Ratio Test and Interval of Convergence for Taylor Series

Math 126

The Ratio Test: For the power series centered at $x = a$

$$P(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots + C_n(x-a)^n + \cdots,$$

suppose that $\lim_{n\to\infty} \frac{|C_n|}{|C_{n+1}|} = R$. Then:

- If $R = \infty$, then the series converges for all x.
- If $0 < R < \infty$, then the series converges for all $|x - a| < R$.
- If $R = 0$, then the series converges only for $x = a$.

We call R the **radius of convergence**.

1. Use the ratio test to compute the radius of convergence for the following power series.

(a) $\sum_{n=0}^{\infty} n! x^n$ $\quad C_n$

$$\lim_{n\to\infty} \frac{|C_n|}{|C_{n+1}|} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

Thus, the radius is $R=0$.

(b) $\sum_{n=0}^{\infty} n!(x+3)^n$ $\quad C_n$

$$\lim_{n\to\infty} \frac{|C_n|}{|C_{n+1}|} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = 0$$

As in part (a), $R=0$.

(c) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$C_n = \frac{1}{n!}$

$$\lim_{n\to\infty} \frac{|C_n|}{|C_{n+1}|} = \lim_{n\to\infty} \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = \infty$$

Thus, the radius is $R = \infty$.

(d) $\sum_{n=1}^{\infty} \frac{n}{2^n}(x+3)^n$ $\quad C_n$

$$\lim_{n\to\infty} \frac{|C_n|}{|C_{n+1}|} = \lim_{n\to\infty} \frac{\frac{n}{2^n}}{\frac{n+1}{2^{n+1}}} = \lim_{n\to\infty} \frac{n}{2^n}\cdot\frac{2^{n+1}}{n+1} = \lim_{n\to\infty} \frac{2n}{n+1} = 2$$

Thus, the radius is $R=2$.

Interval of convergence and their endpoints: For a power series centered at $x = a$, the **interval of convergence** is defined to be all x values for which the series converges. That is, the interval can be a single point, the whole real line $(-\infty, \infty)$ or any of the following:

$$(a-R, a+R), \quad (a-R, a+R], \quad [a-R, a+R), \quad \text{or} \quad [a-R, a+R].$$

To distinguish between these four intervals, you must check convergence at the endpoints directly.

2. Compute the interval of convergence for each series on the previous page.

(a) Since $R=0$, the series converges only at $a=0$.

(b) Since $R=0$, the series converges only at $a=-3$.

(c) Since $R=\infty$, the series converges for all real numbers.

(d) Since $R=2$, we must check convergence at $a \pm R$:

$x = a+R = -3+2 = -1$: $\sum_{n=1}^{\infty} \frac{n}{2^n}(-1+3)^n = \sum_{n=1}^{\infty} \frac{n(2)^n}{2^n} = \sum_{n=1}^{\infty} n = 1+2+3+4+5+\cdots$, which DIVERGES

$x = a-R = -3-2 = -5$: $\sum_{n=1}^{\infty} \frac{n}{2^n}(-5+3)^n = \sum_{n=1}^{\infty} \frac{n}{2^n}(-2)^n = \sum_{n=1}^{\infty} n(-1)^n = -1+2-3+4-5+6-\cdots$, which DIVERGES

Thus, the interval is $(-1,5)$.

3. Find the interval of convergence for $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$. $a=3$, $C_n = \frac{1}{n}$

Ratio Test: $\displaystyle\lim_{n\to\infty} \frac{|C_n|}{|C_{n+1}|} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = \lim_{n\to\infty} \frac{n+1}{n} = 1 = R$

Endpoints: $a \pm R = 3 \pm 1 = 2$ and 4

Alternating harmonic series

$x=2$: $\displaystyle\sum_{n=1}^{\infty} \frac{(2-3)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$ CONVERGES

$x=4$: $\displaystyle\sum_{n=1}^{\infty} \frac{(4-3)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ DIVERGES

Harmonic Series

Thus, the interval of convergence is $[2,4)$.

4. Use the ratio test to show that the Taylor series centered at 0 for $\sin(x)$ converges for all real numbers.

Taylor series for $\sin(x)$ centered at $x=0$:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$C_n = \frac{(-1)^n}{(2n+1)!}$, so $\displaystyle\lim_{n\to\infty} \frac{|C_n|}{|C_{n+1}|} = \lim_{n\to\infty} \frac{\left|\frac{(-1)^n}{(2n+1)!}\right|}{\left|\frac{(-1)^{n+1}}{(2n+3)!}\right|} = \lim_{n\to\infty} \frac{(2n+3)!}{(2n+1)!} = \lim_{n\to\infty} (2n+3)(2n+2) = \infty$

Thus, the radius is $R=\infty$, and the series converges for all real x.

(This conclusion holds even though the series is $\sum C_n x^{2n+1}$ instead of $\sum C_n x^n$.)