Universidad de Valparaíso
CHILE

Prueba II
Métodos Matemáticos
Licenciatura en Física - 2017
IPGG

I).- Demuestre que la siguiente función es continua para para cualquier $z_0 = (x_0, y_0) \neq 0$. Utilice la familia de rectas $y = mx + b$ para demostrar lo anterior:

$$f(z) = \frac{[\mathrm{Re}(z)]^2 - [\mathrm{Im}(z)]^2}{|z|^2}$$

II).- Demuestre las siguientes identidades:

- (50%) $|\cos(z)|^2 = \cos^2(x) + \sinh^2(y)$
- (50%) $\cot^{-1}(z) = \frac{i}{2}\log\left(\frac{z-i}{z+i}\right)$

III).- Halle la imagen de la recta $x = x_0$ cuando se utiliza la siguiente regla de transformación:

- $$w = \exp(z)$$

IV).- Si u y v son funciones armónicas, muestre que:

- (40%) $au + bu$ también es armónica ($a, b \in \mathbb{R}$).
- (60%) uv es armónica si u y v son funciones armónicas conjugadas (son la parte real e imaginaria respectivamente de una misma función $f(z)$).

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1) Debe cumplirse que

$$\lim_{z \to z_0} f(z) = L = f(z_0) \qquad \text{con} \quad z_0 = x_0 + i y_0$$

i) $$f(z_0) = \frac{[Re(z_0)]^2 - [Im(z_0)]^2}{x_0^2 - y_0^2} = \frac{x_0^2 - y_0^2}{x_0^2 + y_0^2}$$

ii) $$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} \frac{[Re(z)]^2 - [Im(z)]^2}{|z|^2}$$

$$= \lim_{\substack{x \to x_0 \\ y \to y_0}} \frac{x^2 - y^2}{x^2 + y^2}$$

si $y = mx + b$

$$\lim_{z \to z_0} f(z) = \lim_{x \to x_0} \frac{x^2 - (mx+b)^2}{x^2 + (mx+b)^2} = \frac{x_0^2 - (mx_0+b)^2}{x_0^2 + (mx_0+b)^2}$$

cuando $x \to x_0 \implies mx_0 + b = y_0$.

$\therefore$

$$\lim_{z \to z_0} f(z) = \frac{x_0^2 - y_0^2}{x_0^2 + y_0^2} \;//$$

$\therefore$ $f(z)$ es continua en z_0 //

2) a) $|\cos z|^2 =$

$$= \cos z \cdot \overline{\cos z} = \cos z \cdot \cos \bar{z} = \frac{1}{4}\left(e^{iz}+e^{-iz}\right)\left(e^{-i\bar{z}}+e^{i\bar{z}}\right)$$

$$= \frac{1}{4}\left[e^{i(z-\bar{z})}+e^{i(z+\bar{z})}+e^{-i(z+\bar{z})}+e^{-i(z-\bar{z})}\right]$$

$$= \frac{1}{4}\left[\underbrace{e^{i(z+\bar{z})}+e^{-i(z+\bar{z})}}_{2\cos(z+\bar{z})}+\underbrace{e^{i(z-\bar{z})}+e^{-i(z-\bar{z})}}_{2\cos(z-\bar{z})}\right]$$

$$= \frac{1}{2}\left[\cos(z+\bar{z}) + \cos(z-\bar{z})\right]$$

$$= \frac{1}{2}\left[\cos 2x + \cos(2iy)\right]$$

Utilizando la identidad:

* $\cos 2\theta = \cos^2\theta - \text{sen}^2\theta$.

$$\Rightarrow \cos 2x = \cos^2 x - \text{sen}^2 x$$

$$\cos 2iy = \cos^2 2iy - \text{sen}^2 2iy$$

* **Obs.** $\text{sen}(i\alpha) = i\,\text{senh}\,\alpha$

$\cos(i\alpha) = \cosh\alpha$

Diego

$$|\cos z|^2 = \frac{1}{2}\left[\cos 2x + \cos 2iy\right]$$

$$= \frac{1}{2}\left[\cos^2 x - \operatorname{sen}^2 x + \underbrace{\cos^2(iy)}_{\cosh^2 y} - \underbrace{\operatorname{sen}^2(iy)}_{i^2 \operatorname{senh} y}\right]$$

$$= \frac{1}{2}\left[\cos^2 x - \operatorname{sen}^2 x + \cosh^2 y + \operatorname{senh} y\right]$$

$$= \frac{1}{2}\left[\cos^2 x - (1-\cos^2 x) + (1+\operatorname{senh}^2 y) + \operatorname{senh} y\right]$$

$$= \frac{1}{2}\left[2\cos^2 x - 1 + 1 + 2\operatorname{senh}^2 y\right]$$

$$= \cos^2 x + \operatorname{senh}^2 y$$ //

b) Sea $z = \cot w \longrightarrow w = \cot^{-1} z$

Ahora $\cot w = \dfrac{\cos w}{\operatorname{sen} w} = \dfrac{\dfrac{e^{iw}+e^{-iw}}{2}}{\dfrac{e^{iw}-e^{-iw}}{2i}} = i\,\dfrac{e^{iw}+e^{-iw}}{e^{iw}-e^{-iw}}$

$$\cot w = i\,\frac{\cancel{e^{iw}}(1+e^{-i2w})}{\cancel{e^{iw}}(1-e^{-i2w})} = z$$

hacemos $\xi = e^{i2w} \implies z = i\,\dfrac{(1+\xi)}{(1-\xi)}$

Luego $-iz(1-\xi) = 1+\xi$

$$-iz-1 = \xi - i\xi z$$

$$-1-iz = \xi(1-iz)$$

$$\Downarrow$$

$$\xi = \frac{-1-iz}{1-iz} = -\frac{1+iz}{1-iz} = -\frac{i(-i+z)}{i(-i-z)} = \frac{z-i}{z+i}$$

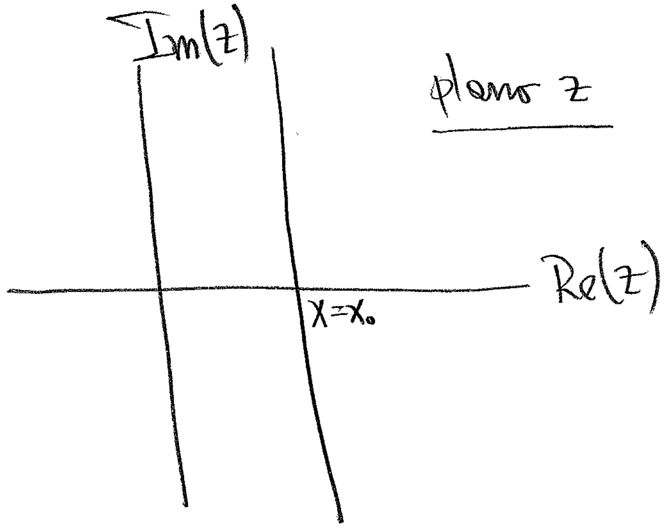
$$\Downarrow$$

$$e^{-i2w} = \frac{z-i}{z+i}$$

$$-i2w = \log\left(\frac{z-i}{z+i}\right)$$

$$w = \frac{1}{2}\, i \log\left(\frac{z-i}{z+i}\right) = \cot^{-1} z$$

3) $w = e^z$



$y \in \mathbb{R}$

$x = x_0$

Ahora $w = e^z = e^{x+iy} = e^x e^{iy} = e^x \cos y + i e^x \operatorname{sen} y$

$$u = e^x \cos y \quad /()^2$$

$$v = e^x \operatorname{sen} y \quad /()^2$$

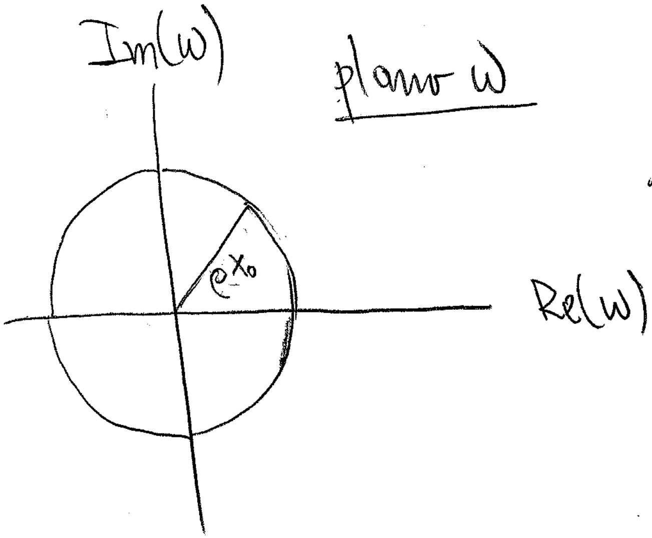
$$\Downarrow$$

$$\left.\begin{array}{l} u^2 = e^{2x} \cos^2 y \\ v^2 = e^{2x} \operatorname{sen}^2 y \end{array}\right\} (+)$$

$$u^2 + v^2 = e^{2x} \quad y \quad x = x_0$$

$$\Downarrow$$

$$u^2 + v^2 = e^{2x_0}$$



4) si $\nabla^2 u = 0 \wedge \nabla^2 v = 0$

a) $f = au + bu \quad / \nabla^2$

$$\nabla^2 f = \nabla^2 (au + bu) = a\cancel{\nabla^2 u}^{0} + b\cancel{\nabla^2 v}^{0} = 0$$

b) $f = uv$; u y v son funciones armónicas conjugadas $\Rightarrow$ Se relacionan por las ecuaciones de Cauchy - Riemann.

luego

$$\nabla^2 f = \nabla^2 (uv) = \nabla \cdot \nabla (uv)$$

$$= \underbrace{\left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath}\right)}_{\nabla} \cdot \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath}\right) uv$$

$$= \nabla \cdot \left(\left(u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}\right)\hat{\imath} + \left(u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}\right)\hat{\jmath}\right)$$

$$= \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \cancel{u\frac{\partial^2 v}{\partial x^2}} + \frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + v\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \cancel{u\frac{\partial^2 v}{\partial y^2}} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2}$$

$$= 2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + u\cancel{\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)}^{0} + v\cancel{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}^{0}$$

$$= 2\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right] ;$$

de las ecs. de Cauchy-Riemann:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

entonces

$$\nabla^2(uv) = 2\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right]$$

$$= 2\left[\frac{\partial v}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right] = 0$$