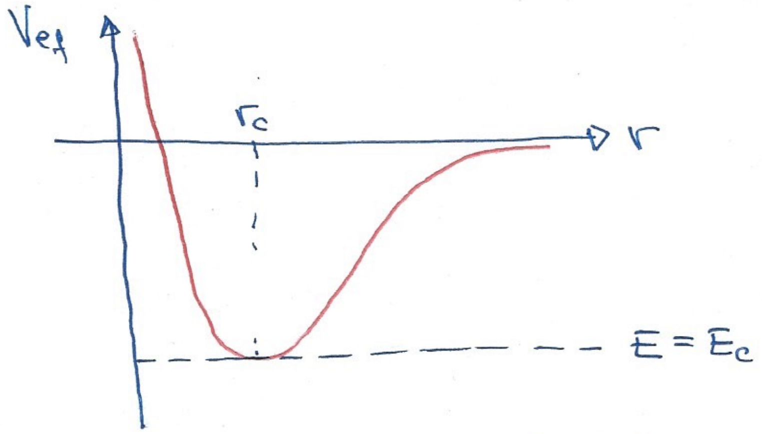
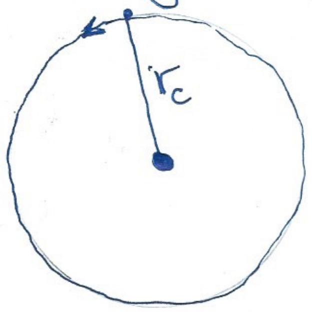


El período en la órbita circular:

Tenemos que

$$L = m r_c^2 \dot{\phi}$$

En un período $t = T$, el cuerpo m completa un ángulo $\phi = 2\pi$



$$\therefore \quad \dot{\phi}\Big|_{\text{período}} = \frac{2\pi}{T} = \frac{L}{m r_c^2}$$

$$\dot{\phi}\Big|_{\text{período}} = \frac{\Delta\phi}{\Delta t} \Rightarrow T = 2\pi \cdot \frac{m r_c^2}{L}$$

$$T = 2\pi \cdot \frac{m}{L} \cdot \left(\frac{L^2}{GMm^2}\right)^2 = 2\pi \cdot \frac{m}{L} \cdot \frac{L^4}{G^2M^2m^4}$$

$$\Rightarrow \boxed{T = 2\pi \frac{L^3}{G^2M^2m^3}}$$

Si $E_c < E < 0$, aún podemos usar la constante de movimiento, pero

$$L = m r^2 \dot{\phi}$$

Y supone que conocemos $r = r(\phi)$ ó $r = r(t)$:

$$\frac{d\phi}{dt} = \frac{L}{m r^2}$$

$r = r(\phi)$: $\quad \frac{L}{m}\int_0^T dt = \int_0^{2\pi} r^2(\phi)\, d\phi$

$r = r(t)$: $\quad \frac{m}{L}\int_0^{2\pi} d\phi = \int_0^T \frac{dt}{r^2(t)}$

$$\boxed{\frac{L}{m} T = \int_0^{2\pi} r^2(\phi)\, d\phi}$$

o bien

$$\frac{m}{L}\cdot 2\pi = \int_0^T \frac{dt}{r^2(t)}$$

¿cómo determinamos r?

Podríamos partir desde

$$E = \frac{1}{2} m \dot{r}^2 + U_{ef}(r)$$

resolver esto implica encontrar $r = r(t)$

$$\dot{r}^2 = \frac{2}{m}\left(E - U_{ef}\right)$$

$$\Rightarrow \dot{r} = \frac{dr}{dt} = \pm \sqrt{\frac{2}{m}}\sqrt{E - U_{ef}(r)}$$

$$\Rightarrow \boxed{\pm \sqrt{\frac{2}{m}} \int dt = \int \frac{dr}{\sqrt{E - U_{ef}(r)}} + cte}$$

Sin embargo, para obtener $r = r(\phi)$ debemos ocupar la otra constante de movimiento:

$$\dot{\phi} = \frac{d\phi}{dt} = \frac{L}{m r^2}$$

$$\dot{r} = \frac{dr}{dt} = \frac{d\phi}{dt}\frac{dr}{d\phi} = \dot{\phi}\frac{dr}{d\phi} = \pm\sqrt{\frac{2}{m}}\sqrt{E-U_{ef}}$$

$$\frac{L}{m}\frac{dr/r^2}{d\phi} = \pm\sqrt{\frac{2}{m}}\sqrt{E-U_{ef}}$$

$$\therefore \int \frac{dr/r^2}{\sqrt{E-U_{ef}}} = \pm\frac{\sqrt{2m}}{L}\int d\phi = \pm\frac{\sqrt{2m}}{L}\cdot\Delta\phi$$

$$E-U_{ef} = E - \frac{L^2}{2mr^2} + \frac{GMm}{r}$$

con $u = 1/r \Rightarrow du = -dr/r^2$

$$\text{de} + \int \frac{du}{\sqrt{E-\frac{L^2}{2m}u^2+GMm\,u}} = \mp\frac{\sqrt{2m}}{L}\cdot\Delta\phi$$

$$E - \frac{L^2}{2m}\left(u^2 - 2GMm^2 u\right)$$

$$E - \frac{L^2}{2m}\left[u^2 - 2\left(GMm^2\right)\cdot u + G^2M^2m^4 - G^2M^2m^4\right]$$

$$\left(E + \frac{G^2L^2M^2m^3}{2}\right) - \frac{L^2}{2m}\left[u^2 - 2\left(GMm^2\right)u + \left(GMm^2\right)^2\right]$$

$$\left(E+\frac{G^2M^2m^3L^2}{2}\right)-\frac{L^2}{2m}\left(\mu-\alpha\right)^2 \quad ; \quad \alpha \equiv GMm^2$$

Luego $x=\mu-\alpha$ quedando

$$\int \frac{dx}{\sqrt{a_1-a_2x^2}}=\frac{1}{\sqrt{a_2}}\int\frac{dx}{\sqrt{\frac{a_1}{a_2}-x^2}}$$

$$x=\frac{a_1}{a_2}\cos\omega \qquad dx=-\frac{a_1}{a_2}\sin\omega\, d\omega$$

$$\therefore -\frac{1}{\sqrt{a_2}}\int\frac{\frac{a_1}{a_2}\sin\omega\, d\omega}{\sqrt{\frac{a_1}{a_2}}\cdot\sin\omega}=-\sqrt{\frac{a_1}{a_2^3}}\int d\omega$$

$$=-\sqrt{\frac{a_1}{a_2^3}}\,\omega=-\sqrt{\frac{a_1}{a_2^3}}\cdot \mathrm{ArcCos}\left(\frac{a_2}{a_1}(\mu-\alpha)\right)$$

$$\therefore -\sqrt{\frac{a_1}{a_2^3}}\,\mathrm{ArcCos}\left[\frac{a_2}{a_1}\left(\frac{1}{r}-\alpha\right)\right]=\mp\frac{\sqrt{2m}}{L}\Delta\phi$$

$$\Rightarrow \mathrm{ArcCos}\left[\frac{a_2}{a_1}\left(\frac{1-\alpha r}{r}\right)\right]=\pm\sqrt{\frac{a_2^3}{a_1}}\cdot\frac{\sqrt{2m}}{L}\cdot(\phi-\phi_i)$$

$$\Rightarrow r(\phi)=\frac{b_1}{1+b_2\cos\left(b_3(\phi-\phi_i)\right)}$$

Entonces, para determinar el periodo

$$\frac{L}{m} \cdot T = \int_0^{2\pi} r^2(\phi)\, d\phi$$

$$T = \frac{m}{L} \cdot b_1^2 \int_0^{2\pi} \frac{d\phi}{\left(1 + b_2 \cos\left(b_3(\phi - \phi_i)\right)\right)^2}$$

TAREA: Determinar, con sus constantes, el período de una órbita elíptica.