Intermission
Page 343, Griffith

All of our cards are now on the table, and in a sense my job is done. In the first seven chapters we assembled electrodynamics piece by piece, and now, with Maxwell's equations in their final form, the theory is complete. There are no more laws to be learned, no further generalizations to be considered, and (with perhaps one exception) no lurking inconsistencies to be resolved. If yours is a one-semester course, this would be a reasonable place to stop.

But in another sense we have just arrived at the starting point. We are at last in possession of a full deck, and we know the rules of the game—it's time to deal. This is the fun part, in which one comes to appreciate the extraordinary power and richness of electrodynamics. In a full-year course there should be plenty of time to cover the remaining chapters, and perhaps to supplement them with a unit on plasma physics, say, or AC circuit theory, or even a little General Relativity. But if you have room only for one topic, I'd recommend Chapter 9, on Electromagnetic Waves (you'll probably want to skim Chapter 8 as preparation). This is the segue to Optics, and is historically the most important application of Maxwell's theory.

Chapter 8. Conservation Laws

(Page 346, Griffith)

Lecture :

Electromagnetic Power Flow

Flow of Electromagnetic Power

- Electromagnetic waves transport throughout space the energy and momentum arising from a set of charges and currents (the sources).
- If the electromagnetic waves interact with another set of charges and currents in a receiver, information (energy) can be delivered from the sources to another location in space.
- The energy and momentum exchange between waves and charges and currents is described by the Lorentz force equation.

- **Poynting's theorem concerns the conservation of energy** for a given volume in space.
- Poynting's theorem is a consequence of Maxwell's equations.

Poynting's Theorem is the "Work-energy theorem" "Conservation of Energy"

"The work done on the charges by the electromagnetic force is equal to the decrease in energy stored in the field, less the energy that flowed out through the surface" .

$$\boxed{\frac{dW}{dt} = \int_V (E \cdot J)\, dv = -\frac{d}{dt}\int_V \left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2\mu}B^2\right) dv - \frac{1}{\mu}\oint_S (E \times B) \cdot ds}$$

Work done by the EM field

Total energy stored in the EM field

Energy flowed out through the surface

$$S \equiv \frac{1}{\mu}(E \times B)$$: Poynting vetor

Poynting's Theorem

$$\boxed{\frac{dW}{dt} = -\frac{dU_{em}}{dt} - \oint_S S \cdot ds}$$

Proof: → Total energy stored in Electromagnetic fields is

$$U_{em} = \int_V \left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2\mu} B^2 \right) dv$$

Suppose we have some charge and current configuration, produces fields E and B at time t.
Work done on a charge *q* in the interval *dt* is

$$dW = F \cdot dl = q(E + v \times B) \cdot vdt = qE \cdot vdt$$

$$Now, \quad q = \rho\, d\tau \;\; and \;\; \rho v = J.$$

$$\frac{dW}{dt} = \int_V (\rho\, d\tau) E \cdot v = \int_V (E \cdot \rho v) d\tau = \int_V \left(E \cdot J \right) d\tau$$

$\left(E \cdot J \right)$ is the work done per unit time, per unit volume (or, power/volume)

Using the Ampere-Maxwell law,

$$E \cdot J = \frac{1}{\mu_0}\mathbf{E} \cdot (\nabla \times \mathbf{B}) - \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B})$$

Invoking Faraday's law ($\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$), it follows that

$$\mathbf{E} \cdot (\nabla \times \mathbf{B}) = -\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{B})$$

$$\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(B^2), \quad \text{and} \quad \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(E^2)$$

$$E \cdot J = \frac{1}{\mu_0}\mathbf{E} \cdot (\nabla \times \mathbf{B}) - \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t}$$

$$= -\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\nabla \cdot (\mathbf{E} \times \mathbf{B})$$

$$\boxed{\frac{dW}{dt} = \int_V (E \cdot J)\, dv = -\frac{d}{dt}\int_V \left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2\mu}B^2\right) dv - \frac{1}{\mu}\oint_S (E \times B) \cdot ds}$$

Note that

$$\frac{dW}{dt} = \frac{d}{dt}\int_{\mathcal{V}} u_{\text{mech}}\, d\tau$$

$$u_{\text{em}} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)$$

$$S \equiv \frac{1}{\mu}(E \times B)$$

$$\boxed{\frac{\partial}{\partial t}(u_{\text{mech}} + u_{\text{em}}) = -\nabla \cdot \mathbf{S}} \quad \Leftrightarrow \quad \frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}$$

Poynting Vector in Time Domain

- We define a new vector called the (instantaneous) ***Poynting vector*** as

$$\overline{S} = \overline{E} \times \overline{H} \qquad (\mathrm{W/m^2})$$

- The Poynting vector has the same direction as the direction of propagation.

- The Poynting vector at a point is equivalent to the power density of the wave at that point.

8.2 Momentum of EM fields (Griffiths)

8.2.1 Newton's Third Law in Electrodynamics

Imagine a point charge *q* traveling In along the *x* axis at a constant speed *v.*
Because it is moving, its electric field is *not* given by Coulomb's law;
nevertheless, E still points radially outward from the instantaneous position of the charge.

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\imath}{(\boldsymbol{\imath} \cdot \mathbf{u})^3}[(c^2 - v^2)\mathbf{u} + \boldsymbol{\imath} \times (\mathbf{u} \times \mathbf{a})] \quad (10.65)$$

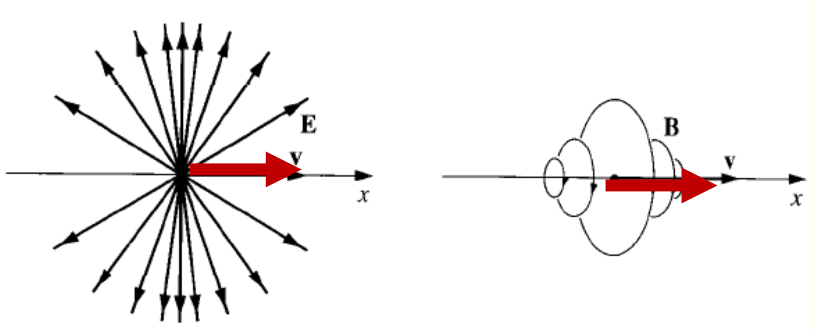
$$\mathbf{u} \equiv c\,\hat{\boldsymbol{\imath}} - \mathbf{v}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{\imath^2}\hat{\boldsymbol{\imath}}$$

If the velocity and acceleration are both zero,

Generalized Coulomb field (velocity field)

Radiation field (acceleration field)

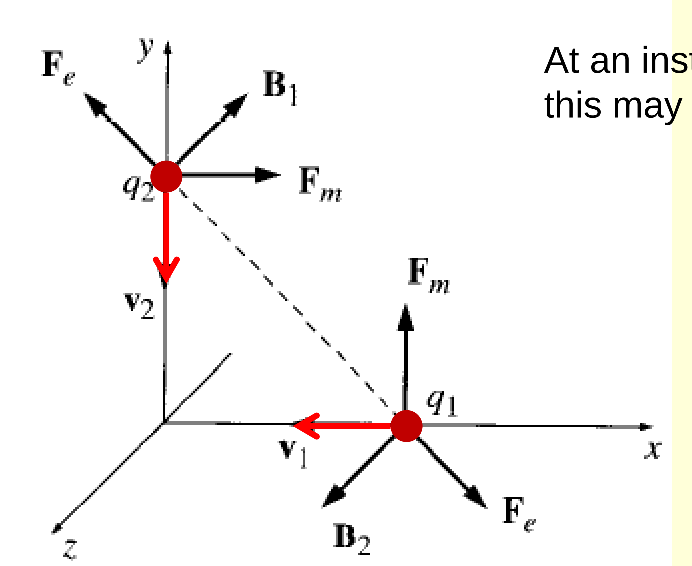


Since, moreover, a moving point charge does not constitute a steady current,
its magnetic field is *not* given by the Biot-Savart law.
Nevertheless, it's a fact that **B** still circles around the axis.

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c}\hat{\boldsymbol{\imath}} \times \mathbf{E}(\mathbf{r}, t) \quad (10.66)$$

Newton's Third Law in Electrodynamics

Now suppose two identical charges, q_1 and q_2, moving with a same speed along the x and *y* axes.



At an instantaneous time t,
this may be regarded as an *electromagnetostatic* case.

$$F_{e,q_2} = -F_{e,q_1}$$

$$F_{m,q_2} \neq -F_{m,q_1}$$

The electromagnetic force of q_1 on q_2 is equal, but not opposite to the force of q_1 and q_2.
➔ **violation of Newton's third law!**

In electrostatics and magnetostatics the third law holds, ***but in electrodynamics it does not.***

➔ **Is it true?**
➔ **The third law must be hold ALL THE TIME!**
➔ **We know that the proof of momentum conservation rest on the cancellation of internal forces.**
➔ **Therefore, let's prove momentum conservation in electrodynamics!**

➔ **The fields themselves carry momentum.**
➔ **Only when the field momentum is added to the mechanical momentum of the charges, momentum conservation (or, the third law) is restored.**

8.2.2 Maxwell's stress tensor

- ➔ **The fields themselves carry momentum.**
- ➔ **To find out the field momentum, let's calculate the total EM force on the charges in volume V in terms of Poynting vector S and stress tensor T:**

$$\mathbf{F} = \int_{\mathcal{V}} (\mathbf{E} + \mathbf{v} \times \mathbf{B})\rho \, d\tau = \int_{\mathcal{V}} (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}) \, d\tau$$

The force per unit volume is

$$\begin{aligned}\mathbf{f} &= \rho\mathbf{E} + \mathbf{J} \times \mathbf{B} \\ &= \epsilon_0(\nabla \cdot \mathbf{E})\mathbf{E} + \left(\frac{1}{\mu_0}\nabla \times \mathbf{B} - \epsilon_0\frac{\partial \mathbf{E}}{\partial t}\right) \times \mathbf{B}\end{aligned}$$

$$\left\{\begin{aligned}&\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) = \left(\frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}\right) + \left(\mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t}\right) \\ &\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}\end{aligned}\right. \quad \Rightarrow \quad \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} = \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) + \mathbf{E} \times (\nabla \times \mathbf{E})$$

$$\begin{aligned}\mathbf{f} &= \epsilon_0[(\nabla \cdot \mathbf{E})\mathbf{E} - \mathbf{E} \times (\nabla \times \mathbf{E})] - \frac{1}{\mu_0}[\mathbf{B} \times (\nabla \times \mathbf{B})] - \epsilon_0\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) \\ &= \underline{\epsilon_0[(\nabla \cdot \mathbf{E})\mathbf{E} + (\mathbf{E} \cdot \nabla)\mathbf{E}] + \frac{1}{\mu_0}[(\nabla \cdot \mathbf{B})\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{B}] - \frac{1}{2}\nabla\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)} - \epsilon_0\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B})\end{aligned}$$

It can be simplified by introducing the ***Maxwell stress tensor T***

$$\epsilon_0\mu_0\frac{\partial \mathbf{S}}{\partial t}$$

Force density in a symmetry form of E and B.

$$\mathbf{f} = \epsilon_0[(\nabla \cdot \mathbf{E})\mathbf{E} + (\mathbf{E} \cdot \nabla)\mathbf{E}] + \frac{1}{\mu_0}[(\nabla \cdot \mathbf{B})\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{B}] - \frac{1}{2}\nabla\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \epsilon_0\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B})$$

Maxwell's stress tensor: $\left(\overleftrightarrow{T}\right)_{ij} \equiv \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)$

$$T_{xx} = \frac{1}{2}\epsilon_0(E_x^2 - E_y^2 - E_z^2) + \frac{1}{2\mu_0}(B_x^2 - B_y^2 - B_z^2),$$

$$T_{xy} = \epsilon_0(E_xE_y) + \frac{1}{\mu_0}(B_xB_y)$$

$$(\nabla \cdot \overleftrightarrow{\mathbf{T}})_j = \epsilon_0\left[(\nabla \cdot \mathbf{E})E_j + (\mathbf{E} \cdot \nabla)E_j - \frac{1}{2}\nabla_j E^2\right] + \frac{1}{\mu_0}\left[(\nabla \cdot \mathbf{B})B_j + (\mathbf{B} \cdot \nabla)B_j - \frac{1}{2}\nabla_j B^2\right]$$

$$\boxed{\mathbf{f} = \nabla \cdot \overleftrightarrow{\mathbf{T}} - \epsilon_0\mu_0\frac{\partial \mathbf{S}}{\partial t}}$$

The total EM force on the charges in volume V is

$$\mathbf{F} = \int_{\mathcal{V}}(\mathbf{E} + \mathbf{v} \times \mathbf{B})\rho\, d\tau = \int_{\mathcal{V}}(\rho\mathbf{E} + \mathbf{J} \times \mathbf{B})\, d\tau$$

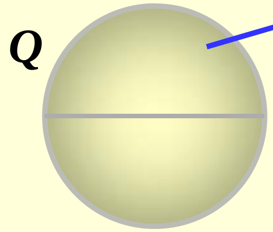
$$\boxed{\mathbf{F} = \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} - \epsilon_0\mu_0\frac{d}{dt}\int_{\mathcal{V}} \mathbf{S}\, d\tau}$$

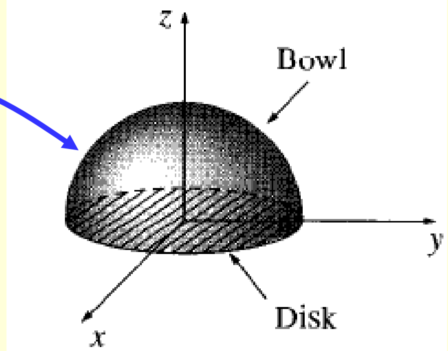
* In *static* case ➔ $\mathbf{F} = \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}$

*$\left(\overleftrightarrow{T}\right)_{ij}$ ➔ The force per unit area (or **stress**) on the surface

Example 8.2 (p. 353)

Find the net force on the northern hemisphere exerted by the southern hemisphere.





For the bowl, $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\,\hat{\mathbf{r}}$ $\quad \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$ $\quad d\mathbf{a} = R^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}$

In *static* case ➔ $\mathbf{F} = \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}$

The net force on the bowl is obviously in the z-direction,

$$(\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a})_z = T_{zx}\,da_x + T_{zy}\,da_y + T_{zz}\,da_z$$

$$\left\{\begin{aligned}
T_{zx} &= \epsilon_0 E_z E_x = \epsilon_0\left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2 \sin\theta\cos\theta\cos\phi \\
T_{zy} &= \epsilon_0 E_z E_y = \epsilon_0\left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2 \sin\theta\cos\theta\sin\phi \\
T_{zz} &= \frac{\epsilon_0}{2}(E_z^2 - E_x^2 - E_y^2) = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2 (\cos^2\theta - \sin^2\theta)
\end{aligned}\right.$$

$$(\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a})_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R}\right)^2 \sin\theta\cos\theta\,d\theta\,d\phi$$

Meanwhile, for the equatorial disk, $(\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a})_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 r^3\,dr\,d\phi$

Combining them, the net force on the northern hemisphere is ➔ $\boxed{F = \frac{1}{4\pi\epsilon_0}\frac{3Q^2}{16R^2}}$

8.2.3 Conservation of momentum for EM fields

According to Newton's second law, **the force on an object** is equal to **the rate of change of its momentum**:

$$\mathbf{F} = \frac{d\mathbf{p}_{\text{mech}}}{dt}$$

When the object has an electric charge distribution in V,

$$\mathbf{F} = \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} - \epsilon_0\mu_0 \frac{d}{dt}\int_{\mathcal{V}} \mathbf{S}\, d\tau$$

$$\boxed{\frac{d\mathbf{p}_{\text{mech}}}{dt} = -\epsilon_0\mu_0 \frac{d}{dt}\int_{\mathcal{V}} \mathbf{S}\, d\tau + \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}}$$

The first integral represents *momentum stored in the electromagnetic fields:*

$$\mathbf{p}_{\text{em}} = \mu_0\epsilon_0 \int_{\mathcal{V}} \mathbf{S}\, d\tau$$

The second integral represents *momentum per unit time flowing in through the surface:*

In differential form,

$$\boxed{\frac{\partial}{\partial t}(\wp_{\text{mech}} + \wp_{\text{em}}) = \nabla \cdot \overleftrightarrow{\mathbf{T}}}$$

$\wp_{\text{mech}}$: densities of mechanical momentum

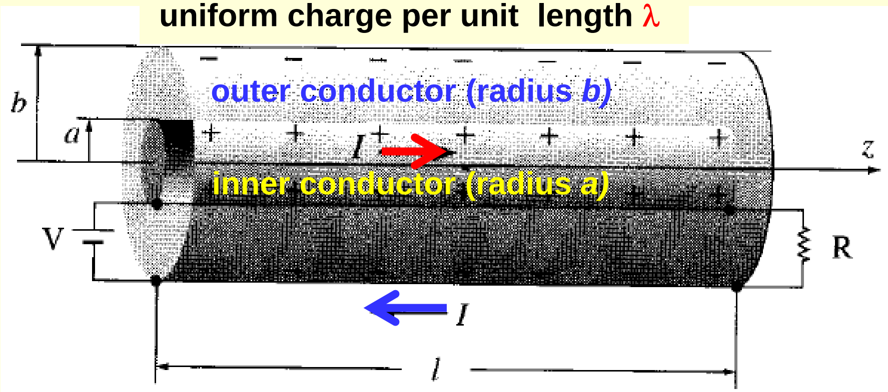
$\wp_{\text{em}} = \mu_0\epsilon_0\mathbf{S}$: densities of field momentum

If V is all of space, then no momentum flows in and out, ➔ $\boxed{P_{mech} + P_{em} = \text{constant}}$

Solve Problems. 8.5 and 8.6

Example 8.3 (p. 356)

What is the electromagnetic momentum stored in the fields?



The fields are

$$\mathbf{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\,\hat{\mathbf{s}}, \qquad \mathbf{B} = \frac{\mu_0}{2\pi}\frac{I}{s}\,\hat{\boldsymbol{\phi}}.$$

The Poynting vector is therefore

$$\mathbf{S} = \frac{\lambda I}{4\pi^2\epsilon_0 s^2}\,\hat{\mathbf{z}}.$$

The *momentum* in the fields is:

$$\mathbf{p}_{\mathrm{em}} = \mu_0\epsilon_0 \int \mathbf{S}\,d\tau = \frac{\mu_0\lambda I}{4\pi^2}\,\hat{\mathbf{z}} \int_a^b \frac{1}{s^2} l 2\pi s\,ds = \frac{\mu_0\lambda I l}{2\pi}\ln(b/a)\,\hat{\mathbf{z}}$$

8.2.4 Angular Momentum

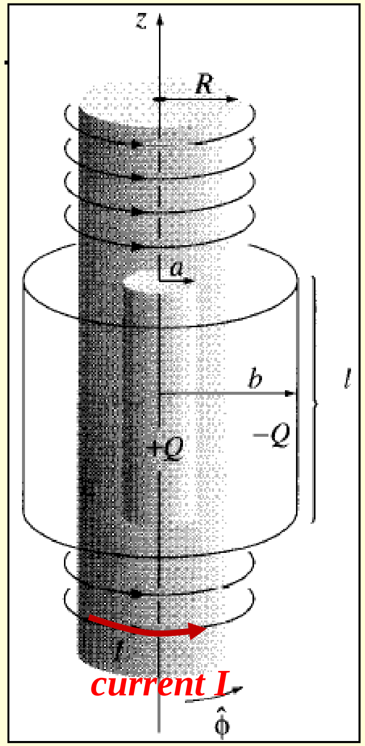
Electromagnetic fields carry energy density and momentum density of

$$u_{em} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) \qquad \boldsymbol{\wp}_{em} = \mu_0\epsilon_0 \mathbf{S} = \epsilon_0(\mathbf{E}\times\mathbf{B})$$

The electromagnetic fields also carry density of angular momentum of

$$\boxed{\boldsymbol{\ell}_{em} = \mathbf{r}\times\boldsymbol{\wp}_{em} = \epsilon_0\left[\mathbf{r}\times(\mathbf{E}\times\mathbf{B})\right]}$$

[Example 8.4] Two long cylindrical shells are coaxial with a solenoid carrying current I. When the current in the solenoid is gradually reduced, the cylinders begin to rotate.
Where does the angular momentum comes from?



$$\mathbf{E} = \frac{Q}{2\pi\epsilon_0 l}\frac{1}{s}\,\hat{\mathbf{s}}\,(a < s < b)$$ in the region between the cylinders

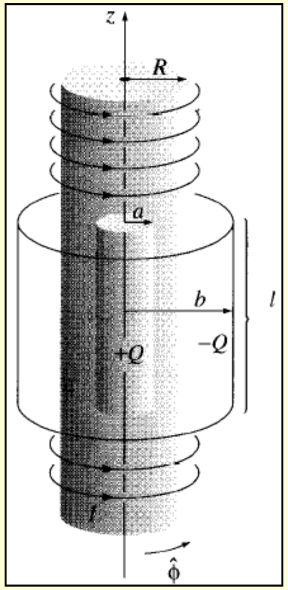
$$\mathbf{B} = \mu_0 n I\,\hat{\mathbf{z}}\,(s < R)$$ inside the solenoid

$$\boldsymbol{\wp}_{em} = \mu_0\epsilon_0\mathbf{S} = \epsilon_0(\mathbf{E}\times\mathbf{B}) \longrightarrow \boldsymbol{\wp}_{em} = -\frac{\mu_0 n I Q}{2\pi l s}\,\hat{\boldsymbol{\phi}}$$ in the region $a < s < R$

$$\boldsymbol{\ell}_{em} = \mathbf{r}\times\boldsymbol{\wp}_{em} = -\frac{\mu_0 n I Q}{2\pi l}\,\hat{\mathbf{z}}$$ = constant over the volume of $\pi(R^2 - a^2)l$:

Total angular momentum in the fields: $\mathbf{L}_{em} = -\frac{1}{2}\mu_0 n I Q(R^2 - a^2)\,\hat{\mathbf{z}}$

[Example 8.4] continued.



When the current is turned off, the changing magnetic field induces a circumferential electric field, given by Faraday's law:

$$\mathbf{E} = \begin{cases} -\frac{1}{2}\mu_0 n \frac{dI}{dt}\frac{R^2}{s}\hat{\boldsymbol{\phi}}, & (s > R) \\ -\frac{1}{2}\mu_0 n \frac{dI}{dt} s\,\hat{\boldsymbol{\phi}}, & (s < R) \end{cases}$$

Torque on the outer cylinder: $\mathbf{N}_b = \mathbf{r} \times (-Q\mathbf{E}) = \frac{1}{2}\mu_0 n Q R^2 \frac{dI}{dt}\hat{\mathbf{z}}$

Angular momentum of the outer cylinder: $\mathbf{L}_b = \frac{1}{2}\mu_0 n Q R^2\,\hat{\mathbf{z}} \int_I^0 \frac{dI}{dt} dt = -\frac{1}{2}\mu_0 n I Q R^2\,\hat{\mathbf{z}}$

Torque on the inner cylinder: $\mathbf{N}_a = -\frac{1}{2}\mu_0 n Q a^2 \frac{dI}{dt}\,\hat{\mathbf{z}}$

Angular momentum of the outer cylinder: $\mathbf{L}_a = \frac{1}{2}\mu_0 n I Q a^2\,\hat{\mathbf{z}}$

➔ Total angular momentum of the inner and outer cylinders:

$$\mathbf{L}_a + \mathbf{L}_b = -\frac{1}{2}\mu_0 n I Q (R^2 - a^2)\,\hat{\mathbf{z}}$$

➔ Which is the same as the angular momentum of the field: $\boxed{\mathbf{L}_{\mathrm{em}} = \mathbf{L}_a + \mathbf{L}_b.}$

Angular momentum *lost* by fields is precisely equal to angular momentum *gained* by the cylinders,

➔ the *total* angular momentum (fields plus matter) is conserved