A rigorous proof of some heuristic rules in the Method of Brackets to evaluate definite integrals

AN ABSTRACT
SUBMITTED ON THE FIFTEENTH DAY OF NOVEMBER, 2018
TO THE DEPARTMENT OF MATHEMATICS
OF THE SCHOOL OF SCIENCE AND ENGINEERING OF
TULANE UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
BY

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Abstract

Many symbolic integration methods and algorithms have been developed to deal with definite integrals such as those of interest to physicists, theoretical chemists and engineers. These have been implemented into mathematical softwares like *Maple* and *Mathematica* to give closed forms of definite integrals. The work presented here introduces and analytically investigates an algorithm called *Method of brackets*.

Method of brackets consists of a small number of rules to transform the evaluation of a definite integral into a problem of solving a system of linear equations. These rules are heuristic so justification is needed to make this method rigorous. Here we use contour integrals to justify the evaluation given by the algorithm.

A RIGOROUS PROOF OF SOME HEURISTIC RULES IN THE METHOD OF BRACKETS TO EVALUATE DEFINITE INTEGRALS

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Contents

Chapter 1

Introduction

1.1 Introduction

The method of brackets (MoB) is an algorithm to evaluate definite integrals. It was developed by I. Gonzalez [1],[2] and further improved in collaboration work with Victor Moll and his students Armin Straub, Lin Jiu and Karen Kohl [3]. The method has its origin on the evaluation of definite integrals arising from the Schwinger parametrization of Feynman diagrams [2].

The method is a generalization of the Ramanujan's Master Theorem which is the following formal identity

$$\int_0^\infty x^{s-1}\left(f(0) - \frac{x}{1!}f(1) + \frac{x^2}{2!}f(2) - \cdots\right)dx = f(-s)\Gamma(s),$$

for $0 < s < 1$. Hardy proved this identity rigorously in [4] by enforcing growth conditions on f. An alternative proof has been presented in [5]. The purpose of the method is to evaluate integrals of the form $\int_0^\infty f(x)dx$. This is done by first rewriting $f(x)$ as a formal expression named `a bracket series`. Then we evaluate the bracket series to get the answer. The whole process is done using a set of rules, whose proofs are the main purpose of this thesis. Below are the ones that we will focus on (newer rules have been added in [6]).

- Rule P_1: Let
$$f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n + \beta - 1},$$
then
$$\int_0^\infty f(x)dx = \int_0^\infty \sum_{n=0}^{\infty} a_n x^{\alpha n+\beta-1} dx =: \sum_{n=0}^{\infty} a_n \langle \alpha n + \beta \rangle.$$
Here the notation $\langle \cdot \rangle$ is called a **bracket**, and a formal sum $\sum_{n=0}^{\infty} c_n \langle \cdot \rangle$ is called a **bracket series**.

- Rule P_2: Let α be a real number. We have
$$(a_1 + \cdots + a_n)^\alpha = \sum_{n_1,\ldots,n_r} \phi_{n_1} \cdots \phi_{n_r} a_1^{n_1} \cdots a_r^{n_r} \frac{\langle -\alpha + n_1 + \cdots + n_r \rangle}{\Gamma(-\alpha)},$$
where
$$\phi_n = \phi(n) = \frac{(-1)^n}{\Gamma(n+1)},$$
which is called the **indicator function**.

- Rule P_3: The **index** of a bracket series is the number of sums minus the number of brackets. For example, the bracket series
$$\sum_{n=0}^{\infty} \phi_n a^n \langle n+1 \rangle$$
has one sum and one bracket so it has index zero. On the other hand, the bracket series
$$\sum_{n_1,n_2,n_3} \phi_{n_1,n_2,n_3} \langle n_1 + n_2 + 1 \rangle \langle n_2 + 2n_3 + 1 \rangle$$
has three sums and two brackets so it has index one.

- Rule E_1: The one-dimensional bracket series is assigned the value

$$\sum_n \phi_n f(n)\langle an+b\rangle := \frac{1}{|a|} f(n^*)\Gamma(-n^*),$$

where n^* solves $an+b=0$.

- Rule E_2: The evaluation of a multi-dimensional bracket series with index zero is given by the the following rule

$$\begin{aligned} & \sum_{n_1,\ldots,n_r} \phi_{\overline{n}_r} f(n_1,\ldots,n_r) \prod_{i=1}^{r} \langle a_{i1}n_1+\cdots+a_{ir}n_r+c_i\rangle \\ = \ & \frac{1}{|\det A|} f(n_1^*,\ldots,n_r^*) \prod_{i=1}^{r} \Gamma(-n_i^*), \end{aligned}$$

where $\phi_{\overline{n}_r} = \phi(n_1)\cdots\phi(n_r)$, A is the matrix of coefficients $\{a_{ij}\}$, and n_i^*'s are the solution to the vanishing of the brackets. No value is assigned if A is singular.

- Rule E_3: To evaluate a multi-dimensional bracket series with positive index, apply rule E_2 to all cases where some of the n_i's take turn to be free. Among these contributions, those that are divergent are discarded. The rest are added to give the answer. However, if the results (from different cases) converge on different regions then they are added based on region of convergence.

1.2 The road so far

Most rules of the method of brackets have not yet been justified. In his PhD thesis in 2016 [7], Lin Jiu justified the evaluation of MoB in cases of zero index. He also showed that the method yields the same answer independently of the factorization of the integrand. He then went further to discuss the use of analytic continuation to make use of divergent series. From this, Jiu proposed a modified rule for E_3 as follows:

Rule $\tilde{E}_3$: the value of a multi-dimensional bracket series of positive index is obtained by computing all the contributions of maximal rank by Rule E_2. These contributions to the integral appear as series in the free parameters. Series converging in a common region are added. *Divergent series are evaluated by their analytic continuation, if such continuations exist. Divergent series having the same analytic continuation are combined and treated as the common analytic continuation on its domain.* Any series producing a non-real contribution is also discarded. There is no assignment to a bracket of negative index.

The use of null and total divergent series it also discussed in Jiu's thesis. We will visit this topic in chapter 4.

1.3 Overview

- **Chapter 2** deals with the case of zero index. Here we start with Ramanujan identity and use it to prove rule E_1, then we move on to justify rule E_2. Afterward, we discuss and prove rule P_2. Finally we point out possible issues that one might see and how to deal with them.

- **Chapter 3** is about positive index cases, namely rule E_3. We treat each bracket as an integral and analyze its evaluation. This leads to the explanation and justification of rule E_3 and how sometimes one case gives the answer while in others we have to add results from different cases. We also introduce the use of extra parameter to help justify the use of MoB.

- **Chapter 4** mentions unique examples. Some integrals require extra care to produce a bracket series. Others require an introduction of additional parameters. Some yield null or divergent series but the MoB still produces the correct answer. Also the use of rule P_2 for multinomial with positive exponent, something that has not been justified, does not prevent the MoB from giving the right answer.

- **Chapter 5** discusses work that still needs to be done to fully justify method of brackets.

1.4 Examples

Here are some examples of how the method of brackets works.

Example 1.4.1. To evaluate the integral

$$\int_0^\infty e^{-ax}dx,$$

we first write the integrand as

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}a^n x^n = \sum_{n=0}^{\infty}\phi_n a^n x^n.$$

Now use rule P_1 to get

$$\sum_{n=0}^{\infty}\phi_n a^n x^n := \sum_{n=0}^{\infty}\phi_n a^n \langle n+1\rangle.$$

The bracket vanishes at $n^* = -1$ so rule E_1 gives this bracket series the value

$$a^{-1}\Gamma(1) = \frac{1}{a}.$$

This agrees with a classical evaluation.

Example 1.4.2. We show

$$\int_0^\infty e^{-\alpha x}\sin(\beta x)dx = \frac{\beta}{\alpha^2+\beta^2}.$$

First write the integrand as

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}(-1)^{n_1+n_2}\frac{\alpha^{n_1}\beta^{2n_2+1}}{\Gamma(n_1+1)\Gamma(2n_2+2)}x^{n_1+2n_2+1}.$$

Thus rule P_1 gives the bracket series

$$\sum_{n_1,n_2}\phi_{n_1}\phi_{n_2}\alpha^{n_1}\beta^{2n_2+1}\frac{\Gamma(n_2+1)}{\Gamma(2n_2+2)}\langle n_1+2n_2+2\rangle.$$

Now rule P_3 says this is an index 1 bracket series. Rule E_3 says we have to consider two cases: n_1 free and n_2 free.

- Case 1: let n_1 free, then the vanishing of the bracket gives $n_2^* = -1-n_1/2$. The bracket now evaluates to

  $$\frac{1}{2}\sum_{n_1=0}^{\infty}\phi_{n_1}\alpha^{n_1}\beta^{-n_1-1}\frac{\Gamma(-n_1/2)}{\Gamma(-n_1)}\Gamma(1+n_1/2).$$

  The summand vanishes with odd n_1. With $n_1 = 2n$ we get

  $$\frac{1}{2}\sum_{n=0}^{\infty}\alpha^{2n}\beta^{-2n-1}\frac{\Gamma(-n)\Gamma(n+1)}{\Gamma(2n+1)\Gamma(-2n)}.$$

  Now use reflection formula for the gamma function (see Appendix) for the numerator and denominator to get

  $$\frac{1}{2}\sum_{n=0}^{\infty}\alpha^{2n}\beta^{-2n-1}\frac{\sin(2\pi n)}{\sin\pi n} = \sum_{n=0}^{\infty}\alpha^{2n}\beta^{-2n-1}\cos(\pi n).$$

  Since $\cos(\pi n) = (-1)^n$, the sum becomes

  $$\frac{1}{\beta}\sum_{n=0}^{\infty}\left(-\frac{\alpha^2}{\beta^2}\right)^n = \frac{1}{\beta}\cdot\frac{1}{1+\frac{\alpha^2}{\beta^2}} = \frac{\beta}{\alpha^2+\beta^2},$$

  where now we need $\alpha < \beta$ for the sum to converge.

- Case 2: let n_2 free, then we get $n_1^* = -2n_2 - 2$. The bracket gives

$$\sum_{n_2=0}^{\infty} (-1)^{n_2} \alpha^{-2-2n_2} \beta^{2n_2+1} = \frac{\beta}{\alpha^2 + \beta^2},$$

  where we need $\beta < \alpha$ for the sum to converge.

Thus the answer is $\beta/(\alpha^2 + \beta^2)$ in any cases.

Example 1.4.3. We will show

$$\int_0^{\infty} \frac{\sin \alpha x}{x(x^2 + \beta^2)} dx = \frac{\pi}{2\beta^2}(1 - e^{-\alpha\beta}).$$

Here we need to use rule P_2 to get

$$\frac{1}{x^2 + \beta^2} = \sum_{n_1, n_2} \phi_{n_1} \phi_{n_2} x^{2n_1} \beta^{2n_2} \langle n_1 + n_2 + 1 \rangle.$$

Combine this with the power series for sine we get the bracket series

$$\sum_{n_1, n_2, n_3} \phi_{n_{123}} \alpha^{2n_3+1} \beta^{2n_2} \frac{\Gamma(n_3 + 1)}{\Gamma(2n_3 + 2)} \langle n_1 + n_2 + 1 \rangle \langle 2n_1 + 2n_3 + 1 \rangle.$$

This is a bracket series with three sums and two brackets. Therefore this is a bracket series of index one.

- Case 1: let n_1 free. Then we get

$$\frac{1}{2} \sum_{n_1=0}^{\infty} \frac{(-1)^{n_1}}{\Gamma(n_1 + 1)} \alpha^{-2n_1} \beta^{-2n_1-2} \frac{\Gamma(-n_1 + 1/2)}{\Gamma(-2n_1 + 1)} \Gamma(n_1 + 1) \Gamma(n_1 + 1/2),$$

  which vanish because of the term $\Gamma(-2n_1 + 1)$ except when $n_1 = 0$, which gives $\frac{\pi}{2\beta^2}$.

- Case 2: let n_2 free, we have

$$-\frac{\pi\alpha^2}{2}\sum_{n_2=0}^{\infty}\frac{(\alpha\beta)^{2n_2}}{\Gamma(2n_2+3)}=\frac{\pi}{2\beta^2}(1-\cosh\alpha\beta).$$

- Case 3: let n_3 free, it gives

$$\frac{\pi\alpha}{2\beta}\sum_{n_3=0}^{\infty}\frac{(\alpha\beta)^{2n_3}}{\Gamma(2n_3+2)}=\frac{\pi}{2\beta^2}\sinh\alpha\beta.$$

Adding the results from all cases gives the desired answer, though according to rule E_3 we should combine the results from all three cases. In chapter 3, an explanation will be given why only case 2 and 3 should be added.

1.5 Notation

Here are some notation that will be used often in our work

- The notation $\mathcal{C}_T(c)$ denotes the open contour

$$[c-iT,-T-iT]\cup[-T-iT,-T+iT]\cup[-T+iT,c+iT],$$

  where the part $[-T-iT,-T+iT]$ will be referred to as the **vertical segment** and the other two as **horizontal segments**.

- The following notation are equivalent

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\equiv\sum_{n_1,n_2}\equiv\sum_{n_{12}}.$$

  In the same way, we also have

$$\phi_{n_1}\phi_{n_2}=\phi_{n_1,n_2}=\phi_{n_{12}}=\frac{(-1)^{n_1+n_2}}{\Gamma(n_1+1)\Gamma(n_2+1)}.$$

- By $f(x) \approx g(x)$ or $f(x) \ll g(x)$ we mean $f(x) = \mathcal{O}(g(x))$ as $x \to \infty$.
- We say a function $f(x,y)$ is **separable** if we could write $f(x,y) = f_1(x)f_2(y)$ for some functions f_1 and f_2. The functions f_1 and f_2 are called the **component functions** of f.

Chapter 2

Zero Index Case

2.1 Introduction

Zero index bracket series are those with the same number of sums as the number of brackets. For example

$$\sum_{n=0}^{\infty} \phi_n 3^n \langle n+1 \rangle$$

is a bracket series with index zero because it has one sum and one bracket. The evaluation of this bracket is given by rule E_1. It will be shown later in this chapter that this rule is equivalent to the Ramanujan identity

$$\int_0^{\infty} \sum_{n=0}^{\infty} x^{s-1} \left[f(0) - xf(1) + x^2 f(2) - \cdots \right] dx = \frac{\pi}{\sin \pi s} f(-s).$$

After we justify rule E_1, we will prove rule E_2 and then P_2. At the end, we will discuss possible issues one might encounter in using the results within this chapter for the justification of MoB in zero index cases.

2.2 Ramanujan identity and rule E_1

In this section, we prove Ramanujan identity and rule E_1. In his proof of this identity, Hardy [4] requires the function f to satisfy

$$|f(u)| < Ce^{Pv+A|w|},$$

where $u = v+iw$, C, P, and A are positive and $A < \pi$. This condition enables him to use certain contour integral to yield the identity. We loosen this condition to define a class of functions of interest: this is the **bracket class**.

Definition 2.2.1. An entire function $f(z)$ is said to belong to the **bracket class**, i.e. $f \in \mathcal{B}$, if

1. f vanishes at negative integers,
2. for any real $x > 0$ and $c \in \mathbb{R}$, the integral

$$\int \frac{f(-z)x^{-z}}{\sin \pi z}\, dz$$

   taken along the contour $\mathcal{C}_T(c)$ vanishes as $T \to \infty$.

The requirement of these functions are kept as minimal as possible. The vanishing of the integral on $\mathcal{C}_T(c)$, is essential for our arguments.

In the applications of the method described here, it is typical that factorials, that is gamma function values, appears often in power series of functions examined. In the application of rule P_2, the next statement is used. This motivates the following lemma.

Lemma 2.2.2. *Let $f(z) = 1/\Gamma(az+b)$ with $0 < b \leq a \leq 2$. If f vanishes at the negative integers, then $f \in \mathcal{B}$. However, for $a = 2$, additional constraints for the constant c are required.*

Proof. The function f is entire because gamma function never vanishes. Thus we only need to verify the second condition.

1. *Vertical segment:* we want to show

$$\int_{-T-iT}^{-T+iT} \frac{f(-z)x^{-z}}{\sin \pi z} dz \to 0,$$

or equivalently

$$\int_{-T}^{T} \frac{f(T-iu)x^{T-iu}}{\sin(\pi T - i\pi u)} du = \int_{-T}^{T} \frac{x^{T-iu}}{\sin(\pi T - i\pi u)\Gamma(aT+b-iau)} du \to 0.$$

Observe that $\sin(\pi T - i\pi u) \approx \exp(\pi|u|) \ll \exp(\pi T)$. On the other hand, $\Gamma(aT) \approx T^T$ and gamma function decays only exponentially on the imaginary axis. This guarantees the vanishing of this segment when $T \to \infty$.

2. *Horizontal segment:* we need to show

$$\int_{-T-iT}^{c-iT} \frac{f(-z)x^{-z}}{\sin \pi z} dz \to 0,$$

or equivalently

$$\int_{-c}^{T} \frac{f(u+iT)x^{u+iT}}{\sin(\pi u + i\pi T)} du = \int_{-c}^{T} \frac{x^{u+iT}}{\sin(\pi u + i\pi T)\Gamma(au+b+iaT)} du \to 0.$$

- If $-c < u < s$: where s is a real number such that $as + b > 3$. For each u in this interval, use the asymptotic identity

$$|\Gamma(u+it)| \ll |t|^{u-1/2} \exp(-\pi|t|/2) \quad \text{as} \quad t \to \infty$$

to get

$$|\Gamma(au+b+iaT)| \ll T^{au+b-1/2} \exp(-\pi a T/2).$$

<u>If $a < 2$</u>: then

$$\left|\frac{x^{u+iT}}{\sin(\pi u + i\pi T)\Gamma(au+b+iaT)}\right| \ll \exp\left(\frac{\pi a T}{2} - \pi T\right).$$

Thus

$$\int_{-c}^{s} \frac{x^{u+iT}du}{\sin(\pi u + i\pi T)\Gamma(au+b+iaT)} \to 0 \quad \text{as} \quad T \to \infty.$$

<u>If $a = 2$</u> then

$$\left|\frac{x^{u+iT}}{\sin(\pi u + i\pi T)\Gamma(au+b+iaT)}\right| \ll T^{-au-b+1/2}.$$

Thus in order to have

$$\int_{-c}^{s} \frac{x^{u+iT}du}{\sin(\pi u + i\pi T)\Gamma(au+b+iaT)} \to 0 \quad \text{as} \quad T \to \infty,$$

we need $-au - b + \frac{1}{2} < 0$ for all $-c < u < s$. For this to be true we need

$$c < \frac{2b-1}{2a}.$$

This requires the extra condition $b > 1/2$.

- If $s < u < T$: then $3 < as + b < au + b = \sigma$. Iterate the recursive formula $\Gamma(z+1) = z\Gamma(z)$ to produce

$$\Gamma(\sigma + iaT) = \Gamma(\sigma^* + iaT) \prod_{k=1}^{\lfloor\sigma\rfloor - 1} (\sigma - k + iaT),$$

with $1 \leq \sigma^* < 2$. Now we break the integrand

$$I = \frac{x^{u+iT}}{\sin(\pi u + i\pi T)\Gamma(au+b+iaT)}$$

into

$$\frac{1}{\sin(\pi u + i\pi T)\Gamma(\sigma^* + iaT)} \cdot \frac{x^{u+iT}}{\prod_{k=1}^{\lfloor\sigma\rfloor-1}(\sigma - k + iaT)} := I_1 \cdot I_2.$$

The first term $I_1 \to 0$ as $T \to \infty$, because

$$\sin(\pi u + i\pi T)\Gamma(\sigma^* + iaT) \ll T^{\sigma^*-1/2} \exp\left(-\frac{\pi aT}{2} + \pi T\right),$$

which tends to zero with $\sigma^* \geq 1$ and $a \leq 2$. To deal with I_2, use the inequality

$$|a + ib| = \sqrt{a^2 + b^2} \geq \sqrt{2|ab|} \geq \sqrt{|a||b|}.$$

This yields

$$\left|\prod_{k=1}^{\lfloor\sigma\rfloor-1}(\sigma - k + iaT)\right| \geq \sqrt{\prod_{k=1}^{\lfloor\sigma\rfloor-1}(\sigma - k)(aT)} = \sqrt{\prod_{k=1}^{\lfloor\sigma\rfloor-1}(\sigma - k) \cdot (aT)^{\lfloor\sigma\rfloor-1}}.$$

The product is at least $\Gamma(\lfloor\sigma\rfloor - 1)$, which is asymptotic to σ^σ, i.e. faster than any exponential of σ. Thus

$$\frac{x^{u+iT}}{a^{\lfloor\sigma\rfloor-1}\sqrt{\prod_{k=1}^{\lfloor\sigma\rfloor-1}(\sigma - k)}} \to 0 \quad \text{as} \quad T \to \infty.$$

Therefore $I_2 \ll T^{1-\lfloor\sigma\rfloor} \ll T^{-2}$, which gives

$$\int_s^T I du \ll TT^{-2} = T^{-1} \to 0 \quad \text{as} \quad T \to \infty.$$

The proof is complete. □

In using the method of brackets, one often sees gamma function appear on the numerator as well. This leads to our next definition.

Definition 2.2.3. We say f has **bracket form** if

$$f(z) = \frac{\alpha^z \prod_{k=1}^m \Gamma(a_k z + b_k)}{\prod_{k=1}^n \Gamma(c_k z + d_k)},$$

where α is positive and a's, b's, c's, d's are real numbers such that

$$\sum_{k=1}^n c_k - \sum_{k=1}^m a_k > 0, \quad \sum |c_k| - \sum |a_k| < 2$$

and f is entire and vanishes at negative integers. The quantity $\sum_{k=1}^n c_k - \sum_{k=1}^m a_k$ is called the **weight** of f. If instead of requiring f to be entire, we only have f to be analytic for the region $\Re(z) > -c$ for some positive c, then we say f has **type two bracket form**.

The conditions in the definition ensure functions having forms belong to the bracket class.

Theorem 2.2.4. *If a function has bracket form, then it is in the bracket class $\mathcal{B}$.*

Proof. Assume f has bracket form, it vanishes at negative integers. Thus we only need to check the second condition.

1. *Vertical segment:* the vanishing of this segment requires

$$\int_{-T}^{T} \frac{f(T-iu)x^{T-iu}}{\sin(\pi T - i\pi u)} du \to 0,$$

which in our case means

$$\int_{-T}^{T} \frac{(\alpha x)^{T-iu} \prod_{k=1}^m \Gamma(a_k T + b_k - i a_k u)}{\sin(\pi T - i\pi u) \prod_{k=1}^n \Gamma(c_k T + d_k - i c_k u)} \to 0.$$

Recall that as $T \to \infty$, the leading term in the asymptotic behavior of $\Gamma(aT+b)$

is T^{aT}. Thus the leading term for the ratio of two products above is

$$T^{\sum Ta_k - \sum Tc_k}.$$

Since $\sum c_k - \sum a_k > 0$, this leading term guarantees the vanishing of this segment because other terms are of exponential order (that is bounded by a multiple of C^T for some constant C).

2. *Horizontal segment:* for this segment we need

$$\int_{-c}^{T} \frac{f(u+iT)x^{u+iT}}{\sin(\pi u + i\pi T)} \to 0,$$

which in our case means

$$\int_{-c}^{T} \frac{(\alpha x)^{u+iT} \prod_{k=1}^{m} \Gamma(a_k u + b_k + ia_k T)}{\sin(\pi u + i\pi T) \prod_{k=1}^{n} \Gamma(c_k u + d_k + ic_k T)} \to 0.$$

Let I be the integrand and P be the ratio of the two products. For a fixed u, when $au + b > 1$ use recursive formula for gamma function repeatedly to get

$$\Gamma(au + b + iaT) = (au + b - 1 + iaT) \cdots (\sigma + iaT)\Gamma(\sigma + iaT),$$

with $0 < \sigma \leq 1$. This implies

$$\Gamma(au + b + iaT) \approx T^{au+b} \cdot \Gamma(\sigma + iaT).$$

Similarly, if $au + b < 0$ we get

$$\Gamma(au + b + iaT) = \frac{\Gamma(\sigma + iaT)}{(au + b + iaT) \cdots (\sigma - 1 + iaT)},$$

with $0 < \sigma \leq 1$. Thus

$$\Gamma(au + b + iaT) \approx T^{au+b} \cdot \Gamma(\sigma + iaT).$$

Evaluating the asymptotic behavior produces

$$\Gamma(a_k u + b_k + ia_k T) \approx T^{a_k u + b_k} \cdot \Gamma(\sigma_{a_k} + ia_k T),$$

$$\Gamma(c_k u + d_k + ic_k T) \approx T^{c_k u + d_k} \cdot \Gamma(\sigma_{c_k} + ic_k T).$$

Therefore

$$P \approx \frac{\prod_{k=1}^{m} \Gamma(\sigma_{a_k} + ia_k T)}{\prod_{k=1}^{n} \Gamma(\sigma_{c_k} + ic_k T)} T^{\sum(a_k u + b_k) - \sum(c_k u + d_k)}.$$

Let P_σ be the ratio of two such products. As $T \to \infty$, the leading term for the decay of $\Gamma(a + ibT)$ is $\exp(-\pi|b|T/2)$. Thus

$$\frac{P_\sigma}{\sin(\pi u + i\pi T)} \ll \exp\left(\frac{\pi T}{2}\left[\sum |c_k| - \sum |a_k| - 2\right]\right).$$

On the other hand,

$$T^{\sum(a_k u + b_k) - \sum(c_k u + d_k)} = T^{-\epsilon_1 u + \epsilon_2},$$

where $\epsilon_1 = \sum c_k - \sum a_k > 0$ and $\epsilon_2 = \sum b_k - \sum d_k$, so for big enough T, $(\alpha x)^{u+iT} T^{\sum(a_k u + b_k) - \sum(c_k u + d_k)}$ is bound by a finite power of T. This and the estimate for P_σ give the vanishing of this segment, because $\sum |c_k| - \sum |a_k| < 2$.

$\square$

A consequence of belonging to the bracket class is to have the following line integral representation, which is crucial in our proof of Ramanujan identity.

Theorem 2.2.5. *Let $f \in \mathcal{B}$. Then*

$$\sum_{n=0}^{\infty}(-1)^n f(n)x^n = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\pi}{\sin \pi u}f(-u)x^{-u}du,$$

for any positive x and c.

Proof. Consider the integrand $f(-u)x^{-u}/\sin \pi u$ on the contour

$$\mathcal{C}_T(c) \cup [c - iT, c + iT].$$

The integrand vanishes on $\mathcal{C}_T(c)$, as $T \to \infty$. This follows from the fact that f belongs to the bracket class. Cauchy's theorem yields the sum on the left-hand-side. The poles of $\sin \pi u$ for positive u are simple and thus are removable singularities because $f(-u)$ vanishes there. □

Theorem 2.2.6. *[Ramanujan's Master Theorem] Let $f \in \mathcal{B}$ and $s > 0$. Then*

$$\int_0^{\infty}\sum_{n=0}^{\infty}(-1)^n f(n)x^{n+s-1}dx = \frac{\pi}{\sin(\pi s)}f(-s).$$

Proof. [Hardy.] Since $f \in \mathcal{B}$, Theorem 2.2.5 gives

$$\Phi(x) := \sum_{n=0}^{\infty}(-1)^n f(n)x^n = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\pi}{\sin(\pi u)}f(-u)x^{-u}du,$$

for any positive c. Choose c_1 and c_2 such that $0 < c_1 < s < c_2$, then we have

$$\begin{aligned}
\int_0^1 \Phi(x)x^{s-1}dx &= \frac{1}{2\pi i}\int_0^1 x^{s-1}dx\int_{c_1-i\infty}^{c_1+i\infty}\frac{\pi}{\sin(\pi u)}f(-u)x^{-u}du \\
&= \frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}\frac{\pi}{\sin(\pi u)}f(-u)du\int_0^1 x^{s-u-1}dx \\
&= \frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}\frac{\pi}{\sin(\pi u)}\frac{f(-u)}{s-u}du.
\end{aligned}$$

Similarly,

$$\begin{aligned}\int_1^\infty \Phi(x)x^{s-1}dx &= \frac{1}{2\pi i}\int_1^\infty x^{s-1}dx\int_{c_2-i\infty}^{c_2+i\infty}\frac{\pi}{\sin(\pi u)}f(-u)x^{-u}du \\ &= \frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty}\frac{\pi}{\sin(\pi u)}f(-u)x^{-u}du\int_1^\infty x^{s-u-1} \\ &= -\frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty}\frac{\pi}{\sin(\pi u)}\frac{f(-u)}{s-u}du.\end{aligned}$$

Combining these we have

$$\begin{aligned}\int_0^\infty \Phi(x)x^{s-1}dx &= \frac{1}{2\pi i}\left(\int_{c_1-i\infty}^{c_1+i\infty}-\int_{c_2-i\infty}^{c_2+i\infty}\right)\frac{\pi}{\sin(\pi u)}\frac{f(-u)}{s-u}du \\ &= \frac{\pi}{\sin(\pi s)}f(-s),\end{aligned}$$

by Cauchy's theorem. The proof is complete. □

Corollary 2.2.7. *Let f be a function such that $f(z)/\Gamma(z+1)$ belongs to the bracket class. Then rule E_1, which says*

$$\int_0^\infty \sum_{n=0}^\infty \phi_n f(n)x^{an+b-1}dx = \sum_{n=0}^\infty \phi_n f(n)\langle an+b\rangle = \frac{1}{|a|}f(n^*)\Gamma(-n^*),$$

where $n^* = -b/a < 0$, is valid.

Proof. Use the change of variable $u = x^a$, $du = ax^{a-1}dx$ to get

$$\frac{1}{|a|}\int_0^\infty \sum_{n=0}^\infty \phi_n f(n)u^{n+\frac{b-1}{a}}\cdot\frac{du}{u^{\frac{a-1}{a}}} = \frac{1}{|a|}\int_0^\infty \sum_{n=0}^\infty \phi_n f(n)u^{n+\frac{b}{a}-1}du,$$

where we have $|a|$ because if $a < 0$ the integral will be from infinity to zero. Recall $\phi_n = (-1)^n/\Gamma(n+1)$. Since $f(n)/\Gamma(n+1)$ belongs to the second bracket class, we can use Ramanujan Master Theorem to get

$$\frac{1}{|a|}\int_0^\infty \sum_{n=0}^\infty (-1)^n\frac{f(n)}{\Gamma(n+1)}u^{n-n^*-1}du = \frac{1}{|a|}\frac{\pi}{\sin(-\pi n^*)}\cdot\frac{f(n^*)}{\Gamma(n^*+1)} = \frac{1}{|a|}f(n^*)\Gamma(-n^*).$$

$\square$

Thus we have provided a rigorous proof of the evaluation rule E_1. This is one of the fundamental results presented in this thesis.

The example presented below illustrates the result presented above. An elementary integral is evaluated by the Method of Brackets. This evaluation is now rigorous.

Example 2.2.1. A simple calculation gives

$$\int_0^\infty e^{-ax}dx = \frac{1}{a} \qquad \text{for} \quad a > 0.$$

Now we prove that method of brackets will give the right answer without requiring further evaluation. The power series of $\exp(-ax)$ yields the bracket series

$$\int_0^\infty \sum_{n=0}^\infty (-1)^n \frac{a^n}{\Gamma(n+1)} x^n dx = \sum_{n=0}^\infty \phi_n a^n \langle n+1 \rangle.$$

Note that from Theorem 2.2.4, $a^n/\Gamma(n+1)$ belongs to the bracket class so Corollary 2.2.7 guarantees, and justifies, that rule E_1 will yield the correct answer. Indeed, the vanishing of the bracket gives $n^* = -1$ so rule E_1 gives $1/a$.

Example 2.2.2. Entry 3.326.2 of [8] gives

$$\int_0^\infty x^m \exp(-\beta x^k)dx = \frac{\Gamma(\gamma)}{k\beta^\gamma}, \qquad \gamma = \frac{m+1}{k}, \ \ \beta > 0, \ \ m > 0, \ \ k > 0.$$

We can rewrite the integral as

$$\int_0^\infty \sum_{n=0}^\infty (-1)^n \frac{\beta^n}{\Gamma(n+1)} x^{nk+m} dx = \sum_n \phi_n \beta^n \langle nk + m + 1 \rangle.$$

Again, Theorem 2.2.4 confirms that $\beta^n/\Gamma(n+1)$ belongs to the bracket class so Corollary 2.2.7 guarantees, and justifies, that rule E_1 will give the correct answer.

Indeed, the vanishing of the bracket gives $n^* = -(m+1)/k = -\gamma$ so rule E_1 yields $\frac{\Gamma(\gamma)}{k\beta^\gamma}$ as expected.

2.3 General case and rule P_2

The discussion presented here was initiated in Jiu [7]. Here we present an extension of his results, follow his method of proof adding growth conditions on the summand in order to present a complete justification of rule E_2.

Theorem 2.3.1. *Rule E_2, which gives the evaluation*

$$\begin{aligned}&\sum_{n_1,\dots,n_r} \phi_{\overline{n}_r} f(n_1,\dots,n_r) \prod_{i=1}^{r} \langle a_{i1}n_1 + \cdots + a_{ir}n_r + c_i\rangle \\ = \quad & \frac{1}{|\det A|} f(n_1^*,\dots,n_r^*) \prod_{i=1}^{r} \Gamma(-n_i^*),\end{aligned}$$

is (rigorously) correct provided the components of the function $g = |\phi| f$ are in the bracket class $\mathfrak{B}$.

Here A is the matrix $\{a_{ij}\}$ of coefficients in the linear systems coming from the bracket series. The values n_i^* is the solution of the vanishing of the brackets.

Proof. [Jiu] It suffices to prove the two-dimensional case, since higher cases are similar. It is required to prove

$$\begin{aligned}&\sum_{n_1,n_2} \phi_{n_1,n_2} f(n_1,n_2) \langle a_{11}n_1 + a_{12}n_2 + c_1\rangle \langle a_{21}n_1 + a_{22}n_2 + c_2\rangle \\ = \quad & \frac{1}{|\det A|} f(n_1^*,n_2^*)\Gamma(-n_1^*)\Gamma(-n_2^*).\end{aligned}$$

Since brackets are just short notation for integrals (in this case a double integrals, say, in x and y), use the change of variables

$$u = x^{a_{11}} y^{a_{21}}, \qquad v = x^{a_{12}} y^{a_{22}}$$

to convert the bracket series into

$$\frac{1}{|\det A|} \int_0^\infty \int_0^\infty \sum_{n_1,n_2} \phi_{n_1,n_2} f(n_1,n_2) u^{n_1-n_1^*-1} v^{n_2-n_2^*-1} du dv.$$

Here n_1^* and n_2^* are solutions to the system

$$\begin{aligned} a_{11}n_1 + a_{12}n_2 + c_1 &= 0 \\ a_{21}n_1 + a_{22}n_2 + c_2 &= 0. \end{aligned}$$

Since both $|\phi| f(n_1,*)$ and $|\phi| f(*,n_2)$ belong to the bracket class, we are allowed to use rule E_1 twice. Thus

$$\begin{aligned} & \frac{1}{|\det A|} \int_0^\infty \int_0^\infty \sum_{n_1,n_2} \phi_{n_1,n_2} f(n_1,n_2) u^{n_1-n_1^*-1} v^{n_2-n_2^*-1} du dv \\ = \; & \frac{1}{|\det A|} f(n_1^*,n_2^*)\Gamma(-n_1^*)\Gamma(-n_2^*). \end{aligned}$$

Notice that we need n_1^* and n_2^* to be negative to use rule E_1. This is the case because otherwise the corresponding integrals (in u and v) diverge. The proof is complete. □

Example 2.3.1. We have the evaluation

$$I = \int_0^\infty \frac{1}{1+x^2} dx = \frac{\pi}{2}.$$

We prove that method of brackets would give the right answer. First we use rule P_2 to get

$$\frac{1}{1+x^2} = \sum_{n_1,n_2} \phi_{n_1,n_2} 1^{n_1} x^{2n_2} \langle n_1 + n_2 + 1 \rangle,$$

thus

$$I = \sum_{n_1,n_2} \phi_{n_1,n_2} \langle n_1 + n_2 + 1 \rangle \langle 2n_2 + 1 \rangle.$$

Notice that we have two sums and two brackets so this bracket series has index zero. Also

$$|\phi_{n_1,n_2}| = \frac{1}{\Gamma(n_1+1)\Gamma(n_2+1)}$$

has each component-function of the form $1/\Gamma(z+1)$ which belongs to the bracket class by Lemma 2.2.2. Therefore, Theorem 2.3.1 guarantees that rule E_2 will give the correct answer. In other words, the method of brackets gives the right evaluation for I.

Indeed, the vanishing of the brackets gives $n_1^* = n_2^* = -1/2$ and the coefficient matrix has determinant 2, so rule E_2 says

$$I = \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right) = \frac{\pi}{2}.$$

Often when we deal with integrand involving rational expressions, we will have to use rule P_2. The rule does not require the exponent α to be negative (i.e. when the multinomial is in the denominator). However, our proof of this rule requires $\alpha < 0$.

Lemma 2.3.2. *For $\alpha < 0$*

$$(a_1 + \cdots + a_n)^\alpha = \sum_{n_1,\ldots,n_r} \phi_{n_1} \cdots \phi_{n_r} a_1^{n_1} \cdots a_r^{n_r} \frac{\langle -\alpha + n_1 + \cdots + n_r \rangle}{\Gamma(-\alpha)}.$$

Proof. Start with the integral representation of gamma function

$$\Gamma(-\alpha) = \int_0^\infty t^{-\alpha-1} e^{-t} dt,$$

(which only holds for negative α), use the change of variable $t = (a_1 + \cdots + a_r)x$ to get

$$\Gamma(-\alpha)(a_1+\cdots+a_r)^{\alpha}=\int_0^{\infty}x^{-\alpha-1}e^{-(a_1+\cdots+a_r)x}dx.$$

Next replace all the terms $\exp(-a_i x)$ with their power series to get

$$\Gamma(-\alpha)(a_1+\cdots+a_r)^{\alpha}=\int_0^{\infty}\sum_{n_1,\ldots,n_r}\phi_{n_1}\cdots\phi_{n_r}a_1^{n_1}\cdots a_r^{n_r}x^{n_1+\cdots+n_r-\alpha-1}dx.$$

Finally, divide both sides by $\Gamma(-\alpha)$ and use rule P_1 to replace the integral with a bracket to complete the proof. □

Example 2.3.2. A higher dimension version of the previous example is

$$I=\int_0^{\infty}\int_0^{\infty}\frac{1}{(1+x^2+y^2)^2}dxdy=\frac{\pi}{4}.$$

Again we prove that the method of brackets would give the correct answer. Rule P_2 gives

$$\frac{1}{(1+x^2+y^2)^2}=\sum_{n_1,n_2,n_3}\phi_{n_1,n_2,n_3}1^{n_1}x^{2n_2}y^{2n_3}\langle n_1+n_2+n_3+2\rangle,$$

and integration gives

$$I=\sum_{n_1,n_2,n_3}\phi_{n_1,n_2,n_3}\langle n_1+n_2+n_3+2\rangle\langle 2n_2+1\rangle\langle 2n_3+1\rangle.$$

This is a zero index bracket series with three sums and three brackets. Furthermore,

$$|\phi_{n_1,n_2,n_3}|=\frac{1}{\Gamma(n_1+1)\Gamma(n_2+1)\Gamma(n_3+1)}$$

has each component-function of the form $1/\Gamma(z+1)$ which belongs to the bracket class by Lemma 2.2.2. Therefore, Theorem 2.3.1 justifies the evaluation of rule E_2. In other words, the method of brackets will give the right answer.

Indeed, the matrix of coefficients has determinant 4 and the solution for the vanishing of the brackets is $n_1^* = -1$, $n_2^* = n_3^* = -1/2$. Thus rule E_2 gives

$$I = \frac{1}{4} \cdot \Gamma(1) \cdot \Gamma\left(\frac{1}{2}\right)^2 = \frac{\pi}{4}.$$

2.4 Possible issues

Recall that Lemma 2.2.2 shows functions of the form $1/\Gamma(az+b)$ belong to the bracket class for $0 < b \leq a \leq 2$. However, if $a = 2$, we need to restrict the constant c in the second bracket class definition by

$$c < \frac{2b-1}{2a}.$$

This, of course, further requires the line integral in Theorem 2.2.5 to follow the same restriction and thus might cause a problem for the proof of Ramanujan identity since we need one of the constants c's to be bigger than s. Such issues are illustrated in Examples 2.4.2 and 2.4.3 below.

Example 2.4.1. We have the evaluation

$$I = \int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

The power series of sine gives us

$$I = \int_0^\infty \sum_{n=0}^\infty (-1)^n \cdot \frac{1}{\Gamma(2n+2)} \cdot x^{2n} dx.$$

Lemma 2.2.2 tells us that the function $1/\Gamma(2n+2)$ (here $a = b = 2$) belongs to the second bracket class with a condition on the constant in the definition being

$$c < \frac{2b-1}{2a} = \frac{3}{4}.$$

Corollary 2.2.7 will justify the evaluation by rule E_1. The only thing we need to check is the use of Ramanujan identity in the proof of this corollary. The s in this case is $-n^* = 1/2$. Thus we have no problem in the proof of Ramanujan Master Theorem when we need to choose two numbers $c_1 < s < c_2$ such that both are less than the bound $3/4$ mentioned above. In short, the method of brackets is justified and will give the right answer.

Indeed, rule E_1 gives the evaluation

$$I = \int_0^\infty \sum_{n=0}^\infty \phi_n \cdot \frac{\Gamma(n+1)}{\Gamma(2n+2)} \langle 2n+1 \rangle = \frac{1}{2} \cdot \frac{\Gamma(1/2)}{\Gamma(1)} \cdot \Gamma(1/2) = \frac{\pi}{2}.$$

Example 2.4.2. We now look at a more general form of the previous integral. This entry 3.761.4 in [8] gives

$$I = \int_0^\infty x^{\mu-1} \sin(ax) dx = \frac{\Gamma(\mu)}{a^\mu} \sin\frac{\mu\pi}{2}, \quad \text{for } a > 0, \quad 0 < \mu < 1.$$

The power series of sine gives us

$$I = \int_0^\infty \sum_{n=0}^\infty (-1)^n \cdot \frac{a^{2n+1}}{\Gamma(2n+2)} \cdot x^{2n+\mu} dx = \sum_{n=0}^\infty \phi_n \frac{a^{2n+1}\Gamma(n+1)}{\Gamma(2n+2)} \langle 2n+\mu+1 \rangle.$$

Here the vanishing of the bracket gives $n^* = -(\mu+1)/2$ so in the use of Ramanujan identity in the proof of Corollary 2.2.7, the number $s = -n^* = (\mu+1)/2$. Thus we need $\mu > -1$. Furthermore, in the proof of Ramanujan identity, we will have to choose $c_1 < s < c_2$ and use Theorem 2.2.5 with each c's. However, by Lemma 2.2.2, the function $a^{2n+1}/\Gamma(2n+2)$ belongs to the second bracket class but with a restriction that the constant c in the definition to be at most $3/4$ (as mentioned in the previous example). In short, if $\mu \geq 1/2$, we would have $s \geq 3/4$ and our proof of MoB in this particular case will not work, though the method does yield correct answer.

Indeed, rule E_1 gives the evaluation

$$I = \frac{1}{2}\frac{a^{-\mu}\Gamma\left(\frac{1-\mu}{2}\right)}{\Gamma(1-\mu)}\Gamma\left(\frac{1+\mu}{2}\right) = \frac{1}{2a^{\mu}\Gamma(1-\mu)} \cdot \frac{\pi}{\sin\pi\frac{1-\mu}{2}}.$$

Now use the reflection formula for gamma again to get

$$I = \frac{1}{2a^{\mu}} \cdot \frac{\Gamma(\mu)\sin\pi\mu}{\cos\frac{\pi\mu}{2}} = \frac{\Gamma(\mu)}{a^{\mu}}\sin\frac{\mu\pi}{2}.$$

Thus the MoB yields the correct answer for all $0 < \mu < 1$. However, we have not yet justified the method in the case $\mu \geq 1/2$.

Example 2.4.3. Another example involving trigonometric functions is entry 3.712.1 in [8]

$$\int_0^{\infty}\sin(ax^p)dx = \frac{\Gamma\left(\frac{1}{p}\right)\sin\frac{\pi}{2p}}{pa^{\frac{1}{p}}}, \qquad \text{for} \quad a > 0,\ p > 1.$$

Power series for sine gives us

$$I = \int_0^{\infty}\sum_{n=0}^{\infty}(-1)^n\frac{a^{2n+1}}{\Gamma(2n+2)}x^{2np+p}dx = \sum_{n=0}^{\infty}\phi_n\frac{a^{2n+1}\Gamma(n+1)}{\Gamma(2n+2)}\langle 2np+p+1\rangle.$$

Similar to the previous two examples, we only need to check the use of Ramanujan identity. The vanishing of the bracket gives $n^* = -\frac{p+1}{2p}$ so the constant s is $\frac{p+1}{2p}$. Thus if $p > 2$ then $s < 3/4$ and we would have no issue choosing $c_1 < s < c_2 < 3/4$ for the proof of Ramanujan identity. However, if $p \leq 2$ then we cannot prove the validity of MoB (though it does give the right answer).

Indeed, rule E_1 gives

$$I = \frac{1}{2p}\frac{a^{-\frac{1}{p}}\Gamma\left(\frac{p-1}{2p}\right)}{\Gamma\left(\frac{p-1}{p}\right)}\Gamma\left(\frac{p+1}{2p}\right) = \frac{\pi}{2pa^{\frac{1}{p}}\Gamma\left(\frac{p-1}{p}\right)\sin\frac{\pi(p-1)}{2p}}.$$

Using reflection formula for gamma one more time gives

$$I = \frac{\Gamma\left(\frac{1}{p}\right) \sin\frac{\pi}{p}}{2pa^{\frac{1}{p}} \cos\frac{\pi}{2p}} = \frac{\Gamma\left(\frac{1}{p}\right) \sin\frac{\pi}{2p}}{pa^{\frac{1}{p}}}.$$

The next example introduces another source of issues that we might face in our project to justify the rules of the method of brackets.

Example 2.4.4. We evaluate the integral

$$I = \int_0^\infty \int_0^\infty \frac{x^a y^b}{(x+y)^c} \exp\left(-\frac{x^d y^d}{(x+y)^d}\right) dx dy.$$

The exponential term gives

$$\sum_{n_1} \phi_{n_1} x^{dn_1} y^{dn_1} (x+y)^{-dn_1},$$

so that

$$\sum_{n_1} \phi_{n_1} x^{dn_1+a} y^{dn_1+b} (x+y)^{-dn_1-c}.$$

Rule P_2 now yields

$$(x+y)^{-dn_1-c} = \sum_{n_2 n_3} \phi_{n_2 n_3} \frac{x^{n_2} y^{n_3}}{\Gamma(dn_1+c)} \langle dn_1 + n_2 + n_3 + c \rangle.$$

This yields the bracket series

$$\sum_{n_{123}} \phi_{n_{123}} \frac{1}{\Gamma(dn_1+c)} \langle dn_1 + n_2 + a + 1 \rangle \langle dn_1 + n_3 + b + 1 \rangle \langle dn_1 + n_2 + n_3 + c \rangle.$$

The solution of the vanishing of the brackets is

$$n_1 = \frac{c-a-b-2}{d}, \quad n_2 = b+1-c, \quad n_3 = a+1-c.$$

This produces

$$I = \frac{1}{d} \cdot \frac{\Gamma\left(\frac{a+b+2-c}{d}\right)\Gamma(c-b-1)\Gamma(c-a-1)}{\Gamma(2c-a-b-2)}.$$

Though this example is a zero index case, we cannot use Theorem 2.3.1 to justify MoB because if $d > 1$, the function $\Gamma(n_1+1)^{-1}\Gamma(dn_1+c)^{-1}$ fails to be in the bracket class.

The next step in the process of making the method of brackets is to show that the arguments given above still work if one component of the functions involved does not belong to $\mathcal{B}$. In order to accomplish this result some results about Mittag-Leffler functions are required. These functions have the form

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)},$$

where α is real and z is complex. This family of functions was introduced by Mittag-Leffler (1903, 1904, 1905) and was investigated by several authors such as Wiman (1905), Pollard (1948), Humbert (1953). These functions are entire and have order $1/\alpha$. Our interest rely on the asymptotic behavior of these functions on the real line.

Lemma 2.4.1. *For $\alpha > 2$, the function $E_\alpha(-z)$ is unbounded as $z \to \infty$ on the real line.*

Proof. The estimate

$$E_\alpha(z) = \frac{1}{\alpha}\sum_m e^{t_m} - \sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(1-\alpha n)} + O\left(|z|^{-N}\right),$$

holds for $z \to \infty$ and $|\arg z| \leq \frac{\alpha\pi}{2}$, [9, page 207 – 208]. Here $t_m = z^{1/\alpha}e^{2\pi i m/\alpha}$ and the summation is over all those integers m such that $|\arg z + 2\pi m| \leq \alpha\pi/2$. Since our goal is to show $E_\alpha(-z) \to \infty$, we focus on the first sum. We have

$$e^{t_m} = \exp\left((-z)^{\frac{1}{\alpha}} e^{2\pi i m/\alpha}\right) = \exp\left(z^{\frac{1}{\alpha}} e^{\pi i(2m+1)/\alpha}\right).$$

We show that $m = 0$ that makes the corresponding e^{t_m} unbounded as $z \to \infty$ on the real line. Indeed,

$$e^{t_0} = \exp\left(z^{\frac{1}{\alpha}} e^{\pi i/\alpha}\right) = \exp\left(z^{\frac{1}{\alpha}} \left[\cos\left(\frac{\pi i}{\alpha}\right) + i \sin\left(\frac{\pi i}{\alpha}\right)\right]\right).$$

Since $\alpha > 2$, $\cos(\pi i/\alpha) > 0$. Thus e^{t_0} unbounded as $z \to \infty$. This completes the proof. □

Theorem 2.4.2. *Theorem 2.3.1 holds when all but one components of the function $g = |\phi| f$ belongs to the bracket class.*

Proof. Without loss of generality, assume n_1 is the component of exception. Since $|\phi| f$ belongs to the bracket class with respect to all other components, we evaluate the integrals involving n_2, n_3, and so on until we get one integral left with the sum over n_1. Our claim is the function we have left, say $h(n_1)$, belongs to the bracket class and thus we could use the Ramanujan Master Theorem to finish our calculation and our proof.

Here we assume that our function f consists of only exponential and gamma functions. The discussion of the general case follows the same pattern. In the case presented here, the function $h(n_1)$ has the same characteristics as f. Assume then that

$$h(n_1) = \frac{\beta^{n_1} \prod_{k=1}^{m} \Gamma(a_k n_1 + b_k)}{\prod_{k=1}^{n} \Gamma(c_k n_1 + d_k)}$$

with positive β, a's, and c's. The integral we have left should be in the form

$$\int_0^\infty z^s \sum_{n_1} h(n_1)(-z)^{n_1} dz$$

For the sum to converge for every z, we need $\sum c_k - \sum a_k > 0$. Now assume $\sum c_k - \sum a_k > 2$. Then for large enough n_1, $h(n_1) \approx 1/\Gamma(\alpha n_1 + 1)$ for some $\alpha > 2$. However, from Lemma 2.4.1 we know

$$\sum_{n=0}^{\infty} \frac{(-z)^n}{\Gamma(\alpha n+1)}$$

grows exponentially as $z \to \infty$ on the real line. For this reason we believe the sum of $h(n)(-z)^n$ is at least unbounded as $z \to \infty$ and thus makes the integral divergent. In short, we should have $\sum c_k - \sum a_k < 2$. Now Theorem 2.2.4 says that $h(n_1)$ belongs to the bracket class, thus completes the proof. $\square$

This result could be used to justify the evaluation of Example 2.4.4. Recall we have the bracket series

$$\sum_{n_{123}} \phi_{n_{123}} \frac{1}{\Gamma(dn_1+c)} \langle dn_1+n_2+a+1\rangle \langle dn_1+n_3+b+1\rangle \langle dn_1+n_2+n_3+c\rangle .$$

Notice that this is nothing more than a triple integral

$$\int_0^\infty \int_0^\infty \int_0^\infty \sum_{n_{123}} \phi_{n_{123}} \frac{1}{\Gamma(dn_1+c)} x_1^{dn_1+n_2+a} x_2^{dn_1+n_3+b} x_3^{dn_1+n_2+n_3+c-1} dx_1 dx_2 dx_3 .$$

We will integrate with respect to x_1 first, that is to evaluate

$$\int_0^\infty \sum_{n_2=0}^{\infty} (-1)^{n_2} \frac{x_3^{n_2}}{\Gamma(n_2+1)} x_1^{n_2+dn_1+a} dx_1 .$$

By Theorem 2.2.4, the function $x_3^{n_2}/\Gamma(n_2+1)$ (as a function of n_2) belongs to the bracket class so we could use Ramanujan theorem to get $x_3^{-dn_1-a-1}\Gamma(dn_1+a+1)$. The triple integral now becomes

$$\int_0^\infty \int_0^\infty \sum_{n_{13}} \phi_{n_{13}} \frac{\Gamma(dn_1+a+1)}{\Gamma(dn_1+c)} x_2^{dn_1+n_3+b} x_3^{n_3+c-a-2} dx_2 dx_3 .$$

Again Theorem 2.2.4 confirms that $1/\Gamma(n_3+1)$ and

$$\frac{\Gamma(dn_1+a+1)}{\Gamma(n_1+1)\Gamma(dn_1+c)}$$

both belong to the bracket class so Theorem 2.3.1 allows us to use rule E_2 now to get the answer.

Remark: Our proof in this case involves using Ramanujan identity with $s = dn_1 + a - 1$, which is typically big as n_1 runs from zero to infinity. Traditionally we only allow $0 < s < 1$ to not pick up unwanted residues. But in this case, the bracket class allows us to work with any positive s.

Chapter 3

Positive Index Case

3.1 Introduction

The index of a bracket series is the difference between the number of sums and the number of brackets. This section discusses series brackets with more sums than brackets, that is, series with positive index. The heuristic Rule E_3 states that in the evaluation of these series, it is required to let some of indices n's to be free. A variety of rules are then imposed to assign a value to these series. To justify this rule we analyze the evaluation of each bracket. This leads to an explanation of why sometimes one case gives the answer, while in others we need to add evaluations from different cases. These two phenomenas correspond to two types of brackets.

Definition 3.1.1. Brackets of the form $\langle a_1n_1+a_2n_2+\ldots+a_kn_k+s\rangle$ with positive a_i's are called **type one** brackets. Brackets of the form $\langle a_1n_1+a_2n_2+\ldots+a_kn_k-bn_{k+1}+s\rangle$ with positive a_i's and b are called **type two** brackets.

Remark 3.1.2. Observe that a single negative index is allowed. The rigorous version of the method of brackets in the presence of more than one such index remains an open problem. This is the subject of current work.

3.2 Type one brackets

We start our analysis with the simplest case of type one brackets, such as $\langle an_1+bn_2+s\rangle$. Then we generalize the results to brackets of the form $\langle a_1n_1+a_2n_2+\ldots+a_kn_k+s\rangle$.

Definition 3.2.1. Consider a bracket series

$$\sum_{n_1,\ldots,n_k} \phi_{n_1,\ldots,n_k} f(n_1,\ldots,n_k)\langle g(n_1,\ldots,n_k)\rangle,$$

where g is a linear function of variables n's. An evaluation of this bracket with respect to n_m, denoted by $\mathcal{E}_m(\langle g\rangle)$, is the value

$$\mathcal{E}_m(\langle g\rangle) = \sum_{\bar{n}_m} \phi_{\bar{n}_m} f(n_1,\ldots,n_m^*,\ldots,n_k)\Gamma(-n_m^*),$$

here n_m^* is the solution of $g(n_1,\ldots,n_k)=0$, solving for n_m and

$$\bar{n}_m = \{n_1,\ldots,n_k\} \setminus \{n_m\}.$$

Example 3.2.1. An evaluation of the bracket series

$$\sum_{n_1,n_2} \phi_{n_1,n_2} a^{n_2}\langle n_1+n_2+1\rangle$$

with respect to n_1 is calculated by solving $n_1+n_2+1=0$ for $n_1^*=-n_2-1$, and thus

$$\mathcal{E}_1\left(\langle n_1+n_2+1\rangle\right) = \sum_{n_2} \phi_{n_2} a^{n_2}\Gamma(n_2+1).$$

On the other hand, an evaluation of this bracket with respect to n_2 is calculated by by solving $n_1+n_2+1=0$ for $n_2=-n_1-1=n_2^*$, and thus

$$\mathcal{E}_2\left(\langle n_1+n_2+1\rangle\right) = \sum_{n_1} \phi_{n_1} a^{-n_1-1}\Gamma(n_1+1).$$

Remark 3.2.2. An evaluation of a bracket series with respect to one of its parameters might not be finite. In the above example,

$$\mathcal{E}_1 = \sum_n (-a)^n \quad \text{while} \quad \mathcal{E}_2 = \sum_n (-a)^{-n}/a.$$

Thus depending on whether $|a| < 1$ or $|a| > 1$, either $\mathcal{E}_1$ or $\mathcal{E}_2$ will be a divergent series. Also, an evaluation of a bracket surely depends on the summand f even though the notation $\mathcal{E}_m(\langle g \rangle)$ does not mention it. This should not cause any confusion since the summand should always be obvious from context.

Definition 3.2.3. Let $f(n_1, \ldots, n_k)$ be a separable function with each component function f_i's having bracket form with weight w_i's, respectively. For the bracket series

$$\sum_{n_1,\ldots,n_k} (-1)^{n_1+\ldots+n_k} f(n_1, \ldots, n_k) \langle a_1 n_1 + \ldots + a_{k-1} n_{k-1} - a_k n_k + s \rangle$$

with positive a_i's, the quantity w_i/a_i, for $1 \leq i \leq k-1$, is called the **weight** of n_i in the bracket.

Lemma 3.2.4. *Assume $f(x, y)$ is separable, i.e. $f(x, y) = f_1(x) f_2(y)$, and each f_i's has bracket form with weight w_i's respectively. Let c and α be positive numbers such that $\alpha w_1 < w_2$. Then*

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\pi f(-\sigma + \alpha u, -u)}{\sin \pi u \sin \pi(\sigma - \alpha u)} du = \sum_{n=0}^{\infty} (-1)^n \frac{f(-\sigma - \alpha n, n)}{\sin \pi(\sigma + \alpha n)},$$

assuming $-\sigma + \alpha c < 0$.

Proof. The usual contour $\mathcal{C}_T(c)$ is used to evaluate the line integral on the left hand side.

1. *Vertical segment:* it is required that

$$\int_{-T-iT}^{-T+iT} \frac{f(-\sigma+\alpha u,-u)}{\sin \pi u \sin \pi(\sigma-\alpha u)} du \to 0$$

or, equivalently,

$$\int_{-T}^{T} \frac{f(-\alpha T-\sigma+i\alpha z, T-iz)}{\sin \pi(-T+iz) \sin \pi(\alpha T+\sigma-i\alpha z)}.$$

Assume

$$f_1(z)=\frac{\beta_1^z \prod \Gamma(a_k z+b_k)}{\prod \Gamma(c_k z+d_k)}, \qquad f_2(z)=\frac{\beta_2^z \prod \Gamma(p_k z+q_k)}{\prod \Gamma(r_k z+s_k)},$$

then our integrand

$$I \approx \frac{\beta_1^{-\alpha T-\sigma+i\alpha z} \beta_2^{T-iz} \prod \Gamma(-\alpha a_k T+i\alpha a_k z) \prod \Gamma(p_k T-i p_k z)}{\prod \Gamma(-\alpha c_k T+i\alpha c_k z) \prod \Gamma(r_k T-i r_k z)},$$

where we omitted irrelevant terms in the arguments of gamma functions and also the sine terms. The leading term in the growth of $\Gamma(az+b)$ is z^{az} so the leading term in the behavior of all the gamma function combined is

$$\exp\left[\left(-\alpha \sum a_k+\alpha \sum c_k+\sum p_k-\sum r_k\right) T \log T\right]=\exp\left[(\alpha w_1-w_2) T \log T\right],$$

which goes to zero since $\alpha w_1 < w_2$. This shows this segment vanishes as $T \to \infty$.

2. *Horizontal segment:* the required estimate now reads

$$\int_{-T-iT}^{c-iT} \frac{f(-\sigma+\alpha u,-u)}{\sin \pi u \sin \pi(\sigma-\alpha u)} du \to 0,$$

or equivalently

$$\int_{-c}^{T} \frac{f(-\alpha z - \sigma - i\alpha T, z + iT)}{\sin \pi(z + iT) \sin \pi(\alpha z + \sigma + i\alpha T)} \to 0,$$

which in our case means

$$\frac{\beta_1^{-\alpha z - \sigma - i\alpha T} \beta_2^{z+iT} \prod \Gamma(-\alpha a_k z - i\alpha a_k T) \prod \Gamma(p_k z + ip_k T)}{\sin \pi(z + iT) \sin \pi(\alpha z + \sigma + i\alpha T) \prod \Gamma(-\alpha c_k z - i\alpha c_k T) \prod \Gamma(r_k z + ir_k T)}.$$

Let P be the quotient of all gamma functions. Now we will adopt the method we used in the proof of Lemma 2.2.4, namely, we will use recursive identity for all gamma functions such that they all have the real part of their arguments to be between 0 and 1. After doing this we have

$$P \approx \frac{\prod \Gamma(\sigma_{a_k} - i\alpha a_k T) \prod \Gamma(\sigma_{p_k} + ip_k T)}{\prod \Gamma(\sigma_{c_k} - i\alpha c_k T) \prod \Gamma(\sigma_{r_k} + ir_k T)} \cdot T^{(\alpha w_1 - w_2) z}.$$

Let P_σ be the quotient of products. Then the ratio of P_σ and the two sine functions is asymptotic to

$$\exp\left[\left(-\alpha \sum \frac{|a_k|}{2} + \alpha \sum \frac{|c_k|}{2} - \sum \frac{|p_k|}{2} + \sum \frac{|r_k|}{2} - 1 - \alpha\right) \pi T\right]$$

which goes to zero since $\sum |c_k| - \sum |a_k| < 2$ and $\sum |r_k| - \sum |p_k| < 2$. On the other hand, since $\alpha w_1 - w_2 < 0$, $\beta_1^{-\alpha z - \sigma - i\alpha T} \beta_2^{z+iT} T^{(\alpha w_1 - w_2) z}$ is bounded for all $z \in (-c, T)$. Thus our integrand decays at least exponentially as $T \to \infty$ so this segment vanishes.

Cauchy theorem now yields the sum of residues from $\sin \pi u$ which is the sum on the right hand side. The poles from $\sin \pi(\sigma - \alpha u)$, appearing from the trigonometric factors in the denominator, are removable since $\sigma - \alpha u > 0$ on the contour and $f_1(z)$ vanishes at negative integers. $\square$

Theorem 3.2.5. *Let $f(x,y)$ be a separable function with both component functions f_1 and f_2 having bracket form with weights w_1 and w_2, respectively. Let s, a, b be positive numbers and assume $bw_1 < aw_2$. Let*

$$F(x) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} (-1)^{n_1+n_2} f(n_1, n_2) x^{an_1+bn_2},$$

then we have the representation

$$\int_0^\infty F(x) x^{s-1} dx = \frac{1}{a} \sum_{n=0}^{\infty} (-1)^n \frac{\pi}{\sin \frac{\pi(s+bn)}{a}} f\left(\frac{-s-bn}{a}, n\right).$$

Proof. Since both f_1 and f_2 have bracket form, they both belong to second bracket class. Using Theorem 2.2.5 for each gives us

$$F(x) = \frac{1}{(2\pi i)^2} \int_{c_1-i\infty}^{c_1+i\infty} \int_{c_2-i\infty}^{c_2+i\infty} \frac{\pi^2 f(-u_1,-u_2)}{\sin \pi u_1 \sin \pi u_2} x^{-au_1-bu_2} du_2 du_1,$$

for any positive c_1 and c_2. To simplify our notation, we will use $\int_c^u$ to denote the line integral $\int_{c-i\infty}^{c+i\infty}$ with u being the variable of integration. With $ac_1 + bc_2 < s$ we have

$$\begin{aligned}
\int_0^1 F(x) x^{s-1} dx &= -\frac{1}{4\pi^2} \int_{c_2}^{u_2} \int_{c_1}^{u_1} \frac{\pi^2 f(-u_1,-u_2)}{\sin \pi u_1 \sin \pi u_2} \int_0^1 x^{s-au_1-bu_2-1} dx \\
&= -\frac{1}{4\pi^2} \int_{c_2}^{u_2} \int_{c_1}^{u_1} \frac{\pi^2}{\sin \pi u_1 \sin \pi u_2} \cdot \frac{f(-u_1,-u_2)}{s-au_1-bu_2}.
\end{aligned}$$

Now choose c_1' such that $ac_1' + bc_2 > s$. Then

$$\begin{aligned}
\int_1^\infty F(x) x^{s-1} dx &= -\frac{1}{4\pi^2} \int_{c_2}^{u_2} \int_{c_1'}^{u_1} \frac{\pi^2 f(-u_1,-u_2)}{\sin \pi u_1 \sin \pi u_2} \int_1^\infty x^{s-au_1-bu_2-1} dx \\
&= \frac{1}{4\pi^2} \int_{c_2}^{u_2} \int_{c_1'}^{u_1} \frac{\pi^2}{\sin \pi u_1 \sin \pi u_2} \cdot \frac{f(-u_1,-u_2)}{s-au_1-bu_2}.
\end{aligned}$$

Combine two cases we have

$$\int_0^\infty F(x)x^{s-1}dx = \int_{c_2}^{u_2}\left(\int_{c_1}^{u_1} - \int_{c_1'}^{u_1}\right)\frac{-f(-u_1,-u_2)}{4(s-au_1-bu_2)\sin\pi u_1 \sin\pi u_2}.$$

For each u_2, our integrand has exactly one simple pole at $s - au_1 - bu_2 = 0$ for $c_1 < \Re(u_1) < c_1'$ so Cauchy theorem gives

$$\int_0^\infty F(x)x^{s-1}dx = \frac{1}{2\pi i a}\int_{c_2-i\infty}^{c_2+i\infty}\frac{\pi^2}{\sin\pi u_2 \sin\frac{\pi(s-bu_2)}{a}} f\left(\frac{-s+bu_2}{a}, -u_2\right).$$

Now use Lemma 3.2.4 with $\sigma = s/a$ and $\alpha = b/a$ and the proof is complete. The condition $\alpha w_1 < w_2$ is guaranteed by $bw_1 < aw_2$ and the condition $-\sigma + \alpha c < 0$, which is equivalent to $bc_2 < s$, is a consequence of $ac_1 + bc_2 < s$ required earlier. The proof is now complete. □

Remark 3.2.6. The previous result gives

$$\sum_{n_1,n_2}(-1)^{n_1+n_2}f(n_1,n_2)\langle an_1+bn_2+s\rangle = \frac{1}{a}\sum_{n=0}^{\infty}(-1)^n\frac{\pi}{\sin\frac{\pi(s+bn)}{a}} f\left(\frac{-s-bn}{a}, n\right),$$

where the right hand side comes from letting n_2 be free in the bracket. The condition $bw_1 < aw_2$ gives $\frac{w_1}{a} < \frac{w_2}{b}$. In other words, n_1 has less weight than n_2 in this bracket. Thus the theorem says we should evaluate the bracket with respect to the n with least weight. From this theorem we get the following more general result.

Corollary 3.2.7. *Let $f(n_1,\ldots,n_k)$ be a separable function with each component function f_i's having bracket form. Let $a_1,\ldots,a_k$ and s be positive. To evaluate the bracket series*

$$\sum_{n_1,\ldots,n_k}(-1)^{n_1+\ldots+n_k}f(n_1,\ldots,n_k)\langle a_1n_1+\ldots+a_{k-1}n_{k-1}+a_kn_k+s\rangle$$

where n_1 has the least weight in the bracket, we should let $n_2,\ldots,n_k$ free.

Example 3.2.2. Entry 3.322.2 of [8] gives

$$\int_0^\infty \exp\left(-\frac{x^2}{4\beta} - \gamma x\right) dx = \sqrt{\pi\beta}\exp(\beta\gamma^2)\left[1 - \Phi\left(\gamma\sqrt{\beta}\right)\right], \qquad \text{for} \quad \beta > 0.$$

Here

$$\Phi(u) = \operatorname{erf}(u) = \frac{2}{\sqrt{\pi}}\int_0^u e^{-x^2}dx,$$

the error function. First we will evaluate the integral using method of brackets. We use the power series

$$\exp\left(-\frac{x^2}{4\beta}\right) = \sum_{n_1=0}^\infty \phi_{n_1}\cdot\frac{1}{4^{n_1}\beta^{n_1}}\cdot x^{2n_1}, \qquad e^{-\gamma x} = \sum_{n_2=0}^\infty \phi_{n_2}\gamma^{n_2}x^{n_2}$$

to get

$$\sum_{n_{12}}(-1)^{n_1+n_2}\frac{\gamma^{n_2}}{4^{n_1}\beta^{n_1}\Gamma(n_1+1)\Gamma(n_2+1)}x^{2n_1+n_2} = \sum_{n_{12}}\phi_{n_{12}}\frac{\gamma^{n_2}}{4^{n_1}\beta^{n_1}}\langle 2n_1+n_2+1\rangle.$$

Here we have two sums and one bracket, so this is an index one bracket series. Let n_1 free gives $n_2^* = -2n_1 - 1$. Then rule E_1 gives

$$\sum_{n_1=0}^\infty(-1)^{n_1}\frac{\gamma^{-2n_1-1}}{4^{n_1}\beta^{n_1}\Gamma(n_1+1)}\Gamma(2n_1+1).$$

This series diverges since $\Gamma(2n_1+1)$ grows faster than the terms in the denominator. Now let n_2 free to obtain $n_1^* = -\frac{n_2+1}{2}$. Then rule E_1 gives

$$\frac{1}{2}\sum_{n_2=0}^\infty\frac{(-1)^{n_2}\gamma^{n_2}}{\Gamma(n_2+1)2^{-n_2-1}\beta^{\frac{-n_2-1}{2}}}\Gamma\left(\frac{n_2+1}{2}\right) = \sqrt{\beta}\sum_{n_2=0}^\infty\frac{(-2\gamma\sqrt{\beta})^{n_2}\Gamma\left(\frac{n_2+1}{2}\right)}{\Gamma(n_2+1)},$$

where the last sum has a closed form in terms of error function, which confirms the identity.

Now we justify the evaluation of the method in this case. Recall the bracket series is

$$\sum_{n_{12}} \frac{(-1)^{n_1+n_2}\gamma^{n_2}}{4^{n_1}\beta^{n_1}\Gamma(n_1+1)\Gamma(n_2+1)}\langle 2n_1+n_2+1\rangle = \sum_{n_{12}}(-1)^{n_1+n_2}f(n_1,n_2)\langle 2n_1+n_2+1\rangle,$$

where f is separable and each component function has bracket form with weight 1. In this bracket, n_1 has weight $\frac{1}{2}$ while n_2 has weight 1 so we should let n_2 free. This explains why only the case n_2 free gave us the answer and also justifies the evaluation.

3.3 Type two brackets

Again we start with the analysis on the simplest case $\langle an_1 - bn_2 + s\rangle$, then generalize to the case $\langle a_1n_1 + a_2n_2 + \ldots + a_kn_k - bn_{k+1} + s\rangle$.

Lemma 3.3.1. *Assume $f(x,y)$ is separable, i.e. $f(x,y) = f_1(x)f_2(y)$, and each f_i has bracket form with weight w_i respectively. Let $c > 0$ and $\alpha < 0$ and $\sigma \in \mathbb{R}$. Assume $-\sigma + \alpha c < 0$. Then*

$$\begin{aligned} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\pi f(-\sigma+\alpha u, -u)}{\sin \pi u \sin \pi(\sigma - \alpha u)}du &= \sum_{n=0}^{\infty} \frac{(-1)^n}{\sin \pi(\sigma + \alpha n)} \cdot f(-\sigma - \alpha n, n) \\ &+ \frac{1}{\alpha}\sum_{n=0}^{\infty} \frac{(-1)^n}{\sin \frac{\pi(\sigma+n)}{\alpha}} \cdot f\left(n, \frac{-\sigma-n}{\alpha}\right). \end{aligned}$$

Proof. The proof is exactly the same as that of Lemma 3.2.4. The fact that α is negative makes the the condition $\alpha w_1 < w_2$ required in Lemma 3.2.4 is automatic. The contour integral in this case gives two sets of residues, one from each sine function, and this produces the right hand side. □

Theorem 3.3.2. *Let $f(x,y)$ be a separable function with both component functions f_1 and f_2 having bracket form. Let a and b be positive numbers and s be any real number. Let*

$$F(x) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}(-1)^{n_1+n_2}f(n_1,n_2)x^{an_1-bn_2},$$

then we have

$$\int_0^\infty F(x)x^{s-1}dx = \frac{1}{a}\sum_{n=0}^{\infty}(-1)^n \frac{\pi}{\sin\frac{\pi(s-bn)}{a}} f\left(\frac{-s+bn}{a}, n\right) + \frac{1}{b}\sum_{n=0}^{\infty}(-1)^n \frac{\pi}{\sin\frac{\pi(-s-an)}{b}} f\left(n, \frac{s+an}{b}\right).$$

Proof. The first part of our proof is exactly like that of Theorem 3.2.5 up to the point of the formula

$$\int_0^\infty F(x)x^{s-1}dx = \frac{1}{2\pi i a}\int_{c_2-i\infty}^{c_2+i\infty} \frac{\pi^2}{\sin\pi u_2 \sin\frac{\pi(s+bu_2)}{a}} f\left(\frac{-s-bu_2}{a}, -u_2\right).$$

To conclude the proof now use Lemma 3.3.1 with $\sigma = s/a$ and $\alpha = -b/a$. The proof is complete. □

Remark 3.3.3. Unlike Theorem 3.2.5 where we need s positive, this theorem does not require s to be positive. The reason we need s to be positive in Theorem 3.2.5 is because we need to be able to choose c_1 and c_2 positive such that $ac_1 + bc_2 < s$. In Theorem 3.3.2 we need $ac_1 - bc_2 < s$ so it is possible even for negative s.

Example 3.3.1. Recall an example we mentioned in the introduction.

$$\int_0^\infty \frac{\sin\alpha x}{x(x^2+\beta^2)}dx = \frac{\pi}{2\beta^2}(1-e^{-\alpha\beta}).$$

Power series for sine and rule P_2 for $1/(x^2+\beta^2)$ give the bracket series

$$\sum_{n_1,n_2,n_3} \phi_{n_{123}} \alpha^{2n_3+1}\beta^{2n_2}\frac{\Gamma(n_3+1)}{\Gamma(2n_3+2)}\langle n_1+n_2+1\rangle\langle 2n_1+2n_3+1\rangle.$$

This is a bracket series with three sums and two brackets so it has index one. We evaluate the second bracket first. In this one, n_1 has weight $1/2$ while n_3 has weight one so we evaluate this with respect to n_1 and get

$$\sum_{n_{23}} \phi_{n_{23}} \alpha^{2n_3+1} \beta^{2n_2} \frac{\Gamma(n_3+1)\Gamma\left(n_3+\frac{1}{2}\right)}{\Gamma(2n_3+2)} \left\langle n_2 - n_3 + \tfrac{1}{2} \right\rangle.$$

Theorem 3.3.2 tells us to add the results from the cases n_2 free and n_3 free. Indeed, this gives the answer.

Now we generalize these results for brackets with more indices.

Theorem 3.3.4. *Let $f(x_1, x_2, x_3)$ be a separable function with all component functions f_1, f_2, and f_3 having bracket form with weights w_1, w_2, and w_3, respectively. Assume a, b, and d are positive numbers and s is any real number such that $bw_1 < aw_2$. Let*

$$F(x) = \sum_{n_1,n_2,n_3} (-1)^{n_1+n_2+n_3} f(n_1, n_2, n_3) x^{an_1+bn_2-dn_3},$$

then we have

$$\int_0^\infty F(x) x^{s-1} dx = \mathcal{E}_1 + \mathcal{E}_3,$$

where $\mathcal{E}_i$ is an evaluation of the bracket $\langle an_1 + bn_2 - dn_3 + s\rangle$ with respect to n_i, and the summand is $(-1)^{n_1+n_2+n_3} f(n_1, n_2, n_3)$.

Proof. The first part of the proof is similar to that of Theorem 3.2.5. First we use Theorem 2.2.5 for each component function to get

$$F(x) = \frac{1}{(2\pi i)^3} \int_{c_3}^{u_3} \int_{c_2}^{u_2} \int_{c_1}^{u_1} \frac{\pi^3 f(-u_1, -u_2, -u_3)}{\sin \pi u_1 \sin \pi u_2 \sin \pi u_3} x^{-au_1-bu_2+du_3}$$

for positive c's such that $ac_1 + bc_2 - dc_3 < s$. Then

$$\int_0^1 F(x) x^{s-1} = \frac{-1}{(2\pi i)^3} \int_{c_3}^{u_3} \int_{c_2}^{u_2} \int_{c_1}^{u_1} \frac{\pi^3 f(-u_1, -u_2, -u_3)}{(s - au_1 - bu_2 + du_3) \sin \pi u_1 \sin \pi u_2 \sin \pi u_3}.$$

Now choose c_1' such that $ac_1' + bc_2 - dc_3 > s$. Then

$$\int_1^\infty F(x)x^{s-1} = \frac{1}{(2\pi i)^3}\int_{c_3}^{u_3}\int_{c_2}^{u_2}\int_{c_1'}^{u_1} \frac{\pi^3 f(-u_1,-u_2,-u_3)}{(s-au_1-bu_2+du_3)\sin\pi u_1 \sin\pi u_2 \sin\pi u_3}.$$

Combine the two we have

$$\int_0^\infty F(x)x^{s-1}dx = \frac{1}{(2\pi i)^2 a}\int_{c_3}^{u_3}\int_{c_2}^{u_2} \frac{\pi^3 f\left(\frac{-s+bu_2-du_3}{a},-u_2,-u_3\right)}{\sin\frac{\pi(s-bu_2+du_3)}{a}\sin\pi u_2\sin\pi u_3}.$$

Now use Lemma 3.2.4 to evaluate $\int_{c_2}^{u_2}$ with $\sigma = (s+du_3)/a$ and $\alpha = b/a$ to get

$$\frac{1}{2\pi i a}\sum_{n_2}(-1)^{n_2}\int_{c_3}^{u_3}\frac{\pi^2 f\left(\frac{-s-bn_2-du_3}{a},n_2,-u_3\right)}{\sin\frac{\pi(s+bn_2+du_3)}{a}\sin\pi u_3}.$$

The condition $-\sigma+\alpha c_2 < 0$ is equivalent to $bc_2 - dc_3 - s < 0$, which is given since $bc_2 - dc_3 - s < -ac_1$. Next use Lemma 3.3.1 to evaluate $\int_{c_3}^{u_3}$ with $\sigma = (s+bn_2)/a$ and $\alpha = -d/a$ to get $A_1 + A_2$ where

$$\begin{aligned} A_1 &= \frac{1}{a}\sum_{n_2,n_3}(-1)^{n_2+n_3}\frac{\pi}{\sin\frac{\pi(s+bn_2-dn_3)}{a}} f\left(\frac{-s-bn_2-dn_3}{a},n_2,n_3\right)\\ A_2 &= \frac{1}{d}\sum_{n_1,n_2}(-1)^{n_1+n_2}\frac{\pi}{\sin\frac{\pi(-s-an_1-bn_2)}{d}} f\left(n_1,n_2,\frac{s+an_1+bn_2}{d}\right). \end{aligned}$$

Here the condition $-\sigma+\alpha c_3 < 0$ is equivalent to $-s-bn_2-dc_3 < 0$, which is valid. Finally use reflection formula for gamma function to confirm that $A_1 = \mathcal{E}_1$ and $A_2 = \mathcal{E}_3$. The proof is complete. $\square$

Remark. This theorem shows that to evaluate the bracket series

$$\sum_{n_1,n_2,n_3}(-1)^{n_1+n_2+n_3} f(n_1,n_2,n_3)\langle an_1+bn_2-dn_3+s\rangle,$$

we should let each n_1 and n_3 free, then add the two results to get the answer. This theorem leads to another more general result.

Corollary 3.3.5. *Let $f(x_1, \ldots, x_k)$ be a separable function with all component functions f_i having bracket form with weights w_i respectively. Let s, $a_1, \ldots, a_k$, be positive numbers such that $\frac{w_1}{a_1} < \frac{w_i}{a_i}$ for all $2 \leq i \leq k-1$. Let*

$$F(x) = \sum_{n_1,\ldots,n_k} (-1)^{n_1+\ldots+n_k} f(n_1, \ldots, n_k) x^{a_1 n_1 + \ldots + a_{k-1} n_{k-1} - a_k n_k},$$

then we have

$$\int_0^\infty F(x) x^{s-1} dx = \mathcal{E}_1 + \mathcal{E}_k,$$

where $\mathcal{E}_i$ is an evaluation of the bracket $\langle a n_1 + \ldots + a_{k-1} n_{k-1} - a_k n_k + s \rangle$ with respect to n_i, and the summand is $(-1)^{n_1+\ldots+n_k} f(n_1, \ldots, n_k)$

Proof. The proof is exactly like that of Theorem 3.3.4. □

Remark. In the language of MoB, this corollary states that if in the evaluation of a bracket series

$$\sum_{n_1,\ldots,n_k} (-1)^{n_1+\ldots+n_k} f(n_1, \ldots, n_k) \langle a n_1 + \ldots + a_{k-1} n_{k-1} - a_k n_k + s \rangle,$$

the index n_1 has the least weight, then it suffices to consider two cases:
the `first one` is when $n_2, \ldots, n_k$ free and
the `second one` is when $n_1, \ldots, n_{k-1}$ free.

The sum of results from these two cases suffices to produce the evaluation of this bracket series.

Example 3.3.2. MoB will be used to evaluate

$$\int_0^\infty x^{\alpha-1} \exp\left(-x^2 - x - \frac{1}{x}\right) dx.$$

Separate the integrand into three exponentials then their power series give

$$\sum_{n_{123}} \phi_{n_{123}} \langle 2n_1 + n_2 - n_3 + \alpha \rangle.$$

In this bracket n_1 has smaller weight than n_2 so Corollary 3.3.5 suggests evaluating this bracket with respect to each n_1 and n_3 and the two results should be added (in this case n_2 free results in a divergent sum). The case n_1 free gives

$$\frac{1}{2} \sum_{n_{23}} \phi_{n_{23}} \Gamma \left(\frac{n_2 - n_3 + \alpha}{2} \right),$$

while the case n_3 free yields

$$\sum_{n_{12}} \phi_{n_{12}} \Gamma(-2n_1 - n_2 - \alpha).$$

Unfortunately, it seems that these sums do not admit elementary closed forms.

3.4 Method of parameters

The results presented in the previous section, as Theorems 3.2.5 and 3.3.2, their application requires to and all of their general results, we need to have one of the n_i's in the bracket to have the least weight. This section presents a mechanism to convent the general situation to the one included in the previous theorems.

Example 3.4.1. Entry 3.362.2 in [8] gives

$$\int_0^\infty \frac{e^{-\mu x} dx}{\sqrt{x + \beta}} = \sqrt{\frac{\pi}{\mu}} e^{\beta\mu} \left[1 - \Phi \left(\sqrt{\beta\mu} \right) \right].$$

Lemma 2.3.2, the bracket version of the binomial theorem, gives

$$\frac{1}{\sqrt{x+\beta}} = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \sum_{n_{12}} \phi_{n_{12}} x^{n_1} \beta^{n_2} \left\langle n_1 + n_2 + \tfrac{1}{2} \right\rangle$$

while the power series of exponential gives

$$e^{-\mu x} = \sum_{n_3=0}^{\infty} \phi_{n_3} \mu^{n_3} x^{n_3}.$$

This produces the series

$$\frac{1}{\sqrt{\pi}} \sum_{n_{123}} \phi_{n_{123}} \beta^{n_2} \mu^{n_3} \langle n_1 + n_3 + 1 \rangle \left\langle n_1 + n_2 + \tfrac{1}{2} \right\rangle.$$

First evaluate the bracket $\langle n_1 + n_3 + 1 \rangle$. However, n_1 and n_3 have the same weight, so Theorem 3.2.5 may not be used directly. To fix this problem, introduce the parameter $\epsilon < 1$ in order to change the weight of n_3. Now evaluate

$$\frac{1}{\sqrt{\pi}} \sum_{n_{123}} \phi_{n_{123}} \beta^{n_2} \mu^{n_3} \langle n_1 + \epsilon n_3 + 1 \rangle \left\langle n_1 + n_2 + \tfrac{1}{2} \right\rangle.$$

Since n_3 now has more weight than n_1 in $\langle n_1 + \epsilon n_3 + 1 \rangle$, Theorem 3.2.5 gives

$$\frac{1}{\sqrt{\pi}} \sum_{n_{23}} \phi_{n_{23}} \beta^{n_2} \mu^{n_3} \Gamma(\epsilon n_3 + 1) \left\langle n_2 - \epsilon n_3 - \tfrac{1}{2} \right\rangle.$$

Finally use Theorem 3.3.2 to obtain

$$\begin{aligned} & \frac{1}{\sqrt{\pi}} \sum_{n_3} \frac{(-1)^{n_3}}{\Gamma(n_3+1)} \beta^{\epsilon n_3 + \frac{1}{2}} \mu^{n_3} \Gamma(\epsilon n_3 + 1) \Gamma\left(-\epsilon n_3 - \frac{1}{2}\right) \\ + \quad & \frac{1}{\epsilon\sqrt{\pi}} \sum_{n_2} \frac{(-1)^{n_2}}{\Gamma(n_2+1)} \beta^{n_2} \mu^{\frac{n_2}{\epsilon} - \frac{1}{2\epsilon}} \; \Gamma\left(n_2 + \frac{1}{2}\right) \Gamma\left(-\frac{n_2}{\epsilon} + \frac{1}{2\epsilon}\right). \end{aligned}$$

The value of the original bracket series is obtained by letting $\epsilon \to 1$. This produces

$$\frac{\sqrt{\beta}}{\sqrt{\pi}} \sum_{n_3} (-\beta\mu)^{n_3} \Gamma\left(-n_3 - \frac{1}{2}\right) + \frac{1}{\sqrt{\pi\mu}} \sum_{n_2} \frac{(-\beta\mu)^{n_2}}{\Gamma(n_2+1)} \Gamma\left(n_2 + \frac{1}{2}\right) \Gamma\left(-n_2 + \frac{1}{2}\right),$$

which has the closed form

$$-\sqrt{\frac{\pi}{\mu}}\, e^{\beta\mu}\, \Phi\left(\sqrt{\beta\mu}\right) + \sqrt{\frac{\pi}{\mu}}\, e^{\beta\mu} = \sqrt{\frac{\pi}{\mu}}\, e^{\beta\mu} \left[1 - \Phi\left(\sqrt{\beta\mu}\right)\right].$$

An implicit continuity in ϵ of the bracket series has been assumed. This remains to be discussed.

Example 3.4.2. We now confirm the identity

$$\int_0^\infty e^{-ax} \sin(bx) dx = \frac{b}{a^2 + b^2} \quad \text{for } a, b > 0.$$

This is a special case of the entry 3.944.11 in [8].

Using the power series representation of the integrand produces

$$\sum_{n_1, n_2} (-1)^{n_1 + n_2} \frac{a^{n_1}}{\Gamma(n_1+1)} \cdot \frac{b^{2n_2+1}}{\Gamma(2n_2+2)} \langle n_1 + 2n_2 + 2 \rangle.$$

Both n_1 and n_2 have weight one in this bracket so, as before, we introduce a new parameter $\epsilon < 1$ to transform the original bracket series into

$$\sum_{n_1, n_2} (-1)^{n_1 + n_2} \frac{a^{n_1}}{\Gamma(n_1+1)} \cdot \frac{b^{2n_2+1}}{\Gamma(2n_2+2)} \langle n_1 + 2\epsilon n_2 + 2 \rangle.$$

Now n_1 has less weight than n_2 so we evaluate the bracket with respect to n_1, i.e. we let n_2 free. The evaluation is

$$\sum_{n_2} (-1)^{n_2} a^{-2\epsilon n_2 - 2} b^{2n_2+1} \frac{\Gamma(2\epsilon n_2 + 2)}{\Gamma(2n_2+2)}.$$

Now let $\epsilon \to 1$ to produce

$$\frac{b}{a^2}\sum_n \left(-\frac{b^2}{a^2}\right)^n = \frac{b}{a^2}\cdot\frac{1}{1+\frac{b^2}{a^2}} = \frac{b}{a^2+b^2}.$$

Here we need $b < a$ for the sum to converge. For $b > a$ then we should introduce the parameter $\epsilon < 1$ to n_1 so we would evaluate the bracket with respect to n_2 and would get the same answer at the end. This agrees with the method of brackets. Each case of n_1 or n_2 free gives a sum that converges with either $a < b$ or $a > b$, but both give the same answer at the end. The introduction of the parameter ϵ is equivalent with evaluating instead

$$\int_0^\infty e^{-ax}\sin\left(bx^\epsilon\right)dx$$

then let $\epsilon = 1$ to get our original integral.

Even if originally the summand in the bracket series has all its component functions with bracket form, the evaluations of some of its brackets might produce some terms that do not have bracket form. This is shown in the following lemma.

Lemma 3.4.1. *Let $f(x,y)$ be separable with both component functions having bracket form with weights w_1 and w_2 respectively. Let $\alpha > 0$ be such that $\alpha w_1 < w_2$. Then, for any $s > 0$, the evaluation*

$$\sum_{n_1,n_2}(-1)^{n_1+n_2}f(n_1,n_2)\langle n_1+\alpha n_2+s\rangle = \sum_n (-1)^n g(n),$$

will result in g having type two bracket form.

Proof. Assume

$$f_1(z) = \frac{\beta_1^z\prod\Gamma(a_kz+b_k)}{\prod\Gamma(c_kz+d_k)}, \qquad f_2(z) = \frac{\beta_2^z\prod\Gamma(p_kz+q_k)}{\prod\Gamma(r_kz+t_k)}.$$

Theorem 3.2.5 gives the evaluation of the bracket in the form

$$\sum_n (-1)^n \beta_1^{-\alpha n - s} \beta_2^n \frac{\prod \Gamma(-\alpha a_k n - s a_k + b_k) \prod \Gamma(p_k n + q_k)}{\prod \Gamma(-\alpha c_k n - s c_k + d_k) \prod \Gamma(r_k n + t_k)} \Gamma(-\alpha n - s + 1)\Gamma(\alpha n + s).$$

Observe that

$$\sum -\alpha c_k + \sum \alpha a_k + \sum r_k - \sum p_k = -\alpha w_1 + w_2 > 0.$$

This confirms one of the requirements for our summand to have type two bracket form. Next we need to confirm that

$$\sum \alpha |c_k| - \sum \alpha |a_k| + \sum |r_k| - \sum |p_k| - 2\alpha < 2.$$

This holds because $\sum |c_k| - \sum |a_k| < 2$ and $\sum |r_k| - \sum |p_k| < 2$. Finally we need to check that

$$g(z) = \beta_1^{-\alpha z - s} \beta_2^z \frac{\prod \Gamma(-\alpha a_k z - s a_k + b_k) \prod \Gamma(p_k z + q_k)}{\prod \Gamma(-\alpha c_k z - s c_k + d_k) \prod \Gamma(r_k z + t_k)} \Gamma(-\alpha z - s + 1)\Gamma(\alpha z + s)$$

is analytic when $\Re(z) > -c$ for some positive c. To see this we write

$$g(z) = f_1(-\alpha z - s) f_2(z) \cdot \frac{\pi}{\sin \pi(\alpha z + s)}.$$

Both f_1 and f_2 are entire so the poles could only come from sine function. The poles $\alpha z + s = n$ of sine when $n > 0$ are removable because $f_1(-\alpha z - s) = f_1(-n) = 0$. Thus g is analytic when $\alpha \Re(z) + s > 0$, i.e. $\Re(z) > -s/\alpha$. Thus g has type two bracket form. □

After each evaluation of a bracket in the bracket series, the n's that have not been evaluated might have their component functions drop from having bracket form to type two bracket form. For this reason we need to establish a similar result to

Theorem 3.2.5 where some component functions are only expected to have type two bracket form. Keep in mind that Lemmas 3.2.4 and 3.3.1 still hold when the second component function f_2 only has type two bracket form instead of bracket form.

Theorem 3.4.2. *Let $f(x, y)$ be a separable function with component function f_1 having bracket form with weight w_1 and f_2 having type two bracket form with weight w_2. Let s, a and b be positive numbers such that $bw_1 < aw_2$. Let*

$$F(x) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} (-1)^{n_1+n_2} f(n_1, n_2) x^{an_1+bn_2},$$

then we have

$$\int_0^\infty F(x) x^{s-1} dx = \frac{1}{a} \sum_{n=0}^{\infty} (-1)^n \frac{\pi}{\sin \frac{\pi(s+bn)}{a}} f\left(\frac{-s-bn}{a}, n\right).$$

Proof. The proof is exactly like that for Theorem 3.2.5. The only difference between bracket form and type two bracket form is type two only requires the function to be analytic for $\Re(z) > -c$ for some positive c. Thus if our contour is within this region, the analysis should be the same as though both component functions have bracket form. Assume we have $f_2(z)$ analytic on $\Re(z) > -c$ for some positive c. Then in the proof of Theorem 3.2.5 we need to choose $c_2 = c$. This is the only modification we need to prove this theorem. $\square$

Next we see how type two bracket form affects Theorem 3.3.2.

Theorem 3.4.3. *Let $f(x, y)$ be a separable function with component function f_1 having bracket form and f_2 having type two bracket form. Let a and b be positive numbers and s be any real number. Let*

$$F(x) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} (-1)^{n_1+n_2} f(n_1, n_2) x^{an_1-bn_2},$$

then we have

$$\int_0^\infty F(x)x^{s-1}dx = \frac{1}{a}\sum_{n=0}^{\infty}(-1)^n\frac{\pi}{\sin\frac{\pi(s-bn)}{a}}f\left(\frac{-s+bn}{a},n\right) + \frac{1}{b}\sum_{n=0}^{\infty}(-1)^n\frac{\pi}{\sin\frac{\pi(-s-an)}{b}}f\left(n,\frac{s+an}{b}\right).$$

Proof. The proof is exactly like that for Theorem 3.3.2. Assume we have $f_2(z)$ analytic for $\Re(z) > -c$ for some positive c. Then we just need to pick $c_2 = c$ to make sure we stay in the region where f_2 is analytic. □

Finally we prove similar result to Theorem 3.3.4.

Theorem 3.4.4. *Let $f(x_1, x_2, x_3)$ be a separable function with component function f_1 having bracket form with weight w_1, f_2 and f_3 having type two bracket form with weights w_2, and w_3, respectively. Let s, a, b, and d be positive numbers such that $bw_1 < aw_2$. Let*

$$F(x) = \sum_{n_1,n_2,n_3}(-1)^{n_1+n_2+n_3}f(n_1,n_2,n_3)x^{an_1+bn_2-dn_3},$$

then we have

$$\int_0^\infty F(x)x^{s-1}dx = \mathcal{E}_1 + \mathcal{E}_3,$$

where $\mathcal{E}_i$ is an evaluation of the bracket $\langle an_1 + bn_2 - dn_3 + s\rangle$ with respect to n_i, and the summand is $(-1)^{n_1+n_2+n_3}f(n_1, n_2, n_3)$.

Proof. The proof is the same as that for Theorem 3.3.4. Assume $f_2(z)$ and $f_3(z)$ are analytic for $\Re(z) > -c$ and $\Re(z) > -c'$, respectively. Then in the proof we only need to choose $c_2 = c$ and $c_3 = c'$. □

3.5 Summary

The results obtained here are now described in the language of brackets:

1. Type one brackets should be evaluated with respect to the parameter n_i that has the least weight.

2. Type two brackets, $\langle a_1n_1 + \ldots + a_kn_k - a_{k+1}n_{k+1} + s\rangle$ should be evaluated with respect to the parameter n_i that has the least weight among $n_1, \ldots, n_k$, then evaluated with respect to n_{k+1}, and then add the two results.

3. `Conjecture`. After an evaluation of a bracket, the parameters that have not been evaluated, will have their corresponding component functions drop from having bracket form to having type two bracket form. Furthermore, these parameters, whose component functions only have type two bracket form, should only be evaluated if they become the negative parameter, like n_{k+1}, in a type two bracket.

Recall example 3.4.1 where a new parameter ϵ was introduced. This gave the series

$$\frac{1}{\sqrt{\pi}}\sum_{n_{123}}\phi_{n_{123}}\beta^{n_2}\mu^{n_3}\langle n_1+\epsilon n_3+1\rangle\left\langle n_1+n_2+\tfrac{1}{2}\right\rangle,$$

where the weights of n_1 and n_3 are now different so we could evaluate the bracket $\langle n_1 + \epsilon n_3 + 1\rangle$ to get

$$\frac{1}{\sqrt{\pi}}\sum_{n_{23}}\phi_{n_{23}}\beta^{n_2}\mu^{n_3}\Gamma(\epsilon n_3+1)\left\langle n_2-\epsilon n_3-\tfrac{1}{2}\right\rangle.$$

The parameter n_3 *survived a bracket evaluation*, so its component function f_3 might not have bracket form anymore. This is actually true in this case because of the term $\Gamma(\epsilon z + 1)$. Thus n_3 should not be evaluated in any brackets unless it is the negative term in a type two bracket, which is the case here. In short, everything works out fine and we have obtained the correct answer, in rigorous form.

On the other hand, one could introduce the parameter $\epsilon < 1$ for n_1 to produce

$$\frac{1}{\sqrt{\pi}} \sum_{n_{123}} \phi_{n_{123}} \beta^{n_2} \mu^{n_3} \langle \epsilon n_1 + n_3 + 1 \rangle \left\langle n_1 + n_2 + \tfrac{1}{2} \right\rangle .$$

Now evaluate the first bracket with respect to n_3 to get

$$\frac{1}{\sqrt{\pi}} \sum_{n_{12}} \phi_{n_{12}} \beta^{n_2} \mu^{-\epsilon n_1 - 1} \Gamma(\epsilon n_1 + 1) \left\langle n_1 + n_2 + \tfrac{1}{2} \right\rangle .$$

Now in this remaining bracket, n_1 has weight $1 - \epsilon < 1$ so we need to evaluate this bracket with respect to n_1. However, because of the term $\Gamma(\epsilon z + 1)$, the function $f_1(z)$ only has type two bracket form. Thus we cannot guarantee the evaluation with respect to n_1 would give the correct answer. Indeed, the evaluation would not give the correct answer, since it gives

$$\frac{1}{\sqrt{\pi}} \sum_{n_2} \phi_{n_2} \beta^{n_2} \mu^{\epsilon n_2 + \frac{\epsilon}{2} - 1} \Gamma\left(-\epsilon n_2 - \frac{\epsilon}{2} + 1 \right) \Gamma\left(n_2 + \frac{1}{2} \right) .$$

Let $\epsilon \to 1$ we get

$$\frac{1}{\sqrt{\pi \mu}} \sum_{n} (-\beta\mu)^n \frac{\Gamma\left(-n + \frac{1}{2}\right) \Gamma\left(n + \frac{1}{2}\right)}{\Gamma(n+1)} .$$

The use of reflection formula for the two gamma functions on the numerator yields

$$\sqrt{\frac{\pi}{\mu}} \sum_{n} \frac{(\beta\mu)^n}{\Gamma(n+1)} = \sqrt{\frac{\pi}{\mu}} e^{\beta\mu} .$$

As shown in example 3.4.1, this is only one part of the correct answer.

Example 3.5.1. Entry 3.725.1 of [8] states that

$$\int_0^\infty \frac{\sin ax}{x(x^2 + b^2)} = \frac{\pi}{2b^2} \left(1 - e^{-ab} \right) .$$

The power series of sine and rule P_2 for $1/(x^2+b^2)$ give

$$\sum_{n_{123}}(-1)^{n_1+n_2+n_3}\frac{b^{2n_2}a^{2n_3+1}}{\Gamma(n_1+1)\Gamma(n_2+1)\Gamma(2n_3+2)}\langle n_1+n_2+1\rangle\langle 2n_1+2n_3+1\rangle,$$

for the integral. In the bracket $\langle 2n_1+2n_3+1\rangle$, n_1 has the smallest weight, so we should evaluate this bracket with respect to n_1 to produce

$$\sum_{n_{23}}(-1)^{n_2+n_3}\frac{b^{2n_2}a^{2n_3+1}}{\Gamma(n_2+1)\Gamma(2n_3+2)}\Gamma\left(n_3+\frac{1}{2}\right)\left\langle n_2-n_3+\tfrac{1}{2}\right\rangle.$$

Now evaluate the bracket with respect to n_2 and then n_3 and add the two results to get the answer. This explains, and proves, why the method of bracket yields the correct answer when we add the two results from the cases n_2 free and n_3 free.

Example 3.5.2. We now compute

$$\int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}.$$

This integral has many interesting properties [10].

Rule P_2 for the whole integrand gives the bracket series

$$\sum_{n_{123}}(-1)^{n_1+n_2+n_3}\frac{(2a)^{n_2}}{\Gamma(n_1+1)\Gamma(n_2+1)\Gamma(n_3+1)}\langle 4n_1+2n_2+1\rangle\langle n_1+n_2+n_3+m+1\rangle.$$

We will evaluate the bracket $\langle 4n_1+2n_2+1\rangle$ first. Here n_1 has less weight so we evaluate the bracket with respect to it to get

$$\sum_{n_{23}}(-1)^{n_2+n_3}\frac{(2a)^{n_2}}{\Gamma(n_2+1)\Gamma(n_3+1)}\Gamma\left(\frac{n_2}{2}+\frac{1}{4}\right)\left\langle \tfrac{n_2}{2}+n_3+m+\tfrac{3}{4}\right\rangle.$$

Both n_2 and n_3 have weight one in this last bracket so we need to introduce a new parameter $\epsilon < 1$ to obtain

$$\sum_{n_{23}} (-1)^{n_2+n_3} \frac{(2a)^{n_2}}{\Gamma(n_2+1)\Gamma(n_3+1)} \Gamma\left(\frac{n_2}{2}+\frac{1}{4}\right) \langle \tfrac{n_2}{2} + \epsilon n_3 + m + \tfrac{3}{4} \rangle .$$

Now n_1 has less weight, so evaluating the bracket with respect to n_1 produces

$$\sum_{n} \frac{(-1)^n}{\Gamma(n+1)} (2a)^{-2\epsilon n - 2m - \frac{3}{2}} \Gamma\left(-\epsilon n - m - \frac{1}{2}\right) \Gamma\left(2\epsilon n + 2m + \frac{3}{2}\right).$$

Now let $\epsilon \to 1$ to produce

$$\sum_{n} \frac{(-1)^n}{\Gamma(n+1)} (2a)^{-2n - 2m - \frac{3}{2}} \Gamma\left(-n - m - \frac{1}{2}\right) \Gamma\left(2n + 2m + \frac{3}{2}\right),$$

which has the closed form

$$(2a)^{-2m-\frac{3}{2}} \Gamma\left(-m - \frac{1}{2}\right) \Gamma\left(2m + \frac{3}{2}\right) {}_2F_1\left(\left. \begin{matrix} m+\frac{3}{4} \quad m+\frac{5}{4} \\ m+\frac{3}{2} \end{matrix} \right| \frac{1}{a^2} \right)$$

This series requires $|a| > 1$ for convergence. Similarly, for $|a| < 1$, let $\epsilon > 1$ and evaluate the bracket with respect to n_3 to produce, after letting $\epsilon \to 1$,

$$\sum_{n} \frac{(-1)^n}{\Gamma(n+1)} (2a)^n \Gamma\left(\frac{2n+1}{4}\right) \Gamma\left(\frac{2n+4m+3}{4}\right).$$

This has the closed form

$$\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{4m+3}{4}\right) {}_2F_1\left(\left. \begin{matrix} \frac{1}{4} \quad m+\frac{3}{4} \\ \frac{1}{2} \end{matrix} \right| a^2 \right) - 2a\Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{4m+5}{4}\right) {}_2F_1\left(\left. \begin{matrix} \frac{3}{4} \quad m+\frac{5}{4} \\ \frac{3}{2} \end{matrix} \right| a^2 \right) \quad (3.1)$$

Here we need $|a| < 1$ for the series to converge. This gives

Theorem 3.5.1. *The quartic integral is given by*

$$\int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} = \Gamma\left(\tfrac{1}{4}\right)\Gamma\left(\tfrac{4m+3}{4}\right) {}_2F_1\left(\begin{matrix}\frac{1}{4} & m+\frac{3}{4} \\ & \frac{1}{2}\end{matrix}\middle| a^2\right)$$
$$- 2a\Gamma\left(\tfrac{3}{4}\right)\Gamma\left(\tfrac{4m+5}{4}\right) {}_2F_1\left(\begin{matrix}\frac{3}{4} & m+\frac{5}{4} \\ & \frac{3}{2}\end{matrix}\middle| a^2\right).$$

Example 3.5.3. Entry 6.554.1 in [8] gives

$$\int_0^\infty xJ_0(xy)\frac{dx}{\sqrt{a^2+x^2}} = y^{-1}e^{-ay},$$

where $J_0(x)$ is the Bessel function of the first kind of order 0, with the power series representation

$$J_0(x) = \sum_{n=0}^\infty \frac{(-1)^n}{n!n!2^{2n}}x^{2n}.$$

Use rule P_2 for $1/\sqrt{a^2+x^2}$ to produce a bracket series for the integral:

$$\frac{1}{\Gamma\left(\frac{1}{2}\right)}\sum_{n_{123}}\phi_{n_{123}}\frac{y^{2n_3}a^{2n_1}}{\Gamma(n_3+1)2^{2n_3}}\langle 2n_2+2n_3+2\rangle\left\langle n_1+n_2+\tfrac{1}{2}\right\rangle.$$

Since n_2 has less weight than n_3 in the first bracket we evaluate this with respect to n_2 to have

$$\frac{1}{2\sqrt{\pi}}\sum_{n_{13}}\phi_{n_{13}}\frac{y^{2n_3}a^{2n_1}}{\Gamma(n_3+1)2^{2n_3}}\Gamma(n_3+1)\left\langle n_1-n_3-\tfrac{1}{2}\right\rangle$$
$$= \frac{1}{2\sqrt{\pi}}\sum_{n_{13}}\phi_{n_{13}}\frac{y^{2n_3}a^{2n_1}}{2^{2n_3}}\left\langle n_1-n_3-\tfrac{1}{2}\right\rangle.$$

Theorem 3.3.2 tells us how to evaluate the bracket with respect to each parameter. Then one has to then add the two results to obtain the value of the integral. With respect to n_1, we have

$$\frac{1}{2\sqrt{\pi}}\sum_{n_3}\frac{(-1)^{n_3}y^{2n_3}a^{2n_3+1}}{\Gamma(n_3+1)2^{2n_3}}\Gamma\left(-n_3-\frac{1}{2}\right)=-\frac{\sinh(ay)}{y}.$$

With respect to n_2 gives

$$\frac{1}{2\sqrt{\pi}}\sum_{n_1}\frac{(-1)^{n_1}y^{2n_1-1}a^{2n_1}}{\Gamma(n_1+1)2^{2n_1-1}}\Gamma\left(-n_1+\frac{1}{2}\right)=\frac{\cosh(ay)}{y}.$$

Adding these two sums produces the stated answer $y^{-1}e^{-ay}$.

Chapter 4

Some heuristic applications of MoB

Here we discuss examples of integrals where the use of MoB has not been completely justified. Some bracket series have non-separable summands, which we have not yet analyzed. Others have null or divergent summands, making the bracket series totally formal. Some integrands do not have power series representations, which require different care before using MoB. Also the application of rule P_2 for multinomial of positive exponent has not been justified. The goal of this section is to show that the method of brackets still gives the right answer.

4.1 Non-separable functions

Example 4.1.1. Entry 3.311.1 of [8] gives

$$\int_0^\infty \frac{dx}{e^{px}+1} = \frac{\ln 2}{p}.$$

Use rule P_2 for the integrand and then power series for the exponential function gives

$$\sum_{n_{123}} (-1)^{n_1+n_2+n_3} \frac{(-n_1)^{n_3} p^{n_3}}{\Gamma(n_1+1)\Gamma(n_2+1)\Gamma(n_3+1)} \langle n_1+n_2+1\rangle\langle n_3+1\rangle.$$

First evaluate the bracket $\langle n_3+1\rangle$ to get

$$-\frac{1}{p}\sum_{n_{12}}(-1)^{n_1+n_2}\frac{1}{n_1\Gamma(n_1+1)\Gamma(n_2+1)}\langle n_1+n_2+1\rangle.$$

Then evaluate this bracket with respect to n_1 to obtain

$$-\frac{1}{p}\sum_{n_2}(-1)^{n_2}\frac{1}{(-n_2-1)\Gamma(n_2+1)}\Gamma(n_2+1)=\frac{1}{p}\sum_{n_2=0}^{\infty}\frac{(-1)^{n_2}}{n_2+1}=\frac{\ln 2}{p}.$$

If we evaluate the last bracket with respect to n_2, there is a new difficulty: the resulting series contains the term $1/n_1$ and n_1 may be have the value 0. A possible correction to this problem is presented next.

To justify the evaluation of the method in this example we need to expand our bracket class.

Lemma 4.1.1. *Assume $f(z)$ has bracket form $\mathcal{B}$. Then, for any $\alpha>0$, the function $(-\alpha)^z f(z)$ belongs to the same bracket class.*

Proof. The proof proceed as in the proof of Lemma 2.2.4. We just need to confirm that the extra term $(-\alpha)^z$ does not jeopardize the vanishing of the function on the contour $\mathcal{C}_T(c)$. The magnitude of $(-\alpha)^z$ is $\alpha^{\Re(z)}\exp(-\pi\Im(z))$.

1. *Vertical segment:* the extra term becomes, for this segment, $(-\alpha)^{T-iu}$ which is of exponential order of T. This cannot prevent the vanishing of this segment because the integrand still has order $T^{-\epsilon T}$.

2. *Horizontal segment:* the extra term in this case contributes $(-\alpha)^{u+iT}$, which has the magnitude $\alpha^u e^{-\pi T}$. Thus, for small u, this term actually helps the decay of the integrand. For large u, this term is of exponential order in u so it is weaker than the decay of $\Gamma(au+b)$.

□

Now back to the example. The lemma helps justify the evaluation of the bracket $\langle n_3+1\rangle$, because if a function belongs to the bracket class, we can use Ramanujan Master Theorem. Notice that the evaluation gives the sum $\sum_{n_1}$ that starts at $n_1=1$, not at zero, because the integrand vanishes when $n_1=0$. Therefore we should let the sum $\sum_{n_1}$ start at one before evaluating the bracket to avoid the divergent issue. In short, the correct evaluation should give

$$-\frac{1}{p}\sum_{n_{12}}(-1)^{n_1+n_2}\frac{1}{n_1\Gamma(n_1+1)\Gamma(n_2+1)}\langle n_1+n_2+1\rangle,$$

where $n_1\geq 1$. To set up for the next evaluation we change the sum of n_1 to start at zero.

$$\frac{1}{p}\sum_{n_{12}}(-1)^{n_1+n_2}\frac{1}{(n_1+1)\Gamma(n_1+2)\Gamma(n_2+1)}\langle n_1+n_2+2\rangle.$$

Here n_1 and n_2 have the same weight so we introduce a parameter to change one of their weights. This is just an artificial mechanism used to avoid the same-weight situation and to justify the evaluation. The answer at the end is exactly like what the method of brackets gives. Now if we evaluate with respect to n_2 we get

$$\frac{1}{p}\sum_{n_1}(-1)^{n_1}\frac{1}{(n_1+1)\Gamma(n_1+2)}\Gamma(n_1+2)=\frac{\ln 2}{p}.$$

If we evaluate with respect to n_1 we get

$$\frac{1}{p}\sum_{n_2}(-1)^{n_2}\frac{\Gamma(-n_2-1)}{(-n_2-1)\Gamma(-n_2)\Gamma(n_2+1)}\Gamma(n_2+2)=\frac{1}{p}\sum_{n_2}\frac{(-1)^{n_2}}{n_2+1}=\frac{\ln 2}{p}.$$

Though the summand in this case is not separable because of the term $n_1^{n_3}$, this does not pose a problem in using our results because n_3 is evaluated first from the bracket $\langle n_3+1\rangle$. Afterwards, the summand is separable in terms of n_1 and n_2. The following example requires further results to justify

Example 4.1.2. Evaluate

$$I(a,b) = \int_0^\infty \int_0^\infty \exp\left(-\frac{x^a y^a}{(x+y)^a}\right) \exp\left(-(x+y)^b\right) dxdy.$$

The first exponential term has the power series

$$\exp\left(-\frac{x^a y^a}{(x+y)^a}\right) = \sum_{n_1} \phi_{n_1} x^{an_1} y^{an_1} (x+y)^{-an_1},$$

and the second exponential gives

$$\exp\left(-(x+y)^b\right) = \sum_{n_2} \phi_{n_2} (x+y)^{bn_2}.$$

Combine the two then rule P_2 says

$$(x+y)^{bn_2 - an_1} = \frac{1}{\Gamma(an_1 - bn_2)} \sum_{n_{34}} \phi_{n_{34}} x^{n_3} y^{n_4} \langle n_3 + n_4 + an_1 - bn_2 \rangle.$$

Thus we have the bracket series

$$\sum_{n_{1234}} \phi_{n_{1234}} \frac{1}{\Gamma(an_1 - bn_2)} \langle an_1 + n_3 + 1 \rangle \langle an_1 + n_4 + 1 \rangle \langle an_1 + n_3 + n_4 - bn_2 \rangle.$$

The summand is non-separable because of $\Gamma(an_1 - bn_2)$ and unlike the previous example, this poses an issue because n_1 and n_2 appear in a same bracket. This bracket series has index one so we will let each n_i free:

- *If* n_1 *free*: MoB gives

$$\frac{1}{b} \sum_{n_1} \frac{(-1)^{n_1} \Gamma(an_1 + 1)^2 \Gamma\left(\frac{an_1 + 2}{b}\right)}{\Gamma(n_1 + 1) \Gamma(2an_1 + 2)},$$

  which requires $a < b$ to converge.

- *If n_2 free*: we have

$$\frac{1}{a}\sum_{n_2}\frac{(-1)^{n_2}\Gamma(-bn_2-1)^2\Gamma\left(\frac{bn_2+2}{a}\right)}{\Gamma(n_2+1)\Gamma(-2bn_2-2)},$$

  which diverges at $n_2 = 0$.

- *If n_3 free*: we get

$$\frac{1}{ab}\sum_{n_3}\frac{(-1)^{n_3}\Gamma(-n_3)\Gamma\left(\frac{1-n_3}{b}\right)\Gamma\left(\frac{n_3+1}{a}\right)}{\Gamma(-2n_3)\Gamma(n_3+1)},$$

  where the poles from $\Gamma(-n_3)$ and $\Gamma(-2n_3)$ might cancel each other but when $n_3 = 1$ we have another pole from $\Gamma\left(\frac{1-n_3}{b}\right)$ so this sum diverges.

- *If n_4 free*: we get

$$\frac{1}{ab}\sum_{n_4}\frac{(-1)^{n_4}\Gamma(-n_4)\Gamma\left(\frac{1-n_4}{b}\right)\Gamma\left(\frac{n_4+1}{a}\right)}{\Gamma(-2n_4)\Gamma(n_4+1)},$$

  where we have the same issue as the previous case so this case is also discarded.

Thus we have

$$I(a,b) = \frac{1}{b}\sum_{n}\frac{(-1)^{n}\Gamma(an+1)^2\Gamma\left(\frac{an+2}{b}\right)}{\Gamma(n+1)\Gamma(2an+2)}.$$

For $a = b = 1$ *Mathematica* does not give a closed form for the integral, but it does give a closed form for the sum

$$I(1,1) = \sum_{n}\frac{(-1)^{n}\Gamma(n+1)^2\Gamma(n+2)}{\Gamma(n+1)\Gamma(2n+2)} = \frac{1}{25}\left(-15+8\sqrt{5}\,\text{arcsinh}\left[\frac{1}{2}\right]\right).$$

Numerical evidence confirms

$$\int_0^\infty\int_0^\infty \exp\left(-\frac{xy}{x+y}\right)e^{-x-y}dxdy = \frac{1}{25}\left(-15+8\sqrt{5}\,\text{arcsinh}\left[\frac{1}{2}\right]\right).$$

For $a = 1, b = 2$ *Mathematica* again fails to yield a closed form answer for the integral, but the sum gives

$$\int_0^\infty \int_0^\infty \exp\left(-\frac{xy}{x+y}\right) e^{-(x+y)^2} dxdy \;=\; \frac{1}{2} {}_2F_2\left(\begin{matrix} 1 & 1 \\ \frac{3}{4} & \frac{5}{4} \end{matrix} \middle| \frac{1}{64}\right) - \frac{\sqrt{\pi}}{24} {}_2F_2\left(\begin{matrix} 1 & \frac{3}{2} \\ \frac{5}{4} & \frac{7}{4} \end{matrix} \middle| \frac{1}{64}\right)$$

This example shows the advantage of evaluating a sum versus integrating a function.

4.2 Integrals with exponential integral function

Example 4.2.1. Entry 6.228.2 of [8] gives

$$\int_0^\infty x^{\nu-1} e^{-\mu x} \operatorname{Ei}(-\beta x) dx = -\frac{\Gamma(\nu)}{\nu(\beta+\mu)^\nu} {}_2F_1\left(\begin{matrix} 1 & \nu \\ \nu+1 & \end{matrix} \middle| \frac{\mu}{\beta+\mu}\right),$$

where $\operatorname{Ei}(-x)$ is the exponential integral function, which has the series representation

$$\operatorname{Ei}(-x) = \gamma + \ln x + \sum_{n_1=1}^{\infty} \phi_{n_1} \frac{x^{n_1}}{n_1}.$$

To use the method of brackets, we use the series $\displaystyle\sum_{n_1=0}^{\infty} \phi_{n_1} \frac{x^{n_1}}{n_1}$, though it is undefined at $n_1 = 0$. To avoid divergent issues, we should evaluate brackets with respect to n_1. Combine this with power series for the exponential term yields

$$\sum_{n_{12}} \phi_{n_{12}} \frac{\beta^{n_1} \mu^{n_2}}{n_1} \langle n_1 + n_2 + \nu \rangle.$$

Both n_1 and n_2 have the same weight so as stated above, we should evaluate this bracket with respect to n_1 to avoid divergent issue. Indeed, the evaluation gives

$$\sum_{n_2} \frac{(-1)^{n_2}}{\Gamma(n_2+1)} \frac{\beta^{-n_2-\nu} \mu^{n_2}}{(-n_2-\nu)} \Gamma(n_2+\nu)$$

which has the closed form

$$-\frac{\Gamma(\nu)}{\nu\beta^{\nu}}{}_2F_1\left(\begin{matrix}\nu & \nu\\ & 1+\nu\end{matrix}\middle|-\frac{\mu}{\beta}\right)$$

Then we can apply Pfaff transformation, formula 9.131.1 in [8],

$${}_2F_1\left(\begin{matrix}a & b\\ & c\end{matrix}\middle|z\right) = (1-z)^{-a}\,{}_2F_1\left(\begin{matrix}a & c-b\\ & c\end{matrix}\middle|\frac{z}{z-1}\right)$$

to obtain the stated answer. If we evaluate the bracket with respect to n_2, to avoid the divergent issue, we need to separate the series of exponential integral into two parts, $\gamma + \ln(\beta x)$ and the sum (where n_1 starts at 1 instead of 0). First we evaluate

$$\int_0^\infty (\gamma + \ln(\beta x))\, x^{\nu-1}e^{-\mu x}dx = \mu^{-\nu}\Gamma(\nu)(\gamma + \ln\beta - \ln\mu + \psi(\nu))$$

where ψ is the digamma function. Now for the sum part, the evaluation with respect to n_2 gives

$$\sum_{n_1=1}^{\infty}\frac{(-1)^{n_1}}{\Gamma(n_1+1)}\frac{\beta^{n_1}\mu^{-n_1-\nu}}{n_1}\Gamma(n_1+\nu) = -\frac{\beta}{\mu^{1+\nu}}\Gamma(\nu+1){}_3F_2\left(\begin{matrix}1 & 1 & 1+\nu\\ & 2 & 2\end{matrix}\middle|-\frac{\beta}{\mu}\right).$$

Then add the two results. This produces the identity

$$\begin{aligned}&\mu^{-\nu}\Gamma(\nu)(\gamma + \ln\beta - \ln\mu + \psi(\nu)) - \frac{\beta}{\mu^{1+\nu}}\Gamma(\nu+1){}_3F_2\left(\begin{matrix}1 & 1 & 1+\nu\\ & 2 & 2\end{matrix}\middle|-\frac{\beta}{\mu}\right)\\ = \quad & -\frac{\Gamma(\nu)}{\nu(\beta+\mu)^{\nu}}{}_2F_1\left(\begin{matrix}1 & \nu\\ & \nu+1\end{matrix}\middle|\frac{\mu}{\beta+\mu}\right)\end{aligned}$$

This identity is now expressed as a conjecture.

`Conjecture`. The identity

$${}_3F_2\left(\begin{matrix}1 & 1 & 1+\nu\\ & 2 & 2\end{matrix}\middle|-a\right) = \frac{\gamma + \ln a + \psi(\nu)}{\nu a} + \frac{1}{a\nu^2(1+a)^{\nu}}\,{}_2F_1\left(\begin{matrix}1 & \nu\\ & 1+\nu\end{matrix}\middle|\frac{1}{1+a}\right),$$

holds.

Example 4.2.2. Entry 6.234 in [8] is

$$\int_0^\infty \mathrm{Ei}(-x)\ln x\, dx = 1+\gamma,$$

where γ is the Euler gamma constant. Here we adopt the method we use to deal with logarithmic function. First we replace $\ln x$ by x^ϵ and then differentiate at $\epsilon = 0$. Thus we evaluate instead

$$\int_0^\infty x^\epsilon\, \mathrm{Ei}(-x)\, dx.$$

Use the series representation for exponential integral discussed earlier we get

$$\sum_n \phi_n \frac{1}{n}\langle n+\epsilon+1\rangle.$$

Evaluation of the bracket gives $-\frac{\Gamma(\epsilon+1)}{\epsilon+1}$. Then we differentiate this with respect to ϵ and let $\epsilon = 0$ to get $1+\gamma$ as expected.

Example 4.2.3. Entry 6.233.1 of [8] gives

$$\int_0^\infty \mathrm{Ei}(-x)e^{-\mu x}\sin\beta x\, dx = -\frac{1}{\beta^2+\mu^2}\left\{\frac{\beta}{2}\ln\left[(1+\mu)^2+\beta^2\right] - \mu\arctan\frac{\beta}{1+\mu}\right\}.$$

Start with the bracket series

$$\sum_{n_{123}} \phi_{n_{123}} \frac{\mu^{n_2}\beta^{2n_3+1}\Gamma(n_3+1)}{n_1\Gamma(2n_3+2)}\langle n_1+n_2+2n_3+2\rangle,$$

which is of index two. All the parameters n_i have the same weight in the bracket so whichever n we decide to evaluate the bracket with respect to, the outcome sum should require some conditions to converge. However, with n_1 in the denominator, evaluating with respect to n_2 and n_3 will result in divergent series. Evaluating with respect to n_1 gives us

$$-\sum_{n_{23}} \frac{(-1)^{n_2+n_3}\mu^{n_2}\beta^{2n_3+1}\Gamma(n_2+2n_3+2)}{(n_2+2n_3+2)\Gamma(n_2+1)\Gamma(2n_3+2)}.$$

To find the closed form of this we look at two cases, n_2 even and odd. The even case gives

$$-\frac{1}{2(\beta^2+\mu^2)}\left[\beta\ln r_1+\beta\ln r_2-\mu\theta_1+\mu\theta_2\right].$$

assuming

$$1+\mu+i\beta = r_1 e^{i\theta_1}, \qquad 1-\mu+i\beta = r_2 e^{i\theta_2}.$$

The odd case gives

$$\frac{1}{2(\beta^2+\mu^2)}\left[-\beta\ln r_1+\beta\ln r_2+\mu\theta_1+\mu\theta_2\right].$$

Adding the two results gives

$$-\frac{1}{\beta^2+\mu^2}\left[\beta\ln r_1-\mu\theta_1\right]$$

which is the desired answer since

$$r_1=\sqrt{(1+\mu)^2+\beta^2}, \qquad \theta_1=\arctan\frac{\beta}{1+\mu}.$$

If we evaluate the bracket with respect to other n's, we need to let the sum of n_1 to start at one, not zero. Evaluating with respect to n_2 gives

$$\sum_{n_1=1}^{\infty}\sum_{n_3=0}^{\infty}(-1)^{n_1+n_3}\frac{\mu^{-n_1-2n_3-2}\beta^{2n_3+1}\Gamma(n_1+2n_3+2)}{n_1\Gamma(n_1+1)\Gamma(2n_3+2)}$$

which has the closed form

$$-\frac{1}{\beta^2+\mu^2}\left[\beta\ln r_1-\beta\ln r_2+\mu\theta_1-\mu\theta_2\right].$$

assuming

$$\beta+(1+\mu)i=r_1e^{i\theta_1}, \qquad \beta+i\mu=r_2e^{i\theta_2}$$

But, as discussed in other examples, adding the two results gives

$$\int_0^\infty(\gamma+\ln x)e^{-\mu x}\sin\beta x\,dx=-\frac{1}{\beta^2+\mu^2}\left[\beta\ln r_2+\mu\theta_2\right].$$

Thus we also get the stated answer. At the end, even though some parts have not been justified yet, the method of brackets produces the correct evaluation.

Example 4.2.4. Entry 6.222 in [8] gives

$$\int_0^\infty \operatorname{Ei}(-px)\operatorname{Ei}(-qx)dx=\left(\frac{1}{p}+\frac{1}{q}\right)\ln(p+q)-\frac{\ln q}{p}-\frac{\ln p}{q}.$$

We have the series bracket

$$\sum_{n_{12}}\phi_{n_{12}}\frac{p^{n_1}q^{n_2}}{n_1n_2}\langle n_1+n_2+1\rangle.$$

In this case, evaluating with respect to either n results in a divergent series. Thus we need to let one of the sum start at one. Evaluating with respect to n_2 gives

$$-\frac{1}{q}\sum_{n_1=1}^\infty\frac{(-1)^{n_1}}{n_1(n_1+1)}\left(\frac{p}{q}\right)^{n_1}=\left(\frac{1}{p}+\frac{1}{q}\right)\ln(p+q)-\left(\frac{1}{p}+\frac{1}{q}\right)\ln q-\frac{1}{q}.$$

Then we have to add

$$\int_0^\infty(\gamma+\ln px)\operatorname{Ei}(-qx)dx=\frac{1-\ln p+\ln q}{q},$$

which gives us the desired answer.

Example 4.2.5. Entry 6.225.1 in [8] states that

$$\int_0^\infty \mathrm{Ei}(-x^2)e^{-\mu x^2}dx = -\sqrt{\frac{\pi}{\mu}}\arcsin\sqrt{\mu} = -\sqrt{\frac{\pi}{\mu}}\ln\left(\sqrt{\mu}+\sqrt{1+\mu}\right).$$

Standard procedure gives the following bracket series

$$\sum_{n_{12}}\phi_{n_{12}}\frac{\mu^{n_2}}{n_1}\langle 2n_1+2n_2+1\rangle.$$

Both n's have same weight so we could evaluate with respect to either. With respect to n_1 we get

$$-\frac{1}{2}\sum_{n_2}\frac{(-1)^{n_2}\mu^{n_2}}{\left(n_2+\frac{1}{2}\right)\Gamma(n_2+1)}\Gamma\left(n_2+\frac{1}{2}\right) = -\sqrt{\frac{\pi}{\mu}}\arcsin\sqrt{\mu}.$$

With respect to n_2 we get a divergent series so we need to let the sum of n_1 to start at one. The evaluation is

$$\frac{1}{2}\sum_{n_1=1}^{\infty}\frac{(-1)^{n_1}\mu^{-n_1-\frac{1}{2}}}{n_1\Gamma(n_1+1)}\Gamma\left(n_1+\frac{1}{2}\right) = -\sqrt{\frac{\pi}{\mu}}\ln\left(\sqrt{\mu}+\sqrt{1+\mu}\right)+\sqrt{\frac{\pi}{\mu}}\ln(2\sqrt{\mu}).$$

Then we need to add this with

$$\int_0^\infty \left(\gamma+\ln x^2\right)e^{-\mu x^2}dx = -\sqrt{\frac{\pi}{\mu}}\ln\left(2\sqrt{\mu}\right)$$

which gives the answer.

4.3 Exponential integral function and an identity

First we evaluate the integral

$$\int_0^\infty \mathrm{Ei}\left(-ax^n\right) e^{-x^n} dx.$$

Start with the bracket series

$$\sum_{n_{12}} \phi_{n_{12}} \frac{a^{n_1}}{n_1} \langle nn_1 + nn_2 + 1 \rangle.$$

Evaluating with respect to n_1 gives

$$-\frac{1}{n} \sum_{n_2} \frac{(-1)^{n_2} a^{-n_2-\frac{1}{n}}}{\left(n_2+\frac{1}{n}\right)\Gamma(n_2+1)} \Gamma\left(n_2+\frac{1}{n}\right) = -a^{-\frac{1}{n}} \Gamma\left(\frac{1}{n}\right) {}_2F_1\left(\left.\begin{matrix} \frac{1}{n} & \frac{1}{n} \\ & 1+\frac{1}{n} \end{matrix}\right| -\frac{1}{a}\right).$$

Now we try to evaluate with respect to n_2. Of course, first we should let the sum of n_1 start at one otherwise we get a divergent sum because of the term $1/n_1$. We get the evaluation

$$\frac{1}{n} \sum_{n_1=1}^{\infty} \frac{(-1)^{n_1} a^{n_1}}{n_1 \Gamma(n_1+1)} \Gamma\left(n_1+\frac{1}{n}\right) = -\frac{a}{n} \cdot \Gamma\left(1+\frac{1}{n}\right) {}_3F_2\left(\left.\begin{matrix} 1 & 1 & 1+\frac{1}{n} \\ & 2 & 2 \end{matrix}\right| -a\right).$$

Then we need to add the integral

$$\int_0^\infty (\gamma + \ln(ax^n)) e^{-x^n} \ dx = \Gamma\left(1+\frac{1}{n}\right)\left(\gamma + \ln a + \psi\left(\frac{1}{n}\right)\right).$$

Indeed, the sum of these values give the same answer as the case n_2 free, i.e.

$$-\frac{\Gamma\left(\frac{1}{n}\right)}{a^{\frac{1}{n}}} {}_2F_1\left(\left.\begin{matrix} \frac{1}{n} & \frac{1}{n} \\ & 1+\frac{1}{n} \end{matrix}\right| -\frac{1}{a}\right) =$$
$$\Gamma\left(1+\tfrac{1}{n}\right)\left(\gamma + \ln a + \psi\left(\tfrac{1}{n}\right)\right) - \tfrac{a}{n} \cdot \Gamma\left(1+\tfrac{1}{n}\right) {}_3F_2\left(\left.\begin{matrix} 1 & 1 & 1+\frac{1}{n} \\ & 2 & 2 \end{matrix}\right| -a\right).$$

This gives us the identity

$$\ln a = \frac{a}{n}{}_3F_2\left(\begin{matrix}1 & 1 & 1+\frac{1}{n} \\ & 2 & 2\end{matrix}\middle| -a\right) - \frac{n}{a^{1/n}}{}_2F_1\left(\begin{matrix}\frac{1}{n} & \frac{1}{n} \\ 1+\frac{1}{n} & \end{matrix}\middle| -\frac{1}{a}\right). \tag{4.1}$$

This identity actually comes naturally as a limiting case of the following identity

$$\begin{aligned} F(\alpha,\beta,\mu,z) &= \frac{\Gamma(\mu)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\mu-\alpha)}(-z)^{-\alpha}F(\alpha,\alpha+1-\mu,\alpha+1-\beta,z^{-1}) \\ &+ \frac{\Gamma(\mu)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\mu-\beta)}(-z)^{-\beta}F(\beta,\beta+1-\mu,\beta+1-\alpha,z^{-1}). \end{aligned}$$

Here and from now on, all hypergeometric functions are ${}_2F_1$ unless specified otherwise. First we let $\beta \to \alpha$. Let $\beta = \alpha + \epsilon$, then two terms of the right hand side become

$$\frac{1}{\epsilon}\cdot\left(\frac{\Gamma(1+\epsilon)}{\Gamma(\alpha+\epsilon)\Gamma(\mu-\alpha)}\right)\cdot(-z)^{-\alpha}\cdot F(\alpha,\alpha+1-\mu,1-\epsilon,z^{-1})$$

and

$$-\frac{1}{\epsilon}\cdot\left(\frac{\Gamma(1-\epsilon)}{\Gamma(\alpha)\Gamma(\mu-\alpha-\epsilon)}\right)\cdot(-z)^{-\alpha-\epsilon}\cdot F(\alpha+\epsilon,\alpha+1-\mu+\epsilon,1+\epsilon,z^{-1}).$$

We need to evaluate the limit of the difference as $\epsilon \to 0$. Notice the difference has the form

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}(a_1b_1c_1 - a_2b_2c_2),$$

where a's denote the gamma terms, b's denote the power of z, and c's denote the hypergeometric terms. We rewrite the difference as

$$\frac{(a_1-a_2)b_1c_1}{\epsilon} + \frac{(b_1-b_2)a_2c_1}{\epsilon} + \frac{(c_1-c_2)a_2b_2}{\epsilon}$$

and evaluate the limit of each term. Using L'Hospital we have

$$\begin{aligned}
\lim_{\epsilon\to 0}\frac{a_1-a_2}{\epsilon} &= \lim_{\epsilon\to 0}\frac{\frac{\Gamma(1+\epsilon)}{\Gamma(\alpha+\epsilon)\Gamma(\mu-\alpha)}-\frac{\Gamma(1-\epsilon)}{\Gamma(\alpha)\Gamma(\mu-\alpha-\epsilon)}}{\epsilon} \\
&= \lim_{\epsilon\to 0}\frac{\Gamma'(1+\epsilon)\Gamma(\alpha+\epsilon)-\Gamma(1+\epsilon)\Gamma'(\alpha+\epsilon)}{\Gamma(\mu-\alpha)\Gamma^2(\alpha+\epsilon)} \\
&\quad -\frac{-\Gamma'(1-\epsilon)\Gamma(\mu-\alpha-\epsilon)+\Gamma(1-\epsilon)\Gamma'(\mu-\alpha-\epsilon)}{\Gamma(\alpha)\Gamma^2(\mu-\alpha-\epsilon)} \\
&= \frac{\Gamma'(1)\Gamma(\alpha)-\Gamma'(\alpha)}{\Gamma(\mu-\alpha)\Gamma^2(\alpha)}-\frac{-\Gamma'(1)\Gamma(\mu-\alpha)+\Gamma'(\mu-\alpha)}{\Gamma(\alpha)\Gamma^2(\mu-\alpha)} \\
&= -\frac{2\gamma}{\Gamma(\alpha)\Gamma(\mu-\alpha)}-\frac{\psi(\alpha)+\psi(\mu-\alpha)}{\Gamma(\alpha)\Gamma(\mu-\alpha)}.
\end{aligned}$$

Similarly

$$\lim_{\epsilon\to 0}\frac{(-z)^{-\alpha}-(-z)^{-\alpha-\epsilon}}{\epsilon}=\lim_{\epsilon\to 0}-(-z)^{\alpha-\epsilon}\ln(-z)(-1)=\ln(-z)(-z)^{-\alpha}.$$

Lastly, to compute the limit of the difference of the two hypergeometric series, we rewrite it as

$$\begin{aligned}
& F(\alpha,\alpha+1-\mu,1-\epsilon)-F(\alpha,\alpha+1-\mu,1+\epsilon) \\
+ \;& F(\alpha,\alpha+1-\mu,1+\epsilon)-F(\alpha+\epsilon,\alpha+1-\mu,1+\epsilon) \\
+ \;& F(\alpha+\epsilon,\alpha+1-\mu,1+\epsilon)-F(\alpha+\epsilon,\alpha+1-\mu+\epsilon,1+\epsilon)
\end{aligned}$$

(we dropped the z^{-1} in these notation for simplicity). The limit of these differences is

$$-2F^c(\alpha,\alpha+1-\mu,1)-F^a(\alpha,\alpha+1-\mu,1)-F^b(\alpha,\alpha+1-\mu,1),$$

where the sup-script a, b, and c denote the partial derivative of the hypergeometric series (in the standard notation)

$${}_2F_1(a,b,c,z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}z^k,$$

with respect to the corresponding parameter. Thus, as $\beta \to \alpha$, yields the identity

$$\begin{aligned} F(\alpha,\alpha,\mu,z) &= \frac{(-z)^{-\alpha}\Gamma(\mu)}{\Gamma(\alpha)\Gamma(\mu-\alpha)}\left(-2\gamma-\psi(\alpha)-\psi(\mu-\alpha)+\ln(-z)\right)F \\ &\quad -\frac{(-z)^{-\alpha}\Gamma(\mu)}{\Gamma(\alpha)\Gamma(\mu-\alpha)}(2F^c+F^a+F^b), \end{aligned}$$

where all the hypergeometric series on the right hand side are evaluated at $(\alpha, \alpha + 1 - \mu, 1, z^{-1})$.

Next we will find the limit of the right hand side when $\mu \to \alpha + 1$. We have

$$F(\alpha,\alpha,1+\alpha,z) = \alpha(-z)^{-\alpha}\left[-\gamma-\psi(\alpha)+\ln(-z)-\lim_{\mu\to\alpha+1}(2F^c+F^a+F^b)\right].$$

For $k \geq 1$ we have

$$\begin{aligned} \frac{d}{da}(a)_k &= \frac{d}{da}\left(\frac{\Gamma(a+k)}{\Gamma(a)}\right) = \frac{\Gamma'(a+k)\Gamma(a)-\Gamma(a+k)\Gamma'(a)}{\Gamma^2(a)} \\ &= (a)_k\psi(a+k)-(a)_k\psi(a) = (a)_kH_{a,k}, \end{aligned}$$

where $H_{a,k} = \sum_{i=0}^{k-1}\frac{1}{a+i}$. Similarly $\frac{d}{db}(b)_k = (b)_kH_{b,k}$, and $\frac{d}{dc}\frac{1}{(c)_k} = -\frac{1}{(c)_k}H_{c,k}$. Thus

$$2F^c+F^a+F^b = \sum_{k=1}^{\infty}\frac{(a)_k(b)_k}{(c)_kk!}(H_{a,k}+H_{b,k}-2H_{c,k})z^{-k},$$

where, in our context, $a = \alpha$, $b = \alpha + 1 - \mu$, and $c = 1$. In our limit, $b \to 0$. Clearly, the series corresponding to $H_{a,k}$ and $H_{c,k}$ vanish, while

$$(b)_kH_{b,k} = b(b+1)\ldots(b+k-1)\left(\frac{1}{b}+\frac{1}{b+1}+\ldots+\frac{1}{b+k-1}\right),$$

which goes to $(k-1)!$ as $b \to 0$. Therefore,

$$2F^c+F^a+F^b \to \sum_{k=1}^{\infty}\frac{(\alpha)_k(k-1)!}{(1)_kk!}z^{-k} = \frac{\alpha}{z}\,{}_3F_2\left(\begin{matrix}1 & 1 & 1+\alpha \\ & 2 & 2\end{matrix}\,\middle|\,z^{-1}\right).$$

This produces the identity

$$F(\alpha,\alpha,1+\alpha,z)=\alpha(-z)^{-\alpha}\left[\ln(-z)-\gamma-\psi(\alpha)-\frac{\alpha}{z}\ {}_3F_2(1,1,1+\alpha;2,2;z^{-1})\right].$$

Now let $\alpha = 1/n$ and $z=-1/a$ we get the next result:

$$\ln a=\frac{a}{n}\ {}_3F_2-\frac{n}{a^{1/n}}\ {}_2F_1-\gamma-\psi\left(\frac{1}{n}\right).$$

Theorem 4.3.1. *For any positive a and n we have*

$$\begin{aligned}\ln a &= \frac{a}{n}{}_3F_2\left(\left.\begin{matrix}1 & 1 & 1+\frac{1}{n}\\ & 2 & 2\end{matrix}\right|-a\right)\\ &\quad -\frac{n}{a^{1/n}}{}_2F_1\left(\left.\begin{matrix}\frac{1}{n} & \frac{1}{n}\\ 1+\frac{1}{n}\end{matrix}\right|-a^{-1}\right)-\gamma-\psi\left(\frac{1}{n}\right).\end{aligned} \tag{4.2}$$

4.4 Integrals with logarithmic functions

Example 4.4.1. Entry 4.254.5 of [8] is

$$\int_0^\infty \frac{x^{p-1}\ln x}{1+x^q}dx=-\frac{\pi^2}{q^2}\frac{\cos\frac{p\pi}{q}}{\sin^2\frac{p\pi}{q}}.$$

As before, to deal with logarithm we replace $\ln x$ with x^ϵ. Then at the end we will differentiate the answer with respect to ϵ and let $\epsilon=0$. Rule P_2 for the denominator gives us

$$\sum_{n_{12}}\phi_{n_{12}}\langle n_1+n_2+1\rangle\langle qn_2+p+\epsilon\rangle.$$

This is a zero index case. The evaluation of this bracket is

$$\frac{1}{q}\Gamma\left(\frac{p+\epsilon}{q}\right)\Gamma\left(1-\frac{p+\epsilon}{q}\right)=\frac{\pi}{q}\frac{1}{\sin\frac{\pi(p+\epsilon)}{q}}.$$

Take the derivative of this and let $\epsilon=0$ we get the desired answer.

Example 4.4.2. We now evaluate

$$\int_0^\infty (\ln x)^2 \operatorname{Ei}(-ax^n)dx.$$

This begins with the evaluation of

$$\int_0^\infty x^{\epsilon_1+\epsilon_2} \operatorname{Ei}(-ax^n)dx.$$

Using the series we have been using for exponential integral function gives

$$\sum_{n_1} \phi_{n_1} \frac{a^{n_1}}{n_1} \langle nn_1 + \epsilon_1 + \epsilon_2 + 1 \rangle$$

which has the value

$$-\frac{\Gamma\left(\frac{\epsilon_1+\epsilon_2+1}{n}\right)}{(\epsilon_1+\epsilon_2+1)\sqrt[n]{a^{\epsilon_1+\epsilon_2+1}}}.$$

Now differentiate this with respect to ϵ_1 and then let $\epsilon_1 = 0$. Then we do the same for ϵ_2 and get

$$\int_0^\infty \ln^2 x \operatorname{Ei}(-ax^n)dx = -\frac{\Gamma\left(\frac{1}{n}\right)}{n^2\sqrt[n]{a}}\left[n^2 + \psi_1\left(\frac{1}{n}\right) + \left(n + \ln a - \psi\left(\frac{1}{n}\right)\right)^2\right]$$

where ψ and ψ_1 are the polygamma functions.

It would be an interesting problem to use generalize this result using the previous method to obtain an analytic expression for

$$\int_0^\infty \ln^m(x) \operatorname{Ei}(-ax^n)dx = f^{(m)}(0)$$

where

$$f(x) = -\frac{\Gamma\left(\frac{x+1}{n}\right)}{(x+1)\sqrt[n]{a^{x+1}}}.$$

Example 4.4.3. Entry 4.222.1 in [8] gives

$$\int_0^\infty \ln \frac{a^2+x^2}{b^2+x^2} dx = (a-b)\pi.$$

Again we replace $\ln u$ by u^ϵ so instead we will evaluate

$$\int_0^\infty \left(\frac{a^2+x^2}{b^2+x^2} \right)^\epsilon dx.$$

The replacement of logarithm now causes the integral to diverge. Fortunately, the method of bracket does not detect this issue so it still yields a finite evaluation, whose derivative at $\epsilon = 0$ gives the right answer! Applying rule P_2 to the numerator and denominator we get the series

$$\frac{1}{\Gamma(-\epsilon)\Gamma(\epsilon)} \sum_{n_{1234}} \phi_{n_{1234}} a^{2n_1} b^{2n_3} \langle n_1 + n_2 - \epsilon \rangle \langle n_3 + n_4 + \epsilon \rangle \langle 2n_2 + 2n_4 + 1 \rangle.$$

This is an index one series of brackets. The case n_1 free gives the evaluation

$$\frac{b\epsilon \sin \pi\epsilon}{2\sqrt{\pi}} \Gamma(-\epsilon) \Gamma\left(\epsilon + \frac{1}{2}\right) {}_2F_1\left(-\frac{1}{2}, -\epsilon, \frac{1}{2} - \epsilon, \frac{a^2}{b^2}\right)$$

where we need $a < b$ for convergence. The derivative of this at $\epsilon = 0$ gives $-b\pi$. The case n_4 free gives

$$-\frac{a^{2\epsilon+1} \sin \pi\epsilon}{2\sqrt{\pi} b^{2\epsilon}} \Gamma(\epsilon+1) \Gamma\left(-\epsilon - \frac{1}{2}\right) 2F_1\left(\frac{1}{2}, \epsilon, \epsilon + \frac{3}{2}, \frac{a^2}{b^2}\right)$$

where we need $a < b$ for convergence. The derivative of this at $\epsilon = 0$ is $a\pi$. Since these two cases require the same condition for convergence, we add the two results and get $(a-b)\pi$ as desired. The cases where n_2 and n_3 free both require $a > b$ to converge. Adding their results also gives $(a-b)\pi$.

Example 4.4.4. Entry 4.267.18 in [8] gives

$$\int_0^\infty \frac{x^{p-1}-x^{q-1}}{(1+x^r)\ln x}dx = \ln\left(\tan\frac{p\pi}{2r}\cot\frac{q\pi}{2r}\right).$$

We will replace $1/\ln x$ by x^ϵ. Then we evaluate the integral to get a function of ϵ. Finally we integrate the result with respect to ϵ and let $\epsilon = 0$. We separate the integral into two parts

$$\int_0^\infty \frac{x^{p-1}}{(1+x^r)\ln x}dx \to \int_0^\infty \frac{x^{p+\epsilon-1}}{1+x^r}dx.$$

We get the following bracket series

$$\sum_{n_{12}} \phi_{n_{12}} \langle n_1+n_2+1\rangle\langle n_2 r+p+\epsilon\rangle,$$

which gives the value

$$\frac{\pi}{r}\csc\left(\frac{\pi(\epsilon+p)}{r}\right).$$

Integrate with respect to ϵ and let $\epsilon = 0$ we get

$$\ln\left(\tan\frac{p\pi}{2r}\right).$$

The same procedure works for the second part. Subtracting the two we get the desired formula.

Example 4.4.5. Entry 4.267.19 in [8] gives

$$\int_0^\infty \frac{x^{p-1}-x^{q-1}}{(1-x^r)\ln x}dx = \ln\left(\frac{\sin\frac{p\pi}{r}}{\sin\frac{q\pi}{r}}\right).$$

We now replace $1/\ln x$ by x^ϵ and evaluate the integral to get a function of ϵ. Then we integrate the result with respect to ϵ and let $\epsilon = 0$. We separate the integral into two parts

$$\int_0^\infty \frac{x^{p-1}}{(1-x^r)\ln x}dx \to \int_0^\infty \frac{x^{p+\epsilon-1}}{1-x^r}dx.$$

We get the following bracket series

$$\sum \phi(-1)^{n_2}\langle n_1+n_2+1\rangle\langle n_2 r+p+\epsilon\rangle,$$

which gives the value

$$\frac{\pi}{r}\left(-i+\cot\left(\frac{\pi(\epsilon+p)}{r}\right)\right).$$

Here we have a complex value due to the term $(-1)^{n_2}$. Fortunately, integrating this with respect to ϵ and let $\epsilon=0$ we get a real answer

$$\ln\left(\sin\frac{p\pi}{r}\right).$$

Use the same procedure for the second part then subtract the two we get the desired formula.

Example 4.4.6. Entry 4.271.7 in [8] gives

$$\int_0^\infty \frac{(\ln x)^{2n+1}}{1+bx+x^2}dx = 0, \qquad |b|<2.$$

First we replace $\ln x$ by x^ϵ and evaluate

$$\int_0^\infty \frac{x^\epsilon}{1+bx+x^2}dx.$$

We get the following bracket series

$$\sum \phi\ b^{n_2}\langle n_1+n_2+n_3+1\rangle\langle n_2+2n_3+\epsilon+1\rangle.$$

Only the case n_2 free gives a convergent series for $|b| < 2$, which is

$$\frac{2\pi \csc(\epsilon\pi)}{\sqrt{4-b^2}} \sin\left[\epsilon \cos^{-1}\left(\frac{b}{2}\right)\right].$$

Now we need to differentiate this $2n+1$ times with respect to ϵ and let $\epsilon = 0$. However, notice that this function is even with respect to ϵ so odd derivatives vanish at 0. This proves the identity.

Example 4.4.7. Entry 4.271.9 in [8] gives

$$\int_0^\infty \frac{(\ln x)^{2n}}{1-x^2} dx = 0.$$

First we evaluate the integral

$$\int_0^\infty \frac{x^\epsilon}{1-x^2}.$$

This gives the following bracket series

$$\sum \phi_{12} \, (-1)^{n_2} \langle n_1 + n_2 + 1 \rangle \langle 2n_2 + \epsilon + 1 \rangle,$$

which gives the value

$$-\frac{\pi i}{2} - \frac{\pi}{2} \tan \frac{\pi\epsilon}{2}.$$

To find the original integral, we differentiate this $2n$ times with respect to ϵ and let $\epsilon = 0$. Since tangent is odd, the integral is zero.

From this we also get the identity

$$\int_0^\infty \frac{(\ln x)^{2n+1}}{1-x^2} dx = \frac{(-\pi^2)^{n+1}}{2n+2} B_{2n+2}(4^{n+1} - 1),$$

where B_n are the Bernoulli numbers.

Example 4.4.8. Entry 4.271.14 in [8] gives

$$\int_0^\infty (\ln x)^n \frac{x^{\nu-1}}{a^2+2ax\cos t+x^2}dx = -\pi\csc(t)\frac{d^n}{d\nu^n}\left[a^{\nu-2}\frac{\sin(\nu-1)t}{\sin(\nu\pi)}\right].$$

First we evaluate

$$\int_0^\infty \frac{x^{\epsilon+\nu-1}}{a^2+2ax\cos t+x^2}dx.$$

The integrand gives us the following bracket series

$$\sum \phi\ a^{2n_1+n_2}2^{n_2}(\cos t)^{n_2}\langle n_1+n_2+n_3+1\rangle\langle n_2+2n_3+\epsilon+\nu\rangle.$$

Only the case n_2 free gives us a convergent series with value

$$-\pi\csc(t)\frac{\sin[(\epsilon+\nu-1)t]}{\sin(\pi(\epsilon+\nu))}a^{\epsilon+\nu-2}.$$

Now we need to differentiate n times and let $\epsilon=0$. But since ϵ and ν have the same role in the expression, we could instead let $\epsilon=0$ and differentiate n times with respect to ν. Thus

$$\int_0^\infty (\ln x)^n \frac{x^{\nu-1}}{a^2+2ax\cos t+x^2}dx = -\pi\csc(t)\frac{d^n}{d\nu^n}\left[a^{\nu-2}\frac{\sin(\nu-1)t}{\sin(\nu\pi)}\right].$$

Example 4.4.9. Consider

$$I=\int_0^\infty\int_0^\infty \log(xy)\sin(x)\exp(-x^ay^b)dxdy.$$

Similar procedure yields the bracket series

$$\sum_{n_{12}}\phi_{n_{12}}\frac{\Gamma(n_1+1)}{\Gamma(2n_1+2)}\langle 2n_1+an_2+\epsilon+2\rangle\langle bn_2+\epsilon+1\rangle,$$

whose evaluation is

$$\frac{\pi\Gamma\left(\frac{\epsilon+1}{b}\right)}{2b\sin\frac{\pi(a\epsilon-b\epsilon+a)}{2b}\Gamma\left(\frac{a\epsilon-b\epsilon+a+b}{b}\right)}.$$

The derivative of this at $\epsilon = 0$ gives the identity

$$I = \frac{-\pi\Gamma\left(\frac{1}{b}\right)}{4ab\Gamma\left(\frac{a}{b}\right)\sin\frac{\pi a}{2b}}\left(-2\psi\left(\frac{1}{b}\right) + (a-b)\left[\pi\cot\frac{\pi a}{2b} + 2\psi\left(\frac{a+b}{b}\right)\right]\right).$$

Example 4.4.10. Entry 4.314.2 in [8] gives

$$\int_0^\infty \left[\frac{(q-1)x}{(1+x)^2} - \frac{1}{x+1} + \frac{1}{(1+x)^q}\right]\frac{dx}{x\ln(1+x)} = \ln\Gamma(q).$$

First we replace $\ln(1+x)$ by $(1+x)^\epsilon$. The first term becomes

$$\int_0^\infty (q-1)(1+x)^{\epsilon-2}dx = \frac{q-1}{1-\epsilon}.$$

The middle term becomes

$$\int_0^\infty \frac{(1+x)^{\epsilon-1}}{x}dx,$$

which is divergent. Therefore, to evaluate the middle and last terms separately we need to introduce a free parameter. Replace the term $x\ln(1+x)$ by $x^\beta\ln(1+x)$. Now the middle term is transformed to

$$\int_0^\infty \frac{(1+x)^{\epsilon-1}}{x^\beta}dx.$$

This gives the bracket series

$$\frac{1}{\Gamma(1-\epsilon)}\sum \phi\langle n_1+n_2-\epsilon+1\rangle\langle n_2-\beta+1\rangle,$$

which gives the value

$$\frac{\Gamma(1-\beta)\Gamma(\beta-\epsilon)}{\Gamma(1-\epsilon)}.$$

Similarly, the third term yields

$$\frac{\Gamma(1-\beta)\Gamma(\beta-\epsilon-1+q)}{\Gamma(q-\epsilon)}.$$

Now we combine these two and find the limit when β goes to 1. The limit is $\psi(1-\epsilon)-\psi(q-\epsilon)$. So combine all three terms we have

$$\frac{q-1}{1-\epsilon}+\psi(1-\epsilon)-\psi(q-\epsilon).$$

Now we just need to integrate this with respect to ϵ and let $\epsilon=0$ to get $\ln\Gamma(q)$. This is elementary.

Example 4.4.11. We now use the method of brackets to produce the value of

$$\int_0^\infty \frac{x\ln(1+2x)}{(1+x)(1+2x+2x^2)}dx=\frac{\pi^2}{16}.$$

First rewrite the integrand as

$$\frac{8x^2\ln(1+2x)}{(1+2x)^4-1}.$$

Next we use rule P_2 for the denominator and replace $\ln(1+2x)$ by $(1+2x)^\epsilon$ to get

$$8\sum_{n_{12}}\phi_{n_{12}}(-1)^{n_2}x^2(1+2x)^{4n_1+\epsilon}\langle n_1+n_2+1\rangle.$$

Rule P_2 again for $1+2x$ we get

$$8\sum_{n_{1234}}\phi_{n_{1234}}\frac{(-1)^{n_2}2^{n_3}}{\Gamma(-4n_1-\epsilon)}\langle n_1+n_2+1\rangle\langle n_3+n_4-4n_1-\epsilon\rangle\langle n_3+3\rangle.$$

This bracket series has index one. The case n_1 free gives

$$-2\sum_{n_1=0}^{\infty}\frac{\Gamma(-4n_1-\epsilon-3)}{\Gamma(-4n_1-\epsilon)}=\frac{1}{2}\left[\ln 2-\psi\left(\frac{1+\epsilon}{2}\right)+\psi\left(\frac{2+\epsilon}{4}\right)\right].$$

Derivative of this at $\epsilon = 0$ is $-\frac{\pi^2}{16}$. The case n_2 free gives

$$2\sum_{n_2=0}^{\infty} \frac{\Gamma(4n_2-\epsilon+1)}{\Gamma(4n_2-\epsilon+4)} = \frac{1}{2}\left[\ln 2 - \psi\left(\frac{1-\epsilon}{2}\right) + \psi\left(\frac{2-\epsilon}{4}\right)\right].$$

Derivative of this at $\epsilon = 0$ is $\frac{\pi^2}{16}$. The case n_4 free gives

$$-\frac{\pi}{2}\sum_{n_4=0}^{\infty} \frac{(-1)^{(3n_4+\epsilon+3)/4}\csc\frac{\pi(n_4-\epsilon+1)}{4}}{\Gamma(3-n_4)\Gamma(n_4+1)}.$$

The summand vanishes for $n_4 \geq 3$ so we only need to sum the first three terms. The derivative of this sum at $\epsilon = 0$ is $\frac{\pi^2}{8}$. The case n_3 free is omitted since the bracket $\langle n_3+3\rangle$ forces $n_3 = -3$. Now if we sum the answers from the above three cases we would get $\frac{\pi^2}{8}$, **which is not the stated answer**.

We now look at the evaluation of each bracket to see why we should not add the results from all three cases. First of all, the evaluation of $\langle n_3+3\rangle$ gives

$$2\sum_{n_{124}} \phi_{n_{124}} \frac{(-1)^{n_2}}{\Gamma(-4n_1-\epsilon)}\langle n_1+n_2+1\rangle\langle n_4-4n_1-\epsilon-3\rangle.$$

Though the second bracket is of type two, we should not evaluate this with respect to both parameters n_1 and n_4 because the term $\Gamma(-4n_1-\epsilon)$ make the component function f_1 not having bracket form. Thus we should only evaluate the second bracket with respect to n_4. This gives us

$$2\sum_{n_{12}} \phi_{n_{12}}(-1)^{n_2}\frac{\Gamma(-4n_1-\epsilon-3)}{\Gamma(-4n_1-\epsilon)}\langle n_1+n_2+1\rangle.$$

The component function f_1 now has bracket form but f_2 does not because of the term $(-1)^{n_2}$. A detail check of the proof of Lemma 3.2.4 shows that the presence of this extra term $(-1)^{n_2}$ forces us to evaluate this bracket with respect to n_1 (to guarantee the vanishing of the contour integral in the proof). Notice in this case n_1 and n_2 have the same weight so the vanishing of the undesired segments comes from this extra term $(-1)^{n_2}$.

Chapter 5

Future Projects

1. **Improvement of rule P_2:** It is fascinating how rule P_2 could also be use for multinomial on numerator. For example, to get

$$\int_0^\infty xe^{-x}\sqrt{1-e^{-x}} = \frac{4}{9}(4 - 3\ln 2),$$

we use rule P_2 for the square root and power series of exponentials to get a bracket series

$$\frac{1}{\Gamma\left(-\frac{1}{2}\right)} \sum_{n_{123}} \phi_{n_{123}} (-1)^{n_2} (1+n_2)^{n_3} \langle n_3 + 2 \rangle \left\langle n_1 + n_2 - \tfrac{1}{2} \right\rangle$$

then let n_2 free we get the answer. Recall rule P_2

$$\frac{1}{(a+b)^\alpha} = \frac{1}{\Gamma(\alpha)} \sum_{n_{12}} \phi_{n_{12}} a^{n_1} b^{n_2} \langle n_1 + n_2 + \alpha \rangle$$

comes from the integral representation of gamma function

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$$

which only makes sense for positive α. Thus, the use of rule P_2 for $\sqrt{1-e^{-x}}$ is completely formal. Yet, the method of brackets still yields the correct answer. This use of rule P_2 is still left to be justified.

2. **Non-separable functions:** So far all the integrands in our results are separable functions. Further analysis is required to justify the evaluation of the method of brackets in cases of non-separable functions. In the previous example, if we use power series for e^{-x} outside of the square root first, then rule P_2, then another power series for the last exponential term, we would get

$$\frac{1}{\Gamma\left(-\frac{1}{2}\right)}\sum_{n_{1234}}\phi_{n_{1234}}(-1)^{n_3}n_3^{n_4}\langle n_1+n_4+2\rangle\left\langle n_2+n_3-\tfrac{1}{2}\right\rangle$$

where if we let n_1 and n_3 free we'd get the correct answer.

3. **Restrictions on trigonometric functions:** As mentioned in Example 2.4.2, the justification for the use of MoB in the identity

$$\int_0^\infty x^{\mu-1}\sin(ax)dx=\frac{\Gamma(\mu)}{a^\mu}\sin\frac{\mu\pi}{2}$$

faces some difficulties. The issue is within the identity

$$\sum_{n=0}^{\infty}(-1)^n f(n)x^n=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\pi}{\sin\pi u}f(-u)x^{-u}du,$$

where if $f(n)$ comes from the power series of sine then we lose the freedom in choosing any positive c. Instead we need $c\le\frac{3}{4}$, which in this case prevents us from justifying the correct evaluation of MoB for $\mu\ge\frac{1}{2}$. Similar issue occurs for integrands involving cosine. Thus, further analysis is needed to loosen the conditions of c in sine and cosine line integral representations, or a new approach is needed to justify the correct evaluation of MoB for integrands involving trigonometric functions.

4. **General Mittag-Leffler functions:** Example 2.4.4 shows us another source of issues. All of our results require each component function of our integrand to have bracket form. So if one of our component function, say $f_1(z)$, has negative weight or weight more than two, we could not use any of our results involving the corresponding parameter, n_1. One way around this problem is trying to evaluate all the brackets not involving n_1 first, as stated in the proof of Theorem 2.4.2. Our claim is if we leave n_1 till the end, the function, now only in terms of n_1, has to have bracket form, otherwise the integral would diverge. This is expected from our knowledge of the growth of Mittag-Leffler functions. However, we still need to justify our claim for a the general form of Mittag-Leffler functions.

5. **Improvements:** the true power of the method of brackets lies in the case of zero index because it gives a closed form answer instead of a sum (or nested sum) which might not have a closed form. Thus a major improvement of this method would be to find a way to change positive index bracket series into zero index ones. This might require an introduction of a new function or integral to help reduce the index. The answer from the method would probably in a closed form of this new function (or integral).

6. **Justification:** there is a pattern in the proofs of results presented here. It lies in the choice of parameter that we evaluate a bracket with respect to. The relative growth of the component functions in the integrand tells us which parameter should be free in the bracket and at the same time it guarantees the convergent of the evaluation. For example, recall Lemma 3.2.4

   Assume $f(x, y)$ is separable, i.e. $f(x, y) = f_1(x)f_2(y)$, and each f_i's has bracket form with weight w_i's respectively. Let c and α be positive numbers such that

$\alpha w_1 < w_2$. Then

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\pi f(-\sigma+\alpha u,-u)}{\sin \pi u \sin \pi(\sigma-\alpha u)}du = \sum_{n=0}^{\infty}(-1)^n \frac{f(-\sigma-\alpha n, n)}{\sin \pi(\sigma+\alpha n)},$$

assuming $-\sigma+\alpha c < 0$.

Here, the growth condition $\alpha w_1 < w_2$ is needed for the sum to converge. In other words, **the very condition for the answer to be finite is sufficient for the justification of the evaluation**. Recall the use of MoB in the evaluation

$$\int_0^\infty \frac{\sin \alpha x}{x(x^2+\beta^2)}dx = \frac{\pi}{2\beta^2}(1-e^{-\alpha\beta}),$$

where we have the bracket series

$$\sum_{n_1,n_2,n_3} \phi_{n_{123}} \alpha^{2n_3+1}\beta^{2n_2}\frac{\Gamma(n_3+1)}{\Gamma(2n_3+2)}\langle n_1+n_2+1\rangle\langle 2n_1+2n_3+1\rangle.$$

If we let n_1 free we get the evaluation

$$\frac{1}{2}\sum_{n_1=0}^{\infty}\frac{(-1)^{n_1}}{\Gamma(n_1+1)}\alpha^{-2n_1}\beta^{-2n_1-2}\frac{\Gamma(-n_1+1/2)}{\Gamma(-2n_1+1)}\Gamma(n_1+1)\Gamma(n_1+1/2).$$

In terms of growth, this summand grows as $\Gamma(2n_1)$, but the sum yields a finite answer thanks to the poles of $\Gamma(-2n_1+1)$. For this reason, the contour integral we use in our proof does not work to justify this evaluation. As it turns out, this evaluation is indeed invalid because we need to evaluate the bracket $\langle 2n_1 + 2n_3+1\rangle$ with respect to n_1 so it cannot be free.

7. A new approach might be needed to complete the justification of method of brackets. Recall example 4.4.3 where we use the method to evaluate the divergent integral

$$\int_0^\infty \left(\frac{a^2+x^2}{b^2+x^2}\right)^\epsilon dx$$

then differentiate the result at $\epsilon = 0$ to get the correct answer. Here we see the advantage of MoB not detecting a divergent integral. This suggests a new approach that might make the justification of this method easier. That is to treat the definite integral as a special case of something else (something that does not have divergent issues like integrals). With divergent issue put aside, it might be easier to completely justify this method.

Appendix A

Gamma function

In an attempt to extend the factorial to non-integer values, Euler introduced the infinite product

$$n! = \prod_{k=1}^{\infty} \frac{\left(1+\frac{1}{k}\right)^n}{1+\frac{n}{k}}$$

to Goldbach in 1729. Then a few months later he announced the integral representation

$$n! = \int_0^1 (-\ln s)^n \, ds,$$

for $n \geq 0$. With the change of variables $t = -\ln s$ we get a more well-known integral representation

$$(n-1)! = \Gamma(n) = \int_0^{\infty} t^{n-1} e^{-t} dt,$$

which is due to Legendre, who also introduced the name gamma function and the symbol Γ.

Gauss rewrote Euler's product as

$$\Gamma(z) = \lim_{n\to\infty} \frac{n^z n!}{z(z+1)\cdots(z+n)}$$

and was the first to consider gamma function as a complex valued function. The integral representation of gamma function could also take complex argument but it is only valid for the half-plane $\Re(z) > 0$. Recursive formula $\Gamma(z+1) = z\Gamma(z)$ could be used to extend gamma function to the other half-plane. This gives gamma function simple poles at non-positive integers due to its original pole at zero.

Weierstrass wrote the reciprocal of gamma function in terms of its zero with the formula

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n},$$

where γ is the Euler constant. Thus reciprocal of gamma function is entire and vanishes at non-positive integers. Some important functional equations of gamma function that we use often in this work are the Euler's reflection formula

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z},$$

and the duplication formula

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z).$$

The gamma function has the following asymptotic behavior according to Stirling's formula

$$\Gamma(z) = \sqrt{2\pi}e^{-z}e^{(z-1/2)\ln z}\left[1+\mathcal{O}\left(\frac{1}{z}\right)\right] \qquad (|z|\to\infty, |\arg(z)|<\pi).$$

Write $z = a+bi$ then for a fixed b, the leading term in the asymptotic behavior of $\Gamma(z)$ as $a\to\infty$ is a^a. In our analysis, we often ignore the exponential term in the formula because the term a^a is sufficient to determine the vanishing of the function of interest. On the other hand, for a fixed a, we have the estimate

$$|\Gamma(a+bi)| = \sqrt{2\pi}|b|^{a-1/2}e^{-a-|b|\pi/2}\left[1+\mathcal{O}\left(\frac{1}{|b|}\right)\right] \qquad (|b|\to\infty).$$

Thus if a is in a finite interval I, we use the following in our analysis

$$|\Gamma(a+bi)| \approx |b|^{a-1/2} e^{-|b|\pi/2}.$$

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Biography

The author was born in Vietnam in 1983. He immigrated to the U.S. in November of 2004 and joined the ESL (English as Second Language) program at Delgado Community College in Spring 2006. He then transferred to University of New Orleans in 2008 and graduated in 2012 with a Bachelor in Mathematics. The author started the Ph.D program at the Tulane University mathematics department in 2013, eventually completing the program in December 2018. Right now, he is an Instructor of Mathematics at University of New Orleans.