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Citation: Am. J. Phys. **61**, 650 (1993); doi: 10.1119/1.17174
View online: http://dx.doi.org/10.1119/1.17174
View Table of Contents: http://ajp.aapt.org/resource/1/AJPIAS/v61/i7
Published by the American Association of Physics Teachers

Elementary derivation of the advance of the perihelion of a planetary orbit

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(Received 15 July 1992; accepted 7 December 1992)

An elementary derivation of the law for the advance of the perihelion of a planet in orbit about the sun is given. This is obtained by comparing a Kepler ellipse in a Lorentz coordinate system, with one in Schwarzschild coordinates, related by the areal constant, and attributing the variation entirely to an increase in the angular coordinate. The result is shown to be entirely in agreement with the classical value.

I. INTRODUCTION

The standard treatment for the derivation of the formula for the advance of the perihelia of planetary orbits in general relativity theory involves the solution of the Euler–Lagrange equations for the variational problem where the line element is given in Schwarzschild coordinates.[1] These reduce to an equation similar in form to that of the classical Kepler problem, with an additional quadratic term that causes the solution, even in the lowest order of approximation, to become excessively complicated. Such simplified treatments that have been presented[2,3] deal with the solution of the differential equation that arises. This paper presents a simplified derivation to the same order of approximation that does not require the solution of a differential equation. We compare the classical Kepler orbit in a Lorentz space with an orbit in Schwarzschild space, related by the invariance of Kepler's second law of equal areas being swept out in equal times. This treatment more clearly demonstrates the action of the perturbation effect of curved space-time due to the presence of a gravitating body.

II. COORDINATE TRANSFORMATION

In unperturbed Lorentz coordinates the line element is given by

$$ds^2=c^2dt^2-dr^2-r^2d\vartheta^2-r^2\sin^2\vartheta\, d\varphi^2, \tag{1}$$

whereas in a gravitational field created by a mass M at the origin, the Schwarzschild line element is

$$\begin{aligned}ds^2=&(1-2KM/c^2r)c^2dt^2-(1-2KM/c^2r)^{-1}dr^2\\&-r^2d\vartheta^2-r^2\sin^2\vartheta\, d\phi^2,\end{aligned} \tag{2}$$

where K is the universal gravitational constant.

Considering only the radial and time coordinates in the binomial approximation, the transformation gives

$$dt'=(1-KM/c^2r)dt \quad \text{and} \quad dr'=(1+KM/c^2r)dr. \tag{3}$$

III. KEPLER AND SCHWARZSCHILD ELLIPSES

We consider two elliptical orbits, one the classical Kepler orbit in r,t space and a "Schwarzschild" orbit in an r',t' space. Then in the Lorentz space $dA = \int_0^R r\, dr\, d\vartheta = R^2 d\vartheta/2$ and hence

$$\frac{dA}{dt}=\frac{1}{2}R^2\frac{d\vartheta}{dt}, \tag{4}$$

which is the areal constant of Kepler's second law.

In the Schwarzschild situation however, we have

$$dA'=\int_0^R r\, dr' d\vartheta,$$

with dr' given by Eq. (3).

$$dA'=\int_0^R \left(r+\frac{KM}{c^2}\right) dr d\vartheta=\frac{1}{2}R^2\left(1+\frac{2KM}{c^2R}\right)d\vartheta. \tag{5}$$

Therefore

$$\begin{aligned}\frac{dA'}{dt'}&=(1/2)R^2\left(1+\frac{2KM}{c^2R}\right)\frac{d\vartheta}{dt'}\\&=(1/2)R^2\left(1+\frac{2KM}{c^2R}\right)\left(1+\frac{KM}{c^2R}\right)\frac{d\vartheta}{dt}\\&=(1/2)R^2\left(1+\frac{3KM}{c^2R}\right)\frac{d\vartheta}{dt},\end{aligned} \tag{6}$$

where the binomial approximation has been applied wherever necessary.

Applying all of this increase to increasing the angle element from $d\vartheta$ to $d\vartheta'$ then for a single orbit

$$\int_0^{\Delta\vartheta'} d\vartheta'=\int_0^{\Delta\vartheta=2\pi}(1+3KM/c^2R)d\vartheta. \tag{7}$$

For an ellipse $R=l/(1+\epsilon\cos\vartheta)$ where ϵ is the eccentricity and (essentially) l is the semi-latus rectum.

Therefore,

$$\begin{aligned}\Delta\vartheta'&=2\pi+\frac{3KM}{lc^2}\int_0^{2\pi}(1+\epsilon\cos\vartheta)d\vartheta\\&=2\pi+6KM\pi/lc^2.\end{aligned} \tag{8}$$

Thus the perihelion advance has the standard value $6KM\pi/lc^2$.

With the known values of the constants concerned, this gives in *seconds per century*

$$\text{perihelion advance (secs per century)}=\frac{573\text{ times orbits per annum}}{\text{semi-latus rectum}}, \tag{9}$$

where the semi-latus rectum is to be measured in units of 10^{11} cm.

A comparison of results is shown in Table I.[4,5]

The perihelic advance as predicted by Einstein's formula varies as the $-5/2$ power of the distance of the planet from the sun (Ref. 1 p. 213). It was used thus to distinguish between the relativistic effect and the quadrupole effect of a flattened sun. The latter varies with a $-7/2$ power law.

Table I. Perihelion advance of the planets.

	Observed shift Seconds per century	Distance Semi-latus rectum $\times 10^{11}$ cm	Orbits per annum	Eq. (9)	Classical value
Mercury	43.11 ± 0.45	55.3	4.15	43	43.03
Venus	8.4 ± 4.8	108	1.622	8.6	8.6
Earth	5 ± 1.2	149	1	3.9	3.8
*Icarus	9.8 ± 0.8	60[a]	0.89	8.52	10.3
Mars	(theoretically only)	227	0.531	1.35	...

[a]Icarus is one of the largest Apollo asteroids and was discovered in 1949. Its orbital data is somewhat tentative since it is greatly perturbed by the presence of all the other planets. It has been included here to show the dependence of its highly elliptical orbit ($\epsilon = 0.83$) on the semi-latus rectum and not the mean distance. The value given above for this has been calculated from the known values of the eccentricity and semi-major axis.

To obtain the $-5/2$ law, we invoke Kepler's third law, which states that the square of the period of rotation is proportional to the cube of the mean distance. Thus the number of orbits per annum $N \approx R^{-3/2}$ and in conjunction with Eq. (9) produces precisely that law.

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[2]N. Gauthier, "Periastron precession in general relativity," Am. J. Phys. **55**, 85–86 (1987).

[3]M. G. Olsson, "General relativistic perihelion precession and the anharmonic oscillator," Am. J. Phys. **56**, 89–91 (1988).

[4]S. Weinberg, *Cosmology and Gravitation* (Wiley, New York, 1972), p. 198.

[5]I. I. Shapiro, W. B. Smith, M. E. Ash, and S. Herrick, "General relativity and the orbit of Icarus," Astrophys. J. **76**, 588–606 (1971).

A computer-based data acquisition laboratory for undergraduates

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(Received 27 July 1992; accepted 9 December 1992)

An electronics course for physics majors is described that introduces students to modern data acquisition methods using computer-based equipment and high level software. The course comprises lectures and extensive laboratory work with standard personal computers, research quality data acquisition hardware, and programmable instruments. The laboratory work consists of a mixture of standard laboratory exercises and open-ended projects.

I. OVERVIEW

The role of computers in all areas of physics has grown rapidly in recent years. For experimental physics the personal computer often provides a suitable platform for use in data acquisition and experiment control, and the ability to employ personal computers in such tasks is now an important skill. This skill in the past has most often been acquired when needed. For undergraduate students such exposure may be available only through participation in an ongoing research project and may be very limited in scope. For graduate students, the learning process usually begins once they are engaged in a research topic. In modernizing a laboratory electronics sequence here at UMD, we have sought to provide a coherent introduction to the essential aspects of the use of computers for data acquisition tasks to better equip our students for both undergraduate research participation and for graduate work or employment. One of the daunting aspects of updating such a laboratory is the frenetic pace at which hardware and software are evolving. Most plans are, more likely than not, at least slightly technologically outdated by the time they are fully implemented. In producing a new laboratory environment we sought to assemble a workstation for student use that would reflect some important changes in the way the experimental physicist uses the personal computer in the lab. Chief among our goals were (1) the transition to the use of commercially available data acquisition boards and programmable instrumentation in place of the chip-level circuit tasks our students had previously encountered, and (2) the use of truly high level integrated data acquisition software as opposed to assembly language or even conventional language coding. In this respect, it differs in flavor from a recently described project with similar intentions.[1] In approaching this project, we hoped to provide an opportunity to do some physics along the way, as the new tools are being learned.

The greatest changes have been implemented in the third course of a three-quarter electronics sequence. These courses attract upper division physics majors and minors, undergraduates from other sciences and various engineering fields, and for the latter parts, physics M.S. students. Each week the course meets for two 50-min lectures and a lab, which is 2 hours in the first quarter and expands to 5 in the latter quarters. The first quarter of the sequence is a fairly standard analog electronics course, with an emphasis on developing the ability to design circuits based on available integrated circuits, primarily operational amplifiers, but including exposure to more sophisticated building blocks such as voltage controlled oscillators, multipliers,