Chapter 7. Electrodynamics

Summary: Electrostatics and Magnetostatics

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0 \longrightarrow \nabla \cdot \mathbf{D} = \rho_f$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$$

$$\mathbf{M} = \frac{1}{\mu_0} \chi_m \mathbf{B}$$

$$\mathbf{D} \equiv \varepsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{H} \equiv \mathbf{B} / \mu_0 - \mathbf{M}$$

$$\nabla \times \mathbf{E} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \longrightarrow \nabla \times \mathbf{H} = \mathbf{J}_f$$

$$\rho_b = -\nabla \cdot \mathbf{P}$$

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$$

$$\mathbf{E} = -\nabla V$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{D} = \varepsilon \mathbf{E} \qquad \varepsilon \equiv \varepsilon_0 (1 + \chi_e)$$

$$\mathbf{B} = \mu \mathbf{H} \qquad \mu \equiv \mu_0 (1 + \chi_m)$$

$$\mathbf{J}_b = \nabla \times \mathbf{M}$$

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$$

$$\mathbf{E}^{\parallel}_{\text{above}} - \mathbf{E}^{\parallel}_{\text{below}} = 0$$

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{1}{\epsilon_0} \sigma$$

$$B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}$$

$$\mathbf{B}^{\parallel}_{\text{above}} - \mathbf{B}^{\parallel}_{\text{below}} = \mu_0 (\mathbf{K} \times \hat{\mathbf{n}})$$

$$\mathbf{D}^{\parallel}_{\text{above}} - \mathbf{D}^{\parallel}_{\text{below}} = \mathbf{P}^{\parallel}_{\text{above}} - \mathbf{P}^{\parallel}_{\text{below}}$$

$$D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f$$

$$H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} = -(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}})$$

$$\mathbf{H}^{\parallel}_{\text{above}} - \mathbf{H}^{\parallel}_{\text{below}} = \mathbf{K}_f \times \hat{\mathbf{n}}.$$

7.1 Electromotive Force

7.1.1 Ohm's Law

Current density **J** is proportional to the force per unit charge, **f**:

$$\mathbf{J} = \sigma \mathbf{f}.$$

***σ* : Conductivity** (not to be confused with surface charge density)

***ρ* = 1/*σ* : Resistivity** (not to be confused with volume charge density)

Perfect conductors: $\sigma = \infty$

Material	Resistivity	Material	Resistivity
Conductors:		*Semiconductors:*	
Silver	1.59×10^{-8}	Salt water (saturated)	4.4×10^{-2}
Copper	1.68×10^{-8}	Germanium	4.6×10^{-1}
Gold	2.21×10^{-8}	Diamond	2.7
Aluminum	2.65×10^{-8}	Silicon	2.5×10^{3}
Iron	9.61×10^{-8}	*Insulators:*	
Mercury	9.58×10^{-7}	Water (pure)	2.5×10^{5}
Nichrome	1.00×10^{-6}	Wood	$10^{8} - 10^{11}$
Manganese	1.44×10^{-6}	Glass	$10^{10} - 10^{14}$
Graphite	1.4×10^{-5}	Quartz (fused)	$\sim 10^{16}$

For an electromagnetic force,

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

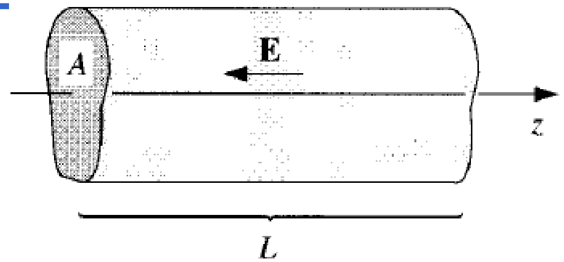
→ Ordinarily, the velocity of the charges is sufficiently small.

$$\boxed{\mathbf{J} = \sigma \mathbf{E}}$$ → **Ohm's law**

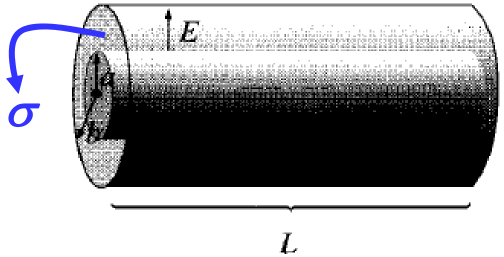
Ohm’s Law

Example 7.1 A cylindrical resistor of cross-sectional area *A* and length *L* is made from material with conductivity σ.

If the potential is constant over each end,
and the potential difference between the ends is *V*,
what current flows?



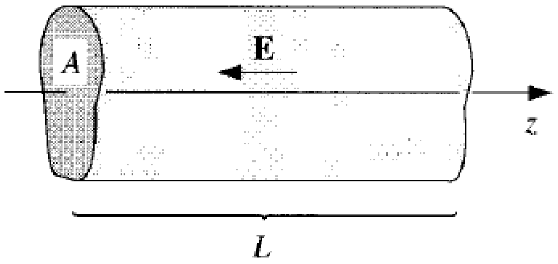
Example 7.2 Two long cylinders (radii ***a* and *b***) are separated by material of conductivity σ.
If they are maintained at a potential difference *V*,
what current flows from one to the other, in a length *L*?



Ohm's Law

Example 7.3 In Example 7.1, we assume that the electric field is *uniform* within the wire, as long as the potential is constant over each end.

Is it true???



Within the cylinder ***V*** obeys Laplace's equation.
What are the boundary conditions?
At the left end the potential is constant-we may as well set it equal to zero.
At the right end the potential is likewise constant → call it V_0.
On the cylindrical surface, $\mathbf{J} \cdot \hat{\mathbf{n}} = 0, \longrightarrow \mathbf{E} \cdot \hat{\mathbf{n}} = 0, \longrightarrow \partial V / \partial n = 0.$

With *V* or its normal derivative specified on all surfaces, the potential is uniquely determined.
But it's *easy* to guess *one* potential that obeys Laplace's equation and fits these boundary conditions:

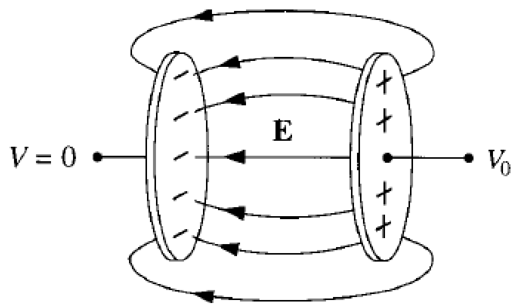
$$V(z) = \frac{V_0 z}{L}$$

→ The uniqueness theorem guarantees that this is *the* solution.

→ The corresponding field is therefore uniform: $\mathbf{E} = -\nabla V = -\frac{V_0}{L}\hat{\mathbf{z}}$

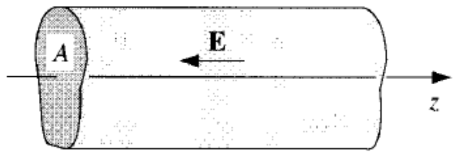
If the conducting material is removed, leaving only a metal plate at either end,

The charge arranges itself over the surface of the wire in just such a way as to produce a nice uniform field within.



Ohm’s Law

If the potential is constant over each end, V,
→ → the corresponding field is therefore uniform.



$$\mathbf{E} = -\nabla V = -\frac{V_0}{L}\hat{\mathbf{z}}$$

A given field E produces a force *qE* (on a charge *q).*
→ *A*ccording to Newton's second law the charge will accelerate.
→ But if the charges are *accelerating,* why doesn't the current *increase* with time, growing larger and larger the longer you leave the field on?
→ Ohm's law implies, on the contrary, that a constant field produces a constant *current,* (constant *velocity).*
→ **Isn't that a contradiction of Newton's law?**

In practice the charges are already moving quite fast because of their thermal energy.

→ The thermal velocities have random directions.

→ Under a field **E** (force ***qE***), the average velocity (**drift velocity)** determines the current density**.**

$$\mathbf{J} = nfq\mathbf{v}_{\text{ave}} = \frac{nfq\lambda}{2v_{\text{thermal}}}\frac{\mathbf{F}}{m} = \left(\frac{nf\lambda q^2}{2m\,v_{\text{thermal}}}\right)\mathbf{E} \;= \sigma\mathbf{E}$$

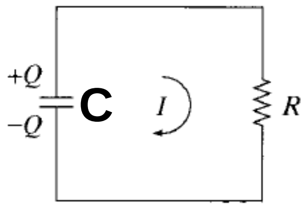
(n, molecules per unit volume; *f,* free electrons per molecule)

➔ As a result of all the collisions; the work done by the electrical force is converted into heat in the resistor.

$P = VI = I^2R$ ➔ **Power (joules per second)** ➔ **Joule heating law**

7.1.1 Ohm's Law

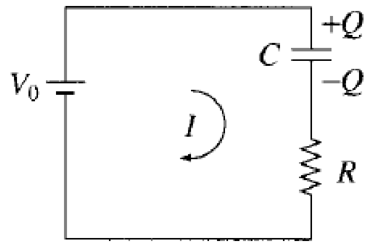
Problem 7.2 A capacitor C has been charged up to potential V_0; at time $t = 0$ it is connected to a resistor R, and begins to discharge.



(a) Determine the charge on the capacitor as a function of time, $Q(t)$. What is the current through the resistor, $I(t)$?

(b) What was the original energy stored in the capacitor? Confirm that the heat delivered to the resistor is equal to the energy lost by the capacitor.

(c) Now imagine *charging up* the capacitor, by connecting it (and the resistor) to a battery of fixed voltage V_0, at time $t = 0$. Determine $Q(t)$ and $I(t)$.



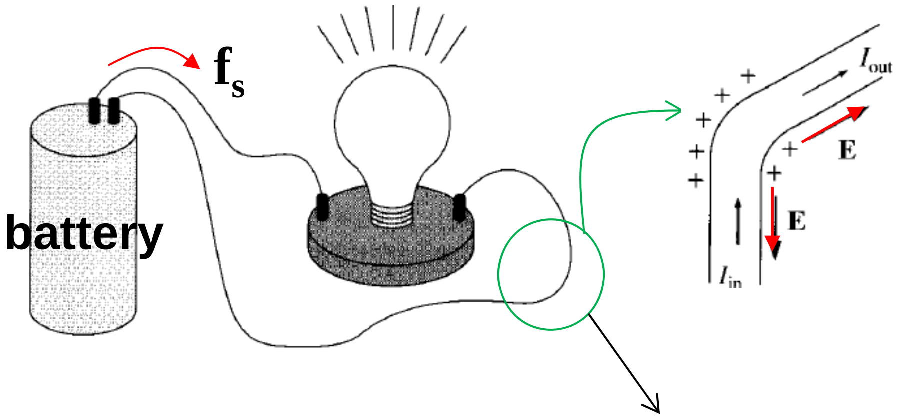
7.1.2 Electromotive Force

In a current circuit, there are really **two forces involved in driving current** :

→ the source, $\mathbf{f}_s$, which is ordinarily confined to one portion of the loop (a battery, say),

→ the electrostatic force, which serves to smooth out the flow and communicate the influence of the source to distant parts of the circuit.

$$\mathbf{f} = \mathbf{f}_s + \mathbf{E}$$



The line integral of **f** around the circuit:

$$\mathcal{E} \equiv \oint \mathbf{f} \cdot d\mathbf{l} = \oint \mathbf{f}_s \cdot d\mathbf{l}$$

($\oint \mathbf{E} \cdot d\mathbf{l} = 0$ for electrostatic fields)

The charge piles up at the "knee," and this produces a field aiming *away* from the kink.
This field *opposes* the current flowing in (slowing it down) and *promotes* the current flowing out (speeding it up) until these currents are equal

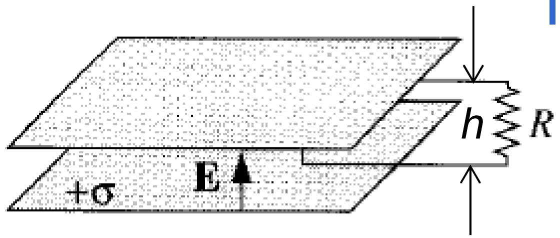
➔ The static E keep the current uniform in a circuit!

$\mathcal{E}$: ***electromotive force (emf)***

➔ work done, per unit charge, by the source

Electromotive Force

Problem 7.6 A rectangular loop of wire is situated so that one end (height h) is between the parallel-plate capacitor, oriented parallel to the field E.
The other end is way outside, where the field is essentially zero.
What is the emf in this loop?
If the total resistance is *R* what current flows?



It looks as though $\mathcal{E} = \oint \mathbf{E} \cdot dl = (\sigma/\epsilon_0)h,$

as would indeed be the case if the field were really just σ/ε_0 inside and zero outside.

But, this situation is electrostatic.

➔ $\nabla \times \mathbf{E} = 0$ → $\mathcal{E} = \oint \mathbf{E} \cdot dl =$ 0 for all electrostatic fields.

➔ A "fringing field" at the edges must be presented.
→ The fringing field is evidently just right to kill off the contribution from the left end of the loop.

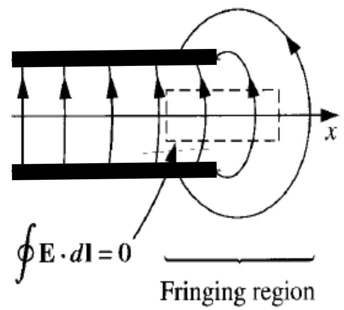


Figure 4.31

➔ **The current on R is zero.**

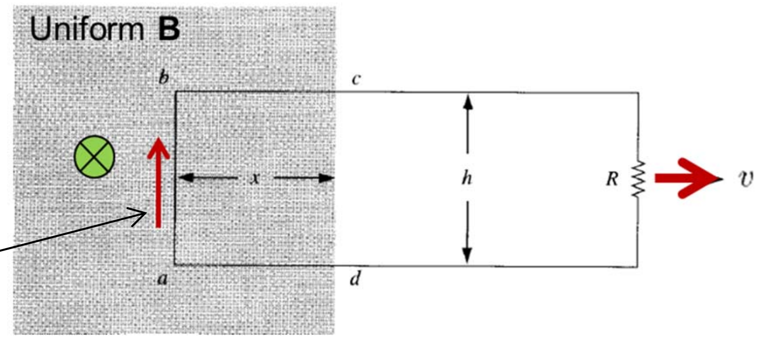
7.1.3 Motional emf

Generator ➔ motional emf

➔ It arises when you move a wire through a magnetic field.

$$\mathcal{E} = \oint \mathbf{f}_{\text{mag}} \cdot d\mathbf{l} = vBh$$

$$\mathbf{f}_{\text{mag}} = qvB$$

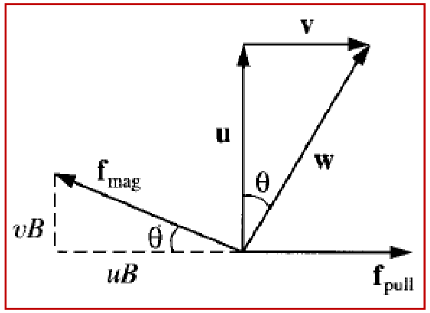


Although the magnetic force is responsible for establishing the emf, it is certainly not doing any work!

→ *Magnetic forces never do work.*

→ *Who, then, is supplying the energy that heats the resistor?*

➔ ***The person who's pulling on the loop!***



With the current flowing,
charges in *ab* have a vertical velocity **u** in addition to the horizontal velocity **v**.

→ Accordingly, the magnetic force has a component ***quB*** **to the left**.
→ To counteract this, the person pulling on the wire must exert a force per unit charge: $f_{\text{pull}} = uB$

The particle is actually *moving* in the direction of the resultant velocity **w**,
→ The distance it goes is *(h/*cosθ*)*.

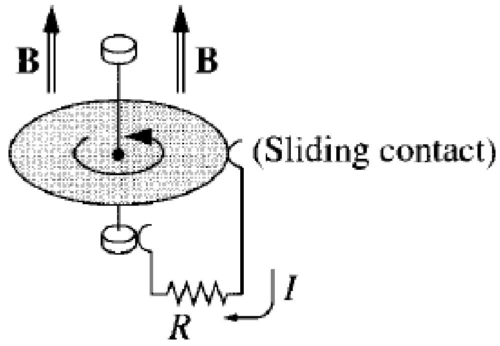
→ **The work done per unit charge is therefore**

$$\int \mathbf{f}_{\text{pull}} \cdot d\mathbf{l} = (uB)\left(\frac{h}{\cos\theta}\right)\sin\theta = vBh = \mathcal{E}$$

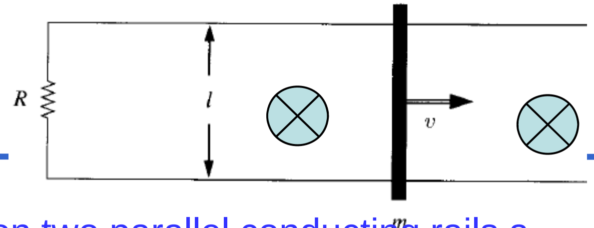
➔ ***The work done per unit charge is exactly equal to the emf.***

Electromotive Force

Example 7.4 A metal disk of radius *a* rotates with angular velocity ω about a vertical axis, through a uniform field B, pointing up. A circuit is made by connecting one end of a resistor to the axle and the other end to a sliding contact, which touches the outer edge of the disk.
Find the current in the resistor.



Electromotive Force



Problem 7.7 A metal bar of mass m slides frictionlessly on two parallel conducting rails a distance $\boldsymbol{l}$ apart. A resistor R is connected across the rails and a uniform magnetic field **B**, pointing into the page, fills the entire region.

(a) If the bar moves to the right at speed $\boldsymbol{v}$, what is the current in the resistor? In what direction does it flow?

(b) What is the magnetic force on the bar? In what direction?

(c) If the bar starts out with speed $\boldsymbol{v_0}$ at time $t = 0$, and is left to slide, what is its speed at a later time t?

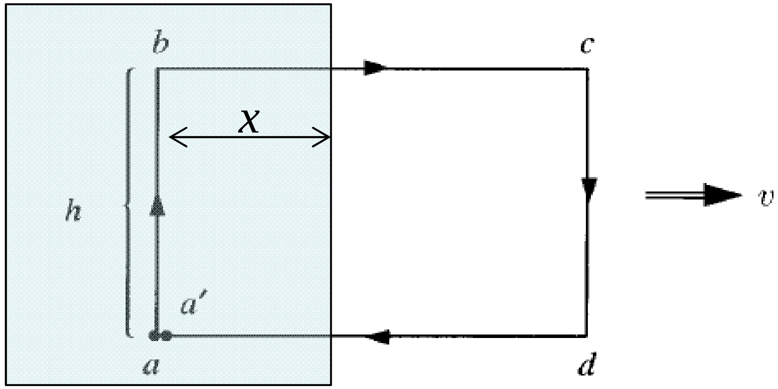
(d) The initial kinetic energy of the bar was, of course, $\frac{1}{2}mv_0{}^2$.

Check that the energy delivered to the resistor is exactly $\frac{1}{2}mv_0{}^2$.

emf and Flux Φ

Define **the flux of B** through the loop:

$$\Phi \equiv \int \mathbf{B} \cdot d\mathbf{a}$$



For the rectangular loop → $\Phi = Bhx$

As the loop moves, the flux decreases → $$\frac{d\Phi}{dt} = Bh\frac{dx}{dt} = -Bhv$$

Note that the emf for a loop was → $$\mathcal{E} = \oint \mathbf{f}_{\text{mag}} \cdot d\mathbf{l} = vBh$$

➔ *The emf generated in the loop is* ***minus the rate of change of flux.***

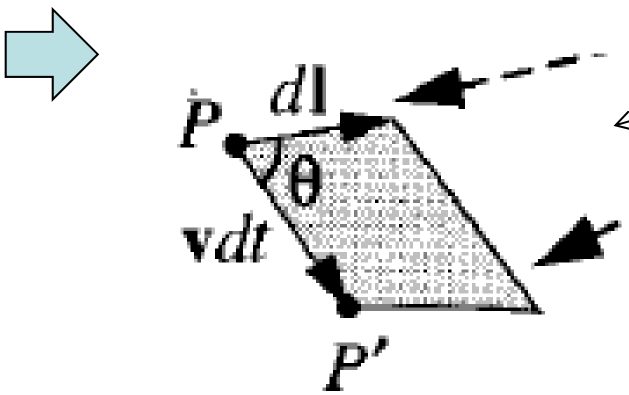
$$\boxed{\mathcal{E} = -\frac{d\Phi}{dt}}$$

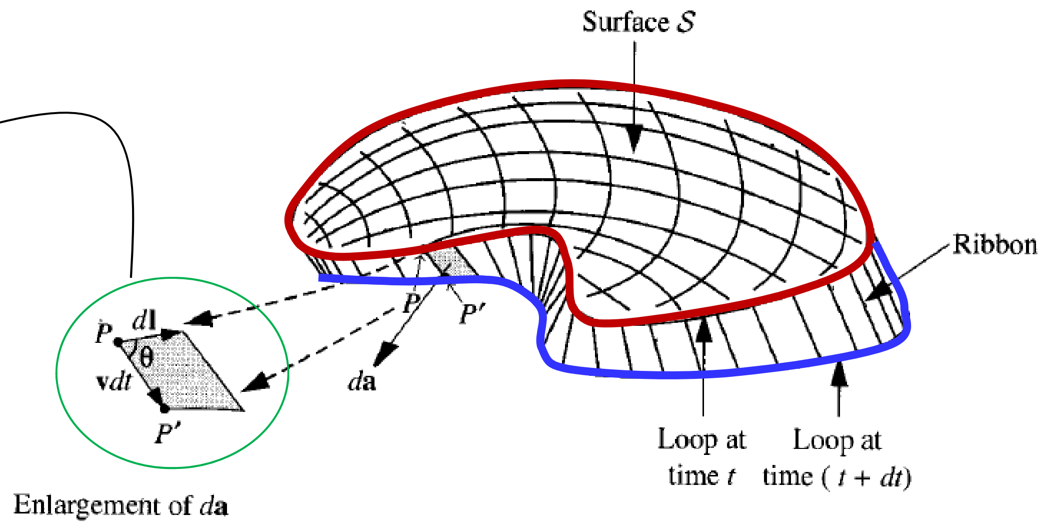
➔ ***Flux rule for motional emf***

emf and Flux Φ

$$\boxed{\mathcal{E} = -\frac{d\Phi}{dt}}$$

→ *In fact, the loop need not even maintain a fixed shape.*
→ *It is **valid in general** for nonrectangular loops moving in arbitrary directions through nonuniform magnetic fields.*





The *change* in flux after a short time dt → $d\Phi = \Phi(t+dt) - \Phi(t) = \Phi_{\text{ribbon}} = \int_{\text{ribbon}} \mathbf{B} \cdot d\mathbf{a}$

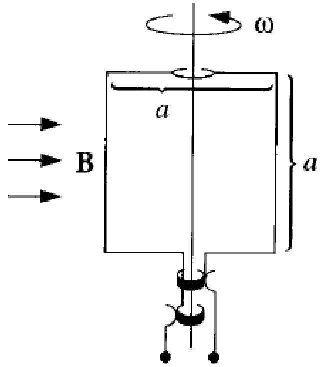
The infinitesimal element of area on the ribbon → $d\mathbf{a} = (\mathbf{v} \times d\mathbf{l})\, dt$

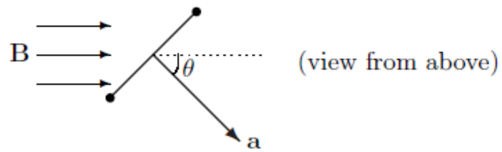
$$\longrightarrow \frac{d\Phi}{dt} = \oint \mathbf{B} \cdot (\mathbf{v} \times d\mathbf{l}) \xrightarrow[\mathbf{B} \cdot (\mathbf{v} \times d\mathbf{l}) = -(\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}]{} \frac{d\Phi}{dt} = -\oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}$$

$$(\mathbf{v} \times \mathbf{B}) = \mathbf{f}_{\text{mag}} \longrightarrow \frac{d\Phi}{dt} = -\oint \mathbf{f}_{\text{mag}} \cdot d\mathbf{l}, \longrightarrow \mathcal{E} = -\frac{d\Phi}{dt}$$

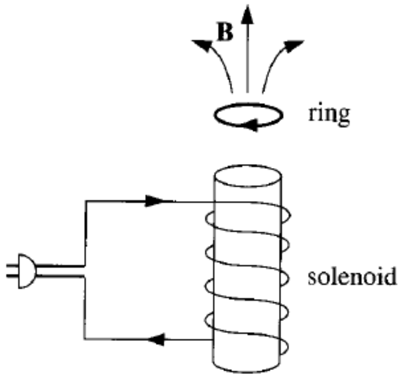
emf and Flux Φ

Problem 7.10 A square loop (side *a)* is mounted on a vertical shaft and rotated at angular velocity ω.
A uniform magnetic field B points to the right.
Find the $\varepsilon(t)$ for the alternating current generator.





Problem 7.12 long solenoid, of radius *a,* is driven by an alternating current, so that the field inside is sinusoidal: $B(t) = B_0 \cos(wt)$.
A circular loop of wire, of radius *al2* and resistance R is paced inside the solenoid, and coaxial with it.
Find the current induced in the loop, as a function of time.



➔ ***A changing magnetic field induces an electric field to generate current!***