$$V_{ef} = 1 + \frac{L^2}{r^2} - \frac{r_s}{r} - \frac{r_s L^2}{r^3}$$

$$\frac{d}{dr} V_{ef} = -\frac{2L^2}{r^3} + \frac{r_s}{r^2} + \frac{3 r_s L^2}{r^4}$$

$$V_{ef}'(r_c) = 0 \quad ; \quad -2L^2 r_c + r_s r_c^2 + 3 r_s L^2 = 0$$

$$r_c^2 - 2 r_c \frac{L^2}{r_s} + 3L^2 = 0 \quad / + \left(\frac{L^4}{r_s^2} - \frac{L^4}{r_s^2}\right)$$

$$\left[r_c - \frac{L^2}{r_s}\right]^2 = L^2\left[\frac{L^2}{r_s^2} - 3\right]$$

$$r_{c1,2} - \frac{L^2}{r_s} = \pm L\sqrt{\frac{L^2}{r_s^2} - 3} = \pm \frac{L^2}{r_s}\sqrt{1 - \frac{3r_s^2}{L^2}}$$

$$\boxed{r_{c1,2} = \frac{L^2}{r_s}\left[1 \pm \sqrt{1 - \frac{3r_s^2}{L^2}}\right]}$$

Luego, existe un valor crítico para el momentum angular

$$L_c = \sqrt{3}\, r_s$$

$$\therefore \quad r_{c1,2} = \frac{L^2}{(L_c/\sqrt{3})}\left[1 \pm \sqrt{1 - \frac{L_c^2}{L^2}}\right]$$

∴ Superficies isobáricas y de densidad constante son superficies equipotenciales de ψ

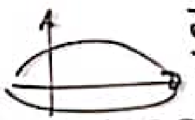
¿Cómo calculamos?

1.- Se escoge una relación de Von Zeipel

$$\Omega = \Omega(s),$$

y calcular V.

2.- Con el potencial Newtoniano Φ desde el objeto central:



$$\Phi = \frac{GM}{r} \quad ; \quad r = \sqrt{s^2+z^2},$$

conocemos ψ y se calculan las equipotenciales

Ej: $\ell = s^2\,\Omega(s)$: cte.

$$\rightarrow \Omega(s) = \frac{\ell}{s^2}$$

$$V = -\int_{s_i}^{s} \frac{\ell^2}{s'^4}\, s'\, ds' + \text{cte.} = \frac{\ell^2}{2s^2} + V_0$$

$$\Rightarrow \psi = -\frac{GM}{\sqrt{s^2+z^2}} + \frac{\ell^2}{2s^2} + \cancel{V_0}^{\,0}$$

$$\Rightarrow z = \pm\sqrt{\left(\frac{GM}{\frac{\ell^2}{2s^2}-\psi}\right)^2 - s^2}$$

$$\frac{1}{\rho}dP = -d\psi$$

$\underline{s}$: $\frac{1}{\rho}\frac{\partial}{\partial}$

$\underline{z}$: $\frac{1}{\rho}\frac{\partial}{\partial z}$

$P = P($

$\therefore \frac{1}{\rho}dP$

$\therefore \frac{1}{\rho}dP$

$\frac{1}{\rho}dP =$

$\frac{1}{\rho}dP =$

caso ba

Relación

Podemos

dV

V

$\Rightarrow$

s: $\frac{1}{\rho}\frac{\partial}{\partial s}P = -\frac{\partial}{\partial s}\Phi + s\,\Omega^2$ $\quad (\Omega(s,z) \to \Omega(s))$

z: $\frac{1}{\rho}\frac{\partial}{\partial z}P = -\frac{\partial}{\partial z}\Phi$

$P = P(z,s) \Rightarrow dP = \frac{\partial P}{\partial z}dz + \frac{\partial P}{\partial s}ds$

$\therefore \frac{1}{\rho}dP = \frac{1}{\rho}\partial_z P\, dz + \frac{1}{\rho}\partial_s P\, ds$

$\therefore \frac{1}{\rho}dP = -\partial_s\Phi\, ds + s\Omega^2 ds - \partial_z\Phi\, dz$

$\frac{1}{\rho}dP = -\left(\frac{\partial\Phi}{\partial s}ds + \frac{\partial\Phi}{\partial z}dz\right) + s\Omega^2 ds$

$\frac{1}{\rho}dP = -d\Phi + s\Omega^2 ds$

Caso barotrópico : $P = P(\rho) \leftarrow$ Ec. de estado.

Relación de Von Zeipel $\Omega = \Omega(s)$.

Podemos introducir el potencial rotacional

$$dV = -\Omega^2(s)\, s\, ds$$

$$\boxed{V = -\int_{s_i}^{s}\Omega^2(s')s'\,ds'}$$

$\frac{1}{\rho}dP = -d\psi \Rightarrow \psi = \Phi + V \rightarrow$

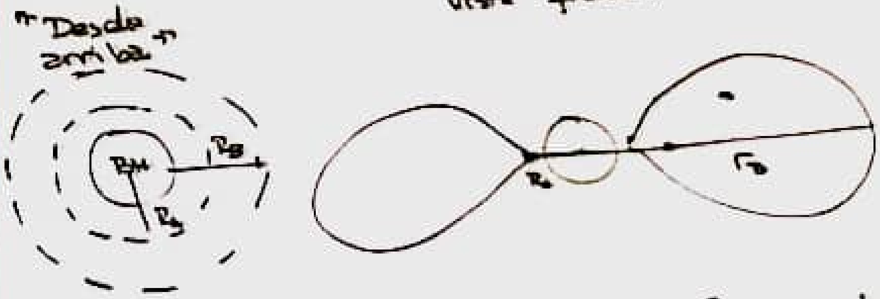
Discos de acreción

- Bajo ciertas circunstancia ($\dot{M} \sim L_{edd}$)

$\dot{M}$: razón de acreción.

$L_{edd} = 1,3 \times 10^{46} \left(\frac{M_{BH}}{10^8 M_\odot}\right) \left[\frac{erg}{s}\right]$: Luminosidad de Eddington

El disco de acreción es delgado y adopta la forma de un toroide



Partamos desde la aproximación Newtoniana en coord. cilíndricas (s, ϕ, z), con el siguiente campo de velocidades:

$$v_r = 0 \;;\; v_z = 0 \;;\; v_\phi = s\,\Omega(s,z)$$

Y el movimiento queda determinado por la ec. de Euler

$$\rho \frac{d\vec{v}}{dt} = -\vec{\nabla} P - \rho \vec{\nabla} \Phi$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}$$: Derivada convectiva

$$\frac{dA'}{dt'} = \frac{1}{2}R^2\frac{d\phi}{dt}\left(1+\frac{3}{2}\frac{r_s}{R}\right)$$

Incremento en el ángulo ϕ respecto ϕ'

$$\int_0^{\Delta\phi'} d\phi' = \int_0^{\Delta\phi = 2\pi} d\phi\left(1+\frac{3}{2}\frac{r_s}{R}\right) = 2\pi + \frac{3r_s}{2}\int_0^{2\pi}\frac{d\phi}{R}$$

Para la elipse $\boxed{R = \frac{l}{1+\epsilon\cos\phi}}$

ϵ : eccentricidad ; l: semi latus-rectum

$$\Delta\phi = 2\pi + \frac{3r_s}{2l}\int_0^{2\pi}(1+\epsilon\cos\phi)\,d\phi$$

$$\Delta\phi = 2\pi + \frac{3\pi r_s}{l}$$

$$\Delta\phi_{RG} = \frac{3\pi r_s}{l} = \frac{6\pi M}{l}$$

$$f(r) = 1 - \frac{r_s}{r} + g(r)$$

$$\boxed{\frac{dA'}{dt'} = \frac{1}{2}R^2\frac{d\phi}{dt}}$$

... per... r

en el espacio-tiempo

$$^2 d\Omega^2$$

$$+ r^2 d\Omega^2$$

$$= 1 - \frac{r_s}{r}$$

$$- \frac{1}{2}\frac{r_s}{r}$$

$$1 + \frac{1}{2}\frac{r_s}{r}$$

En Minkowski:

$$dA = \int_0^R r\,dr\,d\phi = \frac{1}{2}R^2 d\phi$$

$$\Rightarrow \boxed{\frac{dA}{dt} = \frac{1}{2}R^2\frac{d\phi}{dt}}$$ 2ª ley de Kepler.

En Schwarzschild

$$dA' = \int_0^R r\,dr'\,d\phi = \int_0^R r\left(1 + \frac{r_s}{2r}\right)dr\,d\phi$$

$$dA' = d\phi\int_0^R \left(r + \frac{r_s}{2}\right)dr = d\phi\left(\frac{R^2}{2} + \frac{r_s}{2}R\right)$$

$$dA' = \frac{1}{2}R^2 d\phi\left(1 + \frac{r_s}{R}\right)$$

Luego,

$$\frac{dA'}{dt'} = \frac{1}{2}R^2\left(1 + \frac{r_s}{R}\right)\frac{d\phi}{dt'}$$

$$\frac{dA'}{dt'} = \frac{1}{2}R^2\frac{d\phi}{dt}\frac{(1 + r_s/R)}{(1 - r_s/2R)} = \frac{1}{2}R^2\frac{d\phi}{dt}\left(1 + \frac{r_s}{R}\right)\left(1 + \frac{r_s}{2R}\right)$$

$$= \frac{1}{2}R^2\frac{d\phi}{dt}\left(1 + \frac{r_s}{2R} + \frac{r_s}{R} + \mathcal{O}(R^{-2})\right)$$

Derivación Alternativa Precesión de Perihelio. (Combleet): La idea es comparar dos elipses, una en un espacio sin perturbar (Minkoswki), y la otra en el espacio-tiempo perturbado.

$$ds_1^2 = -c^2 dt^2 + dr^2 + r^2 d\Omega^2$$

$$ds_2^2 = -f(r)\, c^2 dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$$

$*\quad (1+x)^m \simeq 1 + mx + \dots$

Schwarzschild: $f(r) = 1 - \frac{r_s}{r}$

i) $\sqrt{f(r)} = \left(1 - \frac{r_s}{r}\right)^{1/2} \simeq 1 - \frac{1}{2}\frac{r_s}{r}$

ii) $\frac{1}{\sqrt{f(r)}} = \left(1 - \frac{r_s}{r}\right)^{-1/2} \simeq 1 + \frac{1}{2}\frac{r_s}{r}$

Tenemos que comparar:

$$dt' = \left(1 - \frac{r_s}{2r}\right) dt$$

$$dr' = \left(1 + \frac{r_s}{2r}\right) dr$$

La relación a): $u_1 + u_2 + u_3 = \frac{1}{r_s}$

$$\frac{2}{R} + u_3 = \frac{1}{r_s} \Rightarrow \boxed{u_3 = \frac{1}{r_s} - \frac{2}{R}}$$

Debemos escribir el polinomio: $g(u) = (u - u_1)(u_2 - u)(u_3 - u)$

$$g(u) = \left[\frac{1 + e\cos x}{R} - \frac{1 - e}{R}\right] \times \left[\frac{1 + e}{R} - \frac{1 + e\cos x}{R}\right] \times \left[\frac{1}{r_s} - \frac{2}{R} - \frac{1 + e\cos x}{R}\right]$$

$$g(x) = \frac{e}{R}(1 + \cos x) \times \frac{e}{R}(1 - \cos x) \times \left(\frac{1}{r_s} - \frac{3 + e\cos x}{R}\right)$$

$$= \operatorname{sen}^2 x \cdot \left(\frac{e}{R}\right)^2 \cdot \frac{1}{r_s}\left(1 - \frac{3r_s + r_s e\cos x}{R}\right)$$

Definimos

$$mu \rightarrow \mu = \frac{r_s}{2R}$$

$$g(x) = \frac{1}{r_s}\left(\frac{e}{R}\right)^2 \operatorname{sen}^2 x \left(1 - 6\mu - 2\mu e\cos x\right).$$

sea $k^2 = \dfrac{4\mu e}{1+2\mu e-6\mu} < 1$ (verificar)

$\mu_2 < \mu_3: \quad \dfrac{1+e}{R} < \dfrac{1}{r_5} - \dfrac{2}{R} \quad / r_5$

$2(1+e)\mu < 1 - 4\mu$

$6\mu < 1 - 2\mu e \quad /- 2\mu e$

$6\mu - 2\mu e < 1 - 4\mu e$

$4\mu e < 1 + 2\mu e - 6\mu \Rightarrow \dfrac{4\mu e}{1+2\mu e-6\mu} < 1 \,//$

$\therefore \; f(\chi) = \sqrt{1+2\mu e-6\mu}\,\sqrt{1-k^2\cos^2\frac{\chi}{2}}$

Hagamos $\gamma = \dfrac{\pi}{2} - \dfrac{\chi}{2} \Rightarrow \cos\dfrac{\chi}{2} = \cos\left(\frac{\pi}{2}-\gamma\right) = s\text{i}$

$d\gamma = -\dfrac{1}{2}d\chi \Rightarrow d\chi = -2d\gamma$

$\dfrac{d\gamma}{d\phi} = \mp \dfrac{\sqrt{1+2\mu e-6\mu}}{2} \cdot \sqrt{1-k^2\sin^2\gamma} \equiv \mp A\sqrt{1-k^2 s\text{i}}$

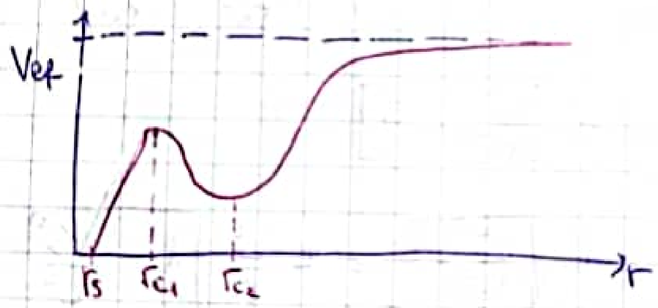
Geodesicas tipo tiempo

El potencial efectivo

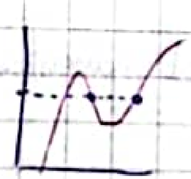
$$V_{ef} = \left(1 - \frac{r_s}{r}\right)\left(h + \frac{L^2}{r^2}\right) \Rightarrow$$ (Graficar en Mathematica con Manipulate (L y r))

Por Normalización, $h = 1$ (partículas masivas)

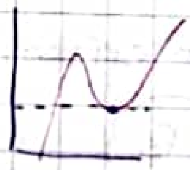
Graficamente



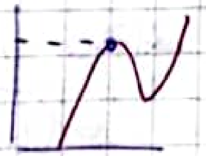
Las orbitas de primera especie se pueden clasificar



Eliptica



Circular



$$\frac{\ldots}{1+2\mu e-6\mu} < 1 \quad \text{(verificar)}$$

$$4\mu e < 1+2\mu e-6\mu \qquad \mu = \frac{1+e\cos\chi}{R}$$

$$2\mu e < 1-6\mu$$

$$\frac{2r_s}{2R}e < 1-\frac{6r_s}{2R}$$

$$\frac{r_s}{R}e < 1-\frac{3r_s}{R}$$

$$\frac{r_s}{R}(e+3) < 1 \Rightarrow \boxed{e+3 < \frac{R}{r_s}}$$

$$\therefore\ f(\chi) = \sqrt{1+2\mu e-6\mu}\,\sqrt{1-k^2\cos^2\frac{\chi}{2}}$$

Hagamos $\gamma = \frac{\pi}{2}-\frac{\chi}{2} \Rightarrow \cos\frac{\chi}{2} = \cos\left(\frac{\pi}{2}-\gamma\right) = \sin\gamma$

$$d\gamma = -\frac{1}{2}d\chi \Rightarrow d\chi = -2d\gamma$$

$$\therefore\ \frac{d\gamma}{d\phi} = \mp\frac{\sqrt{1+2\mu e-6\mu}}{2}\cdot\sqrt{1-k^2\sin^2\gamma} \equiv \mp A\sqrt{1-k^2\sin^2\gamma}$$

En el afelio: $\chi_A = \pi \Rightarrow \gamma_A = 0$ y pongamos $\phi_A = 0$

$$A\int_0^{\phi} d\phi' = \int_0^{\gamma}\frac{d\gamma'}{\sqrt{1-k^2\sin^2\gamma'}} \Rightarrow \boxed{A\phi = F(\gamma,k)} \quad /\operatorname{sn}$$

$$\operatorname{sn}(A\phi) = \sin\gamma = \sin\left(\frac{\pi}{2}-\frac{\chi}{2}\right) = \cos\frac{\chi}{2}$$

$$\sin^2(A\phi) = \cos^2\frac{\chi}{2} = \frac{1+\cos\chi}{2}$$

$$2\sin^2(A\phi) - 1 = \cos\chi \quad / e \;/ +1 \;/ : R$$

$$2e\sin^2(A\phi) - e = e\cos\chi$$

$$(1-e) + 2e\sin^2(A\phi) = 1 + e\cos\chi$$

$$\frac{(1-e) + 2e\overset{1-\cos^2}{\sin^2}(A\phi)}{R} = \mu \qquad \boxed{\mu = \frac{r_s}{2R}}$$

$$\frac{1+e-2e\cos^2(A\phi)}{R} = \mu$$

$$\Rightarrow \quad \boxed{r(\phi) = \frac{R}{1+e-2e\cos^2(A\phi)}}$$

TAREA: Graficar esta órbita.

Nota: Para órbitas planetarias, $0<e<1$.

Veamos que $\quad \cos\chi = 2\cos^2\frac{\chi}{2} - 1$

$$\therefore \quad 1-6\mu-2\mu e\cos\chi = 1-6\mu-2\mu e\left(2\cos^2\frac{\chi}{2}-1\right)$$

Tarea: Usando el método de cardano, encontrar de forma exacta los puntos de retorno

también podemos escribir

$$g(\mu) = (\mu-\mu_1)(\mu-\mu_2)(\mu-\mu_3)$$

$$g(\mu) = \mu^3 - \mu^2(\mu_1+\mu_2+\mu_3) + \mu(\mu_1\mu_2+\mu_1\mu_3+\mu_2\mu_3) - \mu_1\mu_2\mu_3$$

a) $\mu_1+\mu_2+\mu_3 = 1/s$

b) $\mu_1\mu_2+\mu_1\mu_3+\mu_2\mu_3 = \frac{1}{L^2}$

c) $\mu_1\mu_2\mu_3 = \frac{\varepsilon^2}{r_s L^2}$

Hagamos el sig. cambio de variables

$$\mu = \frac{1+e\cos x}{R}$$

Con el orden de jerarquía $0<r_3<r_2<r<r_1$

En términos de μ: $\infty > \mu_3 > \mu_2 > \mu > \mu_1 > 0$

- En el Afelio, $x_A=\pi \Rightarrow \mu_1 = \frac{1-e}{R}$
- En el Perihelio, $x_p=0 \Rightarrow \mu_2 = \frac{1+e}{R}$

$$\boxed{r_{c\,1,2} = \sqrt{3}\,\frac{L^2}{L_c}\left[1 \pm \sqrt{1-\frac{L_c^2}{L^2}}\right]}$$

- Si $L = L_c$ $\quad r_{c1} = r_{c2} = 3 r_s = 6M$
- Si $L < L_c$ $\quad r_{c1} = r_{c2}^* \in \mathbb{C}$
  No hay órbitas circulares
- Si $L > L_c$ $\quad r_{c1} \neq r_{c2} \in \mathbb{R}$

$$r_{c2} = \sqrt{3}\,\frac{L^2}{L_c}\left[1 + \sqrt{1-\frac{L_c^2}{L^2}}\right] \text{ : órbita circular estable}$$

$$r_{c1} = \sqrt{3}\,\frac{L^2}{L_c}\left[1 - \sqrt{1-\frac{L_c^2}{L^2}}\right] \text{ : órbita circular inestable}$$

<u>Órbitas</u> <u>confinadas</u>: Dos condiciones se deben satisfacer:

$$L > L_c \quad \wedge \quad E < 1$$

La ec. de Mov. $\quad \dot{r}^2 = E^2 - V_{ef}$

$$\dot{r}^2 = E^2 - 1 - \frac{L^2}{r^2} + \frac{r_s}{r} + \frac{r_s L^2}{r^3}$$

$$\rightarrow \dot{r}^2 = (E^2 - 1) + \frac{r_s}{r} - \frac{L^2}{r^2} + \frac{r_s L^2}{r^3}$$

Definimos $\xi^2 = 1 - E^2$

$$\therefore \boxed{\dot{r}^2 = \frac{r_s L^2}{r^3} - \frac{L^2}{r^2} + \frac{r_s}{r} - \xi^2}$$

Por otro lado, $r^2 \dot{\phi} = L \Rightarrow \dot{\phi} = \frac{L}{r^2}$

$$\dot{r} = \frac{dr}{d\phi}\dot{\phi} = \frac{L}{r^2}\frac{dr}{d\phi}$$

$$\therefore L^2\left(\frac{1}{r^2}\frac{dr}{d\phi}\right)^2 = r_s\frac{L^2}{r^3} - \frac{L^2}{r^2} + \frac{r_s}{r} - \xi^2$$

Cambio de variable $r = 1/\mu$

$$L^2\left(-\frac{d\mu}{d\phi}\right)^2 = L^2 r_s \mu^3 - L^2\mu^2 + r_s\mu - \xi^2$$

$$\left(-\frac{d\mu}{d\phi}\right)^2 = r_s\mu^3 - \mu^2 + \frac{r_s}{L^2}\mu - \left(\frac{\xi}{L}\right)^2$$

ó bien $\left(-\frac{d\mu}{d\phi}\right)^2 = r_s \, g(\mu)$

donde $\boxed{g(\mu) = \mu^3 - \frac{1}{r_s}\mu^2 + \frac{\mu}{L^2} - \frac{1}{r_s}\left(\frac{\xi}{L}\right)^2}$

Escaneado con CamScanner