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Legendre Polynomials

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21.1 Legendre Polynomials

21.1.1 Definition

$$P_n(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2n-2k)! t^{n-2k}}{2^n k! (n-k)! (n-2k)!}$$

$$[n/2] = \begin{cases} n/2 & n \text{ even} \\ (n-1)/2 & n \text{ odd} \end{cases}$$

21.1.2 Generating Function

$$w(t,s) = \frac{1}{\sqrt{1 - 2st + s^2}} = \begin{cases} \sum_{n=0}^{\infty} P_n(t) s^n & |s| < 1\\ \sum_{n=0}^{\infty} P_n(t) s^{-n-1} & |s| > 1 \end{cases}$$
 generating function
$$w(-t, -s) = w(t, s)$$

21.1.3 Rodrigues Formula

$$P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n} (t^2 - 1)^n \qquad n = 0, 1, 2 \cdots$$

21.1.4 Recursive Formulas

1.
$$(n+1)P_{n+1}(t) - (2n+1)tP_n(t) + nP_{n-1}(t) = 0$$
 $n = 1, 2, \dots$

2.
$$P'_{n+1}(t) - tP'_n(t) = (n+1)P_n(t)$$
 $(P'(t) \doteq \text{derivative of } P(t)$ $n = 0,1,2,\cdots$

3.
$$t P'_n(t) - P'_{n-1}(t) = n P_n(t)$$
 $n = 1, 2, \cdots$
4. $P'_{n+1}(t) - P'_{n-1}(t) = (2n+1)P_n(t)$ $n = 1, 2, \cdots$

5.
$$(t^2 - 1)P'_n(t) = ntP_n(t) - nP_{n-1}(t)$$

6.
$$P_0(t) = 1$$
 $P_1(t) = t$

TABLE 21.1 Legendre Polynomials

$$\begin{split} P_0 &= 1 \\ P_1 &= t \\ P_2 &= \frac{3}{2}t^2 - \frac{1}{2} \\ P_3 &= \frac{5}{2}t^3 - \frac{3}{2}t \\ P_4 &= \frac{35}{8}t^4 - \frac{30}{8}t^2 + \frac{3}{8} \\ P_5 &= \frac{63}{8}t^5 - \frac{70}{8}t^3 + \frac{15}{8}t \\ P_6 &= \frac{231}{16}t^6 - \frac{316}{16}t^4 + \frac{105}{16}t^2 - \frac{5}{16} \\ P_7 &= \frac{429}{16}t^7 - \frac{693}{16}t^5 + \frac{315}{16}t^3 - \frac{35}{16}t \end{split}$$

Figure 21.1 shows a few Legendre functions.

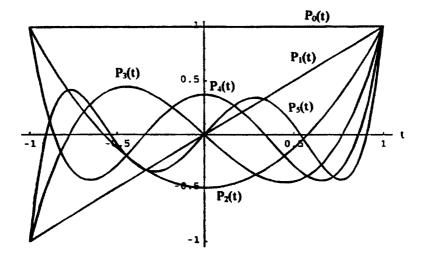


FIGURE 21.1

21.1.5 Legendre Differential Equation

If $y = P_n(x)$ $(n = 0, 1, 2, \dots)$ is a solution to the second-order DE

For
$$t = \cos \varphi$$
:
$$\frac{1}{\sin \varphi} \frac{d}{d\varphi} \left(\sin \varphi \frac{dy}{d\varphi} \right) + n(n+1)y = 0$$

Example

From (21.1.4.4) and t = 1 implies $0 = nP_n(1) - nP_{n-1}(1)$ or $P_n(1) = P_{n-1}(1)$. For n = 1, $P_1(1) = P_0(1) = 1$. For n = 2, $P_2(1) = P_1(1) = 1$ and so forth. Hence $P_n(1) = 1$.

21.1.6 Integral Representation

1. Laplace integral:
$$P_n(t) = \frac{1}{\pi} \int_0^{\pi} [t + \sqrt{t^2 - 1} \cos \varphi]^n d\varphi$$

2. Mehler-Dirichlet formula:
$$P_n(\cos\theta) = \frac{2}{\pi} \int_0^\theta \frac{\cos(n+\frac{1}{2})\psi}{\sqrt{2\cos\psi - \cos\theta}} d\psi \quad 0 < \theta < \pi, n = 0, 1, 2, \dots$$

3. Schläfli integral:
$$P_n(t) = \frac{1}{2\pi j} \int_C \frac{(z^2 - 1)^n}{2^n (z - t)^{n+1}} dz$$

where C is any regular, simple, closed curve surrounding t.

21.1.7 Complete Orthonormal System

$$\{\left[\frac{1}{2}(2n+1)\right]^{1/2}P_n(t)\}$$

The Legendre polynomials are orthogonal in [-1,1]

$$\int_{-1}^{1} P_n(t) P_m(t) dt = 0$$

$$\int_{-1}^{1} [P_n(t)]^2 dt = \frac{2}{2n+1} \qquad n = 0,1,2 \dots$$

and therefore the set

$$\phi_n(t) = \sqrt{\frac{2n+1}{2}} P_n(t) \qquad n = 0,1,2\dots$$

is orthonormal.

21.1.8 Asymptotic Representation:

$$P_n(\cos\theta) \cong \sqrt{\frac{2}{\pi n \sin \theta}} \sin \left[\left(n + \frac{1}{2} \right) \theta + \frac{\pi}{4} \right], \qquad n \to \infty, \quad \delta \le \theta \le \pi - \delta$$

 δ = fixed positive number

21.1.9 Series Expansion

If f(t) is integrable in [-1,1] then

$$f(t) = \sum_{n=0}^{\infty} a_n P_n(t) -1 < t < 1$$

$$a_n = \frac{2n+1}{2} \int_{1}^{1} f(t) P_n(t) dt$$
 $n = 0,1,2\cdots$

For even f(t), the series will contain term $P_n(t)$ of even index; if f(t) is odd, the term of odd index only. If the real function f(t) is piecewise smooth in (-1,1) and if it is square integrable in (-1,1), then the series converges to f(t) at every continuity point of f(t). If there is a discontinuity at t then the series converges at [f(t+0)+f(t-0)]/2.

21.1.10 Change of Range

If a function f(t) is defined in [a,b], it is sometimes necessary in the applications to expand the function in a series in the applications to expand the function in a series of orthogonal polynomials in this interval. Clearly the substitution

$$t = \frac{2}{b-a} \left[x - \frac{b+a}{2} \right], \qquad a < b, \qquad \left[x = \frac{b-a}{2} t + \frac{b+a}{2} \right]$$

transform the interval [a,b] of the x-axis into the interval [-1,1] of the t-axis. It is, therefore, sufficient to consider

$$f\left[\frac{b-a}{2}t + \frac{b+a}{2}\right] = \sum_{n=0}^{\infty} a_n P_n(t)$$
$$a_n = \frac{2n+1}{2} \int_{-1}^{1} f\left[\frac{b-a}{2}t + \frac{b+a}{2}\right] P_n(t) dt$$

The above equation can also be accomplished as follows:

$$f(t) = \sum_{n=0}^{\infty} a_n X_n(t)$$

$$X_n(t) = \frac{1}{n!(b-a)^n} \frac{d^n (t-a)^n (t-b)^n}{dt^n}$$

$$a_n = \frac{2n+1}{b-a} \int_{t-a}^{a} f(t) X_n(t) dt$$

Example

Suppose f(t) is given by

$$f(t) = \begin{cases} 0 & -1 \le t < a \\ 1 & a < t \le 1 \end{cases}$$

Then

$$a_n = \frac{2n+1}{2} \int_a^1 P_n(t) dt$$

Using (21.1.4.4), and noting that $P_n(1) = 1$ we obtain

$$a_n = -\frac{1}{2}[P_{n+1}(a) - P_{n-1}(a)], \qquad a_0 = \frac{1}{2}(1-a)$$

which leads to the expansion

$$f(t) \cong \frac{1}{2}(1-a) - \frac{1}{2} \sum_{n=1}^{\infty} [P_{n+1}(a) - P_{n-1}(a)] P_n(t), \qquad -1 < t < 1$$

Example

Suppose f(t) is given by

$$f(t) = \begin{cases} -1 & -1 \le t < 0 \\ 1 & 0 < t \le 1 \end{cases}$$

The function is an odd function and, therefore, $f(t)P_n(t)$ is an odd function of $P_n(t)$ with even index. Hence a_n are zero for n = 0,2,4,... For odd index n, the product $f(t)P_n(t)$ is even and hence

$$a_n = \left(n + \frac{1}{2}\right) \int_{-1}^{1} f(t) P_n(t) dt = 2\left(n + \frac{1}{2}\right) \int_{0}^{1} P_n(t) dt$$
 $n = 1, 3, 5, \dots$

Using (21.1.4.4) and setting n = 2k + 1, k = 0,1,2... we obtain

$$a_{2k+1} = (4k+3) \int_{0}^{1} P_{2k+1}(t) dt = \int_{0}^{1} [P'_{2k+2}(t) - P'_{2k}(t)] dt$$
$$= [P_{2k+2}(t) - P_{2k}(t)]_{0}^{1} = P_{2k}(0) - P_{2k+2}(0)$$

where we have used the property $P_n(1) = 1$ for all n. But

$$P_{2n}(0) = {-\frac{1}{2} \choose n} = \frac{(-1)^n (2n)!}{2^{2n} (n!)^2}$$

and, thus, we have

$$a_{2k+1} = \frac{(-1)^k (2k)!}{2^{2k} (k!)^2} - \frac{(-1)^{k+1} (2k+2)}{2^{2k+2} [(k+1)!]^2} = \frac{(-1)^k (2k)!}{2^{2k} (k!)^2} \left[1 + \frac{2k+1}{2k+2} \right]$$
$$= \frac{(-1)^k (2k)! (4k+3)}{2^{2k+1} k! (k+1)!}$$

The expansion is

$$f(t) = \sum_{n=0}^{\infty} \frac{(-1)^k (2k)! (4k+3)}{2^{2k+1} k! (k+1)!} P_{2k+1}(t) -1 \le t \le 1$$

21.1.11 Expansion of Polynomials

If $q_m(t) = \sum_{k=0}^m c_k x^k$ is an arbitrary polynomial, then $q_m(t) = c_0 P_0(t) + c_1 P_1(t) + \dots + c_m P_m(t)$ where $c_n = \left(n + \frac{1}{2}\right) \int_{-1}^1 q_m(t) P_n(t) \, dt = 0$, $n = 0, 1, 2 \cdots$. If $q_m(t)$ is a polynomial of degree m and m < r, then $\int_{-1}^1 q_m(t) P_r(t) \, dt = 0$, m < r.

Example

To find $P_{2n}(0)$ we use the summation

$$P_{2n}(t) = \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n} \frac{(-1)^k (2n+2k-1)!}{(2k)!(n+k-1)!(n-k)!} t^{2k}$$

with k = 0. Hence

$$P_{2n}(0) = \frac{(-1)^n (2n-1)!}{2^{2n-1} (n-1)! n!} = \frac{(-1)^n 2n[(2n-1)!]}{2^{2n} n[(n-1)!] n!} = \frac{(-1)^n (2n)!}{2^{2n} (n!)^2}$$

Example

To evaluate $\int_0^1 P_m(t) dt$ for $m \neq 0$ we must consider the two cases: m = odd and m = even.

(a) $m = \text{even and } m \neq 0$

$$\int_{0}^{1} P_{m}(t)dt = \frac{1}{2} \int_{-1}^{1} P_{m}(t)dt = \frac{1}{2} \int_{-1}^{1} P_{m}(t) \cdot 1dt = \frac{1}{2} \int_{-1}^{1} P_{m}(t) P_{0}(t)dt = 0$$

The result is due to the orthogonality principle.

(b) m = odd and $m \neq 0$. From the relation (see Table 21.2)

$$\int_{-1}^{1} P_m(t) dt = \frac{1}{2m+1} [P_{m-1}(t) - P_{m+1}(t)]$$

with t = 0 we obtain

$$\int_{0}^{1} P_{m}(t) dt = \frac{1}{2m+1} [P_{m-1}(0) - P_{m+1}(0)]$$

Using the results of the previous example, we obtain

$$\int_{0}^{1} P_{m}(t) dt = \frac{1}{2m+1} \left[\frac{(-1)^{\frac{m-1}{2}} (m-1)!}{2^{m-1} \left[\left(\frac{m-1}{2} \right)! \right]^{2}} - \frac{(-1)^{\frac{m+1}{2}} (m+1)!}{2^{m+1} \left[\left(\frac{m+1}{2} \right)! \right]^{2}} \right]$$

$$= \frac{(-1)^{\frac{m-1}{2}} (m-1)! (2m+1) (m+1)}{(2m+1) 2^{m+1} \left(\frac{m+1}{2} \right)! \left(\frac{m-1}{2} \right)!} = \frac{(-1)^{\frac{m-1}{2}} (m-1)!}{2^{m} \left(\frac{m+1}{2} \right)! \left(\frac{m-1}{2} \right)!} \qquad m = \text{odd}$$

21.2 Legendre Functions of the Second Kind (Second Solution)

21.2.1 Second Kind:

1.
$$Q_0 = \frac{1}{2} \ln \frac{1+t}{1-t}$$
, $|t| < 1$;

2.
$$Q_1(t) = \frac{1}{2}t \ln \frac{1+t}{1-t} - 1$$
, $|t| < 1$;

3.
$$Q_{n+1}(t) = \frac{2n+1}{n+1}tQ_n(t) - \frac{n}{n+1}Q_{n-1}(t), \quad n = 1, 2, \dots$$

4.
$$Q_n(t) = P_n(t)Q_0(t) - \sum_{k=0}^{\left[\frac{1}{2}(n-1)\right]} \frac{2n-4k-1}{(2k+1)(n-k)} P_{n-2k-1}(t), \quad |t| < 1, \quad n = 1, 2, \dots$$
 for $\left[\frac{1}{2}(n-1)\right]$ see 21.1.1.

21.2.2 Recursions

 $Q_n(t)$ satisfies all the recurrence relations of $P_n(t)$.

21.2.3 Property

$$\frac{1}{x-t} = \sum_{n=0}^{\infty} (2n+1) P_n(t) Q_n(x)$$

21.2.4 Newman Formula

$$Q_n(t) = \frac{1}{2} \int_{-1}^{1} \frac{P_n(x)}{t - x} dx, \qquad n = 0, 1, 2 \dots$$

21.3 Associated Legendre Polynomials

21.3.1 Definition

If m is a positive integer and $-1 \le t \le 1$, then

$$P_n^m(t) = (1 - t^2)^{m/2} \frac{d^m P_n(t)}{dt^m}$$
 $m = 1, 2, \dots, n$

where $P_n^m(t)$ is known as the associated Legendre function or Ferrers' functions.

21.3.2 Rodrigues Formula

$$P_n^m(t) = \frac{(1-t^2)^{m/2}}{2^n n!} \frac{d^{n+m}}{dt^{n+m}} (t^2 - 1)^n, \qquad m = 1, 2, \dots, n; \quad n+m \ge 0$$

21.3.3 Properties

1.
$$P_n^{-m}(t) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(t)$$

2.
$$P_{n}^{0}(t) = P_{n}(t)$$

3.
$$(n-m+1)P_{n+1}^m(t)-(2n+1)tP_n^m(t)+(n+m)P_{n-1}^m(t)=0$$

4.
$$(1-t^2)^{1/2} P_n^m(t) = \frac{1}{2n+1} [P_{n+1}^{m+1}(t) - P_{n-1}^{m+1}(t)]$$

5.
$$(1-t^2)^{1/2} P_n^m(t) = \frac{1}{2n+1} [(n+m)(n+m-1)P_{n-1}^{m-1}(t) - (n-m+1)(n-m+2)P_{n+1}^{m-1}(t)]$$

6.
$$P_n^m(t) = 2mt(1-t^2)^{-1/2}P_n^m(t) - [n(n+1)-m(m-1)]P_n^{m-1}(t)$$

7.
$$P_n^{m+1}(t) = (t^2 - 1)^{-1/2} [(n-m) t P_n^m(t) - (n+m) P_{n-1}^m(t)]$$

8.
$$P_{n+1}^m(t) = P_{n-1}^m(t) + (2n+1)(t^2-1)^{1/2} P_n^{m-1}(t)$$

9.
$$P_n^m(t) = (t^2 - 1)^{m/2} \frac{d^m P_n(t)}{dt^m}$$

10.
$$\int_{-1}^{1} P_n^m(t) P_k^m(t) dt = 0 \qquad k \neq n$$

11.
$$\int_{-1}^{1} \left[P_n^m(t) \right]^2 dt = \frac{2(n+m)!}{(2n+1)(n-m)!}$$

21.3.4 Differential Equation

$$(1-t^2)\frac{d^2 P_n^m(t)}{dt^2} - 2t\frac{d P_n^m(t)}{dt} + \left[n(n+1) - \frac{m^2}{(1-t^2)}\right] P_n^m(t) = 0$$

21.3.5 Schlafli Formula

$$P_n^m(t) = \frac{(n+m)!}{2\pi \, in!} (1-t^2)^{m/2} \oint_C \frac{(x^2-1)^n}{2^n (x-t)^{n+m+1}} dx$$

where C is any regular closed curve surrounding the point t and taking it counterclockwise.

21.4 Bounds for Legendre Polynomials

21.4.1 Stieltjes Theorem

$$|P_n(\cos\gamma)| \le \sqrt{2} \frac{4}{\sqrt{\pi}} \frac{1}{\sqrt{n} \sqrt{\sin\gamma}}, \qquad 0 < \gamma < \pi, \quad n = 1, 2, \dots$$

21.4.2 Second Stieltjes Theorem

$$|P_n(t) - P_{n+2}(t)| < \frac{4}{\sqrt{\pi} \sqrt{n+2}}$$

21.4.3
$$\left| \frac{dP_n(t)}{dt} \right| < \frac{2}{\sqrt{\pi}} \frac{\sqrt{n}}{1 - t^2}, \quad |t| < 1, \quad n = 1, 2, \cdots$$

21.4.4
$$|P_{n+1}(t) + P_n(t)| < \frac{6\sqrt{2}}{\sqrt{\pi}} \frac{1}{\sqrt{n}} \frac{1}{\sqrt{1-t}}, \qquad |t| < 1$$

21.5 Table of Legendre and Associate Legendre Functions

TABLE 21.2 Properties of Legendre and Associate Legendre Functions $[P_n(t)]$ = Legendre Functions, $P_n^m(t)$ = Associate Legendre Functions, $Q_n(t)$ = Legendre Functions of the Second Kind]

1.
$$\frac{1}{\sqrt{1 - 2t \, x + x^2}} = \sum_{n=0}^{\infty} P_n(t) \, x^n \qquad |t| \le 1 \qquad |x| < 1$$

2.
$$P_n(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2n-2k)! t^{n-2k}}{2^n k! (n-k)! (n-2k)!} \qquad \lfloor n/2 \rfloor = \frac{n}{2} \qquad n = \text{even}; \qquad \lfloor n/2 \rfloor = (n-1)/2 \qquad n = \text{odd}$$

3.
$$P_0(t) = 1$$

4.
$$P_{2n}(0) = {\binom{-\frac{1}{2}}{n}} = \frac{(-1)^n (2n)!}{2^{2n} (n!)^2} \qquad n = 1, 2, \dots$$

5.
$$P_{2n+1}(0) = 0$$
 $n = 0, 1, 2 \cdots$

6.
$$P_{2n}(-t) = P_{2n}(t)$$
 $P_{2n+1}(-t) = -P_{2n+1}(t)$ $n = 0, 1, 2 \cdots$

7.
$$P_{\nu}(-t) = (-1)^n P_{\nu}(t)$$
 $n = 0,1,2...$

8.
$$P_n(1) = 1$$
 $n = 0,1,2\cdots$; $P_n(-1) = (-1)^n$ $n = 0,1,2\cdots$

9.
$$P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n} (t^2 - 1)^n = \text{Rodrigues formula}, \quad n = 0, 1, 2 \dots$$

10.
$$(n+1)P_{n+1}(t) - (2n+1)tP_n(t) + nP_{n+1}(t) = 0$$
 $n = 1, 2, \cdots$

11.
$$P'_{n+1}(t) - 2t P'_n(t) + P'_{n-1}(t) - P_n(t) = 0$$
 $n = 1, 2, \dots$

12.
$$P'_{n+1}(t) = P_n(t) + 2t P'_n(t) - P'_{n+1}(t)$$

13.
$$P'_{n+1}(t) = P_n(t) + 2t P'_n(t) - P'_{n-1}(t)$$

14.
$$P'_{n+1}(t) - t P'_{n}(t) = (n+1)P_{n}(t)$$

15.
$$t P'_n(t) - P'_{n-1}(t) = n P_n(t)$$
 $n = 1, 2, \cdots$

16.
$$P'_{n+1}(t) - P'_{n-1}(t) = (2n+1)P_n(t)$$
 $n = 1, 2, \cdots$

17.
$$(1-t^2)P'_n(t) = nP_{n-1}(t) - ntP_n(t)$$

18.
$$|P_n(t)| < 1, -1 < t < 1$$

19.
$$P_{2n}(t) = \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^n \frac{(-1)^k (2n+2k-1)!}{(2k)!(n+k-1)!(n-k)!} t^{2k}$$
 $n = 0,1,2\cdots$

20.
$$(1-t^2)P'_n(t) = (n+1)[tP_n(t) - P_{n+1}(t)]$$
 $n = 0,1,2\cdots$

21.
$$\int_{1}^{1} P_{n}(t)dt = 0 \qquad n = 1, 2, \dots$$

22.
$$|P_{n}(t)| \le 1$$
 $|t| \le 1$

TABLE 21.2 Properties of Legendre and Associate Legendre Functions $[P_n(t)]$ = Legendre Functions, $P_n^m(t)$ = Associate Legendre Functions, $Q_n(t)$ = Legendre Functions of the Second Kind] (continued)

23.
$$\int_{-1}^{1} P_{n}(t)P_{m}(t)dt = 0 \qquad n \neq m$$
24.
$$\int_{-1}^{1} \left[P_{n}(t)\right]^{2} dt = \frac{2}{2n+1} \qquad n = 0.1, 2 \cdots$$
25.
$$\frac{1}{2} \int_{-1}^{1} t^{m} P_{s}(t) dt = \frac{m(m-2)\cdots(m-s+2)}{(m+s+1)(m+s-1)\cdots(m+1)} \qquad m, s = \text{even}$$
26.
$$\frac{1}{2} \int_{-1}^{1} t^{m} P_{s}(t) dt = \frac{(m-1)(m-3)\cdots(m-s+2)}{(m+s+1)(m+s-1)\cdots(m+2)} \qquad m, s = \text{odd}$$
27.
$$\int_{-1}^{1} t P_{n}(t) P_{n-1}(t) dt = \frac{2n}{4n^{2}-1} \qquad n = 1, 2, \cdots$$
28.
$$\int_{-1}^{1} t P_{n}(t) P_{n+1}(t) dt = 2 \qquad n = 0, 1, 2 \cdots$$
29.
$$\int_{-1}^{1} t P_{n}'(t) P_{n}'(t) dt = \frac{2n}{2n+1} \qquad n = 0, 1, 2 \cdots$$
30.
$$\int_{-1}^{1} (1-t^{2}) P_{n}'(t) P_{s}'(t) dt = 0 \qquad k \neq n$$
31.
$$\int_{-1}^{1} (1-t)^{-1/2} P_{n}(t) dt = \frac{2\sqrt{2}}{2n+1} \qquad n = 0, 1, 2 \cdots$$
32.
$$\int_{-1}^{1} t^{2} P_{n+1}(t) P_{n-1}(t) dt = \frac{2n(n+1)}{(4n^{2}-1)(2n+3)} \qquad n = 1, 2, \cdots$$
33.
$$\int_{-1}^{1} (t^{2}-1) P_{n+1}(t) P_{n}'(t) dt = \frac{2n(n+1)}{(2n+1)(2n+3)} \qquad n = 1, 2, \cdots$$
34.
$$\int_{-1}^{1} t^{n} P_{n}(t) dt = \frac{2^{n+1}(n!)^{2}}{(2n+1)!} \qquad n = 0, 1, 2 \cdots$$
35.
$$\int_{-1}^{1} t^{2} [P_{n}(t)]^{2} dt = \frac{2}{(2n+1)^{2}} \left[\frac{(n+1)^{2}}{2n+3} + \frac{n^{2}}{2n-1} \right] \qquad n = 0, 1, 2 \cdots$$
36.
$$P_{n}^{m}(t) = (1-t^{2})^{m/2} \frac{d^{m}}{dt^{m}} P_{n}(t) \qquad m > 0$$
37.
$$P_{n}^{m}(t) = \frac{1}{2^{n} n!} (1-t^{2})^{m/2} \frac{d^{n+m}}{dt^{n+m}} [(t^{2}-1)^{n}] \qquad m+n \geq 0$$
38.
$$P_{n}^{-m}(t) = (-1)^{m} \frac{(n-m)!}{(n+m)!} P_{n}^{m}(t)$$
39.
$$P_{n}^{n}(t) = P_{n}(t), \quad P_{n}^{m}(t) = 0 \text{ for } m > n$$
40.
$$(n-m+1) P_{n+1}^{m}(t) - (2n+1)t P_{n}^{m}(t) + (n+m) P_{n-1}^{m}(t) = 0$$

 $(1-t^2)^{1/2} P_n^m(t) = \frac{1}{2n+1} [P_{n+1}^{m+1}(t) - P_{n-1}^{m+1}(t)]$

TABLE 21.2 Properties of Legendre and Associate Legendre Functions [$P_n(t)$ = Legendre Functions, $P_n^m(t)$ = Associate Legendre Functions, $Q_n(t)$ = Legendre Functions of the Second Kind] (continued)

57.
$$Q_0 = \frac{1}{2} \ln \frac{1+t}{1-t}, \quad |t| < 1$$

TABLE 21.2 Properties of Legendre and Associate Legendre Functions $[P_n(t)]$ = Legendre Functions, $P_n^m(t)$ = Associate Legendre Functions, $Q_n(t)$ = Legendre Functions of the Second Kind] (continued)

58.
$$Q_1(t) = \frac{1}{2}t \ln \frac{1+t}{1-t} - 1 = tQ_0(t) - 1, \quad |t| < 1$$

59.
$$Q_{n+1}(t) = \frac{2n+1}{n+1}tQ_n(t) - \frac{n}{n+1}Q_{n-1}(t), \quad n=1,2,\cdots$$

60.
$$Q_n(t) = P_n(t)Q_0(t) - \sum_{k=0}^{\left[\frac{1}{2}(n-1)\right]} \frac{2n-4k-1}{(2k+1)(n-k)} P_{n-2k-1}(t), \quad |t| < 1$$

61.
$$Q_n(t) = \frac{1}{2} \int_{-1}^1 \frac{P_n(x)}{t - x} dx$$
 $n = 0, 1, 2 \cdots$

62.
$$Q'_{n+1}(t) - 2t Q'_n(t) + Q'_{n-1}(t) - Q_n(t) = 0$$

63.
$$Q'_{n+1}(t) - tQ'_n(t) - (n+1)Q_n(t) = 0$$

64.
$$Q'_{n+1}(t) - Q'_{n-1}(t) = (2n+1)Q_n(t)$$

65.
$$Q_0(-t) = -Q_0(t), \quad Q_n(-t) = (-1)^{n+1}Q_n(t), \quad n = 1, 2, \dots$$

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