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Definite integrals by the method of brackets. Part 1 ☆

Ivan Gonzalez [a], Victor H. Moll [b,*]

[a] Departmento de Fisica, Pontificia Universidad Catolica de Santiago, Chile
[b] Department of Mathematics, Tulane University, New Orleans, LA 70118, United States

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A B S T R A C T

A new heuristic method for the evaluation of definite integrals is presented. This *method of brackets* has its origin in methods developed for the evaluation of Feynman diagrams. We describe the operational rules and illustrate the method with several examples. The method of brackets reduces the evaluation of a large class of definite integrals to the solution of a linear system of equations.

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1. Introduction

The *method of brackets* presented here, even though it is heuristic and still lacking a rigorous description, is a powerful method of integration. It is quite simple to work with: the evaluation of a definite integral is reduced to solving a linear system of equations. Many of the entries of [16] can be derived using this method. The basic idea behind it is the assignment of a *bracket* $\langle a \rangle$ to any parameter a. This is a symbol associated to the divergent integral

$$\int_0^\infty x^{a-1}\,dx. \tag{1.1}$$

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* Corresponding author.
E-mail addresses: ivan.gonzalez@usm.cl (I. Gonzalez), igonzalez@fis.puc.cl (I. Gonzalez), vhm@math.tulane.edu (V.H. Moll).

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The formal rules for operating with these brackets are described in Section 2 and their justification is work-in-progress. The method is reminiscent of the Umbral Calculus as developed in [22]. Examples involving explicit integration of classical polynomials by a method similar to the one presented here can be found in [33].

Given a formal sum

$$f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n+\beta-1} \tag{1.2}$$

we associate to the integral of f a *bracket series* written as

$$\int_0^{\infty} f(x)\,dx \doteq \sum_n a_n \langle \alpha n + \beta \rangle, \tag{1.3}$$

to keep in mind the formality of the method described in this paper. Convergence issues are ignored at the present time. Moreover only integrals over the half-line $[0, \infty)$ will be considered.

Note. In the evaluation of these formal sums, the index $n \in \mathbb{N}$ will be replaced by a number n^* defined by the vanishing of the bracket. Observe that it is possible that $n^* \in \mathbb{C}$. For book-keeping purposes, specially in cases with many indices, we write $\sum_n$ instead of the usual $\sum_{n=0}^{\infty}$. After the brackets are eliminated, those indices that remain recover their original nature.

The rules of operation described below assign a *value* to the bracket series. The claim is that for a large class of integrands, including all the examples described here, this formal procedure provides the actual value of the integral. Many of the examples involve the hypergeometric function

$${}_pF_q(z) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{z^n}{n!}. \tag{1.4}$$

This series converges absolutely for all $z \in \mathbb{C}$ if $p \leqslant q$ and for $|z| < 1$ if $p = q + 1$. The series diverges for all $z \neq 0$ if $p > q + 1$ unless the series terminates. The special case $p = q + 1$ is of great interest. In this special case and with $|z| = 1$, the series

$${}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) \tag{1.5}$$

converges absolutely if $\operatorname{Re}(\sum b_j - \sum a_j) > 0$. The series converges conditionally if $z = e^{i\theta} \neq 1$ and $0 \geqslant \operatorname{Re}(\sum b_j - \sum a_j) > -1$ and the series diverges if $\operatorname{Re}(\sum b_j - \sum a_j) \leqslant -1$.

The last section of this paper employs the method of brackets to evaluate certain definite integrals associated to a Feynman diagram. From the present point of view, a *Feynman diagram* is simply a generic graph G that contains $E + 1$ *external* lines and N *internal* lines or *propagators* and L *loops*. All but one of these external lines are assumed independent. The internal and external lines represent particles that transfer momentum among the vertices of the diagram. Each of these particles carries a *mass* $m_i \geqslant 0$ for $i = 1, \ldots, N$. The vertices represent the interaction of these particles and conservation of momentum at each vertex assigns the momentum corresponding to the internal lines. A Feynman diagram has an associated integral given by the *parametrization* of the diagram. For example, in Fig. 1 we have three external lines represented by the momentum P_1, P_2, P_3 and one

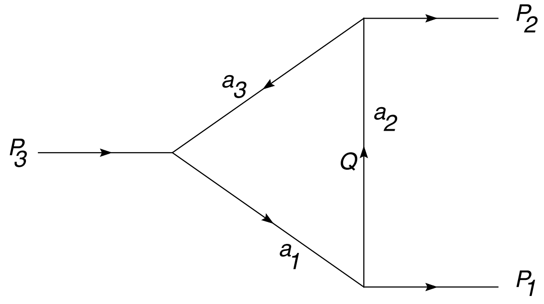

Fig. 1. The triangle.

loop. The parameters a_i are arbitrary real numbers. The integral associated to this diagram is given by

$$G = \frac{(-1)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^\infty \int_0^\infty \int_0^\infty \frac{x_1^{a_1-1} x_2^{a_2-1} x_3^{a_3-1}}{(x_1+x_2+x_3)^{D/2}}$$
$$\times \exp\left(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2\right) \exp\left(-\frac{C_{11} P_1^2 + 2C_{12} P_1 \cdot P_2 + C_{22} P_2^2}{x_1+x_2+x_3}\right) \mathbf{dx},$$

where $\mathbf{dx} = dx_1\, dx_2\, dx_3$. The evaluation of this integral in terms of the variables $P_i \in \mathbb{R}^4$, $m_i \in \mathbb{R}$ and $a_i \in \mathbb{R}$ is the *solution of the Feynman diagram*. The functions C_{ij} are polynomials described in Section 11.

The method of brackets presented here has its origin in quantum field theory (QFT). A version of the method of brackets was developed to address one of the fundamental questions in QFT: the evaluation of loop integrals arising from Feynman diagrams. As described above, these are directed graphs depicting the interaction of particles in the model. The loop integrals depend on the dimension D and one of the (many) intrinsic difficulties is related to their divergence at $D = 4$, the dimension of the physical world. A correction to this problem is obtained by taking $D = 4 - 2\epsilon$ and considering a Laurent expansion in powers of ϵ. This is called the *dimensional regularization* [5] and the parameter ϵ is the *dimensional regulator*.

The method of brackets discussed in this paper is based on previous results by I.G. Halliday, R.M. Ricotta and G.V. Dunne [10,11,17]. The work involves an analytic extension of D to negative values, so the method was labeled NDIM (negative dimensional integration method). The validity of this continuation is based on the observation that the objects associated to a Feynman diagram (loop integrals as well as the functions linked to propagators) are analytic in the dimension D. A.T. Suzuki and A.G.M. Schmidt employed this technique to the evaluation of diagrams with two loops [27,28]; three loops [30]; tensorial integrals [29] and massive integrals with one loop [26,31,32]. An extensive use of this method as well as an analysis of the solutions was provided by C. Anastasiou and E.W.N. Glover in [2] and [3]. The conclusion of these studies is that the NDIM method is inadequate to the evaluation of Feynman diagrams with an arbitrary number of loops. The proposed solutions involve hypergeometric functions with a large number of parameters. By establishing new procedural rules I. Gonzalez and I. Schmidt [14] and [15] have concluded that the modification of the previous procedures permits now the evaluation of more complex Feynman diagrams. One of the results of [14, 15] is the justification of the method of brackets in terms of arguments derived from fractional calculus. The authors have given NDIM the alternative name IBFE (Integration by Fractional Expansion).

From the mathematical point of view, the NDIM method has been used to provide evaluation of a very limited type of integrals [24,25]. The examples presented in this paper show great flexibility of the method of brackets. A systematic study of integrals arising from Feynman diagrams is in preparation.

2. The method of brackets

The method of brackets discussed in this paper is based on the assignment of a *bracket* $\langle a \rangle$ to the parameter a. In the examples presented here $a \in \mathbb{R}$, but the extension to $a \in \mathbb{C}$ is direct. The formal rules for operating with these brackets are described next.

Definition 2.1. Let f be a formal power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n + \beta - 1}. \tag{2.1}$$

The symbol

$$\int_0^\infty f(x)\, dx \doteq \sum_n a_n \langle \alpha n + \beta \rangle \tag{2.2}$$

represents a *bracket series* assignment to the integral on the left. Rule 2.2 describes how to evaluate this series.

Definition 2.2. The symbol

$$\phi_n = \frac{(-1)^n}{\Gamma(n+1)} \tag{2.3}$$

will be called the *indicator of n*.

The symbol ϕ_n gives a simpler form for the bracket series associated to an integral. For example,

$$\int_0^\infty x^{a-1} e^{-x}\, dx \doteq \sum_n \phi_n \langle n + a \rangle. \tag{2.4}$$

The integral is the gamma function $\Gamma(a)$ and the right-hand side its bracket expansion.

Rule 2.1. For $\alpha \in \mathbb{C}$, the expression

$$(a_1 + a_2 + \cdots + a_r)^\alpha \tag{2.5}$$

is assigned to the bracket series

$$\sum_{m_1,\ldots,m_r} \phi_{1,2,\ldots,r} a_1^{m_1} \cdots a_r^{m_r} \frac{\langle -\alpha + m_1 + \cdots + m_r \rangle}{\Gamma(-\alpha)}, \tag{2.6}$$

where $\phi_{1,2,\ldots,r}$ is a short-hand notation for the product $\phi_{m_1}\phi_{m_2}\cdots\phi_{m_r}$.

Rule 2.2. The series of brackets

$$\sum_{n} \phi_n f(n) \langle an+b\rangle \tag{2.7}$$

is given by the *value*

$$\frac{1}{a} f\left(n^*\right)\Gamma\left(-n^*\right), \tag{2.8}$$

where n^* solves the equation $an+b=0$.

Rule 2.3. A two-dimensional series of brackets

$$\sum_{n_1,n_2} \phi_{n_1,n_2} f(n_1,n_2)\langle a_{11}n_1+a_{12}n_2+c_1\rangle\langle a_{21}n_1+a_{22}n_2+c_2\rangle \tag{2.9}$$

is assigned to the *value*

$$\frac{1}{|a_{11}a_{22}-a_{12}a_{21}|} f\left(n_1^*,n_2^*\right)\Gamma\left(-n_1^*\right)\Gamma\left(-n_2^*\right), \tag{2.10}$$

where (n_1^*,n_2^*) is the unique solution to the linear system

$$\begin{aligned} a_{11}n_1+a_{12}n_2+c_1&=0, \\ a_{21}n_1+a_{22}n_2+c_2&=0, \end{aligned} \tag{2.11}$$

obtained by the vanishing of the expressions in the brackets. A similar rule applies to higher dimensional series, that is,

$$\sum_{n_1}\cdots\sum_{n_r} \phi_{1,\dots,r} f(n_1,\dots,n_r)\langle a_{11}n_1+\cdots+a_{1r}n_r+c_1\rangle\cdots\langle a_{r1}n_1+\cdots+a_{rr}n_r+c_r\rangle$$

is assigned to the value

$$\frac{1}{|\det(A)|} f\left(n_1^*,\dots,n_r^*\right)\Gamma\left(-n_1^*\right)\cdots\Gamma\left(-n_r^*\right), \tag{2.12}$$

where A is the matrix of coefficients (a_{ij}) and $\{n_i^*\}$ is the solution of the linear system obtained by the vanishing of the brackets. The value is not defined if the matrix A is not invertible.

Rule 2.4. In the case where the assignment leaves free parameters, any divergent series in these parameters is discarded. In case several choices of free parameters are available, the series that converge in a common region are added to contribute to the integral.

A typical place to apply Rule 2.4 is where the hypergeometric functions ${}_pF_q$, with $p=q+1$, appear. In this case the convergence of the series imposes restrictions on the internal parameters of the problem. Example 11.2, dealing with a Feynman diagram with a *bubble*, illustrates the latter part of this rule.

Note. To motivate Rule 2.1 start with the identity

$$\frac{1}{A^{\alpha}} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha-1} e^{-Ax}\, dx, \tag{2.13}$$

and apply it to $A = a_1 + \cdots + a_r$ to produce

$$\begin{aligned}(a_1 + \cdots + a_r)^{\alpha} &= \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} x^{-\alpha-1} \exp\bigl[-(a_1 + \cdots + a_r)x\bigr]\, dx \\ &= \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} x^{-\alpha-1} e^{-a_1 x} \cdots e^{-a_r x}\, dx.\end{aligned}$$

Expanding the exponentials we obtain

$$(a_1 + \cdots + a_r)^{\alpha} \doteq \frac{1}{\Gamma(-\alpha)} \sum_{m_1} \cdots \sum_{m_r} \phi_{1,\dots,r} a_1^{m_1} \cdots a_r^{m_r} \int_0^{\infty} x^{-\alpha+m_1+\cdots+m_r-1}\, dx$$

and thus

$$(a_1 + \cdots + a_r)^{\alpha} \doteq \sum_{m_1} \cdots \sum_{m_r} \phi_{1,\dots,r} a_1^{m_1} \cdots a_r^{m_r} \frac{\langle -\alpha + m_1 + \cdots + m_r \rangle}{\Gamma(-\alpha)}. \tag{2.14}$$

This is Rule 2.1.

3. Wallis' formula

The evaluation

$$J_{2,m} = \int_0^{\infty} \frac{dx}{(1+x^2)^{m+1}} = \frac{\pi}{2^{2m+1}} \binom{2m}{m} \tag{3.1}$$

is historically one of the earliest closed-form expressions for a definite integral.

The proof of Wallis' formula by the method of brackets starts with the expansion of the integrand as

$$\bigl(1+x^2\bigr)^{-m-1} \doteq \sum_{n_1} \sum_{n_2} \phi_{1,2} \frac{\langle m+1+n_1+n_2 \rangle}{\Gamma(m+1)} x^{2n_2}. \tag{3.2}$$

The corresponding integral $J_{2,m}$ is assigned to the bracket series

$$J_{2,m} \doteq \sum_{n_1} \sum_{n_2} \phi_{1,2} \frac{1}{\Gamma(m+1)} \langle m+1+n_1+n_2 \rangle \langle 2n_2+1 \rangle. \tag{3.3}$$

Rule 2.2 then shows that

$$J_{2,m} = \frac{1}{2}\frac{\Gamma(-n_1^*)\Gamma(-n_2^*)}{\Gamma(m+1)}, \tag{3.4}$$

where (n_1^*, n_2^*) is the solution to the linear system of equations

$$\begin{aligned} m+1+n_1+n_2 &= 0, \\ 2n_2+1 &= 0. \end{aligned} \tag{3.5}$$

Therefore $n_1^* = -(m+\frac{1}{2})$ and $n_2^* = -\frac{1}{2}$. We conclude that

$$J_{2,m} = \frac{\Gamma(m+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(m)}. \tag{3.6}$$

This is exactly the right-hand side of (3.1).

4. A Fresnel integral

In this section we verify the evaluation of Fresnel integral

$$\int_0^\infty \sin\bigl(ax^2\bigr)\,dx = \frac{\pi}{2\sqrt{2a}}. \tag{4.1}$$

The reader will find in [4] the standard evaluation using contour integrals.

In order to apply the method of brackets, use the hypergeometric representation

$$\frac{\sin z}{z} = {}_0F_1\left[-;\frac{3}{2};-\frac{z^2}{4}\right],$$

that can be written as

$$\sin z = \sum_{n=0}^\infty \phi_n \frac{z^{2n+1}}{(\frac{3}{2})_n 4^n}. \tag{4.2}$$

Therefore

$$\int_0^\infty \sin\bigl(ax^2\bigr)\,dx \doteq \sum_n \phi_n \frac{a^{2n+1}}{(\frac{3}{2})_n 4^n}\langle 4n+3\rangle. \tag{4.3}$$

According to Rule 2.2, the assignment of the right-hand side is obtained by evaluating the function

$$g(n) = \frac{a^{2n+1}}{(\frac{3}{2})_n 4^n} \tag{4.4}$$

at the solution of $4n^*+3=0$. Therefore the integral (4.1) has the value

$$\frac{1}{4}g\left(-\frac{3}{4}\right) = \frac{a^{-1/2}\Gamma(\frac{3}{4})}{(\frac{3}{2})_{-3/4}4^{1/4}},$$

where the factor $\frac{1}{4}$ comes from the term $4n+3$ in the bracket. Using $(a)_m = \Gamma(a+m)/\Gamma(a)$, we obtain

$$\left(\frac{3}{2}\right)_{-3/4} = \frac{2\Gamma(\frac{3}{4})}{\sqrt{\pi}}. \tag{4.5}$$

We conclude that the assigned value is $\pi/2\sqrt{2a}$. As expected, this is consistent with (4.1).

The method also give the evaluation of

$$I = \int_0^\infty x^{b-1} \sin\bigl(ax^c\bigr)\, dx. \tag{4.6}$$

The change of variables $t = x^c$ transforms (4.6) into

$$I = \frac{1}{c} \int_0^\infty t^{b/c-1} \sin(at)\, dt, \tag{4.7}$$

and this is formula 3.761.4 in [16] with value

$$\int_0^\infty x^{b-1} \sin\bigl(ax^c\bigr)\, dx = \frac{\Gamma(b/c)}{ca^{b/c}} \sin\left(\frac{\pi b}{2c}\right). \tag{4.8}$$

To verify this result by the method of brackets, start with the expansion

$$x^{b-1} \sin\bigl(ax^c\bigr) = \sum_{n=0}^\infty \phi_n \frac{a^{2n+1}}{(\frac{3}{2})_n 2^{2n}} x^{2nc+c+b-1} \tag{4.9}$$

and associate to it the bracket series

$$\int_0^\infty x^{b-1} \sin\bigl(ax^c\bigr)\, dx \doteq \sum_n \phi_n \frac{a^{2n+1}}{(\frac{3}{2})_n 2^{2n}} \langle 2nc+c+b \rangle. \tag{4.10}$$

Apply Rule 2.2 to obtain

$$I = \frac{1}{2c} \frac{a^{2n_*+1}}{(\frac{3}{2})_{n^*} 2^{2n^*}} \Gamma\bigl(-n^*\bigr), \tag{4.11}$$

where n^* solve $2nc+b+c=0$; that is, $n^* = -1/2 - b/2c$. Then (4.11) yields

$$I = \frac{\Gamma(\frac{3}{2}) 2^{b/c}}{ca^{b/c}\Gamma(1-\frac{b}{2c})} \Gamma\left(\frac{1}{2} + \frac{b}{2c}\right). \tag{4.12}$$

To transform (4.12) into (4.8), simplify (4.12) using the reflection formula

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}, \tag{4.13}$$

and the duplication formula

$$\Gamma\left(x+\frac{1}{2}\right)=\frac{\Gamma(2x)\sqrt{\pi}}{\Gamma(x)2^{2x-1}}, \tag{4.14}$$

with $x=b/2c$.

5. An integral of beta type

In this section we present the evaluation of

$$I=\int_0^\infty \frac{x^a\,dx}{(E+Fx^b)^c}. \tag{5.1}$$

The change of variables $x=C^{1/b}t^{1/b}$, with $C=E/F$, yields

$$I=\frac{C^u}{bE^c}\int_0^\infty \frac{t^{u-1}\,dt}{(1+t)^c}, \tag{5.2}$$

where $u=(a+1)/b$. The new integral evaluates as $B(c-u,u)$ where $B(x,y)$ is the classical beta function; see [16, formula 8.380.3]. We conclude that

$$I=\frac{C^u}{bE^c}B(c-u,u). \tag{5.3}$$

To evaluate this integral by the method of brackets, the integrand $(E+Fx^b)^{-c}$ is expanded as

$$\sum_{n_1}\sum_{n_2}\phi_{1,2}E^{n_1}F^{n_2}x^{bn_2}\frac{\langle c+n_1+n_2\rangle}{\Gamma(c)}. \tag{5.4}$$

Replacing in (5.1) we obtain

$$\begin{aligned}
I &\doteq \sum_{n_1}\sum_{n_2}\phi_{1,2}E^{n_1}F^{n_2}\frac{\langle c+n_1+n_2\rangle}{\Gamma(c)}\int_0^\infty x^{a+bn_2+1-1}\,dx \\
&\doteq \sum_{n_1}\sum_{n_2}\phi_{1,2}E^{n_1}F^{n_2}\frac{1}{\Gamma(c)}\langle c+n_1+n_2\rangle\langle a+bn_2+1\rangle.
\end{aligned}$$

To obtain the value assigned to the two-dimensional sum, solve

$$c+n_1+n_2=0,$$

$$a+bn_2+1=0,$$

to produce the solution $n_1^*=\frac{a+1}{b}-c$ and $n_2^*=-\frac{a+1}{b}$. Therefore

$$I=\frac{1}{b\Gamma(c)}E^{n_1^*}F^{n_2^*}\Gamma\left(-n_1^*\right)\Gamma\left(-n_2^*\right), \tag{5.5}$$

and this reduces to the value in (5.3).

6. A combination of powers and exponentials

In this section we employ the method of brackets and evaluate the integral

$$I = \int_0^\infty \frac{x^{\alpha-1}\,dx}{(A + B\exp(Cx^\beta))^\gamma}, \tag{6.1}$$

with $\alpha, \beta, \gamma, A, B, C \in \mathbb{R}$. To evaluate this integral we consider the bracket series

$$\big(A + B\exp\big(Cx^\beta\big)\big)^{-\gamma} \doteq \sum_{n_1,n_2} A^{n_1} B^{n_2} \exp\big(Cn_2x^\beta\big) \frac{\langle \gamma + n_1 + n_2 \rangle}{\Gamma(\gamma)}. \tag{6.2}$$

The exponential function is expanded as

$$\begin{aligned}\exp\big(Cn_2x^\beta\big) &= \sum_{n_3=0}^\infty \frac{C^{n_3} n_2^{n_3}}{\Gamma(n_3+1)} x^{\beta n_3} \\ &= \sum_{n_3=0}^\infty C^{n_3}(-n_2)^{n_3}\phi_{n_3} x^{n_3}.\end{aligned}$$

Therefore, the integral (6.1) is assigned to the bracket series

$$I \doteq \sum_{n_1,n_2,n_3} \phi_{1,2,3} \frac{A^{n_1} B^{n_2} C^{n_3} (-n_2)^{n_3} \langle \alpha + \beta n_3 \rangle \langle \gamma + n_1 + n_2 \rangle}{\Gamma(\gamma)}.$$

The vanishing of the two brackets leads to the system

$$\begin{aligned}\alpha + \beta n_3 &= 0, \\ \gamma + n_1 + n_2 &= 0,\end{aligned}$$

and we have to choose a free parameter between n_1 and n_2. Observe that $n_3 = -\alpha/\beta$ is determined by the method.

Choice 1. Take n_2 to be free. Then $n_1^* = -\gamma - n_2$ and $n_3^* = -\alpha/\beta$. This leads to

$$I = \sum_{n_2=0}^\infty \frac{B^{n_2}\Gamma(\alpha/\beta)\Gamma(\gamma+n_2)}{A^{\gamma+n_2}C^{\alpha/\beta}\beta\Gamma(\gamma)(-n_2)^{\alpha/\beta}}. \tag{6.3}$$

This is impossible due to the presence of the term $n_2^{\alpha/\beta}$ leading to a divergent series. These divergent series are discarded.

Choice 2. Take n_1 as the free variable. Then $n_3^* = -\alpha/\beta$ and $n_2^* = -\gamma - n_1$. This time we obtain

$$I = \frac{\Gamma(\alpha/\beta)}{\Gamma(\gamma)} \frac{1}{B^\gamma C^{\alpha/\beta}\beta} \sum_{n_1=0}^\infty (-1)^{n_1} \frac{\Gamma(\gamma+n_1)}{\Gamma(1+n_1)} \frac{(A/B)^{n_1}}{(\gamma+n_1)^{\alpha/\beta}}. \tag{6.4}$$

This formula cannot be expressed in term of more elementary special functions.

In the special case $\gamma = 1$ we obtain

$$I = -\frac{\Gamma(\nu)}{a\beta c^{\nu}} \operatorname{PolyLog}(\nu, -a/b), \tag{6.5}$$

with $\nu = \alpha/\beta$. The polylogarithm function appearing here is defined by

$$\operatorname{PolyLog}(z, k) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}. \tag{6.6}$$

Specializing to $A = B = C = \alpha = \gamma = 1$ and $\beta = 2$ we obtain

$$\int_0^{\infty} \frac{dx}{1 + e^{x^2}} = -\frac{\sqrt{\pi}}{2}(\sqrt{2} - 1)\zeta\left(\frac{1}{2}\right). \tag{6.7}$$

Of course, this integral can be evaluated by simply expanding the integrand as a geometric series.

7. The Mellin transform of a quadratic exponential

The Mellin transform of a function $f(x)$ is defined by

$$\mathcal{M}(f)(s) = \int_0^{\infty} x^{s-1} f(x)\, dx. \tag{7.1}$$

Many of the integrals appearing in [16] are of this type. For example, 3.462.1 states that

$$\mathcal{M}\left(e^{-\beta x^2 - \gamma x}\right)(s) = \int_0^{\infty} x^{s-1} e^{-\beta x^2 - \gamma x}\, dx = (2\beta)^{-s/2}\Gamma(s) e^{\gamma^2/(8\beta)} D_{-s}\left(\frac{\gamma}{\sqrt{2\beta}}\right). \tag{7.2}$$

Here $D_p(z)$ is the parabolic cylinder function defined by (formula 9.240 in [16])

$$D_p(z) = 2^{p/2} e^{-z^2/4}\left(\frac{\sqrt{\pi}}{\Gamma((1-p)/2)}\, {}_1F_1\left(-\frac{p}{2}, \frac{1}{2}; \frac{z^2}{2}\right) - \frac{\sqrt{2\pi}\, z}{\Gamma(-p/2)}\, {}_1F_1\left(\frac{1-p}{2}, \frac{3}{2}; \frac{z^2}{2}\right)\right).$$

A direct application of the method of brackets gives

$$\int_0^{\infty} x^{s-1} e^{-\beta x^2 - \gamma x}\, dx \doteq \sum_{n_1} \sum_{n_2} \phi_{1,2} \beta^{n_1} \gamma^{n_2} \langle s + 2n_1 + n_2 \rangle. \tag{7.3}$$

The equation $s + 2n_1 + n_2 = 0$ gives two choices for a free index. Taking $n_2^* = -2n_1 - s$ leads to the series

$$\begin{aligned} \sum_{n_1=0}^{\infty} \frac{1}{\Gamma(n_1+1)}\left(-\frac{\beta}{\gamma^2}\right)^{n_1} (s)_{2n_1} &= \sum_{n_1=0}^{\infty} \frac{1}{\Gamma(n_1+1)}\left(-\frac{4\beta}{\gamma^2}\right)^{n_1} \left(\frac{s}{2}\right)_{n_1} \left(\frac{s+1}{2}\right)_{n_1} \\ &= {}_2F_0\left(\frac{s}{2}, \frac{s+1}{2} \middle| -\frac{4\beta}{\gamma^2}\right). \end{aligned}$$

This choice of a free index is excluded because the resulting series diverges. The second choice is $n_1^* = -n_2/2 - s/2$ and this yields the series

$$\frac{1}{2\beta^{s/2}} \sum_{n_2=0}^{\infty} \frac{\rho^{n_2}}{\Gamma(n_2+1)} \Gamma\left(\frac{n_2}{2} + \frac{s}{2}\right), \tag{7.4}$$

where $\rho = -\gamma/\sqrt{\beta}$. To write (7.4) in hypergeometric form we separate it into two sums according to the parity of n_2 and obtain

$$\frac{1}{2\beta^{s/2}}\left(\Gamma\left(\frac{s}{2}\right)\sum_{n=0}^{\infty}\frac{\rho^{2n}}{(1)_{2n}}\left(\frac{s}{2}\right)_n + \Gamma\left(\frac{1+s}{2}\right)\sum_{n=0}^{\infty}\frac{\rho^{2n+1}}{(2)_{2n}}\left(\frac{1+s}{2}\right)_n\right).$$

The identity

$$(a)_{2n} = 4^n \left(\frac{a}{2}\right)_n \left(\frac{a+1}{2}\right)_n \tag{7.5}$$

gives the final representation of the sum as

$$\frac{1}{2\beta^{s/2}}\left[\Gamma\left(\frac{s}{2}\right) {}_1F_1\left(\frac{s}{2}, \frac{1}{2}; \frac{1}{2}\rho^2\right) + \rho\Gamma\left(\frac{1+s}{2}\right) {}_1F_1\left(\frac{1+s}{2}, \frac{3}{2}; \frac{1}{2}\rho^2\right)\right]. \tag{7.6}$$

This is (7.2).

The special case $s = 1$ gives

$$\int_0^\infty e^{-\beta x^2 - \gamma x}\, dx = \frac{1}{2\sqrt{\beta}}\left[\Gamma\left(\frac{1}{2}\right) {}_1F_1\left(\frac{1}{2}; \frac{1}{2}; \frac{1}{2}\rho^2\right) + \rho\, {}_1F_1\left(1; \frac{3}{2}; \frac{1}{2}\rho^2\right)\right]. \tag{7.7}$$

The first hypergeometric sum evaluates to $e^{\gamma^2/4\beta}$ and using the representation of the *error function*

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \tag{7.8}$$

as

$$\operatorname{erf}(x) = \frac{2x}{\sqrt{\pi}} e^{-x^2} {}_1F_1\left(1; \frac{3}{2}; x^2\right) \tag{7.9}$$

(given as 8.253.1 in [16]), we find the value of the second hypergeometric sum. The conclusion is that

$$\int_0^\infty e^{-\beta x^2 - \gamma x}\, dx = \frac{\sqrt{\pi}}{2\beta} \exp\left(\frac{\gamma^2}{4\beta}\right)\left(1 - \operatorname{erf}\left(\frac{\gamma}{2\sqrt{\beta}}\right)\right). \tag{7.10}$$

This can be checked directly by completing the square in the integrand.

8. A multidimensional integral from Gradshteyn and Ryzhik

The method of brackets can also be used to evaluate some multidimensional integrals. Consider the following integral

$$I_n = \int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1}\, dx_1\, dx_2 \cdots dx_n}{(1 + (r_1x_1)^{q_1} + \cdots + (r_nx_n)^{q_n})^s}, \tag{8.1}$$

which appears as 4.638.3 in [16] with an incorrect evaluation.

The first step in the evaluation of I_n is to expand the denominator of the integrand using Rule 2.1 as

$$\frac{1}{(1 + (r_1x_1)^{q_1} + \cdots + (r_nx_n)^{q_n})^s} \doteq \sum_{k_0,k_1,\ldots,k_n} \phi_{0,\ldots,n} \prod_{j=1}^{n} (r_jx_j)^{q_jk_j} \frac{\langle s + k_0 + \cdots + k_n \rangle}{\Gamma(s)}.$$

Next the integral is assigned to the value

$$I_n \doteq \sum_{k_0,k_1,\ldots,k_n} \phi_{0,\ldots,n} \prod_{j=1}^{n} (r_jx_j)^{q_jk_j} \frac{\langle s + k_0 + \cdots + k_n \rangle}{\Gamma(s)} \prod_{j=1}^{n} \langle p_j + q_jk_j \rangle. \tag{8.2}$$

The evaluation of this bracket sum involves the values

$$k_0 = -s + \sum_{j=1}^{n} \frac{p_j}{q_j} \quad \text{and} \quad k_j = -\frac{p_j}{q_j} \quad \text{for } 1 \leqslant j \leqslant n. \tag{8.3}$$

We conclude that

$$I_n = \frac{1}{\Gamma(s)} \Gamma\left(s - \sum_{j=1}^{n} \frac{p_j}{q_j} \right) \prod_{j=1}^{n} \frac{\Gamma(\frac{p_j}{q_j})}{q_j r_j^{p_j}}. \tag{8.4}$$

The table [16] has the exponents of r_j written as p_jq_j instead of p_j. This has now been corrected.

9. An example involving Bessel functions

The Bessel function $J_\nu(x)$ is defined by the series

$$J_\nu(x) = \frac{1}{2^\nu} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+\nu}}{2^{2k} k! \Gamma(\nu + k + 1)}, \tag{9.1}$$

and it admits the hypergeometric representation

$$J_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(1+\nu)} {}_0F_1\left(\begin{matrix} - \\ 1+\nu \end{matrix} \,\middle|\, \frac{-x^2}{4} \right). \tag{9.2}$$

The method of brackets will now be employed to evaluate the integral

$$I = \int_0^\infty x^{-\lambda} J_\nu(\alpha x) J_\mu(\beta x)\, dx. \tag{9.3}$$

Three integrals of this type form Section 6.574 of [16].

Replacing the hypergeometric form in the integral, we have

$$I \doteq \frac{(\frac{\alpha}{2})^\nu (\frac{\beta}{2})^\mu}{\Gamma(\nu+1)\Gamma(\mu+1)} \times \int_0^\infty \sum_{n_1,n_2} \phi_{1,2} \frac{\alpha^{2n_1}\beta^{2n_2}}{4^{n_1+n_2}(\nu+1)_{n_1}(\mu+1)_{n_2}} x^{2n_1+2n_2-\lambda+\nu+\mu}\, dx.$$

Therefore, the bracket series associated to the integral (9.3) becomes

$$I \doteq \frac{2^{-\nu-\mu}\alpha^\nu\beta^\mu}{\Gamma(\nu+1)\Gamma(\mu+1)} \times \sum_{n_1}\sum_{n_2} \frac{\phi_{1,2}}{4^{n_1+n_2}} \frac{\alpha^{2n_1}\beta^{2n_2}}{(\nu+1)_{n_1}(\mu+1)_{n_2}} \langle 2n_1 + 2n_2 - \lambda + \nu + \mu + 1\rangle.$$

The vanishing of the brackets yields the value $n_1^* = \frac{1}{2}(\lambda - \nu - \mu - 1) - n_2$ and it follows that

$$I = \frac{2^{-\nu-\mu}}{\Gamma(\nu+1)\Gamma(\mu+1)} \sum_{n_2=0}^\infty \frac{\phi_2}{4^{n_1^*+n_2}} \frac{\alpha^{2n_1^*}\beta^{2n_2}}{(\nu+1)_{n_1^*}(\mu+1)_{n_2}} \frac{\Gamma(-n_1^*)}{2}.$$

Writing the Pochhammer symbol $(\nu+1)_{n_1^*}$ in terms of the gamma function we obtain

$$I = \frac{\beta^\mu \alpha^{\lambda-\mu-1}}{2^\lambda \Gamma(\mu++1)} \times \sum_{n_2=0}^\infty \frac{(-1)^{n_2}}{\Gamma(n_2+1)} \frac{(\beta^2/\alpha^2)^{n_2}}{\Gamma(\nu+1+\frac{1}{2}(\lambda-\nu-\mu-1)-n_2)} \frac{\Gamma(\frac{1}{2}(\nu+\mu-\lambda+1)+n_2)}{(\mu+1)_{n_2}}.$$

In order to write this in hypergeometric terms, we start with

$$I = \frac{\beta^\mu \alpha^{\lambda-\mu-1}}{2^\lambda \Gamma(\mu++1)} \times \sum_{n_2=0}^\infty (-1)^{n_2} \frac{(\frac{1}{2}(\nu+\mu-\lambda+1))_{n_2}(\beta^2/\alpha_2)^{n_2}}{(\frac{1}{2}(\lambda+\nu-\mu+1))_{-n_2}(\mu+1)_{n_2}\Gamma(n_2+1)},$$

and use the identity

$$(c)_{-n} = \frac{(-1)^n}{(1-c)_n} \tag{9.4}$$

to obtain

$$I = \frac{\beta^{\mu}\alpha^{\lambda-\mu-1}}{2^{\lambda}} \frac{\Gamma(\frac{1}{2}(\nu+\mu-\lambda+1))}{\Gamma(\mu+1)\Gamma(\frac{1}{2}(\lambda+\nu-\mu+1))} \times \sum_{n_2=0}^{\infty} \left(\frac{1}{2}(1-\lambda-\nu+\mu)\right)_{n_2} \left(\frac{1}{2}(\nu+\mu-\lambda+1)\right)_{n_2} \frac{1}{(\mu+1)_{n_2}\Gamma(n_2+1)} \left(\frac{\beta^2}{\alpha^2}\right)^{n_2},$$

that can be written as

$$I = \frac{\beta^{\mu}\alpha^{\lambda-\mu-1}}{2^{\lambda}} \frac{\Gamma(\frac{1}{2}(\nu+\mu-\lambda+1))}{\Gamma(\mu+1)\Gamma(\frac{1}{2}(\lambda+\nu-\mu+1))} \times {}_2F_1\left(\begin{matrix} \frac{1}{2}(1-\lambda-\nu+\mu) \quad \frac{1}{2}(\nu+\mu-\lambda+1) \\ \mu+1 \end{matrix} \middle| \frac{\beta^2}{\alpha^2} \right).$$

This solution is valid for $|\beta^2/\alpha^2| < 1$ and it corresponds to formula 6.574.3 in [16]. The table contains an error in this formula, the power of β is written as ν instead of μ. To obtain a formula valid for $|\beta^2/\alpha^2| > 1$ we could proceed as before and obtain 6.574.1 in [16]. Alternatively exchange (ν, α) by (μ, β) and use the formula developed above.

10. A new evaluation of a quartic integral

The integral

$$N_{0,4}(a; m) = \int_0^{\infty} \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}} \tag{10.1}$$

is given by

$$N_{0,4}(a, m) = \frac{\pi}{2} \frac{P_m(a)}{[2(a+1)]^{m+\frac{1}{2}}}, \tag{10.2}$$

where P_m is the polynomial

$$P_m(a) = \sum_{l=0}^{m} d_{l,m} a^l, \tag{10.3}$$

with coefficients

$$d_{l,m} = 2^{-2m} \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}. \tag{10.4}$$

The sequence $\{d_{l,m}\colon\ 0 \leqslant l \leqslant m\}$ has remarkable arithmetical and combinatorial properties [21].

The reader will find in [1] a survey of the many different proofs of (10.2) available in the literature. One of these proofs follows from the hypergeometric representation

$$N_{0,4}(a, m) = 2^{m-\frac{1}{2}}(a+1)^{-m-\frac{1}{2}} B\left(2m+\frac{3}{2}\right) {}_2F_1\left(\begin{matrix} -m \quad m+1 \\ m+\frac{3}{2} \end{matrix} \middle| \frac{1-a}{2} \right). \tag{10.5}$$

New proofs of this evaluation keep on appearing. For instance, the survey [1] does not include the recent automatic proof by C. Koutschan and V. Levandovskyy [20]. The goal of this section is to provide yet another proof of the identity (10.2) using the method of brackets.

The bracket series for $I \equiv N_{0,4}(a,m)$ is formed by the usual procedure. The result is

$$I \doteq \frac{1}{\Gamma(m+1)} \sum_{n_1,n_2,n_3} \phi_{1,2,3}(2a)^{n_2} \langle 4n_1+2n_2+1\rangle \langle m+1+n_1+n_2+n_3\rangle. \tag{10.6}$$

The expression (10.6) contains two brackets and three indices. Therefore the final result will be a single series on the free index. We employ the following notation: I is the original bracket series, the symbol I_j denotes the series I after eliminating the index n_j. Similarly $I_{i,j}$ denotes the series I after first eliminating n_i (to produce I_i) and then eliminating n_j.

Case 1. n_3 is the free index. Eliminate first n_1 from the bracket $\langle 4n_1+2n_2+1\rangle$ to obtain $n_1^* = -\frac{1}{2}n_2 - \frac{1}{4}$. The resulting bracket series is

$$I_1 \doteq \sum_{n_2,n_3} \phi_{2,3} \frac{(2a)^{n_2}\Gamma(\frac{1}{2}n_2+\frac{1}{4})}{4\Gamma(m+1)} \left\langle m+\frac{3}{4}+\frac{1}{2}n_2+n_3\right\rangle. \tag{10.7}$$

The next step is to eliminate n_2 to get $n_2^* = -2m - \frac{3}{2} - 2n_3$ and obtain

$$I_{1,2} = \frac{1}{2\Gamma(m+1)(2a)^{2m+3/2}} \sum_{n_3=0}^{\infty} \frac{\phi_3}{(2a)^{n_3}} \Gamma\left(-m-\frac{1}{2}-n_3\right)\Gamma\left(2m+\frac{3}{2}+2n_3\right). \tag{10.8}$$

In order to simplify these expressions, we employ

$$\Gamma(x+m) = (x)_m\Gamma(x), \qquad \Gamma(x-m) = (-1)^m\Gamma(x)/(1-x)_m \tag{10.9}$$

and

$$(x)_{2m} = 2^{2m}\left(\frac{1}{2}x\right)_m\left(\frac{1}{2}(x+1)\right)_m, \tag{10.10}$$

for $x \in \mathbb{R}$ and $m \in \mathbb{N}$. We obtain

$$\Gamma\left(-m-\frac{1}{2}-n_3\right) = \frac{(-1)^{n_3}\Gamma(-\frac{1}{2}-m)}{(\frac{3}{2}+m)_{n_3}}$$

and

$$\Gamma\left(2m+\frac{3}{2}+2n_3\right) = \Gamma\left(2m+\frac{3}{2}\right)\left(m+\frac{3}{4}\right)_{n_3}\left(m+\frac{5}{4}\right)_{n_3} 2^{2n_3}.$$

These yield

$$I_{1,2} = \frac{\Gamma(-\frac{1}{2}-m)\Gamma(2m+\frac{3}{2})}{2\Gamma(m+1)(2a)^{2m+3/2}} \sum_{n_3=0}^{\infty} \frac{(m+3/4)_{n_3}(m+5/4)_{n_3}}{(m+3/2)_{n_3} n_3!} a^{-2n_3}, \tag{10.11}$$

or

$$I_{1,2} = \frac{\Gamma(-\frac{1}{2}-m)\Gamma(2m+\frac{3}{2})}{2\Gamma(m+1)(2a)^{2m+3/2}} {}_2F_1\left(\begin{matrix} m+\frac{3}{4} & & m+\frac{5}{4} \\ & m+\frac{3}{2} & \end{matrix}\middle| \frac{1}{a^2}\right). \tag{10.12}$$

Note. The reader can check that $I_{1,2} = I_{2,1}$, so the value of the sum for the quartic integral does not depend on the order in which the indices n_1 and n_2 are eliminated. The reader can also verify that this occurs in the next two cases described below; that is, $I_{1,3} = I_{3,1}$ and $I_{2,3} = I_{3,2}$.

Case 2. n_1 is the free index. A similar argument yields

$$I_{2,3} = \frac{\Gamma(m+\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(m+1)(2a)^{1/2}} {}_2F_1\left(\begin{matrix} \frac{1}{4} & & \frac{3}{4} \\ & \frac{1}{2}-m & \end{matrix}\middle| \frac{1}{a^2}\right). \tag{10.13}$$

Case 3. n_2 is the free index. Eliminate n_1 from the bracket series (10.6) to produce

$$I_1 \doteq \sum_{n_2,n_3} \phi_{2,3} \frac{(2a)^{n_2}\Gamma(\frac{1}{2}n_2+\frac{1}{4})}{4\Gamma(m+1)} \left\langle m+\frac{3}{4}+\frac{1}{2}n_2+n_3\right\rangle, \tag{10.14}$$

and now eliminate n_3 to obtain $n_3^* = -m - \frac{3}{4} - \frac{1}{2}n_2$. This yields

$$I_{1,3} = \frac{1}{4\Gamma(m+1)} \sum_{n_2=0}^{\infty} (-1)^{n_2} \frac{(2a)^{n_2}}{n_2!} \Gamma\left(\frac{1}{2}n_2+\frac{1}{4}\right)\Gamma\left(m+\frac{3}{4}+\frac{1}{2}n_2\right). \tag{10.15}$$

In order to obtain a hypergeometric representations of these expressions, we separate the last series according to the parity of n_2:

$$\begin{aligned} I_{1,3} &= \frac{1}{4\Gamma(m+1)} \sum_{n_2=0}^{\infty} \frac{(2a)^{2n_2}}{(2n_2)!} \Gamma\left(n_2+\frac{1}{4}\right)\Gamma\left(n_2+m+\frac{3}{4}\right) \\ &\quad - \frac{1}{4\Gamma(m+1)} \sum_{n_2=0}^{\infty} \frac{(2a)^{2n_2+1}}{(2n_2+1)!} \Gamma\left(n_2+\frac{3}{4}\right)\Gamma\left(n_2+m+\frac{5}{4}\right). \end{aligned}$$

Using the standard formulas (10.9) and (10.10), we can write this in the form

$$\begin{aligned} I_{1,3} &= \frac{\Gamma(\frac{1}{4})\Gamma(m+\frac{3}{4})}{4\Gamma(m+1)} {}_2F_1\left(\begin{matrix} \frac{1}{4} & & m+\frac{3}{4} \\ & \frac{1}{2} & \end{matrix}\middle| a^2\right) \\ &\quad - \frac{a\Gamma(\frac{3}{4})\Gamma(m+\frac{5}{4})}{2\Gamma(m+1)} {}_2F_1\left(\begin{matrix} \frac{3}{4} & & m+\frac{5}{4} \\ & \frac{3}{2} & \end{matrix}\middle| a^2\right). \end{aligned}$$

In summary: we have obtained three series related to the integral $N_{0,4}(a,m)$. The series $I_{1,2}$ and $I_{2,3}$ are given in terms of the hypergeometric function ${}_2F_1$ with last argument $1/a^2$. These series converge when $a^2 > 1$. The remaining case $I_{1,3}$ gives ${}_2F_1$ with argument a^2, that is convergent when $a^2 < 1$. Rule 2.4 states that we must add the series $I_{1,2}$ and $I_{2,3}$ to get a valid representation for $a^2 > 1$. In conclusion, the method of brackets shows that

$$
\begin{aligned}
N_{0,4}(a,m) &= \frac{\Gamma(\frac{1}{4})\Gamma(m+\frac{3}{4})}{4\Gamma(m+1)} {}_2F_1\left(\begin{matrix}\frac{1}{4} & & m+\frac{3}{4}\\ & \frac{1}{2} & \end{matrix}\bigg| a^2\right) \\
&\quad - \frac{a\Gamma(\frac{3}{4})\Gamma(m+\frac{5}{4})}{2\Gamma(m+1)} {}_2F_1\left(\begin{matrix}\frac{3}{4} & & m+\frac{5}{4}\\ & \frac{3}{2} & \end{matrix}\bigg| a^2\right) \quad \text{for } a^2<1, \\
&= \frac{\Gamma(\frac{1}{2})\Gamma(m+\frac{1}{2})}{2\sqrt{2a}\Gamma(m+1)} {}_2F_1\left(\begin{matrix}\frac{1}{4} & & \frac{3}{4}\\ & \frac{1}{2}-m & \end{matrix}\bigg| \frac{1}{a^2}\right) \\
&\quad + \frac{\Gamma(-\frac{1}{2})\Gamma(2m+\frac{3}{2})}{2(2a)^{2m+3/2}\Gamma(m+1)} {}_2F_1\left(\begin{matrix}m+\frac{3}{4} & & m+\frac{5}{4}\\ & m+\frac{3}{2} & \end{matrix}\bigg| \frac{1}{a^2}\right) \quad \text{for } a^2>1.
\end{aligned}
$$

The continuity of these expressions at $a=1$ requires the evaluation of ${}_2F_1(a,b;c;1)$. Recall that this is finite only when $c>a+b$. In our case, we have four hypergeometric terms and in each one of them, the corresponding expression $c-(a+b)$ equals $-\frac{1}{2}-m$. Therefore each hypergeometric term blows up as $a\to 1$. This divergence is made evident by employing the relation

$$
{}_2F_1(a,b,c;z) = (1-z)^{c-a-b}{}_2F_1(c-a,c-b,c;z). \tag{10.16}
$$

The expression for $N_{0,4}(a,m)$ given above is transformed into

$$
\begin{aligned}
N_{0,4}(a,m) &= \frac{\Gamma(\frac{1}{4})\Gamma(m+\frac{3}{4})}{4\Gamma(m+1)(1-a^2)^{m+1/2}} {}_2F_1\left(\begin{matrix}\frac{1}{4} & & -m-\frac{1}{4}\\ & \frac{1}{2} & \end{matrix}\bigg| a^2\right) \\
&\quad - \frac{a\Gamma(\frac{3}{4})\Gamma(m+\frac{5}{4})}{2\Gamma(m+1)(1-a^2)^{m+1/2}} {}_2F_1\left(\begin{matrix}\frac{3}{4} & & -m+\frac{1}{4}\\ & \frac{3}{2} & \end{matrix}\bigg| a^2\right) \quad \text{for } a^2<1, \\
&= \frac{\Gamma(\frac{1}{2})\Gamma(m+\frac{1}{2})}{2\sqrt{2a}\Gamma(m+1)(1-a^{-2})^{m+1/2}} {}_2F_1\left(\begin{matrix}\frac{1}{4}-m & & -\frac{1}{4}-m\\ & \frac{1}{2}-m & \end{matrix}\bigg| \frac{1}{a^2}\right) \\
&\quad + \frac{\Gamma(-\frac{1}{2})\Gamma(2m+\frac{3}{2})}{2(2a)^{2m+3/2}\Gamma(m+1)(1-a^{-2})^{m+1/2}} {}_2F_1\left(\begin{matrix}\frac{3}{4} & & \frac{1}{4}\\ & m+\frac{3}{2} & \end{matrix}\bigg| \frac{1}{a^2}\right) \quad \text{for } a^2>1.
\end{aligned}
$$

Introduce the functions

$$
\begin{aligned}
G_1(a,m) &= \left(\frac{3}{4}\right)_m {}_2F_1\left(\begin{matrix}\frac{1}{4} & & -\frac{1}{4}-m\\ & \frac{1}{2} & \end{matrix}\bigg| a^2\right) \\
&\quad - 2a\left(\frac{1}{4}\right)_{m+1} {}_2F_1\left(\begin{matrix}\frac{3}{4} & & \frac{1}{4}-m\\ & \frac{3}{2} & \end{matrix}\bigg| a^2\right)
\end{aligned}
$$

and

$$
\begin{aligned}
G_2(a,m) &= \left(\frac{1}{2}\right)_m (2a)^{2m+1} {}_2F_1\left(\begin{matrix}\frac{1}{4}-m & & -\frac{1}{4}-m\\ & \frac{1}{2}-m & \end{matrix}\bigg| \frac{1}{a^2}\right) \\
&\quad - (-1)^m m! 2^{-2m}\binom{4m+1}{2m} {}_2F_1\left(\begin{matrix}\frac{3}{4} & & \frac{1}{4}\\ & m+\frac{3}{2} & \end{matrix}\bigg| \frac{1}{a^2}\right).
\end{aligned}
$$

Then

$$N_{0,4}(a,m) = \frac{\pi\sqrt{2}}{4m!} \frac{G_1(a,m)}{(1-a^2)^{m+1/2}} \tag{10.17}$$

for $a^2 < 1$ and

$$N_{0,4}(a,m) = \frac{\pi}{2^{2m+5/2}\sqrt{a}m!} \frac{G_2(a,m)}{(a^2-1)^{m+1/2}} \tag{10.18}$$

for $a^2 > 1$. The functions $G_1(a,m)$ and $G_2(a,m)$ match at $a = 1$ to sufficiently high order to verify the continuity at $a = 1$. Moreover, their blow-up at $a = -1$ is a reflection of the fact that the convergence of the integral $N_{0,4}(a,m)$ requires $a > -1$.

It is possible to show that both expressions (10.17) and (10.18) reduce to (10.2). The details will appear elsewhere.

11. Integrals from Feynman diagrams

The flexibility of the method of brackets is now illustrated by evaluating examples of definite integrals appearing in the resolution of Feynman diagrams. The reader will find in [23] and [19] information about these diagrams. The mathematical theory behind Quantum Field Theory and in particular to the role of Feynman diagrams can be obtained from [12]. The reader will find in [7], [18] and [34] different perspectives on these topics.

Example 11.1. Fig. 1 depicts the interaction of three particles corresponding to the three external lines of momentum P_1, P_2, P_3. In this case the Schwinger parametrization provides the integral

$$G = \frac{(-1)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^\infty \int_0^\infty \int_0^\infty \frac{x_1^{a_1-1} x_2^{a_2-1} x_3^{a_3-1}}{(x_1+x_2+x_3)^{D/2}}$$
$$\times \exp\left(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2\right) \exp\left(-\frac{C_{11}P_1^2 + 2C_{12}P_1 \cdot P_2 + C_{22}P_2^2}{x_1+x_2+x_3}\right) dx_1\, dx_2\, dx_3.$$

The algorithm in [13] and [14] gives the coefficients $C_{i,j}$ as

$$C_{11} = x_1(x_2+x_3), \qquad C_{12} = x_1x_3, \qquad C_{22} = x_3(x_1+x_2). \tag{11.1}$$

Conservation of momentum gives $P_3 = P_1 + P_2$ and replacing the coefficients $C_{i,j}$ we obtain

$$G = \frac{(-1)^{-D/2}}{\prod_{j=1}^3 \Gamma(a_j)} \int_0^\infty \int_0^\infty \int_0^\infty x^{a_1-1} x^{a_2-1} x^{a_3-1}$$
$$\times \frac{\exp(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2) \exp(-\frac{x_1x_2P_1^2 + x_2x_3P_2^2 + x_3x_1P_3^2}{x_1+x_2+x_3})}{(x_1+x_2+x_3)^{D/2}} dx_1\, dx_2\, dx_3.$$

To solve the Feynman diagram in Fig. 1 it is required to evaluate the integral G as a function of the variables $P_1, P_2 \in \mathbb{R}^4$, the masses m_i, the dimension D and the parameters a_i.

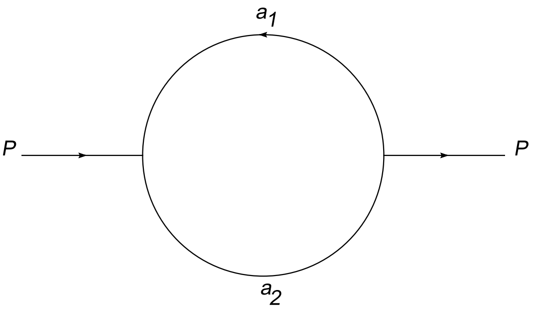

Fig. 2. The bubble.

We now describe the evaluation of the integral G in the special massless situation: $m_1 = m_2 = m_3 = 0$. Moreover we assume that $P_1^2 = P_2^2 = 0$. The integral to be evaluated is then

$$G_1 = \frac{(-1)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int\limits_{\mathbb{R}_+^3} x_1^{a_1-1} x_2^{a_2-1} x_3^{a_3-1} \frac{\exp(-\frac{x_1 x_3}{x_1+x_2+x_3} P_3^2)}{(x_1+x_2+x_3)^{D/2}}\, dx_1\, dx_2\, dx_3.$$

The method of brackets gives

$$G_1 \stackrel{\bullet}{=} \frac{(-1)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \sum_{n_1} \sum_{n_2} \sum_{n_3} \sum_{n_4} \phi_{1234} \big(P_3^2\big)^{n_1} \frac{\Delta_1 \Delta_2 \Delta_3 \Delta_4}{\Gamma(D/2+n_1)}, \tag{11.2}$$

where the brackets Δ_j are given by

$$\Delta_1 = \langle D/2 + n_1 + n_2 + n_3 + n_4 \rangle,$$
$$\Delta_2 = \langle a_1 + n_1 + n_2 \rangle,$$
$$\Delta_3 = \langle a_2 + n_3 \rangle,$$
$$\Delta_4 = \langle a_3 + n_1 + n_4 \rangle.$$

The solution contains no free indices: there are four sums and the linear system corresponding to the vanishing of the brackets eliminates all of them:

$$n_1^* = \frac{D}{2} - a_1 - a_2 - a_3, \qquad n_2^* = -\frac{D}{2} + a_2 + a_3, \qquad n_3^* = -a_2, \qquad n_4^* = -\frac{D}{2} + a_1 + a_2.$$

We conclude that

$$G_1 = \frac{(-1)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \big(P_3^2\big)^{D/2-a_1-a_2-a_3} \times \frac{\Gamma(a_1+a_2+a_3-\frac{D}{2})\Gamma(\frac{D}{2}-a_2-a_3)\Gamma(a_2)\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}-a_1-a_2)}{\Gamma(D-a_1-a_2-a_3)}.$$

Example 11.2. The second example considers the diagram depicted in Fig. 2. The resolution of this diagram is well known and it appears in [6,8,9]. The diagram contains two external lines and two internal lines (propagators) with the same mass m. These propagators are marked 1 and 2.

In momentum variables, the integral representation of this diagram is given by

$$G = \int_{\mathbb{R}^D} \frac{d^D Q}{i\pi^{D/2}} \frac{1}{(Q^2 - m^2)^{a_1}((P - Q)^2 - m^2)^{a_2}}. \tag{11.3}$$

For the diagram considered here, we have $U = x_1 + x_2$ and $F = x_1 x_2 P^2$. The Schwinger representation is given by

$$G = \frac{(-1)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)} \times \int_0^\infty \int_0^\infty \frac{x^{a_1-1} x^{a_2-1}}{(x_1+x_2)^{D/2}} \exp\bigl(m^2(x_1+x_2)\bigr) \exp\left(-\frac{x_1 x_2}{x_1+x_2} P^2\right) dx_1\, dx_2.$$

In order to generate the bracket series for G, we expand first the exponential function to obtain

$$G \doteq \frac{(-1)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)} \sum_{n_1,n_2} \phi_{1,2} \bigl(P^2\bigr)^{n_1} \bigl(-m^2\bigr)^{n_2} \int_{R_+^2} \frac{x_1^{n_1} x_2^{n_2}\, dx_1\, dx_2}{(x_1+x_2)^{D/2+n_1-n_2}}. \tag{11.4}$$

Expanding now the term

$$\frac{1}{(x_1+x_2)^{D/2+n_1-n_2}} \doteq \sum_{n_3,n_4} \phi_{3,4} \frac{x_1^{n_3} x_2^{n_4}}{\Gamma(D/2+n_1-n_2)} \Delta_1, \tag{11.5}$$

with $\Delta_1 = \langle \frac{D}{2} + n_1 - n_2 + n_3 + n_4 \rangle$, and replacing in (11.4) yields

$$G \doteq \frac{(-1)^{-D/2}}{\Gamma(a_1)\Gamma(a_2)} \sum_{n_1,\ldots,n_4} \phi_{1,2,3,4} \frac{(P^2)^{n_1}(-m^2)^{n_2}}{\Gamma(\frac{D}{2}+n_1-n_2)} \Delta_1 \Delta_2 \Delta_3, \tag{11.6}$$

where

$$\begin{aligned} \Delta_1 &= \left\langle \frac{D}{2} + n_1 - n_2 + n_3 + n_4 \right\rangle, \\ \Delta_2 &= \langle a_1 + n_1 + n_3 \rangle, \\ \Delta_3 &= \langle a_2 + n_1 + n_4 \rangle. \end{aligned}$$

The expression for G contains 4 indices and the vanishing of the brackets allows us to express all of them in terms of a single index. We will denote by G_j the expression for G where the index n_j is free.

The sum G_1: in this case the solution of the corresponding linear system is

$$n_2^* = \frac{D}{2} - a_1 - a_2 - n_1, \qquad n_3^* = -a_1 - n_1, \qquad n_4^* = -a_2 - n_1, \tag{11.7}$$

and the sum G_1 becomes

$$G_1 = (-1)^{-D/2} \frac{(-m^2)^{D/2-a_1+a_2}}{\Gamma(a_1)\Gamma(a_2)} \times \sum_{n_1=0}^{\infty} \frac{\Gamma(a_1+a_2-D/2+n_1)\Gamma(a_1+n_1)\Gamma(a_2+n_1)}{\Gamma(a_1+a_2+2n_1)} \frac{(\frac{P^2}{m^2})^{n_1}}{n_1!}.$$

This can be expressed as

$$G_1 = \lambda_1 \left(-m^2\right)^{D/2-a_1+a_2} {}_3F_2 \left(\begin{matrix} a_1+a_2-\frac{D}{2}, & a_1, & a_2 \\ \frac{1}{2}(a_1+a_2+1), & \frac{1}{2}(a_1+a_2) & \end{matrix} \middle| \frac{P^2}{4m^2} \right), \tag{11.8}$$

where

$$\lambda_1 = (-1)^{-D/2} \frac{\Gamma(a_1+a_2-D/2)}{\Gamma(a_1+a_2)}. \tag{11.9}$$

The sum G_2: keeping n_2 as the free index gives

$$n_1^* = \frac{D}{2} - a_1 - a_2 - n_2, \qquad n_3^* = a_2 - \frac{D}{2} + n_2, \qquad n_4^* = a_1 - \frac{D}{2} + n_2,$$

which leads to

$$G_2 = \lambda_2 \left(P_1^2\right)^{D/2-a_1+a_2} {}_3F_2 \left(\begin{matrix} a_1+a_2-\frac{D}{2}, & \frac{1}{2}(1+a_1+a_2-D), & \frac{1}{2}(2+a_1+a_2-D) \\ 1+a_1-\frac{D}{2}, & 1+a_2-\frac{D}{2} & \end{matrix} \middle| \frac{4m^2}{P^2} \right),$$

where the prefactor λ_2 is given by

$$\lambda_2 = (-1)^{-D/2} \frac{\Gamma(a_1+a_2-D/2)\Gamma(\frac{D}{2}-a_1)\Gamma(\frac{D}{2}-a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(D-a_1-a_2)}.$$

The cases G_3 and G_4 are computed by a similar procedure. The results are

$$G_3 = \lambda_3 \left(P_1^2\right)^{-a_1} \left(-m^2\right)^{D/2-a_2} {}_3F_2 \left(\begin{matrix} a_1, & \frac{1}{2}(1+a_1-a_2), & \frac{1}{2}(2+a_1-a_2) \\ 1+a_1-a_2, & 1-a_2+\frac{D}{2} & \end{matrix} \middle| \frac{4m^2}{P^2} \right)$$

and

$$G_4 = \lambda_4 \left(P_1^2\right)^{-a_2} \left(-m^2\right)^{D/2-a_1} {}_3F_2 \left(\begin{matrix} a_2, & \frac{1}{2}(1-a_1+a_2), & \frac{1}{2}(2-a_1+a_2) \\ 1-a_1+a_2, & 1-a_1+\frac{D}{2} & \end{matrix} \middle| \frac{4m^2}{P^2} \right),$$

where the prefactors λ_3 and λ_4 are given by

$$\lambda_3 = (-1)^{-D/2} \frac{\Gamma(a_2-D/2)}{\Gamma(a_2)} \quad \text{and} \quad \lambda_4 = (-1)^{-D/2} \frac{\Gamma(a_1-D/2)}{\Gamma(a_1)}. \tag{11.10}$$

The contributions of these four sums are now classified according to their region of convergence. This is determined by the parameter $\rho = |4m^2/P^2|$. In the region $\rho > 1$, only the sum G_1 converges, therefore $G = G_1$ there. In the region $\rho < 1$ the three remaining sums converge. Therefore, according to Rule 2.4, we have

$$G = \begin{cases} G_1 & \text{for } \rho > 1, \\ G_2 + G_3 + G_4 & \text{for } \rho < 1. \end{cases} \tag{11.11}$$

We have evaluated the Feynman diagram in Fig. 2 and expressed its solution in terms of hypergeometric functions that correspond naturally to the two quotient of the two energy scales presented in the diagram.

12. Conclusions and future work

The method of brackets provides a very effective procedure to evaluate definite integrals over the interval $[0, \infty)$. The method is based on a heuristic list of rules on the bracket series associated to such integrals. In particular we have provided a variety of examples that illustrate the power of this method. A rigorous validation of these rules as well as a systematic study of integrals from Feynman diagrams is in progress.

Traditional methods are capable to compute some of the examples discussed here faster than the method of brackets. These examples are included here simply to illustrate the flexibility of the method of brackets. The main limitation of the method is centered around the fact that it reduces the evaluation of definite integrals to that of a hypergeometric series of several variables.

Future work on the method of brackets will include its mechanization. The rules governing the evaluation of brackets suggest that this is achievable.

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