Chapter 4. Electric Fields in Matter

4.3 The Electric Displacement

4.3.1 Gauss's Law in the Presence of Dielectrics

The effect of polarization (**P**) is to produce accumulations of bound charge:

$\rho_b \equiv -\nabla \cdot \mathbf{P}$: within the dielectric $\qquad$ $\sigma_b \equiv \mathbf{P} \cdot \hat{\mathbf{n}}$: on the surface

If any free charge exists within a dielectric, therefore, the total charge density can be written as

$$\rho = \rho_b + \rho_f$$

➔ Gauss's law reads

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \mathbf{P} + \rho_f \longrightarrow \nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$$

$$\boxed{\mathbf{D} \equiv \varepsilon_0 \mathbf{E} + \mathbf{P} \quad (C/m^2)}$$ ➔ **Electric displacement**

$$\nabla \cdot \mathbf{D} = \rho_f$$

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{enc}}$$

Similar to Gauss's law

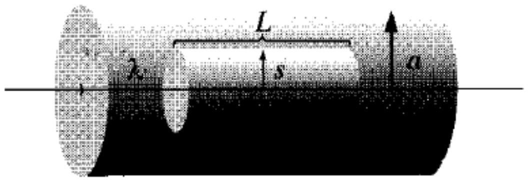
$Q_{f_{enc}}$ denotes the total free charge enclosed in the volume.

In a typical problem, we know ρ_f, but we do not (initially) know ρ_b

Whenever the requisite symmetry is present, we can immediately calculate D by the Gauss's law method, because it makes reference only to free charges.

Gauss's Law: $\nabla \cdot \mathbf{D} = \rho_f$ $\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{enc}}$

Example 4.4 **A long straight wire, carrying uniform line charge** λ, is surrounded by rubber insulation out to a radius ***a***. Find the electric displacement **D** and the field **E**.



➔ Drawing a cylindrical Gaussian surface, of radius ***s*** and length ***L,***

$$D(2\pi s L) = \lambda L \longrightarrow \mathbf{D} = \frac{\lambda}{2\pi s}\hat{\mathbf{s}}.$$

: both within the insulation and outside it.

➔ *Inside* the rubber the electric field cannot be determined, since we do not know P.

➔ *Outside* it, P = 0: $\mathbf{E} = \frac{1}{\epsilon_0}\mathbf{D} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}, \quad \text{for } s > a.$

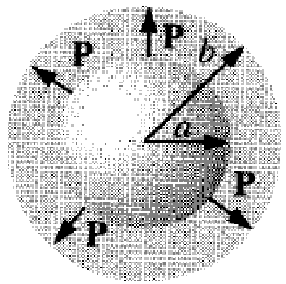
Gauss's Law: $\nabla \cdot \mathbf{D} = \rho_f \qquad \oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{enc}}$

Problem 4.15 A thick **spherical shell (inner radius *a*, outer radius *b*)** is made of dielectric material with a "frozen-in" polarization **P(r).** (No free charge inside)

Find the electric field in all three regions **by two different methods**:

$$\mathbf{P}(\mathbf{r}) = \frac{k}{r}\,\hat{\mathbf{r}}$$

(a) Identify all the bound charge, and use Gauss's law to calculate the field it produces.



$$\rho_b = -\nabla\cdot\mathbf{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{k}{r}\right) = -\frac{k}{r^2};$$

$$\mathbf{P}(\mathbf{r}) = \frac{k}{r}\,\hat{\mathbf{r}}$$

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \left\{\begin{array}{ll} +\mathbf{P}\cdot\hat{\mathbf{r}} = k/b & (\text{at } r=b), \\ -\mathbf{P}\cdot\hat{\mathbf{r}} = -k/a & (\text{at } r=a). \end{array}\right\}$$

For $r < a$, $Q_{\text{enc}} = 0$, so Gauss's law $\Rightarrow$ $\mathbf{E} = 0$.

For $r > b$, $Q_{\text{tot}} = \oint_{\mathcal{S}} \sigma_b\, da + \int_{\mathcal{V}} \rho_b\, d\tau = \oint_{\mathcal{S}} \mathbf{P}\cdot d\mathbf{a} - \int_{\mathcal{V}} \nabla\cdot\mathbf{P}\, d\tau$

(divergence theorem)

$\oint_{\mathcal{S}} \mathbf{P}\cdot d\mathbf{a} = \int_{\mathcal{V}} \nabla\cdot\mathbf{P}\, d\tau$, so $Q_{\text{enc}} = 0$ Gauss's law $\Rightarrow$ $\mathbf{E} = 0$

For $a < r < b$, $Q_{\text{enc}} = \left(\frac{-k}{a}\right)(4\pi a^2) + \int_a^r \left(\frac{-k}{\bar{r}^2}\right) 4\pi \bar{r}^2 d\bar{r} = -4\pi k a - 4\pi k(r-a) = -4\pi k r$; so

$$\mathbf{E} = -(k/\epsilon_0 r)\,\hat{\mathbf{r}}.$$

(b) Use the equations of electric displacement D: $\mathbf{D} \equiv \epsilon_0\mathbf{E} + \mathbf{P}$

$\oint \mathbf{D}\cdot d\mathbf{a} = Q_{f_{\text{enc}}} = 0 \;\Rightarrow\; \mathbf{D} = 0$ everywhere. $\mathbf{E} = (-1/\epsilon_0)\mathbf{P}$, so

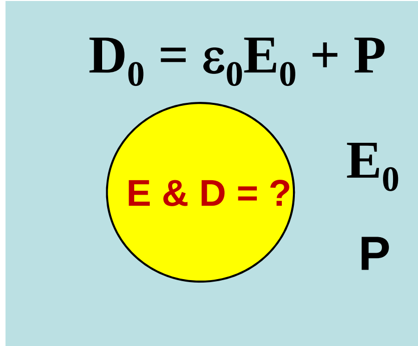
$\mathbf{E} = \mathbf{0}$ (for $r < a$ and $r > b$);

$\mathbf{E} = -(k/\epsilon_0 r)\,\hat{\mathbf{r}}$ (for $a < r < b$)

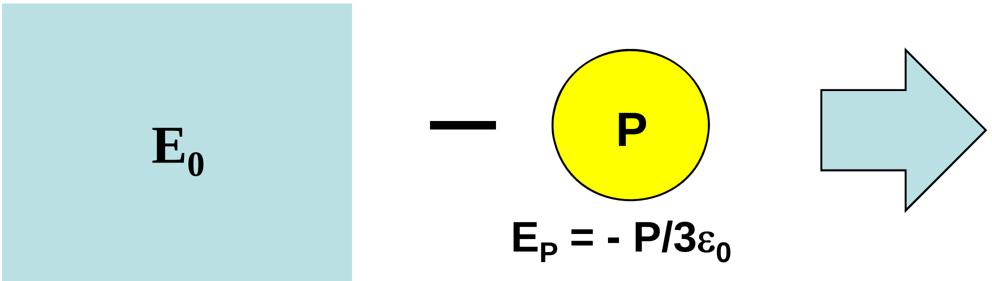
*Much simple to use **D.***

Gauss's Law: $\nabla \cdot \mathbf{D} = \rho_f$ $\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{enc}}$

Problem 4.16 Suppose the field inside a large piece of dielectric is $\mathbf{E}_0$, so that the electric displacement is: $\mathbf{D}_0 = \epsilon_0 \mathbf{E}_0 + \mathbf{P}$
Now a small spherical cavity is hollowed out of the material.
Find the field at the center of the cavity in terms of $\mathbf{E}_0$ and **P**.
Also find the displacement at the center of the cavity in terms of $\mathbf{D}_0$ and **P**.



Assume the cavities are small enough that **P**, $\mathbf{E}_0$, and $\mathbf{D}_0$ are essentially uniform.
Hint ➔ Carving out a cavity is the same as superimposing an object of the same shape but opposite polarization.



$\mathbf{E}_P = -\mathbf{P}/3\varepsilon_0$

$\mathbf{E} = \mathbf{E}_0 - (\mathbf{E}_P)$

$$\boxed{\mathbf{E} = \mathbf{E}_0 + \frac{1}{3\epsilon_0}\mathbf{P}}$$

$$\begin{aligned} \mathbf{D} &= \epsilon_0 \mathbf{E} \\ &= \epsilon_0 \mathbf{E}_0 + \tfrac{1}{3}\mathbf{P} \\ &= \mathbf{D}_0 - \mathbf{P} + \tfrac{1}{3}\mathbf{P} \\ \mathbf{D} &= \mathbf{D}_0 - \tfrac{2}{3}\mathbf{P} \end{aligned}$$

4.3.2 A Deceptive Parallel: Misleading in comparison between E and D

$\nabla \cdot \mathbf{D} = \rho_f$ ← ***Let's compare it with Gauss's law*** → $\epsilon_0 \nabla \cdot \mathbf{E} = \rho$

It looks just like Gauss's law,

→ Only the *total* charge density ρ **is replaced by the *free* charge density** ρ_f, **D is substituted for** ε_o**E.**

→ ***D is "just like" E*** (apart from the factor ε_o), except that its source is ρ_f **instead of** ρ .

Therefore, one may conclude that

"to solve problems involving dielectrics, you just forget all about the bound charge, calculate the field as you ordinarily would, only call the answer D instead of E:"

➔ This conclusion in similarity between E and D is false!

There is **no "Coulomb's law" for D**: $\mathbf{D}(\mathbf{r}) \neq \frac{1}{4\pi} \int \frac{\hat{\boldsymbol{\imath}}}{\imath^2} \rho_f(\mathbf{r}') \, d\tau'$

For the divergence alone is insufficient to determine a vector field; you need to know the curl as well.

In the case of electrostatic fields, **the curl of E is always zero**. $\nabla \times \mathbf{E} = 0$

➔ But **the curl of D is *not* always zero**. $\nabla \times \mathbf{D} = \epsilon_0 (\nabla \times \mathbf{E}) + (\nabla \times \mathbf{P}) = \nabla \times \mathbf{P}$

➔ Because $\nabla \times \mathbf{D} \neq 0$, **D** cannot be expressed as the gradient of a scalar.

→ **there is no "potential" for D**.

Advice**: When you are asked to compute the electric displacement*, ***first look for symmetry.

➔ ***If the problem exhibits symmetry***, then you can ***get D directly by the usual Gauss's law methods***.

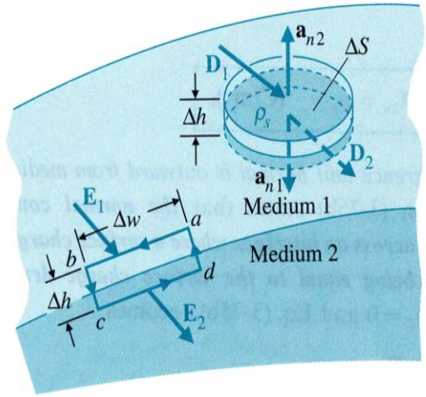
➔ ***If symmetry is absent***, you'll have to think of another approach.

you ***must not assume that D is determined exclusively by the free charge.***

4.3.3 Boundary Conditions

$$\nabla\cdot\mathbf{D}=\rho_f \rightarrow \oint_S D\cdot da=\left(\mathbf{D}_1\cdot\mathbf{a}_{n2}+\mathbf{D}_2\cdot\mathbf{a}_{n1}\right)\Delta S=\mathbf{a}_{n2}\cdot\left(\mathbf{D}_1-\mathbf{D}_2\right)\Delta S=\left(D_{1n}-D_{2n}\right)\Delta S$$

$$\rightarrow \int_V \rho_f\, d\upsilon=\left(\rho_f\Delta h\right)\Delta S$$



$$\boxed{D_{1n}-D_{2n}=\sigma_f} \qquad \sigma_f=\lim_{\Delta h\to 0}\rho_f\Delta h$$

$$\nabla\times D=\nabla\times P\rightarrow \oint_C D\cdot dl=\left(D_1\cdot l_2+D_2\cdot l_1\right)=l_2\cdot\left(D_1-D_2\right)=\left(D_{1t}-D_{2t}\right)$$

$$\boxed{D_{1t}-D_{2t}=P_{1t}-P_{2t}}$$

Note for E:

$$\boxed{E_{1n}-E_{2n}=\frac{\sigma_{enclosed}}{\varepsilon_0}=\frac{\sigma_f+\sigma_b}{\varepsilon_0}}$$

$$\boxed{E_{1t}-E_{2t}=0}$$

Summary of Boundary conditions

$$\mathbf{E}^{\parallel}_{\text{above}}-\mathbf{E}^{\parallel}_{\text{below}}=0 \qquad \mathbf{D}^{\parallel}_{\text{above}}-\mathbf{D}^{\parallel}_{\text{below}}=\mathbf{P}^{\parallel}_{\text{above}}-\mathbf{P}^{\parallel}_{\text{below}}$$

$$E^{\perp}_{\text{above}}-E^{\perp}_{\text{below}}=\frac{1}{\epsilon_0}\sigma \qquad D^{\perp}_{\text{above}}-D^{\perp}_{\text{below}}=\sigma_f$$

$$\left(\sigma=\sigma_f+\sigma_b\right)$$