# The electric field of a dipole

We know that the dipole contribution to the electrostatic potential is

$$V_{\rm dip}(\vec r) = \frac{1}{4\pi\epsilon_0}\frac{\hat r\cdot\vec p}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{\vec r\cdot\vec p}{r^3}\ , \tag{1}$$

where $\vec p$ is the charge distribution's dipole moment, which is an intrinsic (vector) property of the source and does not depend on $\vec r$. What is the corresponding electric field? It is

$$\vec E_{\rm dip} = -\vec\nabla V_{\rm dip}\ . \tag{2}$$

To take the gradient, it is convenient to use the "cartesian tensor" notation, with Einstein's convention that whenever an index appears twice in a product, summation over all possible values that index can take is understood. So, for instance, the potential is

$$V_{\rm dip} = \frac{1}{4\pi\epsilon_0}\frac{r_i\,p_i}{r^3}\ , \tag{3}$$

where $r_i\,p_i \equiv \sum_{i=1}^3 r_i\,p_i = \vec r\cdot\vec p$ . With this notation, the $j$-th component of the electric field is

$$E_j^{\rm dip} = -\frac{\partial}{\partial r_j}V_{\rm dip} = -\frac{1}{4\pi\epsilon_0}\,p_i\,\frac{\partial}{\partial r_j}\Big(\frac{r_i}{r^3}\Big)\ . \tag{4}$$

To compute the partial derivative, we first decompose it into

$$\frac{\partial}{\partial r_j}\Big(\frac{r_i}{r^3}\Big) = \frac{\partial r_i}{\partial r_j}\,\frac{1}{r^3} + r_i\frac{\partial}{\partial r_j}\frac{1}{r^3}\ . \tag{5}$$

As to the first term, we have

$$\frac{\partial r_i}{\partial r_j} = \delta_{ij}\ , \tag{6}$$

where $\delta_{ij}$ is the Kronecker-delta: it is one for $i = j$, and zero otherwise. Eq. (6) is just a fancy way of writing

$$\frac{\partial x}{\partial x} = 1\ , \qquad \frac{\partial x}{\partial y} = 0\ , \qquad \text{etc.} \tag{7}$$

in a compact notation. As to the second term in eq. (5), we have

$$\frac{\partial}{\partial r_j}\frac{1}{r^3} = \Big(\vec\nabla\frac{1}{r^3}\Big)_j = -3\Big(\frac{\hat r}{r^4}\Big)_j = -3\Big(\frac{\vec r}{r^5}\Big)_j = -3\frac{r_j}{r^5}\ . \tag{8}$$

Eq. (5) thus reduces to

$$\frac{\partial}{\partial r_j}\Big(\frac{r_i}{r^3}\Big) = \frac{1}{r^3}\Big(\delta_{ij} - 3\frac{r_ir_j}{r^2}\Big)\ . \tag{9}$$

As a final step, to compute the electric field according to eq. (4), we have to multiply eq. (9) by $p_i$ and, following the summation convention, sum over $i = 1, 2, 3$. We have

$$E_j^{\mathrm{dip}} = -\frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left(p_i\,\delta_{ij} - 3\frac{(p_i\,r_i)r_j}{r^2}\right) . \tag{10}$$

Now, $p_i\delta_{ij} = p_j$, because the Kronecker-delta combined with the summation over all possible $i$'s selects the term with $i = j$. Also $p_i\,r_i = \vec{p}\cdot\vec{r}$. We finally get

$$E_j^{\mathrm{dip}} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left(3\frac{(\vec{p}\cdot\vec{r})r_j}{r^2} - p_j\right) , \tag{11}$$

or, going back to vector notation:

$$\vec{E}_{\mathrm{dip}} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3\frac{(\vec{p}\cdot\vec{r})\vec{r}}{r^2} - \vec{p}\right] \tag{12}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right] \tag{13}$$