Problema: Demuestre la identidad

$$\hat{L}^2 = (\vec{\hat{r}} \times \vec{\hat{p}}) \cdot (\vec{\hat{r}} \times \vec{\hat{p}}) = \hat{r}^2 \hat{p}^2 - \vec{\hat{r}}(\vec{\hat{r}} \cdot \vec{\hat{p}}) \cdot \vec{\hat{p}} + 2i\hbar(\vec{\hat{r}} \cdot \vec{\hat{p}})$$

$$\Downarrow$$

demo:

$$(\vec{\hat{r}} \times \vec{\hat{p}}) \cdot (\vec{\hat{r}} \times \vec{\hat{p}}) = (\vec{\hat{r}} \times \vec{\hat{p}})_i (\vec{\hat{r}} \times \vec{\hat{p}})_i = \epsilon_{ijk} \epsilon_{ilm} \hat{x}_j \hat{p}_k \hat{x}_l \hat{p}_m$$

$$= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) \hat{x}_j \underbrace{\hat{p}_k \hat{x}_l}_{(*)} \hat{p}_m$$

Por otro lado:

$$[\hat{x}_l, \hat{p}_k] = i\hbar \delta_{lk} \Longrightarrow \hat{x}_l \hat{p}_k - \hat{p}_k \hat{x}_l = i\hbar \delta_{kl}$$

$$\Downarrow$$

$$(**) \quad \hat{x}_l \hat{p}_k - i\hbar \delta_{kl} = \hat{p}_k \hat{x}_l$$

reemplazando (**) en (*)

$$\hat{L}^2 = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) \hat{x}_j (\hat{x}_l \hat{p}_k - i\hbar \delta_{kl}) \hat{p}_m$$

$$\Downarrow$$

$$\hat{L}^2 = \delta_{jl}\delta_{km}\,\hat{x}_j\hat{x}_l\hat{p}_k\hat{p}_m - i\hbar\,\delta_{jl}\delta_{km}\delta_{kl}\,\hat{x}_j\hat{p}_m$$

$$- \delta_{jm}\delta_{kl}\,\hat{x}_j\hat{x}_l\hat{p}_k\hat{p}_m + i\hbar\,\delta_{jm}\delta_{kl}\delta_{kl}\,\hat{x}_j\hat{p}_m$$

$$= \underbrace{\hat{x}_j\hat{x}_j}_{\hat{r}^2}\,\underbrace{\hat{p}_m\hat{p}_m}_{\hat{p}^2} - i\hbar\,\underbrace{\hat{x}_j\hat{p}_j}_{\vec{\hat{r}}\cdot\vec{\hat{p}}} - \underbrace{\hat{x}_m\hat{x}_l\hat{p}_l\hat{p}_m}_{\vec{\hat{r}}(\vec{\hat{r}}\cdot\vec{\hat{p}})\vec{\hat{p}}} + i\hbar\,\underset{3}{\delta_{ll}}\,\underbrace{\hat{x}_j\hat{p}_j}_{\vec{\hat{r}}\cdot\vec{\hat{p}}}$$

Finalmente:

$$\hat{L}^2 = \hat{r}^2\hat{p}^2 - \vec{\hat{r}}(\vec{\hat{r}}\cdot\vec{\hat{p}})\vec{\hat{p}} + 2i\hbar\,\vec{\hat{r}}\cdot\vec{\hat{p}}$$

QED