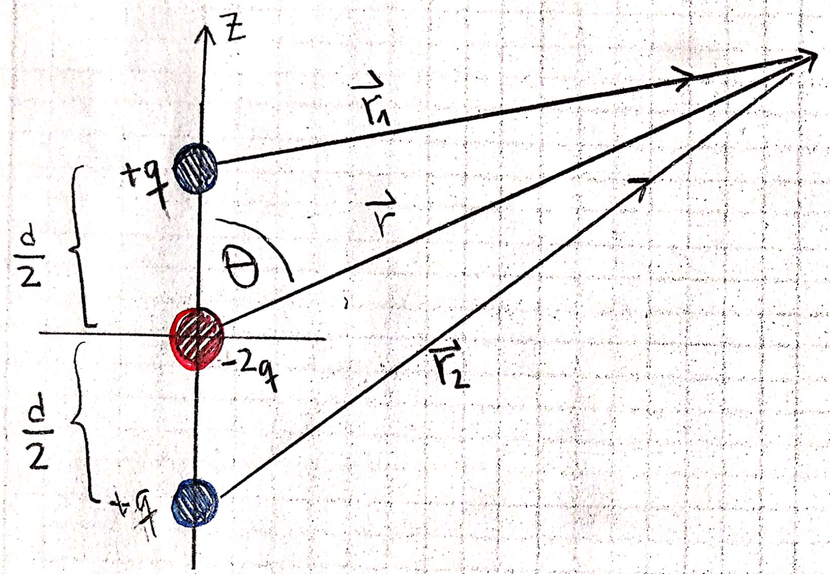
COMPLEMENTO I

POTENCIAL ELÉCTRICO DE UN CUADRUPOLO PUNTUAL



El potencial eléctrico de esta distribución de cargas está dado por la siguiente expresión:

$$\varphi(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left[-\frac{2}{r} + \frac{1}{|\vec{r}_1|} + \frac{1}{|\vec{r}_2|}\right] \quad ; \text{ con } r = |\vec{r}|$$

del diagrama se observa que:

$$\vec{r}_1 = \vec{r} - \frac{d}{2}\hat{k} \qquad y \qquad \vec{r}_2 = \vec{r} + \frac{d}{2}\hat{k}$$

luego

$$|\vec{r}_1| = \left|\vec{r} - \frac{d}{2}\hat{k}\right| = \left[\left(\vec{r} - \frac{d}{2}\hat{k}\right)\cdot\left(\vec{r} - \frac{d}{2}\hat{k}\right)\right]^{1/2}$$

$$|\vec{r}_1| = \left[r^2 - d\,\vec{r}\cdot\hat{k} + \frac{d^2}{4}\right]^{1/2} = \left[r^2 - dr\cos\theta + \frac{d^2}{4}\right]^{1/2}$$

De forma análoga:

$$|\vec{r}_2| = \left[r^2 + d\,\vec{r}\cdot\hat{k} + \frac{d^2}{4}\right]^{1/2} = \left[r^2 + dr\cos\theta + \frac{d^2}{4}\right]^{1/2}$$

* PRIMERA APROXIMACIÓN $r >> d$

$$|\vec{r}_1| = \left[r^2 - dr\cos\theta + \frac{d^2}{4}\right]^{1/2} = r\left[1 - \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right]^{1/2}$$

$$\Downarrow$$

$|\vec{r}_1| \cong r\left(1 - \frac{d}{r}\cos\theta\right)^{1/2}$; se desprecia el término $\frac{d^2}{4r^2} <<< 1$

análogamente

$$|\vec{r}_2| \cong r\left(1 + \frac{d}{r}\cos\theta\right)^{1/2}$$

Con esto se tiene que el potencial $\varphi(\vec{r})$ es ahora:

$$\varphi(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r}\left[\frac{1}{\left(1 - \frac{d}{r}\cos\theta\right)^{1/2}} + \frac{1}{\left(1 + \frac{d}{r}\cos\theta\right)^{1/2}} - 2\right]$$

* SEGUNDA APROXIMACIÓN $\frac{d}{r}\cos\theta << 1$

Sea $\xi = \frac{d}{r}\cos\theta$

luego

$$\varphi(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r}\left[\frac{1}{(1-\xi)^{1/2}} + \frac{1}{(1+\xi)^{1/2}} - 2\right]$$

donde

$$\frac{1}{(1-\xi)^{1/2}} = {}_1F_0\left(\begin{matrix}1/2\\ -\end{matrix}\middle|\,\xi\right) = \sum_{n\geq 0} (1/2)_n \frac{\xi^n}{n!}$$

$$= 1 + (1/2)_1\,\xi + (1/2)_2\,\frac{\xi^2}{2} + O(\xi^3)$$

$$= 1 + \frac{\Gamma(1/2+1)}{\Gamma(1/2)}\,\xi + \frac{\Gamma(1/2+2)}{\Gamma(1/2)}\,\frac{\xi^2}{2} + O(\xi^3)$$

$$\cong 1 + \frac{1}{2}\xi + \frac{3}{8}\xi^2$$

de forma similar se cumple que:

$$\frac{1}{(1+\xi)^{1/2}} \cong 1 - \frac{1}{2}\xi + \frac{3}{8}\xi^2$$

entonces:

$$\varphi(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r}\left[\cancel{1} + \cancel{\frac{1}{2}\xi} + \frac{3}{8}\xi^2 + \cancel{1} - \cancel{\frac{1}{2}\xi} + \frac{3}{8}\xi^2 \cancel{-2}\right]$$

$$\varphi(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r}\left[\frac{3}{4}\xi^2\right] = \frac{3}{4}\,\frac{q}{4\pi\varepsilon_0 r}\,\frac{d^2}{r^2}\cos^2\theta$$

Finalmente

$$\varphi(\vec{r}) = \frac{3}{4}\frac{q}{4\pi\varepsilon_0}\frac{d^2\cos^2\theta}{r^3}$$

obs. $d = |\vec{d}|$ y $\vec{d} = d\hat{k}$

entonces $\vec{r}\cdot\vec{d} = r d\cos\theta$

$$\Downarrow$$

$$d\cos\theta = \frac{\vec{r}\cdot\vec{d}}{r}$$

$$\Downarrow$$

$$d^2\cos^2\theta = \frac{(\vec{r}\cdot\vec{d})^2}{r^2}$$

lo que se resume en la siguiente expresión para el potencial $\varphi(\vec{r})$:

$$\boxed{\varphi(\vec{r}) = \frac{3}{4}\frac{q}{4\pi\varepsilon_0}\frac{(\vec{r}\cdot\vec{d})^2}{r^5}}$$

Una tarea para ti!!!

Determine $\vec{E}(\vec{r}) = -\nabla\varphi(\vec{r})$