Introducción a la mecánica Newtoniana

Gravedad es provocada por una **fuerza**

- Es siempre atractiva
- Es proporcional al producto de las masas de los cuerpos.
- Actúa a lo largo de la línea de unión entre los cuerpos. (Fuerzas centrales)

$$\vec{F} = f\,\hat{r}$$

$(f < 0)$

- Depende únicamente de la distancia variando de forma inversa al cuadrado de ésta.

En términos de la mecánica clásica, es una fuerza conservativa, central:

$$\vec{F} = \underbrace{f(r)}_{\text{función únicamente de } |\vec{r}|} \hat{r}$$

*Puede ser derivada de un potencial

$$\vec{F} = -\frac{dU}{dr}\hat{r} = f(r)\hat{r}$$

$$\Rightarrow \quad U(r) = -\int f(r)\,dr + U_0$$

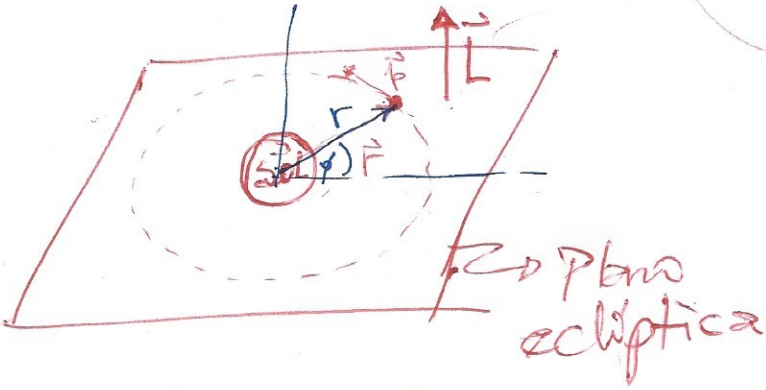
* F. central $\Rightarrow$ El movimiento se realiza en un plano invariante.

$$\vec{\tau}_0 = \vec{r} \times \underbrace{\vec{F}}_{f(r)\hat{r}} = (r\hat{r}) \times (f(r)\hat{r})$$

$$= \vec{0}$$

Pero $\vec{\tau}_0 = \dfrac{d\vec{L}}{dt} = \vec{0} \Rightarrow \underline{\underline{\vec{L} = \vec{cte}}}$

$$\vec{L} = cte \Rightarrow |\vec{L}| = cte \ \wedge \ \hat{L} = \frac{\vec{L}}{|\vec{L}|} = cte$$



* Es natural trabajar en coordenadas polares: $r = \sqrt{x^2+y^2}$

$$\phi = \mathrm{ArcTan}\left(\frac{y}{x}\right)$$

$x = r\cos\phi$

$y = r\sin\phi$

¿Cómo se ve la 2ª ley de Newton en este sistema?

$$m\ddot{\vec{r}} = \vec{F} = f(r)\,\hat{r}$$

$\vec{r} = r\hat{r}$; $\hat{r} = \cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}$

$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\hat{r}}$; $\dot{\hat{r}} = -\sin\phi\,\dot\phi\,\hat{\imath} + \cos\phi\,\dot\phi\,\hat{\jmath}$

$$\dot{\hat{r}} = \dot\phi\,\underbrace{(-\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath})}_{\hat\phi}$$

$$\hat{\phi} = -\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}$$

i) $\hat{r}\cdot\hat{\phi} = 0$

ii) $\hat{r}\times\hat{\phi} = \hat{k}$

$$\dot{\vec{r}} = \dot{r}\,\hat{r} + r\dot{\phi}\,\hat{\phi}$$

$$\therefore\ \ddot{\vec{r}} = \ddot{r}\,\hat{r} + \dot{r}\,\underbrace{\dot{\hat{r}}}_{\dot{\phi}\hat{\phi}} + \dot{r}\dot{\phi}\,\hat{\phi} + r\ddot{\phi}\,\hat{\phi} + r\dot{\phi}\,\dot{\hat{\phi}}$$

$$\ddot{\vec{r}} = \ddot{r}\,\hat{r} + 2\dot{r}\dot{\phi}\,\hat{\phi} + r\ddot{\phi}\,\hat{\phi} + r\dot{\phi}\,\dot{\hat{\phi}}$$

$$* \quad \dot{\hat{\phi}} = -\cos\phi\,\dot{\phi}\,\hat{\imath} - \sin\phi\,\dot{\phi}\,\hat{\jmath}$$

$$\dot{\hat{\phi}} = -\dot{\phi}\,(\cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}) = -\dot{\phi}\,\hat{r}$$

$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\phi}^2)\,\hat{r} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\,\hat{\phi}$$

$$m(\ddot{r} - r\dot{\phi}^2)\,\hat{r} + m(2\dot{r}\dot{\phi} + r\ddot{\phi})\,\hat{\phi} = f(r)\,\hat{r}$$

$\rightarrow\ m(\ddot{r} - r\dot{\phi}^2) = f(r)$

$\rightarrow\ m(2\dot{r}\dot{\phi} + r\ddot{\phi}) = 0$

Para la gravedad Newtoniana,

$$f(r) = -\frac{\cancel{GM_{\bullet}m_2}}{r^2} = -\frac{GMm}{r^2}$$

Además, $\vec{p} = m\dot{\vec{r}}$

$$\vec{p} = m\dot{r}\hat{r} + mr\dot{\phi}\hat{\phi}$$

$$\vec{L} = \vec{r}\times\vec{p} = (r\hat{r})\times[m\dot{r}\hat{r} + mr\dot{\phi}\hat{\phi}]$$

$$\vec{L} = mr^2\dot{\phi}(\hat{r}\times\hat{\phi}) = mr^2\dot{\phi}\hat{k}$$

$$\vec{L} \equiv L\hat{k} \Rightarrow mr^2\dot{\phi} = L \quad (\text{cte})$$

$$\Rightarrow \dot{\phi} = \frac{L}{mr^2} \Rightarrow \dot{\phi}^2 = \frac{L^2}{m^2r^4}$$

$$m\ddot{r} - \cancel{m}\cancel{r}\cdot\frac{L^2}{m^{\cancel{2}}r^{\cancel{4}\,3}} = f(r)$$

$$m\ddot{r} = f(r) + \frac{L^2}{mr^3} \quad / \dot{r}$$

$$m\dot{r}\ddot{r} = \left[f(r) + \frac{d}{dr}\left(-\frac{L^2}{2mr^2}\right)\right]\dot{r}$$

$$= \left[-\frac{dU}{dr} - \frac{d}{dr}\left(\frac{L^2}{2mr^2}\right)\right]\dot{r}$$

⑥ C1

$$m\dot{r}\ddot{r} = \frac{d}{dr}\underbrace{\left[-U(r) - \frac{L^2}{2mr^2}\right]}_{G(r)}\dot{r} = \frac{d}{dr}G(r)\,\dot{r}$$

$$m\dot{r}\ddot{r} = \frac{d}{dr}G(r)\,\dot{r} = \frac{dG}{dr}\dot{r}$$

$$= \frac{dG}{dr}\cdot\frac{dr}{dt} = \frac{dG}{dt}$$

$$m\dot{r}\ddot{r} = \frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right)$$

$$\therefore \quad \frac{d}{dt}\left[\frac{1}{2}m\dot{r}^2\right] = \frac{dG}{dt}$$

$$\Rightarrow \frac{d}{dt}\left[\frac{1}{2}m\dot{r}^2 - G\right] = 0$$

$$\Rightarrow \boxed{\frac{1}{2}m\dot{r}^2 + U(r) + \frac{L^2}{2mr^2} = E \text{ (cte)}}$$