Chapter 4. Electric Fields in Matter

4.1 Polarization

4.1.1 Dielectrics

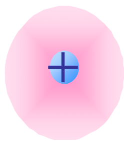
All charges are attached to specific atoms or molecules.

→ They can move a bit *within* the atom or molecule.

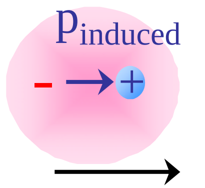
→ Such microscopic displacements account for the characteristic behavior of dielectric materials.

There are actually two principal mechanisms by which electric fields can distort the charge distribution of a dielectric atom or molecule:

➔ ***stretching* and *rotating*.**

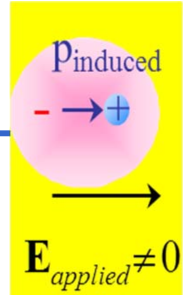


$\mathbf{E}_{applied} = 0$



$\mathbf{E}_{applied} \neq 0$

4.1.2 Induced Dipoles



What happens to a neutral atom when it is placed in an electric field E?

The nucleus is pushed in the direction of the field, and the electrons the opposite way. The two opposing forces reach a balance, leaving the atom **polarized.**
The atom now has a tiny dipole moment **p,** which points in the *same direction as* E.
Typically, this induced dipole moment is approximately proportional to the field (as long as the latter is not too strong):

$$\boxed{\mathbf{p} = \alpha \mathbf{E}}$$

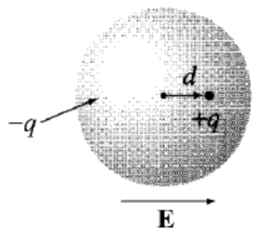
: α is called **atomic polarizability**

Atomic Polarizabilities ($\alpha/4\pi\epsilon_0$, in units of 10^{-30} m^3)

H	He	Li	Be	C	Ne	Na	Ar	K	Cs
0.667	0.205	24.3	5.60	1.76	0.396	24.1	1.64	43.4	59.6

Example 4.1 An atom consists of a point nucleus *(+q)* surrounded by a uniformly charged spherical cloud (*-q)* of radius *a.* Calculate the atomic polarizability of such an atom.

It is reasonable to assume that the electron cloud retains its spherical shape.
Equilibrium occurs when the nucleus is displaced a distance *d* from the center of the sphere and pulling by the internal field produced by the electron cloud, E_e:
At equilibrium, $E = E_e$

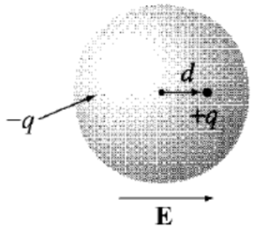


$$E = E_e = \frac{1}{4\pi\epsilon_0}\frac{qd}{a^3} \longrightarrow p = qd = (4\pi\epsilon_0 a^3)E \longrightarrow \alpha = 4\pi\epsilon_0 a^3 = 3\epsilon_0 v$$

Atomic polarizability

Problem 4.2 According to quantum mechanics, the electron cloud **for a hydrogen atom** in the ground state has a charge density.

$$\rho(r) = \frac{q}{\pi a^3} e^{-2r/a} \qquad \text{(not uniformly charged case)}$$



where *q* is the charge of the electron and *a* is the Bohr radius. Find the atomic polarizability of such an atom.

➔ First find the field, *at radius r*, using Gauss' law: $\int \mathbf{E}\cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}} \longrightarrow E = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2} Q_{\text{enc}}$

$$Q_{\text{enc}} = \int_0^r \rho\, d\tau = \frac{4\pi q}{\pi a^3}\int_0^r e^{-2\bar{r}/a}\bar{r}^2\, d\bar{r} = \frac{4q}{a^3}\left[-\frac{a}{2}e^{-2\bar{r}/a}\left(\bar{r}^2 + a\bar{r} + \frac{a^2}{2}\right)\right]\Bigg|_0^r = -\frac{2q}{a^2}\left[e^{-2r/a}\left(r^2 + ar + \frac{a^2}{2}\right) - \frac{a^2}{2}\right]$$

$$= q\left[1 - e^{-2r/a}\left(1 + 2\frac{r}{a} + 2\frac{r^2}{a^2}\right)\right]$$

So the field of the electron cloud is $E_e = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\left[1 - e^{-2r/a}\left(1 + 2\frac{r}{a} + 2\frac{r^2}{a^2}\right)\right]$

The proton will be shifted from $r = 0$ to the point d where $E_e = E$

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{d^2}\left[1 - e^{-2d/a}\left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2}\right)\right]$$

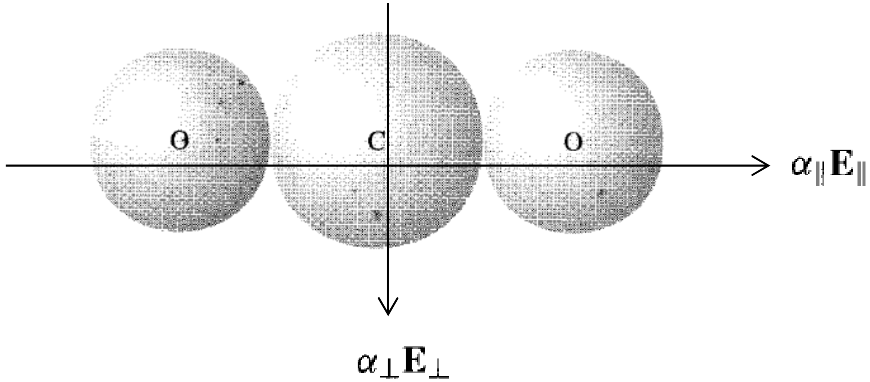
$$1 - e^{-2d/a}\left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2}\right) = 1 - \left(1 - 2\frac{d}{a} + 2\left(\frac{d}{a}\right)^2 - \frac{4}{3}\left(\frac{d}{a}\right)^3 + \cdots\right)\left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2}\right) = \frac{4}{3}\left(\frac{d}{a}\right)^3 + \text{ higher order terms.}$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{d^2}\left(\frac{4}{3}\frac{d^3}{a^3}\right) = \frac{1}{4\pi\epsilon_0}\frac{4}{3a^3}(qd) = \frac{1}{3\pi\epsilon_0 a^3}p \Rightarrow \boxed{\alpha = 3\pi\epsilon_0 a^3} \longleftrightarrow \alpha = 4\pi\epsilon_0 a^3 = 3\epsilon_0 v$$

Polarizability tensor

Carbon dioxide (CO_2), for instance, ***when the field is at some angle to the axis***, you must resolve it into parallel and perpendicular components, and multiply each by the pertinent polarizability:

$$\mathbf{p} = \alpha_{\perp}\mathbf{E}_{\perp} + \alpha_{\parallel}\mathbf{E}_{\parallel} \qquad \alpha_{\parallel} \neq \alpha_{\perp}$$



A most general linear relation between E and p is

$$\left.\begin{aligned} p_x &= \alpha_{xx}E_x + \alpha_{xy}E_y + \alpha_{xz}E_z \\ p_y &= \alpha_{yx}E_x + \alpha_{yy}E_y + \alpha_{yz}E_z \\ p_z &= \alpha_{zx}E_x + \alpha_{zy}E_y + \alpha_{zz}E_z \end{aligned}\right\} \Rightarrow \boxed{\mathbf{p} = \overleftrightarrow{\alpha}\mathbf{E}}$$

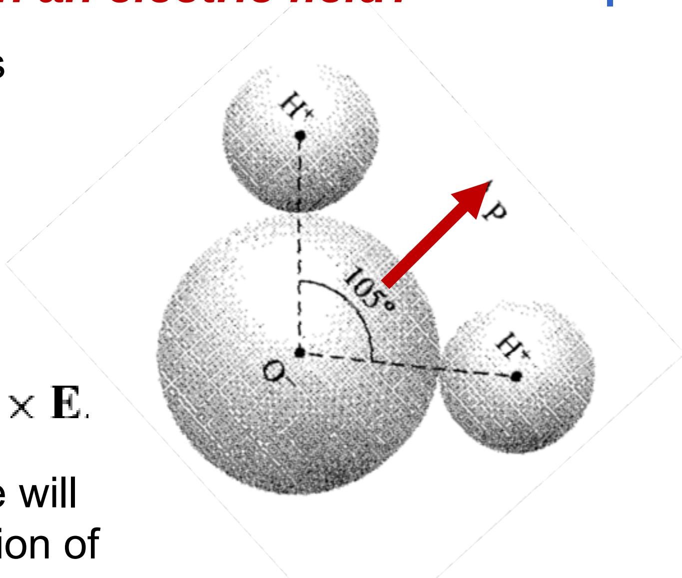
The set of nine constants α_{ij} ➔ **polarizability tensor**

By choose "principal" axes, we can leave just three nonzero polarizabilities: α_{xx} , α_{yy} , and α_{zz}.

4.1.3 Alignment of Polar Molecules

What happens when polar molecules are placed in an electric field?

Polar molecules: molecules having built-in dipole moments (H_2O, for example)

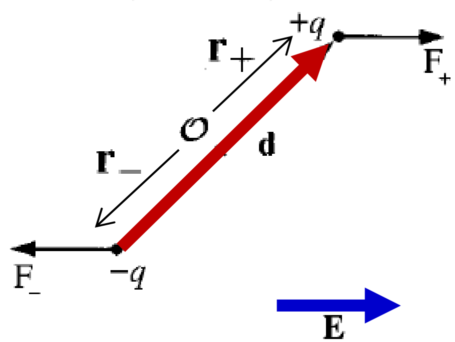


There is a ***torque***:

$$\begin{aligned} \mathbf{N} &= (\mathbf{r}_+ \times \mathbf{F}_+) + (\mathbf{r}_- \times \mathbf{F}_-) \\ &= [(\mathbf{d}/2) \times (q\mathbf{E})] + [(-\mathbf{d}/2) \times (-q\mathbf{E})] = q\mathbf{d} \times \mathbf{E}. \end{aligned}$$

$$\mathbf{N} = \mathbf{p} \times \mathbf{E}$$

→ A polar molecule that is free to rotate will swing around until it points in the direction of the applied field.



If the field is nonuniform, so that F_+ does not exactly balance F_-, there will be a net *force* on the dipole, in addition to the torque.

$$\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_- = q(\mathbf{E}_+ - \mathbf{E}_-) = q(\Delta\mathbf{E})$$

Assuming the dipole is very short, $\Delta\mathbf{E} = \nabla E \cdot \mathbf{d} = (\mathbf{d} \cdot \nabla)\mathbf{E}$

$$\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E}$$

$$\mathbf{N} = (\mathbf{p} \times \mathbf{E}) + (\mathbf{r} \times \mathbf{F})$$

Energy of an ideal dipole p in E

Problem 4.7 Show that the energy of an ideal dipole p in an electric field E is given by

$$U = -\mathbf{p} \cdot \mathbf{E}$$

The energy of a point charge Q is U = QV(r).
➔ For a physical dipole, with −q at r and +q at r + d, the energy is

$$U = qV(\mathbf{r}+\mathbf{d}) - qV(\mathbf{r}) = q\left[-\int_{\mathbf{r}}^{\mathbf{r}+\mathbf{d}} \mathbf{E}\cdot d\mathbf{l}\right]$$

➔ For an ideal dipole the integral reduces to **E · d**,

$$U = -q\mathbf{E}\cdot\mathbf{d} = -\mathbf{p}\cdot\mathbf{E}$$

Problem 4.9 A dipole **p** is a distance *r* from a point charge *q*, and oriented so that **p** makes an θ with the vector ***r*** from q to **p**.

What is the force on **p**?

$$\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E} \longleftarrow \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} = \frac{q}{4\pi\epsilon_0}\frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{(x^2+y^2+z^2)^{3/2}}$$

$$F_x = \left(p_x\frac{\partial}{\partial x} + p_y\frac{\partial}{\partial y} + p_z\frac{\partial}{\partial z}\right)\frac{q}{4\pi\epsilon_0}\frac{x}{(x^2+y^2+z^2)^{3/2}} = \frac{q}{4\pi\epsilon_0}\left[\frac{\mathbf{p}}{r^3} - \frac{3\mathbf{r}(\mathbf{p}\cdot\mathbf{r})}{r^5}\right]_x$$

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\left[\mathbf{p} - 3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}\right]$$

4.1.4 Polarization

What happens to a piece of dielectric material when it is placed in an electric field?

A lot of little dipoles pointing along the direction of the field.
The material becomes **polarized.**

$$\boxed{\mathbf{P} = \lim_{\Delta\upsilon \to 0} \left[\sum_i \mathbf{p}_i / \Delta\upsilon \right] \ \left(\mathrm{C/m^2}\right)}$$

: Dipole moment per unit volume
➔ Polarization

This Polarization vector will displace the electric strength in a dielectric medium:

$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}$$ **: D** is known as the electric displacement

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$: For linear dielectrics

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} = \epsilon_0 (1 + \chi_e) \mathbf{E}$$

$$\mathbf{D} = \epsilon \mathbf{E} \qquad \epsilon \equiv \epsilon_0 (1 + \chi_e)$$: Permittivity

$$\epsilon_r \equiv 1 + \chi_e = \frac{\epsilon}{\epsilon_0}$$: Relative permittivity