Universidad de Valparaíso
CHILE

Prueba I
Métodos Matemáticos de la Física I
Licenciatura en Física - 2017
IPGG

Cálculo de áreas

Determinar el área del cuadrilátero que determinan los siguientes puntos en el plano z:

$$
\begin{aligned}
z_1 &= 1+i \\
z_2 &= 4+2i \\
z_3 &= 4+4i \\
z_4 &= 3+2i
\end{aligned}
$$

Demostraciones

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a).- Verifique que $|z_1 - z_2| \geq ||z_1| - |z_2||$
b).- Demuestre la identidad $\sin^4(\theta) = \dfrac{\cos(4\theta) - 4\cos(2\theta) + 3}{8}$

Geometría analítica

Determine si los siguientes lugares geométricos se intersectan, de ser así determine los puntos donde esto ocurre:

$$\text{Lugar geométrico 1}: \ |z|^2 - 2\,\mathrm{Im}(z) = (\mathrm{Im}(z))^2 + 2\,\mathrm{Im}(z)$$

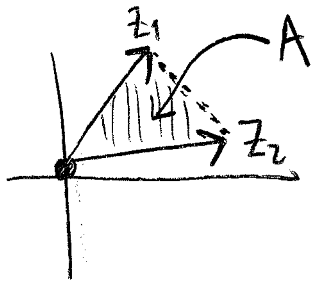
$$\text{Lugar geométrico 2}: \ \mathrm{Re}(z) - \mathrm{Im}(z) = 0$$

Raíces de la unidad

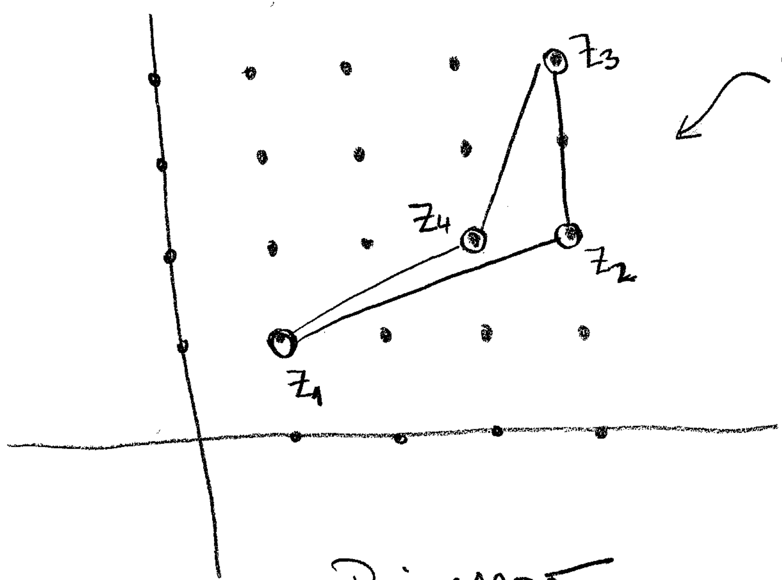
Sean w_1 y w_2 las raíces cúbicas de la unidad distintas de 1. Demuestre que satisfacen:

a).- La ecuación $z^2 + z + 1 = 0$.
b).- $w_1 w_2 = 1$.
c).- $w_1 = w_2^2$.

— Area de una superficie —



$A = \frac{1}{2}\left|\mathrm{Im}(z_1\bar{z}_2)\right|$



2 triángulos

$z_1z_2z_3 \wedge z_2z_3z_4$.

Área total $= A = A_{z_1z_2z_3} + A_{z_2z_3z_4}$

* $A_{z_2z_3z_4} \Rightarrow$ Primero trasladar vértice z_4 a origen

$z'_2 = z_2 - z_4 = 4+2i-(3+2i) = 1$

$z'_3 = z_3 - z_4 = 4+4i-(3+2i) = 1+2i$

$z'_4 = z_4 - z_4 = 0$

Luego $A_{z_2z_3z_4} = A_{z'_2z'_3z'_4} = \frac{1}{2}\left|\mathrm{Im}(z'_2\bar{z}'_3)\right|$

Ahora $z'_2\bar{z}'_3 = (1\,(1-2i)) = 1-2i$

$\Rightarrow \mathrm{Im}(z'_2\bar{z}'_3) = -2 \Rightarrow A_{z'_2z'_3z'_4} = 1$ ///

$* \; A_{z_1 z_2 z_4} \Rightarrow$ Se traslado vértice z_1 al origen

$$z_1' = z_1 - z_1 = 0$$
$$z_2' = z_2 - z_1 = 4+2i-(1+i) = 3+i$$
$$z_4' = z_4 - z_1 = 3+2i-(1+i) = 2+i$$

$$\therefore \; A_{z_1 z_2 z_3} = A_{z_1' z_2' z_3'} = \frac{1}{2}\left|\operatorname{Im}\left(z_2' \overline{z_3'}\right)\right|$$

donde

$$z_2' \overline{z_4'} = (3+i)(2-i) = 6+1+2i-3i$$
$$= 7-i$$

$$\operatorname{Im}(z_2' z_3') = -1$$

$$\therefore \; A_{z_1 z_2 z_3} = \frac{1}{2}|-1| = \frac{1}{2} \,//$$

$$\therefore \; \text{Área total} = A_{z_1 z_2 z_4} + A_{z_2 z_3 z_4} = \frac{1}{2} + 1 = \frac{3}{2} \,//$$

DEMOSTRACIONES!

a) $|z_1 - z_2| \geq ||z_1| - |z_2||$

Desarrollo

$$|z_1 - z_2|^2 = (z_1 - z_2)\overline{(z_1 - z_2)} = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$$

$$= z_1\bar{z}_1 - z_1\bar{z}_2 - \bar{z}_1 z_2 + z_2\bar{z}_2$$

$$= |z_1|^2 - z_1\bar{z}_2 - \overline{z_1\bar{z}_2} + |z_2|$$

$$= |z_1|^2 + |z_2|^2 - 2\,\mathrm{Re}(z_1\bar{z}_2)$$

Obs. $|z|^2 = \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2$

$$\Downarrow$$

$$|z^2| \geq \mathrm{Re}(z)^2$$

$$\Downarrow$$

$$\boxed{|z| \geq \mathrm{Re}(z)}$$

utilizando la propiedad anterior:

$$\mathrm{Re}(z_1\bar{z}_2) \leq |z_1\bar{z}_2|$$

luego

$$|z_1 - z_2|^2 \geq |z_1|^2 + |z_2|^2 - 2|z_1\bar{z}_2|$$

donde $|z_1\bar{z}_2| = |z_1||\bar{z}_2| = |z_1||z_2|$

$\therefore$ $|z_1 - z_2|^2 \geq |z_1|^2 + |z_2|^2 - 2|z_1||z_2| = (|z_1| - |z_2|)^2$

finalmente:

$$|z_1-z_2|^2 \geq (|z_1|-|z_2|)^2 = ||z_1|-|z_2||^2$$

$$\Downarrow$$

$$|z_1-z_2| \geq ||z_1|-|z_2|| \quad /// \quad \text{Q.E.D.}$$

b)

$$\operatorname{sen}\theta = \frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)$$

$$\operatorname{sen}^4\theta = \frac{1}{2^4 i^4}\left(e^{i\theta}-e^{-i\theta}\right)^4$$

$$= \frac{1}{16}\, e^{4i\theta}\left(1-e^{-2i\theta}\right)^4$$

$$= \frac{1}{16}\, e^{4i\theta}\sum_{k=0}^{4}\binom{4}{k}\left(-e^{-2i\theta}\right)^k (1)^{4-k}$$

$$= \frac{1}{16}\, e^{4i\theta}\Big(\cancel{\tbinom{4}{0}}^{1} - \cancel{\tbinom{4}{1}}^{4} e^{-2i\theta} + \cancel{\tbinom{4}{2}}^{6} e^{-4i\theta} - \cancel{\tbinom{4}{3}}^{4} e^{-6i\theta}$$

$$+ \cancel{\tbinom{4}{4}}^{1} e^{-8i\theta}\Big)$$

$$= \frac{1}{16}\left[e^{4i\theta} - 4e^{2i\theta} + 6 - 4e^{-2i\theta} + e^{-4i\theta}\right]$$

$$= \frac{1}{16}\left[e^{4i\theta} + e^{-4i\theta} - 4\left(e^{2i\theta} + e^{-2i\theta}\right) + 6 \right]$$

$$= \frac{1}{16}\left[2\cos(4\theta) - 8\cos(2\theta) + 6 \right]$$

$$= \frac{1}{8}\left[\cos 4\theta - 4\cos 2\theta + 3 \right] \quad //$$

QED.

— Geometría analítica —

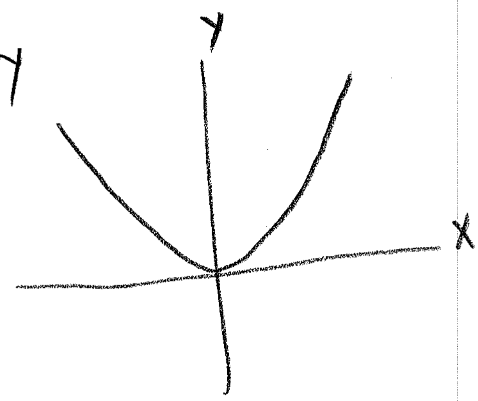
1) $|z|^2 - 2\mathrm{Im}(z) = [\mathrm{Im}(z)]^2 + 2\mathrm{Im}(z)$

$$\Downarrow$$

$$x^2 + \cancel{y^2} - 2y = \cancel{y^2} + 2y$$

$$\Downarrow$$

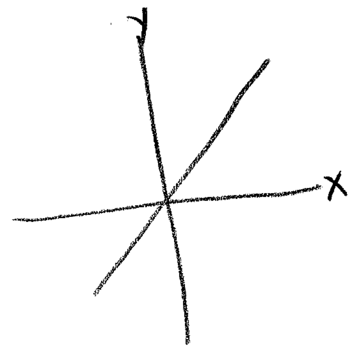
$$\frac{x^2}{4} = y$$

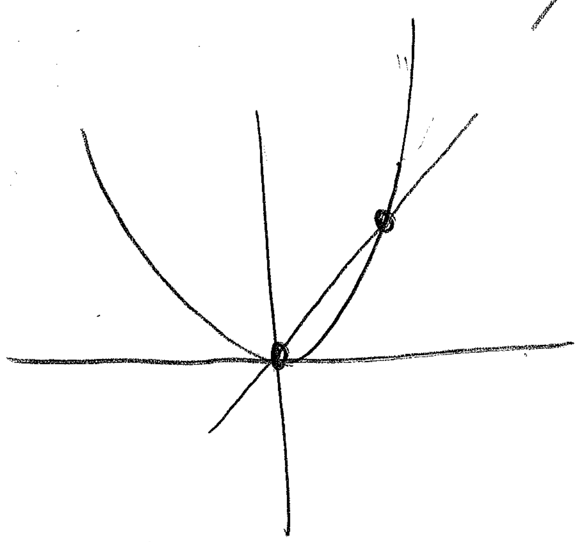


2) $\mathrm{Re}(z) - \mathrm{Im}(z) = 0$

$$\Downarrow$$

$$x = y$$





Los puntos de intersección se obtienen haciendo

$$y = \frac{x^2}{4} = y = x$$

$$\Downarrow$$

$$\frac{x^2}{4} - x = 0$$

$$x\left(\frac{x}{4} - 1\right) = 0$$

soluciones

$x = 0 \rightarrow y = 0$

$x = 4 \rightarrow y = 4$ ///

Raices de la unidad

$$\omega^3 = 1$$

$$\Downarrow$$

$$\omega_k = e^{i\left(\frac{\theta + 2\pi k}{3}\right)} \quad ; \text{ con } \theta_0 = 0.$$

$$\omega_k = e^{i\frac{2\pi k}{3}}$$

$$\Downarrow$$

$$\omega_0 = 1$$

$$\omega_1 = e^{i\frac{2\pi}{3}}$$

$$\omega_2 = e^{i\frac{4\pi}{3}}$$

a) Para ω_1

$$\omega_1^2 + \omega_1 + 1 = \left(e^{i\frac{2\pi}{3}}\right)^2 + e^{i\frac{2\pi}{3}} + 1$$

$$= e^{i\frac{4\pi}{3}} + e^{i\frac{2\pi}{3}} + 1$$

$$= \left(\cos\frac{4\pi}{3} + i\,\text{sen}\frac{4\pi}{3}\right) + \left(\cos\frac{2\pi}{3} + i\,\text{sen}\frac{2\pi}{3}\right) + 1$$

$$= \left(\cancel{-\frac{1}{2}} \cancel{-\frac{\sqrt{3}}{2}}\right) + \left(\cancel{-\frac{1}{2}} \cancel{+ i\frac{\sqrt{3}}{2}}\right) \cancel{+1}$$

$$= 0 \quad // \text{ QED}$$

b) Para ω_2

$$\omega_2^2 + \omega_2 + 1 = e^{i\frac{8\pi}{3}} + e^{i\frac{4\pi}{3}} + 1$$

$$= \left(\cos \frac{8\pi}{3} + i \operatorname{sen} \frac{8\pi}{3}\right) + \left(\cos \frac{4\pi}{3} + i \operatorname{sen} \frac{4\pi}{3}\right) + 1$$

$$= \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) + 1$$

$$= 0 \,//$$ QED

b) $$w_1 w_2 = e^{i\frac{2\pi}{3}} e^{i\frac{4\pi}{3}} = e^{i\frac{6\pi}{3}} = e^{i2\pi} = 1 \,//$$

QED.

c) $$w_2^2 = e^{i\frac{8\pi}{3}} = e^{i\left(\frac{2\pi}{3} + \frac{6\pi}{3}\right)} = e^{i\frac{2\pi}{3}} \underbrace{\cancel{e^{i2\pi}}}_{1} = e^{i\frac{2\pi}{3}} = w_1$$

QED