Transformaciones unitarias

Complemento IV

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Cambio de base (N-dim)

Sean las bases ortonormales | completas:

$$\{|u_i\rangle\} \qquad\qquad \{|\tilde{u}_j\rangle\}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\langle u_i|u_j\rangle = \delta_{ij} \qquad\qquad \langle\tilde{u}_i|\tilde{u}_j\rangle = \delta_{ij}$$

$$\sum_{i=1}^{N} |u_i\rangle\langle u_i| = \hat{\mathbb{1}} \qquad\qquad \sum_{i=1}^{N} |\tilde{u}_i\rangle\langle\tilde{u}_i| = \hat{\mathbb{1}}$$

Def. Sea el elemento de matriz

$$(\hat{U})_{ij} = U_{ij} = \langle u_i|\tilde{u}_j\rangle \implies U_{ij}^* = \langle u_i|\tilde{u}_j\rangle^*$$

$$(\hat{U}^\dagger)_{ji} = \langle\tilde{u}_j|u_i\rangle$$

Obs. $\hat{U}$ es unitario:

Evaluemos $(\hat{U}^\dagger\hat{U})_{ij} = \sum_{l} (\hat{U}^\dagger)_{il}(\hat{U})_{lj}$

$$= \sum_{l} U^*_{li} U_{lj} = \sum_{l} \langle \tilde{u}_i | u_l \rangle \langle u_l | \tilde{u}_j \rangle$$

$$= \langle \tilde{u}_i | \tilde{u}_j \rangle = \delta_{ij}$$

$$\therefore \quad (\hat{U}^\dagger \hat{U})_{ij} = \delta_{ij} \qquad\qquad \hat{U}^\dagger = \hat{U}^{-1}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Uparrow$$

$$\hat{U}^\dagger \hat{U} = \hat{\mathbb{1}}$$ (Demuestre como tarea que $\hat{U}\hat{U}^\dagger = \hat{\mathbb{1}}$)

¿Cómo transforman los vectores?

$\Downarrow$

Transforman como lo hacen sus componentes!

Sea $|\psi\rangle$ arbitrario, en la base $\{|u_i\rangle\}$ este vector esta descrito como:

$$|\psi\rangle_u = \sum_{i=1}^{N} a_i |u_i\rangle \qquad \text{donde}$$

$$a_i = \langle u_i | \psi \rangle$$

análogamente $|\psi\rangle_{\tilde{u}} = \sum_{i=1}^{N} \tilde{a}_i |\tilde{u}_i\rangle$

con $\tilde{a}_i = \langle \tilde{u}_i | \psi \rangle$

Ahora bien

$$a_i = \langle u_i | \underset{\uparrow \hat{\mathbb{1}}}{} \psi \rangle = \sum_{l} \langle u_i | \tilde{u}_l \rangle \langle \tilde{u}_l | \psi \rangle$$

$$a_i = \sum_{l} U_{il} \tilde{a}_l$$

$\therefore$

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Matricialmente

$$\begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} = \begin{pmatrix} U_{11} & \cdots & U_{1N} \\ \vdots & \ddots & \vdots \\ U_{N1} & \cdots & U_{NN} \end{pmatrix} \begin{pmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{a}_N \end{pmatrix}$$

$\Downarrow$

$$\boxed{|\psi\rangle_{u} = \hat{U} |\psi\rangle_{\tilde{u}}}$$

$\Downarrow$

y

$$\boxed{|\psi\rangle_{\tilde{u}} = \hat{U}^{\dagger} |\psi\rangle_{u}}$$

¿Como transforman los operadores?

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como transformen sus componentes!

Sea $\hat{A}$ un operador arbitrario m :

$$\hat{A}_u = \sum_i \sum_j A_{ij} |u_i\rangle\langle u_j| \quad ; \quad A_{ij} = \langle u_i|\hat{A}|u_j\rangle$$

y

$$\hat{A}_{\tilde{u}} = \sum_i \sum_j \tilde{A}_{ij} |\tilde{u}_i\rangle\langle \tilde{u}_j| \quad ; \quad \tilde{A}_{ij} = \langle \tilde{u}_i|\hat{A}|\tilde{u}_j\rangle$$

entonces:

$$A_{ij} = \langle u_i|\hat{A}|u_j\rangle$$

$$\hat{\mathbb{1}} \qquad \hat{\mathbb{1}}$$

$$= \sum_l \sum_m \langle u_i|\tilde{u}_l\rangle\langle \tilde{u}_l|\hat{A}|\tilde{u}_m\rangle\langle \tilde{u}_m|u_j\rangle$$

$$= \sum_{l,m} U_{il} \tilde{A}_{lm} (\hat{U}^\dagger)_{mj}$$

$$A_{ij} = (\hat{A}_u)_{ij} = \sum_{l,m} (\hat{U})_{il} (\hat{A}_{\tilde{u}})_{lm} (\hat{U}^\dagger)_{mj}$$

$$\therefore \quad \boxed{\hat{A}_u = \hat{U}\hat{A}_{\tilde{u}}\hat{U}^\dagger} \Rightarrow \boxed{\hat{A}_{\tilde{u}} = \hat{U}^\dagger \hat{A}_u \hat{U}}$$

I) Producto interno se preserva cuando cambia la base!

En la base $\{|u_i\rangle\}$ se tiene el siguiente producto interno

$$_u\langle\phi|\psi\rangle_u = {}_u\langle\phi|\hat{\mathbb{1}}|\psi\rangle_u = {}_u\langle\phi|\hat{U}\hat{U}^{+}|\psi\rangle_u$$

pero $\hat{U}^{+}|\psi\rangle_u = |\psi\rangle_{\tilde{u}}$

y $\hat{U}^{+}|\phi\rangle_u = |\phi\rangle_{\tilde{u}} \Longrightarrow {}_u\langle\phi|\hat{U} = {}_{\tilde{u}}\langle\phi|$

$\therefore$ $\boxed{{}_u\langle\phi|\psi\rangle_u = {}_{\tilde{u}}\langle\phi|\psi\rangle_{\tilde{u}}}$ $\Longrightarrow$ Producto interno es invariante

II) El determinante de un operador es invariante

$$\det(\hat{A}_u) = \det(\hat{U}\hat{A}_{\tilde{u}}\hat{U}^{+}) = \det\hat{U}\,\det\hat{A}_{\tilde{u}}\det\hat{U}^{+}$$

$$= \overbrace{\det(\hat{U}\hat{U}^{+})}^{1}\det(\hat{A}_{\tilde{u}})$$

$\therefore$ $\boxed{\det(\hat{A}_u) = \det(\hat{A}_{\tilde{u}})}$

III) La traza de un operador es invariante!

$$Tr(\hat{A}_u) = \sum_{l} \langle u_l | \hat{A} | u_l \rangle$$

$$= \sum_{m} \sum_{l} \langle u_l | \tilde{u}_m \rangle \langle \tilde{u}_m | A | u_l \rangle$$

$$= \sum_{m} \sum_{l} \langle \tilde{u}_m | \hat{A} | \underbrace{u_l \rangle \langle u_l}_{} | \tilde{u}_m \rangle$$

$$= \sum_{m} \langle \tilde{u}_m | \hat{A} | \tilde{u}_m \rangle = Tr(\hat{A}_{\tilde{u}})$$

$$\Downarrow$$

$$Tr(\hat{A}_u) = Tr(\hat{A}_{\tilde{u}})$$

IV) Los valores propios son invariantes!

se cumple que

i) $\hat{A}_u |a\rangle_u = a_u |a\rangle_u$

ii) $\hat{A}_{\tilde{u}} |a\rangle_{\tilde{u}} = a_{\tilde{u}} |a\rangle_{\tilde{u}}$

en (ii):

$$\hat{U} \;/\; \hat{A}_{\tilde{\mu}}|a\rangle_{\tilde{\mu}} = a_{\tilde{\mu}}|a\rangle_{\tilde{\mu}}$$

$$\Downarrow$$

$$\hat{U}\hat{A}_{\tilde{\mu}}\underset{\uparrow}{\,}|a\rangle_{\tilde{\mu}} = a_{\tilde{\mu}}\,\hat{U}|a\rangle_{\tilde{\mu}}$$

$$\hat{\mathbb{1}} = \hat{U}^{+}\hat{U}$$

$$\Downarrow$$

$$\underbrace{\hat{U}\hat{A}_{\tilde{\mu}}\hat{U}^{+}}_{\hat{A}_{\mu}}\underbrace{\hat{U}|a\rangle_{\tilde{\mu}}}_{|a\rangle_{\mu}} = a_{\tilde{\mu}}\underbrace{\hat{U}|a\rangle_{\tilde{\mu}}}_{|a\rangle_{\mu}}$$

$\therefore$

$$\hat{A}_{\mu}|a\rangle_{\mu} = a_{\tilde{\mu}}|a\rangle_{\mu}$$

comparando este resultado con (i) se tiene que:

$$a_{\tilde{\mu}} = a_{\mu}$$

El autovalor no depende de la base elegida!