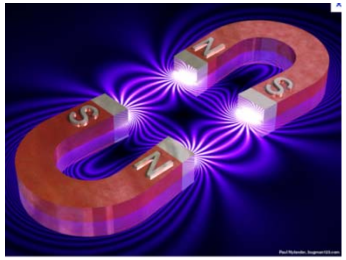
Chapter 5. Magnetostatics



5.2 The Biot-Savart Law

5.2.1 Steady Currents

Steady current ➔ A continuous flow that has been going on forever, without change and without charge piling up anywhere.

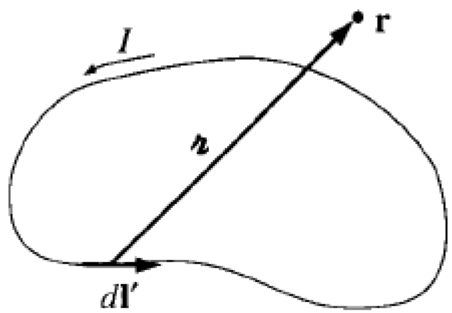
For a steady current, constant magnetic fields: Magnetostatics ➔ $\boxed{\nabla \cdot \mathbf{J} = 0}$

5.2.2 The Magnetic Field of a Steady Current

The integration is along the current path in the direction of the flow.

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \hat{\boldsymbol{\imath}}}{\imath^2}\, dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\imath}}}{\imath^2}$$

➔ Biot-Savart law



(Permeability of free space) $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}$

(Unit of B, Tesla) $1\ \mathrm{T} = 1\ \mathrm{N/(A \cdot m)}$

For surface and volume currents

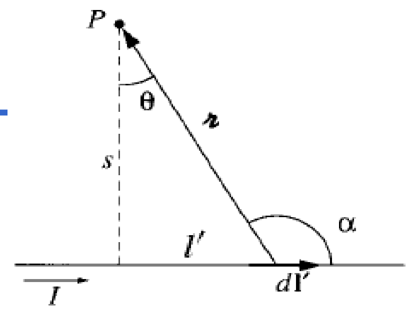
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{r}') \times \hat{\boldsymbol{\imath}}}{\imath^2}\, da' \qquad \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\imath}}}{\imath^2}\, d\tau'$$

Biot-Savart law in magnetostatics plays a role analogous to Coulomb's law in electrostatics

$1/\imath^2$ dependence is common

The Biot-Savart Law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}'\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}.$$



Example 5.5

Find the magnetic field a distance *s* from a long straight wire carrying a steady current *I.*

$(d\mathbf{l}'\times\hat{\boldsymbol{\mathscr{r}}})$ points *out* of the page, and has the magnitude of

$$dl'\sin\alpha = dl'\cos\theta \xrightarrow{l' = s\tan\theta} dl' = \frac{s}{\cos^2\theta}\,d\theta \xrightarrow{s = \mathscr{r}\cos\theta} \frac{1}{\mathscr{r}^2} = \frac{\cos^2\theta}{s^2}$$

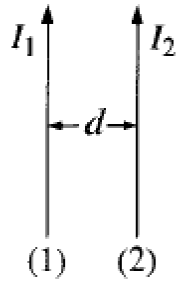
$$B = \frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2}\left(\frac{\cos^2\theta}{s^2}\right)\left(\frac{s}{\cos^2\theta}\right)\cos\theta\,d\theta = \frac{\mu_0 I}{4\pi s}\int_{\theta_1}^{\theta_2}\cos\theta\,d\theta = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1).$$

In the case of an infinite wire, $\theta_1 = -\pi/2$ and $\theta_2 = \pi/2$, $\longrightarrow \boxed{B = \frac{\mu_0 I}{2\pi s}}$

As an application, let's find the force of attraction between two wires carrying currents, I_1 and I_2.

The field at (2) due to (1) is $B = \frac{\mu_0 I_1}{2\pi d}$ → It points into the page.

$$\mathbf{F}_{\text{mag}} = I\int (d\mathbf{l}\times\mathbf{B}) \longrightarrow F = I_2\left(\frac{\mu_0 I_1}{2\pi d}\right)\int dl$$ → It is directed towards (1)

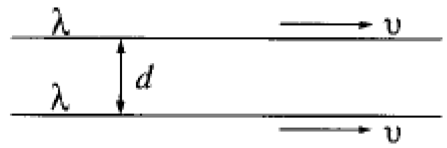


The force per unit length → $f = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}$ → It is attractive to each other.

The Biot-Savart Law $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}\times\hat{\mathbf{r}}}{r^2}\,dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}'\times\hat{\mathbf{r}}}{r^2}.$

Problem 5.13

Suppose you have two infinite straight line charges λ, a distance *d* apart, moving along at a constant speed *v*. How great would v have to be in order for the magnetic attraction to balance the electrical repulsion?



Magnetic attraction per unit length

$$f = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d} \xrightarrow{\quad \mathbf{I} = \lambda\mathbf{v} \quad} f_m = \frac{\mu_0}{2\pi}\frac{\lambda^2 v^2}{d}$$

Electric repulsion per unit length on the other wire

Since the electric field of one wire is $E = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s} \longrightarrow f_e = \frac{1}{2\pi\epsilon_0}\frac{\lambda^2}{d}$

➔ They balance when $\mu_0 v^2 = \frac{1}{\epsilon_0} \longrightarrow v = \frac{1}{\sqrt{\epsilon_0\mu_0}}$

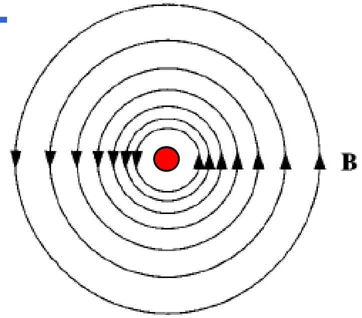
➔ This is precisely the speed of light!

➔ In fact you could never get the wires going fast enough.

➔ The electric force always dominates.

5.3 The Divergence and Curl of B

5.3.1 Straight-Line Currents



The magnetic field of an infinite straight wire looks to have a ***nonzero curl***.
➔ Let's calculate **the Curl of B**.

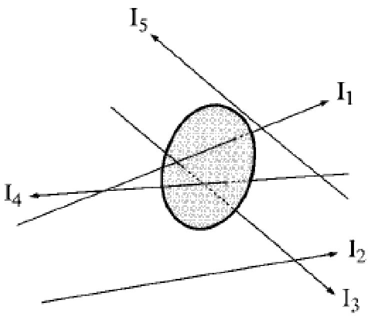
If we use cylindrical coordinates *(s, ϕ, z),* with the current flowing along the *z* axis,

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}} \qquad d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}},$$

$$\Rightarrow \oint \mathbf{B}\cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi}\oint \frac{1}{s}\,s\,d\phi = \frac{\mu_0 I}{2\pi}\int_0^{2\pi} d\phi = \mu_0 I$$

➔ It is independent of *s.*

Now suppose we have a *bundle* of straight wires:



$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \longleftarrow I_{\text{enc}} = \int \mathbf{J}\cdot d\mathbf{a}$$

Applying Stokes' theorem → $\int (\nabla \times \mathbf{B})\cdot d\mathbf{a} = \mu_0 \int \mathbf{J}\cdot d\mathbf{a}$

$$\Rightarrow \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

➔ *But, this derivation is seriously flawed by the restriction to infinite straight line currents.*
➔ *Let's use the Biot-Savart Law for general derivation.*

5.3.2 The Divergence and Curl of B

The Biot-Savart law for the general case of a volume current is

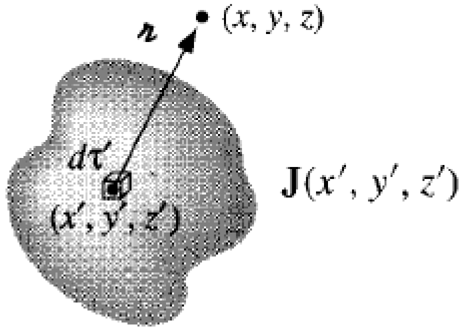
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau'$$

$\mathbf{B}$ is a function of (x, y, z)

$\mathbf{J}$ is a function of (x', y', z')

$\boldsymbol{\mathscr{r}} = (x - x')\,\hat{\mathbf{x}} + (y - y')\,\hat{\mathbf{y}} + (z - z')\,\hat{\mathbf{z}}$

$d\tau' = dx'\,dy'\,dz'$



Applying the divergence with respect to the ***unprimed*** coordinates:

$$\nabla \cdot \mathbf{B} = \frac{\mu_0}{4\pi}\int \nabla \cdot \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) d\tau' \longrightarrow \boxed{\nabla \cdot \mathbf{B} = 0}$$

$$\nabla \cdot \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \cdot (\nabla \times \mathbf{J}) - \mathbf{J} \cdot \left(\nabla \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)$$

$\nabla \times \mathbf{J} = 0$, because $\mathbf{J}$ doesn't depend on the unprimed variables (x, y, z)

$\nabla \times (\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2) = 0$

The Divergence and Curl of B

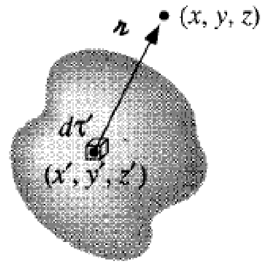
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$$

$\mathbf{B}$ is a function of (x, y, z)

$\mathbf{J}$ is a function of (x', y', z')

$\mathscr{r} = (x - x')\,\hat{\mathbf{x}} + (y - y')\,\hat{\mathbf{y}} + (z - z')\,\hat{\mathbf{z}}$

$d\tau' = dx'\,dy'\,dz'$



Applying the curl with respect to the ***unprimed*** coordinates:

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi}\int \nabla \times \left(\mathbf{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) d\tau' \xrightarrow{\quad ? \quad} \boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}}$$

➔ ***Ampere's Law***

$$\nabla \times \left(\mathbf{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \mathbf{J}\left(\nabla \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) - (\mathbf{J} \cdot \nabla)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\nabla \cdot \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\,\delta^3(\mathscr{r})$$

$$\frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{r}')4\pi\,\delta^3(\mathbf{r} - \mathbf{r}')\, d\tau' = \mu_0 \mathbf{J}(\mathbf{r})$$

$$-(\mathbf{J} \cdot \nabla)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = (\mathbf{J} \cdot \nabla')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

The x component, in particular, is $(\mathbf{J} \cdot \nabla')\left(\frac{x - x'}{\mathscr{r}^3}\right) = \nabla' \cdot \left[\frac{(x - x')}{\mathscr{r}^3}\mathbf{J}\right] - \left(\frac{x - x'}{\mathscr{r}^3}\right)(\nabla' \cdot \mathbf{J})$

For *steady* currents the divergence of J is zero → $\left[-(\mathbf{J} \cdot \nabla)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right]_x = \nabla' \cdot \left[\frac{(x - x')}{\mathscr{r}^3}\mathbf{J}\right]$

Note the switching by putting - sign from ∇ to ∇'

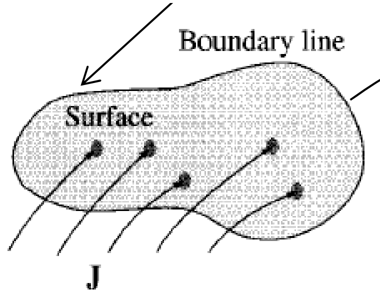
$$\int_{\mathcal{V}} \nabla' \cdot \left[\frac{(x - x')}{\mathscr{r}^3}\mathbf{J}\right] d\tau' = \oint_{\mathcal{S}} \frac{(x - x')}{\mathscr{r}^3}\mathbf{J} \cdot d\mathbf{a}' = 0$$

We can have *the boundary surface large enough, so* the current is zero on the surface (all current is safely *inside)*

5.3.3 Applications of Ampere's Law

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}} \longleftarrow \int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a} \longrightarrow \boxed{\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc}}$$

Total current passing through the surface



Amperian loop

Which direction through the surface corresponds to a positive current?

➔ ***Keep the Right-Hand Rule.***

➔ **If the fingers of your right hand indicate the direction of integration around the boundary, then your thumb defines the direction of a positive current.**

Electrostatics :	**Coulomb law**	**→**	**Gauss's law**
Magnetostatics:	**Biot-Savart law**	**→**	**Ampere's law**

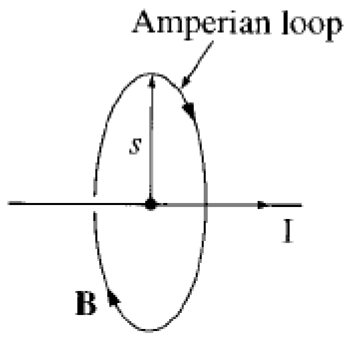
Like Gauss's law, Ampere's law is useful
only when the symmetry of the problem enables you to pull B outside the integral.

Ampere's Law

$$\boxed{\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{enc}}$$

Example 5.7 Find the magnetic field a distance *s* from a long straight wire.

$$\oint \mathbf{B}\cdot d\mathbf{l} = B\oint dl = B2\pi s = \mu_0 I_{\text{enc}} = \mu_0 I. \longrightarrow B = \frac{\mu_0 I}{2\pi s}$$

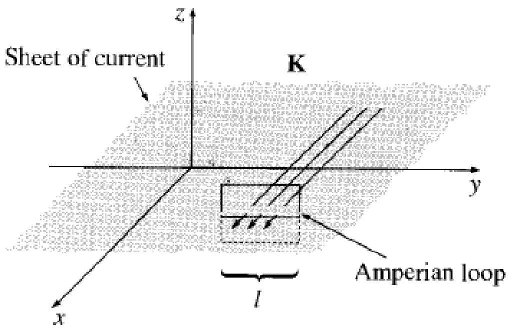


Example 5.8 Find the magnetic field of an infinite uniform surface current **K** , flowing over the xy plane.

B can only have a *y-component:*

$$\oint \mathbf{B}\cdot d\mathbf{l} = 2Bl = \mu_0 I_{\text{enc}} = \mu_0 K l$$

$$\longrightarrow \mathbf{B} = \begin{cases} +(\mu_0/2)K\,\hat{\mathbf{y}} & \text{for } z < 0 \\ -(\mu_0/2)K\,\hat{\mathbf{y}} & \text{for } z > 0 \end{cases}$$

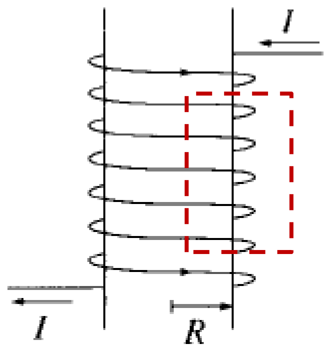


Example 5.9 Find the magnetic field of a very long solenoid, consisting of *n* closely wound turns per unit length on a cylinder of radius *R* and carrying a steady current *I*.

B of an infinite, closely wound solenoid runs parallel to the axis:

$$\oint \mathbf{B}\cdot d\mathbf{l} = BL = \mu_0 I_{\text{enc}} = \mu_0 n I L$$

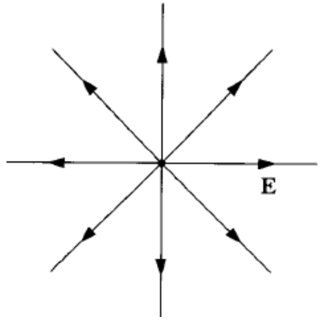
$$\longrightarrow \mathbf{B} = \begin{cases} \mu_0 n I\,\hat{\mathbf{z}}, & \text{inside the solenoid,} \\ 0, & \text{outside the solenoid.} \end{cases}$$



5.3.4 Comparison of Magnetostatics and Electrostatics

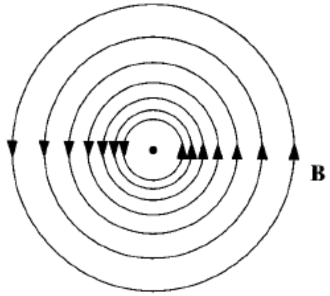
The divergence and curl of the electrostatic field are

$$\begin{cases} \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho, & \text{(Gauss's law);} \\ \nabla \times \mathbf{E} = 0, & \text{(no name).} \end{cases}$$



The divergence and curl of the magnetostatic field are

$$\begin{cases} \nabla \cdot \mathbf{B} = 0, & \text{(no name);} \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J}, & \text{(Ampère's law)} \end{cases}$$



➔ ***These 4 equations are Maxwell's Equations for static electromagnetics***

The force law is $\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

➔ ***Typically, electric forces are enormously larger than magnetic ones.***