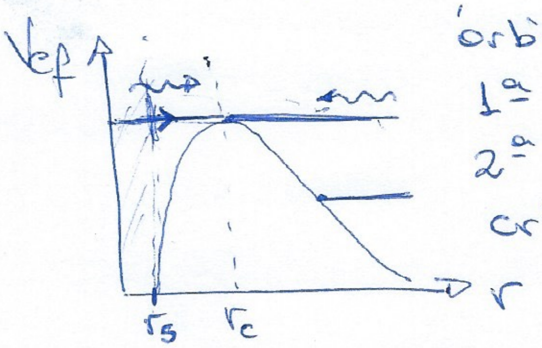
Clase 12 051021 1



órbitas:

1ª especie $r > r_c$

2ª especie $r < r_c$

crítica $r \to r_c$

Ec. de movimiento crítico

$$\frac{du}{d\phi} = \mp\sqrt{r_s}\,\sqrt{(u_c-u)^2(u+|u_2|)}$$

Recordar: $u_0 = u_1 = u_c = \frac{2}{3r_s} = \frac{1}{3M}$; $u_2 = -\frac{1}{6M} = \frac{-1}{3r_s} \to |u_2| = \frac{u_s}{3}$

1ª: $0 < r_c < r \to \infty > u_c > u$

$\to u_c - u > 0$

$$\boxed{\frac{du}{d\phi} = \mp\sqrt{r_s}\,(u_c-u)\sqrt{u+|u_2|}}$$

2ª $0 < r < r_c \to \infty > u > u_c$

$$\frac{du}{d\phi} = \mp\sqrt{r_s}\,(u-u_c)\sqrt{u+|u_2|}$$

mov. hacia la derecha, (−) para la variable u:

$$\frac{du}{d\phi} = \frac{1}{\sqrt{u_s}}\cdot(u_c-u)\sqrt{u+|u_2|}$$

CL 12 ②

$$\boxed{u = -\frac{1}{3r_s} + \frac{1}{r_s}\,\mathrm{Tanh}^2\tfrac{1}{2}(\phi-\phi_0)}$$ 1ª especie

$$u = \frac{3\,\mathrm{Tanh}^2\frac{1}{2}(\phi-\phi_0) - 1}{3r_s} = \frac{1}{r}$$

$$\rightarrow \quad r(\phi) = \frac{3r_s}{3\,\mathrm{Tanh}^2\frac{1}{2}(\phi-\phi_0) - 1}$$

$$\rightarrow \mp \frac{1}{\sqrt{u_s}}(u_e - u)\sqrt{u + |u_2|} =$$

$$= \mp \frac{1}{\sqrt{u_s}}\left(\frac{2u_s}{3} + \frac{u_s}{3} - u_s\,\mathrm{Th}^2\tfrac{1}{2}(\phi-\phi_0)\right) \times$$

$$\left[\cancel{-\frac{u_s}{3}} + u_s\,\mathrm{Th}^2\tfrac{1}{2}(\phi-\phi_0) + \cancel{\frac{u_s}{3}}\right]^{1/2}$$

$$= \mp \frac{1}{\cancel{\sqrt{u_s}}}\, u_s\left(1 - \mathrm{Th}^2\tfrac{1}{2}(\phi-\phi_0)\right)\cancel{\times\sqrt{u_s}}\cdot \mathrm{Th}\tfrac{1}{2}(\phi-\phi_0)$$

$$= \mp\, u_s \cdot \mathrm{sech}^2\tfrac{1}{2}(\phi-\phi_0) \times \mathrm{Tanh}\tfrac{1}{2}(\phi-\phi_0)$$

$$\wedge \quad \frac{d}{d\phi}\left[-\frac{u_s}{3} + u_s\,\mathrm{Tanh}^2\tfrac{1}{2}(\phi-\phi_0)\right] =$$

$$u_s\,\mathrm{Tanh}\tfrac{1}{2}(\phi-\phi_0)\,\mathrm{sech}^2\tfrac{1}{2}(\phi-\phi_0) \;//$$

Cambio de variable para la órbita de 2ª especie:

$$u = \frac{2}{3r_s} + \frac{1}{r_s}\mathrm{Tan}^2\tfrac{1}{2}\xi = \frac{2u_s}{3} + u_s\,\mathrm{Tan}^2\tfrac{1}{2}\xi$$

i) $$\frac{du}{d\xi} = u_s \cdot \cancel{2}\,\mathrm{Tan}\tfrac{1}{2}\xi \cdot \sec^2\tfrac{1}{2}\xi \cdot \tfrac{1}{\cancel{2}}$$

$$\therefore \frac{du}{d\phi} = \frac{du}{d\xi}\cdot\frac{d\xi}{d\phi} = u_s\,\mathrm{Tan}\tfrac{1}{2}\xi\,\sec^2\tfrac{1}{2}\xi\,\frac{d\xi}{d\phi}$$

ii) $$\frac{1}{\sqrt{u_s}}(u-u_c)\sqrt{u+u_s/3} = \frac{1}{\sqrt{u_s}}\left(u-\frac{2u_s}{3}\right)\times\sqrt{u+\frac{u_s}{3}}$$

$$\frac{1}{\sqrt{u_s}}\cdot u_s\,\mathrm{Tan}^2\tfrac{1}{2}\xi\times\sqrt{u_s+u_s\,\mathrm{Tan}^2\tfrac{1}{2}\xi} =$$

$$= u_s\,\mathrm{Tan}^2\tfrac{1}{2}\xi\times\sec\tfrac{1}{2}\xi$$

(i) $=\mp$ (ii): $$\cancel{u_s\,\mathrm{Tan}\tfrac{1}{2}\xi}\,\sec^2\tfrac{1}{2}\xi\,\frac{d\xi}{d\phi} = \mp\cancel{u_s}\,\mathrm{Tan}^2\tfrac{1}{2}\xi\,\cancel{\sec}\tfrac{1}{2}\xi$$

$$\frac{d\xi}{d\phi} = \mp\,\mathrm{Tan}\tfrac{1}{2}\xi\times\cos\tfrac{1}{2}\xi = \mp\sin\tfrac{1}{2}\xi$$

$$\Rightarrow \boxed{\frac{d\xi}{d\phi} = \mp\sin\tfrac{1}{2}\xi}$$

CL 12 (4)

Tenemos

$$\mp d\phi = \frac{d\xi}{\sin\frac{1}{2}\xi} = \frac{1}{2}\frac{d\xi}{\sin\frac{1}{4}\xi\cos\frac{1}{4}\xi}$$

$$= \frac{1}{2}\left(\frac{\cos\frac{1}{4}\xi}{\sin\frac{1}{4}\xi} + \frac{\sin\frac{1}{4}\xi}{\cos\frac{1}{4}\xi}\right)d\xi$$

Integrando (+)

$$\phi = \frac{1}{2}\int\frac{\cos\frac{1}{4}\xi\, d\xi}{\sin\frac{1}{4}\xi} + \frac{1}{2}\int\frac{\sin\frac{1}{4}\xi\, d\xi}{\cos\frac{1}{4}\xi}$$

$$= \frac{1}{2}\int\frac{d(\sin\frac{1}{4}\xi)\cdot 4}{\sin\frac{1}{4}\xi} + \frac{1}{2}\int\frac{d(-\cos\frac{1}{4}\xi)(-4)}{\cos\frac{1}{4}\xi}$$

$$= 2\,\mathrm{Ln}\left[\sin\tfrac{1}{4}\xi\right] - 2\,\mathrm{Ln}\left[\cos\tfrac{1}{4}\xi\right]$$

$$\phi = 2\,\mathrm{Ln}\left[\mathrm{Tan}\tfrac{1}{4}\xi\right]$$

o bien $\quad \mathrm{Tan}\frac{1}{4}\xi = e^{\phi/2}$

$$\rightarrow \boxed{\xi = 4\,\mathrm{ArcTan}\left(e^{\phi/2}\right)}$$

Entonces:

$$\mathrm{Tan}\frac{1}{2}\xi = \mathrm{Tan}\left[2\,\mathrm{ArcTan}\left(e^{\phi/2}\right)\right]$$

$* \quad \tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

$\therefore \tan \frac{1}{2}\xi = \dfrac{2e^{\phi/2}}{1-e^{\phi}}$

$\tan^2 \frac{1}{2}\xi = \dfrac{4e^{\phi}}{(1-e^{\phi})^2}$

$u = \dfrac{1}{r} = \dfrac{2u_s}{3} + \dfrac{3u_s}{3} \times \dfrac{4e^{\phi}}{(1-e^{\phi})^2}$

$\rightarrow \boxed{r(\phi) = \dfrac{3r_s}{2} \cdot \dfrac{(1-e^{\phi})^2}{1+4e^{\phi}+e^{2\phi}}}$ $\quad 0 < r < \dfrac{3r_s}{2}$