Bose–Einstein Condensation and Superfluidity

(Reference: Robert B. Leighton, *Principles of Modern Physics*, McGraw–Hill (1959).)

Interesting things happen at very low temperatures and Bose–Einstein condensation is one of them. Recall that there is no statistical limit to the number bosons that can occupy a single state. In a Bose condensed state, an appreciable fraction of the particles is in the lowest energy level at temperatures below T_C. These particles are in the same state and can be described by the same wavefunction. In other words a macroscopic number of particles are in one coherent state. (We saw this in the case of the photons in a laser beam.) If we write $\psi = |\psi|e^{i\phi}$, then this state is described by a given phase ϕ.

The oldest known physical manifestation of Bose condensation is superfluid ^4He. A ^4He atom has total angular momentum zero and is therefore a boson. At $T_C = 2.18$ K liquid helium becomes superfluid. The transition temperature is called the $\lambda-$point because the shape of the specific heat curve at T_C is shaped like λ. One cools liquid helium by pumping on it to get rid of the hot atoms (evaporative cooling). It boils a little. Then at the transition it boils vigorously and suddenly stops. Eisberg and Resnick has a picture of this on page 403. The reason for this behavior is that the thermal conductivity increases by a factor of about 10^6 at the transition, so that the superfluid is no longer able to sustain a temperature gradient. To make a bubble, heat has to locally vaporize the fluid and make it much hotter than the surrounding fluid. This is no longer possible in the superfluid state.

Perhaps the hallmark of a superfluid is that it has no viscosity. As a result the superfluid can flow through tiny capillary tubes that normal liquid can't get through. Superfluid ^4He is often described by a two–fluid model, i.e., it is thought of as consisting of 2 fluids, one of which is normal and the other is superfluid. It's the superfluid component which is able to flow through the capillary tube. So if you use this method to measure the coefficient of viscosity, you find that it suddenly drops to zero at the $\lambda-$point.

One can see the effect of both components by putting a torsional oscillator consisting of a stack thin, light, closely spaced mica disks immersed in the liquid. If the liquid has a high viscosity, the liquid between the disks is dragged along and contributes significantly to the moment of inertia of the disks. If the viscosity is small, the moment of inertia is more nearly equal to that of the disks alone. Using this method, no discontinuity is found in the coefficient of viscosity at the $\lambda-$point.

Another weird thing that superfluid helium does is escape from a beaker by crawling up the sides, flowing down the outside, and dripping off the bottom. The helium atoms are attracted by the van der Waals forces of the walls of the container, and they are able to flow up the walls because of the lack of viscosity. The rate of flow can be 30 cm per second or more. The superfluid helium can surmount quite a high wall, on the order of several meters in height.

(Brief aside to explain the van der Waals force: As an electron moves in a molecule, there exists at any instant of time a separation of positive and negative charge in the molecule. The latter has, therefore, an electric dipole moment p_1 which varies in time.

If another molecule exists nearby, it will have a dipole moment induced by the first molecule. These two dipoles are attracted to each other. This is the van der Waals force.)

We can show mathematically that there is a macrosopic population of the lowest energy state in the following way. Consider a gas of noninteracting bosons. Let the energy levels be measured from the lowest energy level, i.e., let the zero point energy be the zero of energy. Then the chemical potential μ must be negative, otherwise the Bose–Einstein distribution would be negative for some of the levels. Recall from lecture 3 that the Bose–Einstein distribution gives the average number of particles in state s:

$$< n_s >= \frac{1}{e^{\beta(E_s-\mu)}-1} \tag{1}$$

μ is adjusted so that the total number of particles is N:

$$N = \sum_s < n_s > \tag{2}$$

Let me give a sneak preview: If we assume a continuous distribution of states, μ starts out negative and gets bigger as the temperature decreases. μ equals its upper limit of zero at some temperature T_C, below which we can no longer satisfy (2) because the right hand side can't deliver enough particles. This leads us to treat the lowest energy level separately and we find that we can satisfy (2) by keeping the extra particles we need in the lowest state.

Now let's do the math. In order to turn the sum over s in (2) into an integral, let's assume a continuous density of states. In lecture 1 we found that if k–space is isotropic, i.e., the same in every direction, then the number of states in a spherical shell lying between radii k and $k + dk$ is

$$\rho_k dk = \frac{V}{(2\pi)^3}(4\pi k^2 dk) = \frac{V}{2\pi^2}k^2 dk \tag{3}$$

Now if the energy of the bosons is purely kinetic energy and continuous, then

$$E = \frac{\hbar^2 k^2}{2m} \tag{4}$$

and

$$dE = \frac{\hbar^2 k dk}{m} \tag{5}$$

or

$$k dk = \frac{m dE}{\hbar^2} \tag{6}$$

Also (4) implies that

$$k = \frac{\sqrt{2mE}}{\hbar} \tag{7}$$

Plugging (6) and (7) into (3) yields

$$\rho_E dE = V\left(\frac{m^3}{2}\right)^{1/2}\frac{E^{1/2}}{\pi^2\hbar^3}dE \tag{8}$$

So we can rewrite (2) as

$$\begin{aligned} N &= \int \frac{1}{e^{\beta(E-\mu)}-1}\,\rho_E dE \\ &= \frac{V}{\pi^2\hbar^3}\left(\frac{m^3}{2}\right)^{1/2}\int_0^\infty \frac{E^{1/2}}{e^{\beta(E-\mu)}-1}dE \end{aligned}$$

Now recall that for a geometric series

$$\sum_{p=0}^{\infty} e^{-p\beta(E-\mu)} = \frac{1}{1-e^{-\beta(E-\mu)}} \tag{9}$$

So

$$\frac{1}{e^{\beta(E-\mu)}-1} = \frac{1}{1-e^{-\beta(E-\mu)}}\,\frac{1}{e^{\beta(E-\mu)}} = \frac{1}{e^{\beta(E-\mu)}}\sum_{p=0}^{\infty} e^{-p\beta(E-\mu)} \tag{10}$$

Plugging this into (9) leads to

$$N = \frac{V}{\pi^2\hbar^3}\left(\frac{m^3}{2}\right)^{1/2}\int_0^\infty E^{1/2}e^{-\beta(E-\mu)}\sum_{p=0}^{\infty} e^{-p\beta(E-\mu)}dE \tag{11}$$

Let $x = \beta E$. Then

$$N = \frac{V}{\pi^2\hbar^3}\left(\frac{m^3}{2}\right)^{1/2}(k_BT)^{3/2}\sum_{p=0}^{\infty} e^{\beta(p+1)\mu}\int_0^\infty x^{1/2}e^{-(p+1)x}dx \tag{12}$$

Let $s = p+1$ and $y = sx$. Then

$$N = \frac{V}{\pi^2\hbar^3}\left(\frac{m^3}{2}\right)^{1/2}(k_BT)^{3/2}\sum_{s=1}^{\infty} e^{\beta s\mu}\frac{1}{s^{3/2}}\int_0^\infty y^{1/2}e^{-y}dy \tag{13}$$

The definition of a gamma function is

$$\Gamma(n) = \int_0^\infty y^{n-1}e^{-y}dy \quad n > 0 \tag{14}$$

So

$$\int_0^\infty y^{1/2}e^{-y}dy = \Gamma(\frac{3}{2}) \tag{15}$$

Thus

$$N = \frac{V}{\pi^2\hbar^3}\left(\frac{m^3}{2}\right)^{1/2}\Gamma(\frac{3}{2})(k_BT)^{3/2}\sum_{s=1}^{\infty}\left(\frac{e^{\beta s\mu}}{s^{3/2}}\right) \tag{16}$$

Now on the right hand side, as the temperature T decreases, μ must increase to keep the product constant and equal to N. T can be made as small as we wish, but μ, which we said must be negative, cannot be greater than zero. But the product must be a constant. So (16) is only valid above a certain critical temperature T_C. Below this temperature, our treatment breaks down. Where did we go wrong? The flaw lies in the fact that we assumed that the states are continuously distributed. However, since we are interested in very low temperatures which involves the occupation of the lowest lying energy levels, we may expect that the actual discrete nature of the level distribution might play an essential role in the lowest temperature range. So let us treat the lowest level $E_1 = 0$ separately. We will assume that it is not degenerate with any other levels and we will assume that the remaining levels are continuously distributed from $E = 0$ to $E = \infty$ as described by (8). So in our summation (2) we will treat the lowest level separately:

$$\begin{aligned} N &= \frac{1}{e^{-\beta\mu}-1} + \frac{V}{\pi^2\hbar^3}\left(\frac{m^3}{2}\right)^{1/2}\int_0^\infty \frac{E^{1/2}}{e^{\beta(E-\mu)}-1}dE \\ &= \frac{1}{e^{-\beta\mu}-1} + \left(\frac{T}{T_C}\right)^{3/2} f(\mu)N \qquad (17) \\ & \qquad (18) \end{aligned}$$

where the second quantity on the right is just the right hand side of (16), written in terms of the critical temperature T_C. $f(\mu = 0) = 1$ because T_C is defined such that the right hand side of (16) equals N with $T = T_C$ and $\mu = 0$.

We see that it is now possible to satisfy this new equation with negative values of μ for all temperatures, since the first term becomes infinite as $\mu \to 0$. The inclusion of the lowest energy level as a separate term in our treatment has thus removed the previous difficulty of not being able to account for all of the particles at temperatures below T_C. If we now inquire into what this equation means physically, we see that, at temperatures below T_C, the chemical potential μ will take on such values that those particles which are not included in the continuous distribution will be found in the lowest level. That is, a kind of condensation occurs; it is such that an appreciable fraction of the particles is in the lowest energy level at temperatures below T_C.

If we write (18) as

$$N = n_1 + \left(\frac{T}{T_C}\right)^{3/2} f(\mu)N \tag{19}$$

and realize that $f(\mu \approx 0) \approx 1$ at low temperatures, then we find the population n_1 of the lowest level to be, approximately,

$$n_1 = N\left[1 - \left(\frac{T}{T_C}\right)^{3/2}\right] \tag{20}$$

At $T = T_C$, $n_1 = 0$ while at $T = 0$, $n_1 = N$. Using $n_1 = (e^{-\beta\mu} - 1)^{-1}$, we find that at low temperatures

$$\mu = -k_B T \ln\left(\frac{1}{n_1} + 1\right) \tag{21}$$

Notice that μ is negative. As $T \to 0$, $\mu \to 0$:

$$\mu(T \to 0) \to -0^+ \ln(\frac{1}{N} + 1) \to 0 \quad \text{as } N \to \infty \text{ and } T \to 0 \tag{22}$$

Considering superfluid helium as a 2 component fluid with normal and superfluid components is consistent with having some of the particles in the lowest energy level and the rest in higher energy levels. There is no microscopic theory of superfluid helium, though computer simulations by Ceperley have been quite successful in reproducing its properties. One of the complications is that the helium atoms are so closely packed that they are strongly interacting; they're in a liquid state. It would be closer to the ideal case to have a system of bosons which are weakly interacting.

(Reference: H.–J. Miesner and W. Ketterle, "Bose–Einstein Condensation in Dilute Atomic Gases," Solid State Communications **107**, 629 (998) and references therein.)
This has recently been achieved in the case of alkali atoms such as rubidium, sodium, and lithium. Using a combination of optical and magnetic traps together with laser cooling and evaporative cooling, several research groups have achieved Bose condensation in dilute weakly interacting vapors of alkali atoms. In these systems the thermal deBroglie wavelength exceeds the mean distance between atoms. Nanokelvin temperatures and densities of 10^{15} cm^{-3} have been achieved. (Compare this to a mole of liquid which has a typical density of 10^{23} cm^{-3}.) At nanokelvin temperatures the thermal deBroglie wavelength exceeds 1 μm which is about 10 times the average spacing between atoms. In these experiments they have actually been able to directly observe the macroscopic population of the zero momentum eigenstate. In addition the coherence resulting from being in macroscopic wavefunctions has been demonstrated by observing the interference between two independent condensates. Two spatially separated condensates were released from the magnetic trap and allowed to overlap during ballistic expansion of the gases. Interference patterns were observed that are analogous to the pattern produced in a double–slit experiment in optics.