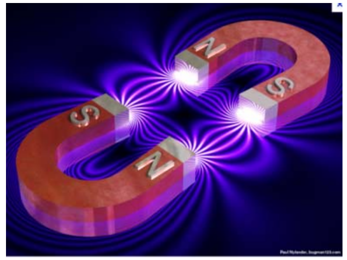
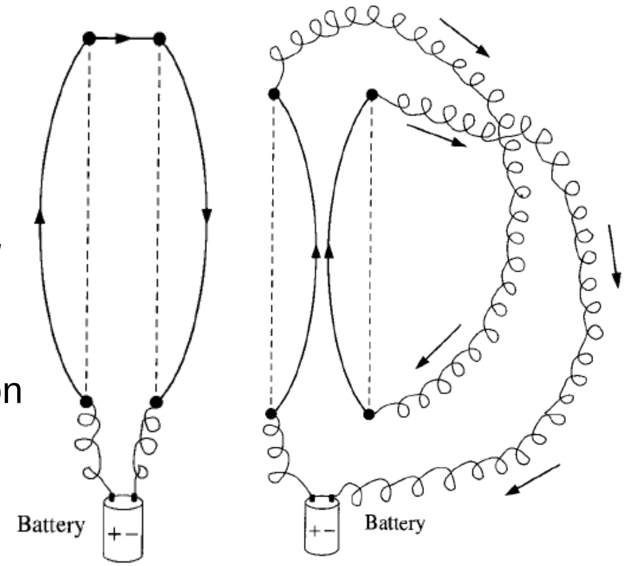
Chapter 5. Magnetostatics



5.1 The Lorentz Force Law

5.1.1 Magnetic Fields



Consider the forces between charges in motion

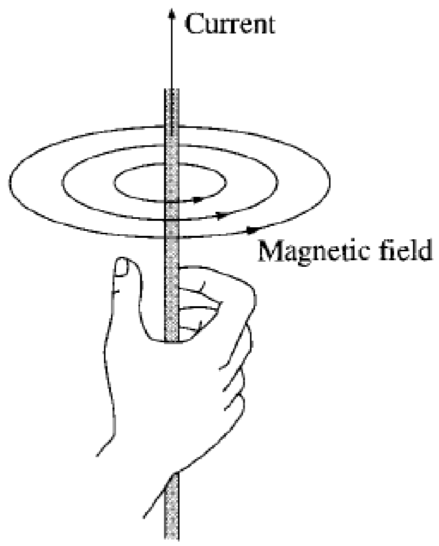
Attraction of parallel currents and Repulsion of antiparallel ones:

➔ ***How do you explain this?***

➔ Does charging up the wires make simply the electrical repulsion of like charges? The wires are in fact electrically neutral!

➔ **It is *not* electrostatic in nature.**

➔ **A moving charge generates a *Magnetic field B*.**



If you grab the wire with your right hand-thumb in the direction of the current-your fingers curl around in the direction of the magnetic field.

How can such a field lead to a force of attraction on a nearby parallel current?

How do you calculate the magnetic field?

5.1.2 Magnetic Forces $\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B})$

In the presence of both ***electric and magnetic fields***, the net force on Q would be

$$\mathbf{F} = Q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})]$$ ➔ **Lorentz force law**

➔ It is a fundamental axiom justified in experiments.

The ***magnetic force*** in a charge Q, moving with **velocity v** in a **magnetic field B**, is

$$\boxed{\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B})}$$ ➔ **Magnetic forces do not work!**

→ For if Q moves an amount ***dl*** = **v** *dt,* the work done is

$$dW_{\text{mag}} = \mathbf{F}_{\text{mag}} \cdot d\mathbf{l} = Q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}\, dt = 0$$

This follows because $(\mathbf{v} \times \mathbf{B})$ is perpendicular to $\mathbf{v}$, so $(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} = 0$.

➔ **Magnetic forces may *alter the direction* in which a particle moves, but they *cannot speed it up or slow it down*.**

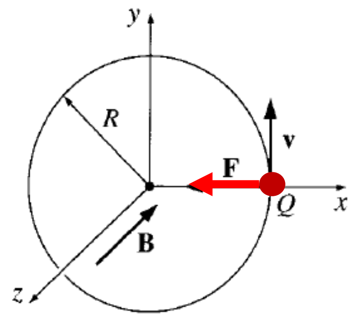
Stationary charges produce electric fields that are constant in time ➔ Electrostatics
Steady currents produce magnetic fields that are constant in time ➔ Magnetostatics

Lorentz Forces: $\mathbf{F} = Q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})]$

Example 5.1 Cyclotron motion

If a charge Q moves counterclockwise, with speed *v,* around a circle of radius *R, in a* plane perpendicular to B. what path will it follow?

The magnetic force points *inward*, and has **a fixed magnitude**, just right to sustain uniform circular motion:



Q moves in a plane perpendicular to B

$$\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B})$$

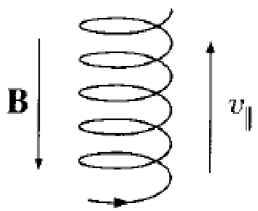
$$\longrightarrow QvB = m\frac{v^2}{R} \longrightarrow \omega \equiv \frac{v}{R} = \frac{QB}{m}$$ **(cyclotron frequency)**

$$\longrightarrow p = mv = QBR$$ ➔ **cyclotron formula**

Cyclotron → the first of the modem particle accelerators.

Cyclotron formula → A simple experimental technique for finding the **momentum of a particle**.
→ Send it through a region of known magnetic field, and measure the radius of its circular trajectory.

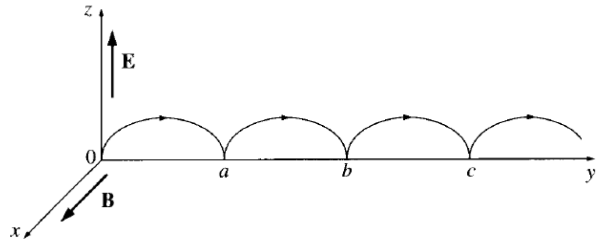
With some additional speed $v_{\parallel}$ *parallel* to B
➔ ***Helix motion***



Lorentz Forces: $\mathbf{F} = Q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})]$

Example 5.2 Cycloid motion

If a charge Q at rest is released from the origin, what path will it follow?



Let's think it through qualitatively, first.

→ Initially, the particle is at rest, so the magnetic force is zero.
→ The electric field accelerates the charge in the z-direction.
→ As it picks up speed, a magnetic force develops.
→ The magnetic field pulls the charge around to the right.
→ The faster it goes, the stronger magnetic force becomes; it curves the particle back around towards the *y* axis.
→ The charge is moving *against* the electrical force, so it begins to slow down.
→ the magnetic force then decreases and the electrical force takes over, bringing the charge to rest at point *a.*
→ There the entire process commences anew, carrying the particle over to point *b,* and so on.

Let's do it quantitatively.

$$\mathbf{v} = (0, \dot{y}, \dot{z}) \longrightarrow \mathbf{v} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & \dot{y} & \dot{z} \\ B & 0 & 0 \end{vmatrix} = B\dot{z}\,\hat{\mathbf{y}} - B\dot{y}\,\hat{\mathbf{z}}$$

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = Q(E\,\hat{\mathbf{z}} + B\dot{z}\,\hat{\mathbf{y}} - B\dot{y}\,\hat{\mathbf{z}}) = m\mathbf{a} = m(\ddot{y}\,\hat{\mathbf{y}} + \ddot{z}\,\hat{\mathbf{z}})$$

$$\begin{aligned} QB\dot{z} &= m\ddot{y} \\ QE - QB\dot{y} &= m\ddot{z} \end{aligned} \xrightarrow[\text{(cyclotron frequency)}]{\omega \equiv \frac{QB}{m}} \begin{aligned} \ddot{y} &= \omega\dot{z} \\ \ddot{z} &= \omega\left(\frac{E}{B} - \dot{y}\right) \end{aligned} \longrightarrow$$

(General solution)

$$\left.\begin{aligned} y(t) &= C_1 \cos\omega t + C_2 \sin\omega t + (E/B)t + C_3, \\ z(t) &= C_2 \cos\omega t - C_1 \sin\omega t + C_4. \end{aligned}\right\}$$

At t = 0: $y(0) = z(0) = 0$, $\dot{y}(0) = \dot{z}(0) = 0$ $\longrightarrow$

$$y(t) = \frac{E}{\omega B}(\omega t - \sin\omega t)$$

$$z(t) = \frac{E}{\omega B}(1 - \cos\omega t)$$

$$\sin^2\omega t + \cos^2\omega t$$

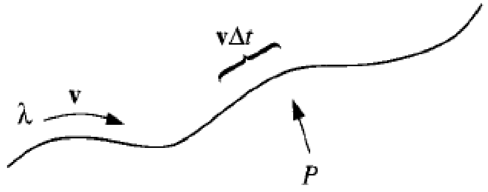
At z =0: $|F_E| = |F_{mag}| \to \upsilon = \frac{E}{B} \to R = \frac{\upsilon}{\omega} = \frac{E}{\omega B} \longrightarrow \sin^2\omega t + \cos^2\omega t = 1 \longrightarrow \boxed{(y - R\omega t)^2 + (z - R)^2 = R^2.}$

This is the formula for a *circle,* of radius *R,* whose center (0, *R*ω*t, R)* travels in the y-direction at a constant speed, v = E/B.

➔ Cycloid motion

5.1.3 Currents ➔ $1\,\text{A} = 1\,\text{C/s}$. ➔ Ampere (A)

When a line charge λ traveling down at velocity *v* ➔ $\mathbf{I} = \lambda \mathbf{v}$



The **magnetic force** on a segment of current-carrying wire is

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\, dq = \int (\mathbf{v} \times \mathbf{B}) \lambda\, dl = \int (\mathbf{I} \times \mathbf{B})\, dl$$

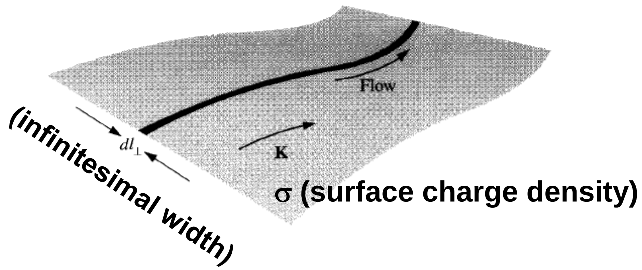
$\mathbf{I}$ and $d\mathbf{l}$ both point in the same direction, $\longrightarrow \mathbf{F}_{\text{mag}} = \int I\,(d\mathbf{l} \times \mathbf{B})$

For a **steady current** (constant in magnitude) along the wire ➔ $\boxed{\mathbf{F}_{\text{mag}} = I \int (d\mathbf{l} \times \mathbf{B})}$

When charge flows over a *surface* *with* surface current density, K:

$$\mathbf{K} \equiv \frac{d\mathbf{I}}{dl_{\perp}} = \sigma \mathbf{v}$$ ➔ ***current per unit width-perpendicular-to-flow***

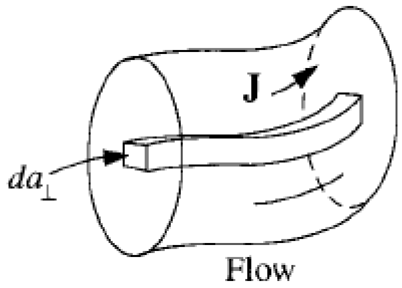
$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\, dq = \int (\mathbf{v} \times \mathbf{B}) \sigma\, da \Rightarrow \boxed{\mathbf{F}_{\text{mag}} = \int (\mathbf{K} \times \mathbf{B})\, da}$$



When charge flows over a *volume* *with* volume current density, J:

$$\mathbf{J} \equiv \frac{d\mathbf{I}}{da_{\perp}} = \rho \mathbf{v}$$ ➔ ***current per unit area-perpendicular-to-flow***

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\, dq = \int (\mathbf{v} \times \mathbf{B}) \rho\, d\tau \Rightarrow \boxed{\mathbf{F}_{\text{mag}} = \int (\mathbf{J} \times \mathbf{B})\, d\tau}$$



Continuity Equation of current density

$$\mathbf{J} \equiv \frac{d\mathbf{I}}{da_\perp} = \rho\mathbf{v}$$

➔ The current crossing a surface *S* can be written as ➔ $I = \int_S J\, da_\perp = \int_S \mathbf{J} \cdot d\mathbf{a}$

The total charge per unit time leaving a volume *V* is

$$\oint_S \mathbf{J} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{J})\, d\tau$$

Because charge is *conserved,* $\int_{\mathcal{V}} (\nabla \cdot \mathbf{J})\, d\tau = -\frac{d}{dt}\int_{\mathcal{V}} \rho\, d\tau = -\int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t}\right) d\tau$

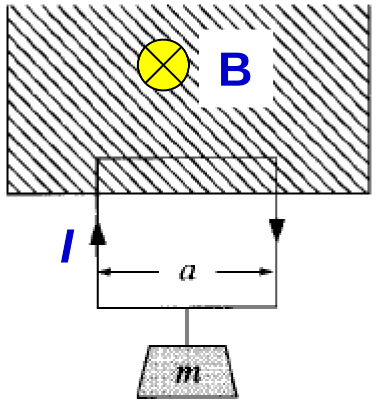
$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$ ➔ **Continuity equation (Local charge conservation)**

For a steady current,

$\partial\rho / \partial t = 0 \Rightarrow \nabla \cdot \mathrm{J} = 0 \rightarrow \oint_S \mathrm{J} \cdot ds = 0 \rightarrow \sum_{j=0} I_j = 0$ ***(Kirchhoff's current law)***

Currents

Example 5.3 For what current *I*, in the loop, would the magnetic force upward exactly balance the gravitational force downward?



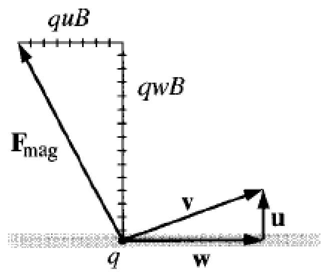
The magnetic forces on the horizontal segment ➔ $\mathbf{F}_{\text{mag}} = I \int (d\mathbf{l} \times \mathbf{B}) = IBa$

(The magnetic forces on the two vertical segments cancel.)

For F_{mag} to balance the weight *(mg)* ➔ $\boxed{I = \dfrac{mg}{Ba}}$

What happens if we now increase the current?

- ➔ The loop rises, lifting the weight.
- ➔ Somebody's doing work → $W_{\text{mag}} = F_{\text{mag}} h = IBah$
- ➔ ***It looks as though the magnetic force is responsible.***
- ➔ ***But, magnetic forces never do work! ➔ What's going on here?***



When the loop starts to rise, the charges in the wire acquires an **upward component *u*** *of velocity,* in addition to the **horizontal component *w*** associated with the current *(I = Aw).*

By the horizontal component *w of velocity* → $F_{\text{vert}} = qwB = \lambda awB = IBa$ (same as before)

By the vertical component *u of velocity* → $F_{\text{horiz}} = quB = \lambda auB$ → It opposes the flow of current.
→ It now push the charges back.
→ But, the current is maintained at constant!
→ Who is in charge of maintaining that current?

Maybe, there is a work done by a battery ➔

In a time *dt* the charges move a (horizontal) distance *w dt* → $W_{\text{battery}} = \lambda aB \int uw\, dt = IBah$

Who did it? The battery! Not B!