Chapter 3. Potentials

3.3 Separation of Variables

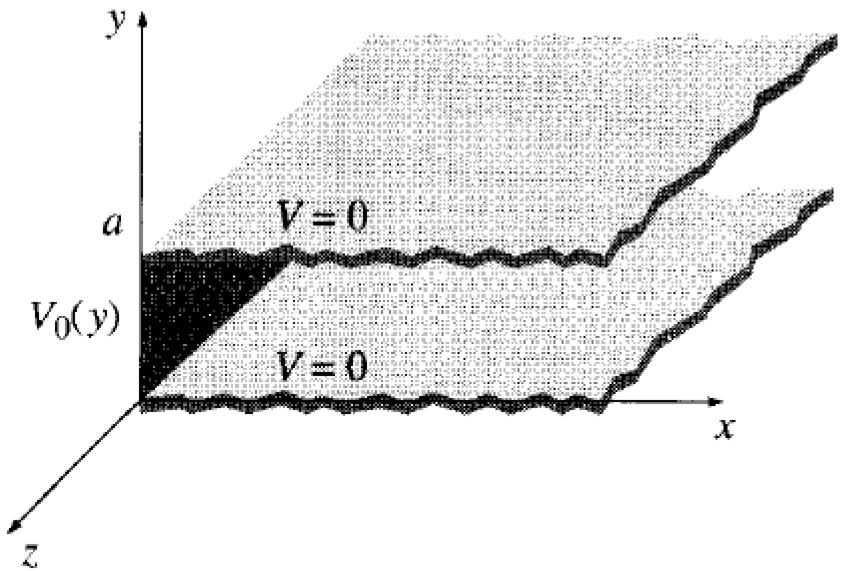
Let's attack Laplace's equation directly using the method of **separation of variables**.

→ by *look for solutions that are* ***products of functions***,
each of which depends on only **one** ***of the coordinates.***

→ Consider two cases: **Cartesian coordinates and Spherical coordinates**

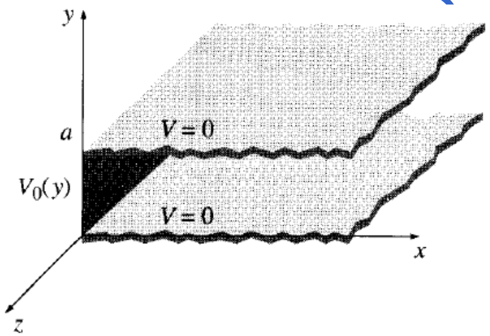
3.3.1 Cartesian Coordinates

Example 3.3 Two infinite grounded metal plates lie parallel.
Find the potential inside this "slot."



3.3.1 Cartesian Coordinates

Example 3.3 Find the potential inside.



(Boundary conditions)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

(i) $V = 0$ when $y = 0$,
(ii) $V = 0$ when $y = a$,
(iii) $V = V_0(y)$ when $x = 0$,
(iv) $V \to 0$ as $x \to \infty$.

Since the potential is specified on all boundaries, the answer is uniquely determined.

The first step is to look for solutions in the form of products:

$$V(x, y) = X(x)Y(y) \longrightarrow Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

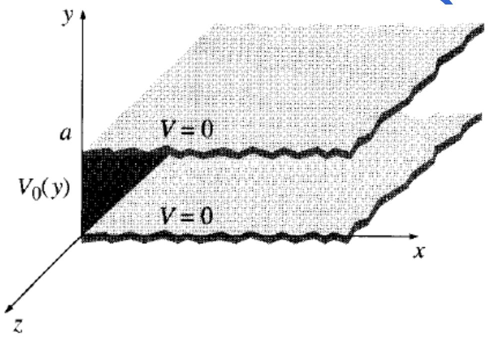
The next step is to "separate the variables"

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0 \longrightarrow \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

→ The first term depends only on *x* and the second only on *y.*
→ ***They must both be constant.***

Cartesian Coordinates $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$

Example 3.3



$$V(x, y) = X(x)Y(y)$$

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$ → ***Must both be constant.***

$$\frac{1}{X}\frac{d^2X}{dx^2} = C_1, \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = C_2 \xrightarrow{C_1 + C_2 = 0}$$

$$\frac{d^2X}{dx^2} = k^2 X \longrightarrow X(x) = Ae^{kx} + Be^{-kx}$$

(k > 0)

$$\frac{d^2Y}{dy^2} = -k^2 Y \longrightarrow Y(y) = C\sin ky + D\cos ky$$

$$\Rightarrow V(x, y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)$$

(iv) $V \to 0$ as $x \to \infty$ ⟶ **A = 0**

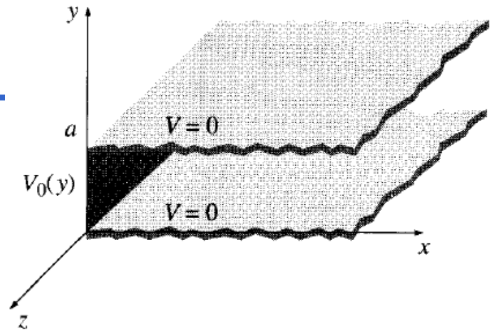
(i) $V = 0$ when $y = 0$ ⟶ **D = 0**

$$\Rightarrow V(x, y) = Ce^{-kx}\sin ky$$

(ii) $V = 0$ when $y = a$ ⟶ $\sin ka = 0$ ⟶ $k = \frac{n\pi}{a} \quad (n = 1, 2, 3, \ldots)$

(iii) $V = V_0(y)$ when $x = 0$

Cartesian Coordinates $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$



Example 3.3

$$V(x, y) = X(x)Y(y) \Rightarrow V(x, y) = C\, e^{-n\pi x/a} \sin(n\pi y/a)$$

$$(n = 1, 2, 3, \ldots)$$

→ An ***infinite set*** **of solutions** (one for each *n)*

→ none of them *by itself* satisfies the final boundary condition

(iii) $V = V_0(y)$ when $x = 0$

Since Laplace's equation is *linear,*

$$V(x, y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a)$$

→ This linear combination is also a solution.

("Completeness" of V function)

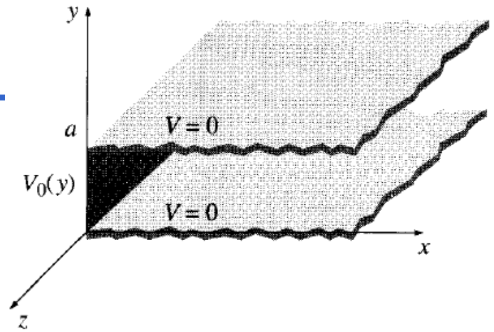
(iii) $V = V_0(y)$ when $x = 0$ ⟶ $V(0, y) = \sum_{n=1}^{\infty} C_n \sin(n\pi y/a) = V_0(y)$

➔ It's a Fourier sine series.

Multiply ***sin(n'πyla)*** (where *n'* is a positive integer), and integrate from 0 to *a:*

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin(n\pi y/a) \sin(n'\pi y/a)\, dy = \int_0^a V_0(y) \sin(n'\pi y/a)\, dy$$

Cartesian Coordinates $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$



Example 3.3

$$V(x, y) = X(x)Y(y) \Longrightarrow V(x, y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a)$$

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin(n\pi y/a) \sin(n'\pi y/a)\, dy = \int_0^a V_0(y) \sin(n'\pi y/a)\, dy$$

$$\Rightarrow \int_0^a \sin(n\pi y/a) \sin(n'\pi y/a)\, dy = \begin{cases} 0, & \text{if } n' \neq n, \\ \frac{a}{2}, & \text{if } n' = n. \end{cases}$$

(Orthogonality)

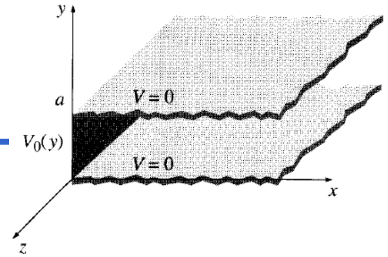
$$\Rightarrow C_n = \frac{2}{a} \int_0^a V_0(y) \sin(n\pi y/a)\, dy$$

Suppose the strip at x = 0 is a metal plate with constant potential V_0.
(insulated from the grounded plates at y = 0 and y = a)

$$C_n = \frac{2V_0}{a} \int_0^a \sin(n\pi y/a)\, dy = \frac{2V_0}{n\pi}(1 - \cos n\pi) = \begin{cases} 0, & \text{if } n \text{ is even}, \\ \frac{4V_0}{n\pi}, & \text{if } n \text{ is odd}. \end{cases}$$

$$\Rightarrow \boxed{V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,5\ldots} \frac{1}{n} e^{-n\pi x/a} \sin(n\pi y/a).}$$

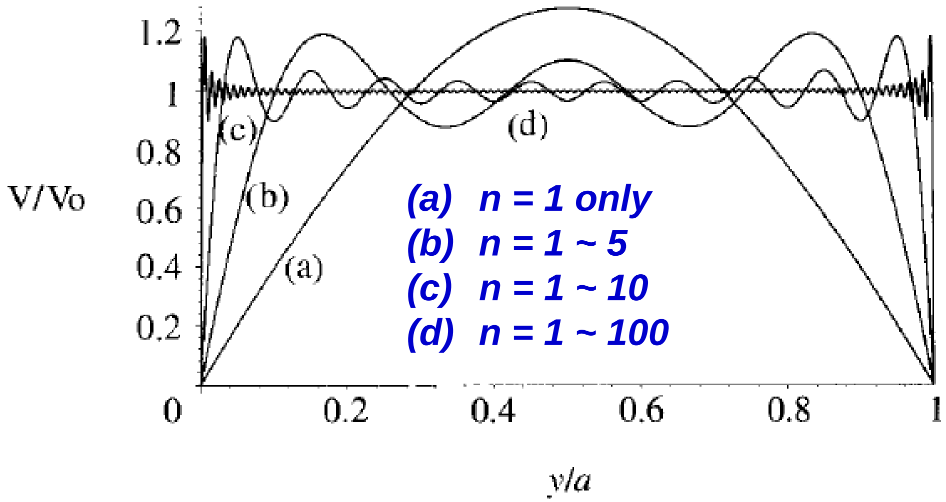
Cartesian Coordinates $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$



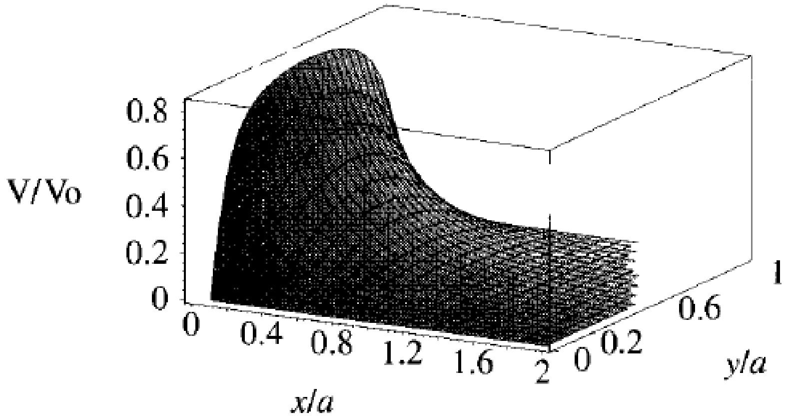
Example 3.3 $V(x, y) = X(x)Y(y)$

$$\boxed{V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,5\ldots} \frac{1}{n} e^{-n\pi x/a} \sin(n\pi y/a).} = \frac{2V_0}{\pi} \tan^{-1}\left(\frac{\sin(\pi y/a)}{\sinh(\pi x/a)}\right)$$

$V(y) = V_0$ when $x = 0$

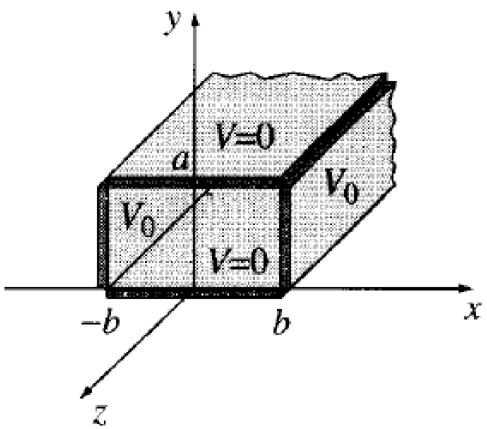


(a) n = 1 only
(b) n = 1 ~ 5
(c) n = 1 ~ 10
(d) n = 1 ~ 100



➔ It shows how the first few terms in the Fourier series combine to make a better and better approximation to the constant v_o.

Cartesian Coordinates



Example 3.4 Two infinitely long grounded metal plates, again at $y = 0$ and $y = a$, are connected at $x = \pm b$ by metal strips maintained at a constant potential V_0. Find the potential inside the resulting rectangular pipe.

(Boundary conditions)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

(i) $V = 0$ when $y = 0$,
(ii) $V = 0$ when $y = a$,
(iii) $V = V_0$ when $x = b$,
(iv) $V = V_0$ when $x = -b$.

$$V(x, y) = X(x)Y(y) \Rightarrow V(x, y) = (Ae^{kx} + Be^{-kx})(C \sin ky + D \cos ky)$$

The situation is *symmetric* with respect to *x,* so *V(-x, y) = V(x, y)* ➔ ***A = B***

$$e^{kx} + e^{-kx} = 2\cosh kx \longrightarrow V(x, y) = \cosh kx\,(C \sin ky + D \cos ky)$$

(i) $V = 0$ when $y = 0$ ⟶ ***D = 0***
(ii) $V = 0$ when $y = a$ ⟶ ***k = nπ/a***

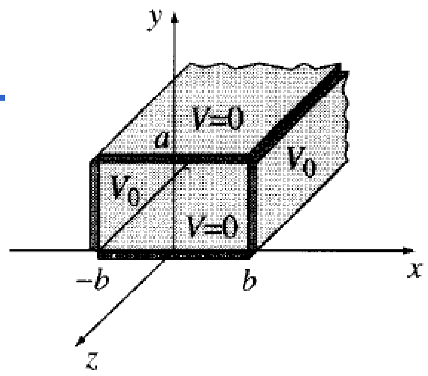
$\Rightarrow V(x, y) = C \cosh(n\pi x/a)\, \sin(n\pi y/a)$

$$\Rightarrow V(x, y) = \sum_{n=1}^{\infty} C_n \cosh(n\pi x/a)\, \sin(n\pi y/a)$$

Cartesian Coordinates $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$

Example 3.4

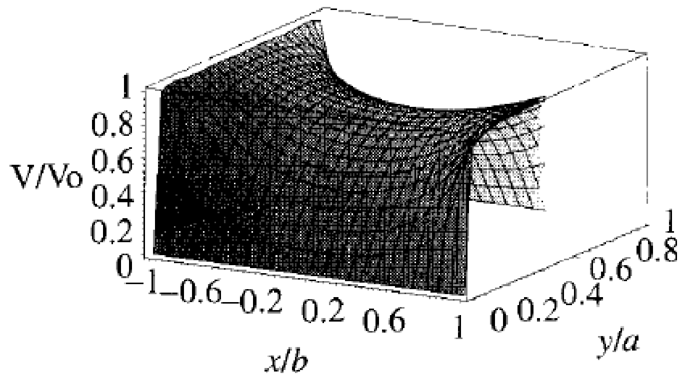
$$V(x, y) = X(x)Y(y) \quad \Rightarrow \quad V(x, y) = \sum_{n=1}^{\infty} C_n \cosh(n\pi x/a) \sin(n\pi y/a)$$



$$\text{(iii)} \quad V = V_0 \text{ when } x = b \longrightarrow V(b, y) = \sum_{n=1}^{\infty} C_n \cosh(n\pi b/a) \sin(n\pi y/a) = V_0$$

→ This is the same problem in Fourier analysis.

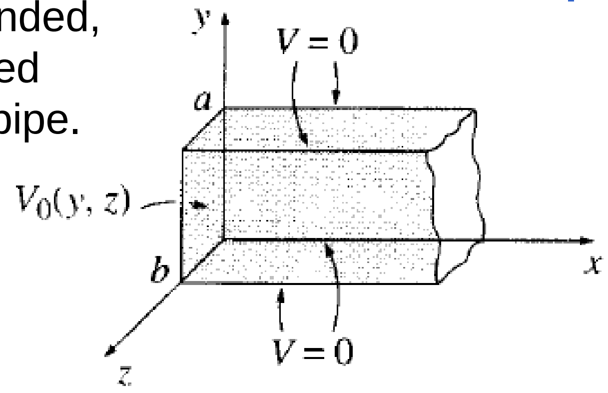
$$C_n \cosh(n\pi b/a) = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4V_0}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$



$$\boxed{V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,5\ldots} \frac{1}{n} \frac{\cosh(n\pi x/a)}{\cosh(n\pi b/a)} \sin(n\pi y/a).}$$

Cartesian Coordinates

Example 3.5 An infinitely long rectangular metal pipe is grounded, but one end, at $x = 0$, is maintained at a specified potential $V_0\ (y, z)$, Find the potential inside the pipe.



➔ ***This is a three-dimensional problem:***

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

(Boundary conditions)

(i) $V = 0$ when $y = 0$,
(ii) $V = 0$ when $y = a$,
(iii) $V = 0$ when $z = 0$,
(iv) $V = 0$ when $z = b$,
(v) $V \to 0$ as $x \to \infty$,
(vi) $V = V_0(y, z)$ when $x = 0$.

$$V(x, y, z) = X(x)Y(y)Z(z) \longrightarrow \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

$$\frac{d^2X}{dx^2} = (k^2 + l^2)X \longrightarrow X(x) = Ae^{\sqrt{k^2+l^2}\,x} + Be^{-\sqrt{k^2+l^2}\,x}$$

$$\frac{d^2Y}{dy^2} = -k^2Y \longrightarrow Y(y) = C \sin ky + D \cos ky$$

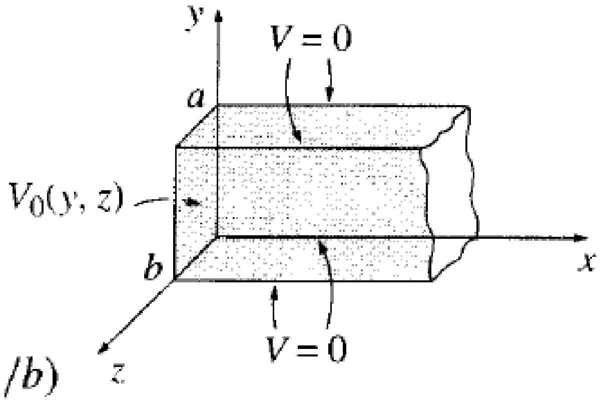
$$\frac{d^2Z}{dz^2} = -l^2Z \longrightarrow Z(z) = E \sin lz + F \cos lz$$

Cartesian Coordinates

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0 \qquad V(x,y,z)=X(x)Y(y)Z(z)$$

Example 3.5

$$\begin{aligned} X(x) &= Ae^{\sqrt{k^2+l^2}\,x}+Be^{-\sqrt{k^2+l^2}\,x} \\ Y(y) &= C\sin ky + D\cos ky \\ Z(z) &= E\sin lz + F\cos lz \end{aligned}$$



From the boundary conditions from (i) to (v),

$$V(x,y,z)=Ce^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x}\sin(n\pi y/a)\,\sin(m\pi z/b)$$

The most general linear combination is a *double* sum:

$$V(x,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{n,m}e^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x}\sin(n\pi y/a)\,\sin(m\pi z/b)$$

(vi) $V=V_0(y,z)$ when $x=0$ $\longrightarrow$ $V(0,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{n,m}\sin(n\pi y/a)\,\sin(m\pi z/b)=V_0(y,z)$

$$\Rightarrow\quad C_{n,m}=\frac{4}{ab}\int_0^a\int_0^b V_0(y,z)\sin(n\pi y/a)\,\sin(m\pi z/b)\,dy\,dz$$

If the end of the tube is a conductor at *constant* potential V_0,

$$\Rightarrow\quad \boxed{V(x,y,z)=\frac{16V_0}{\pi^2}\sum_{n,m=1,3,5\ldots}^{\infty}\frac{1}{nm}e^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x}\sin(n\pi y/a)\,\sin(m\pi z/b).}$$

3.3.2 Spherical Coordinates $\nabla^2 V = 0$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0$$

Assume the problem has **azimuthal symmetry,** so that *V is independent of* ϕ;

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0$$

$$\boxed{V(r,\theta)=R(r)\Theta(\theta)} \Rightarrow \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=0$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=l(l+1) \longrightarrow R(r)=Ar^l+\frac{B}{r^{l+1}}$$

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=-l(l+1) \longrightarrow \Theta(\theta)=P_l(\cos\theta)$$ **(Legendre polynomials)**

The general solution is

$$\Rightarrow \boxed{V(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta).}$$

Spherical Coordinates $V(r,\theta) = R(r)\Theta(\theta)$

Example 3.6 ***$V_0(\theta)$*** **is specified on the surface of a hollow sphere**, of radius *R*. Find the **potential inside the sphere.**

$$\boxed{V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta).}$$

➔ ***$B_l = 0$***; otherwise the potential would blow up at the origin.

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$ ➔ ***At $r = R$***; $$V(R,\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = V_0(\theta)$$

Legendre polynomials (like the sines) are *orthogonal* functions:

$$\int_{-1}^{1} P_l(x) P_{l'}(x)\, dx = \int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta \, d\theta = \begin{cases} 0, & \text{if } l' \neq l, \\ \dfrac{2}{2l+1}, & \text{if } l' = l. \end{cases}$$

Thus, multiplying by $P_{l'}(\cos\theta)\sin\theta$ and integrating,

$$A_l = \frac{2l+1}{2R^l} \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta \longrightarrow V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

Spherical Coordinates $V(r,\theta) = R(r)\Theta(\theta)$

Example 3.7 ***$V_0(\theta)$* is specified on the surface of a hollow sphere**, of radius *R*. Find the **potential outside the sphere.**

$$\boxed{V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta).}$$

➔ ***Now, $A_l = 0$***; otherwise the potential would not go to zero at infinity.

$$V(r,\theta) = \sum_{i=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

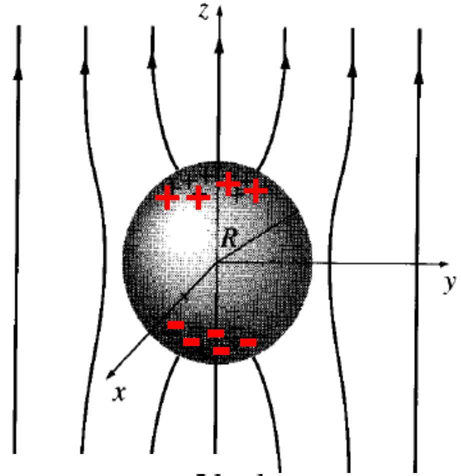
➔ ***At $r = R$***; $V(R,\theta) = \sum_{i=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) = V_0(\theta)$

Thus, multiplying by $P_{l'}(\cos\theta)\sin\theta$ and integrating,

$$B_l = \frac{2l+1}{2} R^{l+1} \int_0^{\pi} V_0(\theta) P_l(\cos\theta)\sin\theta\, d\theta \longrightarrow V(r,\theta) = \sum_{i=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

Spherical Coordinates $V(r,\theta) = R(r)\Theta(\theta)$

Example 3.8 An **uncharged metal sphere** of radius *R* is placed in an otherwise uniform electric field E = $E_0\mathbf{z}$. **The induced charge**, in tum, distorts the field in the neighborhood of the sphere.
Find the potential in the region outside the sphere.



$$\boxed{V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta).}$$

The sphere is an equipotential → we may as well set it to zero.

Far from the sphere the field is $E_0\mathbf{z}$., hence the **boundary conditions** are

(i) $V = 0$ when $r = R$ $\longrightarrow A_l R^l + \frac{B_l}{R^{l+1}} = 0 \longrightarrow B_l = -A_l R^{2l+1}$

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l\left(r^l - \frac{R^{2l+1}}{r^{l+1}}\right)P_l(\cos\theta)$$

(ii) $V \to -E_0 r\cos\theta$ for $r \gg R \longrightarrow \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r\cos\theta$

since $P_1(\cos\theta) = \cos\theta$, $A_1 = -E_0$, all other A_l's zero.

$$\boxed{V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta.}$$

$$\sigma(\theta) = -\epsilon_0 \frac{\partial V}{\partial r}\bigg|_{r=R} = 3\epsilon_0 E_0\cos\theta$$

Spherical Coordinates $V(r,\theta) = R(r)\Theta(\theta)$

Example 3.9 A specified **charge density $\sigma_0(\theta)$ is glued over the surface** of a spherical shell of radius *R.*
Find the potential inside and outside the sphere.

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta). \Rightarrow \quad V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \qquad (r \le R)$$

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \qquad (r \ge R)$$

➔ ***At r = R***; the boundary conditions are:

(1) $$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) \longrightarrow B_l = A_l R^{2l+1}$$

(2) $$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}} \longrightarrow \left(\frac{\partial V_{\text{out}}}{\partial r} - \frac{\partial V_{\text{in}}}{\partial r}\right)\Bigg|_{r=R} = -\frac{1}{\epsilon_0}\sigma_0(\theta)$$

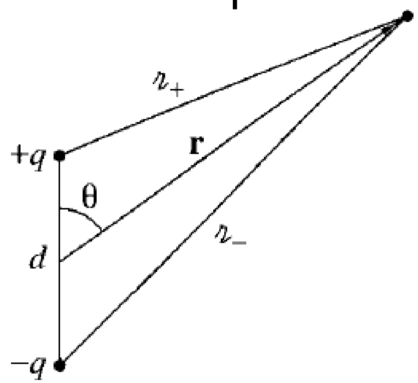
$$\sum_{l=0}^{\infty}(2l+1)A_l R^{l-1} P_l(\cos\theta) = \frac{1}{\epsilon_0}\sigma_0(\theta)$$

$$\Rightarrow \quad \boxed{A_l = \frac{1}{2\epsilon_0 R^{l-1}} \int_0^{\pi} \sigma_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta.} \qquad B_l = A_l R^{2l+1}.$$

3.4 Multipole Expansion

3.4.1 Approximate Potentials at Large Distances

Example 3.10 Find the **approximate potential** at points far from the dipole.

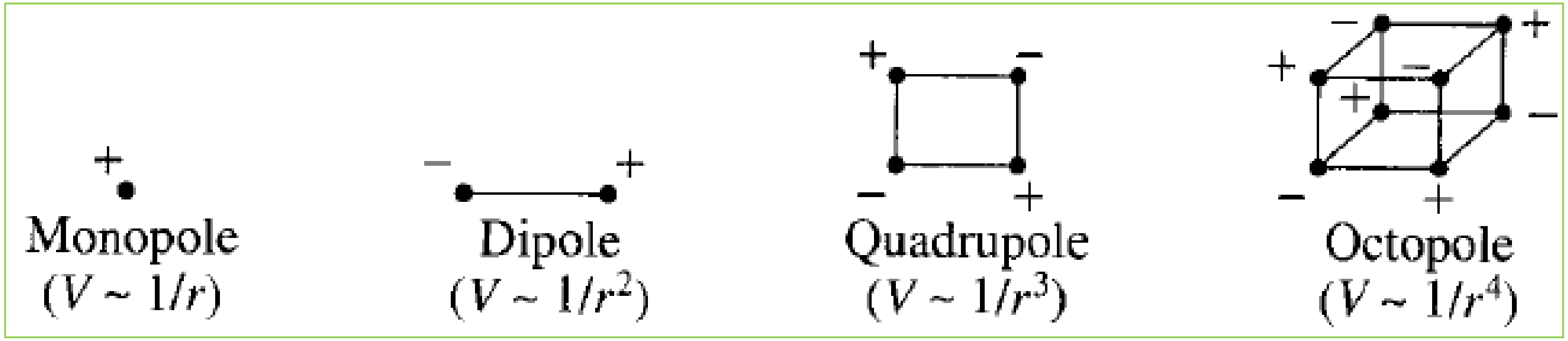


$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}_+} - \frac{q}{\mathscr{r}_-}\right)$$

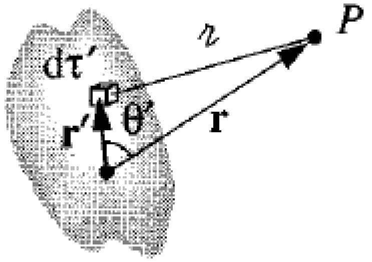
$$\mathscr{r}_\pm^2 = r^2 + (d/2)^2 \mp rd\cos\theta = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)$$

In the regime *r* » *d*,

$$\frac{1}{\mathscr{r}_\pm} \cong \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right) \longrightarrow \boxed{V(\mathbf{r}) \cong \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}.}$$



Approximate Potentials by multiple expansion



Now, develop a *systematic expansion for the potential of an arbitrary localized charge distribution, in powers of 1/ r.*

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\rho(\mathbf{r}')\,d\tau'$$

$$\mathscr{r}^2 = r^2 + (r')^2 - 2rr'\cos\theta' = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta'\right]$$

$$\mathscr{r} = r\sqrt{1+\epsilon} \quad \text{where} \quad \epsilon \equiv \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right)$$ → much less than 1

$$\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\epsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \dots\right)$$

$$= \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\theta'\right)^2 - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\theta'\right)^3 + \dots\right]$$

$$= \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)(\cos\theta') + \left(\frac{r'}{r}\right)^2 (3\cos^2\theta' - 1)/2 + \left(\frac{r'}{r}\right)^3 (5\cos^3\theta' - 3\cos\theta')/2 + \dots\right]$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta') \longrightarrow \boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\theta')\rho(\mathbf{r}')\,d\tau',}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\overbrace{\frac{1}{r}\int \rho(\mathbf{r}')\,d\tau'}^{\textit{monopole}} + \overbrace{\frac{1}{r^2}\int r'\cos\theta'\rho(\mathbf{r}')\,d\tau'}^{\textit{dipole}} + \overbrace{\frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\mathbf{r}')\,d\tau'}^{\textit{quadrupole}} + \dots\right]$$

3.4.2 The Monopole and Dipole Terms

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\underbrace{\frac{1}{r}\int \rho(\mathbf{r}')\,d\tau'}_{\text{monopole}} + \underbrace{\frac{1}{r^2}\int r'\cos\theta'\rho(\mathbf{r}')\,d\tau'}_{\text{dipole}} + \underbrace{\frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\mathbf{r}')\,d\tau'}_{\text{quadrupole}} + \ldots\right]$$

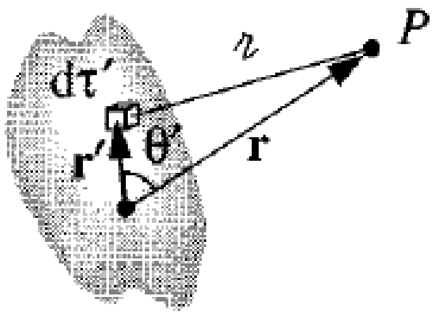
➔ Ordinarily, the **multipole expansion** is dominated (at large *r)* by the **monopole** term.

➔ If the **total charge is zero**, the dominant term in the potential will be the **dipole**.

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\cos\theta'\rho(\mathbf{r}')\,d\tau'$$

$$\longrightarrow \quad r'\cos\theta' = \hat{\mathbf{r}}\cdot\mathbf{r}'$$

$$\longrightarrow \quad V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}\cdot\int \mathbf{r}'\rho(\mathbf{r}')\,d\tau'$$



$$\boxed{\mathbf{p} \equiv \int \mathbf{r}'\rho(\mathbf{r}')\,d\tau'}$$ ➔ **Dipole moment** of the charge distribution

$$\boxed{V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}}$$

3.4.3 Origin of Coordinates in Multipole Expansions

monopole — dipole — quadrupole

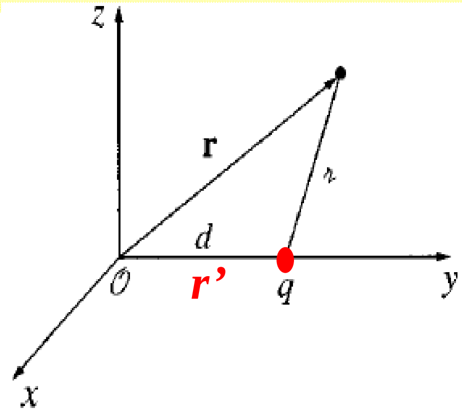
$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(\mathbf{r}')\,d\tau' + \frac{1}{r^2}\int r'\cos\theta'\rho(\mathbf{r}')\,d\tau' + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\mathbf{r}')\,d\tau' + \ldots\right]$$

A point charge at the origin constitutes a "pure" monopole.

If it is not at the origin, it's no longer a pure monopole.

→ The monopole term, $(1/4\pi\epsilon_0)q/r$ is not quite correct.

→ The exact monopole potential is $(1/4\pi\epsilon_0)q/\mathscr{r}$



➔ So, moving the origin can radically alter the expansion.

But there is an important exception:

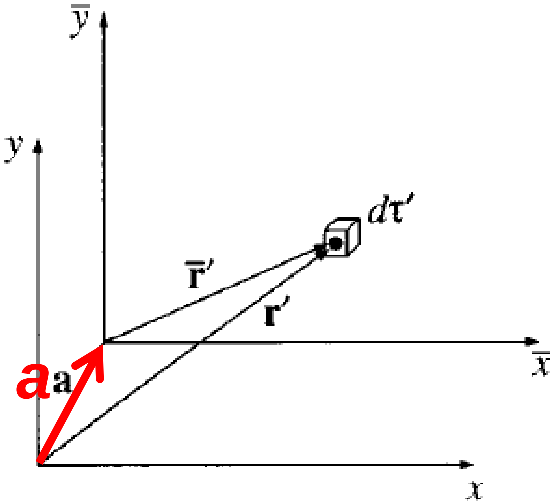
➔ ***If the total charge is zero, the dipole moment is independent of the choice of origin.***

Let's displace the origin by an amount **a**

$$\bar{\mathbf{p}} = \int \bar{\mathbf{r}}'\rho(\mathbf{r}')\,d\tau' = \int (\mathbf{r}' - \mathbf{a})\rho(\mathbf{r}')\,d\tau'$$

$$= \int \mathbf{r}'\rho(\mathbf{r}')\,d\tau' - \mathbf{a}\int \rho(\mathbf{r}')\,d\tau' = \mathbf{p} - Q\mathbf{a}$$

if $Q = 0$, then $\bar{\mathbf{p}} = \mathbf{p}$.



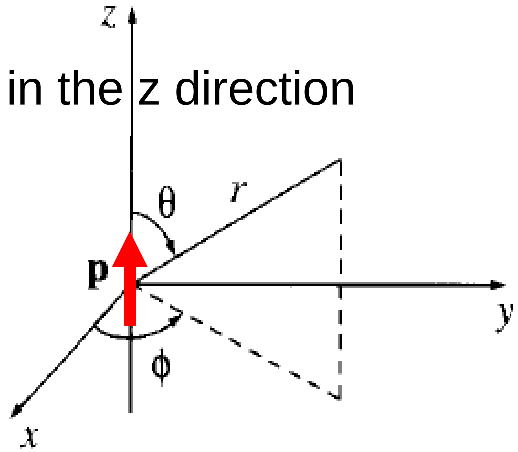
3.4.4 The Electric Field of a Dipole

monopole **dipole** **quadrupole**

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(\mathbf{r}')\,d\tau' + \frac{1}{r^2}\int r'\cos\theta'\rho(\mathbf{r}')\,d\tau' + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\mathbf{r}')\,d\tau' + \dots\right]$$

Now let's calculate the electric field of a (pure, nearly point) dipole.

If we choose coordinates so that ***p*** lies at the origin and points in the z direction

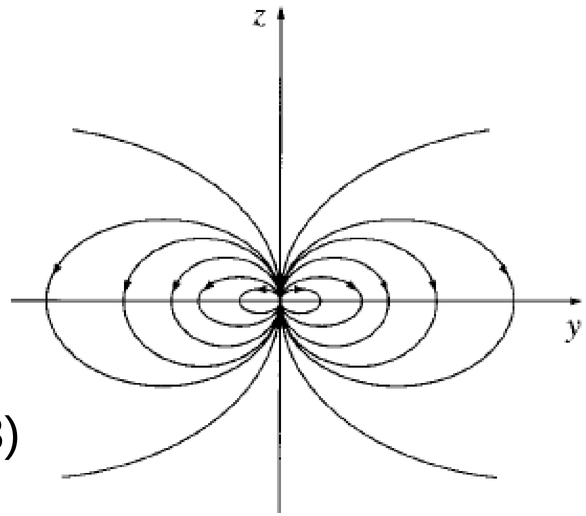


$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3},$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3},$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi} = 0.$$



$$\boxed{\mathbf{E}_{\text{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}).}$$

$$\boxed{\mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}].}$$ (Prob. 3.33)