5.33 Zetillio

Monday, May 8, 2023 9:22 PM

Exercise 5.33 5.31

Consider a spin $\frac{3}{2}$ particle whose Hamiltonian is given by

$$\hat{H} = \frac{\varepsilon_0}{\hbar^2}(\hat{S}_x^2 - \hat{S}_y^2) - \frac{\varepsilon_0}{\hbar^2}\hat{S}_z^2,$$

where ε_0 is a constant having the dimensions of energy.

(a) Find the matrix of the Hamiltonian and diagonalize it to find the energy levels.

(b) Find the eigenvectors and verify that the energy levels are doubly degenerate.

spin $\frac{3}{2} \rightarrow$ $|\xi\rangle = |s, m_s\rangle$

$m_s \in \{-s, -s+1, \ldots, s\}$

$|\xi_1\rangle = |\tfrac{3}{2}, -\tfrac{3}{2}\rangle = |s, m_1\rangle$

$|\xi_2\rangle = |\tfrac{3}{2}, -\tfrac{1}{2}\rangle$

$|\xi_3\rangle = |\tfrac{3}{2}, \tfrac{1}{2}\rangle$

$|\xi_4\rangle = |\tfrac{3}{2}, \tfrac{3}{2}\rangle$

$$\hat{S}_z|\xi_i\rangle = \hbar m_i|\xi_i\rangle \longrightarrow \hat{S}_z^2|\xi_i\rangle = \hbar^2 m_i^2|\xi_i\rangle$$

Luego $S_\pm = S_x + iS_y \rightarrow S_x = \frac{1}{2}(S_+ + S_-)$ $\quad S_y = \frac{1}{2i}(S_+ - S_-)$

$$S_x^2 = \frac{1}{4}(S_+^2 + S_+S_- + S_-S_+ + S_-^2)$$

$$S_y^2 = \frac{-1}{4}(S_+^2 - S_+S_- - S_-S_+ + S_-^2)$$

al diagonalizar $|s, m_i\rangle = |m_i\rangle = |i\rangle$ $\quad \{ S_\pm|m_i\rangle = c_i|m_i \pm 1\rangle$

$$\langle\xi_j|S_x^2|\xi_i\rangle = \langle j|S_x^2|i\rangle =$$

$$= \frac{1}{4}\left\{ \langle m_j | S_+^2 | m_i \rangle + \langle m_j | S_+ S_- | m_i \rangle + \langle m_j | S_- S_+ | m_i \rangle + \langle m_j | S_-^2 | m_i \rangle \right\}$$

$$= \frac{1}{4}\Big\{ C_i C_{i+1} \langle j | i+2 \rangle + \langle j | S_+ S_- | i \rangle + \langle j | S_- S_+ | i \rangle$$

$$+ C_i C_{i-1} \langle j | i-2 \rangle \Big\}$$

$$S_\pm |i\rangle = S_\pm |s, m_i\rangle = \hbar \sqrt{\frac{15}{4} - m_i (m_i \pm 1)}\, \left| \frac{3}{2}, m_i \pm 1 \right\rangle$$

$$S_+ S_- |m_i\rangle = S_+ \hbar \sqrt{\frac{15}{4} - m_i (m_i - 1)}\, \left| \frac{3}{2}, m_i - 1 \right\rangle$$

$$= \hbar \sqrt{\frac{15}{4} - m_i (m_i - 1)}\; \hbar \sqrt{\frac{15}{4} - (m_i - 1)(m_i - 1 + 1)}\, \left| \frac{3}{2}, m_i \right\rangle$$

$$\boxed{\begin{aligned} S_+ S_- |m_i\rangle &= \hbar^2 \left(\frac{15}{4} - m_i (m_i - 1) \right) |m_i\rangle \\ S_- S_+ |m_i\rangle &= \hbar^2 \left(\frac{15}{4} - m_i (m_i + 1) \right) |m_i\rangle \end{aligned}}$$

$$\therefore\; S_+^2 |m_i\rangle = \hbar^2 \sqrt{\frac{15}{4} - m_i (m_i + 1)} \sqrt{\frac{15}{4} - (m_i + 1)(m_i + 2)}\, | m_i + 2 \rangle = M_+ | m_i + 2 \rangle$$

$$S_-^2 |m_i\rangle = \hbar^2 \sqrt{\left(\frac{15}{4} - m_i (m_i - 1) \right) \left(\frac{15}{4} - (m_i - 1)(m_i - 2) \right)}\, | m_i - 2 \rangle = M_- | m_i - 2 \rangle$$

Con esto comencemos a ver como quedan los componentes de la matriz del Hamiltoniano; ya sabemos que Sz^2 fucionara bien, pero aquí:

$$\therefore\; \langle j | S_x^2 - S_y^2 | i \rangle = \frac{1}{4} \langle j | \left(S_+^2 + \cancel{S_+ S_-} + \cancel{S_- S_+} + S_-^2 \right) + \left(S_+^2 - \cancel{S_+ S_-} - \cancel{S_- S_+} + S_-^2 \right) | i \rangle$$

$$= \frac{1}{4} \langle j | 2 S_+^2 + 2 S_-^2 | i \rangle = \frac{1}{2} \langle j | S_+^2 + S_-^2 | i \rangle$$

$$= \frac{1}{2}\left\{ \langle j | S_+^2 | i \rangle + \langle j | S_-^2 | i \rangle \right\} = \frac{1}{2}\left\{ \langle j | M_+ | i+2 \rangle + \langle j | M_- | i-2 \rangle \right\}$$

$$= \frac{1}{2}\left(M_+ \delta_j^{i+2} + M_- \delta_j^{i-2} \right)$$

se generan elementos fuera de la diagonal

$$= \frac{1}{2} \begin{array}{c|cccc|c}
 & j=1 & 2 & 3 & j=4 & \\
\hline
 & 0 & 0 & \langle 3 | S_+^2 | 1 \rangle & 0 & i=1 \\
 & 0 & 0 & 0 & \langle 4 | S_+^2 | 2 \rangle & i=2 \\
 & \langle 1 | S_-^2 | 3 \rangle & 0 & 0 & 0 & 3 \\
 & 0 & \langle 2 | S_-^2 | 4 \rangle & 0 & 0 & i=4
\end{array}$$

$\langle j | i \rangle$

columna

fila

m_1

$m_i = \left\{ -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2} \right\}$

$$S_+^2 | m_i \rangle = \hbar^2 \sqrt{\frac{15}{4} - m_i(m_i+1)} \sqrt{\frac{15}{4} - (m_i+1)(m_i+2)}\, | m_i + 2 \rangle = M_+ | m_i + 2 \rangle$$

$$\langle 3 | S_+^2 | 1 \rangle = \langle 3 | \hbar^2 \sqrt{\left(\frac{15}{4} - \left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)\right)\left(\frac{15}{4} - \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\right)}\, | 3 \rangle$$

$$= \hbar^2 \sqrt{(3)(4)} = \hbar^2 2\sqrt{3}$$

$$\langle 4 | S_+^2 | 2 \rangle = \langle 4 | \hbar^2 \sqrt{\left[\frac{15}{4} - \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\right]\left[\frac{15}{4} - \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\right]}$$

$$= \hbar^2 \sqrt{(4)(3)} = \hbar^2 2\sqrt{3}$$

$$S_-^2 | m_i \rangle = \hbar^2 \sqrt{\left(\frac{15}{4} - m_i(m_i-1)\right)\left(\frac{15}{4} - (m_i-1)(m_i-2)\right)}\, | m_i - 2 \rangle = M_- | m_i - 2 \rangle$$

$$\langle 1|S_-^2|3\rangle = \hbar^2\sqrt{\left[\frac{15}{4}-\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\right]\left[\frac{15}{4}-\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\right]} = \hbar^2 2\sqrt{3}$$

$$\langle 2|S_-|4\rangle = \hbar^2\sqrt{\left[\frac{15}{4}-\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\right]\left[\frac{15}{4}-\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\right]} = \hbar^2 2\sqrt{3}$$

matriz

$$\frac{\varepsilon_0}{\hbar^2}(\hat{S}_x^2-\hat{S}_y^2) \rightarrow \frac{\varepsilon_0}{\hbar^2}\frac{1}{2}\begin{pmatrix} 0 & 0 & \hbar^2 2\sqrt{3} & 0 \\ 0 & 0 & 0 & \hbar^2 2\sqrt{3} \\ \hbar^2 2\sqrt{3} & 0 & 0 & 0 \\ 0 & \hbar^2 2\sqrt{3} & 0 & 0 \end{pmatrix} = \varepsilon_0\begin{pmatrix} 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 \end{pmatrix}$$

Luego las diagonales.

$$\langle m_i|-S_z^2|m_i\rangle = -\hbar^2 m_i^2 = -\hbar^2\left\{\frac{9}{4}, \frac{1}{4}, \frac{1}{4}, \frac{9}{4}\right\}$$

el Hamilton como matriz

$\therefore$

$$\hat{H} = \varepsilon_0\begin{pmatrix} -9/4 & 0 & \sqrt{3} & 0 \\ 0 & -1/4 & 0 & \sqrt{3} \\ \sqrt{3} & 0 & -1/4 & 0 \\ 0 & \sqrt{3} & 0 & -9/4 \end{pmatrix}$$

$$\hat{H} = \mathcal{E}_0 \begin{pmatrix} -9/4 & 0 & \sqrt{3} & 0 \\ 0 & -1/4 & 0 & \sqrt{3} \\ \sqrt{3} & 0 & -1/4 & 0 \\ 0 & \sqrt{3} & 0 & -9/4 \end{pmatrix}$$