Universidad de Valparaíso
CHILE

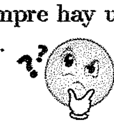
Tarea Voluntaria I
MMF II
Licenciatura en Física - 2020

Halle la solución de la siguiente integral:

$$I = \int_{y=0}^{\infty}\int_{x=0}^{\infty}\int_{z=0}^{\infty} x^{\alpha} y^{\beta} z^{\sigma} \frac{\exp\left(-Bx^{\lambda}\right)}{(x+Ay)^{\mu}\,(xy+Cz)^{\nu}}\, dxdydz$$

Los parámetros α, β, σ, λ, μ y ν asumen los valores necesarios para que la integral exista. Las constantes A, B y C son todas reales y positivas.

Obs.: De acuerdo al teorema de Fubini (investigue sobre esto), la integral es independiente del orden de integración, sin embargo, siempre hay una elección en este orden que hace que la integración sea más simple.



Paso 1: Integración en la variable z

$$\int_0^\infty z^\sigma \frac{dz}{(xy+cz)^\nu} = \frac{1}{(xy)^\nu}\int_0^\infty \frac{z^\sigma \quad dz}{\left(1+\frac{cz}{xy}\right)^\nu} \text{ ; hacemos } t = \frac{cz}{xy}$$

$$\Downarrow$$

$$z = \frac{xy}{c}\,t$$

$$dz = \frac{xy}{c}\,dt$$

$$\therefore = \frac{1}{(xy)^\nu}\,\frac{(xy)^{\sigma+1}}{c^{\sigma+1}} \underbrace{\int_0^\infty \frac{t^\sigma}{(1+t)^\nu}\,dt}_{B(\sigma+1,\,\nu-\sigma-1)}$$

$$= \frac{1}{c^{\sigma+1}}\,(xy)^{\sigma-\nu+1}\,B(\sigma+1,\,\nu-\sigma-1)$$

Luego

$$I = \frac{1}{c^{\sigma+1}}\,B(\sigma+1,\,\nu-\sigma-1)\int_0^\infty\int_0^\infty \frac{x^{\alpha+\sigma-\nu+1}\,y^{\beta+\sigma-\nu+1}\,e^{-Bx^\lambda}}{(x+Ay)^\mu}\,dx\,dy$$

Paso 2: Integración en la variable y

$$\int_0^\infty \frac{y^{\beta+\sigma-\nu+1}}{(x+Ay)^\mu}\,dy = \frac{1}{x^\mu}\int_0^\infty \frac{y^{\beta+\sigma-\nu+1}}{\left(1+\frac{A}{x}y\right)^\mu}\,dy \text{ ; hacemos } t = \frac{A}{x}y$$

$$\Downarrow$$

$$y = \frac{x}{A}\,t$$

$$dy = \frac{x}{A}\,dt$$

$$\int_0^{\infty} \frac{y^{\beta+\sigma-\nu+1}}{(x+Ay)^{\mu}}\,dy = \frac{1}{X^{\mu}}\,\frac{X^{\beta+\sigma-\nu+2}}{A^{\beta+\sigma-\nu+2}}\underbrace{\int_0^{\infty}\frac{t^{\beta+\sigma-\nu+1}}{(1+t)^{\mu}}\,dt}_{B(\beta+\sigma-\nu+2,\ \mu-\beta-\sigma+\nu-2}$$

$$= \frac{1}{A^{\beta+\sigma-\nu+2}}\, X^{\beta+\sigma-\nu-\mu+2}\, B\left(\beta+\sigma-\nu+2,\ \mu-\beta-\sigma+\nu-2\right)$$

$\therefore$

$$I = \frac{1}{C^{\sigma+1}}\,\frac{B(\sigma+1,\ \nu-\sigma-1)\,B(\beta+\sigma-\nu+2,\ \mu-\beta-\sigma+\nu-2)}{A^{\beta+\sigma-\nu+2}}$$

$$\cdot \int_0^{\infty} X^{\alpha+\sigma-\nu+1}\, X^{\beta+\sigma-\nu-\mu+2}\, e^{-BX^{\lambda}}\,dx$$

PASO III

Veamos solo la integral:

$$J = \int_0^{\infty} X^{\alpha+\beta+2\sigma-2\nu-\mu+3}\, e^{-BX^{\lambda}}\,dx \quad , \text{ hacemos } X^{\lambda} = t$$

$$X = t^{1/\lambda}$$

$$dx = \frac{1}{\lambda}\, t^{\frac{1}{\lambda}-1}\,dt$$

$$= \frac{1}{\lambda}\int_0^{\infty} t^{\frac{\alpha+\beta+2\sigma-2\nu-\mu+3}{\lambda}}\, e^{-Bt}\, t^{\frac{1}{\lambda}-1}\,dt$$

$$= \frac{1}{\lambda}\int_0^{\infty} t^{\frac{\alpha+\beta+2\sigma-2\nu-\mu+4}{\lambda}-1}\, e^{-Bt}\,dt$$

luego

$$J = \frac{1}{\lambda} \frac{\Gamma\left(\frac{\alpha+\beta+2\sigma-2\nu-\mu+4}{\lambda}\right)}{B^{\frac{\alpha+\beta+2\sigma-2\nu-\mu+4}{\lambda}}}$$

Finalmente la solución de la integral está dada por:

$$I = \frac{1}{A^{\beta+\sigma-\nu+2}} \times \frac{1}{B^{\frac{\alpha+\beta+2\sigma-2\nu-\mu+4}{\lambda}}} \times \frac{1}{C^{\sigma+1}} \times \frac{1}{\lambda}$$

$$\times\, \Gamma\left(\frac{\alpha+\beta+2\sigma-2\nu-\mu+4}{\lambda}\right) B\left(\beta+\sigma-\nu+2,\ \mu-\beta-\sigma+\nu-2\right)$$

$$\times\, B\left(\sigma+1,\ \nu-\sigma-1\right)$$