Chapter 7. Electrodynamics

7.3 Maxwell's Equations

7.3.1 Electrodynamics Before Maxwell

So far, in the electromagnetic theory

(i) $\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$ (Gauss's law),

(ii) $\nabla \cdot \mathbf{B} = 0$ (no name),

(iii) $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ (Faraday's law),

(iv) $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ (Ampère's law).

(v) $\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$ **(Charge conservation law)**
(→ Continuity equation)

$\nabla \cdot \mathbf{J} = 0$ → For steady current

But, there is a fatal inconsistency

$$\nabla \cdot (\nabla \times \mathbf{E}) = \nabla \cdot \left(-\frac{\partial \mathbf{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B})$$

(Divergence of curl = 0)
($\nabla \cdot \mathbf{B} = 0$) ➔ *It's OK.*

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 (\nabla \cdot \mathbf{J})$$

$\nabla \cdot (\nabla \times \mathbf{B}) = 0$
$\nabla \cdot \mathbf{J} = 0$
➔ *It's OK only for steady current.*

For nonsteady currents,

$\nabla \cdot (\nabla \times \mathbf{B}) = 0$
$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \neq 0$
➔ ***Inconsistent!***

➔ ***Ampere's law cannot be right for nonsteady currents!***

Electrodynamics Before Maxwell (James Clerk Maxwell)

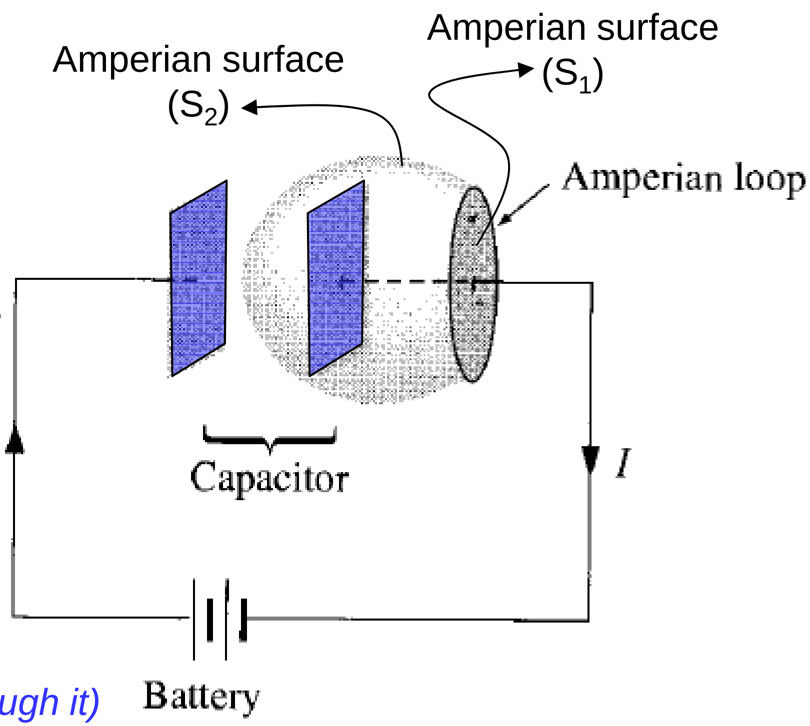
There's another way to see that Ampere's law is bound to fail for nonsteady current.

Consider the process of charging up a capacitor.

Ampere's law reads,

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc}$$

We want to apply it to the Amperian loop or Amperian surface.



→ How do we determine I_{enc}?

→ the total current passing through the loop,
→ or, more precisely, the current piercing a surface (S_1, or S_2) that has the loop for its boundary.

→ *For the surface S_1, $I_{enc} = I$*
→ *For the surface S_2, $I_{enc} = 0$ (No current passes through it)*

The conflict arises only when charge is piling up somewhere (in this case, on the capacitor plates).
➔ For nonsteady currents, "the current enclosed by a loop" is an ill-defined notion, since it depends entirely on what surface you use.

➔ Maxwell fixed it by purely theoretical arguments! (1861)

➔ The real confirmation of Maxwell's theory came in 1888 with Hertz's experiments on electromagnetic waves.

7.3.2 How Maxwell Fixed Ampere's Law

The inconsistent problem arose on Ampere's law when: $\nabla \cdot (\nabla \times \mathbf{B}) \neq \nabla \cdot \mathbf{J}$ for nonsteady currents:

James Clerk Maxwell, "A dynamical theory of the electromagnetic field" (1865).

Applying the continuity equation and Gauss's law,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\epsilon_0 \nabla \cdot \mathbf{E}) = -\nabla \cdot \left(\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

$$\nabla \cdot \left(\mathbf{J} + \varepsilon_0 \frac{\partial E}{\partial t}\right) = 0$$

$$\mathbf{J} \rightarrow \left(\mathbf{J} + \varepsilon_0 \frac{\partial E}{\partial t}\right)$$

$$\nabla \cdot (\nabla \times \mathbf{B}) = \nabla \cdot \left(\mathbf{J} + \varepsilon_0 \frac{\partial E}{\partial t}\right) = 0$$

→ The inconsistency in Ampere's law is now cured.

Ampere's law can generally be expressed as

$$\nabla \times \mathbf{B} = \varepsilon_0 \mathbf{J} \longrightarrow \boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}}$$

How Maxwell Fixed Ampere's Law

Faraday's law: $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ ➔ *A changing magnetic field induces an electric field.*

Ampere's law: $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ ➔ *A steady current induces a magnetic field.*

Maxwell:

$$\mathbf{J} \rightarrow \left(\mathbf{J} + \varepsilon_0 \frac{\partial E}{\partial t} \right)$$

$$\mathbf{J}_d \equiv \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

➔ Displacement current

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}}$$

➔ *A changing electric field induces a magnetic field.*

How Maxwell Fixed Ampere's Law

Let's see now how the displacement current resolves the paradox of the charging capacitor.

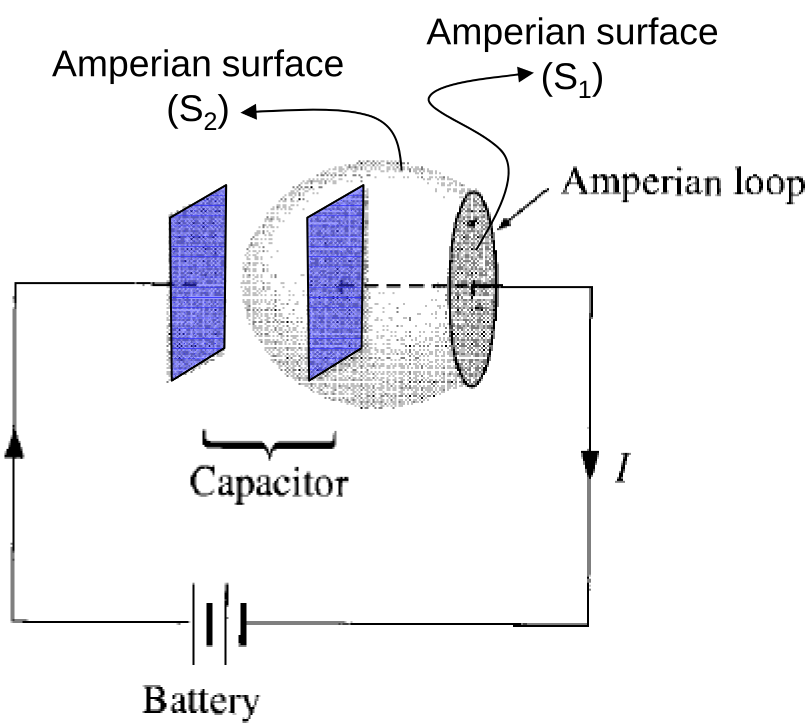
The electric field between the capacitor plates is

$$E = \frac{1}{\epsilon_0}\sigma = \frac{1}{\epsilon_0}\frac{Q}{A}$$

$$\frac{\partial E}{\partial t} = \frac{1}{\epsilon_0 A}\frac{dQ}{dt} = \frac{1}{\epsilon_0 A} I$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\longrightarrow \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} + \mu_0 \epsilon_0 \int \left(\frac{\partial \mathbf{E}}{\partial t}\right) \cdot d\mathbf{a}$$



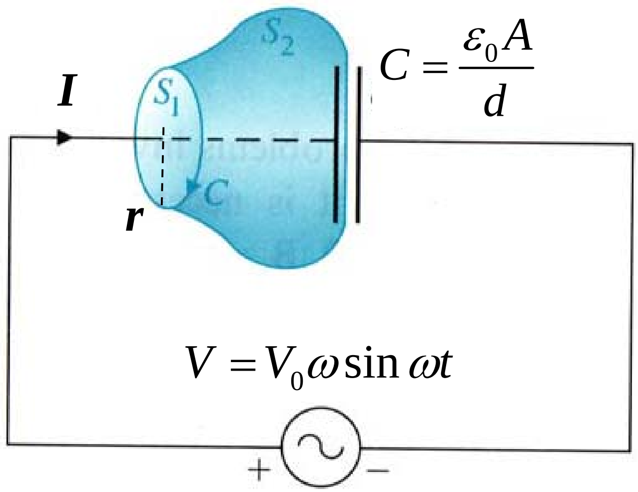
→ *For the flat surface S_1, $\mathbf{E} = 0$ and $I_{enc} = I$.* $\longrightarrow \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I$

→ *For the balloon-shape surface S_2, $I_{enc} = 0$, but* $\int (\partial \mathbf{E}/\partial t) \cdot d\mathbf{a} = I/\epsilon_0$.

➔ The same answer for either surface!

How Maxwell Fixed Ampere's Law

(Example) Verify that the displacement current = conduction current in the wire.



(Example) (a) Determine the magnetic field at a distance r from the wire.

(b) Determine the magnetic field in the capacitor at a distance r from the axis.

*(a) For the flat surface S_1, **E** = 0 and* $I_{enc} = I$,

(b) In the capacitor, the Amperian loop at radius r,

$$J_d = I_d / A = I / A$$

7.3.3 Maxwell's Equations

(i) $$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$$ (Gauss's law),

(ii) $$\nabla \cdot \mathbf{B} = 0$$ (no name),

(iii) $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$ (Faraday's law),

(iv) $$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$ (Ampère's law with Maxwell's correction).

Together with the force law, $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$,

➔ ***they summarize the entire theoretical content of classical electrodynamics***.

(Note again, the continuity equation, $\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$, can be derived from Maxwell's equations.)

Maxwell's Equations

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

➔ Electric fields can be produced either by charges (ρ) *or* by changing magnetic fields.

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

➔ Magnetic fields can be produced either by current (**J**) *or* by changing electric fields.

It may logically be preferable to write with the sources (ρ and J) on the right.

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

➔ All electromagnetic fields (**E** and **B**) are ultimately attributable to charges and currents.

➔ ***Maxwell's equations tell you how (static or dynamic) charges produce fields.***

➔ ***The force law tells you how fields affect charges.***

7.3.4 Magnetic Charge?

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} - \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J} \longrightarrow \nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

➔ ***The symmetry between E and B is spoiled by the charge term and the current.***

If we had ρ_m (the density of magnetic charge) and $\mathbf{J}_m$ (the current of magnetic charge),

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = -\mu_0 \mathbf{J}_m$$

$$\nabla \cdot \mathbf{B} = \mu_0 \rho_m$$

$$\nabla \times \mathbf{B} - \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J} \longrightarrow \nabla \cdot \mathbf{J}_m = -\frac{\partial \rho_m}{\partial t}$$

If we replace $\mathbf{E} \to \mathbf{B}, \quad \mathbf{B} \to -\mu_0\varepsilon_0\mathbf{E}$

➔ ***There could be a pleasing symmetry between E and B.***

In a sense, Maxwell's equations beg for magnetic charge to exist.
And yet, in spite of a diligent search, no one has ever found any charge.
➔ ***Apparently God just didn't make any magnetic charge.***

7.3.5 Maxwell's Equations in Matter

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\mathbf{D} = \epsilon \mathbf{E} \qquad \epsilon \equiv \epsilon_0(1+\chi_e)$$

$$\mathbf{H} = \frac{1}{\mu}\mathbf{B} \qquad \mu \equiv \mu_0(1+\chi_m)$$

$$\nabla \cdot \mathbf{D} = \rho_f$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

The total charge density: $\rho = \rho_f + \rho_b = \rho_f - \nabla \cdot \mathbf{P}$

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}(\rho_f - \nabla \cdot \mathbf{P}) \xrightarrow{\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}} \nabla \cdot \mathbf{D} = \rho_f$$

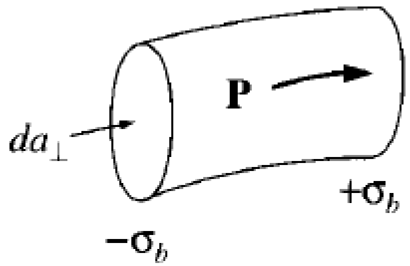
The total current density: $\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b + \mathbf{J}_p = \mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}$

$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t} \rightarrow$???

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \right) + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \xrightarrow{\mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}} \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

Maxwell's Equations in Matter

$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}$$ → ***Polarization current density***



The polarization ***P*** introduces a charge density $\boldsymbol{\sigma_b}$ **=** ***P*** at one end and **-**$\boldsymbol{\sigma_b}$ at the other.

If ***P*** now *increases* a bit, the charge on each end increases accordingly, giving a net current,

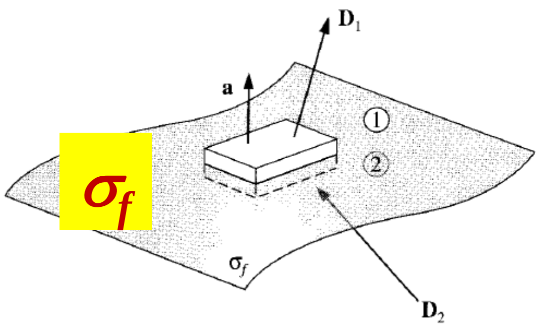
$$dI = \frac{\partial \sigma_b}{\partial t} da_\perp = \frac{\partial P}{\partial t} da_\perp$$ → The current density, therefore, → $$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}$$

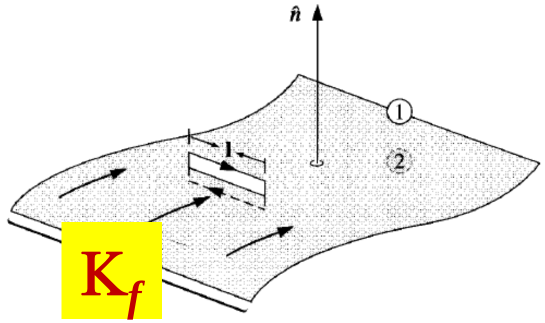
$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}$$ → It is consistent with the continuity equation (Check!)

$$\nabla \cdot \mathbf{J}_p = \nabla \cdot \frac{\partial \mathbf{P}}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{P}) = -\frac{\partial \rho_b}{\partial t}$$ (Conservation of bound charge)

7.3.6 Boundary Conditions

The fields **E, B, D,** and **H** *will be discontinuous* at a boundary between two different media, or at a surface that carries charge density σ or current density **K**.





The integral form of Maxwell's equations can deduct the boundary conditions.

$$\nabla \cdot \mathbf{D} = \rho_f \qquad \oint_S \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \oint_{\mathcal{P}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{a}$$

Integral form

$$\nabla \cdot \mathbf{B} = 0 \qquad \oint_S \mathbf{B} \cdot d\mathbf{a} = 0$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \qquad \oint_{\mathcal{P}} \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}} + \frac{d}{dt} \int_S \mathbf{D} \cdot d\mathbf{a}$$

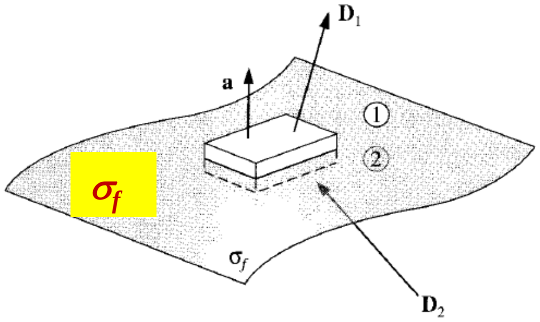
Boundary Conditions

For a tiny, wafer-thin Gaussian pillbox extending just slightly into the material on either side of the boundary,

$$\oint_S \mathbf{D}\cdot d\mathbf{a} = Q_{f_{\text{enc}}} \longrightarrow \mathbf{D}_1\cdot\mathbf{a} - \mathbf{D}_2\cdot\mathbf{a} = \sigma_f\, a$$

$$\longrightarrow \boxed{D_1^{\perp} - D_2^{\perp} = \sigma_f}$$

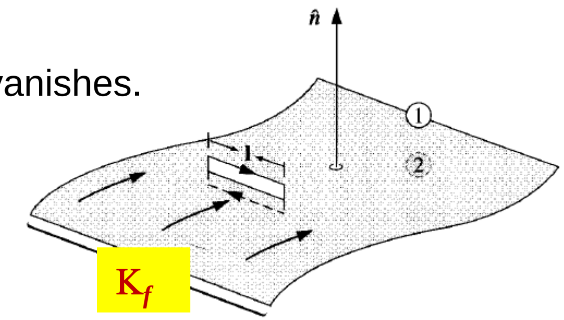
$$\oint_S \mathbf{B}\cdot d\mathbf{a} = 0 \longrightarrow \boxed{B_1^{\perp} - B_2^{\perp} = 0}$$



For a very thin Amperian loop straddling the surface,

$$\oint_{\mathcal{P}} \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{a}$$ → But in the limit as the width of the loop goes to zero, the flux vanishes.

$$\longrightarrow \boxed{\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0}$$



$$\oint_{\mathcal{P}} \mathbf{H}\cdot d\mathbf{l} = I_{f_{\text{enc}}} + \frac{d}{dt}\int_S \mathbf{D}\cdot d\mathbf{a}$$ → But in the limit as the width of the loop goes to zero, the displacement current vanishes.

$$\longrightarrow \mathbf{H}_1\cdot\mathbf{l} - \mathbf{H}_2\cdot\mathbf{l} = I_{f_{\text{enc}}}$$

$$I_{f_{\text{enc}}} = \mathbf{K}_f\cdot(\hat{\mathbf{n}}\times\mathbf{l}) = (\mathbf{K}_f\times\hat{\mathbf{n}})\cdot\mathbf{l}$$

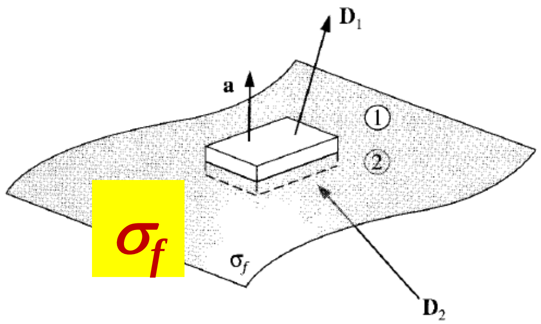
$$\longrightarrow \boxed{\mathbf{H}_1^{\parallel} - \mathbf{H}_2^{\parallel} = \mathbf{K}_f\times\hat{\mathbf{n}}}$$

Boundary Conditions

General boundary conditions for electrodynamics,

$$D_1^{\perp} - D_2^{\perp} = \sigma_f \qquad \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0$$

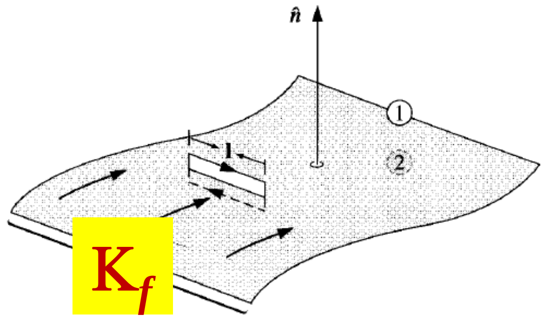
$$B_1^{\perp} - B_2^{\perp} = 0 \qquad \mathbf{H}_1^{\parallel} - \mathbf{H}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}$$



In the case of linear media,

$$\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f \qquad \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0$$

$$B_1^{\perp} - B_2^{\perp} = 0 \qquad \frac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}.$$



If there is no free charge or free current at the interface,

$$\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = 0 \qquad \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0$$

$$B_1^{\perp} - B_2^{\perp} = 0 \qquad \frac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} = 0$$

Intermission on Page 343

(i) $\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$ (Gauss's law),

(ii) $\nabla \cdot \mathbf{B} = 0$ (no name),

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(iv) $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ (Ampère's law with Maxwell's correction).

Together with the force law, $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$,

All of our cards are now on the table, and in a sense my job is done.
In the first seven chapters we assembled electrodynamics piece by piece,
and **now, with Maxwell's equations in their final form, the theory is complete.**
There are no more laws to be learned, no further generalizations to be considered,
and (with perhaps one exception) no lurking inconsistencies to be resolved.
If yours is a one-semester course, this would be a reasonable place to stop.

But in another sense we have just arrived at the starting point.
We are at last in possession of a full deck, and **we know the rules of the game -- it's time to deal.**
This is the fun part, in which one comes to appreciate the extraordinary power and richness of electrodynamics.
In a full-year course **there should be plenty of time to cover the remaining chapters**, and perhaps to supplement them with a unit on plasma physics, say, or AC circuit theory, or even a little General Relativity.
But if you have room only for one topic, I'd recommend Chapter 9, on Electromagnetic Waves (you'll probably want to skim Chapter 8 as preparation).
This is the segue to Optics, and is historically the most important application of Maxwell's theory.