Universidad de Valparaíso
CHILE

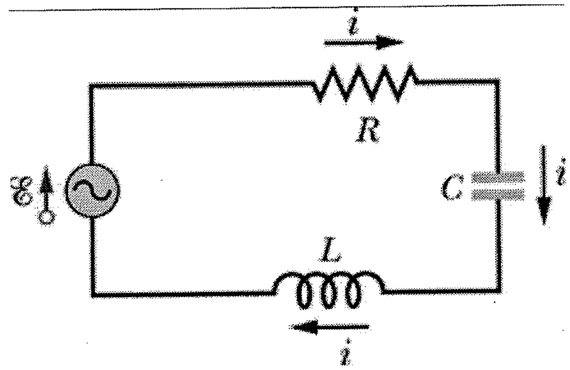
Prueba II
Métodos Matemáticos
Licenciatura en Física - 2016
IPGG

(I) Algebrización de $\frac{d}{dt} = \partial_t$ (25%)

- (10%) Sea $f(\zeta)$ una función tal $f(0)$ es finito. Demuestre que:

$$f\left(\partial_t^k\right)\exp(\alpha t) = f\left(\alpha^k\right)\exp(\alpha t), \text{ con } k \in \mathbb{N}$$

- (15%) El siguiente circuito LRC:



con $\xi(t) = \sin(\omega_0 t)$. Halle la corriente estacionaria (solución particular) $i_p(t)$ que fluye en el circuito, esto es la corriente cuando $t \to \infty$.

(II) IBD basado en transformada de Fourier (25%)

Evalúe la siguiente integral

$$I = \int_{-\infty}^{\infty} \cos(ax)\sin\left(bx^2\right)\, dx, \quad a, b \in \mathbb{R}^+$$

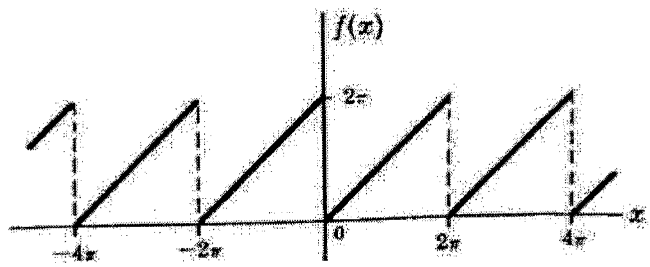
(III) IBD basado en transformada de Laplace (25%)

Evalúe la siguiente integral

$$I = \int_0^\infty \frac{\cos(x)\sin(x)}{x^{\frac{1}{3}}}\, dx$$

(IV) Función de Heaviside y parientes (25%)

- (15%) Escriba la función que representa a la siguiente gráfica:



- (10%) Grafique $f(x) = H(a - x)$

Probl. I)

a) si $f(0)$ es finito es posible expandir

. $f(x)$ en torno a $x=0$

$$\Downarrow$$

$$f(x) = \sum_{n\geq 0} a_m x^n$$

$$\Downarrow$$

$$f(\partial_t^k) = \sum_{n\geq 0} a_n (\partial_t^k)^n$$

luego
$$f(\partial_t^k) e^{\alpha t} = \sum_{n\geq 0} a_m (\partial_t^k)^m e^{\alpha t}$$

Obs.

$$\partial_t e^{\alpha t} = \alpha e^{\alpha t}$$

$$\partial_t^l e^{\alpha t} = \alpha^l e^{\alpha t}$$

$$\vdots$$

$$\partial_t^{kn} e^{\alpha t} = \alpha^{kn} e^{\alpha t}$$

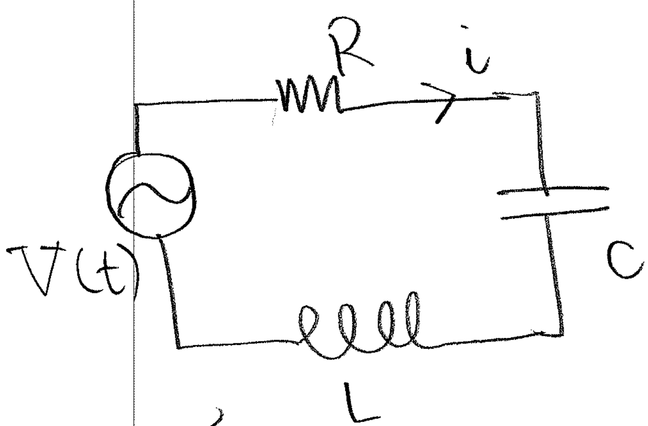
entonces
$$f(\partial_t^k) e^{\alpha t} = \sum_{n\geq 0} a_m (\alpha^k)^n e^{\alpha t}$$

$$\Downarrow$$

$$f(\partial_t^k)\, e^{\alpha t} = e^{\alpha t} \underbrace{\sum_{n \geq 0} a_n (\alpha^k)^n}_{f(\alpha^k)}$$

Q.E.D //

b)



$$V(t) = L\frac{di}{dt} + iR + \frac{q}{C} \quad ; \; V(t) = \operatorname{sen}(\omega_0 t)$$

Para obtener una sola variable dependiente derivamos en t la ecuación diferencial:

$$\omega_0 \cos(\omega_0 t) = L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C} i$$

$$\omega_0 \cos(\omega_0 t) = \left(L\,\partial_t^2 + R\,\partial_t + \frac{1}{C}\right) i(t)$$

$$\Downarrow$$

$$i(t) = \left[\frac{\omega_0}{\left(L\partial_t^2 + R\partial_t + \frac{1}{C}\right)}\right]\cos(\omega_0 t)$$

Si se expande en potencias de ∂_t lo que se obtiene la solución particular para $i(t)$

$$i_p(t) = \frac{1}{2}\frac{\omega_0 C}{(LC\partial_t^2 + RC\partial_t + 1)}\left[e^{i\omega_0 t} + e^{-i\omega_0 t}\right]$$

$$i_p(t) = \frac{1}{2}\omega_0 C\, f(\partial_t)\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right)$$

siendo $f(\partial_t) = \dfrac{1}{LC\partial_2^2 + RC\partial_t + 1}$

$\therefore$ del resultado obtenido en la parte (a)

$$i_p(t) = \frac{\omega_0 C}{2}\left[f(i\omega)\, e^{i\omega_0 t} + f(-i\omega_0)\, e^{-i\omega_0 t}\right]$$

$$\Downarrow$$

$$i_p(t) = \frac{\omega_0 C}{2}\, 2\,\mathrm{Re}\left(f(i\omega)\, e^{i\omega_0 t} \right)$$

$$= \omega_0 C\ \mathrm{Re}\left[\frac{e^{i\omega_0 t}}{LC\,(i\omega_0)^2 + RC(i\omega_0) + 1} \right]$$

$$= \omega_0 C\ \mathrm{Re}\left[\frac{e^{i\omega_0 t}}{(1 - LC\omega_0^2) + i\omega_0 RC} \right]$$

$$= \omega_0 C \left[\frac{e^{i\omega_0 t}\left[(1 - LC\omega_0^2) - i\omega_0 RC\right]}{(1 - LC\omega_0^2)^2 + \omega_0^2 R^2 C^2} \right]$$

$$= \frac{\omega_0 C}{(1 - LC\omega_0^2)^2 + \omega_0^2 R^2 C^2}$$

$$\times \left[(1 - LC\omega_0^2)\cos(\omega_0 t) + \omega_0 RC\,\mathrm{sen}(\omega_0 t) \right]$$

//a.

Probl. II)

$$I = \int_{-\infty}^{\infty} \cos(ax)\, \text{sen}(bx^2)\, dx$$

$$= 2\pi \cos(-i\partial_k) \left. \frac{F(\text{sen}(bx^2))}{\sqrt{2\pi}} \right|_{k=0}.$$

donde $F(\text{sen}(bx^2)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \text{sen}(bx^2)\, dx$

Transf. de Fourier de $\text{sen}(bx^2)$.

Primero analicemos y hallemos

$F(e^{-\alpha x^2})$!!!

$$F\left(e^{-\alpha x^2}\right)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} e^{-\alpha x^2} dx$$

$$= \frac{1}{\sqrt{2\alpha}} e^{-\frac{1}{4}\frac{k^2}{\alpha}}$$

$$\Downarrow$$

$$F\left(\operatorname{sen}(bx^2)\right)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \operatorname{sen}(bx^2)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \frac{\left(e^{ibx^2} - e^{-ibx^2}\right)}{2i} dx$$

$$= \frac{1}{2i} \left[F\left(e^{ibx^2}\right) - F\left(e^{-ibx^2}\right) \right]$$

$$= \frac{1}{2i} \left[\frac{1}{\sqrt{2ib}} e^{-\frac{1}{4}\frac{k^2}{ib}} - \frac{1}{\sqrt{-2ib}} e^{-\frac{1}{4}\frac{k^2}{(-ib)}} \right]$$

$$F\left(\operatorname{sen}(bx^2)\right) = \frac{1}{2i}\, 2i\, \mathrm{Im}\left[\frac{e^{\frac{i}{4}\frac{k^2}{b}}}{\sqrt{2ib}}\right] \neq$$

$$= \mathrm{Im}\left[\frac{e^{i\frac{1}{4}\frac{k^2}{b}}}{\sqrt{2b}\,\sqrt{i}}\right]$$

donde

$$\frac{1}{\sqrt{i}} = \left(\frac{1}{i}\right)^{1/2} = (-i)^{1/2} = i\sqrt{i} = i\,e^{i\pi/4} \;///$$

$$= e^{i\pi/2}e^{i\pi/4} \;!!!$$

$$= e^{i\frac{3}{4}\pi} \;///$$

luego

$$F\left(\operatorname{sen}(bx^2)\right) = \mathrm{Im}\left[\frac{1}{\sqrt{2b}}\, e^{i\left(\frac{1}{4}\frac{k^2}{b} - \frac{\pi}{4}\right)}\right]$$

$$= \frac{1}{\sqrt{2b}} \operatorname{sen}\left(\frac{k^2}{4b} + \frac{3\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2b}} \cos\left(\pi/2 - \frac{k^2}{4b} - \frac{3\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2b}} \cos\left(-\frac{k^2}{4b} - \frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2b}} \cos\left(\frac{k^2}{4b} + \frac{\pi}{4}\right) \;//$$

por otro lado

$$\cos\left(\frac{k^2}{4b}+\frac{\pi}{4}\right)=\cos\left(\frac{k^2}{4b}\right)\cos\left(\frac{\pi}{4}\right)-\operatorname{sen}\left(\frac{k^2}{4b}\right)\operatorname{sen}\left(\frac{\pi}{4}\right)$$

obs. $\cos\frac{\pi}{4}=\operatorname{sen}\frac{\pi}{4}=\frac{1}{\sqrt{2}}$

$$=\frac{1}{\sqrt{2}}\left[\cos\left(\frac{k^2}{4b}\right)-\operatorname{sen}\left(\frac{k^2}{4b}\right)\right]$$

$$\therefore\ F\left(\operatorname{sen}(bx^2)\right)=\frac{1}{2\sqrt{b}}\left[\cos\left(\frac{k^2}{4b}\right)-\operatorname{sen}\left(\frac{k^2}{4b}\right)\right]$$

finalmente

$$I=\int_{-\infty}^{\infty}\cos(ab)\operatorname{sen}(bx^2)\,dx$$

$$=\sqrt{2\pi}\cos(-ia\partial_k)\frac{1}{2\sqrt{b}}\left[\cos\left(\frac{k^2}{4b}\right)-\operatorname{sen}\left(\frac{k^2}{4b}\right)\right]\Bigg|_{k=0}$$

$$=\frac{1}{2}\frac{\sqrt{2\pi}}{\sqrt{b}}\frac{1}{2}\left(e^{a\partial_k}+e^{-a\partial_k}\right)\left(\cos\left(\frac{k^2}{4b}\right)-\operatorname{sen}\left(\frac{k^2}{4b}\right)\right)\Bigg|_{k=0}$$

$$I = \frac{1}{2}\frac{\sqrt{2\pi}}{\sqrt{b}}\frac{1}{2}\left[\cos\left(\frac{(k+a)^2}{4b}\right) - \operatorname{sen}\left(\frac{(k+a)^2}{4b}\right) + \cos\left(\frac{(k-a)^2}{4b}\right) - \operatorname{sen}\left(\frac{(k-a)^2}{4b}\right)\right]\Bigg|_{k=0}$$

$$I = \frac{1}{2}\frac{\sqrt{2\pi}}{\sqrt{b}}\left[\cos\left(\frac{a^2}{4b}\right) - \operatorname{sen}\left(\frac{a^2}{4b}\right)\right]$$

Probl. 3)

$$I = \int_0^\infty \frac{\cos x \ \operatorname{sen} x \, dx}{x^{1/3}}$$

$$= \frac{\cos\left(\frac{1}{\beta}\partial_s\right) \operatorname{sen}\left(\frac{1}{\beta}\partial_s\right)}{\left(\frac{1}{\beta}\partial_s\right)^{1/3}} \frac{1}{s}\Bigg|_{\substack{\beta=-1 \\ s=0}}$$

$$= \cos\left(\frac{1}{\beta}\partial_s\right) \operatorname{sen}\left(\frac{1}{\beta}\partial_s\right) \beta^{1/3} \partial_s^{-1/3} \frac{1}{s}\Bigg|_{\substack{\beta=-1 \\ s=0}}$$

$$= \cos(-\partial_s) \operatorname{sen}(-\partial_s) (-1)^{1/3} \partial_s^{-1/3} \frac{1}{s}\Bigg|_{s=0}$$

donde $\partial_s^{-1/3} \dfrac{1}{s} = (-1)^{-1/3} \dfrac{\Gamma(2/3)}{\Gamma(1)} s^{-2/3}$

$$\therefore I = \cos(-\partial_s) \operatorname{sen}(-\partial_s) \cancel{(-1)^{1/3}} \cancel{(-1)^{-1/3}} \frac{\Gamma(2/3)}{s^{2/3}}\Bigg|_{s=0}$$

Por otro lado:

$$\cos(-\partial s)\,\mathrm{sen}(-\partial s) = \frac{\left(e^{-i\partial s} + e^{i\partial s}\right)}{2}\,\frac{\left(e^{-i\partial s} - e^{i\partial s}\right)}{2i}$$

$$= \frac{e^{-2i\partial s} - e^{2i\partial s}}{4i}$$

luego:

$$I = \frac{\Gamma(2/3)}{4i}\left(e^{-2i\partial s} - e^{2i\partial s}\right)\frac{1}{s^{2/3}}\Bigg|_{s=0}.$$

$$= \frac{\Gamma(2/3)}{4i}\left[\frac{1}{(s-2i)^{2/3}} - \frac{1}{(s+2i)^{2/3}}\right]\Bigg|_{s=0}.$$

$$= \frac{\Gamma(2/3)}{4i}\left[\frac{1}{(-2i)^{2/3}} - \frac{1}{(2i)^{2/3}}\right]$$

$$= \frac{\Gamma(2/3)}{4i}\,2i\,\mathrm{Im}\left[\frac{1}{(-2i)^{2/3}}\right]$$

$$= \frac{\Gamma(2/3)}{2} \, \mathrm{Im}\left[\frac{1}{(-2i)^{2/3}}\right]$$

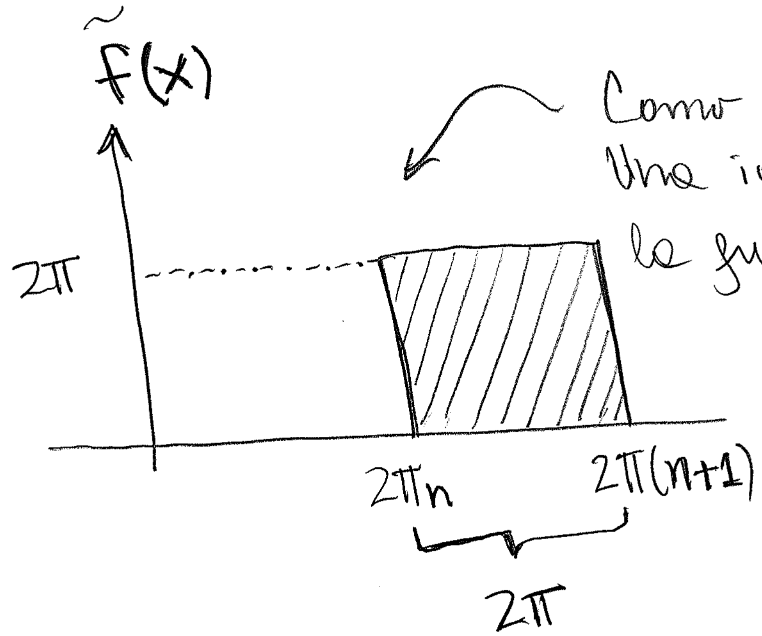
$$= \frac{\Gamma(2/3)}{2} \, \mathrm{Im}\left[\frac{1}{2^{2/3}} \frac{1}{e^{-i\pi/3}}\right]$$

$$= \frac{\Gamma(2/3)}{2^{5/3}} \, \mathrm{Im}\left(e^{i\pi/3}\right)$$

$$= \frac{\Gamma(2/3)}{2^{5/3}} \, \mathrm{sen}\, \pi/3$$

Probl. 4)

a) Obs.



Como para comenzar... Una idea p/ hallar la función pedida.

$\tilde{f}(x)$ en este caso es dado por:

$$\tilde{f}(x) = 4\pi\left[H(x-2\pi n) - H(x-2\pi(n+1))\right]$$

$\boxed{!}$ $\tilde{f}(2\pi n) = 1$

$\tilde{f}(2\pi(n+1)) = 1$

Verificación de lo anterior:

$$\tilde{f}(2\pi n) = 4\pi\left[\cancel{H(0)}^{1/2} - \cancel{H(-2\pi)}^{0}\right]$$

$$= 2\pi \,//$$

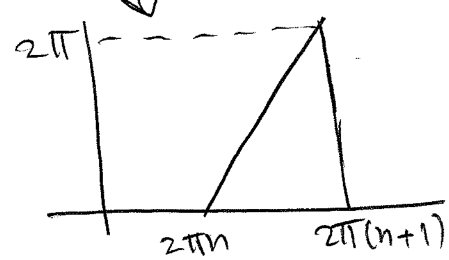
$$\tilde{f}(2\pi(n+1)) = 4\pi\left[\cancel{H(2\pi)}^{1} - \cancel{H(0)}^{1/2}\right]$$

$$= 2\pi \,//$$

En nuestro problema solo requerimos

que $\tilde{f}(2\pi n) = 0$

$\tilde{f}(2\pi(n+1)) = 2\pi$ ||



Propuesta:

$$(*) \quad \tilde{f}(x) = 2(x-2\pi n)\left[H(x-2\pi n) - H(x-2\pi(n+1))\right]$$

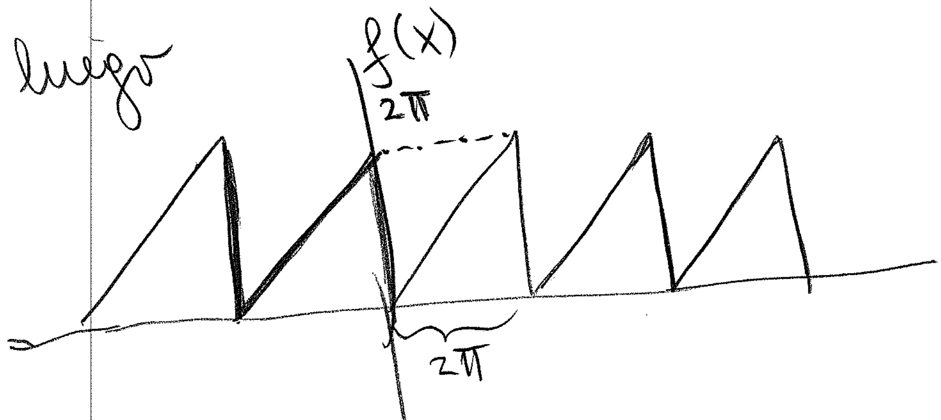
Verificación:

$$\tilde{f}(2\pi n) = 2 \cdot 0 \cdot [\cancel{H(0)}^{1/2} - \cancel{H(-2\pi)}^{0}] = 0$$

$$\tilde{f}(2\pi(n+1)) = 2 \cdot (2\pi) \cdot [\cancel{H(2\pi)}^{1} - \cancel{H(0)}^{1/2}]$$

$$= 2\pi$$

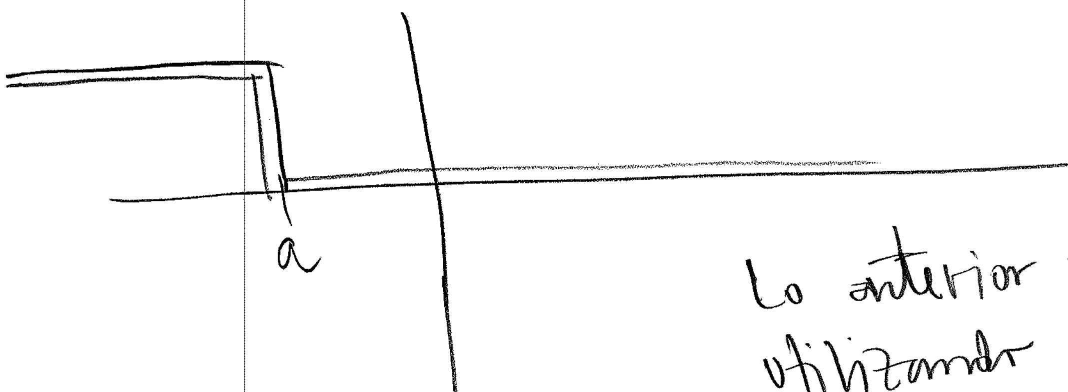
luego



$$f(x) = 2\sum_{n=-\infty}^{\infty} (x-2\pi n)\left[H(x-2\pi n) - H(x-2\pi(n+1))\right]$$ //

b)

$$f(x) = H(a-x) \; ; \; a \in \mathbb{R}$$



Lo anterior se visualiza, utilizando la definición de $H(x)$

$$H(x) = \begin{cases} 1 \; ; & x > 0 \\ 0 \; ; & x < 0. \end{cases}$$