Intermission on Page 343

(i) $\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$ (Gauss's law),

(ii) $\nabla \cdot \mathbf{B} = 0$ (no name),

(iii) $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ (Faraday's law),

(iv) $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ (Ampère's law with Maxwell's correction).

Together with the force law, $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$,

All of our cards are now on the table, and in a sense my job is done.
In the first seven chapters we assembled electrodynamics piece by piece,
and **now, with Maxwell's equations in their final form, the theory is complete.**
There are no more laws to be learned, no further generalizations to be considered,
and (with perhaps one exception) no lurking inconsistencies to be resolved.
If yours is a one-semester course, this would be a reasonable place to stop.

But in another sense we have just arrived at the starting point.
We are at last in possession of a full deck, and **we know the rules of the game -- it's time to deal.**
This is the fun part, in which one comes to appreciate the extraordinary power and richness of electrodynamics.
In a full-year course **there should be plenty of time to cover the remaining chapters**, and perhaps to supplement them with a unit on plasma physics, say, or AC circuit theory, or even a little General Relativity.
But if you have room only for one topic, I'd recommend Chapter 9, on Electromagnetic Waves (you'll probably want to skim Chapter 8 as preparation).
This is the segue to Optics, and is historically the most important application of Maxwell's theory.

Chapter 8. Conservation Laws

8.1 Charge and Energy

8.1.1 Charge conservation (The Continuity Equation)

Let's begin by reviewing the **conservation of *charge***, because it is the paradigm for all conservation laws.

→ What precisely does conservation of charge tell us?

→ If the total charge in some volume changes, then exactly that amount of charge must have passed in or out through the surface.

This local conservation of charge says

$$\frac{dQ}{dt} = -\int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a}$$

divergence theorem

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} \, d\tau = -\int_{\mathcal{V}} \nabla \cdot \mathbf{J} \, d\tau.$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}$$

This is, of course, the continuity equation,

→ ***"the precise mathematical statement of local conservation of charge".***

8.1.2 Poynting's Theorem

In Chapter 2, we found that the **work necessary to assemble a static charge distribution** (against the Coulomb repulsion of like charges) is (Eq. 2.45)

Energy of Continuous Charge Distribution $\Rightarrow$ $W = \frac{1}{2}\int \rho V \, d\tau$ $\Rightarrow$ $W_e = \frac{\epsilon_0}{2}\int E^2 \, d\tau$

Likewise, the **work required to get currents going (against the back emf)** is (Eq. 7.34)

Energy of steady Current flowing $\Rightarrow$ $W = \frac{1}{2} I \oint \mathbf{A} \cdot d\mathbf{l}$ $\Rightarrow$ $W_m = \frac{1}{2\mu_0}\int B^2 \, d\tau$

Therefore, ***the total energy stored in electromagnetic fields*** **is**

$$\Rightarrow \boxed{U_{em} = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) d\tau}$$

➔ Let's derive this total energy stored in EM fields more generally in the context of the energy conservation law for electrodynamics.

➔ "Energy conservation law for electrodynamics": Poynting Theorem

Energy Conservation and Poynting's Theorem

Suppose we have some charge and current configuration which, at time *t,* produces fields **E** and **B**.
In the next instant, ***dt,*** the charges move around a bit.
→ ***How much work, dW, is done by the electromagnetic forces*** acting on these charges in the interval ***dt*** ?

According to the Lorentz force law, the work done on a charge *q* is

$$dW = \mathbf{F} \cdot d\mathbf{l} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}\,dt = q\mathbf{E} \cdot \mathbf{v}\,dt$$

$$q = \rho d\tau \qquad \rho\mathbf{v} = \mathbf{J}$$

$$\boxed{\frac{dW}{dt} = \int_{\mathcal{V}} (\mathbf{E} \cdot \mathbf{J})\,d\tau}$$

$\mathbf{E} \cdot \mathbf{J}$ → **the work done per unit time, per unit volume, or, the *power* delivered per unit volume.**

Ampere-Maxwell law → $\mathbf{E} \cdot \mathbf{J} = \frac{1}{\mu_0}\mathbf{E} \cdot (\nabla \times \mathbf{B}) - \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t}$

$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B})$ and Faraday's law $(\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t)$,

$$\mathbf{E} \cdot (\nabla \times \mathbf{B}) = -\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{B})$$

$$\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(B^2), \quad \text{and} \quad \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(E^2)$$

$$\mathbf{E} \cdot \mathbf{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\nabla \cdot (\mathbf{E} \times \mathbf{B})$$

$$\boxed{\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau - \frac{1}{\mu_0}\oint_{\mathcal{S}} (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{a}}$$

➔ **Poynting's theorem**

➔ **This is “Work-Energy Theorem" or “Energy Conservation Theorem” of Electrodynamics.**

Poynting's Theorem and Poynting Vector

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau - \frac{1}{\mu_0}\oint_{\mathcal{S}} (\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a}$$

➔ *Poynting's theorem*
➔ *Work-Energy Theorem or Energy Conservation Theorem of Electrodynamics.*

The first integral on the right is the total energy stored in the fields → $\int_{\mathcal{V}} \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau = U_{\text{em}}$

The second term evidently represents the rate at which energy is carried out of *V,* across its boundary surface, by the fields. → $\frac{1}{\mu_0}\oint_{\mathcal{S}} (\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a}$

Poynting's theorem says

➔ ***"the work done on the charges by the electromagnetic force is equal to the decrease in energy stored in the field, less the energy that flowed out through the surface."***

The *energy per unit time, per unit area,* transported by the fields is called the *Poynting vector*:

$$\mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) \quad (\mathrm{W/m^2})$$ ➔ ***Poynting vector***

$$\mathbf{S} \equiv (\mathbf{E}\times\mathbf{H})$$ ➔ ***Poynting vector in linear media***

Poynting's theorem ➔ $$\boxed{\frac{dW}{dt} = -\frac{dU_{\text{em}}}{dt} - \oint_{\mathcal{S}} \mathbf{S}\cdot d\mathbf{a}}$$

→ **S** · ***da*** is the energy per unit time crossing the infinitesimal surface ***da***
→ the energy *flux,* if you like (so ***S*** is the energy flux density).

Poynting's Theorem and Poynting Vector

$$\frac{dW}{dt} = -\frac{dU_{\text{em}}}{dt} - \oint_{\mathcal{S}} \mathbf{S} \cdot d\mathbf{a} \qquad \frac{dW}{dt} = \int_{\mathcal{V}} (\mathbf{E} \cdot \mathbf{J})\, d\tau \qquad U_{\text{em}} = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) d\tau \qquad \mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$$

The work ***W*** done on the charges by the fields will increase their mechanical energy (kinetic, potential, or whatever).

→ If we let u_{mech} denote the mechanical energy density,

$$\frac{dW}{dt} = \int_{\mathcal{V}} (\mathbf{E} \cdot \mathbf{J})\, d\tau = \frac{d}{dt}\int_{\mathcal{V}} u_{\text{mech}}\, d\tau$$

→ If we let u_{em} denote the electromagnetic energy density,

$$U_{em} = \int_V u_{em} d\tau \quad \rightarrow \quad u_{em} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) = \frac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}\right)$$

$$\frac{dW}{dt} = -\frac{dU_{\text{em}}}{dt} - \oint_{\mathcal{S}} \mathbf{S} \cdot d\mathbf{a} \longrightarrow \frac{d}{dt}\int_{\mathcal{V}} (u_{\text{mech}} + u_{\text{em}})\, d\tau = -\oint_{\mathcal{S}} \mathbf{S} \cdot d\mathbf{a} = -\int_{\mathcal{V}} (\nabla \cdot \mathbf{S})\, d\tau$$

$$\boxed{\frac{\partial}{\partial t}(u_{\text{mech}} + u_{\text{em}}) = -\nabla \cdot \mathbf{S}}$$ ➔ ***differential version of Poynting's theorem***

➔ Compare it with the continuity equation, expressing conservation of charge: $\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}$

➔ The charge density is replaced by the energy density (mechanical plus electromagnetic),
➔ the current density is replaced by the Poynting vector.

➔***Therefore, Poynting's theorem represents the flow of energy***
in in exactly the same way that **J** in the continuity equation describes the flow of charge.

Poynting's Theorem is the "Work-energy theorem" or "Conservation of Energy"

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau - \frac{1}{\mu_0}\oint_{\mathcal{S}} (\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a} \quad \leftrightarrow \quad \frac{dW}{dt} = -\frac{dU_{\mathrm{em}}}{dt} - \oint_{\mathcal{S}} \mathbf{S}\cdot d\mathbf{a} \quad \leftrightarrow \quad \frac{\partial}{\partial t}(u_{\mathrm{mech}} + u_{\mathrm{em}}) = -\nabla\cdot\mathbf{S}$$

$$\boxed{\frac{dW}{dt} = -\frac{dU_{em}}{dt} - \oint_S S\cdot ds}$$

$S \equiv \frac{1}{\mu}(E\times B)$: Poynting vetor

Work done by the EM field

Energy flowed out through the surface

Total energy stored in the EM field

"The work done on the charges by the electromagnetic force is equal to the decrease in energy stored in the field, less the energy that flowed out through the surface" .

$$\Rightarrow \frac{dW}{dt} = \int_V (E\cdot J)\,dv = -\frac{d}{dt}\int_V \left(\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}B^2\right)dv - \frac{1}{\mu_0}\oint_S (E\times B)\cdot ds$$

$$\frac{\partial u_{em}}{\partial t} = -\nabla\cdot S - E\cdot J$$ ➔ ***differential version of Poynting's theorem***

Poynting's theorem

$$\frac{\partial u_{em}}{\partial t} = -\nabla \cdot \mathbf{S} - \mathbf{E} \cdot \mathbf{J} \qquad \mathbf{S} = \mathbf{E} \times \mathbf{H}$$

Let's prove it directly from Maxwell's equations

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \vec{H} \cdot (\vec{\nabla} \times \vec{E}) = -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \Rightarrow \vec{E} \cdot (\vec{\nabla} \times \vec{H}) = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}$$

$$\vec{E} \cdot (\vec{\nabla} \times \vec{H}) - \vec{H} \cdot (\vec{\nabla} \times \vec{E}) = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t}$$

$$-\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t}$$

$$\vec{S} \equiv \vec{E} \times \vec{H} \longrightarrow \vec{\nabla} \cdot \vec{S} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} = -\vec{E} \cdot \vec{J}$$

$$\boxed{\Rightarrow \frac{\partial u_{em}}{\partial t} + \vec{\nabla} \cdot \vec{S} + \vec{E} \cdot \vec{J} = 0}$$ ***: Poynting's theorem***

For $\vec{J} = 0$ (in free space), $\Rightarrow \dfrac{\partial u_{em}}{\partial t} = -\vec{\nabla} \cdot \vec{S}$

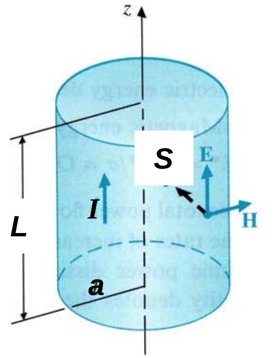
For a steady state $\dfrac{\partial u_{em}}{\partial t} = 0$, $\Rightarrow -\vec{\nabla} \cdot \vec{S} = \vec{E} \cdot \vec{J}$

Poynting's theorem: $\frac{dW}{dt} = -\frac{dU_{em}}{dt} - \oint_S \mathbf{S}\cdot d\mathbf{a}$ $\quad \frac{\partial u_{em}}{\partial t} = -\nabla\cdot\mathbf{S} - \mathbf{E}\cdot\mathbf{J}$ $\quad \mathbf{S} = \mathbf{E}\times\mathbf{H}$

For steady current case, $\frac{\partial u_{em}}{\partial t} = 0 \longrightarrow \frac{dW}{dt} = -\oint_S \mathbf{S}\cdot d\mathbf{a} \quad \left(or,\ \ \mathbf{E}\cdot\mathbf{J} = -\vec{\nabla}\cdot\mathbf{S}\right)$

$$\longrightarrow \frac{dW}{dt} = \mathrm{P\ (in\ Joule\ heating)} = IV = I^2R = -\oint_S \mathrm{S}\cdot da$$

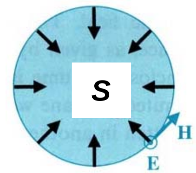
Example 8.1 When current flows down a wire, work is done, which shows up as Joule heating of the wire. Find the Poynting vector (that is, the energy per unit time per unit time delivered to the wire).



The electric field parallel to the wire → $E = \frac{V}{L}$

The magnetic field is "circumferential"; at the surface (radius *a)* → $B = \frac{\mu_0 I}{2\pi a}$

Accordingly, the magnitude of the ***Poynting vector (it points radially inward)*** is



$$\mathbf{S} = \mathbf{E}\times\mathbf{H} \longrightarrow S = -\frac{1}{\mu_0}\frac{V}{L}\frac{\mu_0 I}{2\pi a} = \frac{VI}{2\pi a L}$$

The energy per unit time passing in through the surface of the wire is therefore $\int \mathbf{S}\cdot d\mathbf{a} = S(2\pi a L) = VI$

Poynting's theorem: $\boxed{\dfrac{\partial u_{em}}{\partial t} = -\nabla\cdot\vec{S} - \vec{E}\cdot\vec{J}}$ $\quad u_{em} = \dfrac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}\right), \quad \vec{S} = \vec{E}\times\vec{H}$

From Ohm's law, $E\cdot J = E\cdot(\sigma E) = \sigma E^2 = (J/\sigma)\cdot J = \dfrac{1}{\sigma}J^2$

$$\Rightarrow \boxed{\frac{\partial u_{em}}{\partial t} = -\nabla\cdot\vec{S} - \sigma E^2 = -\nabla\cdot\vec{S} - \frac{1}{\sigma}J^2}$$

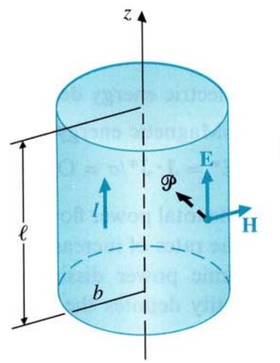
: Poynting's theorem in Ohmic materials

$$I^2R = -\oint_S \mathrm{S}\cdot d\mathrm{a}$$

→ For steady current case

Example. (a) Find the Poynting vector on the surface of a long, straight conducting wire (of radius ***b*** and conductivity ***σ***) that carries a steady current ***I***.

(b) Verify Poynting's theorem.

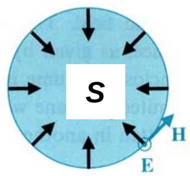


$$J = a_z\frac{I}{\pi b^2}, \quad E = \frac{J}{\sigma} = a_z\frac{I}{\sigma\pi b^2}$$

$H = a_\phi \dfrac{I}{2\pi b}$ on the surface of the wire.

$$S = E\times H = (a_z\times a_\phi)\frac{I^2}{2\sigma\pi^2 b^3} = -a_r\frac{I^2}{2\sigma\pi^2 b^3}$$

which is directed everywhere into the wire surface.



To verify Poynting's theorem,

$$-\oint_S S\cdot d\mathbf{s} = -\oint_S S\cdot\mathbf{a}_r ds = \left(\frac{I^2}{2\sigma\pi^2 b^3}\right)2\pi bl = I^2\left(\frac{l}{\sigma\pi b^2}\right) = I^2R$$

Poynting's vector and Poynting theorem in matter

Problem 8.23 Describe the Poynting's vector and Poynting theorem for the filed in matter.

$$\frac{dW}{dt} = \int_{\mathcal{V}} (\mathbf{E}\cdot\mathbf{J})\,d\tau \longrightarrow \frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau - \frac{1}{\mu_0}\oint_{\mathcal{S}} (\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a}$$ ➔ **Poynting's theorem in vacuum**

The work done on free charges and currents in matter,

$$\frac{dW}{dt} = \int_{\mathcal{V}} (\mathbf{E}\cdot\mathbf{J})\,d\tau \longrightarrow \frac{dW}{dt} = \int_{\mathcal{V}} (\mathbf{E}\cdot\mathbf{J}_f)\,d\tau \xrightarrow{\mathbf{J}_f = \nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t}} \mathbf{E}\cdot\mathbf{J}_f = \mathbf{E}\cdot(\nabla\times\mathbf{H}) - \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}$$

$$\nabla\cdot(\mathbf{E}\times\mathbf{H}) = \mathbf{H}(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{H}),\ \text{while}\ \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \longrightarrow \mathbf{E}\cdot(\nabla\times\mathbf{H}) = -\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} - \nabla\cdot(\mathbf{E}\times\mathbf{H})$$

$$\mathbf{E}\cdot\mathbf{J}_f = -\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} - \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} - \nabla\cdot(\mathbf{E}\times\mathbf{H})$$

$$\longrightarrow \boxed{\frac{dW}{dt} = -\int_{\mathcal{V}}\left(\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\right)d\tau - \oint_{\mathcal{S}}(\mathbf{E}\times\mathbf{H})\cdot d\mathbf{a}}$$ ➔ **Poynting's theorem for the fields in matter**

$$\boxed{\mathbf{S} = \mathbf{E}\times\mathbf{H}}$$ ➔ **Poynting vector in matter**

$$\frac{\partial u_{\text{em}}}{\partial t} = \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} \longrightarrow$$ the rate of change of the electromagnetic energy density

For *linear* media, $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{H} = \frac{1}{\mu}\mathbf{B}$

$$\frac{\partial u_{\text{em}}}{\partial t} = \epsilon\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} + \frac{1}{\mu}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} = \frac{1}{2}\epsilon\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{E}) + \frac{1}{2\mu}\frac{\partial}{\partial t}(\mathbf{B}\cdot\mathbf{B}) = \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$$

$$\boxed{u_{\text{em}} = \tfrac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})}$$ ➔ **Electromagnetic energy density in matter**