Chapter 10. Potentials and Fields

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10.1 The Potential Formulation

10.1.1 Scalar and Vector Potentials

- Vector potential $\mathbf{B} = \nabla \times \mathbf{A} \quad (\mathrm{T}) \quad (\leftarrow \nabla \cdot \mathbf{B} = 0)$

- Electric field for the time-varying case.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) \rightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V$$

$$\rightarrow \quad \mathbf{E} = \underbrace{-\nabla V}_{\text{Due to charge distribution } \rho} \underbrace{- \frac{\partial \mathbf{A}}{\partial t}}_{\text{Due to time-varying current } \mathbf{J}} \quad (\mathrm{V/m})$$

10.1.1 Scalar and Vector Potentials

$$\left(\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}\right) \ \& \ \left(\mathbf{B} = \nabla \times \mathbf{A}\right)$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \quad \Longrightarrow \quad \nabla^2 V + \frac{\partial}{\partial t}\left(\nabla \cdot \mathbf{A}\right) = -\frac{\rho}{\varepsilon_0}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \Longrightarrow \quad \left(\nabla^2 \mathbf{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \mathbf{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J}$$

$\Box^2 \equiv \left(\nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}\right)$: d'Alembertian **(4-dimensional operator, good for special relativity)**

$L \equiv \left(\nabla \cdot \mathbf{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t}\right)$: Gauge (to make *E & B fields unchanged* under transformation)

$$\Longrightarrow \quad \Box^2 V + \frac{\partial L}{\partial t} = -\frac{\rho}{\varepsilon_0}$$

$$\Longrightarrow \quad \Box^2 \mathbf{A} - \nabla L = -\mu_0 \mathbf{J}$$

10.1.3 Coulomb Gauge and Lorentz Gauge

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\varepsilon_0}$$

$$\left(\nabla^2 \mathbf{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \mathbf{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J} \qquad L \equiv \left(\nabla \cdot \mathbf{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t}\right)$$

Coulomb Gauge: $\nabla \cdot \mathbf{A} = 0$

$$\Rightarrow \nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

$$\Rightarrow V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t)}{\mathscr{r}} d\tau'$$

$$\Rightarrow \left(\nabla^2 \mathbf{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) = -\mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \nabla\left(\frac{\partial V}{\partial t}\right)$$

➔ *Advantage* is that the *scalar* potential is particularly simple to calculate;
➔ *Disadvantage* is that A is particularly *difficult* to calculate.

10.1.3 Coulomb Gauge and Lorentz Gauge

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\varepsilon_0}$$

$$\left(\nabla^2 \mathbf{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \mathbf{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J} \qquad L \equiv \left(\nabla \cdot \mathbf{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t}\right)$$

Lorentz Gauge: $\nabla \cdot \mathbf{A} = -\mu_0 \varepsilon_0 \frac{\partial V}{\partial t} \quad [L = 0]$

$$\Rightarrow \nabla^2 V - \mu_0 \varepsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_0} \Rightarrow \Box^2 V = -\frac{\rho}{\varepsilon_0}$$

$$\Rightarrow \nabla^2 \mathbf{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} \Rightarrow \Box^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

➔ *V and **A** have the same differential operator of l'Alembertian (4-dim. operator)*

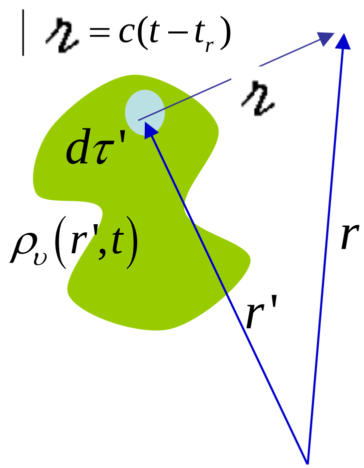
➔ Under the Lorentz gauge, the whole of electrodynamics reduces to the problem of *solving the inhomogeneous wave equation for specified sources.*

10.2 Continuous Distributions

10.2.2 Jefimenko's Equations (Retarded E and B fields)

Time-varying charges and currents generate retarded scalar potential, retarded vector potential.

➔ **Potentials at a distance $\mathscr{r}$ from the source at time t depend on the values of ρ and J at an earlier time (t - $\mathscr{r}$/u)**



➔ **Retarded in time** $\left(t_r = t - \mathscr{r}/u_p = t - \mathscr{r}/c \text{ in vacuum}\right)$

$$V(r,t) = \frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\rho_\upsilon(r',t_r)}{\mathscr{r}}d\tau'$$

$$\mathbf{A}(r,t) = \frac{\mu_0}{4\pi}\int_{V'}\frac{\mathbf{J}(r',t_r)}{\mathscr{r}}d\tau'$$

$$\xrightarrow[\left(t_r = t - \mathscr{r}/c\right)]{\left(\mathscr{r} = |r - r'|\right)}$$

$$\left(\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}\right)$$

$$\left(\mathbf{B} = \nabla \times \mathbf{A}\right)$$

Note, since both the $\mathscr{r}$ and t_r have r dependence, grad(V) to get E is not simple!

$$\nabla V = \frac{1}{4\pi\epsilon_0}\int\left[(\nabla\rho)\frac{1}{\mathscr{r}} + \rho\nabla\left(\frac{1}{\mathscr{r}}\right)\right]d\tau'$$

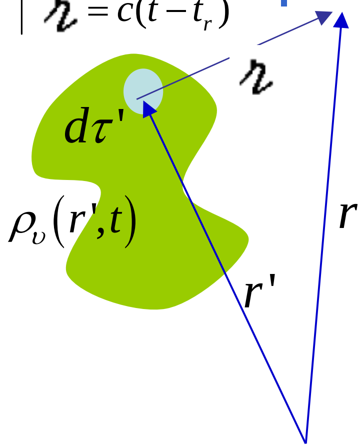
Jefimenko's Equations (Retarded E and B fields)

$$V(r,t)=\frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\rho_\upsilon(r',t_r)}{\mathscr{r}}d\tau'$$

$$\mathbf{A}(r,t)=\frac{\mu_0}{4\pi}\int_{V'}\frac{\mathbf{J}(r',t_r)}{\mathscr{r}}d\tau'$$

$$\xrightarrow[(t_r=t-\mathscr{r}/c)]{(\mathscr{r}=|r-r'|)}\quad \left(\mathbf{E}=-\nabla V-\frac{\partial\mathbf{A}}{\partial t}\right)\quad (\mathbf{B}=\nabla\times\mathbf{A})$$

$\mathscr{r}=c(t-t_r)$



$$\nabla V=\frac{1}{4\pi\epsilon_0}\int\left[(\nabla\rho)\frac{1}{\mathscr{r}}+\rho\nabla\left(\frac{1}{\mathscr{r}}\right)\right]d\tau'$$

$$\nabla\rho=\nabla_r[\rho(r',t_r)]=\frac{\partial\rho}{\partial t_r}\nabla t_r=\frac{\partial\rho}{\partial t}\nabla t_r=\dot{\rho}\nabla t_r=-\frac{1}{c}\dot{\rho}\nabla\mathscr{r}$$

$$\nabla\mathscr{r}=\hat{\boldsymbol{\mathscr{r}}}\text{ and }\nabla(1/\mathscr{r})=-\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$$

$$\nabla V=\frac{1}{4\pi\epsilon_0}\int\left[-\frac{\dot{\rho}}{c}\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}}-\rho\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right]d\tau'$$

$$\frac{\partial\mathbf{A}}{\partial t}=\frac{\mu_0}{4\pi}\int\frac{\dot{\mathbf{J}}}{\mathscr{r}}d\tau' \quad\Rightarrow\quad \boxed{\mathbf{E}(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\int\left[\frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}+\frac{\dot{\rho}(\mathbf{r}',t_r)}{c\mathscr{r}}\hat{\boldsymbol{\mathscr{r}}}-\frac{\dot{\mathbf{J}}(\mathbf{r}',t_r)}{c^2\mathscr{r}}\right]d\tau'}$$

time-dependent generalization of Coulomb's law

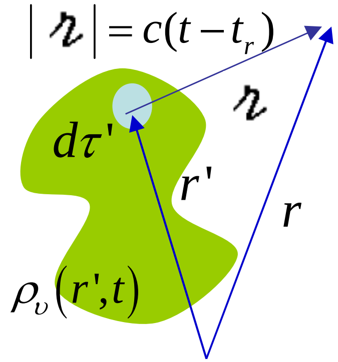
Similarly, $\Rightarrow$

$$\boxed{\mathbf{B}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\int\left[\frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}^2}+\frac{\dot{\mathbf{J}}(\mathbf{r}',t_r)}{c\mathscr{r}}\right]\times\hat{\boldsymbol{\mathscr{r}}}\,d\tau'}$$

time-dependent generalization of the Biot Savart law

Jefimenko's Equations (Retarded E and B fields)

$$\nabla V = \frac{1}{4\pi\epsilon_0}\int\left[-\frac{\dot{\rho}}{c}\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}} - \rho\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right]d\tau'$$



Taking the divergence,

$$\nabla^2 V = \frac{1}{4\pi\epsilon_0}\int\left\{-\frac{1}{c}\left[\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}}\cdot(\nabla\dot{\rho}) + \dot{\rho}\nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}}\right)\right] - \left[\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\cdot(\nabla\rho) + \rho\nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)\right]\right\}d\tau'$$

$$\nabla\dot{\rho} = -\frac{1}{c}\ddot{\rho}\nabla\mathscr{r} = -\frac{1}{c}\ddot{\rho}\,\hat{\boldsymbol{\mathscr{r}}}$$

$$\nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}^2}$$

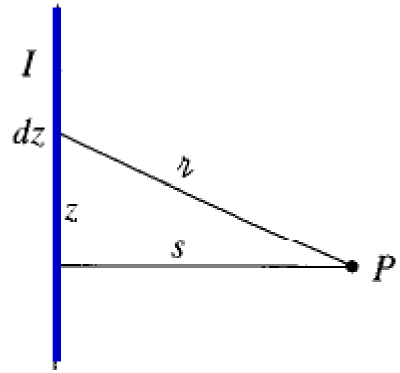
$$\nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}})$$

$$\nabla^2 V = \frac{1}{4\pi\epsilon_0}\int\left[\frac{1}{c^2}\frac{\ddot{\rho}}{\mathscr{r}} - 4\pi\rho\delta^3(\boldsymbol{\mathscr{r}})\right]d\tau' = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \frac{1}{\epsilon_0}\rho(\mathbf{r}, t)$$

➔ **The retarded potential also satisfies the inhomogeneous wave equation.**

Jefimenko's Equations (Retarded E and B fields)

(Example 10.2)



An infinite straight wire carries the current that is turned on abruptly at t = 0.

Find the resulting electric and magnetic fields. $I(t) = \begin{cases} 0, & \text{for } t \leq 0, \\ I_0, & \text{for } t > 0. \end{cases}$

➔ The retarded vector potential at point P is

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t_r)}{\mathscr{r}} d\tau' \longrightarrow \mathbf{A}(s, t) = \frac{\mu_0}{4\pi} \hat{\mathbf{z}} \int_{-\infty}^{\infty} \frac{I(t_r)}{\mathscr{r}} dz$$

For t < s/c, the "news" has not yet reached P, and the potential is zero. For t > s/c, only the segment

$|z| \leq \sqrt{(ct)^2 - s^2}$ contributes (outside this range t_r is negative, so I(t_r) = 0); thus

$$\mathbf{A}(s, t) = \left(\frac{\mu_0 I_0}{4\pi} \hat{\mathbf{z}}\right) 2 \int_0^{\sqrt{(ct)^2 - s^2}} \frac{dz}{\sqrt{s^2 + z^2}}$$

$$= \frac{\mu_0 I_0}{2\pi} \hat{\mathbf{z}} \ln\left(\sqrt{s^2 + z^2} + z\right)\Big|_0^{\sqrt{(ct)^2 - s^2}} = \frac{\mu_0 I_0}{2\pi} \ln\left(\frac{ct + \sqrt{(ct)^2 - s^2}}{s}\right) \hat{\mathbf{z}}$$

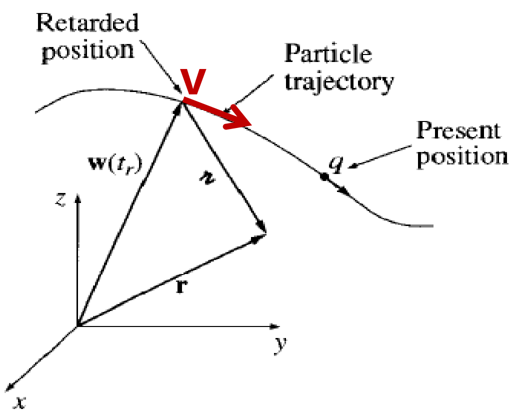
The electric field is $\mathbf{E}(s, t) = -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 I_0 c}{2\pi \sqrt{(ct)^2 - s^2}} \hat{\mathbf{z}}$

The magnetic field is $\mathbf{B}(s, t) = \nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial s} \hat{\boldsymbol{\phi}} = \frac{\mu_0 I_0}{2\pi s} \frac{ct}{\sqrt{(ct)^2 - s^2}} \hat{\boldsymbol{\phi}}$

10.3 (Moving) Point Charges

10.3.1 Lienard-Wiechert Potentials → Potentials of moving point charges

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}}\, d\tau'$$



It might suggest to you that the retarded potential of a point charge is simply

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\frac{1}{\mathscr{r}}\int \rho(\mathbf{r}', t_r)\, d\tau' = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}} \qquad q = \int \rho(\mathbf{r}', t_r)\, d\tau'$$

But, if the source is moving,

$$\int \rho(\mathbf{r}', t_r)\, d\tau' \neq q \longrightarrow \int \rho(\mathbf{r}', t_r)\, d\tau' = \frac{q}{1 - \hat{\mathscr{r}}\cdot\mathbf{v}/c}$$

(Proof: This is a purely *geometrical* effect)

Lienard-Wiechert potentials for a moving point charge

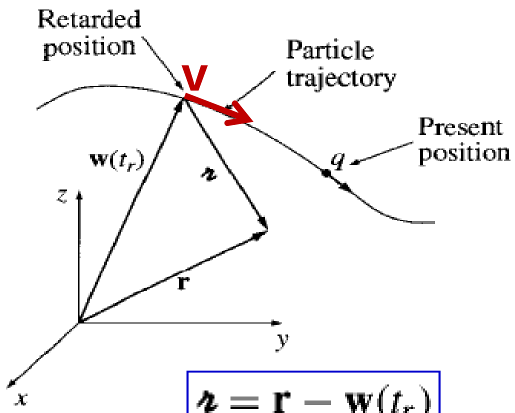
$$\boxed{V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})}}$$

$$\mathbf{J} = \rho\mathbf{v} \longrightarrow \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\rho(\mathbf{r}', t_r)\mathbf{v}(t_r)}{\mathscr{r}}\, d\tau' = \frac{\mu_0}{4\pi}\frac{\mathbf{v}}{\mathscr{r}}\int \rho(\mathbf{r}', t_r)\, d\tau'$$

$$\boxed{\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\frac{qc\mathbf{v}}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})} = \frac{\mathbf{v}}{c^2}V(\mathbf{r}, t)}$$

Lienard-Wiechert potentials for a moving point charge

$$\boxed{V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})}} \quad \boxed{\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\frac{qc\mathbf{v}}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})} = \frac{\mathbf{v}}{c^2}V(\mathbf{r}, t)}$$



$$\boxed{\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{w}(t_r)}$$

$$\boxed{\mathbf{v} = \dot{\mathbf{w}}(t_r)}$$

(Example 10.3)

Find the potentials of a point charge moving with constant velocity.

➔ Consider a point charge *q* that is moving on a specified trajectory

$$\mathbf{w}(t) \equiv \text{position of } q \text{ at time } t$$

For convenience, let's say the particle passes through the origin at time *t* = 0, so

$$\mathbf{w}(t) = \mathbf{v}t$$

The retarded time is determined implicitly by the equation

$$|\mathbf{r} - \mathbf{w}(t_r)| = c(t - t_r) \longleftarrow \boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{w}(t_r)$$

By squaring: $r^2 - 2\mathbf{r}\cdot\mathbf{v}t_r + v^2t_r^2 = c^2(t^2 - 2tt_r + t_r^2)$

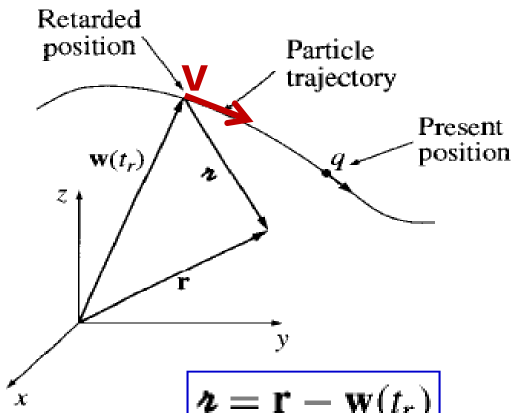
$$\longrightarrow \quad t_r = \frac{(c^2t - \mathbf{r}\cdot\mathbf{v}) \pm \sqrt{(c^2t - \mathbf{r}\cdot\mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}}{c^2 - v^2}$$

To fix the sign, consider the limit *v* = 0: $\longrightarrow \; t_r = t \pm \frac{r}{c} \; \longrightarrow \; t_r$ should be *(t - r/c);*

$$\longrightarrow \quad t_r = \frac{(c^2t - \mathbf{r}\cdot\mathbf{v}) - \sqrt{(c^2t - \mathbf{r}\cdot\mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}}{c^2 - v^2}$$

Lienard-Wiechert potentials for a moving point charge

$$\boxed{V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})}} \quad \boxed{\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{qc\mathbf{v}}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})} = \frac{\mathbf{v}}{c^2}V(\mathbf{r},t)}$$



$$\boxed{\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{w}(t_r)}$$

$$\boxed{\mathbf{v} = \dot{\mathbf{w}}(t_r)}$$

(Example 10.3)

Find the potentials of a point charge moving with constant velocity.

➔ (continued) $$t_r = \frac{(c^2t - \mathbf{r}\cdot\mathbf{v}) - \sqrt{(c^2t - \mathbf{r}\cdot\mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}}{c^2 - v^2}$$

$$\mathscr{r} = c(t - t_r), \quad \text{and} \quad \hat{\boldsymbol{\mathscr{r}}} = \frac{\mathbf{r} - \mathbf{v}t_r}{c(t - t_r)}$$

$$\longrightarrow \quad \mathscr{r}(1 - \hat{\boldsymbol{\mathscr{r}}}\cdot\mathbf{v}/c) = c(t - t_r)\left[1 - \frac{\mathbf{v}}{c}\cdot\frac{(\mathbf{r} - \mathbf{v}t_r)}{c(t - t_r)}\right] - c(t - t_r) - \frac{\mathbf{v}\cdot\mathbf{r}}{c} - \frac{v^2}{c}t_r$$

$$= \frac{1}{c}[(c^2t - \mathbf{r}\cdot\mathbf{v}) - (c^2 - v^2)t_r]$$

$$= \frac{1}{c}\sqrt{(c^2t - \mathbf{r}\cdot\mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}$$

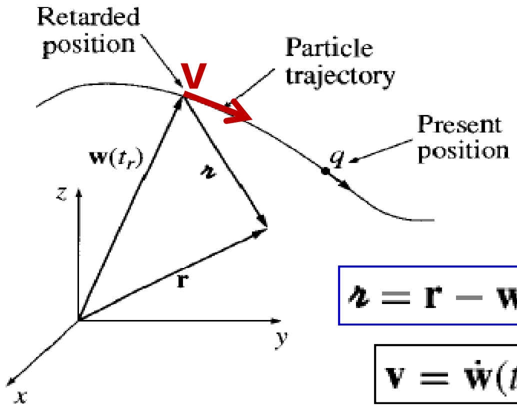
Therefore, $$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{\sqrt{(c^2t - \mathbf{r}\cdot\mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{qc\mathbf{v}}{\sqrt{(c^2t - \mathbf{r}\cdot\mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}}$$

10.3.2 The Fields of a Moving Point Charge

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})}, \qquad \mathbf{A}(\mathbf{r}, t) = \frac{\mathbf{v}}{c^2} V(\mathbf{r}, t)$$



$$\boxed{\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{w}(t_r)}$$

$$\boxed{\mathbf{v} = \dot{\mathbf{w}}(t_r)}$$

Let's begin with the gradient of V: $\nabla V = \frac{qc}{4\pi\epsilon_0} \frac{-1}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^2} \nabla(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})$

Since $\mathscr{r} = c(t - t_r)$, $\nabla \mathscr{r} = -c\nabla t_r$.

$$\nabla(\boldsymbol{\mathscr{r}} \cdot \mathbf{v}) = (\boldsymbol{\mathscr{r}} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\boldsymbol{\mathscr{r}} + \boldsymbol{\mathscr{r}} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \boldsymbol{\mathscr{r}})$$

$$(\boldsymbol{\mathscr{r}} \cdot \nabla)\mathbf{v} = \left(\mathscr{r}_x \frac{\partial}{\partial x} + \mathscr{r}_y \frac{\partial}{\partial y} + \mathscr{r}_z \frac{\partial}{\partial z}\right) \mathbf{v}(t_r) = \mathscr{r}_x \frac{d\mathbf{v}}{dt_r}\frac{\partial t_r}{\partial x} + \mathscr{r}_y \frac{d\mathbf{v}}{dt_r}\frac{\partial t_r}{\partial y} + \mathscr{r}_z \frac{d\mathbf{v}}{dt_r}\frac{\partial t_r}{\partial z} = \mathbf{a}(\boldsymbol{\mathscr{r}} \cdot \nabla t_r)$$

where $\mathbf{a} \equiv \dot{\mathbf{v}}$ is the *acceleration* of the particle at the retarded time.

$$(\mathbf{v} \cdot \nabla)\boldsymbol{\mathscr{r}} = (\mathbf{v} \cdot \nabla)\mathbf{r} - (\mathbf{v} \cdot \nabla)\mathbf{w} = \mathbf{v} - \mathbf{v}(\mathbf{v} \cdot \nabla t_r)$$

$$\nabla \times \mathbf{v} = -\mathbf{a} \times \nabla t_r$$

$$\nabla \times \boldsymbol{\mathscr{r}} = \nabla \times \mathbf{r} - \nabla \times \mathbf{w} = -\mathbf{v} \times \nabla t_r \longleftarrow \nabla \times \mathbf{r} = 0$$

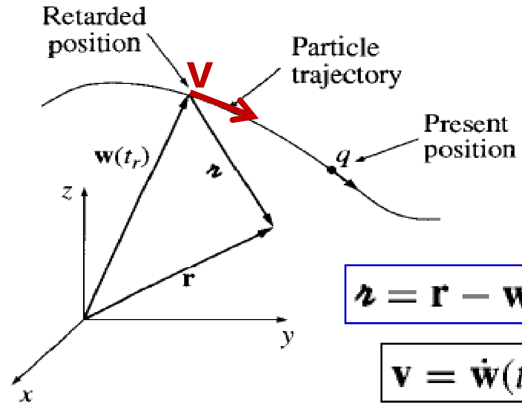
$$\nabla(\boldsymbol{\mathscr{r}} \cdot \mathbf{v}) = \mathbf{a}(\boldsymbol{\mathscr{r}} \cdot \nabla t_r) + \mathbf{v} - \mathbf{v}(\mathbf{v} \cdot \nabla t_r) - \boldsymbol{\mathscr{r}} \times (\mathbf{a} \times \nabla t_r) + \mathbf{v} \times (\mathbf{v} \times \nabla t_r) = \mathbf{v} + (\boldsymbol{\mathscr{r}} \cdot \mathbf{a} - v^2)\nabla t_r$$

$$\nabla V = \frac{qc}{4\pi\epsilon_0} \frac{1}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^2} \left[\mathbf{v} + (c^2 - v^2 + \boldsymbol{\mathscr{r}} \cdot \mathbf{a})\nabla t_r\right]$$

The Fields of a Moving Point Charge

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})}, \qquad \mathbf{A}(\mathbf{r}, t) = \frac{\mathbf{v}}{c^2} V(\mathbf{r}, t)$$



$$\boxed{\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{w}(t_r)}$$

$$\boxed{\mathbf{v} = \dot{\mathbf{w}}(t_r)}$$

$$\nabla V = \frac{qc}{4\pi\epsilon_0} \frac{1}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^2} \left[\mathbf{v} + (c^2 - v^2 + \boldsymbol{\mathscr{r}} \cdot \mathbf{a}) \nabla t_r\right]$$

To complete the calculation, we need to know ∇t_r.

$$-c\nabla t_r = \nabla \mathscr{r} = \nabla\sqrt{\boldsymbol{\mathscr{r}} \cdot \boldsymbol{\mathscr{r}}} = \frac{1}{2\sqrt{\boldsymbol{\mathscr{r}} \cdot \boldsymbol{\mathscr{r}}}} \nabla(\boldsymbol{\mathscr{r}} \cdot \boldsymbol{\mathscr{r}}) = \frac{1}{\mathscr{r}}[(\boldsymbol{\mathscr{r}} \cdot \nabla)\boldsymbol{\mathscr{r}} + \boldsymbol{\mathscr{r}} \times (\nabla \times \boldsymbol{\mathscr{r}})]$$

$$(\boldsymbol{\mathscr{r}} \cdot \nabla)\boldsymbol{\mathscr{r}} = \boldsymbol{\mathscr{r}} - \mathbf{v}(\boldsymbol{\mathscr{r}} \cdot \nabla t_r)$$

$$\nabla \times \boldsymbol{\mathscr{r}} = (\mathbf{v} \times \nabla t_r)$$

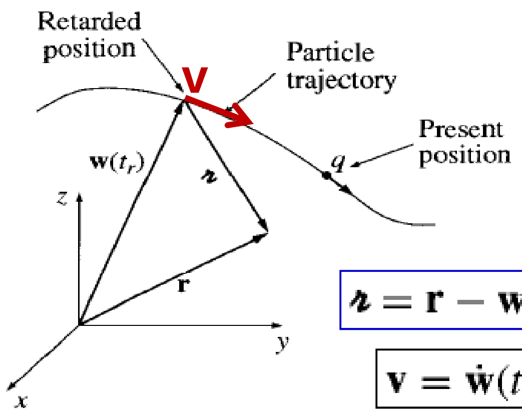
$$-c\nabla t_r = \frac{1}{\mathscr{r}}[\boldsymbol{\mathscr{r}} - \mathbf{v}(\boldsymbol{\mathscr{r}} \cdot \nabla t_r) + \boldsymbol{\mathscr{r}} \times (\mathbf{v} \times \nabla t_r)] = \frac{1}{\mathscr{r}}[\boldsymbol{\mathscr{r}} - (\boldsymbol{\mathscr{r}} \cdot \mathbf{v})\nabla t_r] \longrightarrow \nabla t_r = \frac{-\boldsymbol{\mathscr{r}}}{\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v}}$$

$$\nabla V = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^3} \left[(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})\mathbf{v} - (c^2 - v^2 + \boldsymbol{\mathscr{r}} \cdot \mathbf{a})\boldsymbol{\mathscr{r}}\right]$$

The Fields of a Moving Point Charge

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})}, \qquad \mathbf{A}(\mathbf{r}, t) = \frac{\mathbf{v}}{c^2} V(\mathbf{r}, t)$$



$$\boxed{\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{w}(t_r)}$$

$$\boxed{\mathbf{v} = \dot{\mathbf{w}}(t_r)}$$

$$\nabla V = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^3} \left[(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})\mathbf{v} - (c^2 - v^2 + \boldsymbol{\mathscr{r}} \cdot \mathbf{a})\boldsymbol{\mathscr{r}}\right]$$

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^3} \left[(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})(-\mathbf{v} + \mathscr{r}\mathbf{a}/c) + \frac{\mathscr{r}}{c}(c^2 - v^2 + \boldsymbol{\mathscr{r}} \cdot \mathbf{a})\mathbf{v}\right]$$

$$\Rightarrow \boxed{\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3}[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})] \qquad \mathbf{u} \equiv c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}}$$

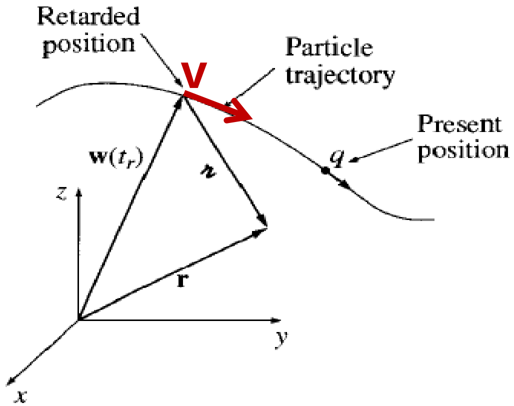
$$\nabla \times \mathbf{A} = \frac{1}{c^2} \nabla \times (V\mathbf{v}) = \frac{1}{c^2}[V(\nabla \times \mathbf{v}) - \mathbf{v} \times (\nabla V)]$$

$$\Rightarrow \boxed{\mathbf{B}(\mathbf{r}, t) = \frac{1}{c} \hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}(\mathbf{r}, t)}$$

The Fields of a Moving Point Charge

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}[(c^2-v^2)\mathbf{u}+\boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})] \qquad \mathbf{u}\equiv c\hat{\boldsymbol{\mathscr{r}}}-\mathbf{v}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{E}(\mathbf{r},t)$$



If the velocity and acceleration are both zero,
E reduces to the old electrostatic result:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}$$

Now, we can say the Lorentz force exerting on a test charge Q by any configuration of a charge (q):

$$\mathbf{F} = Q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) = \frac{qQ}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\left\{[(c^2-v^2)\mathbf{u}+\boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})] + \frac{\mathbf{V}}{c}\times\left[\hat{\boldsymbol{\mathscr{r}}}\times[(c^2-v^2)\mathbf{u}+\boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})]\right]\right\}$$

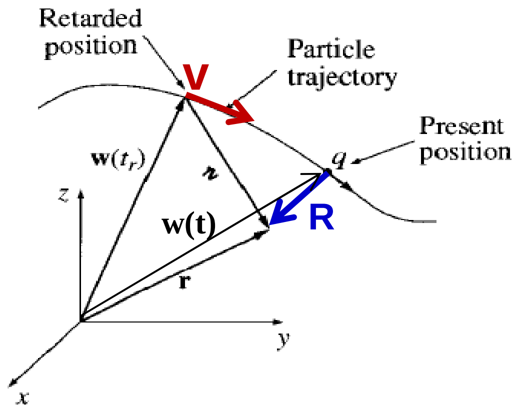
where $\mathbf{V}$ is the velocity of Q, and $\boldsymbol{\mathscr{r}}$, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{a}$ are all evaluated at the retarded time.

➔ The entire theory of classical electrodynamics is contained in that equation.

The Fields of a Moving Point Charge

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}[(c^2-v^2)\mathbf{u} + \boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})] \qquad \mathbf{u} \equiv c\,\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{E}(\mathbf{r},t)$$



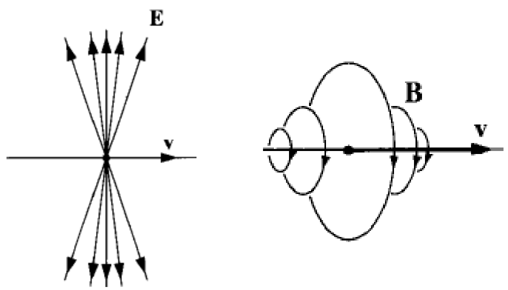
(Example 10.4) Calculate the Electric and magnetic fields of a point charge moving with constant velocity.

$$\mathbf{a} = 0 \longrightarrow \quad \mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{(c^2-v^2)\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\mathbf{u}$$

$$\mathbf{w}(t) = \mathbf{v}t \longrightarrow \quad \mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1-v^2/c^2}{\left(1-v^2\sin^2\theta/c^2\right)^{3/2}}\frac{\hat{\mathbf{R}}}{R^2}$$

where $\mathbf{R} \equiv \mathbf{r} - \mathbf{v}t$ $\quad\theta$ is the angle between $\mathbf{R}$ and $\mathbf{v}$

$$\longrightarrow \quad \mathbf{B} = \frac{1}{c}(\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{E}) = \frac{1}{c^2}(\mathbf{v}\times\mathbf{E})$$



➔ Because of the $\sin^2\theta$ in the denominator, the field of a fast-moving charge is flattened out like a pancake in the direction perpendicular to the motion.

➔ In forward and backward directions E is *reduced* by a factor $(1 - v^2/c^2)$ relative to the field of a charge at rest; in the perpendicular direction it is *enhanced* by a factor $1/\sqrt{1-v^2/c^2}$.

➔ When $v^2 \ll c^2$, they reduce to $\quad \mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\hat{\mathbf{R}}; \qquad \mathbf{B}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{q}{R^2}(\mathbf{v}\times\hat{\mathbf{R}})$

➔ Coulomb's law, Biot-Savart law for a point charge