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Ramanujan's Master Theorem applied to the evaluation of Feynman diagrams

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A B S T R A C T

Ramanujan's Master Theorem is a technique developed by S. Ramanujan to evaluate a class of definite integrals. This technique is used here to produce the values of integrals associated with Feynman diagrams.

1. Introduction

Precise experimental measurements in high energy physics require, in its theoretical counterpart, the development of new techniques for the evaluation of analytic objects

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associated with the corresponding Feynman diagrams. These techniques have lately emphasized the automatization of calculations of multiscale, multiloop diagrams.

Modern numerical methods for the evaluation of Feynman diagrams benefit from analytical techniques employed as preliminary work to detect the presence of divergences. Recent advances include a method based on the Bernstein–Tkachov theorem for the corrections of one and two loop diagrams and methods based on sector-decompositions. New analytic methods to reduce Feynman diagrams to a small number of scalar integrals include integration by parts, the use of Lorenz invariance and other symmetries, Mellin–Barnes transforms and differential equations. The reader is referred to [11] for a description of these and other methods for the evaluation of Feynman diagrams and to [12,13] for readable introductions to the topic.

This paper contains examples of an alternative method for the evaluation of some Feynman diagrams. It is based on the classical Ramanujan's Master Theorem, one of his favorite techniques to evaluate definite integrals. The theoretical aspects of this method are presented in [9] and a collection of examples and some justification of the algorithm is given in [1,4,6]. This technique has also been used in [7] for the evaluation of some multidimensional integrals obtained by the Schwinger parametrization of Feynman diagrams.

The goal of the present work is to illustrate the flexibility of the method by evaluating integrals associated with two and three loop diagrams. Naturally the method works for a large variety of definite integrals and the first example illustrates this by computing the Mellin transform of a Bessel function. The automatization of this process began in [10] and progress is reported in [5].

2. Advantage of Ramanujan's Master Theorem in the evaluation of Feynman diagrams

The integral representation associated with a Feynman diagram associated with a scalar field comes directly from the directed graph G of the diagram. Assume the graph has N propagators (or *internal lines*), L loops (attached to the flux of independent internal momenta) $\underline{q} = (q_1, ..., q_L)$, E external independent momenta (*external lines*) $\underline{p} = (p_1, ..., p_E)$ and each propagator is characterized by its mass $(m_1, ..., m_N)$. The integral representing the diagram in D-dimensions is

$$G = \int \frac{d^D q_1}{i\pi^{D/2}} \cdots \frac{d^D q_L}{i\pi^{D/2}} \frac{1}{(B_1^2 - m_1^2 + i0)^{\nu_1}} \cdots \frac{1}{(B_N^2 - m_N^2 + i0)^{\nu_N}}. \tag{2.1}$$

In this expression, the symbol B_j represents the momentum of the j-th propagator, given as a linear combination of the momenta, both external $\{\underline{p}\}$ and internal $\{\underline{q}\}$. The propagators also have a sequence of arbitrary powers $\underline{\nu} = \{\nu_1, ..., \nu_N\}$, one per propagator.

The evaluation of this integral begins with the so-called *parametrization*. This simple corresponds to a formulation of the problem as an integral in N-dimensional space. The most common forms of parametrization are those developed by Feynman and Schwinger. In the first case, the quotients in the integrand in (2.1) are replaced by

$$\frac{1}{\prod_{j=1}^{N}(B_j^2-m_j^2)^{\nu_j}}=\frac{\Gamma(N_\nu)}{\prod_{j=1}^{N}\Gamma(\nu_j)}\int_0^1\frac{\delta(1-\sum_{j=1}^{N}x_j)}{[\sum_{j=1}^{N}x_jB_j^2-\sum_{j=1}^{N}x_jm_j^2]^{N_\nu}}\,d\vec{x} \tag{2.2}$$

where $d\vec{x}=dx_1...dx_N\prod_{j=1}^{N}x_j^{\nu_j-1}$ and $N_\nu=(\nu_1+...+\nu_N)$. Then (2.1) becomes

$$G=(-1)^{-N_\nu}\frac{\Gamma(N_\nu-\frac{LD}{2})}{\prod_{j=1}^{N}\Gamma(\nu_j)}\int_0^1\delta\left(1-\sum_{j=1}^{N}x_j\right)\frac{U^{N_\nu-(L+1)\frac{D}{2}}}{[U\sum_{j=1}^{N}x_jm_j^2-F]^{N_\nu-\frac{LD}{2}}}\,d\vec{x} \tag{2.3}$$

here U and F are polynomials in the parameters $\{x_1,...,x_N\}$. This is the *Feynman parametrization.*

The second form used to evaluate Feynman diagrams was developed by Schwinger. In this procedure, the integrand in (2.1) is modified using

$$\frac{1}{\prod_{j=1}^{N}(B_j^2-m_j^2)^{\nu_j}}=\frac{1}{\prod_{j=1}^{N}\Gamma(\nu_j)}\int_0^\infty...\int_0^\infty\exp\left(\sum_{j=1}^{N}x_jm_j^2\right)\exp\left(-\sum_{j=1}^{N}x_jB_j^2\right)d\vec{x} \tag{2.4}$$

using the same notation as before. This parametrization provides a new form for G:

$$G=\frac{(-1)^{-\frac{LD}{2}}}{\prod_{j=1}^{N}\Gamma(\nu_j)}\int_0^\infty...\int_0^\infty\frac{\exp(\sum_{j=1}^{N}x_jm_j^2)\exp(-\frac{F}{U})}{U^{\frac{D}{2}}}\,d\vec{x} \tag{2.5}$$

This is the representation of the diagrams used in the current work. Ramanujan's Master Theorem is applied to them directly.

The evaluation of Feynman diagrams requires the computation of definite integrals. One of the most common methods employed for this goal is the technique based on the Mellin–Barnes representation of the integrand, introduced in this context by Boos [2]. This method uses, as a starting point, the integral form (2.3). The method requires the transformation of the integrand to the so-called Mellin–Barnes representation, using

$$\frac{1}{(A+B)^\nu}=\frac{1}{2\pi i}\frac{1}{\Gamma(\nu)}\int_{-i\infty}^{i\infty}\frac{A^z}{B^{\nu+z}}\Gamma(-z)\Gamma(\nu+z)\,dz \tag{2.6}$$

where the contour is chosen so that the poles of $\Gamma(...+z)$ are separated from those of $\Gamma(...-z)$. This method is described in [11]. The general form of the Mellin–Barnes representation of a Feynman diagram is

$$\left(\frac{1}{2\pi i}\right)^n \int\limits_{-i\infty}^{i\infty} \cdots \int\limits_{-i\infty}^{i\infty} \prod_i g(z_1,\dots,z_n,s_1,\dots,s_p,\nu_1,\dots,\nu_q,\epsilon)\frac{\prod_j \Gamma(A_j+V_j+c_jD)}{\prod_k \Gamma(B_j+W_j+d_jD)}\,dz_i \tag{2.7}$$

where $\{s_i\}$ is the set of invariants that characterize the diagram; $\{\nu_i\}$ are the powers of the propagators and $\{A_i\}$, $\{B_i\}$ are linear combinations in the powers $\{\nu_i\}$; $\{V_i, W_i\}$ depend upon $\{z_i\}$ and $\{c_i, d_i\}$ are arbitrary and g is an analytic function.

The advantage of the method of brackets presented here over the Mellin–Barnes procedure is this: this new method behind directly with the Schwinger representation and its application is simply based on the expansion

$$G = \int\limits_0^\infty \frac{dx_1}{x_1}\cdots\int\limits_0^\infty \frac{dx_N}{x_N}\sum_{l_1=0}^\infty\cdots\sum_{l_N=0}^\infty \frac{(-1)^{l_1}}{l_1!}\cdots\frac{(-1)^{l_N}}{l_N!}\varphi$$
$$\times (l_1,\dots,l_N)x_1^{a_{11}l_1+\dots+a_{1N}l_N+\tilde{b}_1}\cdots x_N^{a_{N1}l_1+\dots+a_{NN}l_N+\tilde{b}_N}$$

that is assigned a value via

$$I = \frac{1}{|\det(\mathbf{A})|}\Gamma\left(-l_1^*\right)\dots\Gamma\left(-l_N^*\right)\varphi\left(l_1^*,\dots,l_N^*\right). \tag{2.8}$$

The Mellin–Barnes is very effective in diagrams of high complexity, but at the level of the examples presented here, the method of brackets is simpler to apply.

3. Ramanujan's Master Theorem and its generalization

3.1. The formalism

The Mellin transform

$$\mathcal{M}(f) = \int\limits_0^\infty x^{\nu-1} f(x)\,dx \tag{3.1}$$

may be evaluated by one of Ramanujan's favorite tools; the so-called Ramanujan's Master Theorem. It states that if $f(x)$ admits a series expansion of the form

$$f(x) = \sum_{n=0}^\infty \varphi(n)\frac{(-x)^n}{n!} \tag{3.2}$$

in a neighborhood of $x = 0$, with $f(0) = \varphi(0) \neq 0$, then

$$\int\limits_0^\infty x^{\nu-1} f(x)\,dx = \Gamma(\nu)\varphi(-\nu). \tag{3.3}$$

The term $\varphi(-\nu)$ appearing in (3.3) requires an extension of the function φ, initially defined only for $\nu \in \mathbb{N}$. Details on the natural unique extension of φ are given in [1]. The condition $\varphi(0) \neq 0$ guarantees the convergence of the integral near $x = 0$, when $\nu > 0$. The proof of Ramanujan's Master Theorem and the precise conditions for its application appear in [9].

The *method of brackets* consists of a small number of rules, deduced in heuristic form, some of which are placed on solid ground [1].

For $a \in \mathbb{R}$, the symbol

$$\langle a \rangle \mapsto \int_0^\infty x^{a-1}\, dx \tag{3.4}$$

is the *bracket* associated with the (divergent) integral on the right. The symbol

$$\phi_n := \frac{(-1)^n}{\Gamma(n+1)} \tag{3.5}$$

is called the *indicator* associated with the index n. The notation $\phi_{i_1 i_2 \cdots i_r}$, or simply $\phi_{12\cdots r}$, denotes the product $\phi_{i_1}\phi_{i_2}\cdots\phi_{i_r}$.

Rules for the production of bracket series.

Rule $\boldsymbol{P_1}$. Power series appearing in the integrand are converted into *bracket series* by the procedure

$$\sum_{n=0}^{\infty} a_n x^{\alpha n+\beta-1} \mapsto \sum_{n\geq 0} a_n \langle \alpha n + \beta \rangle. \tag{3.6}$$

Rule $\boldsymbol{P_2}$. For $\alpha \in \mathbb{C}$, the multinomial power $(a_1 + a_2 + \cdots + a_r)^\alpha$ is assigned the r-dimension bracket series

$$\sum_{n_1\geq 0}\sum_{n_2\geq 0}\cdots\sum_{n_r\geq 0} \phi_{n_1 n_2\cdots n_r} a_1^{n_1}\cdots a_r^{n_r} \frac{\langle -\alpha + n_1 + \cdots + n_r\rangle}{\Gamma(-\alpha)}. \tag{3.7}$$

Rule $\boldsymbol{P_3}$. Each representation of an integral by a bracket series has associated an *index of the representation* via

$$\text{index} = \text{number of sums} - \text{number of brackets}. \tag{3.8}$$

It is important to observe that the index is attached to a specific representation of the integral and not just to integral itself. The experience obtained by the authors using this method suggests that, among all representations of an integral as a bracket series, the one with *minimal index* should be chosen.

Rules for the evaluation of a bracket series.

Rule $\boldsymbol{E_1}$. The one-dimensional bracket series is assigned the value

$$\sum_{n\geq 0}\phi_n f(n)\langle an+b\rangle \mapsto \frac{1}{|a|}f\left(n^*\right)\Gamma\left(-n^*\right), \tag{3.9}$$

where n^* is obtained from the vanishing of the bracket; that is, n^* solves $an+b=0$. This is precisely Ramanujan's Master Theorem.

The next rule provides a value for multi-dimensional bracket series of index 0, that is, the number of sums is equal to the number of brackets.

Rule $\boldsymbol{E_2}$. Assuming the matrix $A=(a_{ij})$ is non-singular, then the assignment is

$$\sum_{n_1\geq 0}\cdots\sum_{n_r\geq 0}\phi_{n_1\cdots n_r}f(n_1,\cdots,n_r)\langle a_{11}n_1+\cdots+a_{1r}n_r+c_1\rangle\cdots\langle a_{r1}n_1+\cdots+a_{rr}n_r+c_r\rangle$$
$$\mapsto \frac{1}{|\det(A)|}f\left(n_1^*,\cdots,n_r^*\right)\Gamma\left(-n_1^*\right)\cdots\Gamma\left(-n_r^*\right)$$

where $\{n_i^*\}$ is the (unique) solution of the linear system obtained from the vanishing of the brackets. There is no assignment if A is singular.

Rule $\boldsymbol{E_3}$. The value of a multi-dimensional bracket series of positive index is obtained by computing all the contributions of maximal rank by Rule E_2. These contributions to the integral appear as series in the free parameters. Series converging in a common region are added and divergent series are discarded. Any series producing a non-real contribution is also discarded. There is no assignment to a bracket series of negative index.

3.2. The Mellin transform of a Bessel function

The first example computes an integral involving the Bessel function, with hypergeometric representation

$$J_\alpha(\sqrt{x})=\left(\frac{\sqrt{x}}{2}\right)^\alpha\frac{1}{\Gamma(1+\alpha)}{}_0F_1\left(\left.\begin{matrix} \\ 1+\alpha\end{matrix}\right|-\frac{x}{4}\right), \tag{3.10}$$

with

$${}_0F_1\left(\left.\begin{matrix} \\ a\end{matrix}\right|x\right)=\sum_{n=0}^\infty\frac{1}{(a)_n}\frac{x^n}{n!}, \tag{3.11}$$

and

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)} \tag{3.12}$$

is the Pochhammer symbol. The integral evaluated here

$$I = \int_0^\infty x^{\beta-1} J_\alpha(\sqrt{x})\, dx \tag{3.13}$$

is expressed as

$$\begin{aligned} I &= \int_0^\infty x^{\beta-1} \left(\frac{\sqrt{x}}{2}\right)^\alpha \frac{1}{\Gamma(1+\alpha)} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{1}{(1+\alpha)_n} \frac{x^n}{4^n}\, dx \\ &= \int_0^\infty \sum_{n=0}^\infty \frac{(-1)^n}{n!} \left[\frac{1}{2^{\alpha+2n}\Gamma(1+\alpha+n)}\right] x^{n+(\beta+\frac{\alpha}{2})-1}\, dx. \end{aligned} \tag{3.14}$$

In the notation of (3.2)

$$\varphi(n) = \frac{1}{2^{\alpha+2n}\Gamma(1+\alpha+n)}. \tag{3.15}$$

Therefore

$$I = \frac{\Gamma(n^*)}{2^{\alpha+2n^*}\Gamma(1+\alpha-n^*)}. \tag{3.16}$$

Here $n^* = -(\beta + \frac{\alpha}{2})$, is the solution of

$$n + \beta + \frac{\alpha}{2} = 0. \tag{3.17}$$

Therefore

$$\int_0^\infty x^{\beta-1} J_\alpha(\sqrt{x})\, dx = 2^{2\beta} \frac{\Gamma(\beta + \frac{\alpha}{2})}{\Gamma(1 + \frac{\alpha}{2} - \beta)}. \tag{3.18}$$

This is entry 6.561.14 in the table of integrals [8].

3.3. A second example: the Feynman diagram of a bubble

This is the evaluation a D-dimensional integral corresponding to the massless bubble Feynman diagram. The result is well-known [3]. In momentum space the corresponding integral is given by

$$G := \int \frac{1}{i\pi^{D/2}} \frac{1}{[q^2]^{a_1}[(p-q)^2]^{a_2}} d^D q, \tag{3.19}$$

where the parameters $\{a_i\}$ are arbitrary. The Schwinger representation[1] corresponding to this diagram produces

$$G = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \int_0^\infty x^{a_1-1} y^{a_2-1} \frac{\exp(-\frac{xy}{x+y}p^2)}{(x+y)^{\frac{D}{2}}} \, dx \, dy. \tag{3.20}$$

In order to apply Ramanujan's Master Theorem in iterative form, each term of the integrand is expanded in a Taylor series. In situations where options are available, the optimal course of action seems to be to minimize the number of expansions. This is a heuristic rule and its justification is an open question. In this example, it is convenient to expand first the exponential function

$$\exp\left(-\frac{xy}{x+y}p^2\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(p^2\right)^n \frac{x^n y^n}{(x+y)^n}, \tag{3.21}$$

to produce

$$G = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \int_0^\infty x^{a_1-1} y^{a_2-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(p^2\right)^n \frac{x^n y^n}{(x+y)^{\frac{D}{2}+n}} \, dx \, dy. \tag{3.22}$$

The next step is to expand $(x+y)^{-D/2-n}$ by the binomial theorem

$$(x+y)^{-(D/2+n)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{D}{2}+n\right)_k x^{-D/2-n-k} y^k \tag{3.23}$$

and replace in (3.22) to obtain

$$G = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \int_0^\infty \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(-1)^k}{k!} \left(p^2\right)^n \left(\frac{D}{2}+n\right)_k x^{-k+a_1-\frac{D}{2}} y^{k+n+a_2} \frac{dx}{x} \frac{dy}{y}.$$

The change of variables $x \mapsto 1/x$ produces the alternative expression

$$G = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \int_0^\infty \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(-1)^k}{k!} \left(p^2\right)^n \left(\frac{D}{2}+n\right)_k x^{k-a_1+\frac{D}{2}} y^{k+n+a_2} \frac{dx}{x} \frac{dy}{y}.$$

[1] There is a canonical procedure to associate with each Feynman diagram a multi-dimensional integral. For details, the reader is referred to [11, Chapter 3], under the name *alpha parameters*.

There are several options to employ Ramanujan's Master Theorem to evaluate this integral. Option (a) evaluates first the integral in the x-variable using the expansion in the index k:

$$\int_0^\infty \sum_{k=0}^\infty \cdots \frac{(-x)^k}{k!}\, dx.$$

The value of the integral obtained by this procedure is denoted by G_a. The other two options, labeled G_b and G_c, are produced by replacing the pair (x,k) by (y,k) and (y,n), respectively. It is shown here that each of these options produces the same result.

Solution with option (a). In this case, G is given by

$$G_a = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \left[\int_0^\infty \frac{dx}{x} \sum_{k=0}^\infty \frac{(-1)^k}{k!} \varphi(k) x^{k-a_1+\frac{D}{2}} \right] \frac{dy}{y}, \tag{3.24}$$

where $\varphi(k)$ is

$$\varphi(k) = \sum_{n=0}^\infty \frac{(-1)^n}{n!} \left(p^2\right)^n \left(\frac{D}{2}+n\right)_k y^{k+n+a_2}. \tag{3.25}$$

Ramanujan's Master Theorem now gives

$$G_a = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \Gamma\left(k^*\right) \varphi\left(-k^*\right) \frac{dy}{y}, \quad \text{with } k^* = \frac{D}{2} - a_1.$$

Thus,

$$\begin{aligned} G_a &= \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \Gamma(D/2-a_1) \int_0^\infty \sum_{n=0}^\infty \frac{(-1)^n}{n!} \left(p^2\right)^n \left(\frac{D}{2}+n\right)_{a_1-\frac{D}{2}} y^{a_1+a_2-\frac{D}{2}+n} \frac{dy}{y} \\ &= \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \Gamma(D/2-a_1) \int_0^\infty \sum_{n=0}^\infty \frac{(-1)^n}{n!} \left(p^2\right)^n \frac{\Gamma(a_1+n)}{\Gamma(\frac{D}{2}+n)} y^{n+a_1+a_2-\frac{D}{2}} \frac{dy}{y}. \end{aligned}$$

The last integral is now evaluated using Ramanujan's Master Theorem to obtain

$$G_a = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \Gamma\left(\frac{D}{2}-a_1\right) \Gamma\left(n^*\right) \left(p^2\right)^{-n^*} \frac{\Gamma(a_1-n^*)}{\Gamma(\frac{D}{2}-n^*)},$$

with $n^* = a_1 + a_2 - \frac{D}{2}$. Therefore, option (a) gives the value of G as

$$G_a = (-1)^{-\frac{D}{2}} \left(p^2\right)^{\frac{D}{2}-a_1-a_2} \frac{\Gamma(a_1+a_2-\frac{D}{2})\Gamma(\frac{D}{2}-a_1)\Gamma(\frac{D}{2}-a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(D-a_1-a_2)}. \tag{3.26}$$

Solution with option (b). A similar argument now yields

$$G_b = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \left[\int_0^\infty \sum_{k=0}^\infty \frac{(-1)^k}{k!} \varphi(k) y^{k+n+a_2} \right] \frac{dy}{y} \frac{dx}{x},$$

with

$$\varphi(k) = \sum_{n=0}^\infty \frac{(-1)^n}{n!} \left(p^2\right)^n \left(\frac{D}{2} + n\right)_k x^{k-a_1+\frac{D}{2}}.$$

Therefore

$$G_b = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \sum_{n=0}^\infty \frac{(-1)^n}{n!} \left(p^2\right)^n \frac{\Gamma(n+a_2)\Gamma(\frac{D}{2}-a_2)}{\Gamma(\frac{D}{2}+n)} x^{-n-a_2-a_1+\frac{D}{2}} \frac{dx}{x}.$$

The change of variables $x \mapsto 1/x$ now gives

$$G_b = \frac{(-1)^{-\frac{D}{2}}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \sum_{n=0}^\infty \frac{(-1)^n}{n!} \left(p^2\right)^n \frac{\Gamma(n+a_2)\Gamma(\frac{D}{2}-a_2)}{\Gamma(\frac{D}{2}+n)} x^{n+a_2+a_1-\frac{D}{2}} \frac{dx}{x}$$

and Ramanujan's Master Theorem produces the final result as

$$G_b = (-1)^{-\frac{D}{2}} \left(p^2\right)^{\frac{D}{2}-a_2-a_1} \frac{\Gamma(\frac{D}{2}-a_1)\Gamma(\frac{D}{2}-a_2)\Gamma(a_2+a_1-\frac{D}{2})}{\Gamma(a_1)\Gamma(a_2)\Gamma(D-a_2-a_1)}. \tag{3.27}$$

Observe that $G_a = G_b$. A similar calculation shows that this is also the value of G_c. All choices of indices lead to the same value for the integral G.

4. Some multiloop calculations

This section uses Ramanujan's Master Theorem for the evaluation of two multidimensional integrals of the form

$$I = \int_0^\infty x_1^{\nu_1-1} ... \int_0^\infty x_N^{\nu_N-1} f(x_1,...,x_N) dx_1 \cdots dx_N. \tag{4.1}$$

As in the one-dimensional case, the function f is assumed to admit a Taylor expansion

$$f(x_1,...,x_N) = \sum_{l_1=0}^\infty ... \sum_{l_N=0}^\infty \frac{(-1)^{l_1}}{l_1!} ... \frac{(-1)^{l_N}}{l_N!} \varphi(l_1,...,l_N)$$
$$\times x_1^{a_{11}l_1+...+a_{1N}l_N+b_1} ... x_N^{a_{N1}l_1+...+a_{NN}l_N+b_N},$$

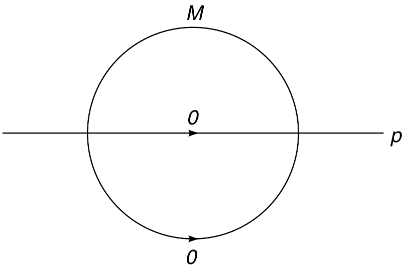

Fig. 1. The sunset diagram.

so that I is expressed as

$$I = \int_0^\infty \cdots \int_0^\infty \sum_{l_1=0}^\infty ... \sum_{l_N=0}^\infty \frac{(-1)^{l_1}}{l_1!} ... \frac{(-1)^{l_N}}{l_N!} \varphi(l_1, ..., l_N)$$
$$\times x_1^{a_{11}l_1+...+a_{1N}l_N+\widetilde{b}_1} ... x_N^{a_{N1}l_1+...+a_{NN}l_N+\widetilde{b}_N} \frac{dx_1}{x_1} \cdots \frac{dx_N}{x_N}$$

with $\widetilde{b}_i = \nu_i + b_i$ $(i = 1, ..., N)$.

Applying Ramanujan's Master Theorem in iterative form gives

$$I = \frac{1}{|\det(\mathbf{A})|} \Gamma(-l_1^*) ... \Gamma(-l_N^*) \varphi(l_1^*, ..., l_N^*) \tag{4.2}$$

where $\mathbf{A} = (a_{ij})$ and $\mathbf{l}^* = (l_1^*, \cdots, l_N^*)$ is the solution of the linear system $\mathbf{A}\mathbf{l}^* = -\widetilde{\mathbf{b}}$. Details of the proof of this result appear in [1].

Example 4.1 *(Massive sunset diagram).* The first example is associated with the diagram shown in Fig. 1. In the momentum space, the integral is given by

$$G := \int \frac{1}{[q^2 - M^2]^{a_1}} \frac{1}{[(q_1 - q_2)^2]^{a_2}} \frac{1}{[(p + q_2)^2]^{a_3}} \frac{d^D q_1}{i\pi^{D/2}} \frac{d^D q_2}{i\pi^{D/2}} \tag{4.3}$$

In terms of the Schwinger parametrization, G becomes

$$G = \frac{(-1)^{-D}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^\infty \int_0^\infty \int_0^\infty \frac{\exp(x_1 M^2) \exp(-\frac{x_1x_2x_3}{x_1x_2+x_1x_3+x_2x_3} p^2)}{(x_1x_2 + x_1x_3 + x_2x_3)^{\frac{D}{2}}} d\vec{x},$$

where $d\vec{x} = x_1^{a_1-1} x_2^{a_2-1} x_3^{a_3-1} dx_1\, dx_2\, dx_3$.

The evaluation is described here only[2] in special case $p^2 = M^2$. The general case is only algebraically more complicated. The integral reduces to

[2] This special case is of physical interest.

$$G = \frac{(-1)^{-D}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^\infty \int_0^\infty \int_0^\infty \frac{\exp[\frac{x_1^2(x_2+x_3)}{x_1(x_2+x_3)+x_2x_3} M^2]}{[x_1(x_2+x_3)+x_2x_3]^{\frac{D}{2}}} d\vec{x}.$$

The expansion of the exponential function yields

$$\exp\left(\frac{x_1^2(x_2+x_3)}{x_1(x_2+x_3)+x_2x_3} M^2\right) = \sum_{n_1=0}^{\infty} \frac{(-1)^{n_1}}{n_1!} \left(-M^2\right)^{n_1} \frac{x_1^{2n_1}(x_2+x_3)^{n_1}}{[x_1(x_2+x_3)+x_2x_3]^{n_1}}$$

so that

$$G = \frac{(-1)^{-D}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^\infty \int_0^\infty \int_0^\infty \sum_{n_1=0}^{\infty} \frac{(-1)^{n_1}}{n_1!} \left(-M^2\right)^{n_1} \frac{x_1^{2n_1}(x_2+x_3)^{n_1}}{[x_1(x_2+x_3)+x_2x_3]^{\frac{D}{2}+n_1}} d\vec{x}. \tag{4.4}$$

The binomial theorem

$$(x+y)^a = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(-a+n)}{\Gamma(-a)} x^{a-n} y^n \tag{4.5}$$

gives

$$\begin{aligned} &\frac{1}{[x_1(x_2+x_3)+x_2x_3]^{\frac{D}{2}+n_1}} \\ &\quad = \sum_{n_2=0}^{\infty} \frac{(-1)^{n_2}}{n_2!} \frac{\Gamma(\frac{D}{2}+n_1+n_2)}{\Gamma(\frac{D}{2}+n_1)} x_1^{-\frac{D}{2}-n_1-n_2} (x_2+x_3)^{-\frac{D}{2}-n_1-n_2} x_2^{n_2} x_3^{n_2}, \end{aligned}$$

and (4.4) becomes

$$\begin{aligned} G = \frac{(-1)^{-D}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} &\int_0^\infty \int_0^\infty \int_0^\infty \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{(-1)^{n_1}}{n_1!} \frac{(-1)^{n_2}}{n_2!} \left(M^2\right)^{n_1} \\ &\times \frac{\Gamma(\frac{D}{2}+n_1+n_2)}{\Gamma(\frac{D}{2}+n_1)} x_1^{n_1-\frac{D}{2}-n_2} x_2^{n_2} x_3^{n_2} (x_2+x_3)^{-\frac{D}{2}-n_2} d\vec{x}. \end{aligned}$$

The final expansion

$$(x_2+x_3)^{-\frac{D}{2}-n_2} = \sum_{n_3=0}^{\infty} \frac{(-1)^{n_3}}{n_3!} \frac{\Gamma(\frac{D}{2}+n_2+n_3)}{\Gamma(\frac{D}{2}+n_2)} x_2^{-\frac{D}{2}-n_2-n_3} x_3^{n_3} \tag{4.6}$$

expresses the integral in the form required for the application of Ramanujan's Master Theorem. This gives

$$G = \int_0^\infty \int_0^\infty \int_0^\infty \sum_{n_1=0}^\infty \sum_{n_2=0}^\infty \sum_{n_3=0}^\infty \frac{(-1)^{n_1}}{n_1!} \frac{(-1)^{n_2}}{n_2!} \frac{(-1)^{n_3}}{n_3!} x_1^{a_1-\frac{D}{2}+n_1-n_2} x_2^{a_2-\frac{D}{2}-n_3} x_3^{a_3+n_2+n_3}$$
$$\times \frac{(-1)^{-D}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \frac{\Gamma(\frac{D}{2}+n_1+n_2)}{\Gamma(\frac{D}{2}+n_1)} \frac{\Gamma(\frac{D}{2}+n_2+n_3)}{\Gamma(\frac{D}{2}+n_2)} \left(-M^2\right)^{n_1} \frac{dx_1}{x_1} \frac{dx_2}{x_2} \frac{dx_3}{x_3}. \tag{4.7}$$

Therefore

$$G = \frac{(-1)^{-D}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \Gamma\left(n_1^*\right)\Gamma\left(n_2^*\right)\Gamma\left(n_3^*\right) \frac{\Gamma(\frac{D}{2}-n_1^*-n_2^*)}{\Gamma(\frac{D}{2}-n_1^*)} \frac{\Gamma(\frac{D}{2}-n_2^*-n_3^*)}{\Gamma(\frac{D}{2}-n_2^*)} \left(-M^2\right)^{-n_1^*}, \tag{4.8}$$

where the indices $\{n_i^*\}$ are given by the unique solution to the linear system

$$\begin{aligned} n_1 - n_2 &= a_1 - D/2, \\ n_3 &= -a_2 + D/2, \\ n_2 + n_3 &= a_3, \end{aligned}$$

associated with (4.7). The solutions are

$$n_1^* = a_1 + a_2 + a_3 - D, \qquad n_2^* = a_2 + a_3 - D/2, \qquad n_3^* = D/2 - a_2. \tag{4.9}$$

The value of the integral G is finally given by

$$G = (-1)^{-D} \frac{\Gamma(a_1+a_2+a_3-D)\Gamma(a_2+a_3-\frac{D}{2})\Gamma(\frac{D}{2}-a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}$$
$$\times \frac{\Gamma(\frac{D}{2}-a_3)\Gamma(2D-a_1-2a_2-2a_3)}{\Gamma(\frac{3D}{2}-a_1-a_2-a_3)\Gamma(D-a_2-a_3)} \left(-M^2\right)^{D-a_1-a_2-a_3}.$$

Example 4.2 *(Massless three loops ladder diagram).* The last example evaluates the integral[3] associated with the diagram seen in Fig. 2

$$G = \frac{(-1)^{-\frac{3D}{2}}}{\Gamma(a_1)...\Gamma(a_{10})} \int_0^\infty ... \int_0^\infty \frac{\exp(-\frac{x_1x_4x_7x_{10}}{U}t)}{U^{\frac{D}{2}}} d\vec{x}. \tag{4.10}$$

Here $d\vec{x} = \prod_{j=1}^{10} dx_j x_j^{a_j-1}$ and U is a polynomial given by

$$U = x_5(x_7+\mathbf{f}_1)(\mathbf{f}_2+x_4) + x_6(x_7+\mathbf{f}_1)(\mathbf{f}_2+x_4) + x_4(x_7+\mathbf{f}_1)\mathbf{f}_2 + x_7(\mathbf{f}_2+x_4)\mathbf{f}_1,$$

[3] In a simplified physical situation, where the conditions $P_i^2 = 0$ for $1 \le i \le 4$ and $s = 0$ are imposed.

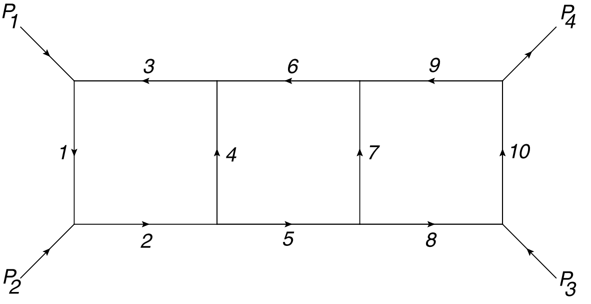

Fig. 2. The 3-loop ladder.

with

$$\mathbf{f}_1 = x_8 + x_9 + x_{10} \quad \text{and} \quad \mathbf{f}_2 = x_1 + x_2 + x_3. \tag{4.11}$$

Expanding the exponential term yields

$$G = \frac{(-1)^{-\frac{3D}{2}}}{\Gamma(a_1)\cdots\Gamma(a_{10})} \int_0^\infty \cdots \int_0^\infty \sum_{n_1=0}^\infty \frac{(-1)^{n_1}}{n_1!} t^{n_1} \frac{x_1^{n_1} x_4^{n_1} x_7^{n_1} x_{10}^{n_1}}{U^{\frac{D}{2}+n_1}} d\vec{x}, \tag{4.12}$$

and expanding U by the multinomial theorem

$$\begin{aligned}(x_1 + ... + x_{k-1} + x_k)^a = &\sum_{n_1=0}^\infty ... \sum_{n_{k-1}=0}^\infty \frac{(-1)^{n_1}}{n_1!} ... \frac{(-1)^{n_{k-1}}}{n_{k-1}!} \frac{\Gamma(-a + n_1 + ... + n_{k-1})}{\Gamma(-a)} \\ &\times x_1^{n_1} ... x_{k-1}^{n_{k-1}} x_k^{a-n_1-...-n_{k-1}}\end{aligned}$$

gives

$$\begin{aligned}U^{-\frac{D}{2}-n_1} = &\sum_{n_2=0}^\infty \sum_{n_3=0}^\infty \sum_{n_4=0}^\infty \frac{(-1)^{n_2}}{n_2!} \frac{(-1)^{n_3}}{n_3!} \frac{(-1)^{n_4}}{n_4!} \frac{\Gamma(\frac{D}{2} + n_1 + n_2 + n_3 + n_4)}{\Gamma(\frac{D}{2} + n_1)} \\ &\times \mathbf{f}_1^{-\frac{D}{2}-n_1-n_2-n_3-n_4} \mathbf{f}_2^{n_2} (x_7 + \mathbf{f}_1)^{n_2+n_3+n_4} (x_4 + \mathbf{f}_2)^{-\frac{D}{2}-n_1-n_2} \\ &\times x_4^{n_2} x_5^{n_3} x_6^{n_4} x_7^{-\frac{D}{2}-n_1-n_2-n_3-n_4}.\end{aligned}$$

Similarly,

$$(x_7 + \mathbf{f}_1)^{n_2+n_3+n_4} = \sum_{n_5=0}^\infty \frac{(-1)^{n_5}}{n_5!} \frac{\Gamma(-n_2 - n_3 - n_4 + n_5)}{\Gamma(-n_2 - n_3 - n_4)} x_7^{n_5} \mathbf{f}_1^{n_2+n_3+n_4-n_5},$$

$$(x_4 + \mathbf{f}_2)^{-\frac{D}{2}-n_1-n_2} = \sum_{n_6=0}^\infty \frac{(-1)^{n_6}}{n_6!} \frac{\Gamma(\frac{D}{2} + n_1 + n_2 + n_6)}{\Gamma(\frac{D}{2} + n_1 + n_2)} x_4^{n_6} \mathbf{f}_2^{-\frac{D}{2}-n_1-n_2-n_6},$$

and

$$\begin{aligned}\mathbf{f}_1^{-\frac{D}{2}-n_1-n_5} &= \sum_{n_7=0}^{\infty}\sum_{n_8=0}^{\infty}\frac{(-1)^{n_7}}{n_7!}\frac{(-1)^{n_8}}{n_9!}\frac{\Gamma(\frac{D}{2}+n_1+n_5+n_7+n_8)}{\Gamma(\frac{D}{2}+n_1+n_5)}\\ &\quad\times x_8^{n_7}x_9^{n_8}x_{10}^{-\frac{D}{2}-n_1-n_5-n_7-n_8},\\ \mathbf{f}_2^{-\frac{D}{2}-n_1-n_6} &= \sum_{n_9=0}^{\infty}\sum_{n_{10}=0}^{\infty}\frac{(-1)^{n_9}}{n_9!}\frac{(-1)^{n_{10}}}{n_{10}!}\frac{\Gamma(\frac{D}{2}+n_1+n_6+n_9+n_{10})}{\Gamma(\frac{D}{2}+n_1+n_6)}\\ &\quad\times x_1^{n_9}x_2^{n_{10}}x_3^{-\frac{D}{2}-n_1-n_6-n_9-n_{10}},\end{aligned}$$

finally produce

$$\begin{aligned}G &= \frac{(-1)^{-\frac{3D}{2}}}{\Gamma(a_1)\cdots\Gamma(a_{10})}\int_0^{\infty}\cdots\int_0^{\infty}\sum_{n_1=0}^{\infty}\ldots\sum_{n_{10}=0}^{\infty}\frac{(-1)^{n_1}}{n_1!}\ldots\frac{(-1)^{n_{10}}}{n_{10}!}\varphi(n_1,\ldots,n_{10})\\ &\quad\times x_1^{a_1+n_1+n_9}x_2^{a_2+n_{10}}x_3^{a_3-\frac{D}{2}-n_1-n_6-n_9-n_{10}}x_4^{a_4+n_1+n_2+n_6}x_5^{a_5+n_3}x_6^{a_6+n_4}\\ &\quad\times x_7^{a_7-\frac{D}{2}-n_2-n_3-n_4+n_5}x_8^{a_8+n_7}x_9^{a_9+n_8}x_{10}^{a_{10}-\frac{D}{2}-n_5-n_7-n_8}\frac{dx_1}{x_1}\cdots\frac{dx_{10}}{x_{10}}\end{aligned}$$

with the notation

$$\begin{aligned}\varphi(n_1,\ldots,n_{10}) &= \frac{\Gamma(\frac{D}{2}+n_1+n_2+n_3+n_4)}{\Gamma(\frac{D}{2}+n_1)}\frac{\Gamma(-n_2-n_3-n_4+n_5)}{\Gamma(-n_2-n_3-n_4)}\\ &\quad\times\frac{\Gamma(\frac{D}{2}+n_1+n_2+n_6)}{\Gamma(\frac{D}{2}+n_1+n_2)}\frac{\Gamma(\frac{D}{2}+n_1+n_5+n_7+n_8)}{\Gamma(\frac{D}{2}+n_1+n_5)}\\ &\quad\times\frac{\Gamma(\frac{D}{2}+n_1+n_6+n_9+n_{10})}{\Gamma(\frac{D}{2}+n_1+n_6)}t^{n_1}.\end{aligned}$$

A direct application of Ramanujan's Master Theorem gives

$$G = \frac{(-1)^{-\frac{3D}{2}}}{\Gamma(a_1)\cdots\Gamma(a_{10})}\frac{1}{|\det(\mathbf{A})|}\varphi\left(-n_1^*,\ldots,-n_{10}^*\right)\prod_{j=1}^{10}\Gamma\left(n_j^*\right) \tag{4.13}$$

where $\{n_i^*\}$ are solutions of a linear system that yields

$$\begin{aligned}n_1^* &= -\frac{3D}{2}+a_1+a_2+a_3+a_4+a_5+a_6+a_7+a_8+a_9+a_{10},\\ n_2^* &= D-a_5-a_6-a_7-a_8-a_9-a_{10},\\ n_3^* &= a_5,\\ n_4^* &= a_6,\end{aligned}$$

$$
\begin{aligned}
n_5^* &= \frac{D}{2} - a_8 - a_9 - a_{10}, \\
n_6^* &= \frac{D}{2} - a_1 - a_2 - a_3, \\
n_7^* &= a_8, \\
n_8^* &= a_9, \\
n_9^* &= \frac{3D}{2} - a_2 - a_3 - a_4 - a_5 - a_6 - a_7 - a_8 - a_9 - a_{10}, \\
n_{10}^* &= a_2.
\end{aligned}
\tag{4.14}
$$

This gives the value of the diagram as

$$
\begin{aligned}
G = (-1)^{-\frac{3D}{2}} & \frac{\Gamma(\frac{D}{2} - a_{89,10})\Gamma(\frac{D}{2} - a_{123})\Gamma(\frac{3D}{2} - a_{23456789,10})\Gamma(\frac{3D}{2} - a_{123456789})}{\Gamma(a_1)\Gamma(a_4)\Gamma(a_7)\Gamma(a_{10})\Gamma(2D - a_{123456789,10})} \\
& \times \frac{\Gamma(a_{123456789,10} - \frac{3D}{2})\Gamma(D - a_{56789,10})\Gamma(D - a_{123456})\Gamma(\frac{D}{2} - a_7)\Gamma(\frac{D}{2} - a_4)}{\Gamma(D - a_{789,10})\Gamma(D - a_{1234})\Gamma(\frac{3D}{2} - a_{1234567})\Gamma(\frac{3D}{2} - a_{456789,10})} \\
& \times t^{\frac{3D}{2} - a_{123456789,10}},
\end{aligned}
$$

with the notation

$$
a_{ijk\ldots} = a_i + a_j + a_k + \ldots . \tag{4.15}
$$

An important special case, when all powers a_i of propagators are 1, is

$$
G = (-1)^{-\frac{3D}{2}} \frac{\Gamma(10 - \frac{3D}{2})\Gamma(\frac{D}{2} - 3)^2\Gamma(\frac{3D}{2} - 9)^2\Gamma(D - 6)^2\Gamma(\frac{D}{2} - 1)^2}{\Gamma(2D - 10)\Gamma(D - 4)^2\Gamma(\frac{3D}{2} - 7)^2} t^{\frac{3D}{2} - 10}.
$$

5. Conclusions

This paper presents a technique for the evaluation of a large variety of integrals coming from Feynman diagrams. The advantage over previous methods is that the evaluation is reduced to series expansions of the integrand coupled with the solution of a linear system of equations.

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