Universidad de Valparaíso
CHILE

PAUTA.

Certámen IV
(Recuperativo)
Métodos Matemáticos de la Física II
Licenciatura en Física - 2017
IPGG

Problema I

Si $\left[\left[\widehat{\mathbf{A}},\widehat{\mathbf{B}}\right],\widehat{\mathbf{A}}\right]=0$, demuestre que:

$$\left[\widehat{\mathbf{A}}^n,\widehat{\mathbf{B}}\right]=n\widehat{\mathbf{A}}^{n-1}\left[\widehat{\mathbf{A}},\widehat{\mathbf{B}}\right]$$

para $n\in\mathbb{N}$.

Problema II

Considere un espacio bidimensional donde un operador hermitiano está definido como $\hat{\mathbf{A}}\left|1\right\rangle=\left|1\right\rangle$ y $\hat{\mathbf{A}}\left|2\right\rangle=-\left|2\right\rangle$, tal que $\left|1\right\rangle$ y $\left|2\right\rangle$ son ortonormales.

a).- Considere el operador $\hat{\mathbf{B}}=\left|1\right\rangle\left\langle 2\right|$. ¿Es hermitiano?. Muestre que $\hat{\mathbf{B}}^2=0$.

b).- Muestre que $\left(\hat{\mathbf{B}}\hat{\mathbf{B}}^\dagger\right)^2=\hat{\mathbf{B}}\hat{\mathbf{B}}^\dagger$.

c).- Muestre que $\hat{\mathbf{B}}\hat{\mathbf{B}}^\dagger-\hat{\mathbf{B}}^\dagger\hat{\mathbf{B}}$ es unitario.

d).- Considere $\hat{\mathbf{C}}=\hat{\mathbf{B}}\hat{\mathbf{B}}^\dagger+\hat{\mathbf{B}}^\dagger\hat{\mathbf{B}}$. Muestre que $\hat{\mathbf{C}}\left|1\right\rangle=\left|1\right\rangle$ y $\hat{\mathbf{C}}\left|2\right\rangle=\left|2\right\rangle$.

Problema III

Evalúe la siguiente integral mediante *IBD*:

$$I=\int_0^\infty\frac{\cos(2x)\sin(x)}{x^{\frac{1}{2}}}\,dx$$

Problema IV

Considere la matriz:

$$\hat{\mathbf{A}}=\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}$$

Demuestre que $\exp\left(x\hat{\mathbf{A}}\right)=\cosh(x)+\hat{\mathbf{A}}\sinh(x)$.

— PAUTA PRUEBA RECUPERATIVA —

PROBLEMA 1)

Si $[[\hat{A},\hat{B}],\hat{A}]=0 \Rightarrow \hat{A}[\hat{A},\hat{B}]=[\hat{A},\hat{B}]\hat{A}$

Conmutan

veamos:

* para $n=2$

$[\hat{A}^2,\hat{B}]=\hat{A}[\hat{A},\hat{B}]+[\hat{A},\hat{B}]\hat{A}$; pero $[\hat{A},\hat{B}]\hat{A}=\hat{A}[\hat{A},\hat{B}]$

$$\therefore \quad [\hat{A}^2,\hat{B}]=2\hat{A}[\hat{A},\hat{B}] \quad (i)$$

* para $n=3$

$$[\hat{A}^3,\hat{B}]=[\hat{A}^2\hat{A},\hat{B}]=\hat{A}^2[\hat{A},\hat{B}]+[\hat{A}^2,\hat{B}]\hat{A}$$

usando ec. (i)

$$=\hat{A}^2[\hat{A},\hat{B}]+2\hat{A}[\hat{A},\hat{B}]\hat{A}$$

$$=\hat{A}^2[\hat{A},\hat{B}]+2\hat{A}\hat{A}[\hat{A},\hat{B}]$$

$$=3\hat{A}^2[\hat{A},\hat{B}] \quad (ii)$$

A partir de aquí se puede generalizar:

$$[\hat{A}^n,\hat{B}]=n\hat{A}^{n-1}[\hat{A},\hat{B}] \quad /\!/. \quad \text{Q.E.D.}$$

PROBLEMA 2)

a) si $\hat{B} = |1\rangle\langle 2|$

$\Downarrow$

$\hat{B}^\dagger = (|1\rangle\langle 2|)^\dagger = |2\rangle\langle 1| \neq \hat{B}$ ($\hat{B}$ no es hermitiano)

Por otro lado

$$\hat{B}^2 = (|1\rangle\langle 2|)(|1\rangle\langle 2|) = |1\rangle \overset{0}{\cancel{\langle 2|1\rangle}} \langle 2| = 0$$

SON ORTONORMALES.

b) $$(\hat{B}\hat{B}^\dagger)^2 = \hat{B}\hat{B}^\dagger\hat{B}\hat{B}^\dagger = \underbrace{|1\rangle\langle 2|}_{\hat{B}}\underbrace{|2\rangle\langle 1|}_{\hat{B}^\dagger}\underbrace{|1\rangle\langle 2|}_{\hat{B}}\underbrace{|2\rangle\langle 1|}_{\hat{B}^\dagger}$$

(with $\langle 2|2\rangle = 1$, $\langle 1|1\rangle = 1$, $\langle 2|2\rangle = 1$)

$$= |1\rangle\langle 1| = |1\rangle \underset{\substack{\uparrow \\ \text{escalar}}}{1} \langle 1|$$

$$= \underbrace{|1\rangle\langle 2|}_{\hat{B}}\underbrace{|2\rangle\langle 1|}_{\hat{B}^\dagger}$$

$$= \hat{B}\hat{B}^\dagger \;///.$$

c) $$(\hat{B}\hat{B}^\dagger - \hat{B}^\dagger\hat{B})^\dagger(\hat{B}\hat{B}^\dagger - \hat{B}^\dagger\hat{B}) = (\hat{B}\hat{B}^\dagger - \hat{B}^\dagger\hat{B})(\hat{B}\hat{B}^\dagger - \hat{B}^\dagger\hat{B})$$

se usó $(\hat{B}^\dagger\hat{B})^\dagger = \hat{B}^\dagger\hat{B}$

$(\hat{B}\hat{B}^\dagger)^\dagger = \hat{B}\hat{B}^\dagger$

Luego expandimos el producto binomial

$$= \hat{B}\hat{B}^\dagger\hat{B}\hat{B}^\dagger - \hat{B}\hat{B}^\dagger\hat{B}^\dagger\hat{B} - \hat{B}^\dagger\hat{B}\hat{B}\hat{B}^\dagger + \hat{B}^\dagger\hat{B}\hat{B}^\dagger\hat{B}$$

$$= (\hat{B}\hat{B}^\dagger)^2 - \hat{B}(\hat{B}^\dagger)^2\hat{B} - \hat{B}^\dagger(\hat{B}^2)\hat{B}^\dagger + (\hat{B}^\dagger\hat{B})^2$$

de (a) $\hat{B}^2 = 0$

de (b) $(\hat{B}\hat{B}^\dagger)^2 = \hat{B}\hat{B}^\dagger$

por otro lado $(\hat{B}^\dagger)^2 = \hat{B}^\dagger\hat{B}^\dagger = |2\rangle\overbrace{\cancel{\langle 1|2\rangle}}^{0}\langle 1| = 0$

y $(\hat{B}^\dagger\hat{B})^2 = \hat{B}^\dagger\hat{B}\hat{B}^\dagger\hat{B} = |2\rangle\cancel{\langle 1|1\rangle}\cancel{\langle 2|2\rangle}\cancel{\langle 1|1\rangle}\langle 2|$

$$= |2\rangle\langle 2| = |2\rangle \underbrace{1}_{\text{escalar}} \langle 2| = |2\rangle\langle 1|1\rangle\langle 2|$$

$$= \hat{B}^\dagger\hat{B}$$

$\therefore$

$$(\hat{B}\hat{B}^\dagger - \hat{B}^\dagger\hat{B})^\dagger(\hat{B}\hat{B}^\dagger - \hat{B}^\dagger\hat{B}) = \hat{B}\hat{B}^\dagger + \hat{B}^\dagger\hat{B} = |1\rangle\langle 2|2\rangle\langle 1| + |2\rangle\langle 1|1\rangle\langle 2|$$

$$= |1\rangle\langle 1| + |2\rangle\langle 2| = \hat{C} \overset{?}{=} \hat{\mathbb{1}}$$

Es la identidad

obs. Matricialmente

$$(\hat{C})_{11} = \langle 1|\hat{C}|1\rangle = 1 = (\hat{C})_{22} = \langle 2|\hat{C}|2\rangle$$

$$(\hat{C})_{12} = \langle 1|\hat{C}|2\rangle = 0 = (\hat{C})_{21} = \langle 2|\hat{C}|1\rangle$$

$\therefore$ $\hat{B}\hat{B}^{+} - \hat{B}^{+}\hat{B}$ es unitario

d) Del item anterior se verifica rápidamente que

si $\hat{C} = \hat{B}\hat{B}^{+} + \hat{B}^{+}\hat{B}$ entonces $\hat{C} = \hat{\mathbb{1}}$

$\therefore$ $\hat{C}|1\rangle = |1\rangle$

$\hat{C}|2\rangle = |2\rangle$ //

PROBLEMA 3

En general

$$I = \int_0^\infty e^{\beta s t} f(t)\,dt = f\left(\frac{1}{\beta}\partial_s\right)\frac{1}{s}\Bigg|_{\beta=-1}$$

ó

$$I = \int_0^\infty f(t)\,dt = f\left(\frac{1}{\beta}\partial_s\right)\frac{1}{s}\Bigg|_{\substack{s=0\\ \beta=-1}}$$

fórmula útil si hay potencias fraccionales de t en la integral.

Obs.

(*) $$\frac{d^n}{ds^n} s^k = \frac{\Gamma(k+1)}{\Gamma(k-n+1)} s^{k-n} \qquad k \geq 0$$

(**) $$\frac{d^n}{ds^n} s^{-k} = (-1)^n \frac{\Gamma(k+n)}{\Gamma(k)} s^{-k-n} \qquad k > 1.$$

se utiliza (**) en esta integral con $k=1$.

luego

$$I = \int_0^\infty \frac{\cos(2x)\,\mathrm{sen}(x)}{x^{1/2}}\,dx = \frac{\cos\left(\frac{2}{\beta}\partial_s\right)\mathrm{sen}\left(\frac{1}{\beta}\partial_s\right)}{\left(\frac{1}{\beta}\partial_s\right)^{1/2}}\frac{1}{s}\Bigg|_{\substack{\beta=-1\\ s=0}}$$

luego

$$I = \beta^{1/2} \cos\left(\tfrac{2}{\beta}\partial s\right) \operatorname{sen}\left(\tfrac{1}{\beta}\partial s\right) \partial_s^{-1/2} \frac{1}{s}\Bigg|_{\substack{s=0\\ \beta=-1}}$$

$$= (-1)^{1/2} \cos(2\partial s)\, \operatorname{sen}(-\partial s)\, \partial_s^{-1/2} \frac{1}{s}\Bigg|_{s=0.}$$

donde $\partial_s^{-1/2} \frac{1}{s} = (-1)^{-1/2} \frac{\Gamma(1-1/2)}{\Gamma(1)} s^{-1+1/2}$

$$= (-1)^{-1/2}\, \Gamma(1/2)\, s^{-1/2}$$

$\therefore$

$$I = \cancel{(-1)^{1/2}} \cos(2\partial_s) \operatorname{sen}(-\partial_s)\, \cancel{(-1)^{-1/2}}\, \Gamma(1/2)\, s^{-1/2}\Big|_{s=0}$$

$$I = -\Gamma(1/2) \cos(2\partial_s)\, \operatorname{sen}(\partial_s)\, s^{-1/2}\Big|_{s=0.}$$

<u>Obs.</u> $\Gamma(1/2) = \sqrt{\pi}$

$\therefore$

$$I = -\sqrt{\pi} \cos(2\partial_s)\, \operatorname{sen}(\partial_s)\, s^{-1/2}\Big|_{s=0}$$

Obs. $\cos(2\partial s) = \frac{1}{2}\left[e^{i2\partial s} + e^{-i2\partial s}\right]$

$$\operatorname{sen}(\partial s) = \frac{1}{2i}\left[e^{i\partial s} - e^{-i\partial s}\right]$$

$$\Downarrow$$

$$\therefore \quad \cos(2\partial s)\operatorname{sen}(\partial s) = \frac{1}{4i}\left[e^{3i\partial s} - e^{i\partial s} + e^{-i\partial s} - e^{-3i\partial s}\right]$$

luego

$$I = -\sqrt{\pi}\,\frac{1}{4i}\underbrace{\left[\left(e^{3i\partial s} - e^{i\partial s}\right) - \left(e^{-3i\partial s} - e^{-i\partial s}\right)\right]}_{2i\,\mathrm{Im}\left(e^{3i\partial s} - e^{i\partial s}\right)}\frac{1}{s^{1/2}}\Bigg|_{s=0}$$

$$= -\frac{\sqrt{\pi}}{2}\,\mathrm{Im}\left[e^{3i\partial s} - e^{i\partial s}\right]\frac{1}{s^{1/2}}\Bigg|_{s=0}$$

$$= -\frac{\sqrt{\pi}}{2}\,\mathrm{Im}\left[\frac{1}{(s+3i)^{1/2}} - \frac{1}{(s+i)^{1/2}}\right]\Bigg|_{s=0}$$

$$= -\frac{\sqrt{\pi}}{2}\,\mathrm{Im}\left[\frac{1}{(3i)^{1/2}} - \frac{1}{i^{1/2}}\right]$$

donde

$$\frac{1}{(3i)^{1/2}} = \frac{1}{(3e^{i\pi/2})^{1/2}} = \frac{1}{\sqrt{3}} e^{-i\pi/4} = \frac{1}{\sqrt{3}}\left(\cos\frac{\pi}{4} - i\,\mathrm{sen}\frac{\pi}{4}\right) \quad \boxed{8}$$

$$\frac{1}{i^{1/2}} = \frac{1}{e^{i\pi/4}} = \cos\pi/4 - i\,\mathrm{sen}\,\pi/4.$$

luego

$$\mathrm{Im}\left[\frac{1}{(3i)^{1/2}} - \frac{1}{i^{1/2}}\right] = -\frac{1}{\sqrt{3}}\,\mathrm{sen}\frac{\pi}{4} + \mathrm{sen}\frac{\pi}{4}.$$

donde $\mathrm{sen}\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

$\therefore$

$$I = -\frac{\sqrt{\pi}}{2}\left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}\sqrt{3}}\right] = -\frac{\sqrt{\pi}}{2}\left[\frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{6}\right]$$

$$= -\frac{\sqrt{\pi}}{12}\left[3\sqrt{2} - \sqrt{6}\right]$$

$$= \frac{\sqrt{\pi}}{12}\left[\sqrt{6} - 3\sqrt{2}\right]$$

PROBLEMA IV)

Obs.

$$e^{x\hat{A}} = \sum_{n=0}^{\infty} x^n \frac{\hat{A}^n}{n!}$$

si $\hat{A} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \Rightarrow \hat{A}^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}.$

$\Downarrow$

$\hat{A}^n = \hat{\mathbb{1}}$ para n PAR

$\hat{A}^n = \hat{A}$ para n IMPAR

$$\therefore e^{x\hat{A}} = \frac{\mathbb{1}}{0!} + \frac{x\hat{A}}{1!} + \frac{x^2\hat{A}^2}{2!} + \frac{x^3\hat{A}^3}{3!} + \frac{x^4\hat{A}^4}{4!} + \frac{x^5\hat{A}^5}{5!} + \ldots$$

$$= \hat{\mathbb{1}} + \frac{x\hat{A}}{1!} + \frac{x^2}{2!} + \frac{x^3\hat{A}}{3!} + \frac{x^4}{4!} + \frac{x^5\hat{A}}{5!} + \ldots$$

$$= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right) + \hat{A}\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)$$

$$= \hat{\mathbb{1}} \cosh(x) + \hat{A}\,\mathrm{senh}(x)$$ //h. QED.