Outline of solutions to Homework 1

Problem 1.3: For convenience we place the cube with one corner at the origin and the cube in the positive quadrant. Then the vectors of the two body diagonals are $\vec{b}_1 = (1,1,1)$ and $\vec{b}_2 = (-1,1,1)$. The angle between the two vectors is found from the definition of the dot product,

$$\vec{b}_1 \cdot \vec{b}_2 = |\vec{b}_1||\vec{b}_2|cos(\theta) \;\text{ so that }\; 1 = 3cos(\theta) \tag{1}$$

Problem 1.4: The two vectors $\vec{v}_1 = (0,2,0) - (1,0,0) = (-1,2,0)$ and $\vec{v}_2 = (0,0,3) - (1,0,0) = (-1,0,3)$ define the plane of the surface of interest. The cross product of these two vectors, $\vec{v}_1 \wedge \vec{v}_2 = (6,-3,2)$, is normal to the plane of interest. The unit normal of interest is then $(6/7, -3/7, 2/7)$.

Problem 1.7: The difference is $\vec{r} - \vec{r}' = (4,6,8) - (2,8,7) = (2,-2,1)$, so the unit vector is $(2/3, -2/3, 1/3)$.

Problem 1.13: In these calculations we can usually just look at one component and then note that the other components have a similar behavior, with the appropriate change of variables. a) The gradient operator is $\vec{\nabla} = \hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z$ and $r^2 = x^2 + y^2 + z^2$. Since $\partial_x(r^2) = 2x$, with similar expressions for the other components, it is clear that $\vec{\nabla}(r^2) = 2\vec{r} = 2r\hat{r}$.
b) We have,

$$\vec{\nabla}(\frac{1}{r}) = (\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z)(\frac{1}{(x^2+y^2+z^2)^{1/2}}) \tag{2}$$

Since $\partial_x(1/r) = -x/r^3$, it is clear that $\vec{\nabla}(\frac{1}{r}) = -\vec{r}/r^3 = -\hat{r}/r^2$. c) Using a similar procedure to that used in b) it can be shown that $\vec{\nabla} r^n = nr^{n-1}\hat{r}$

Problem 1.23: In these proofs we can do the calculation for one component and then extend to the other components. For the first quotient formula we have,

$$\vec{\nabla}(\frac{f}{g}) = \hat{x}\partial_x(\frac{f}{g}) + \hat{y}\partial_y(\frac{f}{g}) + \hat{z}\partial_z(\frac{f}{g}) \tag{3}$$

For the first component we have,

$$\frac{\partial}{\partial x}(\frac{f}{g}) = \frac{g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}}{g^2} = (\frac{1}{g}\vec{\nabla} f - \frac{f}{g^2}\vec{\nabla} g)_x \tag{4}$$

The other components proceed in the same way, completing the proof. The second quotient formula is

$$\vec{\nabla} \cdot (\frac{\vec{A}}{g}) = \partial_x(\frac{A_x}{g}) + \partial_y(\frac{A_y}{g}) + \partial_z(\frac{A_z}{g}) \tag{5}$$

The first term gives,

$$\frac{\partial}{\partial x}(\frac{A_x}{g}) = \frac{g\frac{\partial A_x}{\partial x} - A_x\frac{\partial g}{\partial x}}{g^2} \tag{6}$$

Similar terms occur for the second term, using $x \to y$, and third term, using $x \to z$. It is then evident that this is equivalent to $\vec{\nabla} \cdot \vec{A}/g - \vec{A} \cdot (\vec{\nabla} g)/g^2$ as required. The third quotient rule considers,

$$\vec{\nabla} \wedge (\frac{\vec{A}}{g}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{x}(\partial_y(\frac{A_z}{g}) - \partial_z(\frac{A_y}{g})) - \hat{y}(\partial_x(\frac{A_z}{g}) - \partial_z(\frac{A_x}{g})) + \hat{z}(\partial_x(\frac{A_y}{g}) - \partial_y(\frac{A_x}{g})) \tag{7}$$

The x-component of this expression gives,

$$\frac{g\frac{\partial A_z}{\partial y} - A_z\frac{\partial g}{\partial y}}{g^2} - \frac{g\frac{\partial A_y}{\partial z} - A_y\frac{\partial g}{\partial z}}{g^2} = (\frac{1}{g}(\vec{\nabla} \wedge \vec{A}) + \frac{1}{g^2}\vec{A} \wedge (\vec{\nabla} g))_x \tag{8}$$

The other components follow in the same way.

Problem 1.30: Because any plane obeys $Ax + By + Cz + D$, it is easy to show that the planar surface of the tetrahedon obeys $x + y + z = 1$. It is easiest to do the z-integral last, so doing the x-integral we get,

$$\int_0^1 z^2 dz \int_0^{1-z} dy \int_0^{1-y-z} dx = \int_0^1 z^2 dz \int_0^{1-z} dy(1 - y - z) \tag{9}$$

Now doing the y integral and then the z-integral we find,

$$\int_0^1 dzz^2((1-z)y - y^2/2)|_0^{1-z} = \frac{1}{2}\int_0^1 dzz^2(1-z)^2 = \frac{1}{6} - \frac{1}{4} + \frac{1}{10} = \frac{1}{60} \tag{10}$$

Problem 1.32: We want to demonstrate the divergence theorem $\oint \vec{v} \cdot d\vec{a} = \int (\vec{\nabla} \cdot \vec{v}) d\tau$ for the particular case $\vec{v} = xy\hat{x} + 2yz\hat{y} + 3zx\hat{z}$. First the volume integral where $\vec{\nabla} \cdot \vec{v} = y + 2z + 3x$, so the integral becomes,

$$\int_0^2 dx \int_0^2 dy \int_0^2 dz(y + 2z + 3x) = 4(\int_0^2 ydy + 2\int_0^2 zdz + 3\int_0^2 xdx) = 48 \tag{11}$$

The surface integral must consider all 6 surfaces of the cube. The surface normals point out of the cube and are $\hat{x}, -\hat{x}, \hat{y}, -\hat{y}, \hat{z}, -\hat{z}$. We also have: $\vec{v} \cdot \hat{x} = xy, \vec{v} \cdot \hat{y} = 2yz, \vec{v} \cdot \hat{z} = 3zx$. So the surface integrals to be considered are, for the surfaces with normals in the $\hat{x}$ and $-\hat{x}$ directions,

$$\int_0^2 dy \int_0^2 dzxy|_{x=2} = 8, \quad \text{and} \quad -\int_0^2 dy \int_0^2 dzxy|_{x=0} = 0, \tag{12}$$

and for the surfaces in the $\hat{y}$ and $-\hat{y}$ directions,

$$\int_0^2 dx \int_0^2 dz2yz|_{y=2} = 16, \quad \text{and} \quad -\int_0^2 dx \int_0^2 dz2yz|_{y=0} = 0, \tag{13}$$

and for the surfaces with normals in the $\hat{z}$ and $-\hat{z}$ directions

$$\int_0^2 dy \int_0^2 dz3xz|_{z=2} = 24, \quad \text{and} \quad -\int_0^2 dy \int_0^2 dy3xz|_{z=0} = 0, \tag{14}$$

The sum of the surface integrals is 48 as required to satisfy the divergence theorem.

Problem 1.37 The easiest way to do this problem is to draw the unit vectors and to use geometric reasoning, to find,

$$\hat{r} = sin(\theta)cos(\phi)\hat{x} + sin(\theta)sin(\phi)\hat{y} + cos(\theta)\hat{z}, \tag{15}$$

$$\hat{\theta} = cos(\theta)cos(\phi)\hat{x} + cos(\theta)sin(\phi)\hat{y} - sin(\theta)\hat{z}, \tag{16}$$

and

$$\hat{\phi} = -sin(\phi)\hat{x} + cos(\phi)\hat{y}, \tag{17}$$

It can also be done algebraically, but it does not provide much illumination.

Problem 1.39: We want to verify the divergence theorem, $\oint \vec{v} \cdot d\vec{a} = \int (\vec{\nabla} \cdot \vec{v}) d\tau$ for a hemisphere in the upper half plane, centered at the origin and of radius R. The function is, $\vec{v} = rcos(\theta)\hat{r} + rsin(\theta)\hat{\theta} + rsin(\theta)cos(\phi)\hat{\phi}$. First the volume integral where we first evaluate the divergence in spherical polar co-ordinates,

$$\vec{\nabla} \cdot \vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2(rcos(\theta))) + \frac{1}{rsin(\theta)}\frac{\partial}{\partial \theta}(sin(\theta)(rsin(\theta))) + \frac{1}{rsin(\theta)}\frac{\partial}{\partial \phi}(rsin(\theta)cos(\phi)) = 5cos(\theta) - sin(\phi) \tag{18}$$

The volume integral in spherical polar co-ordinates is then,

$$\int_0^R r^2 dr \int_0^{\pi/2} sin(\theta)d\theta \int_0^{2\pi} d\phi(5cos(\theta) - sin(\phi)) \tag{19}$$

The second term is zero as the integral of $sin(\phi)$ over the interval $[0, 2\pi]$ is zero. The ϕ integral gives 2π for the first term. Also doing the R integral and using a half angle formula for the θ term yields,

$$\frac{2\pi R^3}{3}\frac{5}{2}\int_0^{\pi/2} d\theta sin(2\theta) = \frac{5\pi R^3}{3} \tag{20}$$

The surface normals are $\hat{r}$ for the spherical part and $-\hat{z}$ for the flat part of the hemisphere. The surface integral for the spherical part is then,

$$\int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta R^2 sin(\theta)(Rcos(\theta)) = \pi R^3. \tag{21}$$

Using $\hat{z} = \hat{r}cos(\theta) - \hat{\theta}sin(\theta)$, The integral over the flat part is,

$$\int_0^R \int_0^{2\pi} rdrd\phi(-rcos^2(\theta) + rsin^2(\theta)) = \frac{2\pi R^3}{3} \tag{22}$$

The sum of the surface contributions gives the volume integral, as required.

Problem 1.46: a) $\rho(\vec{r}) = q\delta^3(\vec{r} - \vec{r}')$, b) $\rho(\vec{r}) = q\delta^3(\vec{r} - \vec{r}') - q\delta^3(\vec{r})$, c) $\rho(\vec{r}) = A\delta(r - R)$. To make sure that $\int d\vec{r}\rho(\vec{r}) = Q$, we do the integral in spherical polar co-ordinates,

$$\int_0^R r^2 \int_0^{\pi} sin(\theta)d\theta \int_0^{2\pi} d\phi\delta(r - R) = 4\pi R^2 \tag{23}$$

Therefore $A = Q/(4\pi R^2)$.

Problem 1.54: We need to verify Stokes theorem, $\int(\vec{\nabla} \wedge \vec{v}) \cdot d\vec{a} = \oint \vec{v} \cdot d\vec{l}.$ for the function $\vec{v} = ay\hat{x} + bx\hat{y}$. The contour is a circle of radius R centered at the origin and having unit normal $\hat{z}$. The path integral is,

$$\int_0^{2\pi} (ay\hat{x} + bx\hat{y}) \cdot \hat{\phi}Rd\phi = \int_0^{2\pi} (aRsin(\phi)(\hat{r}cos(\phi) - \hat{\phi}sin(\phi)) + brcos(\theta)(\hat{r}sin(\phi) + \hat{\phi}cos(\phi)) \cdot \hat{\phi}Rd\phi \tag{24}$$

which reduces to,

$$\int_0^{2\pi} (-aR^2 sin^2(\phi) + bR^2 cos^2(\phi))d\phi = \pi R^2(b - a) \tag{25}$$

For the surface integral we use $\vec{\nabla} \wedge \vec{v} = (b - a)\hat{z}$ and $d\vec{a} = rd\phi dr\hat{z}$, so the integral is,

$$\int_0^R rdr \int_0^{2\pi} d\phi(b - a) = \pi R^2(b - a), \tag{26}$$

so Stokes theorem is verified.

Problem 1.58 We need to verify the divergence theorem for $\vec{v} = r^2 sin(\theta)\hat{r} + 4r^2 cos(\theta)\hat{\theta} + r^2 tan(\theta)\hat{\phi}$. For the volume integral, we first find the divergence to be,

$$\vec{\nabla} \cdot \vec{v} = 4rsin(\theta) + 4r\frac{cos(2\theta)}{sin(\theta)} \tag{27}$$

Since there is no dependence on ϕ, the volume integral is,

$$2\pi \int_0^R r^2 \int_0^{\pi/6} d\theta sin(\theta)(4rsin(\theta) + 4r\frac{cos(2\theta)}{sin(\theta)}) = 2\pi R^4(\frac{\pi}{12} + \frac{3^{1/2}}{8}) \tag{28}$$

There are two parts to the surface integral: the spherical cap with unit normal $\hat{r}$ and the cone part with surface normal $\hat{\theta}$. The surface integral of the cone part is,

$$\int_0^R dr \int_0^{2\pi} rsin(\theta)d\phi(4r^2 cos(\theta)) = \frac{1}{2}\pi 3^{1/2} R^4. \tag{29}$$

Note that the integral is in cylindrical co-ordinates but the radius is $rsin(\theta)$. The spherical cap part is,

$$\int_0^{\pi/6} R^2 sin(\theta)d\theta \int_0^{2\pi} d\phi R^2 sin(\theta) = 2\pi R^4(\frac{\pi}{12} - \frac{3^{1/2}}{8}) \tag{30}$$

As required the sum of the surface integrals gives the volume integral.