Modular application of an integration by fractional expansion method to multiloop Feynman diagrams

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We present an alternative technique for evaluating multiloop Feynman diagrams, using the integration by fractional expansion method. Here we consider generic diagrams that contain propagators with radiative corrections which topologically correspond to recursive constructions of bubble type diagrams. The main idea is to reduce these subgraphs, replacing them by their equivalent multiregion expansion. One of the main advantages of this integration technique is that it allows one to reduce massive cases with the same degree of difficulty as in the massless case.

I. INTRODUCTION

The evaluation of multiloop Feynman diagrams is currently one of the most important problems in quantum field theory. Thanks to the development of analytical techniques in high order perturbative calculations, it has become possible to compare precision experimental measurements with the theoretical models that try to explain them. There are several of these techniques, and some of the best known can even be found in textbooks [1,2]. One that is not particularly used but which nevertheless has advantages for the evaluation of some complicated Feynman diagrams is called the negative dimension integration method (NDIM). The basic foundations of NDIM were initially suggested in the work of Halliday and Ricotta [3] which, using the dimensional regularization prescription ($D = 4 - 2\epsilon$), makes an analytical continuation of the dimension D into negative values, something which can be done since the Feynman integrals are in fact analytic in arbitrary dimension D. In a previous work [4] we proposed that a more appropriate name for this technique should be integration by fractional expansion (IBFE), representing better its mathematical and physical basis.

The purpose of the IBFE technique is to transform the Schwinger integral parametric representation of a specific Feynman diagram into an equivalent mathematical structure, which contains several summations and Kronecker deltas and which we call multiregion expansion (MRE) of the diagram. This name comes from the fact that this particular expansion is made around the values zero and infinity simultaneously, and that once the expression is summed using the Kronecker deltas, these expansions become explicitly separated. In general the number of summations is bigger than the number of deltas that are

generated, and in this case there are several different ways of summing using the Kronecker deltas: in fact there are $C_\delta^\sigma = \frac{\sigma!}{\delta!(\sigma-\delta)!}$ different forms, where δ is the number of Kronecker deltas and σ the number of sums. Each one of these forms gives rise in general to generalized and multi-variable hypergeometric functions.

Several authors have used and formalized this method, with a wide variety of applications to mainly 1-loop [5–9] and 2-loop [10–14] diagrams. In a previous work we presented an optimization of this technique [4], which allows one to evaluate more complex diagrams and even certain families of L-loop diagrams. In this work we will show that is possible to improve the efficiency of the IBFE method when it is applied in a modular form to a diagram, that is applying it iteratively to subgraphs or modules that contain one or more contiguous loops of the same diagram. The procedure consists in replacing the Schwinger integral parametric representation of each module by its corresponding MRE, and simultaneously the diagram is reduced topologically into simpler graphs. This reduction allows one to get the MRE of the complete diagram, as a product of functions that we have called n-loop functions (associated to n-loop subgraphs), each of them being an MRE related to one of the modules that are present in the diagram. Since each module can have different masses distributed in different ways in its propagators, there exist several n-loop functions depending on the particular configuration of masses in the propagators.

The optimization that is reached when applying the IBFE method in this way to a diagram consists in a reduction of the obtained MRE with respect to the one that is reached when the diagram is considered with all its loops simultaneously. In order to explain the technique in detail, we start with the analysis of the simplest topology that can constitute a subgraph of a Feynman diagram: the 1-loop module (subgraph), the bubble diagram, which is going to be the basis that will be used to derive 1-loop

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functions, useful for evaluating a variety of Feynman diagrams.

The remainder of the paper is organized as follows. In Sec. II, we describe how to easily obtain the Schwinger parametric representation associated with a Feynman diagram, whose mathematical structure is the starting point for the application of the IBFE technique. Here we present the basic elements of the integration method, and a more rigorous extension is given in the Appendix. In Sec. III we briefly review the conventional way to deal with this type of diagram, which is loop by loop. This will provide a direct test for the method proposed here. Later, in Secs. IV and V we define and deduce the 1-loop functions of the bubble module, for all the possible mass and momentum configurations, which are a total of eight functions: G_k and $\bar{G}_k(k = A, B, C, D)$. Finally, in Sec. VI we develop several applications of the proposed modular procedure, which will be compared with its equivalent MRE obtained from an application of the IBFE technique considering all loops at the same time.

II. MATHEMATICAL FORMALISM

In what follows we will briefly describe the algebraic elements that are needed to understand the technique.

A. Schwinger's parametric representation

The IBFE integration technique is applied directly to the Schwinger parametric representation of a diagram, and therefore it is important to have a simple algorithm to obtain this representation, which is what we will do in this subsection.

Let us consider a generic topology G, which represents a Feynman diagram in a scalar theory, and let us suppose that the graph has: N propagators, each one associated with the masses $\{m_1, \ldots, m_N\}$; L loops, associated with the independent internal momenta $\{q_1, \ldots, q_L\}$); and E independent external momenta $\{p_1, \ldots, p_E\}$.

Using the dimensional regularization prescription we can write the momentum space integral expression that represents the diagram in $D = 4 - 2\epsilon$ dimensions:

$$G = \int \frac{d^D q_1}{i\pi^{D/2}} \cdots \frac{d^D q_L}{i\pi^{D/2}}$$
$$\times \frac{1}{(B_1^2 - m_1^2 + i0)^{\nu_1}} \cdots \frac{1}{(B_N^2 - m_N^2 + i0)^{\nu_N}}. \quad (1)$$

In this expression the B_j symbol represents the momentum of the j propagator, whose dependence in general is given as a linear combination of external and internal momenta. We also define $\{\nu_1, \ldots, \nu_N\}$ as the set of propagator indices or powers, and which are considered to have arbitrary values. After introducing Schwinger's parametrization it is possible to evaluate the momentum integrals as Gaussian integrals, and the result of this operation is Schwinger's parametric representation of (1), which in the general case

is going to be given by an expression of the form

$$G = \frac{(-1)^{LD/2}}{\prod_{j=1}^{N} \Gamma(\nu_j)} \int_0^\infty d\vec{x} \frac{\exp(\sum_{j=1}^{N} x_j m_j^2) \exp(-\frac{F}{U})}{U^{D/2}}. \quad (2)$$

For simplicity we have introduced the notation $d\vec{x} = dx_1 \ldots dx_N \prod_{j=1}^{N} x_j^{\nu_j - 1}$, where F is defined as

$$F = \sum_{i,j=1}^{E} C_{ij} p_i \cdot p_j. \quad (3)$$

The function U and the coefficients C_{ij} are L-*lineal* and $(L + 1)$-*lineal* homogeneous polynomials (Symanzik or Kirchhoff polynomials [15]), respectively, in the Schwinger parameters. Both U and C_{ij} correspond to determinants related to a matrix that we have called matrix of parameters **M** [16]. An important characteristic of the coefficients C_{ij} is that they are symmetric ($C_{ij} = C_{ji}$) which is due to the commutativity of the internal product between the independent external momenta associated with the graph G. These determinants are defined through the following expressions:

$$U = \begin{vmatrix} M_{11} & \cdots & M_{1L} \\ \vdots & & \vdots \\ M_{L1} & \cdots & M_{LL} \end{vmatrix}, \quad (4)$$

$$C_{ij} = \begin{vmatrix} M_{11} & \cdots & M_{1L} & M_{1(L+j)} \\ \vdots & & \vdots & \vdots \\ M_{L1} & \cdots & M_{LL} & M_{L(L+j)} \\ M_{(L+i)1} & \cdots & M_{(L+i)L} & M_{(L+i)(L+j)} \end{vmatrix}. \quad (5)$$

As can be seen, the determinants are associated with submatrices of the matrix of parameters **M**, which is symmetric and of dimension $(L + E) \times (L + E)$. The actual form of this matrix can be easily obtained when (1) is parametrized and the internal products of all the (internal and external) momenta associated with G are expanded, arriving at a quadratic form. The coefficients of such an expansion correspond to the matrix elements M_{ij}. For a better understanding of this process let us define for convenience the momentum

$$Q_j = \begin{cases} q_j & \text{if } L \geq j \geq 1 \\ p_{j-L} & \text{if } (L + E) \geq j > L, \end{cases} \quad (6)$$

with which one can build the $(L + E)$-*vector* $\mathbf{Q} = [Q_1 Q_2 \ldots Q_{(L+E)}]^\mathbf{t}$. Using this definition the following matrix structure is generated in the integral, after the parametrization application and before the loop momenta integration:

$$G = \frac{1}{\prod_{j=1}^{N} \Gamma(\nu_j)} \int_0^\infty d\vec{x} \exp\left(\sum_{j=1}^{N} x_j m_j^2\right) \int \prod_{j=1}^{L} \left(\frac{d^D Q_j}{i\pi^{D/2}}\right)$$

$$\times \exp\left(-\sum_{i=1}^{L+E} \sum_{j=1}^{L+E} Q_i M_{ij} Q_j\right), \qquad (7)$$

and from which we can identify the symmetric matrix **M**.

B. Foundations of the integration method IBFE

1. Notation and fundamental formulae

This technique can be introduced directly by considering the integral expression for the gamma function. The idea is to obtain operational rules, which will allow us later on to work with generalized complicated structures of this type of integral, such as Schwinger's parametric representation of a generic Feynman diagram (2). Let us analyze the following integral structure:

$$\frac{1}{A^\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty dx\, x^{\beta-1} \exp(-Ax), \qquad (8)$$

where the quantities A and β are arbitrary. Expanding the integrand we get

$$\frac{1}{A^\beta} = \frac{1}{\Gamma(\beta)} \sum_n \phi_n A^n \int_0^\infty dx\, x^{\beta+n-1}, \qquad (9)$$

where we have defined the factor

$$\phi_n = \frac{(-1)^n}{\Gamma(n+1)}. \qquad (10)$$

Now the integral evaluation will not be done in the usual way, but we define the following operational relation:

$$\int_0^\infty dx\, x^{\beta+n-1} \equiv \Gamma(\beta) \frac{\Gamma(n+1)}{(-1)^n} \delta_{\beta+n,0}, \qquad (11)$$

which makes (9) to be an identity

$$\frac{1}{A^\beta} = \frac{1}{\Gamma(\beta)} \sum_n \frac{(-1)^n}{\Gamma(n+1)} A^n \int_0^\infty dx\, x^{\beta+n-1}$$

$$= \frac{1}{\Gamma(\beta)} \sum_n \frac{(-1)^n}{\Gamma(n+1)} A^n \left(\Gamma(\beta) \frac{\Gamma(n+1)}{(-1)^n} \delta_{\beta+n,0}\right)$$

$$= \sum_n A^n \delta_{\beta+n,0} = A^{-\beta}. \qquad (12)$$

For convenience we introduce the following notation:

$$\int dx\, x^{(\alpha+\beta)-1} \equiv \langle \alpha + \beta \rangle, \qquad (13)$$

where the parenthesis $\langle \cdot \rangle$ has implicitly the constraint associated with the Kronecker delta. This identity is the first fundamental formula of the integration method IBFE. With the previously defined notation, Eq. (9) can be written in the following way:

$$\frac{1}{A^\beta} = \frac{1}{\Gamma(\beta)} \sum_n \phi_n A^n \langle \beta + n \rangle, \qquad (14)$$

which corresponds to the multiregion expansion (MRE) of the factor $A^{-\beta}$.

On the other hand, starting from the application of Eq. (8) and later Eq. (13) to an arbitrary multinomial, we find the second fundamental formula that the method uses. This expresses the fact that a multinomial of σ terms can be written as a MRE, in such a way that it contains simultaneously all the possible expansions with respect to the ratio of the different terms that are present in the multinomial. This MRE can be written in the following form:

$$(A_1 + \cdots + A_\sigma)^{\pm\nu} = \sum_{n_1} \cdots \sum_{n_\sigma} \phi_{n_1,..,n_\sigma} A_1^{n_1} \ldots A_\sigma^{n_\sigma}$$

$$\times \frac{\langle \mp\nu + n_1 + \cdots + n_\sigma \rangle}{\Gamma(\mp\nu)}, \qquad (15)$$

where the definition of the factor (10) has been generalized to

$$\phi_{n_1,..,n_\sigma} = \phi_{n_1} \ldots \phi_{n_\sigma}$$

$$= (-1)^{n_1+\cdots+n_\sigma} \frac{1}{\Gamma(n_1+1)\ldots\Gamma(n_\sigma+1)}. \qquad (16)$$

2. General form of a diagram MRE and its solutions

Once the parametric representation of the diagram (2) has been obtained, the next step is finding its MRE, and for this it is necessary to expand the integrand starting with the exponentials if they exist, and then expanding all the multinomials that the procedure is generating according to formula (15). This expansion process stops when finally there is only one term, which is a product of all the Schwinger parameters. At this point all that is left is to replace the integrals according to formula (13) into its equivalent $\langle \cdot \rangle$. The result is the MRE of the Feynman integral G considered in (1). In the case of a general topology characterized by M different mass scales, N propagators, L loops, and a minimal quantity P of invariants associated with scalar products of the independent external momenta, then the general form of the MRE of G is given by the following expression:

$$G = (-1)^{LD/2} \sum_{n_1,..,n_\sigma} \phi_{n_1,..,n_\sigma} \prod_{i=1}^{P} (Q_i^2)^{n_i} \prod_{j=P+1}^{P+M} (-m_j^2)^{n_j}$$

$$\times \prod_{k=1}^{N} \frac{\langle \nu_k + \alpha_k \rangle}{\Gamma(\nu_k)} \prod_{l=1}^{K} \frac{\langle \beta_l + \gamma_l \rangle}{\Gamma(\beta_l)}, \qquad (17)$$

where we can identify the following quantities.

(i) $\sigma \Rightarrow$ Multiplicity or number of sums that are present in the MRE of diagram G.

(ii) $Q_j^2 \Rightarrow$ A kinematical invariant, which is a quadratic form of the independent external momenta.

(iii) α_j, β_j, $\gamma_j \Rightarrow$ Linear combinations of the indexes $\{n_1, .., n_\sigma\}$, except for β_1, which is the only one that has dependence on the dimension D:

$$\beta_1 = \frac{D}{2} + n_1 + \cdots + n_P. \quad (18)$$

The coefficients of the indices of the sum $\{n_i\}$ in the linear combinations α_j and γ_j are $(+1)$, and in the case of β_j the indices have coefficients (-1), except for β_1.

(iv) $N \Rightarrow$ Number of propagators or equivalently number of parametric integrations, which the method transforms into N Kronecker deltas.

(v) $K \Rightarrow$ Total number of MREs performed over the integrand of the parametric representation, which in turn generates K constraints or equivalently K Kronecker deltas. This only refers to multinomial expansions that are present in the parametric integral.

In order to find the solutions it is necessary to evaluate the sums that are present in (17), using for this purpose the existing constraints between the sum indices, represented by the $\delta = (N + K)$ Kronecker deltas. There are several ways to do this evaluation, and in fact the number of different ways to evaluate the MRE of G using the Kronecker deltas is given by the combinatorial formula

$$C_\delta^\sigma = \frac{\sigma!}{\delta!(\sigma - \delta)!}. \quad (19)$$

Each of these forms of summing will generate as a result a term in the solution of G, which corresponds to a generalized hypergeometric function, and whose multiplicity is given by

$$\mu = (\sigma - \delta). \quad (20)$$

In general it is not always possible to use the δ Kronecker deltas to evaluate a similar number of sums, since this will depend on the combination of sum indices for which the

sum is going to be done. If this happens, these cases simply do not generate contributions to the solution.

In simple terms, the idea of the method is to generate finally an expansion that represents the diagram G, the MRE of G, characterized by a multiplicity σ in combination with δ Kronecker deltas. From this it is possible to get the solution in terms of generalized hypergeometric functions, series of multiplicity μ whose arguments usually correspond to ratios between two characteristic energy scales of the diagram, or in the more general case include sums of unitary argument.

III. CONVENTIONAL REDUCTION OF DIAGRAMS WITH MASSLESS BUBBLE INSERTIONS

A. Graphical representation of the reduction

Many diagrams contain bubble type insertions in the propagators, which in the massless case can be directly reduced to a propagator and a factor which is a ratio of gamma functions. This result is straightforward and is very useful for an iterative or loop-by-loop reduction of topologies or subtopologies built in terms of bubble type insertions. The fundamental formula for insertions or bubble modules can be deduced directly from the solution of the momentum integral for this diagram,

$$G = \int \frac{d^D q}{i\pi^{D/2}} \frac{1}{(q^2)^{a_1}((q + p)^2)^{a_2}}, \quad (21)$$

where the indices a_1 and a_2 are arbitrary. The explicit solution of this integral is

$$G = g(a_1, a_2) \frac{1}{(p^2)^{a_1 + a_2 - (D/2)}}, \quad (22)$$

where the factor $g(a_1, a_2)$ is given by

$$g(a_1, a_2) = (-1)^{D/2} \frac{\Gamma(a_1 + a_2 - \frac{D}{2})\Gamma(\frac{D}{2} - a_1)\Gamma(\frac{D}{2} - a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(D - a_1 - a_2)}. \quad (23)$$

Usually Eq. (22) is represented pictorially in the following way:

$$(24)$$

Another graphical formula, which is useful when two or more scalar propagators are in series, is
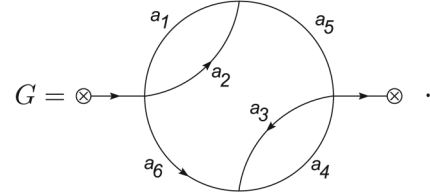
$$(25)$$

This last formula is also valid for a massive theory in the case of two equal masses. These two pictorial expressions are enough to find the solution of this family of diagrams. Finally the operational problem consists in literally reducing series and parallel propagators, and the formulae (24) and (25) represent this operation.
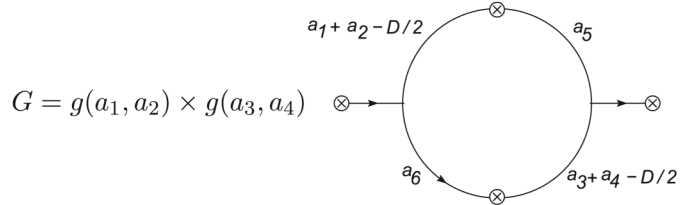
B. A simple application

In order to show the usefulness of the previous pictorial formulae (24) and (25), let us consider the following radiative correction to a propagator through which flows a momentum p

$$G = \otimes \quad \text{(26)}$$

Then, using Eq. (24) we can reduce the two bubble type subgraphs that appear in the diagram, and we get

$$G = g(a_1, a_2) \times g(a_3, a_4) \qquad (27)$$

The resulting loop is easily evaluated after using Eq. (25), and this finally allows us to get the solution of this diagram

$$G = g(a_1, a_2) \times g(a_3, a_4) \times g(a_1 + a_2 + a_5 - \tfrac{D}{2}, a_3 + a_4 + a_6 - \tfrac{D}{2}) \quad \otimes \xrightarrow{\; a_1 + ... + a_6 - \frac{3D}{2}\;} \otimes \; , \qquad (28)$$

which is equivalent to the following expression:

$$G = g(a_1, a_2) \times g(a_3, a_4) \times g\left(a_{125} - \frac{D}{2}, a_{346} - \frac{D}{2}\right)$$
$$\times \frac{1}{(p^2)^{a_1 + ... + a_6 - (3D/2)}}. \qquad (29)$$

To simplify the notation we have defined for the sum of indices the following symbology:

$$a_{ijk...} = a_i + a_j + a_k + ... \qquad (30)$$

Using then formula (23) we finally obtain the explicit solution for the diagram

$$G = (-1)^{3D/2} \frac{\Gamma(a_{12} - \frac{D}{2})\Gamma(\frac{D}{2} - a_1)\Gamma(\frac{D}{2} - a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(D - a_{12})}$$
$$\times \frac{\Gamma(a_{34} - \frac{D}{4})\Gamma(\frac{D}{4} - a_3)\Gamma(\frac{D}{4} - a_4)}{\Gamma(a_3)\Gamma(a_4)\Gamma(D - a_{34})}$$
$$\times \frac{\Gamma(a_{123456} - \frac{3D}{2})\Gamma(D - a_{125})\Gamma(D - a_{346})}{\Gamma(a_{125} - \frac{D}{2})\Gamma(a_{346} - \frac{D}{2})\Gamma(2D - a_{123456})}$$
$$\times \frac{1}{(p^2)^{a_1 + ... + a_6 - (3D/2)}}. \qquad (31)$$

IV. IBFE AND MODULAR REDUCTION OF DIAGRAMS

A. The bubble module

The application of IBFE to the bubble module will allow us to find the MREs that generate the 1-loop functions which identify the different masses and momenta configurations that can be associated with this diagram. We start our analysis by considering this diagram to be associated with the kinematical variable p^2, and then from this we can define the following associated 1-loop functions: G_k and $\bar{G}_k (k = A, B, C, D)$, according to the value of this variable and the corresponding mass configuration for this module (for case $p^2 \neq 0$):

$G_A \Rightarrow$ Massless propagators.
$G_B \Rightarrow$ One massive propagator.
$G_C \Rightarrow$ Equal mass propagators.
$G_D \Rightarrow$ Different mass propagators.

Case $p^2 = 0$ could be obtained from the previous functions, just by taking the kinematical variable to zero. Nevertheless, we will take a different approach, which uses the series multiregion representation obtained directly from the parametric representation of a vacuum bubble. The reason is simple, in general the mathematical expression for the MRE is simpler than the corresponding expression obtained by taking $p^2 = 0$ in the functions G_k. These functions are useful when diagrams associated with

vacuum fluctuations are evaluated. The 1-loop functions that we will define for this case are analogous to the previous ones.

$\bar{G}_A \Rightarrow$ Massless propagators.

$\bar{G}_B \Rightarrow$ One massive propagator.

$\bar{G}_C \Rightarrow$ Equal mass propagators.

$\bar{G}_D \Rightarrow$ Different mass propagators.

For our analysis let us start by writing the general integral representation of this diagram in momentum space

$$G = \int \frac{d^D q}{i\pi^{D/2}} \frac{1}{(q^2 - M_j^2)^{a_j}((p-q)^2 - M_k^2)^{a_k}}. \quad (32)$$

We shall consider general cases, so the powers a_j and a_k are arbitrary. We then obtain Schwinger's parametric representation

$$G = \frac{(-1)^{D/2}}{\Gamma(a_j)\Gamma(a_k)}$$
$$\times \int_0^\infty d\vec{x} \frac{\exp(x_j M_j^2)\exp(x_k M_k^2)\exp(-\frac{x_j x_k}{x_j + x_k} p^2)}{(x_j + x_k)^{D/2}}. \quad (33)$$

Depending on the specific values of the masses $\{M_j, M_k\}$ and of $\{p^2\}$, we can consider eight possible 1-loop functions, which will be discussed in what follows.

B. 1-loop functions for $p^2 \neq 0$

There are four loop functions that can be defined in this case (G_A, G_B, G_C, and G_D), of which only the last two are independent. The loop function G_A is a particular case of G_C or G_D, and in the same way G_B turns out to be a particular case of G_D. This will be shown now.

1. 1-loop function $G_D \Rightarrow M_j \neq M_k \neq 0$

Let us start with the most general case, with propagators with different masses. Then the corresponding parametric integral is given through Eq. (33), and the first step is to fractionally expand the exponential that contains the invariant p^2, which gives

$$G = \frac{(-1)^{D/2}}{\Gamma(a_j)\Gamma(a_k)} \sum_n \phi_n(p^2)^n \int_0^\infty d\vec{x} \exp(x_j M_j^2)\exp(x_k M_k^2)$$
$$\times \frac{x_j^n x_k^n}{(x_j + x_k)^{(D/2)+n}}. \quad (34)$$

Or equivalently

$$G = \sum_n G_D(a_j, a_k; n; M_j^2, M_k^2)(p^2)^n$$
$$= \sum_n G_D(a_j, a_k; n; M_j^2, M_k^2) \frac{1}{(p^2)^{-n}}, \quad (35)$$

where the 1-loop function has been defined as

$$G_D(a_j, a_k; n; M_j^2, M_k^2) = \frac{(-1)^{D/2}}{\Gamma(a_j)\Gamma(a_k)} \phi_n \int_0^\infty d\vec{x} \exp(x_j M_j^2)$$
$$\times \exp(x_k M_k^2) \frac{x_j^n x_k^n}{(x_j + x_k)^{(D/2)+n}}. \quad (36)$$

Remember that the factor ϕ_n is given by $\phi_n = \frac{(-1)^n}{\Gamma(n+1)}$. The next step is finding the MRE of the loop function $G_D(a_j, a_k; n; M_j^2, M_k^2)$, and for this purpose we expand the mass exponentials in (36) and then the binomial in the denominator of the same formula, which gives

$$\frac{1}{(x_j + x_k)^{(D/2)+n}} = \sum_{l_j, l_k} \phi_{l_j, l_k} x_j^{l_j} x_k^{l_k} \frac{\langle \frac{D}{2} + n + l_j + l_k \rangle}{\Gamma(\frac{D}{2} + n)}. \quad (37)$$

Performing the necessary algebra, the MRE for $G_D(a_j, a_k; n; M_j^2, M_k^2)$ is finally given by

$$G_D(a_j, a_k; n; M_j^2, M_k^2) = \frac{(-1)^{D/2}}{\Gamma(a_j)\Gamma(a_k)}$$
$$\times \sum_{m_j, m_k l_j, l_k} \phi_{n, m_j, m_k, l_j, l_k} (-M_j^2)^{m_j}$$
$$\times (-M_k^2)^{m_k} \frac{\Delta_1 \Delta_2 \Delta_3}{\Gamma(\frac{D}{2} + n)} \quad (38)$$

with the constraints being defined by the identities

$$\begin{cases} \Delta_1 = \langle \frac{D}{2} + n + l_j + l_k \rangle, \\ \Delta_2 = \langle a_j + n + m_j + l_j \rangle, \\ \Delta_3 = \langle a_k + n + m_k + l_k \rangle. \end{cases} \quad (39)$$

The process of replacing the graph in terms of this equivalent 1-loop function [see Eq. (35)] can be written symbolically through the following graphical formulation:

$$\sum_n G_D(a_j, a_k; n; M_j^2, M_k^2) \quad \otimes \xrightarrow{-n} \otimes \; . \tag{40}$$

Derivation of G_A from G_D (Case $M_j = M_k = 0$). The loop function G_A turns out to be a particular case of the 1-loop function G_D. The function G_A can be defined as

$$G_A(a_j, a_k; n) = G_D(a_j, a_k; n; 0, 0), \tag{41}$$

and whose equivalent MRE corresponds to the expression

$$G_A(a_j, a_k; n) = \frac{(-1)^{D/2}}{\Gamma(a_j)\Gamma(a_k)} \sum_{l_j, l_k} \phi_{n, l_j, l_k} \frac{\Delta_1 \Delta_2 \Delta_3}{\Gamma(\frac{D}{2} + n)}, \tag{42}$$

where the masses were put to zero in (38) and the sums associated with the indices $\{m_j, m_k\}$ were eliminated, tak-

ing later on these indices to zero. The constraints are now given by the identities

$$\begin{cases} \Delta_1 = \langle \frac{D}{2} + n + l_j + l_k \rangle, \\ \Delta_2 = \langle a_j + n + l_j \rangle, \\ \Delta_3 = \langle a_k + n + l_k \rangle. \end{cases} \tag{43}$$

As before, it is possible to represent the fractional expansion of the graph symbolically through the following graphical equation:

$$\sum_n G_A(a_j, a_k; n) \quad \otimes \xrightarrow{-n} \otimes \; . \tag{44}$$

Derivation of G_B (Case $M_j \neq 0$, $M_k = 0$). The loop function G_B can be written in terms of the loop function G_D as

$$G_B(a_j, a_k; n; M_j^2) = G_D(a_j, a_k; n; M_j^2, 0). \tag{45}$$

The corresponding MRE is obtained just as for G_A

$$G_B(a_j, a_k; n; M_j^2) = \frac{(-1)^{D/2}}{\Gamma(a_j)\Gamma(a_k)} \sum_{m_j, l_j, l_k} \phi_{n, m_j, l_j, l_k} (-M_j^2)^{m_j}$$

$$\times \frac{\Delta_1 \Delta_2 \Delta_3}{\Gamma(\frac{D}{2} + n)}. \tag{46}$$

The constraints now are given by

$$\begin{cases} \Delta_1 = \langle \frac{D}{2} + n + l_j + l_k \rangle, \\ \Delta_2 = \langle a_j + n + m_j + l_j \rangle, \\ \Delta_3 = \langle a_k + n + l_k \rangle. \end{cases} \tag{47}$$

The graphical equation which represents the loop reduction is now

$$\sum_n G_B(a_j, a_k; n; M_j^2) \quad \otimes \xrightarrow{-n} \otimes \; . \tag{48}$$

2. 1-loop function $G_C \Rightarrow M_j = M_k = M$

The loop function G_C does not correspond to a particularization of G_D. To see this we start with the parametric representation already factorized,

$$G = \frac{(-1)^{D/2}}{\Gamma(a_j)\Gamma(a_k)} \int_0^\infty d\vec{x} \frac{\exp(UM^2)\exp(-\frac{x_j x_k}{U}p^2)}{U^{D/2}}, \quad (49)$$

with $U = (x_j + x_k)$. The reason that this loop function cannot be derived from G_D is related to the denominator structure, where the factorization of the polynomial U produces an MRE of (49) which is more reduced than when we take $M_j = M_k$ in G_D.

In Eq. (49) we expand the exponential which contains the momentum, and obtain the following series:

$$G = \sum_n G_C(a_j, a_k; n; M^2) \frac{1}{(p^2)^{-n}}, \quad (50)$$

where we have defined the 1-loop function $G_C(a_j, a_k; n; M^2)$ as

$$G_C(a_j, a_k; n; M^2) = \frac{(-1)^{D/2}}{\Gamma(a_j)\Gamma(a_k)} \phi_n$$
$$\times \int_0^\infty d\vec{x} \exp((x_j + x_k)M^2)$$
$$\times \frac{x_j^n x_k^n}{(x_j + x_k)^{(D/2)+n}}. \quad (51)$$

Expanding the exponential which contains the mass term we have the series

$$G_C(a_j, a_k; n; M^2) = \frac{(-1)^{D/2}}{\Gamma(a_j)\Gamma(a_k)} \phi_n \sum_{l_i} \phi_{l_i}(-M^2)^{l_i}$$
$$\times \int_0^\infty d\vec{x} \frac{x_j^n x_k^n}{(x_j + x_k)^{(D/2)+n-l_i}}. \quad (52)$$

Now we shall find the MRE for the binomial in the integrand denominator

$$\frac{1}{(x_j + x_k)^{(D/2)+n-l_i}} = \sum_{l_j, l_k} \phi_{l_j, l_k} x_j^{l_j} x_k^{l_k} \frac{\langle \frac{D}{2} + n - l_i + l_j + l_k \rangle}{\Gamma(\frac{D}{2} + n - l_i)}, \quad (53)$$

which, replaced in (52) and after some algebra, allows one finally to get the required MRE

$$G_C(a_j, a_k; n; M^2) = \frac{(-1)^{D/2}}{\Gamma(a_j)\Gamma(a_k)} \sum_{l_i, l_j, l_k} \phi_{n, l_i, l_j, l_k}(-M_j^2)^{l_i}$$
$$\times \frac{\Delta_1 \Delta_2 \Delta_3}{\Gamma(\frac{D}{2} + n - l_i)}, \quad (54)$$

where the constraints $\{\Delta_k\}$ are given by

$$\begin{cases} \Delta_1 = \langle \frac{D}{2} + n - l_i + l_j + l_k \rangle, \\ \Delta_2 = \langle a_j + n + l_j \rangle, \\ \Delta_3 = \langle a_k + n + l_k \rangle. \end{cases} \quad (55)$$

Then the loop reduction in this case is represented graphically by the following pictorial equation:

$$(56)$$

Notice that the 1-loop function G_A is also a particular case of G_C

$$G_A(a_j, a_k; n) = G_C(a_j, a_k; n; 0). \quad (57)$$

In all the previous representations or reductions of the bubble diagram one can see the similitude with the conventional Eq. (24). Nevertheless, the differences are important since in the integration by fractional expansion technique (IBFE) massive graphs can be reduced with the same degree of difficulty as in the massless cases.
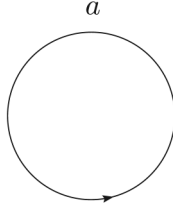
V. MRES FOR 1-LOOP VACUUM DIAGRAMS

The following 1-loop functions are useful when we wish to evaluate certain topologies associated with vacuum fluctuations. These functions will be identified as $\bar{G}_k(K = A, B, C, D)$, in analogy with the ones obtained in the previous section for $p^2 \neq 0$. The corresponding MREs of these new cases will be obtained directly from their momentum integrals, which corresponds to a bubble without external lines. The resulting MRE for each 1-loop function is in fact simpler than the one obtained by making $p^2 = 0$ in the respective loop function G_k, that is, it has a more reduced composition of sums and Kronecker deltas.

A. MRE of a bubble with one propagator

1. Massless propagator

This case does not correspond to any of the bubble diagram formulations previously obtained because it is a bubble with just one propagator. The momentum integral is

$$\left(\begin{array}{c} a \\ \bigcirc \end{array} \right) = \int \frac{d^D q}{i\pi^{\frac{D}{2}}} \frac{1}{(q^2)^a}. \qquad (58)$$
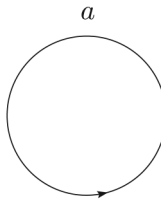
The representation in terms of MRE is for this case easy to find using

$$\int \frac{d^D q}{i\pi^{D/2}} \frac{1}{(q^2)^a} = \frac{1}{\Gamma(\nu)} \int_0^\infty dx x^{a-1} \int \frac{d^D q}{i\pi^{D/2}} \exp(-xq^2)$$
$$= \frac{(-1)^{D/2}}{\Gamma(a)} \left\langle a - \frac{D}{2} \right\rangle, \qquad (59)$$

where we have used the Minskowski space integral identity

$$\int \frac{d^D q}{i\pi^{D/2}} \exp(-\beta q^2) = \frac{(-1)^{D/2}}{\beta^{D/2}}. \qquad (60)$$

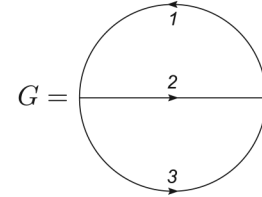Finally the required MRE can be written in terms of a pictorial equation as

$$\left(\begin{array}{c} a \\ \bigcirc \end{array} \right) = \frac{(-1)^{\frac{D}{2}}}{\Gamma(a)} \left\langle a - \frac{D}{2} \right\rangle. \qquad (61)$$

One of the properties of Feynman integrals is that they are invariant under momenta scaling, which allows one to show that the integral in Eq. (58) vanishes; nevertheless, an MRE has been obtained for this diagram. The explanation is quite simple. In fact, first we need to know whether the module constitutes by itself a diagram, or it is *s* product of a reduction of a more complex diagram. For the last option the MRE does not vanish, but has only validity within the integration method that is used. This allows one to formulate a generalization associated with what has been said previously in terms of a theorem:

Any MRE obtained from a Feynman diagram considering all its loops, with a number of constraints (Kronecker deltas) bigger than the number of summations, vanishes identically.

Let us consider a simple example, a 2-loop vacuum fluctuation in a massless theory, whose conventional evaluation is rather obvious, but which nevertheless can be an illustrative example of the theorem:

$$G = \left(\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right). \qquad (62)$$

The parametric representation is given by

$$G = \frac{(-1)^D}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^\infty d\vec{x} \frac{1}{(x_1 x_2 + x_1 x_3 + x_2 x_3)^{D/2}}. \qquad (63)$$

Expanding the integrand denominator

$$\frac{1}{(x_1 x_2 + x_1 x_3 + x_2 x_3)^{D/2}} = \sum_{n_1,n_2,n_3} \phi_{n_1,n_2,n_3} x_1^{n_1+n_2} x_2^{n_1+n_3} x_3^{n_2+n_3} \frac{\left\langle \frac{D}{2} + n_1 + n_2 + n_3 \right\rangle}{\Gamma(\frac{D}{2})}, \qquad (64)$$

and replacing the integral signs by their respective constraints we obtain the following associated MRE:

$$G = \frac{(-1)^D}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \sum_{n_1,n_2,n_3} \phi_{n_1,n_2,n_3} \frac{\left\langle \frac{D}{2} + n_1 + n_2 + n_3 \right\rangle \langle a_1 + n_1 + n_2 \rangle \langle a_2 + n_1 + n_3 \rangle \langle a_3 + n_2 + n_3 \rangle}{\Gamma(\frac{D}{2})}. \qquad (65)$$

The number of constraints (Kronecker deltas) is bigger than the number of summations, and therefore the previously mentioned theorem says that it should vanish,

$$G \equiv 0. \qquad (66)$$

2. Massive propagator

The next degree of difficulty comes from the addition of a mass to the propagator, for which we will use two differ-

ent ways of finding the corresponding MRE. In the first one the usual formalism will be employed, which consists in parametrizing and then replacing the resulting integral for its corresponding MRE. The second relies in expressing the propagator in terms of its own MRE, and then parametrizing the loop integral which now does not contain any mass term.

Alternative I : Direct parameterization of the integral. We want to find the MRE of the following Feynman

integral:

$$a, m$$

$$= \int \frac{d^D q}{i\pi^{\frac{D}{2}}} \frac{1}{(q^2 - m^2)^a}. \quad (67)$$

Performing the corresponding algebra allows us to obtain the required MRE

$$\int \frac{d^D q}{i\pi^{D/2}} \frac{1}{(q^2 - m^2)^a} = \frac{1}{\Gamma(a)} \int_0^\infty dx x^{a-1} \exp(xm^2)$$

$$\times \int \frac{d^D q}{i\pi^{D/2}} \exp(-xq^2)$$

$$= \frac{(-1)^{D/2}}{\Gamma(a)} \sum_n \phi_n (-m^2)^n \left\langle a - \frac{D}{2} + n \right\rangle. \quad (68)$$

Pictorially this is summarized in the following way:

$$a, m$$

$$= \sum_{n_1, n_2} \phi_{n_1, n_2} (-m^2)^{n_1} \frac{\langle a + n_1 + n_2 \rangle}{\Gamma(a)}, \quad \overset{-n_2}{\bigcirc}, \quad (71)$$

where the massless bubble can be evaluated using Eq. (61), and then we finally obtain the MRE for this case

$$a, m$$

$$= \frac{(-1)^{\frac{D}{2}}}{\Gamma(a)} \sum_{n_1, n_2} \phi_{n_1, n_2} (-m^2)^{n_1} \frac{\langle a + n_1 + n_2 \rangle \left\langle -n_2 - \frac{D}{2} \right\rangle}{\Gamma(-n_2)}. \quad (72)$$

Although (69) and (72) are equivalent, it is clear that (69) is a more compact result, which makes actual evaluation easier.

B. MRE of a bubble with two propagators

1. Trivial cases: $\bar{G}_A$, $\bar{G}_C$

There are two cases of vacuum bubbles with two propagators which are reducible to one propagator. This happens when the two propagators which form the bubble have equal mass parameters, in which case it is possible to use the pictorial Eq. (25) in order to simplify the graph.

The first case corresponds to a composition of two massless propagators, which results in an expression for

$$a, m$$

$$= \frac{(-1)^{\frac{D}{2}}}{\Gamma(a)} \sum_n \phi_n (-m^2)^n \left\langle a - \frac{D}{2} + n \right\rangle. \quad (69)$$

Alternative II: *Using the MRE of the massive propagator.* An alternative form for finding the MRE of this module can be implemented using the MRE of the denominator in the loop integral, which extracts the mass out of the integral and leaves the equivalent of a vacuum massless bubble. Meanwhile, the corresponding MRE of a massive propagator can be easily found using the fundamental Eq. (15), that is
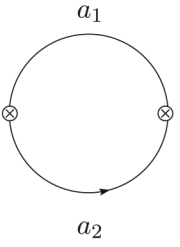
$$\frac{1}{(q^2 - m^2)^a} = \sum_{n_1, n_2} \phi_{n_1, n_2} (-m^2)^{n_1} (q^2)^{n_2} \frac{\langle a + n_1 + n_2 \rangle}{\Gamma(a)}. \quad (70)$$

We can now rewrite the following equation for a massive bubble:

the 1-loop function $\bar{G}_A$ and which is reduced as follows:

$$\bar{G}_A(a_1, a_2) = \overset{a_1}{\underset{a_2}{\bigotimes \qquad \bigotimes}} = \overset{(a_1 + a_2)}{\bigcirc}. \quad (73)$$

With the help of Eq. (61) one then obtains the following expression for $\bar{G}_A$:

$$\overline{G}_A(a_1, a_2) = \qquad = \frac{(-1)^{\frac{D}{2}}}{\Gamma(a_1 + a_2)} \left\langle a_1 + a_2 - \tfrac{D}{2} \right\rangle. \tag{74}$$

Analogously, for the case of two propagators with the same arbitrary mass m, the resulting 1-loop function describes $\bar{G}_C$:

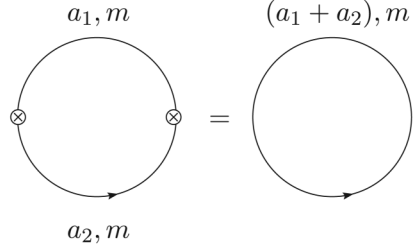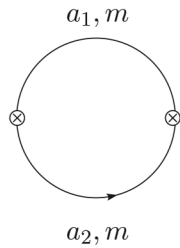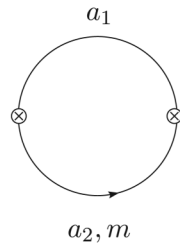$$\overline{G}_C\left(a_1, a_2; m^2\right) = \qquad = \qquad , \tag{75}$$

and then using (69) we get the MRE for this module

$$\overline{G}_C\left(a_1, a_2; m^2\right) = \qquad = \frac{(-1)^{\frac{D}{2}}}{\Gamma(a_1 + a_2)} \sum_n \phi_n \left(-m^2\right)^n \left\langle a_1 + a_2 - \frac{D}{2} + n \right\rangle. \tag{76}$$

2. One massive propagator : $\bar{G}_B$

This case defines the loop function $\bar{G}_B$ and, just as was done before, we will consider two different possible forms to represent the MRE. This module is given by the integral representation

$$\overline{G}_B(a_1, a_2; m^2) = \qquad = \int \frac{d^D q}{i\pi^{\frac{D}{2}}} \frac{1}{(q^2)^{a_1} (q^2 - m^2)^{a_2}}. \tag{77}$$

Alternative I: *Direct parameterization of the integral*. The parametrization and subsequent evaluation of the loop integral gives us

$$\int \frac{d^D q}{i\pi^{D/2}} \frac{1}{(q^2)^{a_1}(q^2 - m^2)^{a_2}} = \frac{1}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty dx dy x^{a_1 - 1} y^{a_2 - 1} \exp(ym^2) \int \frac{d^D q}{i\pi^{D/2}} \exp[-(x + y)q^2]$$

$$= \frac{(-1)^{D/2}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty dx dy x^{a_1 - 1} y^{a_2 - 1} \frac{\exp(ym^2)}{(x + y)^{D/2}}. \tag{78}$$

After some algebra we get the MRE of this case

$$\overline{G}_B(a_1, a_2; m^2) = \quad \text{(diagram)} \quad = \frac{(-1)^{\frac{D}{2}}}{\Gamma(a_1)} \sum_{n_1,n_2,n_3} \phi_{n_1,n_2,n_3} \left(-m^2\right)^{n_1} \frac{\left\langle \frac{D}{2} + n_2 + n_3 \right\rangle \left\langle a_1 + n_2 \right\rangle \left\langle a_2 + n_1 + n_3 \right\rangle}{\Gamma(\frac{D}{2})}.$$

(79)

Alternative II: *Using the MRE of the massive propagator.* The second alternative implies extracting the mass of the loop integral, that is, finding the MRE of the massive propagator. Then we have

$$\overline{G}_B(a_1, a_2; m^2) = \quad \text{(diagram)} \quad = \sum_{n_1,n_2} \phi_{n_1,n_2} \left(-m^2\right)^{n_1} \frac{\left\langle a_2 + n_1 + n_2 \right\rangle}{\Gamma(a_2)} \quad \text{(diagram)}$$

(80)

or equivalently

$$\overline{G}_B(a_1, a_2; m^2) = \quad \text{(diagram)} \quad = \frac{(-1)^{\frac{D}{2}}}{\Gamma(a_2)} \sum_{n_1,n_2} \phi_{n_1,n_2} \left(-m^2\right)^{n_1} \frac{\left\langle a_2 + n_1 + n_2 \right\rangle \left\langle a_1 - n_2 - \frac{D}{2} \right\rangle}{\Gamma(a_1 - n_2)}.$$

(81)

It is evident Eq. (81) is the most compact for function $\bar{G}_B$.

3. Propagators with different masses: $\bar{G}_D$

The next function is associated with a bubble with two propagators of different mass, which defines the loop function $\bar{G}_D$. Again we will deduce this function in two alternative ways, as was done before.

Alternative I: *Expansion of the complete integral.* Applying Schwinger's parametrization allows one to find the following structure for this module:

$$\text{(diagram)} \quad = \frac{1}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty dx\, dy\, x^{a_1-1} y^{a_2-1} \exp(x\, M^2) \exp(y\, m^2) \int \frac{d^D q}{i\pi^{\frac{D}{2}}} \, \exp\left[-(x+y)q^2\right].$$

(82)

Performing the evaluations and replacements already described, we get the MRE for this case

$$\overline{G}_D(a_1, a_2; M^2, m^2) = \begin{array}{c} a_1, M \\ \otimes \quad \bigcirc \quad \otimes \\ a_2, m \end{array} = \frac{(-1)^{\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \sum_{\substack{n_1, n_2 \\ n_3, n_4}} \phi_{n_1, .., n_4} \left(-m^2\right)^{n_1} \left(-M^2\right)^{n_2} \frac{\Delta_1 \Delta_2 \Delta_3}{\Gamma(\frac{D}{2})}, \tag{83}$$

where the associated constraints are

$$\begin{cases} \Delta_1 = \langle \frac{D}{2} + n_3 + n_4 \rangle, \\ \Delta_2 = \langle a_1 + n_1 + n_3 \rangle, \\ \Delta_3 = \langle a_2 + n_2 + n_4 \rangle. \end{cases} \tag{84}$$

Alternative II: *Expansion of massive propagators*. The momentum integral is given by

$$\overline{G}_D(a_1, a_2; M^2, m^2) = \begin{array}{c} a_1, M \\ \otimes \quad \bigcirc \quad \otimes \\ a_2, m \end{array} = \int \frac{d^D q}{i \pi^{\frac{D}{2}}} \frac{1}{(q^2 - M^2)^{a_1} (q^2 - m^2)^{a_2}}, \tag{85}$$

Then we expand each massive propagator

$$\begin{array}{c} a_1, M \\ \otimes \quad \bigcirc \quad \otimes \\ a_2, m \end{array} = \sum_{n_1, n_2} \phi_{n_1, n_2} \left(-M^2\right)^{n_1} \frac{\langle a_1 + n_1 + n_2 \rangle}{\Gamma(a_1)} \sum_{n_3, n_4} \phi_{n_3, n_4} \left(-m^2\right)^{n_3} \frac{\langle a_2 + n_3 + n_4 \rangle}{\Gamma(\nu_2)} \begin{array}{c} (-n_2 - n_4) \\ \bigcirc \end{array}$$

$$\tag{86}$$

and using formula (61) we obtain the loop function $\bar{G}_D$

$$\overline{G}_D(a_1, a_2; M^2, m^2) = \begin{array}{c} a_1, M \\ \otimes \quad \bigcirc \quad \otimes \\ a_2, m \end{array} = \frac{(-1)^{\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \sum_{\substack{n_1, n_2 \\ n_3, n_4}} \phi_{n_1, .., n_4} \left(-M^2\right)^{n_1} \left(-m^2\right)^{n_3} \frac{\Delta_1 \Delta_2 \Delta_3}{\Gamma(-n_2 - n_4)} \tag{87}$$

with the following constraints:

$$\begin{cases} \Delta_1 = \langle a_1 + n_1 + n_2 \rangle, \\ \Delta_2 = \langle a_2 + n_3 + n_4 \rangle, \\ \Delta_3 = \langle -n_2 - n_4 - \frac{D}{2} \rangle. \end{cases} \tag{88}$$

In all the massive cases we can use two equivalent ways of

finding the MRE of the present module, but one of them will produce a mathematical expansion in a more reduced or minimal form. More compact MREs allow a considerable reduction of the possibilities of evaluating the sums of the expansion with the available Kronecker deltas. Remember that in the case of the MRE of a certain Feynman diagram, each alternative way of doing these
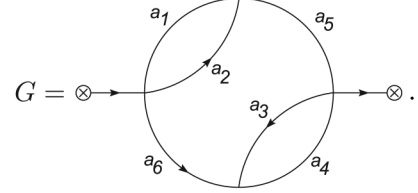
evaluations generates a term (hypergeometric series) of the final solution, and therefore a minimal MRE allows one to eliminate irrelevant or nonexisting terms (cases where it is not possible mathematically to perform the sum using the Kronecker deltas).
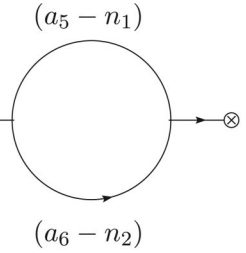
VI. APPLICATIONS

In this section we will use the previously obtained 1-loop functions and present three examples that will show and explain the IBFE methodology applied modularly. Together with this we will be able to visualize the advantages of this procedure compared with IBFE applied to the Schwinger's parametric representation of the complete diagram. In our previous work [4], we show explicitly solutions to start of the MRE equation of a Feynman diagram. In this work, we just compare the ways to obtain the respective MRE associated to a topology.

A. Example I: Radiative correction to the three loop massless propagator
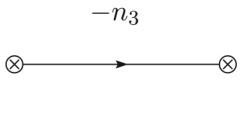
Let us consider as a first example the three loop correction to the massless propagator, with momentum p, as shown in the graphical formula (89), applying the fractional expansion method loop by loop, and using the 1-loop functions previously obtained. In this case it is clear that we need only 1-loop functions of type G_A.

$$G = \otimes \qquad (89)$$

We start by reducing the two bubble insertions directly, using the pictorial formula (44), which allows us to get

$$G = \sum_{n_1,n_2} G_A(a_1, a_2; n_1) \times G_A(a_3, a_4; n_2) \quad \otimes \qquad (90)$$

We can use (25) to sum the propagator indices, and then apply formula (44) for finally getting the diagram MRE in terms of a sum of products of 1-loop functions G_A

$$G = \sum_{n_1,..,n_3} G_A(a_1, a_2; n_1) \times G_A(a_3, a_4; n_2) \times G_A(a_5 - n_1, a_6 - n_2; n_3) \quad \otimes \xrightarrow{\quad -n_3 \quad} \otimes . \qquad (91)$$

This result can be written also as

$$G = \sum_{n_1,..,n_3} G_A(a_1, a_2; n_1) \times G_A(a_3, a_4; n_2)$$
$$\times G_A(a_5 - n_1, a_6 - n_2; n_3)(p^2)^{n_3}, \qquad (92)$$

where the 1-loop functions G_A for each case are determined, according to Eq. (42), by the following expression:

$$G_A(a_1, a_2; n_1) = \frac{(-1)^{D/2}}{\Gamma(a_1)\Gamma(a_2)} \sum_{n_4,n_5} \phi_{n_1,n_4,n_5} \frac{\Delta_1 \Delta_2 \Delta_3}{\Gamma(\frac{D}{2} + n_1)}, \qquad (93)$$

with constraints that are given by the identities

$$\begin{cases} \Delta_1 = \langle \frac{D}{2} + n_1 + n_4 + n_5 \rangle, \\ \Delta_2 = \langle a_1 + n_1 + n_4 \rangle, \\ \Delta_3 = \langle a_2 + n_1 + n_5 \rangle. \end{cases} \qquad (94)$$

Similarly one has

$$G_A(a_3, a_4; n_2) = \frac{(-1)^{D/2}}{\Gamma(a_3)\Gamma(a_4)} \sum_{n_6,n_7} \phi_{n_2,n_6,n_7} \frac{\Delta_4 \Delta_5 \Delta_6}{\Gamma(\frac{D}{2} + n_2)}, \qquad (95)$$

$$\begin{cases} \Delta_4 = \langle \frac{D}{2} + n_2 + n_6 + n_7 \rangle, \\ \Delta_5 = \langle a_3 + n_2 + n_6 \rangle, \\ \Delta_6 = \langle a_4 + n_2 + n_7 \rangle, \end{cases} \qquad (96)$$

TABLE I.   Comparative application of IBFE to example I.

	Complete (better factorization)	Modular
Multiplicity multiregion series (σ)	11	9
Kronecker deltas of the expansion (δ)	11	9
Multiplicity resulting series ($\sigma - \delta$)	0	0
Possible contributions to the solution (C_δ^σ)	1	1

and finally

$$
G_A(a_5 - n_1, a_6 - n_2; n_3) = \frac{(-1)^{D/2}}{\Gamma(a_5 - n_1)\Gamma(a_6 - n_2)}
$$
$$
\times \sum_{n_8,n_9} \phi_{n_3,n_8,n_9} \frac{\Delta_7 \Delta_8 \Delta_9}{\Gamma(\frac{D}{2} + n_3)},
$$

(97)

$$
\begin{cases}
\Delta_7 = \langle \frac{D}{2} + n_3 + n_8 + n_9 \rangle, \\
\Delta_8 = \langle a_5 - n_1 + n_3 + n_8 \rangle, \\
\Delta_9 = \langle a_6 - n_2 + n_3 + n_9 \rangle.
\end{cases}
$$

(98)

Replacing these series in Eq. (92) allows one to get the diagram MRE:

$$
G = \frac{(-1)^{3D/2}}{\prod_{i=1}^{4}\Gamma(a_i)} \sum_{n_1,..,n_9} \phi_{n_1,..,n_9}
$$
$$
\times \frac{(p^2)^{n_3}}{\Gamma(\frac{D}{2} + n_1)\Gamma(\frac{D}{2} + n_2)\Gamma(\frac{D}{2} + n_3)}
$$
$$
\times \frac{\prod_{j=1}^{9} \Delta_j}{\Gamma(a_5 - n_1)\Gamma(a_6 - n_2)}.
$$

(99)

In order to show the advantages of modular IBFE, we show in Table I a comparison of the MRE obtained in this form

with respect to that one that evaluates the complete diagram, that is which uses the parametric representation that incudes all loops simultaneously.

The expression (99) is equivalent to Eq. (29), which was deduced for the same diagram but using the usual 1-loop function (23). The first conclusion is that the fractional expansion technique is, for cases of loop-by-loop reducible massless diagrams, more tedious than the conventional. In spite of this, we will see that for cases in which the diagram contains different mass scales, the present technique is very powerful, and this will become clear in the following example.

B. Example II: Propagator with two mass scales and two loops

Let us consider a graph which consists of a 2-loop propagator with two different mass scales, as shown in the figure (100):

(100)
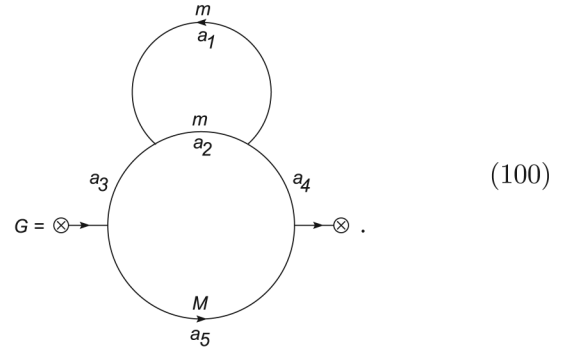
We first reduce the loop associated with the mass m, which uses the 1-loopfunction G_C [Eq. (56)], and then we get

$$
G = \sum_{n_1} G_C\left(a_1, a_2; n_1; m^2\right)
$$

(101)

For the resulting loop reduction we use formula (48), obtaining the massive diagram MRE

TABLE II. Comparative application of IBFE to example II.

	Complete (better factorization)	Modular
Multiplicity multiregion series (σ)	11	8
Kronecker deltas of the expansion (δ)	9	6
Multiplicity resulting series ($\sigma - \delta$)	2	2
Possible contributions to the solution (C_δ^σ)	55	28

$$G = \sum_{n_1,n_2} G_C\left(a_1, a_2; n_1; m^2\right) \times G_B\left(a_5, a_3 + a_4 - n_1; n_2; M^2\right) \quad \otimes\!\!\xrightarrow{\quad -n_2 \quad}\!\!\otimes, \tag{102}$$

which is equivalent to the algebraic expression

$$G = \sum_{n_1,n_2} G_C(a_1, a_2; n_1; m^2)$$
$$\times G_B(a_5, a_3 + a_4 - n_1; n_2; M^2)(p^2)^{n_2}. \tag{103}$$

The 1-loop functions G_C and G_B in (103) are given by

$$G_C(a_1, a_2; n_1; m^2) = \frac{(-1)^{D/2}}{\Gamma(a_1)\Gamma(a_2)} \sum_{n_3,n_4,n_5} \phi_{n_1,n_3,n_4,n_5}(-m^2)^{n_3}$$
$$\times \frac{\Delta_1 \Delta_2 \Delta_3}{\Gamma(\frac{D}{2} + n_1 - n_3)}, \tag{104}$$

where the constraints are

$$\begin{cases} \Delta_1 = \langle \frac{D}{2} + n_1 - n_3 + n_4 + n_5 \rangle, \\ \Delta_2 = \langle a_1 + n_1 + n_4 \rangle, \\ \Delta_3 = \langle a_2 + n_1 + n_5 \rangle, \end{cases} \tag{105}$$

and

$$G_B(a_5, a_3 + a_4 - n_1; n_2; M^2)$$
$$= \frac{(-1)^{D/2}}{\Gamma(a_5)\Gamma(a_3 + a_4 - n_1)}$$
$$\times \sum_{n_6,n_7,n_8} \phi_{n_2,n_6,n_7,n_8}(-M^2)^{n_6} \frac{\Delta_4 \Delta_5 \Delta_6}{\Gamma(\frac{D}{2} + n_2)}, \tag{106}$$

with the constraints

$$\begin{cases} \Delta_4 = \langle \frac{D}{2} + n_2 + n_7 + n_8 \rangle, \\ \Delta_5 = \langle a_5 + n_2 + n_6 + n_7 \rangle, \\ \Delta_6 = \langle a_3 + a_4 - n_1 + n_2 + n_8 \rangle. \end{cases} \tag{107}$$

The replacement in (103) allows one to finally get the MRE of diagram G

$$G = \frac{(-1)^D}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_5)} \sum_{n_1,..,n_8} \phi_{n_1,..,n_8}$$
$$\times \frac{(p^2)^{n_2}(-m^2)^{n_3}(-M^2)^{n_6}}{\Gamma(\frac{D}{2} + n_1 - n_3)\Gamma(\frac{D}{2} + n_2)} \frac{\Delta_1 \Delta_2 \Delta_3 \Delta_4 \Delta_5 \Delta_6}{\Gamma(a_3 + a_4 - n_1)}. \tag{108}$$

Operationally it is not possible to apply directly the conventional 1-loop reduction (23) to this massive example, as has been done here with the fractionally expanded 1-loop functions. Moreover, it is also possible to verify that the diagram MRE obtained loop by loop is far more direct and simple than to get this expansion considering all loops simultaneously. This can be seen from Table II, where we observe that the modular application of the IBFE technique not only reduces the MRE of the diagram in terms of sums and deltas, but also brings in a significant reduction of the number of irrelevant terms and therefore the calculation time of the solution is optimized.

C. Example III: Vacuum fluctuation with three mass scales and five loops

The next diagram that we will analyze is one without external lines, composed of five loops and three different mass scales $\{m_1, m_2, M\}$, arranged as shown here:

$$G = \left[\begin{array}{c} \end{array} \right]. \tag{109}$$

In the first step we reduce the left and right massive loops in terms of the 1-loop functions G_B

$$G = \sum_{n_1,n_2} G_B(a_1, a_2; n_1; m_1^2) \times G_B(a_3, a_4; n_2; m_2^2) \left[\begin{array}{c} \end{array} \right]. \tag{110}$$

Applying formula (25) in order to sum the propagator indices, we get the equivalent equation

$$G = \sum_{n_1,n_2} G_B(a_1, a_2; n_1; m_1^2) \times G_B(a_3, a_4; n_2; m_2^2)$$

$$\times \left[\begin{array}{c} \end{array} \right].$$

Then we eliminate the internal bubble

$$G = \sum_{n_1,..,n_3} G_B(a_1, a_2; n_1; m_1^2) \times G_B(a_3, a_4; n_2; m_2^2) \times G_C(a_7, a_8; n_3; M^2)$$

$$\times \left[\begin{array}{c} \end{array} \right],$$

continuing with a reduction of the left loop

$$G_A(a_{9,10} - n_1, a_{5,6} - n_3; n_4) \times \left[\begin{array}{c} \end{array} \right]. \tag{111}$$

The last loop reduction is represented by the function $\bar{G}_A$, which determines finally the MRE of the diagram

$$
G = \sum_{n_1,..,n_4} G_B(a_1, a_2; n_1; m_1^2) \times G_B(a_3, a_4; n_2; m_2^2)
$$

$$
\times G_C(a_7, a_8; n_3; M^2)
$$

$$
\times G_A(a_{9,10} - n_1, a_{5,6} - n_3; n_4)
$$

$$
\times \bar{G}_A(a_{11,12} - n_2, -n_4). \tag{112}
$$

The corresponding 1-loop functions in this example are given by

$$
G_B(a_1, a_2; n_1; m_1^2) = \frac{(-1)^{D/2}}{\Gamma(a_1)\Gamma(a_2)} \sum_{n_5,..,n_7} \phi_{n_1,n_5,..,n_7} \frac{(-m_1^2)^{n_7}}{\Gamma(\frac{D}{2} + n_1)}
$$

$$
\times \Delta_1 \Delta_2 \Delta_3, \tag{113}
$$

$$
\begin{cases}
\Delta_1 = \langle a_1 + n_1 + n_5 + n_7 \rangle, \\
\Delta_2 = \langle a_2 + n_1 + n_6 \rangle, \\
\Delta_3 = \langle \frac{D}{2} + n_1 + n_5 + n_6 \rangle,
\end{cases} \tag{114}
$$

$$
G_B(a_3, a_4; n_2; m_2^2) = \frac{(-1)^{D/2}}{\Gamma(a_3)\Gamma(a_4)} \sum_{n_8,..,n_{10}} \phi_{n_2,n_8,..,n_{10}} \frac{(-m_2^2)^{n_{10}}}{\Gamma(\frac{D}{2} + n_2)}
$$

$$
\times \Delta_4 \Delta_5 \Delta_6, \tag{115}
$$

$$
\begin{cases}
\Delta_4 = \langle a_3 + n_2 + n_8 + n_{10} \rangle, \\
\Delta_5 = \langle a_4 + n_2 + n_9 \rangle, \\
\Delta_6 = \langle \frac{D}{2} + n_2 + n_8 + n_9 \rangle,
\end{cases} \tag{116}
$$

$$
G_C(a_7, a_8; n_3; M^2) = \frac{(-1)^{D/2}}{\Gamma(a_7)\Gamma(a_8)} \sum_{n_{11},..,n_{13}} \phi_{n_3,n_{11},..,n_{13}}
$$

$$
\times \frac{(-M^2)^{n_{13}}}{\Gamma(\frac{D}{2} + n_3 - n_{13})} \Delta_7 \Delta_8 \Delta_9, \tag{117}
$$

$$
\begin{cases}
\Delta_7 = \langle a_7 + n_3 + n_{11} \rangle, \\
\Delta_8 = \langle a_8 + n_3 + n_{12} \rangle, \\
\Delta_9 = \langle \frac{D}{2} + n_3 - n_{13} + n_{11} + n_{12} \rangle,
\end{cases} \tag{118}
$$

$$
G_A(a_{9,10} - n_1, a_{5,6} - n_3; n_4)
$$

$$
= \frac{(-1)^{D/2}}{\Gamma(a_9 + a_{10} - n_1)\Gamma(a_5 + a_6 - n_3)} \sum_{n_{14},n_{15}} \phi_{n_4,n_{14},n_{15}}
$$

$$
\times \frac{\Delta_{10}\Delta_{11}\Delta_{12}}{\Gamma(\frac{D}{2} + n_4)}, \tag{119}
$$

$$
\begin{cases}
\Delta_{10} = \langle \frac{D}{2} + n_4 + n_{14} + n_{15} \rangle, \\
\Delta_{11} = \langle a_9 + a_{10} - n_1 + n_4 + n_{14} \rangle, \\
\Delta_{12} = \langle a_5 + a_6 - n_3 + n_4 + n_{15} \rangle,
\end{cases} \tag{120}
$$

$$
\bar{G}_A(a_{11} + a_{12} - n_2, -n_4)
$$

$$
= (-1)^{D/2} \frac{\Delta_{13}}{\Gamma(a_{11} + a_{12} - n_2 - n_4)}, \tag{121}
$$

$$
\{\Delta_{13} = \left\langle a_{11} + a_{12} - n_2 - n_4 - \frac{D}{2} \right\rangle. \tag{122}
$$

Finally we obtain the MRE associated to the diagram

$$
G = \frac{(-1)^{5D/2}}{\Gamma(a_7)\Gamma(a_8)\prod_{j=1}^4 \Gamma(a_j)} \sum_{n_1,..,n_{15}} \phi_{n_1,..,n_{17}} (-m_1^2)^{n_7}(-m_2^2)^{n_{10}}(-M^2)^{n_{13}} \frac{1}{\Gamma(\frac{D}{2} + n_1)\Gamma(\frac{D}{2} + n_2)\Gamma(\frac{D}{2} + n_3 - n_{13})\Gamma(\frac{D}{2} + n_4)\Gamma(\frac{D}{2})}
$$

$$
\times \frac{\prod_{k=1}^{13} \Delta_k}{\Gamma(a_9 + a_{10} - n_1)\Gamma(a_5 + a_6 - n_3)\Gamma(a_{11} + a_{12} - n_2 - n_4)}. \tag{123}
$$

TABLE III.   Comparative application of IBFE to example III.

	Complete (better factorization)	Modular
Multiplicity multiregion series (σ)	21	15
Kronecker deltas of the expansion (δ)	19	13
Multiplicity resulting series ($\sigma - \delta$)	2	2
Possible contributions to the solution (C_δ^σ)	210	105

Table III shows again a comparison of the IBFE applied modularly or complete to the diagram.

VII. COMMENTS

A. Other 1-loop functions associated with the bubble module

The 1-loop functions for diagrams that contain the bubble type insertions previously deduced do not contemplate all the possible cases. To see this let us try to solve the following topology:

$$(124)$$
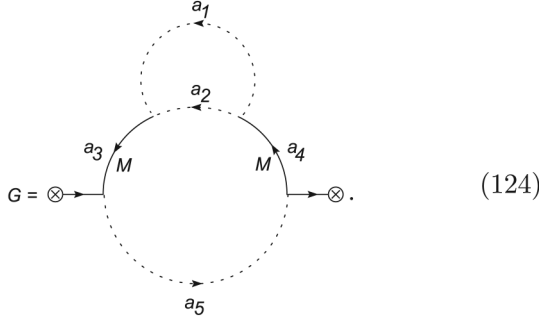
$$G = \frac{(-1)^D}{\prod_{j=1}^5 \Gamma(a_j)} \int_0^\infty d\vec{x} \, \frac{\exp((x_3 + x_4)M^2) \exp\left(-\frac{[x_1 x_2 + (x_1 + x_2)(x_3 + x_4)]x_5}{x_1 x_2 + (x_1 + x_2)(x_3 + x_4) + x_5(x_3 + x_4)} p^2\right)}{[x_1 x_2 + (x_1 + x_2)(x_3 + x_4) + x_5(x_3 + x_4)]^{D/2}}. \tag{125}$$

The integral is written in a factorized form and it is readily evaluated with IBFE. After doing the corresponding expansions we realize that the obtained MRE has ten summations and nine deltas, so the solutions will correspond to one variable hypergeometric function ($_qF_{q-1}$), which is something expected given the type of topology and the number of energy scales present in the problem. Moreover, we will have at most ten possible contributions that distribute themselves in the kinematical regions $\left|\frac{M^2}{p^2}\right| < 1$ and $\left|\frac{p^2}{M^2}\right| < 1$.

Now we will try to solve this diagram, but this time applying the 1-loop functions deduced in this chapter. The upper loop is easily reduced using the 1-loop function G_A, getting then the following reduced diagram:

$$(126)$$
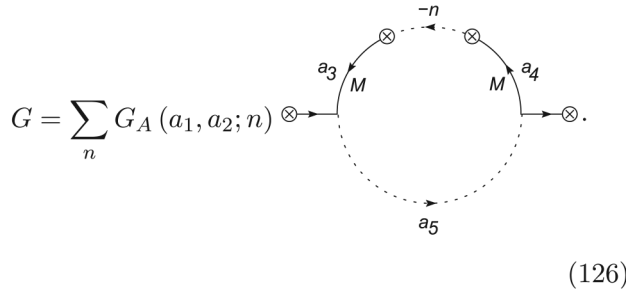
Nevertheless, we obtain one loop which is not possible to reduce with the 1-loop functions previously defined. The reason is the presence of the mixed massive propagator in

This diagram contains two massive propagators, each of them characterized by mass M, arranged as shown (continuous line), and the rest of the propagators are massless (dashed lines). It can be easily evaluated if Schwinger's parametric representation of the whole diagram is integrated. Let us see

the upper branch of the diagram. Thus it is necessary to consider a new loop function that we will call G_E, and which is associated with the generic topology of the form

$$(127)$$
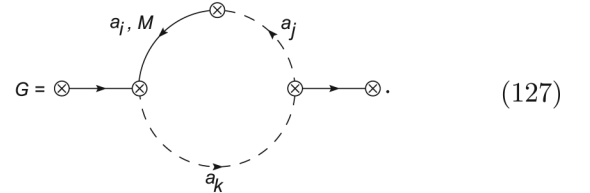
Let us try to find the MRE for this new loop function. For this we start with the corresponding Schwinger parametric representation

$$G = \frac{(-1)^{D/2}}{\Gamma(a_i)\Gamma(a_j)\Gamma(a_k)}$$

$$\times \int_0^\infty d\vec{x} \, \frac{\exp(x_i M^2) \exp\left(-\frac{(x_i + x_j)x_k}{x_i + x_j + x_k} p^2\right)}{(x_i + x_j + x_k)^{D/2}}. \tag{128}$$

When expanding the exponential that contains p^2 the following series is obtained:

$$G = \sum_n G_E(a_i, a_j, a_k; n; M^2) \frac{1}{(p^2)^{-n}}, \tag{129}$$

TABLE IV.   Comparative application of IBFE to the diagram for G.

	Complete (better factorization)	Modular
Multiplicity multiregion series (σ)	10	9
Kronecker deltas of the expansion (δ)	9	8
Multiplicity resulting series ($\sigma - \delta$)	1	1
Possible contributions to the solution (C_δ^σ)	10	9

and just as before the 1-loop function is defined

$$
G_E(a_i, a_j, a_k; n; M^2) = \frac{(-1)^{D/2}}{\Gamma(a_i)\Gamma(a_j)\Gamma(a_k)}\phi_n
$$

$$
\times \int_0^\infty d\vec{x}\,\frac{\exp(x_i M^2)x_k^n(x_i + x_j)^n}{(x_i + x_j + x_k)^{(D/2)+n}}.
$$

(130)

Taking into account the complete expansion procedure finally one finds that the multiregion series for this new 1-loop function is
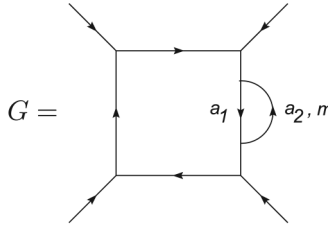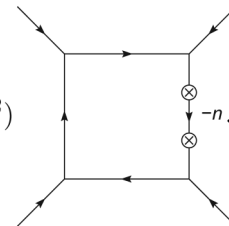
$$
G_E(a_i, a_j, a_k; n; M^2) = \frac{(-1)^{D/2}}{\Gamma(a_i)\Gamma(a_j)\Gamma(a_k)}
$$

$$
\times \sum_{s_1,..,s_5} \phi_{n,s_1,..,s_5}(-M_j^2)^{s_1}
$$

$$
\times \frac{\Delta_1 \dots \Delta_5}{\Gamma(\frac{D}{2} + n)\Gamma(-n - s_2)} \qquad (131)
$$

with the constraints:

$$
\begin{cases}
\Delta_1 = \langle \frac{D}{2} + n + s_2 + s_3 \rangle, \\
\Delta_2 = \langle -n - s_2 + s_4 + s_5 \rangle, \\
\Delta_3 = \langle a_i + s_1 + s_4 \rangle, \\
\Delta_4 = \langle a_j + s_5 \rangle, \\
\Delta_5 = \langle a_k + n + s_3 \rangle.
\end{cases}
\qquad (132)
$$

Graphically this result can be represented as follows:

(133)

This loop contributes with 6Σ and 5δ. Coming back now to our initial problem presented in Eq. (124), and using formula (133) we find the MRE for this problem

$$
G = \sum_{n,l} G_A(a_1, a_2; n) \times G_E(a_3 + a_4, -n, a_5; l; M^2)(p^2)^l,
$$

(134)

which can be compared with the equivalent MRE coming from the parametric representation of the complete diagram (Table IV):

The first thing we should notice is that in this case once again the expression for the MRE obtained in terms of modules is more reduced than the similar expression which evaluates the complete diagram. A second aspect is that,

just as the loop function G_E, there are many others which are variations of the bubble module and that have not been included here, although as has been done above any other configuration has a MRE which can be trivially found.

B. Generic diagrams with bubble type insertions

We have shown that the IBFE technique is very useful and simple to apply in diagrams that are built by successive 1-loop insertions, which can be either massless or massive. Nevertheless, the modular reduction of a diagram using the IBFE technique is not only readily applicable to these types of topologies, but as will be discussed later, the previously deduced formulae are also useful for reducing any bubble type subgraph that is contained within a generic graph. As

an example let us analyze the following diagram:

$$G = \qquad\qquad\qquad a_1 \quad a_2, m. \qquad\qquad (135)$$

Independently of the kinematic characteristics of this diagram, our interest is in the propagator correction with a bubble type insertion and with one of the propagators with mass m. Using the respective loop function in order to reduce the loop, we obtain the following graphical equation:

$$G = \sum_n G_B(a_2, a_1; n; m^2) \qquad\qquad -n. \qquad (136)$$

From a topological point of view one can see that there is a reduction, so to find the MRE of this resulting 1-loop diagram is simpler than the original 2-loop one. Just as in this example, it is possible to use the 1-loop functions in generic diagrams whose propagators contain bubble type corrections and its variations, which simplifies considerably the search of a MRE of certain diagrams and its solutions.

C. n-loop modules and n-loop functions

We have shown that it is possible to apply IBFE to loop-by-loop recursively built subgraphs and find the respective MRE in terms of the 1-loop functions defined above. Nevertheless, the flexibility of IBFE goes beyond that, and it is possible to generate also n-loop functions or equivalently module functions that consider two or more loops simultaneously. Such a module, with n loops, will lead to a growing number of n-loop functions when one takes into account the different possibilities of distributing the propagator masses of the n-loop module. In general to work modularly is advantageous for reducing diagrams, since it simplifies the search of the MRE. Nevertheless, if the idea is to have less algebraic manipulation and a minimal expression for the MRE of the topology, then the previous examples indicate that the MRE is going to be minimal when the modules under consideration are compositions of only one loop.

VIII. CONCLUSIONS

In this work we have described an efficient and easy way to implement the integration technique IBFE, applied to a specific Feynman diagram. The result that has been reached using the concept of modular reduction has two important characteristics: first, in general the diagram multiregion expansion of a given diagram is shorter than if the MRE is obtained from the parametric representation of the complete diagram; second, the modular treatment operationally facilitates obtaining the multiregion expansion. Let us discuss this last point, comparing the modular reduction in cases of loop-by-loop reducible diagrams with 1-loop insertions in a massless theory which are evaluated using conventional calculation. The difference is that with IBFE it is possible to do a calculation with the same degree of simplicity, although for more complex bubble type diagrams it is now possible to include masses.

The modular reduction idea leads in general to a minimization of the MRE of a diagram, which is more evident when the modules to reduce are composed of one loop. In fact, the loop-by-loop reduction optimizes the IBFE application. In more general terms and beyond this work, depending on the 1-loop topology the minimal MRE is reached by a systematic reduction going from the topologically simplest to the more complex: bubbles, triangles, boxes, etc.

The importance of a minimal MRE is of course tied to a minimization of the time of finding the solutions, since the MRE gets the solutions to the parametric integrals evaluating summations with Kronecker deltas in all possible combinations, and now the MRE is shorter (fewer summations and Kronecker deltas, which diminish equally), then there are fewer combinations that need to be taken into account in order to find the solution. Basically what happens is that several combinations which do not lead to relevant terms in the solution are eliminated, and given that with IBFE the integration is replaced by a linear system evaluation, the minimized MRE implies smaller linear systems, and therefore it is simpler to analyze.

The integration technique IBFE is a simple method and at the same time powerful, which can be advantageously compared with other Feynman diagram evaluations, and which does not require great mathematical knowledge. Although here we have concentrated in a scalar theory, this method is directly applicable to other theories by previous scalarization of the tensorial loop integrals, which produces a sum of scalar integrals. In general the most complicated case of an N propagator tensorial integral is precisely the corresponding N propagator scalar integral. This is the reason for improving or creating numerical or analytical techniques for the evaluation of scalar integrals in perturbative studies in field theory.

APPENDIX A: SUMMARY OF FORMULAE FOR THE BUBBLE MODULE

$$= \sum_n G_A\left(a_j, a_k; n\right)$$

$Composition \Longrightarrow 3\Sigma/3\delta.$

$$= \sum_n G_B(a_j, a_k; n; M_j^2)$$

$Composition \Longrightarrow 4\Sigma/3\delta.$

$$= \sum_n G_C\left(a_j, a_k; n; M^2\right)$$

$Composition \Longrightarrow 4\Sigma/3\delta.$

$$= \sum_n G_D(a_j, a_k; n; M_j^2, M_k^2)$$

$Composition \Longrightarrow 5\Sigma/3\delta.$

$$= \overline{G}_A\left(a_j, a_k\right)$$

Composition $\Longrightarrow 0\Sigma/1\delta.$

$$= \overline{G}_B(a_j, a_k; M_j^2)$$

Composition $\Longrightarrow 2\Sigma/2\delta.$

$$= \overline{G}_C\left(a_j, a_k; M^2\right)$$

Composition $\Longrightarrow 2\Sigma/2\delta.$

$$= \overline{G}_D(a_j, a_k; M_j^2, M_k^2)$$

Composition $\Longrightarrow 4\Sigma/3\delta.$

APPENDIX B: HYPERGEOMETRIC FUNCTIONS

In the evaluation of L-loop Feynman diagrams, all the possible solutions that can be found are presented in terms of multivariable generalized hypergeometric series. The actual value of the variables is given in general by ratios of two energy scales associated with the graph [4], including also arguments of value 1. The purpose of this appendix is to provide the necessary information about simple and double hypergeometric functions [17–21], which appear in the solution of the loop integrals, and, in particular, to present their convergence conditions, which are what finally define the different kinematical regions associated with the solution of a specific Feynman diagram.

1. Definition of the generalized hypergeometric function

The functions associated with one variable series solution of a Feynman diagram are always expressible in terms of generalized hypergeometric functions, in particular, the hypergeometric function of order $(q, q - 1)$, which is the only one-variable series that is a solution for the loop integrals. This function is conventionally denoted as

$$
{}_qF_{q-1}(\{a\}; \{b\}; z) \equiv {}_qF_{q-1}\left(\begin{matrix} \{a\} \\ \{b\} \end{matrix} \,\middle|\, z\right)
$$

$$
\equiv \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_q)_k}{(b_1)_k \ldots (b_{q-1})_k} \frac{z^k}{k!}, \quad \text{(B1)}
$$

where the factors $(\alpha)_k$ are called Pochhammer symbols and which are defined by

$$
(\alpha)_k = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)}. \quad \text{(B2)}
$$

For these functions the convergence conditions are:

(i) The generalized hypergeometric functions are given by series defined in the convergence region $|z| < 1$, while for $|z| > 1$ they are defined by analytical continuation.

(ii) If $|z| < 1$ the series converges absolutely. Since the variable z represents the ratio between two energy scales of the topology, what is obtained is one of the limit representations or solutions, an expansion around $z \to 0$.

(iii) If $z = 1$, the necessary requirement for the series convergence is that $\Re(\omega) > 0$, where ω is called parametric excess and it is given by the equation

$$
\omega = \sum_{j=0}^{q} b_j - \sum_{j=0}^{q+1} a_j. \quad \text{(B3)}
$$

(iv) For the convergence in $z = -1$ it is sufficient that $\Re(\omega) > -1$.

2. Some identities of the Pochhammer symbols

The following identities are very useful for building the hypergeometric function starting from the contributions that are obtained of the MRE of an arbitrary diagram G.

Aside from formula (B2), sometimes it is useful to use other identities that are needed when there are factors of the type $\Gamma(a \pm n)$ and $\Gamma(a \pm 2n)$ in the solutions, such as

$$(a)_{-n} = \frac{\Gamma(a - n)}{\Gamma(a)} = \frac{(-1)^n}{(1 - a)_n}, \tag{B4}$$

$$(a)_{2n} = \frac{\Gamma(a + 2n)}{\Gamma(a)} = 4^n \left(\frac{a}{2}\right)_n \left(\frac{a}{2} + \frac{1}{2}\right)_n. \tag{B5}$$

3. Two variable hypergeometric functions

Here we describe two variable hypergeometric functions, which correspond to the Kampé de Fériet generalized double hypergeometric function $F^{p:r:u}_{q:s:v}$ and the generalized hypergeometric $\bar{F}^{p:r:u}_{q:s:v}$.

a. Kampé de Fériet function $F^{p:r:u}_{q:s:v}$

This function is defined as:

$$F^{p:r:u}_{q:s:v} \left(\begin{array}{ccc} \alpha_1, \ldots, \alpha_p & a_1, \ldots, a_r & c_1, \ldots, c_u \\ \beta_1, \ldots, \beta_q & b_1, \ldots, b_s & d_1, \ldots, d_v \end{array} \middle| x, y \right)$$

$$\equiv F^{p:r:u}_{q:s:v} \left(\begin{array}{ccc} \{\alpha\} & \{a\} & \{c\} \\ \{\beta\} & \{b\} & \{d\} \end{array} \middle| x, y \right)$$

$$\equiv \sum_{n,m}^{\infty} \frac{\prod_{j=1}^{p}(\alpha_j)_{n+m} \prod_{j=1}^{r}(a_j)_n \prod_{j=1}^{u}(c_j)_m}{\prod_{j=1}^{q}(\beta_j)_{n+m} \prod_{j=1}^{s}(b_j)_n \prod_{j=1}^{v}(d_j)_m} \frac{x^n}{n!} \frac{y^m}{m!}, \tag{B6}$$

where the convergence conditions of the double series exist if the following relations between the indices are satisfied:

$$p + r \leq q + s + 1, \tag{B7}$$

$$p + u \leq q + v + 1, \tag{B8}$$

and if furthermore, the arguments fulfil the condition:

$$|x|^{1/(p-q)} + |y|^{1/(p-q)} < 1, \qquad \text{if } (p > q),$$

$$\max\{|x|, |y|\} < 1, \qquad \text{if } (p \leq q).$$

b. Function $\bar{F}^{p:r:u}_{q:s:v}$

This series, which appears frequently in the solutions to Feynman diagrams, has the following definition:

$$\bar{F}^{p:r:u}_{q:s:v} \left(\begin{array}{ccc} \alpha_1, \ldots, \alpha_p & a_1, \ldots, a_r & c_1, \ldots, c_u \\ \beta_1, \ldots, \beta_q & b_1, \ldots, b_s & d_1, \ldots, d_v \end{array} \middle| x, y \right)$$

$$\equiv \bar{F}^{p:r:u}_{q:s:v} \left(\begin{array}{ccc} \{\alpha\} & \{a\} & \{c\} \\ \{\beta\} & \{b\} & \{d\} \end{array} \middle| x, y \right)$$

$$\equiv \sum_{n,m}^{\infty} \frac{\prod_{j=1}^{p}(\alpha_j)_{n-m} \prod_{j=1}^{r}(a_j)_n \prod_{j=1}^{u}(c_j)_m}{\prod_{j=1}^{q}(\beta_j)_{n-m} \prod_{j=1}^{s}(b_j)_n \prod_{j=1}^{v}(d_j)_m} \frac{x^n}{n!} \frac{y^m}{m!}, \tag{B9}$$

where the convergence conditions of the double series exist if the following relations between the indices are satisfied:

$$p + r \leq q + s + 1, \tag{B10}$$

$$q + u \leq p + v + 1. \tag{B11}$$

All the series that have been found with the technique used here fulfil this index condition. The determination of the convergence region of the variables can be done using the Horns general convergence theory [21].

APPENDIX C: MATHEMATICAL FORMALISM OF THE INTEGRATION BY FRACTIONAL EXPANSION METHOD

1. Introduction

In section II B we already introduced some algebraic aspects of the IBFE technique, which are generated from the identity associated with the integral parametrization of loops and which is known as Schwinger's parametrization

$$\frac{1}{A^{\beta}} = \frac{1}{\Gamma(\beta)} \int_0^{\infty} dx \, x^{\beta-1} \exp(-Ax). \tag{C1}$$

It is possible to find an operational equivalence between the integral symbol and a Kronecker delta, given by

$$\int dx \, x^{\beta+n-1} \equiv \Gamma(\beta) \frac{\Gamma(n + 1)}{(-1)^n} \delta_{\beta+n,0}. \tag{C2}$$

For simplicity we have eliminated the integral limits, since this identity has only validity in the context of the integrand expansion in (C1). This expression is crucial for the development of the IBFE method, since in the Feynman diagram evaluation the corresponding Schwinger parametric representation is a generalized structure of the expression (C1).

a. Some properties

For the study of some properties of (C2) it is convenient to use the notation defined in (13) in order to help us formalize the mechanism of the IBFE technique. Then let

$$\int dx \, x^{\nu_1+\nu_2-1} \equiv \langle \nu_1 + \nu_2 \rangle, \tag{C3}$$

where ν_1 and ν_2 are indices which can take arbitrary values.

Property l. Commutativity of indices. We can explicitly write (C3) in two possible forms according to formula (C2),

$$\langle \nu_1 + \nu_2 \rangle = \begin{cases} \Gamma(\nu_1) \frac{\Gamma(\nu_2+1)}{(-1)^{\nu_2}} \delta_{\nu_1+\nu_2,0}, \\ \Gamma(\nu_2) \frac{\Gamma(\nu_1+1)}{(-1)^{\nu_1}} \delta_{\nu_1+\nu_2,0}. \end{cases} \tag{C4}$$

Starting from (C1) one can show the equivalence of both forms in (C4), and for this it is enough to expand the integrand exponential and replace $\langle \cdot \rangle$ for each case

$$\frac{1}{A^{\nu_1}} = \frac{1}{\Gamma(\nu_1)} \int_0^{\infty} dx \, x^{\nu_1-1} \exp(-Ax), \tag{C5}$$

which in terms of a series, in the proposed sense (C3), turns out to be

$$\frac{1}{A^{\nu_1}} = \frac{1}{\Gamma(\nu_1)} \sum_{\nu_2} \frac{(-1)^{\nu_2}}{\Gamma(\nu_2 + 1)} A^{\nu_2} \langle \nu_1 + \nu_2 \rangle. \qquad \text{(C6)}$$

Selecting now $\langle \nu_1 + \nu_2 \rangle = \Gamma(\nu_1) \frac{\Gamma(\nu_2+1)}{(-1)^{\nu_2}} \delta_{\nu_1+\nu_2,0}$, we directly obtain the equality in (C5). Analogously selecting $\langle \nu_1 + \nu_2 \rangle = \Gamma(\nu_2) \frac{\Gamma(\nu_1+1)}{(-1)^{\nu_1}} \delta_{\nu_1+\nu_2,0}$ and using then the identity

$$\frac{\Gamma(y)}{\Gamma(y-z)} = (-1)^{-z} \frac{\Gamma(1+z-y)}{\Gamma(1-y)}, \qquad \text{(C7)}$$

with $y = \nu_1$ and $z = 2\nu_1$, the equality (C5) is finally obtained, which shows the equivalence between the two ways of writing $\langle \nu_1 + \nu_2 \rangle$.

Another way of writing (C3), which is useful for the simplification of terms that contain the factor

$$\frac{(-1)^m}{\Gamma(m+1)}, \qquad \text{(C8)}$$

where m is an arbitrary index, is the following:

$$\langle \nu_1 + \nu_2 \rangle = \langle \nu_1 + \nu_2 - m + m \rangle = \langle -m + m \rangle$$
$$= \Gamma(-m) \frac{\Gamma(m+1)}{(-1)^m} \delta_{\nu_1+\nu_2,0}. \qquad \text{(C9)}$$

Notice that explicit use of the Kronecker deltas has been made, in order to simplify the parenthesis $\langle \cdot \rangle$, and since this happens in the context of expansions, the Kronecker deltas remain to indicate the constraints between the indices ν_1 and ν_2.

Property ll. Significance of the Multiregion Expansion MRE. Let us consider the following binomial expansion:

$$(A_1 + A_2)^{\pm \nu}, \qquad \text{(C10)}$$

where the quantities $A_1, A_2,$ and ν can take arbitrary values. In this case there are two possible regions or limits for the expansion: the region where $(A_1 > A_2)$ and the region where $(A_1 < A_2)$. These expansions are respectively:

(1) Region $(A_1 > A_2)$

$$(A_1 + A_2)^{\pm \nu} = A_1^{\pm \nu} \sum_{n=0}^{\infty} \frac{(\mp \nu)_n}{\Gamma(n+1)} \left(-\frac{A_2}{A_1} \right)^n. \quad \text{(C11)}$$

(2) Region $(A_1 < A_2)$

$$(A_1 + A_2)^{\pm \nu} = A_2^{\pm \nu} \sum_{n=0}^{\infty} \frac{(\mp \nu)_n}{\Gamma(n+1)} \left(-\frac{A_1}{A_2} \right)^n. \quad \text{(C12)}$$

The factor $(\nu)_n$ is the Pochhammer symbol and it is given by

$$(\nu)_n = \frac{\Gamma(\nu + n)}{\Gamma(\nu)}. \qquad \text{(C13)}$$

We have obtained in this manner expansions in the two possible limits separately. It is possible, however, to express both results employing a single series which contains simultaneously both regions. In this sense we can say that this type of expansion corresponds to a multiregion series representation of the binomial. To show this, let us express the binomial (C10) using the integral representation of the denominator indicated in (C1). Then we get

$$(A_1 + A_2)^{\pm \nu} = \frac{1}{\Gamma(\mp \nu)} \int_0^{\infty} dx x^{\mp \nu - 1} \exp(-xA_1)$$
$$\times \exp(-xA_2), \qquad \text{(C14)}$$

and the exponentials are expanded separately, obtaining:

$$(A_1 + A_2)^{\pm \nu} = \frac{1}{\Gamma(\mp \nu)} \sum_{n_1} \sum_{n_2} \frac{(-1)^{n_1+n_2}}{\Gamma(n_1 + 1)\Gamma(n_2 + 1)} A_1^{n_1} A_2^{n_2}$$
$$\times \int dx x^{\mp \nu + n_1 + n_2 - 1}. \qquad \text{(C15)}$$

Using the identity (C3) we get the multiregion binomial expansion

$$(A_1 + A_2)^{\pm \nu} = \frac{1}{\Gamma(\mp \nu)} \sum_{n_1} \sum_{n_2} \frac{(-1)^{n_1+n_2}}{\Gamma(n_1 + 1)\Gamma(n_2 + 1)}$$
$$\times A_1^{n_1} A_2^{n_2} \langle \mp \nu + n_1 + n_2 \rangle, \qquad \text{(C16)}$$

where according to property (C4) we can express the parenthesis $\langle \cdot \rangle$ in three different ways. In any of them we will have the same Kronecker delta which eliminates one of the two sums. Let us see

$$\langle \mp \nu + n_1 + n_2 \rangle = \begin{cases} \Gamma(\mp \nu + n_1) \frac{\Gamma(n_2+1)}{(-1)^{n_2}} \delta_{\mp \nu + n_1 + n_2,0} \\ \Gamma(\mp \nu + n_2) \frac{\Gamma(n_1+1)}{(-1)^{n_1}} \delta_{\mp \nu + n_1 + n_2,0} \\ \Gamma(n_2 + n_1) \frac{\Gamma(\mp \nu+1)}{(-1)^{\mp \nu}} \delta_{\mp \nu + n_1 + n_2,0}. \end{cases}$$
$$\text{(C17)}$$

On the other hand, the number of possible ways of summing (C16) using the Kronecker delta can be found in general by evaluating the combinatorial $C_{\text{Deltas}}^{\text{Summations}}$, which in this case is $C_1^2 = 2$. Let us see what happens when we sum with respect to one particular index:

(1) *Sum with respect to n_2.* Let us use for this case the following equality:

$$\langle \mp \nu + n_1 + n_2 \rangle = \Gamma(\mp \nu + n_1)$$
$$\times \frac{\Gamma(n_2 + 1)}{(-1)^{n_2}} \delta_{\mp \nu + n_1 + n_2,0}, \qquad \text{(C18)}$$

and then replacing in (C16) we obtain

$$(A_1 + A_2)^{\pm\nu} = \frac{1}{\Gamma(\mp\nu)} \sum_{n_1} (-1)^{n_1}$$

$$\times \frac{\Gamma(\mp\nu + n_1)}{\Gamma(n_1 + 1)} A_1^{n_1} A_2^{\pm\nu - n_1}, \quad \text{(C19)}$$

or equivalently

$$(A_1 + A_2)^{\pm\nu} = A_2^{\pm\nu} \sum_{n_1=0}^{\infty} \frac{(\mp\nu)_{n_1}}{\Gamma(n_1 + 1)} \left(-\frac{A_1}{A_2}\right)^{n_1}, \quad \text{(C20)}$$

which gives the expansion associated with the region $(A_1 < A_2)$, obtained previously in (C12).

(2) *Sum with respect to n_1.* Analogously, we now use the identity

$$\langle \mp\nu + n_1 + n_2 \rangle = \Gamma(\mp\nu + n_2)$$

$$\times \frac{\Gamma(n_1 + 1)}{(-1)^{n_2}} \delta_{\mp\nu + n_1 + n_2, 0}, \quad \text{(C21)}$$

and replacing in (C16) gives

$$(A_1 + A_2)^{\pm\nu} = A_1^{\pm\nu} \sum_{n_2=0}^{\infty} \frac{(\mp\nu)_{n_2}}{\Gamma(n_2 + 1)} \left(-\frac{A_2}{A_1}\right)^{n_2}, \quad \text{(C22)}$$

the expression that was found in (C11), and valid in the region $(A_1 > A_2)$.

The fundamental idea that has been exposed in the previous demonstration is that using the definition (C2) it is possible to make a binomial expansion which differs from the conventional in the sense that the expansion is now around zero and infinity simultaneously. This can be generalized for multinomials and we can obtain its MRE

$$(A_1 + \cdots + A_l)^{\pm\nu} = \sum_{n_1} \cdots \sum_{n_l} \phi_{n_1,..,n_l} A_1^{n_1} \ldots A_l^{n_l}$$

$$\times \frac{\langle \mp\nu + n_1 + \cdots + n_l \rangle}{\Gamma(\mp\nu)}. \quad \text{(C23)}$$

The number of different expressions that can be extracted starting from Eq. (C23) is given by all the possible forms of evaluating some of the sums, using for this purpose the Kronecker delta generated by the same expansion, that is $C_1^{n_l} = n_l$ possible forms. Generalizing even more, any function expressed of a multiregion series employing σ sums and δ Kronecker deltas, has at most

$$C_\delta^\sigma = \frac{\sigma!}{\delta!(\sigma - \delta)!} \quad \text{(C24)}$$

possible ways of being evaluated, and each of these expansions corresponds to a series of multiplicity $\mu = (\sigma - \delta)$.

All the resulting series are series representations with respect to the ratios between the terms of the multinomial and all of them correspond to multivariable generalizations of the hypergeometric function.

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