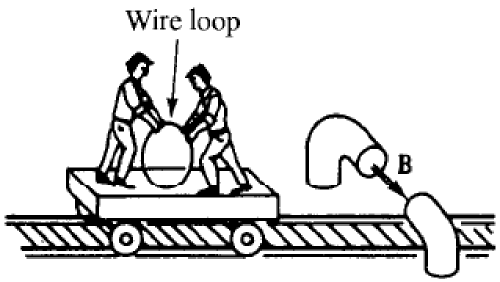
Chapter 12.
Electrodynamics and Relativity

Does the principle of relativity apply to the laws of electrodynamics?

Does the principle of relativity apply to the laws of electrodynamics?

Take, for example, the reciprocal electrodynamic action of a magnet and a conductor:

The observable phenomenon here depends only on the relative motion of the conductor and the magnet.



If the magnet is stationary and the conductor in motion, no electric field arises in the neighborhood of the magnet. In the conductor, however, we find an magnetic force (motional emf)

On the other hand, for someone on the train the magnet is in motion and the conductor at rest. No magnetic force, because the conductor is at rest. But, a changing magnetic field induces an electric field by Faraday's law, resulting in an electric force (induced emf).

➔ The motional emf and the induced emf give rise to electric currents of the same path and intensity.

➔ Was the equality of the two emf's Just a lucky accident?

Einstein could not believe this was a mere coincidence; he took it, rather, as a clue that electromagnetic phenomena, like mechanical ones, obey the principle of relativity.

Einstein proposed his two famous postulates:

1. **The principle of relativity.** (first elevated by Galileo Galilei)
   ➔ The laws of physics apply in all inertial reference systems.
   ➔ It states that there is no absolute rest system.

2. **The universal speed of light.**
   ➔ The speed of light in vacuum is the same for all inertial observers, regardless of the motion of the source.

Einstein's velocity addition rule: $$v_{AC} = \frac{v_{AB} + v_{BC}}{1 + (v_{AB} v_{BC}/c^2)}$$

Galileo's velocity addition rule: $$v_{AC} = v_{AB} + v_{BC}$$

The three most striking geometrical consequences of Einstein's postulates:

➔ **the relativity of simultaneity:** Two events that are simultaneous in one inertial system are not, in general, simultaneolus in another.

➔ **time dilation:** **Moving clocks run slow.**

$$\boxed{\Delta \bar{t} = \sqrt{1 - \upsilon^2 / c^2}\,\Delta t = \frac{1}{\gamma}\Delta t} \qquad \boxed{\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}}$$

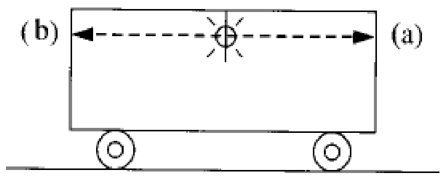
$\Delta \bar{t}$ (measured on car) Δt (measured on ground)

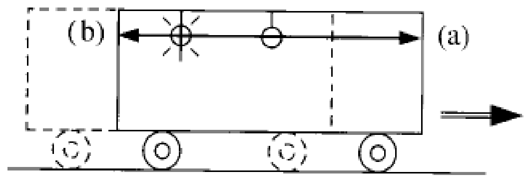
➔ **Lorentz contraction:** **Moving objects looks shorter,** *from ground point of view.*

$$\boxed{\Delta \bar{x} = \frac{1}{\sqrt{1 - \upsilon^2 / c^2}}\Delta x = \gamma \Delta x}$$

12.1 The Special Theory of Relativity

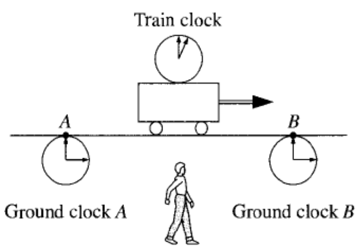
(i) the relativity of simultaneity: Two events that are simultaneous in one inertial system are not, in general, simultaneolus in another.

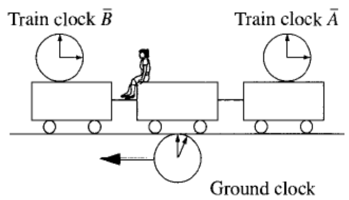


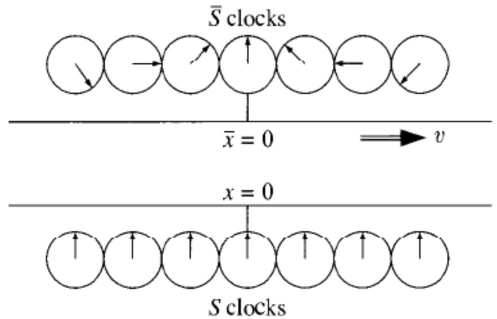


(ii) Time dilation: **Moving clocks run slow.** **The twin paradox.**

$$\boxed{\Delta\bar{t} = \sqrt{1-\upsilon^2/c^2}\,\Delta t}$$





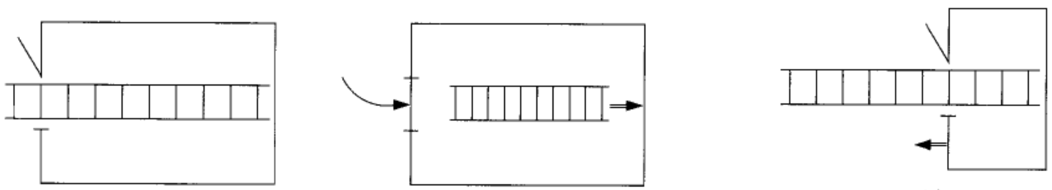


(iii) Lorentz (length) contraction: **Moving objects looks shorter.**

Only along the direction of its motion.

Dimensions perpendicular to the velocity are not contracted.

$$\boxed{\Delta\bar{x} = \frac{1}{\sqrt{1-\upsilon^2/c^2}}\,\Delta x}$$

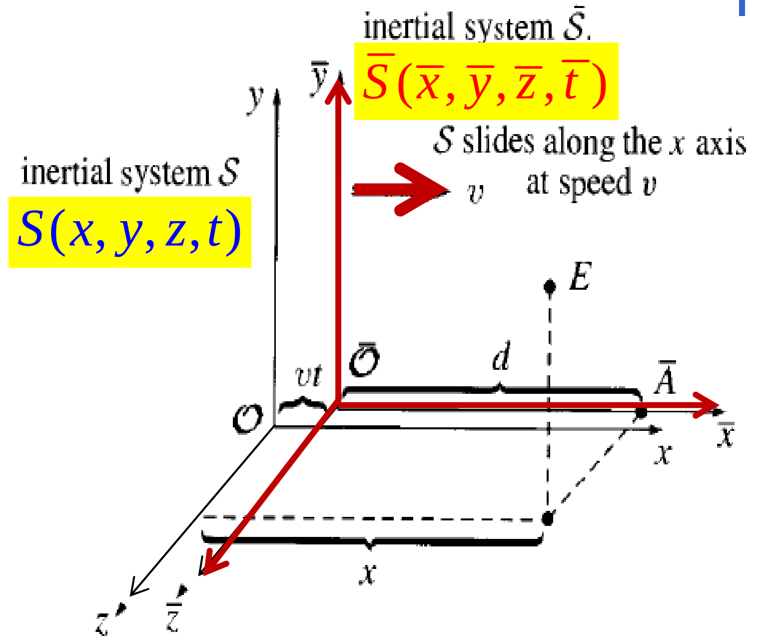


(a) (b) (c)

12.1.3 The Lorentz Transformations

d is the distance from $\bar{\mathcal{O}}$ to $\bar{A}$ *as measured in* $\mathcal{S}$.
$\bar{x}$ is the distance from $\bar{\mathcal{O}}$ to $\bar{A}$ *as measured in* $\bar{\mathcal{S}}$

$$x = d + vt$$



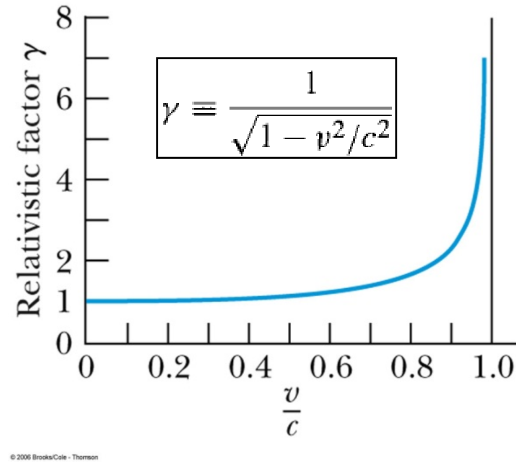
Galilean transformations

$$d = \bar{x}, \quad \Rightarrow \quad \left.\begin{array}{ll} \text{(i)} & \bar{x} = x - vt, \\ \text{(ii)} & \bar{y} = y, \\ \text{(iii)} & \bar{z} = z, \\ \text{(iv)} & \bar{t} = t. \end{array}\right\}$$

Lorentz transformations

$$d = \frac{1}{\gamma}\bar{x} \quad \Rightarrow \quad x = \gamma(\bar{x} + v\bar{t}) \quad \Rightarrow \quad \left.\begin{array}{ll} \text{(i)} & \bar{x} = \gamma(x - vt), \\ \text{(ii)} & \bar{y} = y, \\ \text{(iii)} & \bar{z} = z, \\ \text{(iv)} & \bar{t} = \gamma\left(t - \frac{v}{c^2}x\right) \end{array}\right\}$$

$$\boxed{\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}}$$



12.1.3 The Lorentz Transformations

$$\left.\begin{aligned}
&\text{(i)}\quad \bar{x} = \gamma(x - vt),\\
&\text{(ii)}\quad \bar{y} = y,\\
&\text{(iii)}\quad \bar{z} = z,\\
&\text{(iv)}\quad \bar{t} = \gamma\left(t - \frac{v}{c^2}x\right)
\end{aligned}\right\}$$

$\rightarrow \Delta x = \dfrac{1}{\gamma}\Delta\bar{x}$ → **Moving object looks shorter.**

$$\boxed{\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}}$$

$\rightarrow \Delta t = \dfrac{1}{\gamma}\Delta\bar{t}$ → **Moving clocks run slow.**

(Ex) Einstein's velocity addition rule

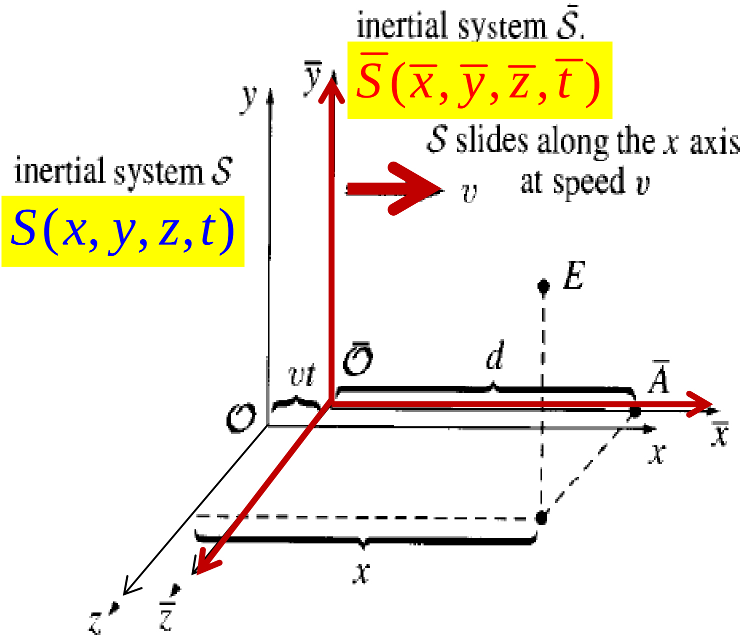
In *S, s*uppose a particle moves a distance *dx* in a time *dt:*

Velocity → $u = \dfrac{dx}{dt}$

In $\bar{\mathcal{S}}$, meanwhile, it has moved

a distance $d\bar{x} = \gamma(dx - v\,dt)$

in a time $d\bar{t} = \gamma\left(dt - \dfrac{v}{c^2}dx\right)$



The velocity in $\bar{\mathcal{S}}$ is therefore

$$\bar{u} = \frac{d\bar{x}}{d\bar{t}} = \frac{\gamma(dx - v\,dt)}{\gamma\left(dt - v/c^2 dx\right)} = \frac{(dx/dt - v)}{1 - v/c^2 dx/dt} = \frac{u - v}{1 - uv/c^2}$$

12.1.4 The Structure of Spacetime

Four-vectors (x^0, x^1, x^2, x^3)

$$x^0 \equiv ct \qquad \text{Using } x^0 \text{ (instead of } t\text{)}$$

→ to changing the unit of time from the *second* to the *meter*

→ *1* meter of x^o corresponds to the time it takes light to travel 1 meter

$$x^1 = x,$$

$$x^2 = y$$

$$x^3 = z$$

$$\beta \equiv \frac{v}{c} \qquad \text{Using } \beta \text{ (instead of } v\text{)}$$

Lorentz transformation matrix: Λ

Λ : **Lorentz transformation matrix**

$$\begin{aligned}
&\text{(i)} \quad \bar{x} = \gamma(x - vt),\\
&\text{(ii)} \quad \bar{y} = y,\\
&\text{(iii)} \quad \bar{z} = z,\\
&\text{(iv)} \quad \bar{t} = \gamma\left(t - \frac{v}{c^2}x\right)
\end{aligned}$$

$$\begin{aligned}
\bar{x}^0 &= \gamma(x^0 - \beta x^1),\\
\bar{x}^1 &= \gamma(x^1 - \beta x^0),\\
\bar{x}^2 &= x^2,\\
\bar{x}^3 &= x^3.
\end{aligned}$$

$$\begin{pmatrix} \bar{x}^0 \\ \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\boxed{\bar{x}^{\mu} = \sum_{\nu=0}^{3} (\Lambda^{\mu}_{\nu}) x^{\nu}}$$

12.1.4 The Structure of Spacetime

Four-vectors = **any set of four components that transform under Lorentz transformations**

$$\bar{a}^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}_{\nu} a^{\nu} \longrightarrow \left.\begin{aligned} \bar{a}^0 &= \gamma(a^0 - \beta a^1), \\ \bar{a}^1 &= \gamma(a^1 - \beta a^0), \\ \bar{a}^2 &= a^2, \\ \bar{a}^3 &= a^3. \end{aligned}\right\}$$

Four-dimensional scalar product: $A \cdot B \equiv -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$

(Prove; Prob. 12.17) The 4-dim. Dot product has the same value in all inertial systems:

$$-\bar{a}^0\bar{b}^0 + \bar{a}^1\bar{b}^1 + \bar{a}^2\bar{b}^2 + \bar{a}^3\bar{b}^3 = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$$

To keep track of the minus sign it is convenient to introduce the **covariant** vector a_μ, which differs from the **contravariant** a^μ only in the sign of the zeroth component:

$$a_\mu = (a_0, a_1, a_2, a_3) \equiv (-a^0, a^1, a^2, a^3)$$

$$-a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 \longrightarrow \sum_{\mu=0}^{3} a_\mu b^\mu \longrightarrow a_\mu b^\mu$$ **: Einstein summation convention**

The invariant interval in spacetime

Suppose event A occurs at $(x_A^0, x_A^1, x_A^2, x_A^3)$, and event B at $(x_B^0, x_B^1, x_B^2, x_B^3)$.

Displacement 4-vector: $\boxed{\Delta x^\mu \equiv x_A^\mu - x_B^\mu}$

Interval between two events: scalar product of Δx^μ with itself

$$\boxed{I \equiv (\Delta x)_\mu (\Delta x)^\mu = -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 = -c^2 t^2 + d^2}$$

where t is the time difference between the two events and d is their spatial separation.
When you transform to a moving system, the *time* between *A* and *B*, and the *spatial separation are altered.*
But, the interval I remains the same!

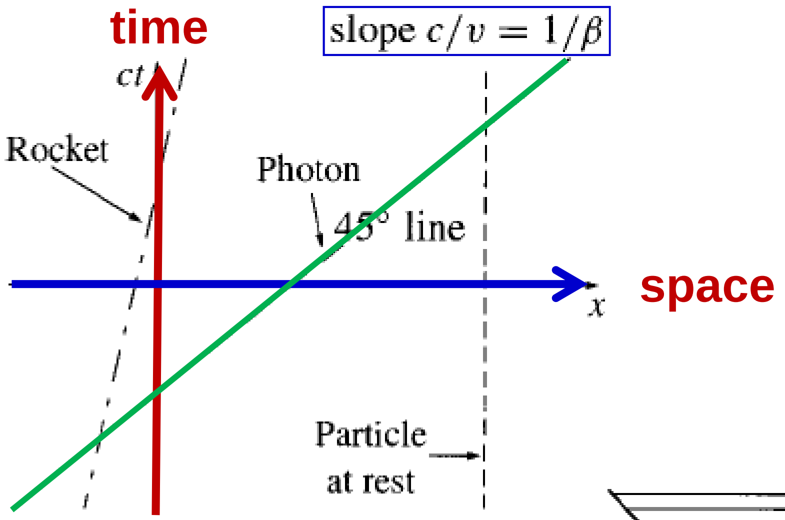
Depending on the two events in question, the interval can be positive, negative, or zero:

timelike 1. If $I < 0$ we call the interval **timelike**, for this is the sign we get when the two occur at the *same place* ($d = 0$), and are separated only temporally.

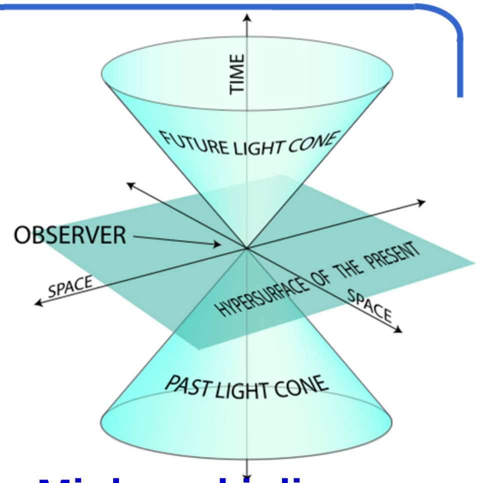
spacelike 2. If $I > 0$ we call the interval **spacelike**, for this is the sign we get when the two occur at the *same time* ($t = 0$), and are separated only spatially.

lightlike 3. If $I = 0$ we call the interval **lightlike**, for this is the relation that holds when the two events are connected by a signal traveling at the speed of light.

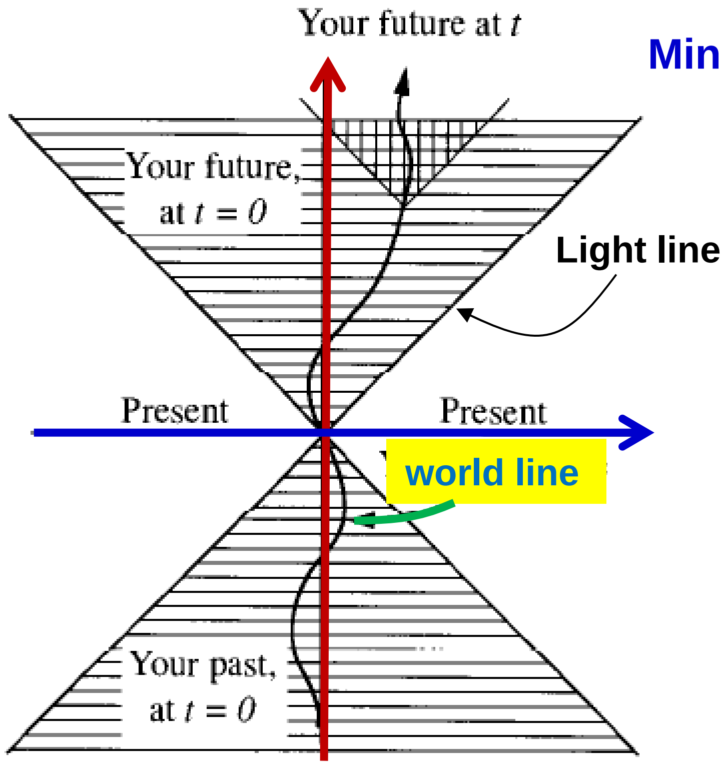
Space-time diagrams (Minkowski diagrams) and World line



timelike (slope greater than 1)
spacelike (slope less than 1)
lightlike (slope 1)



Minkowski diagrams



Trajectory of a particle on a Minkowski diagram ➔ world line.

12.2.1 Proper Time and Proper Velocity

Ordinary velocity (u) ➔ $\mathbf{u} = \frac{d\mathbf{l}}{dt}$: distance measured on the ground, over time measured in the ground

proper time (τ) ➔ **the time associated with me** → $\tau = \bar{t}$ (if I *on* $\bar{S}$); $\tau = t$ (if I *on* S)

(The word suggests a mistranslation of the French *propre,* meaning "own.")

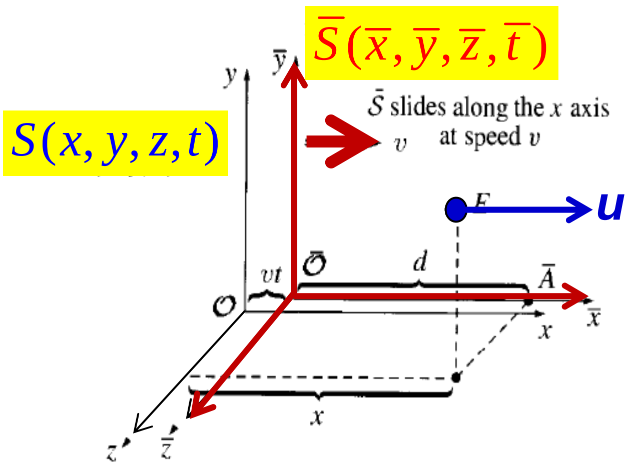
$$d\tau = \sqrt{1 - u^2/c^2}\, dt \quad \longleftarrow \quad \text{if } I \text{ were on } \bar{S};\ \tau = \bar{t}$$

proper velocity (η) ➔ $\boldsymbol{\eta} \equiv \frac{d\mathbf{l}}{d\tau}$: distance (l) measured on S, over the proper time

$$\boldsymbol{\eta} = \frac{1}{\sqrt{1 - u^2/c^2}}\mathbf{u} \longrightarrow \eta^\mu \equiv \frac{dx^\mu}{d\tau}$$ **: proper velocity 4-vector (4-velocity)**

Lorentz transformation of 4-velocity :

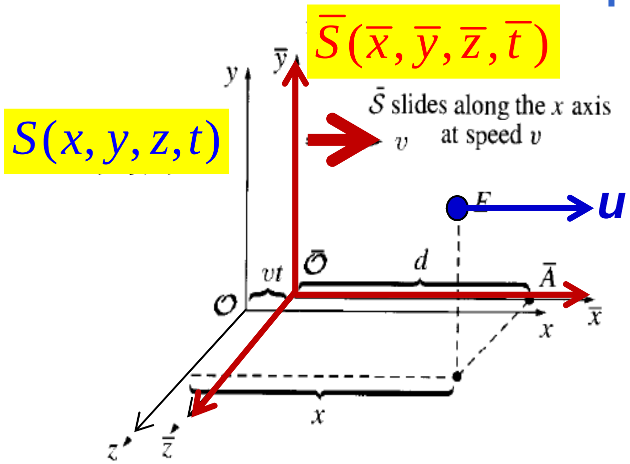
$$\bar{\eta}^\mu = \Lambda^\mu_\nu \eta^\nu \longrightarrow \left.\begin{aligned} \bar{\eta}^0 &= \gamma(\eta^0 - \beta\eta^1), \\ \bar{\eta}^1 &= \gamma(\eta^1 - \beta\eta^0), \\ \bar{\eta}^2 &= \eta^2, \\ \bar{\eta}^3 &= \eta^3. \end{aligned}\right\}$$



Note: the particle velocities under Lorentz transformations

$u = \frac{dx}{dt}$: velocity of the *particle* with respect to S frame

$\bar{u} = \frac{d\bar{x}}{d\bar{t}}$: velocity of the *particle* with respect to $\bar{S}$ frame



$$\left.\begin{aligned}
\bar{u}_x &= \frac{d\bar{x}}{d\bar{t}} = \frac{u_x - v}{(1 - vu_x/c^2)}, \\
\bar{u}_y &= \frac{d\bar{y}}{d\bar{t}} = \frac{u_y}{\gamma(1 - vu_x/c^2)}, \\
\bar{u}_z &= \frac{d\bar{z}}{d\bar{t}} = \frac{u_z}{\gamma(1 - vu_x/c^2)}.
\end{aligned}\right\}$$

$\eta = \frac{dl}{d\tau}$: proper velocity of the *particle* ➔ $\eta = \frac{dx}{d\bar{t}}$ (I on $\bar{\mathrm{S}}$) $\quad \bar{\eta} = \frac{d\bar{x}}{dt}$ (I on S)

$$\left.\begin{aligned}
\bar{\eta}^0 &= \gamma(\eta^0 - \beta\eta^1), \\
\bar{\eta}^1 &= \gamma(\eta^1 - \beta\eta^0), \\
\bar{\eta}^2 &= \eta^2, \\
\bar{\eta}^3 &= \eta^3.
\end{aligned}\right\}$$

12.2.2 Relativistic Energy and Momentum

Relativistic Momentum: $\boxed{\mathbf{p} \equiv m\boldsymbol{\eta} = \frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}}}$ $\longrightarrow$ $p^\mu \equiv m\eta^\mu$ (m: rest mass)

(Prove: Prob. 12.28) The conservation law of momentum would be inconsistent with the principle of relativity if we were to define momentum as $m\mathbf{u}$.

$$p^0 = m\eta^0 = \frac{mc}{\sqrt{1-u^2/c^2}} \longrightarrow m_{\text{rel}} \equiv \frac{m}{\sqrt{1-u^2/c^2}}$$

: Einstein called **relativistic mass**

Relativistic energy: $\boxed{E \equiv \frac{mc^2}{\sqrt{1-u^2/c^2}}}$

$E_{\text{rest}} \equiv mc^2$ ➔ Notice that the relativistic energy is nonzero *even when the object is stationary!* (***rest energy***)

$$E_{\text{kin}} \equiv E - mc^2 = mc^2\left(\frac{1}{\sqrt{1-u^2/c^2}} - 1\right)$$

: kinetic energy

$$= \frac{1}{2}mu^2 + \frac{3}{8}\frac{mu^4}{c^2} + \cdots$$

The scalar product of p^μ with itself: $p^\mu p_\mu = -(p^0)^2 + (\mathbf{p}\cdot\mathbf{p}) = -m^2c^2$

$$\Rightarrow \boxed{E^2 - p^2c^2 = m^2c^4}$$

Total relativistic energy and momentum are conserved.

Relativistic Momentum: $\mathbf{p} \equiv m\boldsymbol{\eta} = \dfrac{m\mathbf{u}}{\sqrt{1-u^2/c^2}}$

Relativistic energy: $E \equiv \dfrac{mc^2}{\sqrt{1-u^2/c^2}}$

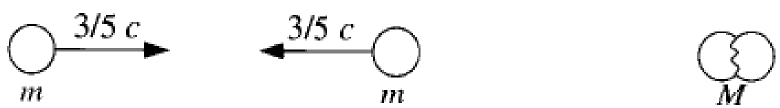
$$E^2 - p^2c^2 = m^2c^4$$

Note the distinction between an **invariant** quantity (same value in all inertial systems) and a **conserved** quantity (same value before and after some process).

→ **rest mass** is invariant, but not conserved;
→ **energy** is conserved but not invariant;
→ **electric charge** is both conserved *and* invariant;
→ **velocity** is neither conserved *nor* invariant.

(Example 12.7)

Two lumps of clay, each of (rest) mass *m,* collide head-on at (3/5)c. They stick together. What is the mass *(M)* of the composite lump?



The energy of each lump: $\dfrac{mc^2}{\sqrt{1-(3/5)^2}} = \frac{5}{4}mc^2$

Conservation energy: $\frac{5}{4}mc^2 + \frac{5}{4}mc^2 = Mc^2$.

$M = \frac{5}{2}m.$

M is greater than the sum of the initial masses!
Mass was not conserved.
Kinetic energy was converted into rest energy,
→ the mass increased.

12.2.3 Relativistic Kinematics

$$\mathbf{p} \equiv m\boldsymbol{\eta} = \frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}} \qquad E \equiv \frac{mc^2}{\sqrt{1-u^2/c^2}} \qquad \boxed{E^2 - p^2c^2 = m^2c^4}$$

For a massless particle (m = 0): photon

➔ If ***u* = c**,

→ then (zero) over (zero), leaving p and *E* indeterminate.

→ therefore, that a massless particle could carry energy and momentum at the speed of light.

→ ***E* = *pc***

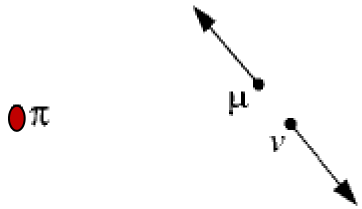
➔ What distinguishes a photon with a *lot* of energy from one with very little?

"they just have the same mass (zero) and the same speed (c)!"

→ Relativity offers no answer to this question.

→ Quantum mechanics *does,* according to the Planck formula, $E = h\nu$

(Example 12.8) A pion at rest decays into a muon and a neutrino.
Find the energy of the outgoing muon,
in terms of the two masses, m_π and m_μ (assume $m_\nu = 0$).

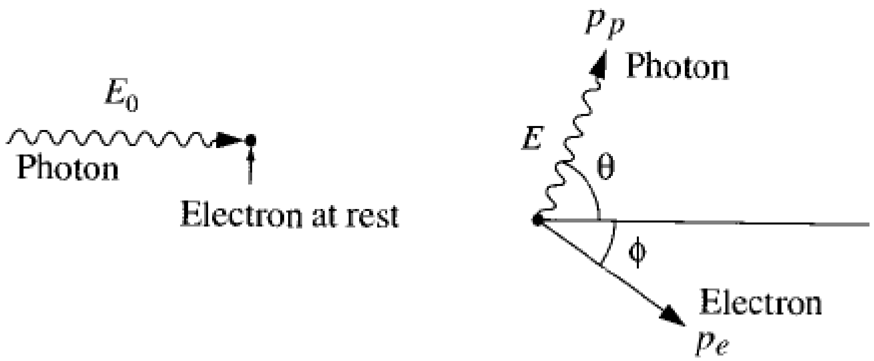


$$E_{\text{before}} = m_\pi c^2, \quad E_{\text{after}} = E_\mu + E_\nu \quad \Rightarrow \quad E_\mu + E_\nu = m_\pi c^2$$

$$\mathbf{p}_{\text{before}} = 0, \quad \mathbf{p}_{\text{after}} = \mathbf{p}_\mu + \mathbf{p}_\nu \quad \Rightarrow \quad \mathbf{p}_\nu = -\mathbf{p}_\mu$$

$$\mathbf{p}_\nu = -\mathbf{p}_\mu, \quad E_\nu = |\mathbf{p}_\nu| c, \quad |\mathbf{p}_\mu| = \sqrt{E_\mu^2 - m_\mu^2 c^4}\,/c \quad \Rightarrow \quad E_\mu + \sqrt{E_\mu^2 - m_\mu^2 c^4} = m_\pi c^2 \quad \Rightarrow \quad \boxed{E_\mu = \frac{(m_\pi^2 + m_\mu^2)c^2}{2m_\pi}}$$

(Example 12.9) Compton scattering: $\lambda = \lambda_0 + \frac{h}{mc}(1 - \cos\theta)$



Conservation of momentum:

$\boxed{\text{vertical}}$ $\quad p_e \sin\phi = p_p \sin\theta \qquad p_p = E/c \longrightarrow \sin\phi = \frac{E}{p_e c}\sin\theta$

$\boxed{\text{horizontal}}$ $\quad \frac{E_0}{c} = p_p\cos\theta + p_e\cos\phi = \frac{E}{c}\cos\theta + p_e\sqrt{1 - \left(\frac{E}{p_e c}\sin\theta\right)^2}$

$$\longrightarrow p_e^2 c^2 = (E_0 - E\cos\theta)^2 + E^2\sin^2\theta = E_0^2 - 2E_0 E\cos\theta + E^2$$

Conservation of energy:

$$E_0 + mc^2 = E + E_e = E + \sqrt{m^2c^4 + p_e^2c^2} = E + \sqrt{m^2c^4 + E_0^2 - 2E_0 E\cos\theta + E^2}$$

$$\Rightarrow E = \frac{1}{(1-\cos\theta)/mc^2 + (1/E_0)} \longrightarrow E = h\nu = \frac{hc}{\lambda} \Rightarrow \boxed{\lambda = \lambda_0 + \frac{h}{mc}(1 - \cos\theta)}$$

(h/mc) is called the **Compton wavelength** of the electron

12.2.4 Relativistic Dynamics

***Newton's first* law is built into the principle of relativity.**

***Newton's second* law:** $\boxed{\mathbf{F} = \frac{d\mathbf{p}}{dt}} \longleftarrow \mathbf{p} \equiv m\boldsymbol{\eta} = \frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}}$

***Newton's third* law does *not,* in general, extend to the relativistic domain.**

→ the third law is incompatible with the relativity of simultaneity:

→ If the two objects in question are separated in space, a moving observer will report the reaction force at different time, therefore, the third law is *violated.*

work-energy theorem: the net work done on a particle equals the increase in its kinetic energy

$$W \equiv \int \mathbf{F} \cdot d\mathbf{l} = \int \frac{d\mathbf{p}}{dt} \cdot d\mathbf{l} = \int \frac{d\mathbf{p}}{dt} \cdot \frac{d\mathbf{l}}{dt}\, dt = \int \frac{d\mathbf{p}}{dt} \cdot \mathbf{u}\, dt$$

$$\frac{d\mathbf{p}}{dt} \cdot \mathbf{u} = \frac{d}{dt}\left(\frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}}\right) \cdot \mathbf{u} = \frac{m\mathbf{u}}{(1-u^2/c^2)^{3/2}} \cdot \frac{d\mathbf{u}}{dt} = \frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-u^2/c^2}}\right) = \frac{dE}{dt}$$

$$\Rightarrow \quad W = \int \frac{dE}{dt}\, dt = E_{\text{final}} - E_{\text{initial}}$$

Minkowski force

Assume that ($\boldsymbol{u} = \boldsymbol{u}_x$):

$$\left.\begin{aligned}&\text{(i)}\ \bar{x} = \gamma(x - vt),\\ &\text{(ii)}\ \bar{y} = y,\\ &\text{(iii)}\ \bar{z} = z,\\ &\text{(iv)}\ \bar{t} = \gamma\left(t - \frac{v}{c^2}x\right)\end{aligned}\right\}$$

$$\bar{F}_y = \frac{d\bar{p}_y}{d\bar{t}} = \frac{dp_y}{\gamma\, dt - \frac{\gamma\beta}{c}\, dx} = \frac{dp_y/dt}{\gamma\left(1 - \frac{\beta}{c}\frac{dx}{dt}\right)} = \frac{F_y}{\gamma(1 - \beta u_x/c)} \qquad \bar{F}_z = \frac{F_z}{\gamma(1 - \beta u_x/c)}$$

$$\bar{F}_x = \frac{d\bar{p}_x}{d\bar{t}} = \frac{\gamma\, dp_x - \gamma\beta\, dp^0}{\gamma\, dt - \frac{\gamma\beta}{c}\, dx} = \frac{\frac{dp_x}{dt} - \beta\frac{dp^0}{dt}}{1 - \frac{\beta}{c}\frac{dx}{dt}} = \frac{F_x - \frac{\beta}{c}\left(\frac{dE}{dt}\right)}{1 - \beta u_x/c} = \frac{F_x - \beta(\mathbf{u}\cdot\mathbf{F})/c}{1 - \beta u_x/c}$$

If the particle is (instantaneously) at rest in S, for example, so that if ***u*** = 0,

$$\bar{\mathbf{F}}_\perp = \frac{1}{\gamma}\mathbf{F}_\perp, \quad \bar{F}_\parallel = F_\parallel$$

➔ the component of *F parallel* to the motion *of S* is unchanged,
➔ Perpendicular components are divided by γ.

Minkowski force: the derivative of momentum with respect to *proper* time:

$$\boxed{K^\mu \equiv \frac{dp^\mu}{d\tau}}$$

the spatial components → $\mathbf{K} = \left(\frac{dt}{d\tau}\right)\frac{d\mathbf{p}}{dt} = \frac{1}{\sqrt{1 - u^2/c^2}}\mathbf{F}$

the zeroth component → $K^0 = \frac{dp^0}{d\tau} = \frac{1}{c}\frac{dE}{d\tau}$

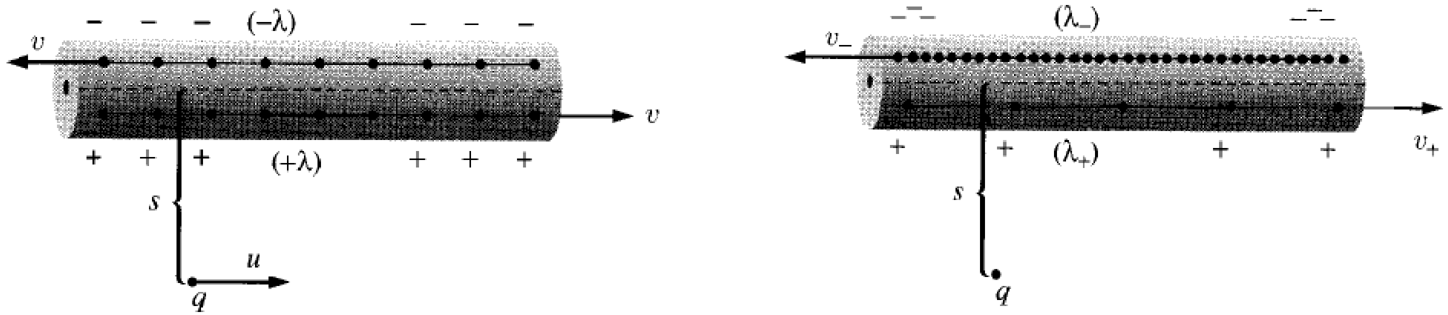
When we wish to generalize some classical force law, such as Lorentz's force, to the relativistic domain, Does the classical formula correspond to the *ordinary* force or to the Minkowski force?

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u}\times\mathbf{B})?$$

$$\mathbf{K} = q(\mathbf{E} + \mathbf{u}\times\mathbf{B})?$$

12.3.1 Magnetism as a Relativistic Phenomenon

net current: $I = 2\lambda v$



In the reference frame where *q* is at rest, system $\bar{S}$,
by the Einstein velocity addition rule, the velocities of the positive and negative lines are

$$v_\pm = \frac{v \mp u}{1 \mp vu/c^2}$$

Because $v_- > v_+$, the Lorentz contraction of the spacing between negative charges is more severe;

→ ***the wire carries a net negative charge!***

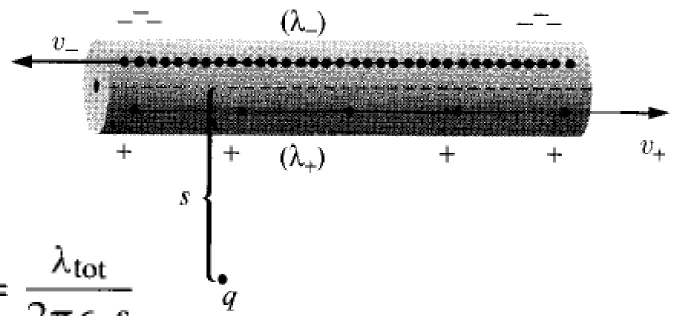
$$\lambda_\pm = \pm(\gamma_\pm)\lambda_0 \longrightarrow \lambda_{tot} = \lambda_+ + \lambda_- = \lambda_0(\gamma_+ - \gamma_-) = \frac{-2\lambda u v}{c^2\sqrt{1-u^2/c^2}}$$

$$\text{where} \quad \gamma_\pm = \frac{1}{\sqrt{1 - v_\pm^2/c^2}} = \gamma\frac{1 \mp uv/c^2}{\sqrt{1-u^2/c^2}}$$

Conclusion: As a result of unequal Lorentz contraction of the positive and negative lines,
a current-carrying wire that is electrically neutral in one inertial system will be charged in another.

Magnetism as a Relativistic Phenomenon

In the reference frame where *q* is at rest, system $\bar{S}$,



$$\lambda_{\text{tot}} = \lambda_+ + \lambda_- = \lambda_0(\gamma_+ - \gamma_-) = \frac{-2\lambda u v}{c^2\sqrt{1-u^2/c^2}}$$

The line charge sets up an *electric* field: $E = \dfrac{\lambda_{\text{tot}}}{2\pi\epsilon_0 s}$

so there is an electrical force on q in $\bar{S}$, $\bar{F} = qE = -\dfrac{\lambda v}{\pi\epsilon_0 c^2 s}\dfrac{qu}{\sqrt{1-u^2/c^2}}$

➔ **In $\bar{S}$ system, the wire is attracted toward the charge by a purely electrical force.**

The force $\bar{F}$ can be transformed into F in S (wire at rest) by (Eq. 12.68)

$$F = \frac{1}{\gamma}\bar{F} = \sqrt{1-u^2/c^2}\,\bar{F} = -\frac{\lambda v}{\pi\epsilon_0 c^2}\frac{qu}{s}$$

But, in the wire frame (S) the total charge is neutral !

➔ **what does the force F imply?**

➔ Electrostatics and relativity imply the existence of another force in view point of S frame.

➔ **magnetic force**.

In fact, by using $c^2 = (\epsilon_0\mu_0)^{-1}$ and $I = 2\lambda v$

$$F = -\frac{\lambda v}{\pi\epsilon_0 c^2}\frac{qu}{s} = -qu\left(\frac{\mu_0 I}{2\pi s}\right)$$, magnetic field, B = $\left(\dfrac{\mu_0 I}{2\pi s}\right)$

➔ **One observer's electric field is another's magnetic field!**

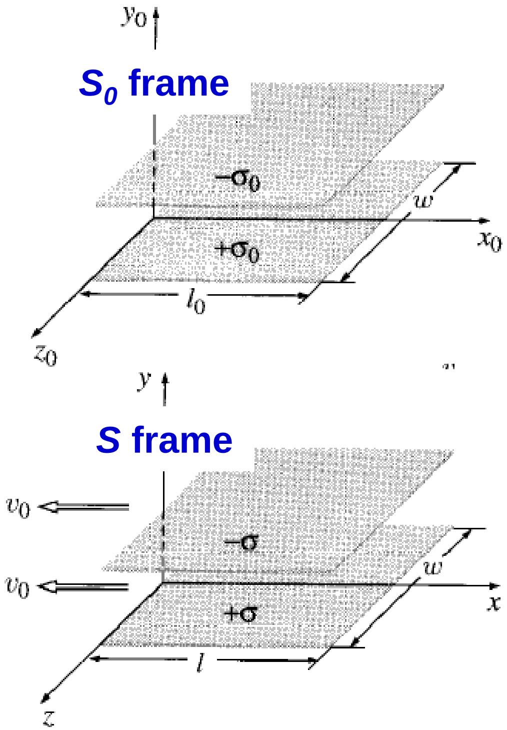
12.3.2 How the Fields Transform

Let's find the general transformation rules for electromagnetic fields:

➔ Given the fields in a frame ($\mathcal{S}$), what are the fields in another frame ($\bar{\mathcal{S}}$)?

consider the *simplest possible* electric field in a large parallel-plate capacitor in ***S*$_0$** frame.

$$\mathbf{E}_0 = \frac{\sigma_0}{\epsilon_0}\,\hat{\mathbf{y}}$$



In the system ***S****,* moving to the right at speed v_0,
the plates are moving to the left with the different surface charge σ:

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\,\hat{\mathbf{y}}$$

The total charge on each plate is invariant, and the *width* (*w)* is unchanged,
but the *length (l)* is Lorentz-contracted by a factor

$$\frac{1}{\gamma_0} = \sqrt{1 - v_0^2/c^2} \longrightarrow \sigma = \gamma_0\sigma_0 \longrightarrow \boxed{\mathbf{E}^{\perp} = \gamma_0\mathbf{E}_0{}^{\perp}}$$

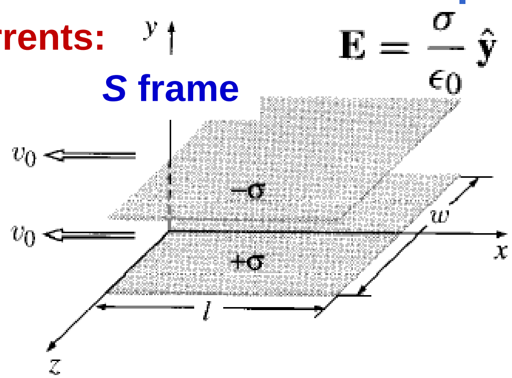
This case is not the most general case: we began with a system *So* in which the charges were at rest and where, consequently, there was no magnetic field.

To derive the *general* rule we must start out in a system with both electric and magnetic fields.

How the Fields Transform

To derive the *general* rule we must start out in a system with both electric and magnetic fields.

Consider the S system, there is also a *magnetic* field due to the surface currents:



$$\mathbf{K}_{\pm} = \mp\sigma v_0\,\hat{\mathbf{x}} \quad (\upsilon_0 : \text{ velocity of } S \text{ relative to } S_0)$$

By the right-hand rule, this field points in the negative *z* direction;

$$B_z = -\mu_0\sigma v_0 \quad \text{by Ampère's law}$$

What we need to derive the *general* rule is an introduction of another frame $\bar{S}$,
then, derivation of the transformation of (E, B) fields in S system into $(\bar{E}, \bar{B})$ fields in $\bar{S}$ system.

In a *third* system, $\bar{S}$, traveling to the right with speed $(\upsilon : \text{ velocity of } \bar{S} \text{ relative to } S)$

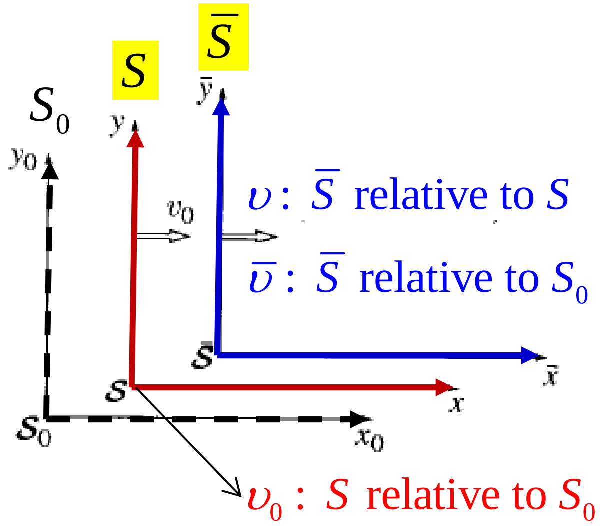
$$\bar{E}_y = \frac{\bar{\sigma}}{\epsilon_0}, \quad \bar{B}_z = -\mu_0\bar{\sigma}\bar{v}$$

$$\bar{v} = \frac{v + v_0}{1 + v v_0/c^2} \quad (\bar{\upsilon} : \text{ velocity of } \bar{S} \text{ relative to } S_0)$$

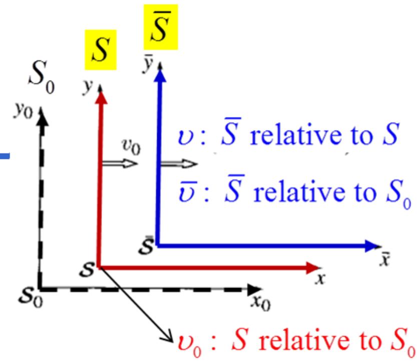
$$\bar{\sigma} = \bar{\gamma}\sigma_0 \quad \bar{\gamma} = \frac{1}{\sqrt{1 - \bar{v}^2/c^2}}$$

also, since $\sigma = \gamma_0\sigma_0 \quad \frac{1}{\gamma_0} = \sqrt{1 - v_0^2/c^2}$

$$\bar{E}_y = \left(\frac{\bar{\gamma}}{\gamma_0}\right)\frac{\sigma}{\epsilon_0}, \quad \bar{B}_z = -\left(\frac{\bar{\gamma}}{\gamma_0}\right)\mu_0\sigma\bar{v}$$



How the Fields Transform



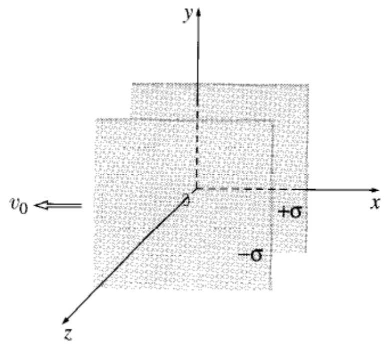
$$\bar{E}_y = \left(\frac{\bar{\gamma}}{\gamma_0}\right)\frac{\sigma}{\epsilon_0}, \quad \bar{B}_z = -\left(\frac{\bar{\gamma}}{\gamma_0}\right)\mu_0\sigma\bar{v}$$

$$\frac{\bar{\gamma}}{\gamma_0} = \frac{\sqrt{1-v_0^2/c^2}}{\sqrt{1-\bar{v}^2/c^2}} = \frac{1+vv_0/c^2}{\sqrt{1-v^2/c^2}} = \gamma\left(1+\frac{vv_0}{c^2}\right) \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

$$\bar{E}_y = \gamma\left(1+\frac{vv_0}{c^2}\right)\frac{\sigma}{\epsilon_0} = \gamma\left(E_y - \frac{v}{c^2\epsilon_0\mu_0}B_z\right) \longleftarrow B_z = -\mu_0\sigma v_0$$

$$\bar{B}_z = -\gamma\left(1+\frac{vv_0}{c^2}\right)\mu_0\sigma\left(\frac{v+v_0}{1+vv_0/c^2}\right) = \gamma(B_z - \mu_0\epsilon_0 v E_y) \longleftarrow \mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{y}}$$

since $\mu_0\epsilon_0 = 1/c^2$,



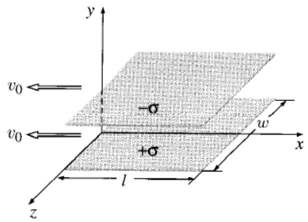
$$\left.\begin{aligned} \bar{E}_y &= \gamma(E_y - vB_z), \\ \bar{B}_z &= \gamma\left(B_z - \frac{v}{c^2}E_y\right). \end{aligned}\right\}$$

Similarly, to do E_z and B_y simply align the same capacitor parallel to *xy* plane instead of *xz* plane

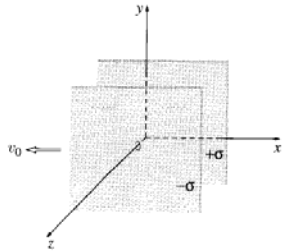
$$\left.\begin{aligned} \bar{E}_z &= \gamma(E_z + vB_y), \\ \bar{B}_y &= \gamma\left(B_y + \frac{v}{c^2}E_z\right). \end{aligned}\right\}$$

How the Fields Transform

$$\bar{E}_y = \gamma(E_y - vB_z), \qquad \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right).$$



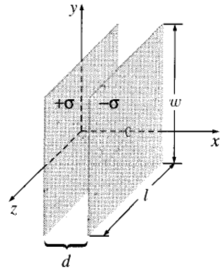
$$\bar{E}_z = \gamma(E_z + vB_y), \qquad \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right).$$

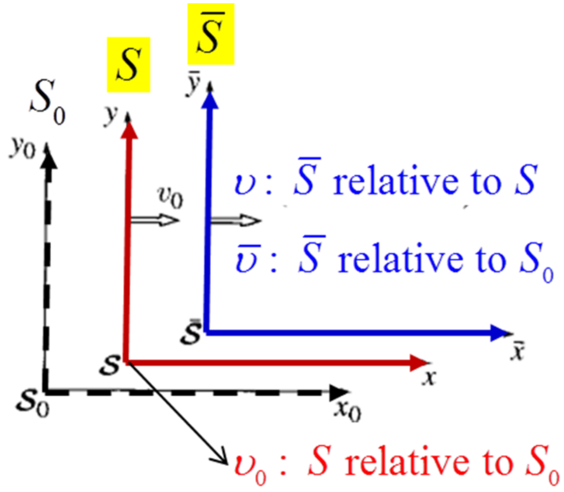


$\bar{E}_x = E_x$

$\bar{B}_x = B_x$

the field component s *parallel* to the motion is unchanged.





$$\bar{E}_x = E_x, \quad \bar{E}_y = \gamma(E_y - vB_z), \quad \bar{E}_z = \gamma(E_z + vB_y),$$
$$\bar{B}_x = B_x, \quad \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), \quad \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

where $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$

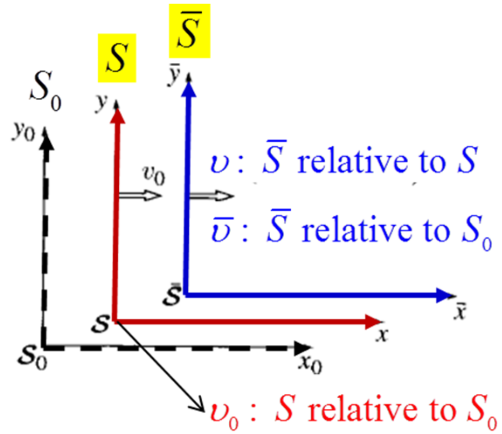
$(\upsilon: \bar{S}$ relative to $S)$

How the Fields Transform

$$\bar{E}_x = E_x, \quad \bar{E}_y = \gamma(E_y - vB_z), \quad \bar{E}_z = \gamma(E_z + vB_y),$$
$$\bar{B}_x = B_x, \quad \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), \quad \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

where $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$

$\left(v:\ \bar{S} \text{ relative to } S\right)$



Two special cases:

(1) *If* $B = 0$ *in* S *frame*, $(E \neq 0)$;

$$\bar{\mathbf{B}} = \gamma \frac{v}{c^2}(E_z\,\hat{\mathbf{y}} - E_y\,\hat{\mathbf{z}})$$

or, since $\mathbf{E}^{\perp} = \gamma_0 \mathbf{E}_0^{\perp} \longrightarrow \bar{\mathbf{B}} = \dfrac{v}{c^2}(\bar{E}_z\,\hat{\mathbf{y}} - \bar{E}_y\,\hat{\mathbf{z}})$

or, since $\mathbf{v} = v\,\hat{\mathbf{x}}, \longrightarrow \bar{\mathbf{B}} = -\dfrac{1}{c^2}(\mathbf{v} \times \bar{\mathbf{E}})$

(2) *If* $E = 0$ *in* S *frame*, $(B \neq 0)$;

$$\bar{\mathbf{E}} = -\gamma v(B_z\,\hat{\mathbf{y}} - B_y\,\hat{\mathbf{z}}) = -v(\bar{B}_z\,\hat{\mathbf{y}} - \bar{B}_y\,\hat{\mathbf{z}}) \longrightarrow \bar{\mathbf{E}} = \mathbf{v} \times \bar{\mathbf{B}}$$

➔ If either E or B is zero (at a particular point) in *one* system,
then in any other system the fields (at that point) are very simply related.

12.3.3 The Field Tensor $F^{\mu\nu}$

$$F^{\mu\nu} = \begin{Bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{Bmatrix} \xrightarrow[\mathbf{B}\to -\mathbf{E}/c]{\mathbf{E}/c\to\mathbf{B}} G^{\mu\nu}$$

$$\boxed{\bar{F}^{\mu\nu} = \Lambda^\mu_\lambda \Lambda^\nu_\sigma F^{\lambda\sigma}}$$

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ Lorentz transformation matrix

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu, \quad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0. \longrightarrow$$ **Maxwell's Equations**

$$J^\mu = (c\rho, J_x, J_y, J_z,)$$

current density 4-vector.

$$K^\mu = q\eta_\nu F^{\mu\nu} \longrightarrow \mathbf{F} = q[\mathbf{E} + (\mathbf{u}\times\mathbf{B})]$$

Minkowski force

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} \longrightarrow \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla\times\mathbf{A}$$

$$A^\mu = (V/c, A_x, A_y, A_z)$$

4-vector potential

$$\Box^2 A^\mu = -\mu_0 J^\mu \xrightarrow[\Box^2 \equiv \frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x^\nu} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}]{\text{(d' Alembertian)}}$$ The most elegant (and the simplest) formulation of Maxwell's equations