Geodésicas en el espacio-tiempo de Kerr.

Debemos hacer una distinción entre órbitas directa y retrogradas, cuya dirección de movimiento coincide ó no con la dirección de rotación de la fuente (Planeta, estrella, agújero negro,)

Vamos a trabajar en el plano ecuatorial $\theta = \pi/2 \rightarrow \dot{\theta} = 0$. EL Lagrangiano

$$2\mathcal{L} = \left(1-\frac{2M}{r}\right)\dot{t}^2 + \frac{4aM}{r}\dot{t}\dot{\phi} - \frac{r^2}{\Delta}\dot{r}^2 - \left(r^2+a^2+\frac{2a^2M}{r}\right)\dot{\phi}^2$$

$$\Delta = r^2 + a^2 - 2Mr$$

notar que la signatura usada es $sig(g) = -2$

Los momenta

$$P_t = \left(1-\frac{2M}{r}\right)\dot{t} + \frac{2aM}{r}\dot{\phi} = E : cte \quad (*)$$

$$P_\phi = \frac{2aM}{r}\dot{t} - \left(r^2+a^2+\frac{2a^2M}{r}\right)\dot{\phi} = -L : cte \quad (**)$$

$$p_r = -\frac{r^2}{\Delta}\dot{r}$$

Veamos el Hamiltoniano

$$\mathcal{H} = p_t\dot{t} + p_\phi\dot{\phi} + p_r\dot{r} - \mathcal{L}$$

$$\boxed{\mathcal{H} = \frac{1}{2}\left(1-\frac{2M}{r}\right)\dot{t}^2 + \frac{2aM}{r}\dot{t}\dot{\phi} - \frac{r^2}{2\Delta}\dot{r}^2 - \frac{1}{2}\left(r^2+a^2+\frac{2a^2M}{r}\right)\dot{\phi}^2}$$

$\mathcal{H}$ no depende explícitamente del tiempo, luego

$$2\mathcal{H} = \left[\left(1-\frac{2M}{r}\right)\dot{t} + \frac{2aM}{r}\dot{\phi}\right]\dot{t} - \left[\left(r^2+a^2+\frac{2a^2M}{r}\right)\dot{\phi} - \frac{2aM}{r}\dot{t}\right]\dot{\phi}$$

$$+\frac{r^2}{\Delta}\dot{r}^2.$$

$$\boxed{2\mathcal{H} = E\dot{t} - L\dot{\phi} - \frac{r^2}{\Delta}\dot{r}^2 = \delta_1 : \text{cte.}}$$

Sin pérdida de generalidad

$$\delta_1 = \begin{cases} 1 & \text{partículas masivas} \\ 0 & \text{partículas sin masa.} \end{cases}$$

<u>Nota</u>: Haciendo $\delta = 1$ para geodésicas tipo-tiempo, requiere que E sea interpretada como energía específica ó energía por unidad de masa.

Resolvamos (*) y (**) para $\dot{t}$ y $\dot{\phi}$

$$\left(1-\frac{2M}{r}\right)\dot{t}+\frac{2aM}{r}\dot{\phi}=E \qquad \Big| \ \text{(i)}\ \frac{2aM}{r} \qquad \Big| \ \text{(ii)}\ \left(r^2+a^2+\frac{2a^2M}{r}\right)$$

$$\frac{2aM}{r}\dot{t}-\left(r^2+a^2+\frac{2a^2M}{r}\right)\dot{\phi}=L \qquad \Big| \ \left(1-\frac{2M}{r}\right) \qquad \Big| \ \frac{2aM}{r}$$

(i) $$\left(1-\frac{2M}{r}\right)\frac{2aM}{r}\dot{t}+\frac{4a^2M^2}{r^2}\dot{\phi}=\frac{2aM}{r}E$$

$$\left(1-\frac{2M}{r}\right)\frac{2aM}{r}\dot{t}-\left(1-\frac{2M}{r}\right)\left(r^2+a^2+\frac{2a^2M}{r}\right)\dot{\phi}=\left(1-\frac{2M}{r}\right)L$$

restando:

$$\left[\cancel{\frac{4a^2M^2}{r^2}}+r^2+a^2+\cancel{\frac{2a^2M}{r}}-2Mr-\cancel{\frac{2a^2M}{r}}-\cancel{\frac{4a^2M^2}{r^2}}\right]\dot{\phi}=$$

$$\frac{2aM}{r}E+\left(1-\frac{2M}{r}\right)L$$

$$\rightarrow \boxed{\dot{\phi}=\frac{1}{\Delta}\left[\frac{2aM}{r}E+\left(1-\frac{2M}{r}\right)L\right]}$$

(ii) $$\overset{=(\cdot)}{\left(1-\frac{2M}{r}\right)\left(r^2+a^2+\frac{2a^2M}{r}\right)}\dot{t}+\frac{2aM}{r}(\cdot)\dot{\phi}=E\cdot(\cdot)$$

$$\frac{4a^2M^2}{r^2}\dot{t}-\frac{2aM}{r}(\cdot)\dot{\phi}=-\frac{2aM}{r}L$$

$$\boxed{\dot{t}=\frac{1}{\Delta}\left[\left(r^2+a^2+\frac{2a^2M}{r}\right)E-\frac{2aM}{r}L\right]}$$

$$E\dot{t} - L\dot{\phi} - \frac{r^2}{\Delta}\dot{r}^2 = \delta_1$$

$$(\cdot)\; E^2 - \frac{2aM}{r}LE - \frac{2aM}{r}LE - \left(1-\frac{2M}{r}\right)L^2 - \frac{r^2}{\cancel{\Delta}}\dot{r}^2 = \delta_1\cdot\Delta$$

$$r^2E^2 + a^2E^2 + \frac{2a^2M}{r}E^2 - \frac{4aM}{r}LE - L^2 + \frac{2M}{r}L^2 - r^2\dot{r}^2 = \delta_1\Delta$$

$$\frac{2M}{r}\left(L^2 - 2aE + a^2E^2\right) - \left(L^2 - a^2E^2\right) + r^2E^2 - r^2\dot{r}^2 = \delta_1\Delta$$

$$r^2E^2 + \frac{2M}{r}(L-aE)^2 - (L^2-a^2E^2) - \delta_1\Delta = r^2\dot{r}^2$$

$$\therefore\; \boxed{\dot{r}^2 = E^2 + \frac{2M}{r^3}(L-aE)^2 - \frac{1}{r^2}(L^2-a^2E^2) - \frac{1}{r^2}\delta_1\cdot\Delta}$$

Geodésicas nulas: $\delta_1 = 0$

$$\boxed{\dot{r}^2 = E^2 + \frac{2M}{r^3}(L-aE)^2 - \frac{1}{r^2}(L^2-a^2E^2)}$$

observar que, si $L = aE$:

i) $\dot{r} = \pm\sqrt{E^2} \rightarrow \boxed{\frac{dr}{d\gamma} = \pm E}$ ← Equivalente al mov. radial en schwarzschild.

ii) $$\dot{\phi} = \frac{1}{\Delta}\left[\frac{2aM}{r}E + \left(1-\frac{2M}{r}\right)\cdot aE\right] =$$

$$= \frac{1}{\Delta}\left[\cancel{\frac{2aM}{r}E} + aE - \cancel{\frac{2aM}{r}E}\right]$$

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$$\dot{\phi} = \frac{1}{\Delta} \cdot a\bar{E} \rightarrow \boxed{\frac{d\phi}{d\tau} = \frac{a\bar{E}}{\Delta}}$$

iii) $\dot{t} = \frac{1}{\Delta}\left[(r^2+a^2)E + \cancel{\frac{2a^2M}{r}E} - \cancel{\frac{2a^2M}{r}E} \right]$

$$\dot{t} = \frac{1}{\Delta} \cdot (r^2+a^2)\bar{E} \rightarrow \boxed{\frac{dt}{d\tau} = \frac{r^2+a^2}{\Delta} \cdot E}$$

Luego, $\frac{dt}{d\tau} = \frac{dr}{d\tau}\frac{dt}{dr} = \pm E \frac{dt}{dr} = \frac{r^2+a^2}{\Delta} \cdot E$.

$$\rightarrow \boxed{\frac{dt}{dr} = \pm \frac{r^2+a^2}{\Delta}}$$

y también $\frac{d\phi}{d\tau} = \frac{dr}{d\tau} \cdot \frac{d\phi}{dr} = \pm E \frac{d\phi}{dr} = \frac{aE}{\Delta}$

$$\rightarrow \boxed{\frac{d\phi}{dr} = \pm \frac{a}{\Delta}}$$