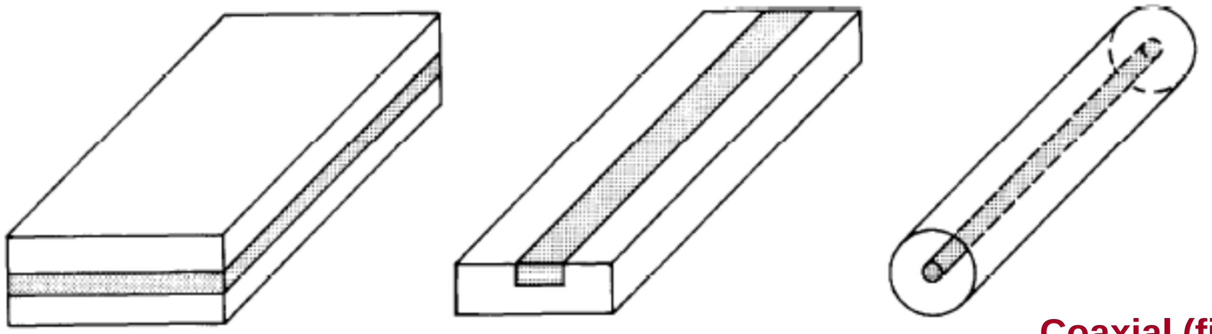
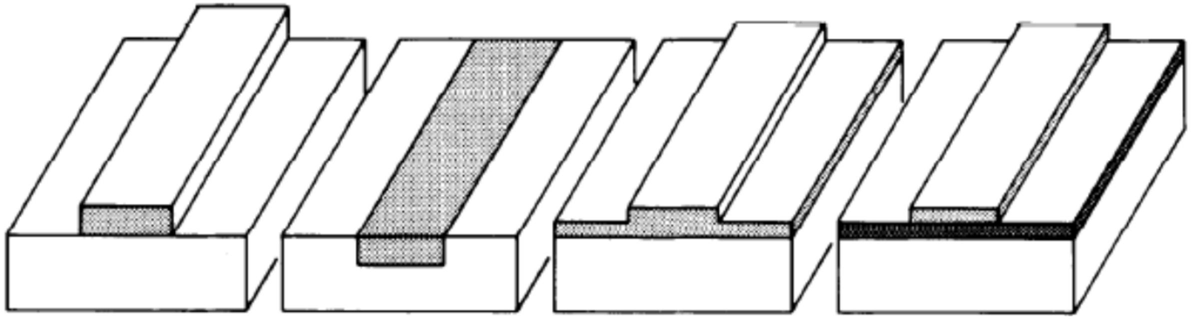
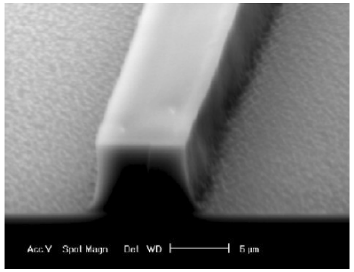
9.5 Guided Waves



slab strip Coaxial (fiber)



strip embedded strip rib or ridge strip loaded



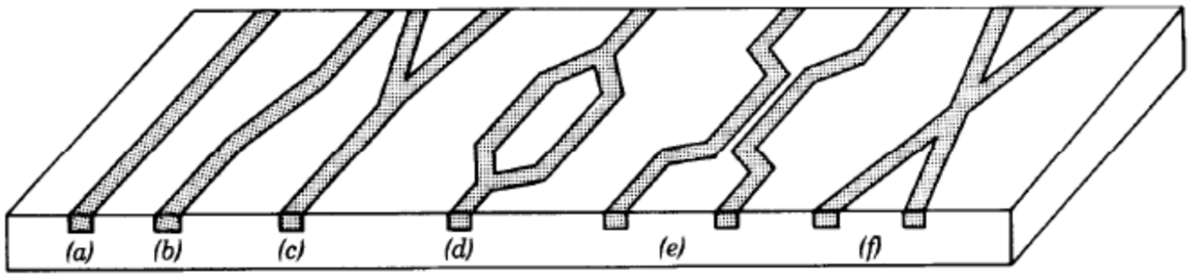


Figure 7.3-4 Different configurations for waveguides: (*a*) straight; (*b*) S bend; (*c*) Y branch; (*d*) Mach–Zehnder; (*e*) directional coupler; (*f*) intersection.

Symmetric & Asymmetric waveguides

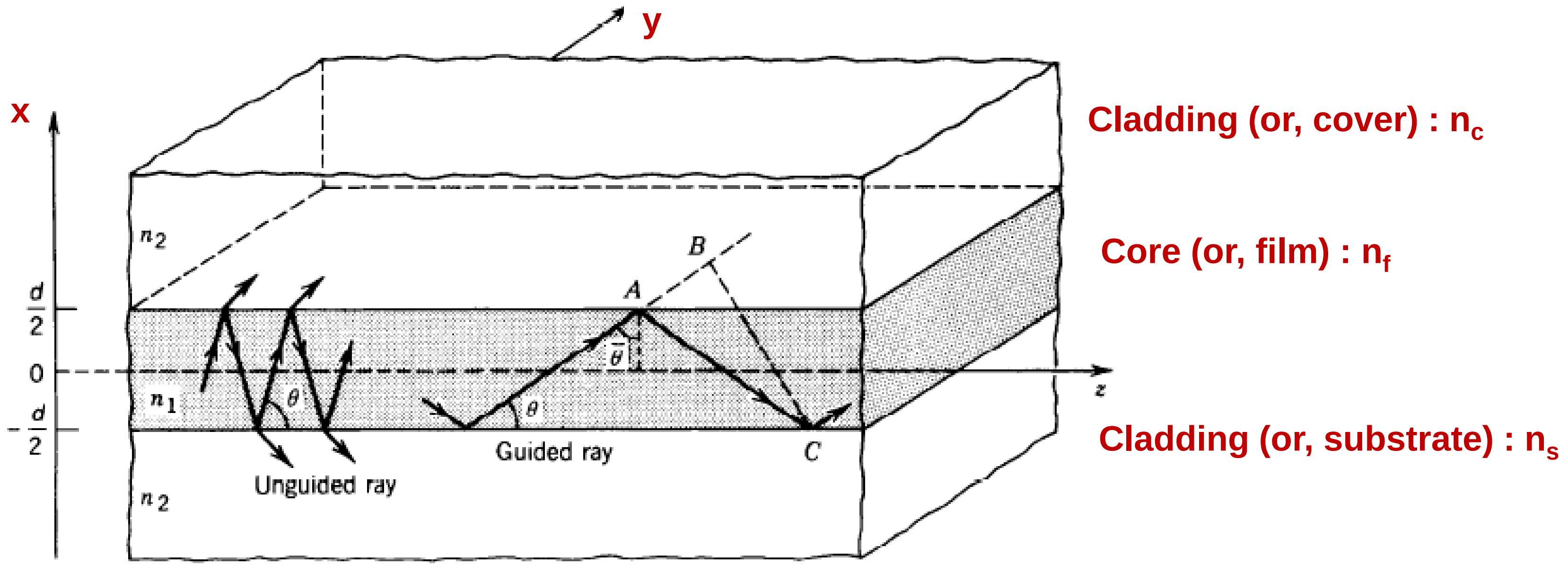


Figure 7.2-1 Planar dielectric waveguide. Rays making an angle $\theta < \bar{\theta}_c = \cos^{-1}(n_2/n_1)$ are guided by total internal reflection.

- $n_c \le n_s < n_f$ $\begin{cases} n_c = n_s \text{ : Symmetric Waveguide} \\ n_c > n_s \text{ : Asymmetric Waveguide} \end{cases}$

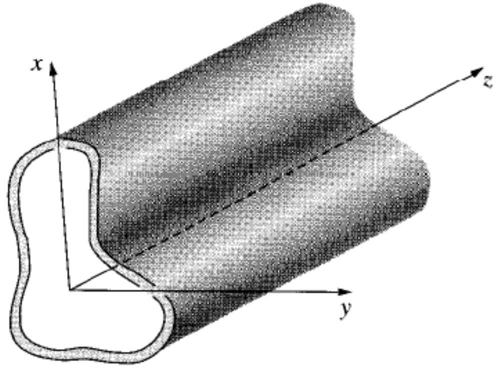
9.5 Guided Waves

9.5.1 Perfect-conductor (or, perfect mirror) waveguides

Consider electromagnetic waves confined to the interior of a hollow pipe, or **wave guide**.

Assume the wave guide is a perfect conductor, E = 0 and B = 0 inside the material itself.

The boundary conditions at the inner wall are $E^{\parallel} = 0,\ B^{\perp} = 0$



For monochromatic waves that propagate down the tube in z direction,

$$\tilde{\mathbf{E}}(x, y, z, t) = \tilde{\mathbf{E}}_0(x, y)e^{i(kz-\omega t)} \qquad \tilde{\mathbf{B}}(x, y, z, t) = \tilde{\mathbf{B}}_0(x, y)e^{i(kz-\omega t)}$$

Confined waves are *not* (in general) transverse; they can include longitudinal components.

$$\tilde{\mathbf{E}}_0 = E_x\,\hat{\mathbf{x}} + E_y\,\hat{\mathbf{y}} + E_z\,\hat{\mathbf{z}}, \qquad \tilde{\mathbf{B}}_0 = B_x\,\hat{\mathbf{x}} + B_y\,\hat{\mathbf{y}} + B_z\,\hat{\mathbf{z}}$$

Putting this into Maxwell' s equations;

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z, \quad \frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x, \quad ikE_x - \frac{\partial E_z}{\partial x} = i\omega B_y$$

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \qquad \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2}E_z, \quad \frac{\partial B_z}{\partial y} - ikB_y = -\frac{i\omega}{c^2}E_x, \quad ikB_x - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c^2}E_y$$

$$\frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x$$

$$ikE_x - \frac{\partial E_z}{\partial x} = i\omega B_y$$

$$\frac{\partial B_z}{\partial y} - ikB_y = -\frac{i\omega}{c^2}E_x$$

$$ikB_x - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c^2}E_y$$

$$E_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right),$$

$$E_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right),$$

$$B_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right)$$

$$B_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right)$$

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

Prove!

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right]E_z = 0$$

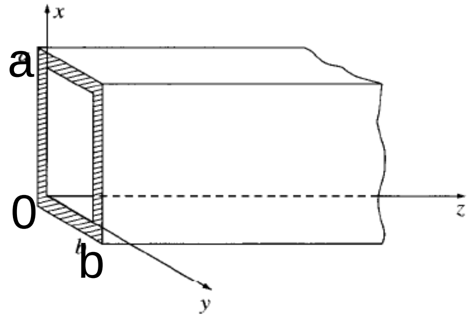
$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right]B_z = 0$$

Problem 9.26

If $E_z = 0$ we call these **TE** ("transverse electric") waves
if $B_z = 0$ they are called **TM** ("transverse magnetic") waves
if both $E_z = 0$ **and** $B_z = 0$, we call them **TEM** waves

Show that TEM waves cannot occur in a hollow wave guide.

9.5.2 TE waves ($E_z = 0$) in a rectangular waveguide



$$\tilde{\mathbf{B}}(x, y, z, t) = \tilde{\mathbf{B}}_0(x, y)e^{i(kz-\omega t)}$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2\right] B_z = 0$$

$$B_z = X(x)Y(y) \Rightarrow Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} + [(\omega/c)^2 - k^2]XY = 0$$

$$\left(\frac{\omega}{c}\right)^2 = k_x^2 + k_y^2 + k^2 (= k_z^2) \Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = -k_x^2 \qquad \frac{1}{Y}\frac{d^2Y}{dy^2} = -k_y^2$$

The general solution is $X(x) = A\sin(k_x x) + B\cos(k_x x)$

The boundary conditions require that $B^\perp = 0 \to B_x = 0$ at x = 0 and x = a

$$B_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right) \longrightarrow dX/dx \text{ vanishes at } x = 0 \text{ and } x = a.$$

$$\longrightarrow A = 0 \ \& \ B\sin(k_x a) = 0$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k_x^2 \longrightarrow k_x = m\pi / a \ (m = 0, 1, 2, \cdots)$$

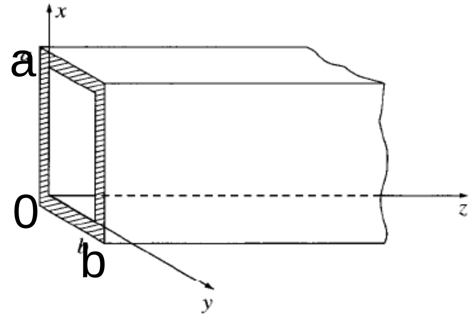
$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -k_y^2 \longrightarrow k_y = n\pi / b \ (n = 0, 1, 2, \cdots)$$

$$\Rightarrow B_z = X(x)Y(y) = B_0\cos(m\pi x / a)\sin(n\pi y / b)$$

In a rectangular waveguide

$$\tilde{\mathbf{E}}(x, y, z, t) = \tilde{\mathbf{E}}_0(x, y)e^{i(kz-\omega t)}$$

$$\tilde{\mathbf{B}}(x, y, z, t) = \tilde{\mathbf{B}}_0(x, y)e^{i(kz-\omega t)}$$



$$(\omega / c)^2 = k_x^2 + k_y^2 + k^2 (= k_z^2)$$

$$k_x = m\pi / a$$

$$k_y = m\pi / b$$

$$k = k_z = \sqrt{(\omega / c)^2 - \pi^2 \left[(m/a)^2 + (n/b)^2 \right]}$$

$$\omega_{mn} = c\pi \sqrt{(m/a)^2 + (n/b)^2}$$

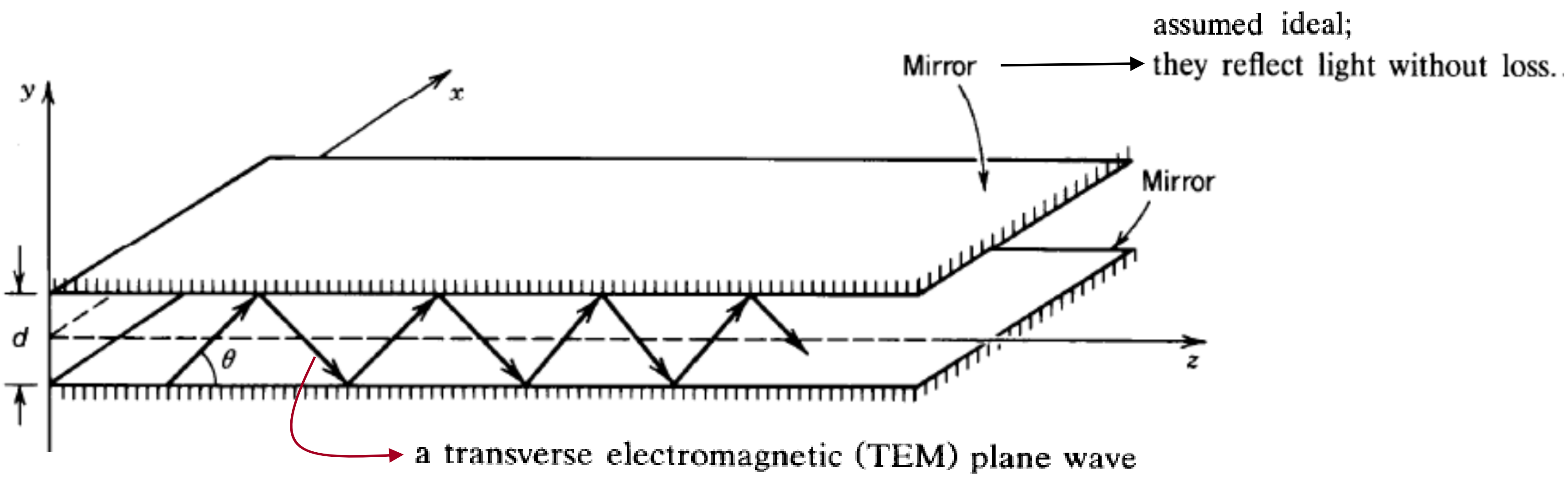
→ **cutoff frequency** of **TE$_{mn}$** mode

The wave number is $k = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}$

The wave (phase) velocity is $v = \frac{\omega}{k} = \frac{c}{\sqrt{1-(\omega_{mn}/\omega)^2}}$ > c

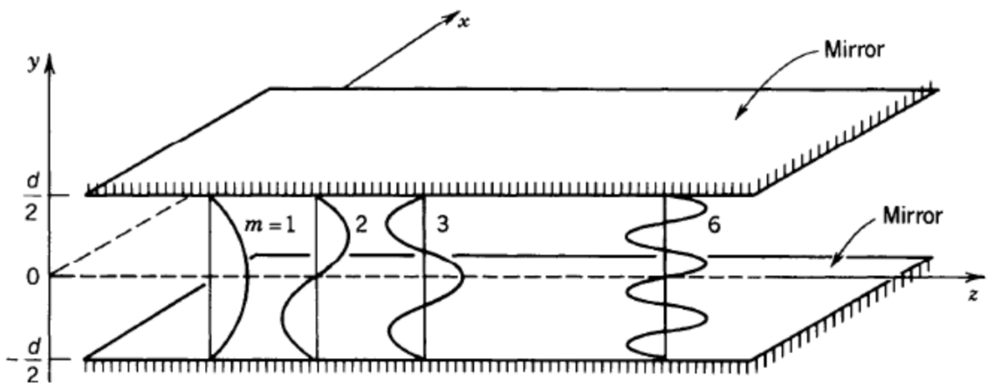
The group velocity is $v_g = \frac{1}{dk/d\omega} = c\sqrt{1-(\omega_{mn}/\omega)^2} < c$

@ Planar perfect-conductor waveguides (1-dimensional waveguides)



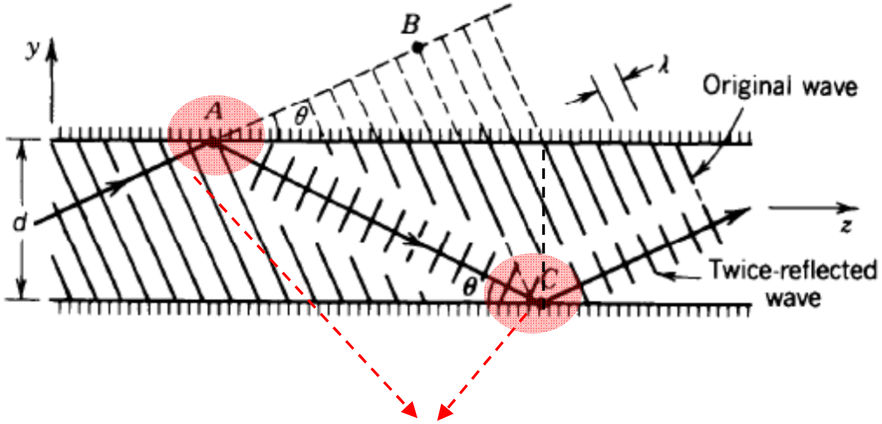
Waveguide modes

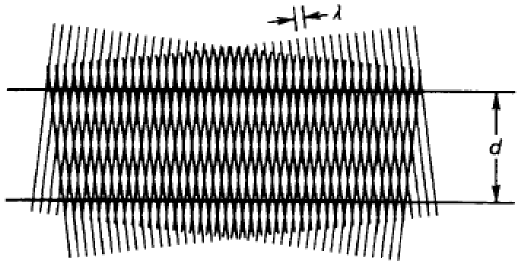
Modes are fields that maintain the same transverse distribution and polarization at all distances along the waveguide axis.



Condition of self-consistency

: The propagation ray picture of wave guidance by multiple reflections





the two waves interfere and create a pattern that does not change with z:

Accounting for a phase shift of π at each reflection,

$$2\pi \overline{AC}/\lambda - 2\pi - 2\pi \overline{AB}/\lambda = 2\pi q, \text{ where } q = 0, 1, 2, \ldots .$$ ← Condition of self-consistency:

$$\overline{AC} - \overline{AB} = 2d \sin\theta$$ ← $AB = AC\cos 2\theta = AC(1-2\sin^2\theta) \Rightarrow AC - AB = 2AC\sin^2\theta = 2d\sin\theta$

$$2\pi(2d\sin\theta)/\lambda = 2\pi(q+1) \rightarrow \frac{2\pi}{\lambda}2d\sin\theta = 2\pi m, \qquad m = 1, 2, \ldots,$$

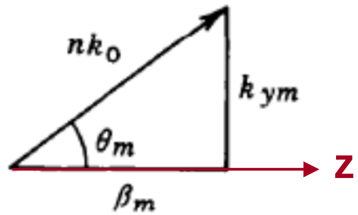
$$\rightarrow \boxed{\sin\theta_m = m\frac{\lambda}{2d}, \qquad m = 1, 2, \ldots .}$$ Bounce angles

Since the y component of the propagation constant is $k_y = nk_o \sin\theta$,

$$\boxed{k_{ym} = m\frac{\pi}{d},} \qquad m = 1, 2, 3, \ldots$$ **Transverse Component of the wavevector**

$$\boxed{\begin{aligned} k_x &= m\pi / a \\ k_y &= m\pi / b \end{aligned}}$$

Propagation constants



$$\sin\theta_m = m\frac{\lambda}{2d}$$

$$\beta = k_z = k\cos\theta$$

$$k_{ym} = m\frac{\pi}{d},$$

$$\beta_m^2 = k^2 - k_{ym}^2$$

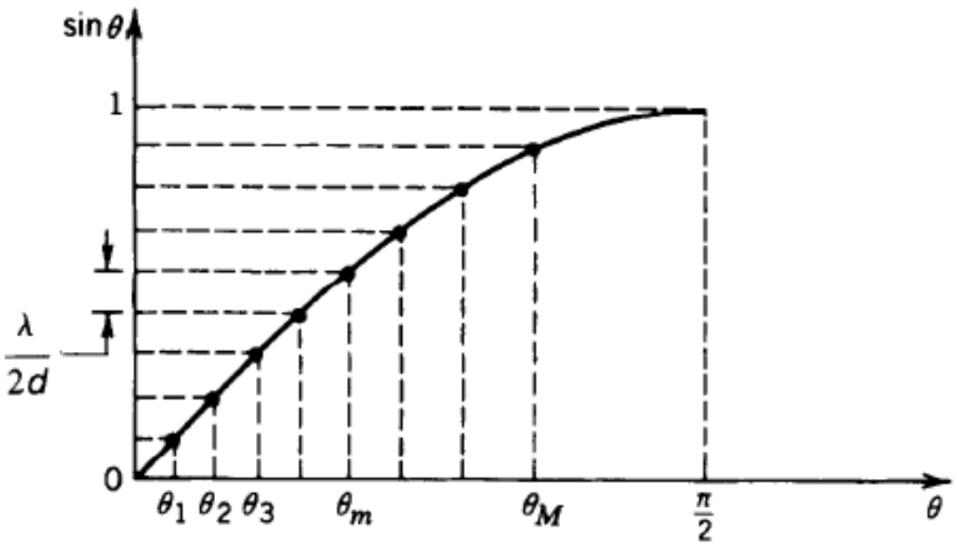
β is quantized $\longrightarrow$ $\beta_m = k\cos\theta_m,$

$$\boxed{\beta_m^2 = k^2(1-\sin^2\theta_m) = k^2 - \frac{m^2\pi^2}{d^2}}$$

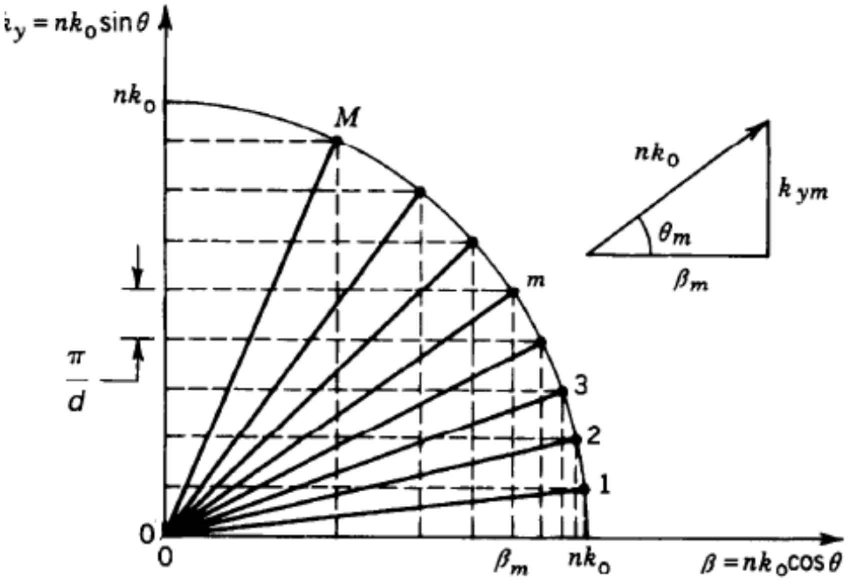
$$\longleftrightarrow \quad \left(\frac{\omega}{c}\right)^2 = k_x^2 + k_y^2 + k_z^2 \rightarrow k_z^2 = \left(\frac{\omega}{c}\right)^2 - k_x^2 - k_y^2$$

$$\frac{\omega}{c} = k$$

Higher-order (more oblique) modes travel with smaller propagation constants.



Bounce angle



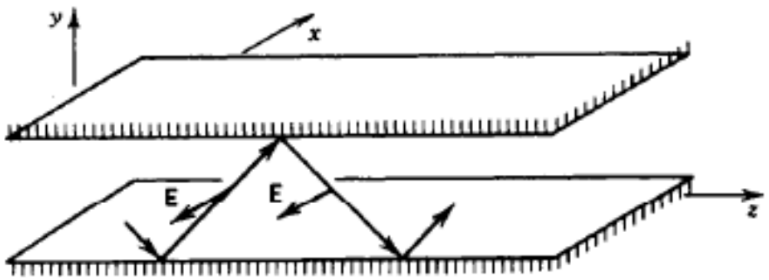
Propagation constant

Field distributions : TE modes

Assume that the bouncing TEM plane wave is polarized in the x direction, the guided wave is a transverse-electric (TE) wave.

The complex amplitude of the total field in the waveguide is the superposition of the two bouncing TEM plane waves :

TE mode



$E_x(y,z) =$ upward wave + downward wave

$$= A_m \exp(-jk_{ym} y - j\beta_m z) + e^{j(m-1)\pi} A_m \exp(+jk_{ym} y - j\beta_m z)$$

$$= \begin{cases} 2A_m \cos(k_{ym} y)\exp(-j\beta_m z) & \text{: symmetric modes, odd modes } m = 1,3,5,\ldots \\ 2jA_m \sin(k_{ym} y)\exp(-j\beta_m z) & \text{: antisymmetric modes, even modes } m = 2,4,6,\ldots \end{cases}$$

$$E_x(y,z) = a_m u_m(y)\exp(-j\beta_m z) \quad \begin{cases} a_m = \sqrt{2d}\,A_m \quad u_m(y) = \sqrt{\dfrac{2}{d}}\cos\dfrac{m\pi y}{d}, & m = 1,3,5,\ldots \\ a_m = j\sqrt{2d}\,A_m \quad u_m(y) = \sqrt{\dfrac{2}{d}}\sin\dfrac{m\pi y}{d}, & m = 2,4,6,\ldots \end{cases}$$

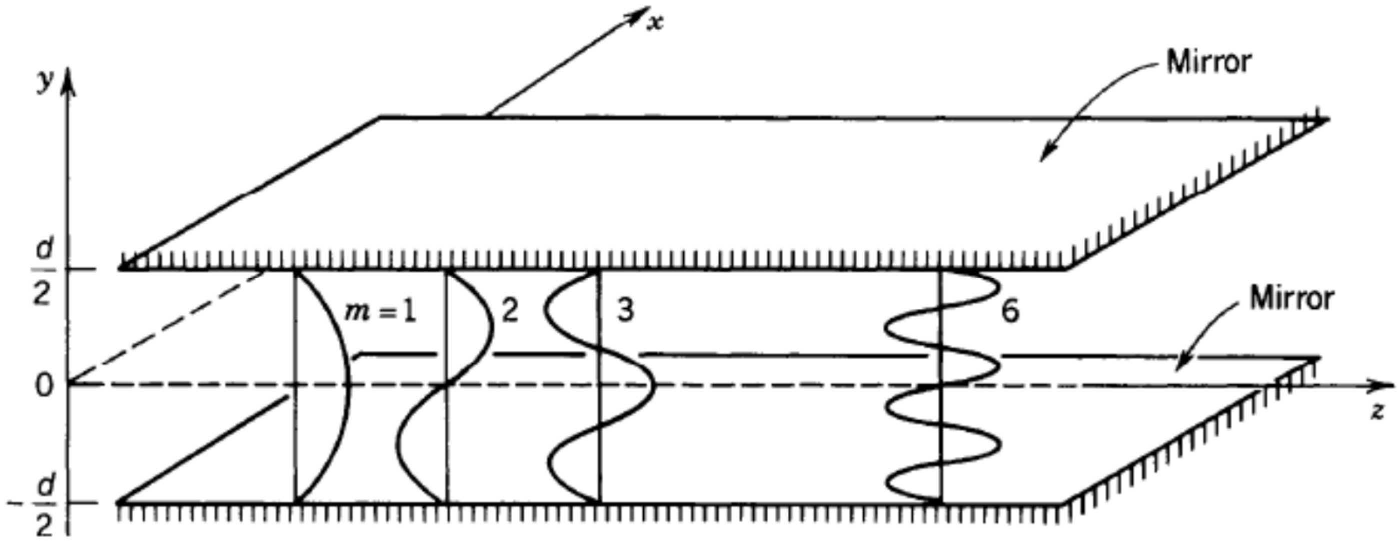
$u_m(y)$ **are normalized** $\displaystyle\int_{-d/2}^{d/2} u_m^2(y)\,dy = 1$

$u_m(y)$ **are orthogonal in [-d/2. d/2] interval** $\displaystyle\int_{-d/2}^{d/2} u_m(y)u_l(y)\,dy = 0, \qquad l \neq m$

[TE guided waves]

$$E_x(y,z) = a_m u_m(y) \exp(-j\beta_m z)$$

$$\begin{cases} a_m = \sqrt{2d}\, A_m \quad u_m(y) = \sqrt{\dfrac{2}{d}} \cos \dfrac{m\pi y}{d}, \quad m = 1,3,5,\ldots \\ a_m = j\sqrt{2d}\, A_m \quad u_m(y) = \sqrt{\dfrac{2}{d}} \sin \dfrac{m\pi y}{d}, \quad m = 2,4,6,\ldots \end{cases}$$



Each mode can be view as a standing waves in the y direction, traveling in the z direction.

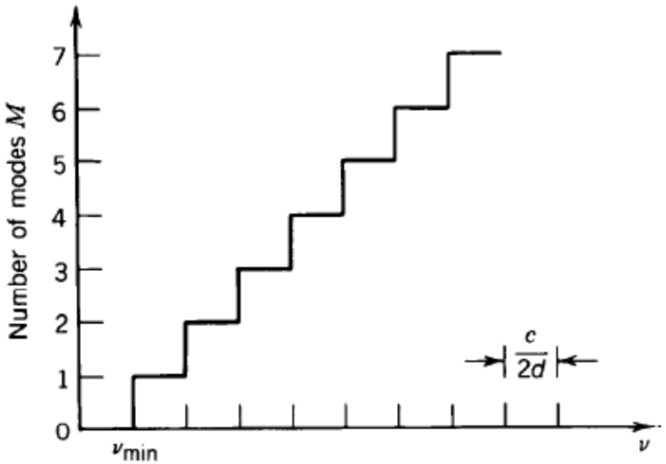
Modes of large m vary in the transverse plane at a greater rate k_y, and travel with a smaller propagation constant β.

The field vanishes at y = +d/2 for all modes, so that the boundary conditions at the surface of the mirrors are always satisfied.

Number of modes

Since $\sin\theta_m = m\lambda/2d$, $m = 1, 2, \ldots$ and for $\sin\theta_m < 1$, the maximum allowed value of m is

$$M \doteq \frac{2d}{\lambda} \doteq \nu/(c/2d),$$ $\doteq$ denotes that $2d/\lambda$ is reduced to the nearest integer.



If $2d/\lambda \le 1$, **(d < λ/2) →** $M = 0$, the waveguide cannot support any modes.

cutoff wavelength $\lambda_{max} = 2d$

cutoff frequency $\nu_{min} = c/2d$ $\longleftrightarrow$ $\omega_{mn} = c\pi\sqrt{(m/a)^2 + (n/b)^2}$

If $1 < 2d/\lambda \le 2$ (i.e., $d \le \lambda < 2d$) **→ (λ/2 < d < λ) → single-mode waveguide**

Group velocities

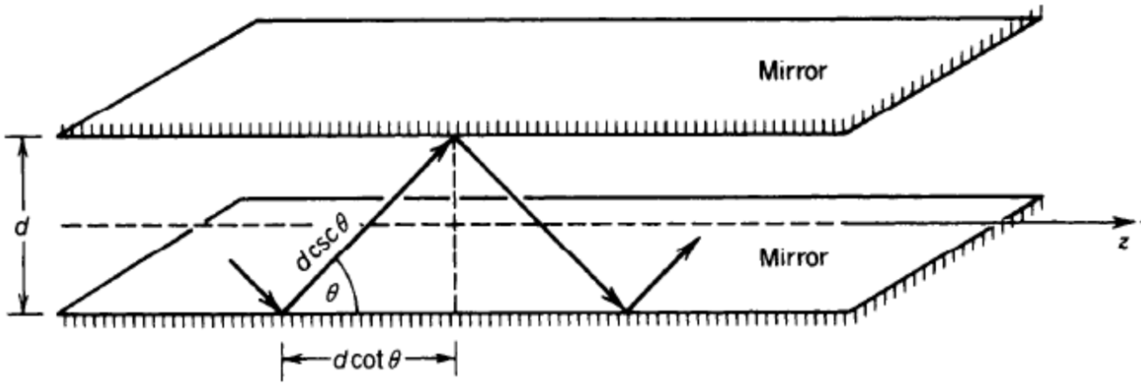
$$v = d\omega/d\beta$$

$$\beta_m^2 = k^2 - \frac{m^2\pi^2}{d^2} = (\omega/c)^2 - m^2\pi^2/d^2$$ **dispersion relation**

$$2\beta_m d\beta_m/d\omega = 2\omega/c^2 \longrightarrow d\omega/d\beta_m = c^2\beta_m/\omega = c^2 k\cos\theta_m/\omega = c\cos\theta_m$$

Group velocity of mode m : $\boxed{v_m = c\cos\theta_m} \longleftrightarrow v_g = \frac{1}{dk/d\omega} = c\sqrt{1-(\omega_{mn}/\omega)^2} < c$

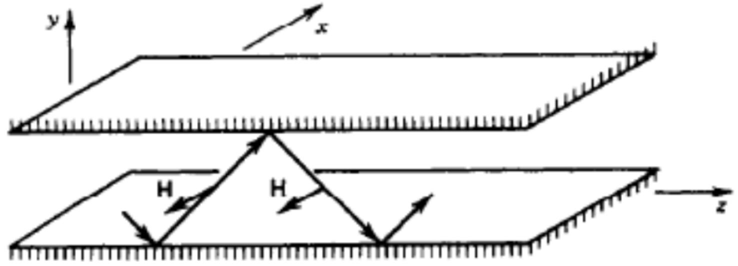
More oblique modes travel with a smaller group velocity
since they are delayed by the longer path of the zigzaging process.



Geometrically, $$v = \frac{\text{distance}}{\text{time}} = \frac{d\cot\theta}{d\csc\theta/c} = c\cos\theta$$

Field distributions : TM modes

Magnetic field is in the x direction,
the guided wave is a transverse-magnetic (TM) wave.



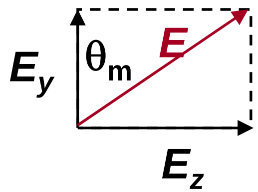
Since the z component of the electric field is parallel to the mirror, it must behave like the x component of the TE mode :

$E_z(y,z) =$ upward wave + downward wave

$$= \begin{cases} a_m\sqrt{\dfrac{2}{d}}\cos\dfrac{m\pi y}{d}\exp(-j\beta_m z), & m = 1,3,5,\ldots \qquad a_m = \sqrt{2d}\,A_m \\ a_m\sqrt{\dfrac{2}{d}}\sin\dfrac{m\pi y}{d}\exp(-j\beta_m z), & m = 2,4,6,\ldots, \qquad a_m = j\sqrt{2d}\,A_m \end{cases}$$

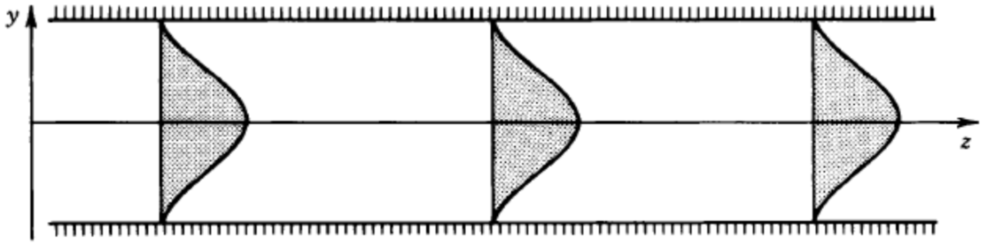
The y components of the electric field of these waves are

$$E_y(y,z) = \begin{cases} a_m\sqrt{\dfrac{2}{d}}\cot\theta_m\cos\dfrac{m\pi y}{d}\exp(-j\beta_m z), & m = 1,3,5,\ldots \\ a_m\sqrt{\dfrac{2}{d}}\cot\theta_m\sin\dfrac{m\pi y}{d}\exp(-j\beta_m z), & m = 2,4,6,\ldots \end{cases}$$

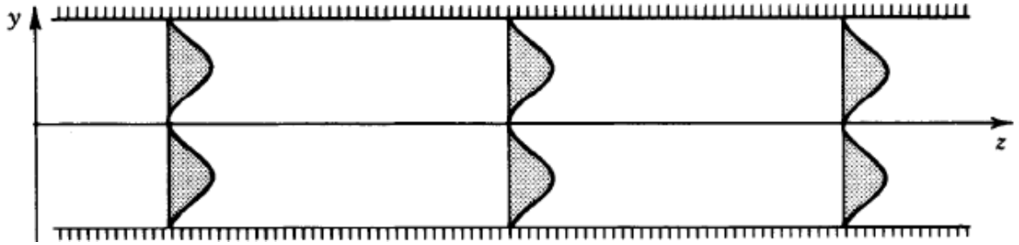


Multimode fields

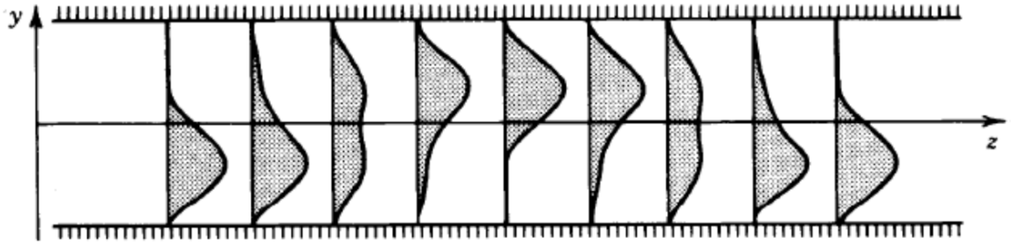
(m = 1) $E(y,z) = u_1(y)\exp(-j\beta_1 z)$, where $u_1(y) = \sqrt{2/d}\,\cos(\pi y/d)$



(m = 2) $E(y,z) = u_2(y)\exp(-j\beta_2 z)$, where $u_2(y) = \sqrt{2/d}\,\sin(2\pi y/d)$



(m = 1 & 2) $E(y,z) = u_1(y)\exp(-j\beta_1 z) + u_2(y)\exp(-j\beta_2 z)$



Since $\beta_1 \neq \beta_2$, the intensity distribution changes with z.