

Solutions Manual for  
**Advanced Engineering Thermodynamics**  
Third Edition

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# **CONTENTS**

<b>Chapter</b>	<b>Page</b>
1. The First Law of Thermodynamics	1 - 1
2. The Second Law of Thermodynamics	2 - 1
3. Entropy Generation, or Exergy Destruction	3 - 1
4. Single-Phase Systems	4 - 1
5. Exergy Analysis	5 - 1
6. Multi-Phase Systems	6 - 1
7. Chemically Reactive Systems	7 - 1
8. Power Generation	8 - 1
9. Solar Power	9 - 1
10. Refrigeration	10 - 1
11. Thermodynamic Optimization	11 - 1
12. Irreversible Thermodynamics	12 - 1
13. The Constructal Law of Flow Configuration Generation	13 - 1

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## Chapter 1

### THE FIRST LAW OF THERMODYNAMICS

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Problem 1.1 (a)  $W_{i-f} = \int_i^f PdV = P_1 V_2$ . Next, to calculate  $T_f$  we note that from (i) to (f) we have

$$\frac{dM}{dt} = \dot{m}$$

$$\frac{dU}{dt} = -\dot{W} + \dot{m}h_1$$

where  $\dot{m}$  is the instantaneous flowrate into the cylinder, and  $M$  and  $U$  are the mass and energy inventories of the system (the "system" is the cylinder volume). Integrating in time,

$$M_f - M_i = \int_i^f \dot{m} dt$$

$$U_f - U_i = -P_1 V_2 + h_1 (M_f - M_i) \quad (1)$$

and recognizing that  $U_i = 0$  and  $M_i = 0$ , the first law reduces to

$$U_f = M_f h_1 - P_1 V_2 \quad (1')$$

For the "ideal gas" working fluid we write

$$U_f = M_f c_v (T_f - T_0)$$

$$h_1 = c_v (T_1 - T_0) + P v_1$$

hence, eq. (1') becomes

$$M_f c_v (T - T_0) = M_f [c_v (T_1 - T_0) + P v_1] - P_1 V_2$$

Noting that  $V_2 = M_f v_f$  and dividing everything by  $M_f$  yields

$$c_v T_f + P_1 v_f = c_v T_1 + P v_1$$

or

$$c_v T_f + R T_f = c_v T_1 + R T_1$$

in other words,  $T_f = T_1$ . The final ideal-gas mass admitted is

$$m_f = M_f = \frac{P_1 V_2}{R T_1}$$

hence the goodness ratio

$$\frac{W_{i-f}}{m_f} = \frac{P_1 V_2}{P_1 V_2 / (RT_1)} = RT_1$$

(b)  $m_1 = P_1 V_1 / RT_1$ , based on the solution for  $m_f$  given in part (a), and

$$W_{i-f} = \int_0^{V_1} P dV + \int_{V_1}^{V_2} P dV \\ = P_1 V_1 + \frac{c_v}{R} (P_1 V_1 - P_2 V_2)$$

The second group of terms on the right-hand-side is the work output during the reversible & adiabatic expansion (path:  $PV^k = \text{constant}$ ). Finally, the goodness ratio is

$$\frac{W_{i-f}}{m_1} = \frac{P_1 V_1 + \frac{c_v}{R} (P_1 V_1 - P_2 V_2)}{P_1 V_1 / (RT_1)} = RT_1 \left[ 1 + \frac{c_v}{R} \left( 1 - \frac{P_2 V_2}{P_1 V_1} \right) \right]$$

(c) The relative goodness is

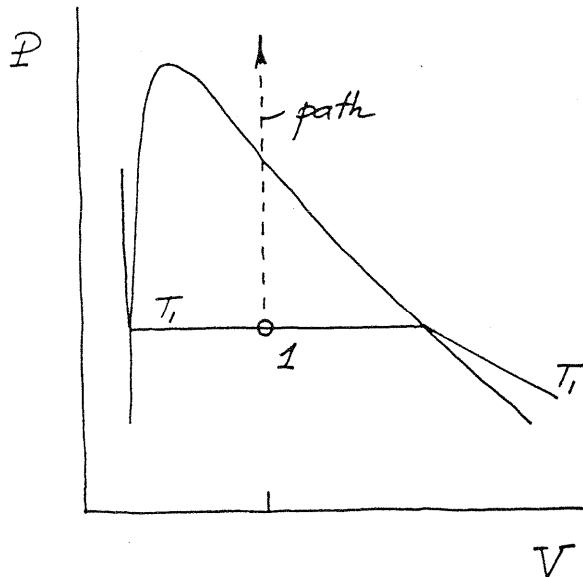
$$\frac{(W_{i-f}/m_1)_{\text{part (b)}}}{(W_{i-f}/m_f)_{\text{part (a)}}} = 1 + \frac{c_v}{R} \left( 1 - \frac{P_2 V_2}{P_1 V_1} \right) \\ = 1 + \frac{c_v}{R} \left[ 1 - \left( \frac{V_1}{V_2} \right)^{k-1} \right]$$

The quantity in the square brackets is positive because  $k > 1$  and  $V_1 < V_2$ , therefore

$$\left( \frac{W_{i-f}}{m_1} \right)_{\text{part (b)}} > \left( \frac{W_{i-f}}{m_f} \right)_{\text{part (a)}}$$


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Problem 1.2 (a) Given are  $m = 1 \text{ kg}$ ,  $T_1 = 100^\circ\text{C}$ , and  $x_1 = 0.5$ . The path is constant volume.



(b) To pinpoint state (2) we must determine two properties at the final state. The first one is the volume

$$\begin{aligned} v_2 &= v_1 = v_{f,T_1} + x_1 v_{fg,T_1} = \\ &= 0.001044 + 0.5 (1.6729 - 0.001044) \\ &= 0.837 \text{ m}^3 / \text{kg} \end{aligned}$$

The second property is the internal energy: this comes from the first law

$$Q_{1-2} - W_{1-2} = m(u_2 - u_1) \quad (1)$$

where  $W_{1-2} = 0$  and

$$u_1 = u_{f,T_1} + x_1 u_{fg,T_1} = 418.94 + (0.5) (2087.6) = 1462.74 \text{ kJ/kg}$$

Equation (1) yields

$$u_2 = u_1 + \frac{1}{m} Q_{1-2} = 3662 \text{ kJ/kg}$$

(c) To find  $T_2$  and  $P_2$  we must first locate state (2) on the  $P(v, t)$  surface (or tables). At state (2) we know  $u_2$  and  $v_2$ , therefore, one way to proceed is to look at the table of superheated steam properties and find the  $u$  values of order 3662 kJ/kg. This is the equivalent of traveling along the  $u = u_2$  line and looking for the  $v$  value that comes closest to  $v_2$ . This search leads to this portion of the table:

T	$P = 0.5 \text{ MPa}$		$P = 0.6 \text{ MPa}$	
	v	u	v	u
...	...	...	...	...
800°C	0.9896	3662.1	0.8245	3661.8
...	...	...	...	...

Fitting  $v_2$  between 0.9896 and 0.8245, we interpolate linearly for pressure and find

$$P_2 \approx 0.592 \text{ MPa}$$

The final temperature is clearly  $T_2 \approx 800^\circ\text{C}$ .

(d) At state (2) the system is superheated steam. This particular fluid approaches ideal gas behavior if near state (2) the following *two conditions* are met:

$$(i) \quad u = u(T)$$

$$(ii) \quad Pv = RT, \text{ i.e. } Pv / T = \text{constant.}$$

Condition (i) is satisfied, as shown by the  $u$  values listed in the preceding table ( $u$  depends on  $T$ , while being practically independent of  $P$ ). As a way of testing condition (ii), we calculate the group  $(Pv/T)$  for the states immediately to the left and right of state (2):

$$\left(\frac{Pv}{T}\right)_{\text{left}} = \frac{(0.5) 10^6 (0.9896)}{273.15 + 800} = 461.1 \frac{\text{Pa m}^3 / \text{kg}}{\text{K}}$$

$$\left(\frac{Pv}{T}\right)_{\text{right}} = \frac{(0.5) 10^6 (0.8245)}{273.15 + 800} = 461.0 \frac{\text{Pa m}^3 / \text{kg}}{\text{K}}$$

Condition (ii) is also satisfied (approximately, or course), therefore, the ideal gas model could be used to describe the behavior of the system at states that are sufficiently close to state (2).

Observation: note the use of absolute temperature in the denominators of the  $(Pv/T)$  calculations presented above.

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Problem 1.3 Taking the  $m$  gas as "system", we write the first law for the process (1) - (2),

$$Q_{1-2} - W_{1-2} = U_2 - U_1$$

which means

$$0 - P_{1A} (V_2 - V_1) = \frac{m}{3} c_v (T_2 - T_1) + \frac{m}{3} c_v (T_2 - T_1) + \frac{m}{3} c_v (T_2 - T_1)$$

or

$$- P_{1A} \left( \frac{mRT_2}{P_2} - \frac{\frac{m}{3} RT_1}{P_{1A}} - \frac{\frac{m}{3} RT_1}{P_{1B}} - \frac{\frac{m}{3} RT_1}{P_{1C}} \right) = mc_v (T_2 - T_1)$$

Noting that  $P_2 = P_{1A}$ , the above statement can be written as

$$\begin{aligned} \frac{T_2}{T_1} &= \frac{1}{1 + R/c_v} + \frac{1 + (P_{1A}/P_{1B}) + (P_{1A}/P_{1C})}{3(1 + R/c_v)} \cdot \frac{R}{c_v} \\ &= \frac{c_v}{c_p} + \frac{R}{3c_p} \left( 1 + \frac{P_{1A}}{P_{1B}} + \frac{P_{1A}}{P_{1C}} \right) \end{aligned}$$


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**Problem 1.4** The process is one of heating at constant volume. Let  $m_f$  and  $m_g$  represent the instantaneous liquid and vapor inventories in the system,

$$m_f + m_g = m, \quad (\text{constant})$$

Furthermore, the constant-volume constraint reads

$$m_f v_f + m_g v_g = V, \quad (\text{constant})$$

The first law of thermodynamics requires on a per unit time basis that

$$\dot{Q} - \dot{W} = \frac{dU}{dt}$$

or, since  $\dot{W} = 0$ ,

$$\begin{aligned} \dot{Q} &= \frac{d}{dt} (m_f u_f + m_g u_g) \\ &= u_f \frac{dm_f}{dt} + m_f \frac{du_f}{dt} + u_g \frac{dm_g}{dt} + m_g \frac{du_g}{dt} \\ &\quad \overbrace{\frac{du_f}{dP} \frac{dP}{dt}} \quad \overbrace{\frac{du_g}{dP} \frac{dP}{dt}} \end{aligned} \quad (3)$$

The time derivatives  $dm_f/dt$  and  $dm_g/dt$  follow from solving the system of two equations

$$\frac{d}{dt} \quad (1) \quad \text{and} \quad \frac{d}{dt} \quad (2)$$

The solution is

$$\frac{dm_f}{dt} = -\frac{A}{v_{fg}} \quad \text{and} \quad \frac{dm_g}{dt} = \frac{A}{v_{fg}} \quad (4)$$

where

$$A = -m_f \frac{dv_f}{dt} - m_g \frac{dv_g}{dt} = -m_f \frac{dv_f}{dP} \frac{dP}{dt} - m_g \frac{dv_g}{dP} \frac{dP}{dt}$$

Combining (3) and (4) we obtain after a few manipulations

$$\frac{dP}{dt} = \frac{\dot{Q}/m}{\frac{du_f}{dP} - \frac{u_{fg}}{v_{fg}} \frac{dv_f}{dP} + x \left( \frac{du_{fg}}{dP} - \frac{u_{fg}}{v_{fg}} \frac{dv_{fg}}{dP} \right)}$$


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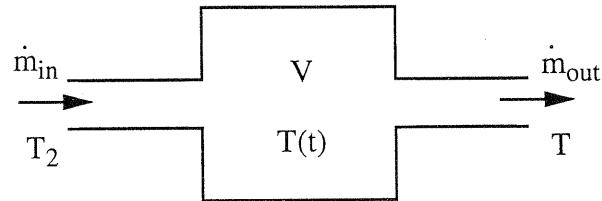
Problem 1.5 After the seal is broken the atmospheric air rushes into the glass tube and the pressure inside the tube becomes atmospheric. If we wait long enough, the temperature inside the tube becomes equal to the atmospheric temperature also (this scenario is described in Example 1.2 in the text). In that example we learned that en route to thermal equilibrium the bottle air rejects heat to the ambient. This means that immediately after we break the seal the air that occupies the glass tube is warmer than the ambient. If we dip the open end of the tube into the pool of water immediately after breaking the seal, the water will rise into the tube as the air mass that is trapped inside the glass tube cools down (and *shrinks*) to atmospheric temperature.

Problem 1.6 (a) Applying the first law to the water container as an open system, we have

$$\frac{d}{dt} (mu) = (\dot{mh})_{in} - (\dot{mh})_{out} \quad (1)$$

where

$$m = \frac{V}{v_w} = \text{constant}$$



Mass conservation dictates

$$\frac{d}{dt} (m) = \dot{m}_{in} - \dot{m}_{out} = 0$$

hence  $\dot{m}_{in} = \dot{m}_{out} = \dot{m}$ . The first law (1) reads finally

$$\frac{V}{v_w} \frac{du}{dt} = \dot{m} (h_{in} - h_{out})$$

For an incompressible fluid we also have

$$du = cdT \quad \text{and} \quad dh = cdT + v_w dP$$

In the present case  $P_{in} = P_{out}$ , therefore

$$h_{in} - h_{out} = c(T_{in} - T_{out}) = c(T_2 - T)$$

Equation (1) becomes

$$\frac{V}{v_w} c \frac{dT}{dt} = \dot{m}c (T_2 - T)$$

which, integrated from 0 to t, means

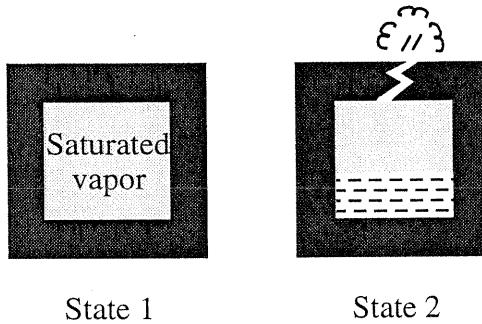
$$\ln \frac{T_2 - T}{T_2 - T_1} = -\frac{\dot{m}v_w}{V} t$$

- (b) The mass of hot water that raises the container water temperature from 10°C to 20°C is

$$\begin{aligned}\dot{m}t &= -\frac{V}{v_w} \ln \frac{T_2 - T}{T_2 - T_1} \\ &= -\frac{m^3}{10^{-3} m^3/kg} \ln \frac{40 - 20}{40 - 10} = 405.5 \text{ kg}\end{aligned}$$


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Problem 1.7 (a) We treat the instantaneous water inventory of the container as an open system operating unsteadily, so that the first law and the principle of mass conservation read



$$\frac{d}{dt} (mu) = \dot{Q} - \dot{W} + (\dot{m}h)_{in} - (\dot{m}h)_{out} \quad (1)$$

$$\frac{d}{dt} (m) = \dot{m}_{in} - \dot{m}_{out} \quad (2)$$

where  $\dot{Q} = \dot{W} = 0$  and  $\dot{m}_{in} = 0$ . Integrating (2) and writing  $m_1$  and  $m_2$  for the initial and final mass inventories of the system yields

$$\int_1^2 \dot{m}_{out} dt = m_1 - m_2$$

The objective is to calculate  $x_2$ , therefore, we focus on pinpointing state (2). We already know one property at that state (namely,  $P_2$ ). To obtain the second property we use the first law (1) in integral form

$$\begin{aligned}m_2 u_2 - m_1 u_1 &= -h_{out} \int_1^2 \dot{m}_{out} dt \\ &= h_{out} (m_2 - m_1)\end{aligned}$$

or the dimensionless form

$$\frac{u_2}{u_1} - \frac{m_1}{m_2} = \frac{h_{out}}{u_1} \left(1 - \frac{m_1}{m_2}\right) \quad (3)$$

Next, we note that the volume of the container is fixed,

$$V = m_1 v_1 = m_2 v_2 \quad (4)$$

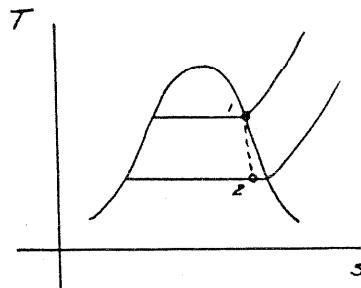
where  $v_2 = v_{f,2} + x_2 v_{fg,2}$ . What follows from eq. (4) is a relationship between  $m_1/m_2$  and  $x_2$ :

$$\begin{aligned} \frac{m_1}{m_2} &= \frac{v_2}{v_1} = \frac{v_{f,2}}{v_1} + x_2 \frac{v_{fg,2}}{v_1} \\ &= \frac{0.001154}{0.03944} + x_2 \frac{0.13064}{0.03944} \\ &= 0.02926 + x_2 (3.312) \end{aligned} \quad (5)$$

A similar relation exists between  $u_2/u_1$  and  $x_2$ ,

$$\begin{aligned} \frac{u_1}{u_2} &= \frac{u_{f,2}}{u_1} + x_2 \frac{u_{fg,2}}{u_1} \\ &= \frac{843.16}{2597.1} + x_2 \frac{1751.3}{2597.1} \\ &= 0.3246 + x_2 (0.6743) \end{aligned} \quad (6)$$

Substituting eqs. (5, 6) into eq. (3) yields an equation for  $x_2$  alone, whose solution is  $x_2 = 0.805$ .



(b) The final vapor / liquid volume ratio is

$$\begin{aligned} \left( \frac{V_g}{V_f} \right)_2 &= \left( \frac{m_g v_g}{m_f v_f} \right)_2 = \frac{x_2}{1 - x_2} \left( \frac{v_g}{v_f} \right)_2 \\ &= \frac{0.805}{1 - 0.805} \left( \frac{0.13177}{0.001154} \right) = 471.4 \end{aligned}$$

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Problem 1.8 Selected for analysis is the system that contains the two masses ( $m_1, m_2$ ). In the initial state (a) the velocities of the two masses are different ( $V_1, V_2$ ), while in the final state (b) mutual friction brings the velocities to the same level ( $V_\infty$ ). Since there are no forces between the system and its environment, the total momentum of the ensemble is conserved,

$$m_1 V_1 + m_2 V_2 = (m_1 + m_2) V_\infty \quad (1)$$

The initial and final kinetic energy inventories of the ensemble are

$$KE_a = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2 \quad (2)$$

$$KE_b = \frac{1}{2} (m_1 + m_2) V_\infty^2 \quad (3)$$

The evolution of the total kinetic energy during the process (a) - (b) is described by the "efficiency" ratio

$$\eta = \frac{KE_b}{KE_a} \quad (4)$$

Eliminating  $V_\infty$  between Eqs. (1) and (3), the efficiency can be expressed in terms of the initial mass and velocity ratios  $m_2 / m_1$  and  $V_2 / V_1$ ,

$$\eta = \frac{\left(1 + \frac{m_2}{m_1} \frac{V_2}{V_1}\right)^2}{\left(1 + \frac{m_2}{m_1}\right) \left[1 + \frac{m_2}{m_1} \left(\frac{V_2}{V_1}\right)^2\right]} < 1 \quad (5)$$

It can be shown analytically that  $\eta$  is less than 1 as soon as  $V_2$  is different than  $V_1$ , for any value of the ratio  $m_2 / m_1$ . Two limits of Eq. (5) are worth noting,

$$\eta = \frac{1}{\frac{m_2}{m_1} + 1} \quad \left( \frac{V_2}{V_1} \rightarrow 0 \right) \quad (6)$$

$$\eta = \frac{1}{\frac{m_1}{m_2} + 1} \quad \left( \frac{V_2}{V_1} \rightarrow \infty \right) \quad (7)$$

with the special case  $\eta = 1$  when  $V_1 = V_2$  for any  $m_2/m_1$ . Equations (5) - (7) show that the order of magnitude of  $\eta$  is 1 when  $m_2 / m_1$  is a number of order 1.

In conclusion, the kinetic energy of the system decreases from state (a) to state (b). According to the first law of thermodynamics, this decrease is balanced by the other energy interactions and energy changes of the system,

$$Q_{a-b} - W_{a-b} = U_b - U_a + KE_b - KE_a \quad (8)$$

where  $W_{a-b} = 0$ . If the process is adiabatic,  $Q_{a-b} = 0$ , then the KE decrease is balanced by an increase in  $U$ ,

$$U_b - U_a = KE_a - KE_b \quad (9)$$

If the system boundary is diathermal, and (a) and (b) are states of thermal equilibrium with the ambient temperature reservoir ( $T_0$ ), then

$$Q_{a-b} = U_b - U_a + KE_b - KE_a \quad (10)$$

If  $m_1$  and  $m_2$  are two incompressible substances then  $U = U(T)$ , and at thermal equilibrium ( $T_0$ ) the energy change  $U_b - U_a$  is zero,

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$$Q_{a-b} = KE_b - KE_a < 0 \quad (11)$$

## Chapter 2

### THE SECOND LAW OF THERMODYNAMICS

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Problem 2.1 With reference to system A sketched below, assume that

$$W > 0 \quad \text{and} \quad Q_2 > 0$$

The first law for one cycle completed by A is

$$Q_1 + Q_2 = W \quad (1)$$

Investigating the possible signs of  $Q_1$  and  $Q_2$  we see three options:

- (i)  $Q_1 < 0$  and  $Q_2 < 0$
- (ii)  $Q_1 > 0$  and  $Q_2 > 0$
- (iii)  $Q_1 Q_2 < 0$

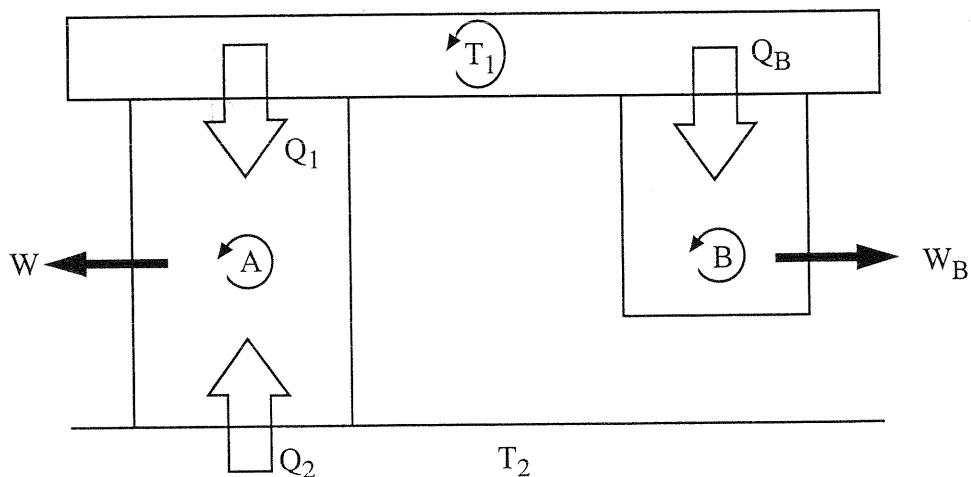
Option (i) is ruled out by the first law (1) and the assumption that  $W$  is positive. Option (ii) is a violation of the Kelvin-Planck statement (2.2). In order to see this violation consider system B which executes one complete cycle while communicating with ( $T_1$ ) such that

$$Q_B = -Q_1$$

Since the net heat transfer interaction experienced by ( $T_1$ ) is zero,  $Q_1 + Q_B = 0$ , the ( $T_1$ ) reservoir completes a cycle at the end of the cycles executed by A and B. The aggregate system [A + B + ( $T_1$ )] also executes a complete cycle. This cycle is executed while making contact with ( $T_2$ ) only. The net heat transfer interaction of this cycle is positive

$$Q_2 > 0$$

which is a clear violation of eq. (2.2). In conclusion the only option possible is (iii):  $Q_1 Q_2 < 0$ .



---

**Problem 2.2** With reference to the system A shown in the preceding figure, we write the first law for one cycle

$$Q_1 + Q_2 = W \quad (1)$$

and assume this time that  $W$  is negative,

$$W < 0$$

There are three options to consider

- (i)  $Q_1 < 0$  and  $Q_2 < 0$
- (ii)  $Q_1 > 0$  and  $Q_2 > 0$
- (iii)  $Q_1 Q_2 < 0$

of which only option (ii) can be ruled out, because it violates the first law. Option (i) is definitely compatible with the sign of eq. (2.27),

$$\frac{Q_1}{T_1} + \frac{Q_2}{T_2} < 0 \quad (2.27)$$

Option (iii), in which  $Q_2$  is the negative of the two heat transfer interactions, produces an analysis identical to the segment contained between eqs. (2.11) and (2.27) in the text. The second law (2.27) is valid therefore for  $W < 0$  and, as shown in the text for  $W > 0$ .

In the special case of  $W = 0$  the first law requires that  $Q_1 = -Q_2$ . The second law (2.27) reduces to

$$Q_1(T_1 - T_2) > 0$$

which means that

- (a) if  $Q_1$  is positive, then  $(T_1 - T_2)$  cannot be negative, or
- (b) if  $Q_1$  is negative, then  $(T_1 - T_2)$  cannot be positive

In less abstract terms, (a) and (b) mean that in the absence of work transfer the heat transfer interaction  $Q_1$  cannot proceed in the direction of higher temperatures.

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**Problem 2.3** According to the problem statement, it is being assumed that two paths ( $1 - 2_{\text{rev}}$  and  $1 - 2'_{\text{rev}}$ ) can be traveled in both directions (see sketch below). The two paths are reversible and adiabatic. This assumption allows us to execute the cycle  $1 - 2'_{\text{rev}} - 2_{\text{rev}} - 1$  in two ways:

(i) clockwise, in which

$$\oint \delta Q = Q_{2'_{\text{rev}} - 2_{\text{rev}}} = U_{2_{\text{rev}}} - U_{2'_{\text{rev}}} < 0$$

(ii) counterclockwise, in which

$$\oint \delta Q = Q_{2_{\text{rev}} - 2'_{\text{rev}}} = U_{2'_{\text{rev}}} - U_{2_{\text{rev}}} > 0$$

Note, however, that the counterclockwise option violates the Kelvin-Planck statement of the second law. This means that the original assumption on which options (i) and (ii) are based is false, i.e. that two reversible and adiabatic paths cannot intersect at state 1.

Is state  $2_{\text{rev}}$  unique on the  $V = V_2$  line? Worth noting is that options (i) and (ii) are both compatible with the Kelvin-Planck statement in the case where state  $2'_{\text{rev}}$  (or, for the matter, any other state  $2'_{\text{rev}}$  on the  $V = V_2$  line) coincides with state  $2_{\text{rev}}$ . In this case the reading of the cycle goes as follows:

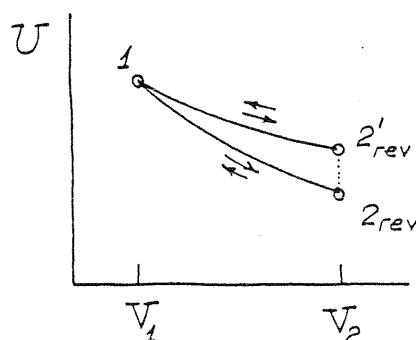
(i) clockwise

$$\oint \delta Q = Q_{2'_{\text{rev}} - 2_{\text{rev}}} = 0$$

(ii) counterclockwise

$$\oint \delta Q = Q_{2_{\text{rev}} - 2'_{\text{rev}}} = 0$$

Geometrically, this second law compatible limit means that state  $2_{\text{rev}}$  is unique, i.e. that there is only one state at  $V = V_2$  that can be reached reversibly and adiabatically from state 1.



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Problem 2.4 (a) With reference to the sketch below assume first that state 2 is such that

$$U_2 < U_{2_{\text{rev}}}$$

Assume further that state 2 is accessible adiabatically from state 1. Then if we execute the cycle 1 - 2 - 2<sub>rev</sub> - 1 clockwise we conclude that

$$\oint Q = Q_{2 - 2_{\text{rev}}} = U_{2_{\text{rev}}} - U_2 > 0$$

which is a violation of the Kelvin-Planck statement.

(b) Consequently we assume that the state 2' that is accessible adiabatically from state 1 is situated above state 2<sub>rev</sub>,

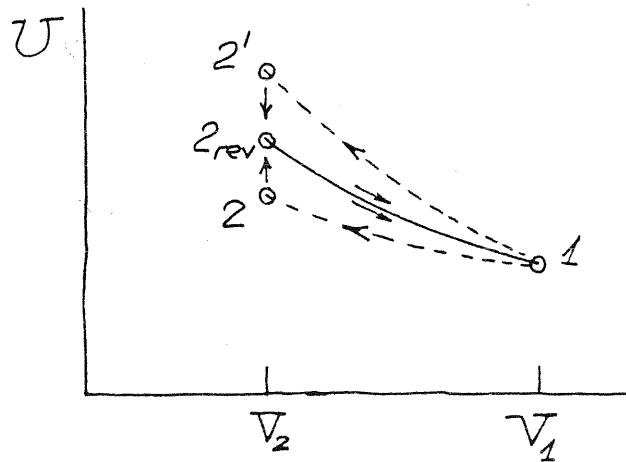
$$U_{2'} < U_{2_{\text{rev}}}$$

Executing the cycle 1 - 2' - 2<sub>rev</sub> - 1 counterclockwise we conclude that

$$\oint \delta Q = Q_{2' - 2_{\text{rev}}} = U_{2_{\text{rev}}} - U_{2'} < 0$$

which is in accord with the Kelvin-Planck Statement.

In conclusion, the states that are accessible adiabatically from state 1 are all situated above state 2<sub>rev</sub>. This conclusion is the same as the one reached in the discussion of Fig. 2.10 in the text.



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Problem 2.5 Consider first the path 1 - a - 2<sub>rev</sub>. During the first leg of this process we recognize the features of "adiabatic" ( $\delta Q = 0$  in the first law for the *aggregate* system A + B)

$$0 - \delta W = dU_A + dU_B \quad (1)$$

and "reversible",

$$\delta W = P_A dV_A$$

The ideal gas model means that  $P_A V_A = (mR)_A T$ , and that  $dU_A = (mc_v)_A dT$  and  $dU_B = (mc_v)_B dT$ . Combining, eq. (1) becomes

$$-\frac{(mR)_A}{(mc_v)_A + (mc_v)_B} \frac{dV_A}{V_A} = \frac{dT}{T} \quad (2)$$

which allows us to calculate the temperature at the end of the first leg:

$$T_a = T_1 \left( \frac{V_{A2}}{V_{A1}} \right)^{-k_A}, \quad \text{where} \quad k_A = \frac{(mR)_A}{(mc_v)_A + (mc_v)_B}$$

The second leg of the same process is similar except that this time it is subsystem B that expands. In place of eq. (2) we obtain

$$-k_B \frac{dV_B}{V_B} = \frac{dT}{T} \quad (3)$$

where

$$k_B = \frac{(mR)_B}{(mc_v)_A + (mc_v)_B}$$

The end temperature  $T_{2_{\text{rev}}}^{(a)}$  is obtained by integrating eq. (3) from (a) to (2<sub>rev</sub>),

$$T_{2_{\text{rev}}}^{(a)} = T_a \left( \frac{V_{B2}}{V_{B1}} \right)^{-k_B}$$

which, in view of the  $T_a$  result obtained earlier, means that

$$T_{2_{\text{rev}}}^{(a)} = T_1 \left( \frac{V_{A2}}{V_{A1}} \right)^{-k_A} \left( \frac{V_{B2}}{V_{B1}} \right)^{-k_B} \quad (4)$$

Considering now the alternate process 1 - b - 2<sub>rev</sub>, we obtain for the end of the first leg

$$T_b = T_1 \left( \frac{V_{B2}}{V_{B1}} \right)^{-k_B} \quad (5)$$

and for the end of the second leg

Combining eqs. (5) and (6) leads to

$$T_{2_{\text{rev}}}^{(b)} = T_1 \left( \frac{V_{B2}}{V_{B1}} \right)^{-k_B} \left( \frac{V_{A2}}{V_{A1}} \right)^{-k_A}$$

which is the same result as eq. (4), in other words

$$T_{2_{\text{rev}}}^{(a)} = T_{2_{\text{rev}}}^{(b)}$$

In conclusion, the state of configuration  $(V_{A2}, B_{B2})$  is unique.

Problem 2.6 Starting with state A, Fig. 2.8, we remove a *single* partition. We have three choices. Labeling the partitions (a), (b) and (c), we analyze each choice and list the results in line with each graph. For example, removing (a) allows the contents of the two leftmost chambers to mix to the new temperature  $\frac{3}{2} T_0$ , which is listed on the drawing. The new entropy inventory of the aggregate system is in this case

$$\frac{S - S_0}{mc} = 2 \ln \frac{3}{2} + \ln 3 + \ln 4 = 3.296$$

This number is listed to the right of each drawing and represents the abscissa values of the points drawn in line with "2 partitions present" in Fig. 2.8.

b	c		
2m	m	m	
$\frac{3}{2} T_0$	$3T_0$	$4T_0$	$S - S_0 = 3.296 \text{ mc}$

a	c		
m	2m	m	
$T_0$	$\frac{5}{2} T_0$	$4T_0$	$S - S_0 = 3.219 \text{ mc}$

a	b		
m	m	2m	
$T_0$	$2T_0$	$\frac{7}{2} T_0$	$S - S_0 = 3.199 \text{ mc}$

Starting again from state A (3 partitions present), we remove *two* partitions at a time. We have the following choices and results:

	c
3m	m
$2T_0$	$4T_0$

$$S - S_0 = 3.466 \text{ mc}$$

	b
2m	2m
$\frac{3}{2} T_0$	$\frac{7}{2} T_0$

$$S - S_0 = 3.316 \text{ mc}$$

	a
m	3m
$T_0$	$3T_0$

$$S - S_0 = 3.296 \text{ mc}$$

Regarding the number of paths from A to B, we can have the following scenarios:

- (i) Removing a *single* partition each time, we have  $3 \times 2 = 6$  choices, therefore 6 possible paths
- (ii) Removing a *single* partition in the first step and *two* partitions in the second, we have  $3 \times 1 = 3$  choices, meaning 3 paths
- (iii) Removing *two* partitions in the first step and a *single* partition in the second, we have again  $3 \times 1 = 3$  choices, hence 3 paths.
- (iv) Removing all three partitions at the same time, one path.

The total number of paths is then  $6 + 3 + 3 + 1 = 13$ .

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**Problem 2.7** Assume first that the end state is (b), in other words, that the piston is pressed against the ceiling of the enclosures. The first law for the (gas + weight) system during the process (a) → (b) is

$$Q_{a-b} - W_{a-b} = \underbrace{(U_b - U_a)_{\text{gas}}}_{0} + \underbrace{(PE_b - PE_a)_{\text{weight}}}_{mc_v(T_2 - T_1)} - \frac{Mg}{A}(V_2 - V_1)$$

This equation delivers the final temperature,

$$T_2 = T_1 - \frac{Mg/A}{mc_v}(V_2 - V_1) \quad (1)$$

For configuration (b) to exist, the final pressure  $P_2$  must be greater than the pressure that could be sustained by the piston weight alone,

$$P_2 > \frac{Mg}{A}$$

hence

$$\frac{mRT_2}{V_2} > \frac{Mg}{A} \quad (2)$$

Combining eqs. (2) and (1), this criterion becomes

$$\frac{1}{c_p \frac{V_2}{V_1} - \frac{R}{c_v}} > \frac{Mg/A}{P_1} \quad (3)$$

The entropy generated during the process (a) → (b) is

$$\begin{aligned} S_{\text{gen},a-b} &= (S_b - S_a)_{\text{gas}} + \underbrace{(S_b - S_a)_{\text{weight}}}_{0} - \underbrace{\int_a^b \frac{\delta Q}{T}}_{0} \\ &= mc_v \ln \frac{T_2}{T_1} + mR \ln \frac{V_2}{V_1} \end{aligned} \quad (4)$$

Using eq. (1), we can put eq. (4) in the following dimensionless form

$$\frac{S_{\text{gen},a-b}}{mc_v} = \ln \left\{ \left[ 1 - \frac{Mg/A}{P_1} \frac{R}{c_v} \left( \frac{V_2}{V_1} - 1 \right) \right] \left( \frac{V_2}{V_1} \right)^{R/c_v} \right\} \quad (5)$$

The objective is to show that the above quantity is positive, i.e. that the quantity between accolades { } is greater than one. The proof that { } > 1 is even stronger if we replace the factor  $(Mg/A)P_1$  by its ceiling value, which is listed on the left side of the inequality (3). In other words, if we prove that

$$\left[ 1 - \frac{\frac{R}{c_v} \left( \frac{V_2}{V_1} - 1 \right)}{\frac{c_p}{c_v} \frac{V_2}{V_1} - 1} \right] \left( \frac{V_2}{V_1} \right)^{R/c_v} > 1 \quad (6)$$

then we can be sure that (5) is positive. To prove the inequality (6) means to prove that

$$\frac{\left( \frac{V_2}{V_1} \right)^{R/c_v}}{R/c_v} > \frac{\frac{V_2}{V_1} - 1}{\frac{V_2}{V_1}} \quad (7)$$

Both sides of the inequality (7) are monotonic in  $(V_2 / V_1)$ . The inequality is clearly correct in the limit  $V_2/V_1 \rightarrow \infty$ . To see its true sign in the opposite limit,  $V_2/V_1 \rightarrow 1$ , let

$$\frac{V_2}{V_1} = 1 + \varepsilon, \text{ where } \varepsilon \ll 1$$

and rewrite (7) side by side as

$$\varepsilon + \left( \frac{R}{c_v} - 1 \right) \frac{\varepsilon^2}{2} \dots > \varepsilon - \varepsilon^2 + \dots$$

This form reduces to

$$\frac{c_p}{c_v} \frac{\varepsilon^2}{2} > 0$$

which certainly validates (7) in the limit  $\varepsilon \rightarrow 0$ . The inequality (7) is true for all values of  $V_2/V_1$  in the range  $(1, \infty)$  because the derivative of the left side of (7) with respect to  $(V_2/V_1)$  is always greater than the derivative of the right side,

$$\left( \frac{V_2}{V_1} \right)^{\frac{R}{c_v} - 1} > \left( \frac{V_2}{V_1} \right)^{-2} \quad \frac{d}{d(V_2/V_1)} \quad [\text{eq. (7)}]$$

for  $(V_2/V_1) > 1$  and  $(R/c_v) > 0$ .

Consider next the process (a) → (c), which occurs when (3) is violated. The first law (1) for this case reads

$$T_3 = T_1 - \frac{Mg/A}{mc_v} (V_3 - V_1)$$

In order to find  $V_3$  we combine (3) with  $P_3V_3 = mRT_3$ ; the result is

$$\frac{V_3}{V_1} = \frac{c_v}{c_p} \left( \frac{P_1}{P_3} + \frac{R}{c_v} \right)$$

where  $P_3 = Mg/A$ . This result makes it easy to verify that when  $P_1 = P_3$  the piston does not move at all,  $V_3 = V_1$ . Finally, the entropy generated during the process (a)  $\rightarrow$  (c) is

$$\begin{aligned} \frac{S_{\text{gen},a-c}}{mc_v} &= \ln \frac{T_3}{T_1} + \frac{R}{c_v} \ln \frac{V_3}{V_1} \\ &= \ln \left[ 1 + \frac{1-x}{\frac{c_p}{R}x - 1} \right] x^{R/c_v} \end{aligned}$$

where  $x$  is shorthand for  $V_2/V_1$ . To prove that the entropy increases from (a) to (c) we must prove

$$\left( 1 + \frac{1-x}{\frac{c_p}{R}x - 1} \right) x^{R/c_v} > 1$$

in other words

$$\frac{x^k - 1}{k} > x - 1 \quad (10)$$

where  $k = c_p/c_v$ . Both sides of the inequality (10) approach zero in the limit  $x \rightarrow 1$ . In the opposite limit ( $x \rightarrow \infty$ ) the inequality is correct. It is correct also at intermediate  $x$ 's, because the same inequality exists between the  $d(\cdot)/dx$  slopes of the left and right sides of (10), respectively,

$$x^{k-1} > 1, \quad \frac{d}{dx} \quad [\text{eq. (10)}]$$

for  $x > 1$  and  $k > 1$ .

Problem 2.8 The initial pressures above and below the partition are

$$P_1 \text{ above} = P_2 = \frac{Mg}{A}, \quad \text{and} \quad P_1 \text{ below} = \frac{mRT_1}{2V_1}$$

The first law for the ( $m + M$ ) system requires

$$\begin{aligned} Q_{a-b} - W_{a-b} &= \underbrace{(U_b - U_a)_{\text{gas}}}_{0} + \underbrace{(PE_b - PE_a)_{\text{weight}}}_{mc_v(T_2 - T_1)} \\ &\quad - \frac{Mg}{A}(V_b - V_a) \end{aligned}$$

where

$$V_b = \frac{mRT_2}{P_2}$$

and

$$V_a = V_1 + \frac{(m/2) RT_1}{Mg/A}$$

Combining these results we obtain the expression for  $T_2$ ,

$$\begin{aligned} T_2 &= \frac{k+1}{2k} T_1 + \frac{P_2 V_1}{mc_P} \\ &= \frac{1}{2} T_1 \left( 1 + \frac{1}{k} + \frac{R}{c_P} \frac{P_2}{P_{1 \text{ below}}} \right) \end{aligned} \quad (1)$$

where  $k = c_p / c_v$ . We can verify at this point that when the initial pressure difference across the partition is zero, the final state (b) is identical to (a): this is in agreement with eq. (1), where  $P_{1 \text{ below}} = P_{1 \text{ above}} (= P_2)$  means that  $T_2 = T_1$ .

The entropy increase from (a) to (b) is associated only with the ideal gas part of the system,

$$\begin{aligned} S_{\text{gen},a-b} &= (S_{\text{gen},a-b})_{\text{gas originally above partition}} + (S_{\text{gen},a-b})_{\text{gas originally below partition}} \\ &= \frac{m}{2} c_P \ln \frac{T_2}{T_1} - \frac{m}{2} R \ln \frac{P_2}{P_{1 \text{ above}}} + \frac{m}{2} c_P \ln \frac{T_2}{T_1} - \frac{m}{2} R \ln \frac{P_2}{P_{1 \text{ below}}} \\ &= m c_P \ln \frac{T_2}{T_1} - \frac{m}{2} R \ln \pi \end{aligned}$$

where  $\pi = P_2/P_{1 \text{ below}}$  is a given constant. We must prove that

$$\frac{S_{\text{gen},a-b}}{mc_P} = \ln \frac{T_2}{T_1} - \frac{R}{2c_P} \ln \pi$$

is positive, i.e. that

$$\frac{1}{2} \left( 1 + \frac{1}{k} + \frac{R}{c_P} \pi \right) \pi^{-R/2c_P} > 1 \quad (2)$$

Assuming the ideal gas constants of air ( $k = 1.4$ ,  $c_p/R = 3.5$ ), we discover numerically that the inequality (2) is satisfied as long as  $\pi \neq 1$ , i.e. when there is an initial pressure difference across the partition:

$\pi$	0.1	0.5	1	2	10
left side of (2)	1.21	1.03	1	1.04	1.64

Problem 2.9 From state (1) to state (2) the trapped gas evolves isentropically,

$$ds = c_P \frac{dT}{T} - R \frac{dP}{P} = 0$$

which means that

$$T P^{-R/c_P} = \text{constant}$$

or

$$T_a P_1^{-R/c_P} = T_2 P_a^{-R/c_P} \quad (1)$$

From state (2) to state (3) the system is being heated at constant volume,

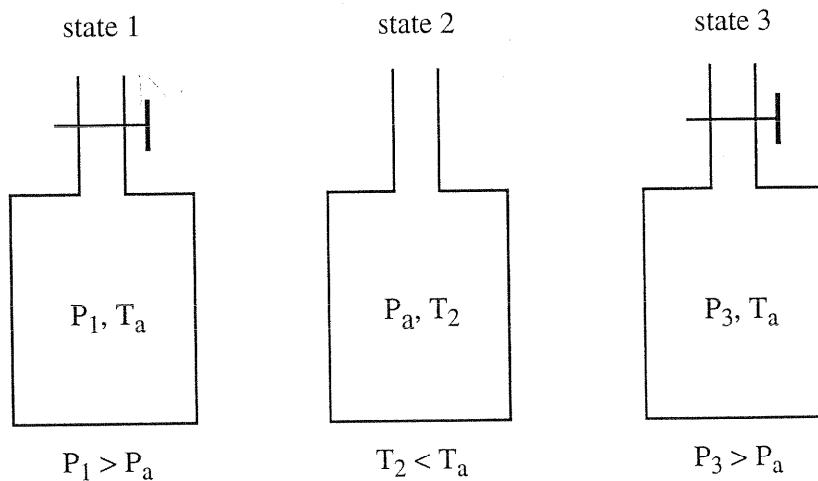
$$\frac{P_a}{T_2} = \frac{P_3}{T_a} \quad (2)$$

Eliminating  $T_2$  between eqs. (1) and (2) yields

$$\frac{R}{c_P} = \frac{\ln(P_3 / P_a)}{\ln(P_1 / P_a)}$$

or, for the ratio of specific heats,

$$\frac{c_P}{c_v} = \frac{\ln(P_1/P_a)}{\ln(P_1/P_3)}$$



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Problem 2.10 (a) The first law for the ideal gas alone is

$$\delta Q - \delta W = dU$$

where

$$\delta Q = " \delta Q_{\text{into ideal gas}} " = - McdT$$

$$\delta W = PdV$$

$$dU = mc_v dT$$

The first law becomes

$$-McdT - PdV = mc_v dT \quad (1)$$

Eliminating T between eq. (1) and  $PV = mRT$ , we obtain

$$0 = \left(1 + \frac{1}{a}\right) \frac{dV}{V} + \frac{dP}{P}, \quad \text{where} \quad a = \frac{mc_v + Mc}{mR}$$

which means that the path is  $PV^n = \text{constant}$ , with

$$n = 1 + \frac{R/c_v}{1 + Mc / (mc_v)} \quad (2)$$

(b) Clearly, n approaches k ( $= c_p/c_v$ ) when

$$\frac{Mc}{mc_v} \rightarrow 0$$

(c)  $dS = dS_{\substack{\text{ideal} \\ \text{gas}}} + dS_{\substack{\text{incompressible} \\ \text{substance}}}$

$$= mc_v \frac{dP}{P} + mc_P \frac{dV}{V} + Mc \frac{dT}{T}$$

(d) The second law for the combined system is

$$dS - \underbrace{\frac{\delta Q}{T}}_0 > 0$$

---

The entropy change  $dS$  can be evaluated using eq. (3) with  $PV^n = \text{constant}$  and  $PV = mRT$ , to conclude that  $dS = 0$ , i.e. that the process is reversible.

Problem 2.11 In general, we can write

$$V = V_a + V_w = \text{constant}, \text{ i. e. } \dot{V}_a + \dot{V}_w = 0$$

$$\frac{dm_w}{dt} = -\dot{m}, \quad m_w = \rho_w V_w, \quad \dot{m} = C(P - P_0)^{1/2}$$

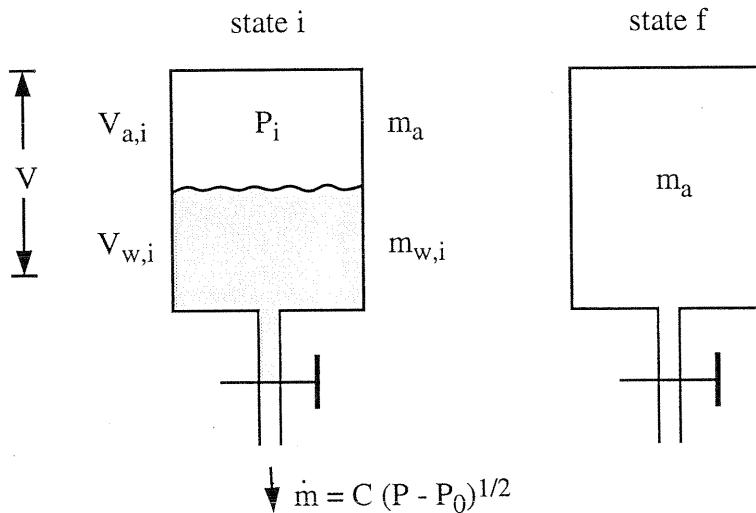
hence

$$\rho_w \frac{dV_w}{dt} = -C(P - P_0)^{1/2}$$

and

$$\rho_w \frac{dV_a}{dt} = C(P - P_0)^{1/2} \quad (1)$$

The problem reduces to finding the dependence of  $V_a$  on  $P$ , so that we might be able to integrate eq. (1).



- (a) If the air/water interface is adiabatic and if the air expansion is slow enough, then the air proceeds at constant entropy (reversibly and adiabatically)

$$PV_a^k = P_i V_{a,i}^k = \text{constant}$$

Equation (1) can be integrated from  $t = 0$  to  $t = t_f$ , by assuming that  $P \gg P_0$ ; the  $T_f$  result is

$$t_{f(a)} = \left[ \left( \frac{V}{V_{a,i}} \right)^{\frac{k}{2} + 1} - 1 \right] \frac{\rho_w V_{a,i}}{\left( \frac{k}{2} + 1 \right) C P_i^{1/2}} \quad (2)$$

(b) If the air/water interface is diathermal, the air and the water have the same instantaneous temperature,  $T(t)$ . They also have the same  $P(t)$  if the process is slow enough. Conclusion: we can model the "contents" (air + water, upstream of the valve) as undergoing a reversible and adiabatic process. This system is open, hence

$$\begin{aligned}\dot{S}_{\text{gen}} = 0 &= \frac{dS}{dt} - \sum \frac{\dot{Q}_i}{T_i} + \dot{m}s_w \\ 0 &= \overbrace{\frac{dS_a}{dt}}^{} + \overbrace{\frac{dS_w}{dt}}^{} + \overbrace{\left( -s_w \frac{dm_w}{dt} \right)}^{}\end{aligned}$$

which in the end yields

$$\begin{aligned}m_a ds_a + m_w ds_w &= 0 \\ m_a \left( c_p \frac{dT}{T} - R \frac{dP}{P} \right) + m_w c \frac{dT}{T} &= 0\end{aligned}\quad (3)$$

Now, combining eq. (3) with  $\left( V_a + \frac{m_w}{\rho_w} = V \right)$  in order to eliminate  $m_w$ , we obtain

$$m_a \left( c_p \frac{dT}{T} - R \frac{dP}{P} \right) + \rho_w (V - V_a) c \frac{dT}{T} = 0$$

Next, we eliminate  $T$  by recalling that  $T = \frac{PV_a}{m_a R}$ ,

$$\frac{c_p}{R} \left[ 1 + \frac{\rho_w C(V - V_a)}{m_a c_p} \right] \left( 1 + \frac{P}{V_a} \frac{dV_a}{dP} \right) = 1 \quad (4)$$

Equation (4) is the  $P(V_a)$  relation needed to integrate eq. (1) numerically. More progress can be made analytically by regarding the quantity inside the square brackets [ ], eq. (4), as constant,

$$[ ] \equiv \begin{cases} 1 + \frac{m_w, i c}{m_a c_p}, & \text{close to } t = 0 \\ 1 & , \text{close to } t = t_f \end{cases} \quad (5)$$

This point of view allows us to see that the integral of eq. (4) is essentially

$$V_a P^{1-A} = \text{constant}$$

where

$$A = \frac{1}{c_p [ ]}, \quad (\text{note: } A < 1)$$

Finally, from eq. (2) with  $1 - A$  in place of  $k$ , we calculate approximately

$$t_{f(b)} \equiv \left[ \left( \frac{V}{V_{a,i}} \right)^{\frac{1-A}{2} + 1} - 1 \right] \frac{\rho_w V_{a,i}}{\left( \frac{1-A}{2} + 1 \right) CP_i^{1/2}} \quad (6)$$

where A (and the quantity [ ]) can have two limiting values, eq. (5).

Now, which discharge mode is faster, (a) or (b)?

$$\frac{t_{f(a)}}{t_{f(b)}} \equiv \frac{\left( \frac{V}{V_{a,i}} \right)^{\frac{k}{2} + 1} - 1}{\left( \frac{V}{V_{a,i}} \right)^{\frac{1-A}{2}} - 1} \left( \frac{3-A}{2+k} \right)$$

Since  $(V/V_{a,i}) > 1$ , the above time ratio is dominated by the  $(V/V_{a,i})^n$  terms. This means that

$$t_{f(a)} > t_{f(b)} \quad (7)$$

if

$$\frac{k}{2} + 1 > \frac{1-A}{2} + 1$$

in other words, if

$$k > 1 - A$$

This last inequality is valid because  $k > 1$ . In conclusion, the relation between  $t_{f(a)}$  and  $t_{f(b)}$  is as shown in eq. (7), if V is considerably larger than  $V_{a,i}$ .

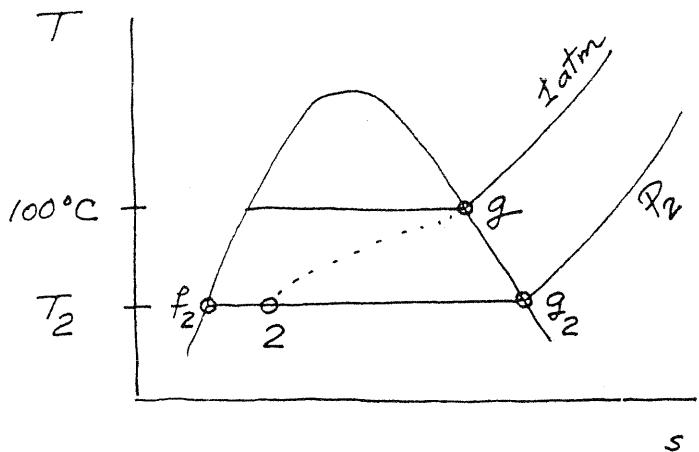
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Problem 2.12 (a) Process (1) → (2) evolves such that  $\delta Q = \delta W = 0$ , which means  $U_2 - U_1 = 0$  (from the first law), and at constant volume,  $V_2 = V_1$ . These two equations lead to the following system:

$$(\Delta m)u_0 + (m)u_g = (\Delta m + m)u_2$$

$$(\Delta m)v_0 + (m)v_g = (\Delta m + m)v_2$$

Let us use  $\Delta m/m = y$ , and introduce  $x_2$  as an unknown,



$$u_2 = u_{f,2} + x_2 u_{fg,2}$$

$$v_2 = v_{f,2} + x_2 v_{fg,2}$$

The system of equations reduces to

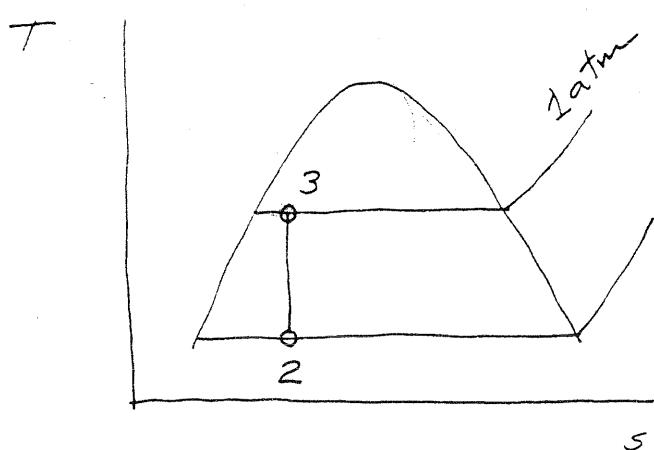
$$y = \frac{\frac{u_g - u_{f,2}}{u_{fg,2}} - x_2}{\frac{u_{f,2} - u_0}{u_{fg,2}} + x_2} = \frac{\frac{v_g - v_{f,2}}{v_{fg,2}} - x_2}{\frac{v_{f,2} - v_0}{v_{fg,2}} + x_2}$$

I have assumed  $P_2 = 50$  kPa; for this choice the above system becomes (numerically)

$$y = \frac{1.011 - x_2}{0.1099 + x_2} = \frac{0.5162 - x_2}{0 + x_2}$$

The solution is  $x_2 = 0.09382$  and  $y = 4.502$ .

(b) Process (2)  $\rightarrow$  (3) is slow (PdV-type work) and adiabatic, hence  $S_3 - S_2 = 0$  (from the second law). Note also that



$$s_3 = s_2 = s_{f,(1 \text{ atm})} + x_3 s_{fg,(1 \text{ atm})}$$

We know all the properties at state (2),

$$s_2 = s_{f_2} + x_2 s_{fg_2} = 1.7011 \frac{\text{kJ}}{\text{kgK}}$$

$$u_2 = u_{f_2} + x_2 u_{fg_2} = 541.52 \frac{\text{kJ}}{\text{kg}}$$

$$v_2 = v_{f_2} + x_2 v_{fg_2} = 0.3049 \frac{\text{m}^3}{\text{kg}}$$

so, we write  $s_2 = s_{f(1 \text{ atm})} + x_3 s_{fg(1 \text{ atm})}$ , or  $1.7011 = 1.3069 + x_3(6.048)$ . We obtain

$$x_3 = 0.06518$$

$$u_3 = 418.94 + x_3 (2087.6) = 555.01 \frac{\text{kJ}}{\text{kg}}$$

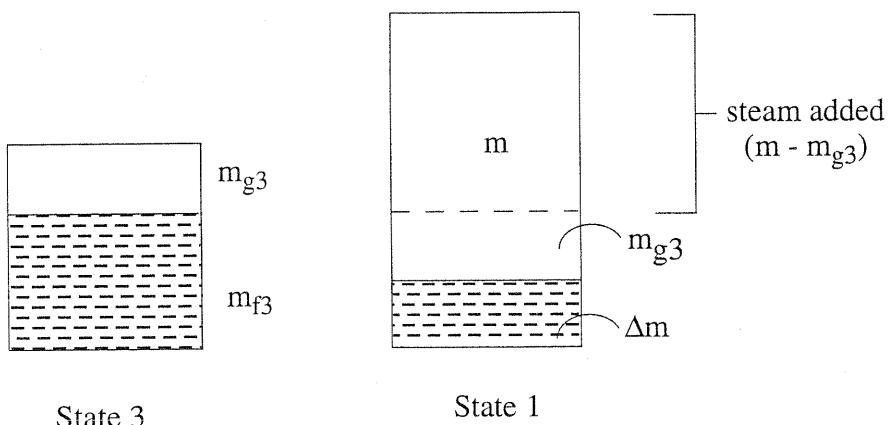
$$v_3 = 0.001044 + x_3 (1.6719) = 0.1102 \frac{\text{m}^3}{\text{kg}}$$

The work required to compress the mixture,  $W_{2-3}$ , follows from the first law,

$$\begin{aligned} \underbrace{Q_{2-3} - W_{2-3}}_0 &= U_3 - U_2 \\ &= (m + \Delta m)(u_3 - u_2) \\ &= (m + \Delta m)(13.49) \frac{\text{kJ}}{\text{kg}} \end{aligned}$$

Note that  $W_{2-3}$  is negative because it is absorbed.

(c) From (3) to (1), we execute the following changes

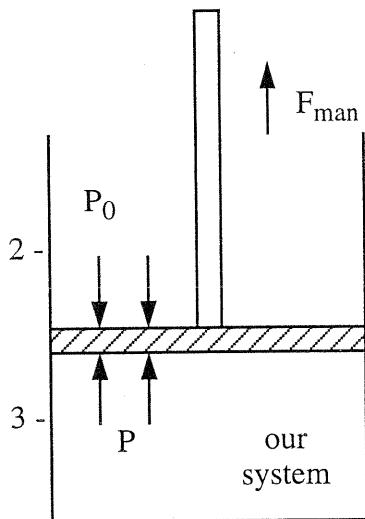


$$m_{f,3} = (1 - x_3)(m + \Delta m) = 0.9348(m + \Delta m)$$

$$m_{g,3} = x_3(m + \Delta m) = 0.06518(m + \Delta m)$$

$$m_{\text{added}} = m - m_{g,3} = m[1 - 0.06518(1 + y)] = 0.6415m$$

(d) The cycle is (1) - (2) - (3) - (1); the only work-transfer process is process (2) - (3). From (2) to (3) the atmosphere expands into "our system" and performs the work



$$\begin{aligned}
 W_{\text{atm}} &= P_0(V_2 - V_3) \\
 &= P_0(m + \Delta m)(v_2 - v_3) \\
 &= (m + \Delta m) 19.753 \frac{\text{kJ}}{\text{kg}}
 \end{aligned}$$

$W_{\text{atm}}$  is positive, because it is written for the atmosphere as a system that performs work. The force balance on the piston shows that  $W_{\text{atm}}$  is used for two purposes

$$W_{\text{atm}} (\text{done by the atmosphere}) \begin{cases} (-W_{23}) \text{ done on our system} \\ W_{\text{man}} (\text{left for man's use}) \end{cases}$$

Conclusion:

$$W_{\text{man}} = W_{\text{atm}} - (-W_{23})$$

or

$$\begin{aligned}
 W_{\text{man}} &= (19.753)(m + \Delta m) - (13.49)(m + \Delta m) \\
 &= (m + \Delta m)(6.2628) \text{ kJ/kg}
 \end{aligned}$$

Problem 2.13 (a) The model suggests that (A) executes a quasistatic and adiabatic compression, whose path is  $PV^k = \text{constant}$ , therefore

$$V_{A2} = V_1 \left( \frac{P_0}{P_2} \right)^{1/k}$$

Likewise, (B) executes a quasistatic and isothermal compression, which means that the path is  $PV = \text{constant}$ , or that

$$V_{B2} = V_1 \frac{P_0}{P_2}$$

Finally, in order to calculate  $T_{A2}$  we note that since  $P_A V_A = mRT_A$  the path of "PV<sup>k</sup> = constant" is the same as

$$T P^{(1-k)/k} = \text{constant}$$

This allows us to conclude that

$$T_{A2} = T_0 \left( \frac{P_2}{P_0} \right)^{(k-1)/k}$$

- (b) In the case of system (A) two walls are moving slowly, therefore

$$W_{A,1-2} = \int_1^2 P_A dV_A, \quad \text{where } P_A V_A^k = \text{constant}$$

Performing the integral and using the  $V_{A2}$  result derived earlier yields in the end

$$W_{A,1-2} = - \frac{P_0 V_1}{k-1} \left[ \left( \frac{P_2}{P_0} \right)^{(k-1)/k} - 1 \right] < 0$$

This quantity is negative because  $P_2 > P_0$  and because it is necessary to invest work into the compression of the trapped air.

- (c) The net work transfer experienced by (A + B) equals the net work experienced by (A) plus the net work experienced by (B):

$$W_{A+B,1-2} = W_{A,1-2} + W_{B,1-2}$$

What we need is  $W_{B,1-2}$ , therefore, we recognize the "PV<sub>B</sub> = constant" path and write

$$\begin{aligned} W_{B,1-2} &= \int_1^2 P_B dV_B = mRT_0 \ln \frac{V_{B2}}{V_{B1}} \\ &= P_0 V_1 \ln \frac{P_0}{P_2} < 0 \end{aligned}$$

- (d) The second law for the *aggregate* system (A + B) is  $S_{\text{gen},1-2} > 0$  where

$$S_{\text{gen},1-2} = (S_2 - S_1)_{A+B} - \frac{(Q_{1-2})_{A+B}}{T_0}$$

and where

$$(S_2 - S_1)_{A+B} = (S_2 - S_1)_A + (S_2 - S_1)_B$$

$$(Q_{1-2})_{A+B} = Q_{A,1-2} + Q_{B,1-2}$$

We have to evaluate four quantities:

$$Q_{A,1-2} = 0, (\text{adiabatic enclosure})$$

$Q_{B,1-2} = W_{B,1-2}$ , (apply the first law to system B)

$$\begin{aligned}(S_2 - S_1)_A &= mc_v \ln \frac{P_2}{P_0} + mc_P \ln \frac{V_{A2}}{V_1} \\ &= mc_v \ln \frac{P_2}{P_0} + mc_P \ln \left( \frac{P_0}{P_2} \right)^{1/k} \\ &= m \left( c_v - c_p \frac{1}{k} \right) \ln \frac{P_2}{P_0} = 0 \\ (S_2 - S_1)_B &= mc_P \ln \frac{T_0}{T_2} - mR \ln \frac{P_2}{P_0} - mR \ln \frac{P_2}{P_0}\end{aligned}$$

Putting everything together we obtain

$$S_{\text{gen},1-2} = -mR \ln \frac{P_2}{P_0} - \frac{1}{T_0} P_0 V_1 \ln \frac{P_0}{P_2} = 0 \quad \left( \text{note: } \frac{P_0 V_1}{T_0} = mR \right)$$

In conclusion, the process 1-2 executed by the aggregate system is reversible, and so are the processes executed by (A) and (B) individually.

---

Problem 2.14 The second law statement for an unsteady open system is

$$\dot{S}_{\text{gen}} = \frac{dS}{dt} - \frac{\dot{Q}_0}{T_0} - \dot{m}s_R > 0$$

or, integrated from state (1) to state (2),

$$S_{\text{gen},1-2} = S_2 - S_1 - \frac{Q_0}{T_0} - (m_2 - m_1)s_R > 0$$

In this expression we recognize that  $S_1 = 0$ ,  $m_1 = 0$  and  $s_R = s_R(T_0, P_R)$ . The heat transfer  $Q_0$  follows from the first law

$$\begin{aligned}\frac{dU}{dt} &= \dot{Q}_0 - P_L \frac{dV}{dt} + \dot{m}h_R \\ U_2 - U_1 &= Q_0 - P_L (V_2 - V_1) + (m_2 - m_1) h_R\end{aligned}$$

in which  $U_1 = 0$  and  $m_1 = 0$ . This means that

$$\begin{aligned}Q_0 &= U_2 + P_L V_2 - m_2 h_R = H_2 - m_2 h_R \\ &= m_2 (h_2 - h_R) = 0\end{aligned}$$

because the enthalpy of an ideal gas is a function of only the temperature [note:  $h_2(T_0)$  and  $h_R(T_0)$ ]. In conclusion, the entropy generation is

$$S_{\text{gen},1-2} = S_2 - m_2 s_R$$

and since  $P_2 = P_L < P_R$ ,

$$S_{\text{gen},1-2} = m_2 R \ln \frac{P_R}{P_L} > 0$$


---

Problem 2.15 (a) During the braking process the total work input to the brake is  $W$ , and the total heat transfer from the brake to the ambient is  $Q$ . The system consists of the brake material and the immediate air layers that are heated by the brake (i.e. the system boundary is a surface of temperature  $T_0$ ). The first law and the second law require (note the directions of  $Q$  and  $W$ )

$$-Q + W = U - U_0 \quad (1)$$

$$S_{\text{gen}} = S - S_0 + \frac{Q}{T_0} \geq 0 \quad (2)$$

where

$$U - U_0 = mc(T - T_0) \quad (3)$$

$$S - S_0 = mc \ln \frac{T}{T_0}. \quad (4)$$

Eliminating  $Q$  between eqs. (1) and (2) we obtain

$$S_{\text{gen}} = \frac{W}{T_0} - mc \left( \frac{T}{T_0} - 1 - \ln \frac{T}{T_0} \right) \geq 0 \quad (5)$$

(b) The quantity in parentheses in eq. (5) is positive, therefore

$$S_{\text{gen}} \leq \frac{W}{T_0} \quad (6)$$

The reason  $S_{\text{gen}}$  is generally less than  $W / T_0$  is that part of the  $W$  is stored in the brake material as nonflow exergy (associated with the temperature  $T$ , which is higher than  $T_0$ ; see Chapter 3). Note that the second term in eq. (5) is precisely this nonflow exergy, divided by  $T_0$ .

(c) After the braking process the cooling continues, and  $T$  returns to  $T_0$ . The second term in equation (5) vanishes as the nonflow exergy is destroyed by the irreversible heat transfer (cooling) that occurs during the post-braking phase.

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**Problem 2.16** The air wheel will not turn, because the air wheel is a closed system in communication with one temperature reservoir. If the wheel executes one turn (one cycle), then the second law requires

$$\oint \delta W \leq 0$$

This means that the wheel will turn only if an external system turns the wheel, which is not the case in this proposed invention. Here the claim is that the air wheel will produce work. This claim is false, because it violates the second law of thermodynamics.

---

**Problem 2.17** (a) The system is open because it has one outlet.

(b) The process is unsteady because (among other things) the mass inventory of the system decreases with time,  $m(t)$ .

(c)  $\frac{dm}{dt} = -\dot{m}$  (0)

$$\frac{dU}{dt} = \dot{Q} - \dot{W} - \dot{m}h_{out} \quad (1)$$

$$\frac{dS}{dt} \geq \frac{\dot{Q}}{T_0} - \dot{m}s_{out} \quad (2)$$

(d) Integrate Eqs. (0) and (1) from state 1 ( $V_1$ ) to state 2 ( $V_2$ ):

$$m_2 - m_1 = - \int_1^2 \dot{m} dt \quad (3)$$

$$U_2 - U_1 = Q_{12} - W_{12} - \int_1^2 \dot{m}h_{out} dt \quad (4)$$

where  $m_2 = 0$ ,  $U_2 = 0$  and

$$W_{12} = \int_1^2 P dV = P(V_2 - V_1) = -PV_1 \quad (5)$$

Because the cylinder air is at  $T_0$ , and because the valve operates at constant-h, the outflowing stream is also at  $T_0$ , and  $h_{out}$  is a constant,  $h(T_0)$ . In conclusion, Eqs. (3) and (4) yield

$$\begin{aligned}
Q_{12} &= -PV_1 - U_1 + h_{out}m_1 \\
-(P_1V_1 + U_1) + h_1m_1 &= \\
-H_1 + H_1 &= 0
\end{aligned} \tag{6}$$

(e) Integrating Eq. (2) we find

$$S_2 - S_1 \geq \frac{Q_{12}}{T_0} - \int_1^2 \dot{m}s_{out}dt \tag{7}$$

where  $S_2 = 0$ ,  $Q_{12} = 0$  and  $S_{out} = S(T_0, P_0)$ , hence, after using Eq. (3) with  $m_2 = 0$ ,

$$S(T_0, P_0)m_1 - S_1 \geq 0 ?$$

$$S(T_0, P_0) - s(T_0, P) \geq 0 ?$$

$$c_p \ln \frac{T_0}{T_0} - R \ln \frac{P_0}{P} \geq 0 ?$$

The conclusion is  $\ln \frac{P}{P_0} > 0$ , positive definitely, because  $P > P_0$ .

(f) Process  $1 \rightarrow 2$  is irreversible, because the inequality sign is  $>$ , not  $=$ .

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## THE TWO LAWS COMBINED: THE DESTRUCTION OF EXERGY

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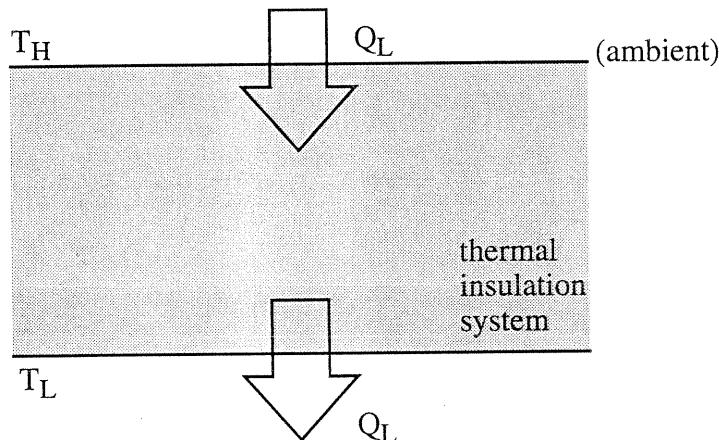
Problem 3.1 It was shown in Fig. 3.9 that the refrigerator deposits exergy in the ( $T_L$ ) space at the rate

$$(-E_{Q_L}) = Q_L \left( \frac{T_H}{T_L} - 1 \right) \quad (1)$$

Looking now at the thermal insulation (the "system") sandwiched between ( $T_H$ ) and ( $T_L$ ), we see that  $Q_L$  enters from  $T_H$  and exits through  $T_L$ . The net exergy flow *into* this system is [see eq. (3.14)]

$$\begin{aligned} \text{net exergy inflow} &= Q_L \left( 1 - \frac{T_H}{T_H} \right) + (-Q_L) \left( 1 - \frac{T_H}{T_L} \right) \\ &= Q_L \left( \frac{T_H}{T_L} - 1 \right) > 0 \end{aligned} \quad (2)$$

The fact that this quantity is positive means that the thermal insulation system acts as a "sink" for exergy. Note that the quantities of eqs. (1) and (2) are equal.




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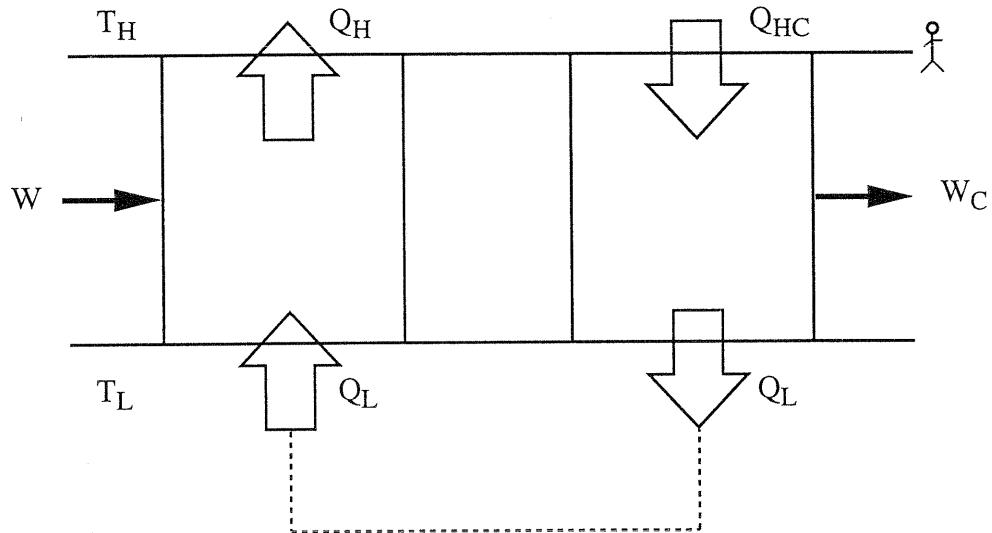
Problem 3.2 Considering first the refrigerator, the refrigeration load could theoretically be provided by the heat rejection from a reversible (C) engine operating between ( $T_H$ ) and ( $T_L$ ). The useful work that would be produced by this engine is

$$W_C = Q_{HC} \left( 1 - \frac{T_L}{T_H} \right)$$

and, since  $Q_{HC}/T_H = Q_L/T_L$ ,

$$W_C = Q_L \left( \frac{T_H}{T_L} - 1 \right)$$

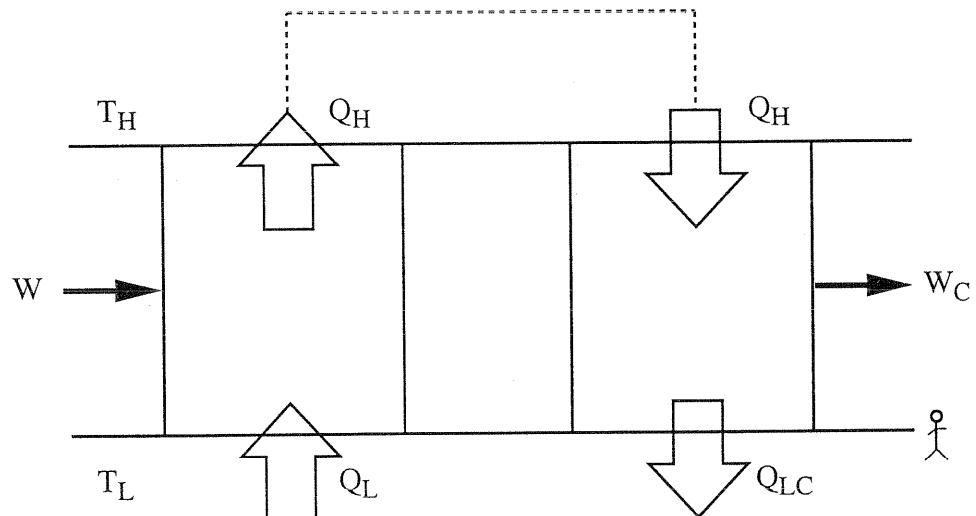
This quantity is exactly the exergy poured by the refrigerator into the ( $T_L$ ) reservoir.



Similarly, the heat transfer pushed up by a heat pump ( $Q_H$ ), could theoretically drive a reversible heat engine between ( $T_H$ ) and ( $T_L$ ). The work produced by this engine

$$W_C = Q_H \left( 1 - \frac{T_L}{T_H} \right)$$

is exactly the exergy that was deposited by the heat pump into the warm space ( $T_H$ ) [see the first term on the right side of eq. (3.26')].



Problem 3.3 Starting with eq. (3.37) and choosing the dimensionless temperature and pressure variables

$$\tau = \frac{T - T_0}{T_0}, \quad \pi = \frac{P - P_0}{P_0}$$

we obtain the dimensionless non-flow exergy of an ideal gas in the limit  $\tau \rightarrow 0$  and  $\pi \rightarrow 0$ :

$$2 \frac{\xi}{c_v T_0} = \tau^2 + \frac{R}{c_v} (\pi - \tau)^2 \quad (3.38)$$

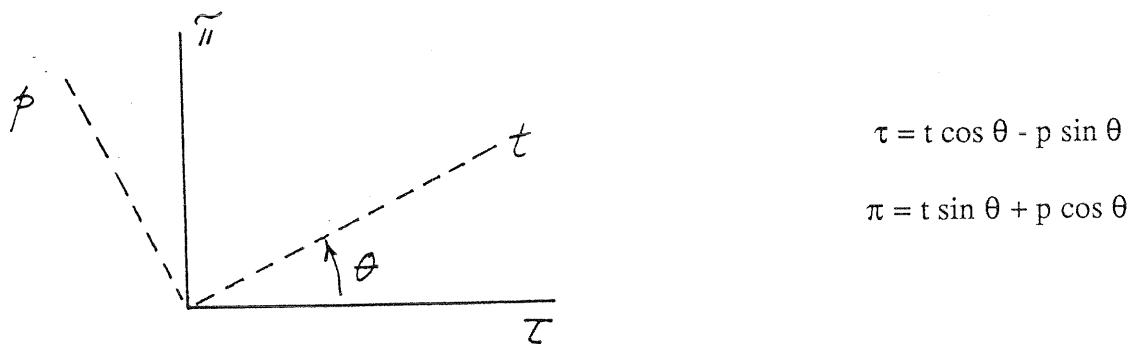
Note that a line of constant  $\xi$  is a circle in the plane in which  $\tau$  and  $(R/c_v)^{1/2}(\pi - \tau)$  are the cartesian coordinates (Fig. 3.13). On the other hand, in the  $\tau - \pi$  plane the constant- $\xi$  line is an ellipse represented by

$$b^2 = \left( \frac{c_v}{R} + 1 \right) \tau^2 - 2\pi\tau + \pi^2 \quad (1)$$

where  $b^2$  is a constant proportional to  $\xi$ . The presence of the " $\pi\tau$ " term on the right side indicates that the axes of this ellipse are not aligned with the  $(\tau, \pi)$  frame. Let  $(t, p)$  be the cartesian system of coordinates that is aligned to the axes of the ellipse, i.e. the frame in which the ellipse equation does not contain the  $tp$  term, say

$$B^2 = k_1 t^2 + k_2 p^2 \quad (2)$$

The  $(t, p)$  frame is rotated by  $\theta$  degrees relative to the original frame  $(\tau, \pi)$  (see the sketch). The relation between the two systems is



Now, in order to obtain eq. (2) we must eliminate  $\tau$  and  $\pi$  from eq. (1) by using the transformation (3) and setting the coefficient of the emerging  $tp$  term equal to zero. This last step yields

$$\tan 2\theta = -2 \frac{R}{C_v}$$

which means that the angle of inclination of the ellipse in the  $\tau - \pi$  plane is a function of  $R/C_v$ . This function is illustrated in a different notation in Fig. 3.13.

---

Problem 3.4 The specific flow exergy is

$$e_x = h - h_0 - T_0(s - s_0)$$

In the case of an incompressible liquid we have, in order,

$$dh = cdT + vdP$$

$$h - h_0 = c(T - T_0) + v(P - P_0)$$

$$ds = c \frac{dT}{T}$$

$$s - s_0 = c \ln \frac{T}{T_0}$$

$$e_x = c T_0 \left( \frac{T}{T_0} - 1 - \ln \frac{T}{T_0} \right) + v (P - P_0) \quad (3.49)$$

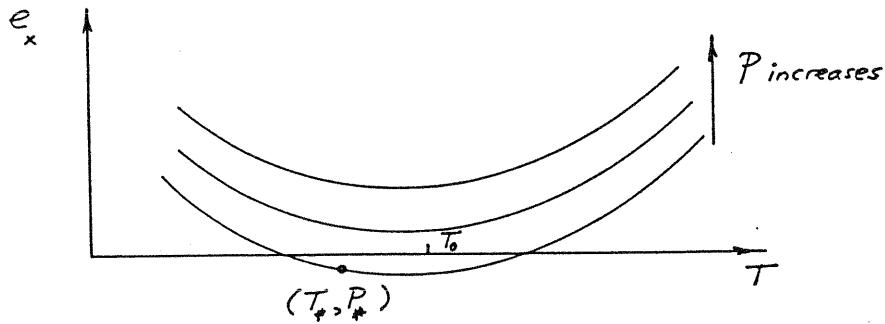
For an ideal gas we write similarly

$$h - h_0 = c_p (T - T_0)$$

$$s - s_0 = c_p \ln \frac{T}{T_0} - R \ln \frac{P}{P_0}$$

$$e_x = c_p T_0 \left( \frac{T}{T_0} - 1 - \ln \frac{T}{T_0} \right) + RT_0 \ln \frac{P}{P_0} \quad (3.51)$$

Both  $e_x$  formulas - eqs. (3.49) and (3.51) - are functions of  $T$  and  $P$ . The main features of these functions are reproduced in the following sketch. At sufficiently low pressures  $P_*$  and at temperatures  $T_*$  near  $T_0$ , the flow exergy becomes negative. The negative value means that if the original state of the fluid in the stream is  $(T_0, P_0)$ , then work (at least as great as  $|e_x|$ ) must be done on the stream in order to bring it to the state  $(T_*, P_*)$ .



Problem 3.5 The non-flow exergy of a closed system is

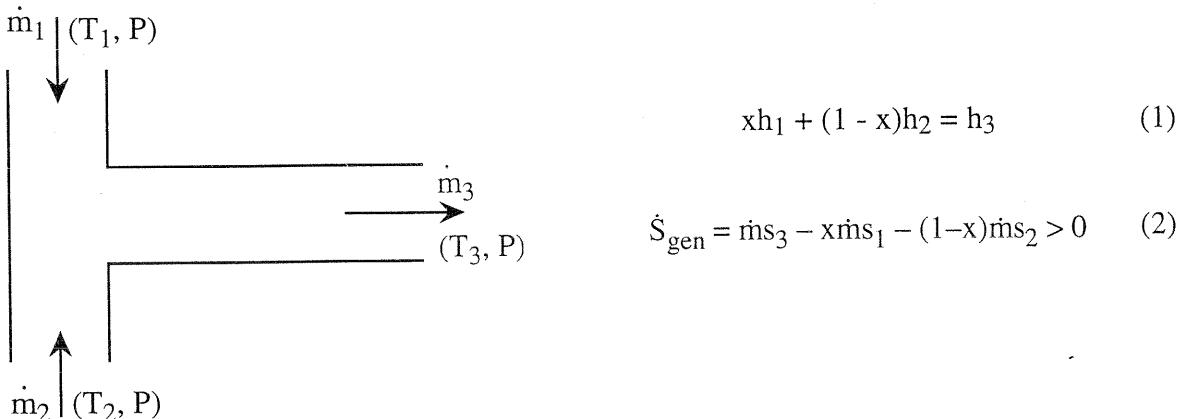
$$\Xi = E - E_0 - T_0(S - S_0) + P_0(V - V_0) \quad (3.32)$$

If the system is isolated, then

- (i)  $E$  is fixed, eq. (2.49)
- (ii)  $V$  is fixed [see the discussion above eq. (2.49)]
- (iii)  $S$  increases or, in the limit, remains the same

In conclusion, if the isolated system undergoes a change, its non-flow exergy  $\Xi$  decreases or, in the limit, remains the same.

Problem 3.6 The first law and the second law for the control volume (mixing chamber) are



where  $\dot{m} = \dot{m}_3 = \dot{m}_1 + \dot{m}_2$ . When the two streams  $\dot{m}_1$  and  $\dot{m}_2$  carry the same ideal gas, eqs. (1) - (2) become

$$x T_1 + (1-x) T_2 = T_3 \quad (3)$$

$$\frac{\dot{S}_{\text{gen}}}{\dot{m} c_P} = x \ln \frac{T_3}{T_1} + (1-x) \ln \frac{T_3}{T_2} > 0 \quad (4)$$

Eliminating  $T_3$  between eqs. (3) - (4) yields

$$\frac{\dot{S}_{\text{gen}}}{\dot{m} c_P} = \ln \left[ \frac{x + \tau(1-x)}{\tau^{1-x}} \right] > 0 \quad (5)$$

in which  $\tau$  is the inlet temperature ratio,

$$\tau = \frac{T_2}{T_1}$$

Solving  $\partial \dot{S}_{\text{gen}} / \partial x = 0$  we obtain the critical mass flowrate fraction for maximum irreversibility,

$$x = \frac{\tau - 1 - \tau \ln \tau}{(1 - \tau) \ln \tau}$$


---

Problem 3.7 The maximum useful work is the non-flow exergy "stored" in the purchased bottle,

$$\Xi = m[u - u_0 - T_0(s - s_0) + P_0(v - v_0)]$$

or, using  $u = h - Pv$ ,

$$\Xi = m[h - h_0 - T_0(s - s_0) + v(P_0 - P)]$$

In this problem  $P_0 - P = 0$  and

$$m = \frac{0.05 \text{ m}^3}{1.24 \times 10^{-3} \text{ m}^3/\text{kg}} = 40.3 \text{ kg}$$

hence

$$\begin{aligned} \Xi &= 40.3 \text{ kg} [-121.5 - 172.1 - 300(2.85 - 6.25)] \frac{\text{kJ}}{\text{kg}} \\ &= 2.93 \times 10^4 \text{ kJ.} \end{aligned}$$


---

Problem 3.8 Consider the aggregate system indicated by the dashed line: this system is closed and contains all the components that operate at temperatures higher than  $T_0$ . We apply the first law to this system, and obtain

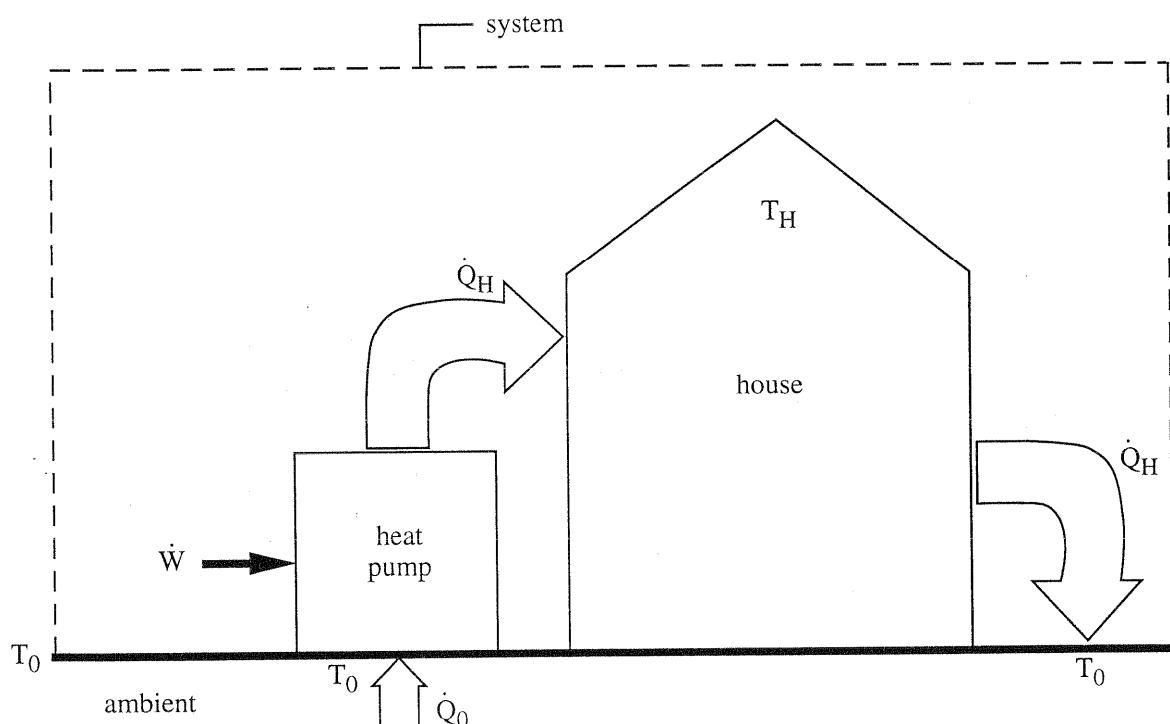
$$\dot{W} = \dot{Q}_H - \dot{Q}_0 = T_0 \left( \frac{\dot{Q}_H}{T_0} - \frac{\dot{Q}_0}{T_0} \right)$$

where the quantity in the parentheses is the total rate of entropy generation of the system:

$$\dot{S}_{\text{gen}} = \frac{\dot{Q}_H}{T_0} - \frac{\dot{Q}_0}{T_0}$$

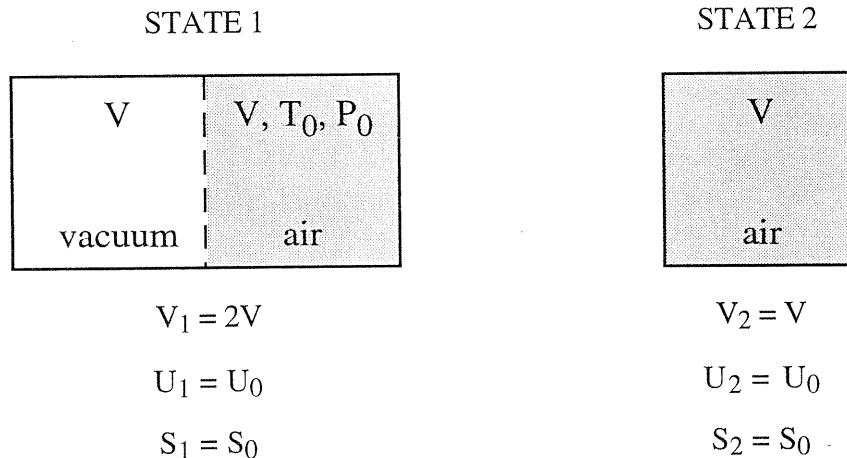
In conclusion, the power required by the heat pump is proportional to the rate of entropy generation in the heat pump and the rest of the space situated at temperatures above  $T_0$ ,

$$\dot{W} = T_0 \dot{S}_{\text{gen}}$$



---

Problem 3.9 The system is closed and occupies the space of the bottle and the mass of atmospheric air that will reside inside the bottle in the final state (state 2). In the initial state (state 1) the total volume of the system is  $2V$ . The end states are



where  $U_0$  and  $S_0$  are the energy and entropy inventories of the mass of atmospheric air ( $V, T_0, P_0$ ). The maximum work that can be extracted from state 1 to state 2 while the system makes contact with the atmospheric reservoir ( $T_0, P_0$ ) is the nonflow exergy

$$\begin{aligned}\Xi &= (Y - T_0 S + P_0 V)_1 - (U - T_0 S + P_0 V)_2 \\ &= P_0 (V_1 - V_2) = P_0 V\end{aligned}$$

This agrees with Example 2.2, where it was shown that  $P_0 V$  is destroyed (during the bottle filling process there is no "user" — no mechanism for extracting and delivering  $\Xi$  to us).

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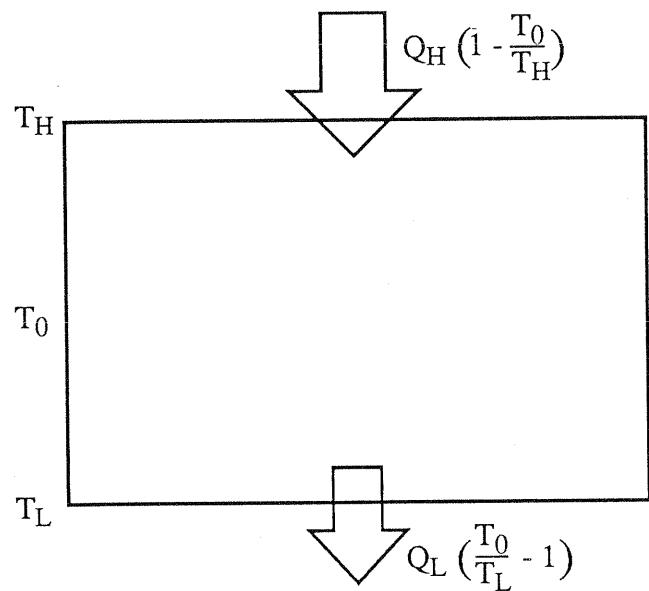
Problem 3.10 The second law efficiency of the power plant (P) is defined as

$$\eta_{II}^{(P)} = \frac{W}{Q_H (1 - T_0/T_H)}$$

Similarly, for the second law efficiency of the refrigeration plant (R) we write

$$\eta_{II}^{(R)} = \frac{Q_L [(T_0/T_L) - 1]}{W}$$

where the numerator is the exergy "deposited" in the cold space. The aggregate system ( $P + R$ ) has one exergy input and one exergy output, as shown in the figure:



The second law efficiency of the aggregate system is the ratio

$$\eta_{II}^{(P+R)} = \frac{\text{exergy output}}{\text{exergy input}} = \frac{Q_L [(T_0/T_L) - 1]}{Q_H (1 - T_0/T_H)} \quad (3)$$

which is equivalent to

$$\eta_{II}^{(P+R)} = \eta_{II}^{(P)} \cdot \eta_{II}^{(R)}$$


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## Chapter 4

### SINGLE-PHASE SYSTEMS

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Problem 4.1 Starting with the Euler equation and fundamental relation in entropy representation

$$S = \frac{1}{T} U + \frac{P}{T} V - \frac{\mu}{T} N \quad (4.30'')$$

$$dS = \frac{1}{T} dU + \frac{P}{T} dV - \frac{\mu}{T} dN \quad (4.18)$$

we differentiate eq. (4.30'')

$$dS = \frac{1}{T} dU + Ud\left(\frac{1}{T}\right) + \frac{P}{T} dV + Vd\left(\frac{P}{T}\right) - \frac{\mu}{T} dN - Nd\left(\frac{\mu}{T}\right) \quad (a)$$

and subtract eq. (a) from eq. (4.18):

$$0 = Ud\left(\frac{1}{T}\right) + Vd\left(\frac{P}{T}\right) - Nd\left(\frac{\mu}{T}\right) \quad (b)$$

The alternate approach is to start with the Gibbs-Duhem relation

$$SdT - VdP + Nd\mu = 0 \quad (4.32)$$

replace  $S$  using eq. (4.30''),

$$\left(\frac{1}{T} U + \frac{P}{T} V - \frac{\mu}{T} N\right) dT - VdP + Nd\mu = 0 \quad (c)$$

divide eq. (c) by  $T$ ,

$$U \frac{dT}{T^2} + V \frac{PdT - TdP}{T^2} + N \frac{Td\mu - \mu dT}{T^2} = 0 \quad (d)$$

and recognize the emergence of the differentials that appear on the right-hand-side of eq. (b).

---

Problem 4.2 We have a system of four equations

$$U(S, V, N) = U_0 \left( \frac{V}{V_0} \right)^{-\bar{R}/\bar{c}_v} \exp \left( \frac{S - S_0}{N\bar{c}_v} \right) \quad (a)$$

$$T(S, V, N) = \left( \frac{\partial U}{\partial S} \right)_{V, N} = U_0 \left( \frac{V}{V_0} \right)^{-\bar{R}/\bar{c}_v} \exp \left( \frac{S - S_0}{N\bar{c}_v} \right) \frac{1}{N\bar{c}_v} \quad (b)$$

$$\mu(S, V, N) = \left( \frac{\partial U}{\partial N} \right)_{S, V} = U_0 \left( \frac{V}{V_0} \right)^{-\bar{R}/\bar{c}_v} \exp \left( \frac{S - S_0}{N\bar{c}_v} \right) \frac{S_0 - S}{N^2\bar{c}_v} \quad (c)$$

$$F_\mu(S, V, N) = U - TS - \mu N = U_0 \left( \frac{V}{V_0} \right)^{-\bar{R}/\bar{c}_v} \exp \left( \frac{S - S_0}{N\bar{c}_v} \right) \left( 1 - \frac{S_0}{N\bar{c}_v} \right) \quad (d)$$

To obtain  $F_\mu$  as a function of  $(T, V, \mu)$  we eliminate  $U, S$  and  $N$  between eqs. (b) - (d). First, dividing (b) and (c) yields

$$\frac{T}{\mu} = \frac{N}{S_0 - S} \quad (e)$$

Second, combining (d) and (e) we obtain the wanted result

$$F_\mu(T, V, \mu) = U_0 \left( \frac{V}{V_0} \right)^{-\bar{R}/\bar{c}_v} \exp \left( -\frac{\mu}{\bar{c}_v T} \right) - S_0 T. \quad (f)$$

**Problem 4.3** In order to derive  $F(T, V, N)$  we begin with three equations (review Table 4.3):

$$U(S, V, N) = U_0 \left( \frac{V}{V_0} \right)^{-\bar{R}/\bar{c}_v} \exp \left( \frac{S - S_0}{N\bar{c}_v} \right) \quad (a)$$

$$T(S, V, N) = \left( \frac{\partial U}{\partial S} \right)_{V, N} = U_0 \left( \frac{V}{V_0} \right)^{-\bar{R}/\bar{c}_v} \exp \left( \frac{S - S_0}{N\bar{c}_v} \right) \frac{1}{N\bar{c}_v} \quad (b)$$

$$F(S, V, N) = U - TS = U_0 \left( \frac{V}{V_0} \right)^{-\bar{R}/\bar{c}_v} \exp \left( \frac{S - S_0}{N\bar{c}_v} \right) \left( 1 - \frac{S}{N\bar{c}_v} \right) \quad (c)$$

The objective is to find  $F = F(T, V, N)$ . First, we combine (b) and (c) to write

$$F = TN\bar{c}_v \left( 1 - \frac{S}{N\bar{c}_v} \right) \quad (d)$$

Second, we eliminate  $S$  between (d) and (b), and obtain the wanted result:

$$F = (T, N, V) = TN\bar{c}_v \left[ 1 - \frac{S_0}{N\bar{c}_v} - \ln \frac{TN\bar{c}_v}{U_0 \left( \frac{V}{V_0} \right)^{-\bar{R}/\bar{c}_v}} \right] \quad (e)$$

The second part of the problem concerns the derivation of  $G(T, P, N)$ . We have a system of four equations, namely (a) and (b) as well as

$$P = \left( \frac{\partial U}{\partial V} \right)_{S, N} = U_0 \left( \frac{V}{V_0} \right)^{-\bar{R}/\bar{c}_v} \frac{(\bar{R}/\bar{c}_v)}{V} \exp \left( \frac{S - S_0}{N\bar{c}_v} \right) \quad (f)$$

$$G(S, V, N) = U - TS + PV = U_0 \left( \frac{V}{V_0} \right)^{-\bar{R}/\bar{c}_v} \exp \left( \frac{S - S_0}{N\bar{c}_v} \right) \left( 1 - \frac{S}{N\bar{c}_v} + \frac{\bar{R}}{\bar{c}_v} \right) \quad (g)$$

First we eliminate  $S$  and  $V$  between (b), (f) and (g). Dividing (b) by (f) yields

$$\frac{T}{P} = \frac{V}{NR} \quad (h)$$

Next, we divide (g) by (b)

$$\frac{G}{T} = N(\bar{c}_v + \bar{R}) - S \quad (i)$$

Using (h) and (i) to eliminate V and S from (g) yields, finally,

$$G = TN(\bar{c}_v + \bar{R}) - S_0 T - TN\bar{c}_v \ln \left[ \frac{TN\bar{c}_v}{U_0} \left( \frac{NRT}{PV_0} \right)^{\bar{R}/\bar{c}_v} \right] = \text{etc.},$$

which is a function of T, P and N.

---

Problem 4.4 The  $(n+2)^{\text{th}}$  Legendre transform is

$$\eta^{(n+2)} = y - \sum_{i=1}^{n+2} \zeta_i x_i$$

The fundamental relation has  $(n+2)$  arguments

$$U = U(S, V, N_1, \dots, N_n), \quad \text{or}$$

$$y = y(x_1, x_2, x_3, \dots, x_{n+2})$$

which means that the  $(n+2)^{\text{th}}$  transform of U is its "total" Legendre transform

$$\eta_U^{(n+2)} = U - TS + PV - \mu_1 N_1 - \dots - \mu_n N_n$$

However, according to eq. (4.30") the right-hand-side is identically equal to zero, hence

$$\eta_U^{(n+2)} \equiv 0$$


---

Problem 4.5 We must prove that

$$\left( \frac{\partial P}{\partial v} \right)_T \frac{c_p}{c_v} = \left( \frac{\partial P}{\partial v} \right)_S \quad (a)$$

Invoking eqs. (4.58) and (4.60) we work successively on the left-hand-side (LHS):

$$\text{LHS} = \left( \frac{\partial P}{\partial v} \right)_T T \left( \frac{\partial S}{\partial T} \right)_P \frac{1}{T} \left( \frac{\partial T}{\partial S} \right)_v \quad (b)$$

Next, we eliminate  $(\partial P / \partial v)_T$  using the cyclical relation between P, v and T,

$$\left(\frac{\partial P}{\partial v}\right)_T \left(\frac{\partial v}{\partial T}\right)_P \left(\frac{\partial T}{\partial P}\right)_v = -1 \quad (c)$$

such that eq. (b) becomes

$$\begin{aligned} \text{LHS} &= -\left(\frac{\partial P}{\partial T}\right)_v \left(\frac{\partial T}{\partial v}\right)_P \left(\frac{\partial s}{\partial T}\right)_P \left(\frac{\partial T}{\partial s}\right)_v \\ &= -\left(\frac{\partial P}{\partial s}\right)_v \left(\frac{\partial s}{\partial v}\right)_P \end{aligned} \quad (d)$$

Putting (d) and (a) together,

$$-\left(\frac{\partial P}{\partial s}\right)_v \left(\frac{\partial s}{\partial v}\right)_P = \left(\frac{\partial P}{\partial v}\right)_s \quad (e)$$

we recognize the cyclical relation for P, v and s,

$$\left(\frac{\partial P}{\partial v}\right)_s \left(\frac{\partial v}{\partial s}\right)_P \left(\frac{\partial s}{\partial P}\right)_v = -1 \quad (f)$$

The fact that (f) is an identity proves the validity of equation (a).

---

#### Problem 4.6

$$\begin{aligned} \text{(a)} \quad \left(\frac{\partial s}{\partial v}\right)_T &= \left(\frac{\partial P}{\partial T}\right)_v && \text{using eq. (4.56f)} \\ &= -\frac{(\partial v/\partial T)_P}{(\partial v/\partial P)_T} && \text{using the cyclical relation for P, v and T} \\ &= \frac{\frac{1}{v} \left(\frac{\partial v}{\partial T}\right)_P}{-\frac{1}{v} \left(\frac{\partial v}{\partial P}\right)_T} = \frac{\beta}{\kappa} \\ \text{(b)} \quad \left(\frac{\partial s}{\partial P}\right)_T &= -\left(\frac{\partial v}{\partial T}\right)_P && \text{using eq. (4.56g)} \\ &= -v \left[ \frac{1}{v} \left(\frac{\partial v}{\partial T}\right)_P \right] = -v\beta \end{aligned}$$

(c) First, a method involving the use of Jacobians:

$$c_v = T \left( \frac{\partial s}{\partial T} \right)_v \quad \text{using eq. (4.58)}$$

$$= T \frac{\partial(s, v)}{\partial(T, v)} \quad \text{using eq. (4.86)}$$

$$= T \frac{\frac{\partial(s, v)}{\partial(T, P)}}{\frac{\partial(T, v)}{\partial(T, T)}} \quad \text{using eq. (4.87)}$$

$$= T \frac{\left( \frac{\partial s}{\partial T} \right)_P \left( \frac{\partial v}{\partial P} \right)_T - \left( \frac{\partial s}{\partial P} \right)_T \left( \frac{\partial v}{\partial T} \right)_P}{\left( \frac{\partial v}{\partial P} \right)_T}$$

$$= T \frac{\frac{c_P}{T} (-\kappa v) - \left[ -\left( \frac{\partial v}{\partial T} \right)_P \right] \left( \frac{\partial v}{\partial T} \right)_P}{-\kappa v} \quad \text{using eq. (4.56g)}$$

$$= c_P - \frac{v}{\kappa} \beta^2 T$$

An alternate proof, one that does not involve the use of Jacobians, begins with recognizing that the objective is to express the difference ( $c_P - c_v$ ) as a function of  $P, v, T$  and their derivatives,

$$c_P - c_v = \frac{v}{\kappa} \beta^2 T = -T \left( \frac{\partial v}{\partial T} \right)_P^2 \left( \frac{\partial P}{\partial v} \right)_T \quad (\text{a})$$

We go back to the entropic definitions of  $c_P$  and  $c_v$  and write

$$\begin{aligned} ds &= \frac{1}{T} c_P dT + \left( \frac{\partial s}{\partial P} \right)_T dP \\ &= \frac{1}{T} c_P dT - \left( \frac{\partial v}{\partial T} \right)_P dP \quad \text{using eq. (4.56g)} \end{aligned}$$

$$\begin{aligned} ds &= \frac{1}{T} c_v dT + \left( \frac{\partial s}{\partial v} \right)_P dv \\ &= \frac{1}{T} c_v dT + \left( \frac{\partial P}{\partial T} \right)_v dv \quad \text{using eq. (4.56f)} \end{aligned}$$

Eliminating  $ds$  between the two expressions developed above, we solve for  $dT$  and find

$$dT = \frac{v}{c_P - c_v} \left( \frac{\partial v}{\partial T} \right)_P dP + \frac{T}{c_P - c_v} \left( \frac{\partial P}{\partial T} \right)_v dv$$

in other words,

$$\frac{T}{c_P - c_v} \left( \frac{\partial v}{\partial T} \right)_P = \left( \frac{\partial T}{\partial P} \right)_v \quad (b)$$

$$\frac{T}{c_P - c_v} \left( \frac{\partial P}{\partial T} \right)_v = \left( \frac{\partial T}{\partial v} \right)_P \quad (c)$$

Equations (b) and (c) are the same equation namely

$$c_P - c_v = T \left( \frac{\partial P}{\partial T} \right)_v \left( \frac{\partial v}{\partial T} \right)_P \quad (d)$$

With an eye on eq. (a), we invoke the cyclical relation

$$\left( \frac{\partial P}{\partial T} \right)_v \left( \frac{\partial T}{\partial v} \right)_P \left( \frac{\partial v}{\partial P} \right)_T = -1 \quad (e)$$

and eliminate  $(\partial P / \partial T)_v$  between (e) and (d):

$$c_P - c_v = -T \left( \frac{\partial v}{\partial T} \right)_P^2 \left( \frac{\partial P}{\partial v} \right)_T = \frac{v}{\kappa} \beta^2 T \quad (a)$$

$$(d) \quad \left( \frac{\partial h}{\partial P} \right)_T = T \left( \frac{\partial s}{\partial P} \right)_T + v \quad \text{using } dh = Tds + vdP$$

$$= -T \left( \frac{\partial v}{\partial P} \right)_T + v \quad \text{using eq. (4.56g)}$$

$$= (-\beta T + 1) v$$

$$(e) \quad \mu_J = \left( \frac{\partial T}{\partial P} \right)_h$$

$$= - \left( \frac{\partial h}{\partial P} \right)_T \left( \frac{\partial T}{\partial h} \right)_P \quad \text{using the cyclical relation for } T, P \text{ and } h$$

$$= \frac{1}{c_P} (\beta T - 1) v \quad \text{using the answer to part (d) above}$$

$$(f) \quad \left( \frac{\partial T}{\partial P} \right)_s = -\frac{T}{c_P} \left( \frac{\partial s}{\partial P} \right)_T \quad \text{using } ds = \frac{c_P}{T} dT + \left( \frac{\partial s}{\partial P} \right)_T dP$$

$$= \frac{T}{c_P} \left( \frac{\partial v}{\partial T} \right)_P \quad \text{using eq. (4.56g)}$$

$$= \frac{T}{c_P} v \beta$$

$$\begin{aligned}
(g) \quad & \left( \frac{\partial c_p}{\partial P} \right)_T = \frac{\partial}{\partial P} \left[ T \left( \frac{\partial s}{\partial T} \right)_P \right]_T \\
& = T \frac{\partial}{\partial T} \left[ \left( \frac{\partial s}{\partial T} \right)_P \right]_P \\
& = T \frac{\partial}{\partial T} \left[ - \left( \frac{\partial v}{\partial T} \right)_P \right]_P = - T \left( \frac{\partial^2 v}{\partial T^2} \right)_P
\end{aligned}$$


---

Problem 4.7 We must prove that

$$\frac{\partial^2 u}{\partial v^2} - \left( \frac{\partial^2 u}{\partial s \partial v} \right)^2 / \frac{\partial^2 u}{\partial s^2} = \frac{\partial^2 f}{\partial v^2} \quad (a)$$

Since  $du = Tds - Pdv$ , we recognize that

$$T = \left( \frac{\partial u}{\partial s} \right)_v \quad \text{and} \quad P = - \left( \frac{\partial u}{\partial v} \right)_s$$

The left-hand-side of eq. (a) becomes

$$\text{LHS} = - \left( \frac{\partial P}{\partial v} \right)_s - \left( \frac{\partial T}{\partial v} \right)_s^2 / \left( \frac{\partial T}{\partial s} \right)_v$$

and, using Table 4.6,

$$\begin{aligned}
\text{LHS} &= - \frac{\left( \frac{\partial P}{\partial v} \right)_s}{\left( \frac{\partial v}{\partial s} \right)_s} - \left[ \frac{\left( \frac{\partial T}{\partial v} \right)_s}{\left( \frac{\partial v}{\partial s} \right)_s} \right]^2 / \frac{\left( \frac{\partial T}{\partial s} \right)_v}{\left( \frac{\partial s}{\partial v} \right)_v} \\
&= - \frac{-c_p/T}{-\beta^2 v^2 + \kappa v c_p/T} - \left( \frac{-\beta v}{-\beta^2 v^2 + \kappa v c_p/T} \right)^2 / \frac{-\kappa v}{\beta^2 v^2 - \kappa v c_p/T} \\
&= \frac{1}{\kappa v}
\end{aligned}$$

For the right-hand-side of eq. (a) we note that  $df = -sdT - Pdv$ , hence

$$\begin{aligned}
\left( \frac{\partial f}{\partial v} \right)_s &= -P \\
\text{RHS} &= - \left( \frac{\partial P}{\partial v} \right)_T = - \frac{\left( \frac{\partial P}{\partial T} \right)_V}{\left( \frac{\partial v}{\partial T} \right)_V} = - \frac{-1}{\kappa v} = \frac{1}{\kappa v}
\end{aligned}$$

In other words, we conclude that  $\text{LHS} = \text{RHS}$ .

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Problem 4.8

$$\begin{aligned}
 du &= \left(\frac{\partial u}{\partial T}\right)_v dT + \left(\frac{\partial u}{\partial v}\right)_T dv \\
 &= c_v dT + \left(\frac{\partial u}{\partial v}\right)_T dv \\
 &= c_v dT + \left[T \left(\frac{\partial s}{\partial v}\right)_T - P\right] dv \quad \text{using } du = Tds - Pdv \\
 &= c_v dT + \left[T \left(\frac{\partial P}{\partial T}\right)_v - P\right] dv
 \end{aligned}$$

$$\begin{aligned}
 dh &= \left(\frac{\partial h}{\partial T}\right)_P dT + \left(\frac{\partial h}{\partial P}\right)_T dP \\
 &= c_P dT + \left(\frac{\partial h}{\partial P}\right)_T dP \\
 &= c_P dT + \left[T \left(\frac{\partial s}{\partial P}\right)_T + v\right] dP \quad \text{using } dh = Tds + vdP \\
 &= c_P dT + \left[-T \left(\frac{\partial v}{\partial T}\right)_P + v\right] dP \\
 &= c_P dT + (-\beta T + 1) vdP
 \end{aligned}$$

$$\begin{aligned}
 ds &= \left(\frac{\partial s}{\partial T}\right)_P dT + \left(\frac{\partial s}{\partial P}\right)_T dP \\
 &= \frac{c_P}{T} dT - \left(\frac{\partial v}{\partial T}\right)_P dP \quad \text{using eq. (4.56g)} \\
 &= \frac{c_P}{T} dT - \beta v dP
 \end{aligned}$$

$$\begin{aligned}
 ds &= \left(\frac{\partial s}{\partial T}\right)_v dT + \left(\frac{\partial s}{\partial v}\right)_T dv \\
 &= \frac{c_v}{T} dT + \left(\frac{\partial P}{\partial T}\right)_v dv \quad \text{using eq. (4.56f)} \\
 &= \frac{c_v}{T} dT - \left(\frac{\partial v}{\partial T}\right)_P \left(\frac{\partial P}{\partial v}\right)_T dv \quad \text{using the cyclical relation} \\
 &= \frac{c_v}{T} dT + \frac{\beta}{\kappa} dv
 \end{aligned}$$

$$\begin{aligned}
ds &= \left(\frac{\partial s}{\partial v}\right)_P dv + \left(\frac{\partial s}{\partial P}\right)_v dP \\
&= \left(\frac{\partial s}{\partial T}\right)_P \left(\frac{\partial T}{\partial v}\right)_P dv + \left(\frac{\partial s}{\partial T}\right)_v \left(\frac{\partial T}{\partial P}\right)_v dP \\
&= \frac{c_P}{T} \frac{1}{\beta v} dv + \frac{c_v}{T} \left(\frac{\partial T}{\partial P}\right)_v dP \\
&= \frac{c_P}{v\beta T} dv - \frac{c_v}{T} \left(\frac{\partial T}{\partial v}\right)_P \left(\frac{\partial v}{\partial P}\right)_T dP \\
&= \frac{c_P}{v\beta T} dv + \frac{c_v}{T} \frac{\kappa}{\beta} dP
\end{aligned}$$


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### Problem 4.9

(a) From Problem 4.6g we learn that

$$\left(\frac{\partial c_P}{\partial P}\right)_T = -T \left(\frac{\partial^2 v}{\partial T^2}\right)_P = -2v_0 \lambda T$$

Integrating at constant temperature we obtain

$$c_P(T, P) - c_P(T, 0) = -2v_0 \lambda T P$$

hence

$$c_P = c_0 - b(T - T_0) - 2\lambda v_0 T P$$

(b) From Table 4.7 we have

$$ds = \frac{c_P}{T} dT - \beta v dP$$

where

$$\beta v = \left(\frac{\partial v}{\partial T}\right)_P = 2\lambda v_0 (T - T_0 + aP)$$

We integrate

$$ds = \left( \frac{c_0 + bT_0}{T} - b - 2\lambda v_0 P \right) dT - 2\lambda v_0 (T - T_0 + aP) dP$$

from the reference state ( $T = T_0, P = 0$ ) to an arbitrary state ( $T, P$ ), by choosing the path  $(T_0, 0) \rightarrow (T, 0)$  followed by  $(T, 0) \rightarrow (T, P)$ . The result is

$$s(T, P) - s(T, 0) = (c_0 + bT_0) \ln \frac{T}{T_0} - b(T - T_0) - 2\lambda v_0 P (T - T_0) - \lambda v_0 a P^2$$

(c) For the Gibbs free energy  $g(T, P)$  we start with (see Table 4.3)

$$dg = -sdT + vdP$$

where we know both  $s(T, P)$  and  $v(T, P)$ . We integrate from  $(T_0, 0)$  to any state  $(T, P)$  along the path  $(T_0, 0) \rightarrow (T, 0) \rightarrow (T, P)$ ,

$$g(T, P) - g(T_0, 0) = - \int_{T_0}^T s(T, 0) dT + \int_0^P v(T, P) dP$$

Setting both  $g(T_0, 0)$  and  $s(T_0, 0)$  equal to zero we obtain

$$\begin{aligned} g(T, P) = & -(c_0 + bT_0)T \ln \frac{T}{T_0} \\ & + (c_0 + bT_0)(T - T_0) + \frac{b}{2}(T - T_0)^2 \\ & + v_0 \left( P - \frac{\kappa_0}{2} P^2 \right) \\ & + \lambda v_0 P \left[ (T - T_0)^2 + aP(T - T_0) + \frac{a^2}{3} P^2 \right] \end{aligned}$$

(d) The incompressible liquid model with constant specific heat means that  $\lambda = 0$ ,  $\kappa_0 = 0$  and  $b = 0$ . In this case the  $g(T, P)$  expression reduces to

$$g(T, P) = -c_0 T \ln \frac{T}{T_0} + c_0(T - T_0) + v_0 P$$


---

Problem 4.10 This process is a zero-work process, therefore the first law requires

$$\delta Q = dU \quad (\text{constant } V)$$

If the process represents "heating", then

$$(dU)_V > 0 \tag{1}$$

Constant-volume pressurization means that

$$(dP)_V > 0 \tag{2}$$

Dividing (1) and (2), we find that the condition for the described process to be a heating process is

$$\left( \frac{\partial U}{\partial P} \right)_V > 0$$

or

$$\left(\frac{\partial U}{\partial T}\right)_V \left(\frac{\partial T}{\partial P}\right)_V > 0$$

$$mc_v \left(\frac{\partial T}{\partial P}\right)_V > 0 \quad (3)$$

Finally, invoking the cyclical relation

$$\left(\frac{\partial T}{\partial P}\right)_V \left(\frac{\partial P}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_P = -1$$

the inequality (3) reduces to

$$c_v \left[ -\left(\frac{\partial V}{\partial P}\right)_T \right] / \left(\frac{\partial V}{\partial T}\right)_P > 0$$

hence

$$c_v \frac{\kappa}{\beta} > 0$$


---

### Problem 4.11

$$(a) \quad a^2 = \left(\frac{\partial P}{\partial \rho}\right)_s = -v^2 \left(\frac{\partial P}{\partial v}\right)_s$$

$$= +v^2 \left(\frac{\partial s}{\partial v}\right)_P \left(\frac{\partial P}{\partial s}\right)_v$$

using the cyclical relation for P, v,  
and s

$$= v^2 \left(\frac{\partial s}{\partial T}\right)_P \left(\frac{\partial T}{\partial v}\right)_P \left(\frac{\partial v}{\partial s}\right)_T \left(\frac{\partial P}{\partial T}\right)_v$$

$$= v^2 \frac{c_P}{T} \left(\frac{\partial T}{\partial v}\right)_P \frac{T}{c_v} \left(\frac{\partial P}{\partial T}\right)_v$$

$$= v^2 \frac{c_P}{c_v} \left[ -\left(\frac{\partial P}{\partial v}\right)_T \right]$$

using the cyclical relation for P, v  
and T

$$= v^2 \frac{c_P}{c_v} \left(\frac{1}{\kappa v}\right) = \frac{v c_P}{\kappa c_v}$$

(b) According to Table 4.5, in the ideal gas limit we have  $\kappa = P^{-1}$ , therefore,

$$a^2 = \frac{v c_P}{P^{-1} c_v} = Pv \frac{c_P}{c_v} = kRT.$$

Alternatively, according to Table 4.7, for an ideal gas we can write

$$ds = \frac{c_P}{v} dv + \frac{c_v}{P} dP$$

$$\left(\frac{dP}{dv}\right)_S = -\frac{c_P}{c_v} \frac{P}{v}$$

and, in conclusion,

$$a^2 = -v^2 \left(\frac{\partial P}{\partial v}\right)_S = -v^2 \left(-k \frac{P}{v}\right) = kPv = kRT$$


---

### Problem 4.12

(i) Regarding  $u$  as a function of  $T$  and  $v$ , we have

$$\begin{aligned} du &= \left(\frac{\partial u}{\partial T}\right)_v dT + \left(\frac{\partial u}{\partial v}\right)_T dv \\ &= c_v dT + \left[T \left(\frac{\partial s}{\partial v}\right)_T - P\right] dv \\ &= c_v dT + \left[T \left(\frac{\partial P}{\partial T}\right)_v - P\right] dv \end{aligned}$$

and, after evaluating  $(\partial P / \partial T)_v$  for a van der Waals gas, we obtain

$$du = c_v dT + \frac{a}{v^2} dv$$

Integrating from  $(T_0, v_0)$  to  $(T, v)$ , we obtain finally

$$u - u_0 = c_v (T - T_0) + a \left( \frac{1}{v_0} - \frac{1}{v} \right)$$

(ii) The same procedure can be used for deriving the function  $s(T, v)$ ,

$$\begin{aligned} ds &= \left(\frac{\partial s}{\partial T}\right)_v dT + \left(\frac{\partial s}{\partial v}\right)_T dv \\ &= \frac{c_v}{T} dT + \left(\frac{\partial P}{\partial T}\right)_v dv \\ &= \frac{c_v}{T} dT + \frac{R}{v-b} dv \end{aligned}$$

yielding finally

$$s - s_0 = c_v \ln \frac{T}{T_0} + R \ln \frac{v-b}{v_0-b}$$

(iii) Using the ds expression obtained above, we write

$$\begin{aligned} dh &= Tds + vdp \\ &= T \left( \frac{c_v}{T} dT + \frac{R}{v-b} dv \right) + v \left[ \frac{R}{v-b} dT - \frac{RT}{(v-b)^2} dv + \frac{2a}{v^3} dv \right] \\ &= \left( c_v + R \frac{v}{v-b} \right) dT + \left[ -\frac{RTb}{(v-b)^2} + \frac{2a}{v^2} \right] dv \end{aligned}$$

The difference  $(h - h_0)$  is independent of the path linking the end states  $(T_0, v_0)$  and  $(T, v)$ . Therefore, we can integrate  $dh$  by first proceeding isothermally from  $(T_0, v_0)$  to  $(T_0, v)$ , and later by keeping the volume constant from  $(T_0, v)$  until  $(T, v)$ :

$$\begin{aligned} h - h_0 &= \int_{v_0}^v \left[ -\frac{RT_0 b}{(v-b)^2} + \frac{2a}{v^2} \right] dv + \int_{T_0}^T \left( c_v + R \frac{v}{v-b} \right) dT \\ &= RT_0 \frac{b(v_0 - v)}{(v-b)(v_0 - b)} - 2a \left( \frac{1}{v} - \frac{1}{v_0} \right) + \left( c_v + R \frac{v}{v-b} \right) (T - T_0) \end{aligned}$$


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### Problem 4.13

$$\begin{aligned} (a) \quad \left( \frac{\partial^2 u}{\partial s^2} \right)_v &= \frac{\partial}{\partial s} \left( \frac{\partial u}{\partial s} \right)_v \\ &= \frac{\partial}{\partial s} (T)_v && \text{because } du = Tds - Pdv \\ &= \frac{T}{c_v} && \text{because } c_v = T \left( \frac{\partial s}{\partial T} \right)_v \end{aligned}$$

$$\begin{aligned} (b) \quad \left( \frac{\partial^2 h}{\partial s^2} \right)_P &= \frac{\partial}{\partial s} \left( \frac{\partial h}{\partial s} \right)_P \\ &= \frac{\partial}{\partial s} (T)_v && \text{because } dh = Tds + vdp \\ &= \frac{T}{c_P} && \text{because } c_P = T \left( \frac{\partial s}{\partial T} \right)_P \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad & \left( \frac{\partial^2 f}{\partial v^2} \right)_T = \frac{\partial}{\partial v} \left( \frac{\partial f}{\partial v} \right)_T \\
 &= \frac{\partial}{\partial v} (-P)_T \quad \text{because } df = -sdT - Pdv \\
 &= \frac{1}{\kappa v} \quad \text{because } \kappa = -\frac{1}{v} \left( \frac{\partial v}{\partial P} \right)_T
 \end{aligned}$$


---

Problem 4.14 The “system” is the total volume  $V$ , which contains two compartments, A and B,

$$V = V_A + V_B \quad (1)$$

or

$$V_A = xV \quad \text{and} \quad V_B = (1-x)V \quad (2)$$

where the volume fraction  $x$  is a number between 0 and 1.

The initial equilibrium state of the system is specified. Chamber A contains the gas mass  $m_A$  at initial pressure  $P_{A1}$  and temperature  $T_{A1}$ , from which follows the ideal gas volume

$$x_1 V = \frac{m_A R T_{A1}}{P_{A1}} \quad (3)$$

Chamber B contains the gas mass  $m_B$  at given initial pressure  $P_{B1}$  and temperature  $T_{B1}$ , therefore

$$(1-x_1)V = \frac{m_B R T_{B1}}{P_{B1}} \quad (4)$$

Dividing eqs. (3) and (4) we conclude that the initial (locked) position of the adiabatic partition ( $x_1$ ) is known:

$$\frac{1-x_1}{x_1} = \frac{m_B T_{B1}}{m_A T_{A1}} \frac{P_{A1}}{P_{B1}} \quad (5)$$

The partition must be “locked” in this position because in general  $P_{A1} \neq P_{B1}$ .

The question is to determine the final equilibrium position of the adiabatic partition ( $x_2$ ) after it is allowed to slide freely. Equilibrium of this kind (mechanical) means that

$$P_{A2} = P_{B2} \quad (6)$$

or, in order,

$$\frac{m_A R T_{A2}}{V_{A2}} = \frac{m_B R T_{B2}}{V_{B2}} \quad (7)$$

$$\frac{1 - x_2}{x_2} = \frac{m_B T_{B2}}{m_A T_{A2}} \quad (8)$$

Equation (8) shows that the unknown  $x_2$  depends on two additional unknowns,  $T_{A2}$  and  $T_{B2}$ . We need two more equations. The first additional equation is the first law of thermodynamics invoked for the entire (isolated) system during process 1-2:

$$Q_{1-2} - W_{1-2} = (U_2 - U_1)_A + (U_2 - U_1)_B \quad (9)$$

in which  $Q_{1-2} = W_{1-2} = 0$ . The result is

$$m_A c_v (T_{A2} - T_{A1}) + m_B c_v (T_{B2} - T_{B1}) = 0 \quad (10)$$

The second additional equation is the subtlety that apparently has eluded thermodynamicists for a hundred years. The partition is massless. This means that when the locking mechanism is removed and the partition slides freely, the pressure *must be the same* on both sides of the partition. Which pressure will that be,  $P_A(t)$  or  $P_B(t)$ ?

Let us convene that subscript A means that initially chamber A is more pressurized than chamber B,

$$P_{A1} > P_{B1} \quad (11)$$

When the partition is free to move (i.e. when the force that used to be provided by the lock is no longer present), the pressure on both sides of the partition will be the *lower* of the two pressures, namely  $P_{B1}$ . This means that a pressure nonuniformity is created inside chamber A, and from

this the opportunity for gas A to expand. The expansion of A means that B will be compressed. Whereas gas A expands as a nonequilibrium (nonuniform  $P_A$ ) subsystem, gas B contracts quasistatically: at every point in time, the pressure on its moving wall,  $P_B(t)$ , is the same as the pressure throughout the gas mass  $m_B$ .

So, the final equation comes from the fact that the adiabatic compression of  $m_B$  is quasistatic. The first law for an infinitesimal change of state of subsystem B requires

$$\delta Q_B - \delta W_B = dU_B \quad (12)$$

where  $\delta Q_B = 0$ , and

$$\delta W_B = P_B dV_B \quad (13)$$

$$dU_B = m_B c_v dT_B \quad (14)$$

From eqs. (12) – (14) follows the conclusion that  $c_v dT_B / T_B + R dV_B / V_B = 0$ , or that the path of the process undergone by  $m_B$  is  $T_B V_B^\alpha = \text{constant}$ , where  $\alpha = R / c_v$ ; from this we obtain the final equation,

$$T_{B2} V_{B2}^\alpha = T_{B1} V_{B1}^\alpha \quad (15)$$

or

$$\left( \frac{1 - x_2}{1 - x_1} \right)^\alpha = \frac{T_{B1}}{T_{B2}} \quad (16)$$

In summary, we have three equations [namely, eqs. (8), (10) and (16)] for three unknowns,  $x_2$ ,  $T_{A2}$  and  $T_{B2}$ . The final state is unique, and it has just been determined. For example, by eliminating  $T_{A2}$  and  $T_{B2}$  from the three equations we obtain the final position of the partition,

$$1 - x_2 = (1 - x_1)^{\frac{\alpha}{1+\alpha}} \beta^{\frac{-1}{1+\alpha}} \quad (17)$$

where

$$\beta = 1 + \frac{m_A T_{A1}}{m_B T_{B1}} \quad (18)$$

The other properties that define the final state are calculated after  $x_2$  is determined:

$$T_{B2} = T_{B1} \left( \frac{1 - x_1}{1 - x_2} \right)^\alpha \quad (19)$$

$$T_{A2} = T_{B1} \left( \frac{1 - x_1}{1 - x_2} \right)^\alpha \frac{m_B}{m_A} \frac{x_2}{1 - x_2} \quad (20)$$

$$P_{B2} = \frac{m_B R T_{B1}}{V(1 - x_2)} \left( \frac{1 - x_1}{1 - x_2} \right)^\alpha = P_{A2} \quad (21)$$

What role does the second law play in all this? It plays the role of test of irreversibility. The solution developed above must not violate the second law, which, for the process 1-2 executed by the entire (isolated) system, requires that

$$S_2 - S_1 \geq \int_1^2 \frac{\delta Q}{T} \quad (22)$$

The right-hand side is zero because the system is isolated ( $\delta Q = 0$ ). The second law becomes

$$(S_2 - S_1)_A + (S_2 - S_1)_B \geq 0 \quad (23)$$

where the B contribution is zero

$$(S_2 - S_1)_B = m_B \left( c_v \ln \frac{T_{B2}}{T_{B1}} + R \ln \frac{V_{B2}}{V_{B1}} \right) = 0 \quad (24)$$

Equation (24) is a reaffirmation of the fact that the process executed by the B chamber is adiabatic and quasistatic, hence isentropic. This means that all the irreversibility of process 1-2 is located in the gas of chamber A. Left to prove is that

$$(S_2 - S_1)_A \geq 0 \quad (25)$$

as long as the assumption of initial mechanical nonequilibrium ( $P_{1A} \geq P_{1B}$ ) is correct. This can be shown numerically, by assigning numerical values to all the symbols, given and calculated. Here I show it analytically, after I make the “convenient” assumptions that the setup is such that  $m_A = m_B$ ,  $T_{A1} = T_{B1}$  and  $\alpha = 1$ . This means that  $\beta = 2$  and, from eq. (5),

$$\frac{1 - x_1}{x_1} = \frac{P_{A1}}{P_{B1}} \geq 1 \quad (26)$$

therefore  $x_1 \leq \frac{1}{2}$ . The second law test (25) becomes

$$\frac{(S_2 - S_1)_A}{m_A c_v} = \ln \frac{T_{A2}}{T_{A1}} + \alpha \ln \frac{V_{A2}}{V_{A1}} = \ln \left[ \frac{T_{A2}}{T_{A1}} \frac{V_{A2}}{V_{A1}} \right] \geq 0 \quad (27)$$

The question is whether the quantity in the square brackets is greater than 1:

$$[ ] = \frac{1 - x_1}{1 - x_2} \frac{x_2}{1 - x_2} \frac{x_2}{x_1} \geq 1 ? \quad (28)$$

This question can be rewritten in several ways until this form:

$$(1 - \sqrt{2x_1})^2 \geq 0 ? \quad (29)$$

This conclusion is correct as soon as  $x_1 \leq \frac{1}{2}$ ; therefore eq. (25) is correct, and finally eq. (22) is correct. Process 1-2 is irreversible because the gas  $m_A$  evolves irreversibly—it executes an expansion process during which it is internally in nonequilibrium (at nonuniform  $P_A$ ).

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## Chapter 5

# EXERGY ANALYSIS

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Problem 5.1 Consider the thick-line control volume defined in Fig. 5.3. We can write the following three statements:

$$\begin{aligned} \frac{dU}{dt} = \dot{Q}_0 + \sum_{\ell=1}^p \dot{Q}_\ell - \dot{E}_W - P_0 \frac{dV}{dt} + \sum_{i=1}^n \bar{h}_{0,i} \dot{N}_{0,i} \\ + \sum_{i=1}^n \bar{h}_{j,i} \dot{N}_{j,i} - \sum_{i=1}^n \bar{h}_{k,i} \dot{N}_{k,i} \end{aligned} \quad (1)$$

$$\begin{aligned} \dot{S}_{gen} = \frac{dS}{dt} - \frac{\dot{Q}_0}{T_0} - \sum_{\ell=1}^p \frac{\dot{Q}_\ell}{T_\ell} - \sum_{i=1}^n \bar{s}_{0,i} \dot{N}_{0,i} \\ - \sum_{i=1}^n \bar{s}_{j,i} \dot{N}_{j,i} + \sum_{i=1}^n \bar{s}_{k,i} \dot{N}_{k,i} \end{aligned} \quad (2)$$

$$\frac{dN_i}{dt} = \dot{N}_{0,i} + \dot{N}_{j,i} - \dot{N}_{k,i}, \quad (i = 1, 2, \dots, n) \quad (3)$$

Next, we evaluate  $\dot{E}_W$  by eliminating  $\dot{Q}_0$  between eqs. (1) and (2),

$$\begin{aligned} \dot{E}_W = -\frac{d}{dt}(U - T_0 S + P_0 V) + \sum_{\ell=1}^p \dot{Q}_\ell \left(1 - \frac{T_0}{T_\ell}\right) - T_0 \dot{S}_{gen} \\ + \sum_{i=1}^n (\bar{h}_{0,i} - T_0 \bar{s}_{0,i}) \dot{N}_{0,i} + \sum_{i=1}^n (\bar{h}_{j,i} - T_0 \bar{s}_{j,i}) \dot{N}_{j,i} \\ - \sum_{i=1}^n (\bar{h}_{k,i} - T_0 \bar{s}_{k,i}) \dot{N}_{k,i} \end{aligned} \quad (4)$$

Later, we replace  $(\bar{h}_{0,i} - T_0 \bar{s}_{0,i})$  with  $\mu_{0,i}$  in the second summation on the right-hand-side. Finally, the elimination of  $\dot{N}_{0,i}$  between eqs. (3) and (4) yields

$$\begin{aligned} \dot{E}_W = -\frac{d}{dt}\left(U - T_0 S + P_0 V - \sum_{i=1}^n \mu_{0,i} \dot{N}_i\right) + \sum_{\ell=1}^p (\dot{E}_Q)_\ell - T_0 \dot{S}_{gen} \\ + \sum_{i=1}^n (\bar{h}_{j,i} - T_0 \bar{s}_{j,i} - \mu_{0,i}) \dot{N}_{j,i} - \sum_{i=1}^n (\bar{h}_{k,i} - T_0 \bar{s}_{k,i} - \mu_{0,i}) \dot{N}_{k,i} \end{aligned} \quad (5)$$

Recognizing the  $\Xi_t$  notation of eq. (5.9') and the  $\bar{e}_t$  notation of eq. (5.22') we rewrite this conclusion as

$$\dot{E}_W = -\frac{d\Xi_t}{dt} + \sum_{\ell=1}^p (\dot{E}_Q)_\ell - T_0 \dot{S}_{gen} + \dot{N}_j \bar{e}_{t,j} - \dot{N}_k \bar{e}_{t,k} \quad (6)$$

In the general case where there are  $q$  inlet ports of type  $j$ , and  $r$  outlet ports of type  $k$ , this result assumes the form listed as eq. (5.30) in the text.

---

**Problem 5.2** (a) This is a good problem on which to practice the general exergy accounting formula (5.30). In this case the system is non-flow ( $q = 0, r = 0$ ),  $T_0$  is the only heat reservoir (hence  $p = 0$ ) and the process is reversible ( $\dot{S}_{\text{gen}} = 0$ ). The system contains initially one mole of  $O_2$  (state 1) in total equilibrium with the atmosphere and, in the end, as a pure batch (state 2). Integrating eq. (5.30) in time from state 1 to state 2 yields

$$E_{W, 1-2} = \Xi_t^{(1)} - \Xi_t^{(2)}$$

Since state 1 is one of total (thermo-mechanical & chemical) equilibrium with the ambient,  $\Xi^{(1)}$  is zero. Next, we find that  $\Xi_t^{(2)}$  is the same as  $\Xi_{\text{ch}}^{(2)}$ , because the physical non-flow exergy at state 2 is zero (recall that  $T_2 = T_0$  and  $P_2 = P_0$ ),

$$\begin{aligned} E_{W, 1-2} &= -\Xi_t^{(2)} = -\Xi^{(2)} - \Xi_{\text{ch}}^{(2)} \\ &= -\Xi_{\text{ch}}^{(2)} \end{aligned}$$

Equation (5.14) in the case of only one component (oxygen) yields further

$$E_{W, 1-2} = -\Xi_{\text{ch}}^{(2)} = -N_{O_2} (\mu^* - \mu_0)_{O_2}$$

where  $\mu^*(T_0, P_0, x_{O_2} = 1)$  and  $\mu_0(T_0, P_0, x_{0,O_2})$ . Invoking eq. (4.139) yields finally

$$\begin{aligned} \frac{E_{W, 1-2}}{N_{O_2}} &= \mu_0 - \mu^* = \bar{R}T_0 \ln x_{0, O_2} \\ &= 8.3143 \frac{\text{kJ}}{\text{kmol K}} \times 298.15 \text{ K} \times \ln (0.2035) \\ &= -3947 \frac{\text{kJ}}{\text{kmol}_{O_2}} \end{aligned}$$

The answer is negative, which means that the physical direction of the arrow in Fig. 5.3 is toward the control volume.

(b) Based on an identical argument, we have

$$\begin{aligned} \frac{E_{W, 1-2}}{N_{N_2}} &= \bar{R}T \ln x_0 \\ &= 8.3143 \frac{\text{kJ}}{\text{kmol K}} \times 298.15 \text{ K} \times \ln (0.7567) \\ &= -691.1 \frac{\text{kJ}}{\text{kmol}_{N_2}} \end{aligned}$$

which shows that the per-mole work requirement now is only 17.5 percent of the value calculated in part (a).

---

Problem 5.3 The maximum work is equal to the total non-flow exergy of the system relative to the dead state specified by the environment, eq. (5.9'),

$$\Xi_{t, \text{vacuum}} = U - T_0 S + P_0 V - \sum_{i=1}^n \mu_{0,i} N_i$$

However, since the system is evacuated, we have  $U = 0$ ,  $S = 0$  and  $N_i = 0$ , ( $i = i, \dots, n$ ) therefore

$$\Xi_{t, \text{vacuum}} = P_0 V$$

This is the work done by the atmosphere and destroyed in the filling process of Examples 1.1 and 1.2.

---

Problem 5.4 (a) In the steady state we account for the conservation of air, water and energy by writing

$$\dot{m}_{a,1} = \dot{m}_{a,3} (= \dot{m}_a) \quad (1)$$

$$\dot{m}_{v,1} + \dot{m}_w = \dot{m}_{v,2} \quad (2)$$

$$\dot{m}_a h_{a,1} + \dot{m}_{v,1} h_{v,1} + \dot{m}_w h_w = \dot{m}_a h_{a,3} + \dot{m}_{v,3} h_{v,3} \quad (3)$$

The last two equations yield, in order,

$$\frac{\dot{m}_w}{\dot{m}_a} = \omega_3 - \omega_1 \quad (4)$$

$$\omega_1 = \frac{h_{a,3} - h_{a,1} + \omega_3(h_{v,3} - h_w)}{h_{v,1} - h_w} \quad (5)$$

Invoking the ideal gas model for dry air and water vapor

$$h_{a,3} - h_{a,1} = c_{P,a}(T_3 - T_1)$$

$$h_{v,3} = h_g(T_3)$$

$$h_{v,1} = h_g(T_1)$$

and the "moderately pressurized" incompressible liquid model for water (state 2),

$$h_w \equiv h_f(T_2)$$

eq. (5) becomes finally

$$\omega_1 \approx \frac{c_{P,a}(T_3 - T_1) + \omega_3 [h_g(T_3) - h_f(T_2)]}{h_g(T_1) - h_f(T_2)}$$

This  $\omega_1$  ( $\omega_3, T_4$ ) relation can be intersected with

$$\omega_3 = \frac{\frac{0.622}{P_3}}{\frac{1}{\phi_3 P_{\text{sat}}(T_3)} - 1}, \quad (\text{with } \phi_3 = 1) \quad (5.42)$$

in order to derive the wanted relation,  $\omega_3(T_3)$ .

(b) Assuming the numerical conditions specified in the problem statement, we can use the preceding solution to calculate the adiabatic saturation temperature (by trial and error):

$$T_3 = 20^\circ\text{C} \rightarrow \omega_1 = 0.0126 > \omega_0$$

$$T_3 = 15^\circ\text{C} \rightarrow \omega_1 = 0.00641 < \omega_0$$

Interpolation results:  $T_3 = 19.43^\circ\text{C}$

$$\omega_3 = 0.0142$$

$$\omega = 0.0229$$

Similar results can be read off a psychrometric chart, in which  $T_3$  is the wet-bulb thermometer temperature corresponding to  $T_0$  and  $\phi_0$ .

The formula for the total flow exergy of liquid water reduces to

$$\begin{aligned} e_{t,w} &= [P_0 - P_{\text{sat}}(T_0)] v_f(T_0) - R_v T_0 \ln \phi_0 \\ &= 0.098 + 70.212 = 70.31 \text{ kJ/kg water liquid} \end{aligned}$$

For the total flow exergy of the inflowing stream of humid air, eq. (5.49), we obtain

$$e_{t,1} = 0$$

because the mixture is at the environmental state  $(T_0, P_0, \phi_0)$ . Finally, for the outflowing stream of humid air we obtain [ note that there are three terms in eq. (5.49)]:

$$e_{t,3} = 0.0542 + 0 + 0.0298 = 0.84 \text{ kJ/kg dry air}$$

Now we can put numbers in place of the following quantities:

$$\text{exergy flowrate out} = \dot{m}_a e_{t,3}$$

$$\text{exergy flowrate in} = \dot{m}_w e_{t,w} = \frac{\dot{m}_w}{\dot{m}_a} \dot{m}_a e_{t,w}$$

$$= (\omega_3 - \omega_1) \dot{m}_a e_{t,w}$$

$$\text{exergy lost, per kg dry air} = (\omega_3 - \omega_1) e_{t,w} - e_{t,3}$$

$$= 0.0777 \text{ kJ/kg dry air}$$

$$\eta_{II} = \frac{\dot{m}_a e_{t,3}}{(\omega_3 - \omega_1) \dot{m}_a e_{t,w}} = \frac{0.084}{(0.0142 - 0.0119) 70.31} = 0.52$$

Note: This problem can be expanded by asking the question of how the irreversibility (or  $\eta_{II}$ ) depends on the original humidity ( $\phi_1$ ). One way to proceed is by repeating parts (a) and (b) assuming a different  $\phi_1$  value (e.g.,  $\phi_1 = 0.5$ ), while keeping  $\phi_0$  and all the other data fixed.

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Problem 5.5 (a) The conservation of water in the two parts of the apparatus requires

$$\dot{m}_{v,1} + \dot{m}_w = \dot{m}_{v,2} \quad \text{and} \quad \dot{m}_{v,2} = \dot{m}_{v,3}$$

or, divided by  $\dot{m}_a$  (the dry air flowrate between 1 and 3)

$$\omega_1 + \frac{\dot{m}_w}{\dot{m}_a} = \omega_2 \quad \text{and} \quad \omega_2 = \omega_3.$$

We conclude that

$$\frac{\dot{m}_w}{\dot{m}_a} = \omega_3 - \omega_1$$

where the humidity ratios are

$$\omega_1 = \frac{0.622}{\frac{P_0}{\phi_1 P_{\text{sat}}(T_0)} - 1} = 0.0139, \text{ or } \tilde{\omega}_1 = 0.0224$$

$$\omega_3 = \frac{0.622}{\frac{P_0}{\phi_3 P_{\text{sat}}(T_0)} - 1} = 0.00788, \text{ or } \tilde{\omega}_3 = 0.0127$$

The answer is

$$\frac{\dot{m}_w}{\dot{m}_a} = 0.00602 \frac{\text{kg water}}{\text{kg dry air}}$$

with the additional observation that  $\omega_2 = 0.00788$  and  $\tilde{\omega}_2 = 0.0127$ .

(b) The intermediate temperature  $T_2$  follows from the observation that  $\phi_2 = 1$ , which provides the equation

$$\omega_2 = \frac{0.622}{\frac{P_0}{P_{\text{sat}}(T_2)} - 1}$$

Knowing  $P_0$  and  $\omega_2$ , we find that  $P_{\text{sat}}(T_2) = 1.268 \text{ kPa}$ , which, via linear interpolation in the steam tables\*, means that

$$T_2 = 10.23^\circ\text{C}$$

(c) Applying the first law to the rectangular box delineating the "first part" of the apparatus we obtain

$$\frac{\dot{Q}_c}{\dot{m}_a} = c_{P,a} (T_0 - T_2) + \omega_1 h_{v,1} - \omega_2 h_{v,2} - (\omega_1 - \omega_2) h_w$$

After recognizing that  $h_{v,1} = h_g(T_0)$ ,  $h_{v,2} = h_g(T_2)$  and  $h_w = h_f(T_2)$ , the first law yields

$$\frac{\dot{Q}_c}{\dot{m}_a} = 30.11 \text{ kJ/kg dry air}$$

(d) Similarly, the first law analysis of the second part of the apparatus requires

$$\frac{\dot{Q}_h}{\dot{m}_a} = c_{P,a} (T_0 - T_2) + \omega_3 h_{v,3} - \omega_2 h_{v,2}$$

in which  $h_{v,3} = h_g(T_0)$  and  $h_{v,2} = h_g(T_2)$ . The numerical results is

$$\frac{\dot{Q}_h}{\dot{m}_a} = 15.03 \text{ kJ/kg dry air}$$

(e) The exergy analysis of the first part of the apparatus consists of calculating

$$e_{t,1} = 0 + 0 + 0.0226 = 0.0226 \text{ kJ/kg dry air} \quad \leftarrow \text{eq. (5.49)}$$

$$e_{t,2} = 0.3851 + 0 + 0.1024 = 0.4875 \text{ kJ/kg dry air} \quad \leftarrow \text{eq. (5.49)}$$

$$e_{t,w} = 1.319 + 0.1 + 70.212 = 71.631 \text{ kJ/kg water} \quad \leftarrow \text{eq. (5.49)}$$

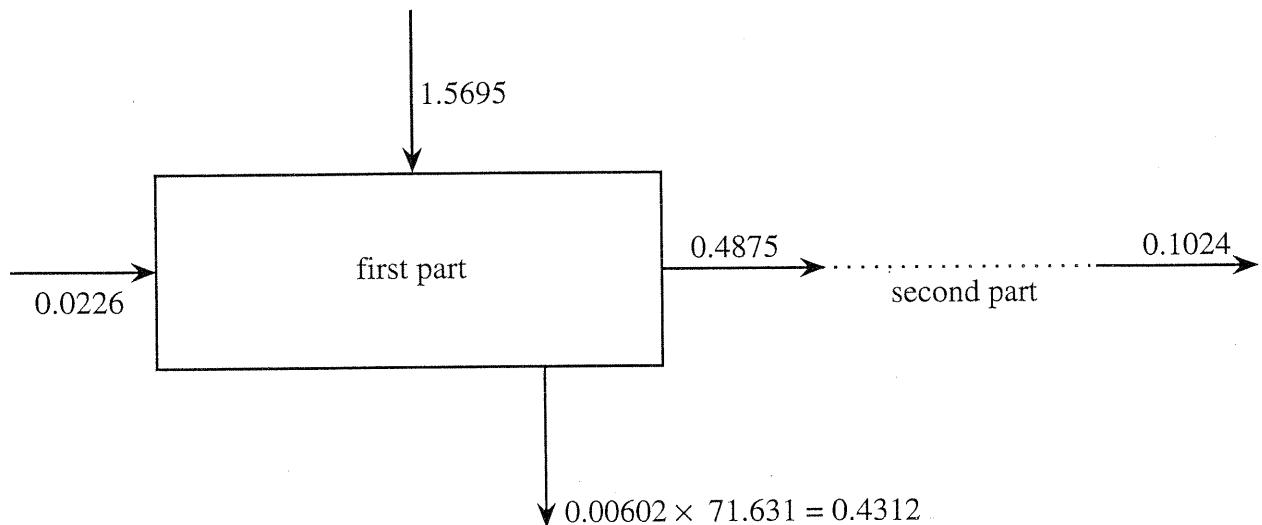
\* The steam tables used throughout this *Solutions Manual* and the calculations reported in *Advanced Engineering Thermodynamics* are the classical *Steam Tables* by J. H. Keenan, F. G. Keyes, P. G. Hill and J. G. Moore (Wiley, New York, 1978).

$$\dot{E}_{Q_c} = (-\dot{Q}_c) \left( 1 - \frac{T_0}{T_2} \right) = +1.5694 \dot{m}_a \text{ kJ/kg dry air}$$

Note that in the calculation of  $e_{t,w}$  the value "1.319 kJ/kg water" represents the sum of the four terms listed on the first (top) line of eq. (5.59). In conclusion, the rate of exergy destruction in the first part of the apparatus is:

$$\begin{aligned} \text{total exergy inflow} - \text{total exergy outflow} &= \dot{m}_a e_{t,1} + \dot{E}_{Q_c} - \dot{m}_a e_{t,2} - \dot{m}_w e_{t,w} \\ &= \dot{m}_a (0.0226 + 1.5694 - 0.4875 - 0.00602 \times 71.631) \\ &= \dot{m}_a (0.673 \text{ kJ/kg dry air}) \end{aligned}$$

The exergy flowrates into and out of the first part of the apparatus are represented by their respective number of kJ per kg dry air in the sketch below.



(f) For the exergy analysis of the second part of the apparatus we calculate

$$e_{t,3} = 0 + 0 + 0.1024 = 0.1024 \text{ kJ/kg dry air} \quad \leftarrow \text{eq. (5.49)}$$

$$\dot{E}_{Q_h} = \dot{Q}_h \left( 1 - \frac{T_0}{T_0} \right) = 0$$

therefore (see the preceding exergy accounting diagram)

$$\begin{aligned} \text{total exergy inflow} - \text{total exergy outflow} &= \dot{m}_a e_{t,2} - \dot{m}_a e_{t,3} = \dot{m}_a (0.4875 - 0.1024) \\ &= \dot{m}_a (0.3851 \text{ kJ/kg dry air}) \end{aligned}$$

(g) The second law efficiency of the *entire* apparatus is

$$\eta_{II} = \frac{\text{total exergy outflow}}{\text{total exergy inflow}} = \frac{0.4312 + 0.1024}{0.0226 + 1.5694} = 0.335$$

Rewriting the second law efficiency as

$$\begin{aligned}\eta_{II} &= 1 - \frac{\text{total exergy lost}}{\text{total exergy inflow}} \\ &= 1 - \frac{(\text{exergy lost})_{1-2}}{\text{total exergy inflow}} - \frac{(\text{exergy lost})_{2-3}}{\text{total exergy inflow}} \\ &= 1 - \frac{0.673 \text{ (see part "e")}}{0.0226 + 1.5694} - \frac{0.3851 \text{ (see part "f")}}{0.0226 + 1.5694} \\ &= 1 - 0.423 - 0.242 \\ &= 0.335\end{aligned}$$

we see that the first part of the apparatus destroys roughly twice as much exergy as the second part,  $0.423/0.242 = 1.748$ . In other words, the larger part of the irreversibility of the apparatus occurs between state 1 and state 2.

---

Problem 5.6 The analysis begins with eq. (5.13'), expressed per mole of humid air mixture:

$$\begin{aligned}\xi_t &= \xi + \xi_{ch} = x_a (\xi_a + \xi_{ch,a}) + x_v (\xi_v + \xi_{ch,v}) \\ &= x_a [\bar{u}_a - \bar{u}_a^* - T_0 (\bar{s}_a - \bar{s}_a^*) + P_0 (\bar{v}_a - \bar{v}_a^*) + \mu_a^* - \mu_{0,a}] \\ &\quad + x_v [\bar{u}_v - \bar{u}_v^* - T_0 (\bar{s}_v - \bar{s}_v^*) + P_0 (\bar{v}_v - \bar{v}_v^*) + \mu_v^* - \mu_{0,v}]\end{aligned}$$

Invoking the ideal gas model for dry air, we write

$$\begin{aligned}\bar{u}_a - \bar{u}_a^* &= \bar{c}_{v,a} (T - T_0) \\ \bar{s}_a - \bar{s}_a^* &= \bar{c}_{P,a} \ln \frac{T}{T_0} - \bar{R} \ln \frac{P}{P_0} \\ \bar{v}_a - \bar{v}_a^* &= \frac{\bar{R}T}{P} - \frac{\bar{R}T_0}{P_0} = \frac{\bar{R}T_0}{P_0} \left( \frac{T/T_0}{P/P_0} - 1 \right) \\ \mu_a^* - \mu_{0,a} &= \bar{R}T_0 \ln \frac{x_a}{x_{0,a}} \\ \bar{c}_{P,a} &= \bar{c}_{v,a} + \bar{R}\end{aligned}$$

The total non-flow exergy per mole of humid air mixture ( $\xi_t$ ) becomes

$$\begin{aligned}\xi_t = & \left( x_a \bar{c}_{v,a} + x_v \bar{c}_{v,v} \right) T_0 \left( \frac{T}{T_0} - 1 - \ln \frac{T}{T_0} \right) \\ & + \bar{R} T_0 \left( \ln \frac{P}{P_0} - \ln \frac{T}{T_0} + \frac{T/T_0}{P/P_0} - 1 \right) \\ & + \bar{R} T_0 \left( x_a \ln \frac{x_a}{x_{0,a}} + x_v \ln \frac{x_v}{x_{0,v}} \right)\end{aligned}$$

The corresponding total non-flow exergy per kilogram of dry air ( $\xi_t$ ) is obtained by recognizing that

$$\xi_t = \frac{\xi_t}{M_a x_a}$$

which yields

$$\begin{aligned}\xi_t = & \left( c_{v,a} + \omega c_{v,v} \right) T_0 \left( \frac{T}{T_0} - 1 - \ln \frac{T}{T_0} \right) \\ & + (1 + \tilde{\omega}) R_a T_0 \left( \ln \frac{P}{P_0} - \ln \frac{T}{T_0} + \frac{T/T_0}{P/P_0} - 1 \right) \\ & + R_a T_0 \left[ (1 + \tilde{\omega}) \ln \frac{1 + \tilde{\omega}_0}{1 + \tilde{\omega}} + \tilde{\omega} \ln \frac{\tilde{\omega}}{\tilde{\omega}_0} \right]\end{aligned}$$

Problem 5.7 We begin with the single-component version of eq. (5.9'), which per mole of water is

$$\xi_{t,w} = \bar{u} - T_0 \bar{s} + P_0 \bar{v} - \mu_{0,w}$$

where  $\mu_{0,w} = \bar{g}_{0,w} = \bar{h}_{0,w} - T_0 \bar{s}_{0,w}$ . Rewritten per kilogram of water, eq. (1) reads

$$\xi_{t,w} = u - T_0 s + P_0 v - h_{0,w} + T_0 s_{0,w}$$

in which, based on the incompressible liquid model, we substitute

$$u \equiv u_f(T)$$

$$s \equiv s_f(T)$$

$$v \equiv v_f(T)$$

$$h_{0,w} \equiv h_g(T_0)$$

$$s_{0,w} \equiv s_g(T_0) - R_v \ln \frac{P_{w,0}}{P_{\text{sat}}(T_0)} = s_g(T_0) - R_v \ln \phi_0$$

The end result is

$$\xi_{t, w} = h_f(T) - h_g(T_0) - T_0 s_f(T) + T_0 s_g(T_0) + (P_0 - P) v_f(T) - R_v T_0 \ln \phi_0$$

where the four terms in the top row add up to zero if  $T = T_0$ . The same result can be expressed as

$$\xi_{t, w} = h_f(T) - h_f(T_0) + T_0 [s_f(T_0) - s_f(T)] + (P_0 - P) v_f(T) - R_v T_0 \ln \phi_0$$

by noting that  $h_g(T_0) = h_f(T_0) + T_0 s_{fg}(T_0)$ .

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## Chapter 6

### MULTIPHASE SYSTEMS

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Problem 6.1 The system evolves at constant S and P. Let "L" denote the half of the system in which the entropy increases, and "R" the half in which the entropy is being depleted,

$$S_L = \frac{1}{2} (S + \delta S)$$

$$S_R = \frac{1}{2} (S - \delta S)$$

Since we will be invoking the H-minimum principle for constant S and P, we evaluate the overall change in the enthalpy inventory of the system. For each half we write

$$H_L = \frac{1}{2} \left[ H + \left( \frac{\partial H}{\partial S} \right)_P \delta S + \frac{1}{2} \left( \frac{\partial^2 H}{\partial S^2} \right)_P (\delta S)^2 \dots \right]$$

$$H_R = \frac{1}{2} \left[ H - \left( \frac{\partial H}{\partial S} \right)_P \delta S + \frac{1}{2} \left( \frac{\partial^2 H}{\partial S^2} \right)_P (\delta S)^2 \dots \right]$$

where H is the original enthalpy inventory of the entire system. Adding  $H_L$  and  $H_R$  we find

$$(H_L + H_R) - H = \frac{1}{2} \left( \frac{\partial^2 H}{\partial S^2} \right)_P (\delta S)^2$$

The original state is one of stable equilibrium if the net enthalpy change  $[(H_L + H_R) - H]$  is positive - this, in accordance with the enthalpy minimum at constant S and P. This requirement translates into

$$\left( \frac{\partial^2 H}{\partial S^2} \right)_P > 0$$

or, since  $dH = TdS + VdP$ , into

$$\left( \frac{\partial T}{\partial S} \right)_P > 0$$

hence

$$c_P > 0$$

However,  $c_P$  is also defined as  $(\partial h / \partial T)_P$ , where  $\Delta h$  equals the infinitesimal heat transfer experienced isobarically as the temperature increases by  $\Delta T$ . Therefore, the stability condition  $c_P > 0$  means that the substance must be one whose temperature increases while it is heated isobarically.

Finally, according to Problem 4.6c, the specific heat at constant pressure always obeys the

relation

$$c_P = c_v + \frac{v}{k} \beta^2 T$$

Clearly, if  $c_v$  and  $\kappa$  are both positive,  $c_P$  is also positive.

---

Problem 6.2 First we recognize the approximation

$$v_{fg} \equiv v_g - v_f \approx v_g$$

and that for an ideal gas

$$v_g = R_g \frac{T}{P}$$

Using these ingredients, we have

$$\begin{aligned} \frac{dP}{dT} &= \frac{h_{fg}}{T v_{fg}} \\ &= \frac{h_{fg}}{R_g} \frac{P}{T^2} \end{aligned}$$

which really means

$$d(\ln P) = -\frac{h_{fg}}{R_g} d\left(\frac{1}{T}\right)$$


---

Problem 6.3 Treating  $s_f$  as the function  $s_f [T, P(T)]$  we write

$$\begin{aligned} ds_f &= \left[ \left( \frac{\partial s_f}{\partial T} \right)_P + \left( \frac{\partial s_f}{\partial P} \right)_T \frac{dP}{dT} \right] dT \\ &= \left( \frac{1}{T} c_{P,f} - \beta_f v_f \frac{dP}{dT} \right) dT \\ &= \left( \frac{1}{T} c_{P,f} - \beta_f v_f \frac{h_{fg}}{T v_{fg}} \right) dT \end{aligned}$$

Recalling the definition  $c_f = T ds_f / dT$ , we conclude that

$$c_f = c_{P,f} - \beta_f v_f \frac{h_{fg}}{v_{fg}} \quad (a)$$

The relevant properties of saturated liquid water at 95°C are

$$\beta_f = 0.00072 \text{ K}^{-1}$$

$$v_f = 0.00104 \text{ m}^3/\text{kg}$$

$$h_{fg} = 2270.2 \text{ kJ/kg}$$

$$v_{fg} = 1.981 \text{ m}^3/\text{kg}$$

$$c_{P,f} = 4.2 \text{ kJ/kg/K}$$

therefore eq. (a) becomes, numerically,

$$c_f = (4.2 - 0.00086) \frac{\text{kJ}}{\text{kg K}}$$

In conclusion,  $c_f \approx c_{P,f}$  for saturated liquid water at common temperatures.

---

Problem 6.4 Repeating the preceding analysis for  $s_g = s_g[T, P(T)]$  we obtain

$$ds_g = \left( \frac{1}{T} c_{P,g} - \beta_g v_g \frac{h_{fg}}{T v_{fg}} \right) dT$$

hence

$$c_g = c_{P,g} - \beta_g v_g \frac{h_{fg}}{v_{fg}} \quad (a)$$

Treating the saturated vapor as an ideal gas we note that

$$\beta_g = \frac{1}{T} \quad (b)$$

Furthermore, a good approximation is

$$v_{fg} = v_g - v_f \approx v_g \quad (c)$$

Equations (a) - (c) yield

$$c_g \approx c_{P,g} - \frac{1}{T} h_{fg} = c_{P,g} - s_{fg} \cdot$$


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Problem 6.5 To find either spinodal line we must solve

$$\left(\frac{\partial P_r}{\partial v_r}\right)_{T_r} = 0$$

in conjunction with eq. (6.35). The result is

$$P_r = \frac{3}{v_r^2} - \frac{2}{v_r^3}$$

therefore, in the high volume limit  $P_r$  varies as  $v_r^{-2}$ .

---

Problem 6.6 The slope of the "ice + water" line at the triple point is given by Clapeyron's relation

$$\begin{aligned} \frac{dP}{dT} &= \frac{s_{\text{water}} - s_{\text{ice}}}{v_{\text{water}} - v_{\text{ice}}} \\ &= \frac{[0 - (-1221)] \text{ J/kgK}}{(0.001 - 0.00109) \text{ m}^3/\text{kg}} = -13.6 \frac{\text{MPa}}{\text{K}} \end{aligned}$$

The negative sign of  $dP/dT$  shows that the "ice + water" line leans to the left starting from the triple point TP. Assuming that the line is straight all the way up to  $P_1 = 10 \text{ MPa}$ , we write

$$\frac{dP}{dT} = \frac{P_1 - P_{\text{TP}}}{T_1 - T_{\text{TP}}} \approx \frac{10 \text{ MPa}}{T_1 - 0.01^\circ\text{C}} = -13.6 \frac{\text{MPa}}{\text{K}}$$

Solving this equation for  $T_1$ , we obtain  $T_1 = -0.73^\circ\text{C}$ .

---

Figure 6.7 The new equation of state is

$$3\phi^3 + (2 + 3\phi)\pi - 8\tau = 0$$

therefore, the critical isochore (the constant-volume curve  $v = v_c$ ) is described by the straight line

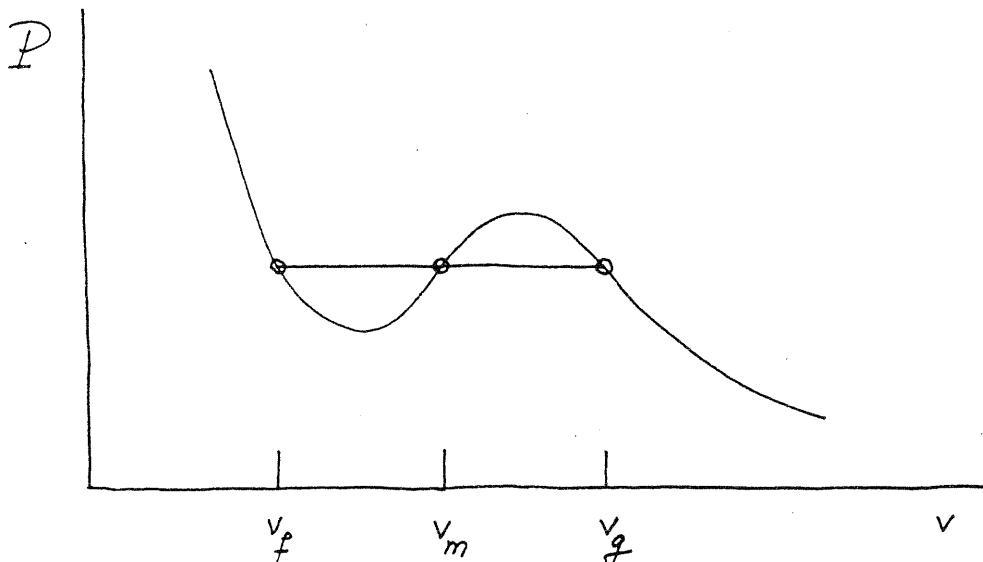
$$2\pi - 8\tau = 0 \quad (\phi = 0)$$

The slope of this line is  $(\partial\pi / \partial\tau)_\phi = 0 = 4$ . All the isochores ( $\phi = \text{constant}$ ) are straight lines.

---

Problem 6.8 This problem can be solved numerically in the following steps:

- (1) Fix the value of  $T_r$  (clearly,  $T_r < 1$ )
- (2) Guess the value of  $P_r$  that might correspond to the selected  $T_r$  (note:  $P_r < 1$ )
- (3) Use eq. (6.46) to calculate the limits of integration  $v_f$  and  $v_g$
- (4) Evaluate the integral  $\int_{v_f}^{v_g} P dv$ , and see whether it is zero. If it is not, go back to step (2) and guess a better  $P_r$  value.



My own results are listed in the following table:

$T_r$	$P_r$	$v_f$	$v_m$	$v_g$	Area I = Area II
0.97	0.884	0.738	1.0235	1.498	0.00559
0.95	0.8119	0.685	1.043	1.727	0.0152
0.9	0.647	0.604	1.091	2.349	0.0618
0.85	0.504	0.554	1.146	3.128	0.141
0.8	0.383	0.517	1.2085	4.172	0.255
0.75	0.282	0.490	1.282	5.642	0.404
0.7	0.2004	0.467	1.367	7.814	0.592
0.65	0.136	0.449	1.469	11.18	0.818
0.6	0.0869	0.433	1.591	16.722	1.087

The Pitzer acentric factor and the  $\alpha_c$  parameter are

$$\omega = -\log_{10}(0.2004) - 1 = -0.302.$$

$$\alpha_c = \frac{1 - 0.884}{1 - 0.97} = 3.87$$


---

Problem 6.9 At the critical point eq. (6.98) requires

$$Z_c^3 - Z_c^2 + (A_c - B_c - B_c^2) Z_c - A_c B_c = 0$$

where

$$A_c = \frac{a P_c}{R^2 T_c^{2.5}} = 0.4275$$

$$B_c = \frac{b P_c}{R T_c} = 0.08664$$

Solving the cubic equation for  $Z_c$  we find  $Z_c = 0.3333$ .

---

Problem 6.10 The slope of the saturation pressure curve  $P(T)$  is given by Clapeyron's relation

$$\frac{dP}{dT} = \frac{s_{fg}}{v_{fg}}$$

In the limit  $T \rightarrow T_c$  the right-hand side of this relation approaches  $(\partial s / \partial v)_T = T_c$ , hence

$$\frac{dP}{dT} = \left( \frac{\partial s}{\partial v} \right)_T, \quad \text{at } T = T_c$$

Finally, in view of Maxwell's relation (4.56f),

$$\left( \frac{\partial s}{\partial v} \right)_T = \left( \frac{\partial P}{\partial T} \right)_v$$

we reach the conclusion that

$$\frac{dP}{dT} = \left( \frac{\partial P}{\partial T} \right)_v, \quad \text{at } T = T_c$$

where the left-hand side refers to the slope of the "L + V" domain projected on Regnault's diagram  $P - T$ .

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Problem 6.11 We start with  $\mu_J = 0$  and the identity (e) of Problem 4.6,

$$\begin{aligned}\mu_J &= \frac{1}{c_P} \left[ T \left( \frac{\partial v}{\partial T} \right)_P - v \right] \\ &= \frac{1}{c_P} \left\{ T \left[ \frac{\partial}{\partial T} \left( \frac{ZRT}{P} \right) \right]_P - \frac{ZRT}{P} \right\} \\ &= \frac{1}{c_P} \left[ T \frac{ZR}{P} + \frac{RT^2}{P} \left( \frac{\partial Z}{\partial T} \right)_P - \frac{ZRT}{P} \right] \\ &= \frac{RT^2}{P c_P} \left( \frac{\partial Z}{\partial T} \right)_P = 0\end{aligned}$$


---

Problem 6.12 We begin with the Clapeyron relation (6.136), in which  $h_{fg} = a + bT$  and  $Z_{fg} \equiv Z_g = \text{constant}$ ,

$$\begin{aligned}\frac{dP}{dT} &= \frac{h_{fg}}{(RT^2/P) Z_{fg}} \\ \frac{dP}{P} &= \frac{1}{RZ_g} \frac{a + bT}{T^2} dT \\ \ln P &= \frac{1}{RZ_g} \left( -\frac{a}{T} + b \ln T + c \right) \\ \ln P_r &= \frac{c_1}{T_r} + c_2 \ln T_r + c_3\end{aligned}$$


---

Problem 6.13 The analysis consists of writing  $g_f = g_g$  and noting that in both states (f and g) the system is a "simple" system. Therefore, according to Table 4.3 we have

$$g_f = u_f - T_f s_f + P_f v_f \quad (1)$$

$$g_g = u_g - T_g s_g + P_g v_g \quad (2)$$

In the present case, however,  $T_f = T_g = T$  and  $P_f = P_g = P$ ; subtracting eq. (1) from eq. (2) yields

$$(u + Pv)_g - (u + Pv)_f = T(s_g - s_f)$$

and, finally,

$$h_{fg} = Ts_{fg}$$


---

Problem 6.14 All along the edge of the dome [e.g. at state (m)] the system is single-phase, therefore

$$dh_m = T_m ds_m + v_m dP_m \quad (1)$$

Integrating from (f) to (g) along the edge of the dome we obtain

$$\begin{aligned} h_g - h_f &= \int_f^g \left( T_m + v_m \frac{dP_m}{dT_m} \frac{dT_m}{ds_m} \right) ds_m \\ &= \int_f^g \left( T_m + v_m \frac{s_{fg}}{v_{fg}} \frac{dT_m}{ds_m} \right) ds_m \end{aligned} \quad (2)$$

Next, we replace the left side with  $T_0 s_{fg}$  and, using the approximation  $s_{fg}/v_{fg} = \text{constant}$ , we obtain

$$\int_f^g (T_m - T_0) ds_m = -\frac{s_{fg}}{v_{fg}} \int_f^g v_m dT_m \quad (3)$$

On the left side of eq. (3) we recognize the area trapped between the dome and the  $T = T_0$  horizontal line. The line integral appearing on the right side can be broken up into two line integrals, one for each side of the critical point,

$$\begin{aligned} \int_f^g v_m dT_m &= \int_f^{CP} v_m dT_m + \int_{CP}^g v_m dT_m \\ &= \int_{T_0}^{T_c} v_f(T) dT + \int_{T_c}^{T_0} v_g(T) dT \\ &= - \int_{T_0}^{T_c} v_{fg}(T) dT \end{aligned} \quad (4)$$

Combining eqs..(3) and (4) we obtain finally

$$\int_f^g (T_m - T_0) ds_m = \frac{s_{fg}(T_0)}{v_{fg}(T_0)} \int_{T_0}^{T_c} v_{fg}(T) dT$$

For the second part of the problem we note that  $g_f(T) = g_g(T)$  means, in order,

$$\int_f^g dg_m = 0$$

$$\int_f^{CP} dg_m + \int_{CP}^g dg_m = 0$$

$$\int_f^{CP} (-s_f dT_m + v_f dP_m) + \int_{CP}^g (-s_g dT_m + v_g dP_m) = 0$$

$$\int_{T_0}^{T_c} s_{fg}(T) dT = \int_{P_0}^{P_c} v_{fg}(P) dP$$


---

Problem 6.15 In the case of the van der Waals fluid we have, in order,

$$\begin{aligned} P_r &= \left[ 1 + \frac{(\rho_r - 1)^3}{1 - \frac{1}{3} \rho_r} \right] T_r - 3\rho_r^2 (1 - T_r) \\ &= \left[ 1 + \frac{(1 - v_r)^3}{v_r^3} \frac{3v_r}{3v_r - 1} \right] T_r - \frac{3}{v_r^2} + \frac{3}{v_r^2} T_r \\ &= \frac{8T_r}{3v_r - 1} - \frac{3}{v_r^2} \end{aligned}$$

The last expression matches eq. (6.46). Considering now Paynter's fluid,

$$P_r = \left[ 1 + \frac{(\rho_r - 1)^3 |\rho_r - 1|}{1 - \frac{1}{4} \rho_r} \right] T_r - 7 \rho_r^2 (1 - T_r)$$

we calculate the Riedel parameter [cf. eq. (6.95)]

$$\alpha_c = \left( \frac{dP_r}{dT_r} \right)_{T_r = \rho_r = 1} = 1 + 7 = 8$$

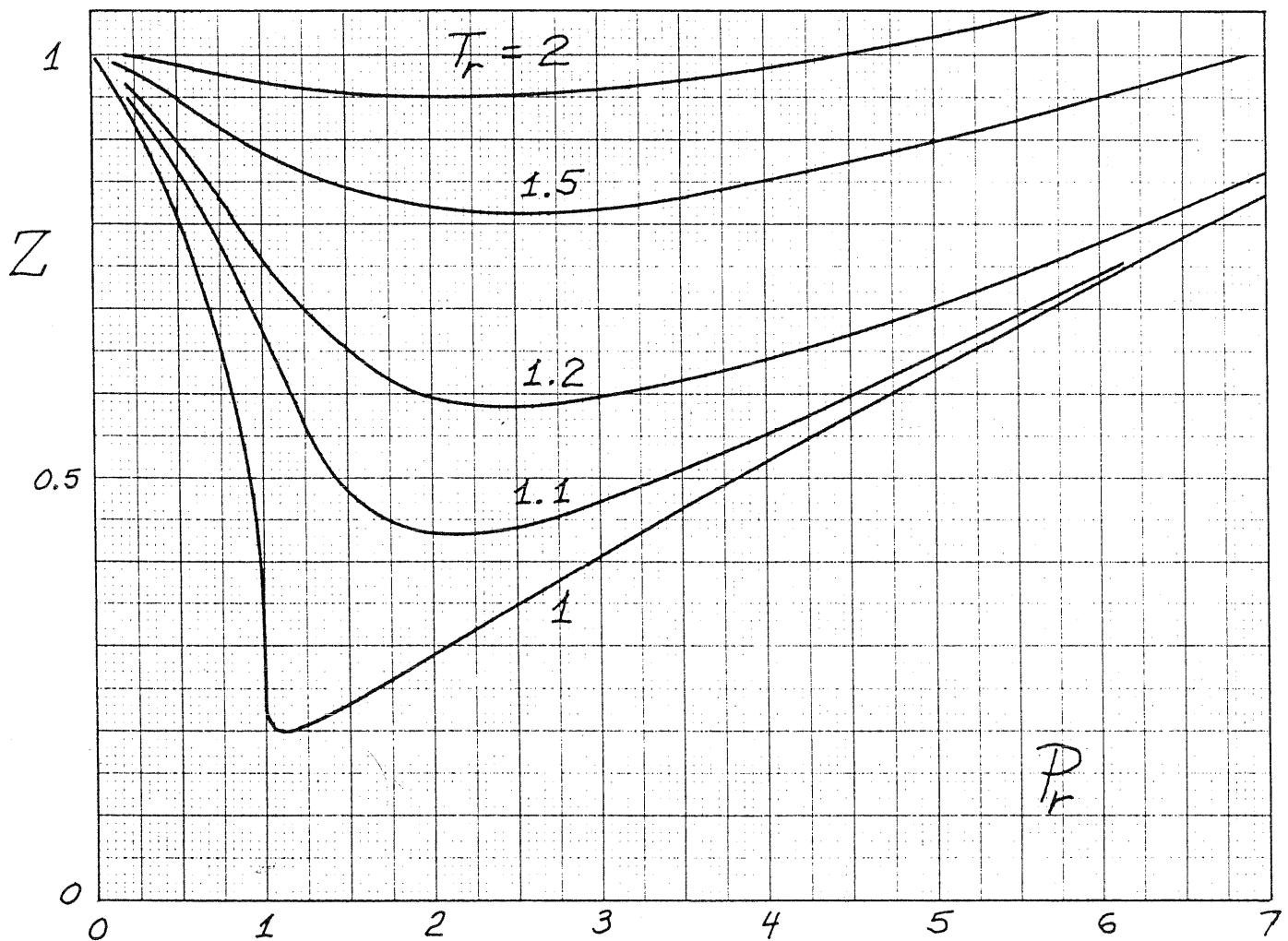
According to Paynter [56] the critical point compressibility factor is

$$Z_c = \left( 4 - \frac{1}{4} \right)^{-1} = \frac{4}{15} = 0.267$$

The  $Z = Z(T_r, P_r)$  chart is obtained by combining Paynter's  $P_r = P_r(T_r, \rho_r)$  equation of state with

$$Z = Z_c \frac{P_r}{T_r \rho_r} \quad (*)$$

I obtained the following chart, first, by fixing the value of  $T_r$ , second, picking a  $\rho_r$  value, third, calculating  $P_r$  from Paynter's relation and, fourth, substituting all three into the  $Z$  formula (\*).



---

Problem 6.16 The linearized Clapeyron relation can be integrated from  $T_{\text{sat}}$  to  $T_H$ , to relate  $P$  to  $T_{\text{sat}}$ ,

$$\frac{dP}{dT_{\text{sat}}} = a \quad (\text{constant}) \quad (1)$$

$$P_{\text{sat},H} - P = a(T_H - T_{\text{sat}}) \quad (2)$$

where  $P_{\text{sat},H}$  is the saturation pressure corresponding to  $T_H$ .

The second relation between  $P$  and  $T_{\text{sat}}$ , which is needed to determine  $P$  and  $T_{\text{sat}}$  uniquely, is provided by the ideal gas model applied to the *nonisothermal* steam layer,

$$T(x) = T_H - (T_H - T_{\text{sat}}) \frac{x}{L} \quad (3)$$

The steam mass  $dm$  contained in an infinitesimal slice of thickness  $dx$  has the pressure  $P$ , temperature  $T(x)$ , and volume  $A dx$ ; we write the equation of state

$$P A dx = (dm) RT \quad (4)$$

and integrate from  $x = 0$  to  $x = L$ ,

$$m = \int_0^L \frac{P A dx}{RT} \quad (5)$$

Combining eqs. (3) and (5) we obtain

$$P = \frac{m}{AL} RT_H \quad \frac{1 - \tau}{-\ln \tau} \quad (6)$$

where  $\tau$  is the dimensionless unknown

$$\tau = \frac{T_{\text{sat}}}{T_H} < 1 \quad (7)$$

The  $(P, T_{\text{sat}})$  solution is determined uniquely by the system of eqs. (2) and (6). For example, eliminating  $P$  we obtain an implicit expression for  $\tau$ ,

$$(1 - \tau) \left( 1 - \frac{\mu}{\ln \tau} \right) = \frac{P_{\text{sat},H}}{a T_H} \quad (8)$$

where  $\mu$  is the dimensionless steam mass,

$$\mu = \frac{m}{AL} \quad \frac{R}{a} \quad (9)$$


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Chapter 7

## CHEMICALLY REACTIVE SYSTEMS

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Problem 7.1 At constant V and T, eq. (7.10u) states that

$$\left(\frac{\partial U}{\partial \zeta}\right)_{T, V} = T \left(\frac{\partial S}{\partial \zeta}\right)_{T, V} - Y$$

or, in view of eq. (7.14),

$$\left(\frac{\partial U}{\partial \zeta}\right)_{T, V} = T \left(\frac{\partial Y}{\partial T}\right)_{V, \zeta} - Y$$

$$\frac{1}{T^2} \left(\frac{\partial U}{\partial \zeta}\right)_{T, V} = \frac{\partial}{\partial T} \left(\frac{Y}{T}\right)_{V, \zeta}$$

Integrating both sides of the last equation at constant V and  $\zeta$  yields

$$\frac{Y(T_2, V, \zeta)}{T_2} - \frac{Y(T_1, V, \zeta)}{T_1} = \int_{T_1}^{T_2} \frac{1}{T^2} \left(\frac{\partial U}{\partial \zeta}\right)_{T, V} dT$$


---

Problem 7.2 Recognizing  $Y/T$  as a function of T, P and  $\zeta$  we write

$$d\left(\frac{Y}{T}\right) = \frac{\partial}{\partial T} \left(\frac{Y}{T}\right)_{P, \zeta} dT + \frac{\partial}{\partial P} \left(\frac{Y}{T}\right)_{T, \zeta} dP + \frac{\partial}{\partial \zeta} \left(\frac{Y}{T}\right)_{T, P} d\zeta$$

The first partial differential coefficient is given by eq. (7.19),

$$\frac{\partial}{\partial T} \left(\frac{Y}{T}\right)_{P, \zeta} = \frac{1}{T^2} \left(\frac{\partial H}{\partial \zeta}\right)_{T, P}$$

For the second coefficient we invoke eq. (7.30),

$$\frac{\partial}{\partial P} \left(\frac{Y}{T}\right)_{T, \zeta} = \frac{1}{T} \left(\frac{\partial Y}{\partial P}\right)_{T, \zeta} = -\frac{1}{T} \left(\frac{\partial V}{\partial \zeta}\right)_{T, P}$$

Finally, the third coefficient is simply

$$\frac{\partial}{\partial \zeta} \left(\frac{Y}{T}\right)_{T, P} = \frac{1}{T} \left(\frac{\partial Y}{\partial \zeta}\right)_{T, P}$$

Put together, the above statements yield

$$d\left(\frac{Y}{T}\right) = \frac{1}{T^2} \left(\frac{\partial H}{\partial \zeta}\right)_{T,P} dT - \frac{1}{T} \left(\frac{\partial V}{\partial \zeta}\right)_{T,P} dP + \frac{1}{T} \left(\frac{\partial Y}{\partial \zeta}\right)_{T,P} d\zeta$$


---

Problem 7.3 We focus on the following portion of the table

T, [K]	$\log_{10} K_P$	$T^{-1}, [K^{-1}]$
298	-35.612	1/298
500	(?)	
1000	-8.646	1/1000

where (?) stands in place of the correct value

$$\log_{10}(500 \text{ K}) = -20.158$$

The numerical results and percentage deviation from the correct value are:

- (1) linear interpolation in T, (?) = -27.846, error = 38.1%
- (2) linear interpolation in  $T^{-1}$ , (?) = -20.090, error = 0.34%

In conclusion, the numerical error can be significant if we do not keep in mind the analytical form of the van't Hoff relation.

---

Problem 7.4 Following Example. 7.1, we conclude the  $K_P = 0.0454$  and

$$0.0454 = \frac{x_{H_2}^{1/2} x_{O_2}^{1/2}}{x_{H_2O}^1} \left(\frac{P}{P_0}\right)^{1+\frac{1}{2}-1}$$

where  $P/P_0 = 0.1$ . In terms of the extent of the reaction  $\zeta$ , the above condition reads

$$0.0454 = \frac{\zeta}{1-\zeta} \left(\frac{\zeta}{2+\zeta}\right)^{1/2} \quad (0.316)$$

Its solution is  $\zeta = 0.288$ , which means that [see eqs. (d) of Example 7.1]

$$x_{H_2O} = 0.622, \quad x_{H_2} = 0.252, \quad x_{O_2} = 0.126.$$

Comparing these values with those found in Example 7.1 for the case when  $P = 1 \text{ atm}$ , we learn to expect that if the pressure were to increase the mole fractions of the products  $H_2$  and  $O_2$  would decrease. The isothermal increase in pressure decreases the extent of the reaction,  $\zeta$ .

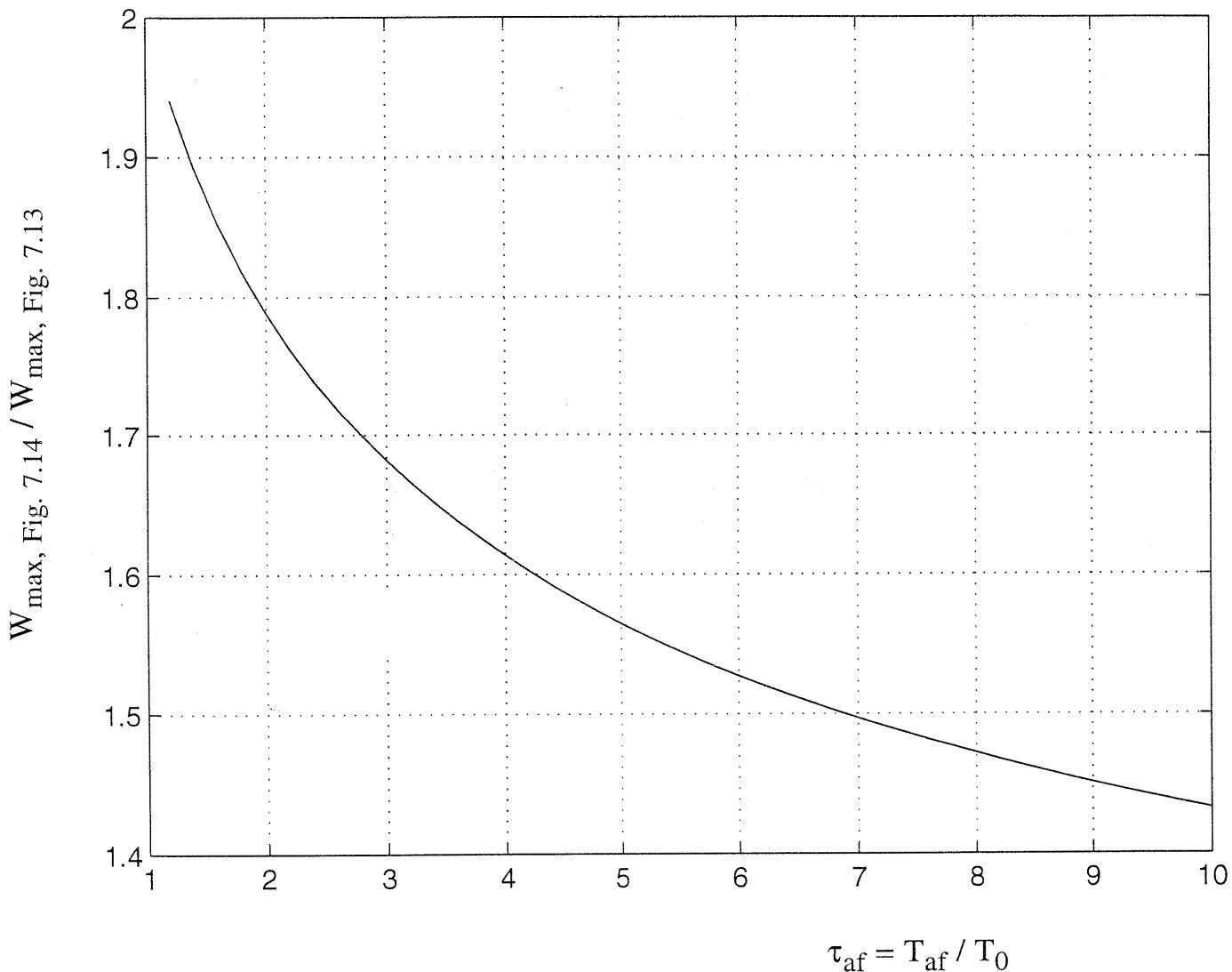
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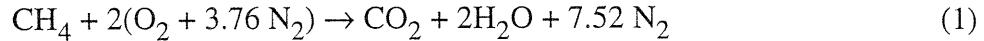
Problem 7.5 The analysis consists of dividing eq. (7.123) by eq. (7.119) and using the shorthand notation  $\tau_{af} = \tau_{af}/T_0$ ,

$$\frac{W_{max, \text{Fig. 7.14}}}{W_{max, \text{Fig. 7.13}}} = \frac{\tau_{af} - 1 - \ln \tau_{af}}{(\tau_{af}^{1/2} - 1)^2}$$

The graph shows that this ratio is greater than one. It increases from a value of 1.43 at  $\tau_{af} = 10$  to the limiting value of 2 at  $\tau_{af} = 1$ . For combustion processes we look at  $\tau_{af}$  values of the order of 10 and see only a modest increase in the W harvest if we optimize the system according to Fig. 7.14. This analysis and its graphical conclusion are only approximate because of the constant specific heat assumption made beginning with eq. (7.117).



Problem 7.6 The theoretical-air chemical equation is



The first law reduces to  $H_r = H_p$ , which is

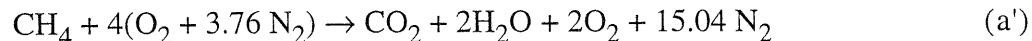
$$\bar{h}_{f, \text{CH}_4}^\circ = [\bar{h}_f^\circ + \Delta\bar{h}(\text{T}_{af})]_{\text{CO}_2} + 2[\bar{h}_f^\circ + \Delta\bar{h}(\text{T}_{af})]_{\text{H}_2\text{O}} + 7.52 \Delta\bar{h}_{\text{N}_2}(\text{T}_{af}) \quad (b)$$

i. e. and implicit equation for  $\text{T}_{af}$ ,

$$802.3 \frac{\text{kJ}}{\text{mol methane}} = \Delta\bar{h}_{\text{CO}_2}(\text{T}_{af}) + 2\Delta\bar{h}_{\text{H}_2\text{O}}(\text{T}_{af}) + 7.52 \Delta\bar{h}_{\text{N}_2}(\text{T}_{af}) \quad (c)$$

The  $\text{T}_{af}$  value that satisfies this equation is  $\text{T}_{af} \approx 2320 \text{ K}$ .

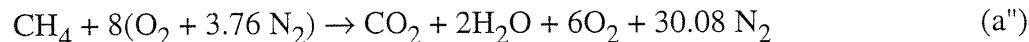
The case of 200% theoretical air is analyzed similarly,



$$802.3 \frac{\text{kJ}}{\text{mol methane}} = \Delta\bar{h}_{\text{CO}_2}(\text{T}_{af}) + 2\Delta\bar{h}_{\text{H}_2\text{O}}(\text{T}_{af}) + 2\Delta\bar{h}_{\text{O}_2}(\text{T}_{af}) + 15.04 \Delta\bar{h}_{\text{N}_2}(\text{T}_{af}) \quad (c')$$

and the answer is  $\text{T}_{af} \approx 1480 \text{ K}$ .

In the case of 400% theoretical air we have



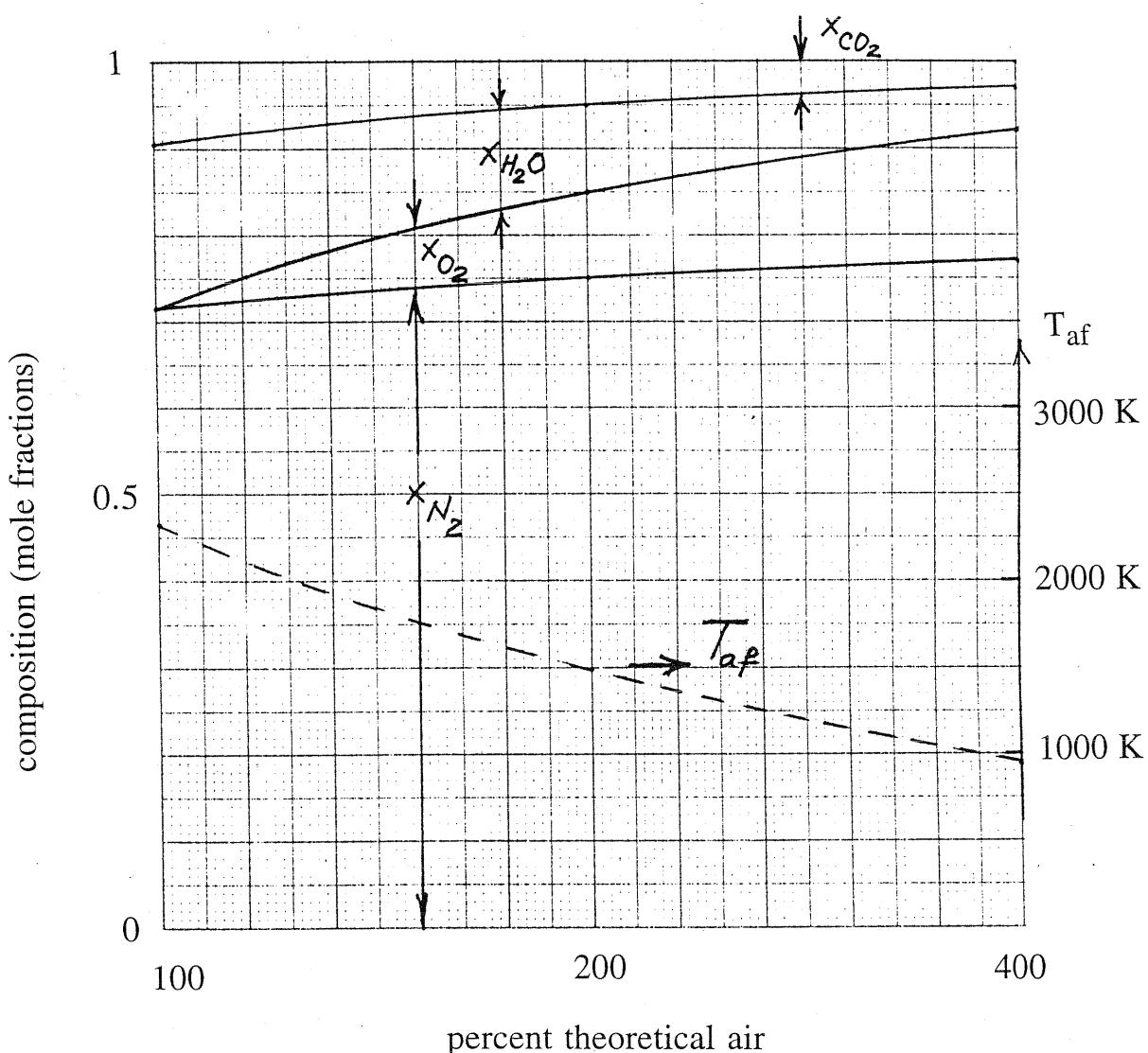
$$802.3 \frac{\text{kJ}}{\text{mol methane}} = \Delta\bar{h}_{\text{CO}_2}(\text{T}_{af}) + 2\Delta\bar{h}_{\text{H}_2\text{O}}(\text{T}_{af}) + 6\Delta\bar{h}_{\text{O}_2}(\text{T}_{af}) + 30.08 \Delta\bar{h}_{\text{N}_2}(\text{T}_{af}) \quad (c'')$$

and, finally,  $\text{T}_{af} \approx 950 \text{ K}$ .

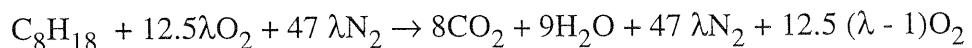
The composition of the products is shown as a function of "percent theoretical air" in the figure. The mole fraction of  $\text{CO}_2$  at 100% theoretical air, for example, is calculated based on eq. (a):

$$x_{\text{CO}_2} = \frac{1}{1 + 2 + 7.52} = 0.095$$

As the amount of air increases the composition becomes dominated by  $\text{N}_2$  and  $\text{O}_2$ . At the same time  $\text{T}_{af}$  decreases.



Problem 7.7 The method of solution is outlined already in Example 7.6. The general chemical equation is



where  $\lambda = 1$  represents the case of theoretical air combustion treated in Example 7.6. Here are the new numerical results\* :

\* Solution worked out by Prof. J. L. Lage, Southern Methodist University, using the numerical and graphical information available in Table 7.2 and Fig. 7.7.

	$\lambda = 2$ (100% excess air, or 200% theoretical air)	$\lambda = 3$ (200% excess air, or 300% theoretical air)
$\frac{\dot{n}_{\text{octane}}}{\dot{m}_{\text{water}}}$	1.499	3.704 $\frac{\text{mol octane}}{\text{kg water}}$
$S_{\text{gen}}$	9 352	9 111 $\frac{\text{kJ/K}}{\text{mol octane}}$
$W_{\text{rev}}$	3 787	3 123 kJ/mol octane
$W_{\text{plant}}$	999	406 kJ/mol octane
$W_{\text{lost}, \Delta T}$	562	227 kJ/mol octane

The splitting of  $W_{\text{rev}}$  into usable work ( $W_{\text{plant}}$ ) and lost work is affected by the amount of air in the following way

	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$
$\eta_{II} = W_{\text{plant}} / W_{\text{rev}}$	0.36	0.264	0.13
$W_{\text{lost}, \Delta T} / W_{\text{rev}}$	0.20	0.148	0.073
$W_{\text{lost by combustion alone}} / W_{\text{rev}}$	0.44	0.588	0.797

The heated water flowrate ( $\dot{m}_{\text{water}}$ ) and, with it, the plant power output and the second law efficiency decrease as  $\lambda$  increases. These three quantities are all zero when  $\lambda \approx 3.68$ .

Another view is to consider varying  $\lambda$  when the water flowrate delivered to the power plant is fixed. In this case the plant work output per kg water ( $W_{\text{plant}} / \dot{m}_{\text{water}}$ ) and the ratio  $W_{\text{lost}, \Delta T} / \dot{m}_{\text{water}}$  are also fixed. The preceding results can be recombined into the following table

	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$	Units
$S_{\text{gen}} / \dot{m}_{\text{water}}$	9 018	14 019	33 747	$\frac{\text{kJ/K}}{\text{kg water}}$
$W_{\text{rev}} / \dot{m}_{\text{water}}$	4 184	5 677	11 566	kJ/kg water
$W_{\text{lost by combustion alone}} / \dot{m}_{\text{water}}$	1 845	3 337	9 221	kJ/kg water

Problem 7.8 (a) The chemical equation is



which means that the total number of moles in the enclosure decreases from  $n_r = 4$  to  $n_p = 3.5$ . The first law pinpoints the final temperature,  $T$ ,

$$\begin{aligned} & \left(\bar{h}_f^\circ + \Delta\bar{h} - \bar{R}T_0\right)_{\text{CO}} + 3\left(\bar{h}_f^\circ + \Delta\bar{h} - \bar{R}T_0\right)_{\text{O}_2} = \\ & \left(\bar{h}_f^\circ + \Delta\bar{h} - \bar{R}T\right)_{\text{CO}_2} + 2.5\left(\bar{h}_f^\circ + \Delta\bar{h} - \bar{R}T\right)_{\text{O}_2} \end{aligned} \quad (\text{b})$$

and reduces to

$$273.1 \frac{\text{kJ}}{\text{mol CO}} = \Delta\bar{h}_{\text{CO}_2}(T) + 2.5 \Delta\bar{h}_{\text{O}_2}(T) - 3.5\bar{R}T \quad (\text{c})$$

Interpolating graphically with the help of Fig. 7.7 yields the temperature of the final products  $T \approx 2705 \text{ K}$ . The final pressure,  $P$ , follows from the constant volume ideal gas condition

$$\frac{n_r \bar{R}T_0}{P_0} = \text{volume} = \frac{n_p \bar{R}T}{P} \quad (\text{d})$$

which means that

$$P = P_0 \frac{n_p}{n_r} \frac{T}{T_0} \approx 7.94 \text{ atm} \quad (\text{e})$$

(b) In the second part of the problem the final temperature is given,  $T = 500 \text{ K}$ , therefore we can calculate the final pressure immediately,

$$P = P_0 \frac{n_p}{n_r} \frac{T}{T_0} = 1.47 \text{ atm} \quad (\text{f})$$

The heat transfer released during the reaction is

$$\begin{aligned} Q_{\text{out}} &= U_r - U_p \\ &= \left(\bar{h}_f^\circ - \bar{R}T_0\right)_{\text{CO}} + 3(-\bar{R}T_0)_{\text{O}_2} \\ &\quad - \left(\bar{h}_f^\circ + \Delta\bar{h} - \bar{R}T\right)_{\text{CO}_2} - 2.5(\Delta\bar{h} - \bar{R}T)_{\text{O}_2} \\ &= (-110.53 - 4 \times 8.314 \times 10^{-3} \times 298.15 \\ &\quad + 393.52 - 8.314 - 2.5 \times 6.09 + 3.5 \times 8.314 \times 10^{-3} \times 500) \text{ kJ/mol} \\ &= 264.1 \text{ kJ/mol.} \end{aligned}$$

---

Problem 7.9 We write the first law for the entire system,

$$\dot{W} = \dot{m}c_P(T_1 - T_2) - \dot{Q}_0 \quad (1)$$

the second law for the lower (reversible) compartment,

$$\frac{\dot{Q}}{T_2} = \frac{\dot{Q}_0}{T_0} \quad (2)$$

and the first law for the upper (irreversible) compartment,

$$\dot{Q} = \dot{m}c_P(T_1 - T_2) \quad (3)$$

Eliminating  $\dot{Q}$  and  $\dot{Q}_0$  between eqs. (1) - (3) we obtain  $\dot{W}$  as a function of  $T_2$ :

$$\frac{\dot{W}}{\dot{m}c_P} = T_1 - T_2 - \frac{T_0 T_1}{T_2} + T_0 \quad (4)$$

Maximized with respect to  $T_2$ , this  $\dot{W}$  expression yields

$$T_{2,\text{opt}} = (T_1 T_0)^{1/2} \quad (5)$$

$$\dot{W}_{\max} = \dot{m}c_P \left( T_1^{1/2} - T_0^{1/2} \right)^2 \quad (6)$$

The efficiency that corresponds to this maximum power design is, cf. eqs. (3), (5), and (6),

$$\eta = \frac{\dot{W}_{\max}}{\dot{Q}} = 1 - \left( \frac{T_0}{T_1} \right)^{1/2} \quad (7)$$


---

Problem 7.10 (a) The instantaneous power output is

$$\dot{W} = \dot{Q} \left( 1 - \frac{T_0}{T_2} \right) \quad (1)$$

where

$$\dot{Q} = \dot{m}c_P(T_1 - T_{\text{out}}) \quad (2)$$

and

$$\frac{T_1 - T_{\text{out}}}{T_1 - T_2} = 1 - \exp(-N_{tu}) = \varepsilon \quad (3)$$

Eliminating  $\dot{Q}$  and  $T_{\text{out}}$ , we rewrite eq. (1) as

$$\dot{W} = \dot{m}c_P(T_1 - T_2)\epsilon \left(1 - \frac{T_0}{T_2}\right) \quad (4)$$

and, solving  $\partial \dot{W} / \partial T_2 = 0$  we obtain

$$T_{2,\text{opt}} = (T_1 T_0)^{1/2} \quad (5)$$

$$\eta = \frac{\dot{W}_{\text{max}}}{\dot{Q}} = 1 - \left(\frac{T_0}{T_1}\right)^{1/2} \quad (6)$$

(b) Because the lower compartment is reversible, the efficiency continues to be given by

$$\eta = 1 - \frac{T_2}{T_0} \quad (7)$$

The same power plant model (well mixed stream + reversible power cycle) was optimized in section 7.3.4. In the present notation, the optimum is described by

$$T_{2,\text{opt}} = (T_1 T_0)^{1/2} \quad (8)$$

and the efficiency at maximum power output becomes

$$\eta = 1 - \left(\frac{T_0}{T_1}\right)^{1/2} \quad (9)$$


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## Chapter 8

### POWER GENERATION

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Problem 8.1 It is more efficient to use the shorthand notation

$$e^{N_H} = x, \quad e^{N_L} = y$$

so that

$$\frac{T_H - T_1}{T_H - T_2} = x \quad (8.20)$$

$$\frac{T_3 - T_L}{T_4 - T_L} = y \quad (8.21)$$

$$\frac{T_1}{T_4} = a, \quad \text{and} \quad \frac{T_2}{T_3} = a \quad (8.22)$$

The total entropy generation rate  $\dot{S}_{\text{gen}}$  is the sum of eqs. (8.18) and (8.19),

$$\frac{\dot{S}_{\text{gen}}}{\dot{m}c_P} = \underbrace{\ln \frac{T_2 T_4}{T_1 T_3}}_0 - \frac{T_2}{T_H} + \frac{T_1}{T_H} - \frac{T_4}{T_L} + \frac{T_3}{T_L} \quad (a)$$

Solving the system (8.20) - (8.22) for the four corner temperatures, we obtain

$$T_2 = \frac{T_H y(x-1) + a T_L (y-1)}{xy - 1} \quad (b)$$

$$T_4 = \frac{\frac{1}{a} T_H (x-1) + x T_L (y-1)}{xy - 1} \quad (c)$$

$$T_1 = a T_4 = \text{etc., using eq. (c)} \quad (d)$$

$$T_3 = \frac{1}{a} T_2 = \text{etc., using eq. (b)} \quad (e)$$

Substituting eqs. (b) - (e) into the  $\dot{S}_{\text{gen}}$  formula (a) yields

$$\frac{\dot{S}_{\text{gen}}}{\dot{m}c_P} = \frac{1}{a T_L T_H} (T_H - a T_L)^2 \frac{(x-1)(y-1)}{xy - 1} \quad (f)$$

which is the same as eq. (8.25) in the text.

In terms of the notation used above, the " $N_H + N_L = \text{constant}$ " constraint reads

$$\ln x + \ln y - \ln C = 0 \quad (g)$$

where  $C$  is a constant. Minimizing (f) subject to the constraint (g) is equivalent to minimizing

$$\Phi = B \frac{(x-1)(y-1)}{xy - 1} + \lambda (\ln x + \ln y - \ln C) \quad (h)$$

subject to no constraints. The system

$$\frac{\partial \Phi}{\partial x} = 0, \quad \frac{\partial \Phi}{\partial y} = 0 \quad (i, j)$$

becomes

$$B \left( \frac{y-1}{xy-1} \right)^2 + \frac{\lambda}{x} = 0 \quad (i')$$

$$B \left( \frac{x-1}{xy-1} \right)^2 + \frac{\lambda}{y} = 0 \quad (j')$$

One immediate result is

$$x = y, \quad (\text{i.e. } N_H = N_L)$$

Finally, to verify eqs. (8.28) and (8.29), we substitute " $x = y$ " into eqs. (b) - (e), and write in order

$$(T_2)_{\text{opt}} = \frac{x T_H + a T_L}{x + 1}$$

$$(T_4)_{\text{opt}} = \frac{\frac{1}{a} T_H + x T_L}{x + 1}$$

$$(T_1 + T_2)_{\text{opt}} = a T_4 + T_2 = \dots = T_H + a T_L \quad (8.28)$$

$$(T_3 + T_4)_{\text{opt}} = \frac{1}{a} T_2 + T_4 = \frac{1}{a} (T_2 + a T_4) = \frac{1}{a} (T_1 + T_2)_{\text{opt}} = \frac{1}{a} T_H + T_L \quad (8.29)$$

**Problem 8.2** Starting with the definition of lost work (lost exergy), eq. (8.40), we write in order

$$\begin{aligned}
 \frac{1}{m} \dot{W}_{\text{lost}} &= \frac{1}{m} \dot{E}_{Q_H} - \frac{1}{m} \dot{W}_{\text{net}} \\
 &= \frac{Q_H}{m} \left( 1 - \frac{T_0}{T_f} \right) - \underbrace{\int_{\text{cycle: 1-2-3-4-1}} T ds}_{\text{cycle: 1-2-3-4-1}} \\
 &= \underbrace{(h_2 - h_1)}_{\text{cycle: 1-2-3-4-1}} \left( 1 - \frac{T_0}{T_f} \right) - \underbrace{\left( \int_1^2 T ds \right)_{P_H}}_{\text{cycle: 1-2-3-4-1}} + \underbrace{\left( \int_4^3 T ds \right)_{P_L}}_{\text{cycle: 1-2-3-4-1}} \\
 &= -\frac{T_0}{T_f} \underbrace{\left( \int_1^2 T ds \right)_{P_H}}_{(s_2 - s_1) T_{H,\text{avg}}} + \underbrace{\left( \int_4^3 T ds \right)_{P_L}}_{\underbrace{(s_3 - s_4)}_{s_2 - s_1} T_{L,\text{avg}}} \quad (a) \\
 &= T_0 (s_2 - s_1) \left( -\frac{T_{H,\text{avg}}}{T_f} + \frac{T_{L,\text{avg}}}{T_0} \right)
 \end{aligned}$$

In order to reproduce eq. (8.38), we start with eq. (a), and add and subtract  $T_0(s_2 - s_1)$ :

$$\begin{aligned}
 \frac{1}{m} W_{\text{lost}} &= -\frac{T_0}{T_f} \underbrace{\left( \int_1^2 T ds \right)_{P_H}}_{s_2 - s_1} + \underbrace{\left( \int_4^3 T ds \right)_{P_L}}_{s_3 - s_4} + T_0 (s_2 - s_1) - T_0 \underbrace{(s_2 - s_1)}_{s_3 - s_4} \\
 &= T_0 \left[ s_2 - s_1 - \frac{1}{T_f} \underbrace{\left( \int_1^2 T ds \right)_{P_H}}_{s_2 - s_1} \right] + T_0 \left[ \frac{1}{T_0} \underbrace{\left( \int_4^3 T ds \right)_{P_L}}_{s_3 - s_4} - (s_3 - s_4) \right] \\
 &= T_0 \frac{1}{m} \dot{S}_{\text{gen},H} + T_0 \frac{1}{m} \dot{S}_{\text{gen},L} \quad (8.38)
 \end{aligned}$$

Problem 8.3 We start with the total area counted on the left side of Fig. 8.7,

$$\begin{aligned}
 \text{left-side area} &= [\text{area}]_L + \underbrace{\oint T ds}_{\text{cycle 1-2-3-4-1}} + \frac{T_0}{T_f} [\text{area}]_H \\
 &= \left( \int_1^2 T ds \right)_{P_H} - T_0(s_2 - s_1) + \frac{T_0}{T_f} \left[ T_f(s_2 - s_1) - \left( \int_1^2 T ds \right)_{P_H} \right] \\
 &= \underbrace{\left( \int_1^2 T ds \right)}_{\text{area on the right side, before shrinking}} \times \underbrace{\left( 1 - \frac{T_0}{T_f} \right)}_{\text{reduction factor}} \\
 &= \text{right-side area (shaded)}
 \end{aligned}$$

Problem 8.4 As shown in the sketch,  $T_L = T_0 = T_{L,\text{avg}}$ . The first-law (heat engine) efficiency is

$$\eta = \frac{\dot{W} / \dot{m}}{\dot{Q}_H / \dot{m}} = \frac{h_2 - h_3 - h_1 + h_4}{h_3 - h_1} \quad (\text{a})$$

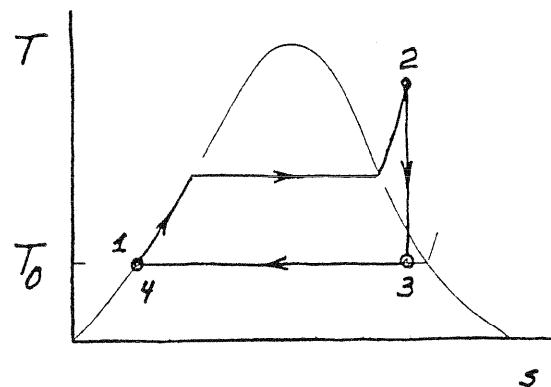
In addition, we have

$$\begin{aligned}
 h_3 - h_4 &= \left( \int_4^3 T ds \right)_{P_L} = T_0(s_3 - s_4) \\
 &= T_0(s_2 - s_1)
 \end{aligned} \quad (\text{b})$$

Combining eqs. (a) and (b), and recalling the flow availability definition

$b = h - T_0 s$ , we write finally:

$$\begin{aligned}
 \eta &= \frac{h_2 - h_1 - (h_3 - h_4)}{h_2 - h_1} \\
 &= \frac{h_2 - h_1 - T_0(s_2 - s_1)}{h_2 - h_1} = \frac{b_2 - b_1}{h_2 - h_1}
 \end{aligned}$$



Problem 8.5 With reference to the left side of Fig. 8.9, we write

$$\eta_p = \frac{h_1 - h_{4'}}{h_1 - h_4}$$

From  $dh = Tds + vdp$ , in which  $v = v_1$  (constant), we deduce that

$$h_1 - h_{4'} = v_1 (P_H - P_L)$$

Integrating  $dh = Tds + vdp$  at constant pressure, from state (1) to state (1') we find

$$h_{1'} - h_1 = \bar{T}_1 (s_{1'} - s_1) \quad (a)$$

where

$$\bar{T}_1 = \frac{\left( \int_1^{1'} T ds \right)}{s_{1'} - s_1} |_{P_H}$$

Adding and subtracting  $h_{4'}$  on the left side of eq. (a) yields, in order,

$$h_{1'} - h_1 + h_{4'} - h_{4'} = \bar{T}_1 (s_{1'} - s_1)$$

$$(h_1 - h_{4'}) \left( -1 + \frac{h_{1'} - h_{4'}}{h_1 - h_{4'}} \right) = \bar{T}_1 (s_{1'} - s_1)$$

$$v_1 (P_H - P_L) \left( -1 + \frac{1}{\eta_p} \right) = \bar{T}_1 (s_{1'} - s_1)$$

in other words,

$$s_{1'} - s_1 = \frac{v_1}{\bar{T}_1} (P_H - P_L) \left( \eta_p^{-1} - 1 \right)$$

Focusing now on the extreme right of Fig. 8.9, the turbine isentropic efficiency is defined as

$$\eta_t = \frac{h_{2'} - h_{3'}}{h_{2'} - h_3}$$

We write in order

$$\begin{aligned}
h_{3'} - h_3 &= h_{3'} - h_3 + h_{2'} - h_{2'} \\
&= (h_{2'} - h_3) \left( 1 - \frac{h_{2'} - h_3}{h_{2'} - h_3} \right) \\
&= (h_{2'} - h_3)(1 - \eta_t) \\
&= c_{P,3}(T_{2'} - T_3)(1 - \eta_t)
\end{aligned} \tag{b}$$

Note the use of the ideal gas and constant- $c_P$  model in eq. (b). The isentropic expansion of such a gas from (2') to (3) also means that

$$\frac{T_{2'}}{T_3} = \left( \frac{P_H}{P_L} \right)^{R/c_P} = \left( \frac{P_H}{P_L} \right)^{(k-1)/k}$$

therefore, eq. (b) becomes

$$h_{3'} = h_3 = c_{P,3} T_3 \left[ \left( \frac{P_H}{P_L} \right)^{(k-1)/k} - 1 \right] (1 - \eta_t)$$

Finally, the ratio describing the power lost by the pump divided by the power lost by the turbine is

$$\begin{aligned}
\frac{\dot{W}_{\text{lost,pump}}}{\text{total } \dot{W}_{\text{lost, expander}}} &= \frac{T_0(s_{1'} - s_{4'})}{\left( \int_3^{3'} T ds \right) P_L} \\
&= \frac{T_0(s_{1'} - s_{4'})}{h_{3'} - h_3} \\
&= \frac{v_1 P_L T_0 (r-1) (\eta_p^{-1} - 1)}{c_{P,3} \bar{T}_1 T_3 [r^{(k-1)/k} - 1] (1 - \eta_t)}
\end{aligned}$$

Problem 8.6 Based on the geometry of the right side of Fig. 7.9, we define the average temperature  $\bar{T}_3$  by writing

$$\left( \int_{3_{\text{ideal}}}^3 T ds \right)_{P_L} = \bar{T}_3 (s_3 - s_{3_{\text{ideal}}}) \tag{a}$$

The entropy difference ( $s_3 - s_{3\text{ideal}}$ ) is also equal to

$$\begin{aligned}
 s_3 - s_{3\text{ideal}} &= s_2 - s_2 \\
 &= -R_2 \ln \frac{P_2'}{P_2} \\
 &= -R_2 \ln \frac{P_H - \Delta P_H}{P_H} \approx R_2 \frac{\Delta P_H}{P_H}
 \end{aligned} \tag{b}$$

Focusing on the numerator of eq. (8.56) we note that

$$\begin{aligned}
 \left( \int_4^{4\text{ideal}} T ds \right)_{P_L} &= h_{4\text{ideal}} - h_4 \\
 &= h_{4\text{ideal}} - h_{4'} \approx \frac{dh_f}{dP} (P_{4\text{ideal}} - P_{4'}) \\
 &\approx \frac{dh_f}{dP} \Delta P_L
 \end{aligned}$$

where  $h_f$  is a function of  $P$  [or of  $T_{\text{sat}}(P)$ ]. Similarly, we write

$$s_{4\text{ideal}} - s_{4'} \approx \frac{ds_f}{dP} \Delta P_L$$

so that the numerator of eq. (8.56) becomes

$$\left( \frac{dh_f}{dP} - T_0 \frac{ds_f}{dP} \right) \Delta P_L = \frac{db_f}{dP} \Delta P_L \tag{c}$$

Combining eqs. (a) - (c), we find that the ratio constructed in eq. (8.56) is

$$\begin{aligned}
 \frac{\text{total } \dot{W}_{\text{lost}} \text{ due to } \Delta P_L}{\text{total } \dot{W}_{\text{lost}} \text{ due to } \Delta P_H} &= \frac{\frac{db_f}{dP} \Delta P_L}{\bar{T}_3 R_2 \Delta P_H / P_H} \\
 &= C \frac{(\Delta P/P)_L}{(\Delta P/P)_H}
 \end{aligned}$$

where

$$C = \frac{P_L}{R_2 \bar{T}_3} \left( \frac{db_f}{dP} \right)_{\substack{\text{near} \\ P=P_L}} \tag{8.57}$$


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Problem 8.7 The preceding two problems provide the ingredients for calculating

$$\frac{\text{total } \dot{W}_{\text{lost}} \text{ due to } \Delta P_H}{\text{total } \dot{W}_{\text{lost}} \text{ due to expander}} = \frac{\left( \int_{P_L}^{P_{\text{ideal}}} T ds \right)_3}{\left( \int_{P_L}^{P_{\text{ideal}}} T ds \right)_3} \approx \frac{\bar{T}_3 R_2 \Delta P_H / P_H}{c_{P,3} T_3 [r^{(k-1)/k} - 1] (1 - \eta_t)}$$

where  $r = P_H / P_L$ . Therefore when  $P_H / P_L$  is fixed, the ratio of the  $\Delta P_H$  and  $\eta_t$  irreversibilities varies as  $(\Delta P_H / P_H) / (1 - \eta_t)$ .

---

Problem 8.8 Let states (a) and (b) represent the reheater inlet and outlet as shown in the sketch. The heat engine efficiency of the cycle is

$$\eta = 1 - \frac{h_3 - h_4}{h_2 - h_1 + h_b - h_a}$$

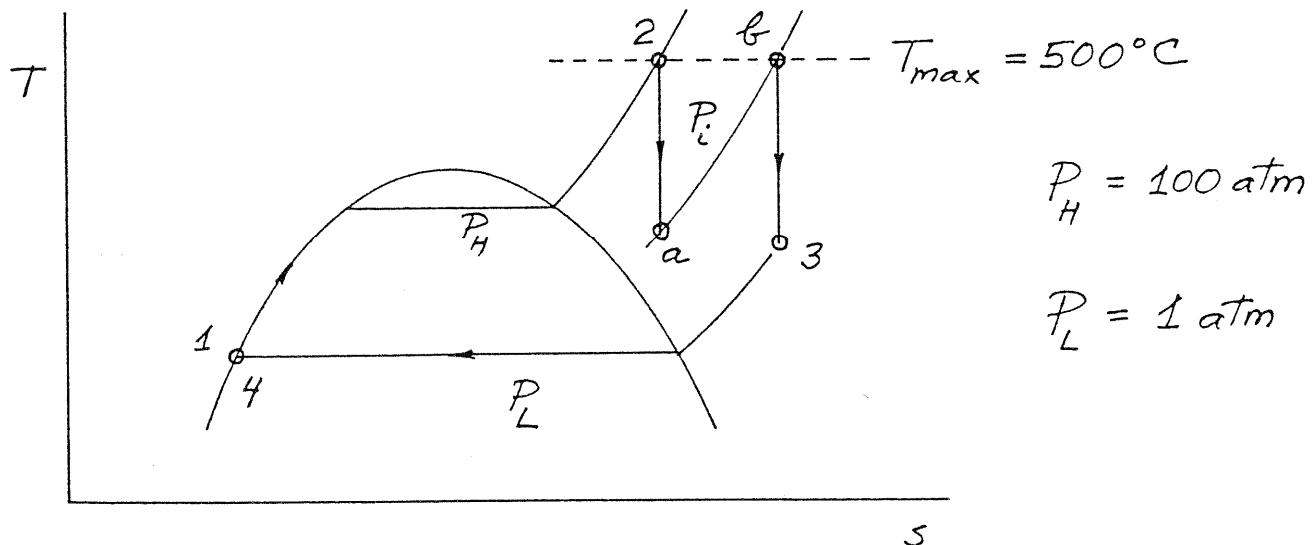
and, using Keenan et al.'s T-s diagram [20],

$$\eta = \frac{3375 \frac{J}{g} - h_3 + h_b - h_a}{2950 \frac{J}{g} + h_b - h_a}$$

Using the same diagram we complete the following table of *approximate* values, where  $P_i = P_a = P_b$ ,

$P_i [\text{atm}]$	$h_a \left[ \frac{\text{J}}{\text{g}} \right]$	$h_b \left[ \frac{\text{J}}{\text{g}} \right]$	$h_3 \left[ \frac{\text{J}}{\text{g}} \right]$	$\eta$
100				0.325
60	3225	3425	2500	0.341
40	3115	3445	2580	0.343
20	2940	3470	2720	0.341
10	2800	3480	2850	0.332
1				0.325

Clearly, there exists an optimal reheater pressure that maximizes the cycle efficiency: that optimal pressure falls in the vicinity of  $P_i \approx 40$  atm.



Problem 8.9 Consider an infinitesimal feed heating stage that raises the temperature of the feed from  $T$  to  $T + \Delta T$ . With reference to the sketch shown next, we write step-by-step

$$\begin{aligned} h_8 - h_7 &= h_4 - h_5 \\ \frac{dh_f}{dT} \Delta T &= T(s_4 - s_5) \\ \frac{dh_f}{dT} \Delta T &= T \frac{ds}{dT} \Delta T \end{aligned} \tag{a}$$

where  $s(T)$  is the sought curve, and where  $h_f = h_f(T)$ . Equation (a) also means that

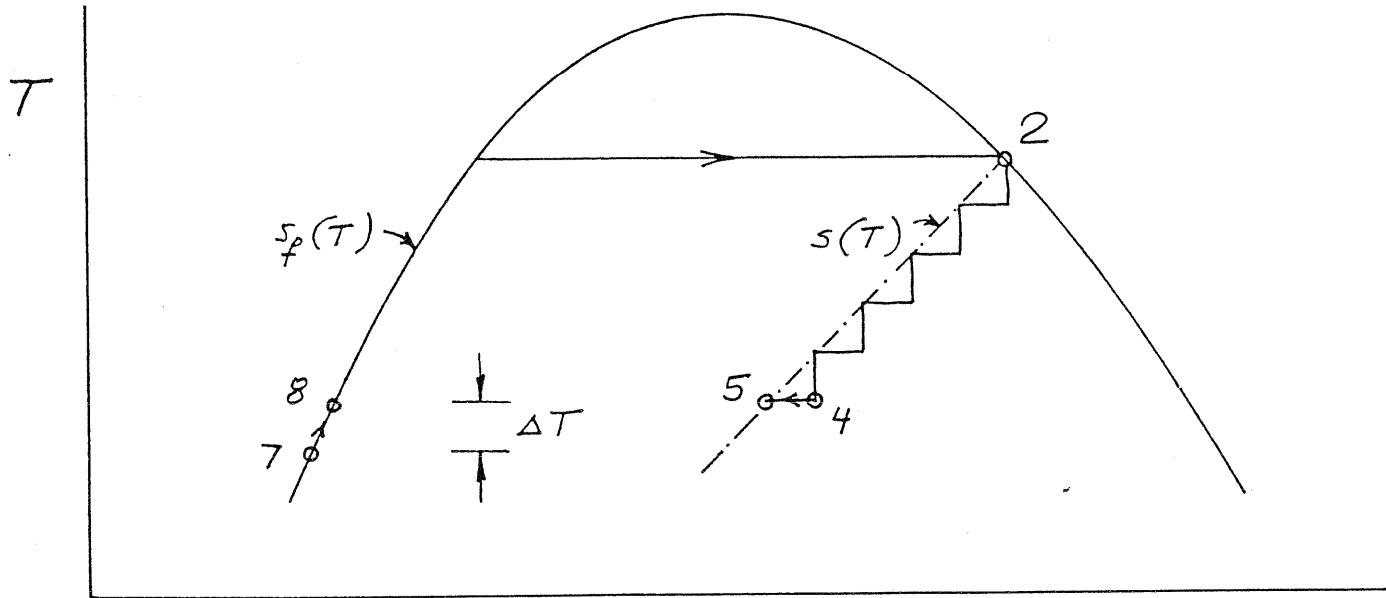
$$ds = \frac{dh_f}{dT} \frac{dT}{T}$$

which can be integrated downward from the known corner state (2),

$$s_2 - s(T) = \int_T^{T_2} \frac{dh_f}{dT} \frac{dT}{T} \tag{b}$$

If  $h_f$  increases linearly with  $T$ , then  $dh_f / dT$  is a constant (called  $\overline{dh_f/dT}$ ). Equation (b) assumes the closed form

$$s(T) = s_2 - \left( \frac{\overline{dh_f}}{dT} \right) \ln \frac{T_2}{T}$$



5

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Problem 8.10 With reference to the T-s diagram of Fig. 8.14, using eq. (8.71) we have

$$\begin{aligned} Y &= \frac{\dot{Q}_H(0) - \dot{Q}_H(n)}{\dot{Q}_H(0) - \dot{Q}_H(\infty)} \\ &= \frac{(h_2 - h_0) - (h_2 - h_n)}{(h_2 - h_0) - (h_2 - h_B)} \\ &= \frac{h_n - h_0}{h_B - h_0} = 1 - \frac{h_B - h_n}{h_B - h_0} \end{aligned}$$

Invoking eq. (8.70) we conclude that the optimal designs must have

$$Y \cong 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

Next, the abscissa parameter defined in eq. (8.72) is equal to

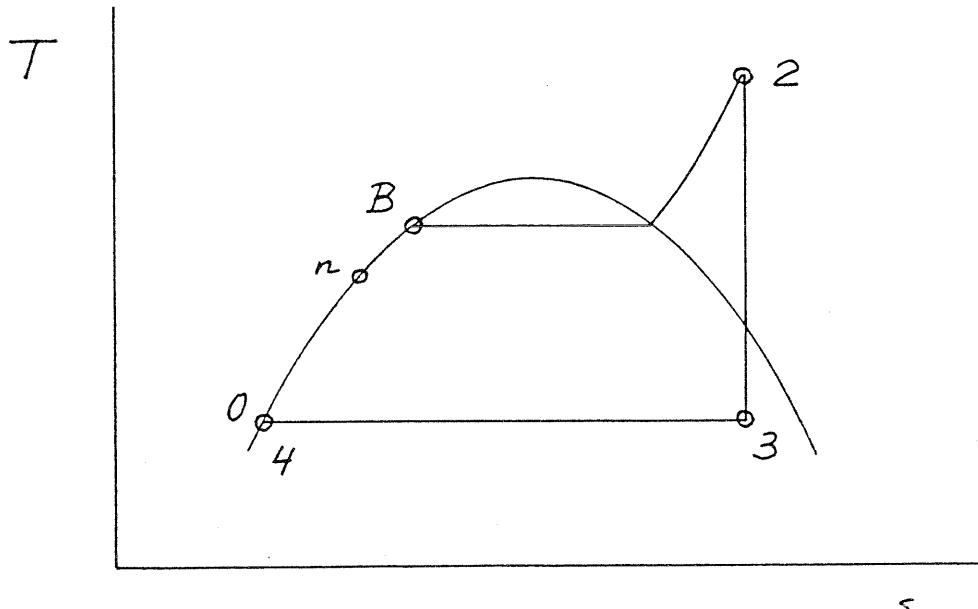
$$X = \frac{h_n - h_0}{h_B - h_0} \equiv \frac{n}{n+1} = Y$$

In conclusion, the maxima of the constant-n curves of Fig. 8.16 must fall close to the  $Y = X$  diagonal.

Problem 8.11 Using the Mollier chart of Keenan et al. [20] for  $P_H = 82.6$  atm (1214 psi),  $P_L = 0.0334$  atm (0.49 psi) and  $T_{max} = 482^\circ\text{C}$  (900 F), we read

$$h_B = 574 \text{ Btu/lbm} \quad h_2 = 1439 \text{ Btu/lbm}$$

$$h_4 = h_0 = 47 \text{ Btu/lbm} \quad h_3 = 851 \text{ Btu/lbm}$$



When no feed heaters are employed the cycle efficiency is

$$\eta_0 = \frac{\dot{m}(h_2 - h_3)}{\dot{m}(h_2 - h_0)} = \frac{1439 - 851}{1439 - 47} = 0.42$$

When  $n$  feed heaters are used, we distinguish between the flowrate through the boiler ( $\dot{m}_n$ ) and the flowrate through the condenser ( $\dot{m}_0$ ),

$$\eta_n = 1 - \frac{\dot{Q}_L}{\dot{Q}_H} = 1 - \frac{\dot{m}_0(h_3 - h_4)}{\dot{m}_n(h_2 - h_n)}$$

where the ratio  $\dot{m}_n / \dot{m}_0$  is given by

$$\frac{\dot{m}_n}{\dot{m}_0} = \prod_{i=1}^n \left(1 + \frac{r_i}{\beta_i}\right) = \left(1 + \frac{r}{\beta}\right)^n$$

The feed enthalpy rise  $r$  is

$$r = \frac{h_B - h_0}{n + 1}$$

The enthalpy drop experienced by the bled steam,  $\beta$ , was assumed constant. Its value can be estimated by imagining the low and high pressure extremes,

$$(\beta)_{P_H} = h_2 - h_B = 865 \text{ Btu/lbm}$$

$$(\beta)_{P_L} = h_3 - h_4 = 804 \text{ Btu/lbm}$$

therefore, the arithmetic average of these two values can be taken to represent the constant  $\beta$ ,

$$\beta \approx 835 \text{ Btu/lbm}$$

Finally, since

$$\frac{h_B - h_n}{h_B - h_0} = \frac{1}{n + 1}$$

the  $\eta_n$  expression becomes

$$\begin{aligned} \eta_n &= 1 - \left[1 + \frac{h_B - h_0}{(n+1)\beta}\right]^{-n} \frac{h_3 - h_4}{h_2 - h_B + \frac{h_B - h_0}{n+1}} \\ &= 1 - \left(1 + \frac{0.631}{n+1}\right)^{-n} \frac{n+1}{1.076n + 1.731} \end{aligned}$$

The calculated  $\eta_n$  values are listed in the following table. In the limit  $n \rightarrow \infty$  we obtain

$$\eta_\infty = 1 - \frac{1}{1.076} e^{-0.631} = 0.507$$

In conclusion, the optimal use of one feed heater leads to an efficiency increase of 9 percent, whereas the use of an infinity of feed-heating stages increases  $\eta$  by 20.7 percent. One feed heater does, roughly half of the job of an infinity of feed heaters: this message is also delivered by Fig. 8.16.

n	$\eta$
0	0.420
1	0.458
2	0.472
3	0.481
4	0.485
5	0.488
6	0.490
8	0.493
10	0.496
$\infty$	0.506

Problem 8.12 Desirable are two independent objectives, the maximization of power output  $\dot{W}$ , and the minimization of irreversibility  $\dot{S}_{\text{gen}}$ . This means that in the vicinity of the optimal regime of operation, we must have, independently,

$$d\dot{W} = 0 \quad (1)$$

$$d\dot{S}_{\text{gen}} = 0 \quad (2)$$

The designer thinks that a larger  $\dot{W}$  and a smaller  $\dot{S}_{\text{gen}}$  mean a larger ratio  $\dot{W}/\dot{S}_{\text{gen}}$ , and to simplify matters he decides to maximize the ratio

$$d\left(\frac{\dot{W}}{\dot{S}_{\text{gen}}}\right) = 0 \quad (3)$$

Is this decision correct? To see the answer, expand eq. (3),

$$\frac{1}{\dot{S}_{\text{gen}}} d\dot{W} - \frac{\dot{W}}{\dot{S}_{\text{gen}}^2} d\dot{S}_{\text{gen}} = 0 \quad (4)$$

and notice that if eqs. (1) and (2) are true then eq. (3) is also true. But, the direction  $(1, 2) \rightarrow (3)$  is not the designer's argument. His direction is the opposite: he claims in fact that if he achieves (3) then he also achieves (1) and (2). His direction of thinking is  $(3) \rightarrow (1, 2)$ : his claim is wrong because eq. (3) does not imply eqs. (1, 2).

The range of validity of eq. (3) is indicated by eq. (3), or by the integral of eq. (4). For example, in eq. (4),  $d\dot{W}$  and  $d\dot{S}_{gen}$  reach zero at the same time only during the special circumstances in which  $\dot{W}$  and  $\dot{S}_{gen}$  are proportional,

$$\frac{\dot{W}}{\dot{S}_{gen}} = C, \text{ constant} \quad (5)$$

This special feature [eq. (5)] does not appear in the original description of the power system eqs. (1, 2), and for this reason the use of eq. (3) is wrong. Another way to see the error is to recognize  $T_0 \dot{S}_{gen}$  as the lost power,  $(\dot{W} + T_0 \dot{S}_{gen})$  as the power in the reversible limit ( $\dot{W}_{rev}$ ), and to recall the definition of second law efficiency

$$\eta_{II} = \frac{\dot{W}}{\dot{W}_{rev}} = \frac{\dot{W}}{\dot{W} + T_0 \dot{S}_{gen}} \quad (6)$$

where  $T_0$  is the absolute temperature of the ambient. In view of eq. (5), we find that eq. (6) states that

$$\eta_{II} = \frac{1}{1 + T_0/C}, \text{ constant} \quad (7)$$

In conclusion, if one maximizes the power/dissipation ratio (3), then one achieves maximum power *and* minimum irreversibility at the same time provided that the second law efficiency of the power plant is fixed, i.e. that *no changes* are taking place in the configuration of the machine. In this way, the claim that by maximizing the power/dissipation ratio one improves (i.e. changes) the design has been reduced to a contradiction.

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Problem 8.13 First we write that

$$\begin{aligned} \dot{S}_{gen,turbine} &= \dot{m}(s_3, -s_2) \\ &= \dot{m}(s_3, -s_3) = \dot{m}c_p \ln \frac{T_3'}{T_3} \end{aligned}$$

In order to evaluate  $T_3'$  and  $T_3$  we recall that

$$\eta_t = \frac{T_2' - T_3'}{T_2 - T_3}, \quad \text{and} \quad \frac{T_2'}{T_3} = \left( \frac{P_H - \Delta P_H}{P_L} \right)^{R/c_p}$$

Combining these two relations we find that

$$\frac{T_3'}{T_3} = 1 + (1 - \eta_t) \left[ \left( \frac{P_H - \Delta P_H}{P_L} \right)^{R/c_P} - 1 \right]$$

which means that in the limit  $\eta_t \rightarrow 1$  and  $\Delta P_H \ll P_H$  the entropy generation rate decreases linearly with  $(1 - \eta_t)$ :

$$\dot{S}_{\text{gen,turbine}} \rightarrow \dot{m}c_P (1 - \eta_t) \left[ \left( \frac{P_H}{P_L} \right)^{R/c_P} - 1 \right]$$


---

Problem 8.14 Let  $\dot{Q}_{2-3}$  and  $\dot{Q}_{4-1}$  be the heat input and heat rejection rates of the cycle,

$$\begin{aligned}\dot{Q}_{2-3} &= \dot{m}RT_{\max} \ln \frac{v_3}{v_2} = \dot{m}RT_{\max} \ln \frac{P_H}{P_L} \\ \dot{Q}_{4-1} &= \dot{m}RT_{\min} \ln \frac{v_1}{v_4} = \dot{m}RT_{\min} \ln \frac{P_L}{P_H}\end{aligned}$$

where both  $\dot{Q}$ 's are positive when going into the working fluid. Note that  $\dot{Q}_{4-1} < 0$ , i.e. that the actual rate of heat rejection is  $-\dot{Q}_{4-1}$ . The heat engine efficiency is

$$\begin{aligned}\eta &= \frac{\dot{W}_{\text{net}}}{\dot{Q}_{2-3}} = \frac{\dot{Q}_{2-3} + \dot{Q}_{4-1}}{\dot{Q}_{2-3}} \\ &= 1 + \frac{\dot{Q}_{4-1}}{\dot{Q}_{2-3}} = 1 - \frac{T_{\min}}{T_{\max}}\end{aligned}$$


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Problem 8.15 Starting with eq. (8.37) we have

$$[\text{area}]_L = \left( \int_4^3 T ds \right)_{P=P_L} - T_0 (s_3 - s_4)$$

In the present case (3)  $\equiv$  (g) and (4)  $\equiv$  (f), which means that

$$[\text{area}]_L = T s_{fg} - T_0 s_{fg}$$

where  $T$  is the saturation temperature corresponding to 30 kPa, namely 69.1°C. At the same pressure,  $s_{fg} = 6.8247 \text{ kJ/kgK}$ ; numerically, the specific exergy lost due to the cold-end irreversibility is

$$[\text{area}]_L = (69.1 - 25) (6.8247) = 300.97 \text{ kJ/kg}$$


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Problem 8.16 Modelling the fluid as an ideal gas we express the total turbine power as follows:

$$\begin{aligned}\dot{W}_t &= \dot{m}(h_1 - h_2) + \dot{m}(h_3 - h_4) \\ &= \dot{m}\eta_{t,H}(h_1 - h_{2,rev}) + \dot{m}\eta_{t,L}(h_3 - h_{4,rev}) \\ &= \dot{m}\eta_{t,H}c_P(T_H - T_{2,rev}) + \dot{m}\eta_{t,L}c_P(T_H - T_{4,rev})\end{aligned}$$

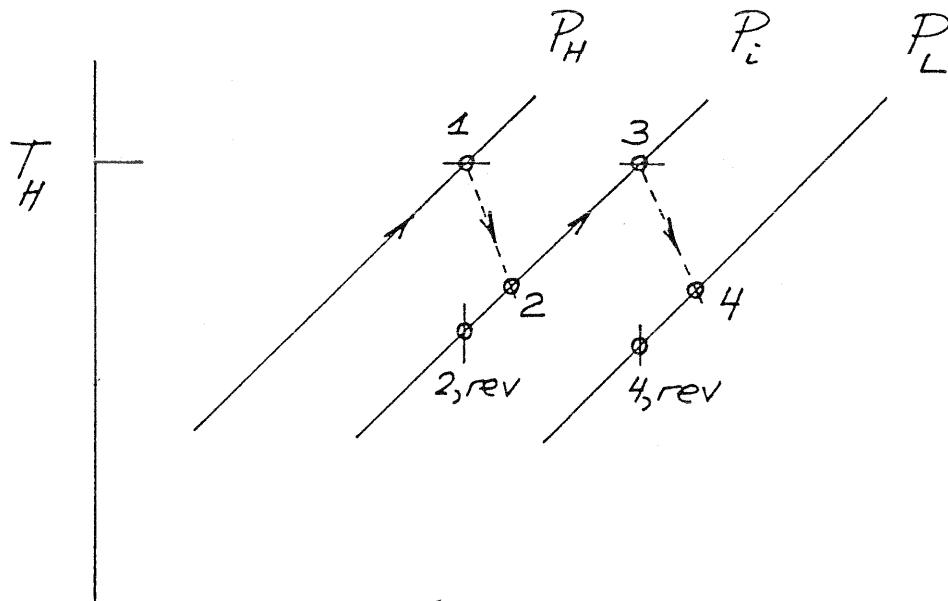
In dimensionless form this result reads

$$\begin{aligned}\frac{\dot{W}_t}{\dot{m}c_P T_H} &= \eta_{t,H}\left(1 - \frac{T_{2,rev}}{T_H}\right) + \eta_{t,L}\left(1 - \frac{T_{4,rev}}{T_H}\right) \\ &= \eta_{t,H}\left[1 - \left(\frac{P_i}{P_H}\right)^\alpha\right] + \eta_{t,L}\left[1 - \left(\frac{P_L}{P_i}\right)^\alpha\right]\end{aligned}$$

where  $\alpha = R/c_P$ . Taking the  $\partial(\ )/\partial P_i$  derivative of the right-hand-side and setting it equal to zero we obtain

$$P_{i,opt} = (P_H P_L)^{1/2} \left(\frac{\eta_{t,L}}{\eta_{t,H}}\right)^{k/2(k-1)}$$

where  $k = c_P / c_v$ . In the special case when  $\eta_{t,H} = \eta_{t,L}$  the optimal reheater pressure is the geometric average of  $P_H$  and  $P_L$ .



Problem 8.17 The design is not possible because it violates the second law. This can be demonstrated by considering the open system situated to the left of the dashed line in the figure. This system has one inlet (in), one outlet (out), and one power interaction ( $\dot{W}$ ) with the environment.

The first law and the second law in the steady state require that

$$\dot{W} = \dot{m}(h_{in} - h_{out}) \quad (1)$$

$$\dot{S}_{gen} = \dot{m}(s_{out} - s_{in}) > 0 \quad (2)$$

We assume that the fluid is an ideal gas with constant  $c_p$  and, optimistically, that  $\dot{m}$  flows through all the piping without pressure drop. In this case eqs. (1) and (2) become

$$\dot{W} = \dot{m}c_p(T_{in} - T_{out}) \quad (1')$$

$$\dot{S}_{gen} = \dot{m}c_p \ln \frac{T_{out}}{T_{in}} \quad (2')$$

and, eliminating  $T_{out}/T_{in}$  from eq. (2'),

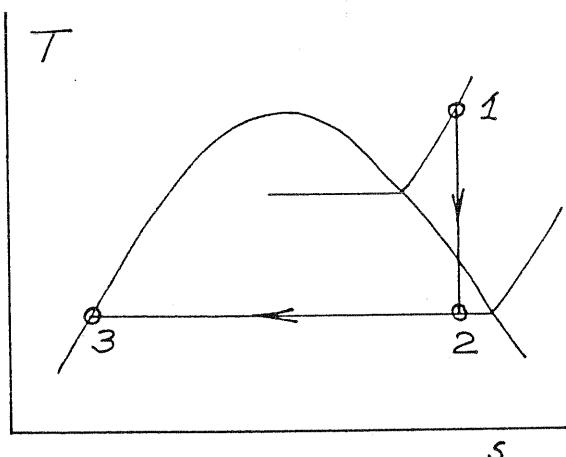
$$\dot{S}_{gen} = \dot{m}c_p \ln \left( 1 - \frac{\dot{W}}{\dot{m}c_p T_{in}} \right) \quad (3)$$

In conclusion, if  $\dot{W}$  is positive then  $\dot{S}_{gen}$  is negative: this means that the overall design violates the second law. Equation (3) also shows the only mode in which the device can function in real life, namely

$$\dot{S}_{gen} > 0, \quad \text{hence} \quad \dot{W} < 0 \quad (4)$$

In other words, it takes work to make the stream  $\dot{m}$  flow through the installation.

Problem 8.18 (a) We start with  $\dot{W}_t / \dot{m} = h_1 - h_2$ , which means that we must find  $h_2$  (or state 2). We proceed in the following sequence:



$$s_2 = s_1 = 7.1706 \text{ kJ/kgK}$$

$$x_2 = \frac{s_2 - s_f(P_2)}{s_{fg}(P_2)} = \frac{7.1706 - 1.3026}{6.0568} = 0.969$$

$$h_2 = h_f(P_2) + x_2 h_{fg}(P_2) = 2605.1 \text{ kJ/kg}$$

$$\begin{aligned} \frac{\dot{W}_t}{\dot{m}} &= h_1 - h_2 = (2860.5 - 2605.1) \text{ kJ/kg} \\ &= 255.4 \text{ kJ/kg} \end{aligned}$$

The first-law analysis of the condenser yields

$$\begin{aligned}\frac{\dot{Q}_C}{\dot{m}} &= h_2 - h_3 = (2605.1 - 417.46) \text{ kJ/kg} \\ &= 2187.6 \text{ kJ/kg}\end{aligned}$$

(b) The second law,

$$\dot{S}_{\text{gen}} = \frac{\dot{Q}_C}{T_C} + \dot{m}(s_3 - s_1) > 0$$

means, in order,

$$\begin{aligned}\frac{\dot{Q}_C}{\dot{m}} &\geq T_C(s_1 - s_3) \\ &\geq T_C(s_2 - s_3) \\ &\geq T_C(s_f + x_2 s_{fg} - s_f) \\ &\geq x_2 T_C s_{fg}\end{aligned}$$

Note that  $T_C s_{fg} = h_{fg}$  and that  $x_2 h_{fg} = h_2 - h_3 = (\dot{Q}_C / \dot{m})_{\text{part (a)}}$ . In conclusion, we have

$$\frac{\dot{Q}_C}{\dot{m}} > \left( \frac{\dot{Q}_C}{\dot{m}} \right)_{\text{part (a)}}$$


---

Problem 8.19 (a)

$$\begin{aligned}\eta_R &= \frac{\dot{W}_{\text{net}}}{\dot{Q}_H + \dot{Q}_R} = 1 - \frac{\dot{Q}_C}{\dot{Q}_H + \dot{Q}_R} \\ &= 1 - \frac{T_3 - T_4}{T_2 - T_1 + T_2 - T_2} \\ &= 1 - \frac{\frac{T_{\max}}{x\tau} - T_{\min}}{2T_{\max} - \tau T_{\min} - xT_{\max}} \\ &= 1 - \frac{\tau_{\max} - x\tau}{x\tau[(2-x)\tau_{\max} - \tau]}\end{aligned}$$

(b) When the reheater is absent we have  $x = 1$ , therefore

$$\eta_B = \eta_R (x = 1) = 1 - \frac{1}{\tau}$$

(c) For this proof we write sequentially

$$\eta_R < \eta_B$$

$$1 - \frac{\tau_{\max} - x\tau}{x\tau[(2-x)\tau_{\max} - \tau]} < 1 - \frac{1}{\tau}$$

and arrive eventually at

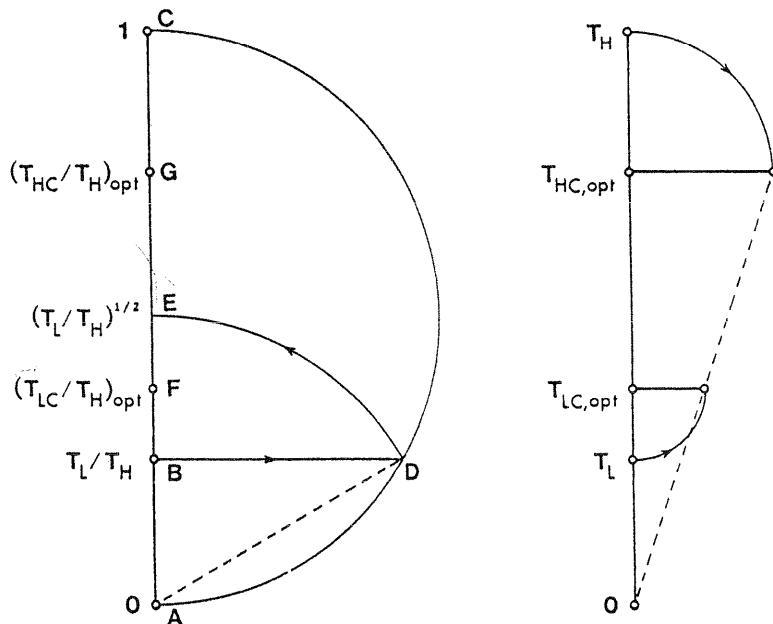
$$0 < (1 - x)^2,$$

which is always true. Now, regarding the paradox, we might have expected the opposite (i.e. that  $\eta_R > \eta_B$ ) because we learned that the introduction of one reheat increases the average heater temperature ( $T_{H,\text{avg}}$ ) in eq. (8.35),

$$\eta = 1 - \frac{T_{L,\text{avg}}}{T_{H,\text{avg}}}$$

The trouble with adding the reheat, however, is that it also increases the average cooler temperature ( $T_{L,\text{avg}}$ ): note the shift of the low pressure outlet state from (3') to (3) on the  $P_L$  isobar. That  $\eta_R$  is smaller than  $\eta_B$  means simply that the damage associated with the increase in  $T_{L,\text{avg}}$  is greater than the benefit due to the increase in  $T_{H,\text{avg}}$ .

Problem 8.20 The solution to this problem is presented in detail in Refs. 3 and 9. The graphic construction of the optimal design is reproduced below:



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Problem 8.21 We begin with the first law and the second law,

$$\dot{W} = \dot{Q}_1 - \dot{Q}_2 \quad (1)$$

$$\frac{\dot{Q}_1}{T_{1C}} = \frac{\dot{Q}_{2,\text{rev}}}{T_2} \quad (2)$$

and use Novikov's expansion irreversibility model,

$$\dot{Q}_2 = (1 + i) \dot{Q}_{2,\text{rev}} \quad (3)$$

and

$$\dot{Q}_1 = UA (T_1 - T_{1C}) \quad (4)$$

Combining eqs. (1) - (4) we obtain

$$\dot{W} = UA (T_1 - T_{1C}) \left[ 1 - (1 + i) \frac{T_2}{T_{1C}} \right] \quad (5)$$

The maximum power design is obtained by solving  $\partial\dot{W} / \partial T_{1C} = 0$ . The results are

$$T_{1C,\text{opt}} = [(1 + i) T_1 T_2]^{1/2}$$

$$\dot{W}_{\text{max}} = UA T_1 \left\{ 1 - \left[ (1 + i) \frac{T_2}{T_1} \right]^{1/2} \right\}^2$$

$$\eta = \frac{\dot{W}_{\text{max}}}{\dot{Q}_1} = 1 - \left[ (1 + i) \frac{T_2}{T_1} \right]^{1/2}$$

When the expansion is executed reversibly, the efficiency is

$$\eta = 1 - \left( \frac{T_2}{T_1} \right)^{1/2}$$

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Problem 8.22 For the two temperature gaps we write the heat transfer rate expressions

$$R \dot{Q}_1 = T_1 - T_{1C} \quad (1)$$

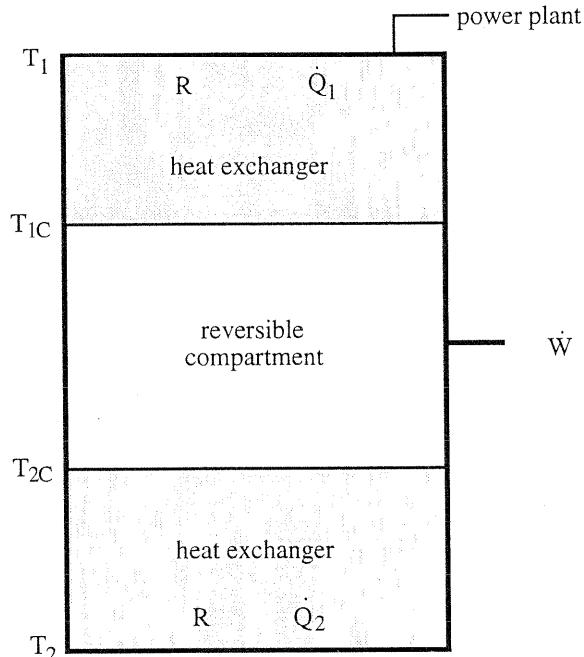
$$R \dot{Q}_2 = T_{2C} - T_2 \quad (2)$$

For the inner compartment we write the first law

$$\dot{W} = \dot{Q}_1 - \dot{Q}_2 \quad (3)$$

and the second law for reversible operation,

$$\frac{\dot{Q}_1}{T_{1C}} = \frac{\dot{Q}_2}{T_{2C}} \quad (4)$$



Combining eqs. (1), (2) and (4) we express  $T_{2C}$  in terms of  $T_{1C}$ ,

$$T_{2C} = \frac{T_2 T_{1C}}{2T_{1C} - T_1} \quad (5)$$

Equations (3), (1), (4) and (5) can be combined to express  $\dot{W}$  as a function of  $T_{1C}$ :

$$\dot{W} = \frac{1}{R} (T_1 - T_{1C}) \left( 1 - \frac{T_2}{2T_{1C} - T_1} \right) \quad (6)$$

Solving  $\partial\dot{W} / \partial T_{1C} = 0$  we obtain

$$T_{1C,\text{opt}} = \frac{1}{2} \left[ T_1 + (T_1 T_2)^{1/2} \right] \quad (7)$$

Working back, we derive the rest of the maximum-power solution:

$$T_{2C,\text{opt}} = \frac{1}{2} \left[ T_2 + (T_1 T_2)^{1/2} \right] \quad (8)$$

$$\left( \frac{T_{1C}}{T_{2C}} \right)_{\text{opt}} = \left( \frac{T_1}{T_2} \right)^{1/2} \quad (9)$$

$$\dot{W}_{\max} = \frac{T_2}{2R} \left[ \left( \frac{T_1}{T_2} \right)^{1/2} - 1 \right]^2 \quad (10)$$

The efficiency at maximum power is

$$\eta = \frac{\dot{W}_{\max}}{\dot{Q}_{1,\text{opt}}} = 1 - \left( \frac{T_2}{T_1} \right)^{1/2} \quad (11)$$

The entropy generation rates of the two heat exchangers (temperature gaps) are

$$\dot{S}_{\text{gen},1} = \dot{Q}_1 \left( \frac{1}{T_{1C}} - \frac{1}{T_1} \right) \quad (12)$$

$$\dot{S}_{\text{gen},2} = \dot{Q}_2 \left( \frac{1}{T_2} - \frac{1}{T_{2C}} \right) \quad (13)$$

At maximum power, the cold-end heat exchanger generates more entropy,

$$\frac{\dot{S}_{\text{gen},1}}{\dot{S}_{\text{gen},2}} = \dots = \left( \frac{T_2}{T_1} \right)^{1/2} < 1 \quad (14)$$

Problem 8.23 The total system shown in the figure has three compartments in which entropy is being generated,

$$\dot{S}_{\text{gen}} = \dot{Q}_H \left( \frac{1}{T_{HC}} - \frac{1}{T_H} \right) + \dot{Q}_L \left( \frac{1}{T_L} - \frac{1}{T_{LC}} \right) + \dot{Q}_e \left( \frac{1}{T_L} - \frac{1}{T_H} \right) \quad (1)$$

Substituting  $\dot{Q}_e = \dot{Q} - \dot{Q}_H$ , and multiplying eq. (1) by  $T_L$  we obtain

$$T_L \dot{S}_{\text{gen}} = - \underbrace{\dot{Q}_H \left( 1 - \frac{T_{LC}}{T_{HC}} \right)}_{\dot{W}(\text{actual})} + \underbrace{\dot{Q} \left( 1 - \frac{T_L}{T_H} \right)}_{\dot{W}_{\text{Carnot}} (\text{constant})} \quad (2)$$

which is the same as  $T_L \dot{S}_{\text{gen}} = \dot{W}_{\text{lost}}$ . Equation (2) shows that to maximize the power output  $\dot{W}$ (actual) is the same as minimizing the total rate of entropy generation. The maximization of  $\dot{W}$  in this model was illustrated in Problem 8.22, except that this time (in general) the thermal conductances are different,  $(UA)_H \neq (UA)_L$ .

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Problem 8.24 For brevity, we write  $C_H$  and  $C_L$  for the two capacity flow rates,

$$C_H = (\dot{m}c_p)_H \quad C_L = (\dot{m}c_p)_L \quad (1)$$

The three compartments of the power plant model are described by the following statements:

$$\dot{Q}_H = C_H (T_H - T_{HC}) \quad (2)$$

$$\dot{Q}_L = C_L (T_{LC} - T_L) \quad (3)$$

$$\frac{\dot{Q}_H}{T_{HC}} = \frac{\dot{Q}_L}{T_{LC}} \quad (4)$$

$$\dot{W} = \dot{Q}_H \left( 1 - \frac{T_{LC}}{T_{HC}} \right) \quad (5)$$

Fixed are the temperature extremes ( $T_H, T_L$ ) and the capacity flow rates ( $C_H, C_L$ ). The objective of the analysis is to maximize  $\dot{W}$  with respect to the design variable  $T_{HC}$  (or, conversely,  $T_{LC}$ ). We use eqs. (2) - (4) to eliminate  $\dot{Q}_H$ ,  $\dot{Q}_L$  and  $T_{LC}$ , such that eq. (5) becomes

$$\frac{\dot{W}}{C_H} = T_H - T_{HC} + \frac{T_L \left( 1 - \frac{T_H}{T_{HC}} \right)}{1 - \frac{C_H}{C_L} \left( \frac{T_H}{T_{HC}} - 1 \right)} \quad (6)$$

Solving  $\partial \dot{W} / \partial T_{HC} = 0$  we obtain

$$T_{HC,\text{opt}} = \frac{\frac{C_H}{C_L} T_H + (T_H T_L)^{1/2}}{\frac{C_H}{C_L} + 1} \quad (7)$$

Equation (7) can be combined with eqs. (2) - (4) to obtain the corresponding cold-end temperature of the reversible compartment,

$$T_{LC,opt} = \left( \frac{T_L}{T_H} \right)^{1/2} \frac{\frac{C_H}{C_L} T_H + (T_H T_L)^{1/2}}{\frac{C_H}{C_L} + 1} \quad (8)$$

which also has the property

$$\frac{T_{LC,opt}}{T_{HC,opt}} = \left( \frac{T_L}{T_H} \right)^{1/2} \quad (9)$$

The maximum power output is obtained by substituting eqs. (7, 8) into eq. (6):

$$\dot{W}_{max} = \frac{(T_H^{1/2} - T_L^{1/2})^2}{\frac{1}{C_H} + \frac{1}{C_L}} \quad (10)$$

This result shows that the maximum power can be increased further by increasing  $C_H$  and/or  $C_L$ . Equation (10) also shows that the maximum power output is limited by the smaller of the two capacity flow rates: to see this, imagine that  $C_H \ll C_L$  such that eq. (10) approaches the limit

---


$$\dot{W}_{max} \approx C_H (T_H^{1/2} - T_L^{1/2})^2$$

Problem 8.25 We continue to write  $C_H$  and  $C_L$  for the two capacity flow rates,  $(\dot{m}c_p)_H$  and  $(\dot{m}c_p)_L$ . Each pump requires a power input that varies as

$$\dot{W}_p = \frac{\dot{m}\Delta P}{\rho\eta_{II}} \quad (1)$$

where  $\eta_{II}$  is the second law efficiency of the pump (assumed constant; also known as pump efficiency,  $\eta_p$ ),  $\dot{m}$  is the mass flow rate,  $\rho$  is the density of the respective incompressible liquid, and  $\Delta P$  is the pressure drop along the duct,

$$\Delta P = r \dot{m}^n \quad (2)$$

In eq. (2) we wrote  $r$  for the effective resistance to fluid flow, and  $n$  to cover the range from laminar flow ( $n = 1$ ) to turbulent flow in the fully rough regime ( $n \geq 2$ ). We treat  $r$  and  $n$  as two constants that are known from the design of the duct. The net power output of the installation is

$$\dot{W}_{net} = \dot{W}_{max} - \dot{W}_{p,H} - \dot{W}_{p,L} \quad (3)$$

where (see the preceding problem),

$$\dot{W}_{\max} = \frac{(T_H^{1/2} - T_L^{1/2})^2}{\frac{1}{C_H} + \frac{1}{C_L}} \quad (4)$$

and, according to eqs. (1, 2),

$$\dot{W}_{p,H} = \left( \frac{r}{\rho \eta_{II}} \right)_H \dot{m}_H^{n+1} \quad (5)$$

$$\dot{W}_{p,L} = \left( \frac{r}{\rho \eta_{II}} \right)_L \dot{m}_L^{n+1} \quad (6)$$

Equation (3) can be written as

$$\dot{W}_{\text{net}} = \frac{\Delta T C_H C_L}{C_H + C_L} - R_H C_H^{n+1} - R_L C_L^{n+1} \quad (7)$$

where  $\Delta T$ ,  $R_H$  and  $R_L$  are known constants:

$$\Delta T = (T_H^{1/2} - T_L^{1/2})^2 \quad (8)$$

$$R_H = \left( \frac{r}{\rho \eta_{II} c_P^{n+1}} \right)_H \quad (9)$$

$$R_L = \left( \frac{r}{\rho \eta_{II} c_P^{n+1}} \right)_L \quad (10)$$

The net power output has maxima with respect to both  $C_H$  and  $C_L$ . Solving the system  $\partial \dot{W}_{\text{net}} / \partial C_H = 0$  and  $\partial \dot{W}_{\text{net}} / \partial C_L = 0$  we obtain

$$C_{H,\text{opt}} = \left[ \frac{\Delta T}{(n+1) R_H (b+1)^2} \right]^{1/n} \quad (11)$$

$$C_{L,\text{opt}} = \left[ \frac{\Delta T}{(n+1) R_L \left( \frac{1}{b} + 1 \right)^2} \right]^{1/n} \quad (12)$$

where  $b$  is shorthand for

$$b = \left( \frac{R_L}{R_H} \right)^{1/(n+2)} \quad (13)$$

Equations (11, 12) show that the optimal capacity flow rates have the property

$$\frac{C_{H,opt}}{C_{L,opt}} = \left(\frac{R_L}{R_H}\right)^{1/(n+2)} \quad (14)$$

The design implication of eq. (14) is that the "easier" flow path (the lower R value) should carry the larger stream (the larger C value). The twice-maximized net power output [eq. (7)] that corresponds to  $C_{H,opt}$  and  $C_{L,opt}$  is

$$\dot{W}_{net,max} = n \left( \frac{\Delta T}{n+1} \right)^{(n+1)/n} [R_H^{1/(n+2)} + R_L^{1/(n+2)}]^{-(n+2)/n} \quad (15)$$

Equation (15) shows that  $\dot{W}_{net,max}$  is larger when  $R_H$  and/or  $R_L$  are smaller, and that the maximum net power is limited by the larger of the two flow resistances.

Note that a small R value means a small flow resistance  $r$  and/or a large pump second law efficiency  $\eta_{II}$ . To achieve these desirable objectives requires engineering effort and capital investment. One way to account for the economic aspect of trying to decrease  $R_H$  and  $R_L$  is through the economic constraint

$$p_H \frac{1}{R_H} + p_L \frac{1}{R_L} = \text{constant} \quad (16)$$

where  $p_H$  and  $p_L$  are the prices of the respective (desirable) flow conductances ( $R_H^{-1}$ ,  $R_L^{-1}$ ). We assume that  $p_H$  and  $p_L$  are known constants, and maximize  $\dot{W}_{net,max}$  of eq. (15) with respect to  $R_H$  (or  $R_L$ ) subject to constraint (16). This operation leads to the optimization rule

$$\frac{R_{L,opt}}{R_{H,opt}} = \left(\frac{p_L}{p_H}\right)^{(n+2)/(n+3)}$$

which says that the two R values should be roughly proportional to the price ratio. In other words, the duct that has the higher price should have the lower conductance (higher flow resistance).

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**Problem 8.26** The net power produced is

$$\dot{W} = \dot{m}(h_3 - h_4) - \dot{m}(h_2 - h_1) \quad (1)$$

$$\frac{\dot{W}}{\dot{m}} = \eta_t (h_3 - h_{4s}) - \frac{1}{\eta_t} (h_{2s} - h_1) \quad (2)$$

$$\frac{\dot{W}}{\dot{m}c_p} = \eta_t T_3 \left( 1 - \frac{T_{4s}}{T_3} \right) - \eta_c^{-1} T_1 \left( \frac{T_{2s}}{T_1} - 1 \right) \quad (3)$$

For brevity, we write  $\tau$  for the isentropic temperature ratio

$$\tau = \frac{T_{2s}}{T_1} = \frac{T_3}{T_{4s}} = \left( \frac{P_H}{P_L} \right)^{R/c_p} \quad (4)$$

where we also used  $ds = 0$  for the adiabatic and reversible flows 1-2s and 3-4s:

$$ds = c_p \frac{dT}{T} - R \frac{dP}{P} \quad (5)$$

$$0 = c_p \frac{dT}{T} - R \frac{dP}{P} \quad (6)$$

$$\frac{T}{P^{R/c_p}} = \text{constant} \quad (7)$$

Equation (4) becomes

$$\frac{\dot{W}}{\dot{m}c_p T_1} = \eta_t \tau_{\max} \left( 1 - \tau^{-1} \right) - \eta_c^{-1} (\tau - 1) \quad (8)$$

where

$$\tau_{\max} = \frac{T_3}{T_1} \quad (9)$$

The power output can be maximized with respect to  $\tau$  (or  $P_H/P_L$ ):

$$\tau_{\text{opt}} = (\eta_t \eta_c \tau_{\max})^{1/2} \quad (10)$$

$$\left( \frac{P_H}{P_L} \right)_{\text{opt}} = (\eta_t \eta_c \tau_{\max})^{c_p/2R} \quad (11)$$

$$\frac{\dot{W}_{\max}}{\dot{m} c_p T_l} = \left[ (\eta_t \tau_{\max})^{1/2} - \eta_c^{-1/2} \right]^2 \quad (12)$$


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**Problem 8.27** (a) The heat engine operates irreversibly, because one of its components operates irreversibly (that component is the conducting strut, which is the locus of heat transfer  $Q_c$  across a finite temperature difference  $T_H - T_L$ ).

(b) The entropy generated by the heat engine defined by the dash-line rectangle is

$$S_{\text{gen, heat engine}} = S_{\text{gen,strut}} + S_{\text{gen, Carnot engine}}$$

$$\begin{aligned} &= \frac{Q_C}{T_L} - \frac{Q_C}{T_H} \\ &= C(T_H - T_L) \left( \frac{1}{T_L} - \frac{1}{T_H} \right). \end{aligned}$$

This quantity is positive because  $T_H > T_L$ , or vice versa:  $T_H$  must be greater than  $T_L$ , because the total entropy generated must be positive.

(c) The mechanical power output of the heat engine is

$$\begin{aligned} W &= \left( \begin{array}{l} \text{heat input to the} \\ \text{Carnot engine} \end{array} \right) \cdot \left( \begin{array}{l} \text{efficiency, of the} \\ \text{Carnot engine} \end{array} \right) \\ &= [Q_H - C(T_H - T_L)] \left( 1 - \frac{T_L}{T_H} \right) \end{aligned}$$

By maximizing  $W$  with respect to  $T_H$ ,

$$\frac{\partial W}{\partial T_H} = -C \left( 1 - \frac{T_L}{T_H} \right) + [Q_H - C(T_H - T_L)] \frac{T_L}{T_H} = 0$$

we obtain

$$T_{H,\text{opt}} = \left( T_L^2 + \frac{Q_H T_L}{C} \right)^{1/2}$$

This result makes sense: it states that an optimum  $T_H$  level exists, because at  $T_H < T_{H,\text{opt}}$  the Carnot efficiency is too low, while at  $T_H > T_{H,\text{opt}}$  too much of the fixed  $Q_H$  bypasses the Carnot engine via the thermal insulation of the engine.

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**Problem 8.28** The first-law efficiency of the gas turbine cycle is

$$\eta_1 = \frac{W_1}{Q_1} \quad (1)$$

The first-law efficiency of the steam turbine cycle (which is driven by the heat rejected from the gas turbine cycle,  $Q_2 = Q_1 - W_1$ ) is

$$\eta_2 = \frac{W_2}{Q_2} \quad (2)$$

The corresponding efficiency of the combined cycle is obtained by combining these efficiency relations,

$$\eta = \frac{W_1 + W_2}{Q_1} = \eta_1 + \eta_2 (1 - \eta_1) \quad (3)$$

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## Chapter 9

### SOLAR POWER

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Problem 9.1 The collector with linear heat loss to the ambient is described by the following equations from the text:

$$Q = \eta_c Q^* \quad (9.116)$$

$$\eta_c = \frac{\theta_{\max} - \theta}{\theta_{\max} - 1} \quad (9.116')$$

where  $\theta = T_c / T_0$  and  $\theta_{\max} = T_{c,\max} / T_0$ . The reversible refrigeration cycle without work input is governed by the first law and the second law:

$$Q + Q_L = Q_r$$

$$\frac{Q}{T_c} + \frac{Q_L}{T_L} = \frac{Q_r}{T_0}$$

where  $Q_r$  is the heat rejected by the cycle to the ambient  $T_0$ . Eliminating  $Q_r$  between the two laws yields

$$Q_L \left( \frac{1}{T_L} - \frac{1}{T_0} \right) = \left( \frac{1}{T_0} - \frac{1}{T_c} \right) \eta_c Q^*$$

The right-hand side of this last equation can be maximized with respect to  $T_c$  using Eq. (9.116'), and the result is

$$T_{c,\text{opt}} = (T_{c,\max} T_0)^{1/2}$$


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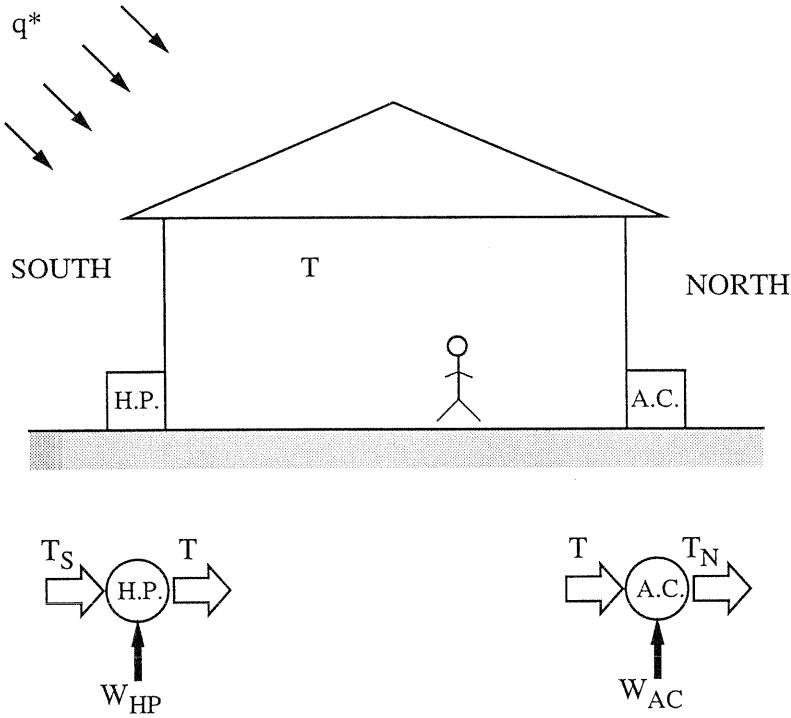
Problem 9.2 On a day that is not too windy, the air bathing the south side of the house is warmer than on the north side,  $T_S > T_N$ . The air conditioner (AC) is needed in the summer, when  $T < T_S$  or  $T_N$ . There are two temperature gaps that can be filled by the AC unit,  $(T_S - T)$  or  $(T_N - T)$ . The AC power requirement will be smaller in the case of the smaller temperature gap,  $T_N - T$ , therefore the AC unit should be installed on the north side.

The heat pump works when

$$T > T_S \text{ or } T_N,$$

The smaller temperature gap is  $(T - T_N)$ , therefore, the heat pump should be installed on the south side.

These conclusions have been noted in the sketch. They certainly depend on the hemisphere: the present drawing was made for the northern hemisphere. For the southern hemisphere, the equivalent drawing can be obtained by switching the position of the SOUTH and NORTH labels.



Problem 9.3 We begin with the first law and the second law for the reversible heat pump,

$$Q_r = Q_c + Q_L \quad (1)$$

$$\frac{Q_r}{T_r} - \frac{Q_c}{T_c} - \frac{Q_L}{T_0} = 0 \quad (2)$$

and continue with the first law and heat loss expressions for the collector:

$$Q_c = Q^* - Q_0 \quad (3)$$

$$Q_0 = U_c A_c (T_c - T_0) \quad (4)$$

Next, we eliminate  $Q_L$ ,  $Q_c$  and  $Q_0$  between eqs. (1) - (4) and obtain the room heat input  $Q_r$  as a function of the collector temperature  $T_c$ :

$$Q_r \left( \frac{1}{T_0} - \frac{1}{T_r} \right) = [Q^* - U_c A_c (T_c - T_0)] \left( \frac{1}{T_0} - \frac{1}{T_c} \right) \quad (5)$$

Note that the factor in parentheses on the left side of eq. (5) is a constant. Finally, we solve the equation  $\partial Q_r / \partial T_c = 0$  to find the optimal collector temperature,

$$T_{c,\text{opt}} = (T_0 T_{c,\text{max}})^{1/2} \quad (6)$$

where  $T_{c,\text{max}}$  is the stagnation temperature of the collector (when  $Q_c = 0$ ):

$$T_{c,\text{max}} = T_0 + \frac{Q^*}{U_c A_c} \quad (7)$$


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**Problem 9.4** Consider a refrigerator where the heat input  $Q_H$  is provided at the temperature level  $T_H$  by a flat plate solar collector. The relation between  $Q_H$  and the collector temperature  $T_H$  can be expressed as

$$Q_H = A_C G_T [a - b(T_H - T_0)] \quad (1)$$

where  $A_C$  is the collector area,  $G_T$  is the irradiance at the collector surface and  $a$  and  $b$  are two constants that can be calculated as shown in Sokolov and Hershgal, *Solar Energy*, Vol. 50, 1993, pp. 507-516. Equation (1) represents a collector with partial heat loss to the ambient. The group  $[a - b(T_H - T_0)]$  is the collector efficiency, and

$$T_{st} = T_0 + \frac{a}{b} \quad (2)$$

is the stagnation (i.e. the ceiling) temperature of the collector. When  $T_H = T_{st}$  the heat input  $Q_H$  is zero.

The rest of the refrigerator model is described by

$$Q_0 = (UA)_0(T_{0C} - T_0) \quad (3)$$

$$Q_L = (UA)_L(T_L - T_{LC}) \quad (4)$$

$$\frac{Q_H}{T_{HC}} + \frac{Q_L}{T_{LC}} = \frac{Q_0}{T_{0C}} \quad (5)$$

$$Q_H + Q_L = Q_0 \quad (6)$$

The total UA inventory is to be shared by the condenser and the evaporator,

$$UA = (UA)_0 + (UA)_L \quad (7)$$

We introduce the dimensionless variables

$$\tau_{0C} = \frac{T_{0C}}{T_0}, \quad \tau_{LC} = \frac{T_{LC}}{T_0} \quad (8)$$

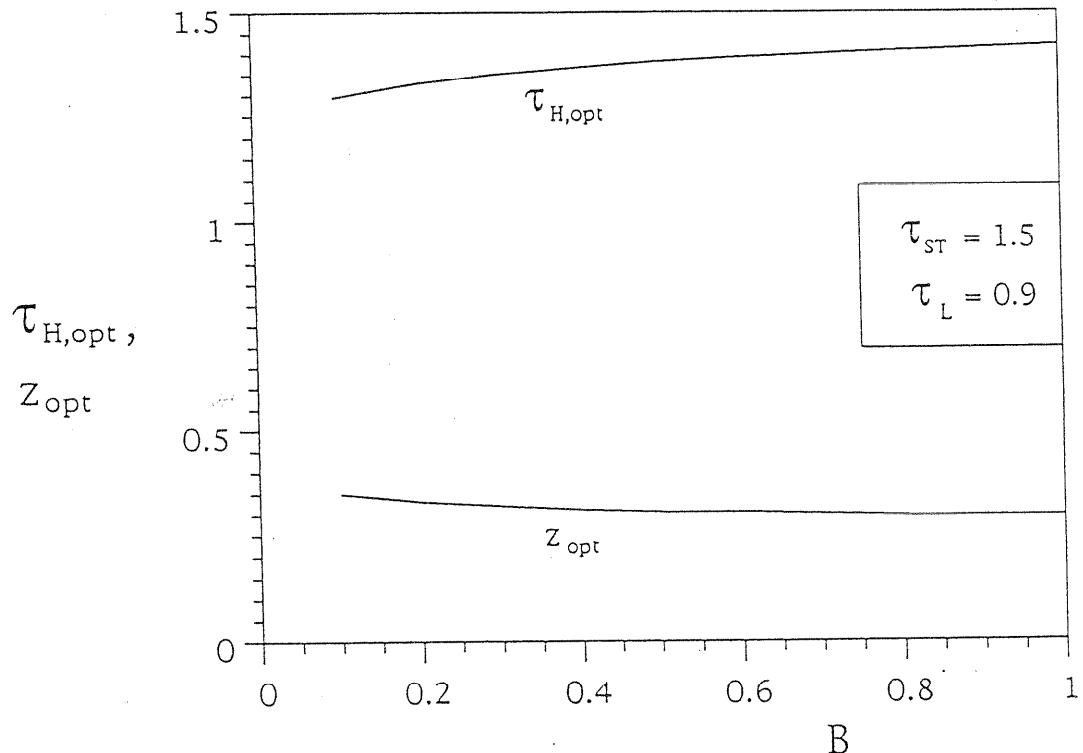
$$\tilde{Q}_H = \frac{Q_H}{UAT_0}, \quad \tilde{Q}_0 = \frac{Q_0}{UAT_0}, \quad \tilde{Q}_L = \frac{Q_L}{UAT_0} \quad (9)$$

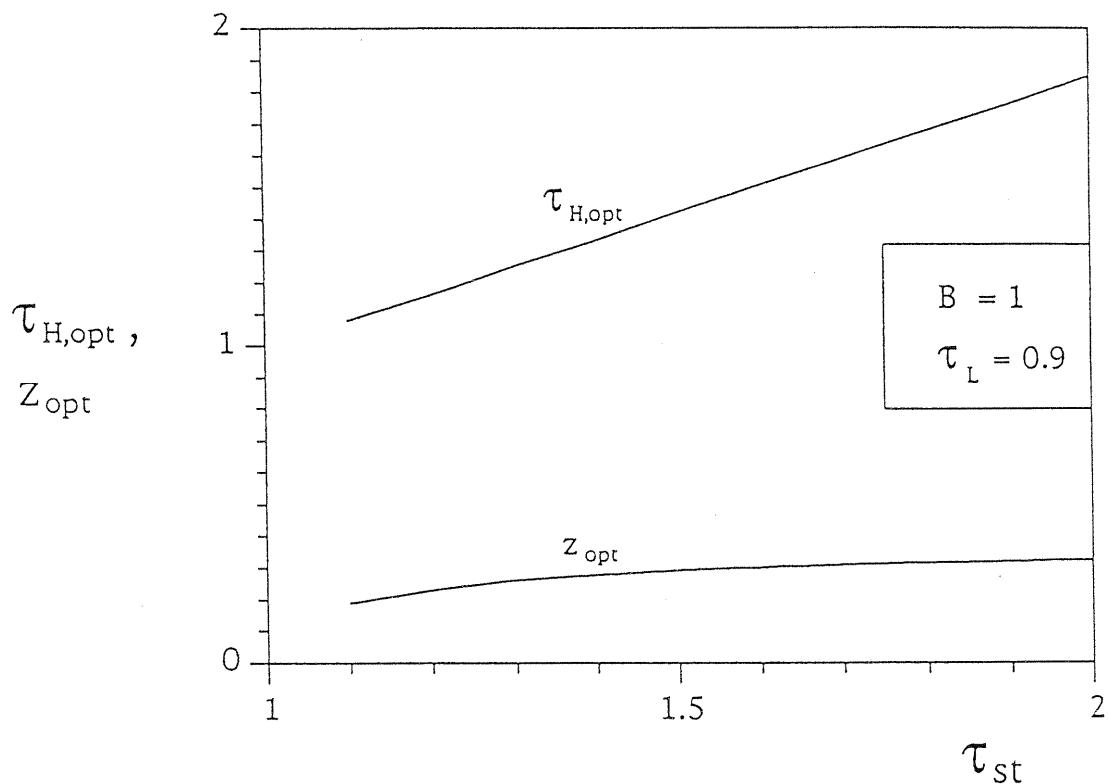
$$\tau_{st} = \frac{T_{st}}{T_0}, \quad B = \frac{bA_c G_T}{UA} \quad (10)$$

and the evaporator conductance allocation ratio

$$z = \frac{(UA)_L}{UA}, \quad 1 - z = \frac{(UA)_0}{UA} \quad (11)$$

By solving eqs. (1) and (3) - (6) numerically, it is possible to determine  $\tilde{Q}_L$  for a given set of values for  $B$ ,  $\tau_H$ ,  $\tau_L$ ,  $\tau_{st}$  and  $z$ . We can vary  $\tau_H$  and  $z$  until we locate the pair  $(\tau_{H,opt}, z_{opt})$  that maximizes  $\tilde{Q}_L$ . The resulting  $\tau_{H,opt}$  and  $z_{opt}$  values are functions of three parameters,  $B$ ,  $\tau_{st}$  and  $\tau_L$ . Some of these results are presented in the attached figures.





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Problem 9.5 (a) The stream flows isothermally

$$T_{in} = T_{out} = T \quad (1)$$

Second, the flow system operates reversibly, i.e., with zero entropy generation

$$\dot{S}_{gen} = \dot{m}(s_{out} - s_{in}) - \frac{\dot{Q}}{T} = 0 \quad (2)$$

Next, we use what is always true. Written for the same system, the first law requires that

$$\dot{Q} - \dot{W} + \dot{m}(h_{in} - h_{out}) = 0 \quad (3)$$

For ideal gases,  $h$  is a function of  $T$  only, and Eq. (1) means that

$$h_{in} = h_{out} \quad (4)$$

Consequently, the first law (3) yields

$$\dot{W} = \dot{Q} \quad (5)$$

The second law (2) yields

$$\dot{Q} = \dot{m}T(s_{out} - s_{in}) \quad (6)$$

Because the stream carries a single-phase fluid, we have

$$dh = Tds + vdp \quad (7)$$

for which

$$dh = 0 \quad (\text{ideal gas, } T = \text{constant}) \quad (8)$$

$$v = RT/P \quad (\text{ideal gas}) \quad (9)$$

Equations (7)-(9) yield

$$s_{out} - s_{in} = -R \ln \frac{P_{out}}{P_{in}} \quad (10)$$

and, in combination with Eqs. (5) and (6):

$$\dot{W} = \dot{Q} = \dot{m}RT \ln \frac{P_{in}}{P_{out}} \quad (11)$$

(b) The cycle executed by the atmospheric air stream has four processes:

1-2 isothermal heating and expansion, at  $T_H$

2-3 isobaric cooling, at  $P_L$

3-4 isothermal cooling and compression, at  $T_L$

4-1 isobaric heating, at  $P_H$

In the ideal limit, the thermal contact between the two isobaric streams is perfect,

$$T_1 = T_2 \quad T_3 = T_4$$

and the regenerative heating released by the  $P_L$  stream is absorbed completely by the  $P_H$  stream,

$$\dot{m}(h_2 - h_3) = \dot{m}(h_1 - h_4)$$

According to the solution to the preceding problem, we can write that

$$\dot{Q}_H = \dot{W}_H = \dot{m}RT_H \ln \frac{P_H}{P_L}$$

$$\dot{Q}_L = \dot{W}_L = \dot{m}RT_L \ln \frac{P_H}{P_L}$$

The net power produced by this flow system is

$$\dot{W}_{\text{net}} = \dot{W}_H - \dot{W}_L = \dot{m}R(T_H - T_L) \ln \frac{P_H}{P_L}$$

The heat engine efficiency is the Carnot efficiency:

$$\eta = \frac{\dot{W}_{\text{net}}}{\dot{Q}_H} = 1 - \frac{T_L}{T_H}$$

This is not a surprise, because the model of this flow system is based on the assumption that irreversible flows are absent. The pressure drop from 4 to 1 is zero, and so is the pressure drop from 2 to 3. Furthermore the regenerative (internal) heat transfer between the  $P_L$  stream (2-3) and the  $P_H$  stream (4-1) occurs with zero temperature difference from stream to stream.

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**Problem 9.6** For a model with one temperature ( $T_1$ ) for the earth surface, we use the start of the analysis of section 9.9 with  $T_H = T_1$  and  $T_L = T_1$ . The heat current received from the sun,

$$q_s = A_p (1 - \rho) f \sigma T_s^4 \quad (1)$$

equals the heat current rejected into space,

$$q_\infty = A (1 - \gamma) \sigma T_1^4 \quad (2)$$

Setting  $q_s = q_\infty$ , and using  $A_p = \pi R^2$  and  $A = 4\pi R^2$ , we obtain the steady-state temperature

$$T_1 = T_s \left( \frac{f}{4} \right)^{1/4} \left( \frac{1 - \rho}{1 - \gamma} \right)^{1/4} \quad (3)$$

If we use the data given in section 9.9,

$$\begin{aligned} T_s &= 5762 \text{ K} & \rho_1 &= 0.3 \\ f &= 2.16 \times 10^{-5} & \gamma_1 &= 0.4 \end{aligned} \quad (4)$$

we find the steady-state temperature

$$T_1 = 288.7 \text{ K} \quad (5)$$


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**Problem 9.7** Because of human activity, the atmospheric radiative properties have changed: the albedo ( $\rho_2$ ) and the greenhouse factor ( $\gamma_2$ ) are larger than their older values ( $\rho_1, \gamma_1$ ). The earth surface temperature has not had time to change: it is still at the  $T_1$  level associated with  $\rho_1$  and  $\gamma_1$ . This creates an imbalance between the heat input

$$q_{s2} = \pi R^2 (1 - \rho_2) f \sigma T_s^4 \quad (1)$$

and the rejected heat current,

$$q_{\infty 2} = 4\pi R^2 (1 - \gamma_2) \sigma T_1^4 \quad (2)$$

The original solar heat input was

$$q_{s1} = \pi R^2 (1 - \rho_1) f \sigma T_s^4 \quad (3)$$

Writing  $\Delta q = q_{s2} - q_{\infty 2}$  for the imbalance current, we use the above equations to nondimensionalize  $\Delta q$  as

$$\frac{\Delta q}{q_{s1}} = \frac{1 - \rho_2}{1 - \rho_1} - \frac{1 - \gamma_2}{1 - \gamma_1} \quad (4)$$

The current global-warming literature reports that  $\Delta q / (4\pi R^2) = 0.85 \text{ W/m}^2$ . The original solar heat flux is

$$\frac{q_{s1}}{4\pi R^2} = \frac{1}{4}(1 - \rho_1)f\sigma T_s^4 \quad (5)$$

$$= 222.46 \text{ W/m}^2 \quad (\rho_1 = 0.3) \quad (6)$$

such that eq. (4) becomes a relationship between  $\rho_2$  and  $\gamma_2$  (a combination of the two changes that together cause  $\Delta q$ ):

$$0.00382 = \frac{1 - \rho_2}{0.7} - \frac{1 - \gamma_2}{0.6} \quad (7)$$

We are interested in the original steady-state temperature ( $T_1, \rho_1, \gamma_1$ ) and the final one ( $T_2, \rho_2, \gamma_2$ ). For the first, we set  $q_{s1} = q_{\infty 1}$ ,

$$\pi R^2 (1 - \rho_1) f \sigma T_s^4 = 4\pi R^2 (1 - \gamma_1) \sigma T_1^4 \quad (8)$$

and obtain

$$T_1 = T_s \left( \frac{f}{4} \right)^{1/4} \left( \frac{1 - \rho_1}{1 - \gamma_1} \right)^{1/4} \quad (9)$$

For the final steady-state temperature, we use the same energy current continuity argument,

$$\pi R^2 (1 - \rho_2) f \sigma T_s^4 = 4\pi R^2 (1 - \gamma_2) \sigma T_2^4 \quad (10)$$

$$T_2 = T_s \left( \frac{f}{4} \right)^{1/4} \left( \frac{1 - \rho_2}{1 - \gamma_2} \right)^{1/4} \quad (11)$$

Combining eqs. (9) and (11), we find

$$\frac{T_2 - T_1}{T_1} \left( \frac{1 - \rho_2}{0.7} \right)^{1/4} \left( \frac{0.6}{1 - \gamma_2} \right)^{1/4} - 1 \quad (12)$$

For analytical ease, we assume small changes in  $\rho$  and  $\gamma$ :

$$\rho_2 = 0.3 + \beta \quad \gamma_2 = 0.4 + \varepsilon \quad (13)$$

where  $\beta \ll 0.3$  and  $\varepsilon \ll 0.4$ . This means that eqs. (7) and (12) become

$$0.00382 = -\frac{\beta}{0.7} + \frac{\varepsilon}{0.6} \quad (14)$$

$$\frac{T_2 - T_1}{T_1} \cong \frac{1}{4} \left( -\frac{\beta}{0.7} + \frac{\varepsilon}{0.6} \right) \quad (15)$$

which yield

$$\frac{T_2 - T_1}{T_1} \cong 0.000955$$

and

$$T_2 - T_1 \cong 0.3 \text{ K.}$$

**Problem 9.8** We rely on the  $(T_H, T_L)$  model of section 9.9, and nondimensionalize all the temperatures by using the temperature scale

$$T_{\text{scale}} = \left( \frac{f}{4} \right)^{1/4} T_s = \left( \frac{2.16 \times 10^{-5}}{4} \right)^{1/4} 5676 \text{ K} = 273.6 \text{ K} \text{ (hey, } 0^\circ \text{C)}$$

hence

$$\tau_H = \frac{T_H}{T_{\text{scale}}} \quad \tau_L = \frac{T_L}{T_{\text{scale}}}$$

Equation (9.165) becomes

$$\tau_H^4 + \left( \frac{1}{x} - 1 \right) \tau_L^4 = \frac{1 - \rho}{1 - \gamma} \quad (1)$$

We invoke eq. (1) for the initial steady state  $(\tau_{H1}, \tau_{L1}, \rho_1, \gamma_1)$ ,

$$\tau_{H1}^4 + \left( \frac{1}{x} - 1 \right) \tau_{L1}^4 = \frac{1 - \rho_1}{1 - \gamma_1} \quad (2)$$

and the final steady state

$$\tau_{H2}^4 + \left( \frac{1}{x} - 1 \right) \tau_{L2}^4 = \frac{1 - \rho_2}{1 - \gamma_2} \quad (3)$$

where we have assumed that the optimized partitioning of the earth surface ( $x = 0.35$ ) remains unchanged. Next, we assume that the temperature changes are relatively small,

$$\tau_{H2} = \tau_{H1} + \lambda_H \quad (\lambda_H \ll \tau_{H1}) \quad (4)$$

$$\tau_{L2} = \tau_{L1} + \lambda_L \quad (\lambda_L \ll \tau_{L1}) \quad (5)$$

Combining eqs. (2) - (5), and using

$$\rho_2 = 0.3 + \beta \quad \gamma_2 = 0.4 + \varepsilon \quad (6)$$

we obtain, after linearization,

$$4\tau_{H1}^3 (\lambda_H + 0.385 \lambda_L) \approx \frac{0.7}{0.6} \left( -\frac{\beta}{0.7} + \frac{\varepsilon}{0.6} \right) \quad (7)$$

A second relation between  $\lambda_H$  and  $\lambda_L$  follows from eq. (9.179) in the text, which means this:

$$(\text{constant}) \frac{(\tau_H - \tau_L)^{3/2}}{1 - \gamma} \sim (1 - x) \tau_L^4 \quad (8)$$

We write eq. (8) twice, for the initial steady state ( $\gamma_1, \tau_{H1}, \tau_{L1}$ ) and the final steady state ( $\gamma_2, \tau_{H2}, \tau_{L2}$ ), divide the two equations, and linearize:

$$\left( 1 + \frac{\lambda_H - \lambda_L}{\tau_{H1} - \tau_{L1}} \right)^{3/2} \left( 1 + \frac{\varepsilon}{0.6} \right) \sim 1 + 4 \frac{\lambda_L}{\tau_{L1}} \quad (9)$$

By using the  $T_H$  and  $T_L$  values listed in Table 9.1,  $T_{H1} = 260.5$  K and  $T_{L1} = 189.5$  K, we find  $\tau_{L1} = 0.695$  and  $\tau_{H1} - \tau_{L1} = 0.26$ . For the final linearization step we assume that

$$\frac{\lambda_H - \lambda_L}{0.26} \ll 1 \quad (10)$$

such that eq. (9) becomes

$$\lambda_H - 2\lambda_L + 0.288\varepsilon \approx 0 \quad (11)$$

Equations (7) and (11) form a system that can be solved for  $\lambda_H$  and  $\lambda_L$ :

$$\lambda_H \approx -0.406\beta + 0.426\varepsilon \quad (12)$$

$$\lambda_L \approx -0.203\beta + 0.357\varepsilon \quad (13)$$

For the wind speed we use eq. (9.171), which shows that  $u$  is proportional to  $(T_H - T_L)^{1/2}$ , therefore

$$\frac{u_2}{u_1} = \left( \frac{\tau_{H1} + \lambda_H - \tau_{L1} - \lambda_L}{\tau_{H1} - \tau_{L1}} \right)^{1/2} \quad (14)$$

$$\frac{u_2 - u_1}{u_1} \approx \frac{1}{2} \left( \frac{\lambda_H - \lambda_L}{\tau_{H1} - \tau_{L1}} \right) = 1.93 (-0.203\beta + 0.069\varepsilon) \quad (15)$$

In conclusion, an increase in albedo ( $\beta$ ) leads to slower winds, while a jump in the greenhouse factor ( $\varepsilon$ ) induces faster winds. The two effects cancel each other if

$$\frac{\beta}{\varepsilon} \approx \frac{1}{3} \quad (16)$$


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**Problem 9.9** The objective is to calculate the time interval between the new steady state and the old steady state. The average excess heat flux absorbed from the sun is  $q'' = 0.85 \text{ W/m}^2$ . The earth temperature increase from the old to the new steady state is  $\Delta T = 0.28 \text{ K}$ . The thermal inertia of the earth's crust is dominated by ocean water, the area-averaged depth (scale) of which is  $H = 1 \text{ km}$ . The first law of thermodynamics requires

$$q'' = \rho H c \frac{\Delta T}{\Delta t} \quad (1)$$

in which  $pc$  is the heat capacity of water. Equation (1) yields

$$\Delta t = 156.7 \text{ years} \frac{H}{1 \text{ km}} \frac{\Delta T}{1 \text{ K}} \quad (2)$$

$$= 44 \text{ years,} \quad \text{if } H = 1 \text{ km and } \Delta T = 0.28 \text{ K.}$$


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## Chapter 10

### REFRIGERATION

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Problem 10.1 Redrawing the T-s diagram of Fig. 10.4, successful refrigeration requires, in order,

$$\begin{aligned}\frac{\dot{Q}_L}{\dot{m}} &= h_1 - h_3 > 0 \\ h_1 - h_{1_{\text{bad}}} &> 0 \\ T_1 - T_{1_{\text{bad}}} &> 0 \quad (\text{constant } P, \text{ constant } c_P)\end{aligned}$$

$$\underbrace{T_1 - T_{1_{\text{ideal}}}} + \underbrace{T_{1_{\text{ideal}}} - T_{1_{\text{bad}}}} > 0$$

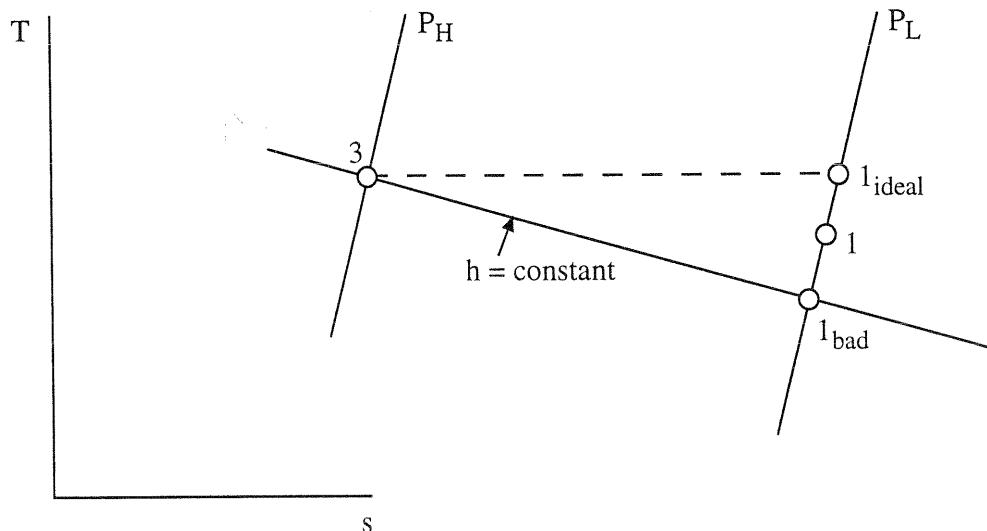
$$-(T_3 - T_1) \quad \underbrace{(T_3 - T_{1_{\text{bad}}})}_{\mu_J(P_H - P_L)} \quad (\text{constant } \mu_J)$$

Note the last observation, namely

$$\mu_J = \left( \frac{\partial T}{\partial P} \right)_h \equiv \frac{T_{1_{\text{bad}}} - T_3}{P_L - P_H}$$

The preceding string of inequalities becomes then

$$\mu_J > \frac{T_3 - T_1}{P_H - P_L}$$



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### Problem 10.2

Nitrogen data:  $k = 1.4$ ,  $\overline{\mu_J c_P} = 0.0075 \text{ m}^3/\text{kg}$ ,  $R \cong 0.3 \text{ kJ/kgK}$

Cycle features:  $P_L = 1 \text{ atm}$ ,  $P_H = 100 \text{ atm}$ ,  $\dot{m}_e/\dot{m} = 0.6$ ,  $T_e = 250 \text{ K}$ ,  $T_3 - T_1 = 25 \text{ K}$ .

$$\frac{\dot{Q}_L}{\dot{m}} \cong \frac{\dot{m}_e k R T_e}{\dot{m}(k-1)} \left[ 1 - \left( \frac{P_L}{P_H} \right)^{(k-1)/k} \right] + \overline{\mu_J c_P} (P_H - P_L) - c_P (T_3 - T_1) \quad (10.9)$$

Calculating the right-hand-side terms, we find:

$$\begin{aligned} \text{the first term} &= \frac{(0.6)(1.4)(0.3) \text{ kJ } (250) \text{ K}}{(1.4 - 1) \text{ kg K}} \left[ 1 - \left( \frac{1}{100} \right)^{0.4/1.4} \right] \\ &= 115.25 \text{ kJ / kg} \end{aligned}$$

$$\begin{aligned} \text{the second term} &= 0.0075 \frac{\text{m}^3}{\text{kg}} (100 - 1) \text{ atm } (1.013) 10^5 \frac{\text{N/m}^2}{\text{atm}} \\ &= 75.2 \text{ kJ/kg} \end{aligned}$$

$$\text{the third term} = - \frac{1.4}{1.4 - 1} (0.3) \frac{\text{kJ}}{\text{kg K}} (25) \text{K} = - 26.25 \text{ kJ/kg}$$

In conclusion, the expander term (the first term) contributes the most to  $\dot{Q}_L$ , especially if  $\mu_J = 0$ .

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### Problem 10.3 The $i^{\text{th}}$ compression requires

$$\begin{aligned} \dot{W}_i &= \frac{1}{\eta_c} \dot{W}_{i, \text{isentropic}} \\ &= \frac{1}{\eta_c} \dot{m} c_P \left( T_{\substack{\text{exit,} \\ \text{isentropic}}} - T_0 \right) \\ &= \frac{1}{\eta_c} \dot{m} c_P T_0 \left[ \left( \frac{P_i}{P_{i-1}} \right)^{(k-1)/k} - 1 \right] \end{aligned}$$

Adopting a shorthand notation for the isentropic temperature ratio for one compression stage,

$$x_i = \left( \frac{P_i}{P_{i-1}} \right)^{(k-1)/k},$$

the total compressor power becomes

$$\dot{W}_{\text{total}} = \frac{1}{\eta_c} \dot{m} c_p T_0 (x_1 + x_2 + \dots + x_n - n) \quad (a)$$

Since  $\dot{m}$  and  $\eta_c$  are fixed, the objective is to minimize the quantity in the brackets, subject to the overall pressure ratio constraint

$$\left( \frac{P_1}{P_0} \right) \left( \frac{P_2}{P_1} \right) \dots \left( \frac{P_n}{P_{n-1}} \right) = \frac{P_H}{P_L} \quad (b)$$

where  $P_H = P_n$  and  $P_L = P_0$ . In terms of the  $x_i$  notation, this constraint reads

$$x_1 x_2 \dots x_n = X, \quad (\text{constant}) \quad (c)$$

where  $X = (P_H / P_L)^{(k-1)/k}$ .

The problem of minimizing (a) subject to constraint (c) is the same as minimizing the function

$$\Phi = x_1 + x_2 + \dots + x_n - n + \lambda (x_1 x_2 \dots x_n - X)$$

subject to no constraints ( $\lambda$  is a Lagrange multiplier - see Appendix). We must solve the system of  $n$  equations for  $n$   $x_i$ 's,

$$\frac{\partial \Phi}{\partial x_1} = 1 + \lambda x_2 x_3 \dots x_n = 0$$

$$\frac{\partial \Phi}{\partial x_n} = 1 + \lambda x_1 x_2 \dots x_{n-1} = 0$$

which can be written as

$$x_2 x_3 \dots x_n = -\frac{1}{\lambda} \quad (d)$$

$$x_1 x_2 \dots x_{n-1} = -\frac{1}{\lambda}$$

Multiplying all  $n$  equations side-by-side,

$$(x_1 x_2 \dots x_n)^{n-1} = \left( -\frac{1}{\lambda} \right)^n$$

we recall eq. (c) and conclude that

$$-\frac{1}{\lambda} = X^{(n-1)/n}$$

The  $i^{\text{th}}$  equation of the system (d)

$$x_1 x_2 \dots x_{i-1} x_{i+1} \dots x_n = -\frac{1}{\lambda}$$

implies that

$$\frac{X}{x_i} = X^{(n-1)/n}$$

hence, the general solution

$$x_i = X^{1/n}$$

or, in terms of pressure ratios,

$$\frac{P_i}{P_{i-1}} = \left(\frac{P_H}{P_L}\right)^{1/n}$$

The optimal (minimum) total compressor power is simply the minimized  $\dot{W}_i$  times  $n$ ,

$$\dot{W}_{\text{total, minimum}} = \frac{n}{\eta_c} \dot{m} c_P T_0 \left[ \left( \frac{P_H}{P_L} \right)^{(k-1)/(nk)} - 1 \right]$$

Finally,  $n \rightarrow \infty$  represents the isothermal compressor limit,

$$\lim_{n \rightarrow \infty} \dot{W}_{\text{total, minimum}} = \frac{\dot{m} c_P T_0}{\eta_c} \lim_{v \rightarrow 0} \frac{p^v - 1}{v}$$

where  $v = 1/n$  and  $p = (P_H/P_L)^{(k-1)/k}$ . However, since

$$\lim_{v \rightarrow 0} \frac{p^v - 1}{v} = \ln p$$

we conclude that

$$\begin{aligned} \dot{W}_{\text{total, isothermal}} &= \frac{\dot{m} c_P T_0}{\eta_c} \ln \left( \frac{P_H}{P_L} \right)^{(k-1)/k} \\ &= \frac{1}{\eta_c} \dot{m} R T_0 \ln \frac{P_H}{P_L} \\ &= \frac{1}{\eta_c} \dot{W}_{\text{isothermal \& reversible}} \end{aligned}$$

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Problem 10.4 In the limit of infinite heat exchanger area, and in the absence of intermediate-temperature expander, eq. (10.9) yields

$$\dot{Q}_L = \dot{m} \mu_J c_P (P_H - P_L) - 0$$

Dividing  $\dot{Q}_L$  by eq. (10.10), we find that  $\dot{Q}_L / \dot{W}_{c,\min}$  increases as  $(\pi - 1) / \ln \pi$ ,

$$\frac{\dot{Q}_L}{\dot{W}_{c,\min}} = \frac{k \mu_J P_L}{(k - 1) T_0} \left( \frac{\pi - 1}{\ln \pi} \right)$$

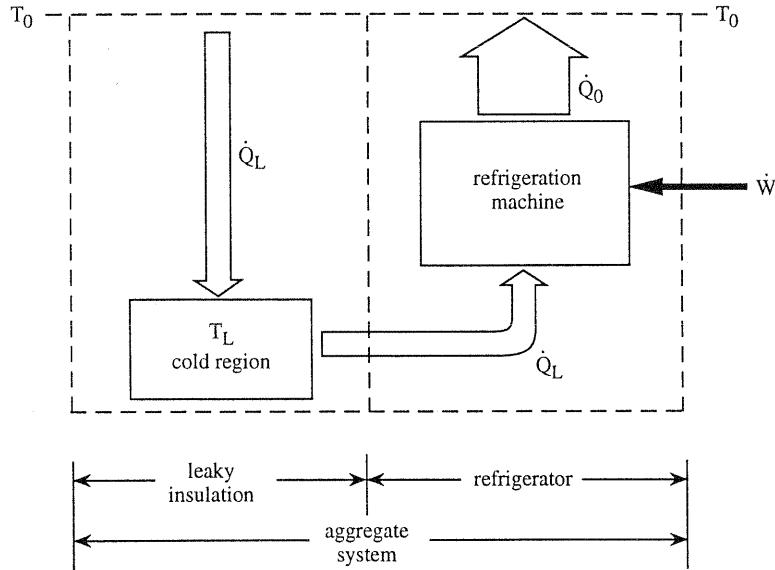
where  $\pi = P_H / P_L$ .

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Problem 10.5 With reference to the aggregate (large) system shown in the figure, we invoke the first law

$$\begin{aligned} \dot{W} &= \dot{Q}_0 - \dot{Q}_L \\ &= T_0 \left( \frac{\dot{Q}_0}{T_0} - \frac{\dot{Q}_L}{T_0} \right) \\ &= T_0 \left( \underbrace{\frac{\dot{Q}_L}{T_L} - \frac{\dot{Q}_L}{T_0}}_{\dot{S}_{\text{gen}} \text{ due to } \dot{Q}_L - \text{leak from } T_0 \text{ to } T_L} + \underbrace{\frac{\dot{Q}_0}{T_0} - \frac{\dot{Q}_L}{T_L}}_{\dot{S}_{\text{gen}} \text{ due to the refrigeration machine alone}} \right) \\ &= T_0 \dot{S}_{\text{gen}} \end{aligned}$$

where  $\dot{S}_{\text{gen}}$  represents the *total* rate of entropy generation.



Problem 10.6 With reference to the detail enlarged on the right of Fig. 10.12, the first law requires

$$\begin{aligned} \dot{m}c_P(T + \Delta T) - \dot{m}c_P T \\ + (\dot{m} + d\dot{m})c_P(T + dT) - (\dot{m} + d\dot{m})c_P[T + \Delta T + dT + d(\Delta T)] \\ + \eta_e(d\dot{m})RT \ln \frac{P_H}{P_L} = 0 \end{aligned}$$

or

$$\left[ \left( \eta_e \frac{R}{c_P} \ln \frac{P_H}{P_L} \right) T - \Delta T \right] d\dot{m} = \dot{m} d(\Delta T) \quad (10.33)$$

The entropy generation for the slice of height  $dT$  is

$$\begin{aligned} d\dot{S}_{gen} &= \dot{m}c_P \ln \frac{T + \Delta T}{T} - \dot{m}R \ln \frac{P_H}{P_L} \\ &\quad - (\dot{m} + d\dot{m})c_P \ln \frac{T + \Delta T + dT + d(\Delta T)}{T + dT} + (\dot{m} + d\dot{m})R \ln \frac{P_H}{P_L} \end{aligned}$$

and, linearizing the logarithms,

$$d\dot{S}_{gen} = \left( R \ln \frac{P_H}{P_L} \right) d\dot{m} + \frac{\Delta T}{T} c_P \left[ -d\dot{m} - \dot{m} \left( \frac{d(\Delta T)}{\Delta T} - \frac{dT}{T} \right) \right]$$

Introducing the dimensionless notation

$$\tau = \frac{\Delta T}{T}$$

where  $\tau = \tau(T)$ , and eliminating  $d\dot{m}$  between the  $d\dot{S}_{gen}$  expression and the first law (10.33) we obtain

$$d\dot{S}_{gen} = \dot{m}c_P \frac{\tau Tb(1-\eta_e) + \tau b - \tau^2}{(\eta_e b - \tau)T} dT$$

where

$$b = \frac{R}{c_P} \ln \frac{P_H}{P_L}$$

In the small temperature difference limit,

$$\tau \ll b$$

the  $d\dot{S}_{gen}$  expression reduces to

$$d\dot{S}_{gen} = \dot{m}c_P \left[ \tau \left( \frac{1}{\eta_e} - 1 \right) + \frac{\tau}{\eta_e T} \right] dT$$

hence

$$\dot{S}_{gen} = \int_{T_L}^{T_H} \dot{m}c_P \left[ \left( \frac{1}{\eta_e} - 1 \right) \frac{d\tau}{dT} + \frac{\tau}{\eta_e T} \right] dT$$

Problem 10.7 Let  $M_f$  and  $M_g$  represent the instantaneous inventories of saturated liquid and saturated vapor in the vessel. Since the total volume occupied by  $M_f$  and  $M_g$  together is constant, we have

$$\dot{M}_f v_f + \dot{M}_g v_g = 0 \quad (a)$$

where  $(\cdot) = d(\cdot)/dt$ . The principle of mass conservation requires

$$\dot{M}_f + \dot{M}_g = -\dot{m} \quad (b)$$

Together, eqs. (a) and (b) deliver

$$\dot{M}_f = -\frac{v_g}{v_{fg}} \dot{m}, \quad \dot{M}_g = \frac{v_f}{v_{fg}} \dot{m} \quad (c)$$

The relation between  $\dot{Q}_L$  and  $\dot{m}$  comes from the first law for the venting vessel as an unsteady open system,

$$\frac{dU}{dt} = \dot{Q}_L - \dot{m} h_g$$

$$\dot{M}_f u_f + \dot{M}_g u_g = \dot{Q}_L - \dot{m} h_g$$

which, in view of eqs. (c) above, means that

$$\dot{Q}_L = \dot{m} \left( h_{fg} - \frac{v_g}{v_{fg}} u_f + \frac{v_f}{v_{fg}} u_g \right) \quad (d)$$

For the entropy generation rate of the open system treated until now we write

$$\begin{aligned} \dot{S}_{gen} &= \frac{dS}{dt} + \dot{m} s_g - \frac{\dot{Q}_L}{T_0} \\ &= \dot{M}_f s_f + \dot{M}_g s_g + \dot{m} s_g - \frac{\dot{Q}_L}{T_0} \end{aligned}$$

and, using eqs. (c) and (d),

$$\dot{S}_{gen} = \frac{v_g}{v_{fg}} s_{fg} - \frac{1}{T_0} \left( h_{fg} - \frac{v_g}{v_{fg}} u_f + \frac{v_f}{v_{fg}} u_g \right) \quad (\text{constant})$$

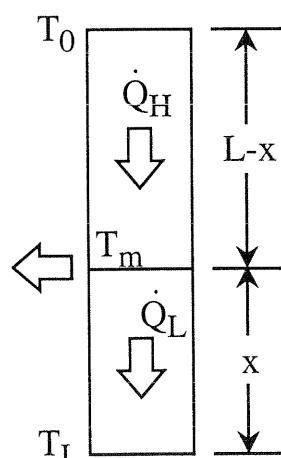
Problem 10.8 The irreversibility of this insulation is due to heat transfer across two temperature gaps,  $\dot{Q}_H$  from  $T_0$  to  $T_m$ , and  $\dot{Q}_L$  from  $T_m$  to  $T_L$ :

$$\dot{S}_{gen} = \dot{Q}_H \left( \frac{1}{T_m} - \frac{1}{T_0} \right) + \dot{Q}_L \left( \frac{1}{T_L} - \frac{1}{T_m} \right)$$

where

$$\dot{Q}_H = k A_c \frac{T_H - T_m}{L - x}$$

$$\dot{Q}_L = k A_c \frac{T_m - T_L}{x}$$



In dimensionless form, we can rewrite  $\dot{S}_{gen}$  as

$$N_S = \frac{\dot{S}_{\text{gen}}}{kA_c/L} = \frac{\tau_0 - \tau_m}{1 - \xi} \left( \frac{1}{\tau_m} - \frac{1}{\tau_0} \right) + \frac{\tau_m - 1}{\xi} \left( 1 - \frac{1}{\tau_m} \right)$$

where

$$\tau_0 = T_0/T_L, \quad \tau_m = T_m/T_L, \quad \xi = x/L$$

In order to minimize  $N_S$  with respect to both  $\tau_m$  and  $\xi$ , we solve simultaneously

$$\frac{\partial N_S}{\partial \tau_m} = 0, \quad \text{and} \quad \frac{\partial N_S}{\partial \xi} = 0$$

Omitting the algebra, these two equations yield, respectively:

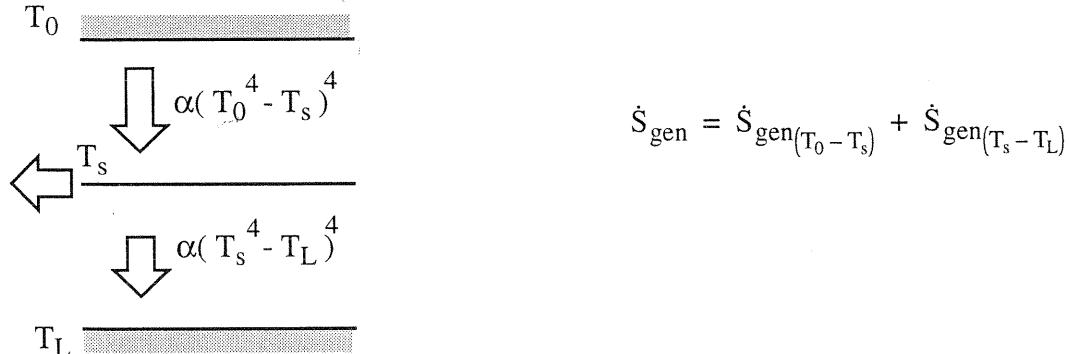
$$\tau_m^2 = \tau_0 \frac{(\tau_0 - 1)\xi + 1}{\xi + \tau_0(1 - \xi)} \quad \text{and} \quad \left( \frac{1}{\xi} - 1 \right)^2 = \frac{\frac{\tau_0}{\tau_m} - 1 + \frac{\tau_m}{\tau_0}}{\tau_m - 2 + \frac{1}{\tau_m}}$$

For a given  $\tau_0$ , the above system can be solved numerically for  $\tau_m$  and  $\xi$ . The solution is possible analytically in the limit  $\tau_0 \gg 1$ , in which the system reduces to

$$\tau_m^2 \approx \frac{\tau_0 \xi}{1 - \xi} \quad \text{and} \quad \left( \frac{1}{\xi} - 1 \right)^2 \approx \frac{\tau_0}{\tau_m^2}$$

hence the optimum values  $\tau_m = \tau_0^{1/2}$  and  $\xi = 0.5$ .

Problem 10.9 The irreversibility of a one-shield insulation is caused by the heat transfer across the two temperature gaps (see the sketch):



$$\dot{S}_{\text{gen}} = \alpha \left( T_0^4 - T_s^4 \right) \left( \frac{1}{T_s} - \frac{1}{T_0} \right) + \alpha \left( T_s^4 - T_L^4 \right) \left( \frac{1}{T_L} - \frac{1}{T_s} \right)$$

Setting  $\partial \dot{S}_{\text{gen}} / \partial T_s = 0$  yields

$$-\frac{T_0^4 + T_L^4}{T_s^2} - 6T_s^2 + 4T_s^3 \left( \frac{1}{T_0} + \frac{1}{T_L} \right) = 0$$

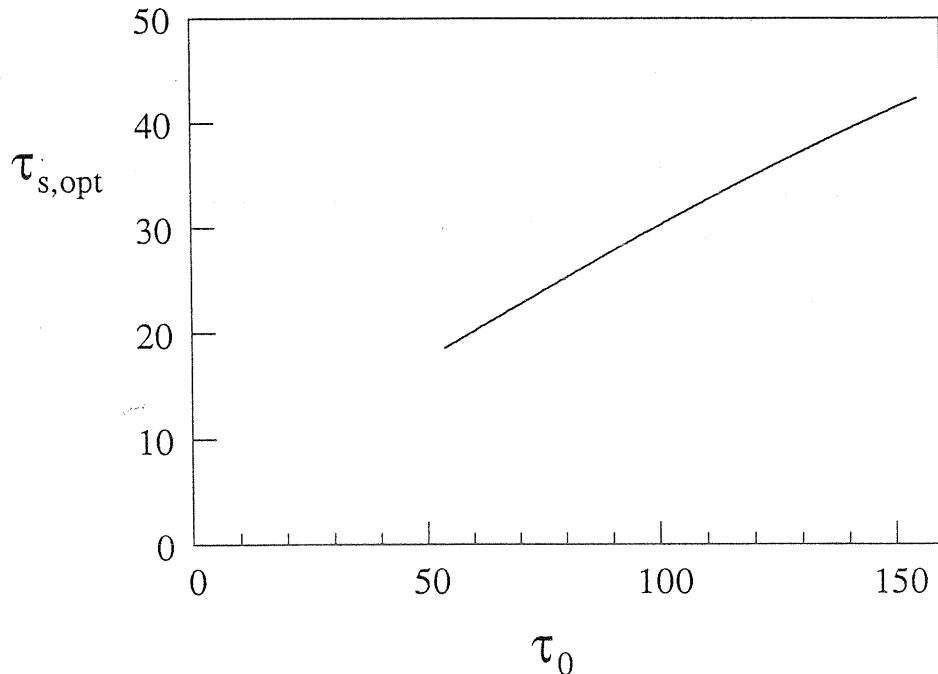
or, in dimensionless form

$$-\frac{\tau_0^4 + 1}{\tau_s^2} - 6\tau_s^2 + 4\tau_s^3 \left( \frac{1}{\tau_0} + 1 \right) = 0$$

where  $\tau_0 = T_0/T_L$  and  $\tau_s = T_s/T_L$ . The last equation is an implicit result for  $\tau_s(\tau_0)$ , which was the unknown in this problem. Plotting the  $\tau_s(\tau_0)$  function becomes easier in the limit  $\tau_0 \gg 1$  where the last equation yields

$$\tau_0 \equiv (4\tau_s^5 - 6\tau_s^4)^{1/4}$$

This result is illustrated below:



---

Problem 10.10 Let  $\dot{W}_{\text{rev}}$  be the mechanical power input to the control volume, and  $\dot{Q}_0$  the heat transfer rate rejected by the control volume to the ambient ( $T_0$ ). In the steady state, the first law and the second law require

$$0 = -\dot{Q}_0 + \dot{W}_{\text{rev}} + \dot{m}_l h_0 - \dot{m}_l h_f$$

$$\dot{S}_{\text{gen}} = \dot{m}_l s_f - \dot{m}_l s_0 + \frac{\dot{Q}_0}{T_0} = 0 \quad (\text{reversible limit})$$

where  $h_0 = h(T_0, P_L)$ ,  $h_f = h_f(P_L)$ ,  $s_0 = s(T_0, P_L)$  and  $s_f = s_f(P_L)$ . Eliminating  $\dot{Q}_0$  between these two equations we find

$$\begin{aligned}\dot{W}_{\text{rev}} &= \dot{m}_l [h_f - h_0 - T_0 (s_f - s_0)] \\ &= \dot{m}_l (b_f - b_0) \\ &= \dot{m}_l e_{x,f}\end{aligned}$$

The work invested in the reversible liquefaction of the  $\dot{m}_l$  stream is conserved (stored) as the flow exergy of the low temperature liquid stream, or as the flow availability of the liquefied stream relative to the restricted dead state stream ( $T_0, P_L$ ). This conclusion is the same as in the reversible limit of Fig. 3.9.

---

Problem 10.11 The total entropy generation rate and the finite size constraint are

$$\dot{S}_{\text{gen}} = \int_{T_L}^{T_H} \frac{\dot{Q}}{T^2} dT \quad (1)$$

$$\frac{L}{A} = \int_{T_L}^{T_H} \frac{k}{\dot{Q}} dT \quad (2)$$

The stack of many radiation shields behaves as a layer with thermal conductivity proportional to  $T^3$ ,

$$k = a T^3 \quad (3)$$

We assume that

$$\dot{Q} = b T^n \quad (4)$$

and, from eqs. (2) - (5), find

$$b = \frac{aA}{(4-n)L} (T_H^{4-n} - T_L^{4-n}) \quad (5)$$

Now we can determine  $\dot{S}_{\text{gen}}$  as a function of  $n$  by combining eqs. (1), (5) and (6),

$$\dot{S}_{\text{gen}} = \frac{a A T_H^3}{L} \frac{1 + \tau^3 - \tau^{4-n} - \tau^{n-1}}{(4-n)(n-1)} \quad (6)$$

where  $\tau = T_L / T_H$  is constant. The denominator in eq. (7) shows that  $\dot{S}_{\text{gen}}$  blows up at  $n = 1$  and  $n = 4$ . The  $\dot{S}_{\text{gen}}$  minimum is located by writing  $\partial \dot{S}_{\text{gen}} / \partial n = 0$ , which becomes:

$$\tau^{n-1} (\tau^{5-2n} - 1) (4-n)(n-1) \ln \tau - (1 + \tau^3 - \tau^{4-n} - \tau^{n-1}) (5-2n) = 0 \quad (7)$$

In eq. (8) both terms vanish when  $(5 - 2n) = 0$ , therefore the optimal cooling of the stack of radiation shields is represented by

$$n_{\text{opt}} = \frac{5}{2}$$

$$\dot{Q}_{\text{opt}} = b T^{5/2}$$

$$\frac{d\dot{Q}_{\text{opt}}}{dT} = \frac{5}{2} b T^{3/2}$$

Problem 10.12 The entropy generation rate due to the counterflow heat exchanger is

$$\dot{S}_{\text{gen, counterflow}} = \dot{Q}_c \left( \frac{1}{T_L} - \frac{1}{T_H} \right) \quad (1)$$

where  $\dot{Q}_c$  is the end-to-end convective heat leak,

$$\dot{Q}_c = \dot{m} c_P \Delta T \quad (2)$$

and  $\Delta T$  is the stream-to-stream temperature difference, in order,

$$\begin{aligned}
\dot{m}c_P dT &= (\pi D_i dx) U_i \Delta T \\
\dot{m}c_P (T_H - T_L) &= \pi D_i L U_i \Delta T \\
\Delta T &= \frac{\dot{m}c_P (T_H - T_L)}{\pi D_i L U_i}
\end{aligned} \tag{3}$$

Putting eqs. (1) - (3) together we obtain:

$$\dot{S}_{\text{gen, counterflow}} = \frac{(\dot{m}c_P)^2}{\pi D_i U_i L} \frac{(T_H - T_L)^2}{T_L T_H} \tag{4}$$

When  $L$  is large enough that we must consider the heat leak from the ambient  $T_0$  to the outer surface of the counterflow heat exchanger [average temperature  $\bar{T} \equiv (T_L + T_H)/2$ ], the entropy generation rate associated with the heat leak is

$$\dot{S}_{\text{gen, heat leak}} = \dot{Q}_0 \left( \frac{1}{\bar{T}} - \frac{1}{T_0} \right)$$

where

$$\dot{Q}_0 = \pi D_0 L U_0 (T_0 - \bar{T}) \tag{5}$$

in other words,

$$\dot{S}_{\text{gen, heat leak}} = \pi D_0 L U_0 \frac{(T_0 - \bar{T})^2}{T_0 \bar{T}} \tag{6}$$

Adding eqs. (4) and (6) we obtain the total entropy generation rate

$$\dot{S}_{\text{gen}} = \frac{a}{L} + bL \tag{7}$$

where

$$a = \frac{(\dot{m}c_P)^2}{\pi D_i U_i} \frac{(T_H - T_L)^2}{T_L T_H}, \quad b = \pi D_0 U_0 \frac{(T_0 - \bar{T})^2}{T_0 \bar{T}} \tag{8}$$

The two-term summation in eq. (7) is based on the assumption that the heat leak from  $T_0$  to the counterflow heat exchanger is small enough such that the adiabatic heat exchanger analysis that yielded eq. (4) is still adequate. Solving  $\partial \dot{S}_{\text{gen}} / \partial L = 0$  using eq. (7) we obtain

$$L_{\text{opt}} = \left( \frac{a}{b} \right)^{1/2} \tag{9}$$

which can be rearranged as an optimal number of heat transfer units based on the average area and average heat transfer coefficient  $(U_i U_0)^{1/2}$ :

$$N_{tu, \text{opt}} = \frac{L_{\text{opt}} \pi (D_i D_0)^{1/2} (U_i U_0)^{1/2}}{\dot{m} c_p} \equiv \left| \frac{T_H - T_L}{T_0 - \bar{T}} \right| \left( \frac{T_0}{\bar{T}} \right)^{1/2} \quad (10)$$

To obtain eq. (10) we assumed that  $\bar{T}$  is approximately the same as  $(T_L T_H)^{1/2}$ .

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**Problem 10.13** Consider the cooling arrangement shown in Fig. P10.13. A certain volume contains electronic packages or windings that operate at the temperature  $T_e$  (below the ambient  $T_0$ ) and generate Joule heating at the rate  $\dot{Q}$ . Flowing through the channels formed between components is a stream of single phase coolant of mass flow rate  $\dot{m}$  and constant specific heat  $c_p$ . The heat transfer area between the heat-releasing surfaces and the coolant is fixed,  $A$ . The overall heat transfer coefficient  $U$  increases with the flow rate as we will see in eq. (6). We assume that the coolant is well mixed at the temperature  $T_{\text{out}}$  inside the cooled volume such that the cooling rate

$$\dot{Q} = UA (T_e - T_{\text{out}}) \quad (1)$$

is matched by the enthalpy gained by the coolant

$$\dot{Q} = \dot{m} c_p (T_{\text{out}} - T_{\text{in}}) \quad (2)$$

The spent stream of temperature  $T_{\text{out}}$  is discharged into the ambient.

The inlet temperature of the coolant,  $T_{\text{in}}$ , is controlled by a steady-state refrigerator that receives the coolant from room temperature,  $T_0$ , and rejects heat to room temperature,  $\dot{Q}_0$ . The refrigerator operates reversibly, and the pressure drop experienced by  $\dot{m}$  between inlet ( $T_0$ ) and outlet ( $T_{\text{out}}$ ) is zero. Accordingly, the first law and second law statements for the refrigerator are

$$\dot{W} = \dot{m} c_p (T_{\text{in}} - T_0) + \dot{Q}_0 \quad (3)$$

$$\dot{S}_{\text{gen}} = \dot{m} c_p \ln \frac{T_{\text{in}}}{T_0} + \frac{\dot{Q}_0}{T_0} = 0 \quad (4)$$

The objective of the following analysis is to determine the minimum refrigerator power  $\dot{W}$  that is required for the removal of  $\dot{Q}$  from  $T_e$ . Combining eqs. (1) and (2) we obtain

$$\dot{Q} = \left( \frac{1}{\dot{m} c_p} + \frac{1}{UA} \right)^{-1} (T_e - T_{\text{in}}) \quad (5)$$

As a model for  $U$ , we assume that the flow passages are sufficiently coarse and rough that the Stanton number ( $St$ ) is approximately independent of the flow rate,

$$St = \frac{U}{\rho c_p \dot{m} / (\rho A_f)} \quad (6)$$

In this expression  $A_f$  and  $\dot{m} / (\rho A_f)$  are the volume averaged flow cross-section and the mean velocity based on  $A_f$ . Equations (5) and (6) yield

$$b \dot{Q} = \dot{m} c_P (T_e - T_{in}) \quad (7)$$

where  $b$  is a constant,

$$b = 1 + \frac{A_f / A}{St} \quad (8)$$

Eliminating  $\dot{Q}_0$  between eqs. (3) and (4) we obtain

$$\dot{W} = \dot{m} c_P \left( T_{in} - T_0 - T_0 \ln \frac{T_{in}}{T_0} \right) \quad (9)$$

It is now convenient to nondimensionalize eqs. (7) and (9) as

$$\tau_{in} = \tau_e - \frac{1}{M} \quad (10)$$

$$\tilde{W} = M (\tau_{in} - 1 - \ln \tau_{in}) \quad (11)$$

where

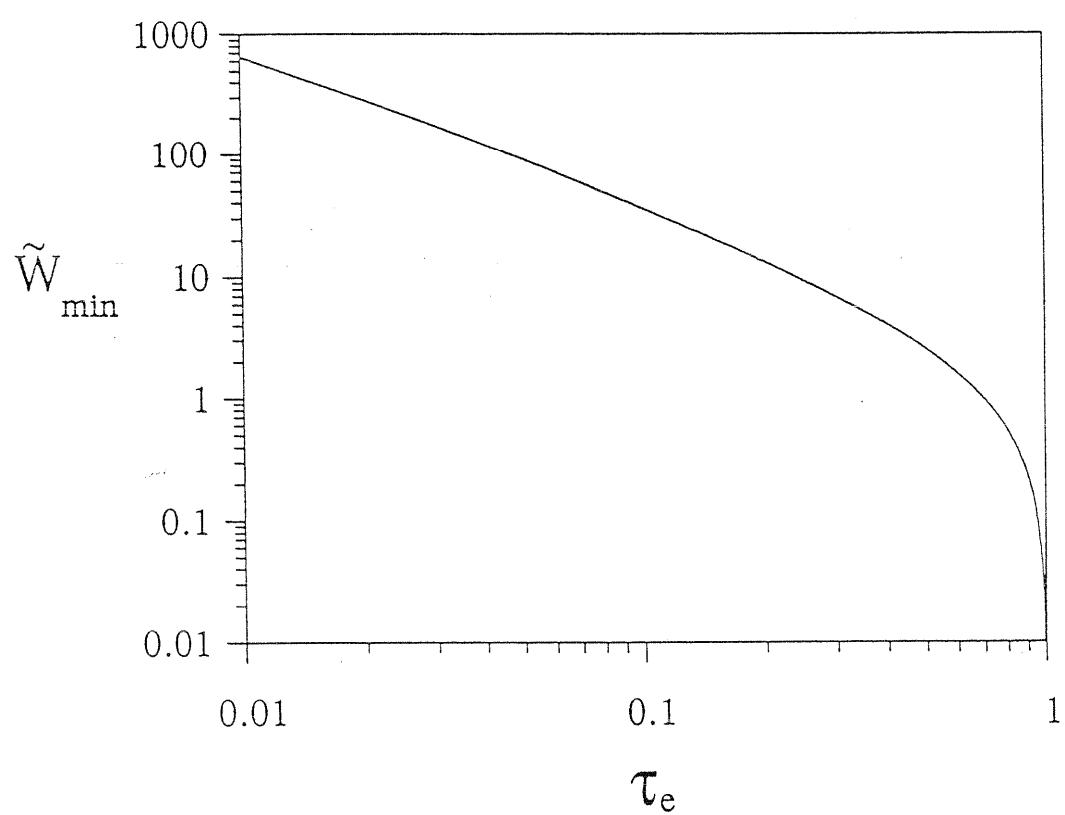
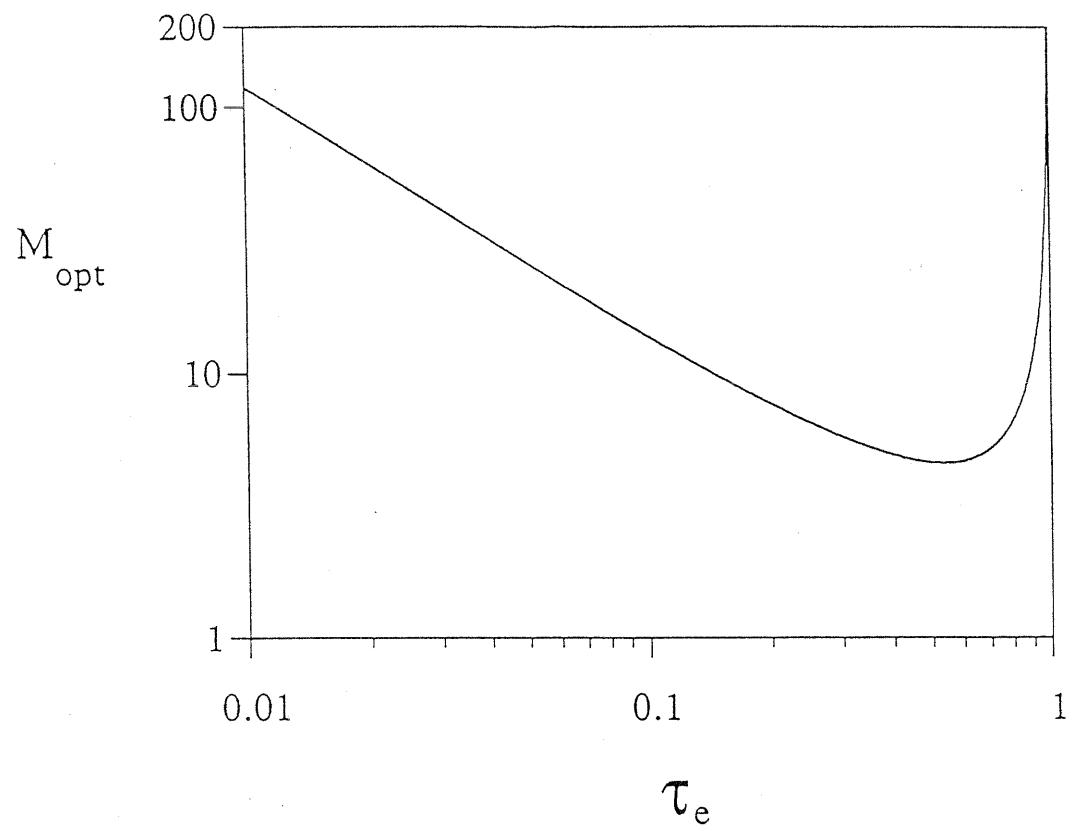
$$\tau_e = \frac{T_e}{T_0}, \quad \tau_{in} = \frac{T_{in}}{T_0} \quad (12)$$

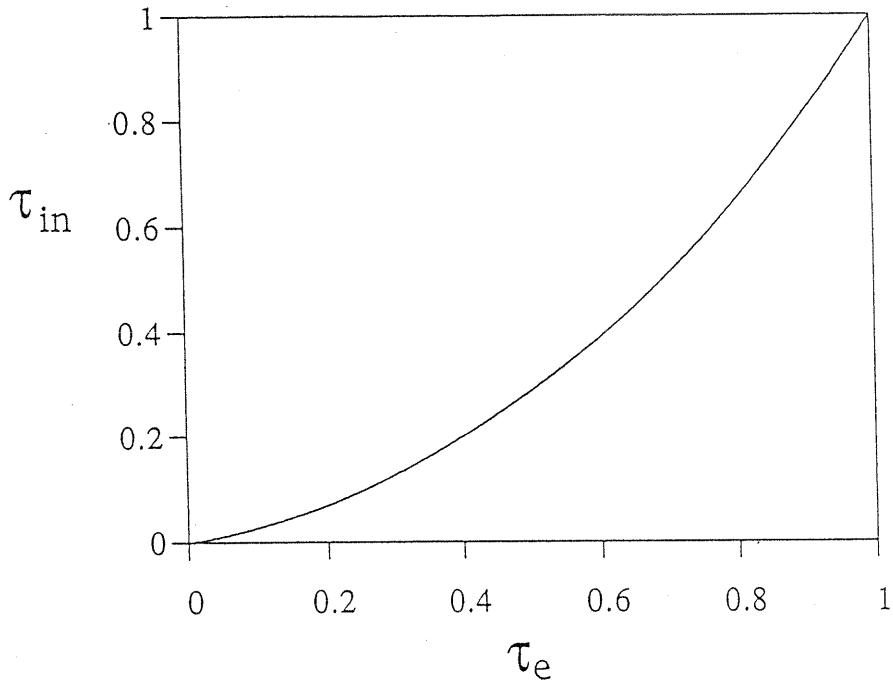
$$M = \frac{\dot{m} c_P T_0}{b \dot{Q}}, \quad \tilde{W} = \frac{\dot{W}}{b \dot{Q}} \quad (13)$$

Equations (10, 11) express the refrigerator power  $\tilde{W}$  as a function of the mass flow number  $M$ . Solving  $\partial \tilde{W} / \partial M = 0$  to minimize  $\tilde{W}$  yields an implicit relation for  $\tau_{in,opt}(\tau_e)$ :

$$\frac{\ln \tau_{in,opt}}{1 - \tau_{in,opt}^{-1}} = \tau_e \quad (14)$$

This can be combined with eq. (10) to plot  $M_{opt}(\tau_e)$  as shown in the first graph. Finally, by substituting  $M_{opt}(\tau_e)$  and  $\tau_{in,opt}(\tau_e)$  into eq. (11) we obtain the function  $\tilde{W}_{min}(\tau_e)$  shown in the second graph. The optimal inlet temperature  $\tau_{in,opt}(\tau_e)$  is also shown.





Problem 10.14 The four compartments of the irreversible refrigerator model are described analytically by the four statements:

$$Q_H = (UA)_H (T_H - T_{HC}) \quad (1)$$

$$Q_0 = (UA)_0 (T_{0C} - T_0) \quad (2)$$

$$Q_L = (UA)_L (T_L - T_{LC}) \quad (3)$$

$$\frac{Q_H}{T_{HC}} + \frac{Q_L}{T_{LC}} = \frac{Q_0}{T_{0C}} \quad (4)$$

In addition, the first law of thermodynamics requires that

$$Q_H + Q_L = Q_0 \quad (5)$$

Each factor of type  $UA$  in eqs. (1) - (3) represents the overall thermal conductance of the respective heat exchanger, or the product between the heat transfer area  $A$  and the overall heat transfer coefficient  $U$  based on  $A$ . The thermal conductances  $(UA)_H$ ,  $(UA)_0$  and  $(UA)_L$  are rough measures of the sizes (hardware) of the three heat exchanges, therefore a reasonable economic constraint is

$$UA = (UA)_H + (UA)_0 + (UA)_L, \quad (\text{constant}) \quad (6)$$

which states that the total thermal conductance inventory  $UA$  is fixed.

The refrigeration plant of Fig. P10.14a is equivalent to the coupling of a power plant and refrigeration plant, as is shown in Fig. P10.14b. The power output ( $W$ ) from the power plant ( $P$ ) drives the refrigeration plant ( $R$ ). The spaces labeled ( $P$ ) and ( $R$ ) are irreversibility free. The heat rejection to the ambient, which in Fig. P10.14a was accommodated by a single heat exchanger,  $(UA)_0$ , is now effected by two heat exchangers operating in parallel,  $(UA)_0^P$  and  $(UA)_0^R$ . The equivalence between Figs. P10.14a and P10.14b is assured by writing

$$Q_0 = Q_0^P + Q_0^R \quad (7)$$

$$(UA)_0 = (UA)_0^P + (UA)_0^R \quad (8)$$

The power plant portion of the Fig. P10.14b model has been optimized in Section 8.1.2. The results that are relevant in the present problem are

$$(UA)_{H, \text{opt}} = (UA)_{0, \text{opt}}^P \quad (9)$$

$$W_{\max} = \frac{1}{4} (UA)_P T_0 (\tau_H^{1/2} - 1) \quad (10)$$

where  $(UA)_P$  is the total thermal conductance inventory of the power plant portion ( $P$ ),

$$(UA)_P = (UA)_H + (UA)_0^P \quad (11)$$

and

$$\tau_H = \frac{T_H}{T_0} > 1 \quad (12)$$

Equation (9) states that  $(UA)_P$  must be divided equally between the two heat exchangers of the power plant portion. The question that remains is how much of the total inventory  $UA$  should be allocated to  $(UA)_P$ .

Similar progress has been made in connection with the refrigerator portion ( $R$ ) of the model, cf. Section 10.6.2. Without repeating the analysis, we know that when  $W$  is given the refrigeration load  $Q_L$  is maximized if

$$(UA)_{0, \text{opt}}^R = (UA)_{L, \text{opt}} \quad (13)$$

while the total thermal conductance inventory of the refrigerator portion is constrained,

$$(UA)_R = (UA)_0^R + (UA)_L \quad (14)$$

It can be shown that the maximum  $Q_L$  that corresponds to the optimization rule (13) is

$$Q_{L, \max} = \frac{1}{8} (UA)_R T_0 \left\{ \left[ (4\tilde{W} - \tau_L + 1)^2 + 16\tau_L \tilde{W} \right]^{1/2} - 4\tilde{W} + \tau_L - 1 \right\} \quad (15)$$

where

$$\tau_L = \frac{T_L}{T_0} < 1 \quad (16)$$

$$\tilde{W} = \frac{W}{(UA)_R T_0} \quad (17)$$

We now return to the original problem formulated in Fig. 10.14a by coupling the optimized (P) and (R) portions of Fig. 10.14b. This amounts to setting  $W = W_{\max}$  in eqs. (15) and (17), for which  $W_{\max}$  is given by eq. (10). The total thermal conductance inventory UA of the entire installation, eq. (6), can also be written in terms of the total inventories of the (P) and (R) portions,

$$UA = (UA)_P + (UA)_R, \quad (\text{constant}) \quad (18)$$

or by introducing the fraction  $x$ :

$$x = \frac{(UA)_P}{UA}, \quad 1 - x = \frac{(UA)_R}{UA} \quad (19)$$

After these substitutions, eq. (15) becomes

$$\frac{Q_{L,\max}}{UAT_0} = \frac{1}{8} (1 - x) \left\{ \left[ (4 \tilde{W}_{\max} - \tau_L + 1)^2 + 16\tau_L \tilde{W}_{\max} \right]^{1/2} - 4 \tilde{W}_{\max} + \tau_L - 1 \right\} \quad (20)$$

where

$$\tilde{W}_{\max} = \frac{x(\tau_H^{1/2} - 1)}{4(1 - x)} \quad (21)$$

The expression listed in eq. (20) has been maximized numerically with respect to  $x$ , while holding the external temperature ratios  $\tau_H$  and  $\tau_L$  fixed. The procedure is illustrated in the first graph. It is interesting to point out that during this maximization the coefficient of performance  $COP = Q_{L,\max}/Q_H$  does not exhibit a maximum with respect to  $x$ .

The optimal allocation fraction  $x_{\text{opt}}(\tau_H, \tau_L)$  obtained based on the first graph is shown in the second graph. The answer sought in connection with eq. (6) is now complete: the optimal three-way allocation of UA between the generator, condenser and evaporator is

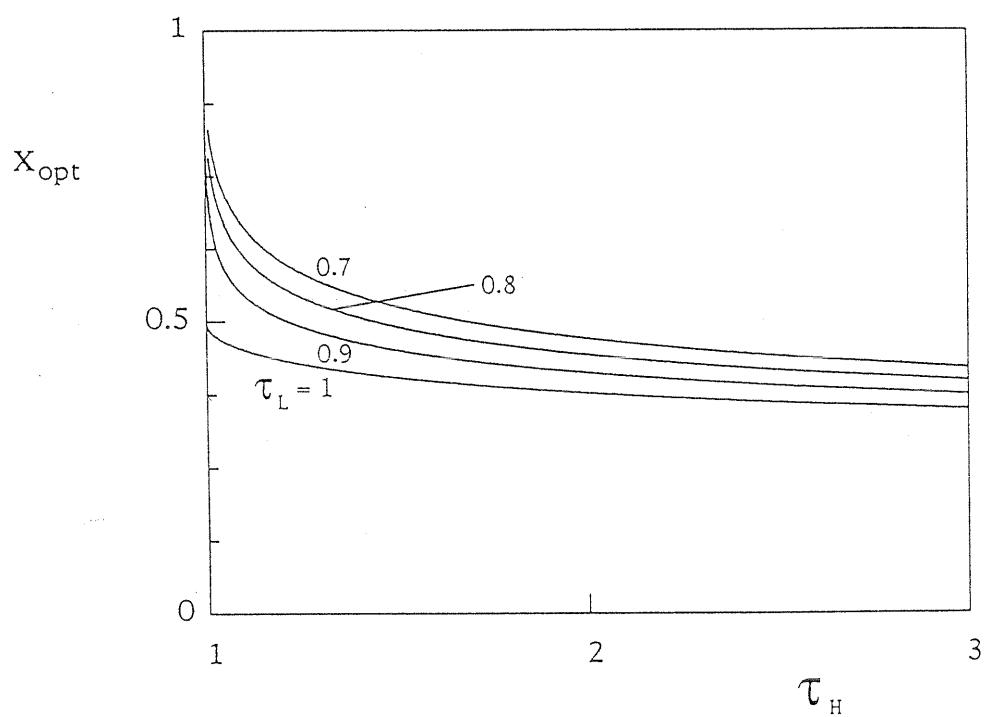
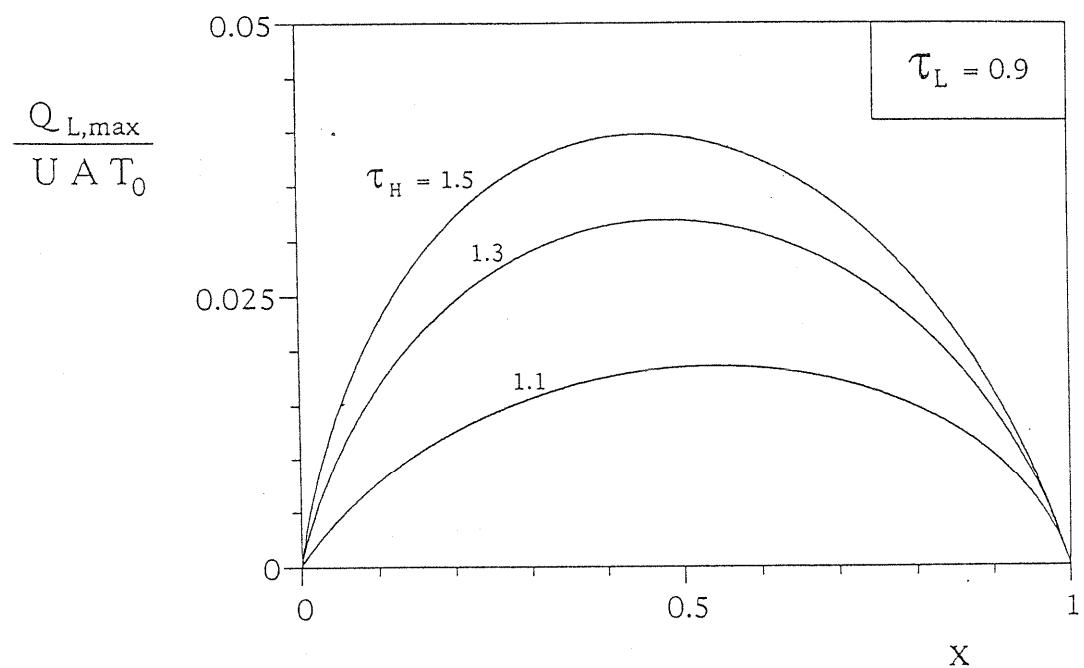
$$(UA)_{H,\text{opt}} = \frac{1}{2} x_{\text{opt}} UA \quad (22)$$

$$(UA)_{0,\text{opt}} = \frac{1}{2} UA \quad (23)$$

$$(UA)_{L,\text{opt}} = \frac{1}{2} (1 - x_{\text{opt}}) UA \quad (24)$$

It is interesting that the optimal condenser always demands half of the total thermal conductance inventory, regardless of the temperature ratios  $T_H / T_0$  and  $T_L / T_0$ . The second graph shows that when  $T_L / T_0$  is not much smaller than 1, the ratio  $x_{\text{opt}}$  is equal to roughly 1/2: this means that the optimal generator and evaporator conductances must have approximately the same size,

$$(UA)_{H,\text{opt}} \approx (UA)_{L,\text{opt}} \approx \frac{1}{4} UA \quad (25)$$



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Problem 10.15 If the power level (Joule heating rate) is the known constant  $q_J$ , eq. (10.100) is replaced by

$$\dot{Q}t_1 = q(t_1 + t_2) + q_J t_2 \quad (1)$$

In place of eq. (10.104), we write that the total work (electrical energy) that is required during the time  $(t_1 + t_2)$  is

$$W = \int_0^{t_1} \dot{W} dt + q_J t_2 \quad (2)$$

If we make these changes, the analytical steps that in the textbook gave us eq. (10.105) lead this time to

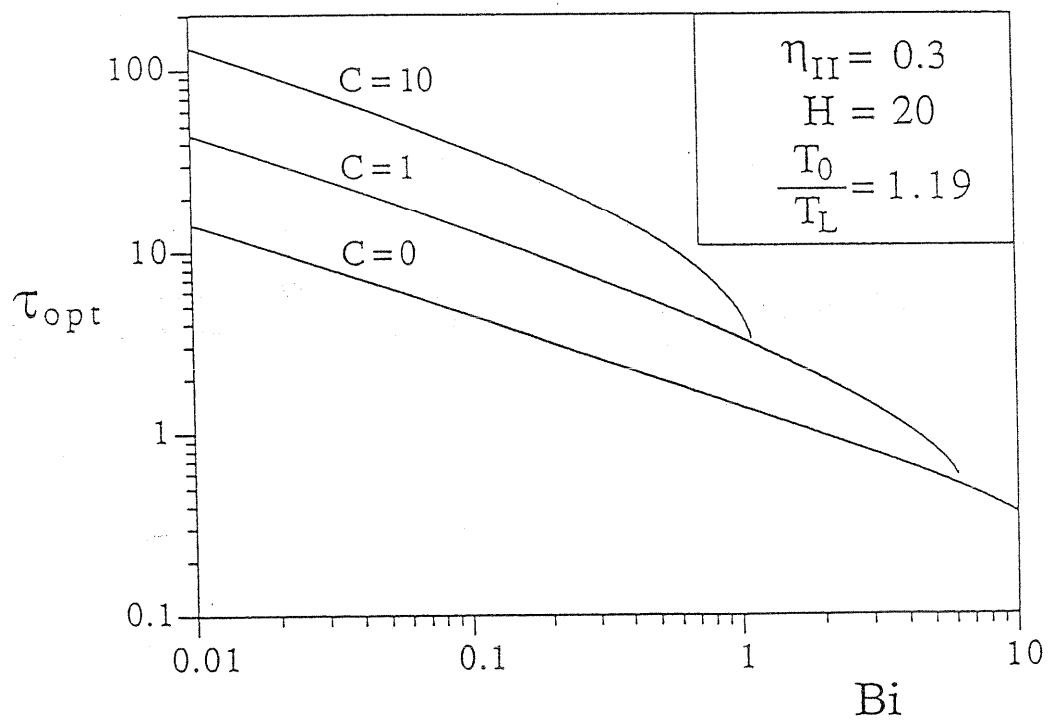
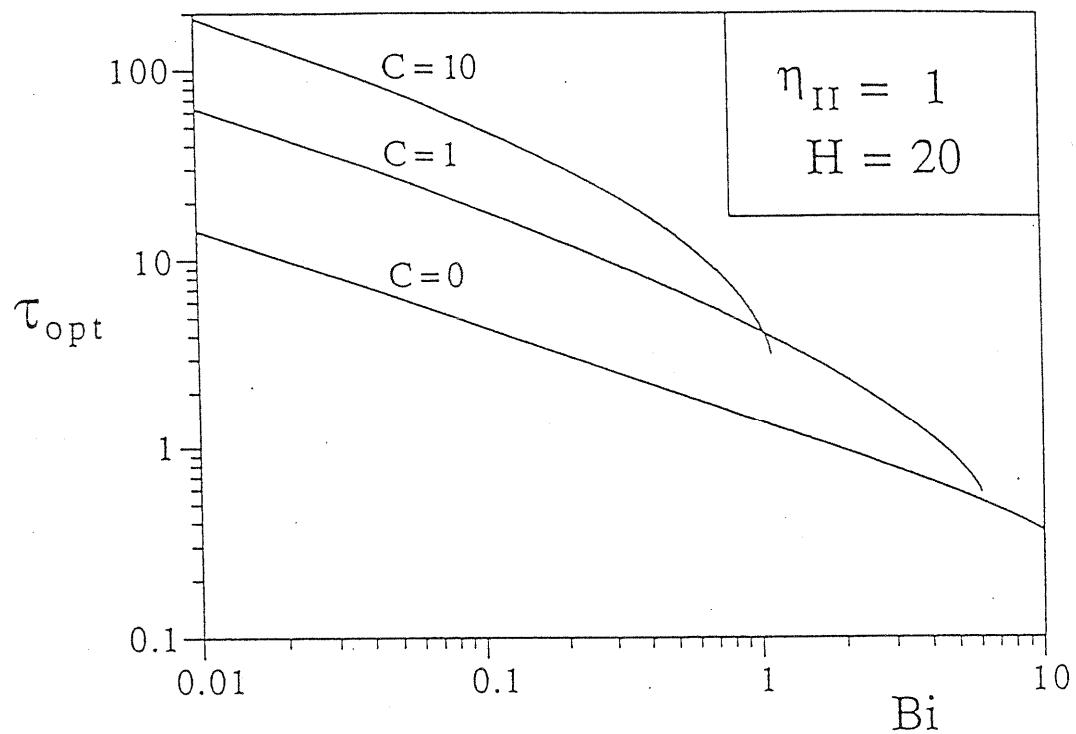
$$\frac{\eta_{II} W}{q(t_1 + t_2)} = \frac{(T_0 / T_L H)}{Bi(1 + \tau)} \ln \left[ \frac{H - 1 - (1 + C)/\tau}{H - 1 - Bi(1 + \tau + C) - (1 + C)/\tau} \right] - 1 - \frac{C(1 - \eta_{II})}{1 + \tau} \quad (3)$$

The new constant  $C$  is shorthand for the ratio

$$C = \frac{q_J}{q} \quad (4)$$

and is assumed known. It is easy to see that eq. (10.105) represents the  $C \rightarrow 0$  limit of the more general eq. (3).

The nondimensional average power requirement expressed by eq. (3) was minimized numerically, and the results are presented in the attached figures. The optimal time of refrigerator operation ( $\tau_{opt} = t_{1,opt} / t_2$ ) decreases with the Biot number, and increases as the  $C$  ratio increases. In the first frame, where  $\eta_{II} = 1$ ,  $\tau_{opt}$  depends only on  $Bi$ ,  $H$  and  $C$ . As shown in the lower frame and eq. (3), when  $\eta_{II} < 1$  the optimal time  $\tau_{opt}$  depends on  $\eta_{II}$  and  $T_0/T_L$ , in addition to  $Bi$ ,  $H$  and  $C$ . The second graph was drawn for  $\eta_{II} = 0.3$ ,  $T_0 = 30^\circ\text{C}$  and  $T_L = -18^\circ\text{C}$ , which means  $T_0/T_L = 1.19$ . Taken together, these two graphs show that  $\eta_{II}$  and  $T_0/T_L$  have only a minor effect on  $\tau_{opt}$  when  $C$  is smaller than 1.



Problem 10.16 We start with the first law:

$$\begin{aligned}
 \delta Q &= dU^* - \mathcal{H}d\mathcal{M} \\
 &= \left( \frac{\partial U^*}{\partial T} \right)_{\mathcal{H}} dT + \left( \frac{\partial U^*}{\partial \mathcal{H}} \right)_T d\mathcal{M} - \mathcal{H} \left[ \left( \frac{\partial \mathcal{M}}{\partial T} \right)_{\mathcal{H}} dT + \left( \frac{\partial \mathcal{M}}{\partial \mathcal{H}} \right)_T d\mathcal{H} \right] \\
 &= \left[ \left( \frac{\partial U^*}{\partial T} \right)_{\mathcal{H}} - \mathcal{H} \left( \frac{\partial \mathcal{M}}{\partial T} \right)_{\mathcal{H}} \right] dT + \left[ \left( \frac{\partial U^*}{\partial \mathcal{H}} \right)_T - \mathcal{H} \left( \frac{\partial \mathcal{M}}{\partial \mathcal{H}} \right)_T \right] d\mathcal{H} \\
 &= \left( \frac{\partial H^*}{\partial T} \right)_{\mathcal{H}} dT + T \left( \frac{\partial S}{\partial \mathcal{H}} \right)_T d\mathcal{H} \\
 &= C_H dT + T \left( \frac{\partial \mathcal{M}}{\partial T} \right)_{\mathcal{H}} d\mathcal{H}
 \end{aligned}$$

The last move consisted of using eqs. (10.115) and (10.112) from the text.

For the second part of the problem we consider the function  $S = S(T, \mathcal{M})$  and write

$$\begin{aligned}
 dS &= \left( \frac{\partial S}{\partial T} \right)_{\mathcal{M}} dT + \left( \frac{\partial S}{\partial \mathcal{M}} \right)_T d\mathcal{M} \\
 &= \frac{C_M}{T} dT - \left( \frac{\partial \mathcal{M}}{\partial T} \right)_{\mathcal{M}} d\mathcal{M}
 \end{aligned}$$

Next, from  $\delta Q = TdS$  follows the second  $\delta Q$  expression given in the problem statement.

Problem 10.17 We start with the definition of magnetic work and exploit the fact that  $dS = 0$ :

$$\begin{aligned}
 (\delta W^*)_S &= -\mathcal{H}(d\mathcal{M})_S \\
 &= -\mathcal{H} \left( \frac{\partial \mathcal{M}}{\partial T} \right)_S dT && \text{using the cyclical relation } S-\mathcal{M}-T \\
 &= \mathcal{H} \left( \frac{\partial S}{\partial T} \right)_{\mathcal{M}} \left( \frac{\partial \mathcal{M}}{\partial S} \right)_T dT && \text{using eqs. (10.114) and (10.121)} \\
 &= \mathcal{H} \frac{C_M}{T} \left( -\frac{\partial T}{\partial \mathcal{M}} \right)_{\mathcal{M}} dT && \text{using the cyclical relation } H-\mathcal{M}-T \\
 &= \mathcal{H} \frac{C_M}{T} \underbrace{\left( \frac{\partial \mathcal{M}}{\partial \mathcal{H}} \right)_T}_{>0} \underbrace{\left( \frac{\partial T}{\partial \mathcal{M}} \right)_{\mathcal{H}}}_{<0} dT && dT > 0
 \end{aligned}$$

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Problem 10.18 To prove the first relation we start with the infinitesimal version of eq. (10.118),

$$\left( \frac{\partial U^*}{\partial H} \right)_T = T \left( \frac{\partial M}{\partial T} \right)_H + H \left( \frac{\partial M}{\partial H} \right)_T = T f' \left( \frac{-H}{T^2} \right) + H f' \frac{1}{T} = 0$$

where  $M = f(H/T)$  and  $f' = df/d(H/T)$ . To prove the second relation we note that the ideal paramagnetic substance equation of state can also be written as

$$H = T \phi(M)$$

Next, at constant temperature eq. (10.121) says

$$\begin{aligned} dS &= 0 - \left( \frac{\partial H}{\partial T} \right)_M dM \\ &= -\phi dM \\ &= -\frac{H}{T} dM \end{aligned}$$


---

Problem 10.19 In the limit  $T \rightarrow 0$  we have:

$$\beta = \frac{1}{V} \left( \frac{\partial V}{\partial T} \right)_P =$$

using Maxwell's relation (4.56g)

$$= -\frac{1}{V} \left( \frac{\partial S}{\partial P} \right)_T \rightarrow 0$$

here  $(\partial S/\partial P)_T$  tends to zero when  $T \rightarrow 0$ , because of eq. (10.134)

$$c_v = T \left( \frac{\partial s}{\partial P} \right)_v \rightarrow 0$$

provided  $(\partial s/\partial T)_v$  is not infinite in the  $T \rightarrow 0$  limit

$$c_P = T \left( \frac{\partial s}{\partial T} \right)_P \rightarrow 0$$

provided  $(\partial s/\partial T)_P$  is not infinite in the  $T \rightarrow 0$  limit

$$\frac{dP}{dT} = \frac{s_f - s_s}{v_f - v_s} = \frac{(\Delta s)_T}{v_f - v_s} \rightarrow 0$$

Clapeyron's relation for liquid (f) - solid (s) mixtures. The numerator  $(\Delta s)_T$  tends to zero according to eq. (10.134)

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Problem 10.20 (a) The power  $\dot{W}$  follows from the first law written for the refrigerator/heat pump system

$$\dot{W} = \dot{Q}_2 - \dot{Q}_0 - \dot{Q}_1 \quad (1)$$

where the  $\dot{Q}$ 's are related through the second law for the reversible limit,

$$\dot{S}_{\text{gen}} = \frac{\dot{Q}_2}{T_2} - \frac{\dot{Q}_0}{T_0} - \frac{\dot{Q}_1}{T_1} = 0 \quad (2)$$

Eliminating  $\dot{Q}_0$  between eqs. (1) and (2) yields

$$\begin{aligned} \dot{W} &= \dot{Q}_2 \left(1 - \frac{T_0}{T_2}\right) - \dot{Q}_1 \left(1 - \frac{T_0}{T_1}\right) \\ &= C_2 T_2 + C_1 T_1 - 2(C_1 + C_2) T_0 + \left(\frac{C_2}{T_2} + \frac{C_1}{T_1}\right) T_0^2 \end{aligned} \quad (3)$$

(b) Solving  $\partial \dot{W} / \partial T_0 = 0$  we obtain

$$T_{0,\text{opt}} = \frac{C_1 + C_2}{\frac{C_1}{T_1} + \frac{C_2}{T_2}} \quad (4)$$

At  $T_0 = T_{0,\text{opt}}$  the power  $\dot{W}$  is minimum because

$$\frac{\partial^2 \dot{W}}{\partial T_0^2} = 2 \left(\frac{C_2}{T_2} + \frac{C_1}{T_1}\right) > 0$$

The minimum  $\dot{W}$  that corresponds to  $T_{0,\text{opt}}$  is

$$\dot{W}_{\min} = \frac{(T_1 - T_2)^2}{\frac{T_1}{C_1} + \frac{T_2}{C_2}}$$

(c) According to the second law (2), the case  $\dot{Q}_0 = 0$  means

$$\frac{\dot{Q}_2}{T_2} = \frac{\dot{Q}_1}{T_1}$$

$$C_2 \left( 1 - \frac{T_0}{T_2} \right) = C_1 \left( \frac{T_0}{T_1} - 1 \right)$$

which means that

$$(T_0)_{Q_0=0} = \frac{C_1 + C_2}{\frac{C_1}{T_1} + \frac{C_2}{T_2}}$$

This ambient temperature is the same as in eq. (4).

(d) The entropy generation rate for the aggregate system (ice storage facility + refrigerator/heat pump + building) is

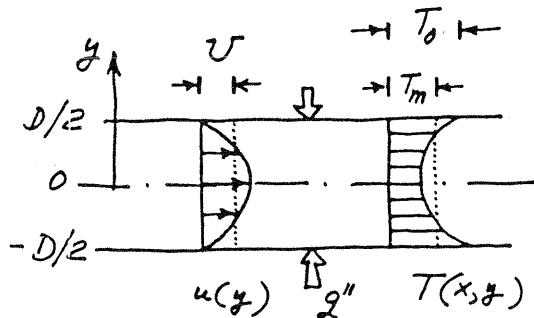
$$\begin{aligned}\dot{S}_{\text{gen}} &= \frac{\dot{Q}_2}{T_0} - \frac{\dot{Q}_0}{T_0} - \frac{\dot{Q}_1}{T_0} \\ &= \frac{1}{T_0} (\dot{Q}_2 - \dot{Q}_0 - \dot{Q}_1)\end{aligned}\tag{5}$$

(e) Combining eqs. (5) with (1) we rediscover the Gouy-Stodola theorem,  $\dot{S}_{\text{gen}} = \dot{W}/T_0$ .

---

**THERMODYNAMIC OPTIMIZATION**

Problem 11.1 The main features of Hagen-Poiseuille flow with uniform heat flux in a parallel-plate channel are shown in the sketch. The velocity profile is well known,



$$u(y) = \frac{3}{2} U \left[ 1 - \left( \frac{y}{D/2} \right)^2 \right]$$

$$U = \frac{D^2}{12\mu} \left( -\frac{dp}{dx} \right)$$

and the temperature profile can be determined from the energy equation

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

↑                      ↑                      ↑  
 constant              zero                    negligible with  
 respect to  $\partial^2 T / \partial y^2$

It can be shown \* that the constant gradient  $\partial T / \partial x$  is

$$\frac{\partial T}{\partial x} = \frac{dT_0}{dx} = \frac{dT_m}{dx}$$

where

$$\rho U \frac{D}{2} c_P dT_m = q'' dx$$

or

$$\frac{dT_m}{dx} = \frac{2}{UD} \frac{q''}{\rho c_P}$$

After these simplifications the energy equation can be written as

\* A. Bejan, *Convection Heat Transfer*, Wiley, New York, 1984, 1989, chapter 3.

$$\frac{3}{2} \frac{U}{\alpha} (1 - n^2) \frac{D^2}{4} \frac{dT_m}{dx} = \frac{\partial^2 T}{\partial n^2}; \quad n = \frac{y}{D/2}$$

and integrated twice, subject to

$$\frac{\partial T}{\partial n} = 0 \text{ at } n = 0, \quad \text{and} \quad T = T_0 \text{ at } n = 1.$$

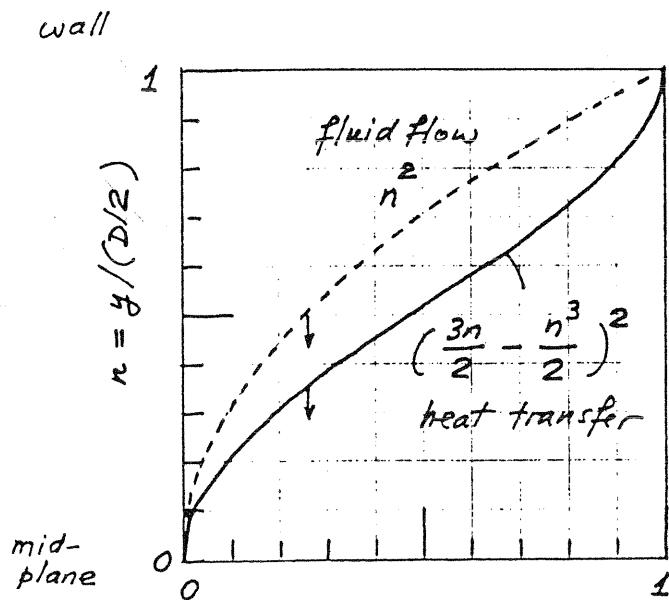
We obtain the temperature distribution

$$T_0 - T = \frac{3}{4\alpha} \frac{q'' D}{\rho c_p} \left( \frac{5}{12} - \frac{n^2}{2} + \frac{n^4}{12} \right).$$

The local rate of entropy generation becomes

$$\begin{aligned} s_{gen}''' &= \frac{k}{T^2} \left( \frac{\partial T}{\partial y} \right)^2 + \frac{\mu}{T} \left( \frac{\partial u}{\partial y} \right)^2 = \dots \\ &= \underbrace{\frac{1}{k} \left( \frac{q''}{T} \right)^2 \left( \frac{3n}{2} - \frac{n^3}{2} \right)^2}_{s_{gen}, \text{heat transfer}} + \underbrace{36 \frac{\mu U^2}{TD^2} n^2}_{s_{gen}, \text{fluid flow}} \end{aligned}$$

The actual  $s_{gen}'''$  profile depends on the relative importance of heat transfer and fluid flow irreversibilities: the dimensionless profiles of each of these contributions are shown below. Regardless of origin,  $s_{gen}'''$  is concentrated near the wall.



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**Problem 11.2** When there are n parallel channels, the entropy generation for the entire passage equals n times the entropy generation of one channel:

$$\begin{aligned}\dot{S}_{\text{gen}} &= n \dot{S}_{\text{gen, one channel}} \\ &= n \left[ \frac{\Delta T q' / n}{T^2} + \frac{\dot{m} / n}{\rho T} \left( -\frac{dP}{dx} \right) \right]\end{aligned}$$

For the heat transfer part, we use the following definitions

$$\Delta T = \frac{q' / n}{W h}, \quad Nu = \frac{h D / n}{k}, \quad W = \text{the width of the stack, normal to both } D \text{ and } \dot{m}$$

For the fluid friction part we have

$$-\frac{dP}{dx} = f \frac{4}{D/n} \frac{\rho u^2}{2},$$

where  $u = \frac{\dot{m}}{\rho DW}$  is independent of n. In the end,  $\dot{S}_{\text{gen}}$  can be expressed as

$$\dot{S}_{\text{gen}} = \frac{(q')^2 D}{n^2 W T^2 N u k} + \frac{4 m f}{\rho T D} n \frac{\rho u^2}{2}.$$

If the flow is laminar,  $Nu = 8.235$  (constant), while the friction factor is given by

$$f = \frac{24}{Re_{D_h}}, \quad \text{where} \quad Re_{D_h} = \frac{u(2D/n)}{v}$$

This formula is based on the assumption that the flow is fully developed with prescribed uniform heat flux. The  $\dot{S}_{\text{gen}}$  expression becomes

$$\dot{S}_{\text{gen}} = \underbrace{\frac{(q')^2 D}{W T^2 N u k}}_A n^{-2} + \underbrace{\frac{24 m v u}{T D^2}}_B n^2$$

To calculate the optimal number of channels we solve

$$\frac{\partial}{\partial n} (An^{-2} + Bn^2) = 0$$

and obtain

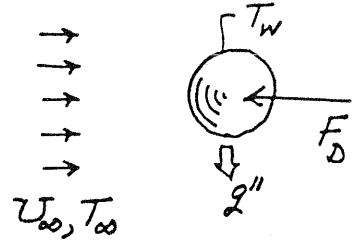
$$n_{\text{opt}} = (A/B)^{1/4}$$

The fully developed turbulent flow regime can be handled in the same way, using appropriate correlations for Nu and f.

---

Problem 11.3 The entropy generated by the convective heat transfer arrangement shown in the sketch is

$$\dot{S}_{\text{gen}} = \frac{\dot{Q}}{T_{\infty}^2} (T_w - T_{\infty}) + \frac{F_D U_{\infty}}{T_{\infty}}$$



The usual definitions

$$\bar{h} = \frac{q}{\Delta T}, \quad q'' = \frac{\dot{Q}}{\pi D^2}, \quad \text{Nu} = \frac{\bar{h}D}{k}$$

$$F_D = C_D \frac{\pi D^2}{4} \frac{\rho U_{\infty}^2}{2}, \quad \text{Re}_D = \frac{U_{\infty} D}{v}$$

allow us to write  $\dot{S}_{\text{gen}}$  in terms of the degree of freedom in the design,  $\text{Re}_D$ ,

$$\dot{S}_{\text{gen}} = \frac{(\dot{Q}/T_{\infty})^2}{\text{Nu} k \pi D} + \frac{\pi}{8} C_D \frac{\rho v^2 U_{\infty}}{T_{\infty}} \text{Re}_D^2$$

where  $C_D \approx 0.5$  and

$$\text{Nu} \approx 0.26 \text{ Pr}^{0.3} \text{ Re}_D^{0.6}, \text{ for gases in the range } 10^3 < \text{Re}_D < 10^5$$

In conclusion

$$\dot{S}_{\text{gen}} = \underbrace{\frac{U_{\infty}}{v} \left( \frac{\dot{Q}}{T_{\infty}} \right)^2 \frac{\text{Pr}^{-0.3}}{(0.26)\pi k}}_A \text{Re}_D^{-1.6} + \underbrace{\frac{\pi}{16} \frac{\rho v^2 U_{\infty}}{T_{\infty}}}_B \text{Re}_D^2$$

Solving  $\partial \dot{S}_{\text{gen}} / \partial \text{Re}_D = 0$  yields the optimal dimensionless sphere diameter

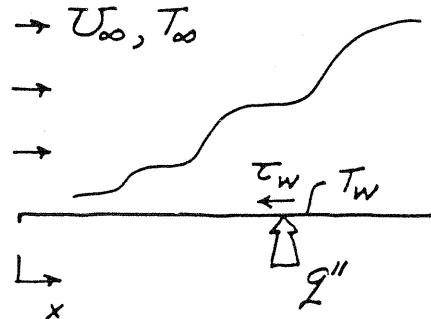
$$\text{Re}_{D,\text{opt}} = \left( 0.8 \frac{A}{B} \right)^{1/3.6}$$

Similar optima for other geometries can be found in Refs. 2 and 12.

---

Problem 11.4 It is sufficient to minimize the irreversibility caused by only one side of the plate fin. Per unit length in the  $x$  direction, the entropy generation is

$$\frac{d\dot{S}_{gen}'}{dx} = \frac{W}{T_{\infty}^2} q'' (T_w - T_{\infty}) + \frac{U_{\infty}}{T_{\infty}} W \tau_w$$



where  $W$  is the width of the fin in the direction normal to the sketch. The product  $W\tau_w$  is the drag force per unit length,  $dF_D/dx$ . Using the heat transfer and skin friction correlations given in Refs. 8 and 13, we have

$$h_x = \frac{C_{f,x}}{2} \rho c_p U_{\infty} \text{Pr}^{-2/3}, \quad \text{where} \quad C_{f,x} = 0.0576 \text{Re}_x^{-1/5} \quad \text{and} \quad \text{Re}_x = U_{\infty} x / v$$

Recognizing that  $q' = q''L$  is fixed, the local entropy generation rate becomes

$$\frac{d\dot{S}_{gen}'}{dx} = \text{Re}_L^{-2} \underbrace{\frac{(q')^2 U_{\infty} W 2 \text{Pr}^{2/3}}{T_{\infty}^2 \rho c_p v^2 (0.0576)}}_a \text{Re}_x^{1/5} + \underbrace{\frac{\rho U_{\infty}^3 W}{2 T_{\infty}} (0.0576)}_b \text{Re}_x^{-1/5}$$

Another way of writing this is

$$\frac{U_{\infty}}{v} \frac{d\dot{S}_{gen}'}{d \text{Re}_x} = a \text{Re}_L^{-2} \text{Re}_x^{1/5} + b \text{Re}_x^{-1/5}$$

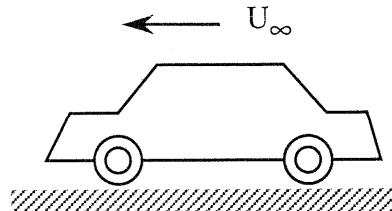
The total entropy generation rate is

$$\begin{aligned} \frac{U_{\infty}}{v} \dot{S}_{gen}' &= \int_0^{\text{Re}_L} (a \text{Re}_L^{-2} \text{Re}_x^{1/5} + b \text{Re}_x^{-1/5}) d \text{Re}_x \\ &= \frac{5a}{6} \text{Re}_L^{-4/5} + \frac{5b}{4} \text{Re}_L^{4/5} \end{aligned}$$

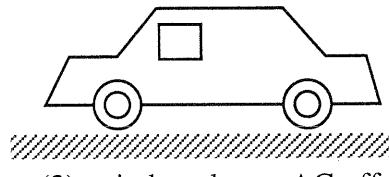
To determine the optimal length, we write  $\partial \dot{S}_{gen}' / \partial \text{Re}_L$  and obtain

$$Re_{L_{opt}} = \left( \frac{16a}{30b} \right)^{5/8}$$

Problem 11.5 The measure of irreversibility in a car is the mechanical power delivered by the car's engine. This power is dissipated fully in order to overcome friction (e.g. ground, air) and/or to run the air-conditioner (the air-conditioner is a refrigeration installation whose power requirement is proportional to the entropy generated in the cold space).



(1) window up; AC on



(2) window down; AC off

We have to compare two regimes of operation, where  $\dot{W}$  is the power output of the engine,

$$\dot{W}_1 = \dot{W}_{AC} + U_\infty \left( C_{D_1} A \frac{\rho U_\infty^2}{2} \right)$$

$$\dot{W}_2 = 0 + U_\infty \left( C_{D_2} A \frac{\rho U_\infty^2}{2} \right)$$

Regime (1) is more economical if  $\dot{W}_1 < \dot{W}_2$ , which means

$$U_\infty > \left[ \frac{2\dot{W}_{AC}}{\rho A (C_{D_2} - C_{D_1})} \right]^{1/3} = 18.4 \frac{m}{s} = 66.2 \text{ km / hr, or } 41.2 \text{ miles per hour}$$

Problem 11.6 First we recognize that

$$C_{min} = (\dot{mc})_{oil}, \text{ and } C_{max} = (\dot{mc})_{water}$$

and that  $\frac{C_{\min}}{C_{\max}} = 0.5$ .

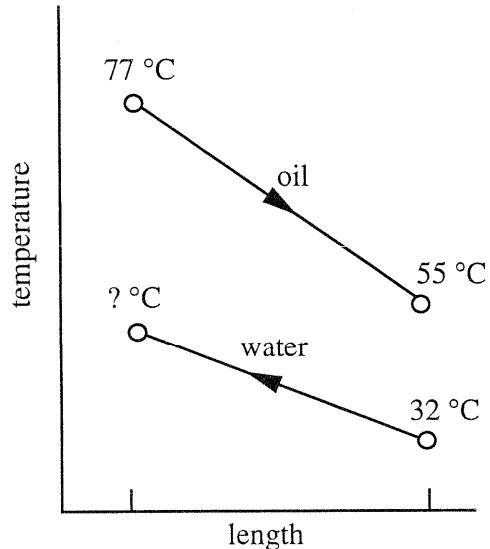
To calculate the effectiveness we write

$$\varepsilon = \frac{C_{\min} (T_{\text{oil,in}} - T_{\text{oil,out}})}{C_{\min} (T_{\text{oil,in}} - T_{\text{water,in}})}$$

maximum  
temperature excursion

hence

$$\varepsilon = \frac{77 - 55}{77 - 32} = 0.49.$$



From the  $\varepsilon(N_{tu})$  relation for counterflow heat exchangers [8]

$$\varepsilon = \frac{1 - \exp \left[ -N_{tu} \left( 1 - \frac{C_{\min}}{C_{\max}} \right) \right]}{1 - \frac{C_{\min}}{C_{\max}} \exp \left[ -N_{tu} \left( 1 - \frac{C_{\min}}{C_{\max}} \right) \right]},$$

we obtain  $N_{tu} = 0.785$ . Neglecting the fluid friction irreversibility, we find that the entropy generation rate for the entire heat exchanger is

$$\dot{S}_{\text{gen}} = \left( \dot{m}c \ln \frac{T_{\text{out}}}{T_{\text{in}}} \right)_{\text{water}} + \left( \dot{m}c \ln \frac{T_{\text{out}}}{T_{\text{in}}} \right)_{\text{oil}}$$

or, numerically,

$$\frac{\dot{S}_{\text{gen}}}{(\dot{m}c)_{\text{oil}}} = 0.0059.$$

In order to calculate the above number, we had to invoke the first law to estimate the water outlet temperature,

$$T_{\text{water,out}} = 43^\circ\text{C}$$

In the second part of the problem the stream-to-stream contact area increases by 20 percent. The new  $N_{tu}$  is therefore

$$N_{tu} = (1.2) (0.785) = 0.942.$$

Repeating step-by-step the preceding analysis we obtain, in order,

$$\varepsilon = 0.546$$

$$T_{\text{oil,out}} = 52.43^\circ\text{C}$$

$$T_{\text{water,out}} = 44.29^\circ\text{C}$$

$$\frac{\dot{S}_{\text{gen}}}{(\dot{m}c)_{\text{oil}}} = 0.00622 \quad \begin{aligned} & \left( \text{note that this value is greater than} \right) \\ & \text{in the original case.} \end{aligned}$$

It might seem paradoxical that increasing the  $N_{tu}$  (i.e. improving the thermal contact between streams) has led to an increase in  $\dot{S}_{\text{gen}}$ . The reason for this increase is that in the latest design the total heat exchange between streams is larger than in the original design (note that the entropy generated by heat transfer across a finite  $\Delta T$  is proportional to both heat transfer rate and  $\Delta T$ ). A correct measure of the thermodynamic improvement associated with increasing the  $N_{tu}$  is the comparison between the following numbers

$$\frac{\dot{S}_{\text{gen}}}{[\dot{m}c(T_{\text{in}} - T_{\text{out}})]_{\text{oil}}} = \begin{cases} 0.00027 \text{ K}^{-1}, & \text{when } N_{tu} = 0.785 \\ 0.00026 \text{ K}^{-1}, & \text{when } N_{tu} = 0.942 \end{cases}$$

Note that  $\dot{S}_{\text{gen}} / [\dot{m}c(T_{\text{in}} - T_{\text{out}})]_{\text{oil}}$  is the entropy generation rate per unit of heat transfer rate: this unit of irreversibility decreases steadily as  $N_{tu}$  increases.

Problem 11.7 The comparison is made for the same number of entropy generation units, meaning that the  $N_{S,1,\min}$  of eq. (11.59) is the same as the  $N_{S,1}$  appearing in the denominator of eq. (11.72),

$$N_{S,1,\min} = 2\tau \left( \frac{R}{c_P} \right)^{1/2} g_1 \left( \frac{f_1}{St_1} \right)^{1/2} = N_{S,1} \quad (\text{a})$$

The dimensionless volume that corresponds to the fixed-g design represented by eq. (11.58) is, cf. eq. (11.68),

$$v_{1,g} = \left( \frac{4L}{D_h} \right)_{1,\text{opt}} \text{Re}_1 g_1^{-2} \quad (\text{b})$$

where  $(4L/D_h)_{1,\text{opt}}$  is given by eq. (11.58). Dividing  $v_{1,g}$  by the  $v_{1,\min}$  of eq. (11.72) we obtain, after using eqs. (a, b),

$$\frac{v_{1,g}}{v_{1,min}} = \frac{32}{27} = 1.185$$

This shows that when the entropy generation number is fixed, the constant-g design yields a volume that exceeds by only 18.5 percent the minimum volume.

---

**Problem 11.8** The dimensionless area in the optimal constant-g design is obtained by writing eq. (11.62),

$$a_{1,g} = \left( \frac{4L}{D_h} \right)_{1,opt} g_1^{-1} \quad (1)$$

in which  $(4L/D_h)_{1,opt}$  is given by eq. (11.58). The equal- $N_{S,1}$  condition means that the  $N_{S,1,min}$  of eq. (11.59) is the same as the  $N_{S,1}$  appearing in eq. (11.66),

$$N_{S,1} = 2\tau \left( \frac{R}{c_P} \right)^{1/2} g_1 \left( \frac{f_1}{St_1} \right)^{1/2} \quad (2)$$

Using eqs. (1) and (2) to evaluate the ratio  $a_{1,g}/a_{1,min}$  we obtain

$$\frac{a_{1,g}}{a_{1,min}} = \left( \frac{27}{16} \right)^{1/2} = 1.299$$

showing that the area of the optimal constant-g design is 30 percent greater than the minimum area that can be achieved for the same number of entropy generation units for that side.

---

**Problem 11.9** Consider the adiabatic mixing of the same two streams that enter the counterflow heat exchanger, that is, the streams  $(\dot{m}c_P, T_1)$  and  $(\dot{m}c_P, T_2)$ . The mixed stream that leaves the adiabatic chamber is  $(2\dot{m}c_P, T_{out})$ . The first law and the second law statements for the mixing chamber are

$$\dot{m}c_P(T_1 - T_0) + \dot{m}c_P(T_2 - T_0) = 2\dot{m}c_P(T_{out} - T_0)$$

$$\begin{aligned} \dot{S}_{gen} &= 2\dot{m}s_{out} - \dot{m}s_1 - \dot{m}s_2 \\ &= \dot{m}c_P \ln \frac{T_{out}}{T_1} + \dot{m}c_P \ln \frac{T_{out}}{T_2} \geq 0 \end{aligned}$$

The first law yields  $T_{out} = (T_1 + T_2)/2$ . Dividing the second law statement by  $\dot{m}c_P$  we obtain, in order,

$$\frac{\dot{S}_{gen}}{\dot{m}c_P} = N_S = \ln \frac{T_{out}^2}{T_1 T_2} = \ln \frac{(T_1 + T_2)^2}{4T_1 T_2} = \ln \left[ \frac{1}{2} + \frac{1}{4} \left( \frac{T_1}{T_2} + \frac{T_2}{T_1} \right) \right]$$

This quantity is the same as the maximum indicated by eq. (11.81). In conclusion, the maximum entropy generation rate of the counterflow heat exchanger treated in the text is the same as in the case of the adiabatic mixer that receives the inlet streams of the heat exchanger.

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Problem 11.10 (a) The volume corresponding to the minimum area design ( $v_{1,a}$ ) is obtained by combining eq. (11.68) with the  $(4L/D_h)_1$  value indicated by eq. (11.62),

$$v_{1,a} = (a_1 g_1) Re_1 g_1^{-2}$$

The minimum volume ( $v_{1,min}$ ) follows from eq. (11.72), in which  $N_{S,1}$  is the same as the  $N_{S,1,min}$  shown in eq. (11.65). Dividing  $v_{1,a}$  by  $v_{1,min}$  and making sure to eliminate  $g_1$  by using eq. (11.64) yields, after some algebra,

$$\frac{v_{1,a}}{v_{1,min}} = \frac{256}{243} = 1.053$$

(b) The dimensionless area of the minimum volume design ( $a_{1,v}$ ) is obtained by substituting into eq. (11.62) the  $(4L/D_h)$  value given by eq. (11.68),

$$a_{1,v} = v_1 g_1 Re_1^{-1}$$

Worth keeping in mind is that  $g_1$  has the value given by eq. (11.70). Finally, we divide  $a_{1,v}$  by the  $a_{1,min}$  of eq. (11.66), from which the  $N_{S,1}$  number is eliminated based on eq. (11.72). In the end we arrive at

$$\frac{a_{1,v}}{a_{1,min}} = \frac{3^{7/2}}{2^{11/2}} = 1.033$$


---

Problem 11.11 We assume that the coolant (air) is well mixed at the temperature  $T_{mix}$  inside the electronic package. The heat transfer from the electronics to the air stream can be expressed in two ways,

$$\dot{Q} = UA (T_e - T_{mix}) \quad (1)$$

$$\dot{Q} = \dot{m}c_p (T_{mix} - T_0) \quad (2)$$

where  $A$  is the total heat transfer area. Eliminating  $T_{mix}$  we obtain the relation between the mass flow rate and the cooling "job" represented by  $(\dot{Q}, T_e)$ :

$$\frac{1}{\dot{m}c_p} + \frac{1}{UA} = \frac{T_e - T_0}{\dot{Q}} \quad (3)$$

The required fan power  $\dot{W}$  is proportional to the product  $\dot{m}\Delta P$ , where  $\Delta P$  is the air pressure drop across the package. We can show this analytically,

$$\begin{aligned}\dot{W} &= \frac{1}{\eta_{II}} \dot{W}_{rev} = \frac{1}{\eta_{II}} \dot{m} c_P (T_{out, rev} - T_{mix}) \\ &= \frac{1}{\eta_{II}} \dot{m} c_P T_{mix} \left( \frac{T_{out, rev}}{T_{mix}} - 1 \right)\end{aligned}\quad (4)$$

where the isentropic temperature ratio across the fan is

$$\frac{T_{out, rev}}{T_{mix}} = \left( \frac{P_0}{P_0 - \Delta P} \right)^{R/c_P} \quad (5)$$

Assuming that  $\Delta P \ll P_0$ , and that  $T_{mix}$  is comparable with  $T_0$ , we linearize eq. (4) and obtain:

$$\dot{W} = \left( \frac{RT_0}{\eta_{II} P_0} \right) \dot{m} \Delta P \quad (6)$$

The pressure drop is related to A and the flow cross-section  $A_f$  by

$$\Delta P = f \frac{A}{A_f} \frac{1}{2} \rho V^2 \quad (7)$$

where

$$V = \frac{\dot{m}}{\rho A_f} \quad (8)$$

Similarly, the overall heat transfer coefficient U is given by  $U = St \rho c_P V$ , which can be written as

$$UA = St \dot{m} c_P \frac{A}{A_f} \quad (9)$$

Combining eqs. (9) and (3) we obtain

$$\dot{m} c_P = \frac{\dot{Q}}{T_e - T_0} \left( 1 + \frac{1}{St} \cdot \frac{A_f}{A} \right) \quad (10)$$

We further assume that the flow passage is extremely "rough" such that  $f$  and  $St$  are relatively insensitive to changes in the flow rate. Substituting these results into eq. (6) we find that  $\dot{W}$  is proportional to

$$\tilde{W} = \dot{m}^3 \frac{A}{A_f^3} \sim \left( 1 + \frac{1}{St} \cdot \frac{A_f}{A} \right)^3 \frac{A}{A_f^3} \quad (11)$$

This quantity decreases monotonically as the flow cross-sectional area  $A_f$  increases. The same expression has a minimum with respect to the heat transfer area: solving  $\partial \tilde{W} / \partial A = 0$  we obtain the optimal heat transfer area for minimum fan power,

$$A_{\text{opt}} = \frac{2}{St} A_f \quad (12)$$

In flow passages with extreme roughness  $St$  is of order  $10^{-2}$ . This means that  $A_{\text{opt}}$  should be of the order of  $100A_f$ .

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Problem 11.12 Let us assume that the heat generation rate  $\dot{Q}$  is distributed uniformly over the length traveled by the stream,

$$q' = \frac{\dot{Q}}{L}, \quad \text{constant} \quad (1)$$

The temperature of the electric components increases in the direction of flow. Their highest temperature ( $T_e$ ) is at the downstream end, where the electronics-coolant temperature difference is related to the local heat generation rate:

$$q' = p U (T_e - T_{\text{out}}) \quad (2)$$

In this expression  $p$  is the "wetted perimeter", i.e. the contact area per unit length,

$$p = \frac{A}{L} \quad (3)$$

Combining eqs. (1) - (3) we obtain

$$\dot{Q} = UA (T_e - T_{\text{out}}) \quad (4)$$

Eliminating  $T_{\text{out}}$  between eq. (4) and the first law for the electronics enclosure

$$\dot{Q} = \dot{m}c_p (T_{\text{out}} - T_0) \quad (5)$$

yields

$$\frac{1}{\dot{m}c_p} + \frac{1}{UA} = \frac{\dot{Q}}{T_e - T_0} \quad (6)$$

Equation (6) is identical to the equation shown as eq. (3) at the start of the solution to the preceding problem. The rest of the solution remains unchanged, because  $\dot{W}$  is proportional to  $\dot{m}\Delta P$ ,

$$\dot{W} = \left( \frac{RT_0}{\eta_{\text{II}} P_0} \right) \dot{m}\Delta P \quad (7)$$

the pressure drop is related to A and the flow cross-section  $A_f$  by

$$\Delta P = f \frac{A}{A_f} \frac{1}{2} \rho V^2 \quad (8)$$

where

$$V = \frac{\dot{m}}{\rho A_f} \quad (9)$$

Similarly, the heat transfer coefficient U is given by  $U = St c_p V$ , which can be written as

$$UA = St \dot{m} c_p \frac{A}{A_f} \quad (10)$$

Combining eqs. (10) and (6) we obtain

$$\dot{m} c_p = \frac{\dot{Q}}{T_e - T_0} \left( 1 + \frac{1}{St} \cdot \frac{A_f}{A} \right) \quad (11)$$

We further assume that the flow passage is extremely "rough" such that  $f$  and  $St$  are relatively insensitive to changes in the flow rate. Substituting these results into eq. (7) we find that  $\tilde{W}$  behaves as

$$\tilde{W} = \dot{m}^3 \frac{A}{A_f^3} \sim \left( 1 + \frac{1}{St} \cdot \frac{A_f}{A} \right)^3 \frac{A}{A_f^3} \quad (12)$$

This quantity decreases monotonically as the flow cross-sectional area  $A_f$  increases. The same expression has a minimum with respect to the heat transfer area: solving  $\partial \tilde{W} / \partial A = 0$  we obtain the optimal heat transfer area for minimum fan power,

$$A_{opt} = \frac{2}{St} A_f \quad (13)$$

In flow passages with extreme roughness  $St$  is of order  $10^{-2}$ . This means that  $A_{opt}$  should be of the order of  $10^2 A_f$ .

Problem 11.13 We assume nonlinear relations between pressure drops and flow rate,

$$\Delta P_1 = r_1 \dot{m}^n \quad (1)$$

$$\Delta P_2 = r_2 \dot{m}^n \quad (2)$$

The power output of the turbine,

$$\dot{W} = \eta_t \dot{m} c_P T_1 \left[ 1 - \left( \frac{P_2 + \Delta P_2}{P_1 - \Delta P_1} \right)^{R/c_p} \right] \quad (3)$$

can be expressed as a function of  $\Delta P_1$  by eliminating  $\Delta P_2$  and  $\dot{m}$  between eqs. (1) - (3). If we further assume that  $\Delta P_1 \ll P_1$  and  $\Delta P_2 \ll P_2$ , the resulting expression is

$$\dot{W} = \eta_t c_P T_1 \left( \frac{P_1}{r_1} \right)^{1/n} y_1^{1/n} (1 - b - aby) \quad (4)$$

where

$$y = \frac{\Delta P_1}{P_1}, \quad a = \frac{R}{c_P} \left( 1 + \frac{r_2 P_1}{r_1 P_2} \right), \quad b = \left( \frac{P_2}{P_1} \right)^{R/c_P} \quad (5)$$

Solving  $\partial \dot{W} / \partial y = 0$  we obtain the condition for operation at maximum power

$$y_{\text{opt}} = \frac{\Delta P_{1,\text{opt}}}{P_1} = \frac{1-b}{ab(1+n)} \ll 1 \quad (6)$$

$$\dot{W}_{\text{max}} = \eta_t c_P T_1 \left( \frac{P_1}{r_1} \right)^{1/n} \frac{n}{(ab)^{1/n}} \left( \frac{1-b}{1+n} \right)^{1+1/n} \quad (7)$$

$$\dot{m}_{\text{opt}} = \left( \frac{\Delta P_{1,\text{opt}}}{r_1} \right)^{1/n}, \quad \Delta P_{2,\text{opt}} = \frac{r_2}{r_1} \Delta P_{1,\text{opt}} \quad (8)$$

It is instructive to compare the maximum power output (7) for finite  $r_1$  and  $r_2$  with the power at the same flowrate ( $\dot{m}_{\text{opt}}$ ) in the limit of zero pressure drops ( $r_1 = r_2 = 0$ ), namely

$$\dot{W}_{r_1=r_2=0} = \eta_t \dot{m}_{\text{opt}} c_P T_1 \left[ 1 - \left( \frac{P_2}{P_1} \right)^{R/c_P} \right] \quad (9)$$

The ratio

$$\frac{\dot{W}_{\text{max}}}{\dot{W}_{r_1=r_2=0}} = \frac{n}{1+n} < 1 \quad (10)$$

is equal to 1/2 when the pressure drop relations (1,2) are linear ( $n = 1$ ).

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**Problem 11.14** The analysis of the ideal-gas compressor with inlet and outlet pressure drops can be carried out by using Fig. P11.14 and the given pressure drop model. The isentropic efficiency of the compression stage (or sequence of stages) is  $\eta_c$ . The compressor power input

$$\dot{W}_c = \frac{1}{\eta_c} \dot{m} c_p T_1 \left[ \left( \frac{P_2 + \Delta P_2}{P_1 - \Delta P_1} \right)^{R/c_p} - 1 \right] \quad (1)$$

can be reduced to

$$\dot{W}_c = \frac{1}{\eta_c} c_p T_1 \left( \frac{P_1}{r_1} \right)^{1/n} y^{1/n} [b - 1 + aby] \quad (2)$$

where  $\Delta P_1 \ll P_1$ ,  $\Delta P_2 \ll P_2$  and

$$y = \frac{\Delta P_1}{P}, \quad a = \frac{R}{c_p} \left( 1 + \frac{r_2 P_1}{r_1 P_2} \right), \quad b = \left( \frac{P_2}{P_1} \right)^{R/c_p} \quad (3)$$

Note that in a compressor  $b$  is greater than 1, and  $\dot{W}_c$  does not have a minimum with respect to  $y$  when  $y$  is positive. The absence of an optimal  $\Delta P_1$ , or an optimal pressure ratio across the inner compartment of the compressor model is analogous to the absence of an optimal temperature ratio across the inner (reversible) compartment of a refrigerator with warm and cold thermal resistances. The common feature of compressors and a refrigerators is that they require power inputs. In contrast to these, turbines and power plants are producers of power and have optimal pressure and temperature ratios.

Conclusions similar to those reached for the compressor are obtained in an analysis of a steady flow pump that operates on an incompressible fluid. When the finite intake and discharge flow resistances are taken into account, it is found that the pump power requirement increases monotonically with the pressure drops, or with the flow rate.

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Problem 11.15 At every location  $x$  along the counterflow, there is heat transfer from the hot stream to the wall

$$q'_h \cong hD(T_h - T_w) \quad (1)$$

from the cold stream to the wall,

$$q'_c \cong hD(T_c - T_w) \quad (2)$$

and from the hot stream to the cold stream,

$$q'_m \cong hD (T_h - T_m) \quad (3)$$

$$q'_m \cong hD (T_m - T_c) \quad (4)$$

where  $T_m$  is the temperature of the tube wall along which the hot and cold tubes touch. Equations (3) and (4) show that

$$T_m = \frac{1}{2} (T_h + T_c) \quad (5)$$

Equations (1)-(4) are based on the assumption that thermal resistances are dominated by convection, where  $h$  is the heat transfer coefficient inside the hot and cold tubes. The tube diameter is the approximate transversal length scale of contact between each stream and the wall, and between the two streams.

Applying the first law of thermodynamics to a slice  $dx$  of the hot tube, we obtain

$$-\dot{m}c_p \frac{dT_h}{dx} = q'_h + q'_m \quad (6)$$

where  $\dot{m}c_p$  is the capacity rate of the hot stream, and  $\dot{m}$  and  $x$  point in the same direction. For the cold stream, the first law requires

$$\dot{m}c_p \frac{dT_c}{dx} = q'_c - q'_m \quad (7)$$

Equations (6) and (7) can be written with the help of eqs. (1)-(4) as

$$-\theta'_h = \theta_h + \frac{1}{2} (\theta_h - \theta_c) \quad (8)$$

$$\theta'_c = \theta_c - \frac{1}{2} (\theta_h - \theta_c) \quad (9)$$

where  $(\cdot)' = d(\cdot)/d\xi$ ,  $\xi = xhD/mc_p$ , and  $\theta = T - T_w$ . Subtracting eqs. (8) and (9) we find  $(\theta_h + \theta_c)' = 0$ , which means that

$$\theta_h + \theta_c = \text{constant} = 2\theta_m \quad (10)$$

Adding eqs. (8) and (9), using eq. (10) and integrating in  $x$ , yields

$$-\theta_h + \theta_c = 2\theta_m\xi + c_1 \quad (11)$$

for which  $c_1$  is determined from the hairpin-end,  $x = L$ , where  $\xi = \xi_L = LhD/mc_p$  and  $\theta_h = \theta_c = \theta_m$ . The constant is  $c_1 = -2\theta_m\xi_L$ .

Combining eq. (9) with eq. (11) to eliminate  $\theta_h$ , we obtain a differential equation for  $\theta_c(\xi)$ . Integrating that equation and evaluating the constant at  $\xi = \xi_L$  where  $\theta_c = \theta_m$ , we find

$$\theta_c = 2\theta_m e^{\xi - \xi_L} + \theta_m (\xi_L - 1 - \xi) \quad (12)$$

The solution for  $\theta_h$  follows from eq. (11):

$$\theta_h = -2\theta_m e^{\xi - \xi_L} + \theta_m (-\xi_L + \xi + 3) \quad (13)$$

The highest temperature in the system is at the hot stream inlet:  $\theta_{in} = \theta_h$  at  $\xi = 0$ . The lowest is at the cold stream outlet,  $\theta_{out} = \theta_c$  at  $\xi = 0$ . We find that  $\theta_{out} > \theta_{in}$  such that

$$\frac{\theta_{in}}{\theta_m} = 1 + [2(1 - e^{-\xi_L}) - \xi_L] \quad (14)$$

$$\frac{\theta_{out}}{\theta_m} = 1 - [2(1 - e^{-\xi_L}) - \xi_L] \quad (15)$$

Equation (14) shows that to specify  $\theta_{in}$  means to specify  $\theta_m$  and  $\theta_{out}$ , because  $\xi_L$  is known. The stream-to-stream temperature difference decreases to zero at the hairpin turn,

$$\theta_h - \theta_c = 2\theta_m (2 + \xi - \xi_L) - 4\theta_m e^{\xi - \xi_L} \quad (16)$$

The rate at which the two streams heat the wall is independent of  $x$ :

$$q'_h + q'_c = 2hD\theta_m, \text{ constant} \quad (17)$$

In conclusion, the two-stream counterflow acts as a line heat source on the wall. The constructal question is how to shape—how to snake—a continuous line of fixed length on a plane wall, such that the wall temperature develops the least excessive cold spots.

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## Chapter 12

# IRREVERSIBLE THERMODYNAMICS

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Problem 12.1 In the case of a single-flux phenomenon the second law (12.16) (i.e. the inequality part of that statement) reads

$$J_1 X_1 \geq 0 \quad (a)$$

or, after the linearization of the  $J_1(X_1)$  relation,  $J_1 = L_{11}X_1$ , eq. (12.25),

$$L_{11} X_1^2 \geq 0 \quad (b)$$

The second law implication is that  $L_{11}$  cannot be negative. It remains to find the relation between  $L_{11}$  and  $k_e$  in the case of pure electrical conduction. For this we compare Ohm's law

$$e J_N = -k_e \frac{d\phi}{dx} \quad (c)$$

with  $J_1 = L_{11}X_1$ , in which  $J_1 \equiv J_N$  and  $X_1 \equiv \frac{\partial}{\partial x} \left( -\frac{e\phi}{T} \right)$ ,

$$\begin{aligned} J_N &= L_{11} \frac{\partial}{\partial x} \left( -\frac{e\phi}{T} \right) \\ &= -\frac{e}{T} L_{11} \frac{\partial \phi}{\partial x} \end{aligned} \quad (d)$$

Put together, eqs. (c) and (d) show that

$$L_{11} \equiv \frac{T}{e^2} k_e$$

proving that the second law (b) requires  $k_e \geq 0$ .

---

Problem 12.2 Having identified the proper pairs of conjugate forces and fluxes, eqs. (12.40) - (12.41), we write according to eq. (12.16):

$$\begin{aligned} s_{\text{gen}} &= J_U \frac{d}{dx} \left( \frac{1}{T} \right) + J_N \frac{d}{dx} \left( -\frac{e\phi}{T} \right) \\ &= (J_U - e\phi J_N) \frac{d}{dx} \left( \frac{1}{T} \right) + \frac{J_N}{T} \frac{d}{dx} (-e\phi) \\ &= q \frac{d}{dx} \left( \frac{1}{T} \right) - \frac{J_N}{T} \frac{d}{dx} (e\phi) \end{aligned} \quad (a)$$

Note that half of the desired expression already appears on the right side of eq. (a). The second term on the right side of (a) refers to the longitudinal gradient of the potential: this quantity is related to the heat flux through the relation used in the derivation of (a),

$$q'' = J_U - e\phi J_N \quad (b)$$

Differentiating eq. (b) and noting that in a one-dimensional conductor both  $J_U$  and  $J_N$  are conserved from one cross-section to the next along the conductor, yields

$$\frac{dq''}{dx} = -J_N \frac{d}{dx}(e\phi) \quad (c)$$

Combining eqs. (a) and (c) we obtain

$$s_{gen}''' = q'' \frac{d}{dx}\left(\frac{1}{T}\right) + \frac{1}{T} \frac{dq''}{dx} \quad (d)$$


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Problem 12.3 Recall the second Kelvin relation (12.64)

$$\pi_\beta - \pi_\alpha = T(\varepsilon_\beta - \varepsilon_\alpha)$$

in which the  $\pi$ 's and  $\varepsilon$ 's depend only on  $T$ . Differentiating with respect to  $T$  and using the Thomson coefficient definition (12.73) we have

$$\begin{aligned} \frac{d}{dT}(\pi_\beta - \pi_\alpha) &= \varepsilon_\beta - \varepsilon_\alpha + T\left(\frac{d\varepsilon_\beta}{dT} - \frac{d\varepsilon_\alpha}{dT}\right) \\ &= \varepsilon_\beta - \varepsilon_\alpha + \tau_\beta - \tau_\alpha \end{aligned}$$


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Problem 12.4 The starting point is that both  $J_U$  and  $eJ_N$  are conserved, or  $x$ -independent:

$$\begin{aligned} \frac{dJ_U}{dx} &\equiv \frac{d}{dx}(q'' + \phi e J_N) \\ &= \frac{dq''}{dx} + e J_N \frac{d\phi}{dx} \\ &= \frac{d}{dx}\left(-k \frac{dT}{dx} + T \varepsilon e J_N\right) + e J_N \left(-\frac{e J_N}{k_e} - \varepsilon \frac{dT}{dx}\right) \\ &= -\frac{d}{dx}\left(k \frac{dT}{dx}\right) + e J_N T \frac{d\varepsilon}{dx} - \frac{(e J_N)^2}{k_e} \\ &\quad e J_N \tau \frac{dT}{dx} \end{aligned}$$

Recognizing that  $dJ_U / dx = 0$  leads to the equation listed in the problem statement.

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Problem 12.5 Starting with eq. (12.89), the efficiency of the infinitesimal two-junction element positioned between  $T$  and  $T + dT$  is

$$\begin{aligned}\eta &= \left(1 - \frac{T}{T + dT}\right) \frac{Z - 1}{Z + T / (T + dT)} \\ &= \frac{Z - 1}{Z + 1} \frac{dT}{T}\end{aligned}\quad (1)$$

Recognizing that

$$\lim_{dT \rightarrow 0} \eta = \frac{d\dot{W}_{el}}{\dot{Q} + d\dot{Q}} = \frac{d\dot{Q}}{\dot{Q} + d\dot{Q}} \rightarrow \frac{d\dot{Q}}{\dot{Q}}$$

and using the notation  $\zeta = (Z - 1) / (Z + 1)$ , eq. (1) reads

$$\frac{d\dot{Q}}{\dot{Q}} = \zeta \frac{dT}{T} \quad (2)$$

The heat current is a function of the temperature level of the element,  $\dot{Q}(T)$ : integrating eq. (2) from  $T_L$  to  $T_H$  we obtain

$$\frac{\dot{Q}_H}{\dot{Q}_L} = \exp \left( \int_{T_L}^{T_H} \zeta \frac{dT}{T} \right) \quad (3)$$

The overall efficiency of the total stack of elements is

$$\eta_{stack} = \frac{\dot{Q}_H - \dot{Q}_L}{\dot{Q}_H} = 1 - \exp \left( - \int_{T_L}^{T_H} \zeta \frac{dT}{T} \right) \quad (4)$$

If  $\zeta$  (or  $Z$ ) is independent of temperature, then eqs. (3) and (4) become

$$\begin{aligned}\frac{\dot{Q}_H}{\dot{Q}_L} &= \left( \frac{T_H}{T_L} \right)^\zeta \\ \eta_{stack} &= 1 - \left( \frac{T_L}{T_H} \right)^\zeta\end{aligned}\quad (5)$$


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Problem 12.6 The analysis consists of transforming the second order derivatives that appear in eq. (12.108):

$$\begin{aligned}\frac{\partial T}{\partial x_1} &= \frac{\partial T}{\partial \xi_1} \frac{\partial \xi_1}{\partial x_1} + \frac{\partial T}{\partial \xi_2} \frac{\partial \xi_2}{\partial x_1} = \frac{\partial T}{\partial \xi_1} \cos \alpha - \frac{\partial T}{\partial \xi_2} \sin \alpha \\ \frac{\partial^2 T}{\partial x_1^2} &= \frac{\partial}{\partial \xi_1} \left( \frac{\partial T}{\partial x_1} \right) \cos \alpha - \frac{\partial}{\partial \xi_2} \left( \frac{\partial T}{\partial x_1} \right) \sin \alpha \\ &= \frac{\partial^2 T}{\partial \xi_1^2} \cos^2 \alpha - \frac{\partial^2 T}{\partial \xi_1 \partial \xi_2} \sin 2\alpha + \frac{\partial^2 T}{\partial \xi_2^2} \sin^2 \alpha\end{aligned}\quad (1)$$

$$\begin{aligned}\frac{\partial T}{\partial x_2} &= \frac{\partial T}{\partial \xi_1} \frac{\partial \xi_1}{\partial x_2} + \frac{\partial T}{\partial \xi_2} \frac{\partial \xi_2}{\partial x_2} = \frac{\partial T}{\partial \xi_1} \sin \alpha + \frac{\partial T}{\partial \xi_2} \cos \alpha \\ \frac{\partial^2 T}{\partial x_2^2} &= \frac{\partial}{\partial \xi_1} \left( \frac{\partial T}{\partial x_2} \right) \sin \alpha + \frac{\partial}{\partial \xi_2} \left( \frac{\partial T}{\partial x_2} \right) \cos \alpha \\ &= \frac{\partial^2 T}{\partial \xi_1^2} \sin^2 \alpha + \frac{\partial^2 T}{\partial \xi_1 \partial \xi_2} \sin 2\alpha + \frac{\partial^2 T}{\partial \xi_2^2} \cos^2 \alpha\end{aligned}\quad (2)$$

$$\begin{aligned}\frac{\partial^2 T}{\partial x_1 \partial x_2} &= \frac{\partial}{\partial \xi_1} \left( \frac{\partial T}{\partial x_2} \right) \cos \alpha - \frac{\partial}{\partial \xi_2} \left( \frac{\partial T}{\partial x_2} \right) \sin \alpha \\ &= \frac{\partial^2 T}{\partial \xi_1^2} \sin \alpha \cos \alpha + \frac{\partial^2 T}{\partial \xi_1 \partial \xi_2} (\cos^2 \alpha - \sin^2 \alpha) - \frac{\partial^2 T}{\partial \xi_2^2} \sin \alpha \cos \alpha\end{aligned}\quad (3)$$

Multiplying eqs. (1), (2) and (3) respectively by  $k_{11}$ ,  $k_{22}$  and  $(k_{12} + k_{21})$ , adding, and collecting the similar terms yields

$$k_1 \frac{\partial^2 T}{\partial \xi_1^2} + K \frac{\partial^2 T}{\partial \xi_1 \partial \xi_2} + k_2 \frac{\partial^2 T}{\partial \xi_2^2} = 0$$

where

$$k_1 = k_{11} \cos^2 \alpha + (k_{12} + k_{21}) \sin \alpha \cos \alpha + k_{22} \sin^2 \alpha \quad (12.112)$$

$$K = -k_{11} \sin 2\alpha + (k_{12} + k_{21}) \cos 2\alpha + k_{22} \sin 2\alpha$$

$$k_2 = k_{11} \sin^2 \alpha - (k_{12} + k_{21}) \sin \alpha \cos \alpha + k_{22} \cos^2 \alpha \quad (12.113)$$

The canonical form condition  $K = 0$  means that

$$\tan 2\alpha = \frac{k_{12} + k_{21}}{k_{11} - k_{22}} \quad (12.114)$$


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Problem 12.7 The relations between the two sets of fluxes are (Fig. 12.10)

$$q_{\xi,1}'' = q_1'' \cos \alpha + q_2'' \sin \alpha$$

$$q_{\xi,2}'' = -q_1'' \sin \alpha + q_2'' \cos \alpha$$

The objective is to express a principal flux such as  $q_{\xi,1}''$  in terms of  $\partial T / \partial \xi_1$  and  $\partial T / \partial \xi_2$ , so that we can compare the resulting expression with eq. (12.115). For this we need  $q_1''$  and  $q_2''$  expressed in terms of the principal gradients  $\partial T / \partial \xi_1$  and  $\partial T / \partial \xi_2$ . We use the reverse transformation  $(x_1, x_2) \leftarrow (\xi_1, \xi_2)$ ,

$$x_1 = \xi_1 \cos \alpha - \xi_2 \sin \alpha$$

$$x_2 = \xi_1 \sin \alpha + \xi_2 \cos \alpha$$

and write (slowly !)

$$q_{\xi,1}'' = - \left( k_{11} \frac{\partial T}{\partial x_1} + k_{12} \frac{\partial T}{\partial x_2} \right) \cos \alpha - \left( k_{21} \frac{\partial T}{\partial x_1} + k_{22} \frac{\partial T}{\partial x_2} \right) \sin \alpha$$

$$= - k_{11} \underbrace{\cos \alpha}_{\left( \frac{\partial T}{\partial \xi_1} \cos \alpha - \frac{\partial T}{\partial \xi_2} \sin \alpha \right)} - k_{12} \underbrace{\cos \alpha}_{\left( \frac{\partial T}{\partial \xi_1} \sin \alpha + \frac{\partial T}{\partial \xi_2} \cos \alpha \right)} \quad \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \quad (1)$$

$$- k_{21} \underbrace{\sin \alpha}_{\left( \frac{\partial T}{\partial \xi_1} \cos \alpha - \frac{\partial T}{\partial \xi_2} \sin \alpha \right)} - k_{22} \underbrace{\sin \alpha}_{\left( \frac{\partial T}{\partial \xi_1} \sin \alpha + \frac{\partial T}{\partial \xi_2} \cos \alpha \right)}$$

Grouping the terms we rediscover that

$$q_{\xi,1}'' = -a_{11} \frac{\partial T}{\partial \xi_1} - a_{12} \frac{\partial T}{\partial \xi_2} \quad (12.115)$$

where

$$a_{11} = k_{11} \cos^2 \alpha + (k_{12} + k_{21}) \sin \alpha \cos \alpha + k_{22} \sin^2 \alpha \quad (12.117)$$

$$a_{12} = k_{12} \cos^2 \alpha - k_{21} \sin^2 \alpha + (k_{22} - k_{11}) \sin \alpha \cos \alpha \quad (12.118)$$

The expression for  $\ddot{q}_{\xi,2}$  is obtained similarly. The starting point is

$$\ddot{q}_{\xi,2} = \left( k_{11} \frac{\partial T}{\partial x_1} + k_{12} \frac{\partial T}{\partial x_2} \right) \sin \alpha - \left( k_{21} \frac{\partial T}{\partial x_1} + k_{22} \frac{\partial T}{\partial x_2} \right) \cos \alpha$$

which develops into a stack similar to eq. (1): the only difference is that the four underlined coefficients are replaced, in order, by  $(-\sin \alpha)$ ,  $(-\sin \alpha)$ ,  $\cos \alpha$  and  $\cos \alpha$ . The end result is

$$\ddot{q}_{\xi,2} = -a_{21} \frac{\partial T}{\partial \xi_1} - a_{22} \frac{\partial T}{\partial \xi_2} \quad (12.116)$$

in which

$$a_{21} = k_{21} \cos^2 \alpha - k_{12} \sin^2 \alpha + (k_{22} - k_{11}) \sin \alpha \cos \alpha \quad (12.119)$$

$$a_{22} = k_{11} \sin^2 \alpha - (k_{12} + k_{21}) \sin \alpha \cos \alpha + k_{22} \cos^2 \alpha \quad (12.120)$$

Finally, in order to prove that  $a_{12} = -a_{21}$ , we calculate the sum

$$\begin{aligned} a_{12} + a_{21} &= (k_{22} - k_{11}) \sin^2 \alpha + (k_{12} + k_{21}) \cos^2 \alpha \\ &= \cos 2\alpha [(k_{22} - k_{11}) \tan 2\alpha + k_{12} + k_{21}] \\ &= \cos 2\alpha \left[ (k_{22} - k_{11}) \frac{k_{12} + k_{21}}{k_{11} - k_{22}} + k_{12} + k_{21} \right] \\ &= 0 \end{aligned}$$

The last step in the above calculation consisted of using eq. (12.114).

---

Problem 12.8 The starting point is eq. (12.118),

$$\begin{aligned} a_{12} &= k_{12} \cos^2 \alpha - k_{21} \sin^2 \alpha + (k_{22} + k_{11}) \sin \alpha \cos \alpha \\ &= \frac{1}{2} [k_{12}(1 + \cos 2\alpha) - k_{21}(1 + \cos 2\alpha) + (k_{22} - k_{11}) \sin 2\alpha] \\ &= \frac{1}{2} \cos 2\alpha \left[ \frac{k_{12} - k_{21}}{\cos 2\alpha} + k_{12} + k_{21} + (k_{22} - k_{11}) \tan 2\alpha \right] \end{aligned}$$

and, using eq. (12.114),

$$a_{12} = \frac{1}{2} \cos 2\alpha \left[ \frac{k_{12} - k_{21}}{\cos 2\alpha} + k_{12} + k_{21} + (k_{22} - k_{11}) \frac{k_{12} + k_{21}}{k_{11} - k_{22}} \right]$$

$$= \frac{1}{2} (k_{12} - k_{21})$$


---

Problem 12.9 The heat flux is related to the energy flux via

$$J_U = q'' + \bar{h}_1 J_{N_1} + \bar{h}_2 J_{N_2}$$

However, since  $J_{N_1} = 0$  and, from eq. (12.166),  $J_{N_2} = 0$ , we have

$$J_U = q'', \quad (J_{N_1} = 0)$$

For the same conditions eqs. (12.177) - (12.178) state

$$q'' = -D_{00} \frac{dT}{dz} - D_{01} \frac{dC_1}{dz}$$

$$0 = -D_{10} \frac{dT}{dz} - D_{11} \frac{dC_1}{dz}$$

and, after eliminating  $dC_1 / dz$ ,

$$q'' = - \left( D_{00} - \frac{D_{01}D_{10}}{D_{11}} \right) \frac{dT}{dz}, \quad (J_{N_1} = 0)$$

The coefficient formed in the brackets is, by definition, the thermal conductivity  $k$ .

---

Problem 12.10 The first law for system I requires

$$dU = - \frac{L_{10}}{L_{11}} dN + \delta Q - \delta W$$

where  $U$  is the internal energy inventory of system I. Since this system is maintained at constant  $T$  and  $P$ , we have, in terms of partial molal properties,

$$dU = - \bar{u} dN$$

(constant  $T$  &  $P$ )

$$dV = - \bar{v} dN$$

which also means that  $\delta W = - \bar{v} P dN$ . The first law reduces to

$$-\bar{u} dN = -\frac{L_{10}}{L_{11}} dN + \delta Q + \bar{v} P dN$$

or, in order,

$$\left( \frac{L_{10}}{L_{11}} - \bar{u} - P \bar{v} \right) dN = \delta Q$$

$$\frac{L_{10}}{L_{11}} - \bar{h} = \frac{\delta Q}{dN}$$

$$Q^* = \frac{\delta Q}{dn}$$

In conclusion,  $Q^*$  is the heat absorbed from the reservoir (T) per mole of fluid transferred, at constant T and P.

---

Problem 12.11 The differential equation derived in Problem 12.4 assumes the form

$$T'' - BT' + C = 0$$

where

$$B = \frac{eJ_N \tau L}{k}$$

$$C = \frac{(eJ_N L)^2}{kk_e}$$

$$( )' = \frac{d}{d(x/L)} ( )$$

The  $T(x)$  solution subject to  $T(0) = T(L) = T_0$  is

$$T - T_0 = \frac{C}{B} \left[ \frac{\exp(Bx/L) - 1}{\exp(B) - 1} - \frac{x}{L} \right]$$

The temperature maximum is found by solving  $dT/dx = 0$ , which yields

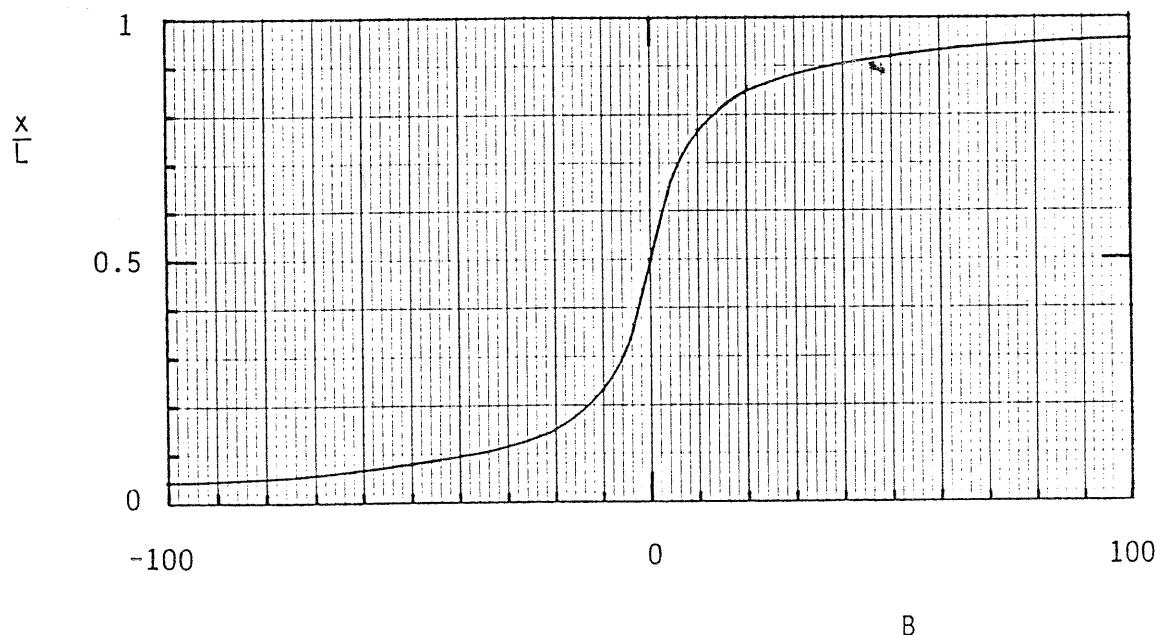
$$\frac{x}{L} = \frac{1}{B} \ln \left[ \frac{\exp(B) - 1}{B} \right]$$

When the electrical current changes direction the dimensionless parameter changes sign (i.e. it passes through zero). Therefore, we study the limit

$$\lim_{B \rightarrow 0} \frac{x}{L} = \frac{1}{B} \ln \left\{ \frac{1}{B} \left[ \left( 1 + B + \frac{B^2}{2} + \frac{B^3}{6} + \dots \right) - 1 \right] \right\}$$

$$= \frac{1}{2} + \frac{B}{6} + \dots$$

which shows that at  $B = 0$  the value of  $x / L$  is  $1/2$ , and that near  $B = 0$  the relation between  $x/L$  and  $B$  is linear. The complete curve  $x/L = \text{function}(B)$  is shown in the attached sketch.





Chapter 13

**THE CONSTRUCTAL LAW**

**OF FLOW CONFIGURATION GENERATION**

---

Problem 13.1 Nondimensionalize all the lengths by using the side R as reference:

$$(\tilde{B}, \tilde{L}_0, \tilde{L}_1, \tilde{x}) = (B, L_0, L_1, x) / R \quad (1)$$

The aspect ratio of the triangle ( $\tilde{B} = B / R$ ) is a given parameter. The tube length  $\tilde{L}_1$  depends on the tube length  $\tilde{L}_0$ ,

$$\tilde{L}_1 = [(a - \tilde{L}_0)^2 + b^2]^{1/2} \quad (2)$$

$$a = \left[ 1 - \left( \frac{B}{2R} \right)^2 \right]^{1/2}, \quad b = \frac{B}{2R} \quad (3)$$

For fully developed laminar flow with negligible junction losses, the pressure drop through one tube varies as

$$\Delta P_i \sim \dot{m}_i \frac{L_i}{D_i^4} \quad (4)$$

The total pressure drop varies as

$$\Delta P = \Delta P_0 + \Delta P_1 \sim \dot{m} \frac{L_0}{D_0^4} + \frac{\dot{m}}{2} \frac{L_1}{D_1^4} \sim \frac{\dot{m}}{D_0^4} \left[ L_0 + \frac{1}{2} \left( \frac{D_0}{D_1} \right)^4 L_1 \right] \quad (5)$$

When the total tube volume is fixed, the total pressure drop  $\Delta P$  is minimum when  $D_0/D_1 = 2^{1/3}$ , and the quantity in square brackets (F) becomes

$$F = L_0 + 2^{1/3} L_1 \quad (6)$$

Eliminating  $L_1$  between Eqs. (2) and (6), and solving  $dF/dL_0 = 0$ , we find the optimal length

$$\tilde{L}_0 = a - \frac{b}{(2^{2/3} - 1)^{1/2}} \quad (7)$$

The angle ( $\beta$ ) between the two  $L_1$  tubes is given by

$$\tan\left(\frac{\beta}{2}\right) = \frac{B/2}{x} = \frac{b}{a - \tilde{L}_0} = (2^{2/3} - 1)^{1/2}$$

which yields

$$\beta = 74.935^\circ \approx 75^\circ$$

This angle is independent of the triangle aspect ratio  $B/R$ . In conclusion, the bifurcation of  $L_0$  into two  $L_1$ 's can be optimized only if  $\tilde{L}_0 > 0$ , that is only if the angle that faces the base  $B$  is smaller than  $75^\circ$ .

---

Problem 13.2 In the case of a cube with four central tubes  $L_0$ , there are four triangles of the type analyzed in the preceding problem. Since  $B$  is the side of the cube,  $R$  is the radius of the spherical surface on which the cube corners are situated, therefore  $B/R = 2/3^{1/2} = 1.1547$ . Substituting this aspect ratio into the tube lengths optimized in the preceding problem, we obtain

$$\frac{L_0}{R} = 0.06319 \quad \frac{L_1}{R} = 0.9491$$

noting again the invariant  $\beta \approx 75^\circ$ . These tube dimensions can also be expressed as

$$\frac{L_0}{B} = 0.05472 \quad \frac{L_1}{B} = 0.8219$$


---

Problem 13.3 Consider one fourth of the cube construct of the preceding problem, namely, a Y-shaped construct with one  $L_0$  tube and two  $L_1$  tubes, and total mass flow rate  $\dot{m}$ . The overall pressure drop is

$$\begin{aligned} \Delta P &= a \left( \dot{m} \frac{L_0}{D_0^4} + \frac{\dot{m}}{2} \frac{L_0}{D_1^4} \right) \\ &= \frac{a \dot{m} R}{D_0^4} \left[ \frac{L_0}{R} + \frac{1}{2} \left( \frac{D_0}{D_1} \right)^4 \frac{L_1}{R} \right] \\ &= \frac{a \dot{m} R}{D_0^4} (0.06319 + 2^{1/3} 0.9491) \end{aligned} \quad (1)$$

where  $a$  is a constant factor. The total tube volume is

$$V = b(D_0^2 L_0 + 2D_1^2 L_1) = b D_0^2 R (0.06319 + 2^{1/3} 0.9491) \quad (2)$$

where  $b = \pi/4$ . Combining eqs. (1) and (2) we obtain

$$\Delta P = ab^2 \frac{\dot{m}R}{V^2} (1.99553) \quad (3)$$

The corresponding analysis for tubes without bifurcation is for one tube of length  $R$ , diameter  $D$ , flow rate  $\dot{m}/2$ , and volume  $V/2$ :

$$\begin{aligned} \Delta P &= a \frac{\dot{m}}{2} \frac{R}{D^4} \\ \frac{V}{2} &= bRD^2 \\ \Delta P &= ab^2 \frac{\dot{m}R^3}{V^2} (2) \end{aligned} \quad (4)$$

Comparing eqs. (3) and (4), we conclude that the performance of the bifurcated design is only 0.22% better than that of the design with eight equidistant radial tubes.

---

Problem 13.4 Consider the horizontal slice of height  $dz$ , where  $z$  increases upward. The upper face is  $A + dA$ , and the lower face is  $A$ . The upper face feels a downward force  $(A + dA)s$ , and the lower face feels an upward force  $As$ . The weight of the slice is  $gpAdz$ . Vertical equilibrium requires

$$gpAdz + sdA = 0 \quad (1)$$

which yields

$$A = C e^{-mz} \quad (2)$$

where  $m = \rho g/s$  and  $C$  is a constant. The base area is  $A_0$  at  $z = 0$ , therefore  $C = A_0$ . The top cross-section is  $A_H$  at  $z = H$ ,

$$A_H = A_0 e^{-mH} \quad (3)$$

The weight of the superstructure supported by  $A_0$  is  $A_0 s$ . The total volume of the structure is

$$V = \int_0^H Adz = \frac{A_0}{m} (1 - e^{-mH}) \quad (4)$$


---

---

Problem 13.5 The comparison to be made is between the drag force experienced by one sphere of diameter  $D_s$ , and one segment of cylinder of length  $S$  and diameter  $D_c$ . The two solid objects have the same volume,

$$\frac{\pi}{6} D_s^3 = \frac{\pi}{4} D_c^2 S \quad (1)$$

The drag force is calculated with the formula

$$F = C_D A \frac{1}{2} \rho V^2 \quad (2)$$

where the frontal areas of the two objects are

$$A_s = \frac{\pi}{4} D_s^2 \quad A_c = D_c S \quad (3)$$

In the range  $10^3 < VD_{s,c}/\nu < 10^5$ , the drag coefficients are

$$C_s \approx 0.4 \quad C_c \approx 1 \quad (4)$$

Writing eq. (2) for the sphere and the cylinder, and dividing the two relations we obtain

$$\frac{F_c}{F_s} \approx 3.2 \frac{D_c S}{D_s^2} \quad (5)$$

Eliminating  $S$  between eqs. (5) and (1) yields

$$\frac{F_c}{F_s} \approx 2.1 \frac{D_s}{D_c} \quad (6)$$

This analysis is more accurate in the limit where there is no interference between the flows around two adjacent spheres, i.e. when

$$D_s \ll S \quad (7)$$

The extreme in which the analysis definitely fails is when the spheres touch,

$$D_s = S \quad (8)$$

Combining eqs. (7, 8) with eq. (1), we find the  $D_s/D_c$  range in which eq. (6) is valid,

$$\frac{D_s}{D_c} > \left(\frac{3}{2}\right)^{1/2} \quad (9)$$

In this range, the  $F_c/F_s$  ratio of eq. (6) takes values greater than 2.6. The reduction in drag from cylinder to sphere is dramatic. In conclusion, the proposal to move from a two-dimensional wing to a three-dimensional one with bulbous features is worth pursuing.

---

Problem 13.6 The drag force per unit length experienced by the tree canopy is

$$F' = C_D D_c \frac{1}{2} \rho V^2 \quad (1)$$

where  $V$  is the horizontal wind speed, and  $D_c(x)$  is the radius of the canopy at the distance  $x$  from the tree top. We assume that the Reynolds number  $VD_c/\nu$  is greater than  $10^3$  so that the drag coefficient  $C_D$  is a constant on the order of 1.

Assume that the canopy is shaped such that

$$D_c = a x^n \quad (2)$$

where  $a$  is a constant, and the shape exponent  $n$  is unknown. The bending moment experienced by the trunk at the distance  $x$  from the tree top is

$$M(x) = \int_0^x F'(x - \xi) d\xi = \frac{a' x^{n+2}}{(n+1)(n+2)} \quad (3)$$

where  $a' = \frac{a}{2} \rho V^2 C_D$ .

We now turn our attention to the maximum bending stresses in the cross-section of the trunk of diameter  $D_t(x)$ ,

$$s_m = \frac{M(x)}{I_t(x)} \frac{D_t(x)}{2} \quad (4)$$

where  $I_t = \pi D_t^4/64$ . Optimal distribution of imperfection means that  $s_m$  must be the same over the entire height of the tree. According to eq. (4), the trunk diameter must vary as

$$D_t(x) = \left[ \frac{32 a' / \pi}{s_m (n+1)(n+2)} \right]^{1/3} x^{(n+2)/3} \quad (5)$$

For example, if the canopy is shaped as a cone ( $n = 1$ ), then the trunk is also shaped as a cone,  $D_t/x = \text{constant}$ .

---

Problem 13.7 The hot spot ( $T_{\max}$ ) forms because it is the farthest from the heat sink ( $T_{\min}$ ). The heat flux ( $q''$ ) is pumped uniformly through the external surfaces for the conducting slab. In particular, the heat current  $q''$  that enters through the hot spot is driven by the temperature difference  $T_{\max} - T_{\min}$ . We can write, approximately,

$$q'' \approx k \frac{T_{\max} - T_{\min}}{L} \quad (1)$$

where the segment  $L$  is an approximation (an underestimate) of the path followed by the heat flux that enters through the hot spot.

To minimize  $T_{\max} - T_{\min}$  is approximately the same as minimizing the distance  $L$ , which is

$$L = \frac{1}{2} (S^2 + H^2)^{1/2} - \frac{1}{2} D \quad (2)$$

where  $S$ ,  $H$  and  $D$  are the spacing between hole centers, the slab thickness (fixed), and the hole diameter. The variables  $S$  and  $D$  are related through the void volume constraint

$$\phi = \frac{\pi}{4} \frac{D}{HS} \ll 1, \text{ constant} \quad (3)$$

Eliminating  $D$  between eqs. (2) and (3) we obtain

$$2 \frac{L}{H} = (x^2 + 1)^{1/2} - \left( \frac{4\phi}{\pi} x \right)^{1/2} \quad (4)$$

where  $x = S/H$ . This dimensionless expression ( $L/H$ ) can be minimized with respect to the spacing  $x$  by solving  $dL/dx = 0$ . The optimal spacing is given by

$$\frac{x_{\text{opt}}^{3/2}}{x_{\text{opt}}^2 + 1} = \left( \frac{\phi}{\pi} \right)^{1/2} \quad (5)$$

In the limit  $\phi \ll 1$ , this result reduces to

$$\frac{S_{\text{opt}}}{H} = \left( \frac{\phi}{\pi} \right)^{1/3} \quad (6)$$

From the  $\phi$  constraint (3) we deduce the corresponding channel diameter

$$\frac{D_{\text{opt}}}{H} = 2 \left( \frac{\phi}{\pi} \right)^{2/3} \quad (7)$$

The largest  $\phi$  value corresponds to the limit where  $S = D = H$ , which substituted in eq. (3) yield

$$\phi \leq \frac{\pi}{4} \quad (8)$$


---

**Problem 13.8** In this design there are two hole sizes,  $D_1$  and  $D_2$ . The  $D_1$  holes have their centers in the center plane of the slab, and their centers are separated by the distance  $S$ . The  $D_2$  holes are placed half-way between the  $D_1$  centers. The role of the  $D_2$  holes is to suppress the hot-spot temperatures that would occur on the external surfaces of the slab in the absence of  $D_2$  holes. The distance between the external surface and the center of the  $D_2$  hole is  $y$ . There are two  $D_2$  holes in every elemental volume  $H \times S$ .

Hot spots will occur on the external surfaces, at distances  $x$  to the right and the left of the transversal plane in which the centers of the  $D_2$  holes are located. The heat flux that enters the wall through the hot spot must be split exactly in half as it flows toward the two heat sinks,  $D_1$  and  $D_2$ . Because we approximate the flow paths as the straight segments  $L_1$  and  $L_2$ ,

$$L_1 = \left[ \left( \frac{H}{2} \right)^2 + \left( \frac{S}{2} - x \right)^2 \right]^{1/2} - \frac{D_1}{2} \quad (1)$$

$$L_2 = \left( x^2 + y^2 \right)^{1/2} - \frac{D_2}{2} \quad (2)$$

the equipartition of  $q''$  means that

$$L_1 = L_2 (= L) \quad (3)$$

The minimization of  $(T_{\max} - T_{\min})$  is the same as the minimization of the path length,  $L_1$  or  $L_2$ . The geometry depends on five variables ( $D_1$ ,  $D_2$ ,  $S$ ,  $x$ ,  $y$ ) and, because of constraints (7,8) there are three degrees of freedom. In the following analysis, we consider  $D_1$ ,  $D_2$  and  $S$  as the degrees of freedom.

I. To begin the numerical minimization of (6) subject to (7) and (8), assume that  $\tilde{x}$  and  $\tilde{y}$  are known, for example

$$\tilde{x} = \frac{\tilde{S}/2}{2} \quad \tilde{y} = \frac{1}{6}$$

II. Later, relax the constant- $\tilde{x}$  assumption, i.e. repeat step I for  $\tilde{x}$  values in the vicinity of  $\tilde{x} = \tilde{S}/4$ , and see whether the minimized  $L$  can be minimized further by varying  $\tilde{x}$ .

III. Repeat loops I-II for  $\tilde{y}$  values in the vicinity of 1/6.

---

---

Problem 13.9 (a) The flow resistance of a round tube with Hagen-Poiseuille flow is governed by one geometric ratio

$$R = \frac{L}{D^4} \quad (1)$$

On the rectangular area  $A = xy$ , constant, we have

$$L = \left( y^2 + \frac{x^2}{4} \right)^{1/2} \quad (2)$$

The total tube volume is fixed, and is proportional to

$$V = D^2 L \quad (3)$$

Eliminating  $D$  between eqs. (1) and (3) we find that

$$R = \frac{L^3}{V^2} \quad (4)$$

which means that the minimization of  $R$  is the same as the minimization of  $L$ . To minimize  $L$  by varying the shape of  $A$ , we combine eqs. (2) and  $A = xy$ ,

$$L = \left( \frac{A^2}{x^2} + \frac{x^2}{4} \right)^{1/2} \quad (5)$$

and solve  $dL/dx = 0$ . We find  $x_{\text{opt}}^2 = 2A$ , which means that

$$\left( \frac{x}{y} \right)_{\text{opt}} = 2 \quad (6)$$

(b) In fully turbulent flow in the fully rough regime ( $f = \text{constant}$ ) and specified flow rate ( $\dot{m}$ ), the flow resistance is proportional to the geometric group

$$R = \frac{L}{D^5} \quad (7)$$

In view of the tube volume constraint (3), the resistance is governed by

$$R = \frac{L^{7/2}}{V^{5/2}} \quad (8)$$

Minimum resistance means minimum L, and the optimal rectangle shape is the same as in eq. (6).

---

Problem 13.10 The quantity to minimize is  $L_1 + 2^{1/3}L_2$ , or

$$R = \lambda_1 + 2^{1/3} \lambda_2 \quad (1)$$

where

$$(\lambda_1, \lambda_2, a, b, h) = \frac{(L_1, L_2, A, B, H)}{S^{1/2}} \quad (2)$$

The area constraint  $AB = S$  becomes

$$ab = 1 \quad (3)$$

The global flow resistance is proportional to

$$R = \left[ (a - h)^2 + \left(\frac{b}{2}\right)^2 \right]^{1/2} + 2^{1/3} \left[ h^2 + \left(\frac{b}{4}\right)^2 \right]^{1/2} \quad (4)$$

In view of constraint (3), the function  $R(a,b)$  has only one degree of freedom. The numerical minimization of R yields the optimal configuration

$$a = 0.875 \quad \alpha_1 = 40.86^\circ$$

$$b = 1.1429 \quad \alpha_2 = 53.11^\circ$$

$$h = 0.2144 \quad R = 1.3235$$


---

Problem 13.11 Because  $L_1$  and  $L_2$  are co-linear, the similarity of the triangles for which  $L_1$  and  $L_2$  are hypotenuses indicates that

$$L_1 = 2L_2 \quad (1)$$

Because the tube diameter ratio has been optimized in accordance with Murray's law, the quantity to minimize is

$$R = L_1 + 2^{1/3}L_2 \quad (2)$$

which in view of eq. (1) becomes

$$R = (1 + 2^{-2/3})L_1 \quad (3)$$

In conclusion, the quantity to minimize is  $L_1$ . Note that if the assembly of tubes is replaced by an assembly of two-dimensional fissures (parallel-plates channels), Murray's law dictates a ratio of  $2^{-1/2}$  between successive channel widths, and the only thing that changes in eq. (1) is the numerical factor in parentheses. The conclusion to minimize  $L_1$  remains.

The area occupied by the construct ( $S$ ) is fixed,

$$S = \frac{3}{2} y_1 4d \quad (4)$$

where

$$y_1 = (L_1^2 - d^2)^{1/2} \quad (5)$$

From eqs. (4) and (5) we find

$$L_1 = \left( d^2 + \frac{S^2}{9d^2} \right)^{1/2} \quad (6)$$

which can be minimized with respect to  $d$ . The results are

$$d = \left( \frac{S}{3} \right)^{1/2} \quad L_1 = \left( \frac{2}{3} S \right)^{1/2} \quad (7)$$

which mean that  $\sin \alpha = d/L_1 = 2^{-1/2}$ , and  $\alpha = 45^\circ$ . The minimized resistance factor is

$$R = (1 + 2^{-2/3}) \left( \frac{2}{3} S \right)^{1/2} = 1.33086 S^{1/2} \quad (8)$$

This is only 0.5 percent higher (i.e. worse) than the performance of the more complicated structure optimized in the preceding problem.

Problem 13.12 The optimized elemental volume has the minimal global thermal resistance

$$\frac{\Delta T_0 k_0}{q'' A_0} = \frac{1}{2(\tilde{k}\phi_0)^{1/2}} \quad (1)$$

where  $q'''$  is the uniform heat generation rate in the  $k_0$  material, and  $q'$  is the fixed heat current generated by the element,

$$q' = q'''A_0 \quad (2)$$

Equation (2) is valid in the limit  $\phi_0 \rightarrow 0$ , where all of  $A_0$  is occupied by heat generating material. When  $\phi_0$  is not negligible, the  $k_0$  heat generation rate  $q'''$  is higher,

$$\begin{aligned} q' &= q'''(A_0 - A_p) \\ &= q'''A_0(1 - \phi) \end{aligned} \quad (3)$$

Substituting this new  $q'''$  into eq. (1), we obtain

$$\frac{\Delta T_0 k_0}{q'} = \frac{1}{(1 - \phi_0) 2 (k\phi_0)^{1/2}} \quad (4)$$

This global thermal resistance is minimum when

$$\phi_0 = \frac{1}{3} \quad (5)$$

In conclusion, there is an optimal balance between the amount of material that generates the current  $q'$ , and the material that collects and guides that current.

---

Problem 13.13 During inhaling, the pressure inside the lungs ( $P$ ) is smaller than outside the lungs ( $P_{atm}$ ). At the start of the inhaling stroke, the thorax muscles pull the external surface of the lungs with the force

$$F = A(P_{atm} - P) > 0 \quad (1)$$

where  $A$  is the thorax-lungs contact area. A pressure difference occurs because of the flow resistance ( $r$ ), in accordance with the general relation

$$P_{atm} - P = r \dot{m}_{in}^n \quad (2)$$

where  $n = 2$  for turbulent flow, and  $n = 1$  for purely laminar flow. The mass flow rate is

$$\dot{m}_{in} = \rho A \left( \frac{dx}{dt} \right)_{x=0^+} \quad (3)$$

where the air density  $\rho$  is assumed to be nearly constant during both inhaling and exhaling.

Next, we make the assumption that the expansion rate  $dx/dt$  is constant during inhaling, i.e., from  $x = 0$  to  $x = L$ ,

$$\frac{dx}{dt} = \left( \frac{dx}{dt} \right)_{x=0^+} = \frac{L}{t_1} \quad (4)$$

where  $t_1$  is the duration of the inhaling stroke. At the end of inhaling the inflow of air stops, and the force of the thorax muscles is balanced by the elastic spring force of the lung tissue,

$$F = kL \quad (5)$$

where  $k$  is the spring constant of lung tissue. Combining eqs. (1)-(5), we find that the inhaling time is

$$t_1 = \rho AL \left( \frac{Ar}{kL} \right)^{1/n} \quad (6)$$

During exhaling the muscles do not work, and the air stored in the lungs is pushed out by the spring force

$$F = kx \quad (7)$$

which decreases from  $kL$  (at  $t = 0$ , or  $x = L$ ) to zero (at  $t = t_2$ , or  $x = 0$ , where  $t_2$  is the exhaling time). The pressure difference now has the opposite sign,

$$P - P_{atm} = r\dot{m}_{out}^n > 0 \quad (8)$$

where

$$\dot{m}_{out} = \rho A \left( -\frac{dx}{dt} \right)^n \quad (9)$$

The instantaneous force balance on the 'piston'  $A$  is

$$F = (P - P_{atm})A \quad (10)$$

Combining eqs. (7)-(10), and integrating the resulting differential equation from  $x = L$  at  $t = 0$  to  $x = 0$  at  $t = t_2$ , we obtain the exhaling time

$$t_2 = \frac{L^{1-n^{-1}}}{1 - n^{-1}} \rho A \left( \frac{Ar}{k} \right)^{1/n} \quad (11)$$

Dividing eqs. (11) and (6) we conclude that  $t_1$  and  $t_2$  are of the same order of magnitude, and that  $t_2$  must be greater than 2 because  $1 < n < 2$ :

$$\frac{t_2}{t_1} = \frac{n}{n - 1} \geq 2 \quad (12)$$

---

Problem 13.14 We treat the Pelton wheel as an adiabatic system in steady flow, with one inlet ( $\dot{m}$ ,  $V_{in}$ ), one outlet ( $\dot{m}$ ,  $V_{out}$ ) and shaft power output ( $\dot{W}$ ). The inlet and outlet water streams are at the same temperature and pressure. The first law of thermodynamics for the wheel system states that the shaft power output equals

$$\dot{W} = \dot{m} \left( h + \frac{V^2}{2} \right)_{in} - \dot{m} \left( h + \frac{V^2}{2} \right)_{out} \quad (1)$$

where  $h_{in} = h_{out}$ . Kinematics requires

$$V_{in} = V_r + V \quad (2)$$

$$V_{out} = V_r - V \quad (3)$$

where  $V$  is the bucket speed, and  $V_r$  is the water speed relative to the bucket (in the reference frame attached to the bucket). Eliminating  $V_r$  and  $V_{out}$  between eqs. (1)-(3), we obtain

$$\dot{W} = \frac{\dot{m}}{2} \left[ V_{in}^2 - (V_{in} - 2V)^2 \right] \quad (4)$$

By solving  $d\dot{W} / dV = 0$ , we find that  $\dot{W}$  is maximal when

$$V_{opt} = \frac{V_{in}}{2} \quad (5)$$

In this case, we also have

$$V_r = \frac{V_{in}}{2} \quad (6)$$

$$V_{out} = 0 \quad (7)$$

In conclusion, maximum power generation occurs when the water stream ( $V_{in}$ ) is brought to rest in the stationary frame ( $V_{out} = 0$ ).

---

Problem 13.15 In this problem we follow the method outlined in chapter 2 of the book A. Bejan, *Shape and Structure, from Engineering to Nature*, Cambridge University Press, Cambridge, UK, 2000. The rotational equilibrium condition, end deflection, and total beam volume are

$$s_{ma}AH = Px \quad (1)$$

$$\delta = \int_0^L \frac{2Px^2dx}{EH^2A} \quad V = \int_0^L 2A dx \quad (2, 3)$$

The beam geometry is not unique, because  $A(x)$  and  $H(x)$  can be specified independently. For example, if we assume the family of designs represented by the power laws

$$A(x) = A_* \left(\frac{x}{L}\right)^m \quad H(x) = H_* \left(\frac{x}{L}\right)^n \quad (4, 5)$$

where  $A_*$  and  $H_*$  are the dimensions in the plane of implantation ( $x = L$ ), eqs. (1)-(3) become

$$s_{ma}A_*H_* = PL \quad m + n = 1 \quad (6, 7)$$

$$\delta = \frac{2PL^3}{(3 - 2n - m)EH_*^2 A_*} \quad (8)$$

$$V = \frac{2LA_*}{m + 1} \quad (9)$$

By using eqs. (6, 7), we substitute  $H_*$  and  $n$  into eq. (8), and obtain

$$\delta = \frac{2s_{ma}^2 LA_*}{(1 + m)EP} \quad (10)$$

Finally, by eliminating  $LA_*$  between eqs. (9) and (10) we find that the figure of merit of the design ( $V$ ) is independent of the assumed beam profile ( $m$ ),

$$V = \frac{EP\delta}{s_{ma}^2} \quad (11)$$

The material required by the designs of eqs. (4, 5) is half of the material needed in Fig. 2.2(3). There is an infinity of such designs, depending on the value chosen for one parameter,  $m$ . They all look different, but they perform at the same level.

**Problem 13.16** Treat the leaning tower of Pisa as a slender cantilever beam implanted in the ground. If the small angle between its axis and the vertical is  $\alpha_0$ , then the component of gravitational acceleration that is perpendicular to the beam is  $g \sin \alpha$ . The beam is placed in pure bending with a transversal load that is uniformly distributed,

$$w = \rho \frac{\pi}{4} D_0^2 g \sin \alpha \quad (1)$$

where  $\rho$  is the beam density, and  $D_0$  is the beam diameter. The bending moment in the  $y =$  constant cross section of the beam is

$$M(y) = \int_y^{H_0} w(b - y) db = w \left( \frac{H_0^2}{2} - H_0 y + \frac{y^2}{2} \right) \quad (2)$$

where  $y = 0$  is the base and  $y = H_0$  the top of the tower. The largest bending moment is at  $y = 0$ ,

$$M(0) = w \frac{H_0^2}{2} \quad (3)$$

The largest tensile stress  $s_b$  occurs in the dorsal fiber of the leaning beam (the generator that is looking at the sky, not the ground)

$$s_b(y) = \frac{M(y)}{I_0} \frac{D_0}{2} \quad (4)$$

where  $I_0 = \pi D_0^4 / 64$ . The tensile stress increases parabolically from the top, where  $s_b = 0$  and  $ds_b/dy = 0$ , to the base, where

$$s_b = \frac{16}{\pi} w \frac{H_0^2}{D_0^3} \quad (5)$$

The tower is a pile of stones held together (in compression) by gravity. The gravitational acceleration component along the tower is  $g \cos \alpha_0 \equiv g$ . The compressive stress in the constant- $y$  cross-section is

$$s_c(y) = \rho g (H_0 - y) \quad (6)$$

For the stones of the dorsal line to stay together,  $s_c$  must be greater than  $s_b$  at every  $y$ . The first  $y$  where this requirement is threatened is  $y = 0$ . The condition for tower survival reads

$$s_c(0) > s_b(0) \quad (7)$$

or, after using the preceding equation,

$$\sin \alpha_0 < \frac{D_0}{4H_0} = 0.07353$$

which means  $\alpha_0 < 4.2^\circ$ .

If we treat the tower as a solid (perfectly fused) cylinder which rests on the ground, then the cylinder will become eligible to tip over when the diagonal of its  $H_0 \times D_0$  longitudinal cross-section becomes aligned with the vertical. The inclination angle in this case is

$$\tan \alpha_0 = \frac{D_0}{H_0} = \frac{1}{3.4} = 0.294$$

which means  $\alpha_0 = 16.4^\circ$ . In conclusion, the leaning tower of Pisa might crumble because its dorsal stones come apart, long before it may tip over like a cylindrical body on a flat table.

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**Problem 13.17** The analysis follows the steps shown in the preceding problem. For a conical tower of density  $\rho$ , angle of inclination  $\alpha$ , and diameter

$$D = c(H - y) \quad (1)$$

the transversely distributed load that places the 'beam' in bending is

$$w = \rho \frac{\pi}{4} D^2 g \sin \alpha \quad (2)$$

The bending moment at the distance  $y$  from the ground is

$$M(y) = \int_y^H w(b - y) db = \rho \frac{\pi}{4} g \sin \alpha \frac{c^2}{4} (H - y)^4 \quad (3)$$

The tensile bending stress along the generator that looks at the sky is

$$s_b(y) = \frac{M(y)}{I(y)} \frac{D(y)}{2} \quad (4)$$

where  $I = \pi D^4 / 64$ , therefore

$$s_b(y) = \rho g \sin \alpha \frac{2}{c} (H - y) \quad (5)$$

The compressive stress  $s_c$  in the constant- $y$  cross-section is determined from the vertical force balance on the conical body located between  $y = y$  and  $y = H$ ,

$$s_c \frac{\pi}{4} D^2 = \int_y^H \rho g \frac{\pi}{4} [D(b)]^2 db \quad (6)$$

which yields

$$s_c(y) = \frac{\rho g}{3} (H - y) \quad (7)$$

Equations (5) and (7) show that the critical condition  $s_b = s_c$  is met at every  $y$  when

$$\sin \alpha = \frac{c}{6} \quad (8)$$

The constant  $c$  is related to the total volume of stones used,

$$V = \int_0^H \frac{\pi}{4} c^2 (H - y)^2 dy \quad (9)$$

$$c = \left( \frac{12V}{\pi H^3} \right)^{1/2} \quad (10)$$

$$\sin \alpha = \frac{1}{6} \left( \frac{12V}{\pi H^3} \right)^{1/2} \quad (11)$$

In the preceding problem, the critical condition for a cylinder of height  $H_0$  and diameter  $D_0$  is

$$\sin \alpha_0 = \frac{D_0}{4 H_0} \quad (12)$$

For the same volume of stones used,  $V = \frac{\pi}{4} D_0^2 H_0$ , eq. (12) becomes

$$\sin \alpha_0 = \frac{1}{4} \left( \frac{4V}{\pi H_0^3} \right)^{1/2} \quad (13)$$

The conical and cylindrical towers can be compared based on eqs. (11) and (13). If they are to lean to the same degree ( $\alpha = \alpha_0$ ), then the conical tower can be taller than the cylindrical tower,

$$\frac{H}{H_0} = \left( \frac{4}{3} \right)^{1/3} = 1.1 \quad (14)$$

If the towers are equally tall ( $H = H_0$ ), then the conical tower can lean more than the cylindrical tower,

$$\frac{\sin \alpha}{\sin \alpha_0} = \frac{2}{3^{1/2}} = 1.155 \quad (15)$$

---

**Problem 13.18** Consider the two-dimensional disc shaped domain of unit thickness, conductivity  $k$ , radius  $R$ , and uniformly distributed heat sink effect  $q''[W/m^3]$ . The  $r = R$  perimeter is insulated. There is a concentric internal ring of radius  $R_m$ , which generates heat at the uniform rate  $q''[W/m^2]$ . The heat generated by the rink ( $2\pi R_m q''$ ) sinks uniformly into the disc ( $q''\pi R^2$ ). The temperature of the disc is the highest ( $T_m$ ) along the  $R_m$  ring. We search for the rink position  $R_m$  that leads to the smallest difference between the highest temperature (at  $r = R_m$ ) and the lowest (at  $r = 0$ , or  $r = R$ ).

The temperature distribution over the disc is governed by

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) - \frac{q'''}{k} = 0 \quad (1)$$

We nondimensionalize this equation by introducing

$$\rho = \frac{r}{R} \quad \theta = \frac{T_m - T}{q''' R^2 / k} \quad (2)$$

where  $\theta$  is positive. The dimensionless eq. (1) becomes

$$\frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{d\theta}{d\rho} \right) + 1 = 0 \quad (3)$$

Integrating it once we find

$$\frac{d\theta}{d\rho} = -\frac{\rho}{2} + \frac{c_1}{\rho} \quad (4)$$

Symmetry requires  $d\theta/d\rho = 0$  at  $\rho = 0$ , hence  $c_1 = 0$ . Integrating eq. (4) one more time yields

$$\theta = -\frac{\rho^2}{4} + c_2 \quad (5)$$

where  $\theta = 0$  at  $\rho = \rho_m (= R_m/R)$ . This means that in the inner region ( $0 < \rho < \rho_m$ ) the temperature distribution is

$$\theta = \frac{1}{4} (\rho_m^2 - \rho^2) \quad (6)$$

The temperature 'depression' in the center of the disc is

$$\theta_c = \frac{1}{4} \rho_m^2 \quad (7)$$

For the temperature distribution in the outer annulus ( $\rho_m < \rho < 1$ ), we start with eq. (4) and the adiabatic conduction on the perimeter,

$$\frac{d\theta}{d\rho} = 0 \quad \text{at} \quad \rho = 1 \quad (8)$$

Equation (4) becomes

$$\frac{d\theta}{d\rho} = \frac{1}{2} \left( \frac{1}{\rho} - \rho \right) \quad (9)$$

Integrating again,

$$\theta = \frac{1}{2} \left( \ln \rho - \frac{\rho^2}{2} - c_2 \right) \quad (10)$$

and invoking the ring temperature condition  $\theta = 0$  at  $\rho = \rho_m$ , we obtain

$$\theta = \frac{1}{2} \ln \frac{\rho}{\rho_m} + \frac{1}{4} (\rho_m^2 - \rho^2) \quad (11)$$

and the temperature depression on the disc perimeter ( $\rho = 1$ ):

$$\theta_p = \frac{1}{2} \ln \frac{1}{\rho_m} + \frac{1}{4} (\rho_m^2 - 1) \quad (12)$$

The objective is to minimize  $\theta$  all over the disc. This means to seek the configuration ( $\rho_m$ ) such that the extreme  $\theta$  values ( $\theta_c$ ,  $\theta_p$ ) are the smallest. The center temperature depression  $\theta_c$  increases monotonically with  $\rho_m$ . The perimeter temperature depression  $\theta_p$  decreases monotonically as  $\rho_m$  increases from 0 to 1. This means that the smallest temperature depression occurs when the curves  $\theta_c(\rho_m)$  and  $\theta_p(\rho_m)$  intersect: eqs. (7) and (12) yield

$$\ln \frac{1}{\rho_m} = \frac{1}{2} \quad (13)$$

with the solution  $\rho_m = 0.6065$ , which means

$$R_{m,opt} \cong 0.61R \quad (14)$$

$$\theta_{min} = 0.092 \quad (15)$$

Problem 13.19 According to constructal theory, "optimal distribution of imperfection" in river basins means (roughly) equipartition (spreading) of flow resistance (elevation drop) from one

pairing level to the next. This means (roughly) a uniform spreading of the water travel time from one pairing level to the next. This also means that in order to estimate the mean residence time scale in a river basin it is sufficient to estimate the time scale of water seepage at the elemental level: from the hill slope to the first channel. The seepage along the hill slope is by Darcy flow through a saturated porous medium of permeability  $K$  and volume averaged velocity  $u$ . If this flow were enclosed in a duct with rigid walls, then it would be driven by an the end-to-end pressure difference  $\Delta P$  such that

$$u = \frac{K}{\mu} \frac{\Delta P}{L} \quad (1)$$

where  $L$  is the duct (hill slope) length, and  $\mu$  is the water density. In reality, the hill slope is at constant pressure, and the height difference between its ends is  $z$ , where that  $z \ll L$ . The slope is  $z/L$ , the gravitational acceleration component that drives the seepage is  $gz/L$ , and the total body force (weight) that pulls the liquid column through the porous layer is  $(\rho AL)gz/L$ , where  $A$  is the flow cross section. This force is balanced by the flow resistance, which according to eq. (1) is  $\Delta PA$ ,

$$\rho ALgz/L = \Delta PA \quad (2)$$

By eliminating  $\Delta P$  between eqs. (1) and (2) we find

$$u = \rho g z \frac{K}{\mu L} \quad (3)$$

The water travel time along the hill slope is

$$t = \frac{L}{u} = \frac{v}{gK} \frac{L^2}{z} \quad (4)$$

The meaning of  $L^2/z$  is purely geometric

$$\frac{L^2}{z} = \frac{L}{z/L} = \frac{\text{flow length}}{\text{gradient}} \quad (5)$$

In conclusion, according to eq. (4) we can expect a proportionality between (flow length)/gradient and travel time

$$\frac{\text{flow length}}{\text{gradient}} = \frac{gK}{v} t \quad (6)$$

For water at 10°C, this means

$$\frac{\text{length/gradient}}{[\text{m}]} \sim 755 \frac{K}{[\text{cm}^2]} \frac{t}{[\text{s}]} \quad (7)$$

where the denominators indicate the units in which to express the numerators. According to the book by A. Bejan, I. Dincer, S. Lorente, A. F. Miguel and A. H. Reis, *Porous and Complex Flow Structures in Modern Technologies*, Springer-Verlag, New York, 2004, p. 9, the permeability of soil is in the range  $K = 2.9 \times 10^{-9} - 1.4 \times 10^{-7} \text{ cm}^2$ , and eq. (7) becomes

$$\frac{\text{length/gradient}}{[\text{m}]} \sim (60 - 3000) \frac{t}{[\text{years}]} \quad (8)$$

This estimate is meant to be approximate, because of the uncertainty in estimating  $K$ , and the approximation discussed in the first two paragraphs. But the proportionality between the left and right sides is demanded (predicted) by theory.

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**Problem 13.20** The pressure drop along a tube with Poiseuille flow is

$$\Delta P_i = c \dot{m}_i \frac{L_i}{D_i^4} \quad (1)$$

where  $c$  is a constant factor. For the three tubes, we write

$$\Delta P_1 = c \dot{m}_1 \frac{L_1}{D_1^4} \quad (2)$$

$$\Delta P_2 = c \dot{m}_2 \frac{L_2}{D_2^4} \quad (3, 4)$$

Note that the pressure drop ( $\Delta P_2$ ) along the  $L_2$  tube is the same as for the  $L_3$  tube. Mass conservation requires

$$\dot{m}_1 = \dot{m}_2 + \dot{m}_3 \quad (5)$$

The overall pressure drop is  $\Delta P_1 + \Delta P_2$ . The overall flow resistance can be written as follows, after using eqs. (1) – (5):

$$\frac{\Delta P_1 + \Delta P_2}{c \dot{m}_1} = x_1^{-1} + (x_2 + x_3)^{-1} \quad (6)$$

where

$$x_i = \frac{D_i^4}{L_i}, \quad i = 1, 2, 3 \quad (7)$$

The total tube volume constraint is

$$V = L_1 D_1^2 + L_2 D_2^2 + L_3 D_3^2 \quad (8)$$

$$= a_1 x_1^{1/2} + a_2 x_2^{1/2} + a_3 x_3^{1/2} \quad (9)$$

where

$$a_i = L_i^{3/2}, \quad i = 1, 2, 3 \quad (10)$$

According to the method of Lagrange multipliers, we seek the extremum of the function  $\Phi$ , which is a linear combination of functions (6) and (9):

$$\Phi = x_1^{-1} + (x_2 + x_3)^{-1} + \lambda (a_1 x_1^{1/2} + a_2 x_2^{1/2} + a_3 x_3^{1/2}) \quad (11)$$

By solving the system

$$\frac{\partial \Phi}{\partial x_i} = 0, \quad i = 1, 2, 3 \quad (12)$$

we determine, in order,

$$\frac{D_2}{D_3} = \frac{L_2}{L_3} \quad (13)$$

$$\frac{D_1}{D_2} = \left[ 1 + \left( \frac{L_3}{L_2} \right)^3 \right]^{1/3} \quad (14)$$

$$\frac{D_1}{D_3} = \left[ 1 + \left( \frac{L_2}{L_3} \right)^3 \right]^{1/3} \quad (15)$$

When  $L_2 = L_3$ , these results reduce to

$$\frac{D_1}{D_2} = \frac{D_1}{D_3} = 2^{1/3} \quad (16)$$

which is Murray's law. The limits of the general solution (13) – (15) are

$$L_3 \gg L_2 : \quad \frac{D_1}{D_2} = \frac{L_3}{L_2}, \quad D_1 \approx D_3$$

$$L_3 \ll L_2 : \quad \frac{D_1}{D_3} = \frac{L_2}{L_3}, \quad D_1 \approx D_2$$

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**Problem 13.21** The momentum equation for fully developed laminar flow is

$$\frac{1}{\mu} \frac{dP}{dx} = \frac{1}{r} \frac{d}{dr} \left( r \frac{du}{dr} \right) = \text{constant} \quad (1)$$

We will write and integrate this equation twice, for the inner region ( $u_1, \mu_1$ ) and the outer region ( $u_2, \mu_2$ ). For the inner region we obtain, after one integration,

$$\frac{r^2}{2\mu_1} \frac{dP}{dx} = r \frac{du_1}{dr} + c_1 \quad (2)$$

Because  $du_1/dr = 0$  at  $r = 0$ ,  $c_1$  must be zero. Integrating eq. (2), we find

$$u_1 = \frac{r^2}{4\mu_1} \frac{dP}{dx} - c_2 \quad (3)$$

For the outer region, the double integration yields

$$u_2 = \frac{r^2}{4\mu_2} \frac{dP}{dx} - c_3 \ln r - c_4 \quad (4)$$

No slip at the wall ( $u_2 = 0$  at  $r = R$ ) requires

$$0 = \frac{R^2}{4\mu_2} \frac{dP}{dx} - c_3 \ln R - c_4 \quad (5)$$

and, after subtracting eq. (5) from eq. (4),

$$u_2 = \frac{r^2 - R^2}{4\mu_2} \frac{dP}{dx} - c_3 \ln \frac{r}{R} \quad (6)$$

Next, we match the  $u_1$  and  $u_2$  solutions at the intermediate radial position  $r = R_i$ , where  $u_{1i}$  must equal  $u_{2i}$ :

$$\frac{R_i^2}{4\mu_1} \frac{dP}{dx} - c_2 = \frac{R_i^2 - R^2}{4\mu_2} \frac{dP}{dx} - c_3 \frac{R_i}{R} \quad (7)$$

In addition, the shear stresses must also match (because there is no external force pulling/pushing along the  $u_1 - u_2$  interface):

$$\mu_1 \frac{du_1}{dr} = \mu_2 \frac{du_2}{dr} \quad (r = R_i) \quad (8)$$

$$\frac{R_i}{2} \frac{dP}{dx} = \frac{R_i}{2} \frac{dP}{dx} - \frac{c_3}{R_i} \quad (9)$$

Equation (9) yields  $c_3 = 0$ , and eq. (7) delivers  $c_2$ :

$$c_2 = \frac{1}{4} \frac{dP}{dx} \left( \frac{R_i^2}{\mu_1} + \frac{R^2 - R_i^2}{\mu_2} \right) \quad (10)$$

$$u_1 = \frac{1}{4} \frac{dP}{dx} \left( \frac{r^2 - R_i^2}{\mu_1} - \frac{R^2 - R_i^2}{\mu_2} \right) \quad (11)$$

The total mass flow rate is

$$\dot{m} = 2\pi\rho \left( \int_0^{R_i} u_1 r dr + \int_{R_i}^R u_2 r dr \right) \quad (12)$$

and, after using eqs. (6) and (11),

$$\dot{m} = \frac{\pi\rho}{8} \left( -\frac{dP}{dx} \right) \left( \frac{R_i^4}{\mu_1} + \frac{R^4}{\mu_2} \right) \quad (13)$$

According to the constructal law (for a specified  $-dP/dx$ ), the flow rate should be greater in the configuration that prevails. We assume that the one that prevails is the one chosen in the problem statement (with  $\mu_1$  in the center), the flow rate of which is proportional to

$$Q_1 = \frac{R_i^4}{\mu_1} + \frac{R^4}{\mu_2} \quad (14)$$

The alternate configuration is the one with  $\mu_2$  in the center. The flow rate in this case would have been proportional to

$$Q_2 = \frac{R_i^4}{\mu_2} + \frac{R^4}{\mu_1} \quad (15)$$

The maximization of flow access means that  $Q_1 > Q_2$ , which is true when  $\mu_1 > \mu_2$ . This was the assumed configuration, and it is correct. Maximum flow access is achieved when the less viscous liquid coats the wall.

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Problem 13.22 The scale of the force exerted by a motor ( $F$ ) is the torque ( $T$ ) divided by the arm ( $a$ ),

$$F \sim \frac{T}{a} \quad (1)$$

The arm  $a$  is the length scale of the motor body. The motor body is turned clockwise ( $-T$ ) by the system attached to its shaft. The motor is stabilized by the counter-clockwise movement provided by its own weight. Rotational equilibrium at the stage where the motor mass  $M$  is just barely large enough (to keep the motor in place on its substrate) yields the scale of the maximum motor force

$$T \sim Mga \quad (2)$$

$$F \sim Mg$$

In conclusion, engineered motors must also have a proportionality between maximum force and motor mass, where the proportionality factor is on the order of  $g$ . This is in agreement with the force-mass proportionality of all animal motors.

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