gravitación

1) la acción puede considerarse como el camino λ que toma, especificamente el largo de ese camino.

$$\boxed{A[r] = \left(1 - \frac{r_s}{r}\right)}$$

$$S = \int\sqrt{ds^2} = \int ds = \int \frac{ds}{d\lambda}\, d\lambda$$

donde

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = A(r)\, dt^2 - \frac{dr^2}{A(r)} - r^2 d\Omega_2$$

y del punto de vista relativo al cuerpo que toma el camino λ

$$ds^2 = c^2 d\tau^2 \quad \rightarrow \quad \frac{d\tau}{d\lambda}d\lambda = \frac{1}{c}\frac{ds}{d\lambda}\, d\lambda$$

chain rul

$$\left(\frac{d\tau}{d\lambda}\right)^2 = \frac{1}{2}\, g_{\mu\nu}\, \frac{dx^\mu}{d\lambda}\, \frac{dx^\nu}{d\lambda}$$

ya que la particula esta libre, seguira el camino más corto en el espacio tiempo curvado; siendo $\lambda = \tau$ (tiempo propio)

$$S = \frac{\epsilon_0}{c}\int \left[g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right]^{1/2} d\tau = \frac{\epsilon_0}{c}\int \left[g_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu \right]^{1/2} d\tau$$

Podemos multiplicar la acción por ϵ de manera que sea una constante que añada las unidades correctas (energia · tiempo)

asi $S = \int \mathcal{L}\, d\tau \Rightarrow$

$$\mathcal{L} = \frac{\epsilon_0}{c}\left[g_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu \right]^{1/2}$$

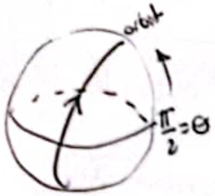
el Lagrangiano

$$\mathcal{L} = \frac{\epsilon_0}{c}\left[\left(1 - \frac{r_s}{r}\right) c^2 (\dot{t})^2 - \frac{(\dot{r})^2}{\left(1 - \frac{r_s}{r}\right)} - r^2 (\dot{\theta})^2 - r^2 \sin^2\theta\, (\dot{\phi})^2 \right]^{1/2}$$

//

explorando las cantidades conservadas.

ya que la metrica de Schwarzschild viene de simetria esferica al M no rotar



siempre sera posible rotar nuestro sistema de coord. para que el mov. ocurra en el plano $\theta = \frac{\pi}{2}$.

$$\therefore \; (\dot{\theta})^2 = \left(\frac{d\theta}{d\tau}\right)^2 = 0 \quad ; \quad \sin^2\theta = \left(\sin\frac{\pi}{2}\right)^2 = 1$$

así el Lagrangiano

$$\mathcal{L} = \frac{\epsilon_0}{c}\left[\left(1-\frac{r_s}{r}\right)c^2(\dot{t})^2 - \frac{(\dot{r})^2}{1-\frac{r_s}{r}} - r^2(\dot{\phi})^2\right]^{1/2} = \frac{\epsilon_0}{c}\Big[\underbrace{g_{\mu\nu}\,\dot{x}^\mu\dot{x}^\nu}_{T}\Big]^{1/2}$$

$$\mathcal{L} = \frac{\epsilon_0}{c}T^{1/2} \quad ; \quad \text{osea que} \quad \frac{\partial \mathcal{L}}{\partial q} = \frac{\epsilon_0}{c}\frac{1}{2\sqrt{T}}\frac{\partial T}{\partial q}.$$

y usando un truco $\sqrt{T} = \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} = \sqrt{c^2\frac{d\tau}{d\tau}\frac{d\tau}{d\tau}} = c$ (objectos masivos)

por tanto para particulas sin masa el Lagangiano y acción $\rightarrow$ $= \sqrt{0}$ (particulas sin masa)

$$S = \frac{\epsilon_0}{c}\int\left[g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right]^{1/2}d\lambda \quad ; \quad \mathcal{L} = \frac{\epsilon_0}{c}\left[g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right]^{1/2}$$

Caso partícula masiva,

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot q}\right)-\frac{\partial L}{\partial q}=0$$

las cantidades conservadas son aquella q con $\frac{\partial L}{\partial q}=0 \rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)=0 \rightarrow \frac{\partial L}{\partial \dot q}=\text{ctte}$

Siendo

$$L=\frac{\epsilon_0}{c}\left[A(r)\,c^2\dot t^2-\frac{\dot r^2}{A(r)}-r^2\dot\phi^2\right]^{1/2} = f[r,\dot t,\dot r,\dot\phi]$$

no hay t o φ

→

$$\therefore \frac{\partial L}{\partial t}=0 \Rightarrow \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot t}\right)=0 \rightarrow \frac{\partial L}{\partial \dot t}=\frac{\epsilon_0}{c}\left[A(r)\,c^2\cancel{2}\dot t\right]\frac{1}{\cancel{2}\sqrt{T}}$$

$$\frac{\partial L}{\partial \dot t}=\frac{\epsilon_0 c\,A(r)\,\dot t}{\sqrt{T}} \quad \Big| \sqrt{T}=\sqrt{c^2}=c \Rightarrow \frac{\partial L}{\partial \dot t}=\epsilon_0\frac{\cancel{c}}{\cancel{c}}\left(1-\frac{r_s}{r}\right)\dot t = \text{ctte.}$$

$\frac{\epsilon_0}{c}$ estaban antes así que ignorandolos

$$\therefore \boxed{c\left(1-\frac{r_s}{r}\right)\dot t=\mathcal{E}}$$

ya que $\dot t=\frac{dt}{d\tau}=\gamma$

y $\lim_{r\to\infty}$

$$\lim_{r\to\infty} c\gamma\left(1-\frac{r_s}{r}\right)=c\gamma=\frac{mc\gamma}{m}=\frac{E}{m}=\varepsilon$$

→

otra cantidad conservada $\frac{\partial L}{\partial \phi}=0 \rightarrow \frac{\partial L}{\partial \dot\phi}=\text{ctte}=\frac{\epsilon_0}{c}\frac{1}{2c}\frac{\partial T}{\partial \dot\phi}=\frac{\epsilon_0}{c}\frac{1}{\cancel{2}c}\left(-2r^2\dot\phi\right)=\text{ctte.}$

$$-\frac{\epsilon_0}{c^2}\left(r^2\dot\phi\right)=\text{ctte} \Rightarrow \boxed{r^2\dot\phi=L}$$

igual que el momentum angular clasico.

caso sin masa

para particulas sin masa; considerar $\dot q=\frac{dq}{d\lambda}$ y estos resultados de cantidad conservadas aplican igual.

ecuaciones de movimiento.

$$\left(\frac{ds}{d\lambda}\right)^2 = \left(1-\frac{r_s}{r}\right)\left(\frac{d\,ct}{d\lambda}\right)^2 - \frac{1}{\left(1-\frac{r_s}{r}\right)}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2$$

con $\varepsilon = \left(1-\frac{r_s}{r}\right)\frac{d\,ct}{d\lambda} = c\left(1-\frac{r_s}{r}\right)\frac{dt}{d\lambda} \rightarrow \frac{d(ct)}{d\lambda} = \varepsilon\frac{1}{\left(1-\frac{r_s}{r}\right)}$

$$L = r^2\frac{d\phi}{d\lambda} \rightarrow \frac{d\phi}{d\lambda} = \frac{L}{r^2}$$

$$\left(\frac{ds}{d\lambda}\right)^2 = \left(1-\frac{r_s}{r}\right)\frac{\varepsilon^2}{\left(1-\frac{r_s}{r}\right)^2} - \frac{1}{\left(1-\frac{r_s}{r}\right)}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{L^2}{r^4}\right)$$

$$\left(\frac{ds}{d\lambda}\right)^2 = \frac{\varepsilon^2}{\left(1-\frac{r_s}{r}\right)} - \frac{(dr/d\lambda)^2}{\left(1-\frac{r_s}{r}\right)} - \frac{L^2}{r^2} \qquad (*1) \qquad /\cdot\left(1-\frac{r_s}{r}\right)$$

para particulas masivas $\lambda \rightarrow \tau$; $ds^2 = c^2 d\tau^2 \rightarrow \left(\frac{ds}{d\tau}\right)^2 = c^2$

$$c^2\left(1-\frac{r_s}{r}\right) = \varepsilon^2 - \left(\frac{dr}{d\tau}\right)^2 - \frac{L^2}{r^2}\left(1-\frac{r_s}{r}\right)$$

$$\boxed{\left(\frac{dr}{d\tau}\right)^2 = \varepsilon^2 - c^2 + \frac{c^2 r_s}{r} - \frac{L^2}{r^2}\left(1-\frac{r_s}{r}\right)}$$

ecs movimiento part. masiva

particulas sin masa $ds^2 = 0$

$$0 = \varepsilon^2 - \left(\frac{dr}{d\lambda}\right)^2 - \frac{L^2}{r^2}\left(1-\frac{r_s}{r}\right)$$

$$\boxed{\left(\frac{dr}{d\lambda}\right)^2 = \varepsilon^2 - \frac{L^2}{r^2}\left(1-\frac{r_s}{r}\right)}$$

ecs movimiento particula sin masa

ahora a redirlo al problema unidimensional equivalente

para ello

$$\frac{dr}{d\lambda} = \frac{dr}{d\phi}\frac{d\phi}{d\lambda} \qquad \text{con} \quad \frac{d\phi}{d\lambda} = \frac{L}{r^2}$$

(caso masiva)

$$\rightarrow \left(\frac{dr}{d\phi}\right)^2\left(\frac{d\phi}{d\lambda}\right)^2 = \left(\frac{dr}{d\phi}\right)^2\left(\frac{L}{r^2}\right)^2 = (\mathcal{E}^2-c^2) + \frac{c^2 r_s}{r} - \frac{L^2}{r^2}\left(1-\frac{r_s}{r}\right)$$

$$\boxed{\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{L^2}(\mathcal{E}^2-c^2) + \frac{r^3c^2r_s}{L^2} - r^2\left(1-\frac{r_s}{r}\right)} \quad //$$

Probl 1D
Part. masiva.

esta ecuación ya es útil para resolver el problema.
si se gusta es posible derivar 1 vez extra, $/\cdot \frac{d}{d\phi}(\,)$

$$2\cancel{\left(\frac{dr}{d\phi}\right)}\frac{d^2r}{d\phi^2} = \left(\frac{4r^3}{L^2}(\mathcal{E}^2-c^2) + 3\frac{r^2c^2r_s}{L^2} - 2r + r_s\right)\cancel{\left(\frac{dr}{d\phi}\right)}$$

$$\boxed{\frac{d^2r}{d\phi^2} = \frac{2r^3}{L^2}(\mathcal{E}^2-c^2) + \frac{3}{2}\frac{r^2c^2r_s}{L^2} - r + \frac{r_s}{2}} \quad //$$

esta es útil
para simulaciones
y considerese
un extra.

Caso sin masa

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{L^2}\left(\mathcal{E}^2 - \frac{L^2}{r^2}\left(1-\frac{r_s}{r}\right)\right) = \frac{r^4\mathcal{E}^2}{L^2} - r^2\left(1-\frac{r_s}{r}\right)$$

$$\boxed{\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{L^2}\mathcal{E}^2 - r^2\left(1-\frac{r_s}{r}\right)} \quad //$$

Probl 1D
part. sin masa.

y como extra. derivando

$$2\cancel{\left(\frac{dr}{d\phi}\right)}\frac{d^2r}{d\phi^2} = \left(4\frac{r^3\mathcal{E}^2}{L^2} - 2r + r_s\right)\cancel{\left(\frac{dr}{d\phi}\right)} \Rightarrow \boxed{\left(\frac{d^2r}{d\phi^2}\right) = \frac{2r^3\mathcal{E}^2}{L^2} - r + \frac{r_s}{2}} \quad //$$

1b) Partículas Masivas

i) encuentre y grafique el potencial efectivo.

$$\left(\frac{dr}{d\tau}\right)^2 = \varepsilon^2 - c^2 + c^2\frac{r_s}{r} - \frac{L^2}{r^2}\left(1-\frac{r_s}{r}\right) \qquad / \cdot \frac{1}{2}m$$

$$\frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2}m\dot{r}^2 = \frac{1}{2}m\left(\varepsilon^2-c^2\right) + \frac{1}{2}m\frac{c^2 r_s}{r} - \frac{1}{2}\frac{L^2 m}{r^2}\left(1-\frac{r_s}{r}\right)$$

$m\varepsilon^2 = m\left(\frac{E}{m}\right)^2 = \frac{E^2}{m}$; mc^2 unidades de energía también : $\frac{1}{2}m\left(\varepsilon^2-c^2\right) = (\tilde{E})$

interpreta como una const. ~ energía

$$\underbrace{\frac{1}{2}m\dot{r}^2}_{\text{energía cinética}} + m\underbrace{\left(-\frac{c^2 r_s}{2r} + \frac{L^2}{2r^2}\left(1-\frac{r_s}{r}\right)\right)}_{\text{Potencial efectivo relativista}} = (\tilde{E})$$

$$\boxed{V_{eff}[r] = -\frac{c^2 r_s}{2r} + \frac{L^2}{2r^2}\left(1-\frac{r_s}{r}\right)}$$

$\{ r_s = \frac{2GM}{c^2}$

$$\boxed{V_{eff}[r] = -\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{L^2 GM}{c^2 r^3}}$$

relativistic

$\{ L = r^2\dot{\phi} = \frac{\ell}{m}$

momento angular por unidad de masa

donde el Newt. a $V^{N}_{eff}[r] = -\frac{GM}{r} + \frac{L^2}{2r^2}$ comparandolos

si $\lim_{c\to\infty} V_{eff}[r] = \lim_{c\to\infty}\left(-\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{L^2 GM}{c^2 r^3}\right) = -\frac{GM}{r} + \frac{L^2}{2r^2} = V^{Newt}_{eff}[r]$ //

// $\lim_{c\to\infty}\frac{1}{c} = 0$; osea $\frac{-L^2 GM}{c^2 r^3}$ es la corrección relativista y es importante a r pequeños (mercurio ej) gráfica del potencial efectivo relativista.

veomos los puntos de estabilidad e inestabilidad, dond $\left.\frac{dV_{eff}(r)}{dr}\right|_{r=r^*}=0$

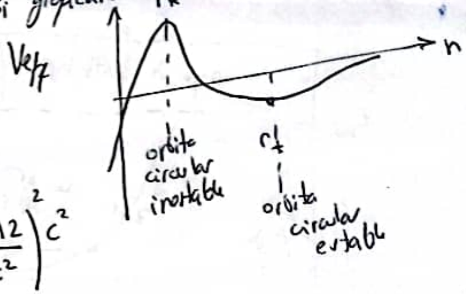
$$\frac{d}{dr}V_{eff}=\frac{GM}{r^2}-\frac{\cancel{2}}{\cancel{2}}\frac{L^2}{r^3}+\frac{3L^2GM}{c^2r^4}=\frac{GM}{r^2}-\frac{L^2}{r^3}+\frac{3L^2GM}{c^2r^4}=0\ /\cdot r^4$$

$$r^2GM-rL^2+3\frac{L^2GM}{c^2}=0 \quad \text{ecs cuadratica}$$

$$r_*=\left(L^2\pm\sqrt{L^4-4GM\cdot\frac{3L^2GM}{c^2}}\right)\frac{1}{2GM}=\frac{L^2}{2GM}\pm\frac{L^2}{2GM}\sqrt{1-\frac{12G^2M^2}{L^2c^2}}$$

$$r_*=\frac{L^2}{2GM}\left(1\pm\sqrt{1-\frac{12G^2M^2}{L^2c^2}}\right) \quad ;$$

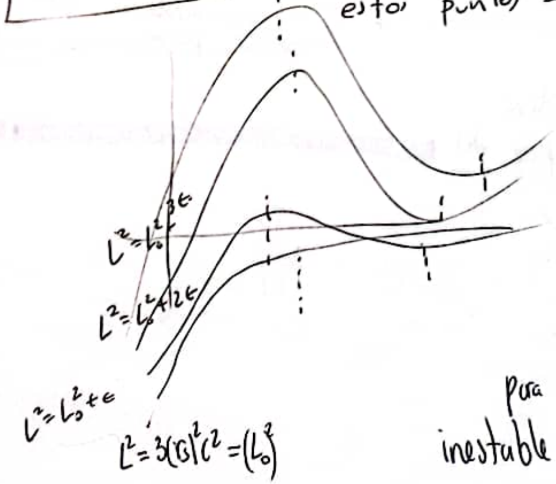
Si graficamos



ocurre que las dos r_* se juntan

si $$1-\frac{12G^2M^2}{L^2c^2}=0\Rightarrow L^2=\frac{12G^2M^2}{c^2}=3\left(\frac{GM2}{c^2}\right)^2c^2$$

$$\boxed{L^2=3(r_s)^2c^2}$$ por tanto para $L^2>3(r_s)^2c^2$ tendriamos estos puntos separados.



boceto de distintos potenciales efectivos variando el momento angular.

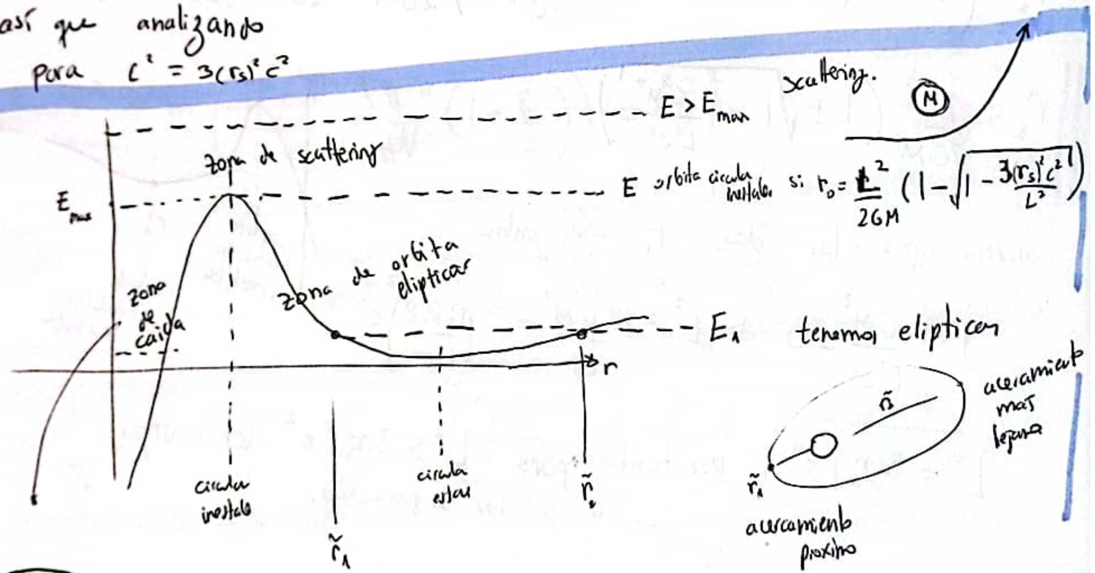
para $L^2=3(r_s)^2c^2$ es altamente inestable y si $r\neq\frac{L^2}{2GM}$ la partícula caerá.

la orbita con $L^2 = 3(r_s)^2c^2$ el radio de la orbita circular es $r_+ = \frac{L^2}{2GM}(1+\sqrt{0}) = \frac{3(r_s)^2c^2}{2GM} = \frac{3(r_s)^2}{r_s} = 3r_s$ (8)

esta es la orbita circular más pequeña, pues mayores valores de $L^2 > 3(r_s)^2c^2$ producen mayor r_+.

$r_+ = 3r_s$ es llamada la orbita estable más cercana o "inner most stable $\underset{\text{circula}}{_\wedge}$ orbit" $r_+ = r_{isco} = 3r_s$.

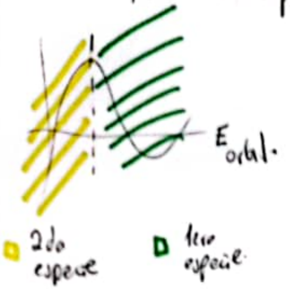
así que analizando para $L^2 = 3(r_s)^2c^2$



analisis cualitativo en base al grafico del Potencial //

1b)-II) determinar y graficar orbitas de primera y 2da especie.

Para orbitas planetarias.



para ello comenzando con

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{L^2}(\mathcal{E}^2-c^2) + \frac{r^3c^2r_s}{L^2} - r^2\left(1-\frac{r_s}{r}\right)$$

$$= \frac{r^4}{L^2}\left\{E^2 + \frac{c^2r_s}{r} - \frac{L^2}{r^2}\left(1-\frac{r_s}{r}\right)\right\}$$

$E^2 = \mathcal{E}^2 - c^2$

$$= r^4\left\{\left(\frac{E}{L}\right)^2 + \frac{c^2r_s}{L^2r} - \frac{1}{r^2}\left(1-\frac{r_s}{r}\right)\right\}$$

$b = \frac{L}{E}$ parametro de impacto. $\left\{\left(\frac{dr}{d\phi}\right)^2 = r^4\left\{\frac{1}{b^2} + \frac{c^2r_s}{L^2r} - \frac{1}{r^2} + \frac{r_s}{r^3}\right\}\right.$

luego hacemos aparecer un polinomio $\left\{\frac{b^2}{L^2} = \frac{1}{E^2}\right.$

$$= \frac{r^4}{b^2}\left\{1 + \frac{b^2c^2r_s}{L^2r} - \frac{b^2}{r^2} + \frac{b^2r_s}{r^3}\right\}$$

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{b^2}\left(\frac{1}{r^3}\right)\underbrace{\left\{r^3 + r^2\frac{c^2r_s}{E^2} - b^2r^1 + b^2r_s\right\}}$$

$$r^3 + r^2\beta - b^2r + b^2r_s = P_3[r].$$

las raices del polinomio representan puntos donde $\frac{dr^2}{d\phi^2} = 0$ osea orbitas circulares,

podemos usar el metodo de Cardano $P_3[r] = (r-r_0)(r-r_1)(r-r_2)$

$r^3 + \beta r^2 - b^2 r + b^2 r_s = 0$ / buscamos las raíces usando método Cardano

#⑩

sustituyendo $z - \frac{\beta}{3} = r$ $\Big\{$ $\left(z - \frac{\beta}{3}\right)^3 + \beta\left(z - \frac{\beta}{3}\right)^2 - b^2\left(z - \frac{\beta}{3}\right) + b^2 r_s = 0$

forma canonica

$$\Rightarrow z^3 - z\left(b^2 + \frac{\beta^2}{3}\right) + \left(b^2 r_s + \frac{b^2\beta}{3} + \frac{2\beta^3}{27}\right) = 0 \quad \Rightarrow z^3 + zp + q = 0$$

$$p = -\frac{3b^2 + \beta^2}{3} \quad ; \quad q = \frac{27b^2 r_s + 9b^2\beta + 2\beta^3}{27}$$

$$4z^3 - \frac{4}{3}\left(3b^2 + \beta^2\right)z + \frac{4}{27}\left[b^2\left(27 r_s + 9\beta\right) + 2\beta^2\right] = 0$$

| forma canonia

$$4z^3 - g_2 z + g_3 = 0$$

#Identidad

$$4\sin^3\theta - 3\sin\theta + \sin 3\theta = 0 \qquad ⊛$$

$\therefore z = w\sin\theta$ e incluyendo un multiplicador de Lagrange.

$$\lambda 4 w^3 \sin^3\theta - g_2\lambda w \sin\theta + \lambda g_3 = 0$$

comparando a ⊛.

$$\lambda w^3 = 1 \rightarrow \boxed{\lambda = 1/w^3} \; ; \quad g_2\lambda w = 3 \rightarrow g_2\frac{w}{w^3} = g_2\frac{1}{w^2} = 3 \Rightarrow \boxed{w = \sqrt{\frac{g_2}{3}}}$$

$$\lambda g_3 = \sin 3\theta \rightarrow \frac{1}{w^3}g_3 = \boxed{\left(\frac{3}{g_2}\right)^{3/2} g_3 = \sin 3\theta}$$

$\lambda w = 1 \rightarrow \boxed{\lambda = \; w} \; / \; g_2 \lambda w \quad g_2 \frac{}{w^3} - g_2 \frac{}{w^2} \quad \boxed{w \quad \sqrt{\frac{}{3}}}$

$$\lambda g_3 = \sin 3\theta \rightarrow \frac{1}{w^3} g_3 = \boxed{\left(\frac{3}{g_2}\right)^{3/2} g_3 = \sin 3\theta}$$

en general $\sin(3\theta \pm 2n\pi) = g_3 \left(\frac{3}{g_2}\right)^{3/2} \Rightarrow 3\theta = \mathrm{ArcSin}\sqrt{\frac{27 (g_3)^2}{(g_2)^3}} + 2n\pi$

$$\theta = \frac{1}{3} \mathrm{ArcSin}\sqrt{\frac{27 (g_3)^2}{(g_2)^3}} + \frac{2n\pi}{3} = \theta_0 + \frac{2n\pi}{3}$$

luego. $g_2 = \frac{4}{3}(3b^2 + \beta^2)$

$g_3 = \frac{4}{27}\left[b^2(27 r_s + 9\beta) + 2\beta^2\right]$

$$w = \sqrt{\frac{4}{9}(3b^2+\beta^2)} = \frac{2}{3}\sqrt{3b^2+\beta^2} \; //$$

$$\theta_0 = \frac{1}{3} \mathrm{ArcSin}\sqrt{\frac{27 \cdot g_3^2}{(g_2)^3}} = \frac{1}{3} \mathrm{ArcSin}\left[\frac{1}{2}\sqrt{\frac{(2\beta^2 + 9b^2(3r_s+\beta))^2}{(3b^2+\beta^2)^3}}\right] = \frac{1}{3}\mathrm{ArcSin}\left[\frac{(2\beta^2 + 9b^2(3r_s+\beta))}{2(3b^2+\beta^2)^{3/2}}\right]$$

y usando $\sin\left[\alpha + n\frac{2\pi}{3}\right] = \sin\alpha \cos\left[n\frac{2\pi}{3}\right] + \sin\left[n\frac{2\pi}{3}\right]\cos\alpha$

$\cos\left[\frac{2\pi}{3}\right] = -\frac{1}{2}$; $\sin\left[\frac{2\pi}{3}\right] = \frac{\sqrt{3}}{2}$; $\sin\left[\frac{4\pi}{3}\right] = -\frac{\sqrt{3}}{2}$

$\cos[0] = 1$; $\sin[0] = 0$; $\cos\left(\frac{4\pi}{3}\right] = -\frac{1}{2}$

#[illegible]

#10.6

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{b^2}\left(\frac{1}{r^3} P_3[r]\right)$$

ecuación de movimiento

$P_3[r] = r^3 + r^2\beta$ [illegible] $b^2 r + b^2 r_s$

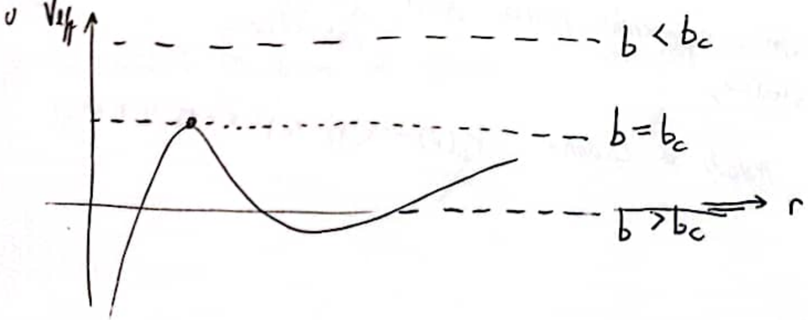
$$= (r-r_0)(r-r_1)(r-r_2)$$

donde $b = \frac{L}{E}$ osea a menor energía mayor b
y a mayor energía menor b. ; $\beta = \frac{c^2 r_s}{E^2}$

Podemos entonces ; donde $\frac{\beta}{b} = \frac{c^2 r_s}{E^2}\frac{E}{L} = \frac{c^2 r_s}{EL} = \alpha^2$

$$\theta_0 = \frac{1}{3}\text{ArcSin}\left[\frac{2\beta^2 + 9b^2(3r_s+\beta)}{2(3b^2+\beta^2)^{3/2}}\right] = \frac{1}{3}\text{ArcSin}\left[\frac{2\beta^2 + 9b^2(3r_s+\beta)}{2b^3\left(3+\frac{\beta^2}{b^2}\right)^{3/2}}\right]$$

$$= \frac{1}{3}\text{ArcSin}\left[\frac{b^2}{b^3}\,\frac{\left(2\frac{\beta^2}{b^2} + 9(3r_s+\beta)\right)}{2\left(3+\frac{\beta^2}{b^2}\right)^{3/2}}\right] = \frac{1}{3}\text{ArcSin}\left[\frac{1}{b}\cdot\underbrace{\frac{2\alpha^2 + 9(3r_s+\beta)}{2(3+\alpha^2)^{3/2}}}_{b_c}\right].$$



#⑪

soluciones $n \in \mathbb{Z}$ $n \in \{0,1,2\}$ $\left\{ Z_n = W \sin\left[\theta + n\frac{2\pi}{3}\right] = \frac{2}{3}\sqrt{3b^2+\beta^2}\left(\sin\theta_0 \cos\left(\frac{2n\pi}{3}\right) + \sin\left(\frac{2n\pi}{3}\right)\cos\theta_0\right)\right.$

devolviendo el cambio: $r_n = Z_n - \frac{\beta}{3}$; recordar $\beta = \frac{c^2 r_s}{E^2}$; $b = \frac{L}{E}$

con $\Theta_0 = \frac{1}{3}\mathrm{ArcSin}\left[\frac{2\beta^2 + 9b^2(3r_s+\beta)}{2(3b^2+\beta^2)^{3/2}}\right] = \frac{1}{3}\mathrm{ArcSin}\left[\frac{1}{b}\frac{2\frac{\beta^2}{b^2}+9(3r_s+\beta)}{2\left(3+\frac{\beta^2}{b^2}\right)^{3/2}}\right]$

$$\boxed{\begin{aligned} r_0 &= \frac{2}{3}\sqrt{3b^2+\beta^2}\,\sin\theta_0 - \frac{\beta}{3} \\ r_1 &= \frac{2}{3}\sqrt{3b^2+\beta^2}\left(-\frac{1}{2}\sin\theta_0 + \frac{\sqrt{3}}{2}\cos\theta_0\right) - \frac{\beta}{3} \\ r_2 &= \frac{2}{3}\sqrt{3b^2+\beta^2}\left(-\frac{1}{2}\sin\theta_0 + \frac{\sqrt{3}}{2}\cos\theta_0\right) - \frac{\beta}{3} \end{aligned}}$$

raices del polinomio.

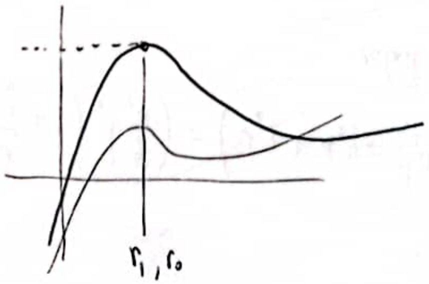
en el caso especial donde $b = b_c \rightarrow \Theta_0 = \frac{1}{3}\mathrm{ArcSin}(1) = \frac{1}{3}\frac{\pi}{2} = \frac{\pi}{6}.$

$$r_0 = \frac{2}{3}\sqrt{3b_c^2+\beta^2}\cdot\sin\frac{\pi}{6} - \frac{\beta}{3} = \frac{1}{3}\sqrt{3b_c^2+\beta^2} - \frac{\beta}{3}$$

$$r_1 = \frac{2}{3}\sqrt{3b_c^2+\beta^2}\left(-\frac{1}{4} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\right) - \frac{\beta}{3} = \frac{2}{3}\sqrt{3b_c^2+\beta^2}\left(\frac{1}{2}\right) - \frac{\beta}{3} = r_0$$

solución repetida.

$$r_2 = \frac{2}{3}\sqrt{3b_c^2+\beta^2}\left(-\frac{1}{4} - \frac{3}{4}\right) - \frac{\beta}{3} = -\frac{2}{3}\sqrt{3b_c^2+\beta^2} - \frac{\beta}{3} < 0$$



Solución o raiz negativa. sin significado físico en este caso.

#12

$$\left(\frac{dr}{d\phi}\right)^2=\left(\frac{r^2}{b}\right)^2\left(\frac{(r-r_0)(r-r_1)(r-r_2)}{r^3}\right)$$ ecs de movimiento

entonces con el cambio y raiz.

$$u=\frac{1}{r}\ ;\ du=-\frac{dr}{r^2}\qquad dr=-du\cdot r^2=-\frac{du}{u^2}\Rightarrow\frac{dr}{d\phi}=-\frac{1}{u^2}\frac{du}{d\phi}$$

$$\frac{dr}{d\phi}=\pm\frac{r^2}{b}\sqrt{\frac{(r-r_0)(r-r_1)(r-r_2)}{r^3}}\ \xrightarrow{r}\ -\frac{1}{u^2}\frac{du}{d\phi}=\pm\frac{1}{u^2}\frac{1}{b}\sqrt{u^3\left(\frac{1}{u}-\frac{1}{u_0}\right)\left(\frac{1}{u}-\frac{1}{u_1}\right)\left(\frac{1}{\ldots}\right.}$$

$$\left(\frac{1}{u}-\frac{1}{u_2}\right)$$

$$\to\frac{du}{d\phi}=\mp\frac{1}{b}\sqrt{u^3\left(\frac{1}{u}-\frac{1}{u_0}\right)\left(\frac{1}{u}-\frac{1}{u_1}\right)\left(\frac{1}{u}+\frac{1}{|u_2|}\right)}$$

raiz negativa ligado a r_2.

$$\frac{du}{d\phi}=\mp\frac{1}{b}\sqrt{u^3\ \frac{(u_0-u)}{u\,u_0}\ \frac{(u_1-u)}{u\,u_1}\ \frac{(|u_2|+u)}{u\,|u_2|}}$$

$$\frac{du}{d\phi}=\mp\frac{1}{b}\sqrt{\frac{(u_0-u)}{u_0}\ \frac{(u_1-u)}{u_1}\ \frac{(|u_2|+u)}{|u_2|}}$$ ecuación de todas las orbitas posibles

es más util como comente previamente volver a diferenciar.
comencemos de:

$$\left(\cancel{\frac{1}{u^2}}\right)^2\left(\frac{du}{d\phi}\right)^2=\left(\cancel{\frac{1}{u^2}}\right)^2\left(\frac{1}{b}\right)^2\left(\frac{(u_0-u)(u_1-u)(|u_2|+u)}{u_0\ u_1\ |u_2|}\right)\ \Big/\ \frac{d}{d\phi}$$

~~$2\left(\frac{du}{d\phi}\right)\frac{d^2u}{d\phi^2}=\frac{1}{b^2}\frac{1}{u_0u_1|u_2|}\Big\{-(u_1-u)(|u_2|+u)\ldots$~~ $\otimes$

o usar la ecs original mejor

$$\left(\frac{dr}{d\phi}\right)^2=\left(\frac{r^2}{b}\right)^2\frac{1}{r^3}\left(r^3+r^2\beta-b^2r+b^2r_s\right)=\left(\frac{r^2}{b}\right)^2\left(1+\frac{\beta}{r}-\frac{b^2}{r^2}+\frac{b^2r_s}{r^3}\right)$$

$$\left(\frac{dr}{d\phi}\right)^2 = \left(\frac{r^2}{b}\right)^2\left(1+\frac{\beta}{r}-\frac{b^2}{r^2}+\frac{b^2 r_s}{r^3}\right) \quad \Big/ \; u=\frac{1}{r} \quad du = -\frac{dr}{r^2} \rightarrow \cancel{\frac{dr}{d\phi} = \frac{-1}{u^2}\frac{du}{d\phi}}$$ ⑬

$$\frac{dr}{d\phi} = -\frac{1}{u^2}\frac{du}{d\phi}$$

$$\left(\frac{du}{d\phi}\right)^2\left(\frac{1}{u^2}\right)^2 = \left(\frac{1}{u^2}\right)^2\left(\frac{1}{b}\right)^2\left(1+\beta u - b^2u^2 + b^2 r_s u^3\right)$$

$$\left(\frac{du}{d\phi}\right)^2 = \left(\frac{1}{b}\right)^2\left(1+\beta u - b^2u^2 + b^2 r_s u^3\right) \quad \Big/ \; \frac{d}{d\phi}$$

$$2\left(\frac{du}{d\phi}\right)\frac{d^2u}{d\phi^2} = \frac{1}{b^2}\left(0+\beta-2b^2u+3b^2r_s u^2\right)\cancel{\left(\frac{du}{d\phi}\right)}$$

$$\frac{d^2u}{d\phi^2} = \frac{1}{b^2}\left(\frac{\beta}{2}-b^2u+\frac{3}{2}b^2r_s u^2\right) = \frac{\beta}{2b^2}-u+\frac{3}{2}r_s u^2$$

$$\boxed{\frac{d^2u}{d\phi^2}+u = \frac{\beta}{2b^2}+\frac{3}{2}r_s u^2} \; \Big/ \cdot\frac{2b^2}{\beta} \quad \Big| \quad y = \frac{2b^2}{\beta}u \rightarrow u = \frac{y\beta}{2b^2}$$

$$\frac{d^2}{d\phi^2}\left(\frac{2b^2}{\beta}u\right)+\frac{2b^2}{\beta}u = 1+\frac{3}{2}r_s\left(\frac{2b^2}{\beta}u\right)u \qquad \left\{ b=\frac{L}{E} \; ; \; \beta = \frac{c^2r_s}{E^2}\right.$$

$$\frac{d^2}{d\phi^2}y + y = 1+\frac{3}{2}r_s\, y\cdot y\frac{\beta}{2b^2} = 1+y^2\frac{3}{4}\frac{r_s\beta}{b^2} = 1+y^2\frac{3}{4}\frac{r_s\,c^2r_s\cancel{E^2}}{\cancel{E^2}L^2}$$

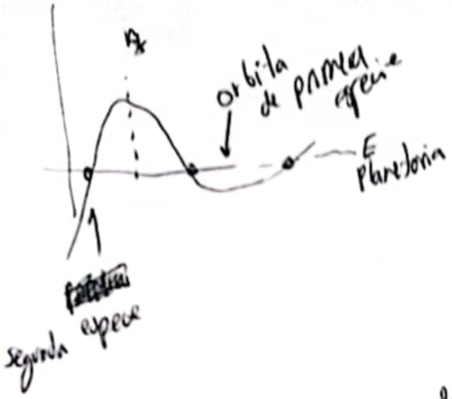
$$\frac{d^2y}{d\phi^2}+y = 1+y^2\frac{3}{4}\frac{c^2(r_s)^2}{L^2} = 1+y^2\alpha^2 \Rightarrow \boxed{\frac{d^2y}{d\phi^2}+y = 1+y^2\alpha^2}$$

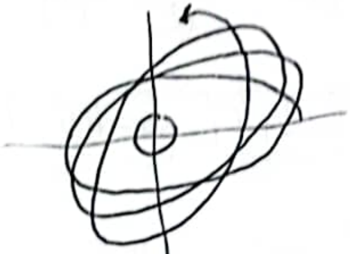
con $y = \frac{2b^2}{\beta}u = \frac{2L^2}{\cancel{E^2}}\frac{\cancel{E^2}}{c^2r_s} = \frac{2L^2}{c^2r_s}\frac{1}{r}$

entonces resolviendo numericamente

para ser capaz de visualizar:

#14





en una orbita de primera especie se tiene el avance del perihelio

y las de 2da especie rapidamente impactan osea que para orbitas se requiere que sean mayor a r_*

usando $L^2 > 3r_s^2c^2$; $M=1$

ver orbita en orbita_masive.ipynb

1c) la ecuación de movimiento para la luz (geodesica nula) $\boxed{\#13}$

$$\left[\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{L^2}\varepsilon^2 - r^2\left(1-\frac{r_s}{r}\right)\right]$$

$u = \frac{1}{r}$ $\rightarrow dr = -\frac{du}{u^2}$

$$\frac{dr}{d\phi}\cancel{d\phi} = -\frac{du}{d\phi}\cancel{d\phi}\frac{1}{u^2}$$

$$\left(-\frac{du}{d\phi}\right)^2\left(\frac{1}{u^2}\right)^2 = \frac{1}{u^4}\frac{\varepsilon^2}{L^2} - \frac{1}{u^2}\left(1-\frac{ur_s}{\cancel{\square}}\right) \quad /\cdot u^4$$

$$\boxed{\left(\frac{du}{d\phi}\right)^2 = \frac{\varepsilon^2}{L^2} - u^2(1-ur_s)}$$

usando

$y = u\left(1-\frac{r_s u}{2}\right)$

$b^{-1} = \frac{\varepsilon}{L}$

$$dy = \left(\left(1-\frac{r_s u}{2}\right) - \frac{ur_s}{2}\right)du = \frac{du}{2}\left(2 - r_s u - ur_s\right) = \frac{du}{2}\left(2-2ur_s\right)$$

$$dy = du\,(1-ur_s)$$ → el cambio $u(y)$.

$$u[y] = y\left(1+y\frac{r_s}{2}\right) + O[r_s^2u^2]$$

$$\left(\frac{du}{d\phi}\right) = \frac{1}{(1-ur_s)}\left(\frac{dy}{d\phi}\right)$$

$$\left(\frac{dy}{d\phi}\right)^2 = (1-ur_s)^2\left(b^{-2} - u^2(1-ur_s)\right) = \left(1-2ur_s+u^2r_s^2\right)\left(b^{-2}-u^2(1-ur_s\right.$$

$$\left(\frac{dy}{d\phi}\right)^2 = (1-ur_s)^2b^{-2} - u^2(1-ur_s)^2 = (1-ur_s)^2\left(b^{-2}-u^2\right)$$

△ donde $(1-ur_s)^2 = \left(1 - r_s y\left(1+\frac{y r_s}{2}\right) + O[r_s^2u^2]\right)^2$

#15.a

$$\left(\frac{du}{d\phi}\right)^2 = b^{-2} - u^2(1 - u\, r_s) \qquad \text{con } y = u\left(1 - r_s \frac{u}{2}\right)$$

$$dy = du\,(1 - u r_s)$$

donde: $y = u - r_s \frac{u^2}{2} \Rightarrow u^2 \frac{r_s}{2} - u + y = 0$

$$u^2 = (y - u)\frac{2}{r_s} \quad /*$$

$$\left(\frac{dy}{d\varphi}\right)^2 \left(\frac{1}{1 - u r_s}\right)^2 = b^{-2} - \frac{2}{r_s}(y - u)(1 - u r_s)$$

$$= b^{-2} - \frac{2}{r_s}\left[y - y u r_s - u + u^2 r_s\right]$$

$$= b^{-2} - \frac{2}{r_s}\left[y - u(y r_s + 1) + \frac{2}{r_s}(y - u)\right]$$

$$= b^{-2} - \frac{2}{r_s}\left[y\left(1 + \frac{2}{r_s}\right) - u\left(y r_s + 1 + \frac{2}{r_s}\right)\right] \quad // u = y\left(1 + \frac{r_s y}{2}\right) + \mathcal{O}(\text{[illegible]})$$

$$= b^{-2} - \frac{2}{r_s}\left[y\left(1 + \frac{2}{r_s}\right) - \left[y\left(1 + \frac{r_s y}{2}\right) + \mathcal{O}^2\right]\left(y r_s + 1 + \frac{2}{r_s}\right)\right]$$

$$= b^{-2} - \frac{2}{r_s}\left\{y\left(1 + \frac{2}{r_s}\right) - y^2 r_s\left(1 + \frac{r_s y}{2}\right) + y\left(1 + \frac{r_s y}{2}\right] + \frac{2}{r_s} y\left(1 + \frac{r_s y}{2}\right) + \mathcal{O}^2 \cdot \left(y r_s + 1 + \frac{2}{r_s}\right)\right\}$$

$$= b^{-2} - \frac{2}{r_s}\left\{y\left[1 + \frac{2}{r_s} - y r_s\left(1 + y\frac{r_s}{2}\right) + \left(1 + \frac{2}{r_s}\right)\left(1 + \frac{y r_s}{2}\right)\right] + \mathcal{O}^2\left(y r_s + 1 + \frac{2}{r_s}\right)\right\}$$

$$= b^{-2} - \frac{2}{r_s}\left\{y\left[\left(1 + \frac{2}{r_s}\right)\left(2 + y\frac{r_s}{2}\right) - y r_s\left(1 + \frac{y r_s}{2}\right)\right] + \mathcal{O}^2 \cdot \left(y r_s + 1 + \frac{2}{r_s}\right)\right\}$$

#TS 6

$$\left(\frac{du}{d\phi}\right)^2 = b^{-2} - u^2\left(1 - \frac{ur_s}{2} - \frac{ur_s}{2}\right) = b^{-2} - u^2\left(1 - \frac{ur_s}{2}\right) + u^2\left(\frac{ur_s}{2}\right)$$

$$\left(\frac{dy}{d\phi}\right)^2 \frac{1}{(1-ur_s)^2} = b^{-2} - u^2(1 - ur_s)$$

$$\left(\frac{d\phi}{dy}\right)^2 = \Big[\underbrace{(1-ur_s)^2\left(b^{-2} - u^2(1-ur_s)\right)}_{\varphi}\Big]^{-1} = \frac{1}{\varphi}$$

$$\varphi = (1-ur_s)^2 b^{-2} - u^2(1-ur_s)^3 = b^{-2} - 2\frac{r_s u}{b^2} + \left(\frac{r_s^2}{b^2} - 1\right)u^2 + 3r_s u^3 - 3r_s^2u^4 + r_s^3u^5$$

$$= \cancel{\left(b^{-2} - u^2(1-ur_s)\right)(1-ur_s)^2}$$

$$\to \cancel{\sqrt{\frac{1}{\varphi}} = \frac{1}{1-ur_s}\,\frac{1}{\sqrt{b^{-2}-u^2(1-ur_s)}} =}$$

colectando por ur_s

$$\varphi = \frac{1}{b^2} - \frac{2(r_s u)}{b^2} - u^2 + \frac{r_s^2u^2}{b^2} + 3(r_s u)u^2 - 3(r_s u)^2u^2 + (r_s u)^3u^2$$

$$\varphi = \frac{1}{b^2} - 2\frac{(r_s u)}{b^2} + u^2\left[-1 + 3(r_s u) - 3(r_s u)^2 + (r_s u)^3\right] + \frac{(r_s u)^2}{b^2}$$

$$\varphi = b^{-2}\left[1 - 2(r_s u) + (r_s u)^2\right] + u^2\left[-1 + 3(r_s u) - 3(r_s u)^2 + (r_s u)^3\right]$$

ahora si tenemos $\varphi\left(\frac{r_s}{r_s}\right)^2 = \frac{1}{r_s^2}(r_s u)^2\left[-1 + 3(r_s u) - 3(r_s u)^2 + (r_s u)^3\right] + b^{-2}$

$$\left[1 - 2(r_s u) + (r_s u)^2\right]$$

usemos $\alpha = (r_s u)$

$$\varphi = \frac{1}{r_s^2}\alpha^2\left[-1 + 3\alpha - 3\alpha^2 + \alpha^3\right] + \frac{1}{b^2}\left[1 - 2\alpha + \alpha^2\right]$$

~~tomemos la expansión de Taylor de.~~ #15_c

~~$\frac{1}{\sqrt{\psi}} =$~~

$$\psi = \frac{\alpha^2}{r_s^2}\left[-1+3\alpha\right] + \frac{1}{b^2}\left[(1-\alpha)^2\right] + \mathcal{O}[\alpha^3]$$

$$\psi = \frac{\alpha^2[3\alpha-1]b^2 + r_s^2[(1-\alpha)^2)}{b^2 \; r_s^2}$$

ii)
$$\frac{d\phi}{dy} = \frac{1 + r_s y}{\sqrt{\frac{1}{b^2} - y^2}} = \frac{1 + r_s y}{\frac{1}{b}\sqrt{1 - b^2 y^2}} = b\,\frac{1 + r_s y}{\sqrt{1 - b^2 y^2}}$$

$$= b \cdot \frac{1}{\sqrt{1 - b^2 y^2}} + b r_s \frac{y}{\sqrt{1 - b^2 y^2}}$$

$1 - \sin^2\theta = \cos^2\theta$

$by = \sin\theta$

$b\,dy = \cos\theta\, d\theta$

$$\int d\phi = b \cdot \int \frac{dy}{\sqrt{1 - b^2 y^2}} + b r_s \int \frac{y\, dy}{\sqrt{1 - b^2 y^2}}$$

$$= b \int \frac{\cos\theta\, d\theta\, \frac{1}{b}}{\cos\theta} + b r_s \int \frac{\frac{\sin\theta}{b}\, \frac{\cancel{\cos\theta}\, d\theta}{b}}{\cancel{\cos\theta}} = \int d\theta + \frac{r_s}{b} \int \sin\theta\, d\theta$$

$$= \mathrm{ArcSin}[by] + \phi_0 + \frac{r_s}{b}(-\cos\theta)$$

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta = 1 - \sin\theta^2$

$\cos\theta = \sqrt{1 - \sin\theta^2}$

$\cos\theta = \sqrt{1 - (by)^2}$

$$= \phi_0 + \mathrm{ArcSin}[by] - \frac{r_s}{b}\sqrt{1 - b^2 y^2}$$

$$\phi = \phi_0 + \mathrm{ArcSin}[by] - r_s \sqrt{\frac{1}{b^2} - y^2}$$

y teniendo en cuenta el factor extra $O(r_s^2 u^2)$

usando $u^2 = y^2\left(1 + \frac{r_s y}{2}\right)^2 + \underset{0}{\cancel{O(r_s^4 u^4)}}$; $y = u\left(1 - \frac{r_s u}{2}\right) = u - \frac{r_s u^2}{2}$

$$\int O(r_s^2 u^2)\, dy = \int r_s^2 y^2 \left(1 + \frac{r_s y}{2}\right)^2 dy = r_s^2 \frac{y^3}{3} + r_s^3 \frac{y^4}{8} + c =$$

incluido en ϕ_0

$$\int O(r_s^3 u^2)\,dy = \frac{r_s^2 y^2 y}{3} + \frac{r_s^3 y^3 y}{8} \quad \Bigg| \quad \begin{aligned} y &= u - \frac{r_s u^2}{2} \\ y^2 &= u^2 - u^3 r_s + \frac{1}{4} r_s^2 u^4 \end{aligned}$$

(#16

con $\boxed{r_s u = \alpha}$

$$= \frac{r_s^2}{3}\left(u^2 - u^3 r_s + \frac{1}{4} r_s^2 u^4\right)\left(u - \frac{r_s u^2}{2}\right) + \frac{r_s^3}{8}\left(u^2 - u^3 r_s + \frac{1}{4} r_s^2 u^4\right)^2$$

$$= \left(\frac{\alpha^2 - \alpha^3 + \frac{1}{4}\alpha^4}{3}\right)\left(\frac{\alpha}{r_s} - \frac{\alpha^2}{2r_s}\right) + \frac{r_s^2}{8}\left(\alpha^2 - \alpha^3 + \frac{1}{4}\alpha^4\right)^2$$

$$= O(r_s^2 u^2) //$$

Por tanto el resultado de integrar

$$\phi = \phi_0 + \mathrm{ArcSin}[by] - r_s\sqrt{\frac{1}{b^2} - y^2} + O(r_s^2 u^2)$$

comparando a la solución que demuestra

$O(r_s^2 u^2)$ debe de tener una forma distinta de $(r_s u)^2$ especificada, para integrar a $\frac{r_s}{b}$.

iii) mostrar que ③ puede ser resuelta con ⊛₂

$$(3) \quad \left(\frac{d\phi}{du}\right)^2 = \frac{1}{b^{-2} - u^2(1-ur_s)} \quad \text{or} \quad \left(\frac{du}{d\phi}\right)^2 = b^{-2} - u^2(1-ur_s)$$

devolviendo el cambio a u ↑

$$*_2) \quad bu = \sin[\phi-\phi_0] + \frac{r_s}{2b}\left(1-\cos[\phi-\phi_0]\right)^2 + O\left[\frac{r_s^2}{b^2}\right]$$

$$\frac{du}{d\phi} = \frac{d}{d\phi}\left\{\frac{1}{b}\sin[\phi-\phi_0] + \frac{r_s}{2b^2}\left(1-\cos[\phi-\phi_0]\right)^2 + \cancel{\frac{1}{b}O\left[\frac{r_s^2}{b^2}\right]}\right\}$$

derivada de constante.

$$= +\frac{\cos[\phi-\phi_0]}{b} + \frac{r_s}{\cancel{2}b^2}\cancel{2}\left(1-\cos[\phi-\phi_0]\right)\left(-\left(-\sin[\phi-\phi_0]\right)\right)$$

$$\frac{du}{d\phi} = \frac{\cos[\phi-\phi_0]}{b} + \frac{r_s}{b^2}\left(1-\cos[\phi-\phi_0]\right)\left(\sin[\phi-\phi_0]\right)$$

$$\left(\frac{du}{d\phi}\right)^2 = \frac{\cos^2[\phi-\phi_0]}{b^2} + \left(\frac{r_s}{b^2}\right)^2\left(1-\cos[\phi-\phi_0]\right)^2\sin^2[\phi-\phi_0]$$

y el lado derecho: $= +b^{-2} - u^2(1-ur_s)$; $\epsilon = O\left[\frac{r_s^2}{b^2}\right]$

$$u^2 = \left(\frac{\sin[\phi-\phi_0]}{b} + \frac{r_s}{2b}\left(1-\cos[\phi-\phi_0]\right)^2 + \epsilon\right)^2$$

$$u^2 = \left(\frac{\sin[\phi-\phi_0]}{b} + \epsilon\right)^2 + \left(\epsilon + \frac{\sin[\phi-\phi_0]}{b}\right)\left(\frac{r_s}{b}\left(1-\cos[\phi-\phi_0]\right)^2\right) + \frac{r_s^2}{4b^2}\left(1-\cos[\phi-\phi_0]\right)^4$$

$$\left(\frac{du}{d\phi}\right)^2 = b^{-2} - u^2(1-ur_s) \qquad // \ \omega = \phi - \phi_0$$

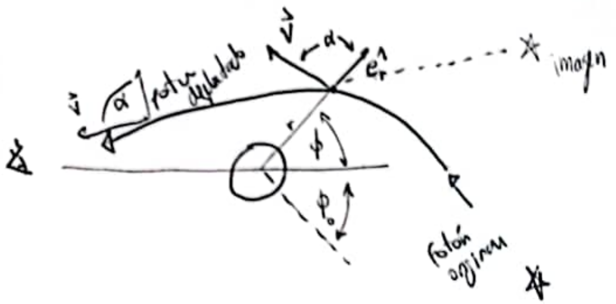
#18

$$(1-ur_s) = 1 - \frac{\sin[\phi-\phi_0]}{b} r_s + \frac{(r_s)^2}{2b}\left(1-\cos[\phi-\phi_0]\right)^2 + \epsilon$$

$$\frac{1^2}{b} - u^2(1-ur_s) = \frac{1}{b^2} - \left(\epsilon + \frac{r_s(1-\cos w)^2}{2b} + \frac{\sin w}{b}\right)^2\left(1 - \xi\left(\epsilon + \frac{r_s(1-\cos[w])^2}{2b} + \frac{\sin w}{b}\right)\right)$$

Comparar las expresiones no hara mucho debido a la enorme cantidad de terminos; así que dejare este problema hasta aqui para seguir con otras. //

iv) tenema el vector: $\vec{v} = \left(0, -1, 0, \frac{-d\phi}{dr}\right)$



$\vec{e}_r = (0,1,0,0) = \delta_{\mu 1}$

$g_{11} = -\frac{1}{(1-\frac{r_s}{r})}$

$$\vec{v}\cdot\vec{e}_r = g_{\mu\nu} V^\mu e_r^\nu = g_{\mu 1} V^\mu \delta_1^\nu = g_{11} V^1 = \frac{-1}{(1-\frac{r_s}{r})}(-1) = \frac{1}{(1-\frac{r_s}{r})}$$

Pra un observadr $\vec{v}$ y $\vec{e}_r$ son paralelos ya que

$$\lim_{r\to\infty} \vec{v}\cdot\hat{e}_r = \lim_{r\to\infty} \frac{1}{(1-\frac{r_s}{r})} = 1$$

Podemos comparar en el observador $\hat{e}_\phi = (0,0,0,1)$

$$\vec{v}\cdot\hat{e}_\phi = g_{\phi\phi}\left(\frac{-d\phi}{dr}\right) = -r^2\left(\frac{-d\phi}{dr}\right) = r^2\frac{d\phi}{dr}$$

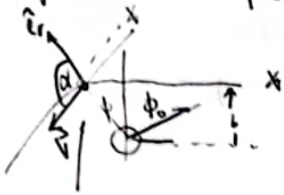
donde

$$\left(\frac{dr}{d\phi}\right)^2 = r^4 b^{-2} - r^2\left(1-\frac{r_s}{r}\right) \rightarrow \left(\frac{d\phi}{dr}\right) = \frac{\pm 1}{r^2\sqrt{\frac{1}{b^2} - \frac{1}{r^2}\left(1-\frac{r_s}{r}\right)}}$$

$$\vec{v}\cdot\vec{e}_\phi = r^2\frac{d\phi}{dr} = \frac{\pm 1}{\sqrt{\frac{1}{b^2} - \frac{1}{r^2}\left(1-\frac{r_s}{r}\right)}}$$

el parametro de impacto.

#26



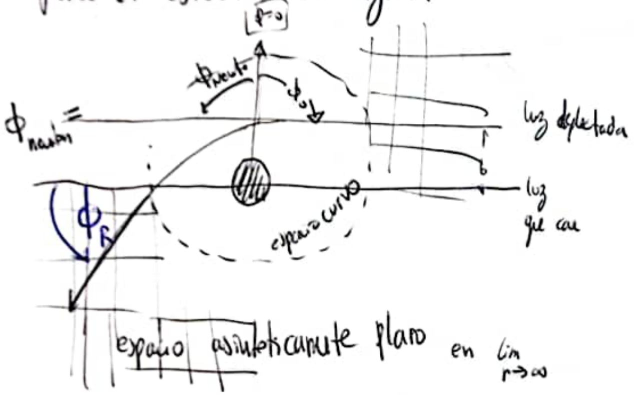
ϕ punto de acercamiento más cercano

$$\vec{v}\cdot\hat{e}_\phi = \frac{\pm 1}{\sqrt{\frac{1}{b^2} - \frac{1}{r^2}\left(1-\frac{r_s}{r}\right)}} = r^2\frac{d\phi}{dr}$$

en el cab

$$\lim_{r\to\infty} \vec{v}\cdot\hat{e}_\phi = \frac{\pm 1}{\sqrt{\frac{1}{b^2}}} = \boxed{\pm b = r^2\frac{d\phi}{dr}}$$

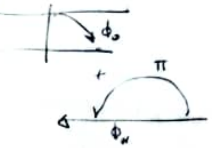
para el estudio de angulos



espacio asintoticamente plano en $\lim_{r\to\infty}$

caso Newton

$$\boxed{\phi_0 + \pi = \phi_N}$$



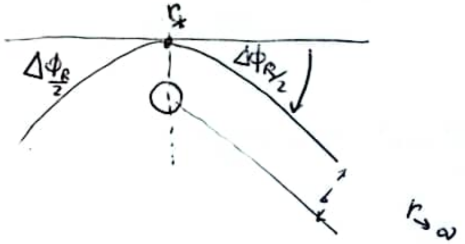
o si vemos $\phi = 0$ en ϕ_0

$\phi_0 = 0$

$$\boxed{\phi_N = \pi}$$

$\phi_R[b]$ el angulo de deflexión relativista dependera de b

el problema es simetrico pues es una hiperbola



es posible calcular

$$\frac{\Delta\phi_R}{2} = \int_{\phi(r=\infty)}^{\phi(r^*)} d\phi = \int_{r=\infty}^{r^*} \frac{(\pm dr)}{r^2\sqrt{\frac{1}{b^2} - \frac{1}{r^2}\left(1-\frac{r_s}{r}\right)}}$$

/ ± el angulo aumenta al disminuir r

$$\frac{\Delta\phi_R}{2} = -\int_{\infty}^{r^*} \frac{dr}{r^2\sqrt{\frac{1}{b^2} - \frac{1}{r^2}\left(1-\frac{r_s}{r}\right)}} = \int_{r^*}^{\infty} \frac{dr}{r^2\sqrt{\frac{1}{b^2} - \frac{1}{r^2}\left(1-\frac{r_s}{r}\right)}} \to \frac{d\phi}{dr} < 0 \quad \left(\text{signo negativo}\right)$$

#21

a partir de la expresión en ii)

$$\phi - \phi_0 = \frac{r_s}{b} + \mathrm{Arcsin}[by] - r_s\sqrt{\frac{1}{b^2} - y^2} = \Delta\phi \qquad (*_3)$$

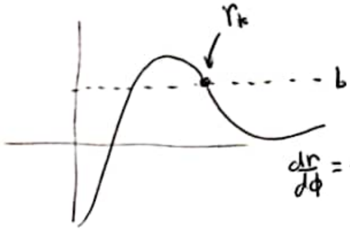
con $y = u\left(1 - \frac{r_s u}{2}\right) = \frac{1}{r}\left(1 - \frac{r_s}{2r}\right)$

evaluamos la integral $\frac{\Delta\phi_B}{2} = \int_{r^*}^{\infty} \frac{dr}{r^2\sqrt{\frac{1}{b^2} - \frac{1}{r^2}(1 - \frac{r_s}{r})}}$ usand $*_3$ como solución

$$\Delta\phi = \lim_{r\to\infty}\left(\frac{r_s}{b} + \mathrm{ArcSin}\left[\frac{1}{r}\left(1 - \frac{r_s}{2r}\right)b\right] - r_s\sqrt{\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{r_s}{2r}\right)^2}\right)$$

$$- \left(\frac{r_s}{b} + \mathrm{ArcSin}\left[\frac{1}{r_*}\left(1 - \frac{r_s}{2r_*}\right)b\right] - r_s\sqrt{\frac{1}{b^2} - \frac{1}{(r_*)^2}\left(1 - \frac{r_s}{2r_*}\right)^2}\right)$$

// r_* es la distancia de maximo acercamiento; osea la raiz más pequeña pra la energía y L dados por b.



asumiendo $r_s u = \frac{r_s}{r} \ll 1$.

es posible resolver raices de

$$\frac{dr}{d\phi} = -r^2\underbrace{\sqrt{\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{r_s}{r}\right)}}_{P_3[r]} \qquad \Bigg| = -r^2\sqrt{\left(1 - \frac{r_s}{r}\right)\left\{\frac{\left(1 - \frac{r_s}{r}\right)^{-1}}{b^2} - \frac{1}{r^2}\right\}}$$

$$P_3[r] = \left(1 - \frac{r_s}{r}\right)\left\{\frac{1}{b^2}\left(1 + \frac{r_s}{r} + \mathcal{O}\left(\frac{r_s}{r}\right)^2\right) - \frac{1}{r^2}\right\} = \left(1 - \frac{r_s}{r}\right)\left\{\frac{1}{b^2} + \frac{1}{r}\frac{r_s}{b^2} - \frac{1}{r^2}\right\}$$

como variable auxiliar $u = \frac{1}{r}$

$$P_3[u] = (1 - u r_s)\left\{\frac{1}{b^2} + u\frac{r_s}{b^2} - u^2\right\}$$

#22

$$P_3(u) = (1-ur_s)\left\{\frac{1}{b^2} + u\frac{r_s}{b^2} - u^2\right\}$$

ecs cuadratica $(u-u_1)(u-u_2)$

$= (\frac{1}{r} - \frac{1}{r_1})(\frac{1}{r} - \frac{1}{r_2})$

$$u = \frac{-\frac{r_s}{b^2} \pm \sqrt{\frac{r_s^2}{b^4} + 4\frac{1}{b^2}}}{-2} = \frac{r_s}{2b^2} \pm \sqrt{\left(\frac{r_s}{2b^2}\right)^2 + \frac{1}{b^2}}$$

$$u = \frac{1}{r} = \frac{r_s}{2b^2}\left(1 \pm \sqrt{1 + \frac{4b^4}{r_s^2 b^2}}\right) = \frac{r_s}{2b^2}\left(1 \pm \sqrt{1 + \frac{4b^2}{r_s^2}}\right) = \left\{\frac{1}{r_1}, \frac{1}{r_2}\right\}$$

2 soluciones

entonces

$$P_3[r] = \left(1 - \frac{r_s}{r}\right)\left(\frac{1}{r} - \frac{1}{(r_1)}\right)\left(\frac{1}{r} + \frac{1}{(r_2)}\right)$$

$\therefore$

esta raiz esta conectada al r_s

esta es la raiz del acercamiento maximo

la raiz negativa pues

$$\frac{1}{r_2} = \frac{r_s}{2b^2}\left(1 - \sqrt{1 + \frac{4b^2}{r_s^2}}\right)$$

$$\boxed{\sqrt{1 + \frac{4b^2}{r_s}} > 1}$$

$$r_1 = r_* = \frac{r_s}{2b^2}\left(1 + \sqrt{1 + \frac{4b^2}{r_s^2}}\right)$$

$$\boxed{P_3(r) = \left(1 - \frac{r_s}{r}\right)\left(\frac{1}{r} - \frac{1}{r_1}\right)\left(\frac{1}{r_2} - \frac{1}{r}\right).}$$

#23

$$r_* = \frac{r_s}{2b^2}\left(1+\sqrt{1+\left(\frac{4b^2}{r_s^2}\right)}\right)$$

en eliminando $r \to \infty$ algo asi como ϕ_0

a

$$\Delta\phi = \left(\mathrm{ArcSin}[0] - \frac{r_s}{b} + \cancel{\frac{0}{b}}\right)$$

$$-\left(\frac{r_s}{b} + \mathrm{ArcSin}\left[\frac{1}{r_*}\left(1-\frac{r_s}{2r_*}\right)b\right] - r_s\sqrt{\frac{1}{b^2}-\frac{1}{(r_*)^2}\left(1-\frac{r_s}{2r_*}\right)^2}\right)$$

$$\Delta\phi = 0 - \left(\frac{r_s}{b} - \mathrm{ArcSin}\left[\frac{2b^2}{r_s}\left(1+\sqrt{1+\left(\frac{4b^2}{r_s}\right)}\right)^{-1}\left(1-\frac{\cancel{r_s}\,2b^2}{\cancel{2}r_s}\right.\right.\right.$$

ya que me queda poco tiempo la ecuación exacta tendra que quedarse para un software matematico

es necesario expresar r_* teniendo en cuenta que r_s puede ser una cantidad pequeña. pues $\propto \frac{G}{c^2} = \frac{10^{-11}}{10^{16}} = 10^{-26}$

$$r_* = \frac{r_s}{2b^2}\left(1+\frac{2b}{r_s}\sqrt{\frac{r_s^2}{4b^2}+1}\right) = \frac{r_s}{2b^2} + \frac{1}{b}\sqrt{\frac{r_s^2}{4b^2}+1}$$

$$r_* = \frac{r_s}{2b^2} + \frac{1}{b}\sqrt{\frac{r_s^2}{4b^2}+1}$$; y necesitamos $\frac{1}{r_*} =$

entonces $(1+x)^\alpha = 1+\alpha x + \mathcal{O}(\alpha^2x^2) \Rightarrow \left(\frac{r_s^2}{4b^2}+1\right)^{1/2} = 1 + \frac{r_s^2}{8b^2} + \mathcal{O}\left(\frac{r_s^2}{4b^2}\right)^2$

$$\frac{1}{r_*} = \left(\frac{r_s}{2b^2} + \frac{1}{b} + \frac{r_s^2}{8b^3}\right)^{-1} = \left[\frac{1}{b}\left(\frac{r_s}{2b} + 1 + \frac{r_s^2}{8b^2}\right)\right]^{-1} = b\left(\frac{4br_s+8b^2+r_s^2}{8b^2}\right)^{-1}$$

$$= b\left(\frac{r_s^2+4br_s+4b^2+4b^2}{8b^2}\right)^{-1} = b\left(\frac{8b^2}{(r_s+2b)^2+4b^2}\right) = \frac{1}{r^*}$$

$$\therefore \Delta\phi = 0 - \left(\frac{r_s}{b} + \text{ArcSin}\left[\frac{1}{r_*}\left(1-\frac{r_s}{2r_*}\right)b\right] - r_s\sqrt{\frac{1}{b^2} - \frac{1}{(r_*)^2}\left(1-\frac{r_s}{2r_*}\right)^2}\right. \quad \#㉔$$

$$\frac{1}{r_*} = \frac{(2b)^3}{(r_s+2b)^2+4b^2} = \frac{\cancel{(2b)^2}\,2b}{\left(\frac{r_s}{2b}+1\right)^2\cancel{(2b)^2}+\cancel{(2b)^2}} = \frac{2b}{\left(1+\frac{r_s}{2b}\right)^2+1}$$

$$\left(\frac{1}{r_*}\left(1-\frac{r_s}{2r_*}\right) = \frac{2b}{\left(1+\frac{r_s}{2b}\right)^2+1}\left(1-\frac{b\,r_s}{\left(1+\frac{r_s}{2b}\right)^2+1}\right)$$

$$\Delta\phi = -\frac{r_s}{b} - \text{ArcSin}\left[\frac{2b^2}{\left(1+\frac{r_s}{2b}\right)^2+1}\left(1-\frac{b\,r_s}{\left(1+\frac{r_s}{2b}\right)^2+1}\right)\right]$$

$$+\frac{r_s}{b}\sqrt{1-\frac{b^2}{\cancel{1}}\left[\frac{2b}{\left(1+\frac{r_s}{2b}\right)^2+1}\left(1-\frac{b\,r_s}{\left(1+\frac{r_s}{2b}\right)^2+1}\right)\right]^2} \quad //.$$

—— no parece muy útil
la aproximación de la solución
provista; así que la construiremos nosotros mismos.

$$\frac{\Delta\phi_R}{2} = \int_{r_*}^{\infty}\frac{dr}{r^2\sqrt{\frac{1}{b^2}-\frac{1}{r^2}\left(1-\frac{r_s}{r}\right)}} \qquad (\text{int}*)$$

donde previamente resolvimos para $r^2\sqrt{\frac{1}{b^2}-\frac{1}{r^2}\left(1-\frac{r_s}{r}\right)}$

$$r^2\sqrt{\left(1-\frac{r_s}{r}\right)\left(\frac{1}{r}-\frac{1}{r_*}\right)\left(-\frac{1}{r}+\frac{1}{r_2}\right)}$$

luego en $(\text{int}*)$ $\left(\frac{r_s}{r} \ll 1\right)$

para $\frac{1}{\sqrt{\left(1-\frac{r_s}{r}\right)}} \approx 1-\frac{1}{2}\left(-\frac{r_s}{r}\right) = 1+\frac{r_s}{2r}$

$$\frac{\Delta\phi_R}{2} = \int_{r_*}^{\infty}\frac{dr\left(1+\frac{r_s}{2r}\right)}{r^2\sqrt{\left(\frac{1}{r}-\frac{1}{r_*}\right)\left(-\frac{1}{r}+\frac{1}{r_2}\right)}}$$

#25

realizamos el cambio $u = \frac{1}{r} \rightarrow du = -\frac{dr}{r^2}$

$$\frac{\Delta\Phi_R}{2} = -\int_{1/r_*}^{0} \frac{du\left(1 + \frac{r_s}{2}u\right)}{\sqrt{\left(u - \frac{1}{r_*}\right)\left(-u + \frac{1}{|r_2|}\right)}}$$

$a = \frac{1}{r_*}$ $\quad m = \frac{r_s}{2}$

$b = \frac{1}{|r_2|}$

$$= -\sqrt{ab} - (a-b)\, m\, \mathrm{ArcTan}\sqrt{\frac{a}{b}} - 2\, \mathrm{ArcTan}\sqrt{\frac{a}{b}}$$

$$= -\left\{\sqrt{ab} + \left(\mathrm{Arctan}\sqrt{\frac{a}{b}}\right)\left((a-b)m + 2\right)\right\}.$$

$$= -\left\{\sqrt{\frac{1}{r_*}\frac{1}{|r_2|}} + \left\{\left(\frac{1}{r_*} - \frac{1}{|r_2|}\right)\frac{r_s}{2} + 2\right) \mathrm{ArcTan}\sqrt{\frac{|r_2|}{r_*}}\right\}$$

y aquí se me acabó el tiempo.

Fin //