Resumen de la notación de Dirac
Mecánica Cuántica I (FIS 321)
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IPGG

Contenido : $\widehat{O}$peradores y $\overrightarrow{V}$ectores *: kets, bras, ketbras, brackets y notación matricial*

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1 Espacio vectorial (kets) sobre el cuerpo complejo $\mathbb{C}$

1.1 Propiedades de kets : $|\cdot\rangle$

- Conmutatividad respecto a la suma:
$$|\boldsymbol{\psi}\rangle + |\boldsymbol{\phi}\rangle = |\boldsymbol{\phi}\rangle + |\boldsymbol{\psi}\rangle$$

- Asociatividad respecto a la suma:
$$(\ |\boldsymbol{\psi}\rangle + |\boldsymbol{\phi}\rangle\ ) + |\boldsymbol{\theta}\rangle = |\boldsymbol{\phi}\rangle + (\ |\boldsymbol{\psi}\rangle + |\boldsymbol{\theta}\rangle\ )$$

- Vector nulo $|\mathbf{Nulo}\rangle \equiv |\mathbf{0}\rangle$, tal que:
$$|\boldsymbol{\psi}\rangle + |\mathbf{0}\rangle = |\boldsymbol{\psi}\rangle$$

- El vector opuesto de $|\boldsymbol{\psi}\rangle$ se define como $|\overline{\boldsymbol{\psi}}\rangle = -|\boldsymbol{\psi}\rangle$ tal que:
$$|\boldsymbol{\psi}\rangle + |\overline{\boldsymbol{\psi}}\rangle = |\mathbf{0}\rangle$$

- Ponderación y distributividad:
$$(\alpha + \beta)(\ |\boldsymbol{\psi}\rangle + |\boldsymbol{\phi}\rangle\ ) = \alpha\,|\boldsymbol{\psi}\rangle + \alpha\,|\boldsymbol{\phi}\rangle + \beta\,|\boldsymbol{\psi}\rangle + \beta\,|\boldsymbol{\phi}\rangle$$

- Equivalencia en la ponderación:
$$\alpha\,|\boldsymbol{\psi}\rangle = |\alpha\boldsymbol{\psi}\rangle$$

1.2 Espacio vectorial dual (bras) : $\langle\cdot|$

1.2.1 Correspondencia

- $|\boldsymbol{\phi}\rangle \iff \langle\boldsymbol{\phi}|$
- $\alpha\,|\boldsymbol{\psi}\rangle + \beta\,|\boldsymbol{\phi}\rangle \iff \alpha^*\,\langle\boldsymbol{\psi}| + \beta^*\,\langle\boldsymbol{\phi}|$
- $|\alpha\boldsymbol{\psi}\rangle \iff \langle\alpha\boldsymbol{\psi}| = \alpha^*\,\langle\boldsymbol{\psi}|$

1.2.2 Relación operacional : Adjuntar

- $(|\cdot\rangle)^{\dagger} = \langle\cdot|$
- $(\langle\cdot|)^{\dagger} = |\cdot\rangle$

1.3 Producto interno o escalar

- Producto interno entre $|\boldsymbol{\phi}\rangle$ y $|\boldsymbol{\psi}\rangle$: $\langle\boldsymbol{\phi}\,|\boldsymbol{\psi}\,\rangle$.
- $\langle\boldsymbol{\psi}\,|\boldsymbol{\psi}\,\rangle \geq 0$ (Módulo de $|\boldsymbol{\psi}\rangle$) $\Longrightarrow$ Es cero solo si $|\boldsymbol{\psi}\rangle = |\mathbf{0}\rangle$.
- $\left|\widetilde{\psi}\right\rangle = \left(\dfrac{1}{\sqrt{\langle\psi\,|\psi\,\rangle}}\right)|\psi\rangle$ (Normalización del ket $\Longrightarrow$ Vector unitario) $\Longrightarrow \left\langle\widetilde{\psi}\,\middle|\widetilde{\psi}\right\rangle = 1$.
- $\langle\boldsymbol{\phi}\,|\boldsymbol{\psi}\,\rangle = 0 \Longrightarrow$ Vectores ortogonales.
- $\langle\alpha\boldsymbol{\phi}\,|\beta\boldsymbol{\psi}\,\rangle = \alpha^{*}\beta\ \langle\boldsymbol{\phi}\,|\boldsymbol{\psi}\,\rangle$
- $\langle\boldsymbol{\phi}\,|\boldsymbol{\psi}\,\rangle^{*} = \langle\boldsymbol{\psi}\,|\boldsymbol{\phi}\,\rangle$
- $\langle\boldsymbol{\phi}\,|\alpha_1\boldsymbol{\psi}_1 + \alpha_2\boldsymbol{\psi}_2\,\rangle = \alpha_1\,\langle\boldsymbol{\phi}\,|\boldsymbol{\psi}_1\,\rangle + \alpha_2\,\langle\boldsymbol{\phi}\,|\boldsymbol{\psi}_2\,\rangle$
- $\langle\alpha_1\boldsymbol{\phi}_1 + \alpha_2\boldsymbol{\phi}_2\,|\boldsymbol{\psi}\,\rangle = \alpha_1^{*}\,\langle\boldsymbol{\phi}_1\,|\boldsymbol{\psi}\,\rangle + \alpha_2^{*}\,\langle\boldsymbol{\phi}_2\,|\boldsymbol{\psi}\,\rangle$

2 Operadores

2.1 Operadores lineales

Cierto operador $\widehat{\mathbf{A}}$ es lineal si cumple lo siguiente:

$$\widehat{\mathbf{A}}\,(\ \alpha\,|\boldsymbol{\psi}\rangle + \beta\,|\boldsymbol{\phi}\rangle\ ) = \alpha\widehat{\mathbf{A}}\,|\boldsymbol{\psi}\rangle + \beta\widehat{\mathbf{A}}\,|\boldsymbol{\phi}\rangle$$

2.2 Producto de operadores

- $\left[\widehat{\mathbf{A}},\widehat{\mathbf{B}}\right] = \widehat{\mathbf{A}}\widehat{\mathbf{B}} - \widehat{\mathbf{B}}\widehat{\mathbf{A}}$ Conmutador
- $\left\{\widehat{\mathbf{A}},\widehat{\mathbf{B}}\right\} = \widehat{\mathbf{A}}\widehat{\mathbf{B}} + \widehat{\mathbf{B}}\widehat{\mathbf{A}}$ Anticonmutador
- Propiedades

  a).- $\left[\widehat{\mathbf{A}},\widehat{\mathbf{B}}\right] = -\left[\widehat{\mathbf{B}},\widehat{\mathbf{A}}\right]$

  b).- $\left[\widehat{\mathbf{A}}, b\widehat{\mathbf{B}} + c\widehat{\mathbf{C}}\right] = b\left[\widehat{\mathbf{A}},\widehat{B}\right] + c\left[\widehat{\mathbf{A}},\widehat{\mathbf{C}}\right]$

  c).- $\left[a\widehat{\mathbf{A}} + b\widehat{\mathbf{B}},\widehat{\mathbf{C}}\right] = a\left[\widehat{\mathbf{A}},\widehat{\mathbf{C}}\right] + b\left[\widehat{\mathbf{B}},\widehat{\mathbf{C}}\right]$

  d).- $\left[\widehat{\mathbf{A}},\widehat{\mathbf{B}}\widehat{\mathbf{C}}\right] = \widehat{\mathbf{B}}\left[\widehat{\mathbf{A}},\widehat{\mathbf{C}}\right] + \left[\widehat{\mathbf{A}},\widehat{\mathbf{B}}\right]\widehat{\mathbf{C}}$

  e).- $\left[\widehat{\mathbf{A}}\widehat{\mathbf{B}},\widehat{\mathbf{C}}\right] = \widehat{\mathbf{A}}\left[\widehat{\mathbf{B}},\widehat{\mathbf{C}}\right] + \left[\widehat{\mathbf{A}},\widehat{\mathbf{C}}\right]\widehat{\mathbf{B}}$

  f).- Identidad de *Jacobi* : $\left[\widehat{\mathbf{A}},\left[\widehat{\mathbf{B}},\widehat{\mathbf{C}}\right]\right] + \left[\widehat{\mathbf{C}},\left[\widehat{\mathbf{A}},\widehat{\mathbf{B}}\right]\right] + \left[\widehat{\mathbf{B}},\left[\widehat{\mathbf{C}},\widehat{\mathbf{A}}\right]\right] = 0\cdot\widehat{\mathbf{1}}$

3 Adjunto de un operador

Se define $\widehat{\mathbf{A}}^{\dagger}$ como el adjunto del operador $\widehat{\mathbf{A}}$.

3.1 Propiedades

- $(\alpha)^{\dagger} = \alpha^{*}$ (Si α es una cantidad escalar)
- $\left(\widehat{\mathbf{A}}^{\dagger}\right)^{\dagger} = \widehat{\mathbf{A}}$
- $\left(\widehat{\mathbf{A}}\widehat{\mathbf{B}}\right)^{\dagger} = \widehat{\mathbf{B}}^{\dagger}\widehat{\mathbf{A}}^{\dagger}$
- $\left(\alpha\widehat{\mathbf{B}}\right)^{\dagger} = \widehat{\mathbf{B}}^{\dagger}\alpha^{*} = \alpha^{*}\widehat{\mathbf{B}}^{\dagger}$
- $\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\right)^{\dagger} = \widehat{\mathbf{A}}^{\dagger} + \widehat{\mathbf{B}}^{\dagger}$
- $\left\langle \boldsymbol{\phi} \middle| \widehat{\mathbf{A}} \middle| \boldsymbol{\psi} \right\rangle = \left\langle \boldsymbol{\phi} \middle| \widehat{\mathbf{A}}\boldsymbol{\psi} \right\rangle = \left\langle \widehat{\mathbf{A}}^{\dagger}\boldsymbol{\phi} \middle| \boldsymbol{\psi} \right\rangle$
- $\left\langle \boldsymbol{\phi} \middle| \widehat{\mathbf{A}} \middle| \boldsymbol{\psi} \right\rangle^{*} = \left\langle \boldsymbol{\psi} \middle| \widehat{\mathbf{A}}^{\dagger} \middle| \boldsymbol{\phi} \right\rangle$
- $\left| \alpha\widehat{\mathbf{B}}\boldsymbol{\phi} \right\rangle = \alpha\widehat{\mathbf{B}} \left|\boldsymbol{\phi}\right\rangle$
- $\left\langle \alpha\widehat{\mathbf{B}}\boldsymbol{\phi} \right| = \left\langle\boldsymbol{\phi}\right| \alpha^{*}\widehat{\mathbf{B}}^{\dagger} = \alpha^{*} \left\langle\boldsymbol{\phi}\right| \widehat{\mathbf{B}}^{\dagger}$

3.2 Hermiticidad

- $\widehat{\mathbf{A}}^{\dagger} = \widehat{\mathbf{A}}$ (Operador hermítico)
- $\widehat{\mathbf{A}}^{\dagger} = -\widehat{\mathbf{A}}$ (Operador antihermítico)
- Si $\widehat{\mathbf{A}}^{\dagger}$es hermítico y $\left|\boldsymbol{\phi}\right\rangle$ es un vector arbitrario, entonces:

  a).- $\left\langle \boldsymbol{\phi} \middle| \widehat{\mathbf{A}}^{2} \middle| \boldsymbol{\phi} \right\rangle \geq 0$

  b).- $\left\langle \boldsymbol{\phi} \middle| \widehat{\mathbf{A}} \middle| \boldsymbol{\phi} \right\rangle = \left\langle \boldsymbol{\phi} \middle| \widehat{\mathbf{A}} \middle| \boldsymbol{\phi} \right\rangle^{*}$

3.3 Casos especiales

- $\widehat{\mathbf{U}}^{\dagger}\widehat{\mathbf{U}} = \widehat{\mathbf{U}}\widehat{\mathbf{U}}^{\dagger}$ (Operador **normal**)
- $\widehat{\mathbf{U}}\widehat{\mathbf{U}}^{\dagger} = \widehat{\mathbf{U}}^{\dagger}\widehat{\mathbf{U}} = \widehat{\mathbf{1}}$ (Operador **unitario**).
  - si $U_{ij} = U_{ij}^{*}$, entonces se cumple que $\widehat{\mathbf{U}}^{T}\widehat{\mathbf{U}} = \widehat{\mathbf{U}}\widehat{\mathbf{U}}^{T} = \widehat{\mathbf{1}}$, este operador es entonces **ortogonal**.
  - $\widehat{\mathbf{U}}$ invertible $\implies$ $\widehat{\mathbf{U}}^{\dagger} = \widehat{\mathbf{U}}^{-1}$
  - $\widehat{\mathbf{U}}\widehat{\mathbf{U}}^{\dagger} = \widehat{\mathbf{1}}$
  - $\widehat{\mathbf{U}}^{\dagger}$ es unitario
  - Las columnas de $\widehat{\mathbf{U}}$ forman un conjunto ortonormal de vectores
  - Las filas de $\widehat{\mathbf{U}}$ forman un conjunto ortonormal de vectores

- $\widehat{\mathbf{P}}_\psi = \dfrac{1}{\langle \boldsymbol{\psi} \,|\boldsymbol{\psi}\,\rangle} \,|\boldsymbol{\psi}\rangle \langle \boldsymbol{\psi}|$ (Operador **proyección**) → Dado un vector arbitrario, el operador proyección extrae la componente de este en la dirección unitaria $\dfrac{1}{\sqrt{\langle \boldsymbol{\psi} \,|\boldsymbol{\psi}\,\rangle}} \,|\boldsymbol{\psi}\rangle$.

4 Kets, Bras y Operadores en términos de una *base ortonormal*

4.1 Vectores y operadores

Sea una base vectorial discreta (recordar que también puede ser continua) $\{|n\rangle\}$ ortonormal, entonces:

$$\langle n \,| l\,\rangle = \delta_{nl}$$

Los vectores arbitrarios $|\boldsymbol{\psi}\rangle$ y $|\boldsymbol{\phi}\rangle$ pueden ser descritos cada uno de ellos como una combinación lineal de los elementos de la base $\{|n\rangle\}$:

$$|\boldsymbol{\psi}\rangle = \sum_n a_n \;|n\rangle \qquad\qquad |\boldsymbol{\phi}\rangle = \sum_n b_n \;|n\rangle$$

siendo $a_n = \langle n \,|\boldsymbol{\psi}\,\rangle$ las componentes del vector $|\boldsymbol{\psi}\rangle$ en cada una de las direcciones $|n\rangle$, de igual manera para b_n.

- $\langle \boldsymbol{\psi}| = |\boldsymbol{\psi}\rangle^\dagger = \sum_n a_n^* \;\langle n|$
- $\langle \boldsymbol{\psi} \,|\boldsymbol{\phi}\,\rangle = \sum_n a_n^* b_n$
- $\langle \boldsymbol{\psi} \,|\boldsymbol{\psi}\,\rangle = \sum_n a_n^* a_n = \sum_n |a_n|^2$

Un operador arbitrario $\widehat{\mathbf{A}}$ está descrito en términos de esta base como:

$$\widehat{\mathbf{A}} = \sum_n \sum_m a_{nm} \,|n\rangle \langle m|$$

siendo $a_{nm} = \left\langle n \,\middle|\widehat{\mathbf{A}}\middle|\, m\right\rangle$.

4.2 Operador unidad $\widehat{\mathbf{1}}$ y Completitud

Existe un operador tal que $\widehat{\mathbf{1}}\,|\boldsymbol{\psi}\rangle = |\boldsymbol{\psi}\rangle$, etc. Dicho operador es definido de la siguiente forma en términos de la base $\{|n\rangle\}$:

$$\widehat{\mathbf{1}} = \sum_n |n\rangle \langle n| \qquad \text{(Relación de completitud)}$$

5 Notación matricial de operadores, bras y kets

5.1 $\widehat{\text{O}}$peradores = Matrices

Un operador $\widehat{\mathbf{A}}$ arbitrario descrito en términos de un set completo ortonormal $\{|n\rangle\}$ tiene la siguiente representación matricial :

$$\widehat{\mathbf{A}} = \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & \\ \vdots & \ddots & \vdots & & \\ a_{i1} & \cdots & a_{ij} & & \\ \vdots & & & \ddots & \\ & & & & \end{pmatrix}$$

siendo:

$$a_{ij} = \left\langle i \left| \widehat{\mathbf{A}} \right| j \right\rangle$$

5.2 Ket = Vector columna $\iff$ Bra = Vector fila

La notación matricial de bras y kets es la siguiente:

$$|\boldsymbol{\psi}\rangle = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{pmatrix} \qquad \text{con } b_i = \langle i | \boldsymbol{\psi}\rangle$$

$$\langle\boldsymbol{\phi}| = \begin{pmatrix} a_1^* \\ a_2^* \\ a_3^* \\ \vdots \end{pmatrix}^T \qquad \text{con } a_i^* = \langle i | \boldsymbol{\phi}\rangle^*$$

5.3 Producto interno

$$\langle \boldsymbol{\phi} | \boldsymbol{\psi} \rangle = \begin{pmatrix} a_1^* & a_2^* & a_3^* & ... \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{pmatrix} = \sum_i a_i^* b_i$$

5.4 Producto externo

$$|\boldsymbol{\psi}\rangle \langle\boldsymbol{\phi}| = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{pmatrix} \begin{pmatrix} a_1^* & a_2^* & a_3^* & ... \end{pmatrix} = \begin{pmatrix} b_1 a_1^* & b_1 a_2^* & ... & \\ b_2 a_1^* & b_2 a_2^* & & \\ \vdots & & \ddots & \\ & & & \end{pmatrix}$$

5.5 Propiedades matriciales (usando convención de suma de Einstein)

- $\left(\widehat{\mathbf{A}}\widehat{\mathbf{B}}\right)_{ij} = \left\langle i \left| \widehat{\mathbf{A}}\widehat{\mathbf{B}} \right| j \right\rangle = A_{ik} B_{kj}$
- $\left(\widehat{\mathbf{A}}\widehat{\mathbf{B}}\widehat{\mathbf{C}}\right)_{ij} = \left\langle i \left| \widehat{\mathbf{A}}\widehat{\mathbf{B}}\widehat{\mathbf{C}} \right| j \right\rangle = A_{ik} B_{kl} C_{lj}$

- $tr\left(\widehat{\mathbf{A}}\right) = \left\langle i \left| \widehat{\mathbf{A}} \right| i \right\rangle = \sum_i A_{ii}$

- $\left(\widehat{\mathbf{A}}^{\dagger}\right)_{ij} = \left\langle i \left| \widehat{\mathbf{A}}^{\dagger} \right| j \right\rangle = \left\langle j \left| \widehat{\mathbf{A}} \right| i \right\rangle^{*} = A^{*}_{ji}$

- $\left(\widehat{\mathbf{1}}\right)_{ij} = \left\langle i \left| \widehat{\mathbf{1}} \right| j \right\rangle = \delta_{ij}$

5.6 Otras propiedades de matrices

- $\left(\widehat{\mathbf{A}}\widehat{\mathbf{B}}\right)^{-1} = \widehat{\mathbf{B}}^{-1}\widehat{\mathbf{A}}^{-1}$

- Si $\widehat{\mathbf{A}} = \widehat{\mathbf{A}}^{\dagger} \implies \widehat{\mathbf{A}}^{-1} = \left(\widehat{\mathbf{A}}^{-1}\right)^{\dagger}$, si $\widehat{\mathbf{A}}^{-1}$ existe.

- si $\widehat{\mathbf{U}}$ es unitaria y $\widehat{\mathbf{B}} = \widehat{\mathbf{U}}^{\dagger}\widehat{\mathbf{A}}\widehat{\mathbf{U}} \implies \widehat{\mathbf{A}} = \widehat{\mathbf{U}}\widehat{\mathbf{B}}\widehat{\mathbf{U}}^{\dagger}$