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QUANTUM PHYSICS

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1-1 INTRODUCTION

At a meeting of the German Physical Society on Dec. 14, 1900, Max Planck read his paper, "On the Theory of the Energy Distribution Law of the Normal Spectrum." This paper, which first attracted little attention, was the start of a revolution in physics. The date of its presentation is considered to be the birthday of quantum physics, although it was not until a quarter of a century later that modern quantum mechanics, the basis of our present understanding, was developed by Schroedinger and others. Many paths converged on this understanding, each showing another aspect of the breakdown of classical physics. In this and the following three chapters we shall examine the major milestones, of what is now called the *old quantum theory*, that led to modern quantum mechanics. The experimental phenomena which we shall discuss in connection with the old quantum theory span all the disciplines of classical physics: mechanics, thermodynamics, statistical mechanics, and electromagnetism. Their repeated contradiction of classical laws, and the resolution of these conflicts on the basis of quantum ideas, will show us the need for quantum mechanics. And our study of the old quantum theory will allow us to more easily obtain a deeper understanding of quantum mechanics when we begin to consider it in the fifth chapter.

As is true of relativity (which is treated briefly in Appendix A), quantum physics represents a generalization of classical physics that includes the classical laws as special cases. Just as relativity extends the range of application of physical laws to the region of high velocities, so quantum physics extends that range to the region of small dimensions. And just as a universal constant of fundamental significance, the velocity of light c, characterizes relativity, so a universal constant of fundamental significance, now called Planck's constant h, characterizes quantum physics. It was while trying to explain the observed properties of thermal radiation that Planck introduced this constant in his 1900 paper. Let us now begin to examine thermal radiation ourselves. We shall be led thereby to Planck's constant and the extremely significant related quantum concept of the discreteness of energy. We shall also find that thermal radiation has considerable importance and contemporary relevance in its own right. For instance, the phenomenon has recently helped astrophysicists decide among competing theories of the origin of the universe. Another example is given by the rapidly developing technology of solar heating, which depends on the thermal radiation received by the earth from the sun.

1-2 THERMAL RADIATION

The radiation emitted by a body as a result of its temperature is called *thermal radiation.* All bodies emit such radiation to their surroundings and absorb such radiation from them. If a body is at first hotter than its surroundings, it will cool off because its rate of emitting energy exceeds its rate of absorbing energy. When thermal equilibrium is reached the rates of emission and absorption are equal.

Matter in a condensed state (i.e., solid or liquid) emits a continuous spectrum of radiation. The details of the spectrum are almost independent of the particular material of which a body is composed, but they depend strongly on the temperature. At ordinary temperatures most bodies are visible to us not by their emitted light but by the light they reflect. If no light shines on them we cannot see them. At very high temperatures, however, bodies are self-luminous. We can see them glow in a darkened room; but even at temperatures as high as several thousand degrees Kelvin well over 90% of the emitted thermal radiation is invisible to us, being in the infrared part of the electromagnetic spectrum. Therefore, self-luminous bodies are quite hot.

Consider, for example, heating an iron poker to higher and higher temperatures in a fire, periodically withdrawing the poker from the fire long enough to observe its properties. When the poker is still at a relatively low temperature it radiates heat, but it is not visibly hot. With increasing temperature the amount of radiation that the

poker emits increases very rapidly and visible effects are noted. The poker assumes a dull red color, then a bright red color, and, at very high temperatures, an intense blue-white color. That is, with increasing temperature the body emits more thermal radiation and the frequency of the most intense radiation becomes higher.

The relation between the temperature of a body and the frequency spectrum of the emitted radiation is used in a device called an optical pyrometer. This is essentially a rudimentary spectrometer that allows the operator to estimate the temperature of a hot body, such as a star, by observing the color, or frequency composition, of the thermal radiation that it emits. There is a continuous spectrum of radiation emitted, the eye seeing chiefly the color corresponding to the most intense emission in the visible region. Familiar examples of objects which emit visible radiation include hot coals, lamp filaments, and the sun.

Generally speaking, the detailed form of the spectrum of the thermal radiation emitted by a hot body depends somewhat upon the composition of the body. However, experiment shows that there is one class of hot bodies that emits thermal spectra of a universal character. These are called *blackbodies*, that is, bodies that have surfaces which absorb all the thermal radiation incident upon them. The name is appropriate because such bodies do not reflect light and appear black when their temperatures are low enough that they are not self-luminous. One example of a (nearly) blackbody would be *any* object coated with a diffuse layer of black pigment, such as lamp black or bismuth black. Another, quite different, example will be described shortly. Independent of the details of their composition, it is found that *all* blackbodies at the same temperature emit thermal radiation with the same spectrum. This general fact can be understood on the basis of classical arguments involving thermodynamic equilibrium. The specific form of the spectrum, however, cannot be obtained from thermodynamic arguments alone. The universal properties of the radiation emitted by blackbodies make them of particular theoretical interest and physicists sought to explain the specific features of their spectrum.

The spectral distribution of blackbody radiation is specified by the quantity $R_T(\nu)$, called the *spectral radiancy*, which is defined so that $R_T(\nu)\,d\nu$ is equal to the energy emitted per unit time in radiation of frequency in the interval ν to $\nu + d\nu$ from a unit area of the surface at absolute temperature T. The earliest accurate measurements of this quantity were made by Lummer and Pringsheim in 1899. They used an instrument essentially similar to the prism spectrometers used in measuring optical spectra, except that special materials were required for the lenses, prisms, etc., so that they would be transparent to the relatively low frequency thermal radiation. The experimentally observed dependence of $R_T(\nu)$ on ν and T is shown in Figure 1-1.

Distribution functions, of which spectral radiancy is an example, are very common in physics. For example, the Maxwellian speed distribution function (which looks rather like one of the curves in Figure 1-1) tells us how the molecules in a gas at a fixed pressure and temperature are distributed according to their speed. Another distribution function that the student has probably already seen is the one (which has the form of a decreasing exponential) specifying the times of decay of radioactive nuclei in a sample containing nuclei of a given species, and he has certainly seen a distribution function for the grades received on a physics exam.

The spectral radiancy distribution function of Figure 1-1 for a blackbody of a given area and a particular temperature, say 1000°K, shows us that: (1) there is very little power radiated in a frequency interval of fixed size $d\nu$ if that interval is at a frequency ν which is very small compared to 10^{14} Hz. The power is zero for ν equal to zero. (2) The power radiated in the interval $d\nu$ increases rapidly as ν increases from very small values. (3) It maximizes for a value of $\nu \simeq 1.1 \times 10^{14}$ Hz. That is, the radiated power is most intense at that frequency. (4) Above $\simeq 1.1 \times 10^{14}$ Hz the radiated power drops slowly but continuously as ν increases. It is zero again when ν approaches infinitely large values.

The two distribution functions for the higher values of temperature, 1500°K and 2000°K, displayed in the figure show us that (5) the frequency at which the radiated power is most

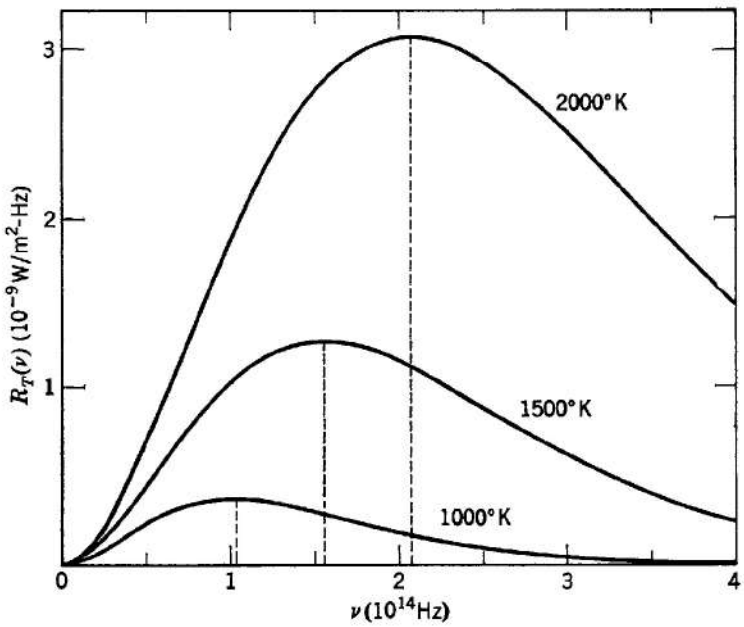

Figure 1-1 The spectral radiancy of a blackbody radiator as a function of the frequency of radiation, shown for temperatures of the radiator of 1000°K, 1500°K, and 2000°K. Note that the frequency at which the maximum radiancy occurs (dashed line) increases linearly with increasing temperature, and that the total power emitted per square meter of the radiator (area under curve) increases very rapidly with temperature.

intense increases with increasing temperature. Inspection will verify that this frequency increases linearly with temperature. (6) The total power radiated in all frequencies increases with increasing temperature, and it does so more rapidly than linearly. The total power radiated at a particular temperature is given simply by the area under the curve for that temperature, $\int_0^\infty R_T(\nu)\,d\nu$, since $R_T(\nu)\,d\nu$ is the power radiated in the frequency interval from ν to $\nu + d\nu$.

The integral of the spectral radiancy $R_T(\nu)$ over all ν is the total energy emitted per unit time per unit area from a blackbody at temperature T. It is called the *radiancy* R_T. That is

$$R_T = \int_0^\infty R_T(\nu)\,d\nu \tag{1-1}$$

As we have seen in the preceding discussion of Figure 1-1, R_T increases rapidly with increasing temperature. In fact, this result is called *Stefan's law*, and it was first stated in 1879 in the form of an empirical equation

$$R_T = \sigma T^4 \tag{1-2}$$

where

$$\sigma = 5.67 \times 10^{-8}\ \text{W/m}^2\text{-°K}^4$$

is called the *Stefan-Boltzmann constant*. Figure 1-1 also shows us that the spectrum shifts toward higher frequencies as T increases. This result is called *Wien's displacement law*

$$\nu_{\max} \propto T \tag{1-3a}$$

where $\nu_{\max}$ is the frequency ν at which $R_T(\nu)$ has its maximum value for a particular T. As T increases, $\nu_{\max}$ is displaced toward higher frequencies. All these results are in agreement with the familiar experiences discussed earlier, namely that the amount of thermal radiation emitted increases rapidly (the poker radiates much more heat energy at higher temperatures), and the principal frequency of the radiation becomes higher (the poker changes color from dull red to blue-white), with increasing temperature.

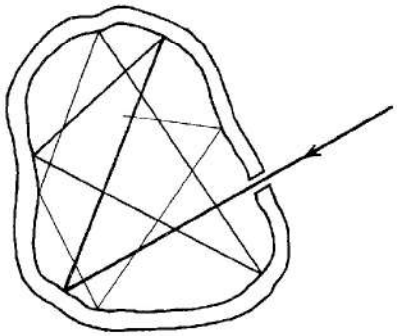

Figure 1-2 A cavity in a body connected by a small hole to the outside. Radiation incident on the hole is completely absorbed after successive reflections on the inner surface of the cavity. The hole absorbs like a blackbody. In the reverse process, in which radiation leaving the hole is built up of contributions emitted from the inner surface, the hole emits like a blackbody.

Another example of a blackbody, which we shall see to be particularly important, can be found by considering an object containing a cavity which is connected to the outside by a small hole, as in Figure 1-2. Radiation incident upon the hole from the outside enters the cavity and is reflected back and forth by the walls of the cavity, eventually being absorbed on these walls. If the area of the hole is very small compared to the area of the inner surface of the cavity, a negligible amount of the incident radiation will be reflected back through the hole. Essentially all the radiation incident upon the hole is absorbed; therefore, the *hole* must have the properties of the surface of a blackbody. Most blackbodies used in laboratory experiments are constructed along these lines.

Now assume that the walls of the cavity are uniformly heated to a temperature T. Then the walls will emit thermal radiation which will fill the cavity. The small fraction of this radiation incident from the inside upon the hole will pass through the hole. Thus the hole will act as an emitter of thermal radiation. Since the hole must have the properties of the surface of a blackbody, the radiation emitted by the hole must have a blackbody spectrum; but since the hole is merely sampling the thermal radiation present inside the cavity, it is clear that the radiation in the cavity must also have a blackbody spectrum. In fact, it will have a blackbody spectrum characteristic of the temperature T on the walls, since this is the only temperature defined for the system. The spectrum emitted by the hole in the cavity is specified in terms of the energy flux $R_T(\nu)$. It is more useful, however, to specify the spectrum of radiation inside the cavity, called *cavity radiation*, in terms of an *energy density*, $\rho_T(\nu)$, which is defined as the energy contained in a unit volume of the cavity at temperature T in the frequency interval ν to $\nu + d\nu$. It is evident that these quantities are proportional to one another; that is

$$\rho_T(\nu) \propto R_T(\nu) \tag{1-4}$$

Hence, the radiation inside a cavity whose walls are at temperature T has the same character as the radiation emitted by the surface of a blackbody at temperature T. It is convenient experimentally to produce a blackbody spectrum by means of a cavity in a heated body with a hole to the outside, and it is convenient in theoretical work to study blackbody radiation by analyzing the cavity radiation because it is possible to apply very general arguments to predict the properties of cavity radiation.

Example 1-1. (a) Since $\lambda\nu = c$, the constant velocity of light, Wien's displacement law (1-3a) can also be put in the form

$$\lambda_{\max} T = \text{const} \tag{1-3b}$$

where $\lambda_{\max}$ is the wavelength at which the spectral radiancy has its maximum value for a particular temperature T. The experimentally determined value of Wien's constant is 2.898×10^{-3} m-°K. If we assume that stellar surfaces behave like blackbodies we can get a good estimate of their temperature by measuring $\lambda_{\max}$. For the sun $\lambda_{\max} = 5100$ Å, whereas for the North Star $\lambda_{\max} = 3500$ Å. Find the surface temperature of these stars. (One *angstrom* = 1 Å $= 10^{-10}$ m.)

▶For the sun, $T = 2.898 \times 10^{-3}$ m-°K/5100×10^{-10} m = 5700°K. For the North Star, $T = 2.898 \times 10^{-3}$ m-°K/3500×10^{-10} m = 8300°K.

At 5700°K the sun's surface is near the temperature at which the greatest part of its radiation lies within the visible region of the spectrum. This suggests that over the ages of human evolution our eyes have adapted to the sun to become most sensitive to those wavelengths which it radiates most intensely. ◀

(b) Using Stefan's law, (1-2), and the temperatures just obtained, determine the power radiated from 1 cm^2 of stellar surface.

▶For the sun

$$R_T = \sigma T^4 = 5.67 \times 10^{-8}\ \text{W/m}^2\text{-°K}^4 \times (5700\text{°K})^4$$
$$= 5.90 \times 10^7\ \text{W/m}^2 \simeq 6000\ \text{W/cm}^2$$

For the North Star

$$R_T = \sigma T^4 = 5.67 \times 10^{-8}\ \text{W/m}^2\text{-°K}^4 \times (8300\text{°K})^4$$
$$= 2.71 \times 10^8\ \text{W/m}^2 \simeq 27{,}000\ \text{W/cm}^2$$ ◀

Example 1-2. Assume we have two small opaque bodies a large distance from one another supported by fine threads in a large evacuated enclosure whose walls are opaque and kept at a constant temperature. In such a case the bodies and walls can exchange energy only by means of radiation. Let e represent the rate of emission of radiant energy by a body and let a represent the rate of absorption of radiant energy by a body. Show that at equilibrium

$$\frac{e_1}{a_1} = \frac{e_2}{a_2} = 1 \tag{1-5}$$

This relation, (1-5), is known as *Kirchhoff's law for radiation.*

▶The equilibrium state is one of constant temperature throughout the enclosed system, and in that state the emission rate necessarily equals the absorption rate for each body. Hence

$$e_1 = a_1 \qquad \text{and} \qquad e_2 = a_2$$

Therefore

$$\frac{e_1}{a_1} = 1 = \frac{e_2}{a_2}$$

If one body, say body 2, is a blackbody, then $a_2 > a_1$ because a blackbody is a better absorber than a non-blackbody. Hence, it follows from (1-5) that $e_2 > e_1$. The observed fact that good absorbers are also good emitters is thus predicted by Kirchhoff's law. ◀

1-3 CLASSICAL THEORY OF CAVITY RADIATION

Shortly after the turn of the present century, Rayleigh, and also Jeans, made a calculation of the energy density of cavity (or blackbody) radiation that points up a serious conflict between classical physics and experimental results. This calculation is similar to calculations that arise in considering many other phenomena (e.g., specific heats of solids) to be treated later. We present the details here, but as an aid in guiding us through the calculations we first outline their general procedure.

Consider a cavity with metallic walls heated uniformly to temperature T. The walls emit electromagnetic radiation in the thermal range of frequencies. We know that this happens, basically, because of the accelerated motions of the electrons in the metallic walls that arise from thermal agitation (see Appendix B). However, it is not necessary to study the behavior of the electrons in the walls of the cavity in detail. Instead, attention is focused on the behavior of the electromagnetic waves in the interior of the cavity. Rayleigh and Jeans proceeded as follows. First, classical electromagnetic theory is used to show that the radiation inside the cavity must exist in the form of standing waves with nodes at the metallic surfaces. By using geometrical arguments, a count is made of the number of such standing waves in the frequency interval ν to $\nu + d\nu$, in order to determine how the number depends on ν. Then a

result of classical kinetic theory is used to calculate the average total energy of these waves when the system is in thermal equilibrium. The average total energy depends, in the classical theory, only on the temperature T. The number of standing waves in the frequency interval times the average energy of the waves, divided by the volume of the cavity, gives the average energy content per unit volume in the frequency interval ν to $\nu + d\nu$. This is the required quantity, the energy density $\rho_T(\nu)$. Let us now do it ourselves.

We assume for simplicity that the metallic-walled cavity filled with electromagnetic radiation is in the form of a cube of edge length a, as shown in Figure 1-3. Then the radiation reflecting back and forth between the walls can be analyzed into three components along the three mutually perpendicular directions defined by the edges of the cavity. Since the opposing walls are parallel to each other, the three components of the radiation do not mix, and we may treat them separately. Consider first the x component and the metallic wall at $x = 0$. All the radiation of this component which is incident upon the wall is reflected by it, and the incident and reflected waves combine to form a standing wave. Now, since electromagnetic radiation is a transverse vibration with the electric field vector **E** perpendicular to the propagation direction, and since the propagation direction for this component is perpendicular to the wall in question, its electric field vector **E** is parallel to the wall. A metallic wall cannot, however, support an electric field parallel to the surface, since charges can always flow in such a way as to neutralize the electric field. Therefore, **E** for this component must always be zero at the wall. That is, the standing wave associated with the x-component of the radiation must have a node (zero amplitude) at $x = 0$. The standing wave must also have a node at $x = a$ because there can be no parallel electric field in the corresponding wall. Furthermore, similar conditions apply to the other two components; the standing wave associated with the y component must have nodes at $y = 0$ and $y = a$, and the standing wave associated with the z component must have nodes at $z = 0$ and $z = a$. These conditions put a limitation on the possible wavelengths, and therefore on the possible frequencies, of the electromagnetic radiation in the cavity.

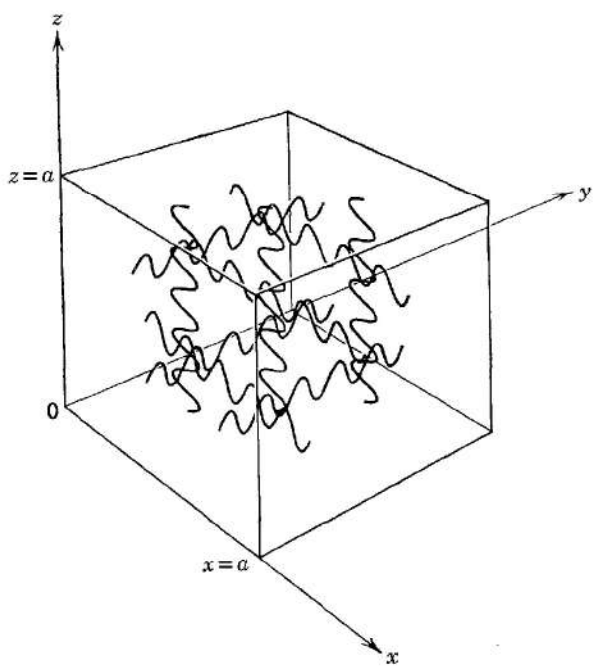

Figure 1-3 A metallic walled cubical cavity filled with electromagnetic radiation, showing three noninterfering components of that radiation bouncing back and forth between the walls and forming standing waves with nodes at each wall.

Now we shall consider the question of counting the number of standing waves with nodes on the surfaces of the cavity, whose wavelengths lie in the interval λ to $\lambda + d\lambda$ corresponding to the frequency interval ν to $\nu + d\nu$. To focus attention on the ideas involved in the calculation, we shall first treat the x component alone; that is, we shall consider the simplified, but artificial, case of a "one-dimensional cavity" of length a. After we have worked through this case, we shall see that the procedure for generalizing to a real three-dimensional cavity is obvious.

The electric field for one-dimensional electromagnetic standing waves can be described mathematically by the function

$$E(x,t) = E_0 \sin(2\pi x/\lambda) \sin(2\pi\nu t) \tag{1-6}$$

where λ is the wavelength of the wave, ν is its frequency, and E_0 is its maximum amplitude. The first two quantities are related by the equation

$$\nu = c/\lambda \tag{1-7}$$

where c is the propagation velocity of electromagnetic waves. Equation (1-6) represents a wave whose amplitude has the sinusoidal space variation $\sin(2\pi x/\lambda)$ and which is oscillating in time sinusoidally with frequency ν like a simple harmonic oscillator. Since the amplitude is obviously zero, at all times t, for positions satisfying the relation

$$2x/\lambda = 0, 1, 2, 3, \ldots \tag{1-8}$$

the wave has fixed nodes; that is, it is a standing wave. In order to satisfy the requirement that the waves have nodes at both ends of the one-dimensional cavity, we choose the origin of the x axis to be at one end of the cavity ($x = 0$) and then require that at the other end ($x = a$)

$$2x/\lambda = n \qquad \text{for } x = a \tag{1-9}$$

where

$$n = 1, 2, 3, 4, \ldots$$

This condition determines a set of allowed values of the wavelength λ. For these allowed values, the amplitude patterns of the standing waves have the appearance shown in Figure 1-4. These patterns may be recognized as the standing wave patterns for vibrations of a string fixed at both ends, a real physical system which also satisfies (1-6). In our case the patterns represent electromagnetic standing waves.

It is convenient to continue the discussion in terms of the allowed frequencies instead of the allowed wavelengths. These frequencies are $\nu = c/\lambda$, where $2a/\lambda = n$. That is

$$\nu = cn/2a \qquad n = 1, 2, 3, 4, \ldots \tag{1-10}$$

We can represent these allowed values of frequency in terms of a diagram consisting of an axis on which we plot a point at every integral value of n. On such a diagram, the value of the allowed frequency ν corresponding to a particular value of n is, by (1-10), equal to $c/2a$ times the distance d from the origin to the appropriate point, or the distance d is $2a/c$ times the frequency ν. These relations are shown in Figure 1-5. Such a diagram is useful in calculating the number of allowed values in frequency

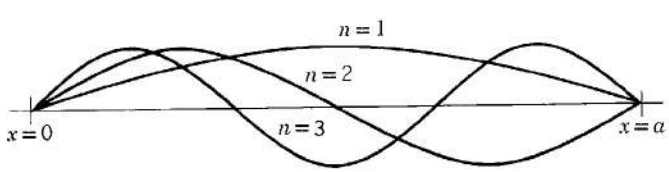

Figure 1-4 The amplitude patterns of standing waves in a one-dimensional cavity with walls at $x = 0$ and $x = a$, for the first three values of the index n.

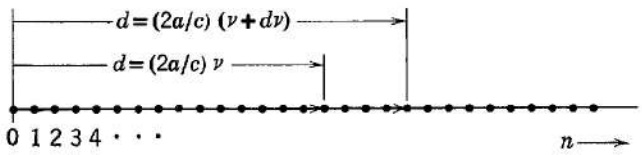

Figure 1-5 The allowed values of the index n, which determines the allowed values of the frequency, in a one-dimensional cavity of length a.

range ν to $\nu + d\nu$, which we call $N(\nu)\,d\nu$. To evaluate this quantity we simply count the number of points on the n axis which fall between two limits which are constructed so as to correspond to the frequencies ν and $\nu + d\nu$, respectively. Since the points are distributed uniformly along the n axis, it is apparent that the number of points falling between the two limits will be proportional to $d\nu$ but will not depend on ν. In fact, it is easy to see that $N(\nu)\,d\nu = (2a/c)\,d\nu$. However, we must multiply this by an additional factor of 2 since, for each of the allowed frequencies, there are actually two independent waves corresponding to the two possible states of polarization of electromagnetic waves. Thus we have

$$N(\nu)\,d\nu = \frac{4a}{c}\,d\nu \tag{1-11}$$

This completes the calculation of the number of allowed standing waves for the artificial case of a one-dimensional cavity.

The above calculation makes apparent the procedures for extending the calculation to the real case of a three-dimensional cavity. This extension is indicated in Figure 1-6. Here the set of points uniformly distributed at integral values along a single n axis is replaced by a uniform three-dimensional array of points whose three coordinates occur at integral values along each of three mutually perpendicular n axes. *Each point* of the array corresponds to a particular allowed three-dimensional

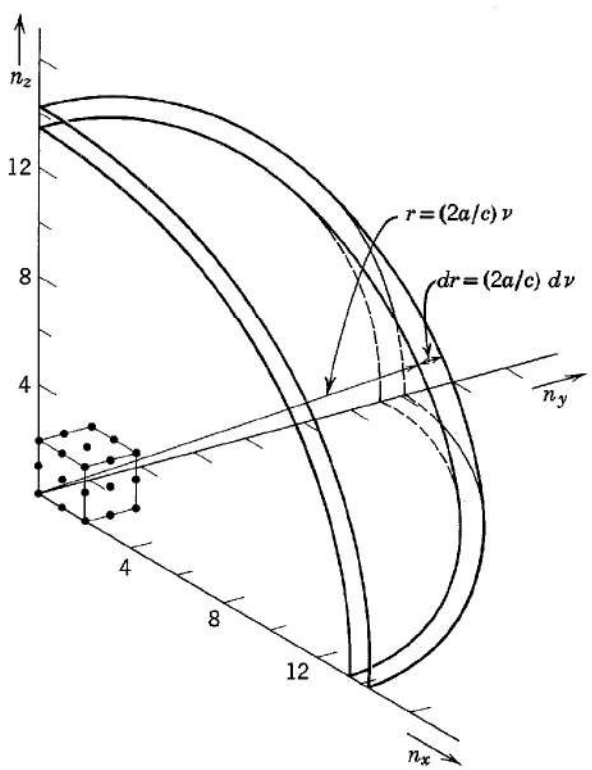

Figure 1-6 The allowed frequencies in a three-dimensional cavity in the form of a cube of edge length a are determined by three indices n_x, n_y, n_z, which can each assume only integral values. For clarity, only a few of the very many points corresponding to sets of these indices are shown.

standing wave. The integral values of n_x, n_y, and n_z specified by each point give the number of nodes of the x, y, and z components, respectively, of the three-dimensional wave. The procedure is equivalent to analyzing a three-dimensional wave (i.e., one propagated in an arbitrary direction) into three one-dimensional component waves. Here the number of allowed frequencies in the frequency interval ν to $\nu + d\nu$ is equal to the number of points contained between shells of radii corresponding to frequencies ν and $\nu + d\nu$, respectively. This will be proportional to the volume contained between these two shells, since the points are uniformly distributed. Thus it is apparent that $N(\nu)\,d\nu$ will be proportional to $\nu^2\,d\nu$, the first factor, ν^2, being proportional to the area of the shells and the second factor, $d\nu$, being the distance between them. In the following example we shall work out the details and find

$$N(\nu)\,d\nu = \frac{8\pi V}{c^3}\,\nu^2\,d\nu \tag{1-12}$$

where $V = a^3$, the volume of the cavity.

Example 1-3. Derive (1-12), which gives the number of allowed electromagnetic standing waves in each frequency interval for the case of a three-dimensional cavity in the form of a metallic-walled cube of edge length a.

►Consider radiation of wavelength λ and frequency $\nu = c/\lambda$, propagating in the direction defined by the three angles α, β, γ, as shown in Figure 1-7. The radiation must be a standing wave since all three of its components are standing waves. We have indicated the locations of some of the fixed nodes of this standing wave by a set of planes perpendicular to the propagation direction α, β, γ. The distance between these nodal planes of the radiation is just $\lambda/2$, where λ is its wavelength. We have also indicated the locations at the three axes of the nodes of the three components. The distances between these nodes are

$$\begin{aligned} \lambda_x/2 &= \lambda/2\cos\alpha \\ \lambda_y/2 &= \lambda/2\cos\beta \\ \lambda_z/2 &= \lambda/2\cos\gamma \end{aligned} \tag{1-13}$$

Let us write expressions for the magnitudes at the three axes of the electric fields of the three components. They are

$$\begin{aligned} E(x,t) &= E_{0_x}\sin(2\pi x/\lambda_x)\sin(2\pi\nu t) \\ E(y,t) &= E_{0_y}\sin(2\pi y/\lambda_y)\sin(2\pi\nu t) \\ E(z,t) &= E_{0_z}\sin(2\pi z/\lambda_z)\sin(2\pi\nu t) \end{aligned}$$

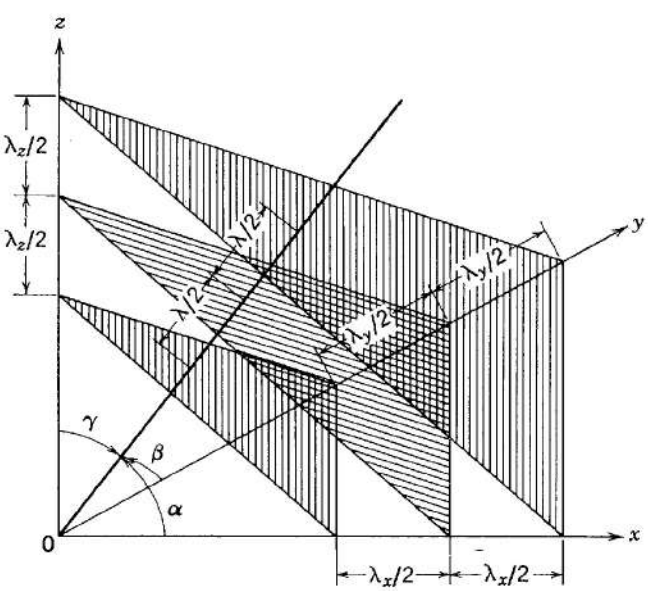

Figure 1-7 The nodal planes of a standing wave propagating in a certain direction in a cubical cavity.

The expression for the x component represents a wave with a maximum amplitude E_{0_x}, with a space variation $\sin(2\pi x/\lambda_x)$, and which is oscillating with frequency ν. As $\sin(2\pi x/\lambda_x)$ is zero for $2x/\lambda_x = 0, 1, 2, 3, \ldots$, the wave is a standing wave of wavelength λ_x because it has fixed nodes separated by the distance $\Delta x = \lambda_x/2$. The expressions for the y and z components represent standing waves of maximum amplitudes E_{0_y} and E_{0_z} and wavelengths λ_y and λ_z, but all three component standing waves oscillate with the frequency ν of the radiation. Note that these expressions automatically satisfy the requirement that the x component have a node at $x = 0$, the y component have a node at $y = 0$, and the z component have a node at $z = 0$. To make them also satisfy the requirement that the x component have a node at $x = a$, the y component have a node at $y = a$, and the z component have a node at $z = a$, set

$$2x/\lambda_x = n_x \qquad \text{for } x = a$$
$$2y/\lambda_y = n_y \qquad \text{for } y = a$$
$$2z/\lambda_z = n_z \qquad \text{for } z = a$$

where $n_x = 1, 2, 3, \ldots; n_y = 1, 2, 3, \ldots; n_z = 1, 2, 3, \ldots$. Using (1-13), these conditions become

$$(2a/\lambda)\cos\alpha = n_x \qquad (2a/\lambda)\cos\beta = n_y \qquad (2a/\lambda)\cos\gamma = n_z$$

Squaring both sides of these equations and adding, we obtain

$$(2a/\lambda)^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = n_x^2 + n_y^2 + n_z^2$$

but the angles α, β, γ have the property

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Thus

$$2a/\lambda = \sqrt{n_x^2 + n_y^2 + n_z^2}$$

where n_x, n_y, n_z take on all possible integral values. This equation describes the limitation on the possible wavelengths of the electromagnetic radiation contained in the cavity.

We again continue the discussion in terms of the allowed frequencies instead of the allowed wavelengths. They are

$$\nu = \frac{c}{\lambda} = \frac{c}{2a}\sqrt{n_x^2 + n_y^2 + n_z^2} \tag{1-14a}$$

Now we shall count the number of allowed frequencies in a given frequency interval by constructing a uniform cubic lattice in one octant of a rectangular coordinate system in such a way that the three coordinates of each point of the lattice are equal to a possible set of the three integers n_x, n_y, n_z (see Figure 1-6). By construction, each lattice point corresponds to an allowed frequency. Furthermore, $N(\nu)d\nu$, the number of allowed frequencies between ν and $\nu + d\nu$, is equal to $N(r)\,dr$, the number of points contained between concentric shells of radii r and $r + dr$, where

$$r = \sqrt{n_x^2 + n_y^2 + n_z^2}$$

From (1-14a), this is

$$r = \frac{2a}{c}\nu \tag{1-14b}$$

Since $N(r)\,dr$ is equal to the volume enclosed by the shells times the density of lattice points, and since, by construction, the density is one, $N(r)\,dr$ is simply

$$N(r)\,dr = \frac{1}{8}4\pi r^2\,dr = \frac{\pi r^2\,dr}{2} \tag{1-15}$$

Setting this equal to $N(\nu)d\nu$, and evaluating $r^2\,dr$ from (1-14b), we have

$$N(\nu)\,d\nu = \frac{\pi}{2}\left(\frac{2a}{c}\right)^3 \nu^2\,d\nu$$

This completes the calculation except that we must multiply these results by a factor of 2 because, for each of the allowed frequencies we have enumerated, there are actually two independent waves corresponding to the two possible states of polarization of electromagnetic radiation. Thus we have derived (1-12). It can be shown that $N(\nu)$ is independent of the assumed shape of the cavity and depends only on its volume. ◀

Note that there is a very significant difference between the results obtained for the case of a real three-dimensional cavity and the results we obtained earlier for the artificial case of a one-dimensional cavity. The factor of ν^2 found in (1-12), but not in (1-11), will be seen to play a fundamental role in the arguments that follow. This factor arises, basically, because we live in a three-dimensional world—the power of ν being one less than the dimensionality. Although Planck, in ultimately resolving the serious discrepancies between classical theory and experiment, had to question certain points which had been considered to be obviously true, neither he nor others working on the problem questioned (1-12). It was, and remains, generally agreed that (1-12) is valid.

We now have a count of the number of standing waves. The next step in the Rayleigh-Jeans classical theory of blackbody radiation is the evaluation of the average total energy contained in each standing wave of frequency ν. According to classical physics, the energy of some particular wave can have any value from zero to infinity, the actual value being proportional to the square of the magnitude of its amplitude constant E_0. However, for a system containing a large number of physical entities of the same kind which are in thermal equilibrium with each other at temperature T, classical physics makes a very definite prediction about the *average* values of the energies of the entities. This applies to our case since the multitude of standing waves, which constitute the thermal radiation inside the cavity, are entities of the same kind which are in thermal equilibrium with each other at the temperature T of the walls of the cavity. Thermal equilibrium is ensured by the fact that the walls of any real cavity will always absorb and reradiate, in different frequencies and directions, a small amount of the radiation incident upon them and, therefore, the different standing waves can gradually exchange energy as required to maintain equilibrium.

The prediction comes from classical kinetic theory, and it is called the *law of equipartition of energy*. This law states that for a system of gas molecules in thermal equilibrium at temperature T, the average kinetic energy of a molecule per degree of freedom is $kT/2$, where $k = 1.38 \times 10^{-23}$ joule/°K is called *Boltzmann's constant*. The law actually applies to any classical system containing, in equilibrium, a large number of entities of the same kind. For the case at hand the entities are standing waves which have one degree of freedom, their electric field amplitudes. Therefore, on the average their *kinetic* energies all have the same value, $kT/2$. However, each sinusoidally oscillating standing wave has a *total* energy which is twice its average kinetic energy. This is a common property of physical systems which have a single degree of freedom that execute simple harmonic oscillations in time; familiar cases are a pendulum or a coil spring. Thus each standing wave in the cavity has, according to the classical equipartition law, an average total energy

$$\bar{\mathscr{E}} = kT \tag{1-16}$$

The most important point to note is that the average total energy $\bar{\mathscr{E}}$ is predicted to have the same value for all standing waves in the cavity, independent of their frequencies.

The energy per unit volume in the frequency interval ν to $\nu + d\nu$ of the blackbody spectrum of a cavity at temperature T is just the product of the average energy per standing wave times the number of standing waves in the frequency interval, divided by the volume of the cavity. From (1-15) and (1-16) we therefore finally obtain the result

$$\rho_T(\nu)\, d\nu = \frac{8\pi\nu^2 kT}{c^3}\, d\nu \tag{1-17}$$

This the *Rayleigh-Jeans formula for blackbody radiation.*

In Figure 1-8 we compare the predictions of this equation with experimental data. The discrepancy is apparent. In the limit of low frequencies, the classical spectrum approaches the experimental results, but, as the frequency becomes large, the theoretical prediction goes to infinity! Experiment shows that the energy density always

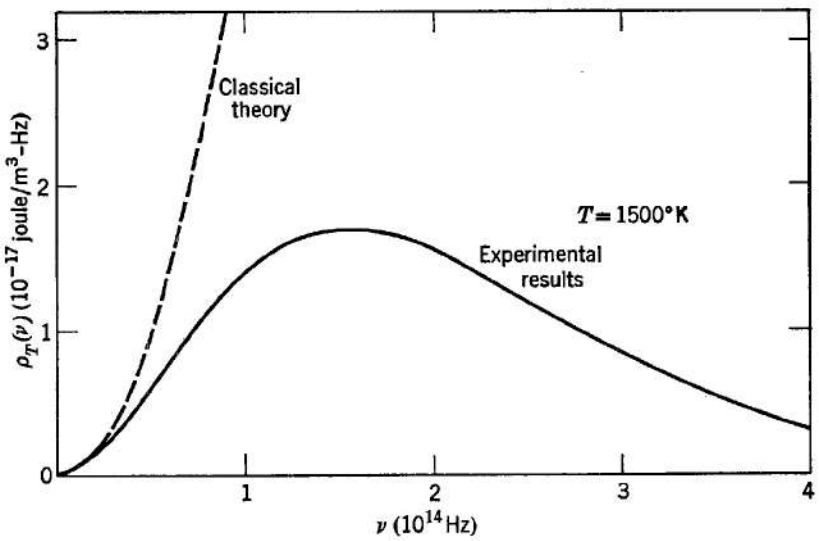

Figure 1-8 The Rayleigh-Jeans prediction (dashed line) compared with the experimental results (solid line) for the energy density of a blackbody cavity, showing the serious discrepancy called the ultraviolet catastrophe.

remains finite, as it obviously must, and, in fact, that the energy density goes to zero at very high frequencies. The grossly unrealistic behavior of the prediction of classical theory at high frequencies is known in physics as the "ultraviolet catastrophe." This term is suggestive of the importance of the failure of the theory.

1-4 PLANCK'S THEORY OF CAVITY RADIATION

In trying to resolve the discrepancy between theory and experiment, Planck was led to consider the possibility of a violation of the law of equipartition of energy on which the theory was based. From Figure 1-8 it is clear that the law gives satisfactory results for small frequencies. Thus we can assume

$$\bar{\mathscr{E}} \xrightarrow[\nu \to 0]{} kT \tag{1-18}$$

that is, the average total energy approaches kT as the frequency approaches zero. The discrepancy at high frequencies could be eliminated if there is, for some reason, a cutoff, so that

$$\bar{\mathscr{E}} \xrightarrow[\nu \to \infty]{} 0 \tag{1-19}$$

that is, if the average total energy approaches zero as the frequency approaches infinity. In other words, Planck realized that, in the circumstances that prevail for the case of blackbody radiation, the average energy of the standing waves is a function of frequency $\bar{\mathscr{E}}(\nu)$ having the properties indicated by (1-18) and (1-19). This is in contrast to the law of equipartition of energy which assigns to the average energy $\bar{\mathscr{E}}$ a value independent of frequency.

Let us look at the origin of the equipartition law. It arises, basically, from a more comprehensive result of classical statistical mechanics called the Boltzmann distribution. (Arguments leading to the Boltzmann distribution are given in Appendix C for students not already familiar with it.) Here we shall use a *special form of the Boltzmann distribution*

$$P(\mathscr{E}) = \frac{e^{-\mathscr{E}/kT}}{kT} \tag{1-20}$$

in which $P(\mathscr{E})\,d\mathscr{E}$ is the probability of finding a given entity of a system with energy in the interval between $\mathscr{E}$ and $\mathscr{E} + d\mathscr{E}$, when the number of energy states for the entity in that interval is independent of $\mathscr{E}$. The system is supposed to contain a large

number of entities of the same kind in thermal equilibrium at temperature T, and k represents Boltzmann's constant. The energies of the entities in the system we are considering, a set of simple harmonic oscillating standing waves in thermal equilibrium in a blackbody cavity, are governed by (1-20).

The Boltzmann distribution function is intimately related to Maxwell's distribution function for the energy of a molecule in a system of molecules in thermal equilibrium. In fact, the exponential in the Boltzmann distribution is responsible for the exponential factor in the Maxwell distribution. The factor of $\mathscr{E}^{1/2}$ that some students may know is also present in the Maxwell distribution results from the circumstance that the number of energy states for a molecule in the interval $\mathscr{E}$ to $\mathscr{E} + d\mathscr{E}$ is not independent of $\mathscr{E}$ but instead increases in proportion to $\mathscr{E}^{1/2}$.

The Boltzmann distribution function provides complete information about the energies of the entities in our system, including, of course, the average value $\bar{\mathscr{E}}$ of the energies. The latter quantity can be obtained from $P(\mathscr{E})$ by using (1-20) to evaluate the integrals in the ratio

$$\bar{\mathscr{E}} = \frac{\int_0^\infty \mathscr{E} P(\mathscr{E})\, d\mathscr{E}}{\int_0^\infty P(\mathscr{E})\, d\mathscr{E}} \tag{1-21}$$

The integrand in the numerator is the energy, $\mathscr{E}$, weighted by the probability that the entity will be found with this energy. By integrating over all possible energies, the average value of the energy is obtained. The denominator is the probability of finding the entity with *any* energy and so should have the value one; it does. The integral in the numerator can be evaluated, and the result is just the law of equipartition of energy

$$\bar{\mathscr{E}} = kT \tag{1-22}$$

Instead of actually carrying through the evaluation here, it will be better, for the purpose of arguments to follow, to look at the graphical presentation of $P(\mathscr{E})$ and $\bar{\mathscr{E}}$ shown in the top half of Figure 1-9. There $P(\mathscr{E})$ is plotted as a function of $\mathscr{E}$. Its maximum value, $1/kT$, occurs at $\mathscr{E} = 0$, and the value of $P(\mathscr{E})$ decreases smoothly with increasing $\mathscr{E}$ to approach zero as $\mathscr{E} \to \infty$. That is, the result that would most probably be found in a measurement of $\mathscr{E}$ is zero. But the average $\bar{\mathscr{E}}$ of the results that would be found in a number of measurements of $\mathscr{E}$ is greater than zero, as is shown on the abscissa of the top figure, since many measurements of $\mathscr{E}$ will lead to values greater than zero. The bottom half of Figure 1–9 indicates the evaluation of $\bar{\mathscr{E}}$ from $P(\mathscr{E})$.

Planck's great contribution came when he realized that he could obtain the required cutoff, indicated in (1-19), if he modified the calculation leading from $P(\mathscr{E})$ to $\bar{\mathscr{E}}$ by treating the energy $\mathscr{E}$ as if it were a *discrete variable* instead of as the *continuous variable* that it definitely is from the point of view of classical physics. Quantitatively, this can be done by rewriting (1-21) in terms of a sum instead of an integral. We shall soon see that this is not too hard to do, but it will be much more instructive for us to study the graphical presentation in Figure 1-10 first.

Planck assumed that the energy $\mathscr{E}$ could take on only certain discrete values, rather than any value, and that the discrete values of the energy were uniformly distributed; that is, he took

$$\mathscr{E} = 0, \Delta\mathscr{E}, 2\Delta\mathscr{E}, 3\Delta\mathscr{E}, 4\Delta\mathscr{E}, \ldots \tag{1-23}$$

as the set of allowed values of the energy. Here $\Delta\mathscr{E}$ is the uniform interval between

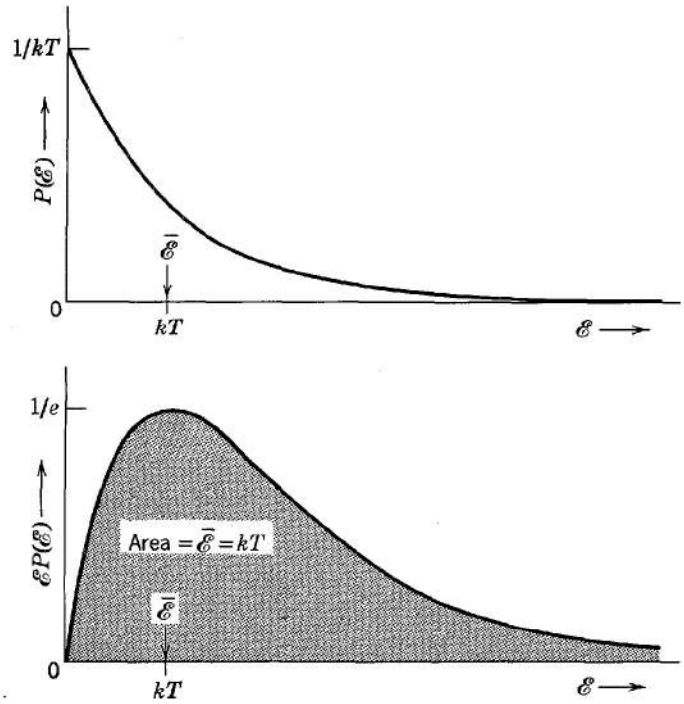

Figure 1-9 *Top:* A plot of the Boltzmann probability distribution $P(\mathscr{E}) = e^{-\mathscr{E}/kT}/kT$. The average value of the energy $\mathscr{E}$ for this distribution is $\bar{\mathscr{E}} = kT$, which is the classical law of equipartition of energy. To calculate this value of $\bar{\mathscr{E}}$, we integrate $\mathscr{E}P(\mathscr{E})$ from zero to infinity. This is just the quantity that is being averaged, $\mathscr{E}$, multiplied by the relative probability $P(\mathscr{E})$ that the value of $\mathscr{E}$ will be found in a measurement of the energy. *Bottom:* A plot of $\mathscr{E}P(\mathscr{E})$. The area under this curve gives the value of $\bar{\mathscr{E}}$.

successive allowed values of the energy. The top part of Figure 1-10 illustrates an evaluation of $\bar{\mathscr{E}}$ from $P(\mathscr{E})$, for a case in which $\Delta\mathscr{E} \ll kT$. In this case the result obtained is $\bar{\mathscr{E}} \simeq kT$. That is, a value essentially equal to the classical result is obtained here since the discreteness $\Delta\mathscr{E}$ is very small compared to the energy range kT in which $P(\mathscr{E})$ changes by a significant amount; it makes no essential difference in this case whether $\mathscr{E}$ is continuous or discrete. The middle part of Figure 1-10 illustrates the case in which $\Delta\mathscr{E} \simeq kT$. Here we find $\bar{\mathscr{E}} < kT$, because most of the entities have energy $\mathscr{E} = 0$ since $P(\mathscr{E})$ has a rather small value at the first allowed nonzero value $\Delta\mathscr{E}$ so $\mathscr{E} = 0$ dominates the calculation of the average value of $\mathscr{E}$ and a smaller result is obtained. The effect of the discreteness is seen most clearly, however, in the lower part of Figure 1-10, which illustrates a case in which $\Delta\mathscr{E} \gg kT$. In this case the probability of finding an entity with any of the allowed energy values greater than zero is negligible, since $P(\mathscr{E})$ is extremely small for all these values, and the result obtained is $\bar{\mathscr{E}} \ll kT$.

Recapitulating, Planck discovered that he could obtain $\bar{\mathscr{E}} \simeq kT$ when the difference in adjacent energies $\Delta\mathscr{E}$ is small, and $\bar{\mathscr{E}} \simeq 0$ when $\Delta\mathscr{E}$ is large. Since he needed to obtain the first result for small values of the frequency ν, and the second result for large values of ν, he clearly needed to make $\Delta\mathscr{E}$ an increasing function of ν. Numerical work showed him that he could take the simplest possible relation between $\Delta\mathscr{E}$ and ν having this property. That is, he assumed these quantities to be proportional

$$\Delta\mathscr{E} \propto \nu \tag{1-24}$$

Written as an equation instead of a proportionality, this is

$$\Delta\mathscr{E} = h\nu \tag{1-25}$$

where h is the proportionality constant.

Further numerical work allowed Planck to determine the value of the constant h by finding the value which produced the best fit of his theory with the experimental

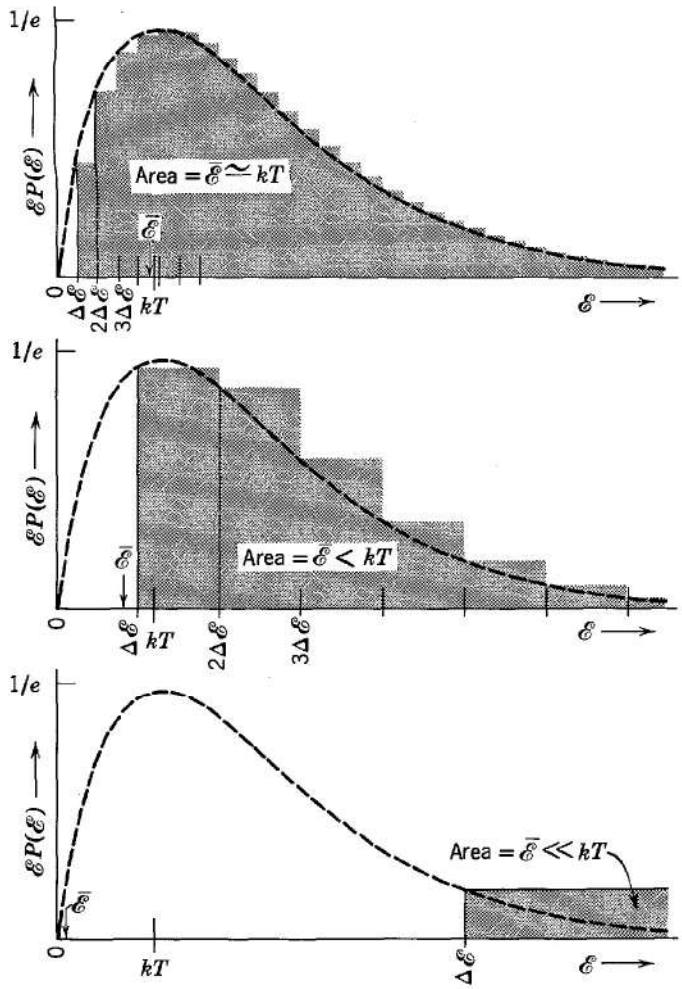

Figure 1-10 *Top:* If the energy $\mathscr{E}$ is not a continuous variable but is instead restricted to discrete values $0, \Delta\mathscr{E}, 2\Delta\mathscr{E}, 3\Delta\mathscr{E}, \ldots$, as indicated by the ticks on the $\mathscr{E}$ axis of the figure, the integral used to calculate the average value $\bar{\mathscr{E}}$ must be replaced by a summation. The average value is thus a sum of areas of rectangles, each of width $\Delta\mathscr{E}$, and with heights given by the allowed values of $\mathscr{E}$ times $P(\mathscr{E})$ at the beginning of each interval. In this figure $\Delta\mathscr{E} \ll kT$, and the allowed energies being closely spaced the area of all the rectangles differs but little from the area under the smooth curve. Thus the average value $\bar{\mathscr{E}}$ is nearly equal to kT, the value found in Figure 1-9. *Middle:* $\Delta\mathscr{E} \simeq kT$, and $\bar{\mathscr{E}}$ has a smaller value than it has in the case of the top figure. *Bottom:* $\Delta\mathscr{E} \gg kT$, and $\bar{\mathscr{E}}$ is further reduced. In all three figures the rectangles show the contribution to the total area of $\mathscr{E}P(\mathscr{E})$ for each allowed energy. The rectangle for $\mathscr{E} = 0$ of course is always of zero height. This will make a large effect on the total area if the widths of the rectangles are large.

data. The value he obtained was very close to the currently accepted value

$$h = 6.63 \times 10^{-34} \text{ joule-sec}$$

This very famous constant is now called *Planck's constant.*

The formula Planck obtained for $\bar{\mathscr{E}}$ by evaluating the summation analogous to the integral in (1-21), and that we shall obtain in Example 1-4, is

$$\bar{\mathscr{E}}(\nu) = \frac{h\nu}{e^{h\nu/kT} - 1} \tag{1-26}$$

Since $e^{h\nu/kT} \to 1 + h\nu/kT$ for $h\nu/kT \to 0$, we see that $\bar{\mathscr{E}}(\nu) \to kT$ in this limit as predicted by (1-18). In the limit $h\nu/kT \to \infty$, $e^{h\nu/kT} \to \infty$, and $\bar{\mathscr{E}}(\nu) \to 0$, in agreement with the prediction of (1-19).

The formula which he then immediately obtained for the energy density in the blackbody spectrum, using his result for $\bar{\mathscr{E}}(\nu)$ rather than the classical value $\bar{\mathscr{E}} = kT$,

is

$$\rho_T(\nu)\,d\nu = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{h\nu/kT}-1}\,d\nu \tag{1-27}$$

This is *Planck's blackbody spectrum.* Figure 1-11 shows a comparison of this result of Planck's theory (expressed in terms of wavelength) with experimental results for a temperature $T = 1595°\text{K}$. The experimental results are in complete agreement with Planck's formula at all temperatures.

We should remember that Planck did not alter the Boltzmann distribution. "All" he did was to treat the energy of the electromagnetic standing waves, oscillating sinusoidally in time, as a discrete instead of a continuous quantity.

Example 1-4. Derive Planck's expression for the average energy $\bar{\mathscr{E}}$ and also his blackbody spectrum.

▶The quantity $\bar{\mathscr{E}}$ is evaluated from the ratio of sums

$$\bar{\mathscr{E}} = \frac{\sum_{n=0}^{\infty} \mathscr{E}P(\mathscr{E})}{\sum_{n=0}^{\infty} P(\mathscr{E})}$$

analogous to the ratio of integrals in (1-21). Sums must be used because with Planck's postulate the energy $\mathscr{E}$ becomes a discrete variable that takes on only the values $\mathscr{E} = 0, h\nu, 2h\nu, 3h\nu, \ldots$. That is, $\mathscr{E} = nh\nu$ where $n = 0, 1, 2, 3, \ldots$. Evaluating the Boltzmann distribution $P(\mathscr{E}) = e^{-\mathscr{E}/kT}/kT$, we have

$$\bar{\mathscr{E}} = \frac{\sum_{n=0}^{\infty} \frac{nh\nu}{kT} e^{-nh\nu/kT}}{\sum_{n=0}^{\infty} \frac{1}{kT} e^{-nh\nu/kT}} = kT\frac{\sum_{n=0}^{\infty} n\alpha e^{-n\alpha}}{\sum_{n=0}^{\infty} e^{-n\alpha}} \qquad \text{where } \alpha = \frac{h\nu}{kT}$$

This, in turn, can be evaluated most easily by noting that

$$-\alpha\frac{d}{d\alpha}\ln\sum_{n=0}^{\infty} e^{-n\alpha} = \frac{-\alpha\frac{d}{d\alpha}\sum_{n=0}^{\infty} e^{-n\alpha}}{\sum_{n=0}^{\infty} e^{-n\alpha}} = \frac{-\sum_{n=0}^{\infty}\alpha\frac{d}{d\alpha}e^{-n\alpha}}{\sum_{n=0}^{\infty} e^{-n\alpha}} = \frac{\sum_{n=0}^{\infty} n\alpha e^{-n\alpha}}{\sum_{n=0}^{\infty} e^{-n\alpha}}$$

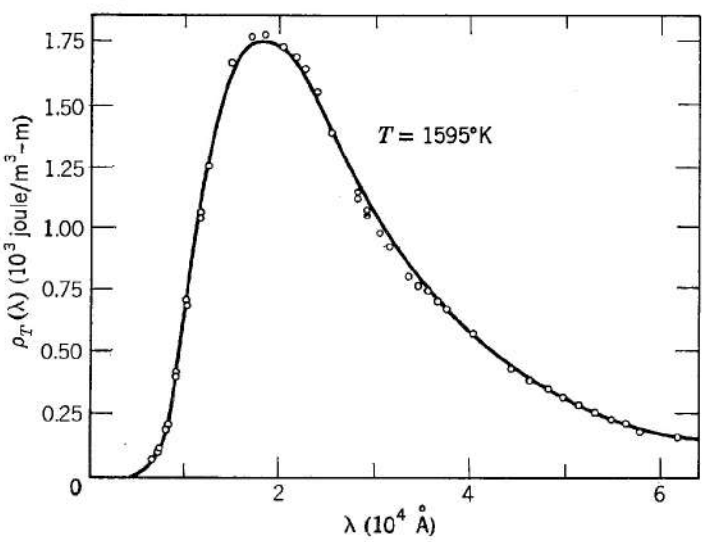

Figure 1-11 Planck's energy density prediction (solid line) compared to the experimental results (circles) for the energy density of a blackbody. The data were reported by Coblentz in 1916 and apply to a temperature of 1595°K. The author remarked in his paper that after drawing the spectral energy curves resulting from his measurements, "owing to eye fatigue it was impossible for months thereafter to give attention to the reduction of the data." The data, when finally reduced, led to a value for Planck's constant of 6.57×10^{-34} joule-sec.

so that

$$\bar{\mathscr{E}} = kT\left(-\alpha \frac{d}{d\alpha} \ln \sum_{n=0}^{\infty} e^{-n\alpha}\right) = -h\nu \frac{d}{d\alpha} \ln \sum_{n=0}^{\infty} e^{-n\alpha}$$

Now

$$\sum_{n=0}^{\infty} e^{-n\alpha} = 1 + e^{-\alpha} + e^{-2\alpha} + e^{-3\alpha} + \cdots$$
$$= 1 + X + X^2 + X^3 + \cdots \qquad \text{where } X = e^{-\alpha}$$

but

$$(1 - X)^{-1} = 1 + X + X^2 + X^3 + \cdots$$

so

$$\bar{\mathscr{E}} = -h\nu \frac{d}{d\alpha} \ln (1 - e^{-\alpha})^{-1}$$
$$= \frac{-h\nu}{(1 - e^{-\alpha})^{-1}} (-1)(1 - e^{-\alpha})^{-2} e^{-\alpha}$$
$$= \frac{h\nu e^{-\alpha}}{1 - e^{-\alpha}} = \frac{h\nu}{e^{\alpha} - 1} = \frac{h\nu}{e^{h\nu/kT} - 1}$$

We have derived (1-26) for the average energy of an electromagnetic standing wave of frequency ν. Multiplying this by (1-12), the number $N(\nu)\,d\nu$ of waves having this frequency derived in Example 1-3, we immediately obtain the Planck blackbody spectrum, (1-27). ◀

Example 1-5. It is convenient in analyzing experimental results, as in Figure 1-11, to express the Planck blackbody spectrum in terms of wavelength λ rather than frequency ν. Obtain $\rho_T(\lambda)$, the wavelength form of Planck's spectrum, from $\rho_T(\nu)$, the frequency form of the spectrum. The quantity $\rho_T(\lambda)$ is defined from the equality $\rho_T(\lambda)\,d\lambda = -\rho_T(\nu)\,d\nu$. The minus sign indicates that, though $\rho_T(\lambda)$ and $\rho_T(\nu)$ are both positive, $d\lambda$ and $d\nu$ have opposite signs. (An increase in frequency gives rise to a corresponding decrease in wavelength.)

▶From the relation $\nu = c/\lambda$ we have $d\nu = -(c/\lambda^2)\,d\lambda$, or $d\nu/d\lambda = -(c/\lambda^2)$, so that

$$\rho_T(\lambda) = -\rho_T(\nu) \frac{d\nu}{d\lambda} = \rho_T(\nu) \frac{c}{\lambda^2}$$

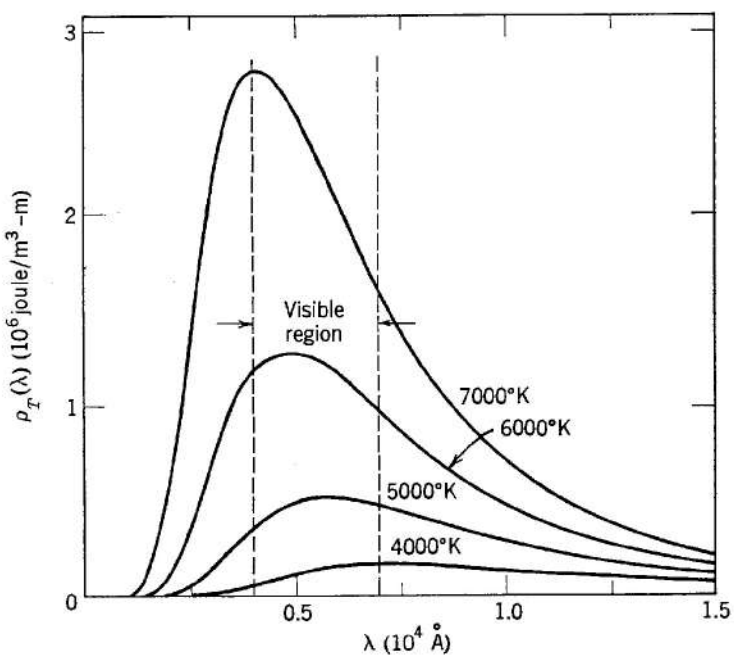

Figure 1-12 Planck's energy density of blackbody radiation at various temperatures as a function of wavelength. Note that the wavelength at which the curve is a maximum decreases as the temperature increases.

If now we set $\nu = c/\lambda$ in (1-27) for $\rho_T(\nu)$ we obtain

$$\rho_T(\lambda)\,d\lambda = \frac{8\pi hc}{\lambda^5}\frac{d\lambda}{e^{hc/\lambda kT} - 1} \tag{1-28}$$

In Figure 1-12 we show $\rho_T(\lambda)$ versus λ for several different temperatures. The trend from "red heat" to "white heat" to "blue heat" radiation with rising temperatures becomes clear as the distribution of radiant energy with wavelength is studied for increasing temperatures. ◀

Stefan's law, (1-2), and Wien's displacement law, (1-3), can be derived from the Planck formula. By fitting them to the experimental results we can determine values of the constants h and k. Stefan's law is obtained by integrating Planck's law over the entire spectrum of wavelengths. The radiancy is found to be proportional to the fourth power of the temperature, the proportionality constant $2\pi^5k^4/15c^2h^3$ being identified with σ, Stefan's constant, which has the experimentally determined value 5.67×10^{-8} W/m^2-°K^4. Wien's displacement law is obtained by setting $d\rho(\lambda)/d\lambda = 0$. We find $\lambda_{\max}T = 0.2014hc/k$ and identify the right-hand side of the equation with Wien's experimentally determined constant 2.898×10^{-3} m-°K. Using these two measured values and assuming a value for the speed of light c, we can calculate the values of h and k. Indeed, this was done by Planck, his values agreeing very well with those obtained subsequently by other methods.

1-5 THE USE OF PLANCK'S RADIATION LAW IN THERMOMETRY

The radiation emitted from a hot body can be used to measure its temperature. If total radiation is used, then, from the Stefan-Boltzmann law, we know that the energies emitted by two sources are in the ratio of the fourth power of the temperature. However, it is difficult to measure total radiation from most sources so that we measure instead the radiancy over a finite wavelength band. Here we use the Planck radiation law which gives the radiancy as a function of temperature and wavelength. For monochromatic radiation of wavelength λ the ratio of the spectral intensities emitted by sources at T_2 °K and T_1 °K is given from Planck's law as

$$\frac{e^{hc/\lambda kT_1} - 1}{e^{hc/\lambda T_2} - 1}$$

If T_1 is taken as a standard reference temperature, then T_2 can be determined relative to the standard from this expression by measuring the ratio experimentally. This procedure is used in the International Practical Temperature Scale, where the normal melting point of gold is taken as the standard fixed point, 1068°C. That is, the primary standard *optical pyrometer* is arranged to compare the spectral radiancy from a blackbody at an unknown temperature $T > 1068$°C with a blackbody at the gold point. Procedures must be adopted, and the theory developed, to allow for the practical circumstances that most sources are not blackbodies and that a finite spectral band is used instead of monochromatic radiation.

Most optical pyrometers use the eye as a detector and call for a large spectral bandwidth so that there will be enough energy for the eye to see. The simplest and most accurate type of instrument used above the gold point is the disappearing filament optical pyrometer (see Figure 1-13). The source whose temperature is to be measured is imaged on the filament of the

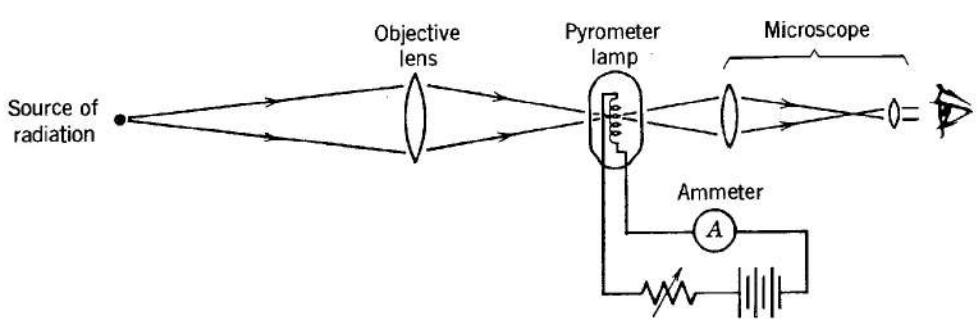

Figure 1-13 Schematic diagram of an optical pyrometer.

pyrometer lamp, and the current in the lamp is varied until the filament seems to disappear into the background of the source image. Careful calibration and precision potentiometers insure accurate measurement of temperature.

A particularly interesting example in the general category of thermometry using blackbody radiation was discovered by Dicke, Penzias, and Wilson in the 1950s. Using a radio telescope operating in the several millimeter to several centimeter wavelength range, they found that a blackbody spectrum of electromagnetic radiation, with a characteristic temperature of about 3°K, is impinging on the earth with equal intensity from all directions. The uniformity in direction indicates that the radiation fills the universe uniformly. Astrophysicists consider these measurements as strong evidence in favor of the so-called *big-bang theory*, in which the universe was in the form of a very dense, and hot, fireball of particles and radiation around 10^{10} years ago. Due to subsequent expansion and the resulting Doppler shift, the temperature of the radiation would be expected to drop by now to something like the observed value of 3°K.

1-6 PLANCK'S POSTULATE AND ITS IMPLICATIONS

Planck's contribution can be stated as a postulate, as follows:

Any physical entity with one degree of freedom whose "coordinate" is a sinusoidal function of time (i.e., executes simple harmonic oscillations) can possess only total energies $\mathscr{E}$ which satisfy the relation

$$\mathscr{E} = nh\nu \qquad n = 0, 1, 2, 3, \ldots$$

where ν is the frequency of the oscillation, and h is a universal constant.

The word coordinate is used in its general sense to mean any quantity which describes the instantaneous condition of the enity. Examples are the length of a coil spring, the angular position of a pendulum bob, and the amplitude of a wave. All these examples happen also to be sinusoidal functions of time.

An energy-level diagram, as shown in Figure 1-14, provides a convenient way of illustrating the behavior of an entity governed by this postulate, and it is also useful in contrasting this behavior with what would be expected on the basis of classical physics. In such a diagram we indicate each of the possible energy states of the entity with a horizontal line. The distance from the line to the zero energy line is proportional to the total energy to which it corresponds. Since the entity may have any energy from zero to infinity according to classical physics, the classical energy-level diagram consists of a continuum of lines extending from zero up. However, the entity executing simple harmonic oscillations can have only one of the discrete total energies $\mathscr{E} = 0, h\nu, 2h\nu, 3h\nu \ldots$ if it obeys Planck's postulate. This is indicated by the discrete set of lines in its energy-level diagram. The energy of the entity obeying Planck's postulate is said to be *quantized*, the allowed energy states are called *quantum states*, and the integer n is called the *quantum number*.

It may have occurred to the student that there are physical systems whose behavior seems to be obviously in disagreement with Planck's postulate. For instance, an ordi-

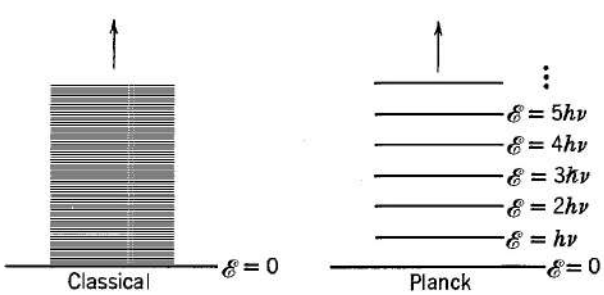

Figure 1-14 *Left:* The allowed energies in a classical system, oscillating sinusoidally with frequency ν, are continuously distributed. *Right:* The allowed energies according to Planck's postulate are discretely distributed since they can only assume the values $nh\nu$. We say that the energy is quantized, n being the quantum number of an allowed quantum state.

nary pendulum executes simple harmonic oscillations, and yet this system certainly appears to be capable of possessing a continuous range of energies. Before we accept this argument, however, we should make some simple numerical calculations concerning such a system.

Example 1-6. A pendulum consisting of a 0.01 kg mass is suspended from a string 0.1 m in length. Let the amplitude of its oscillation be such that the string in its extreme positions makes an angle of 0.1 rad with the vertical. The energy of the pendulum decreases due, for instance, to frictional effects. Is the energy decrease observed to be continuous or discontinuous?

▶The oscillation frequency of the pendulum is

$$\nu = \frac{1}{2\pi}\sqrt{\frac{g}{l}} = \frac{1}{2\pi}\sqrt{\frac{9.8\ \text{m/sec}^2}{0.1\ \text{m}}} = 1.6/\text{sec}$$

The energy of the pendulum is its maximum potential energy

$$\begin{aligned} mgh &= mgl(1 - \cos\theta) = 0.01\ \text{kg} \times 9.8\ \text{m/sec}^2 \times 0.1\ \text{m} \times (1 - \cos 0.1) \\ &= 5 \times 10^{-5}\ \text{joule} \end{aligned}$$

The energy of the pendulum is quantized so that changes in energy take place in discontinuous jumps of magnitude $\Delta E = h\nu$, but

$$\Delta E = h\nu = 6.63 \times 10^{-34}\ \text{joule-sec} \times 1.6/\text{sec} = 10^{-33}\ \text{joule}$$

whereas $E = 5 \times 10^{-5}$ joule. Therefore, $\Delta E/E = 2 \times 10^{-29}$. Hence, to measure the discreteness in the energy decrease we need to measure the energy to better than two parts in 10^{29}. It is apparent that even the most sensitive experimental equipment is totally incapable of this energy resolution. ◀

We conclude that experiments involving an ordinary pendulum cannot determine whether Planck's postulate is valid or not. The same is true of experiments on all other macroscopic mechanical systems. The smallness of h makes the graininess in the energy too fine to be distinguished from an energy continuum. Indeed, h might as well be zero for classical systems and, in fact, one way to reduce quantum formulas to their classical limits would be to let $h \to 0$ in these formulas. Only where we consider systems in which ν is so large and/or $\mathscr{E}$ is so small that $\Delta\mathscr{E} = h\nu$ is of the order of $\mathscr{E}$ are we in a position to test Planck's postulate. One example is, of course, the high-frequency standing waves in blackbody radiation. Many other examples will be considered in following chapters.

1-7 A BIT OF QUANTUM HISTORY

In its original form, Planck's postulate was not so far reaching as it is in the form we have given. Planck's initial work was done by treating, in detail, the behavior of the electrons in the walls of the blackbody and their coupling to the electromagnetic radiation within the cavity. This coupling leads to the same factor ν^2 we obtained in (1-12) from the more general arguments due to Rayleigh and Jeans. Through this coupling, Planck related the energy in a particular frequency component of the blackbody radiation to the energy of an electron in the wall oscillating sinusoidally at the same frequency, and he postulated only that the energy of the oscillating particle is quantized. It was not until later that Planck accepted the idea that the oscillating electromagnetic waves were themselves quantized, and the postulate was broadened to include any entity whose single coordinate oscillates sinusoidally.

At first Planck was unsure whether his introduction of the constant h was only a mathematical device or a matter of deep physical significance. In a letter to R. W. Wood, Planck called his limited postulate "an act of desperation." "I knew," he wrote, "that the problem (of the equilibrium of matter and radiation) is of fundamental significance for physics; I knew the formula that reproduces the energy distribution in the normal spectrum; a theoretical interpretation *had* to be found at any cost, no matter how high." For more than a decade Planck tried to fit the quantum idea into classical theory. With each attempt he appeared to retreat

from his original boldness, but always he generated new ideas and techniques that quantum theory later adopted. What appears to have finally convinced him of the correctness and deep significance of his quantum hypothesis was its support of the definiteness of the statistical concept of entropy and the third law of thermodynamics.

It was during this period of doubt that Planck was editor of the German research journal *Annalen der Physik*. In 1905 he received Einstein's first relativity paper and stoutly defended Einstein's work. Thereafter he became one of young Einstein's patrons in scientific circles, but he resisted for some time the very ideas on the quantum theory of radiation advanced by Einstein that subsequently confirmed and extended Planck's own work. Einstein, whose deep insight into electromagnetism and statistical mechanics was perhaps unequalled by anyone at the time, saw as a result of Planck's work the need for a sweeping change in classical statistics and electromagnetism. He advanced predictions and interpretations of many physical phenomena which were later strikingly confirmed by experiment. In the next chapter we turn to one of these phenomena and follow another road on the way to quantum mechanics.

QUESTIONS

1. Does a blackbody always appear black? Explain the term blackbody.
2. Pockets formed by coals in a coal fire seem brighter than the coals themselves. Is the temperature in such pockets appreciably higher than the surface temperature of an exposed glowing coal?
3. If we look into a cavity whose walls are kept at a constant temperature no details of the interior are visible. Explain.
4. The relation $R_T = \sigma T^4$ is exact for blackbodies and holds for all temperatures. Why is this relation not used as the basis of a definition of temperature at, for instance, 100°C?
5. A piece of metal glows with a bright red color at 1100°K. At this temperature, however, a piece of quartz does not glow at all. Explain. (Hint: Quartz is transparent to visible light.)
6. Make a list of distribution functions commonly used in the social sciences (e.g., distribution of families with respect to income). In each case, state whether the variable whose distribution is described is discrete or continuous.
7. In (1-4) relating spectral radiancy and energy density, what dimensions would a proportionality constant need to have?
8. What is the origin of the ultraviolet catastrophe?
9. The law of equipartition of energy requires that the specific heat of gases be independent of the temperature, in disagreement with experiment. Here we have seen that it leads to the Rayleigh-Jeans radiation law, also in disagreement with experiment. How can you relate these two failures of the equipartition law?
10. Compare the definitions and dimensions of spectral radiancy $R_T(\nu)$, radiancy R_T, and energy density $\rho_T(\nu)$.
11. Why is optical pyrometry commonly used above the gold point and not below it? What objects typically have their temperatures measured in this way?
12. Are there quantized quantities in classical physics? Is energy quantized in classical physics?
13. Does it make sense to speak of charge quantization in physics? How is this different from energy quantization?
14. Elementary particles seem to have a discrete set of rest masses. Can this be regarded as quantization of mass?
15. In many classical systems the allowed frequencies are quantized. Name some of the systems. Is energy quantized there too?
16. Show that Planck's constant has the dimensions of angular momentum. Does this necessarily suggest that angular momentum is a quantized quantity?
17. For quantum effects to be everyday phenomena in our lives, what would be the minimum order of magnitude of h?

18. What, if anything, does the 3°K universal blackbody radiation tell us about the temperature of outer space?

19. Does Planck's theory suggest quantized atomic energy states?

20. Discuss the remarkable fact that discreteness in energy was first found in analyzing a continuous spectrum emitted by interacting atoms in a solid, rather than in analyzing a discrete spectrum such as is emitted by an isolated atom in a gas.

PROBLEMS

1. At what wavelength does a cavity at 6000°K radiate most per unit wavelength?

2. Show that the proportionality constant in (1-4) is $4/c$. That is, show that the relation between spectral radiancy $R_T(\nu)$ and energy density $\rho_T(\nu)$ is $R_T(\nu)\,d\nu = (c/4)\rho_T(\nu)\,d\nu$.

3. Consider two cavities of arbitrary shape and material, each at the same temperature T, connected by a narrow tube in which can be placed color filters (assumed ideal) which will allow only radiation of a specified frequency ν to pass through. (a) Suppose at a certain frequency ν', $\rho_T(\nu')\,d\nu$ for cavity 1 was greater than $\rho_T(\nu')\,d\nu$ for cavity 2. A color filter which passes only the frequency ν' is placed in the connecting tube. Discuss what will happen in terms of energy flow. (b) What will happen to their respective temperatures? (c) Show that this would violate the second law of thermodynamics; hence prove that all blackbodies at the same temperature must emit thermal radiation with the same spectrum independent of the details of their composition.

4. A cavity radiator at 6000°K has a hole 10.0 mm in diameter drilled in its wall. Find the power radiated through the hole in the range 5500–5510 Å. (Hint: See Problem2.)

5. (a) Assuming the surface temperature of the sun to be 5700°K, use Stefan's law, (1-2), to determine the rest mass lost per second to radiation by the sun. Take the sun's diameter to be 1.4×10^9 m. (b) What fraction of the sun's rest mass is lost each year from electromagnetic radiation? Take the sun's rest mass to be 2.0×10^{30} kg.

6. In a thermonuclear explosion the temperature in the fireball is momentarily 10^7 °K. Find the wavelength at which the radiation emitted is a maximum.

7. At a given temperature, $\lambda_{max} = 6500$ Å for a blackbody cavity. What will λ_{max} be if the temperature of the cavity walls is increased so that the rate of emission of spectral radiation is doubled?

8. At what wavelength does the human body emit its maximum temperature radiation? List assumptions you make in arriving at an answer.

9. Assuming that λ_{max} is in the near infrared for red heat and in the near ultraviolet for blue heat, approximately what temperature in Wien's displacement law corresponds to red heat? To blue heat?

10. The *average* rate of solar radiation incident per unit area on the earth is 0.485 cal/cm^2-min (or 338 W/m^2). (a) Explain the consistency of this number with the solar constant (the solar energy falling per unit time at normal incidence on a unit area) whose value is 1.94 cal/cm^2-min (or 1353 W/m^2). (b) Consider the earth to be a blackbody radiating energy into space at this same rate. What surface temperature would the earth have under these circumstances?

11. Attached to the roof of a house are three solar panels, each 1 m × 2 m. Assume the equivalent of 4 hrs of normally incident sunlight each day, and that all the incident light is absorbed and converted to heat. How many gallons of water can be heated from 40°C to 120°C each day?

12. Show that the Rayleigh-Jeans radiation law, (1-17), is not consistent with the Wien displacement law $\nu_{max} \propto T$, (1-3a), or $\lambda_{max}T = \text{const}$, (1-3b).

13. We obtain ν_{max} in the blackbody spectrum by setting $d\rho_T(\nu)/d\nu = 0$ and λ_{max} by setting $d\rho_T(\lambda)/d\lambda = 0$. Why is it not possible to get from $\lambda_{max}T = \text{const}$ to $\nu_{max} = \text{const} \times T$ simply by using $\lambda_{max} = c/\nu_{max}$? That is, why is it wrong to assume that $\nu_{max}\lambda_{max} = c$, where c is the speed of light?

14. Consider the following numbers: 2, 3, 3, 4, 1, 2, 2, 1, 0 representing the number of hits garnered by each member of the Baltimore Orioles in a recent outing. (a) Calculate

directly the average number of hits per man. (b) Let x be a variable signifying the number of hits obtained by a man, and let $f(x)$ be the number of times the number x appears. Show that the average number of hits per man can be written as

$$\bar{x} = \frac{\sum_0^4 xf(x)}{\sum_0^4 f(x)}$$

(c) Let $p(x)$ be the probability of the number x being attained. Show that $\bar{x}$ is given by

$$\bar{x} = \sum_0^4 xp(x)$$

15. Consider the function

$$f(x) = \frac{1}{10}(10 - x)^2 \qquad 0 \le x \le 10$$

$$f(x) = 0 \qquad \text{all other } x$$

(a) From

$$\bar{x} = \frac{\int_{-\infty}^{\infty} xf(x)\,dx}{\int_{-\infty}^{\infty} f(x)\,dx}$$

find the average value of x. (b) Suppose the variable x were discrete rather than continuous. Assume $\Delta x = 1$ so that x takes on only integral values 0, 1, 2, . . . , 10. Compute $\bar{x}$ and compare to the result of part (a). (Hint: It may be easier to compute the appropriate sum directly rather than working with general summation formulas.) (c) Compute $\bar{x}$ for $\Delta x = 5$, i.e. $x = 0, 5, 10$. Compare to the result of part (a). (d) Draw analogies between the results obtained in this problem and the discussion of Section 1-4. Be sure you understand the roles played by $\bar{\mathscr{E}}$, $\Delta\mathscr{E}$, and $P(\mathscr{E})$.

16. Using the relations $P(\mathscr{E}) = e^{-\mathscr{E}/kT}/kT$ and $\int_0^\infty P(\mathscr{E})\,d\mathscr{E} = 1$, evaluate the integral of (1-21) to deduce (1-22), $\bar{\mathscr{E}} = kT$.

17. Use the relation $R_T(\nu)\,d\nu = (c/4)\rho_T(\nu)\,d\nu$ between spectral radiancy and energy density, together with Planck's radiation law, to derive Stefan's law. That is, show that

$$R_T = \int_0^\infty \frac{2\pi h}{c^2}\frac{\nu^3\,d\nu}{e^{h\nu/kT} - 1} = \sigma T^4$$

where $\sigma = 2\pi^5k^4/15c^2h^3$.

$$\left(\text{Hint: } \int_0^\infty \frac{q^3dq}{e^q - 1} = \frac{\pi^4}{15}\right)$$

18. Derive the Wien displacement law, $\lambda_{\max}T = 0.2014\,hc/k$, by solving the equation $d\rho(\lambda)/d\lambda = 0$. (Hint: Set $hc/\lambda kT = x$ and show that the equation quoted leads to $e^{-x} + x/5 = 1$. Then show that $x = 4.965$ is the solution.)

19. To verify experimentally that the 3°K universal background radiation accurately fits a blackbody spectrum, it is decided to measure $R_T(\lambda)$ from a wavelength below $\lambda_{\max}$ where its value is $0.2R_T(\lambda_{\max})$ to a wavelength above $\lambda_{\max}$ where its value is again $0.2R_T(\lambda_{\max})$. Over what range of wavelength must the measurements be made?

20. Show that, at the wavelength $\lambda_{\max}$, where $\rho_T(\lambda)$ has its maximum

$$\rho_T(\lambda_{\max}) = 170\pi(kT)^5/(hc)^4$$

21. Use the result of the preceding problem to find the two wavelengths at which $\rho_T(\lambda)$ has a value one-half the value at λ_{max}. Give answers in terms of λ_{max}.

22. A tungsten sphere 2.30 cm in diameter is heated to 2000°C. At this temperature tungsten radiates only about 30% of the energy radiated by a blackbody of the same size and temperature. (a) Calculate the temperature of a perfectly black spherical body of the same size that radiates at the same rate as the tungsten sphere. (b) Calculate the diameter of a perfectly black spherical body at the same temperature as the tungsten sphere that radiates at the same rate.

23. (a) Show that about 25% of the radiant energy in a cavity is contained within wavelengths zero and λ_{max}; i.e., show that

$$\frac{\int_0^{\lambda_{max}} \rho_T(\lambda)\, d\lambda}{\int_0^{\infty} \rho_T(\lambda)\, d\lambda} \simeq \frac{1}{4}$$

(Hint: $hc/\lambda_{max}kT = 4.965$; hence Wien's approximation is fairly accurate in evaluating the integral in the numerator above.) (b) By what percent does Wien's approximation used over the entire wavelength range overestimate or underestimate the integrated energy density?

24. Find the temperature of a cavity having a radiant energy density at 2000 Å that is 3.82 times the energy density at 4000 Å.

2

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2-1 INTRODUCTION

In this chapter we shall examine processes in which radiation interacts with matter. Three processes (the photoelectric effect, the Compton effect, and pair production) involve the scattering or absorption of radiation in matter. Two processes (bremsstrahlung and pair annihilation) involve the production of radiation. In each case we shall obtain experimental evidence that radiation is particlelike in its interaction with matter, as distinguished from the wavelike nature of radiation when it propagates. In the following chapter we shall study a generalization of this result, due to de Broglie, which leads directly into quantum mechanics. Some of the material of these two chapters may be a review of topics the student has already come across in studying elementary physics.

2-2 THE PHOTOELECTRIC EFFECT

It was in 1886 and 1887 that Heinrich Hertz performed the experiments that first confirmed the existence of electromagnetic waves and Maxwell's electromagnetic theory of light propagation. It is one of those fascinating and paradoxical facts in the history of science that in the course of his experiments Hertz noted the effect that Einstein later used to contradict other aspects of the classical electromagnetic theory. Hertz discovered that an electric discharge between two electrodes occurs more readily when ultraviolet light falls on one of the electrodes. Lenard, following up some experiments of Hallwachs, showed soon after that the ultraviolet light facilitates the discharge by causing electrons to be emitted from the cathode surface. The ejection of electrons from a surface by the action of light is called the *photoelectric effect*. It is the phenomenon underlying the operation of the solar cells being developed to convert thermal energy received from the sun directly into electrical energy.

Figure 2-1 shows an apparatus used to study the photoelectric effect. A glass envelope encloses the apparatus in an evacuated space. Monochromatic light, incident through a quartz window, falls on the metal plate A and liberates electrons,

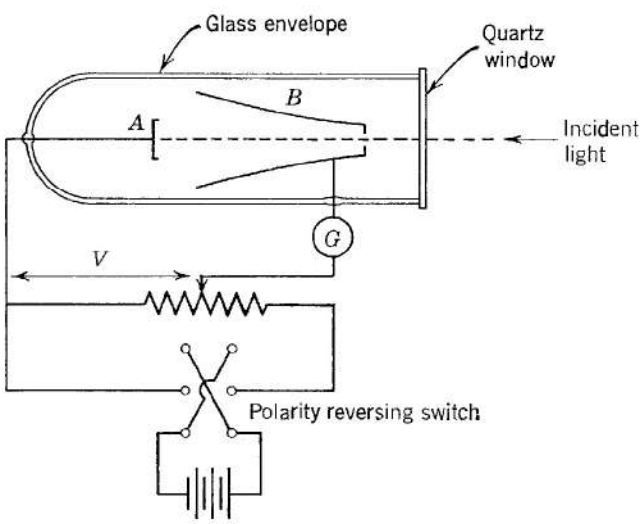

Figure 2-1 An apparatus used to study the photoelectric effect. The potential difference V can be varied continuously in magnitude, and also reversed in sign by the switching arrangement. If the same metal is used to make plate A and cup B then the potential difference between them equals the value of V measured with a voltmeter between the points indicated in the figure. But if this is not the case then the measured value of V must be corrected by adding to it the *contact potential* acting between the two metals in order to obtain the quantity of interest—the potential difference between A and B. The phenomenon of contact potential is explained in Chapter 11.

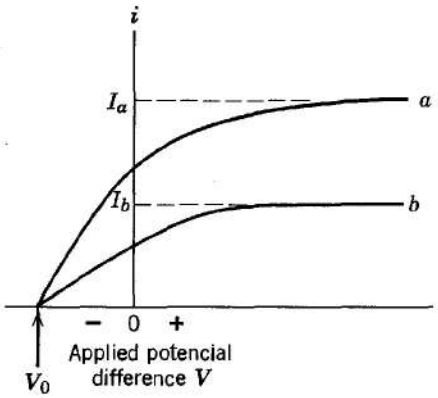

Figure 2-2 Graphs of current i as a function of potential difference V from data taken with the apparatus of Figure 2-1. The applied potential difference V is called positive when the cup B in Figure 2-1 is positive with respect to the photoelectric surface A. In curve b the incident light intensity has been reduced to one-half that of curve a. The stopping potential V_0 is independent of light intensity, but the saturation currents I_a and I_b are directly proportional to it.

called *photoelectrons*. The electrons can be detected as a current if they are attracted to the metal cup B by means of a potential difference V applied between A and B. The sensitive ammeter G serves to measure this photoelectric current.

Curve a of Figure 2-2 is a plot of the photoelectric current, in an apparatus like that of Figure 2-1, as a function of the potential difference V. If V is made large enough, the photoelectric current reaches a certain limiting (saturation) value at which all photoelectrons ejected from A are collected by cup B.

If V is reversed in sign, the photoelectric current does not immediately drop to zero, which suggests that the electrons are emitted from A with kinetic energy. Some will reach cup B in spite of the fact that the electric field opposes their motion. However, if this reversed potential difference is made large enough, a value V_0 called the *stopping potential* is reached at which the photoelectric current does drop to zero. This potential difference V_0, multiplied by electron charge, measures the kinetic energy K_{max} of the *fastest* ejected photoelectron. That is

$$K_{max} = eV_0 \tag{2-1}$$

The quantity K_{max} turns out experimentally to be independent of the intensity of the light, as is shown by curve b in Figure 2-2 in which the light intensity has been reduced to one-half the value used in obtaining curve a.

Figure 2-3 shows the stopping potential V_0 as a function of the frequency of the light incident on sodium. Note that there is a definite *cutoff* frequency ν_0, below which no photoelectric effect occurs. These data were taken in 1914 by Millikan whose painstaking work on the photoelectric effect won him the Nobel prize in 1923. Because the photoelectric effect for visible or near-visible light is largely a surface phenomenon, it is necessary in the experiments to avoid oxide films, grease, or other surface contaminants.

There are three major features of the photoelectric effect that cannot be explained in terms of the classical wave theory of light:

1. Wave theory requires that the oscillating electric vector **E** of the light wave increase in amplitude as the intensity of the light beam is increased. Since the force applied to the electron is $e\mathbf{E}$, this suggests that the *kinetic energy* of the photo-

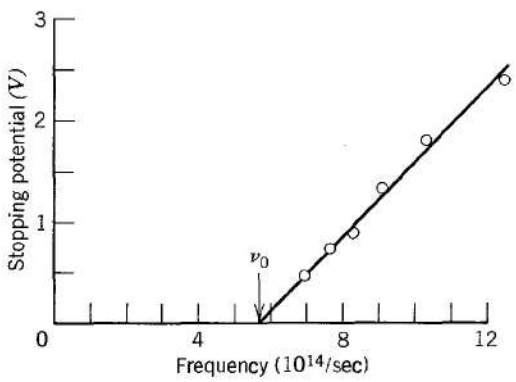

Figure 2-3 The stopping potential at various frequencies for sodium. The points show Millikan's data, except that the correction mentioned in the caption to Figure 2-1 has been recalculated using a recent measurement of the contact potential. The cutoff frequency ν_0 is 5.6×10^{14} Hz.

electrons should also increase as the light beam is made more intense. However, Figure 2-2 shows that K_{max}, which equals eV_0, *is independent of the light intensity.* This has been tested over a range of intensities of 10^7.

2. According to the wave theory the photoelectric effect should occur for any frequency of the light, provided only that the light is intense enough to give the energy needed to eject the photoelectrons. However, Figure 2-3 shows that there exists, for each surface, *a characteristic cutoff frequency* ν_0. *For frequencies less than* ν_0, *the photoelectric effect does not occur, no matter how intense the illumination.*

3. If the energy acquired by a photoelectron is absorbed from the wave incident on the metal plate, the "effective target area" for an electron in the metal is limited, and probably not much more than that of a circle having about an atomic diameter. In the classical theory the light energy is uniformly distributed over the wave front. Thus, if the light is feeble enough, there should be a measurable time lag, which we shall estimate in Example 2-1, between the time when light starts to impinge on the surface and the ejection of the photoelectron. During this interval the electron should be absorbing energy from the beam until it has accumulated enough to escape. *However, no detectable time lag has ever been measured.* This disagreement is particularly striking when the photoelectric substance is a gas; under these circumstances collective absorption mechanisms can be ruled out and the energy of the emitted photoelectron must certainly be soaked out of the light beam by a single atom or molecule.

Example 2-1. A potassium plate is placed 1 m from a feeble light source whose power is 1 W = 1 joule/sec. Assume that an ejected photoelectron may collect its energy from a circular area of the plate whose radius r is, say, one atomic radius: $r \simeq 1 \times 10^{-10}$ m. The energy required to remove an electron through the potassium surface is about 2.1 eV $= 3.4 \times 10^{-19}$ joule. (One electron volt = 1 eV $= 1.60 \times 10^{-19}$ joule is the energy gained by an electron, of charge 1.60×10^{-19} coul, in falling through a potential drop of 1 V.) How long would it take for such a target to absorb this much energy from the light source? Assume the light energy to be spread uniformly over the wave front.

▶The target area is $\pi r^2 = \pi \times 10^{-20}$ m^2. The area of a 1 m sphere centered on the source is $4\pi(1\text{ m})^2 = 4\pi$ m^2. Thus if the source radiates uniformly in all directions (i.e., if the energy is uniformly distributed over spherical wave fronts spreading out from the source, in agreement with classical theory) the rate R at which energy falls on the target is given by

$$R = 1 \text{ joule/sec} \times \frac{\pi \times 10^{-20}\text{ m}^2}{4\pi\text{ m}^2} = 2.5 \times 10^{-21} \text{ joule/sec}$$

Assuming that all this power is absorbed, we may calculate the time required for the electron to acquire enough energy to escape; we find

$$t = \frac{3.4 \times 10^{-19}\text{ joule}}{2.5 \times 10^{-21}\text{ joule/sec}} = 1.4 \times 10^2 \text{ sec} \simeq 2 \text{ min}$$

Of course, we could modify the preceding picture to reduce the calculated time by assuming a larger effective target area. The most favorable assumption, that energy is transferred by a resonance process from light wave to electron, leads to a target area of λ^2, where λ is the wavelength of the light, but we would still obtain a finite time lag which is well within our ability to measure experimentally. (For ultraviolet light of $\lambda = 100$ Å, for example, $t \simeq 10^{-2}$ sec.) However, no time lag has been detected under any circumstances, the early experiments setting an upper limit of 10^{-9} sec on any such possible delay! ◀

2-3 EINSTEIN'S QUANTUM THEORY OF THE PHOTOELECTRIC EFFECT

In 1905 Einstein called into question the classical theory of light, proposed a new theory, and cited the photoelectric effect as one application that could test which theory was correct. This was many years before Millikan's work, but Einstein was influenced by Lenard's experiment. As we have mentioned, Planck originally restricted

his concept of energy quantization to the radiating electron in the walls of a black-body cavity. Planck believed that electromagnetic energy, once radiated, spreads through space like water waves spread through water. Einstein proposed instead that radiant energy is quantized into concentrated bundles which later came to be called *photons*.

Einstein argued that the well-known optical experiments on interference and diffraction of electromagnetic radiation had been performed only in situations involving *very* large numbers of photons. These experiments yield results which are averages of the behaviors of the individual photons. The presence of the photons is not apparent in them any more than the presence of individual droplets of water is apparent in a fine spray from a garden hose, if the number of droplets is very high. Of course the interference and diffraction experiments definitely show that photons do not travel from where they are emitted to where they are absorbed in the simple ways that classical particles, like water droplets, do. They travel like classical waves, in the sense that calculations based on the way such waves propagate (and in particular the way two component waves reinforce or nullify each other depending on their relative phases) correctly explain measurements of the average way photons travel.

Einstein focused his attention not on the familiar wavelike way radiation propagates, but on what he first realized is the particlelike way it is emitted and absorbed. He reasoned that Planck's requirement that the energy content of the electromagnetic waves of frequency ν in a radiant source (e.g., an ultraviolet light source in a photoelectric experiment) can only be 0, or $h\nu$, or $2h\nu, \ldots,$ or $nh\nu, \ldots$ implies that in the process of going from energy state $nh\nu$ to energy state $(n-1)h\nu$ the source would emit a discrete burst of electromagnetic energy of energy content $h\nu$.

Einstein assumed that such a bundle of energy is initially localized in a small volume of space, and that it remains localized as it moves away from the source with velocity c. He assumed that the energy content E of the bundle, or photon, is related to its frequency ν by the equation

$$E = h\nu \tag{2-2}$$

He also assumed that in the photoelectric process one photon is completely absorbed by one electron in the photocathode.

When the electron is emitted from the surface of the metal, its kinetic energy will be

$$K = h\nu - w \tag{2-3}$$

where $h\nu$ is the energy of the absorbed incident photon and w is the work required to remove the electron from the metal. This work is needed to overcome the attractive fields of the atoms in the surface and losses of kinetic energy due to internal collisions of the electron. Some electrons are bound more tightly than others; some lose energy in collisions on the way out. In the case of loosest binding and no internal losses, the photoelectron will emerge with the maximum kinetic energy, K_{max}. Hence

$$K_{max} = h\nu - w_0 \tag{2-4}$$

where w_0, a characteristic energy of the metal called the *work function*, is the minimum energy needed by an electron to pass through the metal surface and escape the attractive forces that normally bind the electron to the metal.

Consider now how Einstein's photon hypothesis meets the three objections raised against the wave theory interpretation of the photoelectric effect. As for objection 1 (the lack of dependence of K_{max} on the intensity of illumination), there is complete agreement of the photon theory with experiment. Doubling the light intensity merely doubles the number of photons and thus doubles the photoelectric current; it does *not* change the energy $h\nu$ of the individual photons or the nature of the individual photoelectric process described by (2-3).

Objection 2 (the existence of a cutoff frequency) is removed at once by (2-4). If K_{max} equals zero we have

$$h\nu_0 = w_0 \tag{2-5}$$

which asserts that a photon of frequency ν_0 has just enough energy to eject the photoelectrons and none extra to appear as kinetic energy. If the frequency is reduced below ν_0, the individual photons, no matter how many of them there are (that is, no matter how intense the illumination), will not have enough energy individually to eject photoelectrons.

Objection 3 (the absence of a time lag) is eliminated in the photon theory because the required energy is supplied in concentrated bundles. It is *not* spread uniformly over a large area, as we assumed in Example 2-1, which is based on the assumption that the classical wave theory is true. If there is any illumination at all incident on the cathode, then there will be at least one photon that hits it; this photon will be immediately absorbed, by *some* atom, leading to the immediate emission of a photoelectron.

Let us rewrite Einstein's photoelectric equation, (2-4), by substituting eV_0 for K_{max} from (2-1). This yields

$$V_0 = \frac{h\nu}{e} - \frac{w_0}{e}$$

Thus Einstein's theory predicts a linear relationship between the stopping potential V_0 and the frequency ν, in complete agreement with experimental results as shown in Figure 2-3. The slope of the experimental curve in the figure should be h/e or, using data from the figure

$$\frac{h}{e} = \frac{2.1\ \text{V} - 0.1\ \text{V}}{11.0 \times 10^{14}/\text{sec} - 6.0 \times 10^{14}/\text{sec}} = 4.0 \times 10^{-15}\ \text{V-sec}$$

We can find h by multiplying this ratio by the electronic charge e. Thus $h = 4.0 \times 10^{-15}$ V-sec $\times\ 1.6 \times 10^{-19}$ coul $= 6.4 \times 10^{-34}$ joule-sec. From a much more careful analysis of these and other data, including data taken with lithium surfaces, Millikan found the value $h = 6.57 \times 10^{-34}$ joule-sec, with an accuracy of about 0.5%. This early measurement was in good agreement with the value of h derived from Planck's radiation formula. The numerical agreement in two determinations of h, using completely different phenomena and theories, is striking. A modern value of h, deduced from diverse experiments, is

$$h = 6.6262 \times 10^{-34}\ \text{joule-sec}$$

To quote Millikan: "The photoelectric effect . . . furnishes a proof which is quite independent of the facts of blackbody radiation of the correctness of the fundamental assumption of the quantum theory, namely, the assumption of a discontinuous or explosive emission of the energy absorbed by the electronic constituents of atoms from . . . waves. It materializes, so to speak, the quantity h discovered by Planck through the study of blackbody radiation and gives us a confidence inspired by no other type of phenomenon that the primary physical conception underlying Planck's work corresponds to reality."

Example 2-2. Deduce the work function for sodium from Figure 2-3.

▶The intersection of the straight line in Figure 2-3 with the horizontal axis is the cutoff frequency, $\nu_0 = 5.6 \times 10^{14}$/sec. Substituting this into (2-5) gives us

$$\begin{aligned} w_0 = h\nu_0 &= 6.63 \times 10^{-34}\ \text{joule-sec} \times 5.6 \times 10^{14}/\text{sec} \\ &= 3.7 \times 10^{-19}\ \text{joule} \times \frac{1\ \text{eV}}{1.60 \times 10^{-19}\ \text{joule}} \\ &= 2.3\ \text{eV} \end{aligned}$$

The same value is obtained from Figure 2-3 as the magnitude of the intercept of the extended line with the vertical axis.

For most conducting metals the value of the work function is of the order of a few electron volts. It is the same as the work function for thermionic emission from these metals. ◀

Example 2-3. At what rate per unit area do photons strike the metal plate in Example 2-1? Assume that the light is monochromatic, of wavelength 5890 Å (yellow light).

▶The rate per unit area at which energy falls on a metal plate 1 m from a 1-W light source (see Example 2-1) is

$$R = \frac{1 \text{ joule/sec}}{4\pi(1 \text{ m})^2} = 8.0 \times 10^{-2} \text{ joule/m}^2\text{-sec}$$
$$= 5.0 \times 10^{17} \text{ eV/m}^2\text{-sec}$$

Each photon has an energy of

$$E = h\nu = \frac{hc}{\lambda} = \frac{6.63 \times 10^{-34} \text{ joule-sec} \times 3.00 \times 10^8 \text{ m/sec}}{5.89 \times 10^{-7} \text{ m}}$$
$$= 3.4 \times 10^{-19} \text{ joule}$$
$$= 2.1 \text{ eV}$$

Thus the rate R at which photons strike a unit area of the plate is

$$R = 5.0 \times 10^{17} \text{ eV/m}^2\text{-sec} \times \frac{1 \text{ photon}}{2.1 \text{ eV}} = 2.4 \times 10^{17} \frac{\text{photon}}{\text{m}^2\text{-sec}}$$

The photoelectric effect is just able to occur because the photon energy just equals the 2.1 eV work function for the potassium surface (see Example 2-1). Note that if the wavelength is slightly increased (that is, if ν is slightly decreased) the photoelectric effect will not occur, no matter how large the rate R might be.

This example suggests that the intensity of light I can be regarded as the product of N, the number of photons per unit area per unit time, and $h\nu$, the energy of a single photon. We see that even at the relatively low intensity here ($\simeq 10^{-1}$ W/m^2) the number N is extremely large ($\simeq 10^{17}$ photons/m^2-sec) so that the energy of any one photon is very small. This accounts for the extreme fineness of the granularity of radiation and suggests why ordinarily it is difficult to detect at all. It is analogous to detecting the atomic structure of bulk matter which for most purposes can be regarded as continuous, the discreteness being revealed only under special circumstances. ◀

In 1921 Einstein received the Nobel Prize for predicting theoretically the law of the photoelectric effect. Before Millikan's complete experimental validation of this law in 1914, Einstein was recommended to membership in the Prussian Academy of Sciences by Planck and others. Their early negative attitude toward the photon hypothesis is revealed in their signed affidavit, praising Einstein, in which they wrote: "Summing up, we may say that there is hardly one among the great problems, in which modern physics is so rich, to which Einstein has not made an important contribution. That he may have sometimes missed the target in his speculations, as, for example, in his hypothesis of light quanta (photons), cannot really be held too much against him, for it is not possible to introduce fundamentally new ideas, even in the most exact sciences, without occasionally taking a risk."

Today the photon hypothesis is used throughout the electromagnetic spectrum, not only in the light region (see Figure 2-4). A microwave cavity, for example, can be said to contain photons. At $\lambda = 10$ cm, a typical microwave wavelength, the photon energy can be computed as above to be 1.20×10^{-5} eV. This energy is much too low to eject photoelectrons from metal surfaces. For x rays, or for energetic γ rays such as are emitted from radioactive nuclei, the photon energy may be 10^6 eV or higher. Such photons can eject electrons bound deep in heavy atoms by energies of the order of 10^5 eV. The photons in the visible region of the electromagnetic spectrum are not energetic enough to do this, the photoelectrons which they eject being the so-called *conduction* electrons which are bound to the metal by energies of only a few electron volts.

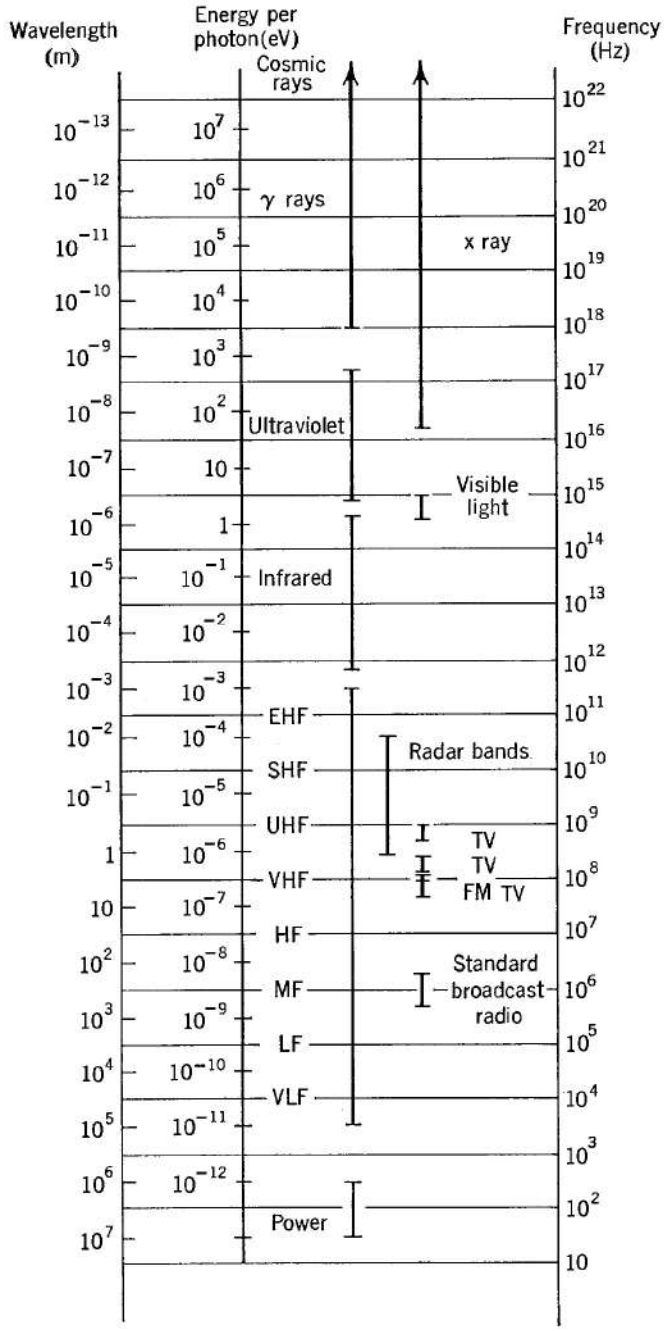

Figure 2-4 The electromagnetic spectrum, showing wavelength, frequency, and energy per photon on a logarithmic scale.

Notice that the photons are absorbed in the photoelectric process. This requires the electrons to be bound to atoms, or solids, for a truly free electron cannot absorb a photon and conserve both total relativistic energy and momentum in the process. We must have a bound electron, therefore, the binding forces serving to transmit momentum to the atom or solid. Due to the large mass of an atom, or solid, compared to the electron, the system can absorb a large amount of momentum without acquiring a significant amount of energy. Our photoelectric energy equation remains valid, the effect being possible only because there is a heavy recoiling particle in addition to an ejected electron. The photoelectric effect is one important way in which photons, of energy up to and including x-ray energies, are absorbed by matter. At higher energies other photon absorption processes, soon to be discussed, become more important.

Finally, it should be emphasized here that in the Einstein picture a photon of frequency ν has exactly the energy $h\nu$; it does *not* have energies that are integral multiples of $h\nu$. Of course, there can be n photons of frequency ν so that the energy at that frequency can be $nh\nu$. In treating blackbody cavity radiation in the Einstein picture, we deal with a "photon gas," because the radiant energy is localized in space in bundles rather than extended through space in standing waves. Years after the Planck deduction of the cavity radiation formula, Bose and Einstein derived the same formula on the basis of a photon gas.

2-4 THE COMPTON EFFECT

The corpuscular (particlelike) nature of radiation received dramatic confirmation in 1923 from the experiments of Compton. He allowed a beam of x rays of sharply defined wavelength λ to fall on a graphite target, as shown in Figure 2-5. For various angles of scattering, he measured the intensity of the scattered x rays as a function of their wavelength. Figure 2-6 shows his experimental results. We see that, although the incident beam consists essentially of a single wavelength λ, the scattered x rays have intensity peaks at *two* wavelengths; one of them is the same as the incident wavelength, the other, λ', being larger by an amount $\Delta\lambda$. This so-called *Compton shift* $\Delta\lambda = \lambda' - \lambda$ varies with the angle at which the scattered x rays are observed.

The presence of scattered wavelength λ' cannot be understood if the incident x radiation is regarded as a classical electromagnetic wave. In the classical model the oscillating electric field vector in the incident wave of frequency ν acts on the free electrons in the scattering target and sets them oscillating at that same frequency. These oscillating electrons, like charges surging back and forth in a small radio transmitting antenna, radiate electromagnetic waves that again have this same frequency ν. Hence, in the classical picture the scattered wave should have the same frequency ν and the same wavelength λ as the incident wave.

Compton (and independently Debye) interpreted his experimental results by postulating that the incoming x-ray beam was not a wave of frequency ν but a collection of photons, each of energy $E = h\nu$, and that these photons collided with free electrons in the scattering target as in a collision between billiard balls. In this view, the "recoil" photons emerging from the target make up the scattered radiation. Since the incident photon transfers some of its energy to the electron with which it collides, the scattered photon must have a lower energy E'; it must therefore have a

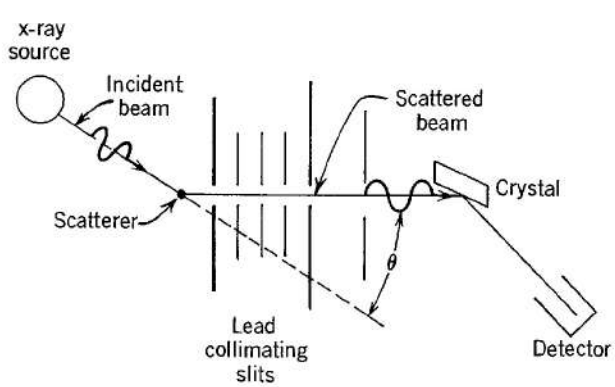

Figure 2-5 Compton's experimental arrangement. Monochromatic x rays of wavelength λ fall on a graphite scatterer. The distribution of intensity with wavelength is measured for x rays scattered at any scattering angle θ. The scattered wavelengths are measured by observing Bragg reflections from a crystal (see Figure 3-3). Their intensities are measured by a detector such as an ionization chamber.

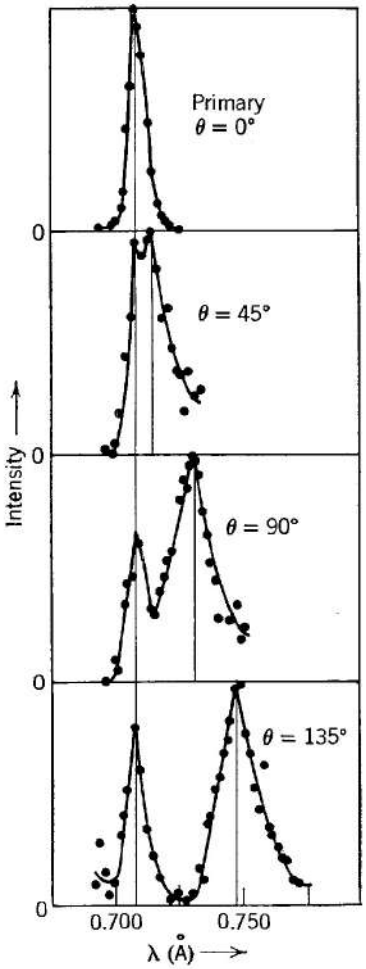

Figure 2-6 Compton's experimental results. The solid vertical line on the left corresponds to the wavelength λ, that on the right to λ'. Results are shown for four different angles of scattering θ. Note that the Compton shift, $\Delta\lambda = \lambda' - \lambda$, for $\theta = 90°$, agrees well with the theoretical prediction $h/m_0c = 0.0243$ Å.

lower frequency $\nu' = E'/h$, which implies a longer wavelength $\lambda' = c/\nu'$. This point of view accounts qualitatively for the wavelength shift, $\Delta\lambda = \lambda' - \lambda$. Notice that in the interaction the x rays are regarded as particles, not as waves, and that, as distinguished from their behavior in the photoelectric process, the x-ray photons are scattered rather than absorbed. Let us now analyze a single photon-electron collision quantitatively.

For x radiation of frequency ν, the energy of a photon in the incident beam is

$$E = h\nu$$

Taking the idea of a photon as a localized bundle of energy quite literally, we shall consider it to be a particle of energy E and momentum p. Such a particle must, however, have certain quite specialized properties. Consider the equation (see Appendix A) giving the total relativistic energy of a particle in terms of its rest mass m_0 and its velocity v

$$E = m_0c^2/\sqrt{1 - v^2/c^2}$$

Since the velocity of a photon equals c, and since its energy content $E = h\nu$ is finite, it is apparent that the rest mass of a photon must be zero. Thus a photon can be considered to be a particle of zero rest mass, and of total relativisitic energy E which is entirely kinetic. The momentum of a photon can be evaluated from the general relation between the total relativistic energy E, momentum p, and rest mass m_0. This is

$$E^2 = c^2p^2 + (m_0c^2)^2 \tag{2-6}$$

For a photon the second term on the right is zero, and we have

$$p = E/c = h\nu/c \tag{2-7}$$

or

$$p = h/\lambda \tag{2-8}$$

where $\lambda = c/\nu$ is the wavelength of the electromagnetic radiation that the photon comprises. It is quite interesting to note that Maxwell's classical wave theory of electromagnetic radiation also leads to an equation $p = E/c$, with p representing the momentum content per unit volume of radiation and E representing its energy content per unit volume.

Now the frequency ν of the scattered radiation was observed to be independent of the material in the foil. This implies that the scattering does not involve entire atoms. Compton assumed that the scattering was due to collisions between the photon and an individual electron in the target. He also assumed that the electrons participating in this scattering process are free and initially stationary. Some *a priori* justification of these assumptions can be found from considering the fact that the energy of an x-ray photon is several orders of magnitude greater than the energy of an ultraviolet photon, and from our discussion of the photoelectric effect it is apparent that the energy of an ultraviolet photon is comparable to the minimum energy with which an electron is bound in a metal.

Consider, then, a collision between a photon and a free stationary electron, as in Figure 2-7. In the diagram on the left, a photon of total relativistic energy E_0 and momentum p_0 is incident on a stationary electron of rest mass energy m_0c^2. In the diagram on the right, the photon is scattered at an angle θ and moves off with total relativistic energy E_1 and momentum p_1, while the electron recoils at an angle φ with kinetic energy K and momentum p. Compton applied the conservation of momentum and total relativistic energy to this collision problem. Relativistic equations were used since the photon always moves at relativistic velocities, and the recoiling electron does too under most circumstances.

Momentum conservation requires

$$p_0 = p_1 \cos\theta + p \cos\varphi$$

and

$$p_1 \sin\theta = p \sin\varphi$$

Squaring these equations, we obtain

$$(p_0 - p_1 \cos\theta)^2 = p^2 \cos^2\varphi$$

and

$$p_1^2 \sin^2\theta = p^2 \sin^2\varphi$$

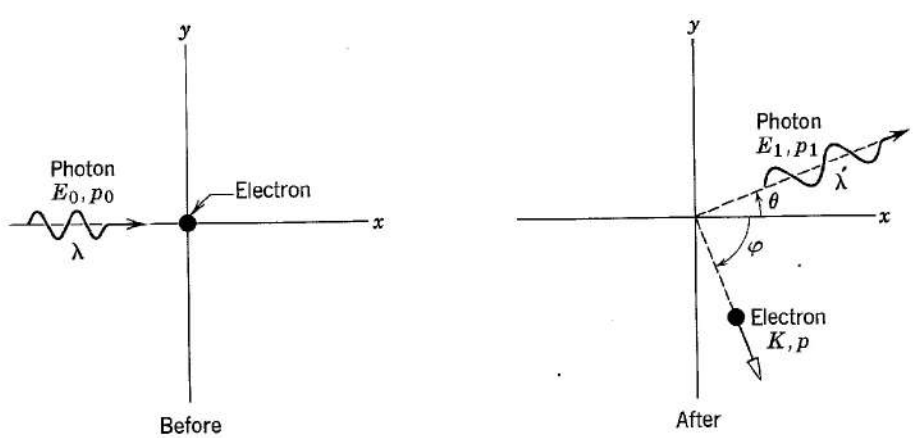

Figure 2-7 Compton's interpretation. A photon of wavelength λ is incident on a free electron at rest. On collision, the photon is scattered at an angle θ with increased wavelength λ', while the electron moves off at angle φ.

Adding, we find

$$p_0^2 + p_1^2 - 2p_0p_1 \cos\theta = p^2 \tag{2-9}$$

Conservation of total relativistic energy requires

$$E_0 + m_0c^2 = E_1 + K + m_0c^2$$

Thus

$$E_0 - E_1 = K$$

According to (2-7), this is

$$c(p_0 - p_1) = K \tag{2-10}$$

Writing $K + m_0c^2$ for E in (2-6), that equation becomes

$$(K + m_0c^2)^2 = c^2p^2 + (m_0c^2)^2$$

which simplifies to

$$K^2 + 2Km_0c^2 = c^2p^2$$

or

$$K^2/c^2 + 2Km_0 = p^2$$

Evaluating p^2 from (2-9) and K from (2-10), we have

$$(p_0 - p_1)^2 + 2m_0c(p_0 - p_1) = p_0^2 + p_1^2 - 2p_0p_1 \cos\theta$$

which reduces to

$$m_0c(p_0 - p_1) = p_0p_1(1 - \cos\theta)$$

or

$$\frac{1}{p_1} - \frac{1}{p_0} = \frac{1}{m_0c}(1 - \cos\theta)$$

Multiplying through by h, and applying (2-8), we obtain the *Compton equation*

$$\Delta\lambda = \lambda_1 - \lambda_0 = \lambda_C(1 - \cos\theta) \tag{2-11}$$

where

$$\lambda_C \equiv h/m_0c = 2.43 \times 10^{-12}\ \text{m} = 0.0243\ \text{Å} \tag{2-12}$$

is the so-called *Compton wavelength.*

Notice that $\Delta\lambda$, the *Compton shift*, depends only on the scattering angle θ, and *not* on the initial wavelength λ. Equation (2-11) predicts the experimentally observed Compton shifts of Figure 2-6 to within the experimental limits of accuracy. In (2-11) we see that $\Delta\lambda$ varies from zero (for $\theta = 0$, corresponding to a "grazing" collision with the incident photon being scarcely deflected) to $2h/m_0c = 0.049$ Å (for $\theta = 180°$, corresponding to a "head-on" collision, the incident photon being reversed in direction). Figure 2-8 is a plot of $\Delta\lambda$ versus θ.

Subsequent experiments (by Compton, Simon, Wilson, Bothe, Geiger, and Blass) detected the recoil electron in the process, showed that it appeared simultaneously with the scattered x ray, and confirmed quantitatively the predicted electron energy and direction of scattering.

The presence of the peak in Figure 2-6 for which the photon wavelength does *not* change on scattering must still be explained. We have assumed heretofore that the electron with which the photon collides is free. Even though the electron is initially bound, this assumption is justifiable if the kinetic energy acquired by the electron in the collision is much larger than its binding energy. If the electron is particularly strongly bound to an atom in the target, however, or if the incident photon energy is very small, there is some chance that the electron will not be ejected from the atom. In this case, the collision can be regarded as taking place between the photon and the whole atom. The ionic core, to which the electron is bound in the scattering

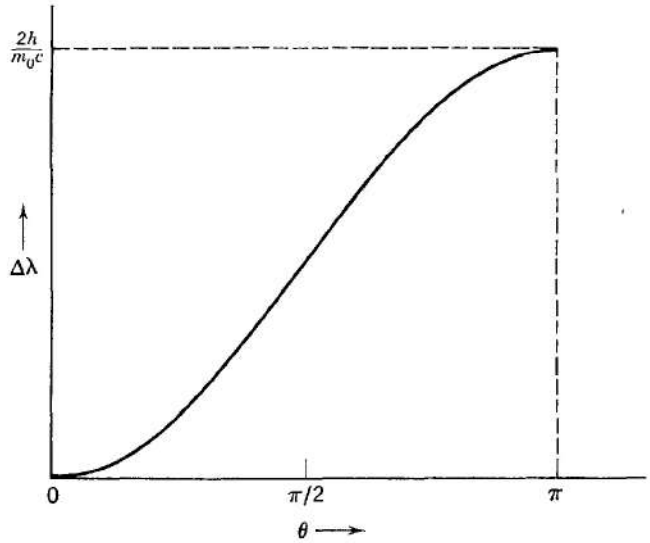

Figure 2-8 Compton's result $\Delta\lambda = (h/m_0c)(1 - \cos\theta)$.

target, recoils as a whole during the collision. Then the mass M of the atom is the characteristic mass for the process, and it must be substituted in the Compton shift equations for the electron mass m_0. Since $M \gg m_0$ ($M \simeq 22{,}000m_0$ for carbon, for instance), the Compton shift for collisions with tightly bound electrons is seen, from (2-11) and (2-12), to be immeasurably small (one millionth of an angstrom for carbon), so that the scattered photon is essentially unmodified in wavelength. To summarize, some photons are scattered from electrons which are freed by the collision; these photons are modified in wavelength. Other photons are scattered from electrons which remain bound during the collision; these photons are not modified in wavelength.

The process that scatters photons without changing their wavelength is called *Rayleigh scattering*, after the physicist who developed a classical theory of the scattering of electromagnetic radiation by atoms around the year 1900. He considered a beam of electromagnetic waves whose oscillating electric field interacts with the charges of the atomic electrons in the target. This interaction produces forces on the electrons which cause oscillating accelerations. As a result of the accelerations, the electrons will radiate electromagnetic waves of the same frequency, and in phase with, the incident waves. (See Appendix B.) Thus the atomic electrons absorb energy from the incident beam of x rays and scatter it in all directions, without modifying the wavelength. Although this classical explanation of Rayleigh scattering is different from the quantum explanation presented in the preceding paragraph, both explain the same feature observed in the measurements. Thus Rayleigh scattering is a case where classical and quantum results merge.

It is interesting to ask in what region of the electromagnetic spectrum Rayleigh scattering will be the dominant process, and in what region Compton scattering will dominate. If the incident radiation is in the visible, microwave, or radio part of the spectrum, then λ is extremely large compared to the Compton shift $\Delta\lambda$, independent of whether an electron or an atomic mass is used in evaluating the Compton wavelength of (2-12). Thus the scattered radiation in this region of the spectrum will in all circumstances have a wavelength which is the same as the wavelength of the incident radiation within experimental accuracy. So, as $\lambda \to \infty$ the quantum results merge with the classical results, and Rayleigh scattering dominates. Moving into the x-ray region of the spectrum, Compton scattering starts to become important, particularly for scattering targets of low atomic number where the atomic electrons are not very tightly bound, and the wavelength shift in scattering from an electron which

is freed in the process becomes easily measurable. In the γ-ray region where $\lambda \to 0$, the photon energy becomes so large, that an electron is always freed in a collision, and Compton scattering dominates.

It is in the short wavelength region that the classical results fail to explain the scattering of radiation, just as in the ultraviolet catastrophe of classical physics where predictions concerning the radiation in a cavity diverged radically from experimental results at short wavelengths. These circumstances are due to the size of Planck's constant h. At long wavelengths the frequency ν is small, and since h is also small the granularity in electromagnetic energy, $h\nu$, is so small as to be virtually indistinguishable from the continuum of classical physics. But at sufficiently short wavelengths, where ν is large enough, $h\nu$ is no longer small enough to be negligible and quantum effects abound.

Example 2-4. Consider an x-ray beam, with $\lambda = 1.00$ Å, and also a γ-ray beam from a Cs^{137} sample, with $\lambda = 1.88 \times 10^{-2}$ Å. If the radiation scattered from free electrons is viewed at 90° to the incident beam: (a) What is the Compton wavelength shift in each case? (b) What kinetic energy is given to a recoiling electron in each case? (c) What percentage of the incident photon energy is lost in the collision in each case?

▶(a) The Compton shift, with $\theta = 90°$, is

$$\Delta\lambda = \frac{h}{m_0 c}(1 - \cos\theta) = \frac{6.63 \times 10^{-34}\ \text{joule-sec}}{9.11 \times 10^{-31}\ \text{kg} \times 3.00 \times 10^{8}\ \text{m/sec}} \times (1 - \cos 90°)$$
$$= 2.43 \times 10^{-12}\ \text{m} = 0.0243\ \text{Å}$$

This result is independent of the incident wavelength, the same for the γ rays as the x rays.

(b) Equation (2-10) can be written as

$$hc/\lambda = hc/\lambda' + K$$

Then, since $\lambda' = \lambda + \Delta\lambda$, we have

$$hc/\lambda = hc/(\lambda + \Delta\lambda) + K$$

so that $K = hc\ \Delta\lambda/\lambda(\lambda + \Delta\lambda)$.

For the x-ray beam, with $\lambda = 1.00$ Å, we have

$$K = \frac{6.63 \times 10^{-34}\ \text{joule-sec} \times 3.00 \times 10^{8}\ \text{m/sec} \times 2.43 \times 10^{-12}\ \text{m}}{1.00 \times 10^{-10}\ \text{m} \times (1.00 + 0.024) \times 10^{-10}\ \text{m}} = 4.73 \times 10^{-17}\ \text{joule}$$
$$= 295\ \text{eV} = 0.295\ \text{keV}$$

For the γ-ray beam, with $\lambda = 1.88 \times 10^{-2}$ Å, we have

$$K = \frac{6.63 \times 10^{-34}\ \text{joule-sec} \times 3.00 \times 10^{8}\ \text{m/sec} \times 2.43 \times 10^{-12}\ \text{m}}{1.88 \times 10^{-12}\ \text{m} \times (0.0188 + 0.0243) \times 10^{-10}\ \text{m}} = 5.98 \times 10^{-14}\ \text{joule}$$
$$= 378\ \text{keV}.$$

(c) The incident x-ray photon energy is

$$E = h\nu = \frac{hc}{\lambda} = \frac{6.63 \times 10^{-34}\ \text{joule-sec} \times 3.00 \times 10^{8}\ \text{m/sec}}{1.00 \times 10^{-10}\ \text{m}} = 1.99 = 10^{-15}\ \text{joule}$$
$$= 12.4\ \text{keV}$$

The energy lost by the photon equals that gained by the electron, or 0.295 keV, so the percentage loss in energy is

$$\frac{0.295\ \text{keV}}{12.4\ \text{keV}} \times 100\% = 2.4\%$$

The incident γ-ray photon energy is

$$E = h\nu = \frac{hc}{\lambda} = \frac{6.63 \times 10^{-34}\ \text{joule-sec} \times 3.00 \times 10^{8}\ \text{m/sec}}{1.88 \times 10^{-12}\ \text{m}} = 1.06 \times 10^{-13}\ \text{joule}$$
$$= 660\ \text{keV}$$

The energy lost by the photon equals that gained by the electron, or 378 keV, so that the percentage loss in energy is

$$\frac{378 \text{ keV}}{660 \text{ keV}} \times 100\% = 57\%$$

Hence, the more energetic photons (which have small wavelengths) experience a larger *percent* loss in energy in Compton scattering. This corresponds to the fact that the photons of smaller wavelengths experience a larger *percent* increase in wavelength on being scattered. This becomes clear from the expression for fractional loss in energy, given simply by

$$\frac{K}{E} = \frac{hc\Delta\lambda/\lambda(\lambda + \Delta\lambda)}{hc/\lambda} = \frac{\Delta\lambda}{\lambda + \Delta\lambda}$$

From this it can be shown that at $\lambda = 5500$ Å, corresponding to visible photons, the percentage loss (for $\theta = 90°$) is less than one-thousandth of 1%, whereas at $\lambda = 1.25 \times 10^{-2}$ Å, corresponding to 1 MeV γ-ray photons, the percentage loss (for $\theta = 90°$) is 67%. ◀

2-5 THE DUAL NATURE OF ELECTROMAGNETIC RADIATION

In his paper, "A Quantum Theory of the Scattering of X-rays by Light Elements," Compton wrote: "The present theory depends essentially upon the assumption that each electron which is effective in the scattering scatters a complete quantum (photon). It involves also the hypothesis that the quanta of radiation are received from definite directions and are scattered in definite directions. The experimental support of the theory indicates very convincingly that a radiation quantum carries with it directed momentum as well as energy."

The need for a photon, or localized particle, interpretation of processes dealing with the interaction between radiation and matter is clear, but at the same time we need a wave theory of radiation to understand interference and diffraction phenomena. The idea that radiation is neither purely a wave phenomenon nor merely a stream of particles must therefore be taken seriously. Whatever radiation is, it behaves wavelike under some circumstances and particlelike under other circumstances. Indeed, the situation is revealed most forcefully in Compton's experimental work where (a) a crystal spectrometer is used to measure x-ray wavelengths, the measurement being interpreted by a wave theory of diffraction and (b) the scattering affects the wavelength in a way that can be understood only by treating the x rays as particles. It is in the very expressions $E = h\nu$ and $p = h/\lambda$ that the wave attributes (ν and λ) and the particle attributes (E and p) are combined.

Although many physicists felt at first very uncomfortable when contemplating the "split personality" of electromagnetic radiation, the broader point of view provided by the development of quantum mechanics has caused the contemporary attitude to be quite different. The duality evident in the wave-particle nature of radiation is no longer considered at all unusual because it is now known to be a general characteristic of all physical entities. We shall see that electrons and protons, for example, have exactly the same dual nature as photons. We shall also see that it is possible to reconcile the existence of the wave aspects with the existence of the particle aspects, for any of these entities, with the aid of quantum mechanics.

2-6 PHOTONS AND X-RAY PRODUCTION

X rays, so named by their discoverer Roentgen because their nature was then unknown, are radiations in the electromagnetic spectrum of wavelength less than about 1.0 Å. They show the typical transverse wave behavior of polarization, interference, and diffraction that is found in light and all other electromagnetic radiation. X rays are produced in the target of an x-ray tube, illustrated in Figure 2-9, when a beam of energetic electrons, accelerated through a potential difference of thousands of volts,

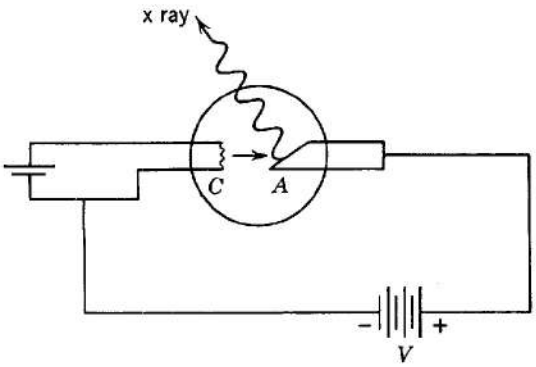

Figure 2-9 An x-ray tube. Electrons are emitted thermally from the heated cathode *C* and are accelerated toward the anode target *A* by the applied potential *V*. X rays are emitted from the target when electrons are stopped by striking it.

is stopped upon striking the target. According to classical physics (see Appendix B), the deceleration of the electrons, brought to rest in the target material, results in the emission of a continuous spectrum of electromagnetic radiation.

Figure 2-10 shows, for four different values of the incident electron energy, how the x rays emerging from a tungsten target are distributed in wavelength. (In addition to the *continuous* x-ray spectrum shown in the figure, x-ray *lines* characteristic of the target material are emitted. We shall discuss the lines in Chapter 9.) The most notable feature of these smooth curves is that, for a given electron energy, there exists a well-defined minimum wavelength λ_{min}; for 40 keV electrons, for instance, λ_{min} is 0.311 Å. Although the shape of the continuous x-ray distribution spectrum depends slightly on the choice of target material as well as on the electron accelerating potential V, the value of λ_{min} depends only on V, being the same for all target materials. Classical electromagnetic theory cannot account for this fact, there being no reason why waves whose wavelength is less than a certain critical value should not emerge from the target.

A ready explanation appears, however, if we regard the x rays as photons. Figure 2-11 shows the elementary process that, on the photon view, is responsible for the continuous x-ray spectrum of Figure 2-10. An electron of initial kinetic energy K is

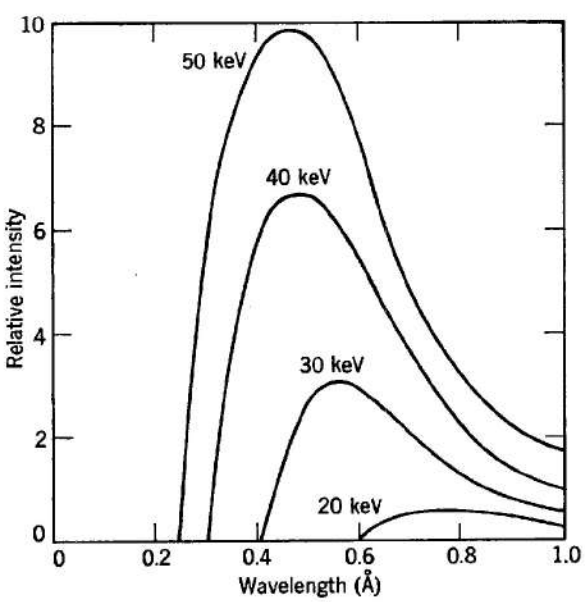

Figure 2-10 The continuous x-ray spectrum emitted from a tungsten target for four different values of eV, the incident electron energy.

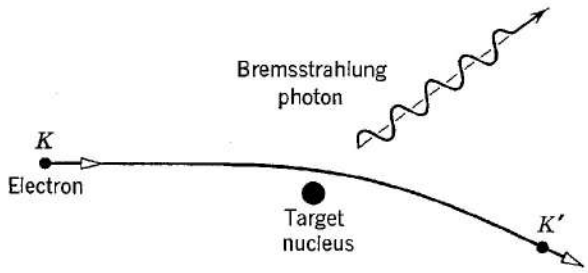

Figure 2-11 The bremsstrahlung process responsible for the production of x rays in the continuous spectrum.

decelerated during an encounter with a heavy target nucleus, the energy it loses appearing in the form of radiation as an x-ray photon. The electron interacts with the charged nucleus via the Coulomb field, transferring momentum to the nucleus. The accompanying deceleration of the electron leads to photon emission. The target nucleus is so massive that the energy it acquires during the collision can safely be neglected. If K' is the kinetic energy of the electron after the encounter, then the energy of the photon is

$$h\nu = K - K'$$

and the photon wavelength follows from

$$hc/\lambda = K - K' \tag{2-13}$$

Electrons in the incident beam can lose different amounts of energy in such encounters and typically a single electron will be brought to rest only after many encounters. The x rays thus produced by many electrons make up the continuous spectrum of Figure 2-10 and are very many discrete photons whose wavelengths vary from $\lambda_{\min}$ to $\lambda \to \infty$, corresponding to the different energy losses in the individual encounters. The shortest wavelength photon would be emitted when an electron loses *all* its kinetic energy in one deceleration process; here $K' = 0$ so that $K = hc/\lambda_{\min}$. Since K equals eV, the energy acquired by the electron in being accelerated through the potential difference V applied to the x-ray tube, we have

$$eV = hc/\lambda_{\min}$$

or

$$\lambda_{\min} = hc/eV \tag{2-14}$$

Thus the minimum wavelength cutoff represents the complete conversion of the electron's kinetic energy to x radiation. Equation (2-14) shows clearly that if $h \to 0$ then $\lambda_{\min} \to 0$, which is the prediction of classical theory. This shows that the very existence of a minimum wavelength is a quantum phenomenon.

The continuous x radiation of Figure 2-10 is often called *bremsstrahlung*, from the German *brems* (= braking, i.e., decelerating) + *strahlung* (= radiation). The bremsstrahlung process occurs not only in x-ray tubes but wherever fast electrons collide with matter, as in cosmic rays, in the van Allen radiation belts which surround the earth, and in the stopping of electrons emerging from accelerators or radioactive nuclei. The bremsstrahlung process can be considered as an inverse photoelectric effect: in the photoelectric effect, a photon is absorbed, its energy and momentum going to an electron and a recoiling nucleus; in the bremsstrahlung process, a photon is created, its energy and momentum coming from a colliding electron and nucleus. We deal with the *creation* of photons in the bremsstrahlung process, rather than with their absorption or scattering by matter.

Example 2-5. Determine Planck's constant h from the fact that the minimum x-ray wavelength produced by 40.0 keV electrons is 3.11×10^{-11} m.

▶ From (2-14), we have

$$h = \frac{eV\lambda_{\min}}{c}$$

$$= \frac{1.60 \times 10^{-19}\ \text{coul} \times 4.00 \times 10^{4}\ \text{V} \times 3.11 \times 10^{-11}\ \text{m}}{3.00 \times 10^{8}\ \text{m/sec}}$$

$$= 6.64 \times 10^{-34}\ \text{joule-sec}$$

This agrees well with the value of h deduced from the photoelectric effect and the Compton effect.

Measurement of V, $\lambda_{\min}$, and c provides one of the most accurate methods for evaluating the ratio h/e. Bearden, Johnson, and Watts at the Johns Hopkins University found in 1951, using this procedure, $h/e = 1.37028 \times 10^{-15}$ joule-sec/coul. This ratio is combined with many other measured combinations of physical constants, the assembly of data being analyzed by elaborate statistical methods to find the "best" value for the various physical constants. The best values change (but usually only within the *a priori* estimates of accuracy) and become increasingly precise as new experimental data and higher precision methods are used. ◀

2-7 PAIR PRODUCTION AND PAIR ANNIHILATION

In addition to the photoelectric and Compton effects there is another process whereby photons lose their energy in interactions with matter, namely the process of *pair production*. Pair production is also an excellent example of the conversion of radiant energy into rest mass energy as well as into kinetic energy. In this process, illustrated schematically in Figure 2-12, a high energy photon loses all of its energy $h\nu$ in an encounter with a nucleus, creating an electron and a positron (the *pair*) and endowing them with kinetic energies. A *positron* is a particle which is identical in all of its properties with an electron, except that the sign of its charge (and of its magnetic moment) is opposite to that of an electron; a positron is a positively charged electron. In pair production the energy taken by the recoil of the nucleus is negligible because it is so massive, and thus the balance of total relativistic energy in the process is simply

$$h\nu = E_- + E_+ = (m_0c^2 + K_-) + (m_0c^2 + K_+) = K_- + K_+ + 2m_0c^2 \quad (2\text{-}15)$$

In this expression E_- and E_+ are the total relativistic energies, and K_- and K_+ are the kinetic energies of the electron and positron, respectively. Both particles have the same rest mass energy m_0c^2. The positron is produced with a slightly larger kinetic energy than the electron because the Coulomb interaction of the pair with the positively charged nucleus leads to an acceleration of the positron and a deceleration of the electron.

In analyzing this process here we ignore the details of the interaction itself, considering only the situation before and after the interaction. Our guiding principles are the conservation of total relativistic energy, conservation of momentum, and conservation of charge. From these conservation laws, it is not difficult to show that a photon cannot simply disappear in empty space, creating a pair as it vanishes. The

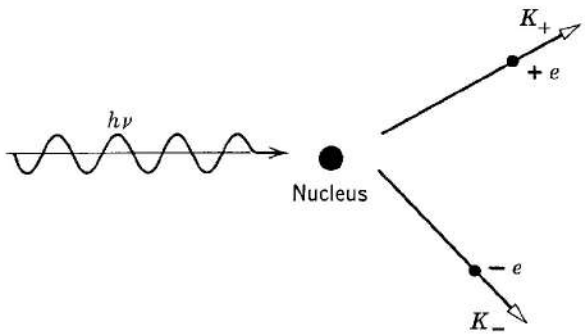

Figure 2-12 The pair production process.

presence of the massive nucleus (which can absorb momentum without appreciably affecting the energy balance) is necessary to allow both energy and momentum to be conserved in the process. Charge is automatically conserved, the photon having no charge and the created pair of particles having no net charge. From (2-15) we see that the minimum, or threshold, energy needed by a photon to create a pair is $2m_0c^2$ or 1.02 MeV (1 MeV = 10^6 eV), which is a wavelength of 0.012 Å. If the wavelength is shorter than this, corresponding to an energy greater than the threshold value, the photon endows the pair with kinetic energy as well as rest mass energy. The pair production phenomenon is a high-energy one, the photons being in the very short x-ray or γ-ray regions of the electromagnetic spectrum (see Figure 2-4), where their energies $h\nu$ are equal to or greater than $2m_0c^2$. As we shall see in the next section, experimental results demonstrate that the absorption of photons in interaction with matter occurs principally by the photoelectric process at low energies, by the Compton effect at medium energies, and by pair production at high energies.

Electron-positron pairs are produced in nature by cosmic-ray photons and in the laboratory by bremsstrahlung photons from particle accelerators. Other particle pairs, such as proton and antiproton, can be produced as well if the initiating photon has sufficient energy. Because the electron and positron have the smallest rest mass of known particles, the threshold energy of their production is the smallest. Experiment verifies the quantum picture of the pair production process. There is no satisfactory explanation whatever of this phenomenon in classical theory.

Example 2-6. Analysis of a bubble chamber photograph (as in Figure 2-13) reveals the creation of an electron-positron pair as photons pass through matter. The electron and positron tracks have opposite curvatures in the uniform magnetic field B of 0.20 weber/m^2, their radii r each being 2.5×10^{-2} m. What was the energy and the wavelength of the pair producing photon?

▶The momentum p of the electron is given by

$$p = eBr = 1.6 \times 10^{-19} \text{ coul} \times 2.0 \times 10^{-1} \text{ weber/m}^2 \times 2.5 \times 10^{-2} \text{ m}$$
$$= 8.0 \times 10^{-22} \text{ kg-m/sec}$$

Its total relativistic energy E_- is given by

$$E_-^2 = c^2p^2 + (m_0c^2)^2$$

Since $m_0c^2 = 0.51$ MeV, and $pc = 8.0 \times 10^{-22}$ kg-m/sec $\times 3.0 \times 10^8$ m/sec $= 2.4 \times 10^{-13}$ joule $= 1.5$ MeV, we have $E_-^2 = (1.5 \text{ MeV})^2 + (0.51 \text{ MeV})^2$ and $E_- = 1.6$ MeV.

The positron's total relativistic energy had the same value since its track had the same radius, so the energy of the photon was

$$h\nu = E_- + E_+ = 3.2 \text{ MeV}$$

The photon's wavelength follows from

$$E = h\nu = hc/\lambda$$

or

$$\lambda = \frac{hc}{E} = \frac{6.6 \times 10^{-34} \text{ joule-sec} \times 3.0 \times 10^8 \text{ m/sec}}{3.2 \times 10^6 \text{ eV} \times 1.6 \times 10^{-19} \text{ joule/eV}} = 3.9 \times 10^{-13} \text{ m} = 0.0039 \text{ Å} \quad ◀$$

Closely related to pair production is the inverse process called *pair annihilation.* An electron and a positron, which are essentially at rest near one another, unite and are annihilated. Matter disappears and in its place we get radiant energy. Since the initial momentum of the system is zero and momentum must be conserved in the process, we cannot have only one photon created because a single photon cannot have zero momentum. The most probable process is the creation of two photons moving with equal momenta in opposite directions. Less probable, but possible, is the creation of three photons.

In the two-photon process illustrated by Figure 2-14, momentum conservation gives $0 = \mathbf{p}_1 + \mathbf{p}_2$ or $\mathbf{p}_1 = -\mathbf{p}_2$ so that the photon momenta are oppositely directed

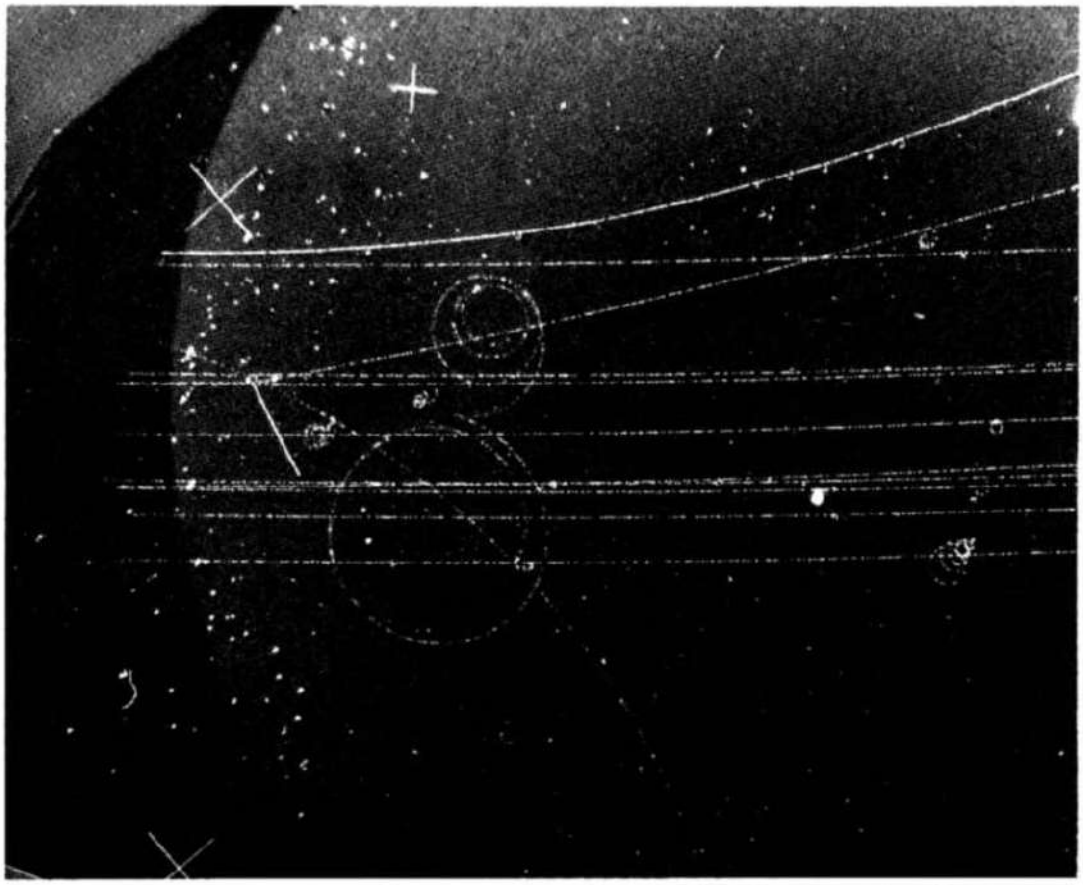

Figure 2-13 Electron pair production, as seen in a bubble chamber. The electron and positron tracks are the two spirals meeting at the point where the production took place in the liquid filling of the chamber. The student can determine which of the two spirals belongs to the positron by knowing that the long tracks are primarily positively charged deuterons which are incident from the left. (Courtesy of C. R. Sun, State University of New York at Albany)

but equal in magnitude. Hence, $p_1 = p_2$ or $h\nu_1/c = h\nu_2/c$ and $\nu_1 = \nu_2 = \nu$. Total relativistic energy conservation then requires that $m_0c^2 + m_0c^2 = h\nu + h\nu$, the positron and electron having no initial kinetic energy and the photon energies being the same. Hence, $h\nu = m_0c^2 = 0.51$ MeV, corresponding to a photon wavelength of 0.024 Å. If the initial pair had some kinetic energy then the photon energy would exceed 0.51 MeV and its wavelength could be less than 0.024 Å.

Positrons are created in the pair production process. On passing through matter a positron loses energy in successive collisions until it combines with an electron to form a bound system called *positronium*. The positronium "atom" is short lived, decaying into photons within about 10^{-10} sec of its formation. The electron and positron presumably move about their common center of mass in a kind of death dance before mutual annihilation.

Example 2-7. (a) Assume that Figure 2-14 represents the annihilation process in a reference frame S, the electron-positron pair being at rest there and the two annihilation photons moving along the x axis. Find the wavelength λ of these photons in terms of m_0, the rest mass of an electron or positron.

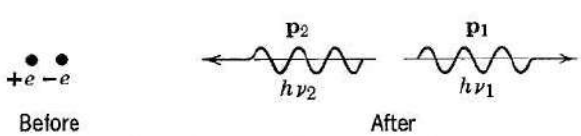

Figure 2-14 Pair annihilation producing two photons.

▶We saw that $p_1 = p_2$ and $h\nu_1 = h\nu_2$. Each photon has the same energy, the same frequency, and the same wavelength. We can drop the subscripts then and from the relation $h\nu = m_0c^2$ and $p = E/c$ we obtain

$$p = E/c = h\nu/c = m_0c^2/c = m_0c$$

But we also have the relation

$$p = h/\lambda$$

so that

$$\lambda = h/p = h/m_0c$$

Hence, in the rest frame of the positronium atom each photon has the same wavelength, $\lambda = h/m_0c$. ◀

(b) Now consider the same annihilation event to be observed in frame S', moving relative to S with a velocity v to the left. What wavelength does this (moving) observer record for the annihilation photons?

▶Here, the pair has initial total relativistic energy $2mc^2$, where m is relativistic mass, rather than merely the rest mass energy $2m_0c^2$, so that conservation of energy in the annihilation process gives us

$$2mc^2 = p'_1c + p'_2c$$

Also, the pair now moves with velocity v along the positive x' axis so that its initial momentum is $2mv$, rather than zero as before. Conservation of momentum now gives us

$$2mv = p'_1 - p'_2$$

the photons moving in opposite directions also the x' axis. Let us combine these two expressions. We multiply the second by c and add it to the first, obtaining, since $m = m_0/\sqrt{1 - v^2/c^2}$

$$p'_1 = m(c + v) = \frac{m_0(c + v)}{\sqrt{1 - v^2/c^2}} = m_0c\sqrt{\frac{c + v}{c - v}}$$

But $p'_1 = h/\lambda'_1$, so that

$$\lambda'_1 = \frac{h}{p'_1} = \frac{h}{m_0c}\sqrt{\frac{c - v}{c + v}} = \lambda\sqrt{\frac{c - v}{c + v}} \tag{2-16a}$$

In a similar manner, by subtracting the second equation from the first, we obtain

$$\lambda'_2 = \frac{h}{p'_2} = \frac{h}{m_0c}\sqrt{\frac{c + v}{c - v}} = \lambda\sqrt{\frac{c + v}{c - v}} \tag{2-16b}$$

The photons do *not* have the same wavelength, but they are *Doppler shifted* from the wavelength λ they had in the rest frame of the source (the positronium atom). If an observer is situated on the x' axis so that the source moves *toward* him, he will receive photon 1, having a frequency *higher* than the "rest" frequency. If an observer is situated on the x' axis so that the source moves *away* from him, he will receive photon 2, having a frequency *lower* than the rest frequency. This Example is actually a derivation of the *longitudinal Doppler shift formula* of relativity theory. ◀

The first experimental evidence for the pair production process, and the existence of positrons, was obtained in 1933 by Anderson during an investigation of the cosmic radiation. This radiation consists of a flux of very high energy photons and charged particles incident upon the earth from extra-terrestrial sources. Anderson was using a cloud chamber containing a thin lead plate, with the entire apparatus in a magnetic field. Upon exposing this apparatus to the cosmic radiation, it was found that very infrequently a pair of charged particles was ejected from some point in the lead plate. These events were assumed to be the result of the interaction of a photon in the lead because no charged particle was seen to strike the point of ejection, whereas a photon, being uncharged, could strike the point of ejection without being seen. The two charged particles ejected in these events were bent in opposite directions by the magnetic field. Therefore their charges were of opposite sign. From other considerations it could be shown that the magnitudes of these charges were equal to one electronic charge and that the masses of the particles were approximately equal to one electronic mass.

The discovery of the pair production process explained the origin of a discrepancy between the then current theory of x-ray attenuation and the measured attenuation coefficients of several materials for 2.6 MeV x rays (γ rays obtained from a radioactive source). As the theory originally did not include pair production, the predicted attenuation was too small; with the inclusion of the pair production process, good agreement is now obtained between experiment and theory. However, the real importance of Anderson's discovery was in the beautiful confirmation which it provided for Dirac's relativistic quantum mechanical theory of the electron.

The Dirac theory leads to the prediction that the allowed values of total relativistic energy E for a free electron are

$$E = \pm\sqrt{c^2p^2 + (m_0c^2)^2} \tag{2-17}$$

where m_0 is the electron rest mass. These are simply the solutions for E of (2-6), but the solution with the minus sign corresponds to a negative total relativistic energy—a concept as foreign to relativistic mechanics as a negative total energy is to classical mechanics. Instead of just throwing away the negative part on the grounds that it is not physically realistic, Dirac pursued the consequences of the entire equation. In doing this he was led to some very interesting conclusions. Consider Figure 2-15, which is an energy-level diagram representing (2-17). If the indicated continuum of negative energy levels exists, all free electrons of positive energy should be able to make transitions into these levels, accompanied by the emission of photons of the appropriate energies. This obviously disagrees with experiment because free electrons are not generally observed to emit spontaneously photons of energy $h\nu \geq 2m_0c^2$. However, Dirac pointed out that this difficulty can be removed by assuming that all the negative energy levels are normally filled at all points in space. According to this assumption, *a vacuum consists of a sea of electrons in negative energy levels.* This does not disagree with experiment. For instance, the negative charge could not be detected, as it is assumed to be uniformly distributed and therefore exerts no force on a charged body. Similar considerations will demonstrate that all the "usual" properties of a sea of negative energy electrons are such that its presence would not be apparent in any of the usual experiments. However, Dirac's theory of the vacuum is not completely vacuous because it predicts certain new properties which can be tested by experiment.

The energy-level diagram for a free electron suggests the possibility of exciting an electron in a negative energy level by the absorption of a photon. Since all the negative energy levels are assumed to be fully occupied, the electron must be excited to one of the unoccupied positive energy levels. The minimum photon energy required for this process is obviously $h\nu = 2m_0c^2$, and the process results in the production of an electron in a positive energy level plus a hole in a negative energy level. We can demonstrate that a hole in a negative electron energy level has all the mechanical and electrical properties of a positron of positive energy. For instance, there is a positive charge $+e$ associated with the absence of an electron of negative charge $-e$. Consequently, this is the pair production process observed experimentally by Anderson three years after its theoretical prediction by Dirac.

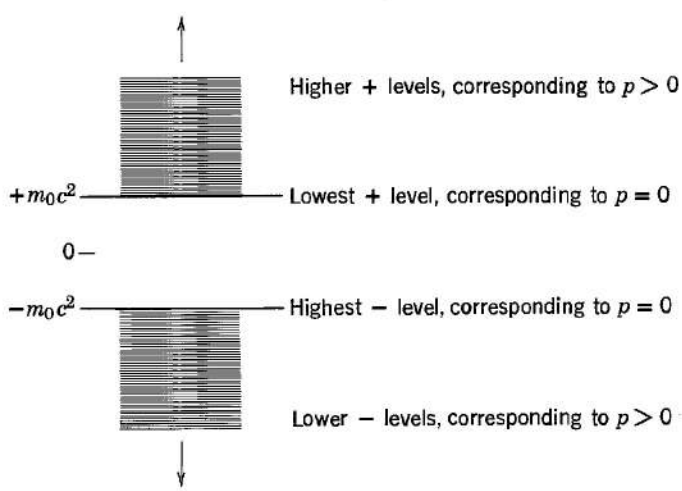

Figure 2-15 The energy levels of a free electron according to Dirac.

2-8 CROSS SECTIONS FOR PHOTON ABSORPTION AND SCATTERING

Consider a parallel beam of photons passing through a slab of matter, as in Figure 2-16. The photons can interact with the atoms in the slab by four different processes: photoelectric, pair production, Rayleigh, and Compton. The first two absorb photons completely, while the last two only scatter them, but all the processes remove photons from the parallel beam. The question of what the chances of these processes happening are, in a given set of circumstances, is one of considerable theoretical and practical significance. For instance, it is very important to a medical physicist designing the shielding for an x-ray machine, or a nuclear engineer designing the shielding for a reactor. The answer to the question is expressed in terms of quantities called cross sections. We first meet cross sections here in connection with photons, but we shall encounter them again in other connections elsewhere in this book.

The probability that a photon of a given energy will be, for example, absorbed by the photoelectric process in passing an atom of the slab is specified by the value of the photoelectric *cross section* σ_{PE}. This measure of the likelihood of the photoelectric process occurring is defined so that the number N_{PE} of photoelectric absorptions occurring is

$$N_{PE} = \sigma_{PE} I n \tag{2-18}$$

when a beam containing I photons is incident on a slab containing n atoms per unit area. It is assumed here that the slab is thin enough that the probability of a given photon being absorbed in passing through the slab is much smaller than one.

The definition of (2-18), which is a prototype of the definitions of all cross sections, is sufficiently important to warrant careful physical interpretation. First note that the number N_{PE} of absorptions should certainly increase in proportion to the number I of photons incident on the slab. Furthermore, it the slab is thin in the sense specified previously the atoms in the slab will not appreciably "shadow" each other, as far as the incident photons are concerned. Then the number N_{PE} of absorptions should also increase in proportion to the number n of target atoms per unit area of the slab. Thus we should have

$$N_{PE} \propto I n$$

If we write this proportionality as an equality, calling the proportionality constant σ_{PE}, we obtain the defining equation for that cross section. Thus we see that the cross section, which has a value depending on both the energy of the photon and the type of atom, measures how effective such atoms are in absorbing those photons by the photoelectric effect. Since the quantities N_{PE} and I in (2-18) are dimensionless, while n has the dimensions of $(\text{area})^{-1}$, it is clear that σ_{PE} must have the dimensions of (area). Thus it is reasonable to use the name cross section for σ_{PE}. It is often given

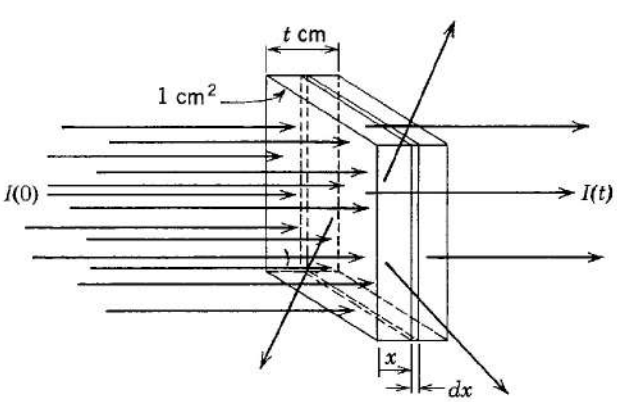

Figure 2-16 A beam of photons passing through a slab.

a geometrical interpretation by imagining that a circle of area σ_{PE} is centered on each atom in the slab in the plane of the slab, with the property that any photon entering the circular area is absorbed by the atom through the photoelectric effect. This geometrical interpretation is convenient for visualization and even for calculation, but it definitely should not be taken to be literally true. A cross section is really just a way of expressing numerically the probability that a certain type of atom will cause a photon of a given energy to undergo a particular process. The definitions and interpretations of the cross sections for the other absorption or scattering processes are completely analogous to those for the example we have considered.

Figure 2-17 shows the measured scattering (σ_S), photoelectric (σ_{PE}), pair production (σ_{PR}), and total (σ) cross sections for a lead atom as a function of the photon energy $h\nu$. The scattering cross section specifies the probability of scattering occurring by either the Rayleigh or the Compton process. For lead, which has a high atomic number and thus tightly bound atomic electrons, Rayleigh scattering dominates Compton scattering when the photon energy is below about $h\nu = 10^5$ eV. The sharp breaks in the photoelectric cross section occur at the binding energies of the different electrons in the lead atom; when $h\nu$ drops below the binding energy of a particular electron a photoelectric process involving it is no longer energetically possible. The pair production cross section rises very rapidly from zero when $h\nu$ exceeds the threshold energy $2m_0c^2 \simeq 10^6$ eV required to materialize a pair. The total cross section σ in Figure 2-17 is the sum of the scattering, photoelectric, and pair production cross sections. This quantity specifies the probability that a photon will make any kind of interaction with the atom. We see from the figure that the energy ranges in which each of the three processes makes the most important contribution to σ are approximately, for lead:

Photoelectric effect:	$h\nu < 5 \times 10^5$ eV
Scattering:	5×10^5 eV $< h\nu < 5 \times 10^6$ eV
Pair production:	5×10^6 eV $< h\nu$

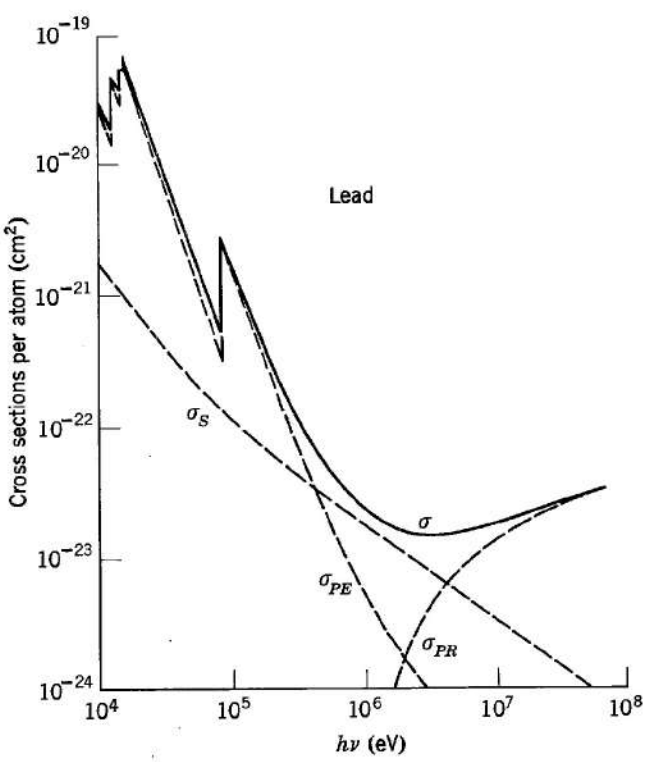

Figure 2-17 The scattering, photoelectric, pair production, and total cross sections for a lead atom.

Because these processes have probabilities with different dependences on atomic number, the energy ranges in which they dominate are quite different for atoms of low atomic number. The energy ranges are approximately, for aluminum:

Photoelectric effect:	$h\nu < 5 \times 10^4$ eV
Scattering:	5×10^4 eV $< h\nu < 1 \times 10^7$ eV
Pair production:	1×10^7 eV $< h\nu$

Example 2-8. Evaluate, in terms of the total cross section σ, the attenuation of a parallel beam of x rays in passing through a *thick* slab of matter.

▶ Referring to Figure 2-16, $I(0)$ photons are in the beam as it is incident on the front face of the slab of thickness t, which contains ρ atoms per cm^3. Assume, for simplicity, that the area of the slab is 1 cm^2. Because of scattering and absorption processes, the parallel beam contains a smaller number $I(x)$ of photons after penetrating x cm into the slab. Consider a thin lamina of the slab, of width dx located at x. The number of atoms per cm^2 in the lamina is ρ times its volume dx, or $\rho\, dx$. The number of beam photons that will be scattered or absorbed in the lamina is specified by the total cross section σ, in a definition analogous to (2-18). It is $\sigma I(x)\rho\, dx$. Thus the number of beam photons emerging from the lamina, $I(x + dx)$, which equals the number incident minus the number removed, is

$$I(x + dx) = I(x) - \sigma I(x)\rho\, dx$$

or

$$dI(x) \equiv I(x + dx) - I(x) = -\sigma I(x)\rho\, dx$$

We find $I(t)$, the number of beam photons emerging from the rear face of the slab, by solving for $dI(x)/I(x)$ and then integrating over x

$$\frac{dI(x)}{I(x)} = -\sigma\rho\, dx$$

$$\int_0^t \frac{dI(x)}{I(x)} = -\sigma\rho \int_0^t dx$$

$$\ln I(x)\Big]_0^t = -\sigma\rho\, t$$

$$\ln \frac{I(t)}{I(0)} = -\sigma\rho\, t$$

$$\frac{I(t)}{I(0)} = e^{-\sigma\rho t}$$

$$I(t) = I(0)e^{-\sigma\rho t} \tag{2-19}$$

The intensity of the beam, as measured by the number I of photons it contains, decreases exponentially as the thickness t of the slab increases. The quantity $\sigma\rho$, which is called the *attenuation coefficient*, has the dimensions (cm^{-1}) and is the reciprocal of the thickness of slab required to attenuate the beam intensity by a factor of e. This thickness is called the *attenuation length* Λ. That is

$$\Lambda = 1/\sigma\rho \tag{2-20}$$

Of course, the attenuation coefficient has the same dependence on photon energy as the total cross section. Figure 2-18 shows measured attenuation coefficients of lead, tin, and aluminum for photons of relatively high energy. ◀

This section summarizes many of the practical aspects of the electromagnetic radiation emission and absorption phenomena we have studied in the present chapter. But the fundamental aspects of these phenomena are better summarized by saying that they show electromagnetic radiation to be quantized into particles of energy called photons. It should also be emphasized that the phenomena of interference and diffraction show photons do not travel through a system from where they are emitted to where they are absorbed in the simple way that classical particles do. Instead, photons act as if they were guided by classical waves because photons travel through

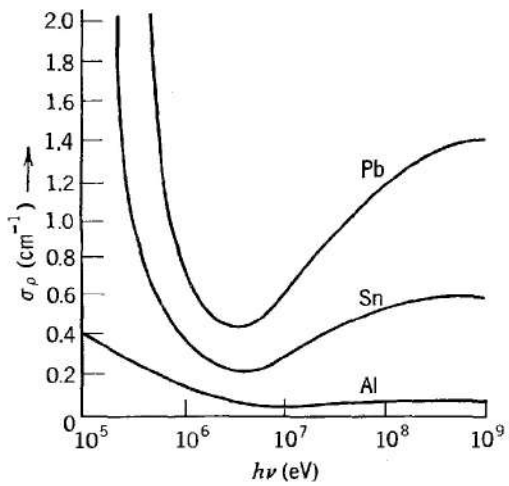

Figure 2-18 The attenuation coefficients for several atoms and a range of photon energies.

a system such as a diffraction apparatus in a way that is best described by the way that classical waves would propagate through the apparatus.

QUESTIONS

1. In the photoelectric experiments, the current (number of electrons emitted per unit time) is proportional to the intensity of light. Can this result alone be used to distinguish between the classical and quantum theories?
2. In Figure 2-2 why does the photoelectric current not rise vertically to its maximum (saturation) value when the applied potential difference is slightly more positive than $-V_0$?
3. Why is it that even for incident radiation that is monochromatic, photoelectrons are emitted with a spread of velocities?
4. The existence of a cutoff frequency in the photoelectric effect is often regarded as the most potent objection to a wave theory. Explain.
5. Why are photoelectric measurements very sensitive to the nature of the photoelectric surface?
6. Do the results of photoelectric experiments invalidate Young's interference experiment?
7. Can you use the device of letting $h \to 0$ to obtain classical results from quantum results in the case of the photoelectric effect? Explain.
8. Assume that the emission of photons from a source of radiation is random in direction. Would you expect the intensity (or energy density) to vary inversely as the square of the distance from the source in the photon theory as it does in the wave theory?
9. Does a photon of energy E have mass? If so, evaluate it.
10. Why, in Compton scattering, would you expect $\Delta\lambda$ to be independent of the materials of which the scatterer is composed?
11. Would you expect to observe the Compton effect more readily with scattering targets composed of atoms with high atomic number or those composed of atoms with low atomic number? Explain.
12. Do you observe a Compton effect with visible light? Why?
13. Would you expect a definite minimum wavelength in the emitted radiation for a given value of the energy of an electron incident on the target of an x-ray tube from the classical electromagnetic theory of the process?
14. Does a television tube emit x rays? Explain.
15. What effect(s) does decreasing the voltage across an x-ray tube have on the resulting x-ray spectrum?

16. Discuss the bremsstrahlung process as the inverse of the Compton process. Of the photoelectric process.
17. Describe several methods that can be used to determine experimentally the value of Planck's constant h.
18. From what factors would you expect to judge whether a photon will lose its energy in interactions with matter by the photoelectric process, the Compton process, or the pair production process?
19. Can you think of experimental evidence contradicting the idea that vacuum is a sea of electrons in negative energy states?
20. Can electron-positron annihilation occur with the creation of *one* photon if a nearby nucleus is available for recoil momentum?
21. Explain how pair annihilation with the creation of *three* photons is possible. Is it possible in principle to create even more than three photons in a single annihilation process?
22. What would be the inverse of the process in which two photons are created in electron-positron annihilation? Can it occur? Is it likely to occur?
23. What is wrong with taking the geometrical interpretation of a cross section as literally true?

PROBLEMS

1. (a) The energy required to remove an electron from sodium is 2.3 eV. Does sodium show a photoelectric effect for yellow light, with $\lambda = 5890$ Å? (b) What is the cutoff wavelength for photoelectric emission from sodium?
2. Light of a wavelength 2000 Å falls on an aluminum surface. In aluminum 4.2 eV are required to remove an electron. What is the kinetic energy of (a) the fastest and (b) the slowest emitted photoelectrons? (c) What is the stopping potential? (d) What is the cutoff wavelength for aluminum? (e) If the intensity of the incident light is 2.0 W/m^2, what is the average number of photons per unit time per unit area that strike the surface?
3. The work function for a clean lithium surface is 2.3 eV. Make a rough plot of the stopping potential V_0 versus the frequency of the incident light for such a surface, indicating its important features.
4. The stopping potential for photoelectrons emitted from a surface illuminated by light of wavelength $\lambda = 4910$ Å is 0.71 V. When the incident wavelength is changed the stopping potential is found to be 1.43 V. What is the new wavelength?
5. In a photoelectric experiment in which monochromatic light and a sodium photocathode are used, we find a stopping potential of 1.85 V for $\lambda = 3000$ Å and of 0.82 V for $\lambda = 4000$ Å. From these data determine (a) a value for Planck's constant, (b) the work function of sodium in electron volts, and (c) the threshold wavelength for sodium.
6. Consider light shining on a photographic plate. The light will be recorded if it dissociates an AgBr molecule in the plate. The minimum energy to dissociate this molecule is of the order of 10^{-19} joule. Evaluate the cutoff wavelength greater than which light will not be recorded.
7. The relativistic expression for kinetic energy should be used for the electron in the photoelectric effect when $v/c > 0.1$, if errors greater than about 1% are to be avoided. For photoelectrons ejected from an aluminum surface ($w_0 = 4.2$ eV) what is the smallest wavelength of an incident photon for which the classical expression may be used?
8. X rays with $\lambda = 0.71$ Å eject photoelectrons from a gold foil. The electrons form circular paths of radius r in a region of magnetic induction B. Experiment shows that $rB = 1.88 \times 10^{-4}$ tesla-m. Find (a) the maximum kinetic energy of the photoelectrons and (b) the work done in removing the electron from the gold foil.
9. (a) Show that a free electron cannot absorb a photon and conserve both energy and momentum in the process. Hence, the photoelectric process requires a bound electron. (b) In the Compton effect, however, the electron *can* be free. Explain.

10. Under ideal conditions the normal human eye will record a visual sensation at 5500 Å if as few as 100 photons are absorbed per second. What power level does this correspond to?

11. An ultraviolet lightbulb, emitting at 4000 Å, and an infrared lightbulb, emitting at 7000 Å, each are rated at 40 W. (a) Which bulb radiates photons at the greater rate, and (b) how many more photons does it produce each second over the other bulb?

12. Solar radiation falls on the earth at a rate of 1.94 cal/cm^2-min on a surface normal to the incoming rays. Assuming an average wavelength of 5500 Å, how many photons per cm^2-min is this?

13. What are the frequency, wavelength, and momentum of a photon whose energy equals the rest mass energy of an electron?

14. In the photon picture of radiation, show that if beams of radiation of two different wavelengths are to have the same intensity (or energy density) then the numbers of the photons per unit cross-sectional area per sec in the beams are in the same ratio as the wavelengths.

15. Derive the relation

$$\cot\frac{\theta}{2} = \left(1 + \frac{h\nu}{m_0c^2}\right)\tan\varphi$$

between the direction of motion of the scattered photon and the recoil electron in the Compton effect.

16. Derive a relation between the kinetic energy K of the recoil electron and the energy E of the incident photon in the Compton effect. One form of the relation is

$$\frac{K}{E} = \frac{\left(\dfrac{2h\nu}{m_0c^2}\right)\sin^2\dfrac{\theta}{2}}{1 + \left(\dfrac{2h\nu}{m_0c^2}\right)\sin^2\dfrac{\theta}{2}}$$

(Hint: See Example 2-4.)

17. Photons of wavelength 0.024 Å are incident on free electrons. (a) Find the wavelength of a photon which is scattered 30° from the incident direction and the kinetic energy imparted to the recoil electron. (b) Do the same if the scattering angle is 120°. (Hint: See Example 2-4.)

18. An x-ray photon of initial energy 1.0×10^5 eV traveling in the $+x$ direction is incident on a free electron at rest. The photon is scattered at right angles into the $+y$ direction. Find the components of momentum of the recoiling electron.

19. (a) Show that $\Delta E/E$, the fractional change in photon energy in the Compton effect, equals $(h\nu'/m_0c^2)(1 - \cos\theta)$. (b) Plot $\Delta E/E$ versus θ and interpret the curve physically.

20. What fractional increase in wavelength leads to a 75% loss of photon energy in a Compton collision?

21. Through what angle must a 0.20 MeV photon be scattered by a free electron so that it loses 10% of its energy?

22. What is the maximum possible kinetic energy of a recoiling Compton electron in terms of the incident photon energy $h\nu$ and the electron's rest energy m_0c^2?

23. Determine the maximum wavelength shift in the Compton scattering of photons from *protons*.

24. (a) Show that the short wavelength cutoff in the x-ray continuous spectrum is given by $\lambda_{min} = 12.4$ Å$/V$, where V is applied voltage in kilovolts. (b) If the voltage across an x-ray tube is 186 kV what is λ_{min}?

25. (a) What is the minimum voltage across an x-ray tube that will produce an x ray having the Compton wavelength? A wavelength of 1 Å? (b) What is the minimum voltage needed

across an x-ray tube if the subsequent bremsstrahlung radiation is to be capable of pair production?

26. A 20 KeV electron emits *two* bremsstrahlung photons as it is being brought to rest in two successive decelerations. The wavelength of the second photon is 1.30 Å longer than the wavelength of the first. (a) What was the energy of the electron after the first deceleration, and (b) what are the wavelengths of the photons?

27. A γ ray creates an electron-positron pair. Show directly that, without the presence of a third body to take up some of the momentum, energy and momentum cannot both be conserved. (Hint: Set the energies equal and show that this leads to unequal momenta before and after the interaction.)

28. A γ ray can produce an electron-positron pair in the neighborhood of an electron at rest as well as a nucleus. Show that in this case the threshold energy is $4m_0c^2$. (Hint: Do not ignore the recoil of the original electron, but assume that all three particles move off together.)

29. A particular pair is produced such that the positron is at rest and the electron has a kinetic energy of 1.0 MeV moving in the direction of flight of the pair-producing photon. (a) Neglecting the energy transferred to the nucleus of the nearby atom, find the energy of the incident photon. (b) What percentage of the photon's momentum is transferred to the nucleus?

30. Assume that an electron-positron pair is formed by a photon having the threshold energy for the process. (a) Calculate the momentum transferred to the nucleus in the process. (b) Assume the nucleus to be that of a lead atom and compute the kinetic energy of the recoil nucleus. Are we justified in neglecting this energy compared to the threshold energy assumed above?

31. An electron-positron pair at rest annihilate, creating two photons. At what speed must an observer move along the line of the photons in order that the wavelength of one photon be twice that of the other?

32. Show that the results of Example 2-8, expressed in terms of ρ and t, are valid independent of the assumed area of the slab.

33. Show that the attenuation length Λ is just equal to the average distance a photon will travel before being scattered or absorbed.

34. Use the data of Figure 2-17 to calculate the thickness of a lead slab which will attenuate a beam of 10 keV x rays by a factor of 100.