

# Method of Brackets and Feynman diagrams evaluation

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## 1. Introduction

The aim of this work is to present an improved and more general technique related with NDIM (Negative Dimensional Integration Method) , [ 1], [ 2], [ 3], [ 4], [ 5], [ 6], which we have called Method of Brackets. This modification to NDIM was originally presented in [ 7] in the context of multiloops Feynman integrals. A complete description of the operational rules of the method, together with a variety of examples related with Feynman diagrams and a generalization to arbitrary integrals, was discussed in [ 8]. In Ref. [ 9] the usefulness of this technique in analytic continuation of trascendental functions was also discussed, specifically for hypergeometric functions of the form  ${}_qF_{q-1}$ . The method of brackets is a versatile and simple technique for evaluating Feynman diagrams up to a certain level of difficulty. The reason for this is the complexity of solutions, which are presented in terms of multiple hypergeometric series in the general case. The method of brackets is a heuristic method for the evaluation of definite integrals, whose great advantage is to reduce the evaluation of a large class of definite integrals to the solution of a linear system of equations.

Until now this technique does not have a rigorous mathematical proof. In this work we show that the method of brackets is a generalization of Ramanujan's master theorem (RMT). However, this theorem is not sufficient to explain mathematically the bracket's technique in complete form.

In our case the application of the method of brackets to Feynman diagrams requires the

Schwinger's parametric representation of diagram. Then through of a systematic procedure it is possible to obtain the analytical solution to this diagram as a sum of hypergeometric functions. In this work we present two simple examples describing, step by step the application of the method of brackets and the Ramanujan's master theorem.

## 2. General momentum representation and Schwinger's representation

An arbitrary diagram with  $L$  loops,  $N$  propagators and  $E$  independent external lines, has the following associated momentum integral in  $D$  dimensions in Minkowski space,

$$G = \int \frac{d^D q_1}{i\pi^{D/2}} \cdots \frac{d^D q_L}{i\pi^{D/2}} \times \frac{1}{(B_1^2 - m_1^2 + i0)^{\nu_1}} \cdots \frac{1}{(B_N^2 - m_N^2 + i0)^{\nu_N}}, \quad (1)$$

where we define (explicit or implicit):

- $B_j \rightarrow$  Momentum of the  $j$ -th ( $j = 1, \dots, N$ ) propagator or internal line. It is a linear combination of external momenta  $\{p\}$  and internal momenta  $\{q\}$ .
- $\nu_j \rightarrow$  Arbitrary indices ( $j = 1, \dots, N$ ).
- $p_k \rightarrow$  External momentum ( $k = 1, \dots, E$ ).
- $q_k \rightarrow$  Internal momentum ( $k = 1, \dots, L$ ).
- $m_j \rightarrow$  Mass associated to the  $j$ -th propagator.

In this case the corresponding Schwinger's parametric representation is given by the equa-

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tion

$$G = \frac{(-1)^{-\frac{LD}{2}}}{\prod_{j=1}^N \Gamma(\nu_j)} \int_0^\infty d\vec{x} \frac{\exp\left(\sum_{j=1}^N x_j m_j^2\right) \exp\left(-\frac{F}{U}\right)}{U^{\frac{D}{2}}}, \quad (2)$$

where  $d\vec{x} = \prod_{j=1}^N dx_j x_j^{\nu_j-1}$ ,  $N_\nu = \nu_1 + \dots + \nu_N$ ,  $U$  and  $F$  are polynomials  $L$ -linear and  $(L+1)$ -linear respectively in Schwinger's parameters. The polynomials  $U$  and  $F$  can be evaluated using the general formula [10]

$$F = \sum_{i,j=1}^E C_{ij} p_i \cdot p_j, \quad (3)$$

$$U = \begin{vmatrix} M_{11} & \cdots & M_{1L} \\ \vdots & & \vdots \\ M_{L1} & \cdots & M_{LL} \end{vmatrix},$$

where the coefficients  $C_{ij}$  are given for the following determinant

$$\begin{vmatrix} M_{11} & \cdots & M_{1L} & M_{1(L+j)} \\ \vdots & & \vdots & \vdots \\ M_{L1} & \cdots & M_{LL} & M_{L(L+j)} \\ M_{(L+i)1} & \cdots & M_{(L+i)L} & M_{(L+i)(L+j)} \end{vmatrix}. \quad (4)$$

The  $\mathbf{M}$  matrix, the parameter matrix, may be evaluated directly from topology of the diagram.

### 3. Rules in the Method of Brackets

In the following we show the fundamental rules of this technique. The technique of brackets transforms the parameter integral into series-like structure called : "brackets expansion". We need only four basic rules for obtaining such an expansion.

#### 3.1. Rule I : Exponential function expansion

To expand the exponential function, we use the "usual" way, this is

$$\exp(-xA) = \sum_n \frac{(-1)^n}{\Gamma(n+1)} x^n A^n, \quad (5)$$

if the argument of exponential function is  $\exp(xA)$ , we expand in this way

$$\exp(xA) = \sum_n \frac{(-1)^n}{\Gamma(n+1)} x^n (-A)^n. \quad (6)$$

The reason for this is to associate to each expansion the factor  $\phi_n = \frac{(-1)^n}{\Gamma(n+1)}$  as a simple convention.

#### 3.2. Rule II : Integration symbol and its equivalent bracket

This rule corresponds to the definition of the bracket symbol. The structure  $\int x^{a_1+a_2+\dots+a_n-1} dx$  is replaced by its respective bracket representation

$$\int x^{a_1+a_2+\dots+a_n-1} dx = \langle a_1 + a_2 + \dots + a_n \rangle. \quad (7)$$

#### 3.3. Rule III : Polynomials expansion

For polynomials we use the following representation in terms of series of brackets

$$(A_1 + \dots + A_r)^{\pm\mu} = \sum_{n_1} \dots \sum_{n_r} \phi_{n_1} \dots \phi_{n_r} (A_1)^{n_1} \dots (A_r)^{n_r} \times \frac{\langle \mp\mu + n_1 + \dots + n_r \rangle}{\Gamma(\mp\mu)}. \quad (8)$$

This rule is derived using rule (I) and (II) after applying the Schwinger's parametrization to this polynomial. An adequate way for expanding repeated polynomials in the integral is described in [7], the idea in this case is to minimize the complexity of the solution.

#### 3.4. Rule IV : Finding the solution

For the case of a generic series of brackets  $J$

$$J = \sum_{n_1} \dots \sum_{n_r} \phi_{n_1} \dots \phi_{n_r} F(n_1, \dots, n_r) \times \langle a_{11}n_1 + \dots + a_{1r}n_r + c_1 \rangle \dots \times \dots \langle a_{r1}n_1 + \dots + a_{rr}n_r + c_r \rangle, \quad (9)$$

the solution is obtained using the general formula

$$J = \frac{1}{|\det(\mathbf{A})|} \Gamma(-n_1^*) \dots \Gamma(-n_r^*) F(n_1^*, \dots, n_r^*) \quad (10)$$

where  $\det(\mathbf{A})$  is evaluated by the following expression

$$\det(\mathbf{A}) = \begin{vmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rr} \end{vmatrix}, \quad (11)$$

and  $\{n_i^*\}$  ( $i = 1, \dots, r$ ) is the solution of the linear system obtained by the vanishing of the brackets

$$\begin{cases} a_{11}n_1 + \dots + a_{1r}n_r = -c_1 \\ \vdots \\ a_{r1}n_1 + \dots + a_{rr}n_r = -c_r. \end{cases} \quad (12)$$

The value of  $J$  is not defined if the matrix  $\mathbf{A}$  is not invertible.

**Note :** In the case where a higher dimensional series has more summation indices than brackets, the appropriate number of free variables is chosen among the indices. For each such choice, Rule IV yields a series. Those converging in a common region are added to evaluate the desired integral.

In the evaluation of these formal sums, the index  $n \in N$  will be replaced by a number  $n^*$  defined by the vanishing of the bracket. Observe that it is possible that  $n^* \in \mathbb{C}$ . For book-keeping purposes, specially in cases with many indices, we write  $\sum_n$  instead of the usual  $\sum_{n=0}^{\infty}$ . After that the brackets are eliminated, those indices that remain recover their original nature.

Some simple examples and their respective expansions in brackets:

- For binomial expression

$$\frac{1}{(A-B)^\beta} = \sum_{n_1} \sum_{n_2} \phi_{n_1, n_2} A^{n_1} (-B)^{n_2} \times \frac{\langle \beta + n_1 + n_2 \rangle}{\Gamma(\beta)}.$$

- For integral

$$\int_0^\infty dx \frac{x^{\alpha-1}}{\exp(Ax)} = \sum_n \phi_n A^n \langle \alpha + n \rangle.$$

#### 4. Ramanujan's Master Theorem (RMT)

In the following we describe the Ramanujan's master theorem and its relation with the method of brackets. The theorem says that for an integral  $J = \int_0^\infty dx_1 x^{\nu-1} f(x)$ , where, we suppose that  $f(x)$  admits a Taylor expansion of the form

$$f(x) = \sum_k F(k) \frac{(-x)^k}{k!} \quad (13)$$

in a neighborhood of  $x = 0$  and  $f(0) = F(0) \neq 0$ , then the solution is given by

$$J = \int_0^\infty dx x^{\nu-1} f(x) = \Gamma(\nu) F(-\nu). \quad (14)$$

This integral corresponds to the Mellin transform of  $f(x)$ . The condition  $F(0) \neq 0$  guarantees the convergence of integral in (14) near  $x = 0$ , for  $\nu > 0$ . This theorem was demonstrated by Hardy [11]. We present a generalization to this theorem when it is applied to the multidimensional integral

$$J = \int_0^\infty dx_1 x_1^{\nu_1-1} \dots \int_0^\infty dx_N x_N^{\nu_N-1} \times f(x_1, \dots, x_N). \quad (15)$$

If  $f(x_1, \dots, x_N)$  is expressible in the form of multidimensional Taylor series as follows

$$f(x_1, \dots, x_N) = \sum_{l_1=0}^{\infty} \dots \sum_{l_N=0}^{\infty} \frac{(-1)^{l_1}}{l_1!} \dots \frac{(-1)^{l_N}}{l_N!} F(l_1, \dots, l_N) \times x_1^{a_{11}l_1 + \dots + a_{1N}l_N + b_1} \dots x_N^{a_{N1}l_1 + \dots + a_{NN}l_N + b_N}$$

then we obtain the expression

$$J = \int_0^\infty \frac{dx_1}{x_1} \dots \int_0^\infty \frac{dx_N}{x_N} \sum_{l_1=0}^{\infty} \dots \sum_{l_N=0}^{\infty} \frac{(-1)^{l_1}}{l_1!} \dots \frac{(-1)^{l_N}}{l_N!} F(l_1, \dots, l_N) \times x_1^{a_{11}l_1 + \dots + a_{1N}l_N + \tilde{b}_1} \dots x_N^{a_{N1}l_1 + \dots + a_{NN}l_N + \tilde{b}_N}$$

being  $\tilde{b}_i = \nu_i + b_i$  ( $i = 1, \dots, N$ ). After applying sistematically Ramanujan's master theorem to the integral, we find by method of induction, the general solution for this integral

$$J = \frac{1}{|\det(\mathbf{A})|} \Gamma(l_1^*) \dots \Gamma(l_N^*) F(-l_1^*, \dots, -l_N^*) \quad (16)$$

where  $\det(\mathbf{A})$  is evaluated by the formula

$$\det(\mathbf{A}) = \begin{vmatrix} a_{11} & \dots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \dots & a_{NN} \end{vmatrix}, \quad (17)$$

and the variables  $l_i^*$  ( $i = 1, \dots, N$ ) are solutions of the following linear system

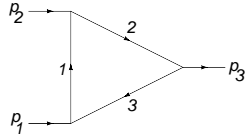
$$\begin{cases} a_{11}l_1 + \dots + a_{1N}l_N = \tilde{b}_1 \\ \vdots \\ a_{N1}l_1 + \dots + a_{NN}l_N = \tilde{b}_N. \end{cases} \quad (18)$$

We have obtained Ramanujan's Master Theorem Generalized (RMTG). This general formula (16) is equivalent to the general formula obtained with the method of brackets for integral  $J$  (10). This result justifies mathematically the method of brackets as a valid method for evaluating multidimensional integrals. Many examples are discussed in references [ 8, 9].

## 5. Applications : Two detailed examples

### 5.1. Using Method of Brackets : Triangle diagram

In the following we discuss two examples. The first example using the method of brackets and a second example using RMTG. We start with the evaluation of the following Feynman diagram


(19)

The momentum integral for this graph is

$$G = \int \frac{d^D q}{i\pi^{D/2}} \frac{1}{((p_1+q)^2 - m_1^2)^{a_1}} \times \frac{1}{((p_1+p_2+q)^2 - m_2^2)^{a_2} (q^2 - m_3^2)^{a_3}}. \quad (20)$$

We also define the set  $\{a_1, \dots, a_N\}$  as the set of powers of the propagators, which in general can take arbitrary values. The Schwinger's parametric representation of the Eq. (20) is the following integral

$$G = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^\infty d\vec{x} \exp\left(\sum_{k=1}^3 x_k m_k^2\right) \times \frac{\exp\left(-\frac{C_{11}x_1^2 + 2C_{12}x_1x_2 + C_{22}x_2^2}{U}\right)}{U^{\frac{D}{2}}}, \quad (21)$$

where  $d\vec{x} = \prod_{j=1}^3 x_j^{a_j-1} dx_j$ . The polynomials  $U = x_1 + x_2 + x_3$  and  $C_{ij}$  are given by the following equations

$$\begin{aligned} C_{11} &= x_3(x_1 + x_2), \\ C_{12} &= x_2x_3, \\ C_{22} &= x_2(x_1 + x_3). \end{aligned} \quad (22)$$

Inserting this in (21) and remembering that  $p_3^2 = (p_1 + p_2)^2 = p_1^2 + 2p_1 \cdot p_2 + p_2^2$ , after a little algebra, we get the Schwinger's parametric representation of (20)

$$G = \frac{(-1)^{-D/2}}{\prod_{j=1}^3 \Gamma(a_j)} \int_0^\infty d\vec{x} \exp\left(\sum_{k=1}^3 x_k m_k^2\right) \times \frac{\exp\left(-\frac{x_3x_1}{U}p_1^2\right) \exp\left(-\frac{x_1x_2}{U}p_2^2\right) \exp\left(-\frac{x_2x_3}{U}p_3^2\right)}{U^{D/2}}. \quad (23)$$

Now, we solve the diagram with the following conditions (for simplicity) :  $m_1 = m_2 = 0$ ,  $m_3 = M$ ,  $p_1^2 = p_2^2 = 0$  and  $p_3 = Q$ , then, we obtain the following integral for this case

$$G = \frac{(-1)^{-D/2}}{\prod_{j=1}^3 \Gamma(a_j)} \int_0^\infty d\vec{x} \exp(x_3 M^2) \times \frac{\exp\left(-\frac{x_2x_3}{U}Q^2\right)}{U^{D/2}}. \quad (24)$$

In the following we obtain the expansion of brackets (step by step). First, we expand the exponential functions using rule (I)

- $\exp(x_3 M^2) = \sum_{n_1} \phi_{n_1} (-M^2)^{n_1} x_3^{n_1}.$
- $\exp\left(-\frac{x_2x_3}{U}Q^2\right) = \sum_{n_2} \phi_{n_2} (Q^2)^{n_2} \frac{x_2^{n_2} x_3^{n_2}}{U^{n_2}}.$

then, we obtain the integral

$$G = \frac{(-1)^{-D/2}}{\prod_{j=1}^3 \Gamma(a_j)} \sum_{n_1, n_2} \phi_{n_1, n_2} (Q^2)^{n_1} \times (M^2)^{n_2} \int_0^\infty d\vec{x} \frac{x_2^{n_2} x_3^{n_1+n_2}}{U^{\frac{D}{2}+n_2}}, \quad (25)$$

now, we expand the polynomial  $U = (x_1 + x_2 + x_3)$  using rule (III)

$$\frac{1}{(x_1 + x_2 + x_3)^{\frac{D}{2} + n_2}} = \sum_{n_3, \dots, n_5} \phi_{n_3, \dots, n_5} \times x_1^{n_3} x_2^{n_4} x_3^{n_5} \frac{\langle \frac{D}{2} + n_2 + n_3 + n_4 + n_5 \rangle}{\Gamma(\frac{D}{2} + n_2)}, \quad (26)$$

then using rule (II) and a little algebra allows us to find the expansion of brackets associated to the integral (24)

$$G = \frac{(-1)^{-D/2}}{\prod_{j=1}^3 \Gamma(a_j)} \sum_{n_1, \dots, n_5} \phi_{n_1, \dots, n_5} (-M^2)^{n_1} \times (Q^2)^{n_2} \frac{\prod_{j=1}^4 \Delta_j}{\Gamma(\frac{D}{2} + n_2)}, \quad (27)$$

where the symbols  $\{\Delta_j\}$  represent the brackets

$$\begin{cases} \Delta_1 = \langle \frac{D}{2} + n_2 + n_3 + n_4 + n_5 \rangle, \\ \Delta_2 = \langle a_1 + n_3 \rangle, \\ \Delta_3 = \langle a_2 + n_2 + n_4 \rangle, \\ \Delta_4 = \langle a_3 + n_1 + n_2 + n_5 \rangle. \end{cases} \quad (28)$$

Using rule (IV) we find finally the solution to (24). In this case exists two kinematical regions :  $\left| \frac{M^2}{Q^2} \right| < 1$  and  $\left| \frac{Q^2}{M^2} \right| < 1$ . The solution in the region  $\left| \frac{M^2}{Q^2} \right| < 1$  is given by the following expression

$$G\left(\frac{M^2}{Q^2}\right) = (-1)^{\frac{D}{2}} (Q^2)^{\frac{D}{2} - a_{123}} \times \frac{\Gamma(\frac{D}{2} - a_{12}) \Gamma(\frac{D}{2} - a_{13}) \Gamma(a_{123} - \frac{D}{2})}{\Gamma(a_2) \Gamma(a_3) \Gamma(D - a_{123})} \times {}_2F_1\left(\begin{matrix} 1 + a_{123} - D, & a_{123} - \frac{D}{2} \\ 1 + a_{13} - \frac{D}{2} \end{matrix} \middle| \frac{M^2}{Q^2}\right). \quad (29)$$

The solution for the region  $\left| \frac{Q^2}{M^2} \right| < 1$  are obtained by analytic continuation of (29).

## 5.2. Using RMTG : Massless bubble diagram

We will solve the following diagram



$$(30)$$

the Schwinger's parametric representation is given by the expression

$$G = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1) \Gamma(a_2)} \int_0^\infty \int_0^\infty dx dy x^{a_1-1} y^{a_2-1} \times \frac{\exp(-\frac{xy}{x+y} p^2)}{(x+y)^{\frac{D}{2}}}. \quad (31)$$

Now, we expand the integrand using conventional mathematics

$$\exp\left(-\frac{xy}{x+y} p^2\right) = \sum_{n=0}^\infty \frac{(-1)^n}{n!} (p^2)^n \frac{x^n y^n}{(x+y)^n}, \quad (32)$$

resulting the following integral

$$G = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1) \Gamma(a_2)} \int_0^\infty \int_0^\infty dx dy x^{a_1-1} y^{a_2-1} \times \sum_{n=0}^\infty \frac{(-1)^n}{n!} (p^2)^n \frac{x^n y^n}{(x+y)^{\frac{D}{2}+n}}, \quad (33)$$

then, we expand the binomial in the denominator

$$\frac{1}{(x+y)^{\frac{D}{2}+n}} = \sum_{k=0}^\infty \frac{(-1)^k}{k!} \left(\frac{D}{2} + n\right)_k x^{-\frac{D}{2}-n-k} y^k, \quad (34)$$

replacing in (33) and doing a change of variables :  $x \rightarrow \frac{1}{x}$ , we obtain finally the optimal structure for applying RMTG, this is

$$G = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1) \Gamma(a_2)} \int_0^\infty \int_0^\infty dx dy x^{-a_1 + \frac{D}{2} - 1} y^{a_2 - 1} \times \sum_{k=0}^\infty \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{(-1)^k}{k!} (p^2)^n \times \left(\frac{D}{2} + n\right)_k x^k y^{k+n}, \quad (35)$$

allows us to obtain the solution of the diagram (30)

$$G = (-1)^{-\frac{D}{2}} (p^2)^{\frac{D}{2} - a_1 - a_2} \times \frac{\Gamma(a_1 + a_2 - \frac{D}{2}) \Gamma(\frac{D}{2} - a_1) \Gamma(\frac{D}{2} - a_2)}{\Gamma(a_1) \Gamma(a_2) \Gamma(D - a_1 - a_2)} \quad (36)$$

## 6. Conclusions

The method of brackets has been presented as an competitive alternative compared with other advanced techniques for evaluating Feynman diagrams. The main advantage of this technique is that it is systematic and it does not require advanced mathematical tools, just linear algebra. Although, this technique is not fully explained by RMTG, the results obtained by method of brackets are identical to the obtained with RMTG when the number of the summation indices is the same as the number of the brackets. If the number of the brackets is less than the number of the summation indices, RMTG is not useful to explain the obtained results by applying method of brackets, although these results are correct. We are currently working on studies to fully validate this technique through the mathematical point of view.

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