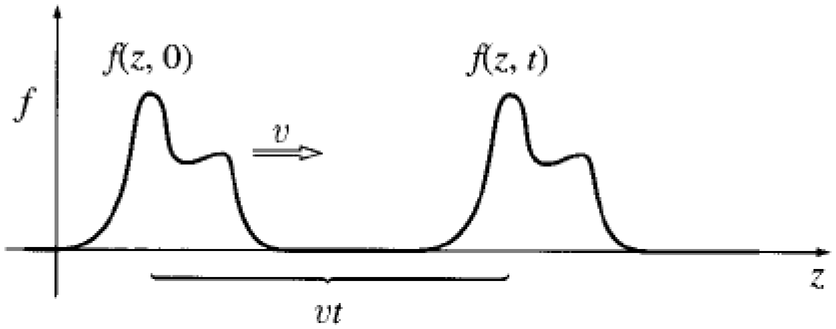
Chapter 9. Electromagnetic Waves

9.1 Waves in One Dimension

9.1.1 The Wave Equation

What is a "wave?"

Let's start with the simple case: fixed shape, constant speed:



How would you represent such a string object mathematically?

***f* (*z*, *t*)** represents the displacement of the string at the point *z,* at time *t.*

Given the *initial* shape of the string, $g(z) \equiv f(z, 0)$,

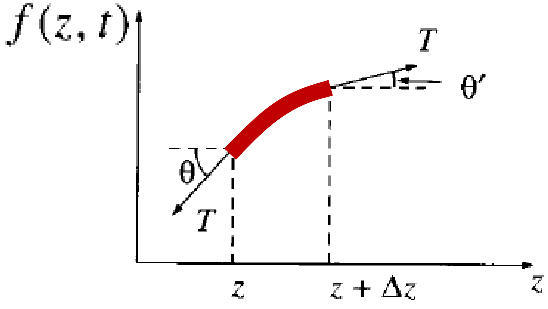
The displacement at point *z,* at the later time *t,* is the same as
the displacement a distance *vt* to the left (i.e. at *z* - *vt*), back at time *t* = 0:

$$f(z, t) = f(z - vt, 0) = g(z - vt)$$

➔ It represents a wave of fixed shape traveling in the +z direction at speed v.

(Mechanical or Classical) Waves Equation

Why does a stretched string support wave motion?
➔ Actually, it follows from Newton's second law.



Consider a stretched string under tension *T.*

The net transverse force on the segment between *z* and (*z* + *Δz*) is

$$\Delta F = T \sin\theta' - T \sin\theta$$

If the distortion of the string is not too great, sinθ ~ tanθ.

$$\Delta F \cong T(\tan\theta' - \tan\theta) = T\left(\left.\frac{\partial f}{\partial z}\right|_{z+\Delta z} - \left.\frac{\partial f}{\partial z}\right|_{z}\right) \cong T\frac{\partial^2 f}{\partial z^2}\Delta z$$

If the mass per unit length is μ, Newton's second law says

$$\Delta F = \mu(\Delta z)\frac{\partial^2 f}{\partial t^2} \Rightarrow \frac{\partial^2 f}{\partial z^2} = \frac{\mu}{T}\frac{\partial^2 f}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

where *v* (which, as we'll soon see, represents the speed of propagation) is $v = \sqrt{\frac{T}{\mu}}$

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

➔ *(classical)* ***Wave Equation***
because it admits as solutions all functions of the form $f(z, t) = g(z - vt)$

➔ (Mechanical or Classical) Waves Equation = Equation of Motion governed by Newton's second law!

Waves Equation

$$f(z,t) = g(z - vt)$$

: All functions that depend on the variables z and *t* in the special combination $u = z - vt$,
represent waves propagating in the z direction with speed *v*.

$$\frac{\partial f}{\partial z} = \frac{dg}{du}\frac{\partial u}{\partial z} = \frac{dg}{du}$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{dg}{du}\right) = \frac{d^2 g}{du^2}\frac{\partial u}{\partial z} = \frac{d^2 g}{du^2}$$

$$\frac{d^2 g}{du^2} = \frac{\partial^2 f}{\partial z^2}$$

$$\frac{\partial f}{\partial t} = \frac{dg}{du}\frac{\partial u}{\partial t} = -v\frac{dg}{du}$$

$$\frac{\partial^2 f}{\partial t^2} = -v\frac{\partial}{\partial t}\left(\frac{dg}{du}\right) = -v\frac{d^2 g}{du^2}\frac{\partial u}{\partial t} = v^2\frac{d^2 g}{du^2}$$

$$\frac{d^2 g}{du^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

➔ Therefore, the **Wave Equation** admits as solutions all functions of the form $f(z,t) = g(z - vt)$

For example, if *A* and *b* are constants (with the appropriate units),

$$f_1(z,t) = Ae^{-b(z-vt)^2}, \quad f_2(z,t) = A\sin[b(z-vt)], \quad f_3(z,t) = \frac{A}{b(z-vt)^2+1}$$ ➔ All represent waves

$$f_4(z,t) = Ae^{-b(bz^2+vt)}, \quad \text{and} \quad f_5(z,t) = A\sin(bz)\cos(bvt)^3$$ ➔ All do not represent waves

(Mechanical or Classical) Waves Equation $\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$

Problem 9.1 By explicit differentiation, check that the functions f_1, f_2, and f_3 satisfy the wave equation. Show that f_4 and f_5 do not.

$$f_1(z,t) = Ae^{-b(z-vt)^2}$$

$$\frac{\partial f_1}{\partial z} = -2Ab(z-vt)e^{-b(z-vt)^2} \qquad \frac{\partial^2 f_1}{\partial z^2} = -2Ab\left[e^{-b(z-vt)^2} - 2b(z-vt)^2 e^{-b(z-vt)^2}\right]$$

$$\frac{\partial f_1}{\partial t} = 2Abv(z-vt)e^{-b(z-vt)^2} \qquad \frac{\partial^2 f_1}{\partial t^2} = 2Abv\left[-ve^{-b(z-vt)^2} + 2bv(z-vt)^2 e^{-b(z-vt)^2}\right]$$

$$\Rightarrow \frac{\partial^2 f_1}{\partial t^2} = v^2\frac{\partial^2 f_1}{\partial z^2}$$

$$f_2(z,t) = A\sin[b(z-vt)]$$

$$\frac{\partial f_2}{\partial z} = Ab\cos[b(z-vt)] \qquad \frac{\partial^2 f_2}{\partial z^2} = -Ab^2\sin[b(z-vt)]$$

$$\frac{\partial f_2}{\partial t} = -Abv\cos[b(z-vt)] \qquad \frac{\partial^2 f_2}{\partial t^2} = -Ab^2v^2\sin[b(z-vt)]$$

$$\Rightarrow \frac{\partial^2 f_2}{\partial t^2} = v^2\frac{\partial^2 f_2}{\partial z^2}$$

$$f_3(z,t) = \frac{A}{b(z-vt)^2+1}$$

$$\frac{\partial f_3}{\partial z} = \frac{-2Ab(z-vt)}{[b(z-vt)^2+1]^2}; \quad \frac{\partial^2 f_3}{\partial z^2} = \frac{-2Ab}{[b(z-vt)^2+1]^2} + \frac{8Ab^2(z-vt)^2}{[b(z-vt)^2+1]^3}$$

$$\frac{\partial f_3}{\partial t} = \frac{2Abv(z-vt)}{[b(z-vt)^2+1]^2}; \quad \frac{\partial^2 f_3}{\partial t^2} = \frac{-2Abv^2}{[b(z-vt)^2+1]^2} + \frac{8Ab^2v^2(z-vt)^2}{[b(z-vt)^2+1]^3}$$

$$\Rightarrow \frac{\partial^2 f_3}{\partial t^2} = v^2\frac{\partial^2 f_3}{\partial z^2}$$

$$f_4(z,t) = Ae^{-b(bz^2+vt)}$$

$$\frac{\partial f_4}{\partial z} = -2Ab^2ze^{-b(bz^2+vt)} \qquad \frac{\partial^2 f_4}{\partial z^2} = -2Ab^2\left[e^{-b(bz^2+vt)} - 2b^2z^2e^{-b(bz^2+vt)}\right]$$

$$\frac{\partial f_4}{\partial t} = -Abve^{-b(bz^2+vt)} \qquad \frac{\partial^2 f_4}{\partial t^2} = Ab^2v^2e^{-b(bz^2+vt)}$$

$$\Rightarrow \frac{\partial^2 f_4}{\partial t^2} \neq v^2\frac{\partial^2 f_4}{\partial z^2}$$

$$f_5(z,t) = A\sin(bz)\cos(bvt)^3$$

$$\frac{\partial f_5}{\partial z} = Ab\cos(bz)\cos(bvt)^3 \qquad \frac{\partial^2 f_5}{\partial z^2} = -Ab^2\sin(bz)\cos(bvt)^3$$

$$\frac{\partial f_5}{\partial t} = -3Ab^3v^3t^2\sin(bz)\sin(bvt)^3 \qquad \frac{\partial^2 f_5}{\partial t^2} = -6Ab^3v^3t\sin(bz)\sin(bvt)^3 - 9Ab^6v^6t^4\sin(bz)\cos(bvt)^3$$

$$\Rightarrow \frac{\partial^2 f_5}{\partial t^2} \neq v^2\frac{\partial^2 f_5}{\partial z^2}$$

(Mechanical or Classical) Waves Equation $\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$

Problem 9.2 Show that the standing wave ***f(z, t) = A sin(kz) cos(kvt)*** satisfies the wave equation, and express it as the sum of a wave traveling to the left and a wave traveling to the right.

$$f(z,t) = g(z-vt) + h(z+vt)$$

$$f(z,t) = A\sin(kz)\cos(kvt)$$

$$\Rightarrow \frac{\partial^2 f}{\partial t^2} = v^2\frac{\partial^2 f}{\partial z^2}$$

By using the trig identity $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

$$f(z,t) = A\sin(kz)\cos(kvt)$$

$$=$$

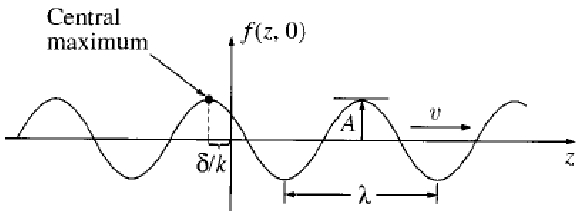
→ the sum of a wave traveling to the left and a wave traveling to the right.

9.1.2 Sinusoidal Waves

(i) Terminology: **Of all possible wave forms, the sinusoidal one is**

$$\boxed{f(z,t) = A\cos[k(z - vt) + \delta]}$$

(At time t = 0)



***A* is the amplitude of the wave**

→ it is positive, and represents the maximum displacement from equilibrium.

The argument of the cosine, $\phi = k(z-vt) + \delta$, is called the phase

→ **δ is the phase constant**

→ Obviously, you can add any integer multiple of 2π to δ without changing $f(z, t)$

→ Ordinarily, one uses a value in the range $0 \le \delta \le 2p$

***k* is the wave number**

→ it is related to the **wavelength** λ by the equation → $\lambda = \frac{2\pi}{k}$

One full cycle in a time period → $\phi = kvT = 2\pi$ → $T = \frac{2\pi}{kv}$

Frequency ν (number of oscillations per unit time) → $\nu = \frac{1}{T} = \frac{kv}{2\pi} = \frac{v}{\lambda}$

Angular frequency ω, the number of radians swept out per unit time → $\omega = 2\pi\nu = kv$

$$\boxed{f(z,t) = A\cos(kz - \omega t + \delta)}$$

→ A sinusoidal waves in terms of k and ω

Sinusoidal Waves

$$f(z,t) = A\cos[k(z - vt) + \delta] \qquad f(z,t) = A\cos(kz - \omega t + \delta)$$

(ii) Complex notation

In view of **Euler's formula**, $e^{i\theta} = \cos\theta + i\sin\theta$

$$f(z,t) = A\cos(kz - \omega t + \delta) \longrightarrow f(z,t) = \mathrm{Re}[Ae^{i(kz-\omega t+\delta)}]$$

Complex wave function:

$$\boxed{\tilde{f}(z,t) \equiv \tilde{A}e^{i(kz-\omega t)}} \qquad \tilde{A} \equiv Ae^{i\delta}$$ → **Complex amplitude**

- ➔ The actual wave function is the real part: $f(z,t) = \mathrm{Re}[\tilde{f}(z,t)]$
- ➔ The advantage of the complex notation is that exponentials are much easier to manipulate than sines and cosines.

Example 9.1 Suppose you want to combine two sinusoidal waves:

$$f_3 = f_1 + f_2 = \mathrm{Re}(\tilde{f}_1) + \mathrm{Re}(\tilde{f}_2) = \mathrm{Re}(\tilde{f}_1 + \tilde{f}_2) = \mathrm{Re}(\tilde{f}_3)$$

In particular, if they have the same frequency and wave number, you just add the (complex) amplitudes.

$$\tilde{f}_3 = \tilde{A}_1 e^{i(kz-\omega t)} + \tilde{A}_2 e^{i(kz-\omega t)} = \tilde{A}_3 e^{i(kz-\omega t)}$$

→ Then take the real part → $f_3(z,t) = A_3\cos(kz - \omega t + \delta_3)$

→ The combined wave still has the same frequency and wavelength.
→ *Without* using the complex notation, you will find yourself looking up trig identities and slogging through nasty algebra.

Sinusoidal Waves $\tilde{f}(z,t) \equiv \tilde{A}e^{i(kz-\omega t)}$ $f(z,t) = \text{Re}[\tilde{f}(z,t)] = A\cos(kz-\omega t+\delta)$

(iii) Linear combinations of sinusoidal waves

Any wave can be expressed as a linear combination of sinusoidal ones:

$$\tilde{f}(z,t) = \int_{-\infty}^{\infty} \tilde{A}(k)e^{i(kz-\omega t)}\,dk \qquad \text{(Fourier transformation)}$$

→ ***k* includes negative values**

→ This does not mean that λ and ω are negative: wavelength and frequency are *always* positive.

→ ***k* to run through negative values** in order to represent waves going in both directions.

Note the point that any wave can be written as a linear combination of sinusoidal waves,

→ therefore if you know how sinusoidal waves behave, you know in principle how any wave behaves.

→ So from now on, we shall confine our attention to sinusoidal waves.

9.1.3 Boundary conditions: Reflection and Transmission

What happens depends a lot on how the string is *attached* at the end.
→ that is, how the wave propagation depends on the specific boundary conditions.

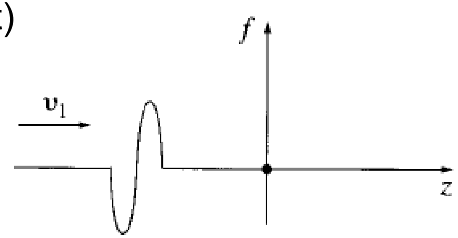
Suppose, for instance, that the string is simply tied onto a second string with different mass at z =0.

(→ Assume the wave velocities v_1 and v_2 are different)

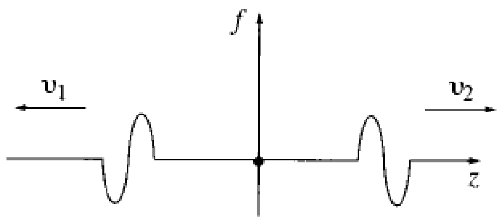
The incident wave → $\tilde{f}_I(z,t) = \tilde{A}_I e^{i(k_1 z - \omega t)}, \quad (z < 0)$

The reflected wave → $\tilde{f}_R(z,t) = \tilde{A}_R e^{i(-k_1 z - \omega t)}, \quad (z < 0)$

The reflected wave → $\tilde{f}_T(z,t) = \tilde{A}_T e^{i(k_2 z - \omega t)}, \quad (z > 0)$



(a) Incident pulse



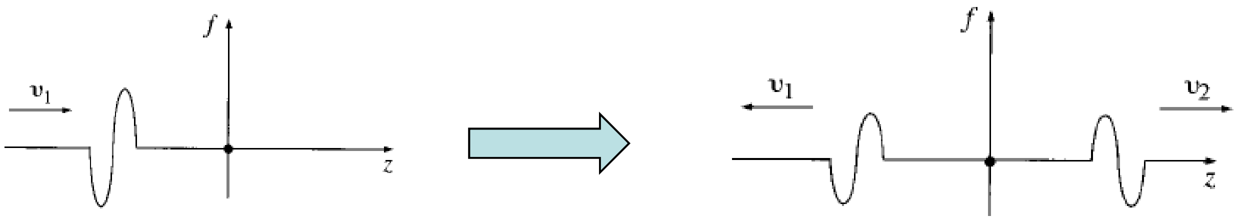
(b) Reflected and transmitted pulses

All parts of the system are oscillating at the same frequency ω
(a frequency determined by the person who is shaking the string)

Since the wave velocities are different in the two strings,
→ the wavelengths and wave numbers are also different:

$$\frac{\lambda_1}{\lambda_2} = \frac{k_2}{k_1} = \frac{v_1}{v_2}$$

Boundary conditions: Reflection and Transmission



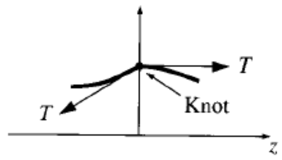
$\tilde{f}_I(z,t) = \tilde{A}_I e^{i(k_1 z - \omega t)}, \quad (z < 0)$ $\qquad$ $\tilde{f}_R(z,t) = \tilde{A}_R e^{i(-k_1 z - \omega t)}$ $\qquad$ $\tilde{f}_T(z,t) = \tilde{A}_T e^{i(k_2 z - \omega t)}$

For a sinusoidal incident wave, then, the net disturbance of the string is:

$$\tilde{f}(z,t) = \begin{cases} \tilde{A}_I e^{i(k_1 z - \omega t)} + \tilde{A}_R e^{i(-k_1 z - \omega t)}, & \text{for } z < 0, \\ \tilde{A}_T e^{i(k_2 z - \omega t)}, & \text{for } z > 0. \end{cases}$$

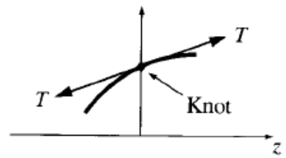
At the join $(z = 0)$,
the displacement and slope just slightly to the left $(z = 0^-)$ must equal those slightly to the right $(z = 0^+)$,
or else there would be a break between the two strings.

Mathematically, $f(z,t)$ is *continuous* at $z = 0$: $\longrightarrow$ $f(0^-, t) = f(0^+, t)$



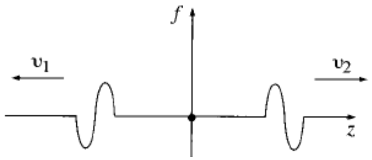
Discontinuous slope; force on knot

the *derivative* of f must *also* be continuous: $\longrightarrow$ $\left.\dfrac{\partial f}{\partial z}\right|_{0^-} = \left.\dfrac{\partial f}{\partial z}\right|_{0^+}$



Continuous slope; no force on knot

Boundary conditions: Reflection and Transmission



$$\tilde{f}(z,t) = \begin{cases} \tilde{A}_I e^{i(k_1 z - \omega t)} + \tilde{A}_R e^{i(-k_1 z - \omega t)}, & \text{for } z < 0, \\ \tilde{A}_T e^{i(k_2 z - \omega t)}, & \text{for } z > 0. \end{cases}$$

$$f(0^-,t) = f(0^+,t) \qquad \left.\frac{\partial f}{\partial z}\right|_{0^-} = \left.\frac{\partial f}{\partial z}\right|_{0^+} \longrightarrow \tilde{f}(0^-,t) = \tilde{f}(0^+,t), \quad \left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^-} = \left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^+}$$

These boundary conditions apply directly to the *real* wave function *f(z, t).*
→ But since the imaginary part differs from the real part only in the replacement of cosine by sine,
→ it follows that the complex wave function obeys the same rules:

$$\tilde{f}(0^-,t) = \tilde{f}(0^+,t), \quad \left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^-} = \left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^+} \longrightarrow \tilde{A}_I + \tilde{A}_R = \tilde{A}_T, \quad k_1(\tilde{A}_I - \tilde{A}_R) = k_2 \tilde{A}_T$$

$$\longrightarrow \tilde{A}_R = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)\tilde{A}_I, \quad \tilde{A}_T = \left(\frac{2k_1}{k_1 + k_2}\right)\tilde{A}_I$$

$$\frac{k_2}{k_1} = \frac{v_1}{v_2} \longrightarrow \tilde{A}_R = \left(\frac{v_2 - v_1}{v_2 + v_1}\right)\tilde{A}_I, \quad \tilde{A}_T = \left(\frac{2v_2}{v_2 + v_1}\right)\tilde{A}_I$$

The real amplitudes and phases, then, are related by $\longrightarrow A_R e^{i\delta_R} = \left(\frac{v_2 - v_1}{v_2 + v_1}\right) A_I e^{i\delta_I}, \quad A_T e^{i\delta_T} = \left(\frac{2v_2}{v_2 + v_1}\right) A_I e^{i\delta_I}$

If the second string is lighter than the first, $(\mu_2 < \mu_1$, so that $v_2 > v_1)$, → all three waves have the same phase angle

$$(\delta_R = \delta_T = \delta_I) \longrightarrow A_R = \left(\frac{v_2 - v_1}{v_2 + v_1}\right) A_I, \quad A_T = \left(\frac{2v_2}{v_2 + v_1}\right) A_I$$

If the second string is heavier than the first, $(v_2 < v_1)$ → the reflected 'Nave is out of phase by *180º*

$$(\delta_R + \pi = \delta_T = \delta_I) \longrightarrow A_R = \left(\frac{v_1 - v_2}{v_2 + v_1}\right) A_I \text{ and } A_T = \left(\frac{2v_2}{v_2 + v_1}\right) A_I$$

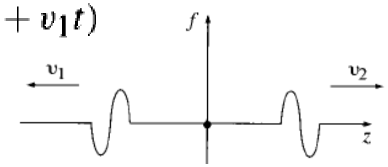
In particular, if the second string is *infinitely* massive, $\longrightarrow A_R = A_I$ and $A_T = 0$

Boundary conditions: $f(0^-,t) = f(0^+,t)$ $\quad \left.\frac{\partial f}{\partial z}\right|_{0^-} = \left.\frac{\partial f}{\partial z}\right|_{0^+}$

Problem 9.5 Suppose you send an incident wave of specified shape, $g_I(z - v_1 t)$, down string number 1. It gives rise to a reflected wave, $h_R(z + v_1 t)$, and a transmitted wave, $g_T(z - v_2 t)$. By imposing the boundary conditions, find h_R and g_T.

$h_R(z + v_1 t)$

$g_I(z - v_1 t) \longrightarrow$



$g_T(z - v_2 t)$

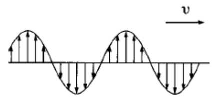
$f(0^-,t) = f(0^+,t)$

$$\left.\frac{\partial f}{\partial z}\right|_{0^-} = \left.\frac{\partial f}{\partial z}\right|_{0^+} \qquad \frac{\partial g_I}{\partial z} = -\frac{1}{v_1}\frac{\partial g_I}{\partial t} \qquad \frac{\partial h_R}{\partial z} = \frac{1}{v_1}\frac{\partial h_R}{\partial t} \qquad \frac{\partial g_T}{\partial z} = -\frac{1}{v_2}\frac{\partial g_T}{\partial t}$$

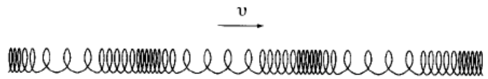
$g_I(z,t)$, $g_T(z,t)$, and $h_R(z,t)$ are each functions of a single variable u
(in the first case $u = z - v_1 t$, in the second $u = z - v_2 t$, and in the third $u = z + v_1 t$)

9.1.4 Polarization

Transverse Waves: the displacement is perpendicular to the direction of propagation

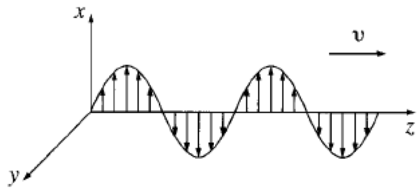


Longitudinal Waves: the displacement is along the direction of propagation

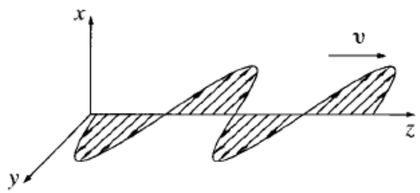


Transverse waves occur in two independent states of polarization

Vertical polarization: $\tilde{\mathbf{f}}_v(z,t) = \tilde{A}e^{i(kz-\omega t)}\,\hat{\mathbf{x}}$

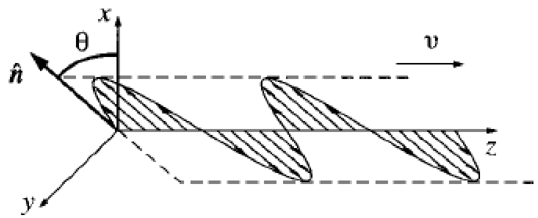


Horizontal polarization: $\tilde{\mathbf{f}}_h(z,t) = \tilde{A}e^{i(kz-\omega t)}\,\hat{\mathbf{y}}$



Any other Polarization: $\tilde{\mathbf{f}}(z,t) = \tilde{A}e^{i(kz-\omega t)}\,\hat{\mathbf{n}}$

Polarization Vector: $\hat{\mathbf{n}}$ $\quad \hat{\mathbf{n}} = \cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}} \quad \hat{\mathbf{n}}\cdot\hat{\mathbf{z}} = 0$



Any transverse wave can be considered a superposition of two waves: one horizontally polarized, the other vertically:

$$\tilde{\mathbf{f}}(z,t) = (\tilde{A}\cos\theta)e^{i(kz-\omega t)}\,\hat{\mathbf{x}} + (\tilde{A}\sin\theta)e^{i(kz-\omega t)}\,\hat{\mathbf{y}}$$