## Derivation of Euler's reflection formula

Since $e^{-t} = \lim_{n\to\infty}\left(1 - \frac{t}{n}\right)^n$,

the gamma function can be represented as

$$\Gamma(z) = \lim_{n\to\infty}\int_0^n t^{z-1}\left(1 - \frac{t}{n}\right)^n dt.$$

Integrating by parts $n$ times yields

$$\Gamma(z) = \lim_{n\to\infty}\frac{n}{nz}\cdot\frac{n-1}{n(z+1)}\cdot\frac{n-2}{n(z+2)}\cdots\frac{1}{n(z+n-1)}\int_0^n t^{z+n-1}\,dt,$$

which is equal to

$$\Gamma(z) = \lim_{n\to\infty}\frac{n!}{n^n}\prod_{k=0}^{n}(z+k)^{-1}n^{z+n}.$$

This can be rewritten as

$$\Gamma(z) = \lim_{n\to\infty}\frac{n^z}{z}\prod_{k=1}^{n}\frac{k}{z+k} = \lim_{n\to\infty}\frac{n^z}{z}\prod_{k=1}^{n}\frac{1}{1+\frac{z}{k}}.$$

Then, using the functional equation of the gamma function, we get

$$-z\Gamma(-z)\Gamma(z) = \Gamma(1-z)\Gamma(z) = \lim_{n\to\infty}\frac{1}{z}\prod_{k=1}^{n}\frac{1}{1-\frac{z^2}{k^2}}.$$

It can be proved that

$$\sin(\pi z) = \pi z\prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2}\right).$$

Then

$$\frac{\pi}{\sin(\pi z)} = \lim_{n\to\infty}\frac{1}{z}\prod_{k=1}^{n}\frac{1}{1-\frac{z^2}{k^2}}.$$

Euler's reflection formula follows:

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}, \qquad z\notin\mathbb{Z}.$$