Legendre Polynomials: a Simple Methodology

View the article online for updates and enhancements.

IOP ebooks™

Bringing you innovative digital publishing with leading voices to create your essential collection of books in STEM research.

Start exploring the collection - download the first chapter of every title for free.

LEGENDRE POLYNOMIALS: A SIMPLE METHODOLOGY

V. Aboites [1]

[1]Centro de Investigaciones en Óptica, Loma del Bosque 115, 37150 León, México

[1]aboites@cio.mx

Abstract. Legendre polynomials are obtained through well-known linear algebra methods based on Sturm-Liouville theory. A matrix corresponding to the Legendre differential operator is found and its eigenvalues are obtained. The elements of the eigenvectors obtained correspond to the Legendre polynomials. This method contrast in simplicity with standard methods based on solving Legendre differential equation by power series, using the Legendre generating function, using the Rodriguez formula for Legendre polynomials, or by a contour integral.

1. Introduction

Legendre polynomials, also known as spherical harmonics or zonal harmonics, were first introduced in 1782 by Adrien-Marie Legendre and are frequently encountered in physics and other technical fields; for instance, the coefficients in the expansion of the Newtonian potential that gives the gravitational potential associated to a point mass or the Coulomb potential associated to a point charge, the gravitational and electrostatic potential inside a spherical shell, steady-state heat conduction problems in spherical problems inside a homogeneous solid sphere, and so forth [1].

Legendre polynomials are solutions of an ordinary differential equation (ODE) which is a hypergeometric equation. In particular, the Legendre ordinary differential equation appear when solving Laplace's equation (and related partial differential equations) in spherical coordinates and also when solving the Schrödinger equation in three dimensions for a central force [2]. A hypergeometric equation may be written as:

$$s(x)F''(x) + t(x)F'(x) + \lambda F(x) = 0 \qquad (1)$$

Where $F(x)$ is a real function of a real variable $F: U \to \mathbb{R}$, where $U \subset \mathbb{R}$ is an open subset of the real line, and $\lambda \in \mathbb{R}$ a corresponding eigenvalue, and the functions $s(x)$ and $t(x)$ are real polynomials of at most second order and first order, respectively.

There are different cases obtained depending on the kind of the $s(x)$ function in eq. (1). When $s(x)$ is a constant, eq. (1) takes the form $F''(x) - 2\alpha x F'(x) + \lambda F(x) = 0$, and if $\alpha = 1$ one obtains the Hermite Polynomials. When $s(x)$ is a polynomial of first degree, eq. (1) takes the form $xF''(x) + (-\alpha x + \beta + 1)F'(x) + \lambda F(x) = 0$, and when $\alpha = 1$ and $\beta = 0$ one obtains the Laguerre Polynomials. There are three different cases when $s(x)$ is a polynomial of second degree. When the second degree polynomial has two different real roots, eq. (1) takes the form $(1 - x^2)F''(x) + [\beta - \alpha - (\alpha + \beta + 2)x]F'(x) + \lambda F(x) = 0$ and that is the Jacobi equation, for different values of α and β one obtains particular cases of polynomials; Gegenbauer Polynomials if $\alpha = \beta$; Tchebycheff I and II if $\alpha = \beta = \pm 1/2$; Legendre Polynomials if $\alpha = \beta = 0$. When the second degree polynomial has one double root, eq. (1) takes the form $x^2 F''(x) + [(\alpha + 2)x + \beta]F'(x) + \lambda F(x) = 0$, and when $\alpha = -1$ and $\beta = 0$ one obtains the Bessel Polynomials. At last, when the second degree polynomial has two complex roots, eq.(1) takes the form $(1 + x)^2 F''(x) + (2\beta x + \alpha)F'(x) + \lambda F(x) = 0$, and that is the Romanovski equation [3].

The Sturm-Liouville Theory is covered in most advanced physics and engineering courses. In this context an eigenvalue equation sometimes takes the more general self-adjoint form: $\mathcal{L}u(x) + \lambda w(x)u(x) = 0$, where $\mathcal{L}$ is a differential operator; $\mathcal{L}u(x) = \frac{d}{dx}\left[p(x)\frac{du(x)}{dx}\right] + q(x)u(x)$, λ an eigenvalue, and $w(x)$ is known as a weight or density function. The analysis of this equation and its solutions is called Sturm-Liouville theory. Specific forms of $p(x)$, $q(x)$, λ and $w(x)$, are given for Legendre, Laguerre, Hermite and other well-known equations in the given references. There, it is also shown the close analogy of this theory with linear algebra concepts. For example, functions here take the role of vectors there, and linear operators here take that of matrices there. Finally, the diagonalization of a real symmetric matrix corresponds to the solution of an ordinary differential equation, defined by a self-adjoint operator $\mathcal{L}$, in terms of its eigenfunctions which are the "continuous" analog of the eigenvectors [3,4].

Legendre polynomials are studied in most science and engineering mathematics courses, mainly in those courses focused on differential equations or special functions. These polynomials are typically obtained as a result of the solution of Legendre differential equation by power series. Usually it is also shown that they can be obtained by a generating function and also by Rodriguez formula for Legendre polynomial. Finally they can also be defined as a contour integral. Most mathematics courses also include a study of the properties of these polynomials such as: orthogonality, completeness, recursion relations, special values, asymptotic expansions and relation to other polynomials and hypergeometric functions [2, 4]. There is no doubt that this is a demanding subject that requires a great deal of attention from most students.

In this paper Legendre polynomials are obtained using basic concepts of linear algebra (which most students are already familiar with) and which contrasts in simplicity with the standard methods as those described in the previously outlined syllabus. In the next section the Legendre differential operator matrix is obtained as well as its eigenvalues and eigenvectors. From the eigenvectors found, the Legendre Polynomials follow. The method

here presented has been applied to other polynomials such as Laguerre, Hermite, Tchebycheff and Gegenbauer [4, 5, 6, 7]

2. Legendre polynomials though matrix algebra

The general algebraic polynomial of degree n with $a_o, a_1, ..., a_n \in \mathbb{R}$,

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots a_nx^n \tag{2}$$

Is represented by the vector:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} \tag{3}$$

The first and second derivative of polynomial (2) are:

$$\frac{d}{dx} \rightarrow \frac{d}{dx}[a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots a_nx^n] =$$
$$= a_1 + 2a_2x + 3a_3x^2 + \cdots na_nx^{n-1} \tag{4}$$

$$\frac{d^2}{dx^2} \rightarrow \frac{d}{dx}[a_1 + 2a_2x + 3a_3x^2 + \cdots na_nx^{n-1}] =$$
$$= 2a_2 + 6a_3x + \cdots n(n-1)a_nx^{n-2} \tag{5}$$

The matrix representation of the first and second derivative operators is:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & .. & 0 & 0 \\ 0 & 0 & 2 & 0 & .. & 0 & 0 \\ 0 & 0 & 0 & 3 & .. & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & .. & 0 & n \\ 0 & 0 & 0 & 0 & .. & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ .. \\ a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ .. \\ na_n \\ 0 \end{bmatrix} \tag{6}$$

Or:

$$\frac{d}{dx} \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 & .. & 0 & 0 \\ 0 & 0 & 2 & 0 & .. & 0 & 0 \\ 0 & 0 & 0 & 3 & .. & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & .. & 0 & n \\ 0 & 0 & 0 & 0 & .. & 0 & 0 \end{bmatrix} \tag{7}$$

And:

$$\begin{bmatrix} 0 & 0 & 2 & 0 & .. & 0 & 0 \\ 0 & 0 & 0 & 6 & .. & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & .. & 0 & n(n-1) \\ 0 & 0 & 0 & 0 & .. & 0 & 0 \\ 0 & 0 & 0 & 0 & .. & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ .. \\ a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} 2a_1 \\ 6a_3 \\ 12a_4 \\ .. \\ n(n-1) \\ 0 \\ 0 \end{bmatrix} \tag{8}$$

Or:

$$\frac{d^2}{dx^2} \rightarrow \begin{bmatrix} 0 & 0 & 2 & 0 & .. & 0 & 0 \\ 0 & 0 & 0 & 6 & .. & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & .. & 0 & n(n-1) \\ 0 & 0 & 0 & 0 & .. & 0 & 0 \\ 0 & 0 & 0 & 0 & .. & 0 & 0 \end{bmatrix} \tag{9}$$

Substituting (7) and (9) in the Legendre differential operator $(1 - x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx}$, the matrix representation of the Legendre operator is obtained:

$$
\begin{bmatrix}
0 & 0 & 2 & 0 & 0 & 0 & .. & 0 \\
0 & -2 & 0 & 6 & 0 & 0 & .. & 0 \\
0 & 0 & -6 & 0 & 12 & 0 & .. & 0 \\
0 & 0 & 0 & -12 & 0 & 20 & .. & 0 \\
0 & 0 & 0 & 0 & -20 & 0 & .. & 0 \\
.. & .. & .. & .. & .. & .. & .. & .. \\
0 & 0 & 0 & 0 & 0 & 0 & .. & n(n-1) \\
0 & 0 & 0 & 0 & 0 & 0 & .. & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & .. & -n(n-1)-2n
\end{bmatrix}
\begin{bmatrix}
a_0 \\ a_1 \\ a_2 \\ a_3 \\ .. \\ a_{n-2} \\ a_{n-1} \\ a_n
\end{bmatrix}
=
\begin{bmatrix}
2a_2 \\
-2a_1 + 6a_3 \\
-6a_2 + 12a_4 \\
-12a_3 + 20a_6 \\
.. \\
n(n-1)a_n \\
0 \\
[-n(n-1)-2n]a_n
\end{bmatrix}
\tag{10}
$$

Therefore:

$$
\left(1-x^2\right)\frac{d^2}{dx^2} - 2x\frac{d}{dx} \rightarrow
\begin{bmatrix}
0 & 0 & 2 & 0 & 0 & 0 & .. & 0 \\
0 & -2 & 0 & 6 & 0 & 0 & .. & 0 \\
0 & 0 & -6 & 0 & 12 & 0 & .. & 0 \\
0 & 0 & 0 & -12 & 0 & 20 & .. & 0 \\
0 & 0 & 0 & 0 & -20 & 0 & .. & 0 \\
.. & .. & .. & .. & .. & .. & .. & .. \\
0 & 0 & 0 & 0 & 0 & 0 & .. & n(n-1) \\
0 & 0 & 0 & 0 & 0 & 0 & .. & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & .. & -n(n-1)-2n
\end{bmatrix}
\tag{11}
$$

As an example and for the sake of simplicity, the Legendre differential operator as a 4x4 matrix is represented by the following matrix:

$$
\begin{bmatrix}
0 & 0 & 2 & 0 \\
0 & -2 & 0 & 6 \\
0 & 0 & -6 & 0 \\
0 & 0 & 0 & -12
\end{bmatrix}
\tag{12}
$$

As it is known the eigenvalues of a matrix M are the values that satisfy the equation Det(M $-\lambda$I) = 0. Since the matrix (12) is a triangular matrix, the eigenvalues λ_i of this matrix are the elements of the diagonal, namely: $\lambda_1 = 0$, $\lambda_2 = -2$, $\lambda_3 = -6$, $\lambda_4 = -12$. The corresponding eigenvectors are the solutions of the equation (M $-\lambda_i$I)•v = 0, where the eigenvector $v = [a_0, a_1, a_2, a_3]^T$.

$$
\begin{bmatrix}
0-\lambda_i & 0 & 2 & 0 \\
0 & -2-\lambda_i & 0 & 6 \\
0 & 0 & -6-\lambda_i & 0 \\
0 & 0 & 0 & -12-\lambda_i
\end{bmatrix}
\begin{bmatrix}
a_0 \\ a_1 \\ a_2 \\ a_3
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0
\end{bmatrix}
\tag{13}
$$

Substituting the eigenvalue λ_1 in the equation (13), the eigenvector v_1 is obtained: $v_1 = [1,0,0,0]^T$. Which corresponds to Legendre polynomial $P_0(x) = 1$. Substituting the eigenvalue λ_2 in the equation (13), the eigenvector v_2 is obtained: $v_2 = [0,1,0,0]^T$. Which corresponds to Legendre polynomial $P_1(x) = x$. Substituting the eigenvalue λ_3 in the equation (13), the eigenvector v_3 is obtained: $v_3 = [1,0,-3,0]^T$. Which corresponds to Legendre polynomial $P_2(x) = (3x^2 - 1)/2$. Substituting the eigenvalue λ_4 in the equation (13), the eigenvector v_3 is obtained: $v_3 = [0,3,0,-5]^T$. Which corresponds to Legendre polynomial $P_3(x) = (5x^3 - 3x)/2$. For any value $n = 0, 1, 2, 3\ldots$, Rodriguez formula follows:

$$P_n = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \qquad (14)$$

3. Conclusion

Legendre polynomials are obtained in a simple and straightforward way using basic linear algebra concepts such the eigenvalue and eigenvector of a matrix. Once the corresponding matrix of the Legendre differential operator is obtained, the eigenvalues of this matrix are found and the elements of its eigenvectors correspond to the coefficients of Legendre Polynomials. This method contrast in simplicity with standard methods based on solving Legendre differential equation by power series, using the Legendre generating function, using the Rodriguez formula for Legendre polynomials, or by a contour integral.

4. References

[1] Jackson, John David, Classical Electrodynamics, New York: Wiley, (1999).
[2] L. C. Andrews, Special functions of mathematics for engineers, SPIE, USA, (1998).
[3] A.P. Raposo et al., Romanovski polynomials in selected physics problems, Central European Journal of Physics, **5**(3), (2007), 253–284
[4] V. Aboites, Laguerre Polynomials and linear algebra, Memorias Sociedad Matemática Mexicana **52** (2017), 3-13.
[5] V. Aboites, Hermite polynomials through linear algebra, International Journal of Pure and Applied Mathematics **114** (2017), 401-406.
[6] V. Aboites, Easy Route to Tchebycheff Polynomials, Rev. Mex. Fis., (2018) (In Press)
[7] M. Ramírez, Simple Approach to Gegenbauer Polynomials, International Journal of Pure and Applied Mathematics, (2018) Ref. AP2017-31-5266, (In press)