Chapter 4. Electric Fields in Matter

4.4 Linear Dielectrics

4.4.1 Susceptibility, Permittivity, Dielectric Constant

$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$ ➔ ***In linear dielectrics,*** **P** is proportional to E, provided ***E*** *is not too strong.*

χ_e **: Electric susceptibility** (It would be a tensor in general cases)

Note that **E is the total field** from free charges and the polarization itself.

➔ If, for instance, we put a piece of dielectric into an external field $\mathbf{E}_0$,

➔ we cannot compute **P** directly from $\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$; $\mathbf{E} \neq \mathbf{E}_0$

➔ $\mathbf{E}_0$ produces **P**, **P** will produce its own field, this in turn modifies **P**, which ... Breaking where?

To calculate **P**, the simplest approach is to begin with the *displacement* ***D,***
at least in those cases where **D** can be deduced directly from the free charge distribution.

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 (1 + \chi_e) \mathbf{E}$$

$$\mathbf{D} = \varepsilon \mathbf{E}$$

$$\varepsilon = \varepsilon_0 (1 + \chi_e)$$ **: Permittivity**

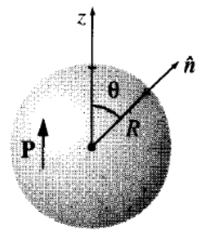
$$\varepsilon_r \equiv 1 + \chi_e = \frac{\varepsilon}{\varepsilon_0}$$ **: Relative permittivity (Dielectric constant)**

Material	Dielectric Constant
Vacuum	1
Helium	1.000065
Neon	1.00013
Hydrogen	1.00025
Argon	1.00052
Air (dry)	1.00054
Nitrogen	1.00055
Water vapor (100° C)	1.00587
Diamond	5.7
Salt	5.9
Silicon	11.8

Susceptibility, Permittivity, Dielectric Constant

Problem 4.41 In a linear dielectric, the polarization is proportional to the field: $\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$. If the material consists of atoms (or nonpolar molecules), the induced dipole moment of each one is likewise proportional to the field $\boldsymbol{p} = \alpha \boldsymbol{E}$. What is the relation between atomic polarizability α and susceptibility χ_e?

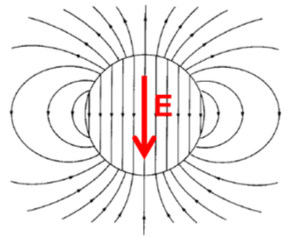
Note that, the atomic polarizability α was defined for an isolated atom subject to an external field coming from somewhere else, $\boldsymbol{E}_{else} \rightarrow \boldsymbol{p} = \alpha \boldsymbol{E}_{else}$



For N atoms in unit volume, the polarization can be set $\rightarrow \mathbf{P} = N\alpha\mathbf{E}_{\text{else}}$

There is another electric field, $\mathbf{E}_{\text{self}}$, produced by the polarization **P:**

$$\mathbf{E}_{\text{self}} = -\frac{1}{4\pi\epsilon_0}\frac{q\mathbf{d}}{R^3} \longrightarrow \mathbf{p} = q\mathbf{d} = (\tfrac{4}{3}\pi R^3)\mathbf{P} \longrightarrow \mathbf{E}_{\text{self}} = -\frac{1}{3\epsilon_0}\mathbf{P}$$



$$\longrightarrow \mathbf{E}_{\text{self}} = -\frac{N\alpha}{3\epsilon_0}\mathbf{E}_{\text{else}}$$

Therefore, the total field is $\mathbf{E} = \mathbf{E}_{\text{self}} + \mathbf{E}_{\text{else}} = \left(1 - \frac{N\alpha}{3\epsilon_0}\right)\mathbf{E}_{\text{else}}$

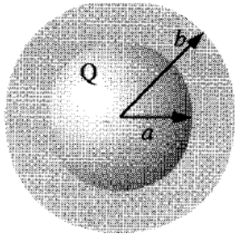
The total field **E** finally produce the polarization **P**:

$$\mathbf{P} = \frac{N\alpha}{(1 - N\alpha/3\epsilon_0)}\mathbf{E} = \epsilon_0\chi_e\mathbf{E} \longrightarrow \boxed{\alpha = \frac{\epsilon_0}{N}\frac{\chi_e}{(1+\chi_e/3)}} \text{ or } \boxed{\alpha = \frac{3\epsilon_0}{N}\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)}$$

➔ ***Clausius-Mossotti formula***

Susceptibility, Permittivity, Dielectric Constant

Example 4.5 **A metal sphere of radius *a* carries a charge *Q*.**
It is surrounded, **out to radius *b***, by linear dielectric material of **permittivity** ε.
Find the potential **V** at the center (relative to infinity).



To compute ***V***, we need to know **E.** → **Because it has spherical symmetry, calculate D first.**

Inside the metal sphere, $\mathbf{E} = \mathbf{P} = \mathbf{D} = 0.$

Outside the metal sphere,

$$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}}, \qquad \text{for all points } r > a. \longrightarrow \mathbf{E} = \begin{cases} \dfrac{Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}, & \text{for } a < r < b, \\[2ex] \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}, & \text{for } r > b. \end{cases}$$

The potential at the center is therefore

$$\begin{aligned} V &= -\int_\infty^0 \mathbf{E}\cdot d\mathbf{l} = -\int_\infty^b \left(\frac{Q}{4\pi\epsilon_0 r^2}\right) dr - \int_b^a \left(\frac{Q}{4\pi\epsilon r^2}\right) dr - \int_a^0 (0)\, dr \\ &= \frac{Q}{4\pi}\left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b}\right). \end{aligned}$$

Note that the polarization **P** in the dielectric is $\mathbf{P} = \epsilon_0\chi_e\mathbf{E} = \dfrac{\epsilon_0\chi_e Q}{4\pi\epsilon r^2}\hat{\mathbf{r}} \quad \text{for } a < r < b$

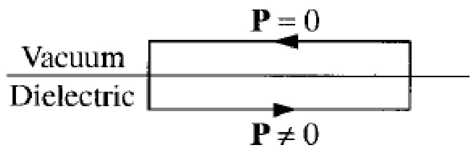
$$\rho_b = -\nabla\cdot\mathbf{P} = 0 \qquad \sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \begin{cases} \dfrac{\epsilon_0\chi_e Q}{4\pi\epsilon b^2}, & \text{at the outer surface,} \\[2ex] \dfrac{-\epsilon_0\chi_e Q}{4\pi\epsilon a^2}, & \text{at the inner surface.} \end{cases}$$

The two vectors of E and D is parallel in linear dielectric.

In linear dielectrics, $\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} \longrightarrow \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 (1 + \chi_e) \mathbf{E}$

$$\nabla \times \mathbf{E} = 0 \rightarrow \nabla \times \mathbf{D} = 0 ?? \text{ since } \mathbf{E} \text{ and } \mathbf{D} \text{ are parallel} ???$$

➔ **Unfortunately, it does *not:***



$$\nabla \times \mathbf{D} = \epsilon_0 (\nabla \times \mathbf{E}) + (\nabla \times \mathbf{P}) = \nabla \times \mathbf{P} \neq 0$$

Of course, **if the space is *entirely* filled with a homogeneous** (χ_e is the same in all position) linear dielectric,

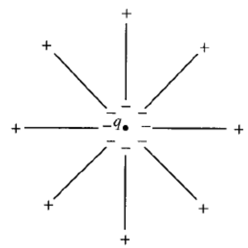
$$\nabla \cdot \mathbf{D} = \rho_f \quad \text{and} \quad \nabla \times \mathbf{D} = 0 \quad \xleftrightarrow{\textbf{Compare}} \quad \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho \quad \textbf{and} \quad \varepsilon_0 (\nabla \times \mathbf{E}) = 0$$

➔ **D** can be found from the free charge just as though the dielectric were not there:

$$\mathbf{D} = \epsilon_0 \mathbf{E}_{\text{vac}} \longrightarrow \mathbf{E} = \frac{1}{\epsilon} \mathbf{D} = \frac{1}{\epsilon_r} \mathbf{E}_{\text{vac}}$$

Conclusion: When all space is filled with a homogeneous linear dielectric,
the field everywhere is simply reduced by a factor of one over the dielectric constant.

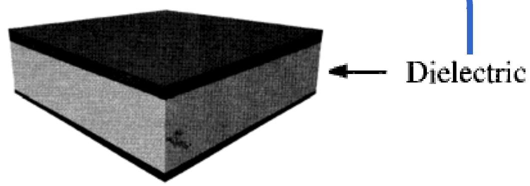
For example, if a free charge q is embedded in a large dielectric ε → $\mathbf{E} = \frac{1}{4\pi\epsilon} \frac{q}{r^2} \hat{\mathbf{r}}$



→ It becomes weaker! Because **P** shield the charge.

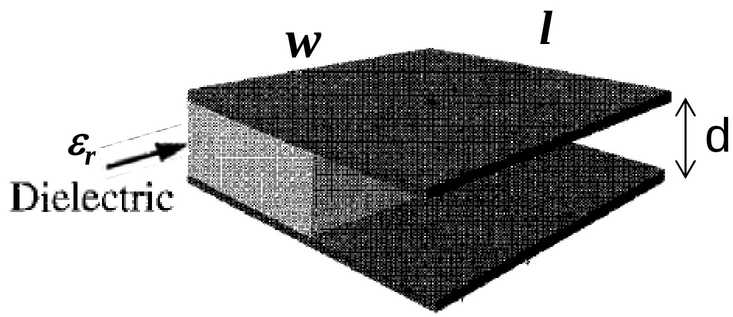
In Linear Dielectrics

Example 4.6 A parallel-plate capacitor is filled with insulating material of dielectric constant ε_r.
What effect does this have on its capacitance?



➔ The dielectric will reduce E, and hence also the potential difference *V,* by a factor *1/* ε_r. Accordingly, the capacitance C = Q/*V* is *increased by a factor of the dielectric constant* ➔ $C = \epsilon_r C_{\text{vac}}$

Problem 4.19 *Half-fill* a parallel-plate capacitor with a given potential difference V.



With no dielectric, $C_0 = A\epsilon_0/d$ $(A = wl)$

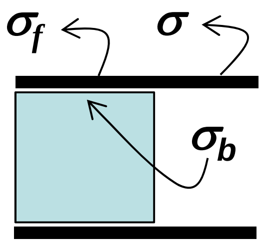
The electric field is uniform in all volume ➔ $E = V/d$

The surface charge density σ is uniform on the parallel plate ➔ $\sigma = \epsilon_0 E = \epsilon_0 V/d$

At the top surface of dielectric, there is a bound surface charge due to **P**:

$$P = \epsilon_0 \chi_e E = \epsilon_0 \chi_e V/d \longrightarrow \sigma_b \equiv \mathbf{P}\cdot\hat{\mathbf{n}} = -\epsilon_0 \chi_e V/d$$

On the top plate above dielectric, a free surface charge σ_f must exist:



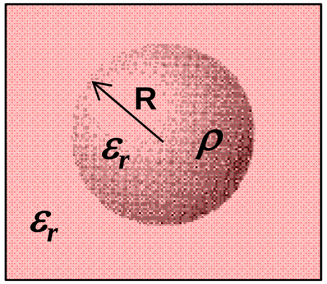
$$\sigma_f = \sigma - \sigma_b = \epsilon_0 V/d + \epsilon_0 \chi_e V/d = \epsilon_0 V(1+\chi_e)/d = \epsilon_0 \epsilon_r V/d$$

$\sigma_f + \sigma_b = \sigma$

$$\Rightarrow C_b = \frac{Q}{V} = \frac{1}{V}\left(\sigma\frac{A}{2} + \sigma_f\frac{A}{2}\right) = \frac{A}{2V}\left(\epsilon_0\frac{V}{d} + \epsilon_0\frac{V}{d}\epsilon_r\right) = \frac{A\epsilon_0}{d}\left(\frac{1+\epsilon_r}{2}\right) \Rightarrow \frac{C_b}{C_0} = \frac{1+\epsilon_r}{2}$$

In Linear Dielectrics

Problem 4.20 A sphere of linear dielectric material with a uniform free charge density ρ has embedded in it without charge. Find **the potential at the center of the sphere** (relative to infinity), if its radius is *R* and its dielectric constant is ε_r.



$$\text{for } r < R; \quad \xrightarrow{\phantom{\oint \mathbf{D}\cdot d\mathbf{a} = Q_{f_{enc}}}} \quad D4\pi r^2 = \rho\tfrac{4}{3}\pi r^3 \longrightarrow D = \tfrac{1}{3}\rho r \longrightarrow \mathbf{E} = (\rho r/3\epsilon)\,\hat{\mathbf{r}}$$

$$\text{for } r > R. \quad \xrightarrow{\oint \mathbf{D}\cdot d\mathbf{a} = Q_{f_{enc}}} \quad D4\pi r^2 = \rho\tfrac{4}{3}\pi R^3 \longrightarrow D = \rho R^3/3r^2 \longrightarrow \mathbf{E} = (\rho R^3/3\epsilon_0 r^2)\,\hat{\mathbf{r}}$$

$$V = -\int_{\infty}^{0} \mathbf{E}\cdot d\mathbf{l}$$

$$= \frac{\rho R^3}{3\epsilon_0}\left.\frac{1}{r}\right|_{\infty}^{R} - \frac{\rho}{3\epsilon}\int_R^0 r\,dr = \frac{\rho R^2}{3\epsilon_0} + \frac{\rho}{3\epsilon}\frac{R^2}{2} = \boxed{\frac{\rho R^2}{3\epsilon_0}\left(1 + \frac{1}{2\epsilon_r}\right).}$$

4.4.2 Boundary Value Problems with Linear Dielectrics

$$\boxed{\rho_b = -\nabla \bullet \mathbf{P}} \Rightarrow \rho_b = -\nabla \cdot \mathbf{P} = -\nabla \cdot \left(\epsilon_0 \frac{\chi_e}{\epsilon} \mathbf{D}\right) = -\left(\frac{\chi_e}{1+\chi_e}\right) \rho_f$$

➔ The bound charge density is proportional to the free charge density.

➔ If $\rho_f = 0$; $\rho_b = 0$ → Any net charge must resident at the surface

➔ **Within a dielectric when $\rho_f = 0$; the potential obeys Laplace's equation.**

$$\boxed{D_{1n} - D_{2n} = \sigma_f} \Rightarrow \boxed{\epsilon_{\text{above}} E^{\perp}_{\text{above}} - \epsilon_{\text{below}} E^{\perp}_{\text{below}} = \sigma_f}$$

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{1}{\epsilon_0}\sigma$$

(If embedded in vacuum)

OR $$\Rightarrow \boxed{\epsilon_{\text{above}} \frac{\partial V_{\text{above}}}{\partial n} - \epsilon_{\text{below}} \frac{\partial V_{\text{below}}}{\partial n} = -\sigma_f}$$

$$V_{\text{above}} - V_{\text{below}} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} \Rightarrow \boxed{V_{\text{above}} = V_{\text{below}}}$$

(As **a** approaches to **b**)

Summary on Boundary Conditions for Electrostatics

Electric field

$$\mathbf{E}^{\parallel}_{\text{above}} - \mathbf{E}^{\parallel}_{\text{below}} = 0$$

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{1}{\epsilon_0}\sigma \qquad \left(\sigma = \sigma_f + \sigma_b\right)$$

Field displacement

$$\mathbf{D}^{\parallel}_{\text{above}} - \mathbf{D}^{\parallel}_{\text{below}} = \mathbf{P}^{\parallel}_{\text{above}} - \mathbf{P}^{\parallel}_{\text{below}}$$

$$D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f \Rightarrow \boxed{\epsilon_{\text{above}} E^{\perp}_{\text{above}} - \epsilon_{\text{below}} E^{\perp}_{\text{below}} = \sigma_f}$$

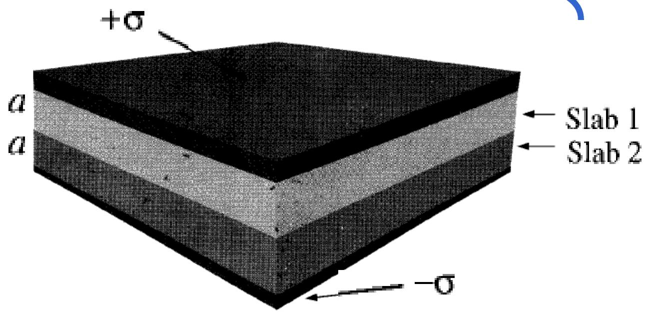
$$\Rightarrow \boxed{\epsilon_{\text{above}} \frac{\partial V_{\text{above}}}{\partial n} - \epsilon_{\text{below}} \frac{\partial V_{\text{below}}}{\partial n} = -\sigma_f}$$

Potential

$$V_{\text{above}} = V_{\text{below}}$$

E, P, and D; Boundary Conditions

Problem 4 18 The space between the plates of a parallel plate capacitor is filled with two slabs of linear dielectric material. Each slab has thickness *a*. Slab 1 has a dielectric constant of $\varepsilon_1 = 2$, and slab 2, $\varepsilon_2 = 1.5$. The free charge density on the top plate is *a* and on the bottom plate σ.



(a) Find the electric displacement D in each slab.

$$\int \mathbf{D}\cdot d\mathbf{a} = Q_{f_{\text{enc}}} \longrightarrow DA = \sigma A \longrightarrow D = \sigma \qquad \mathbf{D} = 0 \text{ inside the metal plate}$$

(b) Find the electric field E in each slab. $\mathbf{D} = \epsilon\mathbf{E} \longrightarrow E = \sigma/\epsilon \longrightarrow \epsilon_1 = 2\epsilon_0 \longrightarrow E_1 = \sigma/2\epsilon_0$

$$\epsilon_2 = \tfrac{3}{2}\epsilon_0 \longrightarrow E_2 = 2\sigma/3\epsilon_0$$

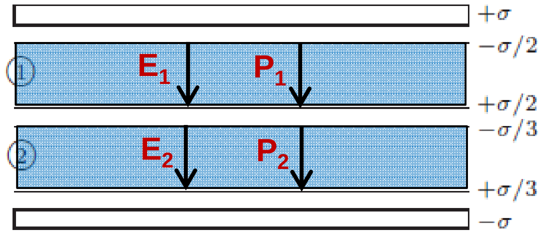
(c) Find the polarization P in each slab.

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E} \longrightarrow P = \epsilon_0\chi_e\ \sigma/(\epsilon_0\epsilon_r) \longrightarrow P = (1-\epsilon_r^{-1})\sigma \longrightarrow P_1 = \sigma/2 \qquad P_2 = \sigma/3$$

(d) Find the potential difference between the plates. $V = E_1 a + E_2 a = (\sigma a/6\epsilon_0)(3+4) = 7\sigma a/6\epsilon_0$

(e) Find the location and amount of all bound charge.

$$\rho_b = 0 \qquad \sigma_b = -P_1 = -\sigma/2 \quad \text{at top of slab (1)}$$
$$\sigma_b = +P_1 = \sigma/2 \quad \text{at bottom of slab (1)}$$
$$\sigma_b = -P_2 = -\sigma/3 \quad \text{at top of slab (2)}$$
$$\sigma_b = +P_2 = \sigma/3 \quad \text{at bottom of slab (2)}$$



(f) Recalculate the electric field in each slab, using the relation of $E = \dfrac{\sigma}{\epsilon_0}$

$$\text{In slab 1: } \begin{cases} \text{total surface charge } \textit{above}\text{: } \sigma - (\sigma/2) = \sigma/2, \\ \text{total surface charge } \textit{below}\text{: } (\sigma/2) - (\sigma/3) + (\sigma/3) - \sigma = -\sigma/2, \end{cases} \Longrightarrow E_1 = \frac{\sigma}{2\epsilon_0}$$

$$\text{In slab 2: } \begin{cases} \text{total surface charge } \textit{above}\text{: } \sigma - (\sigma/2) + (\sigma/2) - (\sigma/3) = 2\sigma/3, \\ \text{total surface charge } \textit{below}\text{: } (\sigma/3) - \sigma = -2\sigma/3, \end{cases} \Longrightarrow E_2 = \frac{2\sigma}{3\epsilon_0}$$

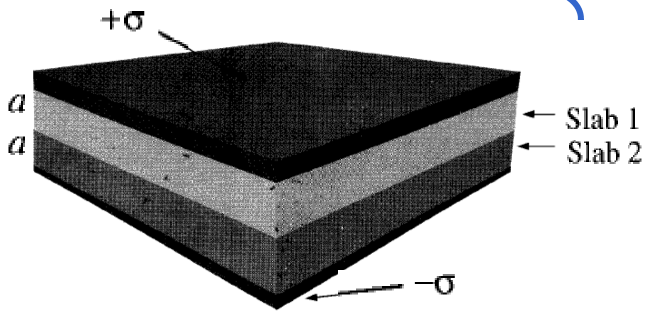
Check up the Boundary Conditions

Problem 4 18

$$D_{\text{above}}^{\perp} - D_{\text{below}}^{\perp} = \sigma_f$$

$$E_{\text{above}}^{\perp} - E_{\text{below}}^{\perp} = \frac{1}{\epsilon_0}(\sigma_f + \sigma_b)$$

$$\epsilon_{\text{above}} E_{\text{above}}^{\perp} - \epsilon_{\text{below}} E_{\text{below}}^{\perp} = \sigma_f$$



$D = \sigma$

D = 0 inside the metal plate

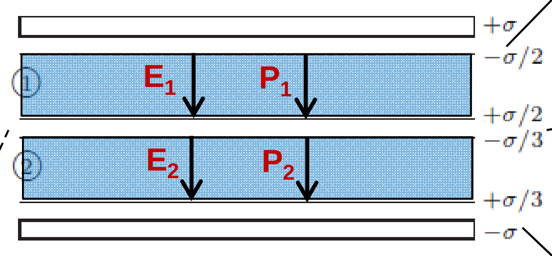
$$D_{\text{above}}^{\perp} = 0$$

$$D_{\text{below}}^{\perp} = \sigma$$

$$D_{\text{above}}^{\perp} - D_{\text{below}}^{\perp} = \sigma$$

$$E_1 = \sigma/2\epsilon_0$$

$$E_2 = 2\sigma/3\epsilon_0$$



$$E_{\text{above}}^{\perp} - E_{\text{below}}^{\perp} = 0 - E_1 = \sigma/2\epsilon_0$$

$$(\sigma_f + \sigma_b) = \sigma - \frac{\sigma}{2} = \frac{\sigma}{2}$$

$$E_{\text{above}}^{\perp} - E_{\text{below}}^{\perp} = E_1 - E_2 = \frac{\sigma}{6\varepsilon_0}$$

$$(\sigma_f + \sigma_b) = \frac{\sigma}{2} - \frac{\sigma}{3} = \frac{\sigma}{6}$$

$$E_{\text{above}}^{\perp} - E_{\text{below}}^{\perp} = E_2 - 0 = \frac{2\sigma}{3\varepsilon_0}$$

$$(\sigma_f + \sigma_b) = \frac{\sigma}{3} - \sigma = -\frac{2\sigma}{3}$$

$$E_{\text{above}}^{\perp} - E_{\text{below}}^{\perp} = \frac{1}{\epsilon_0}(\sigma_f + \sigma_b)$$

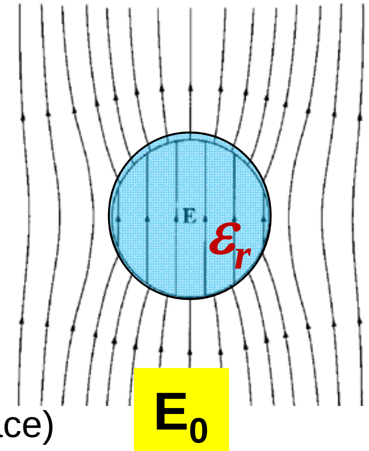
$$\varepsilon_1 E_1 - \varepsilon_2 E_2 = 2\frac{\sigma}{2\varepsilon_0} - 1.5\frac{2\sigma}{3\varepsilon_0} = 0$$

$$\sigma_f = 0$$

$$\epsilon_{\text{above}} E_{\text{above}}^{\perp} - \epsilon_{\text{below}} E_{\text{below}}^{\perp} = \sigma_f$$

Boundary Value Problems with Linear Dielectrics

Example 4.7 A sphere of homogeneous linear dielectric material is placed in an otherwise uniform electric field **E**o. Find the electric field inside the sphere.



The problem is to solve Laplace's equation for ***V (r, θ)***, under the boundary conditions:

(i) $V_{\text{in}} = V_{\text{out}}, \quad \text{at } r = R,$

(ii) $\epsilon \dfrac{\partial V_{\text{in}}}{\partial r} = \epsilon_0 \dfrac{\partial V_{\text{out}}}{\partial r}, \quad \text{at } r = R,$ (since no free charge at the surface)

(iii) $V_{\text{out}} \to -E_0 r \cos\theta, \quad \text{for } r \gg R$

The general solution is $V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \dfrac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$

Inside the sphere, $V_{\text{in}}(r,\theta) = \sum A_l r^l P_l(\cos\theta)$

Outside the sphere, $V_{\text{out}}(r,\theta) = -E_0 r \cos\theta + \sum_{l=0}^{\infty} \dfrac{B_l}{r^{l+1}} P_l(\cos\theta)$

(i) $\longrightarrow$ $\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = -E_0 R \cos\theta + \sum_{l=0}^{\infty} \dfrac{B_l}{R^{l+1}} P_l(\cos\theta)$

(ii) $\longrightarrow$ $\epsilon_r \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = -E_0 \cos\theta - \sum_{l=0}^{\infty} \dfrac{(l+1) B_l}{R^{l+2}} P_l(\cos\theta)$

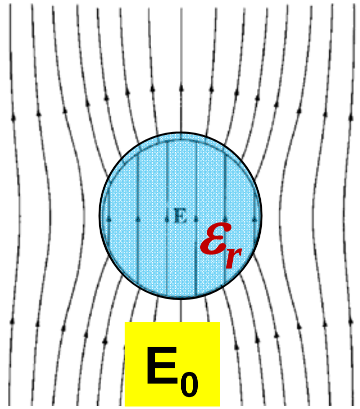
$$\left.\begin{aligned} &A_l = B_l = 0, \qquad \text{for } l \neq 1, \\ &A_1 = -\frac{3}{\epsilon_r + 2} E_0 \qquad B_1 = \frac{\epsilon_r - 1}{\epsilon_r + 2} R^3 E_0. \end{aligned}\right\}$$

$$\Rightarrow V_{\text{in}}(r,\theta) = -\frac{3E_0}{\epsilon_r + 2} r \cos\theta = -\frac{3E_0}{\epsilon_r + 2} z \Rightarrow \boxed{\mathbf{E} = \frac{3}{\epsilon_r + 2} \mathbf{E}_0}$$

➔ The field inside the sphere is (surprisingly) *uniform:*

Boundary Value Problems with Linear Dielectrics

Problem 4.23 Find the electric field inside the sphere.
Use the following method of ***successive approximations***:



→ *First pretend the field inside is just E_0*

→ *Write down the resulting polarization P_0* ⟶ $\mathbf{P}_0 = \epsilon_0 \chi_e \mathbf{E}_0$

→ *P_0 generates a field of its own, E_1* ⟶ $\mathbf{E}_1 = -\frac{1}{3\epsilon_0}\mathbf{P}_0 = -\frac{\chi_e}{3}\mathbf{E}_0$

→ *E_1 modifies the polarization by an amount P_1* ⟶ $\mathbf{P}_1 = \epsilon_0 \chi_e \mathbf{E}_1 = -\frac{\epsilon_0 \chi_e^2}{3}\mathbf{E}_0$

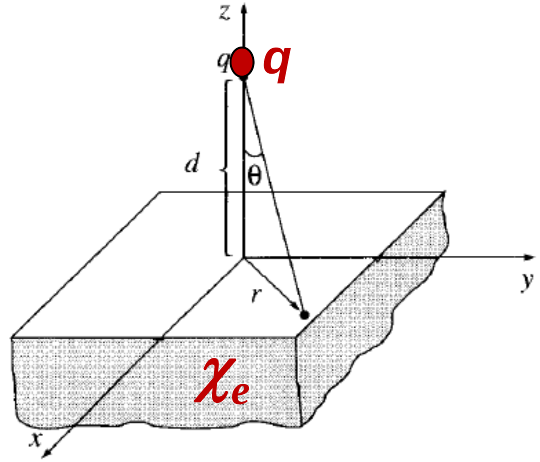
→ *P_1 → E_2* ⟶ $\mathbf{E}_2 = -\frac{1}{3\epsilon_0}\mathbf{P}_1 = \frac{\chi_e^2}{9}\mathbf{E}_0$

→ *and so on.* ⟶ $\mathbf{E}_n = \left(-\frac{\chi_e}{3}\right)^n \mathbf{E}_0$

→ *The resulting field is* ➔ $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1 + \mathbf{E}_2 + \cdots = \left[\sum_{n=0}^{\infty}\left(-\frac{\chi_e}{3}\right)^n\right]\mathbf{E}_0$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \Rightarrow \quad \mathbf{E} = \frac{1}{(1+\chi_e/3)}\mathbf{E}_0 \quad \Rightarrow \quad \boxed{\mathbf{E} = \frac{3}{\epsilon_r + 2}\mathbf{E}_0}$$

Boundary Value Problems



Example 4.8 Calculate the force on a point charge *q* situated a distance *d* above a uniform linear dielectric material of susceptibility χ_e.

$$\rho_b = -\nabla \cdot \mathbf{P} = -\left(\frac{\chi_e}{1+\chi_e}\right)\rho_f = 0$$ (since no free charge inside)

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = P_z = \epsilon_0 \chi_e E_z$$ (E_z is the z-component of **the total field** just inside the dielectric, at z = O. This field is due in part to ***q*** and in part to the bound charge itself.)

E_z field due to **q** is $$-\frac{1}{4\pi\epsilon_0}\frac{q}{(r^2+d^2)}\cos\theta = -\frac{1}{4\pi\epsilon_0}\frac{qd}{(r^2+d^2)^{3/2}}$$

E_z field due to the bound charge σ_b is $$-\sigma_b/2\epsilon_0 \longleftarrow \mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$$

$$\sigma_b = \epsilon_0\chi_e\left[-\frac{1}{4\pi\epsilon_0}\frac{qd}{(r^2+d^2)^{3/2}} - \frac{\sigma_b}{2\epsilon_0}\right] \longrightarrow \sigma_b = -\frac{1}{2\pi}\left(\frac{\chi_e}{\chi_e+2}\right)\frac{qd}{(r^2+d^2)^{3/2}}$$

We could obtain the field of σ_b by direct integration: $$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int\left(\frac{\hat{\boldsymbol{\imath}}}{\imath^2}\right)\sigma_b\,da$$

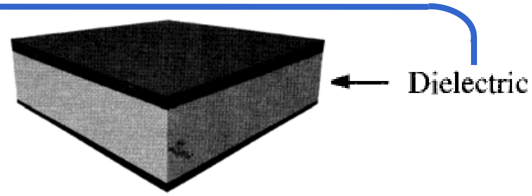
But as in the case of the conducting plane, **there is a nicer solution by the method of images.** Indeed, if we replace the dielectric by a single point charge q_b at the image position (0, 0, *-d),*

$$q_b = \int \sigma_b\,da = -\left(\frac{\chi_e}{\chi_e+2}\right)q \longleftarrow \int_0^{2\pi}\int_0^{\infty}\frac{-qd}{2\pi(r^2+d^2)^{3/2}}r\,dr\,d\phi = \left.\frac{qd}{\sqrt{r^2+d^2}}\right|_0^{\infty} = -q$$

$$\Rightarrow \mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qq_b}{(2d)^2}\hat{\mathbf{z}} = -\frac{1}{4\pi\epsilon_0}\left(\frac{\chi_e}{\chi_e+2}\right)\frac{q^2}{4d^2}\hat{\mathbf{z}}$$

4.4.3 Energy in Dielectric Systems

Note that the work to charge up a capacitor with potential V is



$$W = \frac{1}{2}CV^2 \qquad C = \epsilon_r C_{\text{vac}}$$

➔ Evidently the work necessary to charge a dielectric-filled capacitor is increased by a factor of the dielectric constant.

➔ The reason is pretty clear: you have to pump on more (free) charge to achieve a given potential, because part of the field is canceled off by the bound charges.

A general formula for the energy stored in any electrostatic system was

$$W = \frac{\epsilon_0}{2}\int E^2\, d\tau.$$

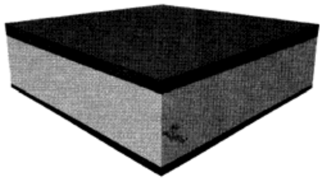
The case of the dielectric-filled capacitor suggests that this should be changed to

$$W = \frac{\epsilon_0}{2}\int \epsilon_r E^2\, d\tau$$

➔ ***The energy, increased by a factor of the dielectric constant, can be stored in a dielectric-filled capacitor.***

Energy in Dielectric Systems

$$W = \frac{\epsilon_0}{2}\int \epsilon_r E^2\, d\tau = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\, d\tau$$



Let's prove it.

As we bring in free charge, a bit at a time, the work done on the increamental free charge $\Delta\rho_f$ is

$$\Delta W = \int (\Delta\rho_f) V\, d\tau$$

$$= \int [\nabla\cdot(\Delta\mathbf{D})] V\, d\tau \longleftarrow \Delta\rho_f = \nabla\cdot(\Delta\mathbf{D}), \quad \text{since} \quad \nabla\cdot\mathbf{D} = \rho_f$$

$$= \int \nabla\cdot[(\Delta\mathbf{D})V]\, d\tau + \int (\Delta\mathbf{D})\cdot\mathbf{E}\, d\tau \longleftarrow \nabla\cdot[(\Delta\mathbf{D})V] = [\nabla\cdot(\Delta\mathbf{D})]V + \Delta\mathbf{D}\cdot(\nabla V)$$

$$= \int (\Delta\mathbf{D})\cdot\mathbf{E}\, d\tau \longleftarrow$$

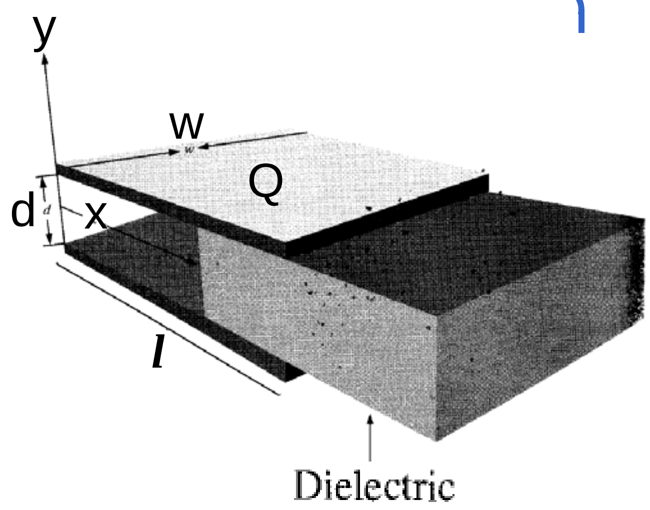
The divergence theorem turns the first term into a surface integral, which vanishes over all of space.

If the medium is a linear dielectric, $\mathbf{D} = \epsilon\mathbf{E} \longrightarrow \frac{1}{2}\Delta(\mathbf{D}\cdot\mathbf{E}) = \frac{1}{2}\Delta(\epsilon E^2) = \epsilon(\Delta\mathbf{E})\cdot\mathbf{E} = (\Delta\mathbf{D})\cdot\mathbf{E}$

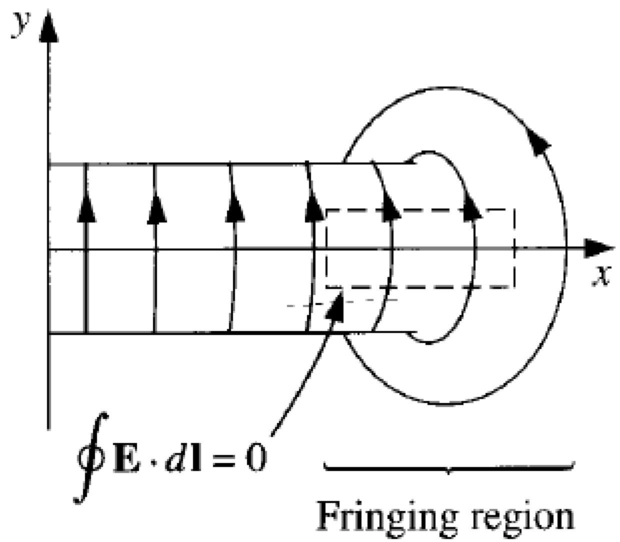
$$\Delta W = \Delta\left(\frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\, d\tau\right) \xrightarrow[\textit{Total work done = Stored energy}]{} \boxed{W = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\, d\tau}$$

4.4.4 Forces on Dielectrics

Consider, the case of a slab of linear dielectric material, partially inserted between the plates of a parallel-plate capacitor with a **total charge Q**.



Inside a parallel-plate capacitor,
→ the field is uniform and zero outside.
→ no net force on the dielectric at all, since the field everywhere would be perpendicular to the plates.



However, there is in reality a **fringing field** around the edges.
➔ This nonuniform fringing field that pulls the dielectric into the capacitor.

➔ Fringing fields are notoriously difficult to calculate.
➔ thus, to calculate the force on dielectric may be too difficult.

➔ **BUT, considering work energy makes it simple!**

Forces on Dielectrics

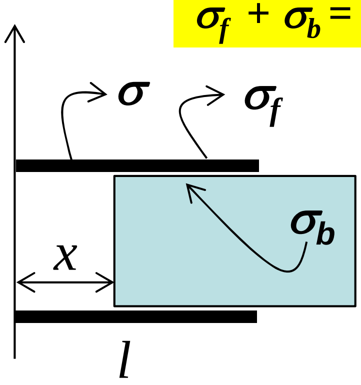
$$\sigma = \epsilon_0 E = \epsilon_0 V/d$$

$$\sigma_f = \epsilon_0 \epsilon_r V/d = \epsilon_r \sigma$$

$$C = \frac{Q}{V} = \frac{Q}{V}\left[wx\sigma + w(l-x)\sigma_f\right]$$

$$= \frac{\varepsilon_0 w}{d}\left[\varepsilon_r l - (\varepsilon_r - 1)x\right] = \frac{\varepsilon_0 w}{d}\left(\varepsilon_r l - \chi_e x\right)$$

$\sigma_f + \sigma_b = \sigma$



The energy stored in the capacitor with **a constant charge Q** is

$$W = \frac{1}{2}\frac{Q^2}{C} \longrightarrow F = -\frac{dW}{dx} = \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx} = \frac{1}{2}V^2\frac{dC}{dx}$$

$$\frac{dC}{dx} = -\frac{\epsilon_0 \chi_e w}{d} \longrightarrow \boxed{F = -\frac{\epsilon_0 \chi_e w}{2d}V^2}$$

(the dielectric is pulled into the capacitor)

Note that, if we use the energy form (with ***V*** **constant**) of

$$W = \tfrac{1}{2}CV^2 \longrightarrow F = -\frac{dW}{dx} = -\frac{1}{2}V^2\frac{dC}{dx}$$ (the sign is opposite!)

To maintain the capacitor at a fixed potential V, we need to connect it up to **a battery**.
But in that case the **battery *also does work*** as the dielectric moves;

$$F = -\frac{dW}{dx} + V\frac{dQ}{dx} = -\frac{1}{2}V^2\frac{dC}{dx} + V^2\frac{dC}{dx} = \frac{1}{2}V^2\frac{dC}{dx} \longrightarrow \boxed{F = -\frac{\epsilon_0 \chi_e w}{2d}V^2}$$ ***(the same sign)***

➔ In conclusion, it's simpler to calculate the force assuming constant Q.