "is NOT a bracket-series!!"
        anslist_final = ['Cannot solve by Method of Brackets']
        return False
    else:
        anslist_final = eval_bracket_series(bracket_series)
    return anslist_final


def get_brackets(summand,xvar):
    # given a summand and a variable of integration,
    # return the bracket in summand from xvar (by def of bracket):
    summand=summand.simplify_exp()
    num_brackets = get_brackets_num(numerator(summand),xvar)
    den_brackets = get_brackets_num(denominator(summand),xvar)
    return num_brackets - den_brackets + 1


def get_brackets_num(summand,xvar):
    # find the brackets resulting from xvar in the numerator of ↩
       summand
    w0 = SR.wild(0); w1 = SR.wild(1); w2 = SR.wild(2);
    w3 = SR.wild(3); w4 = SR.wild(4)
    dblpow = [exp1.operands()[1]*(exp1.operands()[0]).operands()[1] ↩
       for exp1 in summand.find((w2*xvar^(w0))^(w1))]
    sglpow=[exp1.operands()[1] for exp1 in summand.operands() if exp1.↩
       operator()==sage.symbolic.expression.operator.pow and exp1.↩
       operands()[0]==xvar]
    nopow = [len(summand.find(w3*xvar*w4))]
    bracketa = reduce(lambda x,y: x+y, union(dblpow,union(sglpow,nopow↩
       ))) + 1
    return bracketa
```

```
def get_nvars(sum1):
    # read off the list of summation indices from a SUM (sum1)
    if str(sum1.operator())=='SUM':
        return [sum1.operands()[1]]+get_nvars(sum1.operands()[0])
    else:
        return []


def make_sum(expr,nlist):
    # make a (multi-)sum with summand=expr
    #     over summation indices in nlist
    nlist.sort(lambda x, y: cmp(str(x),str(y)))
    if nlist==[]:
        return expr
    else:
        return SUM(make_sum(expr,nlist[1:]),nlist[0])
```

C.2 make_bracket_series.sage

```
def write_as_series(integrand,xvarlist,nviter):
    # INPUT:   1.   integrand
    #          2.   xvarlist:  variables of integration
    #          3.   "n_i" index iterator
    # OUTPUT: list of two items:
    #     (a) True/False - the integrand could be written as a series
    #     (b) if (a): the series expansion; else: 0
    # PROCEDURE:
    # 1. simplify, simplify_exp, collect, group, etc...
    # 2. rank factors that contain a variable of integration
    # 3. expand first-ranked to a series
    # 4. push all others inside this series
    # 5. recurse until no more can be expanded to series
    factors = flatten([rec_rewrite_factor(fac,xvarlist) for fac in ↩
        get_factor_list(integrand,xvarlist)])
    constantfactors = [fac for fac in factors if not any([fac.has(xvar↩
        ) for xvar in xvarlist])]
    varfactors = get_factor_list(simplify_and_group(prod([fac for fac ↩
        in factors if any([fac.has(xvar) for xvar in xvarlist])]), ↩
        xvarlist),xvarlist)
    w0 = SR.wild(0)
    intvarfactors = [fac for fac in varfactors if any([(fac.match(↩
        intvar^w0) or fac==intvar) for intvar in xvarlist])]
    functionfactors = [fac for fac in varfactors if not fac in ↩
        intvarfactors]
    noncompvarfactors = [fac for fac in functionfactors if not ↩
        is_composition(fac)]
    compvarfactors = [fac for fac in functionfactors if not fac in ↩
        noncompvarfactors]
    # rank factors for order of series expansion
    ranked_compvar_factors = rankfactors(compvarfactors)
    ranked_factors = ranked_compvar_factors + noncompvarfactors + ↩
        intvarfactors + constantfactors
    # Go through ranked factors one at a time,
    #   report first that can be converted to a series
    ffse = find_first_series_expansion(ranked_compvar_factors,xvarlist↩
        ,nviter)
    if ffse==[0,0]:  #no factor could be converted to a series
        return [False,0]
    else:
        series1 = ffse[1]
        nvars1 = get_nvars(series1)
        # do #4:  push all others inside this series
        #   replace that one factor with its series
        ranked_factors.remove(ffse[0])
        #   push other factors inside to make summand
        ims = simplify_and_group(innermost_summand(series1),xvarlist)
```

```
        summand = prod(ranked_factors)*ims
        # do #5:  repeat #1 with innermost_summand
        wias2 = write_as_series(summand,xvarlist,nviter)
        series2=wias2[1]
        if wias2 == [False,0]:
            return [True,make_sum(factor(summand),nvars1)]
        else:
            nvars2 = get_nvars(series2)+nvars1
            return [True, make_sum(simplify_and_group(↩
               innermost_summand(series2),xvarlist), nvars2)]

def simplify_and_group(intg,xvarlist):
    # INPUT:  expression and a list of vars of integration
    # OUTPUT:  chosen/best representation of expression
    # Create a list of different representations
    #   choose one that meets the goal of minimizing
    #  ( -[# polynomial^exp factors] + [# terms in these polynomial ↩
       facs])
    # 1.  find all polynomial factors, put each over common ↩
       denominator
    # 2.  for each polynomial factor, find all groupings, try ↩
       factoring each
    factors =  get_factor_list(intg,xvarlist)
    # make sure each polynomial factor is put over common denominator
    intg2 = prod([together(fac) for fac in factors])
    factors2 = get_factor_list(intg2,xvarlist)
    constantfactors2 = [fac for fac in factors2 if not any([fac.has(↩
       xvar) for xvar in xvarlist])]
    varfactors2 =[fac for fac in factors2 if any([fac.has(xvar) for ↩
       xvar in xvarlist])]
    w0 = SR.wild(0)
    intvarfactors2 = [fac for fac in varfactors2 if any([(fac.match(↩
       intvar^w0) or fac==intvar) for intvar in xvarlist])]
    functionfactors2 = [fac for fac in varfactors2 if not fac in ↩
       intvarfactors2]
    noncompvarfactors2 = [fac for fac in functionfactors2 if not ↩
       is_composition(fac)]
    compvarfactors2 = [fac for fac in functionfactors2 if not fac in ↩
       noncompvarfactors2]
    cvf3 = [grouplist(cvf) for cvf in compvarfactors2]
    #make all combinations, selecting one rewrite for each factor:
    combs = [simplify(prod(comb)) for comb in make_combs(cvf3)]
    # then select the one with minimal score:
    bestrep = select_best_rep(combs,xvarlist)
    return bestrep*prod(constantfactors2)*prod(noncompvarfactors2)*↩
       prod(intvarfactors2)


def select_best_rep(combs,xvarlist):
    # INPUT:  list of representations and list of integraion variables
    # OUTPUT: the selected best representation
    # compute count of each representation
```

```
    #     ( - [# polynomial factors + #terms in these polynomial facs)
    if combs==[]:
        return 1
    score_list = [scorerep(expr,xvarlist) for expr in combs]
    minscore = min(score_list)
    allscores = zip(combs,score_list)
    # then minimize:
    minscore_list = [expr for expr in allscores if expr[1] == minscore↩
        ]
    # return first one of the list with minimal score
    #   if more than one, it shouldn't matter which so choose first
    return minscore_list[0][0]


def scorerep(expr,xvarlist):
    # INPUT:  expression and a list of integration variables
    # OUTPUT:  a score for the expression:
    #    sum scores of all polynomials where score for each polynomial↩
        = #terms - 1
    if expr.operator()==sage.symbolic.expression.operator.add:
        terms = expr.operands()
        if any([expr.has(intvar) for intvar in xvarlist]):
            termcounts = [scorerep(term,xvarlist) for term in terms]
            count =  len(terms) - 1 + sum(termcounts)
            return count
        else:
            return 0
    elif expr.operator()==sage.symbolic.expression.operator.mul:
        factors = get_factor_list(expr,xvarlist)
        fac_scores= [scorerep(fac,xvarlist) for fac in factors]
        return sum(fac_scores)
    elif expr.operator()==sage.symbolic.expression.operator.pow:
        if (expr.operands()[0]).operator()==sage.symbolic.expression.↩
            operator.add:
            return scorerep(expr.operands()[0],xvarlist)
        else:
            return 0
    else:
        return 0


def make_combs(faclist):
    # make a list of all combinations of an expression,
    #   using various representations of each factor
    if faclist==[]:
        return []
    if len(faclist) == 1:
        return faclist[0]
    else:
        return [flatten([x,y]) for x in faclist[0]
                  for y in make_combs(faclist[1:])]
```

```
def grouplist(expr):
    #  if a polynomial^exponent kind of factor
    #      return a list of all possible groupings
    # INPUT:  expression and list of integration variables
    if expr.operator()==sage.symbolic.expression.operator.pow:
        if (expr.operands()[0]).operator()==sage.symbolic.expression.↩
            operator.add:
             exponent = expr.operands()[1]
             terms = (expr.operands()[0]).operands()
             mg1 = groupterms(terms)
             return [rep1^exponent for rep1 in mg1]
    # if no grouping possible, return list of original expression
    return [expr]


def groupterms(terms):
    # INPUT:   a list of terms
    # OUTPUT:  a list of all possible groupings of this set of terms
    num_terms=len(terms)
    total = sum(terms)
    fac_total=factor(total)
    if len(list(terms)) < 3:
        # if only 1 or 2 terms, consider just factoring as an option
        return list(Set([fac_total,total]))
    if not (str(fac_total)==str(total)):
        # if can factor totally, consider that as an option
        return list(Set([fac_total,total]))
    parts=Partitions(num_terms).list()[1:]
    # exclude those that end in 2 monomials:
    parts2 = [part for part in parts if not part[-2:]==[1, 1]]
    setparts = reduce(lambda x,y: x+y, [map(sage.combinat.↩
       set_partition.standard_form, SetPartitions(num_terms,part)) for↩
        part in parts2])
    grplist = [sum([factor(sum([terms[idx-1] for idx in part])) for ↩
       part in sp]) for sp in setparts]
    return list(Set(grplist))


def together(expr):
    # Given a polynomial, put its terms over a common denominator,
    #   expanding within:
    if expr.operator()==sage.symbolic.expression.operator.pow:
        if (expr.operands()[0]).operator()==sage.symbolic.expression.↩
            operator.add:
             # common denominator, expand
             polynom = expr.operands()[0]
             exponent = expr.operands()[1]
             expr = (polynom.simplify_rational(method='noexpand',map=↩
                 False))^exponent
    return expr
```

```
def rankfactors(factors):
    # input:  list of factors (all compositions)
    # output:  factors ranked by which series expansion
    #          should be attempted first
    if len(factors)<=1:
        return factors
    else:
        #sort by the "contains"
        factors.sort(lambda x,y: -contains(x,y))
        return factors


def is_composition(expr):
    # returns True if expr is a composition of functions
    return any([opd.operator() for opd in expr.operands()])


def contains(expr1,expr2):
    # input:   2 expressions
    # output:  a boolean defined by whether expr1 contains a
    #           "fuzzy match" of expr2 in composition
    if not is_composition(expr1):
        return False
    opds = expr1.operands()
    return any([bool(opd.operator()==expr2.operator()) for opd in opds↩
       ]) or any([contains(opd,expr2) for opd in opds])


def find_first_series_expansion(factors,xvarlist,nviter):
    # input:
    #  1.  list of factors (ranked in order of attempt for this proc)
    #  2.  list of vars of integration
    #  3.  "n_i" index iterator
    # output:   list of two items:
    #    (a) first factor that can be converted to a series
    #    (b) the series expansion of (a)
    if factors==[]:  # no factors left
        return [0,0]
    # try to expand first factors[0]
    expansion = write_expr_as_series(factors[0],xvarlist,nviter)
    if expansion!=factors[0]:  # expansion worked
        return [factors[0],expansion]
    else:  # recurse
        return find_first_series_expansion(factors[1:], xvarlist,↩
           nviter)



def write_expr_as_series(expr,xvarlist,nviter):
    # input:  expression, list of vars of integration,
    #           n-index iterator
```

```
    # output:  original expr, if it cannot be turned into a series
    #          otherwise, the series representation (one level only)
    possible_xvars = list(set(expr.variables()) & set(xvarlist))
    if possible_xvars == []:
        return expr
    else:
        xvar = possible_xvars[0]
        return series_lookup(expr,xvar,nviter,xvarlist)


def apply_ruleP2(expr,nviter):     # multinomial rule for series ↩
    expansion
    # Input:   expr:   to be converted to a series by Rule P2
    #          nviter: iterator for summation index
    # Output:  the series formed by Rule P2
    sumlist = (expand(expr.operands()[0])).operands()
    alpha1 = expr.operands()[1]
    nvars = create_nvars(nviter,len(sumlist))
    sumd = prod([indicator(nv) for nv in nvars])*simplify(prod([(ai↩
        [0]^ai[1]).simplify_exp() for ai in zip(sumlist,nvars)]))*↩
        bracket(-alpha1+sum(nvars))/gamma(-alpha1)
    return make_sum(sumd,nvars)


def rec_rewrite_factor(expr,xvarlist):
    rf1 = rewrite_factor2(expr,xvarlist)
    if rf1 != expr:
        factors1 = get_factor_list(rf1,xvarlist)
        rflist = [rec_rewrite_factor(fac,xvarlist) for fac in factors1↩
            ]
        return rflist
    else:
        return expr

def rewrite_factor2(expr,xvarlist):
    # rewrite specific types of expressions.
    w0 = SR.wild(0)
    if expr.operator()==sage.symbolic.expression.operator.pow:
        if (expr.operands()[0]).operator()==sage.symbolic.expression.↩
            operator.add:
            return expr
        # if expr is of the form a^n, with n a pos integer, return a ↩
            list of n copies of a
        alpha1 = expr.operands()[1]
        base1 = expr.operands()[0]
        if (bool(alpha1==floor(alpha1))) and alpha1 > 0 and not any([↩
            base1.is_polynomial(xvar) for xvar in xvarlist]):
            explist = []
            for i in range(alpha1):
                explist = explist+[expr.operands()[0]]
            return explist
        elif expr.match(1/cosh(w0)):
```

```
            m1 = expr.match(1/cosh(w0))
            return 2/(e^m1[w0]+e^(-m1[w0]))
        elif expr.match(1/sinh(w0)):
            m1 = expr.match(1/sinh(w0))
            return 2/(e^m1[w0]-e^(-m1[w0]))
        else:
            return expr
    elif str(expr.operator())=="log":  #only for |z|<1
        zvar = 1-expr.operands()[0]
        return zvar*hypergeometric2F1(1/2,1/2,3/2,zvar)
    elif expr.operator()==erf:
        return 2*expr.operands()[0]/sqrt(pi)*hypergeometric1F1↩
            (1/2,3/2,-(expr.operands()[0])^2)
    elif expr.operator() == sin:
        return expr.operands()[0]*hypergeometric0F1(3/2,-(expr.↩
            operands()[0])^2/4)
    elif expr.operator() == cos:
        return hypergeometric0F1(1/2,-(expr.operands()[0])^2/4)
    elif expr.operator() == cosh:
        zvar = expr.operands()[0]
        return hypergeometric0F1(1/2,(zvar/2)^2)
    elif expr.operator() == sinh:
        zvar = expr.operands()[0]
        return zvar*hypergeometric0F1(3/2,(zvar/2)^2)
    elif expr.operator() == tanh:
        zvar = expr.operands()[0]
        return sinh(zvar)/cosh(zvar)
    elif expr.operator()==jacobi_P:
        jn = expr.operands()[0]
        ja = expr.operands()[1]
        jb = expr.operands()[2]
        jx = expr.operands()[3]
        return gamma(ja+1+jn)/(gamma(jn+1)*gamma(ja+1))*↩
            hypergeometric2F1(-jn,1+ja+jb+jn,ja+1,(1-jx)/2)
    else:
        return expr


def get_factor_list(expr,xvarlist):
    # Given an expression, return a list of its factors
    if type(expr)==list:
        return expr
    elif expr.operator()==sage.symbolic.expression.operator.mul:
        factors = expr.operands()
    else:
        factors = [expr]
    if factors == [expr]:
        return factors
    else:
        return flatten([get_factor_list(fac,xvarlist) for fac in ↩
            factors])
```

```
def innermost_summand(ps):
    if type(ps)!=sage.symbolic.expression.Expression:
        return ps
    if ps.operator()==SUM:
        summand = ps.operands()[0]
        if summand.operator()==SUM:
            return innermost_summand(summand)
        else:
            return summand
    else:
        return ps


def series_lookup(expr,xvar,nviter,xvarlist):
    # INPUT:   expression,
    #          the variable for use in the series expansion,
    #          the summation index iterator
    #          and a list of all vars of integration
    # OUTPUT:  the (multi-)series representation of expr, if possible
    #                (only 1 level of expansion)
    #          if no expansion possible, the original expression
    w0 = SR.wild(0); w1 = SR.wild(1)
    if expr.parent()== QQ or expr.parent()==ZZ or (expr.parent()==SR ↩
       and len(expr.variables())==0):
        return expr
    elif expr.operator() == exp:
        return series_lookup(hypergeometric0F0(expr.operands()[0]),↩
            xvar,nviter,xvarlist)
    elif expr.operator()==sage.symbolic.expression.operator.pow:
        if (expr.operands()[0]).operator() == sage.symbolic.expression↩
            .operator.pow:
            # power of a power:    expr = newexpr^newpow
            newpow = expr.operands()[1]*(expr.operands()[0]).operands↩
                ()[1]
            newexpr = (expr.operands()[0]).operands()[0]
            return series_lookup(newexpr^newpow, xvar, nviter, ↩
                xvarlist)
        elif (expr.operands()[0]).operator() == sage.symbolic.↩
            expression.operator.add and any([expr.operands()[0].has(↩
            xvar) for xvar in xvarlist]):
            #  power of a sum"
            alpha1 = expr.operands()[1]    #exponent
            if type(alpha1.parent())==sage.rings.integer_ring.↩
                IntegerRing_class:
                if bool(num <=0):
                    # cannot do a positive integer power of a sum:
                    #    add +eps to exponent
                    alpha1 = alpha1 + eps
            # apply RULE P2:
            return apply_ruleP2(ruleP2(expr.operands()[0],alpha1),↩
                nviter)
```

```
        elif expr.operands()[1]==-1:
            return expr
        else:
            return expr
    elif str(expr.operator())[0:3]=='hyp':    # hypergeometric pFq
        pq = (str(expr.operator())[14:]).split("F")
        p = sage_eval(pq[0])
        q = sage_eval(pq[1])
        arglist = expr.operands()
        alist = []
        if p > 0:
            alist = arglist[0:p]
        blist = arglist[p:-1]
        nvar = create_nvars(nviter,1)[0]
        lvars = {str(xvar):xvar,str(nvar):nvar}
        for vbl in list(set(flatten([list(arg1.variables()) for arg1 ↩
            in arglist]))):
            lvars[str(vbl)] = vbl
        return SUM(sage_eval(get_hyp_term(alist,blist,arglist[-1],nvar↩
            ),locals=lvars).simplify(),nvar)
    elif expr.operator()==sage.symbolic.expression.operator.add:
        alpha1 = 1+eps  #exponent
        # power of 1 of a sum modified so that the exponent is 1+eps
        sumlist = expr.operands()
        nvars = create_nvars(nviter,len(sumlist))
        # make multiD SUM
        sumd_num = (-1)^sum([nv for nv in nvars])*simplify(prod([(ai↩
            [0]^ai[1]).simplify_exp() for ai in zip(sumlist,nvars)]))*↩
            bracket(-alpha1+sum(nvars))
        sumd_den = gamma(-alpha1)*prod([gamma(nv+1) for nv in nvars])
        sumd = sumd_num/sumd_den
        return make_sum(sumd,nvars)
    else:  # no series lookup defined for other expressions
        return expr


def create_nvars(nv,num):
    # INPUT:  iterator (nv) and number (num) of nvars to be created
    # OUTPUT:   a list of new summation indices.
    return [var('n' + str(nv.next())) for i in range(num)]


def get_hyp_term(alist,blist,xexpr,nvar):
    # INPUT:
    #   list of numerator params (alist),
    #   list of denominator params (blist),
    #   the x-variable (xexpr),
    #   and the summation index (nvar)
    # OUTPUT:  a string of hypergeometric term where the term is
    #             written using gammas (rather than pochhammers)
    return reducestr(lambda x,y: x+y, ['('+ str((gamma(a_i+nvar)/gamma↩
        (a_i))) + ')*' for a_i in alist]) + reducestr(lambda x,y: x+y, ↩
```

```
        ['('+ str((gamma(b_i)/gamma(b_i+nvar))) + ')*' for b_i in blist↩
        ])  + '(('+str(xexpr) + ')^(' + str(nvar) + '))/gamma(' + str(↩
        nvar) + '+1)'


def reducestr(function,lst):
    # reduce a list of strings
    return reduce(function,lst,'')
```

C.3 eval_bracket_series.sage

```
from sage.symbolic.relation import solve
from sage.interfaces.mathematica import mathematica


def eval_bracket_series(bseries):
    # INPUT a bracket series
    bseries_sumd = innermost_summand(bseries)
    nvars = get_nvars(bseries)
    # get brackets
    blist = [f2.operands()[0] for f2 in bseries_sumd.operands() if str↩
       (f2.operator())=='bracket']
    num_brackets = len(blist)
    eqns = blist
    print len(nvars), "indices, ", num_brackets, "brackets"
    idx = len(nvars) - num_brackets
    print "Index=", idx
    print "brackets =", eqns
    A = Matrix(SR,[[eqn.coeff(nv1) for nv1 in nvars] for eqn in eqns])
    print "A = \n", A
    summand = (bseries_sumd / prod([bracket(b2) for b2 in blist])).↩
       simplify()
    rankA = rank(A)
    if rankA == len(nvars):  # No free vars:
        ans = det_method(summand,nvars,[],blist,rankA)
        anslist_final = [ans]
    else:  # do have have some free vars
        # determine fixed and then free vars
        ech_form = A.echelon_form()
        print "A's echelon_form = \n", ech_form
        fixedvarlists = determine_fixed_vars(ech_form,[],nvars)
        fixed_free_vars = [[fixedlist,list(set(nvars).difference(↩
           fixedlist))] for fixedlist in fixedvarlists]
        fixed_free_vars.reverse()
        # make the appropriate sums over the free vars:
        anslist = [fixed_free_sum(ffvars,summand,blist,rankA) for ↩
           ffvars in fixed_free_vars]
        anslist = [sum1 for sum1 in anslist if sum1 != 0]
        print "\n\nAnswer list (", len(anslist), "):   (convergence ↩
           NOT yet determined)"
        print anslist, "\n"
        anslist_final = eval_anslist(anslist)
    return anslist_final


def eval_anslist(anslist):
    unknown_iter = iter(range(100))
    unknown_iter.next()
```

```
    num_nvars = len(get_nvars(anslist[0]))
    mmalist = eval_list_in_mma(anslist)
    print "MMA(anslist) = \n", mmalist


def determine_fixed_vars(matrixA,fixedvars,nvars):
    # choose which is fixed in first row--cannot be among those ↩
        already fixed
    rows = matrixA.rows()
    row = rows[0]
    possible_fixed = [nvars[pf] for pf in list(set(row.↩
        nonzero_positions()).difference(fixedvars)) if nvars[pf] not in↩
         fixedvars]
    if possible_fixed == []:
        return []
    elif len(rows) == 1:
        return [fixedvars+[pf] for pf in possible_fixed]
    else:       #recursively determine other fixed
        return flatten([determine_fixed_vars(matrixA[1:],fixedvars+[pf↩
            ],nvars) for pf in possible_fixed],max_level=1)


def fixed_free_sum(fixed_free_vars,summand2,bracket_list,rankA):
    #  Create the output over these free variables
    fixedvars = fixed_free_vars[0]
    freevars = fixed_free_vars[1]
    print "\n\nfixed vars = ", fixedvars, " and free vars = ", ↩
        freevars
    # can do det method now with only fixed vars
    fixed_sum = det_method(summand2,fixedvars,freevars,bracket_list,↩
        rankA)
    if fixed_sum==0: #  any sum of zeros is still zero
        ans = 0
    else:  # return SUM over free vars:
        ans = make_sum(fixed_sum,freevars)
    return ans


def eval_list_in_mma(anslist):
    # evaluate the list of output series in Mathematica
    lst1str = "{"+",".join(map(lambda ps: "{"+repr(((innermost_summand↩
        (ps)).simplify_exp())._mathematica_()).replace('\n','')+","+str↩
        (get_nvars(ps)).replace("[","{").replace("]","}")+"}",anslist))↩
        +"}"
    fullstring = "ToString[ParallelMultiSum["+lst1str+"],InputForm]"
    print "fullstring=", fullstring
    mathematica("<<PSMS.m")
    mma_out = mathematica(fullstring)
    return mma_out


def det_method(expr,fixedvars,freevars,bracket_list,rankA):
```

```
    eqns = [bracketa for bracketa in bracket_list]
    A = Matrix([[eqn.coeff(nv1) for nv1 in fixedvars] for eqn in eqns↩
        ])
    detA = A.determinant()
    print "det(A) = ", detA
    if rankA !=rank(A):
        return 0
    eqns = [bracketa==0 for bracketa in bracket_list]
    for nv in fixedvars:      # fixed n_i's no longer need to be ↩
        positive integers:
        if bool(nv>=0):   #  time-consuming:  ~20 sec for each nv
            forget(nv,'integer')
            forget(nv >= 0)
    for nv in freevars:
        if not bool(nv>=0): # time-consuming:  ~20 sec for each nv
            assume(nv,'integer')
            assume(nv >= 0)
    if len(fixedvars)==1:
        nv = fixedvars[0]
        if nv >=0:
            forget(nv,'integer')
            forget(nv >=0)
        print "only one equation", eqns, "; var = ", nv
        solns = solve(eqns[0],nv,solution_dict=True)
    else:
        solns = solve(eqns,fixedvars,solution_dict=True)
    soln = solns[0]    # should be only one solution!!!
    for idx in soln.keys():
        soln[idx] = expand(soln[idx])
    print "solution:  ", soln
    # identify f by dividing by phi's (indicators)
    f1 = (expr/prod([indicator(idx) for idx in fixedvars])).↩
        simplify_exp()
    SUM_over_Fixed = ((1/abs(detA))*(f1.subs(soln))*prod([gamma(-soln[↩
        idx_star]) for idx_star in soln])).simplify()
    PS = make_sum(SUM_over_Fixed,freevars)
    print "result = ", PS
    return SUM_over_Fixed
```

Appendix D

Mathematica Code

The function `ParallelMultiSum` is called by `eval_list_in_mma` in the Sage file `eval_bracket_series.sage`. This code evaluates a multi-sum with various orderings of the summation indices until one simplification of the sum is found. Since this code takes advantage of parallel processing, the user should set Mathematica preferences so that parallel kernels are launched at startup. The file `PSMS.m` should be located within Mathematica's path.

D.1 PSMS.m

```
(* ::Package:: *)

ParallelMultiSum[sumlist_]:=
ParallelMap[Apply[SMSmain,#] &, sumlist]


SMSmain[expr_,nvars_]:=
Module[{nvperms=Permutations[nvars],i=1,smsout},
smsout=SMS[expr,nvars];
While[smsout[[1]]!=False &&
    Length[Characters[ToString[smsout,InputForm]]]>8 &&
    Characters[ToString[smsout,InputForm]][[1;;3]]=={"S","u","m"},
    i++;
    smsout = SMS[expr,nvperms[[i]]]];
smsout]


SMS[expr_,nvars_]:=Module[
{limitslist = {},sumout,convcond},
```

```
For[i=1,i<=Length[nvars],i++,
     AppendTo[limitslist,{nvars[[i]],0,Infinity}]];
convcond = FullSimplify[SumConvergence[expr,##]& @@ nvars];
If[convcond,
     sumout = FullSimplify[Sum[expr,##]& @@ limitslist],
     sumout=Infinity,
     sumout = FullSimplify[Sum[expr,##]& @@ limitslist]];
If[Length[Characters[ToString[convcond,InputForm]]]>8 &&
     Characters[ToString[convcond,InputForm]][[1;;3]]=={"S","u","m"},
     convcond="Unknown"];
{sumout,convcond}]
```

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Biography

Karen Thompson Kohl is a native of Pass Christian, Mississippi. Karen pursued her undergraduate degree at the Massachusetts Institute of Technology, where she graduated in 1996 with a S.B. in Mathematics and a minor in Linguistics. After working in the software industry, she returned to MIT for graduate study in Computer Science, earning a master's in 1999. She returned to the Gulf Coast in 2006 to pursue her doctoral degree in Mathematics at Tulane University. Completing her program in August 2011, she has accepted an offer as a Visiting Assistant Professor at the Gulf Coast campus of the University of Southern Mississippi.