Chapter 8. Conservation Laws

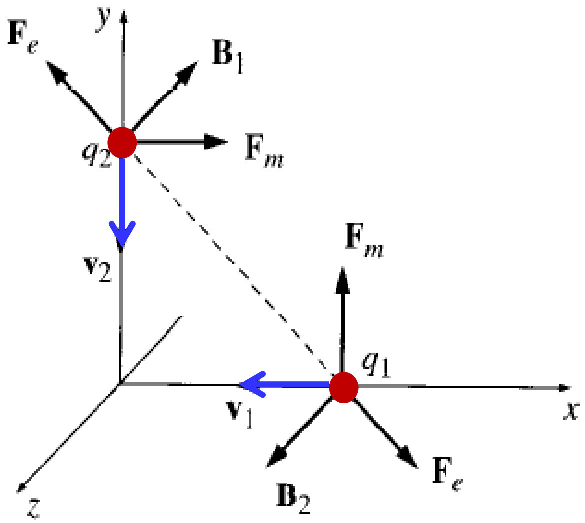
8.2 Momentum of EM fields

8.2.1 Newton's Third Law in Electrodynamics

*Consider two charges, **q_1 and q_2**, moving with speeds **v_1** and **v_2** along the x-axis and y-axis under magnetic fields **B_1 and B_2**.*

At an instantaneous time ***t***, each of the forces on ***q_1 and q_2***, *is a sum of electric and magnetic forces:*

$$\mathbf{F}_{q\,1,2} = \mathbf{F}_e + \mathbf{F}_m$$



$$\mathbf{F}_{q_1} = -\mathbf{F}_{q_2}$$ ➔ **If yes, Newton's third law can also valid in electrodynamics**

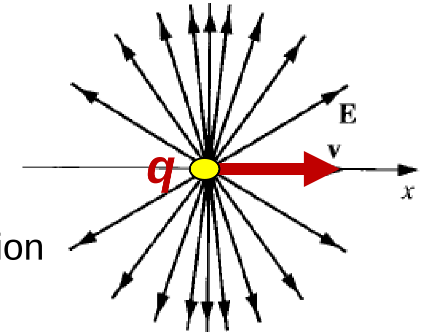
$$\mathbf{F}_{q_1} \neq -\mathbf{F}_{q_2}$$ ➔ **If not, Newton's third law looks not to be valid in electrodynamics**

➔ **However, the third law must be hold ALL THE TIME!**
➔ **If not, therefore, there must be another force hidden elsewhere.**

➔ ***We will see that the fields themselves carry forces (or, momentum)***
➔ **Only when the field momentum is added to the mechanical momentum of the charges, momentum conservation (or, the third law) is restored.**

Newton's Third Law in Electrodynamics

Imagine a point charge q traveling in along the x-axis at a constant speed v.



→ Because it is moving, *its electric field is not* given by Coulomb's law.

→ Nevertheless, ***E*** *still points radially outward* from the instantaneous position of the charge.

(In Chapter 10)

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\left[(c^2-v^2)\mathbf{u} + \boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})\right] \quad (10.65) \longrightarrow \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}$$

Generalized Coulomb field (velocity field)

Radiation field (acceleration field)

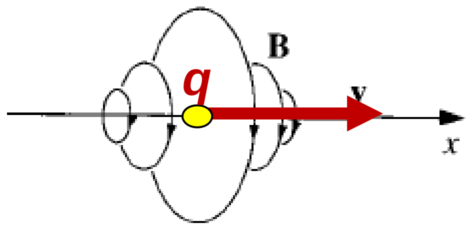
If the velocity and acceleration are both zero,

Since, moreover, a moving point charge does not constitute a steady current,

→ *its magnetic field is not* given by the Biot-Savart law.

→ Nevertheless, it's a fact that ***B*** *still circles around the axis*.

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{E}(\mathbf{r},t) \quad (10.66)$$

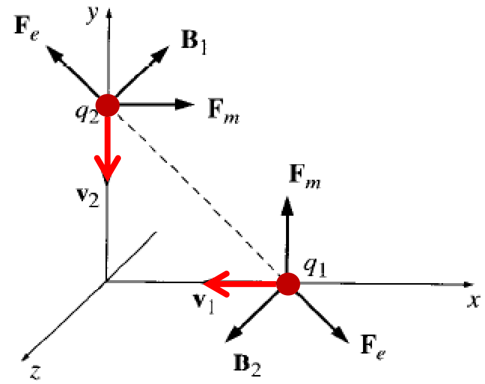


➔ ***Therefore, at an instantaneous time t, a situation with moving point charges at constant velocities may be regarded as an electromagnetostatic case.***

Newton's Third Law in Electrodynamics

Now suppose two identical charges, q_1 and q_2, moving with a same speed along the x and y axes.

(This is an *electromagnetostatic* case at an instantaneous time ***t***.)



➔ The **electrostatic force** between them is equal in magnitude but repulsive in direction: ⇨ $\mathbf{F}_{e,q_2} = -\mathbf{F}_{e,q_1}$

➔ How about **the magnetic force**?

→ The magnetic field $\mathbf{B_1}$ generated by $\mathbf{q_1}$ points into the page (at the position of q_2), so the magnetic force on q_2 is toward the *right.*

→ The magnetic field $\mathbf{B_2}$ generated by $\mathbf{q_2}$ points out of the page (at the position of q_1), so the magnetic force on q_1 is upward.

⇨ $\mathbf{F}_{m,q_2} \neq -\mathbf{F}_{m,q_1}$

The electromagnetic force of q_1 on q_2 is equal, but not opposite to the force of q_1 and q_2.
➔ The result may reveal violation of Newton's third law in electrodynamics!

$\mathbf{F}_{q_1} \neq -\mathbf{F}_{q_2}$

In electrostatics and magnetostatics the third law holds, ***but in electrodynamics it does not.***
➔ **Is it true?**

➔ ***The third law must be hold ALL THE TIME!***
➔ From the third law, we know that the proof of momentum conservation rests on the cancellation of internal forces: $\left(\sum \mathbf{F}_i = d\mathbf{P}/dt = 0\right)$

Therefore, let's prove the momentum conservation in electrodynamics!

➔ **The fields themselves carry momentum.**
➔ Only when the field momentum is added to the mechanical momentum of the charges, momentum conservation (or, the third law) is restored.

8.2.2 Maxwell's stress tensor

- ➔ **Now we know that the fields themselves must carry momentum.**
- ➔ **To find out the field momentum, let's calculate the total EM force on the charges in volume V in terms of Poynting vector S and stress tensor T:**

EM force → $\mathbf{F} = \int_{\mathcal{V}} (\mathbf{E} + \mathbf{v} \times \mathbf{B})\rho \, d\tau = \int_{\mathcal{V}} (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B})\, d\tau$

The force per unit volume is

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$$

$$= \epsilon_0(\nabla \cdot \mathbf{E})\mathbf{E} + \left(\frac{1}{\mu_0}\nabla \times \mathbf{B} - \epsilon_0\frac{\partial \mathbf{E}}{\partial t}\right) \times \mathbf{B}$$

$$\left\{\begin{array}{l} \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) = \left(\frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}\right) + \left(\mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t}\right) \\ \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \end{array}\right. \Rightarrow \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} = \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) + \mathbf{E} \times (\nabla \times \mathbf{E})$$

$$\mathbf{f} = \epsilon_0[(\nabla \cdot \mathbf{E})\mathbf{E} - \mathbf{E} \times (\nabla \times \mathbf{E})] - \frac{1}{\mu_0}[\mathbf{B} \times (\nabla \times \mathbf{B})] - \epsilon_0\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B})$$

A term seems to be "missing" from the symmetry in **E** and **B**, which can be achieved by inserting (∇ • **B**)**B** (= 0)

$$\mathbf{f} = \epsilon_0[(\nabla \cdot \mathbf{E})\mathbf{E} - \mathbf{E} \times (\nabla \times \mathbf{E})] + \frac{1}{\mu_0}[(\nabla \cdot \mathbf{B})\mathbf{B} - \mathbf{B} \times (\nabla \times \mathbf{B})] - \epsilon_0\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B})$$

Using the vector calculus identity of $\frac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A}) = \mathbf{A} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{A}$

$$\mathbf{f} = \underbrace{\epsilon_0[(\nabla \cdot \mathbf{E})\mathbf{E} + (\mathbf{E} \cdot \nabla)\mathbf{E}] + \frac{1}{\mu_0}[(\nabla \cdot \mathbf{B})\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{B}] - \frac{1}{2}\nabla\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)} - \underbrace{\epsilon_0\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B})}_{\epsilon_0\mu_0\frac{\partial \mathbf{S}}{\partial t}}$$

It can be simplified by introducing the *Maxwell stress tensor T*

Maxwell's stress tensor

$$\mathbf{f} = \boxed{\epsilon_0[(\nabla\cdot\mathbf{E})\mathbf{E} + (\mathbf{E}\cdot\nabla)\mathbf{E}] + \frac{1}{\mu_0}[(\nabla\cdot\mathbf{B})\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{B}] - \frac{1}{2}\nabla\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)} - \epsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B})$$

→ It can be written more compactly by introducing the **Maxwell stress tensor**,

Maxwell's stress tensor: $\overleftrightarrow{\mathbf{T}}$

$$\boxed{T_{ij} \equiv \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)}$$

$$T_{xx} = \frac{1}{2}\epsilon_0(E_x^2 - E_y^2 - E_z^2) + \frac{1}{2\mu_0}(B_x^2 - B_y^2 - B_z^2), \quad T_{xy} = \epsilon_0(E_xE_y) + \frac{1}{\mu_0}(B_xB_y),$$ and so on.

The divergence of if has as its *j-th* component,

$$(\nabla\cdot\overleftrightarrow{\mathbf{T}})_j = \epsilon_0\left[(\nabla\cdot\mathbf{E})E_j + (\mathbf{E}\cdot\nabla)E_j - \frac{1}{2}\nabla_jE^2\right] + \frac{1}{\mu_0}\left[(\nabla\cdot\mathbf{B})B_j + (\mathbf{B}\cdot\nabla)B_j - \frac{1}{2}\nabla_jB^2\right]$$

Thus the force per unit volume can be written in the much simpler form:

$$\Rightarrow \quad \mathbf{f} = \nabla\cdot\overleftrightarrow{\mathbf{T}} - \epsilon_0\mu_0\frac{\partial\mathbf{S}}{\partial t}$$

The total EM force on the charges in volume V is therefore

$$\Rightarrow \quad \boxed{\mathbf{F} = \oint_{\mathcal{S}}\overleftrightarrow{\mathbf{T}}\cdot d\mathbf{a} - \epsilon_0\mu_0\frac{d}{dt}\int_{\mathcal{V}}\mathbf{S}\,d\tau}$$

* In *static* case ➔ $\mathbf{F} = \oint_{\mathcal{S}}\overleftrightarrow{\mathbf{T}}\cdot d\mathbf{a}$

* $\left(\overleftrightarrow{T}\right)_{ij}$ ➔ The force per unit area (or, **stress**) on the surface → called by "Stress Tensor"

Electromagnetic Force given by Maxwell's Stress Tensor: $\mathbf{F} = \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} - \epsilon_0 \mu_0 \frac{d}{dt} \int_{\mathcal{V}} \mathbf{S}\, d\tau$

Example 8.2 Find the net force on the northern hemisphere of a uniformly charged solid sphere of radius ***R*** and charge ***Q***, exerted by the southern hemisphere.

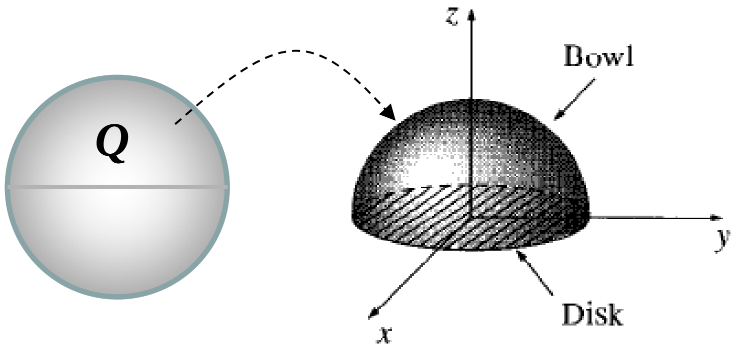
In this static case ➔ $\mathbf{F} = \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}$ $\qquad T_{ij} \equiv \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2\right) = \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2\right)$

The boundary surface consists of two parts:

→ a hemispherical "bowl" at radius *R*, and a circular disk at $\theta = \pi/2$.

For the bowl, $d\mathbf{a} = R^2 \sin\theta\, d\theta\, d\phi\, \hat{\mathbf{r}}$ $\qquad \mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{R^2} \hat{\mathbf{r}}$

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$$



The net force on the bowl is obviously in the z-direction,

$$(\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a})_z = T_{zx}\, da_x + T_{zy}\, da_y + T_{zz}\, da_z$$

$$\begin{cases} T_{zx} = \epsilon_0 E_z E_x = \epsilon_0 \left(\dfrac{Q}{4\pi\epsilon_0 R^2}\right)^2 \sin\theta\cos\theta\cos\phi \\ T_{zy} = \epsilon_0 E_z E_y = \epsilon_0 \left(\dfrac{Q}{4\pi\epsilon_0 R^2}\right)^2 \sin\theta\cos\theta\sin\phi \\ T_{zz} = \dfrac{\epsilon_0}{2}(E_z^2 - E_x^2 - E_y^2) = \dfrac{\epsilon_0}{2}\left(\dfrac{Q}{4\pi\epsilon_0 R^2}\right)^2 (\cos^2\theta - \sin^2\theta) \end{cases}$$

$$(\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a})_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R}\right)^2 \sin\theta\cos\theta\, d\theta\, d\phi$$

Meanwhile, for the equatorial disk, $(\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a})_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 r^3\, dr\, d\phi$

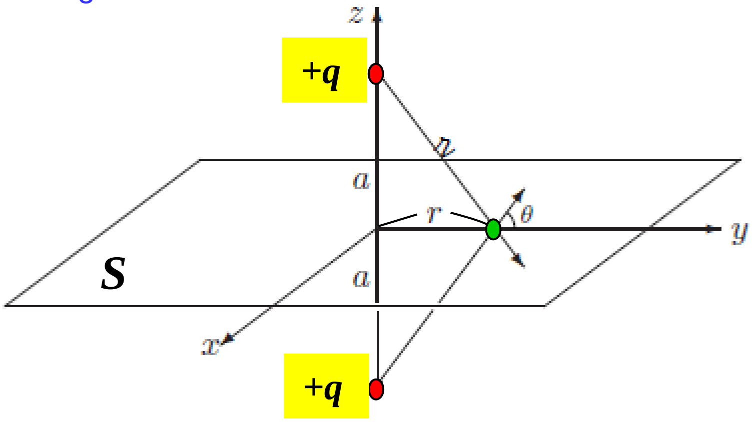
Combining them, the net force on the northern hemisphere is

$$\boxed{F = \frac{1}{4\pi\epsilon_0} \frac{3Q^2}{16R^2}}$$

Electromagnetic Force given by Maxwell's Stress Tensor: $\mathbf{F} = \oint_S \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}$ *(static case)*

Problem 8.4 (a) Consider two equal point charges *q*, separated by a distance *2a.* Construct the plane equidistant from the two charges. By integrating Maxwell's stress tensor over **determine the force of one charge on th**

(b) **Do the same for charges that are opposi**



$$T_{ij} \equiv \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right) = \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right)$$

(a) The force on +q is clearly in the + z direction,

(b) The force on +q is clearly in the - z direction,

8.2.3 Conservation of momentum for EM fields

According to the second law**, the force on an object** is equal to **the rate of change of its momentum**:

$$\mathbf{F}=\frac{d\mathbf{p}_{\mathrm{mech}}}{dt} \longleftrightarrow \mathbf{F}=\oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}}\cdot d\mathbf{a}-\epsilon_0\mu_0\frac{d}{dt}\int_{\mathcal{V}}\mathbf{S}\,d\tau$$

($\mathbf{P}_{\mathrm{mech}}$ is the mechanical momentum of the particles contained in the volume *V.)*

$$\boxed{\frac{d\mathbf{p}_{\mathrm{mech}}}{dt}=-\epsilon_0\mu_0\frac{d}{dt}\int_{\mathcal{V}}\mathbf{S}\,d\tau+\oint_{\mathcal{S}}\overleftrightarrow{\mathbf{T}}\cdot d\mathbf{a}}$$

: Conservation of momentum in electromagnetics

(Poynting theorem for energy conservation)

$$\frac{d\mathbf{p}_{\mathrm{mech}}}{dt}=-\frac{d}{dt}\int_{\mathcal{V}}\left(\epsilon_0\mu_0\mathbf{S}\right)d\tau-\oint_{\mathcal{S}}\left(-\overleftrightarrow{\mathbf{T}}\right)\cdot d\mathbf{a} \longleftrightarrow \frac{dW}{dt}=-\frac{d}{dt}\int_{\mathcal{V}}u_{\mathrm{em}}\,d\tau-\oint_{\mathcal{S}}\mathbf{S}\cdot d\mathbf{a}$$

(In first integral)

$$\mathbf{p}=\int_{\mathcal{V}}\left(\epsilon_0\mu_0\mathbf{S}\right)d\tau$$

➔ ***Momentum stored in the EM fields***

(In first integral)

$$U_{em}=\int_V u_{em}\,d\tau$$

➔ Energy stored in the EM fields

$$\mathbf{g}=\mu_0\varepsilon_0\mathbf{S}=\varepsilon_0\left(\mathbf{E}\times\mathbf{B}\right)$$

➔ ***Momentum density in the EM fields***

$$u_{em}=\frac{1}{2}\left(\varepsilon_0E^2+\frac{1}{\mu_0}B^2\right)$$

➔ Energy density in the EM fields

(In second integral)

$$\left(-\overleftrightarrow{\mathbf{T}}\right)$$

➔ ***Momentum flux density***

(Momentum per unit time, per unit area)

(In second integral)

$$\mathbf{S}=\frac{1}{\mu_0}\left(\mathbf{E}\times\mathbf{B}\right)$$

➔ Energy flux density

(Energy per unit time, per unit area)

$$\left(-\overleftrightarrow{\mathbf{T}}\right)\cdot d\mathbf{a}$$

➔ ***Momentum flux***

(Momentum per unit time passing through da)

$$\mathbf{S}\cdot d\mathbf{a}$$

➔ Energy flux

(Energy per unit time passing through da)

Conservation of momentum for EM fields

$$\frac{d\mathbf{p}_{\text{mech}}}{dt} = -\epsilon_0\mu_0 \frac{d}{dt}\int_{\mathcal{V}} \mathbf{S}\, d\tau + \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}$$

$$\frac{\partial}{dt}\left(\mathbf{P}_{\text{mech}} + \mathbf{P}\right) = \nabla \cdot \overleftrightarrow{\mathbf{T}}$$

(In differential form)

$\mathbf{p} = \int_{\mathcal{V}} \left(\epsilon_0\mu_0 \mathbf{S}\right) d\tau$ → *Total momentum stored in EM fields*

$\left(\mathbf{g} = \mu_0 \varepsilon_0 \mathbf{S}\right)$ → ***Momentum density stored in EM fields***

$\oint_{\mathcal{S}} \left(-\overleftrightarrow{\mathbf{T}}\right) \cdot d\mathbf{a}$ → ***Total momentum per unit time passing through a closed surface***

$\left(-\overleftrightarrow{\mathbf{T}}\right)$ → ***Momentum flux density***
(Momentum per unit time, per unit area)

If V is all of space, no momentum flows in and out → $\nabla \cdot \overleftrightarrow{\mathbf{T}} = 0$ → $\left(\mathbf{P}_{\text{mech}} + \mathbf{P}\right) = \textbf{constant}$

→ ***Total (mech + EM) momentum is conserved.***

If the mechanical momentum in V is not changing (for example, in a region of empty space) → $\frac{\partial}{dt}\left(\mathbf{P}_{\text{mech}}\right) = 0$ → $\int_V \left(\frac{\partial \mathbf{g}}{\partial t}\right) d\tau = \oint_S \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} = \int_V \left(\nabla \cdot \overleftrightarrow{\mathbf{T}}\right) d\tau$

→ $\frac{\partial \mathbf{g}}{\partial t} = \nabla \cdot \overleftrightarrow{\mathbf{T}}$ ***: Continuity equation of EM momentum***

$$\frac{\partial \rho}{\partial t} = -\left(\nabla \cdot \mathbf{J}\right) \quad \text{<----->} \quad \frac{\partial \mathbf{g}}{\partial t} = -\nabla \cdot \left(-\overleftrightarrow{\mathbf{T}}\right)$$

$\left(-\overleftrightarrow{\mathbf{T}}\right)$ *playing the part of J → Local conservation of filed momentum*

Conservation of Energy and Momentum for EM fields

Energy Conservation

$$\boxed{\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} u_{\text{em}}\, d\tau - \oint_{\mathcal{S}} \mathbf{S}\cdot d\mathbf{a}}$$

$$\frac{\partial}{dt}\left(u_{\mathbf{mech}} + u_{\mathbf{em}}\right) = -\nabla\cdot\mathbf{S}$$

$$u_{\text{em}} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)$$

Momentum Conservation

$$\boxed{\frac{d\mathbf{p}_{\text{mech}}}{dt} = -\epsilon_0\mu_0\frac{d}{dt}\int_{\mathcal{V}} \mathbf{S}\, d\tau + \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}}\cdot d\mathbf{a}}$$

$$\frac{\partial}{dt}\left(\mathbf{P}_{\mathbf{mech}} + \mathbf{P}_{\mathbf{em}}\right) = -\nabla\cdot\left(-\overleftrightarrow{\mathbf{T}}\right)$$

$$\mathbf{P}_{\mathbf{em}} = \int_V \left(\varepsilon_0\mu_0\mathbf{S}\right)d\tau = \int_V \varepsilon_0\left(\mathbf{E}\times\mathbf{B}\right)d\tau$$

$$\mathbf{g} = \varepsilon_0\mu_0\mathbf{S} = \varepsilon_0\left(\mathbf{E}\times\mathbf{B}\right)$$

Poynting Vector *S*

S : ***Energy per unit area (Energy flux density), per unit time transport by EM fields***

$\varepsilon_0\mu_0 S$: ***Momentum per unit volume (Momentum density) stored in EM fields***

Stress Tensor $\overleftrightarrow{\mathbf{T}}$

$\overleftrightarrow{\mathbf{T}}$: ***EM field stress (Force per unit area) acting on a surface***

$-\overleftrightarrow{\mathbf{T}}$: ***Flow of momentum (momentum per unit area, unit time) carried by EM fields***

Continuity Equations of EM fields in empty space

$$\frac{\partial\rho}{\partial t} = -\left(\nabla\cdot\mathbf{J}\right)$$

$$\frac{\partial u_{em}}{\partial t} = -\left(\nabla\cdot\mathbf{S}\right)$$

$\left(\mathbf{S}\right)$ *playing the part of **J** → **Local conservation of field energy***

$$\frac{\partial\mathbf{g}}{\partial t} = -\nabla\cdot\left(-\overleftrightarrow{\mathbf{T}}\right)$$

$\left(-\overleftrightarrow{\mathbf{T}}\right)$ *playing the part of **J** → **Local conservation of field momentum***

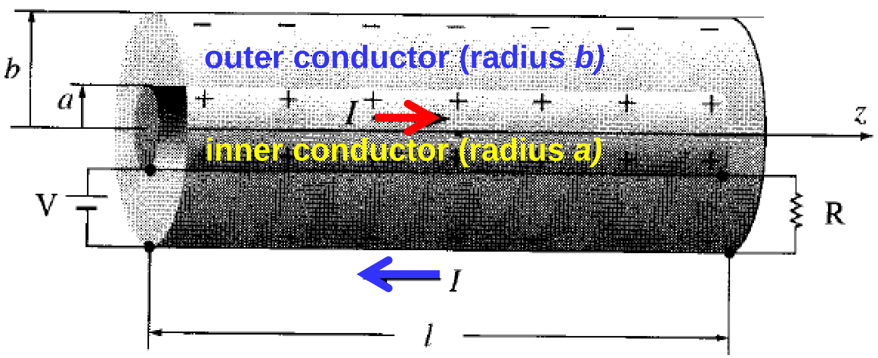
$$\mathbf{p}_{em} = \mu_0\epsilon_0 \int_{\mathcal{V}} \mathbf{S}\, d\tau \qquad \mathbf{F} = \frac{d\mathbf{p}_{mech}}{dt} = -\epsilon_0\mu_0 \frac{d}{dt}\int_{\mathcal{V}} \mathbf{S}\, d\tau + \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}$$

Example 8.3

A long coaxial cable, of length *l*, consists of an inner conductor (radius *a)* and an outer conductor (radius *b).* It is connected to a battery at one end and a resistor at the other. The inner conductor carries a uniform charge per unit length λ, and a steady current *I* to the right; the outer conductor has the opposite charge and current.

What is the electromagnetic momentum stored in the fields?

The fields are $\mathbf{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\hat{\mathbf{s}}$ $\mathbf{B} = \frac{\mu_0}{2\pi}\frac{I}{s}\hat{\boldsymbol{\phi}}$



The Poynting vector is therefore

$$\mathbf{S} = \frac{\lambda I}{4\pi^2\epsilon_0 s^2}\hat{\mathbf{z}}$$

Evidently energy is flowing down the line, from the battery to the resistor.
In fact, the power transported is $P = \int \mathbf{S}\cdot d\mathbf{a} = \frac{\lambda I}{4\pi^2\epsilon_0}\int_a^b \frac{1}{s^2} 2\pi s\, ds = \frac{\lambda I}{2\pi\epsilon_0}\ln(b/a) = IV$

The EM momentum in the fields ➔

- ➔ *This is an astonishing result.*
- ➔ *The cable is not moving, and the fields are static.*
- ➔ *Yet, we are asked to believe that there is momentum in the system.*

- ➔ *The total momentum must be zero.*
- ➔ *In this case it turns out that there is "hidden" mechanical momentum associated with the flow of current, and this exactly cancels the momentum in the fields.*

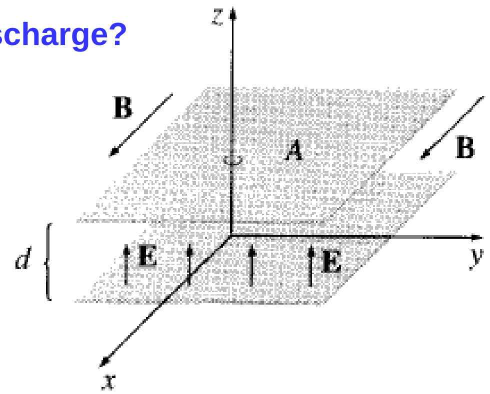
$$\mathbf{P}_{\mathbf{em}} = \int_V \mathbf{g} d\tau \qquad \mathbf{g} = \varepsilon_0 \mu_0 \mathbf{S} = \varepsilon_0 \left(\mathbf{E} \times \mathbf{B}\right)$$ ***: Momentum density stored in EM fields***

Problem 8.5

A charged parallel-plate capacitor (with uniform **E** = *E* **z**) is placed in a uniform magnetic field **B** = B **x**.

(a) **Find the electromagnetic momentum in the space between the plates.**

(b) Now a resistive wire is connected between the plates, along the *z* axis, so that the capacitor slowly discharges.
The current through the wire will experience a magnetic force;
what is the total impulse delivered to the system, during the discharge?



(a) $\mathbf{g}_{\text{em}} = \epsilon_0(\mathbf{E} \times \mathbf{B}) = \epsilon_0 E B\, \hat{\mathbf{y}};$

(b) Impulse: $\mathbf{I} = \int \mathbf{F}\, dt$

$$= \int_0^\infty \mathbf{F}\, dt = \int_0^\infty I(\mathbf{l} \times \mathbf{B})\, dt$$

8.2.4 Angular Momentum

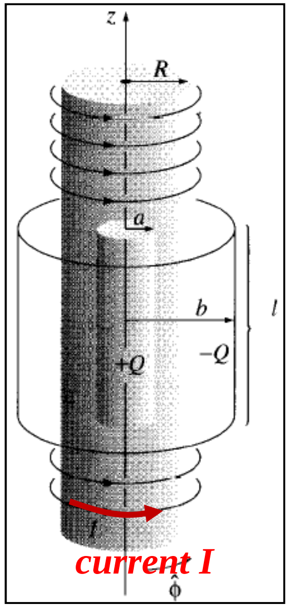
Energy density and Momentum density of Electromagnetic fields

$$u_{\text{em}} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) \qquad \mathbf{g} = \varepsilon_0\mu_0\mathbf{S} = \varepsilon_0\left(\mathbf{E}\times\mathbf{B}\right)$$

Density of angular momentum of electromagnetic fields

$$\ell_{\text{em}} = \mathbf{r}\times\mathbf{g} = \varepsilon_0\left[\mathbf{r}\times\left(\mathbf{E}\times\mathbf{B}\right)\right]$$

Example 8.4 Two long cylindrical shells are coaxial with a solenoid carrying current ***I***.
When the current in the solenoid is gradually reduced, the cylinders begin to rotate.
Where does the angular momentum of the cylinder comes from?



Before the current was switched off, there were an electric field and a magnetic field:

$$\mathbf{E} = \frac{Q}{2\pi\epsilon_0 l}\frac{1}{s}\,\hat{\mathbf{s}}\,(a < s < b) \quad \text{in the region between the cylinders}$$

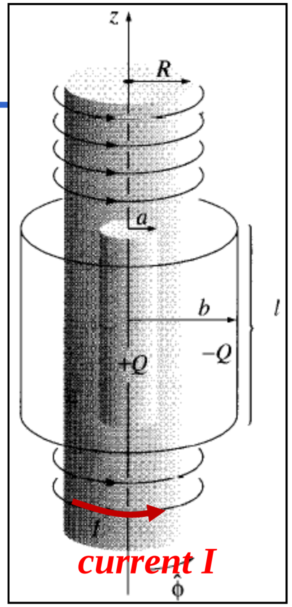
$$\mathbf{B} = \mu_0 n I\,\hat{\mathbf{z}}\,(s < R) \quad \text{inside the solenoid}$$

$$\mathbf{g} = \mu_0\epsilon_0\mathbf{S} = \epsilon_0(\mathbf{E}\times\mathbf{B}) \longrightarrow \mathbf{g} = -\frac{\mu_0 n I Q}{2\pi l s}\,\hat{\boldsymbol{\phi}} \quad \text{in the region } a < s < R$$

$$\ell_{\text{em}} = \mathbf{r}\times\mathbf{g} = \varepsilon_0\left[\mathbf{r}\times\left(\mathbf{E}\times\mathbf{B}\right)\right] = -\frac{\mu_0 n I Q}{2\pi l}\,\hat{\mathbf{z}}$$ → constant over the volume of $\pi(R^2 - a^2)l$:

Total angular momentum in the fields (before switching off): $\mathbf{L}_{\text{em}} = -\frac{1}{2}\mu_0 n I Q(R^2 - a^2)\,\hat{\mathbf{z}}$

Example 8.4 (continued) $\mathbf{L}_{\rm em} = -\frac{1}{2}\mu_0 n I Q (R^2 - a^2)\,\hat{\mathbf{z}}$



When the current is turned off,
the changing magnetic field induces a circumferential electric field, given by Faraday's law:

$$\mathbf{E} = \begin{cases} -\frac{1}{2}\mu_0 n \frac{dI}{dt}\frac{R^2}{s}\,\hat{\boldsymbol{\phi}}, & (s > R) \\ -\frac{1}{2}\mu_0 n \frac{dI}{dt} s\,\hat{\boldsymbol{\phi}}, & (s < R) \end{cases} \qquad \mathbf{B} = \mu_0 n I\,\hat{\mathbf{z}} (s < R)$$

Torque on the outer cylinder: $\mathbf{N}_b = \mathbf{r} \times (-Q\mathbf{E}) = \frac{1}{2}\mu_0 n Q R^2 \frac{dI}{dt}\,\hat{\mathbf{z}}$

→ Angular momentum of the outer cylinder: $\mathbf{L}_b = \frac{1}{2}\mu_0 n Q R^2\,\hat{\mathbf{z}} \int_I^0 \frac{dI}{dt} dt = -\frac{1}{2}\mu_0 n I Q R^2\,\hat{\mathbf{z}}$

Torque on the inner cylinder: $\mathbf{N}_a = -\frac{1}{2}\mu_0 n Q a^2 \frac{dI}{dt}\,\hat{\mathbf{z}}$

→ Angular momentum of the inner cylinder: $\mathbf{L}_a = \frac{1}{2}\mu_0 n I Q a^2\,\hat{\mathbf{z}}$

➔ **Total angular momentum of the inner and outer cylinders:**

$$\mathbf{L}_a + \mathbf{L}_b = -\frac{1}{2}\mu_0 n I Q (R^2 - a^2)\,\hat{\mathbf{z}}$$

Which is the same as the angular momentum of the field: $\boxed{\mathbf{L}_{\rm em} = \mathbf{L}_a + \mathbf{L}_b.}$

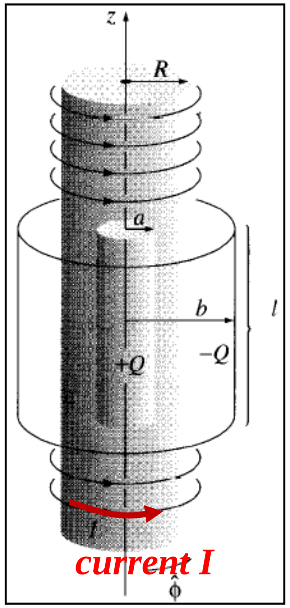
➔ ***The total angular momentum (fields plus matter) is conserved.***
➔ ***Therefore, the angular momentum lost by fields is precisely equal to the angular momentum gained by the cylinders.***

Angular Momentum $\ell_{\mathbf{em}} = \mathbf{r}\times\mathbf{g} = \varepsilon_0\left[\mathbf{r}\times(\mathbf{E}\times\mathbf{B})\right]$

Problem 8.8 In Ex. 8.4, suppose that instead of turning off the *magnetic* field (by reducing I) **we tum off the *electric* field**, by connecting a weakly conducting radial spoke between the cylinders. (We'll have to cut a slot in the solenoid, so the cylinders can still rotate freely.) From the magnetic force on the current in the spoke, **determine the total angular momentum delivered to the cylinders**, as they discharge (they are now rigidly connected, so they rotate together). **Compare the initial angular momentum stored in the fields** (Eq. 8.34). (Notice that the *mechanism* by which angular momentum is transferred from the fields to the cylinders is entirely different in the two cases' in Ex. S.4 it was Faraday's law, but here it is the Lorentz force law.)

In Ex. 8.4 we turned the current off slowly, to keep things quasistatic;
here we reduce the electric field slowly to keep the displacement current negligible.

$$\mathbf{B} = \mu_0 n I\,\hat{\mathbf{z}}\ (\text{for } s < R;\ \text{outside the solenoid } B = 0).$$



The force on a segment ds of spoke is

The torque on the spoke is

Therefore the angular momentum of the cylinders is

→ Same as the initial total angular momentum in the fields (before switching off):

8.3 Magnetic Forces Do No Work

(5.1.2) The ***magnetic force*** in a charge Q, moving with **velocity v** in a **magnetic field B**, is

$$\boxed{\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B})}$$

➔ **Magnetic forces do no work!**

$$dW_{\text{mag}} = \mathbf{F}_{\text{mag}} \cdot d\mathbf{l} = Q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}\, dt = 0$$

➔ Magnetic forces may *alter the direction* in which a particle moves, but they *cannot speed it up or slow it down*.

Poynting's vector and Poynting theorem in matter

Problem 8.23 (a) Describe the Poynting's vector and Poynting theorem for the filed in matter.

$$\frac{dW}{dt} = \int_{\mathcal{V}} (\mathbf{E}\cdot\mathbf{J})\, d\tau \longrightarrow \frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau - \frac{1}{\mu_0}\oint_{\mathcal{S}} (\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a}$$

➔ **Poynting's theorem in vacuum**

The work done on free charges and currents in matter,

$$\frac{dW}{dt} = \int_{\mathcal{V}} (\mathbf{E}\cdot\mathbf{J})\, d\tau \longrightarrow \frac{dW}{dt} = \int_{\mathcal{V}} (\mathbf{E}\cdot\mathbf{J}_f)\, d\tau \xrightarrow{\mathbf{J}_f = \nabla\times\mathbf{H} - \frac{\partial \mathbf{D}}{\partial t}} \mathbf{E}\cdot\mathbf{J}_f = \mathbf{E}\cdot(\nabla\times\mathbf{H}) - \mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla\cdot(\mathbf{E}\times\mathbf{H}) = \mathbf{H}(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{H}), \text{ while } \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \longrightarrow \mathbf{E}\cdot(\nabla\times\mathbf{H}) = -\mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} - \nabla\cdot(\mathbf{E}\times\mathbf{H})$$

$$\mathbf{E}\cdot\mathbf{J}_f = -\mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} - \mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} - \nabla\cdot(\mathbf{E}\times\mathbf{H})$$

$$\boxed{\frac{dW}{dt} = -\int_{\mathcal{V}} \left(\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t}\right) d\tau - \oint_{\mathcal{S}} (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{a}}$$

➔ **Poynting's theorem for the fields in matter**

$$\boxed{\mathbf{S} = \mathbf{E}\times\mathbf{H}}$$

➔ **Poynting vector in matter**

$$\frac{\partial u_{\text{em}}}{\partial t} = \mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} \longrightarrow$$ the rate of change of the electromagnetic energy density

For *linear* media, $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{H} = \frac{1}{\mu}\mathbf{B}$

$$\frac{\partial u_{\text{em}}}{\partial t} = \epsilon\mathbf{E}\cdot\frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\mu}\mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t} = \frac{1}{2}\epsilon\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{E}) + \frac{1}{2\mu}\frac{\partial}{\partial t}(\mathbf{B}\cdot\mathbf{B}) = \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$$

$$\boxed{u_{\text{em}} = \tfrac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})}$$

➔ **Electromagnetic energy density in matter**

EM force and momentum density in matter

Problem 8.23 (b) Describe the Poynting's vector and Poynting theorem for the filed in matter.

The force per unit volume : $\mathbf{f} = \rho\mathbf{E} + \mathbf{J}\times\mathbf{B}$

The momentum density : $\mathbf{g} = \varepsilon_0\mu_0\mathbf{S} = \varepsilon_0(\mathbf{E}\times\mathbf{B})$

In matter,

$$\mathbf{f} = \rho_f\mathbf{E} + \mathbf{J}_f\times\mathbf{B} \xrightarrow{\rho_f = \nabla\cdot\mathbf{D},\ \text{and}\ \mathbf{J}_f = \nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t}} \mathbf{f} = \mathbf{E}(\nabla\cdot\mathbf{D}) + (\nabla\times\mathbf{H})\times\mathbf{B} - \left(\frac{\partial\mathbf{D}}{\partial t}\right)\times\mathbf{B}$$

$$\frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B}) = \frac{\partial\mathbf{D}}{\partial t}\times\mathbf{B} + \mathbf{D}\times\left(\frac{\partial\mathbf{B}}{\partial t}\right),\ \text{and}\ \frac{\partial\mathbf{B}}{\partial t} = -\nabla\times\mathbf{E} \longrightarrow \frac{\partial\mathbf{D}}{\partial t}\times\mathbf{B} = \frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B}) + \mathbf{D}\times(\nabla\times\mathbf{E})$$

$$\mathbf{f} = \mathbf{E}(\nabla\cdot\mathbf{D}) - \mathbf{D}\times(\nabla\times\mathbf{E}) - \mathbf{B}\times(\nabla\times\mathbf{H}) - \frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B})$$

$$\mathbf{f} = \{[\mathbf{E}(\nabla\cdot\mathbf{D}) - \mathbf{D}\times(\nabla\times\mathbf{E})] + [\mathbf{H}(\nabla\cdot\mathbf{B}) - \mathbf{B}\times(\nabla\times\mathbf{H})]\} - \frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B})$$

$$\longleftrightarrow\quad \mathbf{f} = \epsilon_0[(\nabla\cdot\mathbf{E})\mathbf{E} - \mathbf{E}\times(\nabla\times\mathbf{E})] + \frac{1}{\mu_0}[(\nabla\cdot\mathbf{B})\mathbf{B} - \mathbf{B}\times(\nabla\times\mathbf{B})] - \epsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B})$$

$$\mathbf{f} = \nabla\cdot\overleftrightarrow{\mathbf{T}} - \epsilon_0\mu_0\frac{\partial\mathbf{S}}{\partial t}$$

$$\boxed{\mathbf{f} = \nabla\cdot\overleftrightarrow{\mathbf{T}} - \frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B})}$$ ➔ **EM force per unit volume in matter**

$$\boxed{\mathbf{g} = \mathbf{D}\times\mathbf{B} = \varepsilon\mu\mathbf{S}}$$ ➔ **EM momentum density in matter**