Analytical Solution to Lane-Emden Equation

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Abstract

Polytropic gaseous spheres plays an important role and describe a crude but useful approximation in many different astrophysical problems. The technique used to solve equation is based on a heuristic method for evaluation of definite integrals called Method of Brackets [1] and Mellin Transform. Many method have been used to solve this problem numerically or/and (semi)analytically, in this work we presents a general analytical solution for arbitrary index.

1. The Problem

Polytropes are useful as they provide simple solutions to describe the internal internal structure of a star. They are much easy to solve than the full equations of stellar structure, i.e, the mechanical structure of the star describe by mass and momentum conservation and the thermal structure by the energy transport equation. However, in this approximation the only contact between the mechanical and thermal equations is through the temperature dependence of the equation of state and under certain assumptions the pressure becomes independent of the temperate and only depend on density,

$$P = K\rho^{\gamma} = K\rho^{1+1/n}, \tag{1}$$

where is the *polytropic index* and K is a constant. This equation simplify enormously the calculation. First, We assume hydrostatic equilibrium,

$$\frac{dP}{dr} = -\frac{GM}{r^2}\rho, \tag{2}$$

multiplying both sides by r^2/ρ

$$\frac{r^2}{\rho}\frac{dP}{dr} = -\frac{GM}{r^2}\rho\frac{r^2}{\rho} = -GM, \tag{3}$$

If we then take the derivative of both sides and divide by r^2, then we get

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -\frac{G}{r^2}\frac{dM}{dr} = -\frac{G}{r^2}4\pi r^2\rho = -4\pi G\rho, \tag{4}$$

naturally appears the Poisson equation in spherical coordinates. And substituting by the polytropic equation.

$$\frac{dP}{dr} = K\left(1+\frac{1}{n}\right)\rho^{1/n}\frac{d\rho}{dr}, \tag{5}$$

and Poisson's equation becomes

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho}K\left(1+\frac{1}{n}\right)\rho^{1/n}\frac{d\rho}{dr}\right] = -4\pi G\rho, \tag{6}$$

To simplify this expression, let's put it in dimensionless form. Defining the θ variable, such that

$$\theta(r) = (\rho/\rho_c)^{1/n} \implies \rho = \rho_c\theta^n. \tag{7}$$

Replacing (6) in the Poisson equation and simplifying.

$$\frac{(n+1)K\rho_c^{1+1/n}}{4\pi G\rho_c^2}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n, \tag{8}$$

Then define a dimensionless radius, $\eta = r/r_n$, where

$$r_n = \left(\frac{(n+1)K}{4}\right)^{1/2}\rho_c^{(1-n)/2n} = \left(\frac{(n+1)P_c}{4\pi G\rho_c^2}\right). \tag{9}$$

We used eqn.2 to substitute in the central pressure. In this form, the equation takes de form

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n, \tag{10}$$

This is called the Lane-Emden equation. Its solution gives a dimensionless density θ as a function of the dimensionless radius, η. Since it is a second order differential equation, need two boundary conditions. The first is at the center from spherical symmetry, the pressure gradient must be zero. The second condition comes from the stellar surface, where the density goes to zero. Our boundary conditions are:

$$\rho = \rho_c \quad \text{en} \quad r = 0 \quad \text{then} \quad \theta(0) = 1,$$

$$\frac{dP}{dr} = 0 \quad \text{en} \quad r = 0 \quad \Longrightarrow \quad \left.\frac{d\theta}{d\xi}\right|_{\xi=0} = 0,$$

2 Method of Brackets

The procedure used to solve this equation is based in the Mellin transform and an heuristic method for evaluation of definite integrals called Method of Brackets (*For a complete review see Iván González's Talk*).
This method has as its origin in Quantum Field Theory specifically in Feynman Diagrams. In this work, we present an alternative scenario to solve Non-linear differential equations.

2.1 Series Expansion

We assume that an arbitrary function can be expanded in a Taylor series. As follows

$$f(x) = f^{(0)}(x) = \sum_{l\geqslant 0}\phi_l\, F(l+1)\, x^l, \tag{11}$$

where,

$$\phi_l = \frac{(-1)^l}{\Gamma(l+1)}, \tag{12}$$

In general, the n-derivative of series expansion takes de form

$$f^{(n)}(x) = (-1)^n\sum_{l\geqslant 0}\phi_l\, F(l+n)\, x^l, \tag{13}$$

2.2 Mellin Transform

We need introduce The Mellin transform in combination with series expansion to solve Lane-Endem equation.
The transformations shown below will be useful for solving non-linear differential equations.

$$\mathbf{M}[\, f^{(n)}(x)\,](s) = \int_0^\infty x^{s-1}\, f^{(n)}(x)\, dx, \tag{14}$$

using series expansion and Method of Bracket

$$\begin{aligned}\mathbf{M}[\, f^{(n)}(x)\,](s) &= (-1)^n\int_0^\infty x^{s-1}\left[\sum_{l\geqslant 0}\phi_l\, F(l+n)\, x^l\right]dx\,,\\ &= (-1)^n\sum_l\phi_l\, F(l+m)\int_0^\infty x^{s+l-1}\, dx,\\ &= (-1)^n\sum_l F(l+m)\langle s+l\rangle.\end{aligned} \tag{15}$$

Applying the rules

$$\mathbf{M}[\, f^{(n)}(x)\,](s) = (-1)^n\Gamma(s)\, F(-s+n), \tag{16}$$

And

$$\begin{aligned}\mathbf{M}[\, x^r\, f^{(n)}(x)\,](s) &= (-1)^n\int_0^\infty x^{s+r-1}\left[\sum_{l\geqslant 0}\phi_l\, F(l+n)\, x^l\right]dx,\\ &= (-1)^n\sum_l\phi_l\, F(l+m)\int_0^\infty x^{s+r+l-1}\, dx,\\ &= (-1)^n\sum_l F(l+m)\langle s+r+l\rangle.\end{aligned} \tag{17}$$

We found a similar expression

$$\mathbf{M}[\, x^r\, f^n(x)\,](s) = (-1)^n\Gamma(s+r)\, F(-s-r+n), \tag{18}$$

3. Solution to Lane-Emden Equation

Lane-Emden equation is given by (eqn. 10) with their boundary conditions. In right side of the equation we used a change of variable, $f(\xi) = (\theta(\xi))^\gamma$ and deriving. Therefore the equation are linearising, obtaining a set of coupled equations.

$$\frac{d^2\theta}{d\xi^2} + \frac{2}{\xi}\frac{d\theta}{d\xi} = -f(\xi), \tag{19}$$

$$\theta\, f' = \gamma\, f\,\theta', \tag{20}$$

Expanding in power series

$$f(\xi) = \sum\phi_n\, F(n)\,\xi^n, \tag{21}$$

and

$$\theta(\xi) = \sum\phi_n\,\Theta(n)\,\xi^n, \tag{22}$$

Replacing in their respective differential equations and applying the Mellin transform, we obtain respectively,

$$\Theta(n+2)\frac{(n+3)}{(n+1)} = -F(n), \tag{23}$$

$$\sum_{m=0}^{n}\binom{n}{m}\Theta(m)F(n+1-m) = \gamma\sum_{m=0}^{n}\binom{n}{m}F(m)\Theta(n+1-m), \tag{24}$$

This corresponds to a recurrence equations, changing the initial conditions, we have, $\Theta(0) = 0$, $\Theta(1) = 0$ and $F(0) = 1$. Solving the recurrence equation with boundary conditions,

$$\begin{aligned}F(0) &= 1,\\ F(2) &= -\frac{1}{3},\\ F(4) &= \frac{1}{5}\gamma,\\ F(6) &= -\frac{(8\gamma-5)\gamma}{21},\\ F(8) &= \frac{(70-183\gamma+122\gamma^2)\gamma}{81},\end{aligned} \tag{25}$$

For the odd terms $F(1) = F(3) = F(5) = F(7) = F(9) = F(2n+1) = 0$. Finally, the first five terms of the completely analytical solution we get,

$$\begin{aligned}\theta(\xi) = 1 - \frac{1}{6}x^2 + \frac{\gamma}{120}x^4 - \frac{(8\gamma-5)\gamma}{15120}x^6\\ + \frac{(70-183\gamma+122\gamma^2)\gamma}{3265920}x^8 + \ldots\end{aligned} \tag{26}$$

4. Examples

The trivial solution: $\gamma = 0$

$$\theta(\xi) = 1 - \frac{1}{6}x^2, \tag{27}$$

Solution: $\gamma = 1$

$$\begin{aligned}\theta(\xi) &= 1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + \frac{x^8}{3265920} + \ldots, &(28)\\ &= \frac{\sin(x)}{x}, &(29)\end{aligned}$$

5. Conclusions

In this work we present an extension of the Method of Brackets to provide a novel technique to solve non-linear differential equations. In addition we reduce the order of non-linear term. Obtaining a coupled system is formed of two differential equations. Applying Mellin combined by Method of Brackets results in a coupled systems of recurrence equations, easy to program.
Particularly we provide a analytical solution to Lane-Emden equation for an arbitrary index. This technique can be extended to several problems in physics and other areas.

References

[1] I. Gonzalez and V. Moll, *Definite integrals by method of brackets. Part 1*, Advances in Applied Mathematics, Vol. 45, Issue 1, 50-73 (2010).
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