

Chapter 15

Landau-Ginzburg theory

We have seen in Chap. 6.1 that Phase transitions are caused most of the time by the interaction between particles, with an expectation being the Bose-Einstein condensation discussed in Sect. 14.3. One speaks of *cooperative phenomena*. The basis theory for understanding phase transition from a general viewpoint is the *Landau-Ginzburg* theory.

15.1 The Landau model

Order parameter. One assumes that there exist an *order parameter* x that differentiates the disordered phase from the ordered phase, such that

$$x = \begin{cases} \sim (1-t)^\beta & t < 1 \quad \text{viz } T < T_c \\ 0 & t > 1 \quad \text{viz } T > T_c \end{cases}, \quad t = \frac{T}{T_c}, \quad (15.1)$$

where T_c and t are the transition temperature (the *critical* temperature) and respectively the *reduced* temperature.

Critical exponents. The exponent β entering (15.1) is an example of a *critical exponent*. One of the aims of the theory of phase transitions is to classify the critical exponents that characterize the scaling of various observables close to a phase transition.

Free Enthalpy. The Landau model assumes that the

$$G = G(T, P, y; x)$$

is a function of the temperature T , the pressure P , an external field y and of the order parameter x . We have discussed in Sect. 5.5.2 the Principle of minimal Gibbs enthalpy, which states that $G(T, P, y; x)$ is minimal for (irreversible) process whenever T and P are constant. The thermodynamic state is hence determined by minimizing $G(T, P, y; x)$ with respect to the order parameter x .

Small order parameters. The Landau model is based on the assumption, that one may expand $G = G(T, y; x)$ with

$$G(T, y; x) = G_0(T, y) - yx + ax^2 + bx^4 + \dots \quad (15.2)$$

respect to the order parameter x .

- In order to avoid solutions with unbounded $x \rightarrow \infty$ one needs that $b > 0$.
- The linear coupling yx of the order parameter x to the external field y is generic. For magnetic systems it take the form HM , where H is the magnetic field and M the magnetization. Compare Sect. 3.7.
- The Landau-Ginzburg functional (15.2) contains, apart from the coupling to the external field y , only even powers of the order parameter x . This is necessarily the case if the sign of x (like the direction of the magnetization for a magnetic system) does not change the thermodynamic properties of the state.

Variational minimization. The variation δG of the Landau free energy with respect to a variation δx of the order parameter is

$$\delta G = (-y + 2ax + 4bx^3) \delta x, \quad (15.3)$$

when T and y are constant. The stationary condition $\delta G = 0$ then leads to the relation

$$-y + 2ax + 4bx^3 = 0. \quad (15.4)$$

Local stability. The (15.4) corresponds to a local minimum if

$$\delta^2 G = (2a + 12bx^2) (\delta x)^2 > 0. \quad (15.5)$$

We will check this condition further below.

15.1.1 Solution in the absence of an external field

The solution of (15.4) for $y = 0$ is

$$x = \begin{cases} 0 & \text{for } a > 0 \\ \pm \sqrt{-a/(2b)} & \text{for } a < 0 \end{cases} \quad (15.6)$$

where the condition for a results from $b > 0$. The stability condition (15.5) then takes the form

$$\frac{\delta^2 G}{(\delta x)^2} = \begin{cases} 2a & \text{for } a > 0 \\ -4a & \text{for } a < 0 \end{cases} \quad (15.7)$$

Both solutions are hence stable.

Order parameter scaling. The transition from $a > 0$ to $a < 0$ describes with (15.6) a continues transition between a non-ordered to an ordered states. Expanding $a = a(T)$ into a Taylor series around T_c we may hence assume that

$$a = a_0(t - 1), \quad t = \frac{T}{T_c}, \quad (15.8)$$

which leads to the square-root scaling

$$x = \pm \sqrt{\frac{a_0}{2b}} (1 - t), \quad t < 1 \quad (15.9)$$

of the order parameter x below the critical temperature T_c

Rescaling of parameters. Rescaling the parameters entering (15.2) we may rewrite the Landau-Ginzburg functional as

$$G(T, y; x) - G_0(T, y) = -yx + ax^2 + \frac{1}{4}x^4, \quad a = \frac{t-1}{2}, \quad (15.10)$$

where we have set $a_0 = 1/2$ and $b = 1/4$. The solution (15.9) of the order parameter then takes the form

$$x = \pm\sqrt{1-t}, \quad t < 1. \quad (15.11)$$

Ginzburg criterion. The stability condition (15.7) breaks down when the thermal expectation value $\langle(\delta x)^2\rangle$ fluctuation of the order parameter becomes too large with respect to the (quadratic) distance of the two solutions (15.6), viz when

$$\frac{\langle(\delta x)^2\rangle}{\langle(x_+ - x_-)^2\rangle} \approx 1, \quad \langle(\delta x)^2\rangle \approx 4(1-t). \quad (15.12)$$

The *Ginzburg criterion* is always violated for $T \rightarrow T_c$, viz close enough to the critical temperature, that is when thermal fluctuations induce transitions between the two minima x_+ and x_- .

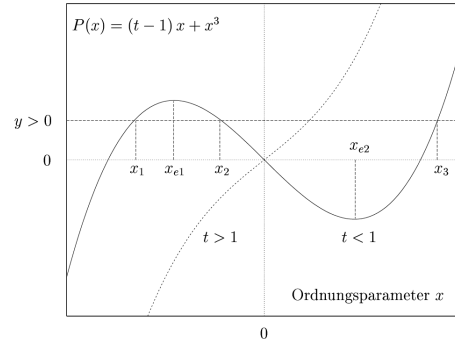
15.1.2 Effect of an external field

We write the rescaled solution of (15.4) as

$$y = P(x) \equiv (t-1)x + x^3,$$

which implies that we need to find the intersection of $P(x)$ with the external field y .

- For $T > T_c$, viz for $t > 1$, there one solution of $y = P(x)$ for every y .
- For $t < 1$, that is for $T < T_c$, there is a range $y \in [-y_e, +y_e]$ for which $y = P(x)$ has three solutions $x_1 < x_2 < x_3$.



Stability. The two extrema von $P(x)$ are given for values of the order parameter x_{e1} and x_{e2} which are

$$x_{e1,e2} = \mp\sqrt{\frac{1-t}{3}}, \quad P(x_{e1,e2}) = \pm y_e,$$

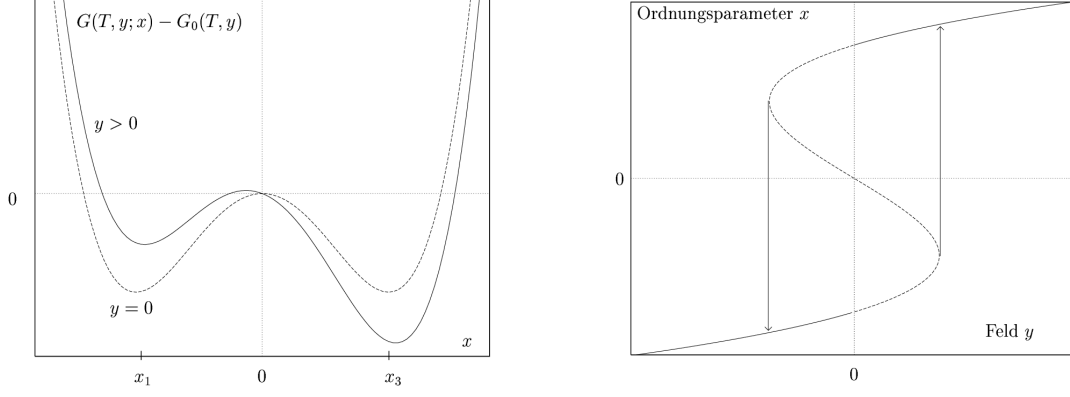
with

$$y_e = \frac{2}{3\sqrt{3}} (1-t)^{3/2}. \quad (15.13)$$

The solutions are stable, according to (15.5), if $\delta^2 G > 0$. For rescaled parameters that implies that

$$t - 1 + 3x^2 > 0, \quad x^2 > \frac{1-t}{3} = x_{e1,2}^2 \quad (15.14)$$

needs to be satisfied. This implies that x_1 and x_3 are stable and x_2 unstable.



Hysteresis. The overlapping coexistence of two solutions implies a discontinuous change of the order parameter x upon increasing the external field y continuously. Upon reaching y_e from below the state of the system jumps from x_{e1} to x_{e2} .

15.1.3 Specific heat

The differential

$$dG = -S dT + V dP + \mu dN, \quad G(T, P; x) = G_0 - xy + a_0 \left(\frac{T}{T_c} - 1 \right) x^2 + b x^4$$

of the free Enthalpy $G(T, P; x)$ implies that the entropy is given by

$$S = - \left(\frac{\partial G}{\partial T} \right)_P = S_0 - \frac{a_0}{T_c} x^2 - \underbrace{\left(\frac{\partial G}{\partial x} \right)_{T,P} \left(\frac{\partial x}{\partial T} \right)_P}_{\equiv 0}, \quad (15.15)$$

where we have pointed out the stationary condition $\partial G / \partial x = 0$. The component of the entropy unaffected by the ordering has been denoted here by $S_0 = S_0(T, P) = -(\partial G_0 / \partial T)_P$.

Entropy in the absence of an external field. Using the solution (15.6) for $y = 0$, that is $x^2 = 0$ for $T > T_c$ and

$$x^2 = \frac{a_0}{2b} \left(1 - \frac{T}{T_c} \right) \quad \text{for} \quad T < T_c,$$

we find

$$\begin{aligned} T > T_c : \quad S &= S_0(T, p), \\ T < T_c : \quad S &= S_0(T, p) - \frac{a_0^2}{2b T_c} \left(1 - \frac{T}{T_c} \right) \end{aligned} \quad (15.16)$$

for the entropy S . The entropy is continuous at $T = T_c$, but not its derivatives, the telltale sign of a phase transition of second order. Compare Sect. 6.3.

Specific heat jump. Using

$$C_P = T \left(\frac{\partial S}{\partial T} \right)_P, \quad C_{P,0} = T \left(\frac{\partial S_0}{\partial T} \right)_P,$$

we obtain from (15.16) the specific heat

$$\begin{aligned} T > T_c : \quad & C_P = C_{P,0}, \\ T < T_c : \quad & C_P = C_{P,0} + \frac{a_0^2}{2bT_c^2} T. \end{aligned} \quad (15.17)$$

The specific heat is discontinuous at $T = T_c$, with the specific heat jump ΔC_P ,

$$\boxed{\Delta C_P = \frac{a_0^2}{2bT_c}} \quad (15.18)$$

being easily measurable quantity.

15.2 Space-dependent order parameter

Till now we assumed the order parameter $x(\mathbf{r})$ to be independent of the position \mathbf{r} . This is however correct only on the average. Interaction with a thermal bath will generically induce excited states for all thermodynamic relevant quantities, with the order parameter being no exception.

Gibbs-Duhem relation. A straightforward generalization of the Gibbs-Duhem relation (5.17) for the Gibbs enthalpy G , which we did derive for homogeneous systems, to the case of a spatially non-constant chemical potential $\mu = \mu(T, P, y(\mathbf{r}); x(\mathbf{r}))$ is

$$G = \mu N \quad \rightarrow \quad G = \int d^3r n(\mathbf{r}) \mu(T, P, y(\mathbf{r}); x(\mathbf{r})), \quad (15.19)$$

where $n(\mathbf{r})$ is the density of particles.

Expansion of the chemical potential. We assume now that $n(\mathbf{r}) \equiv n_0$ is constant, entering (15.19) hence just as an overall multiplicative factor. This is the case e.g. for magnetic order superconducting systems.

For the expansion of the chemical potential μ as a function of the order parameter x we generalize (15.2) to

$$\mu(T, y; x) - \mu_0(T, y) \sim -y x + \frac{t-1}{2} x^2 + \frac{1}{4} x^4 + \frac{1}{2} (\nabla x)^2 + \dots, \quad (15.20)$$

where the gradient term $\sim (\nabla x)^2$ implies that the homogeneous state with a constant $x(\mathbf{r})$ is thermodynamically favored.

Stationarity condition. The Gibbs enthalpy G becomes stationary, with the chemical potential μ given by (15.20), when

$$\delta G \sim \int d^3r [-y + (t-1)x + x^3] \delta x + \int d^3r \frac{1}{2} \delta (\nabla x)^2, \quad (15.21)$$

where the variation $\delta (\nabla x)^2$ of the gradient terms is given by

$$\frac{1}{2} \delta (\nabla x)^2 = \nabla x (\delta \nabla x) = \nabla (\delta x \nabla x) - (\nabla^2 x) \delta x,$$

We have used here that the variation δ and the gradient ∇ are independent operation and that independent operations commute.

For the integration of the variation of the gradient term we use Gauss' theorem,

$$\int d^3r \nabla (\delta x \nabla x) = \oint_{\partial V} d\mathbf{f} \delta x \nabla x = 0, \quad (15.22)$$

and fixed boundary conditions, with the latter implying that value of the order parameter is given and hence fixed at the surface ∂V . The variation δx of the order parameter then vanishes on ∂V .

Determining equation for the order parameter. With (15.22) we obtain

$$\delta G \sim \int d^3r [-y + (t-1)x + x^3 - \nabla^2 x] \delta x \quad (15.23)$$

for δG , which vanishes for arbitrary $\delta x = \delta x(\mathbf{r})$ if the second-order differential equation

$$\boxed{(\nabla^2 - t + 1)x - x^3 = -y} \quad (15.24)$$

is fulfilled.

Fluctuation in the paramagnetic state. The determining equation (15.24) for $x = x(\mathbf{r})$ is both inhomogeneous, due to the term $(-y)$ on the right-hand side, and non-linear, due the term $\sim x^3$. We study here the effect of a localized external perturbation

$$y(\mathbf{r}) \rightarrow y_0 \delta(\mathbf{r})$$

on the paramagnetic state, that is for $T > T_c$. The order parameter then vanishes for $y_0 = 0$ and may expect that it remains small for small y_0 . We may therefore neglect the term $\sim x^3$ (15.24), which then becomes

$$(\nabla^2 - t + 1)x = -y_0 \delta(\mathbf{r}). \quad (15.25)$$

Fourier transformation. Eq. (15.25) is solved by a Fourier transformation,

$$x(\mathbf{r}) = \int d^3k \tilde{x}(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{r}), \quad \tilde{x}(\mathbf{k}) = \frac{1}{(2\pi)^3} \int d^3r x(\mathbf{r}) \exp(-i\mathbf{k} \cdot \mathbf{r}), \quad (15.26)$$

which is based on the representation

$$\delta(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d^3r \exp(i\mathbf{k} \cdot \mathbf{r})$$

of the δ -function. Substituting (15.26) into (15.25) leads to

$$\tilde{x}(\mathbf{k}) = \frac{y_0}{(2\pi)^3} \frac{1}{k^2 + t - 1} . \quad (15.27)$$

Green's function. For the back-transformation

$$x(\mathbf{r}) = \frac{y_0}{(2\pi)^3} \int d^3k \frac{\exp(i\mathbf{k} \cdot \mathbf{r})}{k^2 + t - 1} \quad (15.28)$$

we use spherical coordinates

$$|\mathbf{k}| = k, \quad \mathbf{k} \cdot \mathbf{r} = k r \cos \theta ,$$

such that

$$\begin{aligned} \int d^3k \frac{\exp(i\mathbf{k} \cdot \mathbf{r})}{k^2 + t - 1} &= 2\pi \int_0^\infty dk \frac{k^2}{k^2 + t - 1} \int_{-1}^{+1} d(\cos \theta) \exp(ik \cos \theta) \\ &= \frac{4\pi}{r} \int_0^\infty dk \frac{k \sin kr}{k^2 + t - 1} \\ &= \frac{2\pi^2}{r} \exp(-\sqrt{t-1}r) . \end{aligned}$$

The end result is

$$\boxed{x(\mathbf{r}) = \frac{y_0}{4\pi r} e^{-r/r_0(t)}}, \quad r_0(t) = \frac{1}{\sqrt{t-1}}, \quad t = \frac{T}{T_c}, \quad (15.29)$$

which hold for $T > 1$. The *Green's function* $x(\mathbf{r})$ propagates the disturbance $\propto y_0$ occurring at $\mathbf{r} = 0$ to an arbitrary location \mathbf{r} .

Scale invariant divergence of the correlation length. The *correlation length* r_0 entering (15.29) diverges for $T \rightarrow T_c$ from above. The type of divergence, a powerlaw with a *critical exponent* $\nu = 1/2$, implies *scale invariance*,

$$r(t) \sim t^{-\nu}, \quad (c_s t)^{-\nu} = c_s^{-\nu} t^{-\nu} \sim t^{-\nu} . \quad (15.30)$$

Rescaling the reduced temperature t by a scale factor c_s does not change the functional dependence of the correlation length r_0 on $t = T/T_c$.

Scale invariance and universality. The fact that thermodynamic quantities become scale invariant close to a second order phase transition has profound consequences. It implies with (15.30) that the thermodynamics properties become independent also of the scales of the underlying physics. Length scales like the Bohr radius $4\pi\epsilon_0\hbar^2/(m_e e^2)$ or energy scales like $k_B T$ do not influence the scaling behavior.

UNIVERSALITY The properties of a system close to a second-order phase transition become universal in the sense that the critical exponents depend only on the dimensionality of the system and of the symmetry of the order parameter.

The emergence of universality is a key result of theory of phase transitions.

Fluctuations in the ordered phase. For the ordered phase we write the order parameter as

$$x = \sqrt{1-t} + \Delta x, \quad x^3 = (1-t)^{3/2} + 3(1-t)\Delta x + \dots,$$

where $x = \sqrt{1-t}$ is its value in the absence of an external field. The response $\Delta x = \Delta x(\mathbf{r})$ to a perturbation $y = y_0 \delta(\mathbf{r})$ is determined via (15.24) by

$$(\nabla^2 + 2(t-1)) \Delta x = -y_0 \delta(\mathbf{r}). \quad (15.31)$$

Solving in analogy to (15.24) one finds with

$$r_0 = [2(1-t)]^{-\nu}, \quad \nu = \frac{1}{2} \quad (15.32)$$

the critical same critical exponent $\nu = 1/2$.