CERTÁMEN I
MMF II

PROBLEMA 1

La componente i-ésima está dada por:

$$\left[\nabla\times\nabla\left[e^{-\nabla\cdot(\alpha r^2\vec{r})}\right]\right]_i = \epsilon_{ijk}\,\partial_j\partial_k e^{-\alpha\,\partial_l(r^2 x_l)}$$

Paso I:
$$\partial_l(r^2x_l) = (\partial_l r^2)x_l + r^2\partial_l x_l$$
$$= 2r(\partial_l r)x_l + 3r^2$$
$$= 2r\frac{x_l}{r}x_l + 3r^2$$
$$= 2x_lx_l + 3r^2$$
$$= 2r^2 + 3r^2 = 5r^2.$$

$$\therefore \left[\nabla\times\nabla\left[e^{-\nabla\cdot(\alpha r^2\vec{r})}\right]\right]_i = \epsilon_{ijk}\,\partial_j\partial_k e^{-5\alpha r^2}$$

Obs. $\partial_k e^{-5\alpha r^2} = \frac{\partial u}{\partial x_k}\frac{\partial}{\partial u} e^{-5\alpha u}$; $u = r^2$.

↑ regla de la cadena.

$$= \partial_k(r^2)\left(-5\alpha e^{-5\alpha u}\right)$$
$$= 2r\,\partial_k r\left(-5\alpha e^{-5\alpha u}\right)$$
$$= -10\alpha e^{-5\alpha r^2}\, r\underbrace{\partial_k r}_{\frac{x_k}{r}}$$

$\langle \text{II} \rangle$

$$= -10\alpha\, x_k e^{-5\alpha r^2}$$

$$\Downarrow$$

Finalmente:

$$\partial_k e^{-5\alpha r^2} \equiv -10\alpha\, x_k e^{-5\alpha r^2} \qquad (*)$$

entonces

$$\left[\nabla \times \nabla\left[e^{-\nabla\cdot(\alpha r^2 \vec{r})}\right]\right]_i = -10\alpha\, \epsilon_{ijk}\, \partial_j\left[x_k e^{-5\alpha r^2}\right]$$

$$= -10\alpha\, \epsilon_{ijk}\left[(\partial_j x_k) e^{-5\alpha r^2} + x_k \partial_j\left(e^{-5\alpha r^2}\right)\right]$$

$$= -10\alpha\, \epsilon_{ijk}\left[\delta_{jk} e^{-5\alpha r^2} + x_k\left(-10\alpha\, x_j e^{-5\alpha r^2}\right)\right]$$

$$= -10\alpha \underbrace{\epsilon_{ijk}\,\delta_{jk}}_{0} e^{-5\alpha r^2} + 100\alpha^2 \underbrace{\epsilon_{ijk}\, x_j x_k}_{0} e^{-5\alpha r^2}$$

$$= 0.$$

$$\therefore \nabla \times \nabla\left[e^{-\nabla\cdot(\alpha r^2 \vec{r})}\right] = 0 \quad //$$

PROBLEMA 2

Evaluamos: $\nabla^2 \phi(\vec{r}) = \nabla^2 \left[\frac{1}{r^{1+2\beta}}\right]$

$$\Rightarrow \nabla^2 \phi(\vec{r}) = \nabla \cdot \nabla \left(\frac{1}{r^{1+2\beta}}\right) = \partial_i \partial_i \left[\frac{1}{r^{1+2\beta}}\right]$$

$$= \partial_i \partial_i \left(r^{-1-2\beta}\right)$$

$$= \partial_i \left(-(1+2\beta) r^{-2-2\beta} \underbrace{\partial_i r}_{\frac{x_i}{r}}\right)$$

$$= -(1+2\beta)\, \partial_i \left[\frac{x_i}{r^{3+2\beta}}\right]$$

$$= -(1+2\beta) \left[\frac{(\partial_i x_i)}{r^{3+2\beta}} + x_i\, \partial_i \left(r^{-3-2\beta}\right)\right]$$

$$= -(1+2\beta) \left[\frac{\delta_{ii}}{r^{3+2\beta}} + x_i (-3-2\beta) r^{-4-2\beta} \underbrace{(\partial_i r)}_{\frac{x_i}{r}}\right]$$

$$= -(1+2\beta) \left[\frac{3}{r^{3+2\beta}} - \frac{(3+2\beta)}{r^{5+2\beta}} x_i x_i\right]$$

$$= -(1+2\beta) \left[\frac{3}{r^{3+2\beta}} - \frac{(3+2\beta)\, r^2}{r^{5+2\beta}}\right] = 2(1+2\beta)\beta \,\frac{1}{r^{3+2\beta}}$$

$\beta = 0 \wedge \beta = -\frac{1}{2}$ permite $\nabla^2 \phi = 0$.

$\beta = -\frac{1}{2} \Rightarrow \phi = 1 = cte$

γ_1

Problema 3

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'}\left[\frac{1}{2}\left(3\frac{(\vec{r}\cdot\vec{r}')^2}{r^5} - \frac{r'^2}{r^3}\right)\right]\rho(\vec{r}')\,dV'$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{3}{2}\int_{V'}\frac{(\vec{r}\cdot\vec{r}')^2}{r^5}\rho(\vec{r}')\,dV' - \frac{1}{4\pi\varepsilon_0}\frac{1}{2}\int_{V'}\frac{r'^2}{r^3}\rho(\vec{r}')\,dV'$$

luego

$$\vec{E}(\vec{r}) = -\nabla\phi(\vec{r})$$

$$\Downarrow$$

$$\vec{E}(\vec{r}) = -\frac{3}{2}\frac{1}{4\pi\varepsilon_0}\int_{V'}\nabla\left[\frac{(\vec{r}\cdot\vec{r}')^2}{r^5}\right]\rho(\vec{r}')\,dV'$$

$$+\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\int_{V'}\nabla\left[\frac{r'^2}{r^3}\right]\rho(\vec{r}')\,dV'$$

$$\Downarrow$$

Para la componente i-ésima se tiene que:

$$E_i(\vec{r}) = -\frac{3}{2}\frac{1}{4\pi\varepsilon_0}\int_{V'}\left(\nabla\left[\frac{(\vec{r}\cdot\vec{r}')^2}{r^5}\right]\right)_i\rho(\vec{r}')\,dV'$$

$$+\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\int_{V'}\left(\nabla\left[\frac{r'^2}{r^3}\right]\right)_i\rho(\vec{r}')\,dV'$$

fórmulas útiles:

$\circledast$ $\partial_i r = \frac{x_i}{r}$

$\circledast$ $\partial_i (x_\ell x_\ell) = 2 x_i$

luego

$$\left[\nabla\left[\frac{(\vec{r}\cdot\vec{r}\,')^2}{r^5}\right]\right]_i = \partial_i\left[x_\ell x'_\ell x_k x'_k r^{-5}\right]$$

$$= x'_\ell x'_k\, \partial_i\left(x_\ell x_k r^{-5}\right)$$

$$= x'_\ell x'_k\left[(\partial_i x_\ell)\, x_k r^{-5} + x_\ell (\partial_i x_k) r^{-5} + x_\ell x_k\, \partial_i (r^{-5})\right]$$

$$= x'_\ell x'_k\left[\delta_{i\ell}\frac{x_k}{r^5} + \delta_{ik}\frac{x_\ell}{r^5} + (-5) r^{-6} x_\ell x_k \underbrace{(\partial_i r)}_{\frac{x_i}{r}}\right]$$

$$= x'_\ell x'_k\left[\delta_{i\ell}\frac{x_k}{r^5} + \delta_{ik}\frac{x_\ell}{r^5} - 5\frac{x_\ell x_k}{r^6}\frac{x_i}{r}\right]$$

$$= \delta_{i\ell}\frac{x'_\ell x'_k x_k}{r^5} + \delta_{ik}\frac{x'_\ell x'_k x_\ell}{r^5} - 5\frac{x'_\ell x'_k x_\ell x_k x_i}{r^7}$$

$$= \frac{(x'_k x_k)\, x'_i}{r^5} + \frac{(x'_\ell x_\ell)\, x'_i}{r^5} - 5\frac{(x'_\ell x_\ell)(x'_k x_k)}{r^7} x_i$$

$$= \frac{(\vec{r}\cdot\vec{r}\,')\, x'_i}{r^5} + \frac{(\vec{r}\cdot\vec{r}\,')\, x'_i}{r^5} - 5\frac{(\vec{r}\cdot\vec{r}\,')^2}{r^7} x_i$$

$$= 2\frac{(\vec{r}\cdot\vec{r}\,')}{r^5} x'_i - 5\frac{(\vec{r}\cdot\vec{r}\,')^2}{r^7} x_i \quad //$$

$\overline{\gamma_3}$

también se tiene que:

$$\left[\nabla\left(\frac{r'^2}{r^3}\right)\right]_i = r'^2\left[\nabla\left(\frac{1}{r^3}\right)\right]_i$$

$$= r'^2\,\partial_i(r^{-3})$$

$$= r'^2(-3)r^{-4}\partial_i r$$

$$= -3\frac{r'^2}{r^4}\frac{x_i}{r} = -3\frac{r'^2}{r^5}x_i \;//$$

Entonces:

$$E_i(\vec{r}) = -\frac{3}{2}\frac{1}{4\pi\varepsilon_0}\left[\int_{V'}\frac{2(\vec{r}\cdot\vec{r}')}{r^5}x'_i\,\rho(\vec{r}')dV' - \int_{V'}\frac{5(\vec{r}\cdot\vec{r}')^2}{r^7}x_i\,\rho(\vec{r}')dV'\right]$$

$$-\frac{3}{2}\frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{r'^2}{r^5}x_i\,\rho(\vec{r}')dV'$$

Finalmente:

$$\vec{E}(\vec{r}) = \frac{15}{2}\frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{(\vec{r}\cdot\vec{r}')^2}{r^7}\vec{r}\,\rho(\vec{r}')dV' - \frac{3}{4\pi\varepsilon_0}\int_{V'}\frac{(\vec{r}\cdot\vec{r}')\vec{r}'}{r^5}\rho(\vec{r}')dV'$$

$$-\frac{3}{2}\frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{r'^2}{r^5}\vec{r}\,\rho(\vec{r}')dV'$$