Nuestra ecuación cúbica a resolver

$$\wp_3(r) = E^2 r^3 - L^2 r + r_s L^2$$

$$= E^2 \left[r^3 - \left(\frac{L}{E}\right)^2 r + \left(\frac{L}{E}\right)^2 r_s \right]$$

Definición: $\boxed{b \equiv \frac{L}{E}}$: parámetro de impacto.

$$\wp_3(r) = E^2 \left[r^3 - b^2 r + b^2 r_s \right]$$

Luego $\wp_3^*(r) = r^3 - b^2 r + b^2 r_s$

$$\therefore \quad 4r^3 - g_2 r - g_3 = 0$$

donde $g_2 = 4b^2 > 0$

$g_3 = -4b^2 r_s < 0$

ó bien $\boxed{4r^3 - g_2 r + |g_3| = 0}$

Identidades fundamentales

1.- $4\cos^3 x - 3\cos x - \cos 3x = 0$ $(\cancel{|x|<1})$ $|\cos x| < 1$

2.- $4\cosh^3 x - 3\cosh x - \cosh 3x = 0$ $(\cancel{|x|<\infty})$ $|\cosh x < \infty|$

3.- $4\sin^3 x - 3\sin x + \sin 3x = 0$

4.- $4\sinh^3 x + 3\sinh x - \sinh 3x = 0$

cl10 (2)

cambio de variable:

$$r = \tilde{r}_0 \sin\theta$$

$$4\lambda\tilde{r}_0^3\sin^3\theta - g_2\lambda\tilde{r}_0\sin\theta + \lambda|g_3| = 0$$

$$4\sin^3\theta - 3\sin\theta + \sin 3\theta = 0$$

$$\Rightarrow \quad 4\lambda\tilde{r}_0^3 = 4 \rightarrow \lambda = \frac{1}{\tilde{r}_0^3}$$

$$g_2\lambda\tilde{r}_0 = 3$$

$$\lambda|g_3| = \sin 3\theta$$

$$g_2 \cdot \frac{1}{\tilde{r}_0^3} \cdot \tilde{r}_0 = 3 \Rightarrow \boxed{\tilde{r}_0 = \sqrt{\frac{g_2}{3}}}$$

$$\therefore \boxed{\lambda = \sqrt{\frac{27}{g_2^3}}}$$

$$\therefore \quad \sin 3\theta = \sqrt{\frac{27|g_3|^2}{g_2^3}}$$

$$\therefore \quad 3\theta = \sqrt{\frac{27g_3^2}{g_2^3}} + 2m\pi \qquad m \in \mathbb{Z}$$

$$\Rightarrow \boxed{\theta_m = \frac{1}{3}\sqrt{\frac{27g_3^2}{g_2^3}} + \frac{2m\pi}{3}}$$

$$\theta_0 = \frac{1}{3}\sqrt{\frac{27g_3^2}{g_2^3}} = \frac{1}{3}\left[\frac{27\cdot\cancel{16}\,\cancel{b^6}\,b^4\, r_s^2}{4\cdot\cancel{16}\,\cancel{b^6}\,b^2}\right]^{1/2} = \frac{1}{3}\left[\frac{27}{4}\,\frac{r_s^2}{b^2}\right]^{1/2}$$

$$\boxed{\theta_0 = \frac{\sqrt{3}}{2}\,\frac{r_s}{b}}$$

4to (3)

$$\tilde{r}_m = \tilde{r}_0 \sin\left(\theta_0 + \frac{2}{3}\pi m\right) \qquad ; \quad m = 0, 1, 2$$

$$\tilde{r}_0 = \sqrt{\frac{g_2}{3}} = \sqrt{\frac{4b^2}{3}} = \frac{2\sqrt{3}}{3}\, b$$

Así

$$r_0 = \frac{2\sqrt{3}}{3}\, b \sin\left[\frac{\sqrt{3}}{2}\frac{r_s}{b}\right]$$

$$r_1 = \frac{2\sqrt{3}}{3}\, b \sin\left[\frac{\sqrt{3}}{2}\frac{r_s}{b} + \frac{2\pi}{3}\right]$$

$$r_2 = \frac{2\sqrt{3}}{3}\, b \sin\left[\frac{\sqrt{3}}{2}\frac{r_s}{b} + \frac{4\pi}{3}\right]$$

recordemos:

$$\sin(a+b) = \sin a \cos b + \cos a \sin b.$$

$$\therefore \sin\left(\theta_0 + \frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}\cos\theta_0 - \frac{1}{2}\sin\theta_0$$

$$\therefore \sin\left(\theta_0 + \frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}\cos\theta_0 - \frac{1}{2}\sin\theta_0$$

Existe un valor crítico para el parámetro de impacto: $b = b_c$.

En efecto,

$$r_c^3 - b_c^2 r_c + b_c^2 r_s = 0 \qquad \left(r_c = \frac{3}{2} r_s\right)$$

$$\frac{27}{8} r_s^3 - b_c^2\left(\frac{3}{2} r_s - r_s\right) = 0$$

$$\frac{27}{\cancel{8}\,4} r_s^{\cancel{3}\,2} = b_c^2\left(\frac{\cancel{r_s}}{\cancel{2}}\right) \Rightarrow \boxed{b_c = \frac{3\sqrt{3}}{2} r_s}$$