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# INVERSE STURM-LIOUVILLE PROBLEMS AND THEIR APPLICATIONS

G. Freiling and V. Yurko

## PREFACE

This book presents the main results and methods on inverse spectral problems for Sturm-Liouville differential operators and their applications. Inverse problems of spectral analysis consist in recovering operators from their spectral characteristics. Such problems often appear in mathematics, mechanics, physics, electronics, geophysics, meteorology and other branches of natural sciences. Inverse problems also play an important role in solving nonlinear evolution equations in mathematical physics. Interest in this subject has been increasing permanently because of the appearance of new important applications, and nowadays the inverse problem theory develops intensively all over the world.

The greatest success in spectral theory in general, and in particular in inverse spectral problems has been achieved for the Sturm-Liouville operator

$$\ell y := -y'' + q(x)y, \tag{1}$$

which also is called the one-dimensional Schrödinger operator. The first studies on the spectral theory of such operators were performed by D. Bernoulli, J. d'Alembert, L. Euler, J. Liouville and Ch. Sturm in connection with the solution of the equation describing the vibration of a string. An intensive development of the spectral theory for various classes of differential and integral operators and for operators in abstract spaces took place in the XX-th century. Deep ideas here are due to G. Birkhoff, D. Hilbert, J. von Neumann, V. Steklov, M. Stone, H. Weyl and many other mathematicians. The main results on inverse spectral problems appear in the second half of the XX-th century. We mention here the works by R. Beals, G. Borg, L.D. Faddeev, M.G. Gasymov, I.M. Gelfand, B.M. Levitan, I.G. Khachatryan, M.G. Krein, N. Levinson, Z.L. Leibenson, V.A. Marchenko, L.A. Sakhnovich, E. Trubowitz, V.A. Yurko and others (see Section 1.9 for details). An important role in the inverse spectral theory for the Sturm-Liouville operator was played by the transformation operator method (see [mar1], [lev2] and the references therein). But this method turned out to be unsuitable for many important classes of inverse problems being more complicated than the Sturm-Liouville operator. At present time other effective methods for solving inverse spectral problems have been created. Among them we point out the method of spectral mappings connected with ideas of the contour integral method. This method seems to be perspective for inverse spectral problems. The created methods allowed to solve a number of important problems in various branches of natural sciences.

In recent years there appeared new areas for applications of inverse spectral problems. We mention a remarkable method for solving some nonlinear evolution equations of mathematical physics connected with the use of inverse spectral problems (see [abl1], [abl3], [zak1] and Sections 4.1-4.2 for details). Another important class of inverse problems, which often appear in applications, is the inverse problem of recovering differential equations from incomplete spectral information when only a part of the spectral information is available for measurement. Many applications are connected with inverse problems for differential equations having singularities and turning points, for higher-order differential operators, for differential operators with delay or other types of "aftereffect".

The main goals of this book are as follows:

- To present a fairly elementary and complete introduction to the inverse problem theory for ordinary differential equations which is suitable for the "first reading" and accessible not only to mathematicians but also to physicists, engineers and students. Note that the book requires knowledge only of classical analysis and the theory of ordinary linear differential equations.
- To describe the main ideas and methods in the inverse problem theory using the Sturm-Liouville operator as a model. Up to now many of these ideas, in particular those which appeared in recent years, are presented in journals only. It is very important that the methods, provided in this book, can be used (and have been already used) not only for Sturm-Liouville operators, but also for solving inverse problems for other more complicated classes of operators such as differential operators of arbitrary orders, differential operators with singularities and turning points, pencils of operators, integro-differential and integral operators and others.
- To reflect various applications of the inverse spectral problems in natural sciences and engineering.

The book is organized as follows. In Chapter 1, Sturm-Liouville operators (1) on a finite interval are considered. In Sections 1.1-1.3 we study properties of spectral characteristics and eigenfunctions, and prove a completeness theorem and an expansion theorem. Sections 1.4-1.8 are devoted to the inverse problem theory. We prove uniqueness theorems, give algorithms for the solution of the inverse problems considered, provide necessary and sufficient conditions for their solvability, and study stability of the solutions. We present several methods for solving inverse problems. The transformation operator method, in which the inverse problem is reduced to the solution of a linear integral equation, is described in Section 1.5. In Section 1.6 we present the method of spectral mappings, in which ideas of the contour integral method are used. The central role there is played by the so-called main equation of the inverse problem which is a linear equation in a corresponding Banach space. We give a derivation of the main equation, prove its unique solvability and provide explicit formulae for the solution of the inverse problem. At present time the contour integral method seems to be the most universal tool in the inverse problem theory for ordinary differential operators. It has a wide area for applications in various classes of inverse problems (see, for example, [bea1], [yur1] and the references therein). In the method of standard models, which is described in Section 1.7, a sequence of model differential operators approximating the unknown operator are constructed. In Section 1.8 we provide the method for the local solution of the inverse

problem from two spectra which is due to G. Borg [Bor1]. In this method, the inverse problem is reduced to a special nonlinear integral equation, which can be solved locally.

Chapter 2 is devoted to Sturm-Liouville operators on the half-line. First we consider non-selfadjoint operators with integrable potentials. Using the contour integral method we study the inverse problem of recovering the Sturm-Liouville operator from its Weyl function. Then locally integrable complex-valued potentials are studied, and the inverse problem is solved by specifying the generalized Weyl function. In Chapter 3 Sturm-Liouville operators on the line are considered, and the inverse scattering problem is studied. In Chapter 4 we provide a number of applications of the inverse problem theory in natural sciences and engineering: we consider the solution of the Korteweg-de Vries equation on the line and on the half-line, solve the inverse problem of constructing parameters of a medium from incomplete spectral information, and study boundary value problems with aftereffect, inverse problems in elasticity theory and others.

There exists an extensive literature devoted to inverse spectral problems. Instead of trying to give a complete list of all relevant contributions, we mention only monographs, survey articles and the most important papers, and refer to the references therein. In Section 1.9 we give a short review on literature on the inverse problem theory.

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# I. STURM-LIOUVILLE OPERATORS ON A FINITE INTERVAL

In this chapter we present an introduction to the spectral theory for Sturm-Liouville operators on a finite interval. Sections 1.1-1.2 are devoted to the so-called direct problems of spectral analysis. In Section 1.1 we introduce and study the main spectral characteristics for Sturm-Liouville boundary value problems. In particular, the theorem on the existence and the asymptotic behavior of the eigenvalues and eigenfunctions is proved. In Section 1.2 properties of the eigenfunctions are investigated. It is proved that the system of the eigenfunctions is complete and forms an orthogonal basis in  $L_2(0, \pi)$ . An expansion theorem in the uniform norm is provided. We also investigate oscillation properties of the eigenfunctions, and prove that the  $n$ -th eigenfunction has exactly  $n$  zeros inside the interval  $(0, \pi)$ . In Section 1.3 we study the so-called transformation operators which give us an effective tool in the Sturm-Liouville spectral theory.

Sections 1.4-1.8 are devoted to constructing the inverse problem theory for Sturm-Liouville operators on a finite interval. In Section 1.4 we give various formulations of the inverse problems and prove the corresponding uniqueness theorems. In Sections 1.5-1.8 we present several methods for constructing the solution of the inverse problems considered. The variety of ideas in these methods allows one to use them for many classes of inverse problems and their applications. Section 1.5 deals with the Gelfand-Levitan method [gel1], in which the transformation operators are used. The method of spectral mappings, considered in Section 1.6, leans on ideas of the contour integral method. By these two methods we obtain algorithms for the solution of the inverse problems, and provide necessary and sufficient conditions for their solvability. In Section 1.7 we describe the method of standard models which allows one to obtain effective algorithms for the solution of a wide class of inverse problems. The method for the local solution of the inverse problem, which is due to Borg [Bor1], is provided in Section 1.8. A short historical review on the inverse problems of spectral analysis for ordinary differential operators is given in Section 1.9.

## 1.1. BEHAVIOR OF THE SPECTRUM

Let us consider the boundary value problem  $L = L(q(x), h, H)$  :

$$\ell y := -y'' + q(x)y = \lambda y, \quad 0 < x < \pi, \quad (1.1.1)$$

$$U(y) := y'(0) - hy(0) = 0, \quad V(y) := y'(\pi) + Hy(\pi) = 0. \quad (1.1.2)$$

Here  $\lambda$  is the spectral parameter;  $q(x)$ ,  $h$  and  $H$  are real;  $q(x) \in L_2(0, \pi)$ . We shall subsequently refer to  $q$  as the potential. The operator  $\ell$  is called *the Sturm-Liouville operator*.

We are interested in non-trivial solutions of the boundary value problem (1.1.1)-(1.1.2).

**Definition 1.1.1.** The values of the parameter  $\lambda$  for which  $L$  has nonzero solutions are called *eigenvalues*, and the corresponding nontrivial solutions are called *eigenfunctions*. The set of eigenvalues is called *the spectrum* of  $L$ .

In this section we obtain the simplest spectral properties of  $L$ , and study the asymptotic behavior of eigenvalues and eigenfunctions. We note that similarly one can also study other types of separated boundary conditions, namely:

- (i)  $U(y) = 0, y(\pi) = 0;$
- (ii)  $y(0) = 0, V(y) = 0;$
- (iii)  $y(0) = y(\pi) = 0.$

Let  $C(x, \lambda), S(x, \lambda), \varphi(x, \lambda), \psi(x, \lambda)$  be solutions of (1.1.1) under the initial conditions

$$C(0, \lambda) = 1, \quad C'(0, \lambda) = 0, \quad S(0, \lambda) = 0, \quad S'(0, \lambda) = 1,$$

$$\varphi(0, \lambda) = 1, \quad \varphi'(0, \lambda) = h, \quad \psi(\pi, \lambda) = 1, \quad \psi'(\pi, \lambda) = -H.$$

For each fixed  $x$ , the functions  $\varphi(x, \lambda), \psi(x, \lambda), C(x, \lambda), S(x, \lambda)$  are entire in  $\lambda$ . Clearly,

$$U(\varphi) := \varphi'(0, \lambda) - h\varphi(0, \lambda) = 0, \quad V(\psi) := \psi'(\pi, \lambda) + H\psi(\pi, \lambda) = 0. \quad (1.1.3)$$

Denote

$$\Delta(\lambda) = \langle \psi(x, \lambda), \varphi(x, \lambda) \rangle, \quad (1.1.4)$$

where

$$\langle y(x), z(x) \rangle := y(x)z'(x) - y'(x)z(x)$$

is the Wronskian of  $y$  and  $z$ . By virtue of Liouville's formula for the Wronskian [cod1, p.83],  $\langle \psi(x, \lambda), \varphi(x, \lambda) \rangle$  does not depend on  $x$ . The function  $\Delta(\lambda)$  is called *the characteristic function* of  $L$ . Substituting  $x = 0$  and  $x = \pi$  into (1.1.4), we get

$$\Delta(\lambda) = V(\varphi) = -U(\psi). \quad (1.1.5)$$

The function  $\Delta(\lambda)$  is entire in  $\lambda$ , and it has an at most countable set of zeros  $\{\lambda_n\}$ .

**Theorem 1.1.1.** *The zeros  $\{\lambda_n\}$  of the characteristic function coincide with the eigenvalues of the boundary value problem  $L$ . The functions  $\varphi(x, \lambda_n)$  and  $\psi(x, \lambda_n)$  are eigenfunctions, and there exists a sequence  $\{\beta_n\}$  such that*

$$\psi(x, \lambda_n) = \beta_n \varphi(x, \lambda_n), \quad \beta_n \neq 0. \quad (1.1.6)$$

*Proof.* 1) Let  $\lambda_0$  be a zero of  $\Delta(\lambda)$ . Then, by virtue of (1.1.3)-(1.1.5),  $\psi(x, \lambda_0) = \beta_0 \varphi(x, \lambda_0)$ , and the functions  $\psi(x, \lambda_0), \varphi(x, \lambda_0)$  satisfy the boundary conditions (1.1.2). Hence,  $\lambda_0$  is an eigenvalue, and  $\psi(x, \lambda_0), \varphi(x, \lambda_0)$  are eigenfunctions related to  $\lambda_0$ .

2) Let  $\lambda_0$  be an eigenvalue of  $L$ , and let  $y_0$  be a corresponding eigenfunction. Then  $U(y_0) = V(y_0) = 0$ . Clearly  $y_0(0) \neq 0$  (if  $y_0(0) = 0$  then  $y_0'(0) = 0$  and, by virtue of the uniqueness theorem for the equation (1.1.1) (see [cod1, Ch.1]),  $y_0(x) \equiv 0$ ). Without loss of generality we put  $y_0(0) = 1$ . Then  $y_0'(0) = h$ , and consequently  $y_0(x) \equiv \varphi(x, \lambda_0)$ . Therefore, (1.1.5) yields  $\Delta(\lambda_0) = V(\varphi(x, \lambda_0)) = V(y_0(x)) = 0$ . We have also proved that for each eigenvalue there exists only one (up to a multiplicative constant) eigenfunction.  $\square$

Throughout the whole chapter we use the notation

$$\alpha_n := \int_0^\pi \varphi^2(x, \lambda_n) dx. \quad (1.1.7)$$

The numbers  $\{\alpha_n\}$  are called *the weight numbers*, and the numbers  $\{\lambda_n, \alpha_n\}$  are called *the spectral data* of  $L$ .

**Lemma 1.1.1.** *The following relation holds*

$$\beta_n \alpha_n = -\dot{\Delta}(\lambda_n), \quad (1.1.8)$$

where the numbers  $\beta_n$  are defined by (1.1.6), and  $\dot{\Delta}(\lambda) = \frac{d}{d\lambda} \Delta(\lambda)$ .

*Proof.* Since

$$-\psi''(x, \lambda) + q(x)\psi(x, \lambda) = \lambda\psi(x, \lambda), \quad -\varphi''(x, \lambda_n) + q(x)\varphi(x, \lambda_n) = \lambda_n\varphi(x, \lambda_n),$$

we get

$$\frac{d}{dx} \langle \psi(x, \lambda), \varphi(x, \lambda_n) \rangle = \psi(x, \lambda) \varphi''(x, \lambda_n) - \psi''(x, \lambda) \varphi(x, \lambda_n) = (\lambda - \lambda_n) \psi(x, \lambda) \varphi(x, \lambda_n),$$

and hence with the help of (1.1.5),

$$\begin{aligned} (\lambda - \lambda_n) \int_0^\pi \psi(x, \lambda) \varphi(x, \lambda_n) dx &= \langle \psi(x, \lambda), \varphi(x, \lambda_n) \rangle \Big|_0^\pi \\ &= \varphi'(\pi, \lambda_n) + H\varphi(\pi, \lambda_n) + \psi'(0, \lambda) - h\psi(0, \lambda) = -\Delta(\lambda). \end{aligned}$$

For  $\lambda \rightarrow \lambda_n$ , this yields

$$\int_0^\pi \psi(x, \lambda_n) \varphi(x, \lambda_n) dx = -\dot{\Delta}(\lambda_n).$$

Using (1.1.6) and (1.1.7) we arrive at (1.1.8).  $\square$

**Theorem 1.1.2.** *The eigenvalues  $\{\lambda_n\}$  and the eigenfunctions  $\varphi(x, \lambda_n), \psi(x, \lambda_n)$  are real. All zeros of  $\Delta(\lambda)$  are simple, i.e.  $\dot{\Delta}(\lambda_n) \neq 0$ . Eigenfunctions related to different eigenvalues are orthogonal in  $L_2(0, \pi)$ .*

*Proof.* Let  $\lambda_n$  and  $\lambda_k$  ( $\lambda_n \neq \lambda_k$ ) be eigenvalues with eigenfunctions  $y_n(x)$  and  $y_k(x)$  respectively. Then integration by parts yields

$$\int_0^\pi \ell y_n(x) y_k(x) dx = \int_0^\pi y_n(x) \ell y_k(x) dx,$$

and hence

$$\lambda_n \int_0^\pi y_n(x) y_k(x) dx = \lambda_k \int_0^\pi y_n(x) y_k(x) dx,$$

or

$$\int_0^\pi y_n(x) y_k(x) dx = 0.$$

Further, let  $\lambda^0 = u + iv, v \neq 0$  be a non-real eigenvalue with an eigenfunction  $y^0(x) \neq 0$ . Since  $q(x), h$  and  $H$  are real, we get that  $\bar{\lambda}^0 = u - iv$  is also the eigenvalue with the eigenfunction  $\overline{y^0(x)}$ . Since  $\lambda^0 \neq \bar{\lambda}^0$ , we derive as before

$$\|y^0\|_{L_2}^2 = \int_0^\pi y^0(x) \overline{y^0(x)} dx = 0,$$



which is impossible. Thus, all eigenvalues  $\{\lambda_n\}$  of  $L$  are real, and consequently the eigenfunctions  $\varphi(x, \lambda_n)$  and  $\psi(x, \lambda_n)$  are real too. Since  $\alpha_n \neq 0$ ,  $\beta_n \neq 0$ , we get by virtue of (1.1.8) that  $\Delta(\lambda_n) \neq 0$ .  $\square$

**Example 1.1.1.** Let  $q(x) = 0$ ,  $h = 0$  and  $H = 0$ , and let  $\lambda = \rho^2$ . Then one can check easily that

$$C(x, \lambda) = \varphi(x, \lambda) = \cos \rho x, \quad S(x, \lambda) = \frac{\sin \rho x}{\rho}, \quad \psi(x, \lambda) = \cos \rho(\pi - x),$$

$$\Delta(\lambda) = -\rho \sin \rho \pi, \quad \lambda_n = n^2 \quad (n \geq 0), \quad \varphi(x, \lambda_n) = \cos nx, \quad \beta_n = (-1)^n.$$

**Lemma 1.1.2.** For  $|\rho| \rightarrow \infty$ , the following asymptotic formulae hold

$$\left. \begin{aligned} \varphi(x, \lambda) &= \cos \rho x + O\left(\frac{1}{|\rho|} \exp(|\tau|x)\right) = O(\exp(|\tau|x)), \\ \varphi'(x, \lambda) &= -\rho \sin \rho x + O(\exp(|\tau|x)) = O(|\rho| \exp(|\tau|x)), \end{aligned} \right\} \quad (1.1.9)$$

$$\left. \begin{aligned} \psi(x, \lambda) &= \cos \rho(\pi - x) + O\left(\frac{1}{|\rho|} \exp(|\tau|(\pi - x))\right) = O(\exp(|\tau|(\pi - x))), \\ \psi'(x, \lambda) &= \rho \sin \rho(\pi - x) + O(\exp(|\tau|(\pi - x))) = O(|\rho| \exp(|\tau|(\pi - x))), \end{aligned} \right\} \quad (1.1.10)$$

uniformly with respect to  $x \in [0, \pi]$ . Here and in the sequel,  $\lambda = \rho^2$ ,  $\tau = \operatorname{Im} \rho$ , and  $o$  and  $O$  denote the Landau symbols.

*Proof.* Let us show that

$$\varphi(x, \lambda) = \cos \rho x + h \frac{\sin \rho x}{\rho} + \int_0^x \frac{\sin \rho(x-t)}{\rho} q(t) \varphi(t, \lambda) dt. \quad (1.1.11)$$

Indeed, the Volterra integral equation

$$y(x, \lambda) = \cos \rho x + h \frac{\sin \rho x}{\rho} + \int_0^x \frac{\sin \rho(x-t)}{\rho} q(t) y(t, \lambda) dt$$

has a unique solution (for the theory of Volterra integral equations see [Kre1]). On the other hand, if a certain function  $y(x, \lambda)$  satisfies this equation, then we get by differentiation

$$y''(x, \lambda) + \rho^2 y(x, \lambda) = q(x) y(x, \lambda), \quad y(0, \lambda) = 1, \quad y'(0, \lambda) = h,$$

i.e.  $y(x, \lambda) \equiv \varphi(x, \lambda)$ , and (1.1.11) is valid.

Differentiating (1.1.11) we calculate

$$\varphi'(x, \lambda) = -\rho \sin \rho x + h \cos \rho x + \int_0^x \cos \rho(x-t) q(t) \varphi(t, \lambda) dt. \quad (1.1.12)$$

Denote  $\mu(\lambda) = \max_{0 \leq x \leq \pi} (|\varphi(x, \lambda)| \exp(-|\tau|x))$ . Since  $|\sin \rho x| \leq \exp(|\tau|x)$  and  $|\cos \rho x| \leq \exp(|\tau|x)$ , we have from (1.1.11) that for  $|\rho| \geq 1$ ,  $x \in [0, \pi]$ ,

$$|\varphi(x, \lambda)| \exp(-|\tau|x) \leq 1 + \frac{1}{|\rho|} \left( h + \mu(\lambda) \int_0^x |q(t)| dt \right) \leq C_1 + \frac{C_2}{|\rho|} \mu(\lambda),$$

and consequently

$$\mu(\lambda) \leq C_1 + \frac{C_2}{|\rho|} \mu(\lambda).$$

For sufficiently large  $|\rho|$ , this yields  $\mu(\lambda) = O(1)$ , i.e.  $\varphi(x, \lambda) = O(\exp(|\tau|x))$ . Substituting this estimate into the right-hand sides of (1.1.11) and (1.1.12), we arrive at (1.1.9). Similarly one can derive (1.1.10).

We note that (1.1.10) can be also obtained directly from (1.1.9). Indeed, since

$$-\psi''(x, \lambda) + q(x)\psi(x, \lambda) = \lambda\psi(x, \lambda), \quad \psi(\pi, \lambda) = 1, \quad \psi'(\pi, \lambda) = -H,$$

the function  $\tilde{\varphi}(x, \lambda) := \psi(\pi - x, \lambda)$  satisfies the following differential equation and the initial conditions

$$-\tilde{\varphi}''(x, \lambda) + q(\pi - x)\tilde{\varphi}(x, \lambda) = \lambda\tilde{\varphi}(x, \lambda), \quad \tilde{\varphi}(0, \lambda) = 1, \quad \tilde{\varphi}'(0, \lambda) = H.$$

Therefore, the asymptotic formulae (1.1.9) are also valid for the function  $\tilde{\varphi}(x, \lambda)$ . From this we arrive at (1.1.10).  $\square$

The main result of this section is the following theorem on the existence and the asymptotic behavior of the eigenvalues and the eigenfunctions of  $L$ .

**Theorem 1.1.3.** *The boundary value problem  $L$  has a countable set of eigenvalues  $\{\lambda_n\}_{n \geq 0}$ . For  $n \geq 0$ ,*

$$\rho_n = \sqrt{\lambda_n} = n + \frac{\omega}{\pi n} + \frac{\kappa_n}{n}, \quad \{\kappa_n\} \in l_2, \quad (1.1.13)$$

$$\varphi(x, \lambda_n) = \cos nx + \frac{\xi_n(x)}{n}, \quad |\xi_n(x)| \leq C, \quad (1.1.14)$$

where

$$\omega = h + H + \frac{1}{2} \int_0^\pi q(t) dt.$$

Here and everywhere below one and the same symbol  $\{\kappa_n\}$  denotes various sequences from  $l_2$ , and the symbol  $C$  denotes various positive constants which do not depend on  $x, \lambda$  and  $n$ .

*Proof.* 1) Substituting the asymptotics for  $\varphi(x, \lambda)$  from (1.1.9) into the right-hand sides of (1.1.11) and (1.1.12), we calculate

$$\left. \begin{aligned} \varphi(x, \lambda) &= \cos \rho x + q_1(x) \frac{\sin \rho x}{\rho} + \frac{1}{2\rho} \int_0^x q(t) \sin \rho(x - 2t) dt + O\left(\frac{\exp(|\tau|x)}{\rho^2}\right), \\ \varphi'(x, \lambda) &= -\rho \sin \rho x + q_1(x) \cos \rho x + \frac{1}{2} \int_0^x q(t) \cos \rho(x - 2t) dt + O\left(\frac{\exp(|\tau|x)}{\rho}\right), \end{aligned} \right\} \quad (1.1.15)$$

where

$$q_1(x) = h + \frac{1}{2} \int_0^x q(t) dt.$$

According to (1.1.5),  $\Delta(\lambda) = \varphi'(\pi, \lambda) + H\varphi(\pi, \lambda)$ . Hence by virtue of (1.1.15),

$$\Delta(\lambda) = -\rho \sin \rho \pi + \omega \cos \rho \pi + \kappa(\rho), \quad (1.1.16)$$

where

$$\kappa(\rho) = \frac{1}{2} \int_0^\pi q(t) \cos \rho(\pi - 2t) dt + O\left(\frac{1}{\rho} \exp(|\tau|\pi)\right).$$

2) Denote  $G_\delta = \{\rho : |\rho - k| \geq \delta, k = 0, \pm 1, \pm 2, \dots\}$ ,  $\delta > 0$ . Let us show that

$$|\sin \rho\pi| \geq C_\delta \exp(|\tau|\pi), \quad \rho \in G_\delta, \quad (1.1.17)$$

$$|\Delta(\lambda)| \geq C_\delta |\rho| \exp(|\tau|\pi), \quad \rho \in G_\delta, |\rho| \geq \rho^*, \quad (1.1.18)$$

for sufficiently large  $\rho^* = \rho^*(\delta)$ .

Let  $\rho = \sigma + i\tau$ . It is sufficient to prove (1.1.17) for the domain

$$D_\delta = \{\rho : \sigma \in [-\frac{1}{2}, \frac{1}{2}], \tau \geq 0, |\rho| \geq \delta\}.$$

Denote  $\theta(\rho) = |\sin \rho\pi| \exp(-|\tau|\pi)$ . Let  $\rho \in D_\delta$ . For  $\tau \leq 1$ ,  $\theta(\rho) \geq C_\delta$ . Since  $\sin \rho\pi = (\exp(i\rho\pi) - \exp(-i\rho\pi))/(2i)$ , we have for  $\tau \geq 1$ ,  $\theta(\rho) = |1 - \exp(2i\sigma\pi) \exp(-2\tau\pi)|/2 \geq 1/4$ . Thus, (1.1.17) is proved. Further, using (1.1.16) we get for  $\rho \in G_\delta$ ,

$$\Delta(\lambda) = -\rho \sin \rho\pi \left(1 + O\left(\frac{1}{\rho}\right)\right),$$

and consequently (1.1.18) is valid.

3) Denote

$$\Gamma_n = \{\lambda : |\lambda| = (n + 1/2)^2\}.$$

By virtue of (1.1.16),

$$\Delta(\lambda) = f(\lambda) + g(\lambda), \quad f(\lambda) = -\rho \sin \rho\pi, \quad |g(\lambda)| \leq C \exp(|\tau|\pi).$$

According to (1.1.17),  $|f(\lambda)| > |g(\lambda)|$ ,  $\lambda \in \Gamma_n$ , for sufficiently large  $n$  ( $n \geq n^*$ ). Then by Rouché's theorem [con1, p.125], the number of zeros of  $\Delta(\lambda)$  inside  $\Gamma_n$  coincides with the number of zeros of  $f(\lambda) = -\rho \sin \rho\pi$ , i.e. it equals  $n+1$ . Thus, in the circle  $|\lambda| < (n+1/2)^2$  there exist exactly  $n+1$  eigenvalues of  $L$ :  $\lambda_0, \dots, \lambda_n$ . Applying now Rouché's theorem to the circle  $\gamma_n(\delta) = \{\rho : |\rho - n| \leq \delta\}$ , we conclude that for sufficiently large  $n$ , in  $\gamma_n(\delta)$  there is exactly one zero of  $\Delta(\rho^2)$ , namely  $\rho_n = \sqrt{\lambda_n}$ . Since  $\delta > 0$  is arbitrary, it follows that

$$\rho_n = n + \varepsilon_n, \quad \varepsilon_n = o(1), \quad n \rightarrow \infty. \quad (1.1.19)$$

Substituting (1.1.19) into (1.1.16) we get

$$0 = \Delta(\rho_n^2) = -(n + \varepsilon_n) \sin(n + \varepsilon_n)\pi + \omega \cos(n + \varepsilon_n)\pi + \kappa_n,$$

and consequently

$$-n \sin \varepsilon_n \pi + \omega \cos \varepsilon_n \pi + \kappa_n = 0. \quad (1.1.20)$$

Then  $\sin \varepsilon_n \pi = O\left(\frac{1}{n}\right)$ , i.e.  $\varepsilon_n = O\left(\frac{1}{n}\right)$ . Using (1.1.20) once more we obtain more precisely  $\varepsilon_n = \frac{\omega}{\pi n} + \frac{\kappa_n}{n}$ , i.e. (1.1.13) is valid. Substituting (1.1.13) into (1.1.15) we arrive at (1.1.14), where

$$\xi_n(x) = \left(h + \frac{1}{2} \int_0^x q(t) dt - x \frac{\omega}{\pi} - x \kappa_n\right) \sin nx + \frac{1}{2} \int_0^x q(t) \sin n(x - 2t) dt + O\left(\frac{1}{n}\right). \quad (1.1.21)$$

Consequently  $|\xi_n(x)| \leq C$ , and Theorem 1.1.3 is proved.  $\square$

By virtue of (1.1.6) with  $x = \pi$ ,

$$\beta_n = (\varphi(\pi, \lambda_n))^{-1}.$$

Then, using (1.1.7), (1.1.8), (1.1.14) and (1.1.21) one can calculate

$$\alpha_n = \frac{\pi}{2} + \frac{\kappa_n}{n}, \quad \beta_n = (-1)^n + \frac{\kappa_n}{n}, \quad \dot{\Delta}(\lambda_n) = (-1)^{n+1} \frac{\pi}{2} + \frac{\kappa_n}{n}. \quad (1.1.22)$$

Since  $\Delta(\lambda)$  has only simple zeros, we have  $\text{sign } \dot{\Delta}(\lambda_n) = (-1)^{n+1}$  for  $n \geq 0$ .

Denote by  $W_2^N$  the Sobolev space of functions  $f(x)$ ,  $x \in [0, \pi]$ , such that  $f^{(j)}(x)$ ,  $j = 0, N-1$  are absolutely continuous, and  $f^{(N)}(x) \in L_2(0, \pi)$ .

**Remark 1.1.1.** If  $q(x) \in W_2^N$ ,  $N \geq 1$ , then one can obtain more precise asymptotic formulae as before. In particular,

$$\left. \begin{aligned} \rho_n &= n + \sum_{j=1}^{N+1} \frac{\omega_j}{n^j} + \frac{\kappa_n}{n^{N+1}}, \quad \omega_{2p} = 0, \quad p \geq 0, \quad \omega_1 = \frac{\omega}{\pi}, \\ \alpha_n &= \frac{\pi}{2} + \sum_{j=1}^{N+1} \frac{\omega_j^+}{n^j} + \frac{\kappa_n}{n^{N+1}}, \quad \omega_{2p+1}^+ = 0, \quad p \geq 0, \quad \alpha_n > 0. \end{aligned} \right\} \quad (1.1.23)$$

Indeed, let  $q(x) \in W_2^1$ . Then integration by parts yields

$$\left. \begin{aligned} \frac{1}{2} \int_0^x q(t) \cos \rho(x-2t) dt &= \frac{\sin \rho x}{4\rho} (q(x) + q(0)) + \frac{1}{4\rho} \int_0^x q'(t) \sin \rho(x-2t) dt, \\ \frac{1}{2\rho} \int_0^x q(t) \sin \rho(x-2t) dt &= \frac{\cos \rho x}{4\rho^2} (q(x) - q(0)) - \frac{1}{4\rho^2} \int_0^x q'(t) \cos \rho(x-2t) dt. \end{aligned} \right\} \quad (1.1.24)$$

It follows from (1.1.15) and (1.1.24) that

$$\varphi(x, \lambda) = \cos \rho x + \left( h + \frac{1}{2} \int_0^x q(t) dt \right) \frac{\sin \rho x}{\rho} + O\left( \frac{\exp(|\tau|x)}{\rho^2} \right).$$

Substituting this asymptotics into the right-hand sides of (1.1.11)-(1.1.12) and using (1.1.24) and (1.1.5), one can obtain more precise asymptotics for  $\varphi^{(\nu)}(x, \lambda)$  and  $\Delta(\lambda)$  than (1.1.15)-(1.1.16):

$$\begin{aligned} \varphi(x, \lambda) &= \cos \rho x + q_1(x) \frac{\sin \rho x}{\rho} + q_{20}(x) \frac{\cos \rho x}{\rho^2} - \frac{1}{4\rho^2} \int_0^x q'(t) \cos \rho(x-2t) dt + O\left( \frac{\exp(|\tau|x)}{\rho^3} \right), \\ \varphi'(x, \lambda) &= -\rho \sin \rho x + q_1(x) \cos \rho x + q_{21}(x) \frac{\sin \rho x}{\rho} + \frac{1}{4\rho} \int_0^x q'(t) \sin \rho(x-2t) dt + O\left( \frac{\exp(|\tau|x)}{\rho^2} \right), \\ \Delta(\lambda) &= -\rho \sin \rho \pi + \omega \cos \rho \pi + \omega_0 \frac{\sin \rho \pi}{\rho} + \frac{\kappa_0(\rho)}{\rho}, \end{aligned} \quad (1.1.25)$$

where

$$q_1(x) = h + \frac{1}{2} \int_0^x q(t) dt, \quad \omega_0 = q_{21}(\pi) + H q_1(\pi),$$

$$q_{2j}(x) = \frac{1}{4} \left( q(x) + (-1)^{j+1} q(0) \right) + \frac{(-1)^{j+1}}{2} \int_0^x q(t) q_1(t) dt, \quad j = 0, 1,$$

$$\kappa_0(\rho) = \frac{1}{4} \int_0^\pi q'(t) \sin \rho(\pi - 2t) dt + O\left(\frac{\exp(|\tau|\pi)}{\rho}\right).$$

From (1.1.25), by the same arguments as above, we deduce

$$\rho_n = n + \varepsilon_n, \quad -n \sin \varepsilon_n \pi + \omega \cos \varepsilon_n \pi + \frac{\kappa_n}{n} = 0.$$

Hence

$$\rho_n = n + \frac{\omega}{\pi n} + \frac{\kappa_n}{n^2}, \quad \{\kappa_n\} \in l_2.$$

Other formulas in (1.1.23) can be derived similarly.

**Theorem 1.1.4.** *The specification of the spectrum  $\{\lambda_n\}_{n \geq 0}$  uniquely determines the characteristic function  $\Delta(\lambda)$  by the formula*

$$\Delta(\lambda) = \pi(\lambda_0 - \lambda) \prod_{n=1}^{\infty} \frac{\lambda_n - \lambda}{n^2}. \quad (1.1.26)$$

*Proof.* It follows from (1.1.16) that  $\Delta(\lambda)$  is entire in  $\lambda$  of order  $1/2$ , and consequently by Hadamard's factorization theorem [con1, p.289],  $\Delta(\lambda)$  is uniquely determined up to a multiplicative constant by its zeros:

$$\Delta(\lambda) = C \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right) \quad (1.1.27)$$

(the case when  $\Delta(0) = 0$  requires minor modifications). Consider the function

$$\tilde{\Delta}(\lambda) := -\rho \sin \rho \pi = -\lambda \pi \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{n^2}\right).$$

Then

$$\frac{\Delta(\lambda)}{\tilde{\Delta}(\lambda)} = C \frac{\lambda - \lambda_0}{\lambda_0 \pi \lambda} \prod_{n=1}^{\infty} \frac{n^2}{\lambda_n} \prod_{n=1}^{\infty} \left(1 + \frac{\lambda_n - n^2}{n^2 - \lambda}\right).$$

Taking (1.1.13) and (1.1.16) into account we calculate

$$\lim_{\lambda \rightarrow -\infty} \frac{\Delta(\lambda)}{\tilde{\Delta}(\lambda)} = 1, \quad \lim_{\lambda \rightarrow -\infty} \prod_{n=1}^{\infty} \left(1 + \frac{\lambda_n - n^2}{n^2 - \lambda}\right) = 1,$$

and hence

$$C = \pi \lambda_0 \prod_{n=1}^{\infty} \frac{\lambda_n}{n^2}.$$

Substituting this into (1.1.27) we arrive at (1.1.26).  $\square$

**Remark 1.1.2.** Analogous results are valid for Sturm-Liouville operators with other types of separated boundary conditions. Let us briefly formulate these results which will be used below. The proof is recommended as an exercise.

(i) Consider the boundary value problem  $L_1 = L_1(q(x), h)$  for equation (1.1.1) with the boundary conditions  $U(y) = 0$ ,  $y(\pi) = 0$ . The eigenvalues  $\{\mu_n\}_{n \geq 0}$  of  $L_1$  are simple and coincide with zeros of the characteristic function  $d(\lambda) := \varphi(\pi, \lambda)$ , and

$$d(\lambda) = \prod_{n=0}^{\infty} \frac{\mu_n - \lambda}{(n + 1/2)^2}. \quad (1.1.28)$$

For the spectral data  $\{\mu_n, \alpha_{n1}\}_{n \geq 0}$ ,  $\alpha_{n1} := \int_0^\pi \varphi^2(x, \mu_n) dx$  of  $L_1$  we have the asymptotic formulae:

$$\sqrt{\mu_n} = n + \frac{1}{2} + \frac{\omega_1}{\pi n} + \frac{\kappa_n}{n}, \quad \{\kappa_n\} \in l_2, \quad (1.1.29)$$

$$\alpha_{n1} = \frac{\pi}{2} + \frac{\kappa_{n1}}{n}, \quad \{\kappa_{n1}\} \in l_2, \quad (1.1.30)$$

where  $\omega_1 = h + \frac{1}{2} \int_0^\pi q(t) dt$ .

(ii) Consider the boundary value problem  $L^0 = L^0(q(x), H)$  for equation (1.1.1) with the boundary conditions  $y(0) = V(y) = 0$ . The eigenvalues  $\{\lambda_n^0\}_{n \geq 0}$  of  $L^0$  are simple and coincide with zeros of the characteristic function  $\Delta^0(\lambda) := \psi(0, \lambda) = S'(\pi, \lambda) + HS(\pi, \lambda)$ . The function  $S(x, \lambda)$  satisfies the Volterra integral equation

$$S(x, \lambda) = \frac{\sin \rho x}{\rho} + \int_0^x \frac{\sin \rho(x-t)}{\rho} q(t) S(t, \lambda) dt, \quad (1.1.31)$$

and for  $|\rho| \rightarrow \infty$ ,

$$\left. \begin{aligned} S(x, \lambda) &= \frac{\sin \rho x}{\rho} + O\left(\frac{1}{|\rho|^2} \exp(|\tau|x)\right) = O\left(\frac{1}{|\rho|} \exp(|\tau|x)\right), \\ S'(x, \lambda) &= \cos \rho x + \left(\frac{1}{|\rho|} \exp(|\tau|x)\right) = O(\exp(|\tau|x)), \\ \Delta^0(\lambda) &= \cos \rho \pi + \left(\frac{1}{|\rho|} \exp(|\tau|\pi)\right), \quad \tau = \operatorname{Im} \rho. \end{aligned} \right\} \quad (1.1.32)$$

Moreover,

$$\Delta^0(\lambda) = \prod_{n=0}^{\infty} \frac{\lambda_n^0 - \lambda}{(n + 1/2)^2}.$$

$$\sqrt{\lambda_n^0} = n + \frac{1}{2} + \frac{\omega^0}{\pi n} + \frac{\kappa_n}{n}, \quad \{\kappa_n\} \in l_2,$$

where  $\omega^0 = H + \frac{1}{2} \int_0^\pi q(t) dt$ .

(iii) Consider the boundary value problem  $L_1^0 = L_1^0(q(x))$  for equation (1.1.1) with the boundary conditions  $y(0) = y(\pi) = 0$ . The eigenvalues  $\{\mu_n^0\}_{n \geq 1}$  of  $L_1^0$  are simple and coincide with zeros of the characteristic function  $d^0(\lambda) := S(\pi, \lambda)$ , and

$$d^0(\lambda) = \pi \prod_{n=1}^{\infty} \frac{\mu_n^0 - \lambda}{n^2}.$$

$$\sqrt{\mu_n^0} = n + \frac{\omega_1^0}{\pi n} + \frac{\kappa_n}{n}, \quad \{\kappa_n\} \in l_2,$$

where  $\omega_1^0 = \frac{1}{2} \int_0^\pi q(t) dt$ .

**Lemma 1.1.3.** *The following relation holds*

$$\lambda_n < \mu_n < \lambda_{n+1}, \quad n \geq 0. \quad (1.1.33)$$

*i.e. the eigenvalues of two boundary value problems  $L$  and  $L_1$  are alternating.*

*Proof.* As in the proof of Lemma 1.1.1 we get

$$\frac{d}{dx} \langle \varphi(x, \lambda), \varphi(x, \mu) \rangle = (\lambda - \mu) \varphi(x, \lambda) \varphi(x, \mu), \quad (1.1.34)$$

and consequently,

$$\begin{aligned} (\lambda - \mu) \int_0^\pi \varphi(x, \lambda) \varphi(x, \mu) dx &= \langle \varphi(x, \lambda), \varphi(x, \mu) \rangle \Big|_0^\pi \\ &= \varphi(\pi, \lambda) \varphi'(\pi, \mu) - \varphi'(\pi, \lambda) \varphi(\pi, \mu) = d(\lambda) \Delta(\mu) - d(\mu) \Delta(\lambda). \end{aligned}$$

For  $\mu \rightarrow \lambda$  we obtain

$$\int_0^\pi \varphi^2(x, \lambda) dx = \dot{d}(\lambda) \Delta(\lambda) - d(\lambda) \dot{\Delta}(\lambda) \quad \text{with} \quad \dot{\Delta}(\lambda) = \frac{d}{d\lambda} \Delta(\lambda), \quad \dot{d}(\lambda) = \frac{d}{d\lambda} d(\lambda).$$

In particular, this yields

$$\begin{aligned} \alpha_n &= -\dot{\Delta}(\lambda_n) d(\lambda_n), \quad (1.1.35) \\ \frac{1}{d^2(\lambda)} \int_0^\pi \varphi^2(x, \lambda) dx &= -\frac{d}{d\lambda} \left( \frac{\Delta(\lambda)}{d(\lambda)} \right), \quad -\infty < \lambda < \infty, \quad d(\lambda) \neq 0. \end{aligned}$$

Thus, the function  $\frac{\Delta(\lambda)}{d(\lambda)}$  is monotonically decreasing on  $\mathbb{R} \setminus \{\mu_n \mid n \geq 0\}$  with

$$\lim_{\lambda \rightarrow \mu_n \pm 0} \frac{\Delta(\lambda)}{d(\lambda)} = \pm \infty.$$

Consequently in view of (1.1.13) and (1.1.29), we arrive at (1.1.33).  $\square$

## 1.2. PROPERTIES OF EIGENFUNCTIONS

**1.2.1. Completeness and expansion theorems.** In this subsection we prove that the system of the eigenfunctions of the Sturm-Liouville boundary value problem  $L$  is complete and forms an orthogonal basis in  $L_2(0, \pi)$ . This theorem was first proved by Steklov at the end of XIX-th century. We also provide sufficient conditions under which the Fourier series for the eigenfunctions converges uniformly on  $[0, \pi]$ . The completeness and expansion theorems are important for solving various problems in mathematical physics by the Fourier method, and also for the spectral theory itself. In order to prove these theorems we apply the contour integral method of integrating the resolvent along expanding contours in the complex plane of the spectral parameter (since this method is based on Cauchy's theorem, it sometimes called Cauchy's method).

**Theorem 1.2.1.** (i) The system of eigenfunctions  $\{\varphi(x, \lambda_n)\}_{n \geq 0}$  of the boundary value problem  $L$  is complete in  $L_2(0, \pi)$ .

(ii) Let  $f(x)$ ,  $x \in [0, \pi]$  be an absolutely continuous function. Then

$$f(x) = \sum_{n=0}^{\infty} a_n \varphi(x, \lambda_n), \quad a_n = \frac{1}{\alpha_n} \int_0^{\pi} f(t) \varphi(t, \lambda_n) dt, \quad (1.2.1)$$

and the series converges uniformly on  $[0, \pi]$ .

(iii) For  $f(x) \in L_2(0, \pi)$ , the series (1.2.1) converges in  $L_2(0, \pi)$ , and

$$\int_0^{\pi} |f(x)|^2 dx = \sum_{n=0}^{\infty} \alpha_n |a_n|^2 \quad (\text{Parseval's equality}). \quad (1.2.2)$$

There are several methods to prove Theorem 1.2.1. We shall use the contour integral method which plays an important role in studying direct and inverse problems for various differential, integro-differential and integral operators.

*Proof.* 1) Denote

$$G(x, t, \lambda) = -\frac{1}{\Delta(\lambda)} \begin{cases} \varphi(x, \lambda) \psi(t, \lambda), & x \leq t, \\ \varphi(t, \lambda) \psi(x, \lambda), & x \geq t, \end{cases}$$

and consider the function

$$\begin{aligned} Y(x, \lambda) &= \int_0^{\pi} G(x, t, \lambda) f(t) dt \\ &= -\frac{1}{\Delta(\lambda)} \left( \psi(x, \lambda) \int_0^x \varphi(t, \lambda) f(t) dt + \varphi(x, \lambda) \int_x^{\pi} \psi(t, \lambda) f(t) dt \right). \end{aligned}$$

The function  $G(x, t, \lambda)$  is called *Green's function* for  $L$ .  $G(x, t, \lambda)$  is the kernel of the inverse operator for the Sturm-Liouville operator, i.e.  $Y(x, \lambda)$  is the solution of the boundary value problem

$$\ell Y - \lambda Y + f(x) = 0, \quad U(Y) = V(Y) = 0; \quad (1.2.3)$$

this is easily verified by differentiation. Taking (1.1.6) into account and using Theorem 1.1.2 we calculate

$$\begin{aligned} \text{Res}_{\lambda=\lambda_n} Y(x, \lambda) &= -\frac{1}{\dot{\Delta}(\lambda_n)} \left( \psi(x, \lambda_n) \int_0^x \varphi(t, \lambda_n) f(t) dt + \varphi(x, \lambda_n) \int_x^{\pi} \psi(t, \lambda_n) f(t) dt \right) \\ &= -\frac{\beta_n}{\dot{\Delta}(\lambda_n)} \varphi(x, \lambda_n) \int_0^{\pi} f(t) \varphi(t, \lambda_n) dt, \end{aligned}$$

and by virtue of (1.1.8),

$$\text{Res}_{\lambda=\lambda_n} Y(x, \lambda) = \frac{1}{\alpha_n} \varphi(x, \lambda_n) \int_0^{\pi} f(t) \varphi(t, \lambda_n) dt. \quad (1.2.4)$$

2) Let  $f(x) \in L_2(0, \pi)$  be such that

$$\int_0^{\pi} f(t) \varphi(t, \lambda_n) dt = 0, \quad n \geq 0.$$



Then in view of (1.2.4),  $\operatorname{Res}_{\lambda=\lambda_n} Y(x, \lambda) = 0$ , and consequently (after extending  $Y(x, \lambda)$  continuously to the whole  $\lambda$ -plane) for each fixed  $x \in [0, \pi]$ , the function  $Y(x, \lambda)$  is entire in  $\lambda$ . Furthermore, it follows from (1.1.9), (1.1.10) and (1.1.18) that for a fixed  $\delta > 0$  and sufficiently large  $\rho^* > 0$ :

$$|Y(x, \lambda)| \leq \frac{C_\delta}{|\rho|}, \quad \rho \in G_\delta, \quad |\rho| \geq \rho^*.$$

Using the maximum principle [con1, p.128] and Liouville's theorem [con1, p.77] we conclude that  $Y(x, \lambda) \equiv 0$ . From this and (1.2.3) it follows that  $f(x) = 0$  a.e. on  $(0, \pi)$ . Thus (i) is proved.

3) Let now  $f \in AC[0, \pi]$  be an arbitrary absolutely continuous function. Since  $\varphi(x, \lambda)$  and  $\psi(x, \lambda)$  are solutions of (1.1.1), we transform  $Y(x, \lambda)$  as follows

$$\begin{aligned} Y(x, \lambda) = & -\frac{1}{\lambda \Delta(\lambda)} \left( \psi(x, \lambda) \int_0^x (-\varphi''(t, \lambda) + q(t)\varphi(t, \lambda)) f(t) dt \right. \\ & \left. + \varphi(x, \lambda) \int_x^\pi (-\psi''(t, \lambda) + q(t)\psi(t, \lambda)) f(t) dt \right). \end{aligned}$$

Integration of the terms containing second derivatives by parts yields in view of (1.1.4),

$$Y(x, \lambda) = \frac{f(x)}{\lambda} - \frac{1}{\lambda} (Z_1(x, \lambda) + Z_2(x, \lambda)), \quad (1.2.5)$$

where

$$\begin{aligned} Z_1(x, \lambda) = & \frac{1}{\Delta(\lambda)} \left( \psi(x, \lambda) \int_0^x g(t) \varphi'(t, \lambda) dt + \varphi(x, \lambda) \int_x^\pi g(t) \psi'(t, \lambda) dt \right), \quad g(t) := f'(t), \\ Z_2(x, \lambda) = & \frac{1}{\Delta(\lambda)} \left( h f(0) \psi(x, \lambda) + H f(\pi) \varphi(x, \lambda) \right. \\ & \left. + \psi(x, \lambda) \int_0^x q(t) \varphi(t, \lambda) f(t) dt + \varphi(x, \lambda) \int_x^\pi q(t) \psi(t, \lambda) f(t) dt \right). \end{aligned}$$

Using (1.1.9), (1.1.10) and (1.1.18), we get for a fixed  $\delta > 0$ , and sufficiently large  $\rho^* > 0$ :

$$\max_{0 \leq x \leq \pi} |Z_2(x, \lambda)| \leq \frac{C}{|\rho|}, \quad \rho \in G_\delta, \quad |\rho| \geq \rho^*. \quad (1.2.6)$$

Let us show that

$$\lim_{\substack{|\rho| \rightarrow \infty \\ \rho \in G_\delta}} \max_{0 \leq x \leq \pi} |Z_1(x, \lambda)| = 0. \quad (1.2.7)$$

First we assume that  $g(x)$  is absolutely continuous on  $[0, \pi]$ . In this case another integration by parts yields

$$\begin{aligned} Z_1(x, \lambda) = & \frac{1}{\Delta(\lambda)} \left( \psi(x, \lambda) g(t) \varphi(t, \lambda) \Big|_0^x + \varphi(x, \lambda) g(t) \psi(t, \lambda) \Big|_x^\pi \right. \\ & \left. - \psi(x, \lambda) \int_0^x g'(t) \varphi(t, \lambda) dt - \varphi(x, \lambda) \int_x^\pi g'(t) \psi(t, \lambda) dt \right). \end{aligned}$$

By virtue of (1.1.9), (1.1.10) and (1.1.18), we infer

$$\max_{0 \leq x \leq \pi} |Z_1(x, \lambda)| \leq \frac{C}{|\rho|}, \quad \rho \in G_\delta, \quad |\rho| \geq \rho^*.$$

Let now  $g(t) \in L(0, \pi)$ . Fix  $\varepsilon > 0$  and choose an absolutely continuous function  $g_\varepsilon(t)$  such that

$$\int_0^\pi |g(t) - g_\varepsilon(t)| dt < \frac{\varepsilon}{2C^+},$$

where

$$C^+ = \max_{0 \leq x \leq \pi} \sup_{\rho \in G_\delta} \frac{1}{|\Delta(\lambda)|} \left( |\psi(x, \lambda)| \int_0^x |\varphi'(t, \lambda)| dt + |\varphi(x, \lambda)| \int_x^\pi |\psi'(t, \lambda)| dt \right).$$

Then, for  $\rho \in G_\delta$ ,  $|\rho| \geq \rho^*$ , we have

$$\max_{0 \leq x \leq \pi} |Z_1(x, \lambda)| \leq \max_{0 \leq x \leq \pi} |Z_1(x, \lambda; g_\varepsilon)| + \max_{0 \leq x \leq \pi} |Z_1(x, \lambda; g - g_\varepsilon)| \leq \frac{\varepsilon}{2} + \frac{C(\varepsilon)}{|\rho|}.$$

Hence, there exists  $\rho^0 > 0$  such that  $\max_{0 \leq x \leq \pi} |Z_1(x, \lambda)| \leq \varepsilon$  for  $|\rho| > \rho^0$ . By virtue of the arbitrariness of  $\varepsilon > 0$ , we arrive at (1.2.7).

Consider the contour integral

$$I_N(x) = \frac{1}{2\pi i} \int_{\Gamma_N} Y(x, \lambda) d\lambda,$$

where  $\Gamma_N = \{\lambda : |\lambda| = (N + 1/2)^2\}$  (with counterclockwise circuit). It follows from (1.2.5)-(1.2.7) that

$$I_N(x) = f(x) + \varepsilon_N(x), \quad \lim_{N \rightarrow \infty} \max_{0 \leq x \leq \pi} |\varepsilon_N(x)| = 0. \quad (1.2.8)$$

On the other hand, we can calculate  $I_N(x)$  with the help of the residue theorem [con1, p.112]. By virtue of (1.2.4),

$$I_N(x) = \sum_{n=0}^N a_n \varphi(x, \lambda_n), \quad a_n = \frac{1}{\alpha_n} \int_0^\pi f(t) \varphi(t, \lambda_n) dt.$$

Comparing this with (1.2.8) we arrive at (1.2.1), where the series converges uniformly on  $[0, \pi]$ , i.e. (ii) is proved.

4) Since the eigenfunctions  $\{\varphi(x, \lambda_n)\}_{n \geq 0}$  are complete and orthogonal in  $L_2(0, \pi)$ , they form an orthogonal basis in  $L_2(0, \pi)$ , and Parseval's equality (1.2.2) is valid.  $\square$

**1.2.2. Oscillations of the eigenfunctions.** Consider the boundary value problem (1.1.1)-(1.1.2) with  $q(x) \equiv 0$ ,  $h = H = 0$ . In this case, according to Example 1.1.1, the eigenfunctions are  $\cos nx$ , and we have that the  $n$ -th eigenfunction has exactly  $n$  zeros inside the interval  $(0, \pi)$ . It turns out that this property of the eigenfunctions holds also in the general case. In other words, the following oscillation theorem, which is due to Sturm, is valid.

**Theorem 1.2.2.** *The eigenfunction  $\varphi(x, \lambda_n)$  of the boundary value problem  $L$  has exactly  $n$  zeros in the interval  $0 < x < \pi$ .*

First we prove several auxiliary assertions.

**Lemma 1.2.1.** *Let  $u_j(x)$ ,  $j = 1, 2$ ,  $x \in [a, b]$  be solutions of the equations*

$$u_j'' + g_j(x)u_j = 0, \quad g_1(x) < g_2(x), \quad j = 1, 2, \quad x \in [a, b]. \quad (1.2.9)$$

*Suppose that for certain  $x_1, x_2 \in [a, b]$ ,  $u_1(x_1) = u_1(x_2) = 0$ , and  $u_1(x) \neq 0$ ,  $x \in (x_1, x_2)$ . Then there exists  $x^* \in (x_1, x_2)$  such that  $u_2(x^*) = 0$ . In other words, the function  $u_2(x)$  has at least one zero lying between any two zeros of  $u_1(x)$ .*

*Proof.* Suppose, on the contrary, that  $u_2(x) \neq 0$  for  $x \in (x_1, x_2)$ . Without loss of generality we assume that  $u_j(x) > 0$  for  $x \in (x_1, x_2)$ ,  $j = 1, 2$ . By virtue of (1.2.9),

$$\frac{d}{dx}(u_1' u_2 - u_2' u_1) = (g_2 - g_1)u_1 u_2,$$

and consequently

$$\int_{x_1}^{x_2} (g_2 - g_1)u_1 u_2 dx = (u_1' u_2 - u_2' u_1) \Big|_{x_1}^{x_2} = u_1'(x_2)u_2(x_2) - u_1'(x_1)u_2(x_1). \quad (1.2.10)$$

The integral in (1.2.10) is strictly positive. On the other hand, since  $u_1'(x_1) > 0$ ,  $u_1'(x_2) < 0$ , and  $u_2(x) \geq 0$  for  $x \in [x_1, x_2]$ , the right-hand side of (1.2.10) is non-positive. This contradiction proves the lemma.  $\square$

**Corollary 1.2.1.** *Let  $g_1(x) < -\gamma^2 < 0$ . Then each non-trivial solution of the equation  $u_1'' + g_1(x)u_1 = 0$  cannot have more than one zero.*

Indeed, this follows from Lemma 1.2.1 with  $g_2(x) = -\gamma^2$  since the equation  $u_2'' - \gamma^2 u_2 = 0$  has the solution  $u_2(x) = \exp(\gamma x)$ , which has no zeros.

We consider the function  $\varphi(x, \lambda)$  for real  $\lambda$ . Zeros of  $\varphi(x, \lambda)$  with respect to  $x$  are functions of  $\lambda$ . We show that these zeros are continuous functions of  $\lambda$ .

**Lemma 1.2.2.** *Let  $\varphi(x_0, \lambda_0) = 0$ . For each  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $|\lambda - \lambda_0| < \delta$ , then the function  $\varphi(x, \lambda)$  has exactly one zero in the interval  $|x - x_0| < \varepsilon$ .*

*Proof.* A zero  $x_0$  of a solution  $\varphi(x, \lambda_0)$  of differential equation (1.1.1) is a simple zero, since if we had  $\varphi'(x_0, \lambda_0) = 0$ , it would follow from the uniqueness theorem for equation (1.1.1) (see [cod1, Ch.1]) that  $\varphi(x, \lambda_0) \equiv 0$ . Hence,  $\varphi'(x_0, \lambda_0) \neq 0$ , and for definiteness let us assume that  $\varphi'(x_0, \lambda_0) > 0$ . Choose  $\varepsilon_0 > 0$  such that  $\varphi'(x, \lambda_0) > 0$  for  $|x - x_0| \leq \varepsilon_0$ . Then  $\varphi(x, \lambda_0) < 0$  for  $x \in [x_0 - \varepsilon_0, x_0)$ , and  $\varphi(x, \lambda_0) > 0$  for  $x \in (x_0, x_0 + \varepsilon_0]$ . Take  $\varepsilon \leq \varepsilon_0$ . By continuity of  $\varphi(x, \lambda)$  and  $\varphi'(x, \lambda)$ , there exists  $\delta > 0$  such that for  $|\lambda - \lambda_0| < \delta$ ,  $|x - x_0| < \varepsilon$  we have  $\varphi'(x, \lambda) > 0$ ,  $\varphi(x_0 - \varepsilon, \lambda) < 0$ ,  $\varphi(x_0 + \varepsilon, \lambda) > 0$ . Consequently, the function  $\varphi(x, \lambda)$  has exactly one zero in the interval  $|x - x_0| < \varepsilon$ .  $\square$

**Lemma 1.2.3.** *Suppose that for a fixed real  $\mu$ , the function  $\varphi(x, \mu)$  has  $m$  zeros in the interval  $0 < x \leq a$ . Let  $\lambda > \mu$ . Then the function  $\varphi(x, \lambda)$  has not less than  $m$  zeros in the same interval, and the  $k$ -th zero of  $\varphi(x, \lambda)$  is less than the  $k$ -th zero of  $\varphi(x, \mu)$ .*

*Proof.* Let  $x_1 > 0$  be the smallest positive zero of  $\varphi(x, \mu)$ . By virtue of Lemma 1.2.1, it is sufficient to prove that  $\varphi(x, \lambda)$  has at least one zero in the interval  $0 < x < x_1$ .

Suppose, on the contrary, that  $\varphi(x, \lambda) \neq 0$ ,  $x \in [0, x_1]$ . Since  $\varphi(0, \lambda) = 1$ , we have  $\varphi(x, \lambda) > 0$ ,  $\varphi(x, \mu) > 0$ ,  $x \in [0, x_1]$ ;  $\varphi(x_1, \mu) = 0$ ,  $\varphi'(x_1, \mu) < 0$ . It follows from (1.1.34) that

$$(\lambda - \mu) \int_0^{x_1} \varphi(x, \lambda) \varphi(x, \mu) dx = \langle \varphi(x, \lambda), \varphi(x, \mu) \rangle \Big|_0^{x_1} = \varphi(x_1, \lambda) \varphi'(x_1, \mu) \leq 0.$$

But the integral in the left-hand side is strictly positive. This contradiction proves the lemma.  $\square$

*Proof of Theorem 1.2.2.* Let us consider the function  $\varphi(x, \lambda)$  for real  $\lambda$ . By virtue of (1.1.9), the function  $\varphi(x, \lambda)$  has no zeros for sufficiently large negative  $\lambda$ :  $\varphi(x, \lambda) > 0$ ,  $\lambda \leq -\lambda^* < 0$ ,  $x \in [0, \pi]$ . On the other hand,  $\varphi(\pi, \mu_n) = 0$ , where  $\{\mu_n\}_{n \geq 0}$  are eigenvalues of the boundary value problem  $L_1$ .

Using Lemmas 1.2.2-1.2.3 we get that if  $\lambda$  moves from  $-\infty$  to  $\infty$ , then the zeros of  $\varphi(x, \lambda)$  on the interval  $[0, \pi]$  continuously move to the left. New zeros can appear only through the point  $x = \pi$ . This yields:

- (i) The functions  $\varphi(x, \mu_n)$  has exactly  $n$  zeros in the interval  $x \in [0, \pi]$ .
- (ii) If  $\lambda \in (\mu_{n-1}, \mu_n)$ ,  $n \geq 1$ ,  $\mu_1 := -\infty$ , then the function  $\varphi(x, \lambda)$  has exactly  $n$  zeros in the interval  $x \in [0, \pi]$ .

According to (1.1.33),

$$\lambda_0 < \mu_0 < \lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \dots$$

Consequently, the function  $\varphi(x, \lambda_n)$  has exactly  $n$  zeros in the interval  $[0, \pi]$ .  $\square$

### 1.3. TRANSFORMATION OPERATORS

An important role in the inverse problem theory for Sturm-Liouville operators is played by the so-called transformation operator. It connects solutions of two different Sturm-Liouville equations for all  $\lambda$ . In this section we construct the transformation operator and study its properties. The transformation operator first appeared in the theory of generalized translation operators of Delsarte and Levitan (see [lev1]). Transformation operators for arbitrary Sturm-Liouville equations were constructed by Povzner [pov1]. In the inverse problem theory the transformation operators were used by Gelfand, Levitan and Marchenko (see Section 1.5 and the monographs [mar1], [lev2]).

**Theorem 1.3.1.** *For the function  $C(x, \lambda)$  the following representation holds*

$$C(x, \lambda) = \cos \rho x + \int_0^x K(x, t) \cos \rho t dt, \quad \lambda = \rho^2, \quad (1.3.1)$$

where  $K(x, t)$  is a real continuous function, and

$$K(x, x) = \frac{1}{2} \int_0^x q(t) dt. \quad (1.3.2)$$

*Proof.* It follows from (1.1.11) for  $h = 0$ , that the function  $C(x, \lambda)$  is the solution of the following integral equation

$$C(x, \lambda) = \cos \rho x + \int_0^x \frac{\sin \rho(x-t)}{\rho} q(t) C(t, \lambda) dt. \quad (1.3.3)$$

Since

$$\frac{\sin \rho(x-t)}{\rho} = \int_t^x \cos \rho(s-t) ds,$$

(1.3.3) becomes

$$C(x, \lambda) = \cos \rho x + \int_0^x q(t) C(t, \lambda) \left( \int_t^x \cos \rho(s-t) ds \right) dt,$$

and hence

$$C(x, \lambda) = \cos \rho x + \int_0^x \left( \int_0^t q(\tau) C(\tau, \lambda) \cos \rho(t-\tau) d\tau \right) dt.$$

The method of successive approximations gives

$$C(x, \lambda) = \sum_{n=0}^{\infty} C_n(x, \lambda), \quad (1.3.4)$$

$$C_0(x, \lambda) = \cos \rho x, \quad C_{n+1}(x, \lambda) = \int_0^x \left( \int_0^t q(\tau) C_n(\tau, \lambda) \cos \rho(t-\tau) d\tau \right) dt. \quad (1.3.5)$$

Let us show by induction that the following representation holds

$$C_n(x, \lambda) = \int_0^x K_n(x, t) \cos \rho t dt, \quad (1.3.6)$$

where  $K_n(x, t)$  does not depend on  $\lambda$ .

First we calculate  $C_1(x, \lambda)$ , using  $\cos \alpha \cos \beta = \frac{1}{2} (\cos(\alpha + \beta) + \cos(\alpha - \beta))$  :

$$\begin{aligned} C_1(x, \lambda) &= \int_0^x \left( \int_0^t q(\tau) \cos \rho \tau \cos \rho(t-\tau) d\tau \right) dt \\ &= \frac{1}{2} \int_0^x \cos \rho t \left( \int_0^t q(\tau) d\tau \right) dt + \frac{1}{2} \int_0^x \left( \int_0^t q(\tau) \cos \rho(t-2\tau) d\tau \right) dt. \end{aligned}$$

The change of variables  $t - 2\tau = s$  in the second integral gives

$$C_1(x, \lambda) = \frac{1}{2} \int_0^x \cos \rho t \left( \int_0^t q(\tau) d\tau \right) dt + \frac{1}{4} \int_0^x \left( \int_{-t}^t q\left(\frac{t-s}{2}\right) \cos \rho s ds \right) dt.$$

Interchanging the order of integration in the second integral we obtain

$$\begin{aligned} C_1(x, \lambda) &= \frac{1}{2} \int_0^x \cos \rho t \left( \int_0^t q(\tau) d\tau \right) dt + \frac{1}{4} \int_0^x \cos \rho s \left( \int_s^x q\left(\frac{t-s}{2}\right) dt \right) ds \\ &\quad + \frac{1}{4} \int_{-x}^0 \cos \rho s \left( \int_{-s}^x q\left(\frac{t-s}{2}\right) dt \right) ds = \frac{1}{2} \int_0^x \cos \rho t \left( \int_0^t q(\tau) d\tau \right) dt \\ &\quad + \frac{1}{4} \int_0^x \cos \rho s \left( \int_s^x \left( q\left(\frac{t-s}{2}\right) + q\left(\frac{t+s}{2}\right) \right) dt \right) ds. \end{aligned}$$

Thus, (1.3.6) is valid for  $n = 1$ , where

$$\begin{aligned} K_1(x, t) &= \frac{1}{2} \int_0^t q(\tau) d\tau + \frac{1}{4} \int_t^x \left( q\left(\frac{s-t}{2}\right) + q\left(\frac{s+t}{2}\right) \right) ds \\ &= \frac{1}{2} \int_0^{\frac{x+t}{2}} q(\xi) d\xi + \frac{1}{2} \int_0^{\frac{x-t}{2}} q(\xi) d\xi, \quad t \leq x. \end{aligned} \quad (1.3.7)$$

Suppose now that (1.3.6) is valid for a certain  $n \geq 1$ . Then substituting (1.3.6) into (1.3.5) we have

$$\begin{aligned} C_{n+1}(x, \lambda) &= \int_0^x \int_0^t q(\tau) \cos \rho(t - \tau) \int_0^\tau K_n(\tau, s) \cos \rho s \, ds \, d\tau \, dt \\ &= \frac{1}{2} \int_0^x \int_0^t q(\tau) \int_0^\tau K_n(\tau, s) (\cos \rho(s + t - \tau) + \cos \rho(s - t + \tau)) \, ds \, d\tau \, dt. \end{aligned}$$

The changes of variables  $s + t - \tau = \xi$  and  $s - t + \tau = \xi$ , respectively, lead to

$$\begin{aligned} C_{n+1}(x, \lambda) &= \frac{1}{2} \int_0^x \int_0^t q(\tau) \int_{t-\tau}^t K_n(\tau, \xi + \tau - t) \cos \rho \xi \, d\xi \, d\tau \, dt \\ &\quad + \frac{1}{2} \int_0^x \int_0^t q(\tau) \int_{\tau-t}^{2\tau-t} K_n(\tau, \xi + t - \tau) \cos \rho \xi \, d\xi \, d\tau \, dt. \end{aligned}$$

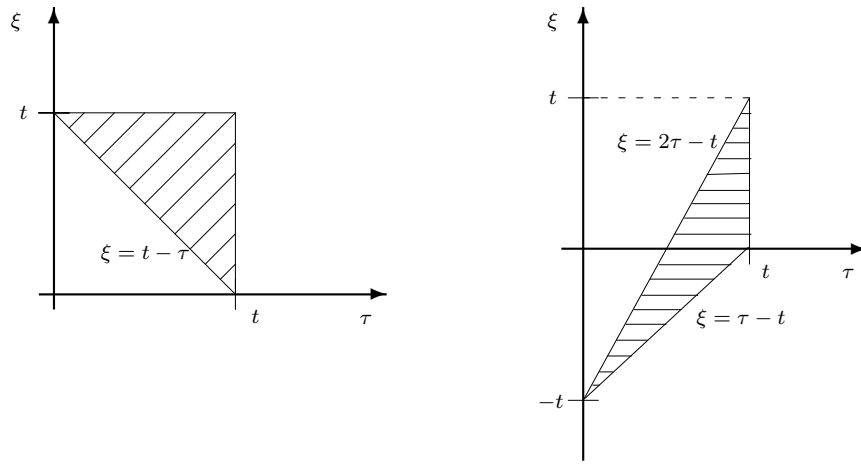


fig. 1.3.1

Interchanging the order of integration (see fig. 1.3.1) we obtain

$$C_{n+1}(x, \lambda) = \int_0^x K_{n+1}(x, t) \cos \rho t \, dt,$$

where

$$\begin{aligned} K_{n+1}(x, t) &= \frac{1}{2} \int_t^x \left( \int_{\xi-t}^\xi q(\tau) K_n(\tau, t + \tau - \xi) \, d\tau \right. \\ &\quad \left. + \int_{\frac{\xi+t}{2}}^\xi q(\tau) K_n(\tau, t - \tau + \xi) \, d\tau + \int_{\frac{\xi-t}{2}}^{\xi-t} q(\tau) K_n(\tau, -t - \tau + \xi) \, d\tau \right) d\xi. \end{aligned} \quad (1.3.8)$$

Substituting (1.3.6) into (1.3.4) we arrive at (1.3.1), where

$$K(x, t) = \sum_{n=1}^{\infty} K_n(x, t). \quad (1.3.9)$$

It follows from (1.3.7) and (1.3.8) that

$$|K_n(x, t)| \leq (Q(x))^n \frac{x^{n-1}}{(n-1)!}, \quad Q(x) := \int_0^x |q(\xi)| \, d\xi.$$

Indeed, (1.3.7) yields for  $t \leq x$ ,

$$|K_1(x, t)| \leq \frac{1}{2} \int_0^{\frac{x+t}{2}} |q(\xi)| d\xi + \frac{1}{2} \int_0^{\frac{x-t}{2}} |q(\xi)| d\xi \leq \int_0^x |q(\xi)| d\xi = Q(x).$$

Furthermore, if for a certain  $n \geq 1$ , the estimate for  $|K_n(x, t)|$  is valid, then by virtue of (1.3.8),

$$\begin{aligned} |K_{n+1}(x, t)| &\leq \frac{1}{2} \int_t^x \left( \int_{\frac{\xi+t}{2}}^{\xi} |q(\tau)| (Q(\tau))^n \frac{\tau^{n-1}}{(n-1)!} d\tau + \int_{\frac{\xi-t}{2}}^{\xi} |q(\tau)| (Q(\tau))^n \frac{\tau^{n-1}}{(n-1)!} d\tau \right) d\xi \\ &\leq \int_0^x \int_0^{\xi} |q(\tau)| (Q(\tau))^n \frac{\tau^{n-1}}{(n-1)!} d\tau d\xi \leq \int_0^x (Q(\xi))^{n+1} \frac{\xi^{n-1}}{(n-1)!} d\xi \leq (Q(x))^{n+1} \frac{x^n}{n!}. \end{aligned}$$

Thus, the series (1.3.9) converges absolutely and uniformly for  $0 \leq t \leq x \leq \pi$ , and the function  $K(x, t)$  is continuous. Moreover, it follows from (1.3.7)-(1.3.9) that the smoothness of  $K(x, t)$  is the same as the smoothness of the function  $\int_0^x q(t) dt$ . Since according to (1.3.7) and (1.3.8)

$$K_1(x, x) = \frac{1}{2} \int_0^x q(t) dt, \quad K_{n+1}(x, x) = 0, \quad n \geq 1,$$

we arrive at (1.3.2). □

The operator  $T$ , defined by

$$Tf(x) = f(x) + \int_0^x K(x, t)f(t) dt,$$

transforms  $\cos \rho x$ , which is the solution of the equation  $-y'' = \lambda y$  with zero potential, to the function  $C(x, \lambda)$ , which is the solution of equation (1.1.1) satisfying the same initial condition (i.e.  $C(x, \lambda) = T(\cos \rho x)$ ). The operator  $T$  is called *the transformation operator* for  $C(x, \lambda)$ . It is important that the kernel  $K(x, t)$  does not depend on  $\lambda$ .

Analogously one can obtain the transformation operators for the functions  $S(x, \lambda)$  and  $\varphi(x, \lambda)$ :

**Theorem 1.3.2.** *For the functions  $S(x, \lambda)$  and  $\varphi(x, \lambda)$  the following representations hold*

$$S(x, \lambda) = \frac{\sin \rho x}{\rho} + \int_0^x P(x, t) \frac{\sin \rho t}{\rho} dt, \quad (1.3.10)$$

$$\varphi(x, \lambda) = \cos \rho x + \int_0^x G(x, t) \cos \rho t dt, \quad (1.3.11)$$

where  $P(x, t)$  and  $G(x, t)$  are real continuous functions with the same smoothness as  $\int_0^x q(t) dt$ , and

$$G(x, x) = h + \frac{1}{2} \int_0^x q(t) dt. \quad (1.3.12)$$

$$P(x, x) = \frac{1}{2} \int_0^x q(t) dt. \quad (1.3.13)$$

*Proof.* The function  $S(x, \lambda)$  satisfies (1.1.31), and hence

$$S(x, \lambda) = \frac{\sin \rho x}{\rho} + \int_0^x \int_0^t q(\tau) S(\tau, \lambda) \cos \rho(t - \tau) d\tau dt.$$

The method of successive approximations gives

$$S(x, \lambda) = \sum_{n=0}^{\infty} S_n(x, \lambda), \quad (1.3.14)$$

$$S_0(x, \lambda) = \frac{\sin \rho x}{\rho}, \quad S_{n+1}(x, \lambda) = \int_0^x \int_0^t q(\tau) S_n(\tau, \lambda) \cos \rho(t - \tau) d\tau dt.$$

Acting in the same way as in the proof of Theorem 1.3.1, we obtain the following representation

$$S_n(x, \lambda) = \int_0^x P_n(x, t) \frac{\sin \rho t}{\rho} dt,$$

where

$$\begin{aligned} P_1(x, t) &= \frac{1}{2} \int_0^{\frac{x+t}{2}} q(\xi) d\xi - \frac{1}{2} \int_0^{\frac{x-t}{2}} q(\xi) d\xi, \\ P_{n+1}(x, t) &= \frac{1}{2} \int_t^x \left( \int_{\xi-t}^{\xi} q(\tau) P_n(\tau, t + \tau - \xi) d\tau \right. \\ &\quad \left. + \int_{\frac{\xi+t}{2}}^{\xi} q(\tau) P_n(\tau, t - \tau + \xi) d\tau - \int_{\frac{\xi-t}{2}}^{\xi-t} q(\tau) P_n(\tau, -t - \tau + \xi) d\tau \right) d\xi, \end{aligned}$$

and

$$|P_n(x, t)| \leq (Q(x))^n \frac{x^{n-1}}{(n-1)!}, \quad Q(x) := \int_0^x |q(\xi)| d\xi,$$

Hence, the series (1.3.14) converges absolutely and uniformly for  $0 \leq t \leq x \leq \pi$ , and we arrive at (1.3.10) and (1.3.13).

The relation (1.3.11) can be obtained directly from (1.3.1) and (1.3.10):

$$\varphi(x, \lambda) = C(x, \lambda) + hS(x, \lambda) = \cos \rho x + \int_0^x K(x, t) \cos \rho t dt + h \int_0^x \cos \rho t dt +$$

$$h \int_0^x P(x, t) \left( \int_0^t \cos \rho \tau d\tau \right) dt = \cos \rho x + \int_0^x G(x, t) \cos \rho t dt,$$

where

$$G(x, t) = K(x, t) + h + h \int_t^x P(x, \tau) d\tau.$$

Taking here  $t = x$  we get (1.3.12). □

#### 1.4. UNIQUENESS THEOREMS

Let us go on to inverse problems of spectral analysis for Sturm-Liouville operators. In this section we give various formulations of these inverse problems and prove the corresponding uniqueness theorems. We present several methods for proving these theorems. These



methods have a wide area for applications and allow one to study various classes of inverse spectral problems.

**1.4.1. Ambarzumian's theorem.** The first result in the inverse problem theory is due to Ambarzumian [amb1]. Consider the boundary value problem  $L(q(x), 0, 0)$ , i.e.

$$-y'' + q(x)y = \lambda y, \quad y'(0) = y'(\pi) = 0. \quad (1.4.1)$$

Clearly, if  $q(x) = 0$  a.e. on  $(0, \pi)$ , then the eigenvalues of (1.4.1) have the form  $\lambda_n = n^2$ ,  $n \geq 0$ . Ambarzumian proved the inverse assertion:

**Theorem 1.4.1.** *If the eigenvalues of (1.4.1) are  $\lambda_n = n^2$ ,  $n \geq 0$ , then  $q(x) = 0$  a.e. on  $(0, \pi)$ .*

*Proof.* It follows from (1.1.13) that  $\omega = 0$ , i.e.  $\int_0^\pi q(x) dx = 0$ . Let  $y_0(x)$  be an eigenfunction for the first eigenvalue  $\lambda_0 = 0$ . Then

$$y_0''(x) - q(x)y_0(x) = 0, \quad y_0'(0) = y_0'(\pi) = 0.$$

According to Theorem 1.2.2, the function  $y_0(x)$  has no zeros in the interval  $x \in [0, \pi]$ . Taking into account the relation

$$\frac{y_0''(x)}{y_0(x)} = \left(\frac{y_0'(x)}{y_0(x)}\right)^2 + \left(\frac{y_0'(x)}{y_0(x)}\right)',$$

we get

$$0 = \int_0^\pi q(x) dx = \int_0^\pi \frac{y_0''(x)}{y_0(x)} dx = \int_0^\pi \left(\frac{y_0'(x)}{y_0(x)}\right)^2 dx.$$

Thus,  $y_0'(x) \equiv 0$ , i.e.  $y_0(x) \equiv \text{const}$ ,  $q(x) = 0$  a.e. on  $(0, \pi)$ .  $\square$

**Remark 1.4.1.** Actually we have proved a more general assertion than Theorem 1.4.1, namely:

If  $\lambda_0 = \frac{1}{\pi} \int_0^\pi q(x) dx$ , then  $q(x) = \lambda_0$  a.e. on  $(0, \pi)$ .

**1.4.2. Uniqueness of the recovery of differential equations from the spectral data.** Ambarzumian's result is an exception from the rule. In general, the specification of the spectrum does not uniquely determine the operator. In Subsections 1.4.2-1.4.4 we provide three uniqueness theorems where we point out spectral characteristics which uniquely determine the operator.

Consider the following inverse problem:

**Inverse Problem 1.4.1.** Given the spectral data  $\{\lambda_n, \alpha_n\}_{n \geq 0}$ , construct the potential  $q(x)$  and the coefficients  $h$  and  $H$  of the boundary conditions.

The goal of this subsection is to prove the uniqueness theorem for the solution of Inverse Problem 1.4.1.

We agree that together with  $L$  we consider here and in the sequel a boundary value problem  $\tilde{L} = L(\tilde{q}(x), \tilde{h}, \tilde{H})$  of the same form but with different coefficients. If a certain symbol  $\gamma$  denotes an object related to  $L$ , then the corresponding symbol  $\tilde{\gamma}$  with tilde denotes the analogous object related to  $\tilde{L}$ , and  $\hat{\gamma} := \gamma - \tilde{\gamma}$ .

**Theorem 1.4.2.** *If  $\lambda_n = \tilde{\lambda}_n$ ,  $\alpha_n = \tilde{\alpha}_n$ ,  $n \geq 0$ , then  $L = \tilde{L}$ , i.e.  $q(x) = \tilde{q}(x)$  a.e. on  $(0, \pi)$ ,  $h = \tilde{h}$  and  $H = \tilde{H}$ . Thus, the specification of the spectral data  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  uniquely determines the potential and the coefficients of the boundary conditions.*

We give here two proofs of Theorem 1.4.2. The first proof is due to Marchenko [mar4] and uses the transformation operator and Parseval's equality (1.2.2). This method also works for Sturm-Liouville operators on the half-line, and gives us an opportunity to prove the uniqueness theorem of recovering an operator from its spectral function. The second proof is due to Levinson [Lev1] and uses the contour integral method. We note that Levinson first applied the ideas of the contour integral method to inverse spectral problems. In Section 1.6 one can see a development of these ideas for constructing the solution of the inverse problem.

*Marchenko's proof.* According to (1.3.11),

$$\varphi(x, \lambda) = \cos \rho x + \int_0^x G(x, t) \cos \rho t \, dt, \quad \tilde{\varphi}(x, \lambda) = \cos \rho x + \int_0^x \tilde{G}(x, t) \cos \rho t \, dt.$$

In other words,

$$\varphi(x, \lambda) = (E + G) \cos \rho x, \quad \tilde{\varphi}(x, \lambda) = (E + \tilde{G}) \cos \rho x,$$

where

$$(E + G)f(x) = f(x) + \int_0^x G(x, t)f(t) \, dt, \quad (E + \tilde{G})f(x) = f(x) + \int_0^x \tilde{G}(x, t)f(t) \, dt.$$

One can consider the relation  $\tilde{\varphi}(x, \lambda) = (E + \tilde{G}) \cos \rho x$  as a Volterra integral equation (see [jer1] for the theory of integral equations) with respect to  $\cos \rho x$ . Solving this equation we get

$$\cos \rho x = \tilde{\varphi}(x, \lambda) + \int_0^x \tilde{H}(x, t)\tilde{\varphi}(t, \lambda) \, dt,$$

where  $\tilde{H}(x, t)$  is a continuous function which is the kernel of the inverse operator:

$$(E + \tilde{H}) = (E + \tilde{G})^{-1}, \quad \tilde{H}f(x) = \int_0^x \tilde{H}(x, t)f(t) \, dt.$$

Consequently

$$\varphi(x, \lambda) = \tilde{\varphi}(x, \lambda) + \int_0^x Q(x, t)\tilde{\varphi}(t, \lambda) \, dt, \tag{1.4.2}$$

where  $Q(x, t)$  is a real continuous function.

Let  $f(x) \in L_2(0, \pi)$ . It follows from (1.4.2) that

$$\int_0^\pi f(x)\varphi(x, \lambda) \, dx = \int_0^\pi g(x)\tilde{\varphi}(x, \lambda) \, dx,$$

where

$$g(x) = f(x) + \int_x^\pi Q(t, x)f(t) \, dt.$$

Hence, for all  $n \geq 0$ ,

$$a_n = \tilde{b}_n, \\ a_n := \int_0^\pi f(x)\varphi(x, \lambda_n) \, dx, \quad \tilde{b}_n := \int_0^\pi g(x)\tilde{\varphi}(x, \lambda_n) \, dx.$$

Using Parseval's equality (1.2.2), we calculate

$$\int_0^\pi |f(x)|^2 dx = \sum_{n=0}^\infty \frac{|a_n|^2}{\alpha_n} = \sum_{n=0}^\infty \frac{|\tilde{b}_n|^2}{\alpha_n} = \sum_{n=0}^\infty \frac{|\tilde{b}_n|^2}{\tilde{\alpha}_n} = \int_0^\pi |g(x)|^2 dx,$$

i.e.

$$\|f\|_{L_2} = \|g\|_{L_2}. \quad (1.4.3)$$

Consider the operator

$$Af(x) = f(x) + \int_x^\pi Q(t, x)f(t) dt.$$

Then  $Af = g$ . By virtue of (1.4.3),  $\|Af\|_{L_2} = \|f\|_{L_2}$  for any  $f(x) \in L_2(0, \pi)$ . Consequently,  $A^* = A^{-1}$ , but this is possible only if  $Q(x, t) \equiv 0$ . Thus,  $\varphi(x, \lambda) \equiv \tilde{\varphi}(x, \lambda)$ , i.e.  $q(x) = \tilde{q}(x)$  a.e. on  $(0, \pi)$ ,  $h = \tilde{h}$ ,  $H = \tilde{H}$ .

*Levinson's proof.* Let  $f(x)$ ,  $x \in [0, \pi]$  be an absolutely continuous function. Consider the function

$$Y^0(x, \lambda) = -\frac{1}{\Delta(\lambda)} \left( \psi(x, \lambda) \int_0^x f(t) \tilde{\varphi}(t, \lambda) dt + \varphi(x, \lambda) \int_x^\pi f(t) \tilde{\psi}(t, \lambda) dt \right)$$

and the contour integral

$$I_N^0(x) = \frac{1}{2\pi i} \int_{\Gamma_N} Y^0(x, \lambda) d\lambda.$$

The idea used here comes from the proof of Theorem 1.2.1 but here the function  $Y^0(x, \lambda)$  is constructed from solutions of two boundary value problems.

Repeating the arguments of the proof of Theorem 1.2.1 we calculate

$$Y^0(x, \lambda) = \frac{f(x)}{\lambda} - \frac{Z^0(x, \lambda)}{\lambda},$$

where

$$\begin{aligned} Z^0(x, \lambda) = & \frac{1}{\Delta(\lambda)} \left\{ f(x) [\varphi(x, \lambda) (\tilde{\psi}'(x, \lambda) - \psi'(x, \lambda)) - \psi(x, \lambda) (\tilde{\varphi}'(x, \lambda) - \varphi'(x, \lambda))] \right. \\ & + \tilde{h} f(0) \psi(x, \lambda) + \tilde{H} f(\pi) \varphi(x, \lambda) + \psi(x, \lambda) \int_0^x (\tilde{\varphi}'(t, \lambda) f'(t) + \tilde{q}(t) \tilde{\varphi}(t, \lambda) f(t)) dt \\ & \left. + \varphi(x, \lambda) \int_x^\pi (\tilde{\psi}'(t, \lambda) f'(t) + \tilde{q}(t) \tilde{\psi}(t, \lambda) f(t)) dt \right\}. \end{aligned}$$

The asymptotic properties for  $\tilde{\varphi}(x, \lambda)$  and  $\tilde{\psi}(x, \lambda)$  are the same as for  $\varphi(x, \lambda)$  and  $\psi(x, \lambda)$ . Therefore, by similar arguments as in the proof of Theorem 1.2.1 one can obtain

$$I_N^0(x) = f(x) + \varepsilon_N^0(x), \quad \lim_{N \rightarrow \infty} \max_{0 \leq x \leq \pi} |\varepsilon_N^0(x)| = 0.$$

On the other hand, we can calculate  $I_N^0(x)$  with the help of the residue theorem:

$$I_N^0(x) = \sum_{n=0}^N \left( -\frac{1}{\Delta(\lambda_n)} \right) \left( \psi(x, \lambda_n) \int_0^x f(t) \tilde{\varphi}(t, \lambda_n) dt + \varphi(x, \lambda_n) \int_x^\pi f(t) \tilde{\psi}(t, \lambda_n) dt \right).$$

It follows from Lemma 1.1.1 and Theorem 1.1.4 that under the hypothesis of Theorem 1.4.2 we have  $\beta_n = \tilde{\beta}_n$ . Consequently,

$$I_N^0(x) = \sum_{n=0}^N \frac{1}{\alpha_n} \varphi(x, \lambda_n) \int_0^\pi f(t) \tilde{\varphi}(t, \lambda_n) dt.$$

If  $N \rightarrow \infty$  we get

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{\alpha_n} \varphi(x, \lambda_n) \int_0^\pi f(t) \tilde{\varphi}(t, \lambda_n) dt.$$

Together with (1.2.1) this gives

$$\int_0^\pi f(t) (\varphi(t, \lambda_n) - \tilde{\varphi}(t, \lambda_n)) dt = 0.$$

Since  $f(x)$  is an arbitrary absolutely continuous function we conclude that  $\varphi(x, \lambda_n) = \tilde{\varphi}(x, \lambda_n)$  for all  $n \geq 0$  and  $x \in [0, \pi]$ . Consequently,  $q(x) = \tilde{q}(x)$  a.e. on  $(0, \pi)$ ,  $h = \tilde{h}$ ,  $H = \tilde{H}$ .  $\square$

**Symmetrical case.** Let  $q(x) = q(\pi - x)$ ,  $H = h$ . In this case in order to construct the potential  $q(x)$  and the coefficient  $h$ , it is sufficient to specify the spectrum  $\{\lambda_n\}_{n \geq 0}$  only.

**Theorem 1.4.3.** *If  $q(x) = q(\pi - x)$ ,  $H = h$ ,  $\tilde{q}(x) = \tilde{q}(\pi - x)$ ,  $\tilde{H} = \tilde{h}$  and  $\lambda_n = \tilde{\lambda}_n$ ,  $n \geq 0$ , then  $q(x) = \tilde{q}(x)$  a.e. on  $(0, \pi)$  and  $h = \tilde{h}$ .*

*Proof.* If  $q(x) = q(\pi - x)$ ,  $H = h$  and  $y(x)$  is a solution of (1.1.1), then  $y_1(x) := y(\pi - x)$  also satisfies (1.1.1). In particular, we have  $\psi(x, \lambda) = \varphi(\pi - x, \lambda)$ . Using (1.1.6) we calculate

$$\psi(x, \lambda_n) = \beta_n \varphi(x, \lambda_n) = \beta_n \psi(\pi - x, \lambda_n) = \beta_n^2 \varphi(\pi - x, \lambda_n) = \beta_n^2 \psi(x, \lambda_n).$$

Hence,  $\beta_n^2 = 1$ .

On the other hand, it follows from (1.1.6) that  $\beta_n \varphi(\pi, \lambda_n) = 1$ . Applying Theorem 1.2.2 we conclude that

$$\beta_n = (-1)^n.$$

Then (1.1.8) implies

$$\alpha_n = (-1)^{n+1} \Delta(\lambda_n).$$

Under the assumptions of Theorem 1.4.3 we obtain  $\alpha_n = \tilde{\alpha}_n$ ,  $n \geq 0$ , and by Theorem 1.4.2,  $q(x) = \tilde{q}(x)$  a.e. on  $(0, \pi)$  and  $h = \tilde{h}$ .  $\square$

#### 1.4.3. Uniqueness of the recovery of differential equations from two spectra.

G. Borg [Bor1] suggested another formulation of the inverse problem: Reconstruct a differential operator from two spectra of boundary value problems with a common differential equation and one common boundary condition. For definiteness, let a boundary condition at  $x = 0$  be common.

Let  $\{\lambda_n\}_{n \geq 0}$  and  $\{\mu_n\}_{n \geq 0}$  be the eigenvalues of  $L$  and  $L_1$  ( $L_1$  is defined in Remark 1.1.2) respectively. Consider the following inverse problem

**Inverse Problem 1.4.2.** Given two spectra  $\{\lambda_n\}_{n \geq 0}$  and  $\{\mu_n\}_{n \geq 0}$ , construct the potential  $q(x)$  and the coefficients  $h$  and  $H$  of the boundary conditions.

The goal of this subsection is to prove the uniqueness theorem for the solution of Inverse Problem 1.4.2.

**Theorem 1.4.4.** *If  $\lambda_n = \tilde{\lambda}_n$ ,  $\mu_n = \tilde{\mu}_n$ ,  $n \geq 0$ , then  $q(x) = \tilde{q}(x)$  a.e. on  $(0, \pi)$ ,  $h = \tilde{h}$  and  $H = \tilde{H}$ . Thus, the specification of two spectra  $\{\lambda_n, \mu_n\}_{n \geq 0}$  uniquely determines the potential and the coefficients of the boundary conditions.*

*Proof.* By virtue of (1.1.26) and (1.1.28), the characteristic functions  $\Delta(\lambda)$  and  $d(\lambda)$  are uniquely determined by their zeros, i.e. under the hypothesis of Theorem 1.4.4 we have

$$\Delta(\lambda) \equiv \tilde{\Delta}(\lambda) \quad d(\lambda) \equiv \tilde{d}(\lambda).$$

Moreover, it follows from (1.1.13) and (1.1.29) that

$$H = \tilde{H}, \quad \hat{h} + \frac{1}{2} \int_0^\pi \hat{q}(x) dx = 0, \quad (1.4.4)$$

where  $\hat{h} = h - \tilde{h}$ ,  $\hat{q}(x) = q(x) - \tilde{q}(x)$ . Hence

$$\varphi(\pi, \lambda) = \tilde{\varphi}(\pi, \lambda), \quad \varphi'(\pi, \lambda) = \tilde{\varphi}'(\pi, \lambda). \quad (1.4.5)$$

Since

$$-\varphi''(x, \lambda) + q(x)\varphi(x, \lambda) = \lambda\varphi(x, \lambda), \quad -\tilde{\varphi}''(x, \lambda) + \tilde{q}(x)\tilde{\varphi}(x, \lambda) = \lambda\tilde{\varphi}(x, \lambda),$$

we get

$$\int_0^\pi \hat{q}(x)\varphi(x, \lambda)\tilde{\varphi}(x, \lambda) dx = \left|_0^\pi (\varphi'(x, \lambda)\tilde{\varphi}(x, \lambda) - \varphi(x, \lambda)\tilde{\varphi}'(x, \lambda)) \right|, \quad \hat{q} := q - \tilde{q}.$$

Taking (1.4.4) and (1.4.5) into account we calculate

$$\int_0^\pi \hat{q}(x) \left( \varphi(x, \lambda)\tilde{\varphi}(x, \lambda) - \frac{1}{2} \right) dx = 0. \quad (1.4.6)$$

Let us show that (1.4.6) implies  $\hat{q} = 0$ . For this we use the transformation operator (1.3.11). However, in order to prove this fact one can also apply the original Borg method which is presented in Section 1.8.

Using the transformation operator (1.3.11) we rewrite the function  $\varphi(x, \lambda)\tilde{\varphi}(x, \lambda) - \frac{1}{2}$  as follows:

$$\begin{aligned} \varphi(x, \lambda)\tilde{\varphi}(x, \lambda) - \frac{1}{2} &= \frac{1}{2} \cos 2\rho x + \int_0^x (G(x, t) + \tilde{G}(x, t)) \cos \rho x \cos \rho t dt \\ &+ \int_0^x \int_0^x G(x, t)\tilde{G}(x, s) \cos \rho t \cos \rho s dt ds = \frac{1}{2} \cos 2\rho x \\ &+ \frac{1}{2} \int_{-x}^x (G(x, t) + \tilde{G}(x, t)) \cos \rho(x - t) dt + \frac{1}{4} \int_{-x}^x \int_{-x}^x G(x, t)\tilde{G}(x, s) \cos \rho(t - s) dt ds \end{aligned}$$

(here  $G(x, -t) = G(x, t)$ ,  $\tilde{G}(x, -t) = \tilde{G}(x, t)$ ). The changes of variables  $\tau = (x - t)/2$  and  $\tau = (s - t)/2$ , respectively, yield

$$\varphi(x, \lambda)\tilde{\varphi}(x, \lambda) - \frac{1}{2} = \frac{1}{2} \left( \cos 2\rho x + 2 \int_0^x (G(x, x - 2\tau) + \tilde{G}(x, x - 2\tau)) \cos 2\rho\tau d\tau \right)$$

$$+ \int_{-x}^x \tilde{G}(x, s) \left( \int_{(s-x)/2}^{(s+x)/2} G(x, s-2\tau) \cos 2\rho\tau \, d\tau \right) ds,$$

and hence

$$\varphi(x, \lambda) \tilde{\varphi}(x, \lambda) - \frac{1}{2} = \frac{1}{2} \left( \cos 2\rho x + \int_0^x V(x, \tau) \cos 2\rho\tau \, d\tau \right), \quad (1.4.7)$$

where

$$\begin{aligned} V(x, \tau) &= 2(G(x, x-2\tau) + \tilde{G}(x, x-2\tau)) \\ &+ \int_{2\tau-x}^x \tilde{G}(x, s) G(x, s-2\tau) \, ds + \int_{-x}^{x-2\tau} \tilde{G}(x, s) G(x, s+2\tau) \, ds. \end{aligned} \quad (1.4.8)$$

Substituting (1.4.7) into (1.4.6) and interchanging the order of integration we obtain

$$\int_0^\pi \left( \hat{q}(\tau) + \int_\tau^\pi V(\tau, x) \hat{q}(x) \, dx \right) \cos 2\rho\tau \, d\tau \equiv 0.$$

Consequently,

$$\hat{q}(\tau) + \int_\tau^\pi V(\tau, x) \hat{q}(x) \, dx = 0.$$

This homogeneous Volterra integral equation has only the trivial solution, i.e.  $\hat{q}(x) = 0$  a.e. on  $(0, \pi)$ . From this and (1.4.4) it follows that  $h = \tilde{h}$ ,  $H = \tilde{H}$ .  $\square$

**Remark 1.4.2.** Clearly, Borg's result is also valid when instead of  $\{\lambda_n\}$  and  $\{\mu_n\}$ , the spectra  $\{\lambda_n\}$  and  $\{\lambda_n^0\}$  of  $L$  and  $L^0$  ( $L^0$  is defined in Remark 1.1.2) are given, i.e. it is valid for the following inverse problem.

**Inverse Problem 1.4.3.** Given two spectra  $\{\lambda_n\}_{n \geq 0}$  and  $\{\lambda_n^0\}_{n \geq 0}$ , construct the potential  $q(x)$  and the coefficients  $h$  and  $H$  of the boundary conditions.

The uniqueness theorem for Inverse Problem 1.4.3 has the form

**Theorem 1.4.5.** *If  $\lambda_n = \tilde{\lambda}_n$ ,  $\lambda_n^0 = \tilde{\lambda}_n^0$ ,  $n \geq 0$ , then  $q(x) = \tilde{q}(x)$  a.e. on  $(0, \pi)$ ,  $h = \tilde{h}$  and  $H = \tilde{H}$ . Thus, the specification of two spectra  $\{\lambda_n, \lambda_n^0\}_{n \geq 0}$  uniquely determines the potential and the coefficients of the boundary conditions.*

We note that Theorem 1.4.4 and Theorem 1.4.5 can be reduced each to other by the replacement  $x \rightarrow \pi - x$ .

**1.4.4. The Weyl function.** Let  $\Phi(x, \lambda)$  be the solution of (1.1.1) under the conditions  $U(\Phi) = 1$ ,  $V(\Phi) = 0$ . We set  $M(\lambda) := \Phi(0, \lambda)$ . The functions  $\Phi(x, \lambda)$  and  $M(\lambda)$  are called the *Weyl solution* and the *Weyl function* for the boundary value problem  $L$ , respectively. The Weyl function was introduced first (for the case of the half-line) by H. Weyl. For further discussions on the Weyl function see, for example, [lev3]. Clearly,

$$\Phi(x, \lambda) = -\frac{\psi(x, \lambda)}{\Delta(\lambda)} = S(x, \lambda) + M(\lambda)\varphi(x, \lambda), \quad (1.4.9)$$

$$M(\lambda) = -\frac{\Delta^0(\lambda)}{\Delta(\lambda)}, \quad (1.4.10)$$

$$\langle \varphi(x, \lambda), \Phi(x, \lambda) \rangle \equiv 1, \quad (1.4.11)$$

where  $\Delta^0(\lambda)$  is defined in Remark 1.1.2. Thus, the Weyl function is meromorphic with simple poles in the points  $\lambda = \lambda_n$ ,  $n \geq 0$ .

**Theorem 1.4.6.** *The following representation holds*

$$M(\lambda) = \sum_{n=0}^{\infty} \frac{1}{\alpha_n(\lambda - \lambda_n)}. \quad (1.4.12)$$

*Proof.* Since  $\Delta^0(\lambda) = \psi(0, \lambda)$ , it follows from (1.1.10) that  $|\Delta^0(\lambda)| \leq C \exp(|\tau|\pi)$ . Then, using (1.4.10) and (1.1.18), we get for sufficiently large  $\rho^* > 0$ ,

$$|M(\lambda)| \leq \frac{C_\delta}{|\rho|}, \quad \rho \in G_\delta, \quad |\rho| \geq \rho^*. \quad (1.4.13)$$

Further, using (1.4.10) and Lemma 1.1.1, we calculate

$$\operatorname{Res}_{\lambda=\lambda_n} M(\lambda) = -\frac{\Delta^0(\lambda_n)}{\dot{\Delta}(\lambda_n)} = -\frac{\beta_n}{\dot{\Delta}(\lambda_n)} = \frac{1}{\alpha_n}. \quad (1.4.14)$$

Consider the contour integral

$$J_N(\lambda) = \frac{1}{2\pi i} \int_{\Gamma_N} \frac{M(\mu)}{\lambda - \mu} d\mu, \quad \lambda \in \operatorname{int} \Gamma_N.$$

By virtue of (1.4.13),  $\lim_{N \rightarrow \infty} J_N(\lambda) = 0$ . On the other hand, the residue theorem and (1.4.14) yield

$$J_N(\lambda) = -M(\lambda) + \sum_{n=0}^N \frac{1}{\alpha_n(\lambda - \lambda_n)},$$

and Theorem 1.4.6 is proved.  $\square$

In this subsection we consider the following inverse problem:

**Inverse Problem 1.4.4.** Given the Weyl function  $M(\lambda)$ , construct  $L(q(x), h, H)$ .

Let us prove the uniqueness theorem for Inverse Problem 1.4.4.

**Theorem 1.4.7.** *If  $M(\lambda) = \tilde{M}(\lambda)$ , then  $L = \tilde{L}$ . Thus, the specification of the Weyl function uniquely determines the operator.*

*Proof.* Let us define the matrix  $P(x, \lambda) = [P_{jk}(x, \lambda)]_{j,k=1,2}$  by the formula

$$P(x, \lambda) \begin{bmatrix} \tilde{\varphi}(x, \lambda) & \tilde{\Phi}(x, \lambda) \\ \tilde{\varphi}'(x, \lambda) & \tilde{\Phi}'(x, \lambda) \end{bmatrix} = \begin{bmatrix} \varphi(x, \lambda) & \Phi(x, \lambda) \\ \varphi'(x, \lambda) & \Phi'(x, \lambda) \end{bmatrix}. \quad (1.4.15)$$

Using (1.4.11) and (1.4.15) we calculate

$$\left. \begin{aligned} P_{j1}(x, \lambda) &= \varphi^{(j-1)}(x, \lambda) \tilde{\Phi}'(x, \lambda) - \Phi^{(j-1)}(x, \lambda) \tilde{\varphi}'(x, \lambda), \\ P_{j2}(x, \lambda) &= \Phi^{(j-1)}(x, \lambda) \tilde{\varphi}(x, \lambda) - \varphi^{(j-1)}(x, \lambda) \tilde{\Phi}(x, \lambda), \end{aligned} \right\} \quad (1.4.16)$$

$$\left. \begin{aligned} \varphi(x, \lambda) &= P_{11}(x, \lambda) \tilde{\varphi}(x, \lambda) + P_{12}(x, \lambda) \tilde{\varphi}'(x, \lambda), \\ \Phi(x, \lambda) &= P_{11}(x, \lambda) \tilde{\Phi}(x, \lambda) + P_{12}(x, \lambda) \tilde{\Phi}'(x, \lambda). \end{aligned} \right\} \quad (1.4.17)$$

It follows from (1.4.16), (1.4.9) and (1.4.11) that

$$P_{11}(x, \lambda) = 1 + \frac{1}{\Delta(\lambda)} \left( \psi(x, \lambda) (\tilde{\varphi}'(x, \lambda) - \varphi'(x, \lambda)) - \varphi(x, \lambda) (\tilde{\psi}'(x, \lambda) - \psi'(x, \lambda)) \right),$$

$$P_{12}(x, \lambda) = \frac{1}{\Delta(\lambda)} (\varphi(x, \lambda) \tilde{\psi}(x, \lambda) - \psi(x, \lambda) \tilde{\varphi}(x, \lambda)).$$

By virtue of (1.1.9), (1.1.10) and (1.1.18), this yields

$$|P_{11}(x, \lambda) - 1| \leq \frac{C_\delta}{|\rho|}, \quad |P_{12}(x, \lambda)| \leq \frac{C_\delta}{|\rho|}, \quad \rho \in G_\delta, \quad |\rho| \geq \rho^*, \quad (1.4.18)$$

$$|P_{22}(x, \lambda) - 1| \leq \frac{C_\delta}{|\rho|}, \quad |P_{21}(x, \lambda)| \leq C_\delta, \quad \rho \in G_\delta, \quad |\rho| \geq \rho^*. \quad (1.4.19)$$

According to (1.4.9) and (1.4.16),

$$P_{11}(x, \lambda) = \varphi(x, \lambda) \tilde{S}'(x, \lambda) - S(x, \lambda) \tilde{\varphi}'(x, \lambda) + (\tilde{M}(\lambda) - M(\lambda)) \varphi(x, \lambda) \tilde{\varphi}'(x, \lambda),$$

$$P_{12}(x, \lambda) = S(x, \lambda) \tilde{\varphi}(x, \lambda) - \varphi(x, \lambda) \tilde{S}(x, \lambda) + (M(\lambda) - \tilde{M}(\lambda)) \varphi(x, \lambda) \tilde{\varphi}(x, \lambda).$$

Thus, if  $M(\lambda) \equiv \tilde{M}(\lambda)$ , then for each fixed  $x$ , the functions  $P_{11}(x, \lambda)$  and  $P_{12}(x, \lambda)$  are entire in  $\lambda$ . Together with (1.4.18) this yields  $P_{11}(x, \lambda) \equiv 1$ ,  $P_{12}(x, \lambda) \equiv 0$ . Substituting into (1.4.17) we get  $\varphi(x, \lambda) \equiv \tilde{\varphi}(x, \lambda)$ ,  $\Phi(x, \lambda) \equiv \tilde{\Phi}(x, \lambda)$  for all  $x$  and  $\lambda$ , and consequently,  $L = \tilde{L}$ .  $\square$

**Remark 1.4.3.** According to (1.4.12), the specification of the Weyl function  $M(\lambda)$  is equivalent to the specification of the spectral data  $\{\lambda_n, \alpha_n\}_{n \geq 0}$ . On the other hand, by virtue of (1.4.10) zeros and poles of the Weyl function  $M(\lambda)$  coincide with the spectra of the boundary value problems  $L$  and  $L^0$ , respectively. Consequently, the specification of the Weyl function  $M(\lambda)$  is equivalent to the specification of two spectra  $\{\lambda_n\}$  and  $\{\lambda_n^0\}$ . Thus, the inverse problems of recovering the Sturm-Liouville equation from the spectral data and from two spectra are particular cases of Inverse Problem 1.4.4 of recovering the Sturm-Liouville equation from the given Weyl function, and we have several independent methods for proving the uniqueness theorems. The Weyl function is a very natural and convenient spectral characteristic in the inverse problem theory. Using the concept of the Weyl function and its generalizations we can formulate and study inverse problems for various classes of operators. For example, the inverse problem of recovering higher-order differential operators from the Weyl functions has been studied in [yur1]. We will also use the Weyl function in Chapter 2 for the investigation of the Sturm-Liouville operator on the half-line.

## 1.5. GELFAND-LEVITAN METHOD

In Sections 1.5-1.8 we present various methods for constructing the solution of inverse problems. In this section we describe the Gelfand-Levitan method ([gel1], [lev2], [mar1]) in which the transformation operators constructed in Section 1.3 are used. By the Gelfand-Levitan method we obtain algorithms for the solution of inverse problems and provide necessary and sufficient conditions for their solvability. The central role in this method is played by a linear integral equation with respect to the kernel of the transformation operator (see Theorem 1.5.1). The main results of Section 1.5 are stated in Theorems 1.5.2 and 1.5.4.

**1.5.1. Auxiliary propositions.** In order to present the transformation operator method we first prove several auxiliary assertions.



**Lemma 1.5.1.** *In a Banach space  $B$ , consider the equations*

$$(E + A_0)y_0 = f_0,$$

$$(E + A)y = f,$$

where  $A$  and  $A_0$  are linear bounded operators, acting from  $B$  to  $B$ , and  $E$  is the identity operator. Suppose that there exists the linear bounded operator  $R_0 := (E + A_0)^{-1}$ , this yields in particular that the equation  $(E + A_0)y_0 = f_0$  is uniquely solvable in  $B$ . If

$$\|A - A_0\| \leq (2\|R_0\|)^{-1},$$

then there exists the linear bounded operator  $R := (E + A)^{-1}$  with

$$R = R_0 \left( E + \sum_{k=1}^{\infty} ((A_0 - A)R_0)^k \right),$$

and

$$\|R - R_0\| \leq 2\|R_0\|^2 \|A - A_0\|.$$

Moreover,  $y$  and  $y_0$  satisfy the estimate

$$\|y - y_0\| \leq C_0(\|A - A_0\| + \|f - f_0\|),$$

where  $C_0$  depends only on  $\|R_0\|$  and  $\|f_0\|$ .

*Proof.* We have

$$E + A = (E + A_0) + (A - A_0) = (E + (A - A_0)R_0)(E + A_0).$$

Under the assumptions of the lemma it follows that  $\|(A - A_0)R_0\| \leq 1/2$ , and consequently there exists the linear bounded operator

$$R := (E + A)^{-1} = R_0 \left( E + (A - A_0)R_0 \right)^{-1} = R_0 \left( E + \sum_{k=1}^{\infty} ((A_0 - A)R_0)^k \right).$$

This yields in particular that  $\|R\| \leq 2\|R_0\|$ . Using again the assumption on  $\|A - A_0\|$  we infer

$$\|R - R_0\| \leq \|R_0\| \frac{\|(A - A_0)R_0\|}{1 - \|(A - A_0)R_0\|} \leq 2\|R_0\|^2 \|A - A_0\|.$$

Furthermore,

$$y - y_0 = Rf - R_0f_0 = (R - R_0)f_0 + R(f - f_0).$$

Hence

$$\|y - y_0\| \leq 2\|R_0\|^2 \|f_0\| \|A - A_0\| + 2\|R_0\| \|f - f_0\|.$$

□

The following lemma is an obvious corollary of Lemma 1.5.1 (applied in a corresponding function space).

**Lemma 1.5.2.** *Consider the integral equation*

$$y(t, \alpha) + \int_a^b A(t, s, \alpha) y(s, \alpha) ds = f(t, \alpha), \quad a \leq t \leq b, \quad (1.5.1)$$

where  $A(t, s, \alpha)$  and  $f(t, \alpha)$  are continuous functions. Assume that for a fixed  $\alpha = \alpha_0$  the homogeneous equation

$$z(t) + \int_a^b A_0(t, s)z(s) ds = 0, \quad A_0(t, s) := A(t, s, \alpha_0)$$

has only the trivial solution. Then in a neighbourhood of the point  $\alpha = \alpha_0$ , equation (1.5.1) has a unique solution  $y(t, \alpha)$ , which is continuous with respect to  $t$  and  $\alpha$ . Moreover, the function  $y(t, \alpha)$  has the same smoothness as  $A(t, s, \alpha)$  and  $f(t, \alpha)$ .

**Lemma 1.5.3.** Let  $\varphi_j(x, \lambda)$ ,  $j \geq 1$ , be the solution of the equation

$$-y'' + q_j(x)y = \lambda y, \quad q_j(x) \in L_2(0, \pi)$$

under the conditions  $\varphi_j(0, \lambda) = 1$ ,  $\varphi_j'(0, \lambda) = h_j$ , and let  $\varphi(x, \lambda)$  be the function defined in Section 1.1. If  $\lim_{j \rightarrow \infty} \|q_j - q\|_{L_2} = 0$ ,  $\lim_{j \rightarrow \infty} h_j = h$ , then

$$\lim_{j \rightarrow \infty} \max_{0 \leq x \leq \pi} \max_{|\lambda| \leq r} |\varphi_j(x, \lambda) - \varphi(x, \lambda)| = 0.$$

*Proof.* One can verify by differentiation that the function  $\varphi_j(x, \lambda)$  satisfies the following integral equation

$$\varphi_j(x, \lambda) = \varphi(x, \lambda) + (h_j - h)S(x, \lambda) + \int_0^x g(x, t, \lambda)(q_j(t) - q(t))\varphi_j(t, \lambda) dt,$$

where  $g(x, t, \lambda) = C(t, \lambda)S(x, \lambda) - C(x, \lambda)S(t, \lambda)$  is the Green function for the Cauchy problem:

$$-y'' + q(x)y - \lambda y = f, \quad y(0) = y'(0) = 0.$$

Fix  $r > 0$ . Then for  $|\lambda| \leq r$  and  $x \in [0, \pi]$ ,

$$|\varphi_j(x, \lambda) - \varphi(x, \lambda)| \leq C(|h_j - h| + \|q_j - q\|_{L_2} \max_{0 \leq x \leq \pi} \max_{|\lambda| \leq r} |\varphi_j(x, \lambda)|).$$

In particular, this yields

$$\max_{0 \leq x \leq \pi} \max_{|\lambda| \leq r} |\varphi_j(x, \lambda)| \leq C,$$

where  $C$  does not depend on  $j$ . Hence

$$\max_{0 \leq x \leq \pi} \max_{|\lambda| \leq r} |\varphi_j(x, \lambda) - \varphi(x, \lambda)| \leq C(|h_j - h| + \|q_j - q\|_{L_2}) \rightarrow 0 \quad \text{for } j \rightarrow \infty.$$

□

**Lemma 1.5.4.** Let numbers  $\{\rho_n, \alpha_n\}_{n \geq 0}$  of the form

$$\rho_n = n + \frac{\omega}{\pi n} + \frac{\kappa_n}{n}, \quad \alpha_n = \frac{\pi}{2} + \frac{\kappa_{n1}}{n}, \quad \{\kappa_n\}, \{\kappa_{n1}\} \in l_2, \quad \alpha_n \neq 0 \quad (1.5.2)$$

be given. Denote

$$a(x) = \sum_{n=0}^{\infty} \left( \frac{\cos \rho_n x}{\alpha_n} - \frac{\cos nx}{\alpha_n^0} \right), \quad (1.5.3)$$

where

$$\alpha_n^0 = \begin{cases} \frac{\pi}{2}, & n > 0, \\ \pi, & n = 0. \end{cases}$$

Then  $a(x) \in W_2^1(0, 2\pi)$ .

*Proof.* Denote  $\delta_n = \rho_n - n$ . Since

$$\begin{aligned} \frac{\cos \rho_n x}{\alpha_n} - \frac{\cos nx}{\alpha_n^0} &= \frac{1}{\alpha_n^0} (\cos \rho_n x - \cos nx) + \left( \frac{1}{\alpha_n} - \frac{1}{\alpha_n^0} \right) \cos \rho_n x, \\ \cos \rho_n x - \cos nx &= \cos(n + \delta_n)x - \cos nx = -\sin \delta_n x \sin nx - 2 \sin^2 \frac{\delta_n x}{2} \cos nx \\ &= -\delta_n x \sin nx - (\sin \delta_n x - \delta_n x) \sin nx - 2 \sin^2 \frac{\delta_n x}{2} \cos nx, \end{aligned}$$

we have

$$a(x) = A_1(x) + A_2(x),$$

where

$$\begin{aligned} A_1(x) &= -\frac{\omega x}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{n} = -\frac{\omega x}{\pi} \cdot \frac{\pi - x}{2}, \quad 0 < x < 2\pi, \\ A_2(x) &= \sum_{n=0}^{\infty} \left( \frac{1}{\alpha_n} - \frac{1}{\alpha_n^0} \right) \cos \rho_n x + \frac{1}{\pi} (\cos \rho_0 x - 1) - x \sum_{n=1}^{\infty} \kappa_n \frac{\sin nx}{n} \\ &\quad - \sum_{n=1}^{\infty} (\sin \delta_n x - \delta_n x) \sin nx - 2 \sum_{n=1}^{\infty} \sin^2 \frac{\delta_n x}{2} \cos nx. \end{aligned} \quad (1.5.4)$$

Since

$$\delta_n = O\left(\frac{1}{n}\right), \quad \frac{1}{\alpha_n} - \frac{1}{\alpha_n^0} = \frac{\gamma_n}{n}, \quad \{\gamma_n\} \in l_2,$$

the series in (1.5.4) converge absolutely and uniformly on  $[0, 2\pi]$ , and  $A_2(x) \in W_2^1(0, 2\pi)$ . Consequently,  $a(x) \in W_2^1(0, 2\pi)$ .  $\square$

By similar arguments one can prove the following more general assertions.

**Lemma 1.5.5.** *Let numbers  $\{\rho_n, \alpha_n\}_{n \geq 0}$  of the form (1.1.23) be given. Then  $a(x) \in W_2^{N+1}(0, 2\pi)$ .*

*Proof.* Indeed, substituting (1.1.23) into (1.5.3) we obtain two groups of series. The series of the first group can be differentiated  $N + 1$  times. The series of the second group are

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^{2k+1}}, \quad \sum_{n=1}^{\infty} \frac{\cos nx}{n^{2k}}.$$

Differentiating these series  $2k$  and  $2k - 1$  times respectively, we obtain the series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n} = \frac{\pi - x}{2}, \quad 0 < x < 2\pi.$$

$\square$

**Lemma 1.5.6.** *Let numbers  $\{\rho_n, \alpha_n\}_{n \geq 0}$  of the form (1.5.2) be given. Fix  $C_0 > 0$ . If certain numbers  $\{\tilde{\rho}_n, \tilde{\alpha}_n\}_{n \geq 0}$ ,  $\tilde{\alpha}_n \neq 0$  satisfy the condition*

$$\Omega := \left( \sum_{n=0}^{\infty} ((n+1)\xi_n)^2 \right)^{1/2} \leq C_0, \quad \xi_n := |\tilde{\rho}_n - \rho_n| + |\tilde{\alpha}_n - \alpha_n|,$$

then

$$\hat{a}(x) := \sum_{n=0}^{\infty} \left( \frac{\cos \tilde{\rho}_n x}{\tilde{\alpha}_n} - \frac{\cos \rho_n x}{\alpha_n} \right) \in W_2^1(0, 2\pi), \quad (1.5.5)$$

and

$$\max_{0 \leq x \leq 2\pi} |\hat{a}(x)| \leq C \sum_{n=0}^{\infty} \xi_n, \quad \|\hat{a}(x)\|_{W_2^1} \leq C\Omega,$$

where  $C$  depends on  $\{\rho_n, \alpha_n\}_{n \geq 0}$  and  $C_0$ .

*Proof.* Obviously,

$$\sum_{n=0}^{\infty} \xi_n \leq C\Omega.$$

We rewrite (1.5.5) in the form

$$\begin{aligned} \hat{a}(x) &:= \sum_{n=0}^{\infty} \left( \left( \frac{1}{\tilde{\alpha}_n} - \frac{1}{\alpha_n} \right) \cos \rho_n x + \frac{1}{\tilde{\alpha}_n} (\cos \tilde{\rho}_n x - \cos \rho_n x) \right) \\ &= \sum_{n=0}^{\infty} \left( \frac{\alpha_n - \tilde{\alpha}_n}{\tilde{\alpha}_n \alpha_n} \cos \rho_n x + \frac{2}{\tilde{\alpha}_n} \sin \frac{(\rho_n - \tilde{\rho}_n)x}{2} \sin \frac{(\rho_n + \tilde{\rho}_n)x}{2} \right). \end{aligned}$$

This series converges absolutely and uniformly,  $\hat{a}(x)$  is a continuous function, and

$$|\hat{a}(x)| \leq C \sum_{n=0}^{\infty} \xi_n.$$

Differentiating (1.5.5) we calculate analogously

$$\begin{aligned} \hat{a}'(x) &:= - \sum_{n=0}^{\infty} \left( \frac{\tilde{\rho}_n \sin \tilde{\rho}_n x}{\tilde{\alpha}_n} - \frac{\rho_n \sin \rho_n x}{\alpha_n} \right) = - \sum_{n=0}^{\infty} \left( \left( \frac{\tilde{\rho}_n}{\tilde{\alpha}_n} - \frac{\rho_n}{\alpha_n} \right) \sin \rho_n x + \frac{\tilde{\rho}_n}{\tilde{\alpha}_n} (\sin \tilde{\rho}_n x - \sin \rho_n x) \right) \\ &= \sum_{n=0}^{\infty} \left( \frac{\tilde{\alpha}_n \rho_n - \alpha_n \tilde{\rho}_n}{\tilde{\alpha}_n \alpha_n} \sin \rho_n x + \frac{2\tilde{\rho}_n}{\tilde{\alpha}_n} \sin \frac{(\rho_n - \tilde{\rho}_n)x}{2} \cos \frac{(\rho_n + \tilde{\rho}_n)x}{2} \right) \\ &= \sum_{n=0}^{\infty} \left( \frac{\tilde{\alpha}_n \rho_n - \alpha_n \tilde{\rho}_n}{\tilde{\alpha}_n \alpha_n} \sin \rho_n x + x \frac{\tilde{\rho}_n}{\tilde{\alpha}_n} (\rho_n - \tilde{\rho}_n) \cos \frac{(\rho_n + \tilde{\rho}_n)x}{2} \right. \\ &\quad \left. + \frac{\tilde{\rho}_n}{\tilde{\alpha}_n} \left( 2 \sin \frac{(\rho_n - \tilde{\rho}_n)x}{2} - x(\rho_n - \tilde{\rho}_n) \right) \cos \frac{(\rho_n + \tilde{\rho}_n)x}{2} \right) = A_1(x) + A_2(x), \end{aligned}$$

where

$$\begin{aligned} A_1(x) &= \sum_{n=0}^{\infty} \frac{\tilde{\alpha}_n \rho_n - \alpha_n \tilde{\rho}_n}{\tilde{\alpha}_n \alpha_n} \sin \rho_n x + x \sum_{n=0}^{\infty} \frac{\tilde{\rho}_n}{\tilde{\alpha}_n} (\rho_n - \tilde{\rho}_n) \cos \rho_n x, \\ A_2(x) &= \sum_{n=0}^{\infty} \frac{\tilde{\rho}_n}{\tilde{\alpha}_n} \left( 2 \sin \frac{(\rho_n - \tilde{\rho}_n)x}{2} - x(\rho_n - \tilde{\rho}_n) \right) \cos \frac{(\rho_n + \tilde{\rho}_n)x}{2} \end{aligned} \quad (1.5.6)$$

$$+x \sum_{n=0}^{\infty} \frac{\tilde{\rho}_n}{\tilde{\alpha}_n} (\rho_n - \tilde{\rho}_n) \left( \cos \frac{(\rho_n + \tilde{\rho}_n)x}{2} - \cos \rho_n x \right). \quad (1.5.7)$$

The series in (1.5.7) converge absolutely and uniformly, and

$$|A_2(x)| \leq C \sum_{n=0}^{\infty} |\rho_n - \tilde{\rho}_n|.$$

The series in (1.5.6) converge in  $L_2(0, 2\pi)$ , and

$$\|A_1(x)\|_{L_2(0, 2\pi)} \leq C\Omega.$$

These estimates imply the assertions of the lemma.  $\square$

**1.5.2. Recovery of differential operators from the spectral data.** Let us consider the boundary value problem  $L = L(q(x), h, H)$ . Let  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  be the spectral data of  $L$ ,  $\rho_n = \sqrt{\lambda_n}$ . We shall solve the inverse problem of recovering  $L$  from the given spectral data  $\{\lambda_n, \alpha_n\}_{n \geq 0}$ . It was shown in Section 1.1 that the spectral data have the properties:

$$\rho_n = n + \frac{\omega}{\pi n} + \frac{\kappa_n}{n}, \quad \alpha_n = \frac{\pi}{2} + \frac{\kappa_{n1}}{n}, \quad \{\kappa_n\}, \{\kappa_{n1}\} \in \ell_2, \quad (1.5.8)$$

$$\alpha_n > 0, \quad \lambda_n \neq \lambda_m \quad (n \neq m). \quad (1.5.9)$$

More precisely

$$\kappa_n = \frac{1}{2\pi} \int_0^\pi q(t) \cos 2nt \, dt + O\left(\frac{1}{n}\right), \quad \kappa_{n1} = -\frac{1}{2} \int_0^\pi (\pi - t) q(t) \sin 2nt \, dt + O\left(\frac{1}{n}\right),$$

i.e. the main terms depend linearly on the potential.

Consider the function

$$F(x, t) = \sum_{n=0}^{\infty} \left( \frac{\cos \rho_n x \cos \rho_n t}{\alpha_n} - \frac{\cos nx \cos nt}{\alpha_n^0} \right), \quad (1.5.10)$$

where

$$\alpha_n^0 = \begin{cases} \frac{\pi}{2}, & n > 0, \\ \pi, & n = 0. \end{cases}$$

Since  $F(x, t) = (a(x+t) + a(x-t))/2$ , then by virtue of Lemma 1.5.4,  $F(x, t)$  is continuous, and  $\frac{d}{dx} F(x, x) \in L_2(0, \pi)$ .

**Theorem 1.5.1.** *For each fixed  $x \in (0, \pi]$ , the kernel  $G(x, t)$  appearing in representation (1.3.11) satisfies the linear integral equation*

$$G(x, t) + F(x, t) + \int_0^x G(x, s) F(s, t) \, ds = 0, \quad 0 < t < x. \quad (1.5.11)$$

*This equation is called the Gelfand-Levitan equation.*

Thus, Theorem 1.5.1 allows one to reduce our inverse problem to the solution of the Gelfand-Levitan equation (1.5.11). We note that (1.5.11) is a Fredholm type integral equation in which  $x$  is a parameter.

*Proof.* One can consider the relation (1.3.11) as a Volterra integral equation with respect to  $\cos \rho x$ . Solving this equation we obtain

$$\cos \rho x = \varphi(x, \lambda) + \int_0^x H(x, t) \varphi(t, \lambda) dt, \quad (1.5.12)$$

where  $H(x, t)$  is a continuous function. Using (1.3.11) and (1.5.12) we calculate

$$\begin{aligned} \sum_{n=0}^N \frac{\varphi(x, \lambda_n) \cos \rho_n t}{\alpha_n} &= \sum_{n=0}^N \left( \frac{\cos \rho_n x \cos \rho_n t}{\alpha_n} + \frac{\cos \rho_n t}{\alpha_n} \int_0^x G(x, s) \cos \rho_n s ds \right), \\ \sum_{n=0}^N \frac{\varphi(x, \lambda_n) \cos \rho_n t}{\alpha_n} &= \sum_{n=0}^N \left( \frac{\varphi(x, \lambda_n) \varphi(t, \lambda_n)}{\alpha_n} + \frac{\varphi(x, \lambda_n)}{\alpha_n} \int_0^t H(t, s) \varphi(s, \lambda_n) ds \right). \end{aligned}$$

This yields

$$\Phi_N(x, t) = I_{N1}(x, t) + I_{N2}(x, t) + I_{N3}(x, t) + I_{N4}(x, t),$$

where

$$\begin{aligned} \Phi_N(x, t) &= \sum_{n=0}^N \left( \frac{\varphi(x, \lambda_n) \varphi(t, \lambda_n)}{\alpha_n} - \frac{\cos nx \cos nt}{\alpha_n^0} \right), \\ I_{N1}(x, t) &= \sum_{n=0}^N \left( \frac{\cos \rho_n x \cos \rho_n t}{\alpha_n} - \frac{\cos nx \cos nt}{\alpha_n^0} \right), \\ I_{N2}(x, t) &= \sum_{n=0}^N \frac{\cos nt}{\alpha_n^0} \int_0^x G(x, s) \cos ns ds, \\ I_{N3}(x, t) &= \sum_{n=0}^N \int_0^x G(x, s) \left( \frac{\cos \rho_n t \cos \rho_n s}{\alpha_n} - \frac{\cos nt \cos ns}{\alpha_n^0} \right) ds, \\ I_{N4}(x, t) &= - \sum_{n=0}^N \frac{\varphi(x, \lambda_n)}{\alpha_n} \int_0^t H(t, s) \varphi(s, \lambda_n) ds. \end{aligned}$$

Let  $f(x) \in AC[0, \pi]$ . According to Theorem 1.2.1,

$$\lim_{N \rightarrow \infty} \max_{0 \leq x \leq \pi} \int_0^\pi f(t) \Phi_N(x, t) dt = 0;$$

furthermore, uniformly with respect to  $x \in [0, \pi]$ ,

$$\begin{aligned} \lim_{N \rightarrow \infty} \int_0^\pi f(t) I_{N1}(x, t) dt &= \int_0^\pi f(t) F(x, t) dt, \\ \lim_{N \rightarrow \infty} \int_0^\pi f(t) I_{N2}(x, t) dt &= \int_0^x f(t) G(x, t) dt, \\ \lim_{N \rightarrow \infty} \int_0^\pi f(t) I_{N3}(x, t) dt &= \int_0^\pi f(t) \left( \int_0^x G(x, s) F(s, t) ds \right) dt, \\ \lim_{N \rightarrow \infty} \int_0^\pi f(t) I_{N4}(x, t) dt &= - \lim_{N \rightarrow \infty} \sum_{n=0}^N \frac{\varphi(x, \lambda_n)}{\alpha_n} \int_0^\pi \varphi(s, \lambda_n) \left( \int_s^\pi H(t, s) f(t) dt \right) ds \\ &= - \int_x^\pi f(t) H(t, x) dt. \end{aligned}$$

Extend  $G(x, t) = H(x, t) = 0$  for  $x < t$ . Then, in view of the arbitrariness of  $f(x)$ , we derive

$$G(x, t) + F(x, t) + \int_0^x G(x, s)F(s, t) ds - H(t, x) = 0.$$

For  $t < x$ , this yields (1.5.11).  $\square$

The next theorem, which is the main result of Section 1.5, gives us an algorithm for the solution of the inverse problem as well as necessary and sufficient conditions for its solvability.

**Theorem 1.5.2.** *For real numbers  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  to be the spectral data for a certain boundary value problem  $L(q(x), h, H)$  with  $q(x) \in L_2(0, \pi)$ , it is necessary and sufficient that the relations (1.5.8)-(1.5.9) hold. Moreover,  $q(x) \in W_2^N$  iff (1.1.23) holds. The boundary value problem  $L(q(x), h, H)$  can be constructed by the following algorithm:*

**Algorithm 1.5.1.** (i) From the given numbers  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  construct the function  $F(x, t)$  by (1.5.10).

(ii) Find the function  $G(x, t)$  by solving equation (1.5.11).

(iii) Calculate  $q(x)$ ,  $h$  and  $H$  by the formulae

$$q(x) = 2 \frac{d}{dx} G(x, x), \quad h = G(0, 0), \quad (1.5.13)$$

$$H = \omega - h - \frac{1}{2} \int_0^\pi q(t) dt. \quad (1.5.14)$$

The necessity part of Theorem 1.5.2 was proved above, here we prove the sufficiency. Let real numbers  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  of the form (1.5.8)-(1.5.9) be given. We construct the function  $F(x, t)$  by (1.5.10) and consider equation (1.5.11).

**Lemma 1.5.7.** *For each fixed  $x \in (0, \pi]$ , equation (1.5.11) has a unique solution  $G(x, t)$  in  $L_2(0, x)$ .*

*Proof.* Since (1.5.11) is a Fredholm equation it is sufficient to prove that the homogeneous equation

$$g(t) + \int_0^x F(s, t)g(s) ds = 0 \quad (1.5.15)$$

has only the trivial solution  $g(t) = 0$ .

Let  $g(t)$  be a solution of (1.5.15). Then

$$\int_0^x g^2(t) dt + \int_0^x \int_0^x F(s, t)g(s)g(t) ds dt = 0$$

or

$$\int_0^x g^2(t) dt + \sum_{n=0}^{\infty} \frac{1}{\alpha_n} \left( \int_0^x g(t) \cos \rho_n t dt \right)^2 - \sum_{n=0}^{\infty} \frac{1}{\alpha_n^0} \left( \int_0^x g(t) \cos nt dt \right)^2 = 0.$$

Using Parseval's equality

$$\int_0^x g^2(t) dt = \sum_{n=0}^{\infty} \frac{1}{\alpha_n^0} \left( \int_0^x g(t) \cos nt dt \right)^2,$$

for the function  $g(t)$ , extended by zero for  $t > x$ , we obtain

$$\sum_{n=0}^{\infty} \frac{1}{\alpha_n} \left( \int_0^x g(t) \cos \rho_n t dt \right)^2 = 0.$$

Since  $\alpha_n > 0$ , then

$$\int_0^x g(t) \cos \rho_n t \, dt = 0, \quad n \geq 0.$$

The system of functions  $\{\cos \rho_n t\}_{n \geq 0}$  is complete in  $L_2(0, \pi)$  (see Levinson's theorem [you1, p.118] or Proposition 1.8.6 in Section 1.8). This yields  $g(t) = 0$ .  $\square$

Let us return to the proof of Theorem 1.5.2. Let  $G(x, t)$  be the solution of (1.5.11). The substitution  $t \rightarrow tx$ ,  $s \rightarrow sx$  in (1.5.11) yields

$$F(x, xt) + G(x, xt) + x \int_0^1 G(x, xs) F(xt, xs) \, ds = 0, \quad 0 \leq t \leq 1. \quad (1.5.16)$$

It follows from (1.5.11), (1.5.16) and Lemma 1.5.2 that the function  $G(x, t)$  is continuous, and has the same smoothness as  $F(x, t)$ . In particular,  $\frac{d}{dx} G(x, x) \in L_2(0, \pi)$ .

We construct the function  $\varphi(x, \lambda)$  by (1.3.11) and the function  $q(x)$  and the number  $h$  by (1.5.13).

**Lemma 1.5.8.** *The following relations hold*

$$-\varphi''(x, \lambda) + q(x)\varphi(x, \lambda) = \lambda\varphi(x, \lambda), \quad (1.5.17)$$

$$\varphi(0, \lambda) = 1, \quad \varphi'(0, \lambda) = h. \quad (1.5.18)$$

*Proof.* 1) First we assume that  $a(x) \in W_2^2(0, 2\pi)$ , where  $a(x)$  is defined by (1.5.3). Differentiating the identity

$$J(x, t) := F(x, t) + G(x, t) + \int_0^x G(x, s) F(s, t) \, ds \equiv 0, \quad (1.5.19)$$

we calculate

$$J_t(x, t) = F_t(x, t) + G_t(x, t) + \int_0^x G(x, s) F_t(s, t) \, ds = 0, \quad (1.5.20)$$

$$J_{tt}(x, t) = F_{tt}(x, t) + G_{tt}(x, t) + \int_0^x G(x, s) F_{tt}(s, t) \, ds = 0, \quad (1.5.21)$$

$$\begin{aligned} J_{xx}(x, t) &= F_{xx}(x, t) + G_{xx}(x, t) + \frac{dG(x, x)}{dx} F(x, t) + G(x, x) F_x(x, t) \\ &\quad + \frac{\partial G(x, t)}{\partial x} \Big|_{t=x} F(x, t) + \int_0^x G_{xx}(x, s) F(s, t) \, ds = 0. \end{aligned} \quad (1.5.22)$$

According to (1.5.10),  $F_{tt}(s, t) = F_{ss}(s, t)$  and  $F_t(x, t)|_{t=0} = 0$ . Then, (1.5.20) for  $t = 0$  gives

$$\frac{\partial G(x, t)}{\partial t} \Big|_{t=0} = 0. \quad (1.5.23)$$

Moreover, integration by parts in (1.5.21) yields

$$\begin{aligned} J_{tt}(x, t) &= F_{tt}(x, t) + G_{tt}(x, t) + G(x, x) \frac{\partial F(s, t)}{\partial s} \Big|_{s=x} \\ &\quad - \frac{\partial G(x, s)}{\partial s} \Big|_{s=x} F(x, t) + \int_0^x G_{ss}(x, s) F(s, t) \, ds = 0. \end{aligned} \quad (1.5.24)$$



It follows from (1.5.19), (1.5.22), (1.5.24) and the equality

$$J_{xx}(x, t) - J_{tt}(x, t) - q(x)J(x, t) \equiv 0,$$

that

$$(G_{xx}(x, t) - G_{tt}(x, t) - q(x)G(x, t)) + \int_0^x (G_{xx}(x, s) - G_{ss}(x, s) - q(x)G(x, s))F(s, t) ds \equiv 0.$$

According to Lemma 1.5.7, this homogeneous equation has only the trivial solution, i.e.

$$G_{xx}(x, t) - G_{tt}(x, t) - q(x)G(x, t) = 0 \quad 0 < t < x. \quad (1.5.25)$$

Differentiating (1.3.11) twice, we get

$$\varphi'(x, \lambda) = -\rho \sin \rho x + G(x, x) \cos \rho x + \int_0^x G_x(x, t) \cos \rho t dt, \quad (1.5.26)$$

$$\begin{aligned} \varphi''(x, \lambda) &= -\rho^2 \cos \rho x - G(x, x) \rho \sin \rho x \\ &+ \left( \frac{dG(x, x)}{dx} + \frac{\partial G(x, t)}{\partial x} \Big|_{t=x} \right) \cos \rho x + \int_0^x G_{xx}(x, t) \cos \rho t dt. \end{aligned} \quad (1.5.27)$$

On the other hand, integrating by parts twice, we obtain

$$\begin{aligned} \lambda \varphi(x, \lambda) &= \rho^2 \cos \rho x + \rho^2 \int_0^x G(x, t) \cos \rho t dt = \rho^2 \cos \rho x + G(x, x) \rho \sin \rho x \\ &+ \frac{\partial G(x, t)}{\partial t} \Big|_{t=x} \cos \rho x - \frac{\partial G(x, t)}{\partial t} \Big|_{t=0} - \int_0^x G_{tt}(x, t) \cos \rho t dt. \end{aligned}$$

Together with (1.3.11) and (1.5.27) this gives

$$\begin{aligned} \varphi''(x, \lambda) + \lambda \varphi(x, \lambda) - q(x) \varphi(x, \lambda) &= \left( 2 \frac{dG(x, x)}{dx} - q(x) \right) \cos \rho x - \frac{\partial G(x, t)}{\partial t} \Big|_{t=0} \\ &+ \int_0^x (G_{xx}(x, t) - G_{tt}(x, t) - q(x)G(x, t)) \cos \rho t dt. \end{aligned}$$

Taking (1.5.13), (1.5.23) and (1.5.25) into account, we arrive at (1.5.17). The relations (1.5.18) follow from (1.3.11) and (1.5.26) for  $x = 0$ .

2) Let us now consider the general case when (1.5.8)-(1.5.9) hold. Then according to Lemma 1.5.4,  $a(x) \in W_2^1(0, 2\pi)$ . Denote by  $\tilde{\varphi}(x, \lambda)$  the solution of equation (1.1.1) under the conditions  $\tilde{\varphi}(0, \lambda) = 1$ ,  $\tilde{\varphi}'(0, \lambda) = h$ . Our goal is to prove that  $\tilde{\varphi}(x, \lambda) \equiv \varphi(x, \lambda)$ .

Choose numbers  $\{\rho_{n,(j)}, \alpha_{n,(j)}\}_{n \geq 0, j \geq 1}$  of the form

$$\rho_{n,(j)} = n + \frac{\omega}{\pi n} + \frac{\kappa_{n,(j)}}{n^2}, \quad \alpha_{n,(j)} = \frac{\pi}{2} + \frac{\omega_{n,(j)}}{n^2} + \frac{\kappa_{n1,(j)}}{n^2}, \quad \{\kappa_{n,(j)}\}, \{\kappa_{n1,(j)}\} \in \ell_2,$$

such that for  $j \rightarrow \infty$ ,

$$\Omega_j := \left( \sum_{n=0}^{\infty} |(n+1)\xi_{n,(j)}|^2 \right)^{1/2} \rightarrow 0, \quad \xi_{n,(j)} := |\rho_{n,(j)} - \rho_n| + |\alpha_{n,(j)} - \alpha_n|.$$

Denote

$$a_j(x) := \sum_{n=0}^{\infty} \left( \frac{\cos \rho_{n,(j)} x}{\alpha_{n,(j)}} - \frac{\cos nx}{\alpha_n^0} \right) \quad j \geq 1.$$

By virtue of Lemma 1.5.5,  $a_j(x) \in W_2^2(0, 2\pi)$ . Let  $G_j(x, t)$  be the solution of the Gelfand-Levitan equation

$$G_j(x, t) + F_j(x, t) + \int_0^x G_j(x, s) F_j(s, t) ds = 0, \quad 0 < t < x,$$

where  $F_j(x, t) = (a_j(x+t) + a_j(x-t))/2$ . Take

$$\begin{aligned} q_j(x) &:= 2 \frac{d}{dx} G_j(x, x), \quad h_j := G_j(0, 0), \\ \varphi_j(x, \lambda) &= \cos \rho x + \int_0^x G_j(x, t) \cos \rho t dt. \end{aligned} \quad (1.5.28)$$

Since  $a_j(x) \in W_2^2(0, 2\pi)$ , it follows from the first part of the proof of Lemma 1.5.8 that

$$-\varphi_j''(x, \lambda) + q_j(x) \varphi_j(x, \lambda) = \lambda \varphi_j(x, \lambda), \quad \varphi_j(0, \lambda) = 1, \quad \varphi_j'(0, \lambda) = h_j.$$

Further, by virtue of Lemma 1.5.6,

$$\lim_{j \rightarrow \infty} \|a_j(x) - a(x)\|_{W_2^1} = 0.$$

From this, taking Lemma 1.5.1 into account, we get

$$\lim_{j \rightarrow \infty} \max_{0 \leq t \leq x \leq \pi} |G_j(x, t) - G(x, t)| = 0, \quad (1.5.29)$$

$$\lim_{j \rightarrow \infty} \|q_j - q\|_{L_2} = 0, \quad \lim_{j \rightarrow \infty} h_j = h. \quad (1.5.30)$$

It follows from (1.3.11), (1.5.28) and (1.5.29) that

$$\lim_{j \rightarrow \infty} \max_{0 \leq x \leq \pi} \max_{|\lambda| \leq r} |\varphi_j(x, \lambda) - \varphi(x, \lambda)| = 0.$$

On the other hand, according to Lemma 1.5.3 and (1.5.30),

$$\lim_{j \rightarrow \infty} \max_{0 \leq x \leq \pi} \max_{|\lambda| \leq r} |\varphi_j(x, \lambda) - \tilde{\varphi}(x, \lambda)| = 0.$$

Consequently,  $\tilde{\varphi}(x, \lambda) \equiv \varphi(x, \lambda)$ , and Lemma 1.5.8 is proved.  $\square$

**Lemma 1.5.9** *The following relation holds*

$$H(x, t) = F(x, t) + \int_0^t G(t, u) F(x, u) du, \quad 0 \leq t \leq x, \quad (1.5.31)$$

where  $H(x, t)$  is defined in (1.5.12).

*Proof.* 1) First we assume that  $a(x) \in W_2^2(0, 2\pi)$ . Differentiating (1.5.12) twice, we calculate

$$-\rho \sin \rho x = \varphi'(x, \lambda) + H(x, x) \varphi(x, \lambda) + \int_0^x H_t(x, t) \varphi(t, \lambda) dt, \quad (1.5.32)$$

$$\begin{aligned}
-\rho^2 \cos \rho x &= \varphi''(x, \lambda) + H(x, x)\varphi'(x, \lambda) \\
&+ \left( \frac{dH(x, x)}{dx} + \frac{\partial H(x, t)}{\partial x} \Big|_{t=x} \right) \varphi(x, \lambda) + \int_0^x H_{xx}(x, t) \varphi(t, \lambda) dt.
\end{aligned} \tag{1.5.33}$$

On the other hand, it follows from (1.5.12) and (1.5.17) that

$$-\rho^2 \cos \rho x = \varphi''(x, \lambda) - q(x)\varphi(x, \lambda) + \int_0^x H(x, t)(\varphi''(t, \lambda) - q(t)\varphi(t, \lambda)) dt.$$

Integrating by parts twice and using (1.5.18), we infer

$$\begin{aligned}
-\rho^2 \cos \rho x &= \varphi''(x, \lambda) + H(x, x)\varphi'(x, \lambda) - \left( \frac{\partial H(x, t)}{\partial t} \Big|_{t=x} + q(x) \right) \varphi(x, \lambda) \\
&+ \left( \frac{\partial H(x, t)}{\partial t} \Big|_{t=0} - hH(x, 0) \right) + \int_0^x (H_{tt}(x, t) - q(t)H(x, t)) \varphi(t, \lambda) dt.
\end{aligned}$$

Together with (1.5.33) and

$$\frac{dH(x, x)}{dx} = \left( \frac{\partial H(x, t)}{\partial x} + \frac{\partial H(x, t)}{\partial t} \right) \Big|_{t=x}$$

this yields

$$b_0(x) + b_1(x)\varphi(x, \lambda) + \int_0^x b(x, t)\varphi(t, \lambda) dt = 0, \tag{1.5.34}$$

where

$$\left. \begin{aligned} b_0(x) &:= - \left( \frac{\partial H(x, t)}{\partial t} \Big|_{t=0} - hH(x, 0) \right), & b_1(x) &:= 2 \frac{dH(x, x)}{dx} + q(x), \\ b(x, t) &:= H_{xx}(x, t) - H_{tt}(x, t) + q(t)H(x, t). \end{aligned} \right\} \tag{1.5.35}$$

Substituting (1.3.11) into (1.5.34) we obtain

$$b_0(x) + b_1(x) \cos \rho x + \int_0^x B(x, t) \cos \rho t dt = 0, \tag{1.5.36}$$

where

$$B(x, t) = b(x, t) + b_1(x)G(x, t) + \int_t^x b(x, s)G(s, t) ds. \tag{1.5.37}$$

It follows from (1.5.36) for  $\rho = \left(n + \frac{1}{2}\right) \frac{\pi}{x}$  that

$$b_0(x) + \int_0^x B(x, t) \cos \left(n + \frac{1}{2}\right) \frac{\pi t}{x} dt = 0.$$

By the Riemann-Lebesgue lemma, the integral here tends to 0 as  $n \rightarrow \infty$ ; consequently  $b_0(x) = 0$ . Further, taking in (1.5.36)  $\rho = \frac{2n\pi}{x}$ , we get

$$b_1(x) + \int_0^x B(x, t) \cos \frac{2n\pi t}{x} dt = 0,$$

therefore, similarly,  $b_1(x) = 0$ . Hence, (1.5.36) takes the form

$$\int_0^x B(x, t) \cos \rho t \, dt = 0, \quad \rho \in \mathbb{C},$$

and consequently  $B(x, t) = 0$ . Therefore, (1.5.37) implies

$$b(x, t) + \int_t^x b(x, s) G(s, t) \, ds = 0.$$

From this it follows that  $b(x, t) = 0$ . Taking in (1.5.32)  $x = 0$ , we get

$$H(0, 0) = -h. \quad (1.5.38)$$

Since  $b_0(x) = b_1(x) = b(x, t) = 0$ , we conclude from (1.5.35) and (1.5.38) that the function  $H(x, t)$  solves the boundary value problem

$$\left. \begin{aligned} H_{xx}(x, t) - H_{tt}(x, t) + q(t)H(x, t) &= 0, \quad 0 \leq t \leq x, \\ H(x, x) &= -h - \frac{1}{2} \int_0^x q(t) \, dt, \quad \frac{\partial H(x, t)}{\partial t} \Big|_{t=0} - hH(x, 0) = 0. \end{aligned} \right\} \quad (1.5.39)$$

The inverse assertion is also valid, namely: If a function  $H(x, t)$  satisfies (1.5.39) then (1.5.12) holds.

Indeed, denote

$$\gamma(x, \lambda) := \varphi(x, \lambda) + \int_0^x H(x, t) \varphi(t, \lambda) \, dt.$$

By similar arguments as above one can calculate

$$\begin{aligned} \gamma''(x, \lambda) + \lambda \gamma(x, \lambda) &= \left( 2 \frac{dH(x, x)}{dx} + q(x) \right) \varphi(x, \lambda) - \left( \frac{\partial H(x, t)}{\partial t} \Big|_{t=0} - hH(x, 0) \right) \\ &\quad + \int_0^x (H_{xx}(x, t) - H_{tt}(x, t) + q(t)H(x, t)) \varphi(t, \lambda) \, dt. \end{aligned}$$

In view of (1.5.39) we get  $\gamma''(x, \lambda) + \lambda \gamma(x, \lambda) = 0$ . Clearly,  $\gamma(0, \lambda) = 1$ ,  $\gamma'(0, \lambda) = 0$ . Hence  $\gamma(x, \lambda) = \cos \rho x$ , i.e. (1.5.12) holds.

Denote

$$\tilde{H}(x, t) := F(x, t) + \int_0^t G(t, u) F(x, u) \, du. \quad (1.5.40)$$

Let us show that the function  $\tilde{H}(x, t)$  satisfies (1.5.39).

(i) Differentiating (1.5.40) with respect to  $t$  and then taking  $t = 0$ , we get

$$\frac{\partial \tilde{H}(x, t)}{\partial t} \Big|_{t=0} = G(0, 0) F(x, 0) = hF(x, 0).$$

Since  $\tilde{H}(x, 0) = F(x, 0)$ , this yields

$$\frac{\partial \tilde{H}(x, t)}{\partial t} \Big|_{t=0} - h\tilde{H}(x, 0) = 0.$$

(ii) It follows from (1.3.11) and (1.5.40) that

$$\tilde{H}(x, x) = F(x, x) + \int_0^x G(x, u)F(x, u) du = -G(x, x),$$

i.e. according to (1.5.13)

$$\tilde{H}(x, x) = -h - \frac{1}{2} \int_0^x q(t) dt.$$

(iii) Using (1.5.40) again, we calculate

$$\begin{aligned} \tilde{H}_{tt}(x, t) &= F_{tt}(x, t) + \frac{dG(t, t)}{dt} F(x, t) + G(t, t) F_t(x, t) \\ &\quad + \frac{\partial G(t, u)}{\partial t} \Big|_{u=t} F(x, t) + \int_0^t G_{tt}(t, u) F(x, u) du, \\ \tilde{H}_{xx}(x, t) &= F_{xx}(x, t) + \int_0^t G(t, u) F_{xx}(x, u) du \\ &= F_{xx}(x, t) + \int_0^t G(t, u) F_{uu}(x, u) du = F_{xx}(x, t) + G(t, t) F_t(x, t) \\ &\quad - \frac{\partial G(t, u)}{\partial u} \Big|_{u=t} F(x, t) + \int_0^t G_{uu}(t, u) F(x, u) du. \end{aligned}$$

Consequently

$$\begin{aligned} \tilde{H}_{xx}(x, t) - \tilde{H}_{tt}(x, t) + q(t) \tilde{H}(x, t) &= \left( q(t) - 2 \frac{dG(t, t)}{dt} \right) F(x, t) \\ &\quad - \int_0^t (G_{tt}(t, u) - G_{uu}(t, u) - q(t) G(t, u)) dt. \end{aligned}$$

In view of (1.5.13) and (1.5.25) this yields

$$\tilde{H}_{xx}(x, t) - \tilde{H}_{tt}(x, t) + q(t) \tilde{H}(x, t) = 0.$$

Since  $\tilde{H}(x, t)$  satisfies (1.5.39), then, as was shown above,

$$\cos \rho x = \varphi(x, \lambda) + \int_0^x \tilde{H}(x, t) \varphi(t, \lambda) dt.$$

Comparing this relation with (1.5.12) we conclude that

$$\int_0^x (\tilde{H}(x, t) - H(x, t)) \varphi(t, \lambda) dt = 0 \quad \text{for all } \lambda,$$

i.e.  $\tilde{H}(x, t) = H(x, t)$ .

2) Let us now consider the general case when (1.5.8)-(1.5.9) hold. Then  $a(x) \in W_2^1(0, 2\pi)$ . Repeating the arguments of the proof of Lemma 1.5.8, we construct numbers  $\{\rho_{n,(j)}, \alpha_{n,(j)}\}_{n \geq 0, j \geq 1}$ , and functions  $a_j(x) \in W_2^2(0, 2\pi)$ ,  $j \geq 1$ , such that

$$\lim_{j \rightarrow \infty} \|a_j(x) - a(x)\|_{W_2^1} = 0.$$

Then

$$\lim_{j \rightarrow \infty} \max_{0 \leq t \leq x \leq \pi} |F_j(x, t) - F(x, t)| = 0,$$

and (1.5.29) is valid. Similarly,

$$\lim_{j \rightarrow \infty} \max_{0 \leq t \leq x \leq \pi} |H_j(x, t) - H(x, t)| = 0.$$

It was proved above that

$$H_j(x, t) = F_j(x, t) + \int_0^t G_j(t, u) F_j(x, u) du.$$

As  $j \rightarrow \infty$ , we arrive at (1.5.31), and Lemma 1.5.9 is proved.  $\square$

**Lemma 1.5.10.** *For each function  $g(x) \in L_2(0, \pi)$ ,*

$$\int_0^\pi g^2(x) dx = \sum_{n=0}^{\infty} \frac{1}{\alpha_n} \left( \int_0^\pi g(t) \varphi(t, \lambda_n) dt \right)^2. \quad (1.5.41)$$

*Proof.* Denote

$$Q(\lambda) := \int_0^\pi g(t) \varphi(t, \lambda) dt.$$

It follows from (1.3.11) that

$$Q(\lambda) = \int_0^\pi h(t) \cos \rho t dt,$$

where

$$h(t) = g(t) + \int_t^\pi G(s, t) g(s) ds. \quad (1.5.42)$$

Similarly, in view of (1.5.12),

$$g(t) = h(t) + \int_t^\pi H(s, t) h(s) ds. \quad (1.5.43)$$

Using (1.5.42) we calculate

$$\begin{aligned} \int_0^\pi h(t) F(x, t) dt &= \int_0^\pi \left( g(t) + \int_t^\pi G(u, t) g(u) du \right) F(x, t) dt \\ &= \int_0^\pi g(t) \left( F(x, t) + \int_0^t G(t, u) F(x, u) du \right) dt \\ &= \int_0^x g(t) \left( F(x, t) + \int_0^t G(t, u) F(x, u) du \right) dt + \int_x^\pi g(t) \left( F(x, t) + \int_0^t G(t, u) F(x, u) du \right) dt. \end{aligned}$$

From this, by virtue of (1.5.31) and (1.5.11), we derive

$$\int_0^\pi h(t) F(x, t) dt = \int_0^x g(t) H(x, t) dt - \int_x^\pi g(t) G(t, x) dt. \quad (1.5.44)$$

It follows from (1.5.10) and Parseval's equality that

$$\int_0^\pi h^2(t) dt + \int_0^\pi \int_0^\pi h(x) h(t) F(x, t) dx dt$$

$$\begin{aligned}
&= \int_0^\pi h^2(t) dt + \sum_{n=0}^\infty \left( \frac{1}{\alpha_n} \left( \int_0^\pi h(t) \cos \rho_n t dt \right)^2 - \frac{1}{\alpha_n^0} \left( \int_0^\pi h(t) \cos nt dt \right)^2 \right) \\
&= \sum_{n=0}^\infty \frac{Q^2(n^2)}{\alpha_n^0} + \sum_{n=0}^\infty \left( \frac{Q^2(\lambda_n)}{\alpha_n} - \frac{Q^2(n^2)}{\alpha_n^0} \right) = \sum_{n=0}^\infty \frac{Q^2(\lambda_n)}{\alpha_n}.
\end{aligned}$$

Using (1.5.44) we get

$$\begin{aligned}
\sum_{n=0}^\infty \frac{Q^2(\lambda_n)}{\alpha_n} &= \int_0^\pi h^2(t) dt + \int_0^\pi h(x) \left( \int_0^x g(t) H(x, t) dt \right) dx - \int_0^\pi h(x) \left( \int_x^\pi g(t) G(t, x) dt \right) dx \\
&= \int_0^\pi h^2(t) dt + \int_0^\pi g(t) \left( \int_t^\pi H(x, t) h(x) dx \right) dt - \int_0^\pi h(x) \left( \int_x^\pi g(t) G(t, x) dt \right) dx.
\end{aligned}$$

In view of (1.5.42) and (1.5.43) we infer

$$\sum_{n=0}^\infty \frac{Q^2(\lambda_n)}{\alpha_n} = \int_0^\pi h^2(t) dt + \int_0^\pi g(t)(g(t) - h(t)) dt - \int_0^\pi h(x)(h(x) - g(x)) dx = \int_0^\pi g^2(t) dt,$$

i.e. (1.5.41) is valid.  $\square$

**Corollary 1.5.1.** *For arbitrary functions  $f(x), g(x) \in L_2(0, \pi)$ ,*

$$\int_0^\pi f(x)g(x) dx = \sum_{n=0}^\infty \frac{1}{\alpha_n} \int_0^\pi f(t)\varphi(t, \lambda_n) dt \int_0^\pi g(t)\varphi(t, \lambda_n) dt. \quad (1.5.45)$$

Indeed, (1.5.45) follows from (1.5.41) applied to the function  $f + g$ .

**Lemma 1.5.11** *The following relation holds*

$$\int_0^\pi \varphi(t, \lambda_k)\varphi(t, \lambda_n) dt = \begin{cases} 0, & n \neq k, \\ \alpha_n, & n = k. \end{cases} \quad (1.5.46)$$

*Proof.* 1) Let  $f(x) \in W_2^2[0, \pi]$ . Consider the series

$$f^*(x) = \sum_{n=0}^\infty c_n \varphi(x, \lambda_n), \quad (1.5.47)$$

where

$$c_n := \frac{1}{\alpha_n} \int_0^\pi f(x)\varphi(x, \lambda_n) dx. \quad (1.5.48)$$

Using Lemma 1.5.8 and integration by parts we calculate

$$\begin{aligned}
c_n &= \frac{1}{\alpha_n \lambda_n} \int_0^\pi f(x) \left( -\varphi''(x, \lambda_n) + q(x)\varphi(x, \lambda_n) \right) dx \\
&= \frac{1}{\alpha_n \lambda_n} \left( hf(0) - f'(0) + \varphi(\pi, \lambda_n)f'(\pi) - \varphi'(\pi, \lambda_n)f(\pi) + \int_0^\pi \varphi(x, \lambda_n)(-f''(x) + q(x)f(x)) dx \right).
\end{aligned}$$

Applying the asymptotic formulae (1.5.8) and (1.1.15) one can check that for  $n \rightarrow \infty$ ,

$$c_n = O\left(\frac{1}{n^2}\right), \quad \varphi(x, \lambda_n) = O(1),$$

uniformly for  $x \in [0, \pi]$ . Consequently the series (1.5.47) converges absolutely and uniformly on  $[0, \pi]$ . According to (1.5.45) and (1.5.48),

$$\begin{aligned} \int_0^\pi f(x)g(x) dx &= \sum_{n=0}^{\infty} c_n \int_0^\pi g(t)\varphi(t, \lambda_n) dt \\ &= \int_0^\pi g(t) \sum_{n=0}^{\infty} c_n \varphi(t, \lambda_n) dt = \int_0^\pi g(t)f^*(t) dt. \end{aligned}$$

Since  $g(x)$  is arbitrary, we obtain  $f^*(x) = f(x)$ , i.e.

$$f(x) = \sum_{n=0}^{\infty} c_n \varphi(x, \lambda_n). \quad (1.5.49)$$

2) Fix  $k \geq 0$ , and take  $f(x) = \varphi(x, \lambda_k)$ . Then, by virtue of (1.5.49)

$$\varphi(x, \lambda_k) = \sum_{n=0}^{\infty} c_{nk} \varphi(x, \lambda_n), \quad c_{nk} = \frac{1}{\alpha_n} \int_0^\pi \varphi(x, \lambda_k) \varphi(x, \lambda_n) dx.$$

Further, the system  $\{\cos \rho_n x\}_{n \geq 0}$  is minimal in  $L_2(0, \pi)$  (see Proposition 1.8.6 in Section 1.8), and consequently, in view of (1.3.11), the system  $\{\varphi(x, \lambda_n)\}_{n \geq 0}$  is also minimal in  $L_2(0, \pi)$ . Hence  $c_{nk} = \delta_{nk}$  ( $\delta_{nk}$  is the Kronecker delta), and we arrive at (1.5.46).  $\square$

**Lemma 1.5.12.** For all  $n, m \geq 0$ ,

$$\frac{\varphi'(\pi, \lambda_n)}{\varphi(\pi, \lambda_n)} = \frac{\varphi'(\pi, \lambda_m)}{\varphi(\pi, \lambda_m)}. \quad (1.5.50)$$

*Proof.* It follows from (1.1.34) that

$$(\lambda_n - \lambda_m) \int_0^\pi \varphi(x, \lambda_n) \varphi(x, \lambda_m) dx = \left( \varphi(x, \lambda_n) \varphi'(x, \lambda_m) - \varphi'(x, \lambda_n) \varphi(x, \lambda_m) \right) \Big|_0^\pi.$$

Taking (1.5.46) into account, we get

$$\varphi(\pi, \lambda_n) \varphi'(\pi, \lambda_m) - \varphi'(\pi, \lambda_n) \varphi(\pi, \lambda_m) = 0. \quad (1.5.51)$$

Clearly,  $\varphi(\pi, \lambda_n) \neq 0$  for all  $n \geq 0$ . Indeed, if we suppose that  $\varphi(\pi, \lambda_m) = 0$  for a certain  $m$ , then  $\varphi'(\pi, \lambda_m) \neq 0$ , and in view of (1.5.51),  $\varphi(\pi, \lambda_n) = 0$  for all  $n$ , which is impossible since  $\varphi(\pi, \lambda_n) = (-1)^n + O\left(\frac{1}{n}\right)$ .

Since  $\varphi(\pi, \lambda_n) \neq 0$  for all  $n \geq 0$ , (1.5.50) follows from (1.5.51).  $\square$

Denote

$$\tilde{H} = -\frac{\varphi'(\pi, \lambda_n)}{\varphi(\pi, \lambda_n)}.$$

Notice that (1.5.50) yields that  $\tilde{H}$  is independent of  $n$ . Hence

$$\varphi'(\pi, \lambda_n) + \tilde{H} \varphi(\pi, \lambda_n) = 0, \quad n \geq 0.$$



Together with Lemma 1.5.8 and (1.5.46) this gives that the numbers  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  are the spectral data for the constructed boundary value problem  $L(q(x), h, \tilde{H})$ . Clearly,  $\tilde{H} = H$  where  $H$  is defined by (1.5.14). Thus, Theorem 1.5.2 is proved.  $\square$

**Example 1.5.1.** Let  $\lambda_n = n^2$  ( $n \geq 0$ ),  $\alpha_n = \frac{\pi}{2}$  ( $n \geq 1$ ), and let  $\alpha_0 > 0$  be an arbitrary positive number. Denote  $a := \frac{1}{\alpha_0} - \frac{1}{\pi}$ . Let us use Algorithm 1.5.1:

- 1) By (1.5.10),  $F(x, t) \equiv a$ .
- 2) Solving equation (1.5.11) we get easily

$$G(x, t) = -\frac{a}{1 + ax}.$$

- 3) By (1.5.13)-(1.5.14),

$$q(x) = \frac{2a^2}{(1 + ax)^2}, \quad h = -a, \quad H = \frac{a}{1 + a\pi} = \frac{a\alpha_0}{\pi}.$$

By (1.3.11),

$$\varphi(x, \lambda) = \cos \rho x - \frac{a}{1 + ax} \frac{\sin \rho x}{\rho}.$$

**Remark 1.5.1.** Analogous results are also valid for other types of separated boundary conditions, i.e. for the boundary value problems  $L_1$ ,  $L^0$  and  $L_1^0$ . In particular, the following theorem holds.

**Theorem 1.5.3.** For real numbers  $\{\mu_n, \alpha_{n1}\}_{n \geq 0}$  to be the spectral data for a certain boundary value problem  $L_1(q(x), h)$  with  $q(x) \in L_2(0, \pi)$ , it is necessary and sufficient that  $\mu_n \neq \mu_m$  ( $n \neq m$ ),  $\alpha_{n1} > 0$ , and that (1.1.29)-(1.1.30) hold.

**1.5.3. Recovery of differential operators from two spectra.** Let  $\{\lambda_n\}_{n \geq 0}$  and  $\{\mu_n\}_{n \geq 0}$  be the eigenvalues of  $L$  and  $L_1$  respectively. Then the asymptotics (1.1.13) and (1.1.29) hold, and we have the representations (1.1.26) and (1.1.28) for the characteristic functions  $\Delta(\lambda)$  and  $d(\lambda)$  respectively.

**Theorem 1.5.4.** For real numbers  $\{\lambda_n, \mu_n\}_{n \geq 0}$  to be the spectra for certain boundary value problems  $L$  and  $L_1$  with  $q(x) \in L_2(0, \pi)$ , it is necessary and sufficient that (1.1.13), (1.1.29) and (1.1.33) hold. The function  $q(x)$  and the numbers  $h$  and  $H$  can be constructed by the following algorithm:

- (i) From the given numbers  $\{\lambda_n, \mu_n\}_{n \geq 0}$  calculate the numbers  $\alpha_n$  by (1.1.35), where  $\Delta(\lambda)$  and  $d(\lambda)$  are constructed by (1.1.26) and (1.1.28).
- (ii) From the numbers  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  construct  $q(x)$ ,  $h$  and  $H$  by Algorithm 1.5.1.

The necessity part of Theorem 1.5.4 was proved above, here we prove the sufficiency. Let real numbers  $\{\lambda_n, \mu_n\}_{n \geq 0}$  be given satisfying the conditions of Theorem 1.5.4. We construct the functions  $\Delta(\lambda)$  and  $d(\lambda)$  by (1.1.26) and (1.1.28), and calculate the numbers  $\alpha_n$  by (1.1.35).

Our plan is to use Theorem 1.5.2. For this purpose we should obtain the asymptotics for the numbers  $\alpha_n$ . This seems to be difficult because the functions  $\Delta(\lambda)$  and  $d(\lambda)$  are by construction infinite products. But fortunately, for calculating the asymptotics of  $\alpha_n$  one can also use Theorem 1.5.2, as an auxiliary assertion. Indeed, by virtue of Theorem 1.5.2

there exists a boundary value problem  $\tilde{L} = L(\tilde{q}(x), \tilde{h}, \tilde{H})$  with  $\tilde{q}(x) \in L_2(0, \pi)$  (not unique) such that  $\{\lambda_n\}_{n \geq 0}$  are the eigenvalues of  $\tilde{L}$ . Then  $\Delta(\lambda)$  is the characteristic function of  $\tilde{L}$ , and consequently, according to (1.1.22),

$$\dot{\Delta}(\lambda_n) = (-1)^{n+1} \frac{\pi}{2} + \frac{\kappa_n}{n}, \quad \{\kappa_n\} \in l_2.$$

Moreover,  $\text{sign } \dot{\Delta}(\lambda_n) = (-1)^{n+1}$ . Similarly, using Theorem 1.5.3, one can prove that

$$d(\lambda_n) = (-1)^n + \frac{\kappa_n}{n}, \quad \{\kappa_n\} \in l_2.$$

Moreover, taking (1.1.33) into account, we get  $\text{sign } d(\lambda_n) = (-1)^n$ . Hence by (1.1.35),

$$\alpha_n > 0, \quad \alpha_n = \frac{\pi}{2} + \frac{\kappa_{n1}}{n}, \quad \{\kappa_{n1}\} \in l_2.$$

Then, by Theorem 1.5.2 there exists a boundary value problem  $L = L(q(x), h, H)$  with  $q(x) \in L_2(0, \pi)$  such that  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  are spectral data of  $L$ . Denote by  $\{\tilde{\mu}_n\}_{n \geq 0}$  the eigenvalues of the boundary value problem  $L_1(q(x), h)$ . It remains to show that  $\mu_n = \tilde{\mu}_n$  for all  $n \geq 0$ .

Let  $\tilde{d}(\lambda)$  be the characteristic function of  $L_1$ . Then, by virtue of (1.1.35),  $\alpha_n = -\dot{\Delta}(\lambda_n)\tilde{d}(\lambda_n)$ . But, by construction,  $\alpha_n = -\dot{\Delta}(\lambda_n)d(\lambda_n)$ , and we conclude that

$$d(\lambda_n) = \tilde{d}(\lambda_n), \quad n \geq 0.$$

Consequently, the function

$$Z(\lambda) := \frac{d(\lambda) - \tilde{d}(\lambda)}{\Delta(\lambda)}$$

is entire in  $\lambda$  (after extending it continuously to all removable singularities). On the other hand, by virtue of (1.1.9),

$$|d(\lambda)| \leq C \exp(|\tau|\pi), \quad |\tilde{d}(\lambda)| \leq C \exp(|\tau|\pi).$$

Taking (1.1.18) into account, we get for a fixed  $\delta > 0$ ,

$$|Z(\lambda)| \leq \frac{C}{|\rho|}, \quad \lambda \in G_\delta, \quad |\rho| \geq \rho^*.$$

Using the maximum principle [con1, p.128] and Liouville's theorem [con1, p.77], we infer  $Z(\lambda) \equiv 0$ , i.e.  $d(\lambda) \equiv \tilde{d}(\lambda)$ , and consequently  $\mu_n = \tilde{\mu}_n$  for all  $n \geq 0$ .  $\square$

## 1.6. THE METHOD OF SPECTRAL MAPPINGS

The method of spectral mappings presented in this section is an effective tool for investigating a wide class of inverse problems not only for Sturm-Liouville operators, but also for other more complicated classes of operators such as differential operators of arbitrary orders, differential operators with singularities and/or turning points, pencils of operators and others. In the method of spectral mappings we use ideas of the contour integral method.

Moreover, we can consider the method of spectral mappings as a variant of the contour integral method which is adapted specially for solving inverse problems. In this section we apply the method of spectral mappings to the solution of the inverse problem for the Sturm-Liouville operator on a finite interval. The results obtained in Section 1.6 by the contour integral method are similar to the results of Section 1.5 obtained by the transformation operator method. However, the contour integral method is a more universal and perspective tool for the solution of various classes of inverse spectral problems.

In the method of spectral mappings we start from Cauchy's integral formula for analytic functions [con1, p.84]. We apply this theorem in the complex plane of the spectral parameter for specially constructed analytic functions having singularities connected with the given spectral characteristics (see the proof of Lemma 1.6.3). This allows us to reduce the inverse problem to the so-called *main equation* which is a linear equation in a corresponding Banach space of sequences. In Subsection 1.6.1 we give a derivation of the main equation, and prove its unique solvability. Using the solution of the main equation we provide an algorithm for the solution of the inverse problem. In Subsection 1.6.2 we give necessary and sufficient conditions for the solvability of the inverse problem by the method of spectral mappings. In Subsection 1.6.3 the inverse problem for the non-selfadjoint Sturm-Liouville operator is studied.

**1.6.1. The main equation of the inverse problem.** Consider the boundary value problem  $L = L(q(x), h, H)$  of the form (1.1.1)-(1.1.2) with real  $q(x) \in L_2(0, \pi)$ ,  $h$  and  $H$ . Let  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  be the spectral data of  $L$ ,  $\rho_n = \sqrt{\lambda_n}$ . Then (1.5.8)-(1.5.9) are valid.

In this section we shall solve the inverse problem of recovering  $L$  from the given spectral data by using ideas of the contour integral method.

We note that if  $y(x, \lambda)$  and  $z(x, \mu)$  are solutions of the equations  $\ell y = \lambda y$  and  $\ell z = \mu z$  respectively, then

$$\frac{d}{dx} \langle y, z \rangle = (\lambda - \mu)yz, \quad \langle y, z \rangle := yz' - y'z. \quad (1.6.1)$$

Denote

$$D(x, \lambda, \mu) := \frac{\langle \varphi(x, \lambda), \varphi(x, \mu) \rangle}{\lambda - \mu} = \int_0^x \varphi(t, \lambda) \varphi(t, \mu) dt. \quad (1.6.2)$$

The last identity follows from (1.6.1).

Let us choose a model boundary value problem  $\tilde{L} = L(\tilde{q}(x), \tilde{h}, \tilde{H})$  with real  $\tilde{q}(x) \in L_2(0, \pi)$ ,  $\tilde{h}$  and  $\tilde{H}$  such that  $\tilde{\omega} = \omega$  (take, for example,  $\tilde{q}(x) \equiv 0$ ,  $\tilde{h} = 0$ ,  $\tilde{H} = \omega$ , or  $\tilde{h} = \tilde{H} = 0$ ,  $\tilde{q}(x) \equiv 2\omega/\pi$ ). Let  $\{\tilde{\lambda}_n, \tilde{\alpha}_n\}_{n \geq 0}$  be the spectral data of  $\tilde{L}$ .

**Remark 1.6.1.** Without loss of generality one can assume that  $\omega = 0$ . This can be achieved by the shift of the spectrum  $\{\lambda_n\} \rightarrow \{\lambda_n + C\}$ , since if  $\{\lambda_n\}$  is the spectrum of  $L(q(x), h, H)$ , then  $\{\lambda_n + C\}$  is the spectrum of  $L(q(x) + C, h, H)$ . In this case  $\tilde{\omega} = 0$ , and one can take  $\tilde{q}(x) \equiv 0$ ,  $\tilde{h} = \tilde{H} = 0$ . However we will consider the general case when  $\omega$  is arbitrary.

Let

$$\xi_n := |\rho_n - \tilde{\rho}_n| + |\alpha_n - \tilde{\alpha}_n|.$$

Since  $\omega = \tilde{\omega}$ , it follows from (1.5.8) and the analogous formulae for  $\tilde{\rho}_n$  and  $\tilde{\alpha}_n$  that

$$\Omega := \left( \sum_{n=0}^{\infty} ((n+1)\xi_n)^2 \right)^{1/2} < \infty, \quad \sum_n \xi_n < \infty. \quad (1.6.3)$$

Denote in this section

$$\lambda_{n0} = \lambda_n, \lambda_{n1} = \tilde{\lambda}_n, \alpha_{n0} = \alpha_n, \alpha_{n1} = \tilde{\alpha}_n, \varphi_{ni}(x) = \varphi(x, \lambda_{ni}), \tilde{\varphi}_{ni}(x) = \tilde{\varphi}(x, \lambda_{ni}),$$

$$P_{ni,kj}(x) = \frac{1}{\alpha_{kj}} D(x, \lambda_{ni}, \lambda_{kj}), \tilde{P}_{ni,kj}(x) = \frac{1}{\alpha_{kj}} \tilde{D}(x, \lambda_{ni}, \lambda_{kj}), i, j \in \{0, 1\}, n, k \geq 0. \quad (1.6.4)$$

Then, according to (1.6.2),

$$P_{ni,kj}(x) = \frac{\langle \varphi_{ni}(x), \varphi_{kj}(x) \rangle}{\alpha_{kj}(\lambda_{ni} - \lambda_{kj})} = \frac{1}{\alpha_{kj}} \int_0^x \varphi_{ni}(t) \varphi_{kj}(t) dt.$$

$$\tilde{P}_{ni,kj}(x) = \frac{\langle \tilde{\varphi}_{ni}(x), \tilde{\varphi}_{kj}(x) \rangle}{\alpha_{kj}(\lambda_{ni} - \lambda_{kj})} = \frac{1}{\alpha_{kj}} \int_0^x \tilde{\varphi}_{ni}(t) \tilde{\varphi}_{kj}(t) dt.$$

Clearly,

$$P'_{ni,kj}(x) = \frac{1}{\alpha_{kj}} \varphi_{ni}(x) \varphi_{kj}(x), \quad \tilde{P}'_{ni,kj}(x) = \frac{1}{\alpha_{kj}} \tilde{\varphi}_{ni}(x) \tilde{\varphi}_{kj}(x). \quad (1.6.5)$$

Below we shall use the following version of Schwarz's lemma (see [con1, p.130]):

**Lemma 1.6.1.** *Let  $f(\rho)$  be an analytic function for  $|\rho - \rho^0| < a$  such that  $f(\rho^0) = 0$  and  $|f(\rho)| \leq A$ ,  $|\rho - \rho^0| < a$ . Then*

$$|f(\rho)| \leq \frac{A}{a} |\rho - \rho^0| \quad \text{for } |\rho - \rho^0| < a.$$

In order to obtain the solution of the inverse problem we need several auxiliary propositions.

**Lemma 1.6.2.** *The following estimates are valid for  $x \in [0, \pi]$ ,  $n, k \geq 0$ ,  $i, j, \nu = 0, 1$ ,*

$$|\varphi_{ni}^{(\nu)}(x)| \leq C(n+1)^\nu, \quad |\varphi_{n0}^{(\nu)}(x) - \varphi_{n1}^{(\nu)}(x)| \leq C\xi_n(n+1)^\nu, \quad (1.6.6)$$

$$\left. \begin{aligned} |P_{ni,kj}(x)| &\leq \frac{C}{|n-k|+1}, \quad |P_{ni,kj}^{(\nu+1)}(x)| \leq C(k+n+1)^\nu, \\ |P_{ni,k0}(x) - P_{ni,k1}(x)| &\leq \frac{C\xi_k}{|n-k|+1}, \quad |P_{n0,kj}(x) - P_{n1,kj}(x)| \leq \frac{C\xi_n}{|n-k|+1}, \\ |P_{n0,k0}(x) - P_{n1,k0}(x) - P_{n0,k1}(x) + P_{n1,k1}(x)| &\leq \frac{C\xi_n\xi_k}{|n-k|+1}. \end{aligned} \right\} \quad (1.6.7)$$

The analogous estimates are also valid for  $\tilde{\varphi}_{ni}(x)$ ,  $\tilde{P}_{ni,kj}(x)$ .

*Proof.* It follows from (1.1.9) and (1.5.8) that

$$|\varphi^{(\nu)}(x, \lambda_{ni})| \leq C(n+1)^\nu.$$

Moreover, for a fixed  $a > 0$ ,

$$|\varphi^{(\nu)}(x, \lambda)| \leq C(n+1)^\nu, \quad |\rho - \rho_{n1}| \leq a.$$

Applying Schwarz's lemma in the  $\rho$  - plane to the function  $f(\rho) := \varphi^{(\nu)}(x, \lambda) - \varphi^{(\nu)}(x, \lambda_{n1})$  with fixed  $\nu, n, x$  and  $a$ , we get

$$|\varphi^{(\nu)}(x, \lambda) - \varphi^{(\nu)}(x, \lambda_{n1})| \leq C(n+1)^\nu |\rho - \rho_{n1}|, \quad |\rho - \rho_{n1}| \leq a.$$

Consequently,

$$|\varphi_{n0}^{(\nu)}(x) - \varphi_{n1}^{(\nu)}(x)| \leq C(n+1)^\nu |\rho_{n0} - \rho_{n1}|,$$

and (1.6.6) is proved.

Let us show that

$$|D(x, \lambda, \lambda_{kj})| \leq \frac{C \exp(|\tau|x)}{|\rho \mp k| + 1}, \quad \lambda = \rho^2, \quad \pm \operatorname{Re} \rho \geq 0, \quad \tau := \operatorname{Im} \rho, \quad k \geq 0. \quad (1.6.8)$$

For definiteness, let  $\sigma := \operatorname{Re} \rho \geq 0$ . Take a fixed  $\delta_0 > 0$ . For  $|\rho - \rho_{kj}| \geq \delta_0$ , we have by virtue of (1.6.2), (1.1.9) and (1.5.8),

$$|D(x, \lambda, \lambda_{kj})| = \frac{|\langle \varphi(x, \lambda), \varphi(x, \lambda_{kj}) \rangle|}{|\lambda - \lambda_{kj}|} \leq C \exp(|\tau|x) \frac{|\rho| + |\rho_{kj}|}{|\rho^2 - \rho_{kj}^2|}.$$

Since

$$\frac{|\rho| + |\rho_{kj}|}{|\rho + \rho_{kj}|} \leq \frac{\sqrt{\sigma^2 + \tau^2} + |\rho_{kj}|}{\sqrt{\sigma^2 + \tau^2 + \rho_{kj}^2}} \leq \sqrt{2}$$

(it is used here that  $(a+b)^2 \leq 2(a^2 + b^2)$  for all real  $a, b$ ), we get

$$|D(x, \lambda, \lambda_{kj})| \leq \frac{C \exp(|\tau|x)}{|\rho - \rho_{kj}|}.$$

For  $|\rho - \rho_{kj}| \geq \delta_0$ ,

$$\frac{|\rho - k| + 1}{|\rho - \rho_{kj}|} \leq 1 + \frac{|\rho_{kj} - k| + 1}{|\rho - \rho_{kj}|} \leq C,$$

and consequently

$$\frac{1}{|\rho - \rho_{kj}|} \leq \frac{C}{|\rho - k| + 1}.$$

This yields (1.6.8) for  $|\rho - \rho_{kj}| \geq \delta_0$ . For  $|\rho - \rho_{kj}| \leq \delta_0$ ,

$$|D(x, \lambda, \lambda_{kj})| = \left| \int_0^x \varphi(t, \lambda) \varphi(t, \lambda_{kj}) dt \right| \leq C \exp(|\tau|x),$$

i.e. (1.6.8) is also valid for  $|\rho - \rho_{kj}| \leq \delta_0$ . Similarly one can show that

$$|D(x, \lambda, \mu)| \leq \frac{C \exp(|\tau|x)}{|\rho \mp \theta| + 1}, \quad \mu = \theta^2, \quad |\operatorname{Im} \theta| \leq C_0, \quad \pm \operatorname{Re} \rho \operatorname{Re} \theta \geq 0.$$

Using Schwarz's lemma we obtain

$$|D(x, \lambda, \lambda_{k1}) - D(x, \lambda, \lambda_{k0})| \leq \frac{C \xi_k \exp(|\tau|x)}{|\rho \mp k| + 1}, \quad \pm \operatorname{Re} \rho \geq 0, \quad k \geq 0. \quad (1.6.9)$$

In particular, this yields

$$|D(x, \lambda_{ni}, \lambda_{k1}) - D(x, \lambda_{ni}, \lambda_{k0})| \leq \frac{C \xi_k}{|n - k| + 1}.$$

Symmetrically,

$$|D(x, \lambda_{n1}, \lambda_{kj}) - D(x, \lambda_{n0}, \lambda_{kj})| \leq \frac{C \xi_n}{|n - k| + 1}.$$

Applying Schwarz's lemma to the function

$$Q_k(x, \lambda) := D(x, \lambda, \lambda_{k1}) - D(x, \lambda, \lambda_{k0})$$

for fixed  $k$  and  $x$ , we get

$$|D(x, \lambda_{n0}, \lambda_{k0}) - D(x, \lambda_{n1}, \lambda_{k0}) - D(x, \lambda_{n0}, \lambda_{k1}) + D(x, \lambda_{n1}, \lambda_{k1})| \leq \frac{C\xi_n\xi_k}{|n-k|+1}.$$

These estimates together with (1.6.4), (1.6.5) and (1.5.8) imply (1.6.7).  $\square$

**Lemma 1.6.3.** *The following relations hold*

$$\tilde{\varphi}(x, \lambda) = \varphi(x, \lambda) + \sum_{k=0}^{\infty} \left( \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k0}(x) \rangle}{\alpha_{k0}(\lambda - \lambda_{k0})} \varphi_{k0}(x) - \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle}{\alpha_{k1}(\lambda - \lambda_{k1})} \varphi_{k1}(x) \right), \quad (1.6.10)$$

$$\begin{aligned} & \frac{\langle \varphi(x, \lambda), \varphi(x, \mu) \rangle}{\lambda - \mu} - \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle}{\lambda - \mu} + \sum_{k=0}^{\infty} \left( \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k0}(x) \rangle}{\alpha_{k0}(\lambda - \lambda_{k0})} \frac{\langle \varphi_{k0}(x), \varphi(x, \mu) \rangle}{(\lambda_{k0} - \mu)} \right. \\ & \quad \left. - \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle}{\alpha_{k1}(\lambda - \lambda_{k1})} \frac{\langle \varphi_{k1}(x), \varphi(x, \mu) \rangle}{(\lambda_{k1} - \mu)} \right) = 0. \end{aligned} \quad (1.6.11)$$

Both series converge absolutely and uniformly with respect to  $x \in [0, \pi]$  and  $\lambda, \mu$  on compact sets.

*Proof.* 1) Denote  $\lambda' = \min_{ni} \lambda_{ni}$  and take a fixed  $\delta > 0$ . In the  $\lambda$ -plane we consider closed contours  $\gamma_N$  (with counterclockwise circuit) of the form:  $\gamma_N = \gamma_N^+ \cup \gamma_N^- \cup \gamma' \cup \Gamma'_N$ , where

$$\gamma_N^\pm = \{\lambda : \pm \operatorname{Im} \lambda = \delta, \operatorname{Re} \lambda \geq \lambda', |\lambda| \leq \left(N + \frac{1}{2}\right)^2\},$$

$$\gamma' = \{\lambda : \lambda - \lambda' = \delta \exp(i\alpha), \alpha \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)\},$$

$$\Gamma'_N = \Gamma_N \cap \{\lambda : |\operatorname{Im} \lambda| \leq \delta, \operatorname{Re} \lambda > 0\}, \quad \Gamma_N = \{\lambda : |\lambda| = \left(N + \frac{1}{2}\right)^2\}.$$

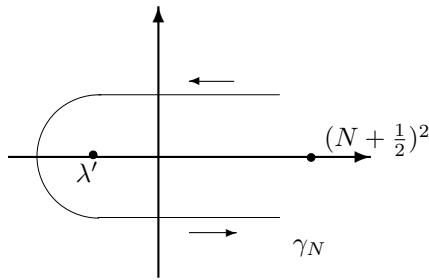


fig. 1.6.1

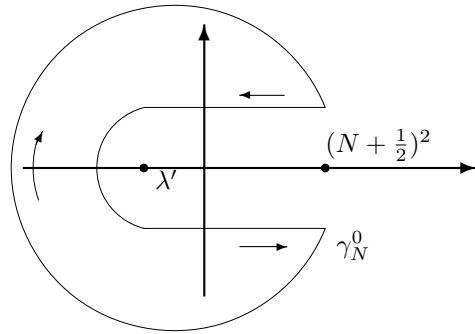


fig. 1.6.2

Denote  $\gamma_N^0 = \gamma_N^+ \cup \gamma_N^- \cup \gamma' \cup (\Gamma_N \setminus \Gamma'_N)$  (with clockwise circuit). Let  $P(x, \lambda) = [P_{jk}(x, \lambda)]_{j,k=1,2}$  be the matrix defined by (1.4.15). It follows from (1.4.16) and (1.4.9)

that for each fixed  $x$ , the functions  $P_{jk}(x, \lambda)$  are meromorphic in  $\lambda$  with simple poles  $\{\lambda_n\}$  and  $\{\tilde{\lambda}_n\}$ . By Cauchy's integral formula [con1, p.84],

$$P_{1k}(x, \lambda) - \delta_{1k} = \frac{1}{2\pi i} \int_{\gamma_N^0} \frac{P_{1k}(x, \xi) - \delta_{1k}}{\lambda - \xi} d\xi, \quad k = 1, 2,$$

where  $\lambda \in \text{int } \gamma_N^0$  and  $\delta_{jk}$  is the Kronecker delta. Hence

$$P_{1k}(x, \lambda) - \delta_{1k} = \frac{1}{2\pi i} \int_{\gamma_N} \frac{P_{1k}(x, \xi)}{\lambda - \xi} d\xi - \frac{1}{2\pi i} \int_{\Gamma_N} \frac{P_{1k}(x, \xi) - \delta_{1k}}{\lambda - \xi} d\xi,$$

where  $\Gamma_N$  is used with counterclockwise circuit. Substituting into (1.4.17) we obtain

$$\varphi(x, \lambda) = \tilde{\varphi}(x, \lambda) + \frac{1}{2\pi i} \int_{\gamma_N} \frac{\tilde{\varphi}(x, \lambda)P_{11}(x, \xi) + \tilde{\varphi}'(x, \lambda)P_{12}(x, \xi)}{\lambda - \xi} d\xi + \varepsilon_N(x, \lambda),$$

where

$$\varepsilon_N(x, \lambda) = -\frac{1}{2\pi i} \int_{\Gamma_N} \frac{\tilde{\varphi}(x, \lambda)(P_{11}(x, \xi) - 1) + \tilde{\varphi}'(x, \lambda)P_{12}(x, \xi)}{\lambda - \xi} d\xi.$$

By virtue of (1.4.18),

$$\lim_{N \rightarrow \infty} \varepsilon_N(x, \lambda) = 0 \quad (1.6.12)$$

uniformly with respect to  $x \in [0, \pi]$  and  $\lambda$  on compact sets. Taking (1.4.16) into account we calculate

$$\begin{aligned} \varphi(x, \lambda) &= \tilde{\varphi}(x, \lambda) + \frac{1}{2\pi i} \int_{\gamma_N} \left( \tilde{\varphi}(x, \lambda)(\varphi(x, \xi)\tilde{\Phi}'(x, \xi) - \Phi(x, \xi)\tilde{\varphi}'(x, \xi)) + \right. \\ &\quad \left. \tilde{\varphi}'(x, \lambda)(\Phi(x, \xi)\tilde{\varphi}(x, \xi) - \varphi(x, \xi)\tilde{\Phi}(x, \xi)) \right) \frac{d\xi}{\lambda - \xi} + \varepsilon_N(x, \lambda). \end{aligned}$$

In view of (1.4.9), this yields

$$\tilde{\varphi}(x, \lambda) = \varphi(x, \lambda) + \frac{1}{2\pi i} \int_{\gamma_N} \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \xi) \rangle}{\lambda - \xi} \hat{M}(\xi) \varphi(x, \xi) d\xi + \varepsilon_N(x, \lambda), \quad (1.6.13)$$

where  $\hat{M}(\lambda) = M(\lambda) - \tilde{M}(\lambda)$ , since the terms with  $S(x, \xi)$  vanish by Cauchy's theorem. It follows from (1.4.14) that

$$\text{Res}_{\xi=\lambda_{kj}} \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \xi) \rangle}{\lambda - \xi} \hat{M}(\xi) \varphi(x, \xi) = \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{kj}(x) \rangle}{\alpha_{kj}(\lambda - \lambda_{kj})} \varphi_{kj}(x).$$

Calculating the integral in (1.6.13) by the residue theorem [con1, p.112] and using (1.6.12) we arrive at (1.6.10).

2) Since

$$\frac{1}{\lambda - \mu} \left( \frac{1}{\lambda - \xi} - \frac{1}{\mu - \xi} \right) = \frac{1}{(\lambda - \xi)(\xi - \mu)},$$

we have by Cauchy's integral formula

$$\frac{P_{jk}(x, \lambda) - P_{jk}(x, \mu)}{\lambda - \mu} = \frac{1}{2\pi i} \int_{\gamma_N^0} \frac{P_{jk}(x, \xi)}{(\lambda - \xi)(\xi - \mu)} d\xi, \quad k, j = 1, 2; \quad \lambda, \mu \in \text{int } \gamma_N^0.$$

Acting in the same way as above and using (1.4.18)-(1.4.19) we obtain

$$\frac{P_{jk}(x, \lambda) - P_{jk}(x, \mu)}{\lambda - \mu} = \frac{1}{2\pi i} \int_{\gamma_N} \frac{P_{jk}(x, \xi)}{(\lambda - \xi)(\xi - \mu)} d\xi + \varepsilon_{Njk}(x, \lambda, \mu), \quad (1.6.14)$$

where  $\lim_{N \rightarrow \infty} \varepsilon_{Njk}(x, \lambda, \mu) = 0$ ,  $j, k = \overline{1, n}$ .

From (1.4.16) and (1.4.11) it follows that

$$\left. \begin{aligned} P_{11}(x, \lambda)\varphi'(x, \lambda) - P_{21}(x, \lambda)\varphi(x, \lambda) &= \tilde{\varphi}'(x, \lambda), \\ P_{22}(x, \lambda)\varphi(x, \lambda) - P_{12}(x, \lambda)\varphi'(x, \lambda) &= \tilde{\varphi}(x, \lambda), \end{aligned} \right\} \quad (1.6.15)$$

$$P(x, \lambda) \begin{bmatrix} y(x) \\ y'(x) \end{bmatrix} = \langle y(x), \tilde{\Phi}(x, \lambda) \rangle \begin{bmatrix} \varphi(x, \lambda) \\ \varphi'(x, \lambda) \end{bmatrix} - \langle y(x), \tilde{\varphi}(x, \lambda) \rangle \begin{bmatrix} \Phi(x, \lambda) \\ \Phi'(x, \lambda) \end{bmatrix} \quad (1.6.16)$$

for any  $y(x) \in C^1[0, 1]$ . Taking (1.6.14) and (1.6.16) into account, we calculate

$$\begin{aligned} \frac{P(x, \lambda) - P(x, \mu)}{\lambda - \mu} \begin{bmatrix} y(x) \\ y'(x) \end{bmatrix} &= \frac{1}{2\pi i} \int_{\gamma_N} \left( \langle y(x), \tilde{\Phi}(x, \xi) \rangle \begin{bmatrix} \varphi(x, \xi) \\ \varphi'(x, \xi) \end{bmatrix} - \right. \\ &\quad \left. \langle y(x), \tilde{\varphi}(x, \xi) \rangle \begin{bmatrix} \Phi(x, \xi) \\ \Phi'(x, \xi) \end{bmatrix} \right) \frac{d\xi}{(\lambda - \xi)(\xi - \mu)} + \varepsilon_N^0(x, \lambda, \mu), \quad \lim_{N \rightarrow \infty} \varepsilon_N^0(x, \lambda, \mu) = 0. \end{aligned} \quad (1.6.17)$$

According to (1.4.15),

$$P(x, \lambda) \begin{bmatrix} \tilde{\varphi}(x, \lambda) \\ \tilde{\varphi}'(x, \lambda) \end{bmatrix} = \begin{bmatrix} \varphi(x, \lambda) \\ \varphi'(x, \lambda) \end{bmatrix}.$$

Therefore,

$$\det \left( P(x, \lambda) \begin{bmatrix} \tilde{\varphi}(x, \lambda) \\ \tilde{\varphi}'(x, \lambda) \end{bmatrix}, \begin{bmatrix} \varphi(x, \mu) \\ \varphi'(x, \mu) \end{bmatrix} \right) = \langle \varphi(x, \lambda), \varphi(x, \mu) \rangle.$$

Furthermore, using (1.6.15) we get

$$\begin{aligned} &\det \left( P(x, \mu) \begin{bmatrix} \tilde{\varphi}(x, \mu) \\ \tilde{\varphi}'(x, \mu) \end{bmatrix}, \begin{bmatrix} \varphi(x, \mu) \\ \varphi'(x, \mu) \end{bmatrix} \right) \\ &= \tilde{\varphi}(x, \mu) (P_{11}(x, \mu)\varphi'(x, \mu) - P_{21}(x, \mu)\varphi(x, \mu)) - \tilde{\varphi}'(x, \mu) (P_{22}(x, \mu)\varphi(x, \mu) - P_{12}(x, \mu)\varphi'(x, \mu)) \\ &= \langle \tilde{\varphi}(x, \mu), \tilde{\varphi}(x, \mu) \rangle. \end{aligned}$$

Thus,

$$\begin{aligned} &\det \left( (P(x, \lambda) - P(x, \mu)) \begin{bmatrix} \tilde{\varphi}(x, \lambda) \\ \tilde{\varphi}'(x, \lambda) \end{bmatrix}, \begin{bmatrix} \varphi(x, \mu) \\ \varphi'(x, \mu) \end{bmatrix} \right) = \\ &\quad \langle \varphi(x, \lambda), \varphi(x, \mu) \rangle - \langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle. \end{aligned}$$

Consequently, (1.6.17) for  $y(x) = \tilde{\varphi}(x, \lambda)$  yields

$$\begin{aligned} &\frac{\langle \varphi(x, \lambda), \varphi(x, \mu) \rangle}{\lambda - \mu} - \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle}{\lambda - \mu} = \\ &\frac{1}{2\pi i} \int_{\gamma_N} \left( \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\Phi}(x, \xi) \rangle \langle \varphi(x, \xi), \varphi(x, \mu) \rangle}{(\lambda - \xi)(\xi - \mu)} - \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \xi) \rangle \langle \Phi(x, \xi), \varphi(x, \mu) \rangle}{(\lambda - \xi)(\xi - \mu)} \right) d\xi \end{aligned}$$



$$+\varepsilon_N^1(x, \lambda, \mu), \quad \lim_{N \rightarrow \infty} \varepsilon_N^1(x, \lambda, \mu) = 0.$$

By virtue of (1.4.9), (1.4.14) and the residue theorem, we arrive at (1.6.11).  $\square$

Analogously one can obtain the following relation

$$\tilde{\Phi}(x, \lambda) = \Phi(x, \lambda) + \sum_{k=0}^{\infty} \left( \frac{\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}_{k0}(x) \rangle}{\alpha_{k0}(\lambda - \lambda_{k0})} \varphi_{k0}(x) - \frac{\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle}{\alpha_{k1}(\lambda - \lambda_{k1})} \varphi_{k1}(x) \right). \quad (1.6.18)$$

It follows from the definition of  $\tilde{P}_{ni,kj}(x)$ ,  $P_{ni,kj}(x)$  and (1.6.10)-(1.6.11) that

$$\tilde{\varphi}_{ni}(x) = \varphi_{ni}(x) + \sum_{k=0}^{\infty} (\tilde{P}_{ni,k0}(x) \varphi_{k0}(x) - \tilde{P}_{ni,k1}(x) \varphi_{k1}(x)), \quad (1.6.19)$$

$$P_{ni,\ell j}(x) - \tilde{P}_{ni,\ell j}(x) + \sum_{k=0}^{\infty} (\tilde{P}_{ni,k0}(x) P_{k0,\ell j}(x) - \tilde{P}_{ni,k1}(x) P_{k1,\ell j}(x)) = 0. \quad (1.6.20)$$

Denote

$$\varepsilon_0(x) = \sum_{k=0}^{\infty} \left( \frac{1}{\alpha_{k0}} \tilde{\varphi}_{k0}(x) \varphi_{k0}(x) - \frac{1}{\alpha_{k1}} \tilde{\varphi}_{k1}(x) \varphi_{k1}(x) \right), \quad \varepsilon(x) = -2\varepsilon'_0(x). \quad (1.6.21)$$

**Lemma 1.6.4.** *The series in (1.6.21) converges absolutely and uniformly on  $[0, \pi]$ . The function  $\varepsilon_0(x)$  is absolutely continuous, and  $\varepsilon(x) \in L_2(0, \pi)$ .*

*Proof.* We rewrite  $\varepsilon_0(x)$  to the form

$$\varepsilon_0(x) = A_1(x) + A_2(x), \quad (1.6.22)$$

where

$$\left. \begin{aligned} A_1(x) &= \sum_{k=0}^{\infty} \left( \frac{1}{\alpha_{k0}} - \frac{1}{\alpha_{k1}} \right) \tilde{\varphi}_{k0}(x) \varphi_{k0}(x), \\ A_2(x) &= \sum_{k=0}^{\infty} \frac{1}{\alpha_{k1}} \left( (\tilde{\varphi}_{k0}(x) - \tilde{\varphi}_{k1}(x)) \varphi_{k0}(x) + \tilde{\varphi}_{k1}(x) (\varphi_{k0}(x) - \varphi_{k1}(x)) \right). \end{aligned} \right\} \quad (1.6.23)$$

It follows from (1.5.8), (1.6.3) and (1.6.6) that the series in (1.6.23) converge absolutely and uniformly on  $[0, \pi]$ , and

$$|A_j(x)| \leq C \sum_{k=0}^{\infty} \xi_k \leq C\Omega, \quad j = 1, 2. \quad (1.6.24)$$

Furthermore, using the asymptotic formulae (1.1.9), (1.5.8) and (1.6.3) we calculate

$$A'_1(x) = \sum_{k=0}^{\infty} \left( \frac{1}{\alpha_{k0}} - \frac{1}{\alpha_{k1}} \right) \frac{d}{dx} \left( \tilde{\varphi}_{k0}(x) \varphi_{k0}(x) \right) = \sum_{k=0}^{\infty} \gamma_k \left( \sin 2kx + \frac{\eta_k(x)}{k+1} \right),$$

where  $\{\gamma_k\} \in l_2$ , and  $\max_{0 \leq x \leq \pi} |\eta_k(x)| \leq C$  for  $k \geq 0$ . Hence  $A_1(x) \in W_2^1(0, \pi)$ . Similarly, we get  $A_2(x) \in W_2^1(0, \pi)$ , and consequently  $\varepsilon_0(x) \in W_2^1(0, \pi)$ .  $\square$

**Lemma 1.6.5.** *The following relations hold*

$$q(x) = \tilde{q}(x) + \varepsilon(x), \quad (1.6.25)$$

$$h = \tilde{h} - \varepsilon_0(0), \quad H = \tilde{H} + \varepsilon_0(\pi), \quad (1.6.26)$$

where  $\varepsilon(x)$  and  $\varepsilon_0(x)$  are defined by (1.6.21).

*Proof.* Differentiating (1.6.10) twice with respect to  $x$  and using (1.6.1) and (1.6.21) we get

$$\begin{aligned} & \tilde{\varphi}'(x, \lambda) - \varepsilon_0(x)\tilde{\varphi}(x, \lambda) = \varphi'(x, \lambda) \\ & + \sum_{k=0}^{\infty} \left( \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k0}(x) \rangle}{\alpha_{k0}(\lambda - \lambda_{k0})} \varphi'_{k0}(x) - \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle}{\alpha_{k1}(\lambda - \lambda_{k1})} \varphi'_{k1}(x) \right), \quad (1.6.27) \\ & \tilde{\varphi}''(x, \lambda) = \varphi''(x, \lambda) + \sum_{k=0}^{\infty} \left( \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k0}(x) \rangle}{\alpha_{k0}(\lambda - \lambda_{k0})} \varphi''_{k0}(x) - \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle}{\alpha_{k1}(\lambda - \lambda_{k1})} \varphi''_{k1}(x) \right) \\ & + 2\tilde{\varphi}(x, \lambda) \sum_{k=0}^{\infty} \left( \frac{1}{\alpha_{k0}} \tilde{\varphi}_{k0}(x) \varphi'_{k0}(x) - \frac{1}{\alpha_{k1}} \tilde{\varphi}_{k1}(x) \varphi'_{k1}(x) \right) + \\ & \sum_{k=0}^{\infty} \left( \frac{1}{\alpha_{k0}} (\tilde{\varphi}(x, \lambda) \tilde{\varphi}_{k0}(x))' \varphi_{k0}(x) - \frac{1}{\alpha_{k1}} (\tilde{\varphi}(x, \lambda) \tilde{\varphi}_{k1}(x))' \varphi_{k1}(x) \right). \end{aligned}$$

We replace here the second derivatives, using equation (1.1.1), and then replace  $\varphi(x, \lambda)$ , using (1.6.10). This yields

$$\begin{aligned} \tilde{q}(x)\tilde{\varphi}(x, \lambda) &= q(x)\tilde{\varphi}(x, \lambda) + \sum_{k=0}^{\infty} \left( \frac{1}{\alpha_{k0}} \langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k0}(x) \rangle \varphi_{k0}(x) - \frac{1}{\alpha_{k1}} \langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle \varphi_{k1}(x) \right) \\ &+ 2\tilde{\varphi}(x, \lambda) \sum_{k=0}^{\infty} \left( \frac{1}{\alpha_{k0}} \tilde{\varphi}_{k0}(x) \varphi'_{k0}(x) - \frac{1}{\alpha_{k1}} \tilde{\varphi}_{k1}(x) \varphi'_{k1}(x) \right) \\ &+ \sum_{k=0}^{\infty} \left( \frac{1}{\alpha_{k0}} (\tilde{\varphi}(x, \lambda) \tilde{\varphi}_{k0}(x))' \varphi_{k0}(x) - \frac{1}{\alpha_{k1}} (\tilde{\varphi}(x, \lambda) \tilde{\varphi}_{k1}(x))' \varphi_{k1}(x) \right). \end{aligned}$$

After canceling terms with  $\tilde{\varphi}'(x, \lambda)$  we arrive at (1.6.25).

Denote  $h_0 = -h$ ,  $h_\pi = H$ ,  $U_0 = U$ ,  $U_\pi = V$ . In (1.6.10) and (1.6.27) we put  $x = 0$  and  $x = \pi$ . Then

$$\begin{aligned} & \tilde{\varphi}'(a, \lambda) + (h_a - \varepsilon_0(a))\tilde{\varphi}(a, \lambda) = U_a(\varphi(x, \lambda)) \\ & + \sum_{k=0}^{\infty} \left( \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k0}(x) \rangle|_{x=a}}{\alpha_{k0}(\lambda - \lambda_{k0})} U_a(\varphi_{k0}) - \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle|_{x=a}}{\alpha_{k1}(\lambda - \lambda_{k1})} U_a(\varphi_{k1}) \right), \quad a = 0, \pi. \quad (1.6.28) \end{aligned}$$

Let  $a = 0$ . Since  $U_0(\varphi(x, \lambda)) = 0$ ,  $\tilde{\varphi}(0, \lambda) = 1$ ,  $\tilde{\varphi}'(0, \lambda) = -\tilde{h}_0$ , we get  $h_0 - \tilde{h}_0 - \varepsilon_0(0) = 0$ , i.e.  $h = \tilde{h} - \varepsilon_0(0)$ . Let  $a = \pi$ . Since

$$U_\pi(\varphi(x, \lambda)) = \Delta(\lambda), \quad U_\pi(\varphi_{k0}) = 0, \quad U_\pi(\varphi_{k1}) = \Delta(\lambda_{k1}),$$

$$\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle = \tilde{\varphi}(\pi, \lambda) \tilde{\Delta}(\mu) - \tilde{\varphi}(\pi, \mu) \tilde{\Delta}(\lambda),$$

it follows from (1.6.28) that

$$\tilde{\varphi}'(\pi, \lambda) + (h_\pi - \varepsilon_0(\pi))\tilde{\varphi}(\pi, \lambda) = \Delta(\lambda) + \sum_{k=0}^{\infty} \frac{\tilde{\varphi}_{k1}(\pi) \tilde{\Delta}(\lambda)}{\alpha_{k1}(\lambda - \lambda_{k1})} \Delta(\lambda_{k1}).$$

For  $\lambda = \lambda_{n1}$  this yields

$$\tilde{\varphi}'_{n1}(\pi) + (h_\pi - \varepsilon_0(\pi))\tilde{\varphi}_{n1}(\pi) = \Delta(\lambda_{n1})\left(1 + \frac{1}{\alpha_{n1}}\tilde{\varphi}_{n1}(\pi)\tilde{\Delta}_1(\lambda_{n1})\right),$$

where  $\tilde{\Delta}_1(\lambda) := \frac{d}{d\lambda}\tilde{\Delta}(\lambda)$ . By virtue of (1.1.6) and (1.1.8),

$$\tilde{\varphi}_{n1}(\pi)\tilde{\beta}_n = 1, \quad \tilde{\beta}_n\tilde{\alpha}_n = -\tilde{\Delta}_1(\lambda_{n1}),$$

i.e.

$$\frac{1}{\alpha_{n1}}\tilde{\varphi}_{n1}(\pi)\tilde{\Delta}_1(\lambda_{n1}) = -1,$$

Then

$$\tilde{\varphi}'_{n1}(\pi) + (h_\pi - \varepsilon_0(\pi))\tilde{\varphi}_{n1}(\pi) = 0.$$

On the other hand,

$$\tilde{\varphi}'_{n1}(\pi) + \tilde{h}_\pi\tilde{\varphi}_{n1}(\pi) = \tilde{\Delta}(\lambda_{n1}) = 0.$$

Thus,  $(h_\pi - \varepsilon_0(\pi) - \tilde{h}_\pi)\tilde{\varphi}_{n1}(\pi) = 0$ , i.e.  $h_\pi - \tilde{h}_\pi = \varepsilon_0(\pi)$ .  $\square$

**Remark 1.6.2.** For each fixed  $x \in [0, \pi]$ , the relation (1.6.19) can be considered as a system of linear equations with respect to  $\varphi_{ni}(x)$ ,  $n \geq 0$ ,  $i = 0, 1$ . But the series in (1.6.19) converges only "with brackets". Therefore, it is not convenient to use (1.6.19) as a main equation of the inverse problem. Below we will transfer (1.6.19) to a linear equation in a corresponding Banach space of sequences (see (1.6.33) or (1.6.34)).

Let  $V$  be a set of indices  $u = (n, i)$ ,  $n \geq 0$ ,  $i = 0, 1$ . For each fixed  $x \in [0, \pi]$ , we define the vector

$$\psi(x) = [\psi_u(x)]_{u \in V} = \begin{bmatrix} \psi_{n0}(x) \\ \psi_{n1}(x) \end{bmatrix}_{n \geq 0} = [\psi_{00}, \psi_{01}, \psi_{10}, \psi_{11}, \dots]^T$$

by the formulae

$$\begin{bmatrix} \psi_{n0}(x) \\ \psi_{n1}(x) \end{bmatrix} = \begin{bmatrix} \chi_n & -\chi_n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varphi_{n0}(x) \\ \varphi_{n1}(x) \end{bmatrix} = \begin{bmatrix} \chi_n(\varphi_{n0}(x) - \varphi_{n1}(x)) \\ \varphi_{n1}(x) \end{bmatrix},$$

$$\chi_n = \begin{cases} \xi_n^{-1}, & \xi_n \neq 0, \\ 0, & \xi_n = 0. \end{cases}$$

We also define the block matrix

$$H(x) = [H_{u,v}(x)]_{u,v \in V} = \begin{bmatrix} H_{n0,k0}(x) & H_{n0,k1}(x) \\ H_{n1,k0}(x) & H_{n1,k1}(x) \end{bmatrix}_{n,k \geq 0}, \quad u = (n, i), \quad v = (k, j)$$

by the formulae

$$\begin{bmatrix} H_{n0,k0}(x) & H_{n0,k1}(x) \\ H_{n1,k0}(x) & H_{n1,k1}(x) \end{bmatrix} = \begin{bmatrix} \chi_n & -\chi_n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_{n0,k0}(x) & P_{n0,k1}(x) \\ P_{n1,k0}(x) & P_{n1,k1}(x) \end{bmatrix} \begin{bmatrix} \xi_k & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \xi_k \chi_n (P_{n0,k0}(x) - P_{n1,k0}(x)) & \chi_n (P_{n0,k0}(x) - P_{n0,k1}(x) - P_{n1,k0}(x) + P_{n1,k1}(x)) \\ \xi_k P_{n1,k0}(x) & P_{n1,k0}(x) - P_{n1,k1}(x) \end{bmatrix}.$$

Analogously we define  $\tilde{\psi}(x)$ ,  $\tilde{H}(x)$  by replacing in the previous definitions  $\varphi_{ni}(x)$  by  $\tilde{\varphi}_{ni}(x)$  and  $P_{ni,kj}(x)$  by  $\tilde{P}_{ni,kj}(x)$ . It follows from (1.6.6) and (1.6.7) that

$$|\psi_{ni}^{(\nu)}(x)| \leq C(n+1)^\nu, \quad |H_{ni,kj}(x)| \leq \frac{C\xi_k}{(|n-k|+1)}, \quad |H_{ni,kj}^{(\nu+1)}(x)| \leq C(n+k+1)^\nu \xi_k. \quad (1.6.29)$$

Similarly,

$$|\tilde{\psi}_{ni}^{(\nu)}(x)| \leq C(n+1)^\nu, \quad |\tilde{H}_{ni,kj}(x)| \leq \frac{C\xi_k}{(|n-k|+1)}, \quad |\tilde{H}_{ni,kj}^{(\nu+1)}(x)| \leq C(n+k+1)^\nu \xi_k. \quad (1.6.30)$$

and also

$$|\tilde{H}_{ni,kj}(x) - \tilde{H}_{ni,kj}(x_0)| \leq C|x - x_0|\xi_k, \quad x, x_0 \in [0, \pi], \quad (1.6.31)$$

where  $C$  does not depend on  $x, x_0, n, i, j$  and  $k$ .

Let us consider the Banach space  $m$  of bounded sequences  $\alpha = [\alpha_u]_{u \in V}$  with the norm  $\|\alpha\|_m = \sup_{u \in V} |\alpha_u|$ . It follows from (1.6.29) and (1.6.30) that for each fixed  $x \in [0, \pi]$ , the operators  $E + \tilde{H}(x)$  and  $E - H(x)$  (here  $E$  is the identity operator), acting from  $m$  to  $m$ , are linear bounded operators, and

$$\|H(x)\|, \|\tilde{H}(x)\| \leq C \sup_n \sum_k \frac{\xi_k}{|n-k|+1} < \infty. \quad (1.6.32)$$

**Theorem 1.6.1.** *For each fixed  $x \in [0, \pi]$ , the vector  $\psi(x) \in m$  satisfies the equation*

$$\tilde{\psi}(x) = (E + \tilde{H}(x))\psi(x) \quad (1.6.33)$$

*in the Banach space  $m$ . Moreover, the operator  $E + \tilde{H}(x)$  has a bounded inverse operator, i.e. equation (1.6.33) is uniquely solvable.*

*Proof.* We rewrite (1.6.19) in the form

$$\begin{aligned} \tilde{\varphi}_{n0}(x) - \tilde{\varphi}_{n1}(x) &= \varphi_{n0}(x) - \varphi_{n1}(x) + \sum_{k=0}^{\infty} \left( (\tilde{P}_{n0,k0}(x) - \tilde{P}_{n1,k0}(x))(\varphi_{k0}(x) - \varphi_{k1}(x)) \right. \\ &\quad \left. + (\tilde{P}_{n0,k0}(x) - \tilde{P}_{n1,k0}(x) - \tilde{P}_{n0,k1}(x) + \tilde{P}_{n1,k1}(x))\varphi_{k1}(x) \right), \\ \tilde{\varphi}_{n1}(x) &= \varphi_{n1}(x) + \sum_{k=0}^{\infty} \left( \tilde{P}_{n1,k0}(x)(\varphi_{k0}(x) - \varphi_{k1}(x)) + (\tilde{P}_{n1,k0}(x) - \tilde{P}_{n1,k1}(x))\varphi_{k1}(x) \right). \end{aligned}$$

Taking into account our notations, we obtain

$$\tilde{\psi}_{ni}(x) = \psi_{ni}(x) + \sum_{k,j} \tilde{H}_{ni,kj}(x) \psi_{kj}(x), \quad (n, i), (k, j) \in V, \quad (1.6.34)$$

which is equivalent to (1.6.33). The series in (1.6.34) converges absolutely and uniformly for  $x \in [0, \pi]$ . Similarly, (1.6.20) takes the form

$$H_{ni,kj}(x) - \tilde{H}_{ni,kj}(x) + \sum_{\ell,s} (\tilde{H}_{ni,\ell s}(x) H_{\ell s,kj}(x)) = 0, \quad (n, i), (k, j), (\ell, s) \in V,$$

or

$$(E + \tilde{H}(x))(E - H(x)) = E.$$

Interchanging places for  $L$  and  $\tilde{L}$ , we obtain analogously

$$\psi(x) = (E - H(x))\tilde{\psi}(x), \quad (E - H(x))(E + \tilde{H}(x)) = E.$$

Hence the operator  $(E + \tilde{H}(x))^{-1}$  exists, and it is a linear bounded operator.  $\square$

Equation (1.6.33) is called *the main equation* of the inverse problem. Solving (1.6.33) we find the vector  $\psi(x)$ , and consequently, the functions  $\varphi_{ni}(x)$ . Since  $\varphi_{ni}(x) = \varphi(x, \lambda_{ni})$  are the solutions of (1.1.1), we can construct the function  $q(x)$  and the coefficients  $h$  and  $H$ . Thus, we obtain the following algorithm for the solution of Inverse Problem 1.4.1.

**Algorithm 1.6.1.** *Given the numbers  $\{\lambda_n, \alpha_n\}_{n \geq 0}$ .*

- (i) *Choose  $\tilde{L}$  such that  $\tilde{\omega} = \omega$ , and construct  $\tilde{\psi}(x)$  and  $\tilde{H}(x)$ .*
- (ii) *Find  $\psi(x)$  by solving equation (1.6.33).*
- (iii) *Calculate  $q(x)$ ,  $h$  and  $H$  by (1.6.21), (1.6.25)-(1.6.26).*

**1.6.2. Necessary and sufficient conditions.** In this item we provide necessary and sufficient conditions for the solvability of Inverse Problem 1.4.1.

**Theorem 1.6.2.** *For real numbers  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  to be the spectral data for a certain  $L(q(x), h, H)$  with  $q(x) \in L_2(0, \pi)$ , it is necessary and sufficient that the relations (1.5.8)-(1.5.9) hold. Moreover,  $q(x) \in W_2^N$  iff (1.1.23) holds. The boundary value problem  $L(q(x), h, H)$  can be constructed by Algorithm 1.6.1.*

The necessity part of Theorem 1.6.2 was proved above. We prove here the sufficiency. Let numbers  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  of the form (1.5.8)-(1.5.9) be given. Choose  $\tilde{L}$ , construct  $\tilde{\psi}(x)$ ,  $\tilde{H}(x)$  and consider the equation (1.6.33).

**Lemma 1.6.6.** *For each fixed  $x \in [0, \pi]$ , the operator  $E + \tilde{H}(x)$ , acting from  $m$  to  $m$ , has a bounded inverse operator, and the main equation (1.6.33) has a unique solution  $\psi(x) \in m$ .*

*Proof.* It is sufficient to prove that the homogeneous equation

$$(E + \tilde{H}(x))\beta(x) = 0, \tag{1.6.35}$$

where  $\beta(x) = [\beta_u(x)]_{u \in V}$ , has only the zero solution. Let  $\beta(x) \in m$  be a solution of (1.6.35), i.e.

$$\beta_{ni}(x) + \sum_{k,j} \tilde{H}_{ni,kj}(x)\beta_{kj}(x) = 0, \quad (n, i), (k, j) \in V. \tag{1.6.36}$$

Denote  $\gamma_{n1}(x) = \beta_{n1}(x)$ ,  $\gamma_{n0}(x) = \beta_{n0}(x)\xi_n + \beta_{n1}(x)$ . Then (1.6.36) takes the form

$$\gamma_{ni}(x) + \sum_{k=0}^{\infty} (\tilde{P}_{ni,k0}(x)\gamma_{k0}(x) - \tilde{P}_{ni,k1}(x)\gamma_{k1}(x)) = 0, \tag{1.6.37}$$

and

$$|\gamma_{ni}(x)| \leq C(x), \quad |\gamma_{n0}(x) - \gamma_{n1}(x)| \leq C(x)\xi_n. \tag{1.6.38}$$

Construct the functions  $\gamma(x, \lambda)$ ,  $\Gamma(x, \lambda)$  and  $B(x, \lambda)$  by the formulas

$$\gamma(x, \lambda) = - \sum_{k=0}^{\infty} \left( \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k0}(x) \rangle}{\alpha_{k0}(\lambda - \lambda_{k0})} \gamma_{k0}(x) - \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle}{\alpha_{k1}(\lambda - \lambda_{k1})} \gamma_{k1}(x) \right), \tag{1.6.39}$$

$$\Gamma(x, \lambda) = - \sum_{k=0}^{\infty} \left( \frac{\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}_{k0}(x) \rangle}{\alpha_{k0}(\lambda - \lambda_{k0})} \gamma_{k0}(x) - \frac{\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle}{\alpha_{k1}(\lambda - \lambda_{k1})} \gamma_{k1}(x) \right), \quad (1.6.40)$$

$$B(x, \lambda) = \overline{\gamma(x, \bar{\lambda})} \Gamma(x, \lambda). \quad (1.6.41)$$

The idea to consider  $B(x, \lambda)$  comes from the construction of Green's function for  $t = x$ :  $G(x, x, \lambda) = \varphi(x, \bar{\lambda}) \Phi(x, \lambda)$ .

In view of (1.6.2) the function  $\gamma(x, \lambda)$  is entire in  $\lambda$  for each fixed  $x$ . The functions  $\Gamma(x, \lambda)$  and  $B(x, \lambda)$  are meromorphic in  $\lambda$  with simple poles  $\lambda_{ni}$ . According to (1.6.2) and (1.6.37),  $\gamma(x, \lambda_{ni}) = \gamma_{ni}(x)$ . Let us show that

$$\operatorname{Res}_{\lambda=\lambda_{n0}} B(x, \lambda) = \frac{1}{\alpha_{n0}} |\gamma_{n0}(x)|^2, \quad \operatorname{Res}_{\lambda=\lambda_{n1}} B(x, \lambda) = 0. \quad (1.6.42)$$

Indeed, by virtue of (1.6.39), (1.6.40) and (1.4.9) we get

$$\Gamma(x, \lambda) = \tilde{M}(\lambda) \gamma(x, \lambda) - \sum_{k=0}^{\infty} \left( \frac{\langle \tilde{S}(x, \lambda), \tilde{\varphi}_{k0}(x) \rangle}{\alpha_{k0}(\lambda - \lambda_{k0})} \gamma_{k0}(x) - \frac{\langle \tilde{S}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle}{\alpha_{k1}(\lambda - \lambda_{k1})} \gamma_{k1}(x) \right). \quad (1.6.43)$$

Since  $\langle \tilde{S}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle|_{x=0} = -1$ , it follows from (1.6.1) that

$$\frac{\langle \tilde{S}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle}{\lambda - \mu} = -\frac{1}{\lambda - \mu} + \int_0^x \tilde{S}(t, \lambda) \tilde{\varphi}(t, \mu) dt.$$

Therefore, according to (1.6.43),

$$\operatorname{Res}_{\lambda=\lambda_{n0}} \Gamma(x, \lambda) = \frac{1}{\alpha_{n0}} \gamma_{n0}(x), \quad \operatorname{Res}_{\lambda=\lambda_{n1}} \Gamma(x, \lambda) = \frac{1}{\alpha_{n1}} \gamma_{n1}(x) - \frac{1}{\alpha_{n1}} \gamma_{n1}(x) = 0.$$

Together with (1.6.41) this yields (1.6.42).

Further, consider the integral

$$I_N^0(x) = \frac{1}{2\pi i} \int_{\Gamma_N} B(x, \lambda) d\lambda, \quad (1.6.44)$$

where  $\Gamma_N = \{\lambda : |\lambda| = (N + 1/2)^2\}$ . Let us show that

$$\lim_{N \rightarrow \infty} I_N^0(x) = 0. \quad (1.6.45)$$

Indeed, it follows from (1.6.2) and (1.6.39) that

$$\begin{aligned} -\gamma(x, \lambda) &= \sum_{k=0}^{\infty} \frac{1}{\alpha_{k0}} \tilde{D}(x, \lambda, \lambda_{k0}) (\gamma_{k0}(x) - \gamma_{k1}(x)) \\ &+ \sum_{k=0}^{\infty} \left( \frac{1}{\alpha_{k0}} - \frac{1}{\alpha_{k1}} \right) \tilde{D}(x, \lambda, \lambda_{k0}) \gamma_{k1}(x) + \sum_{k=0}^{\infty} \frac{1}{\alpha_{k1}} \left( \tilde{D}(x, \lambda, \lambda_{k0}) - \tilde{D}(x, \lambda, \lambda_{k1}) \right) \gamma_{k1}(x). \end{aligned}$$

For definiteness, let  $\sigma := \operatorname{Re} \rho \geq 0$ . By virtue of (1.5.8), (1.6.38), (1.6.8) and (1.6.9), we get

$$|\overline{\gamma(x, \bar{\lambda})}| = |\gamma(x, \lambda)| \leq C(x) \exp(|\tau|x) \sum_{k=0}^{\infty} \frac{\xi_k}{|\rho - k| + 1}, \quad \sigma \geq 0, \quad \tau = \operatorname{Im} \rho. \quad (1.6.46)$$

Similarly, using (1.6.40) we obtain for sufficiently large  $\rho^* > 0$ :

$$|\Gamma(x, \lambda)| \leq \frac{C(x)}{|\rho|} \exp(-|\tau|x) \sum_{k=0}^{\infty} \frac{\xi_k}{|\rho - k| + 1}, \quad \sigma \geq 0, \quad \rho \in G_\delta, \quad |\rho| \geq \rho^*.$$

Then

$$\begin{aligned} |B(x, \lambda)| &\leq \frac{C(x)}{|\rho|} \left( \sum_{k=0}^{\infty} \frac{\xi_k}{|\rho - k| + 1} \right)^2 \\ &\leq \frac{C(x)}{|\rho|} \sum_{k=0}^{\infty} ((k+1)\xi_k)^2 \cdot \sum_{k=0}^{\infty} \frac{1}{(|\rho - k| + 1)^2 (k+1)^2}. \end{aligned}$$

In view of (1.6.3), this implies

$$|B(x, \lambda)| \leq \frac{C(x)}{|\rho|} \sum_{k=0}^{\infty} \frac{1}{|\rho - k|^2 (k+1)^2}, \quad |\rho| = N + \frac{1}{2}, \quad \operatorname{Re} \rho \geq 0. \quad (1.6.47)$$

Since  $\rho = (N + 1/2) \exp(i\theta)$ ,  $\theta \in [-\pi/2, \pi/2]$ , we calculate

$$|\rho - k| = \left(N + \frac{1}{2}\right)^2 + k^2 - 2\left(N + \frac{1}{2}\right)k \cos \theta \geq \left(N + \frac{1}{2} - k\right)^2. \quad (1.6.48)$$

Furthermore,

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{1}{(k+1)^2 (N + \frac{1}{2} - k)^2} &\leq \sum_{k=0}^N \frac{1}{(k+1)^2 (N + \frac{1}{2} - k)^2} + \frac{1}{(N+2)^2} \sum_{k=1}^{\infty} \frac{1}{(k - \frac{1}{2})^2} \\ &= \sum_{k=1}^{N-1} \frac{1}{k^2 (N - k)^2} + O\left(\frac{1}{N^2}\right). \end{aligned}$$

Since

$$\frac{1}{k^2 (N - k)^2} = \frac{2}{kN^3} + \frac{1}{k^2 N^2} + \frac{2}{N^3 (N - k)} + \frac{1}{N^2 (N - k)^2},$$

we get

$$\sum_{k=1}^{N-1} \frac{1}{k^2 (N - k)^2} = \frac{4}{N^3} \sum_{k=1}^{N-1} \frac{1}{k} + \frac{2}{N^2} \sum_{k=1}^{N-1} \frac{1}{k^2}.$$

Hence

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)^2 (N + \frac{1}{2} - k)^2} = O\left(\frac{1}{N^2}\right).$$

Together with (1.6.47) and (1.6.48) this yields

$$|B(x, \lambda)| \leq \frac{C(x)}{|\rho|^3}, \quad \lambda \in \Gamma_N.$$

Substituting this estimate into (1.6.44) we arrive at (1.6.45).

On the other hand, calculating the integral in (1.6.44) by the residue theorem and taking (1.6.42) into account, we arrive at

$$\lim_{N \rightarrow \infty} \sum_{n=0}^N \frac{1}{\alpha_{n0}} |\gamma_{n0}(x)|^2 = 0.$$

Since  $\alpha_{n0} > 0$ , we get  $\gamma_{n0}(x) = 0$ .

Construct the functions

$$\begin{aligned}\Delta(\lambda) &:= \pi(\lambda_0 - \lambda) \prod_{n=1}^{\infty} \frac{\lambda_n - \lambda}{n^2}, \quad \lambda_n = \rho_n^2, \\ \tilde{\Delta}(\lambda) &:= -\pi\lambda \prod_{n=1}^{\infty} \frac{n^2 - \lambda}{n^2} = -\rho \sin \rho\pi, \quad \lambda = \rho^2, \\ f(x, \lambda) &:= \frac{\gamma(x, \lambda)}{\Delta(\lambda)}.\end{aligned}$$

It follows from the relation  $\gamma(x, \lambda_{n0}) = \gamma_{n0}(x) = 0$  that for each fixed  $x \in [0, \pi]$  the function  $f(x, \lambda)$  is entire in  $\lambda$ . On the other hand, we have

$$\frac{\tilde{\Delta}(\lambda)}{\Delta(\lambda)} = \prod_{n=0}^{\infty} \left(1 + \frac{\lambda_n - n^2}{\lambda - \lambda_n}\right).$$

Fix  $\delta > 0$ . By virtue of (1.1.17) and (1.5.8), the following estimates are valid in the sector  $\arg \lambda \in [\delta, 2\pi - \delta]$ :

$$|\tilde{\Delta}(\lambda)| \geq C|\rho| \exp(|\tau|\pi), \quad \left| \frac{\lambda_n - n^2}{\lambda - \lambda_n} \right| \leq \frac{C}{n^2}, \quad \tau := \operatorname{Im} \rho.$$

Consequently

$$|\Delta(\lambda)| \geq C|\rho| \exp(|\tau|\pi), \quad \arg \lambda \in [\delta, 2\pi - \delta].$$

Taking (1.6.46) into account we obtain

$$|f(x, \lambda)| \leq \frac{C(x)}{|\rho|}, \quad \arg \lambda \in [\delta, 2\pi - \delta].$$

From this, using the Phragmen-Lindelöf theorem [you1, p.80] and Liouville's theorem [con1, p.77] we conclude that  $f(x, \lambda) \equiv 0$ , i.e.  $\gamma(x, \lambda) \equiv 0$  and  $\gamma_{n1}(x) = \gamma(x, \lambda_{n1}) = 0$ . Hence  $\beta_{ni}(x) = 0$  for  $n \geq 0$ ,  $i = 0, 1$ , and Lemma 1.6.6 is proved.  $\square$

Let  $\psi(x) = [\psi_u(x)]_{u \in V}$  be the solution of the main equation (1.6.33).

**Lemma 1.6.7.** *For  $n \geq 0$ ,  $i = 0, 1$ , the following relations hold*

$$\psi_{ni}(x) \in C^1[0, \pi], \quad |\psi_{ni}^{(\nu)}(x)| \leq C(n+1)^\nu, \quad \nu = 0, 1, \quad x \in [0, \pi], \quad (1.6.49)$$

$$|\psi_{ni}(x) - \tilde{\psi}_{ni}(x)| \leq C\Omega\eta_n, \quad x \in [0, \pi], \quad (1.6.50)$$

$$|\psi'_{ni}(x) - \tilde{\psi}'_{ni}(x)| \leq C\Omega, \quad x \in [0, \pi], \quad (1.6.51)$$

where  $\Omega$  is defined by (1.6.3), and

$$\eta_n := \left( \sum_{k=0}^{\infty} \frac{1}{(k+1)^2(|n-k|+1)^2} \right)^{1/2}.$$

Here and below, one and the same symbol  $C$  denotes various positive constants which depend here only on  $\tilde{L}$  and  $C_0$ , where  $C_0 > 0$  is such that  $\Omega \leq C_0$ .



*Proof.* 1) Denote  $\tilde{R}(x) = (E + \tilde{H}(x))^{-1}$ . Fix  $x_0 \in [0, \pi]$  and consider the main equation (1.6.33) for  $x = x_0$ :

$$\tilde{\psi}(x_0) = (E + \tilde{H}(x_0))\psi(x_0).$$

By virtue of (1.6.31),

$$\|\tilde{H}(x) - \tilde{H}(x_0)\| \leq C|x - x_0| \sum_k \xi_k \leq C_1|x - x_0|, \quad x, x_0 \in [0, \pi], \quad C_1 > 0.$$

Take

$$\omega(x_0) := \frac{1}{2C_1\|\tilde{R}(x_0)\|}.$$

Then, for  $|x - x_0| \leq \omega(x_0)$ ,

$$\|\tilde{H}(x) - \tilde{H}(x_0)\| \leq \frac{1}{2\|\tilde{R}(x_0)\|}.$$

Using Lemma 1.5.1 we get for  $|x - x_0| \leq \omega(x_0)$ ,

$$\tilde{R}(x) - \tilde{R}(x_0) = \sum_{k=1}^{\infty} \left( \tilde{H}(x_0) - \tilde{H}(x) \right)^k \left( \tilde{R}(x_0) \right)^{k+1},$$

$$\|\tilde{R}(x) - \tilde{R}(x_0)\| \leq 2C_1\|\tilde{R}(x_0)\|^2|x - x_0|.$$

Consequently,  $\tilde{R}(x)$  is continuous with respect to  $x \in [0, \pi]$ , and

$$\|\tilde{R}(x)\| \leq C, \quad x \in [0, \pi].$$

We represent  $\tilde{R}(x)$  in the form  $\tilde{R}(x) = E - H(x)$ . Then,

$$\|H(x)\| \leq C, \quad x \in [0, \pi], \quad (1.6.52)$$

and

$$(E - H(x))(E + \tilde{H}(x)) = (E + \tilde{H}(x))(E - H(x)) = E. \quad (1.6.53)$$

In coordinates (1.6.53) takes the form

$$H_{ni,kj}(x) = \tilde{H}_{ni,kj}(x) - \sum_{\ell,s} H_{ni,\ell s}(x) \tilde{H}_{\ell s,kj}(x), \quad (n, i), (k, j), (\ell, s) \in V, \quad (1.6.54)$$

$$H_{ni,kj}(x) = \tilde{H}_{ni,kj}(x) - \sum_{\ell,s} \tilde{H}_{ni,\ell s}(x) H_{\ell s,kj}(x), \quad (n, i), (k, j), (\ell, s) \in V. \quad (1.6.55)$$

The functions  $H_{ni,kj}(x)$  are continuous for  $x \in [0, \pi]$ , and by virtue of (1.6.52), (1.6.54) and (1.6.30) we have

$$|H_{ni,kj}(x)| \leq C\xi_k, \quad x \in [0, \pi]. \quad (1.6.56)$$

Substituting this estimate into the right-hand side of (1.6.54) and (1.6.55) and using (1.6.30) we obtain more precisely,

$$|H_{ni,kj}(x)| \leq C\xi_k \left( \frac{1}{|n - k| + 1} + \Omega\eta_k \right), \quad x \in [0, \pi], \quad (n, i), (k, j) \in V, \quad (1.6.57)$$

$$|H_{ni,kj}(x)| \leq C\xi_k \left( \frac{1}{|n - k| + 1} + \Omega\eta_n \right), \quad x \in [0, \pi], \quad (n, i), (k, j) \in V. \quad (1.6.58)$$

We note that since

$$\begin{aligned} \sum_{k=0}^{\infty} |\eta_k|^2 &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(k+1)^2(|n-k|+1)^2} = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{1}{(k+1)^2(n-k+1)^2} \\ &+ \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} \frac{1}{(k+1)^2(k-n+1)^2} \leq 2 \left( \sum_{k=1}^{\infty} \frac{1}{k^2} \right)^2, \end{aligned}$$

it follows that  $\{\eta_k\} \in l_2$ .

Solving the main equation (1.6.33) we infer

$$\psi_{ni}(x) = \tilde{\psi}_{ni}(x) - \sum_{k,j} H_{ni,kj}(x) \tilde{\psi}_{kj}(x), \quad x \in [0, \pi], \quad (n, i), (k, j) \in V. \quad (1.6.59)$$

According to (1.6.30) and (1.6.58), the series in (1.6.59) converges absolutely and uniformly for  $x \in [0, \pi]$ ; the functions  $\psi_{ni}(x)$  are continuous for  $x \in [0, \pi]$ ,

$$|\psi_{ni}(x)| \leq C, \quad x \in [0, \pi], \quad (n, i) \in V,$$

and (1.6.50) is valid.

2) It follows from (1.6.53) that

$$H'(x) = (E - H(x)) \tilde{H}'(x) (E - H(x)), \quad (1.6.60)$$

the functions  $H_{ni,kj}(x)$  are continuously differentiable, and

$$|H'_{ni,kj}(x)| \leq C \xi_k. \quad (1.6.61)$$

Differentiating (1.6.59) we calculate

$$\psi'_{ni}(x) = \tilde{\psi}'_{ni}(x) - \sum_{k,j} H_{ni,kj}(x) \tilde{\psi}'_{kj}(x) - \sum_{k,j} H'_{ni,kj}(x) \tilde{\psi}_{kj}(x). \quad (1.6.62)$$

By virtue of (1.6.30), (1.6.57) and (1.6.61), the series in (1.6.62) converge absolutely and uniformly for  $x \in [0, \pi]$ ;  $\psi_{ni}(x) \in C^1[0, \pi]$ ,

$$|\psi'_{ni}(x)| \leq C(n+1), \quad x \in [0, \pi], \quad (n, i) \in V,$$

and (1.6.51) is valid.  $\square$

**Remark 1.6.3.** The estimates for  $\psi'_{ni}(x)$  can be also obtained in the following way. Differentiating (1.6.34) formally we get

$$\tilde{\psi}'_{ni}(x) - \sum_{k,j} \tilde{H}'_{ni,kj}(x) \psi_{kj}(x) = \psi'_{ni}(x) + \sum_{k,j} \tilde{H}_{ni,kj}(x) \psi'_{kj}(x). \quad (1.6.63)$$

Define

$$z(x) = [z_u(x)]_{u \in V}, \quad \tilde{z}(x) = [\tilde{z}_u(x)]_{u \in V}, \quad \tilde{A}(x) = [\tilde{A}_{u,v}(x)]_{u,v \in V}, \quad u = (n, i), v = (k, j),$$

by the formulas

$$z_{ni}(x) = \frac{1}{n+1} \psi'_{ni}(x), \quad \tilde{z}_{ni}(x) = \frac{1}{n+1} \left( \tilde{\psi}'_{ni}(x) - \sum_{k,j} \tilde{H}'_{ni,kj}(x) \psi_{kj}(x) \right),$$

$$\tilde{A}_{ni,kj}(x) = \frac{k+1}{n+1} \tilde{H}_{ni,kj}(x).$$

Then (1.6.63) takes the form

$$\tilde{z}(x) = (E + \tilde{A}(x))z(x)$$

or

$$\tilde{z}_{ni}(x) = z_{ni}(x) + \sum_{k,j} \tilde{A}_{ni,kj}(x) z_{kj}(x). \quad (1.6.64)$$

It follows from (1.6.30) that

$$\sum_{k,j} |\tilde{A}_{ni,kj}(x)| \leq C \sum_{k=0}^{\infty} \frac{(k+1)\xi_k}{(n+1)(|n-k|+1)} \leq \frac{C\Omega}{n+1} \left( \sum_{k=0}^{\infty} \frac{1}{(|n-k|+1)^2} \right)^{1/2}.$$

Since

$$\sum_{k=0}^{\infty} \frac{1}{(|n-k|+1)^2} = \sum_{k=0}^n \frac{1}{(n-k+1)^2} + \sum_{k=n+1}^{\infty} \frac{1}{(k-n+1)^2} \leq 2 \sum_{k=1}^{\infty} \frac{1}{k^2}, \quad (1.6.65)$$

we infer

$$\sum_{k,j} |\tilde{A}_{ni,kj}(x)| \leq \frac{C\Omega}{n+1}.$$

For each fixed  $x \in [0, \pi]$ , the operator  $\tilde{A}(x)$  is a linear bounded operator, acting from  $m$  to  $m$ , and

$$\|\tilde{A}(x)\| \leq C\Omega.$$

The homogeneous equation

$$u_{ni}(x) + \sum_{k,j} \tilde{A}_{ni,kj}(x) u_{kj}(x) = 0, \quad [u_{ni}(x)] \in m, \quad (1.6.66)$$

has only the trivial solution. Indeed, let  $[u_{ni}(x)] \in m$  be a solution of (1.6.66). Then the functions  $\beta_{ni}(x) := (n+1)u_{ni}(x)$  satisfy (1.6.36). Moreover, (1.6.36) gives

$$|\beta_{ni}(x)| \leq C(x) \sum_{k,j} |\tilde{H}_{ni,kj}(x)| (k+1) \leq C(x) \sum_{k=0}^{\infty} \frac{(k+1)\xi_k}{|n-k|+1} \leq C(x)\Omega \left( \sum_{k=0}^{\infty} \frac{1}{(|n-k|+1)^2} \right)^{1/2}.$$

Hence, in view of (1.6.65),  $|\beta_{ni}(x)| \leq C(x)$ , i.e.  $[\beta_{ni}(x)] \in m$ . By Lemma 1.6.6,  $\beta_{ni}(x) = 0$ , i.e.  $u_{ni}(x) = 0$ .

Using (1.6.64), by the same arguments as above, one can show that  $\psi_{ni}(x) \in C^1[0, \pi]$  and  $|\psi'_{ni}(x)| \leq C(n+1)$ , i.e. (1.6.49) is proved. Hence, it follows from (1.6.63) that

$$|\psi'_{ni}(x) - \tilde{\psi}'_{ni}(x)| \leq C \sum_{k=0}^{\infty} \frac{(k+1)\xi_k}{|n-k|+1} \leq C\Omega \left( \sum_{k=0}^{\infty} \frac{1}{(|n-k|+1)^2} \right)^{1/2}.$$

Taking (1.6.65) into account we arrive at (1.6.51).  $\square$

We define the functions  $\varphi_{ni}(x)$  by the formulae

$$\varphi_{n1}(x) = \psi_{n1}(x), \quad \varphi_{n0}(x) = \psi_{n0}(x)\xi_n + \psi_{n1}(x). \quad (1.6.67)$$

Then (1.6.19) and (1.6.6) are valid. By virtue of (1.6.67) and Lemma 1.6.7, we have

$$|\varphi_{ni}^{(\nu)}(x)| \leq C(n+1)^\nu, \quad \nu = 0, 1, \quad (1.6.68)$$

$$|\varphi_{ni}(x) - \tilde{\varphi}_{ni}(x)| \leq C\Omega\eta_n, \quad |\varphi'_{ni}(x) - \tilde{\varphi}'_{ni}(x)| \leq C\Omega. \quad (1.6.69)$$

Furthermore, we construct the functions  $\varphi(x, \lambda)$  and  $\Phi(x, \lambda)$  via (1.6.10), (1.6.18), and the boundary value problem  $L(q(x), h, H)$  via (1.6.21), (1.6.25)-(1.6.26).

Clearly,  $\varphi(x, \lambda_{ni}) = \varphi_{ni}(x)$ .

**Lemma 1.6.8.** *The following relation holds:*

$$q(x) \in L_2(0, \pi).$$

*Proof.* Here we follow the proof of Lemma 1.6.4 with necessary modifications. According to (1.6.22),  $\varepsilon_0(x) = A_1(x) + A_2(x)$ , where the functions  $A_j(x)$  are defined by (1.6.23). It follows from (1.5.8), (1.6.3) and (1.6.6) that the series in (1.6.23) converge absolutely and uniformly on  $[0, \pi]$ , and (1.6.24) holds. Furthermore,

$$\begin{aligned} A'_1(x) &= \sum_{k=0}^{\infty} \left( \frac{1}{\alpha_{k0}} - \frac{1}{\alpha_{k1}} \right) \frac{d}{dx} (\tilde{\varphi}_{k0}(x) \varphi_{k0}(x)) \\ &= 2 \sum_{k=0}^{\infty} \left( \frac{1}{\alpha_{k0}} - \frac{1}{\alpha_{k1}} \right) \tilde{\varphi}_{k0}(x) \tilde{\varphi}'_{k0}(x) + A(x) = \sum_{k=0}^{\infty} \gamma_k \left( \sin 2kx + \frac{\eta_k(x)}{k+1} \right) + A(x), \end{aligned}$$

where

$$A(x) = \sum_{k=0}^{\infty} \left( \frac{1}{\alpha_{k0}} - \frac{1}{\alpha_{k1}} \right) (\tilde{\varphi}_{k0}(x) (\varphi'_{k0}(x) - \tilde{\varphi}'_{k0}(x)) + \tilde{\varphi}'_{k0}(x) (\varphi_{k0}(x) - \tilde{\varphi}_{k0}(x))), \quad (1.6.70)$$

$$\{\gamma_k\} \in l_2, \quad \left( \sum_{k=0}^{\infty} |\gamma_k|^2 \right)^{1/2} \leq C\Omega, \quad \max_{0 \leq x \leq \pi} |\eta_k(x)| \leq C.$$

By virtue of (1.6.68), (1.6.69), (1.6.6) and (1.5.8), the series in (1.6.70) converges absolutely and uniformly on  $[0, \pi]$ , and

$$|A(x)| \leq C\Omega \left( \sum_{k=0}^{\infty} \xi_k + \sum_{k=0}^{\infty} (k+1) \xi_k \eta_k \right) \leq C\Omega^2 \left( 1 + \left( \sum_{k=0}^{\infty} |\eta_k|^2 \right)^{1/2} \right) \leq C\Omega^2,$$

since  $\{\eta_k\} \in l_2$ . Hence  $A_1(x) \in W_2^1[0, \pi]$ . Similarly we get  $A_2(x) \in W_2^1[0, \pi]$ . Consequently,  $\varepsilon_0(x) \in W_2^1(0, \pi)$ ,  $\varepsilon(x) \in L_2(0, \pi)$ , i.e.  $q(x) \in L_2(0, \pi)$ .  $\square$

**Lemma 1.6.9.** *The following relations hold:*

$$\ell \varphi_{ni}(x) = \lambda_{ni} \varphi_{ni}(x), \quad \ell \varphi(x, \lambda) = \lambda \varphi(x, \lambda), \quad \ell \Phi(x, \lambda) = \lambda \Phi(x, \lambda). \quad (1.6.71)$$

$$\varphi(0, \lambda) = 1, \quad \varphi'(0, \lambda) = h, \quad U(\Phi) = 1, \quad V(\Phi) = 0. \quad (1.6.72)$$

*Proof.* 1) Using (1.6.10), (1.6.18) and acting in the same way as in the proof of Lemma 1.6.5, we get

$$\tilde{U}_a(\tilde{\varphi}(x, \lambda)) = U_a(\varphi(x, \lambda))$$

$$+ \sum_{k=0}^{\infty} \left( \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k0}(x) \rangle_{|x=a}}{\alpha_{k0}(\lambda - \lambda_{k0})} U_a(\varphi_{k0}) - \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle_{|x=a}}{\alpha_{k1}(\lambda - \lambda_{k1})} U_a(\varphi_{k1}) \right), \quad a = 0, \pi, \quad (1.6.73)$$

$$\tilde{U}_a(\tilde{\Phi}(x, \lambda)) = U_a(\Phi(x, \lambda))$$

$$+ \sum_{k=0}^{\infty} \left( \frac{\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}_{k0}(x) \rangle_{|x=a}}{\alpha_{k0}(\lambda - \lambda_{k0})} U_a(\varphi_{k0}) - \frac{\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle_{|x=a}}{\alpha_{k1}(\lambda - \lambda_{k1})} U_a(\varphi_{k1}) \right), \quad a = 0, \pi, \quad (1.6.74)$$

where  $U_0 = U$ ,  $U_\pi = V$ .

Since  $\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{kj}(x) \rangle_{|x=0} = 0$ , it follows from (1.6.10) and (1.6.73) for  $x = 0$  that  $\varphi(0, \lambda) = 1$ ,  $U_0(\varphi) = 0$ , and consequently  $\varphi'(0, \lambda) = h$ . In (1.6.74) we put  $a = 0$ . Since  $U_0(\varphi) = 0$  we get  $U_0(\Phi) = 1$ .

2) In order to prove (1.6.71) we first assume that

$$\Omega_1 := \left( \sum_{k=0}^{\infty} ((k+1)^2 \xi_k)^2 \right)^{1/2} < \infty. \quad (1.6.75)$$

In this case, one can obtain from (1.6.60) by differentiation that

$$\begin{aligned} H''(x) &= (E - H(x)) \tilde{H}''(x) (E - H(x)) \\ &\quad - H'(x) \tilde{H}'(x) (E - H(x)) - (E - H(x)) \tilde{H}'(x) H(x). \end{aligned} \quad (1.6.76)$$

It follows from (1.6.30), (1.6.56) and (1.6.61) that the series in (1.6.76) converge absolutely and uniformly for  $x \in [0, \pi]$ ,  $H_{ni,kj}(x) \in C^2[0, \pi]$ , and

$$|H''_{ni,kj}(x)| \leq C(n+k+1)\xi_k. \quad (1.6.77)$$

Furthermore, using (1.6.59) we calculate

$$\tilde{\ell}\psi_{ni}(x) = \tilde{\ell}\tilde{\psi}_{ni}(x) + \sum_{k,j} H_{ni,kj}(x) \tilde{\ell}\tilde{\psi}_{kj}(x) + 2 \sum_{k,j} H'_{ni,kj}(x) \tilde{\psi}'_{kj}(x) + \sum_{k,j} H''_{ni,kj}(x) \tilde{\psi}_{kj}(x). \quad (1.6.78)$$

Since  $\tilde{\ell}\tilde{\varphi}_{ni}(x) = \lambda_{ni}\tilde{\varphi}_{ni}(x)$  it follows that

$$\tilde{\ell}\tilde{\psi}_{n0}(x) = \lambda_{n0}\tilde{\psi}_{n0}(x) + \chi_n(\lambda_{n0} - \lambda_{n1})\tilde{\varphi}_{n1}(x), \quad \tilde{\ell}\tilde{\psi}_{n1}(x) = \lambda_{n1}\tilde{\psi}_{n1}(x),$$

hence

$$\tilde{\ell}\tilde{\psi}_{ni}(x) \in C[0, \pi], \quad |\tilde{\ell}\tilde{\psi}_{ni}(x)| \leq C(n+1)^2, \quad (n, i) \in V.$$

From this and from (1.6.57), (1.6.61), (1.6.65), (1.6.77) and (1.6.30) we deduce that the series in (1.6.78) converge absolutely and uniformly for  $x \in [0, \pi]$ , and

$$\tilde{\ell}\psi_{ni}(x) \in C[0, \pi], \quad |\tilde{\ell}\psi_{ni}(x)| \leq C(n+1)^2, \quad (n, i) \in V.$$

On the other hand, it follows from the proof of Lemma 1.6.8 and from (1.6.75) that in our case  $q(x) - \tilde{q}(x) \in C[0, \pi]$ ; hence

$$\ell\psi_{ni}(x) \in C[0, \pi], \quad |\ell\psi_{ni}(x)| \leq C(n+1)^2, \quad (n, i) \in V.$$

Together with (1.6.67) this implies that

$$\ell\varphi_{ni}(x) \in C[0, \pi], \quad |\ell\varphi_{ni}(x)| \leq C(n+1)^2, \quad |\ell\varphi_{n0}(x) - \ell\varphi_{n1}(x)| \leq C\xi_n(n+1)^2, \quad (n, i) \in V.$$

Using (1.6.19) we calculate

$$\begin{aligned}
-\tilde{\varphi}_{ni}''(x) + q(x)\tilde{\varphi}_{ni}(x) &= \ell\varphi_{ni}(x) + \sum_{k=0}^{\infty} \left( \tilde{P}_{ni,k0}(x)\ell\varphi_{k0}(x) - \tilde{P}_{ni,k1}(x)\ell\varphi_{k1}(x) \right) \\
&\quad - 2\tilde{\varphi}_{ni}(x) \sum_{k=0}^{\infty} \left( \frac{1}{\alpha_{k0}}\tilde{\varphi}_{k0}(x)\varphi'_{k0}(x) - \frac{1}{\alpha_{k1}}\tilde{\varphi}_{k1}(x)\varphi'_{k1}(x) \right) \\
&\quad - \sum_{k=0}^{\infty} \left( \frac{1}{\alpha_{k0}}(\tilde{\varphi}_{ni}(x)\tilde{\varphi}_{k0}(x))'\varphi_{k0}(x) - \frac{1}{\alpha_{k1}}(\tilde{\varphi}_{ni}(x)\tilde{\varphi}_{k1}(x))'\varphi_{k1}(x) \right).
\end{aligned}$$

Taking (1.6.21) and (1.6.25) into account we derive

$$\begin{aligned}
\tilde{\ell}\tilde{\varphi}_{ni}(x) &= \ell\varphi_{ni}(x) + \sum_{k=0}^{\infty} \left( \tilde{P}_{ni,k0}(x)\ell\varphi_{k0}(x) - \tilde{P}_{ni,k1}(x)\ell\varphi_{k1}(x) \right) + \\
&\quad + \sum_{k=0}^{\infty} \left( \frac{1}{\alpha_{k0}}\langle \tilde{\varphi}_{ni}(x), \tilde{\varphi}_{k0}(x) \rangle \varphi_{k0}(x) - \frac{1}{\alpha_{k1}}\langle \tilde{\varphi}_{ni}(x), \tilde{\varphi}_{k1}(x) \rangle \varphi_{k1}(x) \right). \tag{1.6.79}
\end{aligned}$$

Using (1.6.10) and (1.6.18) we calculate similarly

$$\begin{aligned}
\tilde{\ell}\tilde{\varphi}(x, \lambda) &= \ell\varphi(x, \lambda) + \sum_{k=0}^{\infty} \left( \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k0}(x) \rangle}{\alpha_{k0}(\lambda - \lambda_{k0})} \ell\varphi_{k0}(x) - \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle}{\alpha_{k1}(\lambda - \lambda_{k1})} \ell\varphi_{k1}(x) \right) + \\
&\quad \sum_{k=0}^{\infty} \left( \frac{1}{\alpha_{k0}}\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k0}(x) \rangle \varphi_{k0}(x) - \frac{1}{\alpha_{k1}}\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle \varphi_{k1}(x) \right), \tag{1.6.80}
\end{aligned}$$

$$\begin{aligned}
\tilde{\ell}\tilde{\Phi}(x, \lambda) &= \ell\Phi(x, \lambda) + \sum_{k=0}^{\infty} \left( \frac{\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}_{k0}(x) \rangle}{\alpha_{k0}(\lambda - \lambda_{k0})} \ell\varphi_{k0}(x) - \frac{\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle}{\alpha_{k1}(\lambda - \lambda_{k1})} \ell\varphi_{k1}(x) \right) + \\
&\quad \sum_{k=0}^{\infty} \left( \frac{1}{\alpha_{k0}}\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}_{k0}(x) \rangle \varphi_{k0}(x) - \frac{1}{\alpha_{k1}}\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle \varphi_{k1}(x) \right). \tag{1.6.81}
\end{aligned}$$

It follows from (1.6.79) that

$$\begin{aligned}
\lambda_{ni}\tilde{\varphi}_{ni}(x) &= \ell\varphi_{ni}(x) + \sum_{k=0}^{\infty} \left( \tilde{P}_{ni,k0}(x)\ell\varphi_{k0}(x) - \tilde{P}_{ni,k1}(x)\ell\varphi_{k1}(x) \right) + \\
&\quad \sum_{k=0}^{\infty} \left( (\lambda_{ni} - \lambda_{k0})\tilde{P}_{ni,k0}(x)\varphi_{k0}(x) - (\lambda_{ni} - \lambda_{k1})\tilde{P}_{ni,k1}(x)\varphi_{k1}(x) \right),
\end{aligned}$$

and consequently we arrive at (1.6.37) where

$$\gamma_{ni}(x) := \ell\varphi_{ni}(x) - \lambda_{ni}\varphi_{ni}(x).$$

Then (1.6.36) is valid, where

$$\beta_{n1}(x) = \gamma_{n1}(x), \quad \beta_{n0}(x) = \chi_n(\gamma_{n0}(x) - \gamma_{n1}(x)), \quad \chi_n := \begin{cases} \xi_n^{-1}, & \xi_n \neq 0, \\ 0, & \xi_n = 0. \end{cases}$$

Moreover,

$$|\gamma_{ni}(x)| \leq C(n+1)^2, \quad |\gamma_{n0}(x) - \gamma_{n1}(x)| \leq C\xi_n(n+1)^2,$$

and hence

$$|\beta_{ni}(x)| \leq C(n+1)^2. \quad (1.6.82)$$

It follows from (1.6.36), (1.6.30), (1.6.82) and (1.6.75) that

$$|\beta_{ni}(x)| \leq \sum_{k,j} |H_{ni,kj}(x)\beta_{kj}(x)| \leq C \sum_{k=0}^{\infty} \frac{(k+1)^2 \xi_k}{|n-k|+1}.$$

Since

$$\frac{k+1}{(n+1)(|n-k|+1)} \leq 1,$$

we obtain

$$|\beta_{ni}(x)| \leq C(n+1) \sum_{k=0}^{\infty} (k+1)\xi_k \leq C(n+1). \quad (1.6.83)$$

Using (1.6.83) instead of (1.6.82) and repeating the arguments, we infer

$$|\beta_{ni}(x)| \leq \sum_{k,j} |H_{ni,kj}(x)\beta_{kj}(x)| \leq C \sum_{k=0}^{\infty} \frac{(k+1)\xi_k}{|n-k|+1} \leq C.$$

Then, by virtue of Lemma 1.6.6,  $\beta_{ni}(x) = 0$ , and consequently  $\gamma_{ni}(x) = 0$ . Thus,  $\ell\varphi_{ni}(x) = \lambda_{ni}\varphi_{ni}(x)$ .

Furthermore, it follows from (1.6.80) that

$$\begin{aligned} \lambda\tilde{\varphi}(x, \lambda) &= \ell\varphi(x, \lambda) + \sum_{k=0}^{\infty} \left( \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k0}(x) \rangle}{\alpha_{k0}(\lambda - \lambda_{k0})} \lambda_{k0}\varphi_{k0}(x) - \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle}{\alpha_{k1}(\lambda - \lambda_{k1})} \lambda_{k1}\varphi_{k1}(x) \right) + \\ &\quad \sum_{k=0}^{\infty} \left( \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k0}(x) \rangle}{\alpha_{k0}(\lambda - \lambda_{k0})} (\lambda - \lambda_{k0})\varphi_{k0}(x) - \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle}{\alpha_{k1}(\lambda - \lambda_{k1})} (\lambda - \lambda_{k1})\varphi_{k1}(x) \right). \end{aligned}$$

From this, by virtue of (1.6.10), it follows that  $\ell\varphi(x, \lambda) = \lambda\varphi(x, \lambda)$ . Analogously, using (1.6.81), we obtain  $\ell\Phi(x, \lambda) = \lambda\Phi(x, \lambda)$ . Thus, (1.6.71) is proved for the case when (1.6.75) is fulfilled.

3) Let us now consider the general case when (1.6.3) holds. Choose numbers  $\{\rho_{n,(j)}, \alpha_{n,(j)}\}_{n \geq 0, j \geq 1}$ , such that

$$\Omega_{1,(j)} := \left( \sum_{k=0}^{\infty} (k+1)^2 \xi_{k,(j)} \right)^{1/2} < \infty,$$

$$\Omega_{0,(j)} := \left( \sum_{k=0}^{\infty} ((k+1)\eta_{k,(j)})^2 \right)^{1/2} \rightarrow 0 \text{ as } j \rightarrow \infty,$$

where

$$\xi_{k,(j)} := |\rho_{k,(j)} - \tilde{\rho}_k| + |\alpha_{k,(j)} - \tilde{\alpha}_k|, \eta_{k,(j)} := |\rho_{k,(j)} - \rho_k| + |\alpha_{k,(j)} - \alpha_k|.$$

For each fixed  $j \geq 1$ , we solve the corresponding main equation:

$$\tilde{\psi}_{(j)}(x) = (E + \tilde{H}_{(j)}(x))\psi_{(j)}(x),$$

and construct the functions  $\varphi_{(j)}(x, \lambda)$  and the boundary value problem  $L(q_{(j)}(x), h_{(j)}, H_{(j)})$ . Using Lemma 1.5.1 one can show that

$$\lim_{j \rightarrow \infty} \|q_{(j)} - q\|_{L_2} = 0, \quad \lim_{j \rightarrow \infty} h_{(j)} = h,$$

$$\lim_{j \rightarrow \infty} \max_{0 \leq x \leq \pi} |\varphi_{(j)}(x, \lambda) - \varphi(x, \lambda)| = 0. \quad (1.6.84)$$

Denote by  $\varphi_0(x, \lambda)$  the solution of equation (1.1.1) under the conditions  $\varphi_0(0, \lambda) = 1$ ,  $\varphi'_0(0, \lambda) = h$ . According to Lemma 1.5.3,

$$\lim_{j \rightarrow \infty} \max_{0 \leq x \leq \pi} |\varphi_{(j)}(x, \lambda) - \varphi_0(x, \lambda)| = 0.$$

Comparing this relation with (1.6.84) we conclude that  $\varphi_0(x, \lambda) = \varphi(x, \lambda)$ , i.e.  $\ell\varphi(x, \lambda) = \lambda\varphi(x, \lambda)$ . Similarly we get  $\ell\Phi(x, \lambda) = \lambda\Phi(x, \lambda)$ .

4) Denote  $\Delta(\lambda) := V(\varphi)$ . It follows from (1.6.73) and (1.6.74) for  $a = \pi$  that

$$\tilde{\Delta}(\lambda) = \Delta(\lambda) + \sum_{k=0}^{\infty} \left( \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k0}(x) \rangle_{|x=\pi}}{\alpha_{k0}(\lambda - \lambda_{k0})} \Delta(\lambda_{k0}) - \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle_{|x=\pi}}{\alpha_{k1}(\lambda - \lambda_{k1})} \Delta(\lambda_{k1}) \right), \quad (1.6.85)$$

$$0 = V(\Phi) + \sum_{k=0}^{\infty} \left( \frac{\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}_{k0}(x) \rangle_{|x=\pi}}{\alpha_{k0}(\lambda - \lambda_{k0})} \Delta(\lambda_{k0}) - \frac{\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle_{|x=\pi}}{\alpha_{k1}(\lambda - \lambda_{k1})} \Delta(\lambda_{k1}) \right). \quad (1.6.86)$$

In (1.6.85) we set  $\lambda = \lambda_{n1}$  :

$$0 = \Delta(\lambda_{n1}) + \sum_{k=0}^{\infty} \left( \tilde{P}_{n1,k0}(\pi) \Delta(\lambda_{k0}) - \tilde{P}_{n1,k1}(\pi) \Delta(\lambda_{k1}) \right) = 0.$$

Since  $\tilde{P}_{n1,k1}(\pi) = \delta_{nk}$  we get

$$\sum_{k=0}^{\infty} \tilde{P}_{n1,k0}(\pi) \Delta(\lambda_{k0}) = 0.$$

Then, by virtue of Lemma 1.6.6,  $\Delta(\lambda_{k0}) = 0$ ,  $k \geq 0$ .

Substituting this into (1.6.86) and using the relation  $\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle_{|x=\pi} = 0$ , we obtain  $V(\Phi) = 0$ , i.e. (1.6.72) is valid. Notice that we additionally proved that  $\Delta(\lambda_n) = 0$ , i.e. the numbers  $\{\lambda_n\}_{n \geq 0}$  are eigenvalues of  $L$ .  $\square$

It follows from (1.6.18) for  $x = 0$  that

$$M(\lambda) = \tilde{M}(\lambda) + \sum_{k=0}^{\infty} \left( \frac{1}{\alpha_{k0}(\lambda - \lambda_{k0})} - \frac{1}{\alpha_{k1}(\lambda - \lambda_{k1})} \right).$$

According to Theorem 1.4.6,

$$\tilde{M}(\lambda) = \sum_{k=0}^{\infty} \frac{1}{\alpha_{k1}(\lambda - \lambda_{k1})},$$

hence

$$M(\lambda) = \sum_{k=0}^{\infty} \frac{1}{\alpha_k(\lambda - \lambda_k)}.$$

Thus, the numbers  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  are the spectral data for the constructed boundary value problem  $L$ . Notice that if (1.1.23) is valid we can choose  $\tilde{L} = L(\tilde{q}(x), \tilde{h}, \tilde{H})$  with  $\tilde{q}(x) \in W_2^N$  such that  $\{n^{N+1}\xi_n\} \in l_2$ , and obtain that  $q(x) \in W_2^N$ . This completes the proof of Theorem 1.6.2.  $\square$



**Example 1.6.1.** Let  $\lambda_n = n^2$  ( $n \geq 0$ ),  $\alpha_n = \frac{\pi}{2}$  ( $n \geq 1$ ), and let  $\alpha_0 > 0$  be an arbitrary positive number. We choose  $\tilde{L}$  such that  $\tilde{q}(x) = 0$ ,  $\tilde{h} = \tilde{H} = 0$ . Then  $\tilde{\lambda}_n = 0$  ( $n \geq 0$ ),  $\tilde{\alpha}_n = \frac{\pi}{2}$  ( $n \geq 1$ ), and  $\tilde{\alpha}_0 = \pi$ . Denote  $a := \frac{1}{\alpha_0} - \frac{1}{\pi}$ . Clearly,  $a > -\frac{1}{\pi}$ .

Then, (1.6.19) and (1.6.21) yield

$$1 = (1 + ax)\varphi_{00}(x), \quad \varepsilon_0(x) = a\varphi_{00}(x),$$

and consequently,

$$\varphi_{00}(x) = \frac{1}{1 + ax}, \quad \varepsilon_0(x) = \frac{a}{1 + ax}.$$

Using (1.6.25) and (1.6.26) we calculate

$$q(x) = \frac{2a^2}{(1 + ax)^2}, \quad h = -a, \quad H = \frac{a}{1 + a\pi} = \frac{a\alpha_0}{\pi}. \quad (1.6.87)$$

By (1.6.10),

$$\varphi(x, \lambda) = \cos \rho x - \frac{a}{1 + ax} \frac{\sin \rho x}{\rho}.$$

**1.6.3. The nonselfadjoint case.** In the nonselfadjoint case Theorem 1.6.1 remains valid, i.e. by necessity the main equation (1.6.33) is uniquely solvable. Hence, the results of Subsection 1.6.1 are valid, and Algorithm 1.6.1 can be applied for the solution of the inverse problem also in the nonselfadjoint case, but the sufficiency part in Theorem 1.6.2 must be corrected, since Lemma 1.6.6 is not valid in the general case. In the nonselfadjoint case we must require that the main equation is uniquely solvable (see Condition S in Theorem 1.6.3). As it is shown in Example 1.6.2, Condition S is essential and cannot be omitted. On the other hand, we provide here classes of operators for which the unique solvability of the main equation can be proved.

Let us consider a boundary value problem  $L$  of the form (1.1.1)-(1.1.2) where  $q(x) \in L_2(0, \pi)$  is a complex-valued function, and  $h, H$  are complex numbers. For simplicity, we confine ourselves to the case when all zeros of the characteristic function  $\Delta(\lambda)$  are simple. In this case we shall write  $L \in V$ . If  $L \in V$  we have

$$\alpha_n \neq 0, \quad \lambda_n \neq \lambda_k \quad (n \neq k), \quad (1.6.88)$$

and (1.5.8) holds.

**Theorem 1.6.3** *For complex numbers  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  to be the spectral data for a certain  $L(q(x), h, H) \in V$  with  $q(x) \in L_2(0, \pi)$ , it is necessary and sufficient that*

- 1) *the relations (1.6.88) and (1.5.8) hold;*
- 2) *(Condition S) for each fixed  $x \in [0, \pi]$ , the linear bounded operator  $E + \tilde{H}(x)$ , acting from  $m$  to  $m$ , has a unique inverse operator. Here  $\tilde{L} \in V$  is taken such that  $\tilde{\omega} = \omega$ .*

*The boundary value problem  $L(q(x), h, H)$  can be constructed by Algorithm 1.6.1.*

The proof of Theorem 1.6.3 is essentially the same as the proof of Theorem 1.6.2, but only Lemma 1.6.6 must be omitted.

**Example 1.6.2.** Consider Example 1.6.1, but let  $\alpha_0$  now be an arbitrary complex number. Then the main equation of the inverse problem takes the form

$$1 = (1 + ax)\varphi_{00}(x),$$

and Condition S means that  $1 + ax \neq 0$  for all  $x \in [0, \pi]$ , i.e.  $a \notin (-\infty, -1/\pi]$ . Hence, Condition S is fulfilled if and only if  $\alpha_0 \notin (-\infty, 0]$ . Thus, the numbers  $\{n^2, \alpha_n\}_{n \geq 0}$ ;  $\alpha_n = \frac{\pi}{2} (n \geq 1)$ , are the spectral data if and only if  $\alpha_0 \notin (-\infty, 0]$ . The boundary value problem can be constructed by (1.6.87). For the selfadjoint case we have  $\alpha_0 > 0$ , and Condition S is always fulfilled.

In Theorem 1.6.3 one of the conditions under which arbitrary complex numbers  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  are the spectral data for a boundary value problem  $L$  of the form (1.1.1)-(1.1.2) is the requirement that the main equation is uniquely solvable (Condition S). This condition is difficult to check in the general case. In this connection it is important to point out classes of operators for which the unique solvability of the main equation can be checked. Here we single out three such classes which often appear in applications.

(i) *The selfadjoint case.* In this case Condition S is fulfilled automatically (see Theorem 1.6.2 above).

(ii) *Finite-dimensional perturbations.* Let a model boundary value problem  $\tilde{L}$  with the spectral data  $\{\tilde{\lambda}_n, \tilde{\alpha}_n\}_{n \geq 0}$  be given. We change a finite subset of these numbers. In other words, we consider numbers  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  such that  $\lambda_n = \tilde{\lambda}_n$  and  $\alpha_n = \tilde{\alpha}_n$  for  $n > N$ , and arbitrary in the rest. Then, according to (1.6.19), the main equation becomes the following linear algebraic system:

$$\tilde{\varphi}_{ni}(x) = \varphi_{ni}(x) + \sum_{k=0}^N (\tilde{P}_{ni,k0}(x)\varphi_{k0}(x) - \tilde{P}_{ni,k1}(x)\varphi_{k1}(x)), \quad n = \overline{0, N}, \quad i = 0, 1, \quad x \in [0, \pi],$$

and Condition S is equivalent to the condition that the determinant of this system differs from zero for each  $x \in [0, \pi]$ . Such perturbations are very popular in applications. We note that for the selfadjoint case the determinant is always nonzero.

(iii) *Local solvability of the main equation.* For small perturbations of the spectral data Condition S is fulfilled automatically. More precisely, the following theorem is valid.

**Theorem 1.6.4** *Let  $\tilde{L} = L(\tilde{q}(x), \tilde{h}, \tilde{H}) \in V$  be given. There exists a  $\delta > 0$  (which depends on  $\tilde{L}$ ) such that if complex numbers  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  satisfy the condition  $\Omega < \delta$ , then there exists a unique boundary value problem  $L(q(x), h, H) \in V$  with  $q(x) \in L_2(0, \pi)$  for which the numbers  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  are the spectral data, and*

$$\|q - \tilde{q}\|_{L_2(0, \pi)} < C\Omega, \quad |h - \tilde{h}| < C\Omega, \quad |H - \tilde{H}| < C\Omega, \quad (1.6.89)$$

where  $C$  depends on  $\tilde{L}$  only.

*Proof.* In this proof  $C$  denotes various constants which depend on  $\tilde{L}$  only. Since  $\Omega < \infty$ , the asymptotical formulae (1.5.8) are fulfilled. Choose  $\delta_0 > 0$  such that if  $\Omega < \delta_0$  then (1.6.88) is valid.

According to (1.6.32),

$$\|\tilde{H}(x)\| \leq C \sup_n \sum_k \frac{\xi_n}{|n-k|+1} < C\Omega.$$

Choose  $\delta \leq \delta_0$  such that if  $\Omega < \delta$ , then  $\|\tilde{H}(x)\| \leq 1/2$  for  $x \in [0, \pi]$ . In this case there exists  $(E + \tilde{H}(x))^{-1}$ , and

$$\|(E + \tilde{H}(x))^{-1}\| \leq 2.$$

Thus, all conditions of Theorem 1.6.3 are fulfilled. By Theorem 1.6.3, there exists a unique boundary value problem  $L(q(x), h, H) \in V$  for which the numbers  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  are the spectral data. Moreover, (1.6.68) and (1.6.69) are valid. Repeating now the arguments in the proof of Lemma 1.6.8 we get

$$\max_{0 \leq x \leq \pi} |\varepsilon_0(x)| \leq C\Omega, \quad \|\varepsilon(x)\|_{L_2(0, \pi)} \leq C\Omega.$$

Together with (1.6.25)-(1.6.26) this yields (1.6.89).  $\square$

Similarly we also can obtain the stability of the solution of the inverse problem *in the uniform norm*; more precisely we get:

**Theorem 1.6.5.** *Let  $\tilde{L} = L(\tilde{q}(x), \tilde{h}, \tilde{H}) \in V$  be given. There exists  $\delta > 0$  (which depends on  $\tilde{L}$ ) such that if complex numbers  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  satisfy the condition  $\Omega_1 < \delta$ , then there exists a unique boundary value problem  $L(q(x), h, H) \in V$  for which the numbers  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  are the spectral data. Moreover, the function  $q(x) - \tilde{q}(x)$  is continuous on  $[0, \pi]$ , and*

$$\max_{0 \leq x \leq \pi} |q - \tilde{q}| < C\Omega_1, \quad |h - \tilde{h}| < C\Omega_1, \quad |H - \tilde{H}| < C\Omega_1,$$

where  $C$  depends on  $\tilde{L}$  only.

**Remark 1.6.4.** Using the method of spectral mappings we can solve the inverse problem not only in  $L_2(0, \pi)$  and  $W_2^N(0, \pi)$  but also for other classes of potentials. As example we formulate here the following theorem for the non-selfadjoint case.

**Theorem 1.6.6.** *For complex numbers  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  to be the spectral data for a certain  $L(q(x), h, H) \in V$  with  $q(x) \in D \subset L(0, \pi)$ , it is necessary and sufficient that*

- 1) (1.6.88) holds,
- 2) (Asymptotics) there exists  $\tilde{L} = L(\tilde{q}(x), \tilde{h}, \tilde{H}) \in V$  with  $\tilde{q}(x) \in D$  such that  $\{n\xi_n\} \in l_2$ ,
- 3) (Condition S) for each fixed  $x \in [0, \pi]$ , the linear bounded operator  $E + \tilde{H}(x)$ , acting from  $m$  to  $m$ , has a unique inverse operator,
- 4)  $\varepsilon(x) \in D$ , where the function  $\varepsilon(x)$  is defined by (1.6.21).

*The boundary value problem  $L(q(x), h, H)$  can be constructed by Algorithm 1.6.1.*

## 1.7. THE METHOD OF STANDARD MODELS

In the method of standard models we construct a sequence of model operators which approximate in an adequate sense the unknown operator and allow us to construct the potential step by step. The method of standard models provides an effective algorithm for

the solution of inverse problems. This method has a wide area for applications. It works for many important classes of inverse problems when other methods turn out to be unsuitable. For example, in [yur9] the so-called incomplete inverse problems for higher-order differential operators were studied when only some part of the spectral information is accessible to measurement and when there is a priori information about the operator or its spectrum. In [yur27] the method of standard models was applied for solving the inverse problem for systems of differential equations with nonlinear dependence on the spectral parameter. The inverse problem for integro-differential operators was treated in [yur13]. This method was also used for studying an inverse problem in elasticity theory [yur12].

However the method of standard models requires rather strong restrictions on the operator. For example, for the Sturm-Liouville operator the method works in the classes of analytic or piecewise analytic potentials on the interval  $[0, \pi]$ . We can also deal with more general classes, for example, the class of piecewise operator-analytic functions (see [fag2]) or other classes of functions which can be expanded in series which generalize the Taylor series.

In this section we present the method of standard models using the Sturm-Liouville operator as a model. In order to show ideas we confine ourselves to the case when the potential  $q(x)$  in the boundary value problem (1.1.1)-(1.1.2) is an analytic function on  $[0, \pi]$ . Other, more complicated cases can be treated similarly.

Let  $\Phi(x, \lambda)$  be the Weyl solution of (1.1.1) which satisfies the conditions  $U(\Phi) = 1$ ,  $V(\Phi) = 0$ , and let  $M(\lambda) := \Phi(0, \lambda)$  be the Weyl function for the boundary value problem  $L$  (see Subsection 1.4.4). From several equivalent formulations of the inverse problems which were studied in Section 1.4 we will deal with here Inverse Problem 1.4.4 of recovering  $L$  from the Weyl function.

Let the Weyl function  $M(\lambda)$  of the boundary value problem  $L$  be given. Our goal is to construct the potential  $q(x)$  which is an analytic function on  $[0, \pi]$  (to simplify calculations we assume that the coefficients  $h$  and  $H$  are known). We take a model boundary value problem  $\tilde{L} = L(\tilde{q}(x), h, H)$  with an analytic potential  $\tilde{q}(x)$ . Since

$$-\Phi''(x, \lambda) + q(x)\Phi(x, \lambda) = \lambda\Phi(x, \lambda), \quad -\tilde{\Phi}''(x, \lambda) + \tilde{q}(x)\tilde{\Phi}(x, \lambda) = \lambda\tilde{\Phi}(x, \lambda),$$

we get

$$(q(x) - \tilde{q}(x))\Phi(x, \lambda)\tilde{\Phi}(x, \lambda) = (\Phi'(x, \lambda)\tilde{\Phi}(x, \lambda) - \Phi(x, \lambda)\tilde{\Phi}'(x, \lambda))',$$

and consequently

$$\int_0^\pi \hat{q}(x)\Phi(x, \lambda)\tilde{\Phi}(x, \lambda) dx = \hat{M}(\lambda), \quad (1.7.1)$$

where  $\hat{q}(x) = q(x) - \tilde{q}(x)$ ,  $\hat{M}(\lambda) = M(\lambda) - \tilde{M}(\lambda)$ .

We are interested in the asymptotics of the terms in (1.7.1). First we prove an auxiliary assertion. Denote  $Q = \{\rho : \arg \rho \in [\delta_0, \pi - \delta_0]\}$ ,  $\delta_0 > 0$ . Then there exists  $\varepsilon_0 > 0$  such that

$$|Im \rho| \geq \varepsilon_0 |\rho| \quad \text{for } \rho \in Q. \quad (1.7.2)$$

**Lemma 1.7.1.** *Let*

$$r(x) = \frac{x^k}{k!}(\gamma + p(x)), \quad H(x, \rho) = \exp(2i\rho x) \left(1 + \frac{\xi(x, \rho)}{\rho}\right), \quad x \in [0, a],$$

where  $p(x) \in C[0, a]$ ,  $p(0) = 0$ , and where the function  $\xi(x, \rho)$  is continuous and bounded for  $x \in [0, a]$ ,  $\rho \in Q$ ,  $|\rho| \geq \rho^*$ . Then for  $\rho \rightarrow \infty$ ,  $\rho \in Q$ ,

$$\int_0^a r(x)H(x, \rho) dx = \frac{(-1)^k}{(2i\rho)^{k+1}}(\gamma + o(1)).$$

*Proof.* We calculate

$$(2i\rho)^{k+1}(-1)^k \int_0^a r(x)H(x, \rho) dx = I_1(\rho) + I_2(\rho) + I_3(\rho),$$

where

$$I_1(\rho) = \gamma(2i\rho)^{k+1}(-1)^k \int_0^a \frac{x^k}{k!} \exp(2i\rho x) dx,$$

$$I_2(\rho) = (2i\rho)^{k+1}(-1)^k \int_0^a \frac{x^k}{k!} p(x) \exp(2i\rho x) dx,$$

$$I_3(\rho) = (2i)^{k+1}(-\rho)^k \int_0^a r(x) \exp(2i\rho x) \xi(x, \rho) dx.$$

Since

$$\int_0^\infty \frac{x^k}{k!} \exp(2i\rho x) dx = \frac{(-1)^k}{(2i\rho)^{k+1}}, \quad \rho \in Q,$$

it follows that  $I_1(\rho) - \gamma \rightarrow 0$  as  $|\rho| \rightarrow \infty$ ,  $\rho \in Q$ .

Furthermore, take  $\varepsilon > 0$  and choose  $\delta = \delta(\varepsilon)$  such that for  $x \in [0, \delta]$ ,  $|p(x)| < \frac{\varepsilon}{2}\varepsilon_0^{k+1}$ , where  $\varepsilon_0$  is defined in (1.7.2). Then, using (1.7.2) we infer

$$\begin{aligned} |I_2(\rho)| &< \frac{\varepsilon}{2}(2|\rho|\varepsilon_0)^{k+1} \int_0^\delta \frac{x^k}{k!} \exp(-2\varepsilon_0|\rho|x) dx + (2|\rho|)^{k+1} \int_\delta^a \frac{x^k}{k!} |p(x)| \exp(-2\varepsilon_0|\rho|x) dx \\ &< \frac{\varepsilon}{2} + (2|\rho|)^{k+1} \exp(-2\varepsilon_0|\rho|\delta) \int_0^{a-\delta} \frac{(x+\delta)^k}{k!} |p(x+\delta)| \exp(-2\varepsilon_0|\rho|x) dx. \end{aligned}$$

By arbitrariness of  $\varepsilon$  we obtain that  $I_2(\rho) \rightarrow 0$  for  $|\rho| \rightarrow \infty$ ,  $\rho \in Q$ .

Since  $|(\gamma + p(x))\xi(x, \rho)| < C$ , then for  $|\rho| \rightarrow \infty$ ,  $\rho \in Q$ ,

$$|I_3(\rho)| < C|\rho|^k \int_0^a \frac{x^k}{k!} \exp(-2\varepsilon_0|\rho|x) dx \leq \frac{C}{|\rho|\varepsilon_0^{k+1}} \rightarrow 0.$$

□

Suppose that for a certain fixed  $k \geq 0$  the Taylor coefficients  $q_j := q^{(j)}(0)$ ,  $j = \overline{0, k-1}$ , have been already found. Let us choose a model boundary value problem  $\tilde{L}$  such that the first  $k$  Taylor coefficients of  $q$  and  $\tilde{q}$  coincide, i.e.  $\tilde{q}_j = q_j$ ,  $j = \overline{0, k-1}$ . Then, using (1.7.1) and Lemma 1.7.1 we can calculate the next Taylor coefficient  $q_k = q^{(k)}(0)$ . Namely, the following assertion is valid.

**Lemma 1.7.2.** *Fix  $k$ . Let the functions  $q(x)$  and  $\tilde{q}(x)$  be analytic for  $x \in [0, a]$ ,  $a > 0$ , with  $\hat{q}_j := q_j - \tilde{q}_j = 0$  for  $j = \overline{0, k-1}$ . Then*

$$\hat{q}_k = \frac{1}{4}(-1)^k \lim_{\substack{|\rho| \rightarrow \infty \\ \rho \in Q}} (2i\rho)^{k+3} \hat{M}(\lambda). \quad (1.7.3)$$

*Proof.* It follows from (1.1.10), (1.1.16) and (1.4.9) that

$$\Phi(x, \lambda) = \frac{1}{i\rho} \exp(i\rho x) \left(1 + O\left(\frac{1}{\rho}\right)\right).$$

Hence, taking (1.7.1) into account, we have

$$\hat{M}(\lambda) = \int_0^\pi \hat{q}(x) \Phi(x, \lambda) \tilde{\Phi}(x, \lambda) dx = \int_0^a \frac{x^k}{k!} \left( \hat{q}^{(k)}(0) + p(x) \right) \frac{1}{(i\rho)^2} \exp(2i\rho x) \left( 1 + \frac{\xi(x, \rho)}{\rho} \right) dx + \int_a^\pi \hat{q}(x) \Phi(x, \lambda) \tilde{\Phi}(x, \lambda) dx.$$

Using Lemma 1.7.1 we obtain for  $|\rho| \rightarrow \infty$ ,  $\rho \in Q$ :

$$\hat{M}(\lambda) = \frac{4(-1)^k}{(2i\rho)^{k+3}} \left( \hat{q}^{(k)}(0) + o(1) \right).$$

□

Thus, we arrive at the following algorithm for the solution of Inverse Problem 1.4.4 in the class of analytic potentials.

**Algorithm 1.7.1.** *Let the Weyl function  $M(\lambda)$  of the boundary value problem (1.1.1)-(1.1.2) with an analytic potential be given. Then:*

(i) *We calculate  $q_k = q^{(k)}(0)$ ,  $k \geq 0$ . For this purpose we successively perform the following operations for  $k = 0, 1, 2, \dots$ : We construct a model boundary value problem  $\tilde{L}$  with an analytic potential  $\tilde{q}(x)$  such that  $\tilde{q}_j = q_j$ ,  $j = \overline{0, k-1}$  and arbitrary in the rest, and we calculate  $q_k = q^{(k)}(0)$  by (1.7.3).*

(ii) *We construct the function  $q(x)$  by the formula*

$$q(x) = \sum_{k=0}^{\infty} q_k \frac{x^k}{k!}, \quad 0 < x < R,$$

where

$$R = \left( \overline{\lim_{k \rightarrow \infty}} \left( \frac{|q_k|}{k!} \right)^{1/k} \right)^{-1}.$$

If  $R < \pi$  then for  $R < x < \pi$  the function  $q(x)$  (which is analytic on  $[0, \pi]$ ) is constructed by analytic continuation or by the step method which is described below in Remark 1.7.1.

**Remark 1.7.1.** The method of standard models also works when  $q(x)$  is a piecewise analytic function. Indeed, the functions

$$\Phi_a(x, \lambda) = \frac{\Phi(x, \lambda)}{\Phi'(a, \lambda)}, \quad M_a(\lambda) = \frac{\Phi(a, \lambda)}{\Phi'(a, \lambda)} = \frac{S(a, \lambda) + M(\lambda)\varphi(a, \lambda)}{S'(a, \lambda) + M(\lambda)\varphi'(a, \lambda)}$$

are the Weyl solution and the Weyl function for the interval  $x \in [a, \pi]$  respectively. Suppose that  $q(x)$  has been already constructed on  $[0, a]$ . Then we can calculate  $M_a(\lambda)$  and go on to the interval  $[a, \pi]$ .

## 1.8. LOCAL SOLUTION OF THE INVERSE PROBLEM

In this section we present a method for the local solution of Inverse Problem 1.4.2 of recovering Sturm-Liouville operators from two spectra. Without loss of generality we will consider the case when  $H = 0$ . In this method, which is due to Borg [Bor1], the inverse

problem is reduced to a nonlinear integral equation (see (1.8.13)) which can be solved locally. An important role in Borg's method is played by products of eigenfunctions of the boundary value problems considered. In order to derive and study Borg's nonlinear equation one must prove that such products are complete or form a Riesz basis in  $L_2(0, \pi)$ . For uniqueness theorems it is sufficient to prove the completeness of the products (see Subsection 1.4.3). For the local solution of the inverse problem and for studying stability of the solution such products must be a Riesz basis. For convenience of the reader we provide in Subsection 1.8.5 necessary information about Riesz bases in a Hilbert space.

We note that for Sturm-Liouville operators Borg's method is weaker than the Gelfand-Levitan method and the method of spectral mappings. However, Borg's method turns out to be useful when other methods do not work (see, for example, [kha1],[dub1],[yur13] and the references therein).

**1.8.1. Derivation of the Borg equation.** Let  $\lambda_{ni} = \rho_{ni}^2$ ,  $n \geq 0$ ,  $i = 1, 2$ , be the eigenvalues of the boundary value problems  $L_i$ :

$$\ell y := -y'' + q(x)y = \lambda y, \quad (1.8.1)$$

$$y'(0) - hy(0) = y^{(i-1)}(\pi) = 0, \quad (1.8.2)$$

where  $q(x) \in L_2(0, \pi)$  is a real function, and  $h$  is a real number. Then (see Section 1.1)

$$\rho_{n1} = \left(n + \frac{1}{2}\right) + \frac{a}{n} + \frac{\kappa_{n1}}{n}, \quad \rho_{n2} = n + \frac{a}{n} + \frac{\kappa_{n2}}{n}, \quad (1.8.3)$$

where

$$\{\kappa_{ni}\} \in l_2, \quad a = \frac{1}{\pi} \left( h + \frac{1}{2} \int_0^\pi q(t) dt \right).$$

Let  $L_i$  and  $\tilde{L}_i$ ,  $i = 1, 2$ , be such that  $a = \tilde{a}$ , then

$$\Lambda := \left( \sum_{n=0}^{\infty} (|\lambda_{n1} - \tilde{\lambda}_{n1}|^2 + |\lambda_{n2} - \tilde{\lambda}_{n2}|^2) \right)^{1/2} < \infty.$$

Denote

$$y_{ni}(x) = \varphi(x, \tilde{\lambda}_{ni}), \quad \tilde{y}_{ni}(x) = \tilde{\varphi}(x, \tilde{\lambda}_{ni}), \quad s_{ni}(x) = S(x, \tilde{\lambda}_{ni}), \quad \tilde{s}_{ni}(x) = \tilde{S}(x, \tilde{\lambda}_{ni}),$$

$$G_{ni}(x, t) = \begin{cases} S(x, \tilde{\lambda}_{ni})C(t, \tilde{\lambda}_{ni}) - S(t, \tilde{\lambda}_{ni})C(x, \tilde{\lambda}_{ni}), & 0 \leq t \leq x \leq \pi, \\ 0, & 0 \leq x < t \leq \pi, \end{cases}$$

where  $\varphi(x, \lambda)$ ,  $C(x, \lambda)$  and  $S(x, \lambda)$  are solutions of (1.8.1) under the conditions  $C(0, \lambda) = \varphi(0, \lambda) = S'(0, \lambda) = 1$ ,  $C''(0, \lambda) = S(0, \lambda) = 0$ ,  $\varphi'(0, \lambda) = h$ .

Since

$$\ell y_{ni}(x) = \tilde{\lambda}_{ni} y_{ni}(x), \quad \tilde{\ell} \tilde{y}_{ni}(x) = \tilde{\lambda}_{ni} \tilde{y}_{ni}(x),$$

we get

$$\begin{aligned} \int_0^\pi r(x) y_{ni}(x) \tilde{y}_{ni}(x) dx &= \int_0^\pi (y_{ni}(x) \tilde{y}_{ni}''(x) - y_{ni}''(x) \tilde{y}_{ni}(x)) dx \\ &= \left( y_{ni}(x) \tilde{y}_{ni}'(x) - y_{ni}'(x) \tilde{y}_{ni}(x) \right) \Big|_0^\pi, \end{aligned} \quad (1.8.4)$$

where  $r := \tilde{q} - q$ . Let

$$p = -\frac{1}{2} \int_0^\pi r(x) dx. \quad (1.8.5)$$

From the equality  $a = \tilde{a}$ , it follows that  $p = \tilde{h} - h$ ; hence we infer from the boundary conditions that (1.8.4) takes the form

$$\int_0^\pi r(x) y_{ni}(x) \tilde{y}_{ni}(x) dx = y_{ni}(\pi) \tilde{y}'_{ni}(\pi) - y'_{ni}(\pi) \tilde{y}_{ni}(\pi) - p, \quad n \geq 0, i = 1, 2. \quad (1.8.6)$$

Since  $\tilde{y}_{ni}(x)$  are eigenfunctions for the boundary value problems  $\tilde{L}_i$ , we have

$$\tilde{y}_{n1}(\pi) = 0, \quad \tilde{y}'_{n2}(\pi) = 0, \quad n \geq 0. \quad (1.8.7)$$

Furthermore, one can easily verify by differentiation that the function  $\tilde{y}_{ni}(x)$  satisfies the following Volterra integral equation

$$\tilde{y}_{ni}(x) = y_{ni}(x) + p s_{ni}(x) + \int_0^\pi G_{ni}(x, t) r(t) \tilde{y}_{ni}(t) dt, \quad n \geq 0, i = 1, 2. \quad (1.8.8)$$

Solving (1.8.8) by the method of successive approximations we deduce

$$\tilde{y}_{ni}(x) = y_{ni}(x) + p s_{ni}(x) + \varphi_{ni}(x), \quad n \geq 0, i = 1, 2, \quad (1.8.9)$$

where

$$\begin{aligned} \varphi_{ni}(x) = & \sum_{j=1}^{\infty} \underbrace{\int_0^\pi \dots \int_0^\pi}_{j} G_{ni}(x, t_1) G_{ni}(t_1, t_2) \dots G_{ni}(t_{j-1}, t_j) \times \\ & r(t_1) \dots r(t_j) (y_{ni}(t_j) + p s_{ni}(t_j)) dt_1 \dots dt_j. \end{aligned} \quad (1.8.10)$$

Substituting (1.8.9)-(1.8.10) into (1.8.6) and using (1.8.5) and (1.8.7), we get

$$\int_0^\pi r(x) \xi_{ni}(x) dx = \omega_{ni}, \quad n \geq 0, i = 1, 2, \quad (1.8.11)$$

where

$$\xi_{ni}(x) = y_{ni}^2(x) - \frac{1}{2},$$

$$\omega_{n1} = y_{n1}(\pi)(y'_{n1}(\pi) + p s'_{n1}(\pi) + \varphi'_{n1}(\pi)) - \int_0^\pi p r(x) s_{n1}(x) y_{n1}(x) dx - \int_0^\pi r(x) y_{n1}(x) \varphi_{n1}(x) dx,$$

$$\omega_{n2} = -y'_{n2}(\pi)(y_{n2}(\pi) + p s_{n2}(\pi) + \varphi_{n2}(\pi)) - \int_0^\pi p r(x) s_{n2}(x) y_{n2}(x) dx - \int_0^\pi r(x) y_{n2}(x) \varphi_{n2}(x) dx.$$

It follows from (1.4.7) that

$$\xi_{ni}(x) = \frac{1}{2} \left( \cos 2\tilde{\rho}_{ni}x + \int_0^x V_1(x, t) \cos 2\tilde{\rho}_{ni}t dt \right), \quad (1.8.12)$$

where  $V_1(x, t)$  is a continuous function. We introduce the functions  $\{\eta_n(x)\}_{n \geq 0}$  and the numbers  $\{z_n\}_{n \geq 0}$  by the formulae

$$\eta_{2n}(x) = \xi_{n2}(x), \quad \eta_{2n+1}(x) = \xi_{n1}(x), \quad z_{2n} = 2\tilde{\rho}_{n2}, \quad z_{2n+1} = 2\tilde{\rho}_{n1}.$$



Then, according to (1.8.3) and (1.8.12),

$$z_n = n + \frac{a}{n} + \frac{\kappa_n}{n}, \quad \{\kappa_n\} \in l_2,$$

$$\eta_n(x) = \frac{1}{2} \left( \cos z_n x + \int_0^x V_1(x, t) \cos z_n t \, dt \right).$$

Hence, by virtue of Propositions 1.8.6 and 1.8.2, the set of functions  $\{\eta_n(x)\}_{n \geq 0}$  forms a Riesz basis in  $L_2(0, \pi)$ . Denote by  $\{\chi_n(x)\}_{n \geq 0}$  the corresponding biorthonormal basis. Then from (1.8.11) we infer

$$r(x) = \sum_{n=0}^{\infty} \left( \omega_{n2} \chi_{2n}(x) + \omega_{n1} \chi_{2n+1}(x) \right),$$

and consequently, in view of (1.8.5),

$$r(x) = f(x) + \sum_{j=1}^{\infty} \underbrace{\int_0^{\pi} \dots \int_0^{\pi}}_j H_j(x, t_1, \dots, t_j) r(t_1) \dots r(t_j) \, dt_1 \dots dt_j, \quad (1.8.13)$$

where

$$f(x) = \sum_{n=0}^{\infty} (-y'_{n2}(\pi) y_{n2}(\pi) \chi_{2n}(x) + y'_{n1}(\pi) y_{n1}(\pi) \chi_{2n+1}(x)), \quad (1.8.14)$$

$$\begin{aligned} H_1(x, t_1) = & \sum_{n=0}^{\infty} \left( -y'_{n2}(\pi) \left( G_{n2}(\pi, t_1) y_{n2}(t_1) - \frac{1}{2} s_{n2}(\pi) \right) \chi_{2n}(x) \right. \\ & \left. + y_{n1}(\pi) \left( \frac{\partial G_{n1}(x, t_1)}{\partial x} \Big|_{x=\pi} y_{n1}(t_1) - \frac{1}{2} s'_{n1}(\pi) \right) \chi_{2n+1}(x) \right), \end{aligned} \quad (1.8.15)$$

$$\begin{aligned} H_j(x, t_1, \dots, t_j) = & \sum_{n=0}^{\infty} \left( - \left( y_{n2}(t_1) + y'_{n2}(\pi) G_{n2}(\pi, t_1) \right) G_{n2}(t_1, t_2) \dots G_{n2}(t_{j-2}, t_{j-1}) \times \right. \\ & \left. \left( G_{n2}(t_{j-1}, t_j) y_{n2}(t_j) - \frac{1}{2} s_{n2}(t_{j-1}) \right) \chi_{2n}(x) - \left( y_{n1}(t_1) - y_{n1}(\pi) \frac{\partial G_{n1}(x, t_1)}{\partial x} \Big|_{x=\pi} \right) \times \right. \end{aligned}$$

$$\left. G_{n1}(t_1, t_2) \dots G_{n1}(t_{j-2}, t_{j-1}) \left( G_{n1}(t_{j-1}, t_j) y_{n1}(t_j) - \frac{1}{2} s_{n1}(t_{j-1}) \right) \chi_{2n+1}(x) \right), \quad j \geq 2. \quad (1.8.16)$$

Equation (1.8.13) is called the *Borg equation*.

**Remark 1.8.1.** In order to prove that the system of functions  $\{\eta_n(x)\}_{n \geq 0}$  forms a Riesz basis in  $L_2(0, \pi)$  we used above the transformation operator. However in many cases of practical interest the Borg method can be applied, although the transformation operator does not work. In Subsection 1.8.3 we describe Borg's original idea which allows us to prove that the products of eigenfunctions form a Riesz basis in  $L_2(0, \pi)$ .

**1.8.2. The main theorem.** Using the Borg equation we can prove the following theorem on the local solution of the inverse problem and on the stability of the solution.

**Theorem 1.8.1.** *For the boundary value problems  $L_i$  of the form (1.8.1)-(1.8.2), there exists  $\delta > 0$  (which depends on  $L_i$ ) such that if real numbers  $\{\tilde{\lambda}_{ni}\}_{n \geq 0}$ ,  $i = 1, 2$ , satisfy*

the condition  $\Lambda < \delta$ , then there exists a unique real pair  $\tilde{q}(x) \in L_2(0, \pi)$  and  $\tilde{h}$ , for which the numbers  $\{\tilde{\lambda}_{ni}\}_{n \geq 0}$ ,  $i = 1, 2$ , are the eigenvalues of  $\tilde{L}_i$ . Moreover,

$$\|q - \tilde{q}\|_{L_2} < C\Lambda, \quad |h - \tilde{h}| < C\Lambda. \quad (1.8.17)$$

Here and below  $C$  denotes various positive constants which depend on  $L_i$ .

*Proof.* One can choose  $\delta_1 > 0$  such that if real numbers  $\{\tilde{\lambda}_{ni}\}_{n \geq 0}$ ,  $i = 1, 2$ , satisfy the condition  $\Lambda < \delta_1$ , then  $\tilde{\lambda}_{ni} \neq \tilde{\lambda}_{kj}$  for  $(n, i) \neq (k, j)$ , and

$$y_{n2}(\pi) \neq 0, \quad y'_{n1}(\pi) \neq 0 \quad \text{for all } n \geq 0. \quad (1.8.18)$$

Indeed, according to Theorem 1.1.1,  $\varphi(x, \lambda_{n2})$  is an eigenfunction of the boundary value problem  $L_2$ , and  $\varphi'(\pi, \lambda_{n2}) = 0$ ,  $\varphi(\pi, \lambda_{n2}) \neq 0$  for all  $n \geq 0$ . Moreover, by virtue of (1.1.9) and (1.8.3),

$$\varphi(\pi, \lambda_{n2}) = \cos n\pi + O\left(\frac{1}{n}\right) = (-1)^n + O\left(\frac{1}{n}\right).$$

Thus,

$$|\varphi(\pi, \lambda_{n2})| \geq C > 0.$$

On the other hand, using (1.3.11) we calculate

$$\varphi(\pi, \tilde{\lambda}_{n2}) - \varphi(\pi, \lambda_{n2}) = (\cos \tilde{\rho}_{n2}\pi - \cos \rho_{n2}\pi) + \int_0^\pi G(\pi, t)(\cos \tilde{\rho}_{n2}t - \cos \rho_{n2}t) dt,$$

and consequently

$$|\varphi(\pi, \tilde{\lambda}_{n2}) - \varphi(\pi, \lambda_{n2})| < C|\tilde{\rho}_{n2} - \rho_{n2}|.$$

Then for sufficiently small  $\Lambda$  we have

$$y_{n2}(\pi) := \varphi(\pi, \tilde{\lambda}_{n2}) \neq 0 \quad \text{for all } n \geq 0.$$

Similarly, one can derive the second inequality in (1.8.18).

It is easy to verify that the following estimates are valid for  $n \geq 0$ ,  $i = 0, 1$ ,  $0 \leq x, t \leq \pi$ ,

$$|y_{ni}(x)| < C, \quad |y'_{n1}(\pi)| < C(n+1), \quad \left|\frac{\partial G_{ni}(x, t)}{\partial x}\right| < C, \quad |G_{ni}(x, t)| < \frac{C}{n+1},$$

$$|y'_{n2}(\pi)| < C|\tilde{\lambda}_{n2} - \lambda_{n2}|, \quad |y_{n1}(\pi)| < \frac{C}{n+1}|\tilde{\lambda}_{n1} - \lambda_{n1}|, \quad |s'_{n1}(\pi)| < \frac{C}{n+1}.$$

Indeed, the estimate  $|y_{ni}(x)| < C$  follows from (1.1.9) and (1.8.3). Since  $\varphi'(\pi, \lambda_{n2}) = 0$ , we get

$$y'_{n2}(\pi) = \varphi'(\pi, \tilde{\lambda}_{n2}) - \varphi'(\pi, \lambda_{n2}).$$

By virtue of (1.3.11),

$$\varphi'(x, \lambda) = -\rho \sin \rho x + G(x, x) \cos \rho x + \int_0^x \frac{\partial G(x, t)}{\partial x} \cos \rho t dt.$$

Hence

$$\varphi'(\pi, \tilde{\lambda}_{n2}) - \varphi'(\pi, \lambda_{n2}) = -\tilde{\rho}_{n2} \sin \tilde{\rho}_{n2}\pi + \rho_{n2} \sin \rho_{n2}\pi$$

$$+G(\pi, t)(\cos \tilde{\rho}_{n2}\pi - \cos \rho_{n2}\pi) + \int_0^\pi \frac{\partial G(x, t)}{\partial x} \Big|_{x=\pi} (\cos \tilde{\rho}_{n2}t - \cos \rho_{n2}t) dt,$$

and consequently

$$|y'_{n2}(\pi)| < C|\tilde{\lambda}_{n2} - \lambda_{n2}|.$$

The other estimates are proved similarly.

Using (1.8.14)-(1.8.16) we obtain

$$\|f\| < C\Lambda, \quad \|H_1\| < C\Lambda, \quad \|H_j\| < C^j, \quad (j \geq 2), \quad (1.8.19)$$

where  $\|\cdot\|$  is the norm in  $L_2$  with respect to all of the arguments.

Indeed, according to (1.8.14),

$$f(x) = \sum_{n=0}^{\infty} f_n \chi_n(x),$$

where

$$f_{2n} = -y'_{n2}(\pi)y_{n2}(\pi), \quad f_{2n+1} = y'_{n1}(\pi)y_{n1}(\pi).$$

Then, using the preceding estimates for  $y_{ni}(x)$ , we get

$$|f_{2n}| \leq C|\tilde{\lambda}_{n2} - \lambda_{n2}|, \quad |f_{2n+1}| \leq C|\tilde{\lambda}_{n1} - \lambda_{n1}|.$$

By virtue of (1.8.58) this yields  $\|f\| < C\Lambda$ . Other estimates in (1.8.19) can be verified similarly.

Consider in  $L_2(0, \pi)$  the nonlinear integral equation (1.8.13). We rewrite (1.8.13) in the form

$$r = f + Ar,$$

where

$$Ar = \sum_{j=1}^{\infty} A_j r,$$

$$(A_j r)(x) = \underbrace{\int_0^\pi \dots \int_0^\pi}_j H_j(x, t_1, \dots, t_j) r(t_1) \dots r(t_j) dt_1 \dots dt_j.$$

It follows from (1.8.19) that

$$\left. \begin{aligned} \|A_1 r\| &\leq C\Lambda \|r\|, \quad \|A_1 r - A_1 \tilde{r}\| \leq C\Lambda \|r - \tilde{r}\|, \\ \|A_j r\| &\leq (C\|r\|)^j, \quad \|A_j r - A_j \tilde{r}\| \leq \|r - \tilde{r}\| (C \max(\|r\|, \|\tilde{r}\|))^{j-1}, \quad j \geq 2, \\ \|f\| &< C\Lambda. \end{aligned} \right\} \quad (1.8.20)$$

Fix  $C \geq 1/2$  such that (1.8.20) is valid. If

$$\|r\| \leq \frac{1}{2C}, \quad \|\tilde{r}\| \leq \frac{1}{2C},$$

then (1.8.20) implies

$$\|Ar\| \leq C\Lambda \|r\| + 2C^2 \|r\|^2,$$

$$\|Ar - A\tilde{r}\| \leq C\|r - \tilde{r}\|(\Lambda + 2\max(\|r\|, \|\tilde{r}\|)).$$

Moreover, if

$$\Lambda \leq \frac{1}{4C}, \quad \|r\| \leq \frac{1}{8C^2}, \quad \|\tilde{r}\| \leq \frac{1}{8C^2},$$

then

$$\|Ar\| \leq \frac{1}{2}\|r\|, \quad \|Ar - A\tilde{r}\| \leq \frac{1}{2}\|r - \tilde{r}\|. \quad (1.8.21)$$

Therefore, there exists sufficiently small  $\delta_2 > 0$  such that if  $\Lambda < \delta_2$ , then equation (1.8.13) can be solved by the method of successive approximations:

$$r_0 = f, \quad r_{k+1} = f + Ar_k, \quad k \geq 0,$$

$$r = r_0 + \sum_{k=0}^{\infty} (r_{k+1} - r_k). \quad (1.8.22)$$

Indeed, put  $\delta_2 = \frac{1}{16C^3}$ . If  $\Lambda < \delta_2$ , then  $\|f\| \leq \frac{1}{16C^2}$ ,  $\Lambda \leq \frac{1}{4C}$ . Using (1.8.21), by induction we get

$$\|r_k\| \leq 2\|f\|, \quad \|r_{k+1} - r_k\| \leq \frac{1}{2^{k+1}}\|f\|, \quad k \geq 0.$$

Consequently, the series (1.8.22) converges in  $L_2(0, \pi)$  to the solution of equation (1.8.13). Moreover, for this solution the estimate  $\|r\| \leq 2\|f\|$  is valid.

Let  $\delta = \min(\delta_1, \delta_2)$ , and let  $r(x)$  be the above constructed solution of (1.8.13). Define  $p$  via (1.8.5) and  $\tilde{q}(x)$ ,  $\tilde{h}$  by the formulas  $\tilde{q} = q + r$ ,  $\tilde{h} = h + p$ . Clearly, (1.8.17) is valid. Thus, we have constructed the boundary value problems  $\tilde{L}_i$ .

It remains to be shown that the numbers  $\{\tilde{\lambda}_{ni}\}_{n \geq 0}$  are the eigenvalues of the constructed boundary value problems  $\tilde{L}_i$ . For this purpose we consider the functions  $\tilde{y}_{ni}(x)$ , which are solutions of equation (1.8.8). Then (1.8.9)-(1.8.10) hold. Clearly,

$$-\tilde{y}_{ni}''(x) + \tilde{q}(x)\tilde{y}_{ni}(x) = \tilde{\lambda}_{ni}\tilde{y}_{ni}(x), \quad \tilde{y}_{ni}(0) = 1, \quad \tilde{y}_{ni}'(0) = \tilde{h},$$

and consequently (1.8.6) is valid.

Furthermore, multiplying (1.8.13) by  $\eta_n(x)$ , integrating with respect to  $x$  and taking (1.8.9)-(1.8.10) into account, we obtain

$$\int_0^\pi r(x)y_{n1}(x)\tilde{y}_{n1}(x)dx = y_{n1}(\pi)\tilde{y}_{n1}'(\pi) - p,$$

$$\int_0^\pi r(x)y_{n2}(x)\tilde{y}_{n2}(x)dx = -y_{n2}'(\pi)\tilde{y}_{n2}(\pi) - p.$$

Comparing these relations with (1.8.6) and using (1.8.18), we conclude that (1.8.7) holds, i.e.  $\tilde{y}_{ni}(x)$  are eigenfunctions, and  $\{\tilde{\lambda}_{ni}\}_{n \geq 0}$  are eigenvalues for  $\tilde{L}_i$ . The uniqueness follows from Borg's theorem (see Theorem 1.4.4).  $\square$

**1.8.3. The case of Dirichlet boundary conditions.** Similar results are also valid for Dirichlet boundary conditions; in this subsection we prove the corresponding theorem (see Theorem 1.8.2). The most interesting part of this subsection is Borg's original method for proving that products of eigenfunctions form a Riesz basis in  $L_2(0, \pi)$  (see Remark 1.8.1).

Let  $\lambda_{ni}^0 = (\rho_{ni}^0)^2$ ,  $n \geq 1$ ,  $i = 1, 2$ , be the eigenvalues of the boundary value problems  $L_i^0$  of the form

$$-y'' + q(x)y = \lambda y, \quad q(x) \in L_2(0, \pi), \quad (1.8.23)$$

$$y(0) = y^{(i-1)}(\pi) = 0 \quad (1.8.24)$$

with real  $q$ . Then (see Section 1.1)

$$\rho_{n1}^0 = n + \frac{a^0}{n} + \frac{\kappa_{n1}^0}{n}, \quad \rho_{n2}^0 = \left(n - \frac{1}{2}\right) + \frac{a^0}{n} + \frac{\kappa_{n2}^0}{n}, \quad \{\kappa_{ni}^0\} \in l_2, \quad a^0 = \frac{1}{2\pi} \int_0^\pi q(t) dt.$$

Let  $L_i^0$  and  $\tilde{L}_i^0$ ,  $i = 1, 2$  be such that  $a^0 = \tilde{a}^0$ , then

$$\Lambda^0 := \left( \sum_{n=1}^{\infty} (|\lambda_{n1}^0 - \tilde{\lambda}_{n1}^0|^2 + |\lambda_{n2}^0 - \tilde{\lambda}_{n2}^0|^2) \right)^{1/2} < \infty.$$

Hence

$$\int_0^\pi r(t) dt = 0, \quad (1.8.25)$$

where  $r = \tilde{q} - q$ .

Acting in the same way as in Subsection 1.8.1 and using the same notations we get

$$\int_0^\pi r(x) s_{ni}(x) \tilde{s}_{ni}(x) dx = s_{ni}(\pi) \tilde{s}'_{ni}(\pi) - s'_{ni}(\pi) \tilde{s}_{ni}(\pi), \quad n \geq 1, i = 1, 2. \quad (1.8.26)$$

$$\tilde{s}_{n1}(\pi) = 0, \quad \tilde{s}'_{n2}(\pi) = 0, \quad n \geq 1. \quad (1.8.27)$$

$$\tilde{s}_{ni}(x) = s_{ni}(x) + \int_0^\pi G_{ni}(x, t) r(t) \tilde{s}_{ni}(t) dt, \quad n \geq 1, i = 1, 2. \quad (1.8.28)$$

Solving (1.8.28) by the method of successive approximations we obtain

$$\tilde{s}_{ni}(x) = s_{ni}(x) + \psi_{ni}(x), \quad (1.8.29)$$

where

$$\begin{aligned} \psi_{ni}(x) = \sum_{j=1}^{\infty} \underbrace{\int_0^\pi \dots \int_0^\pi}_{j} G_{ni}(x, t_1) G_{ni}(t_1, t_2) \dots G_{ni}(t_{j-1}, t_j) \times \\ r(t_1) \dots r(t_j) s_{ni}(t_j) dt_1 \dots dt_j. \end{aligned} \quad (1.8.30)$$

Substituting (1.8.29)-(1.8.30) into (1.8.26) and using (1.8.27), we get

$$\int_0^\pi r(x) u_{ni}(x) dx = \omega_{ni}^0, \quad n \geq 1, i = 1, 2, \quad (1.8.31)$$

where

$$\begin{aligned} u_{ni}^0(x) = 2n^2 s_{ni}^2(x), \\ \left. \begin{aligned} \omega_{n1}^0 &= 2n^2 \left( s_{n1}(\pi) (s'_{n1}(\pi) + \psi'_{n1}(\pi)) - \int_0^\pi r(x) s_{n1}(x) \psi_{n1}(x) dx \right), \\ \omega_{n2}^0 &= 2n^2 \left( -s'_{n2}(\pi) (s_{n2}(\pi) + \psi_{n2}(\pi)) - \int_0^\pi r(x) s_{n2}(x) \psi_{n2}(x) dx \right). \end{aligned} \right\} \end{aligned} \quad (1.8.32)$$

We introduce the functions  $\{u_n(x)\}_{n \geq 1}$  and the numbers  $\{\omega_n\}_{n \geq 0}$  by the formulae

$$u_{2n}(x) := u_{n1}^0(x), \quad u_{2n-1}(x) := u_{n2}^0(x), \quad \omega_{2n} := \omega_{n1}^0, \quad \omega_{2n-1} := \omega_{n2}^0, \quad (n \geq 1), \quad \omega_0 := 0.$$

Then (1.8.31) takes the form

$$\int_0^\pi r(x)u_n(x) dx = \omega_n, \quad n \geq 1. \quad (1.8.33)$$

Denote

$$v_n(x) := \frac{1}{n}u'_n(x), \quad w_n(x) := 1 - u_n(x) \quad (n \geq 1), \quad w_0(x) := 1.$$

Using the results of Section 1.1, one can easily calculate

$$v_n(x) = \sin nx + O\left(\frac{1}{n}\right), \quad w_n(x) = \cos nx + O\left(\frac{1}{n}\right), \quad n \rightarrow \infty. \quad (1.8.34)$$

By virtue of (1.8.25) and (1.8.33) we have

$$\int_0^\pi r(x)w_n(x) dx = -\omega_n, \quad n \geq 0. \quad (1.8.35)$$

**Lemma 1.8.1.** *The sets of the functions  $\{v_n(x)\}_{n \geq 1}$  and  $\{w_n(x)\}_{n \geq 0}$  are both Riesz bases in  $L_2(0, \pi)$ .*

*Proof.* In order to prove the lemma we could use the transformation operator as above, but we provide here another method which is due to G. Borg [Bor1].

Let  $q(x) \in W_2^1$ , and let  $y$  be a solution of (1.8.23) satisfying the condition  $y(0) = 0$ . Similarly, let  $\tilde{y}$  be such that  $-\tilde{y}'' + \tilde{q}(x)\tilde{y} = \lambda\tilde{y}$ ,  $\tilde{y}(0) = 0$ , where  $\tilde{q}(x) \in W_2^1$ . Denote

$$u = y\tilde{y}.$$

Then

$$\begin{aligned} u' &= y\tilde{y}' + y'\tilde{y}, \quad u'' = -2\lambda u + (q + \tilde{q})u + 2y'\tilde{y}', \\ u''' + 4\lambda u' - (q + \tilde{q})u' - (q' + \tilde{q}')u &= 2(qy\tilde{y}' + \tilde{q}y'\tilde{y}). \end{aligned}$$

From this, taking into account the relations

$$2(qy\tilde{y}' + \tilde{q}y'\tilde{y}) = (q + \tilde{q})u' + (q - \tilde{q})(y\tilde{y}' - y'\tilde{y}), \quad (y\tilde{y}' - y'\tilde{y})(x) = \int_0^x (\tilde{q}(s) - q(s))u(s) ds,$$

we derive

$$u''' + 4\lambda u' - 2(q(x) + \tilde{q}(x))u' - (q'(x) + \tilde{q}'(x))u = (q(x) - \tilde{q}(x)) \int_0^x (\tilde{q}(s) - q(s))u(s) ds.$$

Denote  $v = u'$ . Since  $u(0) = u'(0) = 0$ , we have  $v(0) = 0$ ,  $u(x) = \int_0^x v(s) ds$ , and consequently

$$-v'' + 2(q(x) + \tilde{q}(x))v + \int_0^x N(x, s)v(s) ds = 4\lambda v, \quad (1.8.36)$$

where

$$N(x, s) = q'(x) + \tilde{q}'(x) + (q(x) - \tilde{q}(x)) \int_s^x (\tilde{q}(\xi) - q(\xi)) d\xi.$$

In particular, for  $\tilde{q} = q$  we obtain that the functions  $\{v_n(x)\}_{n \geq 1}$  are the eigenfunctions for the boundary value problem

$$-v'' + p(x)v + \int_0^x M(x, s)v(s) ds = 4\lambda v, \quad v(0) = v(\pi) = 0,$$

where  $p(x) = 4q(x)$ ,  $M(x, t) = 2q'(x)$ . Then for the sequence  $\{v_n(x)\}_{n \geq 1}$ , there exists the biorthonormal sequence  $\{v_n^*(x)\}_{n \geq 1}$  in  $L_2(0, \pi)$  ( $v_n^*(x)$  are eigenfunctions of the adjoint boundary value problem

$$-v^{*''} + p(x)v^* + \int_x^\pi M(s, x)v^*(s) ds = 4\lambda v^*, \quad v^*(0) = v^*(\pi) = 0.$$

By virtue of (1.8.34) and Proposition 1.8.4 this gives that  $\{v_n(x)\}_{n \geq 1}$  is a Riesz basis in  $L_2(0, \pi)$ .

Furthermore, let us show that the system of the functions  $\{w_n(x)\}_{n \geq 0}$  is complete in  $L_2(0, \pi)$ . Indeed, suppose that

$$\int_0^\pi f(x)w_n(x) dx = 0, \quad n \geq 0, \quad f(x) \in L_2(0, \pi).$$

This yields

$$\int_0^\pi f(x) dx = 0,$$

and consequently

$$\int_0^\pi f(x)u_n(x) dx = 0, \quad n \geq 1.$$

Integrating by parts we infer, using  $v_n(0) = v_n(\pi) = 0$ , that

$$\int_0^\pi v_n(x) dx \int_x^\pi f(t) dt = 0, \quad n \geq 1.$$

The system  $\{v_n(x)\}_{n \geq 1}$  is complete in  $L_2(0, \pi)$ . Hence  $\int_x^\pi f(t) dt = 0$  for  $x \in [0, \pi]$ , i.e.  $f(x) = 0$  a.e. on  $(0, \pi)$ . Therefore  $\{w_n(x)\}_{n \geq 0}$  is complete in  $L_2(0, \pi)$ . Since  $\{w_n(x)\}_{n \geq 0}$  is, in view of (1.8.34), quadratically close to the Riesz basis  $\{\cos nx\}_{n \geq 0}$ , it follows from Proposition 1.8.5 that  $\{w_n(x)\}_{n \geq 0}$  is a Riesz basis in  $L_2(0, \pi)$ .  $\square$

**Remark 1.8.2.** With the help of equation (1.8.36) one can also prove the completeness of the products of eigenfunctions  $s_{ni}(x)\tilde{s}_{ni}(x)$ , and hence provide another method of proving Borg's uniqueness theorem (see Section 1.4) without using the transformation operator.

Denote by  $\{v_n^*(x)\}_{n \geq 1}$  and  $\{w_n^*(x)\}_{n \geq 0}$  the biorthonormal bases to  $\{v_n(x)\}_{n \geq 1}$  and  $\{w_n(x)\}_{n \geq 0}$  respectively. Then from (1.8.35) we infer (like in Subsection 1.8.1)

$$r(x) = f^0(x) + \sum_{j=1}^{\infty} \underbrace{\int_0^\pi \dots \int_0^\pi}_{j} H_j^0(x, t_1, \dots, t_j) r(t_1) \dots r(t_j) dt_1 \dots dt_j, \quad (1.8.37)$$

where

$$\begin{aligned} f^0(x) &= - \sum_{n=1}^{\infty} 2n^2 (s'_{n1}(\pi) s_{n1}(\pi) v_{2n}^*(x) - s'_{n2}(\pi) s_{n2}(\pi) v_{2n-1}^*(x)), \\ H_1^0(x, t_1) &= - \sum_{n=1}^{\infty} 2n^2 \left( s_{n1}(\pi) \frac{\partial G_{n1}(x, t_1)}{\partial x} \Big|_{x=\pi} s_{n1}(t_1) v_{2n}^*(x) - s'_{n2}(\pi) G_{n2}(\pi, t_1) s_{n2}(t_1) v_{2n-1}^*(x) \right), \\ H_j^0(x, t_1, \dots, t_j) &= - \sum_{n=1}^{\infty} 2n^2 \left( \left( s_{n1}(\pi) \frac{\partial G_{n1}(x, t_1)}{\partial x} \Big|_{x=\pi} - s_{n1}(t_1) \right) G_{n1}(t_1, t_2) \dots G_{n1}(t_{j-1}, t_j) \right. \end{aligned}$$

$$s_{n1}(t_j)v_{2n}^*(x) - \left(s'_{n2}(\pi)G_{n2}(\pi, t_1) + s_{n2}(t_1)\right)G_{n2}(t_1, t_2) \dots G_{n2}(t_{j-1}, t_j)s_{n2}(t_j)v_{2n-1}^*(x), \quad j \geq 2.$$

Using the nonlinear equation (1.8.37), by the same arguments as in Subsection 1.8.2, we arrive at the following theorem.

**Theorem 1.8.2.** *For the boundary value problems  $L_i^0$  of the form (1.8.23)-(1.8.24), there exists  $\delta > 0$  such that if real numbers  $\{\tilde{\lambda}_{ni}^0\}_{n \geq 1}$ ,  $i = 1, 2$ , satisfy the condition  $\Lambda^0 < \delta$ , then there exists a unique real function  $\tilde{q}(x) \in L_2(0, \pi)$ , for which the numbers  $\{\tilde{\lambda}_{ni}^0\}_{n \geq 1}$ ,  $i = 1, 2$ , are the eigenvalues of  $\tilde{L}_i^0$ . Moreover,*

$$\|q - \tilde{q}\|_{L_2} < C\Lambda^0.$$

**Remark 1.8.3.** Using the Riesz basis  $\{v_n(x)\}_{n \geq 1}$  we can also derive another nonlinear equation. Indeed, denote

$$z(x) := \int_x^\pi r(t) dt.$$

After integration by parts, (1.8.33) takes the form

$$\int_0^\pi z(x)v_n(x) dx = \frac{\omega_n}{n}, \quad n \geq 1.$$

Since  $\{v_n(x)\}_{n \geq 1}$  and  $\{v_n^*(x)\}_{n \geq 1}$  are biorthonormal bases, this yields

$$z(x) = \sum_{n=1}^{\infty} \frac{\omega_n^0}{n} v_n^*(x)$$

or

$$\int_x^\pi r(t) dt = \sum_{n=1}^{\infty} \frac{\omega_n}{n} v_n^*(x), \quad r(x) = - \sum_{n=1}^{\infty} \frac{\omega_n}{n} \frac{d}{dx} v_n^*(x).$$

From this, in view of (1.8.30) and (1.8.32), we get analogously to Subsection 1.8.1

$$r(x) = f_1(x) + \sum_{j=1}^{\infty} \underbrace{\int_0^\pi \dots \int_0^\pi}_{j} H_{j1}(x, t_1, \dots, t_j) r(t_1) \dots r(t_j) dt_1 \dots dt_j,$$

where

$$f_1(x) = - \sum_{n=1}^{\infty} 2n(s'_{n1}(\pi)s_{n1}(\pi) \frac{d}{dx} v_{2n}^*(x) - s'_{n2}(\pi)s_{n2}(\pi) \frac{d}{dx} v_{2n-1}^*(x)),$$

$$H_{11}(x, t_1) = - \sum_{n=1}^{\infty} 2n \left( s_{n1}(\pi) \frac{\partial G_{n1}(x, t_1)}{\partial x} \Big|_{x=\pi} s_{n1}(t_1) \frac{d}{dx} v_{2n}^*(x) - s'_{n2}(\pi) G_{n2}(\pi, t_1) s_{n2}(t_1) \frac{d}{dx} v_{2n-1}^*(x) \right),$$

$$H_{j1}(x, t_1, \dots, t_j) = - \sum_{n=1}^{\infty} 2n \left( \left( s_{n1}(\pi) \frac{\partial G_{n1}(x, t_1)}{\partial x} \Big|_{x=\pi} - s_{n1}(t_1) \right) G_{n1}(t_1, t_2) \dots G_{n1}(t_{j-1}, t_j) s_{n1}(t_j) \frac{d}{dx} v_{2n}^*(x) \right.$$

$$\left. - \left( s'_{n2}(\pi) G_{n2}(\pi, t_1) + s_{n2}(t_1) \right) G_{n2}(t_1, t_2) \dots G_{n2}(t_{j-1}, t_j) s_{n2}(t_j) \frac{d}{dx} v_{2n-1}^*(x) \right), \quad j \geq 2.$$



**1.8.4. Stability of the solution of the inverse problem in the uniform norm.** Let  $\lambda_{ni} = \rho_{ni}^2$ ,  $n \geq 0$ ,  $i = 1, 2$ , be the eigenvalues of the boundary value problems  $L_i$  of the form (1.8.1)-(1.8.2), where  $q$  is a real continuous function, and  $h$  is a real number. The eigenvalues  $\{\lambda_{ni}\}$  coincide with zeros of the characteristic functions  $\Delta_i(\lambda) := \varphi^{(i-1)}(\pi, \lambda)$ ,  $i = 1, 2$ , where  $\varphi(x, \lambda)$  is the solution of (1.8.1) under the conditions  $\varphi(0, \lambda) = 1$ ,  $\varphi'(0, \lambda) = h$ . Without loss of generality we shall assume in the sequel that in (1.8.3)  $a = 0$ . In this case, by virtue of (1.8.3), we have

$$\sum_{n=0}^{\infty} (|\rho_{n1} - (n + 1/2)| + |\rho_{n2} - n|) < \infty. \quad (1.8.38)$$

Let the boundary value problems  $\tilde{L}_i$  be chosen such that

$$\Lambda_1 := \sum_{n=0}^{\infty} (|\lambda_{n1} - \tilde{\lambda}_{n1}| + |\lambda_{n2} - \tilde{\lambda}_{n2}|) < \infty. \quad (1.8.39)$$

The quantity  $\Lambda_1$  describes the distance of the spectra.

**Theorem 1.8.3.** *There exists  $\delta > 0$  (which depends on  $L_i$ ) such that if  $\Lambda_1 < \delta$  then*

$$\max_{0 \leq x \leq \pi} |q(x) - \tilde{q}(x)| < C\Lambda_1, \quad |h - \tilde{h}| < C\Lambda_1.$$

Here and below,  $C$  denotes various positive constants which depend on  $L_i$ .

We first prove some auxiliary propositions. Let

$$\alpha_n := \int_0^\pi \varphi^2(x, \lambda_{n2}) dx$$

be the weight numbers for  $L_2$ .

**Lemma 1.8.2.** *There exists  $\delta_1 > 0$  such that if  $\Lambda_1 < \delta_1$  then*

$$\sum_{n=0}^{\infty} |\alpha_n - \tilde{\alpha}_n| < C\Lambda_1. \quad (1.8.40)$$

*Proof.* According to (1.1.38),

$$\alpha_n = -\dot{\Delta}_2(\lambda_{n2})\Delta_1(\lambda_{n2}), \quad (1.8.41)$$

where  $\dot{\Delta}_2(\lambda) = \frac{d}{d\lambda}\Delta_2(\lambda)$ .

The functions  $\Delta_i(\lambda)$  are entire in  $\lambda$  of order  $1/2$ , and consequently by Hadamard's factorization theorem [con1, p.289], we have

$$\Delta_i(\lambda) = B_i \prod_{k=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_{ki}}\right) \quad (1.8.42)$$

(the case when  $\lambda = 0$  is the eigenvalue of  $L_i$  requires minor modifications). Then

$$\frac{\tilde{\Delta}_i(\lambda)}{\Delta_i(\lambda)} = \frac{\tilde{B}_i}{B_i} \prod_{k=0}^{\infty} \frac{\lambda_{ki}}{\tilde{\lambda}_{ki}} \prod_{k=0}^{\infty} \left(1 + \frac{\tilde{\lambda}_{ki} - \lambda_{ki}}{\lambda_{ki} - \lambda}\right).$$

Since

$$\lim_{\lambda \rightarrow -\infty} \frac{\tilde{\Delta}_i(\lambda)}{\Delta_i(\lambda)} = 1, \quad \lim_{\lambda \rightarrow -\infty} \prod_{k=0}^{\infty} \left(1 + \frac{\tilde{\lambda}_{ki} - \lambda_{ki}}{\lambda_{ki} - \lambda}\right) = 1,$$

it follows that

$$\frac{\tilde{B}_i}{B_i} \prod_{k=0}^{\infty} \frac{\lambda_{ki}}{\tilde{\lambda}_{ki}} = 1, \quad (1.8.43)$$

and consequently

$$\frac{\tilde{\Delta}_i(\lambda)}{\Delta_i(\lambda)} = \prod_{k=0}^{\infty} \frac{\tilde{\lambda}_{ki} - \lambda}{\lambda_{ki} - \lambda}.$$

Furthermore, it follows from (1.8.42) that

$$\dot{\Delta}_2(\lambda_{n2}) = -\frac{B_2}{\lambda_{n2}} \prod_{\substack{k=0 \\ k \neq n}}^{\infty} \left(1 - \frac{\lambda_{n2}}{\lambda_{k2}}\right).$$

Therefore, taking (1.8.43) into account we calculate

$$\frac{\dot{\tilde{\Delta}}_2(\lambda_{n2})}{\dot{\Delta}_2(\lambda_{n2})} = \prod_{\substack{k=0 \\ k \neq n}}^{\infty} \frac{\tilde{\lambda}_{k2} - \tilde{\lambda}_{n2}}{\lambda_{k2} - \lambda_{n2}}.$$

Then, by virtue of (1.8.41),

$$\frac{\tilde{\alpha}_n}{\alpha_n} = \prod_{k=0}^{\infty} \frac{\tilde{\lambda}_{k1} - \tilde{\lambda}_{n2}}{\lambda_{k1} - \lambda_{n2}} \prod_{\substack{k=0 \\ k \neq n}}^{\infty} \frac{\tilde{\lambda}_{k2} - \tilde{\lambda}_{n2}}{\lambda_{k2} - \lambda_{n2}}$$

or

$$\frac{\tilde{\alpha}_n}{\alpha_n} = \prod_{k=0}^{\infty} (1 - \theta_{kn}^{(1)}) \prod_{\substack{k=0 \\ k \neq n}}^{\infty} (1 - \theta_{kn}^{(2)}), \quad (1.8.44)$$

where

$$\theta_{kn}^{(i)} = \frac{\lambda_{ki} - \tilde{\lambda}_{ki}}{\lambda_{ki} - \lambda_{n2}} + \frac{\tilde{\lambda}_{n2} - \lambda_{n2}}{\lambda_{ki} - \lambda_{n2}}.$$

Denote

$$\theta_n = \sum_{k=0}^{\infty} |\theta_{kn}^{(1)}| + \sum_{\substack{k=0 \\ k \neq n}}^{\infty} |\theta_{kn}^{(2)}|.$$

Then

$$\begin{aligned} \sum_{n=0}^{\infty} \theta_n &\leq \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left( \frac{|\lambda_{k1} - \tilde{\lambda}_{k1}|}{|\lambda_{k1} - \lambda_{n2}|} + \frac{|\tilde{\lambda}_{n2} - \lambda_{n2}|}{|\lambda_{k1} - \lambda_{n2}|} \right) \\ &\quad + \sum_{n=0}^{\infty} \sum_{\substack{k=0 \\ k \neq n}}^{\infty} \left( \frac{|\lambda_{k2} - \tilde{\lambda}_{k2}|}{|\lambda_{k2} - \lambda_{n2}|} + \frac{|\tilde{\lambda}_{n2} - \lambda_{n2}|}{|\lambda_{k2} - \lambda_{n2}|} \right) \\ &\leq \sum_{n=0}^{\infty} |\lambda_{n2} - \tilde{\lambda}_{n2}| \left( 2 \sum_{\substack{k=0 \\ k \neq n}}^{\infty} \frac{1}{|\lambda_{k2} - \lambda_{n2}|} + \sum_{k=0}^{\infty} \frac{1}{|\lambda_{k1} - \lambda_{n2}|} \right) \end{aligned}$$

$$+ \sum_{n=0}^{\infty} |\lambda_{n1} - \tilde{\lambda}_{n1}| \sum_{k=0}^{\infty} \frac{1}{|\lambda_{n1} - \lambda_{k2}|}. \quad (1.8.45)$$

Using the asymptotics of the eigenvalues (see Section 1.1), we obtain

$$\frac{1}{|\lambda_{k2} - \lambda_{n2}|} < \frac{C}{|k^2 - n^2|}, \quad k \neq n.$$

Since

$$\sum_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{1}{|k^2 - n^2|} < 1, \quad n \geq 1,$$

we have

$$\sum_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{1}{|\lambda_{k2} - \lambda_{n2}|} < C. \quad (1.8.46)$$

Similarly,

$$\sum_{k=0}^{\infty} \left( \frac{1}{|\lambda_{k1} - \lambda_{n2}|} + \frac{1}{|\lambda_{n1} - \lambda_{k2}|} \right) < C. \quad (1.8.47)$$

It follows from (1.8.45)-(1.8.47) that

$$\sum_{n=0}^{\infty} \theta_n < C\Lambda_1. \quad (1.8.48)$$

Choose  $\delta_1 > 0$  such that if  $\Lambda_1 < \delta_1$  then  $\theta_n < 1/4$ . Since for  $|\xi| < 1/2$ ,

$$|\ln(1 - \xi)| \leq \sum_{k=1}^{\infty} \frac{|\xi|^k}{k} \leq \sum_{k=1}^{\infty} |\xi|^k \leq 2|\xi|,$$

it follows from (1.8.44) that

$$\left| \ln \frac{\tilde{\alpha}_n}{\alpha_n} \right| \leq \sum_{k=0}^{\infty} |\ln(1 - \theta_{kn}^{(1)})| + \sum_{\substack{k=0 \\ k \neq n}}^{\infty} |\ln(1 - \theta_{kn}^{(2)})| < 2\theta_n.$$

Using the properties of  $\ln$ , we get  $\left| \frac{\tilde{\alpha}_n}{\alpha_n} - 1 \right| < 4\theta_n$  or  $|\tilde{\alpha}_n - \alpha_n| < C\theta_n$ . Using (1.8.48) we arrive at (1.8.40).  $\square$

**Lemma 1.8.3.** *There exists  $\delta_2 > 0$  such that if  $\Lambda_1 < \delta_2$  then*

$$|G(x, t) - \tilde{G}(x, t)| < C\Lambda_1, \quad 0 \leq x \leq t \leq \pi, \quad (1.8.49)$$

$$|\varphi'(\pi, \lambda_{n1})\tilde{\varphi}(\pi, \lambda_{n1})| < |\lambda_{n1} - \tilde{\lambda}_{n1}|, \quad (1.8.50)$$

$$|\varphi'(\pi, \tilde{\lambda}_{n2})\tilde{\varphi}(\pi, \tilde{\lambda}_{n2})| < C|\lambda_{n2} - \tilde{\lambda}_{n2}|. \quad (1.8.51)$$

*Proof.* The function  $G(x, t)$  is the solution of the Gelfand-Levitan equation (1.5.11). By virtue of Lemma 1.5.6,  $|\hat{F}(x, t)| < C\Lambda_1$ . Hence, from the unique solvability of the Gelfand-Levitan equation and from Lemma 1.5.1 we obtain (1.8.49).

Furthermore, it follows from (1.1.9) and (1.8.3) that

$$|\varphi'(\pi, \lambda_{n1})| < C(n+1).$$

Using (1.5.11) we get

$$\tilde{\varphi}(\pi, \lambda_{n1}) = \tilde{\varphi}(\pi, \lambda_{n1}) - \tilde{\varphi}(\pi, \tilde{\lambda}_{n1}) = (\cos \rho_{n1}\pi - \cos \tilde{\rho}_{n1}\pi) + \int_0^\pi \tilde{G}(\pi, t)(\cos \rho_{n1}t - \cos \tilde{\rho}_{n1}t) dt.$$

Hence, for  $\Lambda_1 < \delta_2$  we have in view of (1.8.49) that

$$|\tilde{\varphi}(\pi, \lambda_{n1})| < C|\rho_{n1} - \tilde{\rho}_{n1}|,$$

and we arrive at (1.8.50). Similarly we obtain (1.8.51).  $\square$

**Lemma 1.8.4.** *Let  $g(x)$  be a continuous function on  $[0, \pi]$ , and let  $\{z_n\}_{n \geq 0}$  be numbers such that*

$$\sum_{n=0}^{\infty} |z_n - n| < \infty.$$

*Suppose that*

$$\Theta := \sum_{n=0}^{\infty} |\varepsilon_n| < \infty, \quad \varepsilon_n := \int_0^\pi g(x) \cos z_n x dx.$$

*Then  $|g(x)| < M\Theta$ , where the constant  $M$  depends only on the set  $\{z_n\}_{n \geq 0}$ .*

*Proof.* Since the system of the functions  $\{\cos z_n x\}_{n \geq 0}$  is complete in  $L_2(0, \pi)$ , it follows that  $\varepsilon_n$  uniquely determine the function  $g(x)$ . From the equality

$$\int_0^\pi g(x) \cos nx dx = \varepsilon_n + \int_0^\pi g(x)(\cos nx - \cos z_n x) dx$$

we obtain

$$g(x) = \sum_{n=0}^{\infty} \frac{\varepsilon_n}{\alpha_n^0} \cos nx + \sum_{n=0}^{\infty} \frac{1}{\alpha_n^0} \cos nx \int_0^\pi (\cos nt - \cos z_n t) g(t) dt.$$

Hence, the function  $g(x)$  is the solution of the integral equation

$$g(x) = \varepsilon(x) + \int_0^\pi H(x, t) g(t) dt, \tag{1.8.52}$$

where

$$\varepsilon(x) = \sum_{n=0}^{\infty} \frac{\varepsilon_n}{\alpha_n^0} \cos nx, \quad H(x, t) = \sum_{n=0}^{\infty} \frac{1}{\alpha_n^0} \cos nx (\cos nt - \cos z_n t).$$

The series converge absolutely and uniformly for  $0 \leq x, t \leq \pi$ , and

$$|\varepsilon(x)| < \frac{2}{\pi} \Theta, \quad |H(x, t)| < C \sum_{n=0}^{\infty} |z_n - n|.$$

Let us show that the homogeneous equation

$$y(x) = \int_0^\pi H(x, t) y(t) dt, \quad y(x) \in C[0, \pi] \tag{1.8.53}$$

has only the trivial solution. Indeed, it follows from (1.8.53) that

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{\alpha_n^0} \cos nx \int_0^\pi (\cos nt - \cos z_n t) y(t) dt.$$

Hence

$$\int_0^\pi y(t) \cos nt \, dt = \int_0^\pi (\cos nt - \cos z_n t) y(t) \, dt$$

or

$$\int_0^\pi y(t) \cos z_n t \, dt = 0.$$

From this, in view of the completeness of the system  $\{\cos z_n x\}_{n \geq 0}$ , follows that  $y(x) = 0$ .

Thus, the integral equation (1.8.52) is uniquely solvable, and hence  $|g(x)| < M\Theta$ .  $\square$

**Remark 1.8.4.** It is easy to see from the proof of Lemma 1.8.4 that if we consider numbers  $\{\tilde{z}_n\}$  such that

$$\sum_{n=0}^{\infty} |\tilde{z}_n - z_n| < \delta$$

for sufficiently small  $\delta$ , then the constant  $M$  will not depend on  $\tilde{z}_n$ .

*Proof of Theorem 1.8.3.* Using

$$-\varphi''(x, \lambda) + q(x)\varphi(x, \lambda) = \lambda\varphi(x, \lambda), \quad -\tilde{\varphi}''(x, \lambda) + \tilde{q}(x)\tilde{\varphi}(x, \lambda) = \lambda\tilde{\varphi}(x, \lambda),$$

we get

$$\int_0^\pi \hat{q}(x)\varphi(x, \lambda)\tilde{\varphi}(x, \lambda) \, dx = \varphi'(\pi, \lambda)\tilde{\varphi}(\pi, \lambda) - \varphi(\pi, \lambda)\tilde{\varphi}'(\pi, \lambda) + \tilde{h} - h.$$

Since

$$\hat{h} + \frac{1}{2} \int_0^\pi \hat{q}(x) \, dx = 0,$$

we get

$$\int_0^\pi \hat{q}(x) \left( \varphi(x, \lambda)\tilde{\varphi}(x, \lambda) - \frac{1}{2} \right) \, dx = \varphi'(\pi, \lambda)\tilde{\varphi}(\pi, \lambda) - \varphi(\pi, \lambda)\tilde{\varphi}'(\pi, \lambda). \quad (1.8.54)$$

Substituting (1.4.7) into (1.8.54) we obtain for  $\lambda = \lambda_{n1}$  and  $\lambda = \tilde{\lambda}_{n2}$ ,

$$\int_0^\pi g(x) \cos 2\rho_{n1} x \, dx = \varphi'(\pi, \lambda_{n1})\tilde{\varphi}(\pi, \lambda_{n1}),$$

$$\int_0^\pi g(x) \cos 2\tilde{\rho}_{n2} x \, dx = \varphi'(\pi, \tilde{\lambda}_{n2})\tilde{\varphi}(\pi, \tilde{\lambda}_{n2}),$$

where

$$g(x) = 2 \left( \hat{q}(x) + \int_x^\pi V(x, t)\hat{q}(t) \, dt \right). \quad (1.8.55)$$

Taking (1.4.8) and (1.8.49) into account, we get for  $\Lambda_1 < \delta_2$ ,

$$|V(x, t)| < C. \quad (1.8.56)$$

We use Lemma 1.8.4 for  $z_{2n+1} = 2\rho_{n1}$ ,  $z_{2n} = 2\tilde{\rho}_{n2}$ ,  $\varepsilon_{2n+1} = \varphi'(\pi, \lambda_{n1})\tilde{\varphi}(\pi, \lambda_{n1})$ ,  $\varepsilon_{2n} = \varphi'(\pi, \tilde{\lambda}_{n2})\tilde{\varphi}(\pi, \tilde{\lambda}_{n2})$ . It follows from (1.8.35), (1.8.36), (1.8.50) and (1.8.51) that for  $\Lambda_1 < \delta_2$ ,  $|g(x)| < C\Lambda_1$ . Since  $\hat{q}(x)$  is the solution of the Volterra integral equation (1.8.55), it follows from (1.8.56) that  $|\hat{q}(x)| < C\Lambda_1$ , and hence  $|\hat{h}| < C\Lambda_1$ .  $\square$

**1.8.5. Riesz bases.** 1) For convenience of the reader we provide here some facts about Riesz bases in a Hilbert space. For further discussion and proofs see [goh1], [you1] and [Bar1]. Let  $B$  be a Hilbert space with the scalar product  $(\cdot, \cdot)$ .

**Definition 1.8.1.** A sequence  $\{f_j\}_{j \geq 1}$  of vectors of a Hilbert space  $B$  is called a *basis* of this space if every vector  $f \in B$  can be expanded in a unique way in a series

$$f = \sum_{j=1}^{\infty} c_j f_j, \quad (1.8.57)$$

which converges in the norm of the space  $B$ .

Clearly, if  $\{f_j\}_{j \geq 1}$  is a basis then  $\{f_j\}_{j \geq 1}$  is complete and minimal in  $B$ . We remind that " $\{f_j\}_{j \geq 1}$  is complete" means that the closed linear span of  $\{f_j\}_{j \geq 1}$  coincides with  $B$ , and " $\{f_j\}_{j \geq 1}$  is minimal" means that each element of the sequence lies outside the closed linear span of the others.

**Definition 1.8.2.** Two sequences  $\{f_j\}_{j \geq 1}$  and  $\{\chi_j\}_{j \geq 1}$  of  $B$  are called *biorthonormal* if  $(f_j, \chi_k) = \delta_{jk}$  ( $\delta_{jk}$  is the Kronecker delta).

If  $\{f_j\}_{j \geq 1}$  is a basis then the biorthonormal sequence  $\{\chi_j\}_{j \geq 1}$  exists and it is defined uniquely. Moreover,  $\{\chi_j\}_{j \geq 1}$  is also a basis of  $B$ . In the expansion (1.8.57) the coefficients  $c_j$  have the form

$$c_j = (f, \chi_j). \quad (1.8.58)$$

**Definition 1.8.3.** A sequence  $\{f_j\}_{j \geq 1}$  is called *almost normalized* if

$$\inf_j \|f_j\| > 0 \quad \text{and} \quad \sup_j \|f_j\| < \infty.$$

If a basis  $\{f_j\}_{j \geq 1}$  is almost normalized then the biorthonormal basis  $\{\chi_j\}_{j \geq 1}$  is also almost normalized.

**Definition 1.8.4.** A sequence  $\{e_j\}_{j \geq 1}$  is called *orthogonal* if  $(e_j, e_k) = 0$  for  $j \neq k$ . A sequence  $\{e_j\}_{j \geq 1}$  is called *orthonormal* if  $(e_j, e_k) = \delta_{jk}$ .

If  $\{e_j\}_{j \geq 1}$  is orthonormal and complete in  $B$  then  $\{e_j\}_{j \geq 1}$  is a basis of  $B$ . For an orthonormal basis  $\{e_j\}_{j \geq 1}$  the relations (1.8.57)-(1.8.58) take the form

$$f = \sum_{j=1}^{\infty} (f, e_j) e_j, \quad (1.8.59)$$

and for each vector  $f \in B$  the following Parseval's equality holds

$$\|f\|^2 = \sum_{j=1}^{\infty} |(f, e_j)|^2.$$

2) Let now  $\{e_j\}_{j \geq 1}$  be an orthonormal basis of the Hilbert space  $B$ , and let  $A : B \rightarrow B$  be a bounded linear invertible operator. Then  $A^{-1}$  is bounded, and according to (1.8.59) we have for any vector  $f \in B$ ,

$$A^{-1}f = \sum_{j=1}^{\infty} (A^{-1}f, e_j) e_j = \sum_{j=1}^{\infty} (f, A^{*-1}e_j) e_j.$$

Consequently

$$f = \sum_{j=1}^{\infty} (f, \chi_j) f_j,$$

where

$$f_j = Ae_j, \quad \chi_j = A^{*-1}e_j. \quad (1.8.60)$$

Obviously  $(f_j, \chi_k) = \delta_{jk}$  ( $j, k \geq 1$ ), i.e. the sequences  $\{f_j\}_{j \geq 1}$  and  $\{\chi_j\}_{j \geq 1}$  are biorthonormal. Therefore if (1.8.57) is valid then  $c_j = (f, \chi_j)$ , i.e. the expansion (1.8.57) is unique. Thus, every bounded linear invertible operator transforms any orthonormal basis into some other basis of the space  $B$ .

**Definition 1.8.5.** A basis  $\{f_j\}_{j \geq 1}$  of  $B$  is called a *Riesz basis* if it is obtained from an orthonormal basis by means of a bounded linear invertible operator.

According to (1.8.60), a basis which is biorthonormal to a Riesz basis is a Riesz basis itself. Using (1.8.60) we calculate

$$\inf_j \|f_j\| \geq \|A^{-1}\| \quad \text{and} \quad \sup_j \|f_j\| \leq \|A\|,$$

i.e. every Riesz basis is almost normalized.

For the Riesz basis  $\{f_j\}_{j \geq 1}$  ( $f_j = Ae_j$ ) the following inequality is valid for all vectors  $f \in B$ :

$$C_1 \sum_{j=1}^{\infty} |(f, \chi_j)|^2 \leq \|f\|^2 \leq C_2 \sum_{j=1}^{\infty} |(f, \chi_j)|^2, \quad (1.8.61)$$

where  $\{\chi_j\}_{j \geq 1}$  is the corresponding biorthonormal basis ( $\chi_j = A^{*-1}e_j$ ), and the constants  $C_1, C_2$  depend only on the operator  $A$ .

Indeed, using Parseval's equality and (1.8.60) we calculate

$$\|A^{-1}f\|^2 = \sum_{j=1}^{\infty} |(A^{-1}f, e_j)|^2 = \sum_{j=1}^{\infty} |(f, \chi_j)|^2.$$

Since

$$\|f\| = \|AA^{-1}f\| \leq \|A\| \cdot \|A^{-1}f\|, \quad \|A^{-1}f\| \leq \|A^{-1}\| \cdot \|f\|,$$

we get

$$C_1 \|A^{-1}f\|^2 \leq \|f\|^2 \leq C_2 \|A^{-1}f\|^2,$$

where

$$C_1 = \|A^{-1}\|^{-2}, \quad C_2 = \|A\|^2,$$

and consequently (1.8.61) is valid.

The following assertions are obvious.

**Proposition 1.8.1.** If  $\{f_j\}_{j \geq 1}$  is a Riesz basis of  $B$  and  $\{\gamma_j\}_{j \geq 1}$  are complex numbers such that  $0 < C_1 \leq |\gamma_j| \leq C_2 < \infty$ , then  $\{\gamma_j f_j\}_{j \geq 1}$  is a Riesz basis of  $B$ .

**Proposition 1.8.2.** If  $\{f_j\}_{j \geq 1}$  is a Riesz basis of  $B$  and  $A$  is a bounded linear invertible operator, then  $\{Af_j\}_{j \geq 1}$  is a Riesz basis of  $B$ .

3) Now we present several theorems which give us sufficient conditions on a sequence  $\{f_j\}_{j \geq 1}$  to be a Riesz basis of  $B$ . First we formulate the definitions:

**Definition 1.8.6.** Two sequences of vectors  $\{f_j\}_{j \geq 1}$  and  $\{g_j\}_{j \geq 1}$  from  $B$  are called *quadratically close* if

$$\sum_{j=1}^{\infty} \|g_j - f_j\|^2 < \infty.$$

**Definition 1.8.7.** A sequence of vectors  $\{g_j\}_{j \geq 1}$  is called  $\omega$  - linearly independent if the equality

$$\sum_{j=1}^{\infty} c_j g_j = 0$$

is possible only for  $c_j = 0$  ( $j \geq 1$ ).

**Assumption 1.8.1.** Let  $f_j = Ae_j$ ,  $j \geq 1$  be a Riesz basis of  $B$ , where  $\{e_j\}_{j \geq 1}$  is an orthonormal basis of  $B$ , and  $A$  is a bounded linear invertible operator. Let  $\{g_j\}_{j \geq 1}$  be such that

$$\Omega := \left( \sum_{j=1}^{\infty} \|g_j - f_j\|^2 \right)^{1/2} < \infty,$$

i.e.  $\{g_j\}_{j \geq 1}$  is quadratically close to  $\{f_j\}_{j \geq 1}$ .

**Proposition 1.8.3 (stability of bases).** *Let Assumption 1.8.1 hold. If*

$$\Omega < \frac{1}{\|A^{-1}\|},$$

*then  $\{g_j\}_{j \geq 1}$  is a Riesz basis of  $B$ .*

*Proof.* Consider the operator  $T$  :

$$T\left(\sum_{j=1}^{\infty} c_j f_j\right) = \sum_{j=1}^{\infty} c_j (f_j - g_j), \quad \{c_j\} \in l_2. \quad (1.8.62)$$

In other words,  $Te_j = f_j - g_j$ . Clearly,  $T$  is a bounded linear operator and  $\|T\| \leq \Omega$ . Moreover,  $\sum_{j=1}^{\infty} \|Te_j\|^2 < \infty$ . Since

$$A - T = (E - TA^{-1})A, \quad \text{and} \quad \|TA^{-1}\| < 1,$$

then  $A - T$  is a linear bounded invertible operator. On the other hand,

$$(A - T)e_j = g_j,$$

and consequently  $\{g_j\}_{j \geq 1}$  is a Riesz basis of  $B$ . □

**Proposition 1.8.4 (Bari [Bar1]).** *Let Assumption 1.8.1 hold. If  $\{g_j\}_{j \geq 1}$  is  $\omega$ -linearly independent then  $\{g_j\}_{j \geq 1}$  is a Riesz basis of  $B$ .*

*Proof.* Define the operator  $T$  by (1.8.62). The equation  $(A - T)f = 0$  has only the trivial solution. Indeed, if  $(A - T)f = 0$  then from

$$(A - T)f = \sum_{j=1}^{\infty} (f, e_j) f_j - \sum_{j=1}^{\infty} (f, e_j) (f_j - g_j) = \sum_{j=1}^{\infty} (f, e_j) g_j$$

it follows that

$$\sum_{j=1}^{\infty} (f, e_j) g_j = 0.$$



Hence  $(f, e_j) = 0$ ,  $j \geq 1$ , by the  $\omega$ -linearly independence of  $\{g_j\}_{j \geq 1}$ . From this we get  $f = 0$ . Thus, the operator  $A - T$  is a linear bounded invertible operator. Since  $(A - T)e_j = g_j$ , the sequence  $\{g_j\}_{j \geq 1}$  is a Riesz basis of  $B$ .  $\square$

**Proposition 1.8.5.** *Let Assumption 1.8.1 hold. If  $\{g_j\}_{j \geq 1}$  is complete in  $B$  then  $\{g_j\}_{j \geq 1}$  is a Riesz basis of  $B$ .*

*Proof.* Choose  $N$  such that

$$\left( \sum_{j=N+1}^{\infty} \|g_j - f_j\|^2 \right)^{1/2} < \frac{1}{\|A^{-1}\|},$$

and consider the sequence  $\{\psi_j\}_{j \geq 1}$ :

$$\psi_j = \begin{cases} f_j, & j = \overline{1, N}, \\ g_j, & j > N. \end{cases}$$

By virtue of Proposition 1.8.3, the sequence  $\{\psi_j\}_{j \geq 1}$  is a Riesz basis of  $B$ . Let  $\{\psi_j^*\}_{j \geq 1}$  be the biorthonormal basis to  $\{\psi_j\}_{j \geq 1}$ . Denote

$$D := \det[(g_j, \psi_n^*)]_{j,n=\overline{1, N}}.$$

Let us show that  $D \neq 0$ . Suppose, on the contrary, that  $D = 0$ . Then the linear algebraic system

$$\sum_{n=1}^N \beta_n (g_j, \psi_n^*) = 0, \quad j = \overline{1, N},$$

has a non-trivial solution  $\{\beta_n\}_{n=\overline{1, N}}$ . Consider the vector

$$f := \sum_{n=1}^N \overline{\beta_n} \psi_n^*.$$

Since  $\{\psi_n^*\}_{n \geq 1}$  is a Riesz basis we get  $f \neq 0$ . On the other hand, we calculate:

$$(i) \text{ for } j = \overline{1, N}: \quad (g_j, f) = \sum_{n=1}^N \beta_n (g_j, \psi_n^*) = 0;$$

$$(ii) \text{ for } j > N: \quad (g_j, f) = (\psi_j, f) = \sum_{n=1}^N \beta_n (\psi_j, \psi_n^*) = 0.$$

Thus,  $(g_j, f) = 0$  for all  $j \geq 1$ . Since  $\{g_j\}_{j \geq 1}$  is complete we get  $f = 0$ . This contradiction shows that  $D \neq 0$ .

Let us show that  $\{g_j\}_{j \geq 1}$  is  $\omega$ -linearly independent. Indeed, let  $\{c_j\}_{j \geq 1}$  be complex numbers such that

$$\sum_{j=1}^{\infty} c_j g_j = 0. \tag{1.8.63}$$

Since  $g_j = \psi_j$  for  $j > N$  we get

$$\sum_{j=1}^N c_j (g_j, \psi_n^*) = 0, \quad n = \overline{1, N}.$$

The determinant of this linear system is equal to  $D \neq 0$ . Consequently,  $c_j = 0$ ,  $j = \overline{1, N}$ . Then (1.8.63) takes the form

$$\sum_{j=N+1}^{\infty} c_j \psi_j = 0.$$

Since  $\{\psi_j\}_{j \geq 1}$  is a Riesz basis we have  $c_j = 0$ ,  $j > N$ . Thus, the sequence  $\{g_j\}_{j \geq 1}$  is  $\omega$ -linearly independent, and by Proposition 1.8.4  $\{g_j\}_{j \geq 1}$  is a Riesz basis of  $B$ .  $\square$

**Proposition 1.8.6.** *Let numbers  $\{\rho_n\}_{n \geq 0}$ ,  $\rho_n \neq \rho_k$  ( $n \neq k$ ) of the form*

$$\rho_n = n + \frac{a}{n} + \frac{\kappa_n}{n}, \quad \{\kappa_n\} \in l_2, \quad a \in \mathbb{C} \quad (1.8.64)$$

*be given. Then  $\{\cos \rho_n x\}_{n \geq 0}$  is a Riesz basis in  $L_2(0, \pi)$ .*

*Proof.* First we show that  $\{\cos \rho_n x\}_{n \geq 0}$  is complete in  $L_2(0, \pi)$ . Let  $f(x) \in L_2(0, \pi)$  be such that

$$\int_0^\pi f(x) \cos \rho_n x \, dx = 0, \quad n \geq 0. \quad (1.8.65)$$

Consider the functions

$$\Delta(\lambda) := \pi(\lambda_0 - \lambda) \prod_{n=1}^{\infty} \frac{\lambda_n - \lambda}{n^2}, \quad \lambda_n = \rho_n^2,$$

$$F(\lambda) = \frac{1}{\Delta(\lambda)} \int_0^\pi f(x) \cos \rho x \, dx, \quad \lambda = \rho^2, \quad \lambda \neq \lambda_n.$$

It follows from (1.8.65) that  $F(\lambda)$  is entire in  $\lambda$  (after removing singularities). On the other hand, it was shown in the proof of Lemma 1.6.6 that

$$|\Delta(\lambda)| \geq C|\rho| \exp(|\tau|\pi), \quad \arg \lambda \in [\delta, 2\pi - \delta], \quad \delta > 0, \quad \tau := \operatorname{Im} \rho,$$

and consequently

$$|F(\lambda)| \leq \frac{C}{|\rho|}, \quad \arg \lambda \in [\delta, 2\pi - \delta].$$

From this, using the Phragmen-Lindelöf theorem [you1, p.80] and Liouville's theorem [con1, p.77] we conclude that  $F(\lambda) \equiv 0$ , i.e.

$$\int_0^\pi f(x) \cos \rho x \, dx = 0 \quad \text{for} \quad \rho \in \mathbb{C}.$$

Hence  $f = 0$ . Thus,  $\{\cos \rho_n x\}_{n \geq 0}$  is complete in  $L_2(0, \pi)$ . We note that completeness of  $\{\cos \rho_n x\}_{n \geq 0}$  follows independently also from Levinson's theorem [you1, p.118]. Furthermore, it follows from (1.8.64) that  $\{\cos \rho_n x\}_{n \geq 0}$  is quadratically close to the orthogonal basis  $\{\cos nx\}_{n \geq 0}$ . Then, by virtue of Proposition 1.8.5,  $\{\cos \rho_n x\}_{n \geq 0}$  is a Riesz basis in  $L_2(0, \pi)$ .  $\square$

## 1.9. REVIEW OF THE INVERSE PROBLEM THEORY

In this section we give a short review of results on inverse problems of spectral analysis for ordinary differential equations. We describe briefly only the main directions of this theory, mention the most important monographs and papers, and refer to them for details.

The greatest success in the inverse problem theory was achieved for the Sturm-Liouville operator

$$\ell y := -y'' + q(x)y. \quad (1.9.1)$$

The first result in this direction is due to Ambarzumian [amb1]. He showed that if the eigenvalues of the boundary value problem

$$-y'' + q(x)y = \lambda y, \quad y'(0) = y'(\pi) = 0$$

are  $\lambda_k = k^2$ ,  $k \geq 0$ , then  $q = 0$ . But this result is an exception from the rule, and in general the specification of the spectrum does not uniquely determine the operator (1.9.1). Afterwards Borg [Bor1] proved that the specification of two spectra of Sturm-Liouville operators uniquely determines the potential  $q$ . Levinson [Lev1] suggested a different method to prove Borg's results. Tikhonov in [Tik1] obtained the uniqueness theorem for the inverse Sturm-Liouville problem on the half-line from the given Weyl function.

An important role in the spectral theory of Sturm-Liouville operators was played by the transformation operator. Marchenko [mar3]-[mar4] first applied the transformation operator to the solution of the inverse problem. He proved that a Sturm-Liouville operator on the half-line or a finite interval is uniquely determined by specifying the spectral function. For a finite interval this corresponds to the specification of the spectral data (see Subsection 1.4.2). Transformation operators were also used in the fundamental paper of Gelfand and Levitan [gel1], where they obtained necessary and sufficient conditions along with a method for recovering a Sturm-Liouville operator from its spectral function. In [gas1] similar results were established for the inverse problem of recovering Sturm-Liouville operators on a finite interval from two spectra. Another approach for the solution of inverse problems was suggested by Krein [kre1], [kre2]. Blokh [blo1] and Rofe-Beketov [rof1] studied the inverse problem on the line with given spectral matrix. The inverse scattering theory on the half-line and on the line was constructed in [agr1], [cha1], [deg1], [dei1], [fad1], [fad2], [kay1], [kur1] and other works. An interesting approach connected with studying isospectral sets for the Sturm-Liouville operators on a finite interval was described in [car1], [dah1], [isa1], [isa2], [mcl2], [pos1] and [shu1]. The created methods also allow one to study stability of the solution of the inverse problem for (1.9.1) (see [ale1], [Bor1], [dor1], [hoc1], [hoc2], [mar6], [miz1], [rja1] and [yur3]).

In the last twenty years there appeared many new areas for applications of inverse Sturm-Liouville problems; we briefly mention some of them.

Boundary value problems with discontinuity conditions inside the interval are connected with discontinuous material properties. For example, discontinuous inverse problems appear in electronics for constructing parameters of heterogeneous electronic lines with desirable technical characteristics ([lit1], [mes1]). Spectral information can be used to reconstruct the permittivity and conductivity profiles of a one-dimensional discontinuous medium ([kru1], [she1]). Boundary value problems with discontinuities in an interior point also appear in geophysical models for oscillations of the Earth ([And1], [lap1]). Here the main discontinuity is caused by reflection of the shear waves at the base of the crust. Discontinuous inverse problems (in various formulations) have been considered in [hal1], [kob1], [kob2], [kru1], [pro1] and [she1].

Many further applications connect with the differential equation of the form

$$-y'' + q(x)y = \lambda r(x)y \quad (1.9.2)$$

with turning points when the function  $r(x)$  has zeros or (and) changes sign. For example, turning points connect with physical situations in which zeros correspond to the limit of motion of a wave mechanical particle bound by a potential field. Turning points appear also in elasticity, optics, geophysics and other branches of natural sciences. Moreover, a wide class of differential equations with Bessel-type singularities and their perturbations can be reduced to differential equations having turning points. Inverse problems for equations with turning points and singularities help to study blow-up solutions for some nonlinear integrable evolution equations of mathematical physics (see [Con1]). Inverse problems for the equation (1.9.1) with singularities and for the equation (1.9.2) with turning points and singularities have been studied in [Bel1], [Bou1], [car2], [elr1], [FY1], [gas2], [pan1], [sta1], [yur28], [yur29] and other works. Some aspects of the turning point theory and a number of applications are described in [ebe1]-[ebe4], [gol1], [mch1] and [was1].

The inverse problem for the Sturm-Liouville equation in the impedance form

$$-(p(x)y')' = \lambda p(x)y, \quad p(x) > 0, \quad 0 < x < 1, \quad (1.9.3)$$

when  $p'/p \in L_2(0,1)$  has been studied in [and1], [and2] and [col1]. Equation (1.9.3) can be transformed to (1.9.1) when  $p'' \in L_2(0,1)$  but not when only  $p'/p \in L_2(0,1)$ . The lack of the smoothness leads to additional technical difficulties in the investigation of the inverse problem. In [akt2], [dar1], [gri1] and [yak1] the inverse problem for the equation (1.9.2) with a lack of the smoothness of  $r$  is treated.

In [bro1], [hal2], [hal3], [mcl4], [She1], [yan1] and other papers, the so-called nodal inverse problem is considered when the potential is to be reconstructed from nodes of the eigenfunctions. Many works are devoted to incomplete inverse problems when only a part of the spectral information is available for measurement and (or) there is a priori information about the operator or its spectrum (see [akt1], [akt4], [FY2], [ges2], [gre1], [kli1], [mcl3], [run1], [sac2], [sad1], [tik1], [Zho1] and the references therein). Sometimes in incomplete inverse problems we have a lack of information which leads to nonuniqueness of the solution of the inverse problem (see, for example, [FY2] and [tik1]). We also mention inverse problems for boundary value problems with nonseparated boundary conditions ([Gus1], [mar7], [pla1], [yur2] and [yur20]) and with nonlocal boundary conditions of the form

$$\int_0^T y(x) d\sigma_j(x) = 0$$

(see [kra1]). Some aspects of numerical solutions of the inverse problems have been studied in [ahm1], [bAr1], [fab1], [kHa1], [kno1], [low1], [mue1], [neh1], [pai1] and [zhi1].

Numerous applications of the inverse problem theory connect with higher-order differential operators of the form

$$ly := y^{(n)} + \sum_{k=0}^{n-2} p_k(x)y^{(k)}, \quad n > 2. \quad (1.9.4)$$

In contrast to the case of Sturm-Liouville operators, inverse problems for the operator (1.9.4) turned out to be much more difficult for studying. Fage [fag2] and Leontjev [leo1] determined that for  $n > 2$  transformation operators have a more complicated structure than for

S Sturm-Liouville operators which makes it more difficult to use them for solving the inverse problem. However, in the case of analytic coefficients the transformation operators have the same "triangular" form as for Sturm-Liouville operators (see [kha2], [mat1] and [sak1]). Sakhnovich [sak2]-[sak3] and Khachatryan [kha3]-[kha4] used a "triangular" transformation operator to investigate the inverse problem of recovering selfadjoint differential operators on the half-line with analytic coefficients from the spectral function, as well as the inverse scattering problem.

A more effective and universal method in the inverse spectral theory is the method of spectral mappings connected with ideas of the contour integral method. Levinson [Lev1] first applied ideas of the contour integral method for investigating the inverse problem for the Sturm-Liouville operator (1.9.1). Developing Levinson's ideas Leibenson in [lei1]-[lei2] investigated the inverse problem for (1.9.4) on a finite interval under the restriction of "separation" of the spectrum. The spectra and weight numbers of certain specially chosen boundary value problems for the differential operators (1.9.4) appeared as the spectral data of the inverse problem. The inverse problem for (1.9.4) on a finite interval with respect to a system of spectra was investigated under various conditions on the operator in [bar1], [mal1], [yur4], [yur8] and [yur10]. Things are more complicated for differential operators on the half-line, especially for the nonselfadjoint case, since the spectrum can have rather "bad" behavior. Moreover, for the operator (1.9.4) even a formulation of the inverse problem is not obvious since the spectral function turns out to be unsuitable for this purpose. On a finite interval the formulation of the inverse problem is not obvious as well, since waiving the requirement of "separation" of the spectra leads to a violation of uniqueness for the solution of the inverse problem. In [yur1], [yur16] and [yur21] the so-called Weyl matrix was introduced and studied as the main spectral characteristics for the nonselfadjoint differential operator (1.9.4) on the half-line and on a finite interval. This is a generalization of the classical Weyl function for the Sturm-Liouville operator. The concept of the Weyl matrix and a development of ideas of the contour integral method enable one to construct a general theory of inverse problems for nonselfadjoint differential operators (1.9.4) (see [yur1] for details). The inverse scattering problem on the line for the operator (1.9.4) has been treated in various settings in [bea1], [bea2], [Cau1], [dei2], [dei3], [kau1], [kaz1], [kaz2], [suk1] and other works. We note that the use of the Riemann problem in the inverse scattering theory can be considered as a particular case of the contour integral method (see, for example, [bea1]).

The inverse problem for the operator (1.9.4) with locally integrable coefficients has been studied in [yur14]. There the generalized Weyl functions are introduced, and the Riemann-Fage formula [fag1] for the solution of the Cauchy problem for higher-order partial differential equations is used.

Inverse problems for higher-order differential operators with singularities have been studied in [kud1], [yur17], [yur18] and [yur22]. Incomplete inverse problems for higher-order differential operators and their applications are considered in [kha1], [bek1], [elc1], [mal1], [mcl1], [pap1], [yur8]-[yur10] and [yur26]. In particular, in [yur9] the inverse problem of recovering a part of the coefficients of the differential operator (1.9.4) from a part of the Weyl matrix has been studied (the rest of the coefficients are known a priori). For such inverse problems the method of standard models was suggested and a constructive solution for a wide class of incomplete inverse problems was given. This method was also applied for the solution of the inverse problem of elasticity theory when parameters of a beam are to be determined from given frequencies of its natural oscillations (see [yur12]). This problem can

be reduced to the inverse problem of recovering the fourth-order differential operator

$$(h^\mu(x)y'')'' = \lambda h(x)y, \quad \mu = 1, 2, 3$$

from the Weyl function.

Many papers are devoted to inverse problems for systems of differential equations (see [alp1], [amo1], [aru1], [bea3], [bea4], [bou1], [Cha1], [Che1], [cox1], [gas3], [gas4], [goh2], [hin1], [lee1], [li1], [mal2], [pal1], [sak5], [sAk1], [sha1], [sha2], [win1], [yam1], [zam1], [zho1], [zho2], [zho3] and the references therein). Some systems can be treated similarly to the Sturm-Liouville operator. As example we mention inverse problems for the Dirac system ([lev4], [aru1], [gas3], [gas4], [zam1]). But in the general case, the inverse problems for systems deal with difficulties like those for the operator (1.9.4). Such systems have been studied in [bea3], [bea4], [lee1], [zho1], [zho2] and [zho3].

We mention also inverse spectral problems for discrete operators (see [atk1], [ber1], [gla1], [gus1], [gus2], [Kha1], [nik1], [yur23], [yur24] and [yur26]), for differential operators with delay ([pik1], [pik2]), for nonlinear differential equations ([yur25]), for integro-differential and integral operators ([ere1], [yur6], [yur7], [yur13]), for pencils of operators ([akt3], [bea5], [bro2], [Chu1], [gas5], [yur5], [yur27], [yur31], [yur32]) and others.

Extensive literature is devoted to one more wide area of the inverse problem theory. In 1967 Gardner, Green, Kruskal and Miura [gar1] found a remarkable method for solving some important nonlinear equations of mathematical physics (the Korteweg-de Vries equation, the nonlinear Schrödinger equation, the Boussinesq equation and many others) connected with the use of inverse spectral problems. This method is described in [abl1], [lax1], [tak1], [zak1] and other works (see also Sections 4.1-4.2).

Many contributions are devoted to the inverse problem theory for partial differential equations. This direction is reflected fairly completely in [ang1], [ani1], [ber1], [buk1], [cha1], [cha3], [Isa1], [kir1], [lav1], [new1], [niz1], [Pri1], [Pri2], [ram1], [rom1] and [rom2]. In Section 2.4 we study an inverse problem for a wave equation as a model inverse problem for partial differential equations and show connections with inverse spectral problems for ordinary differential equations.

## II. STURM-LIOUVILLE OPERATORS ON THE HALF-LINE

In this chapter we present an introduction to the inverse problem theory for Sturm-Liouville operators on the half-line. First, in Sections 2.1-2.2 nonselfadjoint operators with integrable complex-valued potentials are considered. We introduce and study the Weyl function as the main spectral characteristic, prove an expansion theorem and solve the inverse problem of recovering the Sturm-Liouville operator from its Weyl function. For this purpose we use ideas of the contour integral method and develop the method of spectral mappings presented in Chapter I. Moreover connections with the transformation operator method are established. In Section 2.3 the most important particular cases, which often appear in applications, are considered, namely: selfadjoint operators, nonselfadjoint operators with a simple spectrum and perturbations of the discrete spectrum of a model operator. We introduce here the so-called spectral data which describe the set of singularities of the Weyl function. The solution of the inverse problem of recovering the Sturm-Liouville operator from its spectral data is provided. In Sections 2.4-2.5 locally integrable complex-valued potentials are studied. In Section 2.4 we consider an inverse problem for a wave equation. In Section 2.5 the generalized Weyl function is introduced as the main spectral characteristic. We prove an expansion theorem and solve the inverse problem of recovering the Sturm-Liouville operator from its generalized Weyl function. The inverse problem to construct  $q$  from the Weyl sequence is considered in Section 2.6.

### 2.1. PROPERTIES OF THE SPECTRUM. THE WEYL FUNCTION.

We consider the differential equation and the linear form  $L = L(q(x), h)$  :

$$\ell y := -y'' + q(x)y = \lambda y, \quad x > 0, \quad (2.1.1)$$

$$U(y) := y'(0) - hy(0), \quad (2.1.2)$$

where  $q(x) \in L(0, \infty)$  is a complex-valued function, and  $h$  is a complex number. Let  $\lambda = \rho^2$ ,  $\rho = \sigma + i\tau$ , and let for definiteness  $\tau := \operatorname{Im} \rho \geq 0$ . Denote by  $\Pi$  the  $\lambda$ -plane with the cut  $\lambda \geq 0$ , and  $\Pi_1 = \bar{\Pi} \setminus \{0\}$ ; notice that here  $\Pi$  and  $\Pi_1$  must be considered as subsets of the Riemann surface of the squareroot-function. Then, under the map  $\rho \rightarrow \rho^2 = \lambda$ ,  $\Pi_1$  corresponds to the domain  $\Omega = \{\rho : \operatorname{Im} \rho \geq 0, \rho \neq 0\}$ . Put  $\Omega_\delta = \{\rho : \operatorname{Im} \rho \geq 0, |\rho| \geq \delta\}$ . Denote by  $W_N$  the set of functions  $f(x)$ ,  $x \geq 0$  such that the functions  $f^{(j)}(x)$ ,  $j = \overline{0, N-1}$  are absolutely continuous on  $[0, T]$  for each fixed  $T > 0$ , and  $f^{(j)}(x) \in L(0, \infty)$ ,  $j = \overline{0, N}$ .

**2.1.1. Jost and Birkhoff solutions.** In this subsection we construct a special fundamental system of solutions for equation (2.1.1) in  $\Omega$  having asymptotic behavior at infinity like  $\exp(\pm i\rho x)$ .

**Theorem 2.1.1.** *Equation (2.1.1) has a unique solution  $y = e(x, \rho)$ ,  $\rho \in \Omega$ ,  $x \geq 0$ , satisfying the integral equation*

$$e(x, \rho) = \exp(i\rho x) - \frac{1}{2i\rho} \int_x^\infty (\exp(i\rho(x-t)) - \exp(i\rho(t-x))) q(t) e(t, \rho) dt. \quad (2.1.3)$$

The function  $e(x, \rho)$  has the following properties:

(i<sub>1</sub>) For  $x \rightarrow \infty$ ,  $\nu = 0, 1$ , and each fixed  $\delta > 0$ ,

$$e^{(\nu)}(x, \rho) = (i\rho)^\nu \exp(i\rho x)(1 + o(1)), \quad (2.1.4)$$

uniformly in  $\Omega_\delta$ . For  $\text{Im } \rho > 0$ ,  $e(x, \rho) \in L_2(0, \infty)$ . Moreover,  $e(x, \rho)$  is the unique solution of (2.1.1) (up to a multiplicative constant) having this property.

(i<sub>2</sub>) For  $|\rho| \rightarrow \infty$ ,  $\rho \in \Omega$ ,  $\nu = 0, 1$ ,

$$e^{(\nu)}(x, \rho) = (i\rho)^\nu \exp(i\rho x) \left(1 + \frac{\omega(x)}{i\rho} + o\left(\frac{1}{\rho}\right)\right), \quad \omega(x) := -\frac{1}{2} \int_x^\infty q(t) dt, \quad (2.1.5)$$

uniformly for  $x \geq 0$ .

(i<sub>3</sub>) For each fixed  $x \geq 0$ , and  $\nu = 0, 1$ , the functions  $e^{(\nu)}(x, \rho)$  are analytic for  $\text{Im } \rho > 0$ , and are continuous for  $\rho \in \Omega$ .

(i<sub>4</sub>) For real  $\rho \neq 0$ , the functions  $e(x, \rho)$  and  $e(x, -\rho)$  form a fundamental system of solutions for (2.1.1), and

$$\langle e(x, \rho), e(x, -\rho) \rangle = -2i\rho, \quad (2.1.6)$$

where  $\langle y, z \rangle := yz' - y'z$  is the Wronskian.

The function  $e(x, \rho)$  is called the Jost solution for (2.1.1).

*Proof.* We transform (2.1.3) by means of the replacement

$$e(x, \rho) = \exp(i\rho x)z(x, \rho) \quad (2.1.7)$$

to the equation

$$z(x, \rho) = 1 - \frac{1}{2i\rho} \int_x^\infty (1 - \exp(2i\rho(t-x))) q(t) z(t, \rho) dt, \quad x \geq 0, \rho \in \Omega. \quad (2.1.8)$$

The method of successive approximations gives

$$z_0(x, \rho) = 1, \quad z_{k+1}(x, \rho) = -\frac{1}{2i\rho} \int_x^\infty (1 - \exp(2i\rho(t-x))) q(t) z_k(t, \rho) dt, \quad (2.1.9)$$

$$z(x, \rho) = \sum_{k=0}^{\infty} z_k(x, \rho). \quad (2.1.10)$$

Let us show by induction that

$$|z_k(x, \rho)| \leq \frac{(Q_0(x))^k}{|\rho|^k k!}, \quad \rho \in \Omega, \quad x \geq 0, \quad (2.1.11)$$

where

$$Q_0(x) := \int_x^\infty |q(t)| dt.$$

Indeed, for  $k = 0$ , (2.1.11) is obvious. Suppose that (2.1.11) is valid for a certain fixed  $k \geq 0$ . Since  $|1 - \exp(2i\rho(t-x))| \leq 2$ , (2.1.9) implies

$$|z_{k+1}(x, \rho)| \leq \frac{1}{|\rho|} \int_x^\infty |q(t) z_k(t, \rho)| dt. \quad (2.1.12)$$



Substituting (2.1.11) into the right-hand side of (2.1.12) we calculate

$$|z_{k+1}(x, \rho)| \leq \frac{1}{|\rho|^{k+1} k!} \int_x^\infty |q(t)| (Q_0(t))^k dt = \frac{(Q_0(t))^{k+1}}{|\rho|^{k+1} (k+1)!}.$$

It follows from (2.1.11) that the series (2.1.10) converges absolutely for  $x \geq 0$ ,  $\rho \in \Omega$ , and the function  $z(x, \rho)$  is the unique solution of the integral equation (2.1.8). Moreover, by virtue of (2.1.10) and (2.1.11),

$$|z(x, \rho)| \leq \exp(Q_0(x)/|\rho|), \quad |z(x, \rho) - 1| \leq (Q_0(x)/|\rho|) \exp(Q_0(x)/|\rho|). \quad (2.1.13)$$

In particular, (2.1.13) yields for each fixed  $\delta > 0$ ,

$$z(x, \rho) = 1 + o(1), \quad x \rightarrow \infty, \quad (2.1.14)$$

uniformly in  $\Omega_\delta$ , and

$$z(x, \rho) = 1 + O\left(\frac{1}{\rho}\right), \quad |\rho| \rightarrow \infty, \quad \rho \in \Omega, \quad (2.1.15)$$

uniformly for  $x \geq 0$ . Substituting (2.1.15) into the right-hand side of (2.1.8) we obtain

$$z(x, \rho) = 1 - \frac{1}{2i\rho} \int_x^\infty (1 - \exp(2i\rho(t-x))) q(t) dt + O\left(\frac{1}{\rho^2}\right), \quad |\rho| \rightarrow \infty, \quad (2.1.16)$$

uniformly for  $x \geq 0$ .

**Lemma 2.1.1.** *Let  $q(x) \in L(0, \infty)$ , and denote*

$$J_q(x, \rho) := \int_x^\infty q(t) \exp(2i\rho(t-x)) dt, \quad \rho \in \Omega. \quad (2.1.17)$$

*Then*

$$\lim_{|\rho| \rightarrow \infty} \sup_{x \geq 0} |J_q(x, \rho)| = 0. \quad (2.1.18)$$

*Proof.* 1) First we assume that  $q(x) \in W_1$ . Then integration by parts in (2.1.17) yields

$$J_q(x, \rho) = -\frac{q(x)}{2i\rho} - \frac{1}{2i\rho} \int_x^\infty q'(t) \exp(2i\rho(t-x)) dt,$$

and consequently

$$\sup_{x \geq 0} |J_q(x, \rho)| \leq \frac{C_q}{|\rho|}.$$

2) Let now  $q(x) \in L(0, \infty)$ . Fix  $\varepsilon > 0$  and choose  $q_\varepsilon(x) \in W_1$  such that

$$\int_0^\infty |q(t) - q_\varepsilon(t)| dt < \frac{\varepsilon}{2}.$$

Then

$$|J_q(x, \rho)| \leq |J_{q_\varepsilon}(x, \rho)| + |J_{q-q_\varepsilon}(x, \rho)| \leq \frac{C_{q_\varepsilon}}{|\rho|} + \frac{\varepsilon}{2}.$$

Hence, there exists  $\rho^0 > 0$  such that  $\sup_{x \geq 0} |J_q(x, \rho)| \leq \varepsilon$  for  $|\rho| \geq \rho^0$ ,  $\rho \in \Omega$ . By virtue of arbitrariness of  $\varepsilon > 0$  we arrive at (2.1.18).  $\square$

Let us return to the proof of Theorem 2.1.1. It follows from (2.1.16) and Lemma 2.1.1 that

$$z(x, \rho) = 1 + \frac{\omega(x)}{i\rho} + o\left(\frac{1}{\rho}\right), \quad |\rho| \rightarrow \infty, \quad \rho \in \Omega, \quad (2.1.19)$$

uniformly for  $x \geq 0$ . From (2.1.7), (2.1.9)-(2.1.11), (2.1.14) and (2.1.19) we derive  $(i_1) - (i_3)$  for  $\nu = 0$ . Furthermore, (2.1.3) and (2.1.7) imply

$$e'(x, \rho) = (i\rho) \exp(i\rho x) \left(1 - \frac{1}{2i\rho} \int_x^\infty (1 + \exp(2i\rho(t-x))) q(t) z(t, \rho) dt\right). \quad (2.1.20)$$

Using (2.1.20) we get  $(i_1) - (i_3)$  for  $\nu = 1$ . It is easy to verify by differentiation that the function  $e(x, \rho)$  is a solution of (2.1.1). For real  $\rho \neq 0$ , the functions  $e(x, \rho)$  and  $e(x, -\rho)$  satisfy (2.1.1), and by virtue of (2.1.4),  $\lim_{x \rightarrow \infty} \langle e(x, \rho), e(x, -\rho) \rangle = -2i\rho$ . Since the Wronskian  $\langle e(x, \rho), e(x, -\rho) \rangle$  does not depend on  $x$ , we arrive at (2.1.6).  $\square$

**Remark 2.1.1.** If  $q(x) \in W_N$ , then there exist functions  $\omega_{s\nu}(x)$  such that for  $\rho \rightarrow \infty$ ,  $\rho \in \Omega$ ,  $\nu = 0, 1, 2$ ,

$$e^{(\nu)}(x, \rho) = (i\rho)^\nu \exp(i\rho x) \left(1 + \sum_{s=1}^{N+1} \frac{\omega_{s\nu}(x)}{(i\rho)^s} + o\left(\frac{1}{\rho^{N+1}}\right)\right), \quad \omega_{1\nu}(x) = \omega(x). \quad (2.1.21)$$

Indeed, let  $q(x) \in W_1$ . Substituting (2.1.19) into the right-hand side of (2.1.8) we get

$$\begin{aligned} z(x, \rho) &= 1 - \frac{1}{2i\rho} \int_x^\infty (1 - \exp(2i\rho(t-x))) q(t) dt \\ &\quad - \frac{1}{2(i\rho)^2} \int_x^\infty (1 - \exp(2i\rho(t-x))) q(t) \omega(t) dt + o\left(\frac{1}{\rho^2}\right), \quad |\rho| \rightarrow \infty. \end{aligned}$$

Integrating by parts and using Lemma 2.1.1, we obtain

$$z(x, \rho) = 1 + \frac{\omega(x)}{i\rho} + \frac{\omega_{20}(x)}{(i\rho)^2} + o\left(\frac{1}{\rho^2}\right), \quad |\rho| \rightarrow \infty, \quad \rho \in \Omega, \quad (2.1.22)$$

where

$$\omega_{20}(x) = -\frac{1}{4}q(x) + \frac{1}{4} \int_x^\infty q(t) \left( \int_t^\infty q(s) ds \right) dt = -\frac{1}{4}q(x) + \frac{1}{8} \left( \int_x^\infty q(t) dt \right)^2.$$

By virtue of (2.1.7) and (2.1.22), we arrive at (2.1.21) for  $N = 1$ ,  $\nu = 0$ . Using induction one can prove (2.1.21) for all  $N$ .

If we additionally assume that  $xq(x) \in L(0, \infty)$ , then the Jost solution  $e(x, \rho)$  exists also for  $\rho = 0$ . More precisely, the following theorem is valid.

**Theorem 2.1.2.** Let  $(1+x)q(x) \in L(0, \infty)$ . The functions  $e^{(\nu)}(x, \rho)$ ,  $\nu = 0, 1$  are continuous for  $\operatorname{Im} \rho \geq 0$ ,  $x \geq 0$ , and

$$|e(x, \rho) \exp(-i\rho x)| \leq \exp(Q_1(x)), \quad (2.1.23)$$

$$|e(x, \rho) \exp(-i\rho x) - 1| \leq \left(Q_1(x) - Q_1\left(x + \frac{1}{|\rho|}\right)\right) \exp(Q_1(x)), \quad (2.1.24)$$

$$|e'(x, \rho) \exp(-i\rho x) - i\rho| \leq Q_0(x) \exp(Q_1(x)), \quad (2.1.25)$$

where

$$Q_1(x) := \int_x^\infty Q_0(t) dt = \int_x^\infty (t-x)|q(t)| dt.$$

First we prove an auxiliary assertion.

**Lemma 2.1.2.** Assume that  $c_1 \geq 0$ ,  $u(x) \geq 0$ ,  $v(x) \geq 0$ ,  $(a \leq x \leq T \leq \infty)$ ,  $u(x)$  is bounded, and  $(x-a)v(x) \in L(a, T)$ . If

$$u(x) \leq c_1 + \int_x^T (t-x)v(t)u(t) dt, \quad (2.1.26)$$

then

$$u(x) \leq c_1 \exp\left(\int_x^T (t-x)v(t) dt\right). \quad (2.1.27)$$

*Proof.* Denote

$$\xi(x) = c_1 + \int_x^T (t-x)v(t)u(t) dt.$$

Then

$$\xi(T) = c_1, \quad \xi'(x) = -\int_x^T v(t)u(t) dt, \quad \xi'' = v(x)u(x),$$

and (2.1.26) yields

$$0 \leq \xi''(x) \leq \xi(x)v(x).$$

Let  $c_1 > 0$ . Then  $\xi(x) > 0$ , and

$$\frac{\xi''(x)}{\xi(x)} \leq v(x).$$

Hence

$$\left(\frac{\xi'(x)}{\xi(x)}\right)' \leq v(x) - \left(\frac{\xi'(x)}{\xi(x)}\right)^2 \leq v(x).$$

Integrating this inequality twice we get

$$-\frac{\xi'(x)}{\xi(x)} \leq \int_x^T v(t) dt, \quad \ln \frac{\xi(x)}{\xi(T)} \leq \int_x^T (t-x)v(t) dt,$$

and consequently,

$$\xi(x) \leq c_1 \exp\left(\int_x^T (t-x)v(t) dt\right).$$

According to (2.1.26),  $u(x) \leq \xi(x)$ , and we arrive at (2.1.27).

If  $c_1 = 0$ , then  $\xi(x) = 0$ . Indeed, suppose on the contrary that  $\xi(x) \neq 0$ . Since  $\xi(x) \geq 0$ ,  $\xi'(x) \leq 0$ , there exists  $T_0 \leq T$  such that  $\xi(x) > 0$  for  $x < T_0$ , and  $\xi(x) \equiv 0$  for  $x \in [T_0, T]$ . Repeating these arguments we get for  $x < T_0$  and sufficiently small  $\varepsilon > 0$ ,

$$\ln \frac{\xi(x)}{\xi(T_0 - \varepsilon)} \leq \int_x^{T_0 - \varepsilon} (t-x)v(t) dt \leq \int_x^{T_0} (t-x)v(t) dt,$$

which is impossible. Thus,  $\xi(x) \equiv 0$ , and (2.1.27) becomes obvious.  $\square$

*Proof of Theorem 2.1.2.* For  $\operatorname{Im} \rho \geq 0$  we have

$$\left| \frac{\sin \rho}{\rho} \exp(i\rho) \right| \leq 1. \quad (2.1.28)$$

Indeed, (2.1.28) is obvious for real  $\rho$  and for  $|\rho| \geq 1$ ,  $\operatorname{Im} \rho \geq 0$ . Then, by the maximum principle [con1, p.128], (2.1.28) is also valid for  $|\rho| \leq 1$ ,  $\operatorname{Im} \rho \geq 0$ .

It follows from (2.1.28) that

$$\left| \frac{1 - \exp(2i\rho x)}{2i\rho} \right| \leq x \quad \text{for } \operatorname{Im} \rho \geq 0, x \geq 0. \quad (2.1.29)$$

Using (2.1.9) and (2.1.29) we infer

$$|z_{k+1}(x, \rho)| \leq \int_x^\infty t |q(t) z_k(t, \rho)| dt, \quad k \geq 0, \operatorname{Im} \rho \geq 0, x \geq 0,$$

and consequently by induction

$$|z_k(x, \rho)| \leq \frac{1}{k!} \left( \int_x^\infty t |q(t)| dt \right)^k, \quad k \geq 0, \operatorname{Im} \rho \geq 0, x \geq 0.$$

Then, the series (2.1.10) converges absolutely and uniformly for  $\operatorname{Im} \rho \geq 0$ ,  $x \geq 0$ , and the function  $z(x, \rho)$  is continuous for  $\operatorname{Im} \rho \geq 0$ ,  $x \geq 0$ . Moreover,

$$|z(x, \rho)| \leq \exp \left( \int_x^\infty t |q(t)| dt \right), \quad \operatorname{Im} \rho \geq 0, x \geq 0. \quad (2.1.30)$$

Using (2.1.7) and (2.1.20) we conclude that the functions  $e^{(\nu)}(x, \rho)$ ,  $\nu = 0, 1$  are continuous for  $\operatorname{Im} \rho \geq 0$ ,  $x \geq 0$ .

Furthermore, it follows from (2.1.8) and (2.1.29) that

$$|z(x, \rho)| \leq 1 + \int_x^\infty (t - x) |q(t) z(t, \rho)| dt, \quad \operatorname{Im} \rho \geq 0, x \geq 0.$$

By virtue of Lemma 2.1.2, this implies

$$|z(x, \rho)| \leq \exp(Q_1(x)), \quad \operatorname{Im} \rho \geq 0, x \geq 0, \quad (2.1.31)$$

i.e. (2.1.23) is valid. We note that (2.1.31) is more precise than (2.1.30).

Using (2.1.8), (2.1.29) and (2.1.31) we calculate

$$|z(x, \rho) - 1| \leq \int_x^\infty (t - x) |q(t)| \exp(Q_1(t)) dt \leq \exp(Q_1(x)) \int_x^\infty (t - x) |q(t)| dt,$$

and consequently,

$$|z(x, \rho) - 1| \leq Q_1(x) \exp(Q_1(x)), \quad \operatorname{Im} \rho \geq 0, x \geq 0. \quad (2.1.32)$$

More precisely,

$$|z(x, \rho) - 1| \leq \int_x^{x + \frac{1}{|\rho|}} (t - x) |q(t)| \exp(Q_1(t)) dt + \frac{1}{|\rho|} \int_{x + \frac{1}{|\rho|}}^\infty |q(t)| \exp(Q_1(t)) dt$$

$$\begin{aligned}
&\leq \exp(Q_1(x)) \left( \int_x^\infty (t-x)|q(t)| dt - \int_{x+\frac{1}{|\rho|}}^\infty \left(t-x-\frac{1}{|\rho|}\right)|q(t)| dt \right) \\
&= \left( Q_1(x) - Q_1\left(x+\frac{1}{|\rho|}\right) \right) \exp(Q_1(x)),
\end{aligned}$$

i.e. (2.1.24) is valid. At last, from (2.1.20) and (2.1.31) we obtain

$$|e'(x, \rho) \exp(-i\rho x) - i\rho| \leq \int_x^\infty |q(t)| \exp(Q_1(t)) dt \leq \exp(Q_1(x)) \int_x^\infty |q(t)| dt,$$

and we arrive at (2.1.25). Theorem 2.1.2 is proved.  $\square$

**Remark 2.1.2.** Consider the function

$$q(x) = \frac{2a^2}{(1+ax)^2},$$

where  $a$  is a complex number such that  $a \notin (-\infty, 0]$ . Then  $q(x) \in L(0, \infty)$ , but  $xq(x) \notin L(0, \infty)$ . The Jost solution has in this case the form (see Example 2.3.1)

$$e(x, \rho) = \exp(i\rho x) \left( 1 - \frac{a}{i\rho(1+ax)} \right),$$

i.e.  $e(x, \rho)$  has a singularity at  $\rho = 0$ , hence we cannot omit the integrability condition in Theorem 2.1.2.

For the Jost solution  $e(x, \rho)$  there exists a transformation operator. More precisely, the following theorem is valid.

**Theorem 2.1.3.** *Let  $(1+x)q(x) \in L(0, \infty)$ . Then the Jost solution  $e(x, \rho)$  can be represented in the form*

$$e(x, \rho) = \exp(i\rho x) + \int_x^\infty A(x, t) \exp(i\rho t) dt, \quad \text{Im } \rho \geq 0, \quad x \geq 0, \quad (2.1.33)$$

where  $A(x, t)$  is a continuous function for  $0 \leq x \leq t < \infty$ , and

$$A(x, x) = \frac{1}{2} \int_x^\infty q(t) dt. \quad (2.1.34)$$

$$|A(x, t)| \leq \frac{1}{2} Q_0\left(\frac{x+t}{2}\right) \exp\left(Q_1(x) - Q_1\left(\frac{x+t}{2}\right)\right), \quad (2.1.35)$$

$$1 + \int_x^\infty |A(x, t)| dt \leq \exp(Q_1(x)), \quad \int_x^\infty |A(x, t)| dt \leq Q_1(x) \exp(Q_1(x)). \quad (2.1.36)$$

Moreover, the function  $A(x_1, x_2)$  has first derivatives  $\frac{\partial A}{\partial x_i}$ ,  $i = 1, 2$ ; the functions

$$\frac{\partial A(x_1, x_2)}{\partial x_i} + \frac{1}{4} q\left(\frac{x_1 + x_2}{2}\right)$$

are absolutely continuous with respect to  $x_1$  and  $x_2$ , and satisfy the estimates

$$\left| \frac{\partial A(x_1, x_2)}{\partial x_i} + \frac{1}{4} q\left(\frac{x_1 + x_2}{2}\right) \right|$$

$$\leq \frac{1}{2} Q_0(x_1) Q_0\left(\frac{x_1 + x_2}{2}\right) \exp\left(Q_1(x_1) - Q_1\left(\frac{x_1 + x_2}{2}\right)\right), \quad i = 1, 2. \quad (2.1.37)$$

*Proof.* The arguments for proving this theorem are similar to those in Section 1.3 with adequate modifications. This theorem also can be found in [mar1] and [lev2].

According to (2.1.7) and (2.1.10) we have

$$e(x, \rho) = \sum_{k=0}^{\infty} \varepsilon_k(x, \rho), \quad \varepsilon_k(x, \rho) = z_k(x, \rho) \exp(i\rho x). \quad (2.1.38)$$

Let us show by induction that the following representation is valid

$$\varepsilon_k(x, \rho) = \int_x^{\infty} a_k(x, t) \exp(i\rho t) dt, \quad k \geq 1, \quad (2.1.39)$$

where the functions  $a_k(x, t)$  do not depend on  $\rho$ .

First we calculate  $\varepsilon_1(x, \rho)$ . By virtue of (2.1.9) and (2.1.38),

$$\varepsilon_1(x, \rho) = \int_x^{\infty} \frac{\sin \rho(s-x)}{\rho} \exp(i\rho s) q(s) ds = \frac{1}{2} \int_x^{\infty} q(s) \left( \int_x^{2s-x} \exp(i\rho t) dt \right) ds.$$

Interchanging the order of integration we obtain that (2.1.39) holds for  $k = 1$ , where

$$a_1(x, t) = \frac{1}{2} \int_{(t+x)/2}^{\infty} q(s) ds.$$

Suppose now that (2.1.39) is valid for a certain  $k \geq 1$ . Then

$$\begin{aligned} \varepsilon_{k+1}(x, \rho) &= \int_x^{\infty} \frac{\sin \rho(s-x)}{\rho} q(s) \varepsilon_k(s, \rho) ds \\ &= \int_x^{\infty} \frac{\sin \rho(s-x)}{\rho} q(s) \left( \int_s^{\infty} a_k(s, u) \exp(i\rho u) du \right) ds \\ &= \frac{1}{2} \int_x^{\infty} q(s) \left( \int_s^{\infty} a_k(s, u) \left( \int_{-s+u+x}^{s+u-x} \exp(i\rho t) dt \right) du \right) ds. \end{aligned}$$

We extend  $a_k(s, u)$  by zero for  $u < s$ . For  $s \geq x$  this yields

$$\int_s^{\infty} a_k(s, u) \left( \int_{-s+u+x}^{s+u-x} \exp(i\rho t) dt \right) du = \int_x^{\infty} \exp(i\rho t) \left( \int_{t-s+x}^{t+s-x} a_k(s, u) du \right) dt.$$

Therefore

$$\varepsilon_{k+1}(x, \rho) = \frac{1}{2} \int_x^{\infty} \exp(i\rho t) \left( \int_x^{\infty} q(s) \left( \int_{t-s+x}^{t+s-x} a_k(s, u) du \right) ds \right) dt = \int_x^{\infty} a_{k+1}(x, t) \exp(i\rho t) dt,$$

where

$$a_{k+1}(x, t) = \frac{1}{2} \int_x^{\infty} q(s) \left( \int_{t-s+x}^{t+s-x} a_k(s, u) du \right) ds, \quad t \geq x.$$

Changing the variables according to  $u + s = 2\alpha$ ,  $u - s = 2\beta$ , we obtain

$$a_{k+1}(x, t) = \int_{(t+x)/2}^{\infty} \left( \int_0^{(t-x)/2} q(\alpha - \beta) a_k(\alpha - \beta, \alpha + \beta) d\beta \right) d\alpha.$$

Taking  $H_k(\alpha, \beta) = a_k(\alpha - \beta, \alpha + \beta)$ ,  $t + x = 2u$ ,  $t - x = 2v$ , we calculate for  $0 \leq v \leq u$ ,

$$H_1(u, v) = \frac{1}{2} \int_u^\infty q(s) ds, \quad H_{k+1}(u, v) = \int_u^\infty \left( \int_0^v q(\alpha - \beta) H_k(\alpha, \beta) d\beta \right) d\alpha. \quad (2.1.40)$$

It can be shown by induction that

$$|H_{k+1}(u, v)| \leq \frac{1}{2} Q_0(u) \frac{(Q_1(u - v) - Q_1(u))^k}{k!}, \quad k \geq 0, \quad 0 \leq v \leq u. \quad (2.1.41)$$

Indeed, for  $k = 0$ , (2.1.41) is obvious. Suppose that (2.1.41) is valid for  $H_k(u, v)$ . Then (2.1.40) implies

$$|H_{k+1}(u, v)| \leq \frac{1}{2} \int_u^\infty Q_0(\alpha) \left( \int_0^v |q(\alpha - \beta)| \frac{(Q_1(\alpha - \beta) - Q_1(\alpha))^{k-1}}{(k-1)!} d\beta \right) d\alpha.$$

Since the functions  $Q_0(x)$  and  $Q_1(x)$  are monotonic, we get

$$\begin{aligned} |H_{k+1}(u, v)| &\leq \frac{1}{2} \cdot \frac{Q_0(u)}{(k-1)!} \int_u^\infty (Q_1(\alpha - v) - Q_1(\alpha))^{k-1} (Q_0(\alpha - v) - Q_0(\alpha)) d\alpha \\ &= \frac{1}{2} Q_0(u) \frac{(Q_1(u - v) - Q_1(u))^k}{k!}, \end{aligned}$$

i.e. (2.1.41) is proved. Therefore, the series

$$H(u, v) = \sum_{k=1}^{\infty} H_k(u, v)$$

converges absolutely and uniformly for  $0 \leq v \leq u$ , and

$$H(u, v) = \frac{1}{2} \int_u^\infty q(s) ds + \int_u^\infty \left( \int_0^v q(\alpha - \beta) H(\alpha, \beta) d\beta \right) d\alpha, \quad (2.1.42)$$

$$|H(u, v)| \leq \frac{1}{2} Q_0(u) \exp(Q_1(u - v) - Q_1(u)). \quad (2.1.43)$$

Put

$$A(x, t) = H\left(\frac{t+x}{2}, \frac{t-x}{2}\right). \quad (2.1.44)$$

Then

$$A(x, t) = \sum_{k=1}^{\infty} a_k(x, t),$$

the series converges absolutely and uniformly for  $0 \leq x \leq t$ , and (2.1.33)-(2.1.35) are valid. Using (2.1.35) we calculate

$$\begin{aligned} \int_x^\infty |A(x, t)| dt &\leq \exp(Q_1(x)) \int_x^\infty Q_0(\xi) \exp(-Q_1(\xi)) d\xi \\ &= \exp(Q_1(x)) \int_x^\infty \frac{d}{d\xi} \left( \exp(-Q_1(\xi)) \right) d\xi = \exp(Q_1(x)) - 1, \end{aligned}$$

and we arrive at (2.1.36).

Furthermore, it follows from (2.1.42) that

$$\frac{\partial H(u, v)}{\partial u} = -\frac{1}{2}q(u) - \int_0^v q(u - \beta)H(u, \beta) d\beta, \quad (2.1.45)$$

$$\frac{\partial H(u, v)}{\partial v} = \int_u^\infty q(\alpha - v)H(\alpha, v) d\alpha. \quad (2.1.46)$$

It follows from (2.1.45)-(2.1.46) and (2.1.43) that

$$\left| \frac{\partial H(u, v)}{\partial u} + \frac{1}{2}q(u) \right| \leq \frac{1}{2} \int_0^v |q(u - \beta)| Q_0(u) \exp(Q_1(u - \beta) - Q_1(u)) d\beta,$$

$$\left| \frac{\partial H(u, v)}{\partial v} \right| \leq \frac{1}{2} \int_u^\infty |q(\alpha - v)| Q_0(\alpha) \exp(Q_1(\alpha - v) - Q_1(\alpha)) d\alpha.$$

Since

$$\int_0^v |q(u - \beta)| d\beta = \int_{u-v}^u |q(s)| ds \leq Q_0(u - v),$$

$$Q_1(\alpha - v) - Q_1(\alpha) = \int_{\alpha-v}^\alpha Q_0(t) dt \leq \int_{u-v}^u Q_0(t) dt = Q_1(u - v) - Q_1(u), \quad u \leq \alpha,$$

we get

$$\begin{aligned} \left| \frac{\partial H(u, v)}{\partial u} + \frac{1}{2}q(u) \right| &\leq \frac{1}{2} Q_0(u) \exp(Q_1(u - v) - Q_1(u)) \int_0^v |q(u - \beta)| d\beta \\ &\leq \frac{1}{2} Q_0(u - v) Q_0(u) \exp(Q_1(u - v) - Q_1(u)), \end{aligned} \quad (2.1.47)$$

$$\begin{aligned} \left| \frac{\partial H(u, v)}{\partial v} \right| &\leq \frac{1}{2} Q_0(u) \exp(Q_1(u - v) - Q_1(u)) \int_u^\infty |q(\alpha - v)| d\alpha \\ &\leq \frac{1}{2} Q_0(u - v) Q_0(u) \exp(Q_1(u - v) - Q_1(u)). \end{aligned} \quad (2.1.48)$$

By virtue of (2.1.44),

$$\frac{\partial A(x, t)}{\partial x} = \frac{1}{2} \left( \frac{\partial H(u, v)}{\partial u} - \frac{\partial H(u, v)}{\partial v} \right), \quad \frac{\partial A(x, t)}{\partial t} = \frac{1}{2} \left( \frac{\partial H(u, v)}{\partial u} + \frac{\partial H(u, v)}{\partial v} \right),$$

where

$$u = \frac{t + x}{2}, \quad v = \frac{t - x}{2}.$$

Hence,

$$\begin{aligned} \frac{\partial A(x, t)}{\partial x} + \frac{1}{4}q\left(\frac{x + t}{2}\right) &= \frac{1}{2} \left( \frac{\partial H(u, v)}{\partial u} + \frac{1}{2}q(u) \right) - \frac{1}{2} \frac{\partial H(u, v)}{\partial v}, \\ \frac{\partial A(x, t)}{\partial t} + \frac{1}{4}q\left(\frac{x + t}{2}\right) &= \frac{1}{2} \left( \frac{\partial H(u, v)}{\partial u} + \frac{1}{2}q(u) \right) + \frac{1}{2} \frac{\partial H(u, v)}{\partial v}. \end{aligned}$$

Taking (2.1.47) and (2.1.48) into account we arrive at (2.1.37).  $\square$

Let  $(1 + x)q(x) \in L(0, \infty)$ . We introduce the potentials

$$q_r(x) = \begin{cases} q(x), & x \leq r, \\ 0, & x > r, \end{cases} \quad r \geq 0 \quad (2.1.49)$$



and consider the corresponding Jost solutions

$$e_r(x, \rho) = \exp(i\rho x) + \int_x^\infty A_r(x, t) \exp(i\rho t) dt. \quad (2.1.50)$$

According to Theorems 2.1.2 and 2.1.3,

$$\left. \begin{aligned} |e_r(x, \rho) \exp(-i\rho x)| &\leq \exp(Q_1(x)), \\ |e_r(x, \rho) \exp(-i\rho x) - 1| &\leq Q_1(x) \exp(Q_1(x)), \\ |e'_r(x, \rho) \exp(-i\rho x) - i\rho| &\leq Q_0(x) \exp(Q_1(x)), \end{aligned} \right\} \quad (2.1.51)$$

$$|A_r(x, t)| \leq \frac{1}{2} Q_0\left(\frac{x+t}{2}\right) \exp\left(Q_1(x) - Q_1\left(\frac{x+t}{2}\right)\right). \quad (2.1.52)$$

Moreover,

$$\left. \begin{aligned} e_r(x, \rho) &\equiv \exp(i\rho x) \quad \text{for } x > r, \\ A_r(x, t) &\equiv 0 \quad \text{for } x + t > 2r. \end{aligned} \right\} \quad (2.1.53)$$

**Lemma 2.1.3.** *Let  $(1+x)q(x) \in L(0, \infty)$ . Then for  $\operatorname{Im} \rho \geq 0$ ,  $x \geq 0$ ,  $r \geq 0$ ,*

$$|(e_r(x, \rho) - e(x, \rho)) \exp(-i\rho x)| \leq \int_r^\infty t |q(t)| dt \exp(Q_1(0)), \quad (2.1.54)$$

$$|(e'_r(x, \rho) - e'(x, \rho)) \exp(-i\rho x)| \leq \left(Q_0(r) + \int_r^\infty t |q(t)| dt Q_0(0)\right) \exp(Q_1(0)). \quad (2.1.55)$$

*Proof.* Denote

$$z_r(x, \rho) = e_r(x, \rho) \exp(-i\rho x), \quad z(x, \rho) = e(x, \rho) \exp(-i\rho x), \quad u_r(x, \rho) = |z_r(x, \rho) - z(x, \rho)|.$$

It follows from (2.1.8) and (2.1.29) that

$$u_r(x, \rho) \leq \int_x^\infty (t-x) |q_r(t) z_r(t, \rho) - q(t) z(t, \rho)| dt, \quad \operatorname{Im} \rho \geq 0, \quad x \geq 0, \quad r \geq 0. \quad (2.1.56)$$

Let  $x \geq r$ . By virtue of (2.1.23), (2.1.49) and (2.1.56),

$$u_r(x, \rho) \leq \int_x^\infty (t-x) |q(t) z(t, \rho)| dt \leq \int_x^\infty (t-x) |q(t)| \exp(Q_1(t)) dt.$$

Since the function  $Q_1(x)$  is monotonic we get

$$u_r(x, \rho) \leq Q_1(x) \exp(Q_1(x)) \leq Q_1(r) \exp(Q_1(0)), \quad x \geq r. \quad (2.1.57)$$

For  $x \leq r$ , (2.1.56) implies

$$u_r(x, \rho) \leq \int_r^\infty (t-x) |q(t) z(t, \rho)| dt + \int_x^r (t-x) |q(t)| u_r(t, \rho) dt.$$

Using (2.1.23) we infer

$$u_r(x, \rho) \leq \exp(Q_1(r)) \int_r^\infty t |q(t)| dt + \int_x^r (t-x) |q(t)| u_r(t, \rho) dt.$$

According to Lemma 2.1.2,

$$u_r(x, \rho) \leq \exp(Q_1(r)) \int_r^\infty t|q(t)| dt \exp\left(\int_x^r (t-x)|q(t)| dt\right), \quad x \leq r,$$

and consequently

$$u_r(x, \rho) \leq \exp(Q_1(x)) \int_r^\infty t|q(t)| dt \leq \exp(Q_1(0)) \int_r^\infty t|q(t)| dt, \quad x \leq r.$$

Together with (2.1.57) this yields (2.1.54).

Denote

$$v_r(x, \rho) = |(e'_r(x, \rho) - e'(x, \rho)) \exp(-i\rho x)|, \quad \text{Im } \rho \geq 0, \quad x \geq 0, \quad r \geq 0.$$

It follows from (2.1.20) that

$$v_r(x, \rho) \leq \int_x^\infty |q_r(t)z_r(t, \rho) - q(t)z(t, \rho)| dt, \quad \text{Im } \rho \geq 0, \quad x \geq 0, \quad r \geq 0. \quad (2.1.58)$$

Let  $x \geq r$ . By virtue of (2.1.23), (2.1.49) and (2.1.58),

$$\begin{aligned} v_r(x, \rho) &\leq \int_x^\infty |q(t)| \exp(Q_1(t)) dt \\ &\leq Q_0(x) \exp(Q_1(x)) \leq Q_0(r) \exp(Q_1(0)), \quad \text{Im } \rho \geq 0, \quad 0 \leq r \leq x. \end{aligned} \quad (2.1.59)$$

For  $x \leq r$ , (2.1.58) gives

$$v_r(x, \rho) \leq \int_r^\infty |q(t)z(t, \rho)| dt + \int_x^r |q(t)|u_r(t, \rho) dt.$$

Using (2.1.23) and (2.1.54) we infer

$$v_r(x, \rho) \leq \int_r^\infty |q(t)| \exp(Q_1(t)) dt + \exp(Q_1(0)) \int_r^\infty s|q(s)| ds \int_x^r |q(t)| dt,$$

and consequently

$$v_r(x, \rho) \leq (Q_0(r) + Q_0(0) \int_r^\infty t|q(t)| dt) \exp(Q_1(0)), \quad 0 \leq x \leq r, \quad \text{Im } \rho \geq 0.$$

Together with (2.1.59) this yields (2.1.55).  $\square$

**Theorem 2.1.4.** For each  $\delta > 0$ , there exists  $a = a_\delta \geq 0$  such that equation (2.1.1) has a unique solution  $y = E(x, \rho)$ ,  $\rho \in \Omega_\delta$ , satisfying the integral equation

$$E(x, \rho) = \exp(-i\rho x) + \frac{1}{2i\rho} \int_a^x \exp(i\rho(x-t))q(t)E(t, \rho) dt + \frac{1}{2i\rho} \int_x^\infty \exp(i\rho(t-x))q(t)E(t, \rho) dt. \quad (2.1.60)$$

The function  $E(x, \rho)$ , called the Birkhoff solution for (2.1.1), has the following properties:

- (i<sub>1</sub>)  $E^{(\nu)}(x, \rho) = (-i\rho)^\nu \exp(-i\rho x)(1 + o(1))$ ,  $x \rightarrow \infty$   $\nu = 0, 1$ , uniformly for  $|\rho| \geq \delta$ ,  $\text{Im } \rho \geq \alpha$ , for each fixed  $\alpha > 0$ ;
- (i<sub>2</sub>)  $E^{(\nu)}(x, \rho) = (-i\rho)^\nu \exp(-i\rho x)(1 + O(\rho^{-1}))$ ,  $|\rho| \rightarrow \infty$   $\rho \in \Omega$ , uniformly for  $x \geq a$ ;
- (i<sub>3</sub>) for each fixed  $x \geq 0$ , the functions  $E^{(\nu)}(x, \rho)$  are analytic for  $\text{Im } \rho > 0$ ,  $|\rho| \geq \delta$ , and are continuous for  $\rho \in \Omega_\delta$ ;

- (i<sub>4</sub>) the functions  $e(x, \rho)$  and  $E(x, \rho)$  form a fundamental system of solutions for (2.1.1), and  $\langle e(x, \rho), E(x, \rho) \rangle = -2i\rho$ .  
 (i<sub>5</sub>) If  $\delta \geq Q_0(0)$ , then one can take above  $a = 0$ .

*Proof.* For fixed  $\delta > 0$  choose  $a = a_\delta \geq 0$  such that  $Q_0(a) \leq \delta$ . We transform (2.1.60) by means of the replacement  $E(x, \rho) = \exp(-i\rho x)\xi(x, \rho)$  to the equation

$$\xi(x, \rho) = 1 + \frac{1}{2i\rho} \int_a^x \exp(2i\rho(x-t))q(t)\xi(t, \rho) dt + \frac{1}{2i\rho} \int_x^\infty q(t)\xi(t, \rho) dt. \quad (2.1.61)$$

The method of successive approximations gives

$$\begin{aligned} \xi_0(x, \rho) &= 1, \quad \xi_{k+1}(x, \rho) = \frac{1}{2i\rho} \int_a^x \exp(2i\rho(x-t))q(t)\xi_k(t, \rho) dt + \frac{1}{2i\rho} \int_x^\infty q(t)\xi_k(t, \rho) dt, \\ \xi(x, \rho) &= \sum_{k=0}^{\infty} \xi_k(x, \rho). \end{aligned}$$

This yields

$$|\xi_{k+1}(x, \rho)| \leq \frac{1}{2|\rho|} \int_a^\infty |q(t)\xi_k(t, \rho)| dt,$$

and hence

$$|\xi_k(x, \rho)| \leq \left( \frac{Q_0(a)}{2|\rho|} \right)^k.$$

Thus for  $x \geq a$ ,  $|\rho| \geq Q_0(a)$ , we get

$$|\xi(x, \rho)| \leq 2, \quad |\xi(x, \rho) - 1| \leq \frac{Q_0(a)}{|\rho|}.$$

It follows from (2.1.60) that

$$E'(x, \rho) = \exp(-i\rho x) \left( -i\rho + \frac{1}{2} \int_a^x \exp(2i\rho(x-t))q(t)\xi(t, \rho) dt - \frac{1}{2} \int_x^\infty q(t)\xi(t, \rho) dt \right). \quad (2.1.62)$$

Since  $|\xi(x, \rho)| \leq 2$  for  $x \geq a$ ,  $\rho \in \Omega_\delta$ , it follows from (2.1.61) and (2.1.62) that

$$\begin{aligned} |E^{(\nu)}(x, \rho)(-i\rho)^{-\nu} \exp(i\rho x) - 1| &\leq \frac{1}{|\rho|} \left( \int_a^x \exp(-2\tau(x-t))|q(t)| dt + \int_x^\infty |q(t)| dt \right) \\ &\leq \frac{1}{|\rho|} \left( \exp(-\tau x) \int_a^{x/2} |q(t)| dt + \int_{x/2}^\infty |q(t)| dt \right), \end{aligned}$$

and consequently (i<sub>1</sub>) – (i<sub>2</sub>) are proved.

The other assertions of Theorem 2.1.4 are obvious. □

### 2.1.2. Properties of the spectrum. Denote

$$\Delta(\rho) = e'(0, \rho) - h e(0, \rho). \quad (2.1.63)$$

By virtue of Theorem 2.1.1, the function  $\Delta(\rho)$  is analytic for  $\operatorname{Im} \rho > 0$ , and continuous for  $\rho \in \Omega$ . It follows from (2.1.5) that for  $|\rho| \rightarrow \infty$ ,  $\rho \in \Omega$ ,

$$e(0, \rho) = 1 + \frac{\omega_1}{i\rho} + o\left(\frac{1}{\rho}\right), \quad \Delta(\rho) = (i\rho) \left( 1 + \frac{\omega_{11}}{i\rho} + o\left(\frac{1}{\rho}\right) \right), \quad (2.1.64)$$

where  $\omega_1 = \omega(0)$ ,  $\omega_{11} = \omega(0) - h$ . Using (2.1.7), (2.1.16) and (2.1.20) one can obtain more precisely

$$\left. \begin{aligned} e(0, \rho) &= 1 + \frac{\omega_1}{i\rho} + \frac{1}{2i\rho} \int_0^\infty q(t) \exp(2i\rho t) dt + O\left(\frac{1}{\rho^2}\right), \\ \Delta(\rho) &= (i\rho) \left(1 + \frac{\omega_{11}}{i\rho} - \frac{1}{2i\rho} \int_0^\infty q(t) \exp(2i\rho t) dt + O\left(\frac{1}{\rho^2}\right)\right). \end{aligned} \right\} \quad (2.1.65)$$

Denote

$$\begin{aligned} \Lambda &= \{\lambda = \rho^2 : \rho \in \Omega, \Delta(\rho) = 0\}, \\ \Lambda' &= \{\lambda = \rho^2 : \operatorname{Im} \rho > 0, \Delta(\rho) = 0\}, \\ \Lambda'' &= \{\lambda = \rho^2 : \operatorname{Im} \rho = 0, \rho \neq 0, \Delta(\rho) = 0\}. \end{aligned}$$

Obviously,  $\Lambda = \Lambda' \cup \Lambda''$  is a bounded set, and  $\Lambda'$  is a bounded and at most countable set.

Denote

$$\Phi(x, \lambda) = \frac{e(x, \rho)}{\Delta(\rho)}. \quad (2.1.66)$$

The function  $\Phi(x, \lambda)$  satisfies (2.1.1) and on account of (2.1.63) and Theorem 2.1.1 also the conditions

$$U(\Phi) = 1, \quad (2.1.67)$$

$$\Phi(x, \lambda) = O(\exp(i\rho x)), \quad x \rightarrow \infty, \rho \in \Omega, \quad (2.1.68)$$

where  $U$  is defined by (2.1.2). The function  $\Phi(x, \lambda)$  is called the *Weyl solution* for  $L$ . Note that (2.1.1), (2.1.67) and (2.1.68) uniquely determine the Weyl solution.

Denote  $M(\lambda) := \Phi(0, \lambda)$ . The function  $M(\lambda)$  is called the *Weyl function* for  $L$ . It follows from (2.1.66) that

$$M(\lambda) = \frac{e(0, \rho)}{\Delta(\rho)}. \quad (2.1.69)$$

Clearly,

$$\Phi(x, \lambda) = S(x, \lambda) + M(\lambda)\varphi(x, \lambda), \quad (2.1.70)$$

where the functions  $\varphi(x, \lambda)$  and  $S(x, \lambda)$  are solutions of (2.1.1) under the initial conditions

$$\varphi(0, \lambda) = 1, \varphi'(0, \lambda) = h, \quad S(0, \lambda) = 0, S'(0, \lambda) = 1.$$

We recall that the Weyl function plays an important role in the spectral theory of Sturm-Liouville operators (see [lev3] for more details).

By virtue of Liouville's formula for the Wronskian [cod1, p.83],  $\langle \varphi(x, \lambda), \Phi(x, \lambda) \rangle$  does not depend on  $x$ . Since for  $x = 0$ ,

$$\langle \varphi(x, \lambda), \Phi(x, \lambda) \rangle|_{x=0} = U(\Phi) = 1,$$

we infer

$$\langle \varphi(x, \lambda), \Phi(x, \lambda) \rangle \equiv 1. \quad (2.1.71)$$

**Theorem 2.1.5.** *The Weyl function  $M(\lambda)$  is analytic in  $\Pi \setminus \Lambda'$  and continuous in  $\Pi_1 \setminus \Lambda$ . The set of singularities of  $M(\lambda)$  (as an analytic function) coincides with the set  $\Lambda_0 := \{\lambda : \lambda \geq 0\} \cup \Lambda$ .*

Theorem 2.1.5 follows from (2.1.63), (2.1.69) and Theorem 2.1.1. By virtue of (2.1.70), the set of singularities of the Weyl solution  $\Phi(x, \lambda)$  coincides with  $\Lambda_0$  for all  $x \geq 0$ , since the functions  $\varphi(x, \lambda)$  and  $S(x, \lambda)$  are entire in  $\lambda$  for each fixed  $x \geq 0$ .

**Definition 2.1.1.** The set of singularities of the Weyl function  $M(\lambda)$  is called the spectrum of  $L$ . The values of the parameter  $\lambda$ , for which equation (2.1.1) has nontrivial solutions satisfying the conditions  $U(y) = 0$ ,  $y(\infty) = 0$  (i.e.  $\lim_{x \rightarrow \infty} y(x) = 0$ ), are called eigenvalues of  $L$ , and the corresponding solutions are called eigenfunctions.

**Remark 2.1.3.** One can introduce the operator

$$L^o : D(L^o) \rightarrow L_2(0, \infty), y \rightarrow -y'' + q(x)y$$

with the domain of definition  $D(L^o) = \{y : y \in L_2(I) \cap AC_{loc}(I), y' \in AC_{loc}(I), L^o y \in L_2(I), U(y) = 0\}$ , where  $I := [0, \infty)$ . It is easy to verify that the spectrum of  $L^o$  coincides with  $\Lambda_0$ . For the Sturm-Liouville equation there is no difference between working either with the operator  $L^o$  or with the pair  $L$ . However, for generalizations for many other classes of inverse problems, from methodical point of view it is more natural to consider the pair  $L$  (see, for example, [yur1]).

**Theorem 2.1.6.**  $L$  has no eigenvalues  $\lambda > 0$ .

*Proof.* Suppose that  $\lambda_0 = \rho_0^2 > 0$  is an eigenvalue, and let  $y_0(x)$  be a corresponding eigenfunction. Since the functions  $\{e(x, \rho_0), e(x, -\rho_0)\}$  form a fundamental system of solutions of (2.1.1), we have  $y_0(x) = Ae(x, \rho_0) + Be(x, -\rho_0)$ . For  $x \rightarrow \infty$ ,  $y_0(x) \sim 0$ ,  $e(x, \pm\rho_0) \sim \exp(\pm i\rho_0 x)$ . But this is possible only if  $A = B = 0$ .  $\square$

**Theorem 2.1.7.** Let  $\lambda_0 \notin [0, \infty)$ . For  $\lambda_0$  to be an eigenvalue, it is necessary and sufficient that  $\Delta(\rho_0) = 0$ . In other words, the set of nonzero eigenvalues coincides with  $\Lambda'$ . For each eigenvalue  $\lambda_0 \in \Lambda'$  there exists only one (up to a multiplicative constant) eigenfunction, namely,

$$\varphi(x, \lambda_0) = \beta_0 e(x, \rho_0), \quad \beta_0 \neq 0. \quad (2.1.72)$$

*Proof.* Let  $\lambda_0 \in \Lambda'$ . Then  $U(e(x, \rho_0)) = \Delta(\rho_0) = 0$  and, by virtue of (2.1.4),  $\lim_{x \rightarrow \infty} e(x, \rho_0) = 0$ . Thus,  $e(x, \rho_0)$  is an eigenfunction, and  $\lambda_0 = \rho_0^2$  is an eigenvalue. Moreover, it follows from (2.1.66) and (2.1.71) that  $\langle \varphi(x, \lambda), e(x, \rho) \rangle = \Delta(\rho)$ , and consequently (2.1.72) is valid.

Conversely, let  $\lambda_0 = \rho_0^2$ ,  $Im \rho_0 > 0$  be an eigenvalue, and let  $y_0(x)$  be a corresponding eigenfunction. Clearly,  $y_0(0) \neq 0$ . Without loss of generality we put  $y_0(0) = 1$ . Then  $y_0'(0) = h$ , and hence  $y_0(x) \equiv \varphi(x, \lambda_0)$ . Since the functions  $E(x, \rho_0)$  and  $e(x, \rho_0)$  form a fundamental system of solutions of equation (2.1.1), we get  $y_0(x) = \alpha_0 E(x, \rho_0) + \beta_0 e(x, \rho_0)$ . As  $x \rightarrow \infty$ , we calculate  $\alpha_0 = 0$ , i.e.  $y_0(x) = \beta_0 e(x, \rho_0)$ . This yields (2.1.72). Consequently,  $\Delta(\rho_0) = U(e(x, \rho_0)) = 0$ , and  $\varphi(x, \lambda_0)$  and  $e(x, \rho_0)$  are eigenfunctions.  $\square$

Thus, the spectrum of  $L$  consists of the positive half-line  $\{\lambda : \lambda \geq 0\}$ , and the discrete set  $\Lambda = \Lambda' \cup \Lambda''$ . Each element of  $\Lambda'$  is an eigenvalue of  $L$ . According to Theorem 2.1.6, the points of  $\Lambda''$  are not eigenvalues of  $L$ , they are called *spectral singularities* of  $L$ .

**Example 2.1.1.** Let  $q(x) \equiv 0$ ,  $h = i\theta$ , where  $\theta$  is a real number. Then  $\Delta(\rho) = i\rho - h$ , and  $\Lambda' = \emptyset$ ,  $\Lambda'' = \{\theta\}$ , i.e.  $L$  has no eigenvalues, and the point  $\rho_0 = \theta$  is a spectral singularity for  $L$ .

It follows from (2.1.5), (2.1.64), (2.1.66) and (2.1.69) that for  $|\rho| \rightarrow \infty$ ,  $\rho \in \Omega$ ,

$$M(\lambda) = \frac{1}{i\rho} \left( 1 + \frac{m_1}{i\rho} + o\left(\frac{1}{\rho}\right) \right), \quad (2.1.73)$$

$$\Phi^{(\nu)}(x, \lambda) = (i\rho)^{\nu-1} \exp(i\rho x) \left( 1 + \frac{B(x)}{i\rho} + o\left(\frac{1}{\rho}\right) \right), \quad (2.1.74)$$

uniformly for  $x \geq 0$ ; here

$$m_1 = h, \quad B(x) = h + \frac{1}{2} \int_0^x q(s) ds.$$

Taking (2.1.65) into account one can derive more precisely

$$M(\lambda) = \frac{1}{i\rho} \left( 1 + \frac{m_1}{i\rho} + \frac{1}{i\rho} \int_0^\infty q(t) \exp(2i\rho t) dt + O\left(\frac{1}{\rho^2}\right) \right), \quad |\rho| \rightarrow \infty, \rho \in \Omega. \quad (2.1.75)$$

Moreover, if  $q(x) \in W_N$ , then by virtue of (2.1.21) we get

$$M(\lambda) = \frac{1}{i\rho} \left( 1 + \sum_{s=1}^{N+1} \frac{m_s}{(i\rho)^s} + o\left(\frac{1}{\rho^{N+1}}\right) \right), \quad |\rho| \rightarrow \infty, \rho \in \Omega, \quad (2.1.76)$$

where  $m_1 = h$ ,  $m_2 = -\frac{1}{2}q(0) + h^2, \dots$ .

Denote

$$V(\lambda) = \frac{1}{2\pi i} \left( M^-(\lambda) - M^+(\lambda) \right), \quad \lambda > 0, \quad (2.1.77)$$

where

$$M^\pm(\lambda) = \lim_{z \rightarrow 0, \operatorname{Re} z > 0} M(\lambda \pm iz).$$

It follows from (2.1.73) and (2.1.77) that for  $\rho > 0$ ,  $\rho \rightarrow +\infty$ ,

$$V(\lambda) = \frac{1}{2\pi i} \left( -\frac{1}{i\rho} \left( 1 - \frac{m_1}{i\rho} + o\left(\frac{1}{\rho}\right) \right) - \frac{1}{i\rho} \left( 1 + \frac{m_1}{i\rho} + o\left(\frac{1}{\rho}\right) \right) \right),$$

and consequently

$$V(\lambda) = \frac{1}{\pi\rho} \left( 1 + o\left(\frac{1}{\rho}\right) \right), \quad \rho > 0, \rho \rightarrow +\infty. \quad (2.1.78)$$

In view of (2.1.75), we calculate more precisely

$$V(\lambda) = \frac{1}{\pi\rho} \left( 1 + \frac{1}{\rho} \int_0^\infty q(t) \sin 2\rho t dt + O\left(\frac{1}{\rho^2}\right) \right), \quad \rho > 0, \rho \rightarrow +\infty. \quad (2.1.79)$$

Moreover, if  $q(x) \in W_{N+1}$ , then (2.1.76) implies

$$V(\lambda) = \frac{1}{\pi\rho} \left( 1 + \sum_{s=1}^{N+1} \frac{V_s}{\rho^s} + o\left(\frac{1}{\rho^{N+1}}\right) \right), \quad \rho > 0, \rho \rightarrow +\infty, \quad (2.1.80)$$

where  $V_{2s} = (-1)^s m_{2s}$ ,  $V_{2s+1} = 0$ .

**Remark 2.1.4.** Analogous results are also valid if we replace  $U(y) = 0$  by the boundary condition  $U_0(y) := y(0) = 0$ . In this case, the Weyl solution  $\Phi_0(x, \lambda)$  and the Weyl function  $M_0(\lambda)$  are defined by the conditions  $\Phi_0(0, \lambda) = 1$ ,  $\Phi_0(x, \lambda) = O(\exp(i\rho x))$ ,  $x \rightarrow \infty$ ,  $M_0(\lambda) := \Phi'_0(0, \lambda)$ , and

$$\Phi_0(x, \lambda) = \frac{e(x, \rho)}{e(0, \rho)} = C(x, \lambda) + M_0(\lambda)S(x, \lambda), \quad M_0(\lambda) = \frac{e'(0, \rho)}{e(0, \rho)}, \quad (2.1.81)$$

where  $C(x, \lambda)$  is a solution of (2.1.1) under the conditions  $C(0, \lambda) = 1$ ,  $C'(0, \lambda) = 0$ .

**2.1.3. An expansion theorem.** In the  $\lambda$  - plane we consider the contour  $\gamma = \gamma' \cup \gamma''$  (with counterclockwise circuit), where  $\gamma'$  is a bounded closed contour encircling the set  $\Lambda \cup \{0\}$ , and  $\gamma''$  is the two-sided cut along the arc  $\{\lambda : \lambda > 0, \lambda \notin \text{int } \gamma'\}$ .

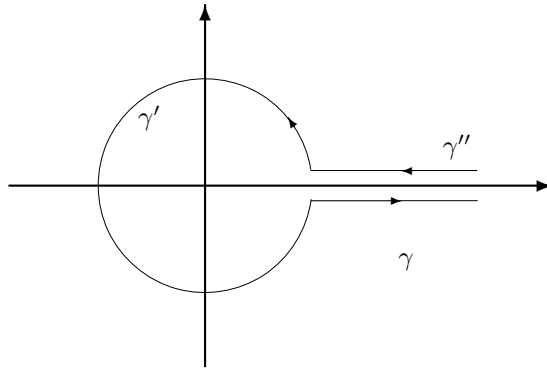


fig. 2.1.1.

**Theorem 2.1.8.** Let  $f(x) \in W_2$ . Then, uniformly for  $x \geq 0$ ,

$$f(x) = \frac{1}{2\pi i} \int_{\gamma} \varphi(x, \lambda) F(\lambda) M(\lambda) d\lambda, \quad (2.1.82)$$

where

$$F(\lambda) := \int_0^{\infty} \varphi(t, \lambda) f(t) dt.$$

*Proof.* We will use the contour integral method. For this purpose we consider the function

$$Y(x, \lambda) = \Phi(x, \lambda) \int_0^x \varphi(t, \lambda) f(t) dt + \varphi(x, \lambda) \int_x^{\infty} \Phi(t, \lambda) f(t) dt. \quad (2.1.83)$$

Since the functions  $\varphi(x, \lambda)$  and  $\Phi(x, \lambda)$  satisfy (2.1.1) we transform  $Y(x, \lambda)$  as follows

$$\begin{aligned} Y(x, \lambda) &= \frac{1}{\lambda} \Phi(x, \lambda) \int_0^x (-\varphi''(t, \lambda) + q(t)\varphi(t, \lambda)) f(t) dt \\ &\quad + \frac{1}{\lambda} \varphi(x, \lambda) \int_x^{\infty} (-\Phi''(t, \lambda) + q(t)\Phi(t, \lambda)) f(t) dt. \end{aligned}$$

Two-fold integration by parts of terms with second derivatives yields in view of (2.1.71)

$$Y(x, \lambda) = \frac{1}{\lambda} (f(x) + Z(x, \lambda)), \quad (2.1.84)$$

where

$$\begin{aligned} Z(x, \lambda) &= (f'(0) - hf(0))\Phi(x, \lambda) + \Phi(x, \lambda) \int_0^x \varphi(t, \lambda) \ell f(t) dt \\ &\quad + \varphi(x, \lambda) \int_x^\infty \Phi(t, \lambda) \ell f(t) dt. \end{aligned} \quad (2.1.85)$$

Similarly,

$$F(\lambda) = -\frac{1}{\lambda} (f'(0) - hf(0)) + \frac{1}{\lambda} \int_0^\infty \varphi(t, \lambda) \ell f(t) dt, \quad \lambda > 0. \quad (2.1.86)$$

The function  $\varphi(x, \lambda)$  satisfies the integral equation (1.1.11). Denote

$$\mu_T(\lambda) = \max_{0 \leq x \leq T} (|\varphi(x, \lambda)| \exp(-|\tau|x)), \quad \tau := \operatorname{Im} \rho.$$

Then (1.1.11) gives for  $|\rho| \geq 1$ ,  $x \in [0, T]$ ,

$$|\varphi(x, \lambda)| \exp(-|\tau|x) \leq C + \frac{\mu_T(\lambda)}{|\rho|} \int_0^T |q(t)| dt,$$

and consequently

$$\mu_T(\lambda) \leq C + \frac{\mu_T(\lambda)}{|\rho|} \int_0^T |q(t)| dt \leq C + \frac{\mu_T(\lambda)}{|\rho|} \int_0^\infty |q(t)| dt.$$

From this we get  $|\mu_T(\lambda)| \leq C$  for  $|\rho| \geq \rho^*$ . Together with (1.1.12) this yields for  $\nu = 0, 1$ ,  $|\rho| \geq \rho^*$ ,

$$|\varphi^{(\nu)}(x, \lambda)| \leq C|\rho|^\nu \exp(|\tau|x), \quad (2.1.87)$$

uniformly for  $x \geq 0$ . Furthermore, it follows from (2.1.74) that for  $\nu = 0, 1$ ,  $|\rho| \geq \rho^*$ ,

$$|\Phi^{(\nu)}(x, \lambda)| \leq C|\rho|^{\nu-1} \exp(-|\tau|x), \quad (2.1.88)$$

uniformly for  $x \geq 0$ . By virtue of (2.1.85), (2.1.87) and (2.1.88) we get

$$Z(x, \lambda) = O\left(\frac{1}{\rho}\right), \quad |\lambda| \rightarrow \infty,$$

uniformly for  $x \geq 0$ . Hence, (2.1.84) implies

$$\lim_{R \rightarrow \infty} \sup_{x \geq 0} \left| f(x) - \frac{1}{2\pi i} \int_{|\lambda|=R} Y(x, \lambda) d\lambda \right| = 0, \quad (2.1.89)$$

where the contour in the integral is used with counterclockwise circuit. Consider the contour  $\gamma_R^0 = (\gamma \cap \{\lambda : |\lambda| \leq R\}) \cup \{\lambda : |\lambda| = R\}$  (with clockwise circuit).



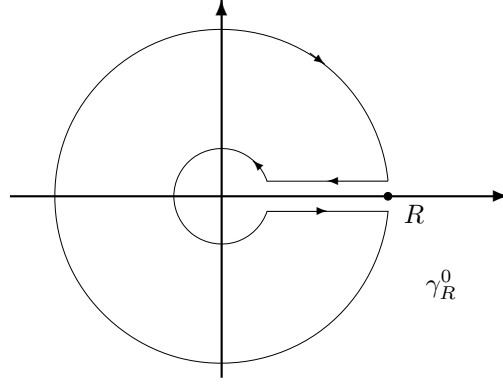


fig. 2.1.2

By Cauchy's theorem [con1, p.85]

$$\frac{1}{2\pi i} \int_{\gamma_R^0} Y(x, \lambda) d\lambda = 0.$$

Taking (2.1.89) into account we obtain

$$\lim_{R \rightarrow \infty} \sup_{x \geq 0} \left| f(x) - \frac{1}{2\pi i} \int_{\gamma_R} Y(x, \lambda) d\lambda \right| = 0,$$

where  $\gamma_R = \gamma \cap \{\lambda : |\lambda| \leq R\}$  (with counterclockwise circuit). From this, using (2.1.83) and (2.1.70), we arrive at (2.1.82), since the terms with  $S(x, \lambda)$  vanish by Cauchy's theorem. We note that according to (2.1.79), (2.1.86) and (2.1.87),

$$F(\lambda) = O\left(\frac{1}{\lambda}\right), \quad M(\lambda) = O\left(\frac{1}{\rho}\right), \quad \varphi(x, \lambda) = O(1), \quad x \geq 0, \lambda > 0, \lambda \rightarrow \infty,$$

and consequently the integral in (2.1.82) converges absolutely and uniformly for  $x \geq 0$ .  $\square$

**Remark 2.1.5.** If  $q(x)$  and  $h$  are real, and  $(1+x)q(x) \in L(0, \infty)$ , then (see Section 2.3)  $\Lambda'' = \emptyset$ ,  $\Lambda' \subset (-\infty, 0)$  is a finite set of simple eigenvalues,  $V(\lambda) > 0$  for  $\lambda > 0$  ( $V(\lambda)$  is defined by (2.1.77)), and  $M(\lambda) = O(\rho^{-1})$  as  $\rho \rightarrow 0$ . Then (2.1.82) takes the form

$$f(x) = \int_0^\infty \varphi(x, \lambda) F(\lambda) V(\lambda) d\lambda + \sum_{\lambda_j \in \Lambda'} \varphi(x, \lambda_j) F(\lambda_j) Q_j, \quad Q_j := \operatorname{Res}_{\lambda=\lambda_j} M(\lambda).$$

or

$$f(x) = \int_{-\infty}^\infty \varphi(x, \lambda) F(\lambda) d\sigma(\lambda),$$

where  $\sigma(\lambda)$  is the spectral function of  $L$  (see [lev3]). For  $\lambda < 0$ ,  $\sigma(\lambda)$  is a step-function; for  $\lambda > 0$ ,  $\sigma(\lambda)$  is an absolutely continuous function, and  $\sigma'(\lambda) = V(\lambda)$ .

**Remark 2.1.6.** It follows from the proof that Theorem 2.1.8 remains valid also for  $f(x) \in W_1$ .

## 2.2. RECOVERY OF THE DIFFERENTIAL EQUATION FROM THE WEYL FUNCTION.

In this section we study the inverse problem of recovering the pair  $L = L(q(x), h)$  of the form (2.1.1)-(2.1.2) from the given Weyl function  $M(\lambda)$ . For this purpose we will use the method of spectral mappings described in Section 1.6 for Sturm-Liouville operators on a finite interval. Since for the half-line the arguments are more or less similar, the proofs in this section are shorter than in Section 1.6.

First, let us prove the uniqueness theorem for the solution of the inverse problem. We agree, as in Chapter 1, that together with  $L$  we consider here and in the sequel a pair  $\tilde{L} = L(\tilde{q}(x), \tilde{h})$  of the same form but with different coefficients. If a certain symbol  $\gamma$  denotes an object related to  $L$ , then the corresponding symbol  $\tilde{\gamma}$  with tilde denotes the analogous object related to  $\tilde{L}$ , and  $\hat{\gamma} := \gamma - \tilde{\gamma}$ .

**Theorem 2.2.1** *If  $M(\lambda) = \tilde{M}(\lambda)$ , then  $L = \tilde{L}$ . Thus, the specification of the Weyl function uniquely determines  $q(x)$  and  $h$ .*

*Proof.* Let us define the matrix  $P(x, \lambda) = [P_{jk}(x, \lambda)]_{j,k=1,2}$  by the formula

$$P(x, \lambda) \begin{bmatrix} \tilde{\varphi}(x, \lambda) & \tilde{\Phi}(x, \lambda) \\ \tilde{\varphi}'(x, \lambda) & \tilde{\Phi}'(x, \lambda) \end{bmatrix} = \begin{bmatrix} \varphi(x, \lambda) & \Phi(x, \lambda) \\ \varphi'(x, \lambda) & \Phi'(x, \lambda) \end{bmatrix}.$$

By virtue of (2.1.71), this yields

$$\left. \begin{aligned} P_{j1}(x, \lambda) &= \varphi^{(j-1)}(x, \lambda) \tilde{\Phi}'(x, \lambda) - \Phi^{(j-1)}(x, \lambda) \tilde{\varphi}'(x, \lambda) \\ P_{j2}(x, \lambda) &= \Phi^{(j-1)}(x, \lambda) \tilde{\varphi}(x, \lambda) - \varphi^{(j-1)}(x, \lambda) \tilde{\Phi}(x, \lambda) \end{aligned} \right\}, \quad (2.2.1)$$

$$\left. \begin{aligned} \varphi(x, \lambda) &= P_{11}(x, \lambda) \tilde{\varphi}(x, \lambda) + P_{12}(x, \lambda) \tilde{\varphi}'(x, \lambda) \\ \Phi(x, \lambda) &= P_{11}(x, \lambda) \tilde{\Phi}(x, \lambda) + P_{12}(x, \lambda) \tilde{\Phi}'(x, \lambda) \end{aligned} \right\}. \quad (2.2.2)$$

Using (2.2.1), (2.1.87)-(2.1.88) we get for  $|\lambda| \rightarrow \infty$ ,  $\lambda = \rho^2$ ,

$$P_{jk}(x, \lambda) - \delta_{jk} = O(\rho^{-1}), \quad j \leq k; \quad P_{21}(x, \lambda) = O(1). \quad (2.2.3)$$

If  $M(\lambda) \equiv \tilde{M}(\lambda)$ , then in view of (2.2.1) and (2.1.70), for each fixed  $x$ , the functions  $P_{jk}(x, \lambda)$  are entire in  $\lambda$ . Together with (2.2.3) this yields  $P_{11}(x, \lambda) \equiv 1$ ,  $P_{12}(x, \lambda) \equiv 0$ . Substituting into (2.2.2) we get  $\varphi(x, \lambda) \equiv \tilde{\varphi}(x, \lambda)$ ,  $\Phi(x, \lambda) \equiv \tilde{\Phi}(x, \lambda)$  for all  $x$  and  $\lambda$ , and consequently,  $L = \tilde{L}$ .  $\square$

Let us now start to construct the solution of the inverse problem. We shall say that  $L \in V_N$  if  $q(x) \in W_N$ . We shall subsequently solve the inverse problem in the classes  $V_N$ .

Let a model pair  $\tilde{L} = L(\tilde{q}(x), \tilde{h})$  be chosen such that

$$\int_{\rho^*}^{\infty} \rho^4 |\hat{V}(\lambda)|^2 d\rho < \infty, \quad \hat{V} := V - \tilde{V} \quad (2.2.4)$$

for sufficiently large  $\rho^* > 0$ . The condition (2.2.4) is needed for technical reasons. In principal, one could take any  $\tilde{L}$  (for example, with  $\tilde{q}(x) = \tilde{h} = 0$ ) but generally speaking proofs would become more complicated. On the other hand, (2.2.4) is not a very strong restriction, since by virtue of (2.1.79),

$$\rho^2 \hat{V}(\lambda) = \frac{1}{\pi} \int_0^{\infty} \hat{q}(t) \sin 2\rho t dt + O\left(\frac{1}{\rho}\right).$$

In particular, if  $q(x) \in L_2$ , then (2.2.4) is fulfilled automatically for any  $\tilde{q}(x) \in L_2$ . Hence, for  $N \geq 1$ , the condition (2.2.4) is fulfilled for any model  $\tilde{L} \in V_N$ .

It follows from (2.2.4) that

$$\int_{\lambda^*}^{\infty} |\hat{V}(\lambda)| d\lambda = 2 \int_{\rho^*}^{\infty} \rho |\hat{V}(\lambda)| d\rho < \infty, \quad \lambda^* = (\rho^*)^2, \quad \lambda = \rho^2. \quad (2.2.5)$$

Denote

$$\left. \begin{aligned} D(x, \lambda, \mu) &= \frac{\langle \varphi(x, \lambda), \varphi(x, \mu) \rangle}{\lambda - \mu} = \int_0^x \varphi(t, \lambda) \varphi(t, \mu) dt, & r(x, \lambda, \mu) &= D(x, \lambda, \mu) \hat{M}(\mu), \\ \tilde{D}(x, \lambda, \mu) &= \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle}{\lambda - \mu} = \int_0^x \tilde{\varphi}(t, \lambda) \tilde{\varphi}(t, \mu) dt, & \tilde{r}(x, \lambda, \mu) &= \tilde{D}(x, \lambda, \mu) \hat{M}(\mu). \end{aligned} \right\} \quad (2.2.6)$$

**Lemma 2.2.1.** *The following estimate holds*

$$|D(x, \lambda, \mu)|, |\tilde{D}(x, \lambda, \mu)| \leq \frac{C_x \exp(|\operatorname{Im} \rho| x)}{|\rho \mp \theta| + 1}, \quad \lambda = \rho^2, \quad \mu = \theta^2 \geq 0, \quad \pm \theta \operatorname{Re} \rho \geq 0. \quad (2.2.7)$$

*Proof.* Let  $\rho = \sigma + i\tau$ . For definiteness, let  $\theta \geq 0$  and  $\sigma \geq 0$ . All other cases can be treated in the same way. Take a fixed  $\delta_0 > 0$ . For  $|\rho - \theta| \geq \delta_0$  we have by virtue of (2.2.6) and (2.1.87),

$$|D(x, \lambda, \mu)| = \left| \frac{\langle \varphi(x, \lambda), \varphi(x, \mu) \rangle}{\lambda - \mu} \right| \leq C \exp(|\tau| x) \frac{|\rho| + |\theta|}{|\rho^2 - \theta^2|}. \quad (2.2.8)$$

Since

$$\frac{|\rho| + |\theta|}{|\rho - \theta|} = \frac{\sqrt{\sigma^2 + \tau^2} + \theta}{\sqrt{(\sigma + \theta)^2 + \tau^2}} \leq \frac{\sqrt{\sigma^2 + \tau^2} + \theta}{\sqrt{\sigma^2 + \tau^2 + \theta^2}} \leq \sqrt{2}$$

(here we use that  $(a + b)^2 \leq 2(a^2 + b^2)$  for all real  $a, b$ ), (2.2.8) implies

$$|D(x, \lambda, \mu)| \leq \frac{C \exp(|\tau| x)}{|\rho - \theta|}. \quad (2.2.9)$$

For  $|\rho - \theta| \geq \delta_0$ , we get

$$\frac{|\rho - \theta| + 1}{|\rho - \theta|} \leq 1 + \frac{1}{\delta_0},$$

and consequently

$$\frac{1}{|\rho - \theta|} \leq \frac{C_0}{|\rho - \theta| + 1} \quad \text{with} \quad C_0 = \frac{\delta_0 + 1}{\delta_0}.$$

Substituting this estimate into the right-hand side of (2.2.9) we obtain

$$|D(x, \lambda, \mu)| \leq \frac{C \exp(|\tau| x)}{|\rho - \theta| + 1},$$

and (2.2.7) is proved for  $|\rho - \theta| \geq \delta_0$ .

For  $|\rho - \theta| \leq \delta_0$ , we have by virtue of (2.2.6) and (2.1.87),

$$|D(x, \lambda, \mu)| \leq \int_0^x |\varphi(t, \lambda) \varphi(t, \mu)| dt \leq C_x \exp(|\tau| x),$$

i.e. (2.2.7) is also valid for  $|\rho - \theta| \leq \delta_0$ .  $\square$

**Lemma 2.2.2.** *The following estimates hold*

$$\int_1^\infty \frac{d\theta}{\theta(|R - \theta| + 1)} = O\left(\frac{\ln R}{R}\right), \quad R \rightarrow \infty, \quad (2.2.10)$$

$$\int_1^\infty \frac{d\theta}{\theta^2(|R - \theta| + 1)^2} = O\left(\frac{1}{R^2}\right), \quad R \rightarrow \infty. \quad (2.2.11)$$

*Proof.* Since

$$\frac{1}{\theta(R - \theta + 1)} = \frac{1}{R + 1} \left( \frac{1}{\theta} + \frac{1}{R - \theta + 1} \right), \quad \frac{1}{\theta(\theta - R + 1)} = \frac{1}{R - 1} \left( \frac{1}{\theta - R + 1} - \frac{1}{\theta} \right),$$

we calculate for  $R > 1$ ,

$$\begin{aligned} \int_1^\infty \frac{d\theta}{\theta(|R - \theta| + 1)} &= \int_1^R \frac{d\theta}{\theta(R - \theta + 1)} + \int_R^\infty \frac{d\theta}{\theta(\theta - R + 1)} \\ &= \frac{1}{R + 1} \int_1^R \left( \frac{1}{\theta} + \frac{1}{R - \theta + 1} \right) d\theta + \frac{1}{R - 1} \int_R^\infty \left( \frac{1}{\theta - R + 1} - \frac{1}{\theta} \right) d\theta \\ &= \frac{2 \ln R}{R + 1} + \frac{\ln R}{R - 1}, \end{aligned}$$

i.e. (2.2.10) is valid. Similarly, for  $R > 1$ ,

$$\begin{aligned} \int_1^\infty \frac{d\theta}{\theta^2(|R - \theta| + 1)^2} &= \int_1^R \frac{d\theta}{\theta^2(R - \theta + 1)^2} + \int_R^\infty \frac{d\theta}{\theta^2(\theta - R + 1)^2} \\ &\leq \frac{1}{(R + 1)^2} \int_1^R \left( \frac{1}{\theta} + \frac{1}{R - \theta + 1} \right)^2 d\theta + \frac{1}{R^2} \int_R^\infty \frac{d\theta}{(\theta - R + 1)^2} \\ &= \frac{2}{(R + 1)^2} \left( \int_1^R \frac{d\theta}{\theta^2} + \int_1^R \frac{d\theta}{\theta(R - \theta + 1)} \right) + \frac{1}{R^2} \int_1^\infty \frac{d\theta}{\theta^2} = O\left(\frac{1}{R^2}\right), \end{aligned}$$

i.e. (2.2.11) is valid.  $\square$

In the  $\lambda$  - plane we consider the contour  $\gamma = \gamma' \cup \gamma''$  (with counterclockwise circuit), where  $\gamma'$  is a bounded closed contour encircling the set  $\Lambda \cup \tilde{\Lambda} \cup \{0\}$ , and  $\gamma''$  is the two-sided cut along the arc  $\{\lambda : \lambda > 0, \lambda \notin \text{int } \gamma'\}$  (see fig. 2.1.1).

**Theorem 2.2.2.** *The following relations hold*

$$\tilde{\varphi}(x, \lambda) = \varphi(x, \lambda) + \frac{1}{2\pi i} \int_\gamma \tilde{r}(x, \lambda, \mu) \varphi(x, \mu) d\mu, \quad (2.2.12)$$

$$r(x, \lambda, \mu) - \tilde{r}(x, \lambda, \mu) + \frac{1}{2\pi i} \int_\gamma \tilde{r}(x, \lambda, \xi) r(x, \xi, \mu) d\xi = 0. \quad (2.2.13)$$

Equation (2.2.12) is called the main equation of the inverse problem.

*Proof.* It follows from (2.1.73), (2.1.87), (2.2.6) and (2.2.7) that for  $\lambda, \mu \in \gamma$ ,  $\pm \text{Re } \rho \text{ Re } \theta \geq 0$ ,

$$|r(x, \lambda, \mu)|, |\tilde{r}(x, \lambda, \mu)| \leq \frac{C_x}{|\mu|(|\rho \mp \theta| + 1)}, \quad |\varphi(x, \lambda)| \leq C. \quad (2.2.14)$$

In view of (2.2.10), it follows from (2.2.14) that the integrals in (2.2.12) and (2.2.13) converge absolutely and uniformly on  $\gamma$  for each fixed  $x \geq 0$ .

Denote  $J_\gamma = \{\lambda : \lambda \notin \gamma \cup \text{int } \gamma'\}$ . Consider the contour  $\gamma_R = \gamma \cap \{\lambda : |\lambda| \leq R\}$  with counterclockwise circuit, and also consider the contour  $\gamma_R^0 = \gamma_R \cup \{\lambda : |\lambda| = R\}$  with clockwise circuit (see fig 2.1.2). By Cauchy's integral formula [con1, p.84],

$$P_{1k}(x, \lambda) - \delta_{1k} = \frac{1}{2\pi i} \int_{\gamma_R^0} \frac{P_{1k}(x, \mu) - \delta_{1k}}{\lambda - \mu} d\mu, \quad \lambda \in \text{int } \gamma_R^0,$$

$$\frac{P_{jk}(x, \lambda) - P_{jk}(x, \mu)}{\lambda - \mu} = \frac{1}{2\pi i} \int_{\gamma_R^0} \frac{P_{jk}(x, \xi)}{(\lambda - \xi)(\xi - \mu)} d\xi, \quad \lambda, \mu \in \text{int } \gamma_R^0.$$

Using (2.2.3) we get

$$\lim_{R \rightarrow \infty} \int_{|\mu|=R} \frac{P_{1k}(x, \mu) - \delta_{1k}}{\lambda - \mu} d\mu = 0, \quad \lim_{R \rightarrow \infty} \int_{|\xi|=R} \frac{P_{jk}(x, \xi)}{(\lambda - \xi)(\xi - \mu)} d\xi = 0,$$

and consequently

$$P_{1k}(x, \lambda) = \delta_{1k} + \frac{1}{2\pi i} \int_\gamma \frac{P_{1k}(x, \mu)}{\lambda - \mu} d\mu, \quad \lambda \in J_\gamma, \quad (2.2.15)$$

$$\frac{P_{jk}(x, \lambda) - P_{jk}(x, \mu)}{\lambda - \mu} = \frac{1}{2\pi i} \int_\gamma \frac{P_{jk}(x, \xi)}{(\lambda - \xi)(\xi - \mu)} d\xi, \quad \lambda, \mu \in J_\gamma. \quad (2.2.16)$$

Here (and everywhere below, where necessary) the integral is understood in the principal value sense:  $\int_\gamma = \lim_{R \rightarrow \infty} \int_{\gamma_R}$ .

By virtue of (2.2.2) and (2.2.15),

$$\varphi(x, \lambda) = \tilde{\varphi}(x, \lambda) + \frac{1}{2\pi i} \int_\gamma \frac{\tilde{\varphi}(x, \lambda)P_{11}(x, \mu) + \tilde{\varphi}'(x, \lambda)P_{12}(x, \mu)}{\lambda - \mu} d\mu, \quad \lambda \in J_\gamma.$$

Taking (2.2.1) into account we get

$$\begin{aligned} \varphi(x, \lambda) = \tilde{\varphi}(x, \lambda) + \frac{1}{2\pi i} \int_\gamma (\tilde{\varphi}(x, \lambda)(\varphi(x, \mu)\tilde{\Phi}'(x, \mu) - \Phi(x, \mu)\tilde{\varphi}'(x, \mu)) + \\ \tilde{\varphi}'(x, \lambda)(\Phi(x, \mu)\tilde{\varphi}(x, \mu) - \varphi(x, \mu)\tilde{\Phi}(x, \mu)) \frac{d\mu}{\lambda - \mu}). \end{aligned}$$

In view of (2.1.70), this yields (2.2.12), since the terms with  $S(x, \mu)$  vanish by Cauchy's theorem.

Using (2.2.16) and acting in the same way as in the proof of Lemma 1.6.3, we arrive at

$$\begin{aligned} D(x, \lambda, \mu) - \tilde{D}(x, \lambda, \mu) = \\ \frac{1}{2\pi i} \int_\gamma \left( \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\Phi}(x, \xi) \rangle \langle \varphi(x, \xi), \varphi(x, \mu) \rangle}{(\lambda - \xi)(\xi - \mu)} - \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \xi) \rangle \langle \Phi(x, \xi), \varphi(x, \mu) \rangle}{(\lambda - \xi)(\xi - \mu)} \right) d\xi. \end{aligned}$$

In view of (2.1.70) and (2.2.6), this yields (2.2.13).  $\square$

Analogously one can obtain the relation

$$\tilde{\Phi}(x, \lambda) = \Phi(x, \lambda) + \frac{1}{2\pi i} \int_\gamma \frac{\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle}{\lambda - \mu} \hat{M}(\mu) \varphi(x, \mu) d\mu, \quad \lambda \in J_\gamma. \quad (2.2.17)$$

Let us consider the Banach space  $C(\gamma)$  of continuous bounded functions  $z(\lambda)$ ,  $\lambda \in \gamma$ , with the norm  $\|z\| = \sup_{\lambda \in \gamma} |z(\lambda)|$ .

**Theorem 2.2.3.** *For each fixed  $x \geq 0$ , the main equation (2.2.12) has a unique solution  $\varphi(x, \lambda) \in C(\gamma)$ .*

*Proof.* For a fixed  $x \geq 0$ , we consider the following linear bounded operators in  $C(\gamma)$  :

$$\tilde{A}z(\lambda) = z(\lambda) + \frac{1}{2\pi i} \int_{\gamma} \tilde{r}(x, \lambda, \mu) z(\mu) d\mu,$$

$$Az(\lambda) = z(\lambda) - \frac{1}{2\pi i} \int_{\gamma} r(x, \lambda, \mu) z(\mu) d\mu.$$

Then

$$\begin{aligned} \tilde{A}Az(\lambda) &= z(\lambda) + \frac{1}{2\pi i} \int_{\gamma} \tilde{r}(x, \lambda, \mu) z(\mu) d\mu - \frac{1}{2\pi i} \int_{\gamma} r(x, \lambda, \mu) z(\mu) d\mu \\ &\quad - \frac{1}{2\pi i} \int_{\gamma} \tilde{r}(x, \lambda, \xi) \left( \frac{1}{2\pi i} \int_{\gamma} r(x, \xi, \mu) z(\mu) d\mu \right) d\xi \\ &= z(\lambda) - \frac{1}{2\pi i} \int_{\gamma} \left( r(x, \lambda, \mu) - \tilde{r}(x, \lambda, \mu) + \frac{1}{2\pi i} \int_{\gamma} \tilde{r}(x, \lambda, \xi) r(x, \xi, \mu) d\xi \right) z(\mu) d\mu. \end{aligned}$$

By virtue of (2.2.13) this yields

$$\tilde{A}Az(\lambda) = z(\lambda), \quad z(\lambda) \in C(\gamma).$$

Interchanging places for  $L$  and  $\tilde{L}$ , we obtain analogously  $A\tilde{A}z(\lambda) = z(\lambda)$ . Thus,  $\tilde{A}A = A\tilde{A} = E$ , where  $E$  is the identity operator. Hence the operator  $\tilde{A}$  has a bounded inverse operator, and the main equation (2.2.12) is uniquely solvable for each fixed  $x \geq 0$ .  $\square$

Denote

$$\varepsilon_0(x) = \frac{1}{2\pi i} \int_{\gamma} \tilde{\varphi}(x, \mu) \varphi(x, \mu) \hat{M}(\mu) d\mu, \quad \varepsilon(x) = -2\varepsilon'_0(x). \quad (2.2.18)$$

**Theorem 2.2.4.** *The following relations hold*

$$q(x) = \tilde{q}(x) + \varepsilon(x), \quad (2.2.19)$$

$$h = \tilde{h} - \varepsilon_0(0). \quad (2.2.20)$$

*Proof.* Differentiating (2.2.12) twice with respect to  $x$  and using (2.2.6) and (2.2.18) we get

$$\tilde{\varphi}'(x, \lambda) - \varepsilon_0(x) \tilde{\varphi}(x, \lambda) = \varphi'(x, \lambda) + \frac{1}{2\pi i} \int_{\gamma} \tilde{r}(x, \lambda, \mu) \varphi'(x, \mu) d\mu, \quad (2.2.21)$$

$$\begin{aligned} \tilde{\varphi}''(x, \lambda) &= \varphi(x, \lambda) + \frac{1}{2\pi i} \int_{\gamma} \tilde{r}(x, \lambda, \mu) \varphi''(x, \mu) d\mu \\ &+ \frac{1}{2\pi i} \int_{\gamma} 2\tilde{\varphi}(x, \lambda) \tilde{\varphi}(x, \mu) \hat{M}(\mu) \varphi'(x, \mu) d\mu + \frac{1}{2\pi i} \int_{\gamma} (\tilde{\varphi}(x, \lambda) \tilde{\varphi}(x, \mu))' \hat{M}(\mu) \varphi(x, \mu) d\mu. \end{aligned} \quad (2.2.22)$$

In (2.2.22) we replace the second derivatives using equation (2.1.1), and then we replace  $\varphi(x, \lambda)$  using (2.2.12). This yields

$$\begin{aligned} \tilde{q}(x)\tilde{\varphi}(x, \lambda) &= q(x)\tilde{\varphi}(x, \lambda) + \frac{1}{2\pi i} \int_{\gamma} \langle \varphi(x, \lambda), \varphi(x, \mu) \rangle \hat{M}(\mu) \varphi(x, \mu) d\mu \\ &+ \frac{1}{2\pi i} \int_{\gamma} 2\tilde{\varphi}(x, \lambda)\tilde{\varphi}(x, \mu)\hat{M}(\mu)\varphi'(x, \mu) d\mu + \frac{1}{2\pi i} \int_{\gamma} (\tilde{\varphi}(x, \lambda)\tilde{\varphi}(x, \mu))'\hat{M}(\mu)\varphi(x, \mu) d\mu. \end{aligned}$$

After canceling terms with  $\tilde{\varphi}'(x, \lambda)$  we arrive at (2.2.19). Taking  $x = 0$  in (2.2.21) we get (2.2.20).  $\square$

Thus, we obtain the following algorithm for the solution of the inverse problem.

**Algorithm 2.2.1.** *Let the function  $M(\lambda)$  be given. Then*

- (1) *Choose  $\tilde{L} \in V_N$  such that (2.2.4) holds.*
- (2) *Find  $\varphi(x, \lambda)$  by solving equation (2.2.12).*
- (3) *Construct  $q(x)$  and  $h$  via (2.2.18)-(2.2.20).*

Let us now formulate necessary and sufficient conditions for the solvability of the inverse problem. Denote in the sequel by  $\mathbf{W}$  the set of functions  $M(\lambda)$  such that

- (i) the functions  $M(\lambda)$  are analytic in  $\Pi$  with the exception of an at most countable bounded set  $\Lambda'$  of poles, and are continuous in  $\Pi_1$  with the exception of bounded set  $\Lambda$  (in general,  $\Lambda$  and  $\Lambda'$  are different for each function  $M(\lambda)$ );
- (ii) for  $|\lambda| \rightarrow \infty$ , (2.1.73) holds.

**Theorem 2.2.5.** *For a function  $M(\lambda) \in \mathbf{W}$  to be the Weyl function for a certain  $L \in V_N$ , it is necessary and sufficient that the following conditions hold:*

- 1) *(Asymptotics) There exists  $\tilde{L} \in V_N$  such that (2.2.4) holds;*
- 2) *(Condition S) For each fixed  $x \geq 0$ , equation (2.2.12) has a unique solution  $\varphi(x, \lambda) \in C(\gamma)$ ;*
- 3)  *$\varepsilon(x) \in W_N$ , where the function  $\varepsilon(x)$  is defined by (2.2.18).*

*Under these conditions  $q(x)$  and  $h$  are constructed via (2.2.19)-(2.2.20).*

As it is shown in Example 2.3.1, conditions 2) and 3) are essential and cannot be omitted. On the other hand, in Section 2.3 we provide classes of operators for which the unique solvability of the main equation can be proved.

The *necessity* part of Theorem 2.2.5 was proved above. We prove now the *sufficiency*. Let a function  $M(\lambda) \in \mathbf{W}$ , satisfying the hypothesis of Theorem 2.2.5, be given, and let  $\varphi(x, \lambda)$  be the solution of the main equation (2.2.12). Then (2.2.12) gives us the analytic continuation of  $\varphi(x, \lambda)$  to the whole  $\lambda$ -plane, and for each fixed  $x \geq 0$ , the function  $\varphi(x, \lambda)$  is entire in  $\lambda$  of order  $1/2$ . Using Lemma 1.5.1 one can show that the functions  $\varphi^{(\nu)}(x, \lambda)$ ,  $\nu = 0, 1$ , are absolutely continuous with respect to  $x$  on compact sets, and

$$|\varphi^{(\nu)}(x, \lambda)| \leq C|\rho|^\nu \exp(|\tau|x). \quad (2.2.23)$$

We construct the function  $\Phi(x, \lambda)$  via (2.2.17), and  $L = L(q(x), h)$  via (2.2.19)-(2.2.20). Obviously,  $L \in V_N$ .

**Lemma 2.2.3.** *The following relations hold*

$$\ell\varphi(x, \lambda) = \lambda\varphi(x, \lambda), \quad \ell\Phi(x, \lambda) = \lambda\Phi(x, \lambda).$$

*Proof.* For simplicity, let

$$\int_{\lambda^*}^{\infty} \rho |\hat{V}(\lambda)| d\lambda < \infty$$

(the general case requires minor modifications). Then (2.2.23) is valid for  $\nu = 0, 1, 2$ . Differentiating (2.2.12) twice with respect to  $x$  we obtain (2.2.21) and (2.2.22). It follows from (2.2.22) and (2.2.12) that

$$\begin{aligned} -\tilde{\varphi}''(x, \lambda) + q(x)\tilde{\varphi}(x, \lambda) &= \ell\varphi(x, \lambda) + \frac{1}{2\pi i} \int_{\gamma} \tilde{r}(x, \lambda, \mu) \ell\varphi(x, \mu) d\mu \\ &+ \frac{1}{2\pi i} \int_{\gamma} \langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle \hat{M}(\mu) \varphi(x, \mu) d\mu - 2\tilde{\varphi}(x, \lambda) \frac{1}{2\pi i} \int_{\gamma} (\tilde{\varphi}(x, \lambda) \tilde{\varphi}(x, \mu))' \hat{M}(\mu) \varphi(x, \mu) d\mu. \end{aligned}$$

Taking (2.2.19) into account we get

$$\begin{aligned} \tilde{\ell}\tilde{\varphi}(x, \lambda) &= \ell\varphi(x, \lambda) + \frac{1}{2\pi i} \int_{\gamma} \tilde{r}(x, \lambda, \mu) \ell\varphi(x, \mu) d\mu \\ &+ \frac{1}{2\pi i} \int_{\gamma} \langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle \hat{M}(\mu) \varphi(x, \mu) d\mu. \end{aligned} \quad (2.2.24)$$

Using (2.2.17) we calculate similarly

$$\tilde{\Phi}'(x, \lambda) - \varepsilon_0(x)\tilde{\Phi}(x, \lambda) = \Phi'(x, \lambda) + \frac{1}{2\pi i} \int_{\gamma} \frac{\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle}{\lambda - \mu} \hat{M}(\mu) \varphi'(x, \mu) d\mu, \quad (2.2.25)$$

$$\begin{aligned} \tilde{\ell}\tilde{\Phi}(x, \lambda) &= \ell\Phi(x, \lambda) + \frac{1}{2\pi i} \int_{\gamma} \frac{\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle}{\lambda - \mu} \hat{M}(\mu) \ell\varphi(x, \mu) d\mu \\ &+ \frac{1}{2\pi i} \int_{\gamma} \langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle \hat{M}(\mu) \varphi(x, \mu) d\mu. \end{aligned} \quad (2.2.26)$$

It follows from (2.2.24) that

$$\lambda\tilde{\varphi}(x, \lambda) = \ell\varphi(x, \lambda) + \frac{1}{2\pi i} \int_{\gamma} \tilde{r}(x, \lambda, \mu) \ell\varphi(x, \mu) d\mu + \frac{1}{2\pi i} \int_{\gamma} (\lambda - \mu) \tilde{r}(x, \lambda, \mu) \varphi(x, \mu) d\mu.$$

Taking (2.2.12) into account we deduce for a fixed  $x \geq 0$ ,

$$\eta(x, \lambda) + \frac{1}{2\pi i} \int_{\gamma} \tilde{r}(x, \lambda, \mu) \eta(x, \mu) d\mu = 0, \quad \lambda \in \gamma, \quad (2.2.27)$$

where  $\eta(x, \lambda) := \ell\varphi(x, \lambda) - \lambda\varphi(x, \lambda)$ . According to (2.2.23) we have

$$|\eta(x, \lambda)| \leq C_x |\rho|^2, \quad \lambda \in \gamma. \quad (2.2.28)$$

By virtue of (2.2.27), (2.2.6) and (2.2.7),

$$|\eta(x, \lambda)| \leq C_x \left( 1 + \int_{\lambda^*}^{\infty} \frac{|\hat{V}(\mu)|}{|\rho - \theta| + 1} |\eta(x, \mu)| d\mu \right), \quad \lambda \in \gamma, \quad \theta > 0, \quad \operatorname{Re} \rho \geq 0. \quad (2.2.29)$$

Substituting (2.2.28) into the right-hand side of (2.2.29) we get

$$|\eta(x, \lambda)| \leq C_x \left( 1 + \int_{\lambda^*}^{\infty} \frac{\theta^2 |\hat{V}(\mu)|}{|\rho - \theta| + 1} d\mu \right), \quad \lambda \in \gamma, \quad \theta > 0, \quad \operatorname{Re} \rho \geq 0.$$



Since

$$\frac{\theta}{\rho(|\rho - \theta| + 1)} \leq 1 \quad \text{for } \theta, \rho \geq 1,$$

this yields

$$|\eta(x, \lambda)| \leq C_x |\rho|, \quad \lambda \in \gamma. \quad (2.2.30)$$

Using (2.2.30) instead of (2.2.28) and repeating the preceding arguments we infer

$$|\eta(x, \lambda)| \leq C_x, \quad \lambda \in \gamma.$$

According to Condition S of Theorem 2.2.5, the homogeneous equation (2.2.27) has only the trivial solution  $\eta(x, \lambda) \equiv 0$ . Consequently,

$$\ell\varphi(x, \lambda) = \lambda\varphi(x, \lambda). \quad (2.2.31)$$

It follows from (2.2.26) and (2.2.31) that

$$\begin{aligned} \lambda\tilde{\Phi}(x, \lambda) &= \ell\Phi(x, \lambda) + \frac{1}{2\pi i} \int_{\gamma} \frac{\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle}{\lambda - \mu} \hat{M}(\mu) \mu \varphi(x, \mu) d\mu \\ &\quad + \frac{1}{2\pi i} \int_{\gamma} (\lambda - \mu) \frac{\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle}{\lambda - \mu} \hat{M}(\mu) \varphi(x, \mu) d\mu. \end{aligned}$$

Together with (2.2.17) this yields  $\ell\Phi(x, \lambda) = \lambda\Phi(x, \lambda)$ . □

**Lemma 2.2.4** *The following relations hold*

$$\varphi(0, \lambda) = 1, \quad \varphi'(0, \lambda) = h. \quad (2.2.32)$$

$$U(\Phi) = 1, \quad \Phi(0, \lambda) = M(\lambda), \quad (2.2.33)$$

$$\Phi(x, \lambda) = O(\exp(i\rho x)), \quad x \rightarrow \infty. \quad (2.2.34)$$

*Proof.* Taking  $x = 0$  in (2.2.12) and (2.2.21) and using (2.2.20) we get

$$\varphi(0, \lambda) = \tilde{\varphi}(0, \lambda) = 1,$$

$$\varphi'(0, \lambda) = \tilde{\varphi}'(0, \lambda) - \varepsilon_0(0)\tilde{\varphi}(0, \lambda) = \tilde{h} + h - \tilde{h} = h,$$

i.e. (2.2.32) is valid. Using (2.2.17) and (2.2.25) we calculate

$$\Phi(0, \lambda) = \tilde{\Phi}(0, \lambda) + \frac{1}{2\pi i} \int_{\gamma} \frac{\hat{M}(\mu)}{\lambda - \mu} d\mu, \quad (2.2.35)$$

$$\Phi'(0, \lambda) = \tilde{\Phi}'(0, \lambda) - \tilde{\Phi}(0, \lambda)\varepsilon_0(0) + \frac{h}{2\pi i} \int_{\gamma} \frac{\hat{M}(\mu)}{\lambda - \mu} d\mu.$$

Consequently,

$$U(\Phi) = \Phi'(0, \lambda) - h\Phi(0, \lambda) = \tilde{\Phi}'(0, \lambda) - (\varepsilon_0(0) + h)\tilde{\Phi}(0, \lambda) =$$

$$\tilde{\Phi}'(0, \lambda) - \tilde{h}\tilde{\Phi}(0, \lambda) = \tilde{U}(\tilde{\Phi}) = 1.$$

Furthermore, since  $\langle y, z \rangle = yz' - y'z$ , we rewrite (2.2.17) in the form

$$\Phi(x, \lambda) = \tilde{\Phi}(x, \lambda) + \frac{1}{2\pi i} \int_{\gamma} \frac{\tilde{\Phi}'(x, \lambda) \tilde{\varphi}(x, \mu) - \tilde{\Phi}(x, \lambda) \tilde{\varphi}'(x, \mu)}{\lambda - \mu} \hat{M}(\mu) \varphi(x, \mu) d\mu, \quad (2.2.36)$$

where  $\lambda \in J_{\gamma}$ . The function  $\varphi(x, \lambda)$  is the solution of the Cauchy problem (2.2.31)-(2.2.32). Therefore, according to (2.1.87),

$$|\varphi^{(\nu)}(x, \mu)| \leq C|\theta|^{\nu}, \quad \mu = \theta^2 \in \gamma, \quad x \geq 0, \quad \nu = 0, 1. \quad (2.2.37)$$

Moreover, the estimates (2.1.87)-(2.1.88) are valid for  $\tilde{\varphi}(x, \lambda)$  and  $\tilde{\Phi}(x, \lambda)$ , i.e.

$$|\tilde{\varphi}^{(\nu)}(x, \mu)| \leq C|\theta|^{\nu}, \quad \mu = \theta^2 \in \gamma, \quad x \geq 0, \quad \nu = 0, 1, \quad (2.2.38)$$

$$|\tilde{\Phi}^{(\nu)}(x, \lambda)| \leq C|\rho|^{\nu-1} \exp(-|Im \rho| x), \quad x \geq 0, \quad \rho \in \Omega. \quad (2.2.39)$$

By virtue of (2.1.73),

$$\hat{M}(\lambda) = O\left(\frac{1}{\lambda}\right), \quad |\rho| \rightarrow \infty, \quad \rho \in \Omega. \quad (2.2.40)$$

Fix  $\lambda \in J_{\gamma}$ . Taking (2.2.37)-(2.2.40) into account we get from (2.2.36) that

$$|\Phi(x, \lambda) \exp(-i\rho x)| \leq C \left(1 + \int_{\rho^*}^{\infty} \frac{d\theta}{\theta|\lambda - \mu|}\right) \leq C_1,$$

i.e. (2.2.34) is valid.

Furthermore, it follows from (2.2.35) that

$$\Phi(0, \lambda) = \tilde{M}(\lambda) + \frac{1}{2\pi i} \int_{\gamma} \frac{\hat{M}(\mu)}{\lambda - \mu} d\mu.$$

By Cauchy's integral formula

$$\hat{M}(\lambda) = \frac{1}{2\pi i} \int_{\gamma_R^0} \frac{\hat{M}(\mu)}{\lambda - \mu} d\mu, \quad \lambda \in \text{int } \gamma_R^0.$$

Since

$$\lim_{R \rightarrow \infty} \frac{1}{2\pi i} \int_{|\mu|=R} \frac{\hat{M}(\mu)}{\lambda - \mu} d\mu = 0,$$

we get

$$\hat{M}(\lambda) = \frac{1}{2\pi i} \int_{\gamma} \frac{\hat{M}(\mu)}{\lambda - \mu} d\mu, \quad \lambda \in J_{\gamma}.$$

Consequently,  $\Phi(0, \lambda) = \tilde{M}(\lambda) + \hat{M}(\lambda) = M(\lambda)$ , i.e. (2.2.33) is valid.  $\square$

Thus,  $\Phi(x, \lambda)$  is the Weyl solution, and  $M(\lambda)$  is the Weyl function for the constructed pair  $L(q(x), h)$ , and Theorem 2.2.5 is proved.  $\square$

**The Gelfand-Levitan method.** For Sturm-Liouville operators on a finite interval, the Gelfand-Levitan method was presented in Section 1.5. For the case of the half-line there are similar results. Therefore here we confine ourselves to the derivation of the Gelfand-Levitan equation. For further discussion see [mar1], [lev2] and [lev4].

Consider the differential equation and the linear form  $L = L(q(x), h)$  of the form (2.2.1)-(2.2.2). Let  $\tilde{q}(x) = 0$ ,  $\tilde{h} = 0$ . Denote

$$F(x, t) = \frac{1}{2\pi i} \int_{\gamma} \cos \rho x \cos \rho t \hat{M}(\lambda) d\lambda, \quad (2.2.41)$$

where  $\gamma$  is the contour defined in Section 2.1 (see fig. 2.1.1). We note that by virtue of (2.1.77) and (2.2.5),

$$\frac{1}{2\pi i} \int_{\gamma''} \cos \rho x \cos \rho t \hat{M}(\lambda) d\lambda = \int_{\lambda^*}^{\infty} \cos \rho x \cos \rho t \hat{V}(\lambda) d\lambda < \infty.$$

Let  $G(x, t)$  and  $H(x, t)$  be the kernels of the transformation operators (1.3.11) and (1.5.12) respectively.

**Theorem 2.2.6.** For each fixed  $x$ , the function  $G(x, t)$  satisfies the following linear integral equation

$$G(x, t) + F(x, t) + \int_0^x G(x, s) F(s, t) ds = 0, \quad 0 < t < x. \quad (2.2.42)$$

Equation (2.2.42) is called the Gelfand-Levitan equation.

*Proof.* Using (1.3.11) and (1.5.12) we calculate

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma_R} \varphi(x, \lambda) \cos \rho t M(\lambda) d\lambda &= \frac{1}{2\pi i} \int_{\gamma_R} \cos \rho x \cos \rho t M(\lambda) d\lambda \\ &+ \frac{1}{2\pi i} \int_{\gamma_R} \left( \int_0^x G(x, s) \cos \rho s ds \right) \cos \rho t M(\lambda) d\lambda, \\ \frac{1}{2\pi i} \int_{\gamma_R} \varphi(x, \lambda) \cos \rho t M(\lambda) d\lambda &= \frac{1}{2\pi i} \int_{\gamma_R} \varphi(x, \lambda) \varphi(t, \lambda) M(\lambda) d\lambda \\ &+ \frac{1}{2\pi i} \int_{\gamma_R} \left( \int_0^t H(t, s) \varphi(s, \lambda) ds \right) \varphi(x, \lambda) M(\lambda) d\lambda, \end{aligned}$$

where  $\gamma_R = \gamma \cap \{\lambda : |\lambda| \leq R\}$ . This yields

$$\Phi_R(x, t) = I_{R1}(x, t) + I_{R2}(x, t) + I_{R3}(x, t) + I_{R4}(x, t),$$

where

$$\begin{aligned} \Phi_R(x, t) &= \frac{1}{2\pi i} \int_{\gamma_R} \varphi(x, \lambda) \varphi(t, \lambda) M(\lambda) d\lambda - \frac{1}{2\pi i} \int_{\gamma_R} \cos \rho x \cos \rho t \tilde{M}(\lambda) d\lambda, \\ I_{R1}(x, t) &= \frac{1}{2\pi i} \int_{\gamma_R} \cos \rho x \cos \rho t \hat{M}(\lambda) d\lambda, \\ I_{R2}(x, t) &= \int_0^x G(x, s) \left( \frac{1}{2\pi i} \int_{\gamma_R} \cos \rho t \cos \rho s \hat{M}(\lambda) d\lambda \right) ds, \\ I_{R3}(x, t) &= \frac{1}{2\pi i} \int_{\gamma_R} \cos \rho t \left( \int_0^x G(x, s) \cos \rho s ds \right) \tilde{M}(\lambda) d\lambda, \\ I_{R4}(x, t) &= -\frac{1}{2\pi i} \int_{\gamma_R} \varphi(x, \lambda) \left( \int_0^t H(t, s) \varphi(s, \lambda) ds \right) M(\lambda) d\lambda. \end{aligned}$$

Let  $\xi(t)$ ,  $t \geq 0$  be a twice continuously differentiable function with compact support. By Theorem 2.1.8,

$$\begin{aligned}\lim_{R \rightarrow \infty} \int_0^\infty \xi(t) \Phi_R(x, t) dt &= 0, \quad \lim_{R \rightarrow \infty} \int_0^\infty \xi(t) I_{R1}(x, t) dt = \int_0^\infty \xi(t) F(x, t) dt, \\ \lim_{R \rightarrow \infty} \int_0^\infty \xi(t) I_{R2}(x, t) dt &= \int_0^\infty \xi(t) \left( \int_0^x G(x, s) F(s, t) ds \right) dt, \\ \lim_{R \rightarrow \infty} \int_0^\infty \xi(t) I_{R3}(x, t) dt &= \int_0^x \xi(t) G(x, t) dt, \\ \lim_{R \rightarrow \infty} \int_0^\infty \xi(t) I_{R4}(x, t) dt &= - \int_x^\infty \xi(t) H(t, x) dt.\end{aligned}$$

Put  $G(x, t) = H(x, t) = 0$  for  $x < t$ . In view of the arbitrariness of  $\xi(t)$ , we derive

$$G(x, t) + F(x, t) + \int_0^x G(x, s) F(s, t) ds - H(t, x) = 0.$$

For  $t < x$ , this gives (2.2.42). □

Thus, in order to solve the inverse problem of recovering  $L$  from the Weyl function  $M(\lambda)$  one can calculate  $F(x, t)$  by (2.2.41), find  $G(x, t)$  by solving the Gelfand-Levitan equation (2.2.42) and construct  $q(x)$  and  $h$  by (1.5.13).

**Remark 2.2.1.** We show the connection between the Gelfand-Levitan equation and the main equation of inverse problem (2.2.12). For this purpose we use the cosine Fourier transform. Let  $\tilde{q}(x) = \tilde{h} = 0$ . Then  $\tilde{\varphi}(x, \lambda) = \cos \sqrt{\lambda} x$ . Multiplying (2.2.42) by  $\cos \sqrt{\lambda} t$  and integrating with respect to  $t$ , we obtain

$$\begin{aligned}\int_0^x G(x, t) \cos \sqrt{\lambda} t dt + \int_0^x \cos \sqrt{\lambda} t \left( \frac{1}{2\pi i} \int_\gamma \cos \sqrt{\mu} x \cos \sqrt{\mu} t \hat{M}(\mu) d\mu \right) dt + \\ \int_0^x \cos \sqrt{\lambda} t \int_0^x G(x, s) \left( \frac{1}{2\pi i} \int_\gamma \cos \sqrt{\mu} t \cos \sqrt{\mu} s \hat{M}(\mu) d\mu \right) ds = 0.\end{aligned}$$

Using (1.3.11) we arrive at (2.2.12).

**Remark 2.2.2.** If  $q(x)$  and  $h$  are real, and  $(1+x)q(x) \in L(0, \infty)$ , then (2.2.41) takes the form

$$F(x, t) = \int_{-\infty}^\infty \cos \rho x \cos \rho t d\hat{\sigma}(\lambda),$$

where  $\hat{\sigma} = \sigma - \tilde{\sigma}$ , and  $\sigma$  and  $\tilde{\sigma}$  are the spectral functions of  $L$  and  $\tilde{L}$  respectively.

### 2.3. RECOVERY OF THE DIFFERENTIAL EQUATION FROM THE SPECTRAL DATA.

In Theorem 2.2.5, one of the conditions under which an arbitrary function  $M(\lambda)$  is the Weyl function for a certain pair  $L = L(q(x), h)$  of the form (2.1.1)-(2.1.2) is the requirement that the main equation is uniquely solvable. This condition is difficult to check in the general case. In this connection it is important to point out classes of operators for which the unique solvability of the main equation can be proved. The class of selfadjoint operators is one of the most important classes having this property. For this class we introduce the

so-called spectral data which describe the set of singularities of the Weyl function  $M(\lambda)$ , i.e. the discrete and continuous parts of the spectrum. In this section we investigate the inverse problem of recovering  $L$  from the given spectral data. We show that the specification of the spectral data uniquely determines the Weyl function. Thus, the inverse problem of recovering  $L$  from the Weyl function is equivalent to the inverse problem of recovering  $L$  from the spectral data. The main equation, obtained in Section 2.2, can be constructed directly from the spectral data on the set  $\{\lambda : \lambda \geq 0\} \cup \Lambda$ . We prove a unique solvability of the main equation in a suitable Banach space (see Theorem 2.3.9). At the end of the section we consider the inverse problem from the spectral data also for the nonselfadjoint case.

**2.3.1. The selfadjoint case.** Consider the differential equation and the linear form  $L = L(q(x), h)$  of the form (2.1.1)-(2.1.2) and assume that  $q(x)$  and  $h$  are real. This means that the operator  $L^\circ$ , defined in Remark 2.1.3, is a selfadjoint operator. For the selfadjoint case one can obtain additional properties of the spectrum to those which we established in Section 2.1.

**Theorem 2.3.1.** *Let  $q(x)$  and  $h$  be real. Then  $\Lambda'' = \emptyset$ , i.e. there are no spectral singularities.*

*Proof.* Since  $q(x)$  and  $h$  are real it follows from Theorem 2.1.1 and (2.1.63) that for real  $\rho$ ,

$$\overline{e(x, \rho)} = e(x, -\rho), \quad \overline{\Delta(\rho)} = \Delta(-\rho). \quad (2.3.1)$$

Suppose  $\Lambda'' \neq \emptyset$ , i.e. for a certain real  $\rho^0 \neq 0$ ,  $\Delta(\rho^0) = 0$ . Then, according to (2.3.1),

$$\Delta(-\rho^0) = \overline{\Delta(\rho^0)} = 0.$$

Together with (2.1.6) this yields

$$-2i\rho^0 = \langle e(x, \rho^0), e(x, -\rho^0) \rangle_{|x=0} = e(0, \rho^0)\Delta(-\rho^0) - e(0, -\rho^0)\Delta(\rho^0) = 0,$$

which is impossible. □

**Theorem 2.3.2.** *Let  $q(x)$  and  $h$  be real. Then the non-zero eigenvalues  $\lambda_k$  are real and negative, and*

$$\Delta(\rho_k) = 0 \quad (\text{where } \lambda_k = \rho_k^2 \in \Lambda'). \quad (2.3.2)$$

*The eigenfunction  $e(x, \rho_k)$  and  $\varphi(x, \lambda_k)$  are real, and*

$$e(x, \rho_k) = e(0, \rho_k)\varphi(x, \lambda_k), \quad e(0, \rho_k) \neq 0. \quad (2.3.3)$$

*Moreover, eigenfunctions related to different eigenvalues are orthogonal in  $L_2(0, \infty)$ .*

*Proof.* It follows from (2.1.66) and (2.1.71) that

$$\langle \varphi(x, \lambda), e(x, \rho) \rangle = \Delta(\rho). \quad (2.3.4)$$

According to Theorem 2.1.7, (2.3.2) holds; therefore (2.3.4) implies

$$e(x, \rho_k) = c_k \varphi(x, \lambda_k), \quad c_k \neq 0.$$

Taking here  $x = 0$  we get  $c_k = e(0, \rho_k)$ , i.e. (2.3.3) is valid.

Let  $\lambda_n$  and  $\lambda_k$  ( $\lambda_n \neq \lambda_k$ ) be eigenvalues with eigenfunctions  $y_n(x) = e(x, \rho_n)$  and  $y_k(x) = e(x, \rho_k)$  respectively. Then integration by parts yields

$$\int_0^\infty \ell y_n(x) y_k(x) dx = \int_0^\infty y_n(x) \ell y_k(x) dx,$$

and hence

$$\lambda_n \int_0^\infty y_n(x) y_k(x) dx = \lambda_k \int_0^\infty y_n(x) y_k(x) dx,$$

consequently

$$\int_0^\infty y_n(x) y_k(x) dx = 0.$$

Assume now that  $\lambda^0 = u + iv$ ,  $v \neq 0$  is a non-real eigenvalue with an eigenfunction  $y^0(x) \not\equiv 0$ . Since  $q(x)$  and  $h$  are real, we get that  $\bar{\lambda}^0 = u - iv$  is also the eigenvalue with the eigenfunction  $\overline{y^0(x)}$ . Since  $\lambda^0 \neq \bar{\lambda}^0$ , we derive as before

$$\|y^0\|_{L_2}^2 = \int_0^\infty y^0(x) \overline{y^0(x)} dx = 0,$$

which is impossible. Thus, all eigenvalues  $\lambda_k$  of  $L$  are real, and consequently the eigenfunctions  $\varphi(x, \lambda_k)$  and  $e(x, \rho_k)$  are real too. Together with Theorem 2.1.6 this yields  $\Lambda' \subset (-\infty, 0)$ .  $\square$

**Theorem 2.3.3.** *Let  $q(x)$  and  $h$  be real, and let  $\Lambda' = \{\lambda_k\}$ ,  $\lambda_k = \rho_k^2 < 0$ . Denote  $\Delta_1(\rho) := \frac{d}{d\lambda} \Delta(\rho)$  ( $\lambda = \rho^2$ ). Then*

$$\Delta_1(\rho_k) \neq 0. \quad (2.3.5)$$

*Proof.* Since  $e(x, \rho)$  satisfies (2.1.1) we have

$$\frac{d}{dx} \langle e(x, \rho), e(x, \rho_k) \rangle = (\rho^2 - \rho_k^2) e(x, \rho) e(x, \rho_k).$$

Using (2.1.4), (2.1.63) and (2.3.2) we calculate

$$(\rho^2 - \rho_k^2) \int_0^\infty e(x, \rho) e(x, \rho_k) dx = \langle e(x, \rho), e(x, \rho_k) \rangle \Big|_0^\infty = e(0, \rho_k) \Delta(\rho), \quad \text{Im } \rho > 0.$$

As  $\rho \rightarrow \rho_k$ , this gives

$$\int_0^\infty e^2(x, \rho_k) dx = \frac{1}{2\rho_k} e(0, \rho_k) \left( \frac{d}{d\rho} \Delta(\rho) \right) \Big|_{\rho=\rho_k} = e(0, \rho_k) \Delta_1(\rho_k). \quad (2.3.6)$$

Since

$$\int_0^\infty e^2(x, \rho_k) dx > 0,$$

we arrive at (2.3.5).  $\square$

Let  $\Lambda' = \{\lambda_k\}$ ,  $\lambda_k = \rho_k^2 < 0$ . It follows from (2.1.69), (2.3.2) and (2.3.5) that the Weyl function  $M(\lambda)$  has simple poles at the points  $\lambda = \lambda_k$ , and

$$\alpha_k := \text{Res}_{\lambda=\lambda_k} M(\lambda) = \frac{e(0, \rho_k)}{\Delta_1(\rho_k)}. \quad (2.3.7)$$

Taking (2.3.3) and (2.3.6) into account we infer

$$\alpha_k = \left( \int_0^\infty \varphi^2(x, \lambda_k) dx \right)^{-1} > 0.$$

Furthermore, the function  $V(\lambda)$ , defined by (2.1.77), is continuous for  $\lambda = \rho^2 > 0$ , and by virtue of (2.1.69),

$$V(\lambda) := \frac{1}{2\pi i} (M^-(\lambda) - M^+(\lambda)) = \frac{1}{2\pi i} \left( \frac{e(0, -\rho)}{\Delta(-\rho)} - \frac{e(0, \rho)}{\Delta(\rho)} \right).$$

Using (2.3.1) and (2.1.6) we calculate

$$V(\lambda) = \frac{\rho}{\pi |\Delta(\rho)|^2} > 0, \quad \lambda = \rho^2 > 0. \quad (2.3.8)$$

Denote by  $W'_N$  the set of functions  $q(x) \in W_N$  such that

$$\int_0^\infty (1+x)|q(x)| dx < \infty, \quad (2.3.9)$$

We shall say that  $L \in V'_N$  if  $q(x)$  and  $h$  are real, and  $q(x) \in W'_N$ .

**Theorem 2.3.4.** *Let  $L \in V'_N$ . Then  $\Lambda'$  is a finite set.*

*Proof.* For  $x \geq 0$ ,  $\tau \geq 0$ , the function  $e(x, i\tau)$  is real and by virtue of (2.1.24)

$$|e(x, i\tau) \exp(\tau x) - 1| \leq Q_1(x) \exp(Q_1(x)), \quad x \geq 0, \tau \geq 0, \quad (2.3.10)$$

where

$$Q_1(x) = \int_x^\infty (t-x)|q(t)| dt.$$

In particular, it follows from (2.3.10) that there exists  $a > 0$  such that

$$e(x, i\tau) \exp(\tau x) \geq \frac{1}{2} \quad \text{for } x \geq a, \tau \geq 0. \quad (2.3.11)$$

Suppose that  $\Lambda' = \{\lambda_k\}$  is an infinite set. Since  $\Lambda'$  is bounded and  $\lambda_k = \rho_k^2 < 0$ , it follows that  $\rho_k = i\tau_k \rightarrow 0$ ,  $\tau_k > 0$ . Using (2.3.11) we calculate

$$\begin{aligned} \int_a^\infty e(x, \rho_k) e(x, \rho_n) dx &\geq \frac{1}{4} \int_a^\infty \exp(-(\tau_k + \tau_n)x) dx \\ &= \frac{\exp(-(\tau_k + \tau_n)a)}{4(\tau_k + \tau_n)} \geq \frac{\exp(-2aT)}{8T}, \end{aligned} \quad (2.3.12)$$

where  $T = \max_k \tau_k$ . Since the eigenfunctions  $e(x, \rho_k)$  and  $e(x, \rho_n)$  are orthogonal in  $L_2(0, \infty)$  we get

$$\begin{aligned} 0 &= \int_0^\infty e(x, \rho_k) e(x, \rho_n) dx = \int_a^\infty e(x, \rho_k) e(x, \rho_n) dx \\ &+ \int_0^a e^2(x, \rho_k) dx + \int_0^a e(x, \rho_k) (e(x, \rho_n) - e(x, \rho_k)) dx. \end{aligned} \quad (2.3.13)$$

Taking (2.3.12) into account we get

$$\int_a^\infty e(x, \rho_k) e(x, \rho_n) dx \geq C_0 > 0, \quad \int_0^a e^2(x, \rho_k) dx \geq 0. \quad (2.3.14)$$

Let us show that

$$\int_0^a e(x, \rho_k)(e(x, \rho_n) - e(x, \rho_k)) dx \rightarrow 0 \quad \text{as } k, n \rightarrow \infty. \quad (2.3.15)$$

Indeed, according to (2.1.23),

$$|e(x, \rho_k)| \leq \exp(Q_1(0)), \quad x \geq 0.$$

Then, using (2.1.33) we calculate

$$\begin{aligned} \left| \int_0^a e(x, \rho_k)(e(x, \rho_n) - e(x, \rho_k)) dx \right| &\leq \exp(Q_1(0)) \left( \int_0^a |\exp(-\tau_n x) - \exp(-\tau_k x)| dx \right. \\ &\quad \left. + \int_0^a \left( \int_x^\infty |A(x, t)(\exp(-\tau_n x) - \exp(-\tau_k x))| dt \right) dx \right). \end{aligned} \quad (2.3.16)$$

By virtue of (2.1.35),

$$|A(x, t)| \leq \frac{1}{2} Q_0\left(\frac{x+t}{2}\right) \exp(Q_1(x)) \leq \frac{1}{2} Q_0\left(\frac{t}{2}\right) \exp(Q_1(0)), \quad 0 \leq x \leq t,$$

and consequently, (2.3.16) yields

$$\begin{aligned} \left| \int_0^a e(x, \rho_k)(e(x, \rho_n) - e(x, \rho_k)) dx \right| &\leq C \left( \int_0^a |\exp(-\tau_n x) - \exp(-\tau_k x)| dx \right. \\ &\quad \left. + \int_0^\infty Q_0\left(\frac{t}{2}\right) |\exp(-\tau_n t) - \exp(-\tau_k t)| dt \right). \end{aligned} \quad (2.3.17)$$

Clearly,

$$\int_0^a |\exp(-\tau_n x) - \exp(-\tau_k x)| dx \rightarrow 0 \quad \text{as } k, n \rightarrow \infty. \quad (2.3.18)$$

Take a fixed  $\varepsilon > 0$ . There exists  $a_\varepsilon > 0$  such that

$$\int_{a_\varepsilon}^\infty Q_0\left(\frac{t}{2}\right) dt < \frac{\varepsilon}{2}.$$

On the other hand, for sufficiently large  $k$  and  $n$ ,

$$\int_0^{a_\varepsilon} Q_0\left(\frac{t}{2}\right) |\exp(-\tau_n t) - \exp(-\tau_k t)| dt \leq Q_0(0) \int_0^{a_\varepsilon} |\exp(-\tau_n t) - \exp(-\tau_k t)| dt < \frac{\varepsilon}{2}.$$

Thus, for sufficiently large  $k$  and  $n$ ,

$$\int_0^\infty Q_0\left(\frac{t}{2}\right) |\exp(-\tau_n t) - \exp(-\tau_k t)| dt < \varepsilon.$$

By virtue of arbitrariness of  $\varepsilon$  we obtain

$$\int_0^\infty Q_0\left(\frac{t}{2}\right) |\exp(-\tau_n t) - \exp(-\tau_k t)| dt \rightarrow 0 \quad \text{as } k, n \rightarrow \infty.$$



Together with (2.3.17) and (2.3.18) this implies (2.3.15). The relations (2.3.13)-(2.3.15) give us a contradiction. This implies that  $\Lambda'$  is a finite set.  $\square$

**Theorem 2.3.5.** *Let  $L \in V'_N$ . Then*

$$\frac{\rho}{\Delta(\rho)} = O(1), \quad \rho \rightarrow 0, \quad \text{Im } \rho \geq 0. \quad (2.3.19)$$

*Proof.* Denote

$$g(\rho) = \frac{2i\rho}{\Delta(\rho)}.$$

By virtue of (2.1.6) and (2.1.63),

$$\Delta(\rho)e(0, -\rho) - e(0, \rho)\Delta(-\rho) = 2i\rho.$$

Hence, for real  $\rho \neq 0$ ,

$$g(\rho) = e(0, -\rho) - \xi(\rho)e(0, \rho), \quad (2.3.20)$$

where

$$\xi(\rho) := \frac{\Delta(-\rho)}{\Delta(\rho)}.$$

It follows from (2.3.1) that

$$|\xi(\rho)| = 1 \quad \text{for real } \rho \neq 0. \quad (2.3.21)$$

Let  $\Lambda' = \{\lambda_k\}_{k=\overline{1,m}}$ ,  $\lambda_k = \rho_k^2$ ,  $\rho_k = i\tau_k$ ,  $0 < \tau_1 < \dots < \tau_m$ . Denote

$$D = \{\rho : \text{Im } \rho > 0, |\rho| < \tau^*\}, \quad (2.3.22)$$

where  $\tau^* = \tau_1/2$ . The function  $g(\rho)$  is analytic in  $D$  and continuous in  $\bar{D} \setminus \{0\}$ . By virtue of (2.3.20), (2.3.21) and (2.1.23),

$$|g(\rho)| \leq C \quad \text{for real } \rho \neq 0. \quad (2.3.23)$$

Suppose that  $\Delta(\rho)$  is analytic in the origin. Then, using (2.3.23), we get that the function  $g(\rho)$  has a removable singularity in the origin, and consequently (after extending  $g(\rho)$  continuously to the origin)  $g(\rho)$  is continuous in  $\bar{D}$ , i.e. (2.3.19) is proved.

In the general case we cannot use these arguments. Therefore, we introduce the potentials  $q_r(x)$  by (2.1.49) and the corresponding Jost solutions by (2.1.50). Denote

$$\Delta_r(\rho) = e'_r(0, \rho) - h e_r(0, \rho), \quad r \geq 0.$$

By virtue of (2.1.53), the functions  $\Delta_r(\rho)$  are entire in  $\rho$ , and according to Lemma 2.1.3,

$$\lim_{r \rightarrow \infty} \Delta_r(\rho) = \Delta(\rho) \quad (2.3.24)$$

uniformly for  $\text{Im } \rho \geq 0$ . Let  $\delta_r$  be the infimum of the distances between the zeros of  $\Delta_r(\rho)$  in the upper half-plane  $\text{Im } \rho \geq 0$ . Let us show that

$$\delta^* := \inf_{r > 0} \delta_r > 0. \quad (2.3.25)$$

Indeed, suppose on the contrary that there exists a sequence  $r_k \rightarrow \infty$  such that  $\delta_{r_k} \rightarrow 0$ . Let  $\rho_{k1} = i\tau_{k1}$ ,  $\rho_{k2} = i\tau_{k2}$  ( $\tau_{k1}, \tau_{k2} \geq 0$ ) be zeros of  $\Delta_{r_k}(\rho)$  such that  $\rho_{k1} - \rho_{k2} \rightarrow 0$  as  $k \rightarrow \infty$ . It follows from (2.1.51) that there exists  $a > 0$  such that

$$e_r(x, i\tau) \exp(\tau x) \geq \frac{1}{2} \quad \text{for } x \geq a, \tau \geq 0, r \geq 0. \quad (2.3.26)$$

Similarly to (2.3.13) we can write

$$\begin{aligned} 0 &= \int_0^\infty e_{r_k}(x, \rho_{k1}) e_{r_k}(x, \rho_{k2}) dx = \int_a^\infty e_{r_k}(x, \rho_{k1}) e_{r_k}(x, \rho_{k2}) dx \\ &+ \int_0^a e_{r_k}^2(x, \rho_{k1}) dx + \int_0^a e_{r_k}(x, \rho_{k1}) (e_{r_k}(x, \rho_{k2}) - e_{r_k}(x, \rho_{k1})) dx. \end{aligned}$$

Taking (2.3.26) into account we get as before

$$\int_a^\infty e_{r_k}(x, \rho_{k1}) e_{r_k}(x, \rho_{k2}) dx \geq \frac{\exp(-(\tau_{k1} + \tau_{k2})a)}{4(\tau_{k1} + \tau_{k2})}.$$

By virtue of (2.3.24),  $|\tau_{k1}|, |\tau_{k2}| \leq C$ , and consequently,

$$\int_a^\infty e_{r_k}(x, \rho_{k1}) e_{r_k}(x, \rho_{k2}) dx \geq C_0 > 0.$$

Moreover,

$$\int_0^a e_{r_k}^2(x, \rho_{k1}) dx \geq 0.$$

On the other hand, using (2.1.50) and (2.1.52) one can easily verify (similar to the proof of (2.3.15)) that

$$\int_0^a e_{r_k}(x, \rho_{k1}) (e_{r_k}(x, \rho_{k2}) - e_{r_k}(x, \rho_{k1})) dx \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

and we get a contradiction. This means that (2.3.25) is valid.

Define  $D$  via (2.3.22) with  $\tau^* = \min(\frac{\tau_1}{2}, \frac{\delta^*}{2})$ . Then, the function  $\Delta_r(\rho)$  has in  $\bar{D}$  at most one zero  $\rho = i\tau_r^0$ ,  $0 \leq \tau_r^0 \leq \tau^*$ . Consider the function

$$\gamma_r(\rho) := g_r(\rho) g_r^0(\rho),$$

where

$$g_r(\rho) = \frac{2i\rho}{\Delta_r(\rho)}, \quad g_r^0(\rho) = \frac{\rho - i\tau_r^0}{\rho + i\tau_r^0}$$

(if  $\Delta_r(\rho)$  has no zeros in  $\bar{D}$  we put  $g_r^0(\rho) := 1$ ). Clearly

$$|g_r^0(\rho)| \leq 1 \quad \text{for } \rho \in \bar{D}. \quad (2.3.27)$$

Analogously to (2.3.20)-(2.3.21), for real  $\rho \neq 0$ ,

$$g_r(\rho) = e_r(0, -\rho) - \xi_r(\rho) e_r(0, \rho), \quad |\xi_r(\rho)| = 1. \quad (2.3.28)$$

It follows from (2.3.24), (2.3.28) and (2.1.51) that

$$|g_r(\rho)| \leq C \quad \text{for } \rho \in \partial D, r \geq 0,$$

where  $\partial D = \bar{D} \setminus D$  is the boundary of  $D$ , and  $C$  does not depend on  $r$ . Together with (2.3.27) this yields

$$|\gamma_r(\rho)| \leq C \text{ for } \rho \in \partial D, r \geq 0.$$

Since the functions  $\gamma_r(\rho)$  are analytic in  $\bar{D}$ , we have by the maximum principle [con1, p.128],

$$|\gamma_r(\rho)| \leq C \text{ for } \rho \in \bar{D}, r \geq 0, \quad (2.3.29)$$

where  $C$  does not depend on  $r$ .

Fix  $\delta \in (0, \tau^*)$  and denote  $D_\delta := \{\rho : \operatorname{Im} \rho > 0, \delta < |\rho| < \tau^*\}$ . By virtue of (2.3.24),

$$\lim_{r \rightarrow \infty} g_r(\rho) = g(\rho) \text{ uniformly in } \bar{D}_\delta.$$

Moreover, (2.3.24) implies  $\lim_{r \rightarrow \infty} \tau_r^0 = 0$ , and consequently,

$$\lim_{r \rightarrow \infty} \gamma_r(\rho) = g(\rho) \text{ uniformly in } \bar{D}_\delta.$$

Then, it follows from (2.3.29) that

$$|g(\rho)| \leq C \text{ for } \rho \in \bar{D}_\delta.$$

By virtue of arbitrariness of  $\delta$  we get

$$|g(\rho)| \leq C \text{ for } \rho \in \bar{D} \setminus \{0\},$$

i.e. (2.3.19) is proved. □

**Theorem 2.3.6.** *Let  $L \in V'_N$ . Then  $\lambda = 0$  is not an eigenvalue of  $L$ .*

*Proof.* The function  $e(x) := e(x, 0)$  is a solution of (2.1.1) for  $\lambda = 0$ , and according to Theorem 2.1.2,

$$\lim_{x \rightarrow \infty} e(x) = 1. \quad (2.3.30)$$

Take  $a > 0$  such that

$$e(x) \geq \frac{1}{2} \text{ for } x \geq a, \quad (2.3.31)$$

and consider the function

$$z(x) := e(x) \int_a^x \frac{dt}{e^2(t)}. \quad (2.3.32)$$

It is easy to check that

$$z''(x) = q(x)z(x)$$

and

$$e(x)z'(x) - e'(x)z(x) \equiv 1.$$

It follows from (2.3.30)-(2.3.32) that

$$\lim_{x \rightarrow \infty} z(x) = +\infty. \quad (2.3.33)$$

Suppose that  $\lambda = 0$  is an eigenvalue, and let  $y_0(x)$  be a corresponding eigenfunction. Since the functions  $\{e(x), z(x)\}$  form a fundamental system of solutions of (2.1.1) for  $\lambda = 0$ , we have

$$y_0(x) = C_1^0 e(x) + C_2^0 z(x).$$

By virtue of (2.3.30) and (2.3.33), this is possible only if  $C_1^0 = C_2^0 = 0$ .  $\square$

**Remark 2.3.1.** Let

$$q(x) = \frac{2a^2}{(1+ax)^2}, \quad h = -a,$$

where  $a$  is a complex number such that  $a \notin (-\infty, 0]$ . Then  $q(x) \in L(0, \infty)$ , but  $xq(x) \notin L(0, \infty)$ . In this case  $\lambda = 0$  is an eigenvalue, and by differentiation one verifies that

$$y(x) = \frac{1}{1+ax}$$

is the corresponding eigenfunction.

According to Theorem 2.1.2,  $e(0, \rho) = O(1)$  as  $\rho \rightarrow 0$ ,  $\operatorname{Im} \rho \geq 0$ . Therefore, Theorem 2.3.5 together with (2.1.69) yields

$$M(\lambda) = O(\rho^{-1}), \quad |\rho| \rightarrow 0. \quad (2.3.34)$$

It is shown below in Example 2.3.1 that if the condition (2.3.9) is not fulfilled, then (2.3.34) is not valid in general.

Combining the results obtained above we arrive at the following theorem.

**Theorem 2.3.7.** *Let  $L \in V'_N$ , and let  $M(\lambda)$  be the Weyl function for  $L$ . Then  $M(\lambda)$  is analytic in  $\Pi$  with the exception of an at most finite number of simple poles  $\Lambda' = \{\lambda_k\}_{k=\overline{1, m}}$ ,  $\lambda_k = \rho_k^2 < 0$ , and continuous in  $\Pi_1 \setminus \Lambda'$ . Moreover,*

$$\alpha_k := \operatorname{Res}_{\lambda=\lambda_k} M(\lambda) > 0, \quad k = \overline{1, m},$$

*and (2.3.34) is valid. The function  $\rho V(\lambda)$  is continuous and bounded for  $\lambda = \rho^2 > 0$ , and*

$$V(\lambda) > 0 \quad \text{for } \lambda = \rho^2 > 0.$$

**Definition 2.3.1.** The data  $S := (\{V(\lambda)\}_{\lambda>0}, \{\lambda_k, \alpha_k\}_{k=\overline{1, m}})$  are called the *spectral data* of  $L$ .

**Lemma 2.3.1.** *The Weyl function is uniquely determined by the specification of the spectral data  $S$  via the formula*

$$M(\lambda) = \int_0^\infty \frac{V(\mu)}{\lambda - \mu} d\mu + \sum_{k=1}^m \frac{\alpha_k}{\lambda - \lambda_k}, \quad \lambda \in \Pi \setminus \Lambda'. \quad (2.3.35)$$

*Proof.* Consider the function

$$I_R(\lambda) := \frac{1}{2\pi i} \int_{|\mu|=R} \frac{M(\mu)}{\lambda - \mu} d\mu.$$

It follows from (2.1.73) that

$$\lim_{R \rightarrow \infty} I_R(\lambda) = 0 \quad (2.3.36)$$

uniformly on compact subsets of  $\Pi \setminus \Lambda'$ . On the other hand, moving the contour  $|\mu| = R$  to the real axis and using the residue theorem, we get

$$I_R(\lambda) = -M(\lambda) + \int_0^R \frac{V(\mu)}{\lambda - \mu} d\mu + \sum_{k=1}^m \frac{\alpha_k}{\lambda - \lambda_k}.$$

Together with (2.3.36) this yields (2.3.35).  $\square$

*It follows from Theorem 2.2.1 and Lemma 2.3.1 that the specification of the spectral data uniquely determines the potential  $q(x)$ ,  $x \geq 0$  and the coefficient  $h$ .*

Let  $L \in V'_N$ , and let  $\tilde{L} \in V'_N$  be chosen such that (2.2.4) holds. Denote

$$\lambda_{n0} = \lambda_n, \lambda_{n1} = \tilde{\lambda}_n, \alpha_{n0} = \alpha_n, \alpha_{n1} = \tilde{\alpha}_n, \varphi_{ni}(x) = \varphi(x, \lambda_{ni}), \tilde{\varphi}_{ni}(x) = \tilde{\varphi}(x, \lambda_{ni}),$$

$$p = m + \tilde{m}, \quad \theta(x) = [\theta_k(x)]_{k=\overline{1,p}},$$

$$\theta_k(x) = \varphi_{k0}(x), \quad k = \overline{1, m}, \quad \theta_{k+m}(x) = \varphi_{k1}(x), \quad k = \overline{1, \tilde{m}}.$$

Analogously we define  $\tilde{\theta}(x)$ . It follows from (2.2.12), (2.2.17) and (2.2.18) that

$$\begin{aligned} \tilde{\varphi}(x, \lambda) &= \varphi(x, \lambda) + \int_0^\infty \tilde{D}(x, \lambda, \mu) \hat{V}(\mu) \varphi(x, \mu) d\mu + \\ &\sum_{k=1}^m \tilde{D}(x, \lambda, \lambda_{k0}) \alpha_{k0} \varphi_{k0}(x) - \sum_{k=1}^{\tilde{m}} \tilde{D}(x, \lambda, \lambda_{k1}) \alpha_{k1} \varphi_{k1}(x), \end{aligned} \quad (2.3.37)$$

$$\begin{aligned} \tilde{\varphi}_{ni}(x) &= \varphi_{ni}(x) + \int_0^\infty \tilde{D}(x, \lambda_{ni}, \mu) \hat{V}(\mu) \varphi(x, \mu) d\mu + \\ &\sum_{k=1}^m \tilde{D}(x, \lambda_{ni}, \lambda_{k0}) \alpha_{k0} \varphi_{k0}(x) - \sum_{k=1}^{\tilde{m}} \tilde{D}(x, \lambda_{ni}, \lambda_{k1}) \alpha_{k1} \varphi_{k1}(x), \end{aligned} \quad (2.3.38)$$

$$\begin{aligned} \tilde{\Phi}(x, \lambda) &= \Phi(x, \lambda) + \int_0^\infty \frac{\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle}{\lambda - \mu} \hat{V}(\mu) \varphi(x, \mu) d\mu + \\ &\sum_{k=1}^m \frac{\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}_{k0}(x) \rangle}{\lambda - \lambda_{k0}} \alpha_{k0} \varphi_{k0}(x) - \sum_{k=1}^{\tilde{m}} \frac{\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle}{\lambda - \lambda_{k1}} \alpha_{k1} \varphi_{k1}(x), \end{aligned} \quad (2.3.39)$$

$$\begin{aligned} \varepsilon_0(x) &= \int_0^\infty \varphi(x, \mu) \tilde{\varphi}(x, \mu) \hat{V}(\mu) d\mu + \sum_{k=1}^m \varphi_{k0}(x) \tilde{\varphi}_{k0}(x) \alpha_{k0} - \sum_{k=1}^{\tilde{m}} \varphi_{k1}(x) \tilde{\varphi}_{k1}(x) \alpha_{k1}, \\ \varepsilon(x) &= -2\varepsilon'_0(x). \end{aligned} \quad (2.3.40)$$

For each fixed  $x \geq 0$ , the relations (2.3.37)-(2.3.38) can be considered as a system of linear equations with respect to  $\varphi(x, \lambda)$ ,  $\lambda > 0$  and  $\theta_k(x)$ ,  $k = \overline{1, p}$ . We rewrite (2.3.37)-(2.3.38) as a linear equation in a corresponding Banach space.

Let  $C = C(0, \infty)$  be the Banach space of continuous bounded functions  $f : [0, \infty) \rightarrow \mathbb{C}$ ,  $\lambda \mapsto f(\lambda)$  on the half-line  $\lambda \geq 0$  with the norm  $\|f\|_C = \sup_{\lambda \geq 0} |f_\lambda|$ . Obviously, for each fixed  $x \geq 0$ ,  $\varphi(x, \cdot), \tilde{\varphi}(x, \cdot) \in C$ . Consider the Banach space  $\tilde{B}$  of vectors

$$F = \begin{bmatrix} f \\ f^0 \end{bmatrix}, \quad f \in C, \quad f^0 = [f_k^0]_{k=\overline{1,p}} \in \mathbb{R}^p$$

with the norm  $\|F\|_B = \max(\|f\|_C, \|f^0\|_{\mathbb{R}^p})$ . Denote

$$\psi(x) = \begin{bmatrix} \varphi(x, \cdot) \\ \theta(x) \end{bmatrix}, \quad \tilde{\psi}(x) = \begin{bmatrix} \tilde{\varphi}(x, \cdot) \\ \tilde{\theta}(x) \end{bmatrix}.$$

Then  $\psi(x), \tilde{\psi}(x) \in B$  for each fixed  $x \geq 0$ . Let

$$\begin{aligned} \tilde{H}_{\lambda, \mu}(x) &= \tilde{D}(x, \lambda, \mu) \hat{V}(\mu), \\ \tilde{H}_{\lambda, k}(x) &= \tilde{D}(x, \lambda, \lambda_{k0}) \alpha_{k0}, \quad k = \overline{1, m}, \quad \tilde{H}_{\lambda, k+m}(x) = -\tilde{D}(x, \lambda, \lambda_{k1}) \alpha_{k1}, \quad k = \overline{1, \tilde{m}}, \\ \tilde{H}_{n, \mu}(x) &= \tilde{D}(x, \lambda_{n0}, \mu) \hat{V}(\mu), \quad n = \overline{1, m}, \quad \tilde{H}_{n+m, \mu}(x) = \tilde{D}(x, \lambda_{n1}, \mu) \hat{V}(\mu), \quad n = \overline{1, \tilde{m}}, \\ \tilde{H}_{n, k}(x) &= \tilde{D}(x, \lambda_{n0}, \lambda_{k0}) \alpha_{k0}, \quad n, k = \overline{1, m}, \\ \tilde{H}_{n, k+m}(x) &= -\tilde{D}(x, \lambda_{n0}, \lambda_{k1}) \alpha_{k1}, \quad n = \overline{1, m}, \quad k = \overline{1, \tilde{m}}, \\ \tilde{H}_{n+m, k}(x) &= \tilde{D}(x, \lambda_{n1}, \lambda_{k0}) \alpha_{k0}, \quad n = \overline{1, \tilde{m}}, \quad k = \overline{1, m}, \\ \tilde{H}_{n+m, k+m}(x) &= -\tilde{D}(x, \lambda_{n1}, \lambda_{k1}) \alpha_{k1}, \quad n, k = \overline{1, \tilde{m}}. \end{aligned}$$

Denote by  $\tilde{H} : B \rightarrow B$  the operator defined by

$$\begin{aligned} \tilde{F} &= \tilde{H}F, \quad F = \begin{bmatrix} f \\ f^0 \end{bmatrix} \in B, \quad \tilde{F} = \begin{bmatrix} \tilde{f} \\ \tilde{f}^0 \end{bmatrix} \in B, \\ \tilde{f}(\lambda) &= \int_0^\infty \tilde{H}_{\lambda, \mu} f(\mu) d\mu + \sum_{k=1}^p \tilde{H}_{\lambda, k} f_k^0, \\ \tilde{f}_n^0 &= \int_0^\infty \tilde{H}_{n, \mu} f(\mu) d\mu + \sum_{k=1}^p \tilde{H}_{n, k} f_k^0. \end{aligned}$$

Then for each fixed  $x \geq 0$ , the operator  $E + \tilde{H}(x)$  (here  $E$  is the identity operator), acting from  $B$  to  $B$ , is a linear bounded operator. Taking into account our notations we can rewrite (2.3.37)-(2.3.38) to the form

$$\tilde{\psi}(x) = (E + \tilde{H}(x))\psi(x). \quad (2.3.41)$$

Thus, we proved the following assertion.

**Theorem 2.3.8.** *For each fixed  $x \geq 0$ , the vector  $\psi(x) \in B$  is a solution of equation (2.3.41).*

Now let us go on to necessary and sufficient conditions for the solvability of the inverse problem under consideration. Denote by  $\mathbf{W}'$  the set of vectors  $S = (\{V(\lambda)\}_{\lambda>0}, \{\lambda_k, \alpha_k\}_{k=\overline{1, m}})$  (in general,  $m$  is different for each  $S$ ) such that

- (1)  $\alpha_k > 0$ ,  $\lambda_k < 0$ , for all  $k$ , and  $\lambda_k \neq \lambda_s$  for  $k \neq s$ ;
- (2) the function  $\rho V(\lambda)$  is continuous and bounded for  $\lambda > 0$ ,  $V(\lambda) > 0$ , and  $M(\lambda) = O(\rho^{-1})$ ,  $\rho \rightarrow 0$ , where  $M(\lambda)$  is defined by (2.3.35);
- (3) there exists  $\tilde{L}$  such that (2.2.4) holds.

Clearly, if  $S$  is the vector of spectral data for a certain  $L \in V'_N$ , then  $S \in \mathbf{W}'$ .

**Theorem 2.3.9.** *Let  $S \in \mathbf{W}'$ . Then for each fixed  $x \geq 0$ , equation (2.3.41) has a unique solution in  $B$ , i.e. the operator  $E + \tilde{H}(x)$  is invertible.*

*Proof.* As in the proof of Lemma 1.6.6 it is sufficient to prove that for each fixed  $x \geq 0$  the homogeneous equation

$$(E + \tilde{H}(x))\beta(x) = 0, \quad \beta(x) \in B, \quad (2.3.42)$$

has only the zero solution. Let

$$\beta(x) = \begin{bmatrix} \beta(x, \cdot) \\ \beta^0(x) \end{bmatrix} \in B, \quad \beta^0(x) = [\beta_k^0(x)]_{k=\overline{1,p}},$$

be a solution of (2.3.42), i.e.

$$\begin{aligned} & \beta(x, \lambda) + \int_0^\infty \tilde{D}(x, \lambda, \mu) \hat{V}(\mu) \beta(x, \mu) d\mu \\ & + \sum_{k=1}^m \tilde{D}(x, \lambda, \lambda_{k0}) \alpha_{k0} \beta_{k0}(x) - \sum_{k=1}^{\tilde{m}} \tilde{D}(x, \lambda, \lambda_{k1}) \alpha_{k1} \beta_{k1}(x) = 0, \end{aligned} \quad (2.3.43)$$

$$\begin{aligned} & \beta_{ni}(x) + \int_0^\infty \tilde{D}(x, \lambda_{ni}, \mu) \hat{V}(\mu) \beta(x, \mu) d\mu \\ & + \sum_{k=1}^m \tilde{D}(x, \lambda_{ni}, \lambda_{k0}) \alpha_{k0} \beta_{k0}(x) - \sum_{k=1}^{\tilde{m}} \tilde{D}(x, \lambda_{ni}, \lambda_{k1}) \alpha_{k1} \beta_{k1}(x) = 0, \end{aligned} \quad (2.3.44)$$

where  $\beta_{k0}(x) = \beta_k^0(x)$ ,  $k = \overline{1, m}$ ,  $\beta_{k1}(x) = \beta_{k+m}^0(x)$ ,  $k = \overline{1, \tilde{m}}$ . Then (2.3.43) gives us the analytic continuation of  $\beta(x, \lambda)$  to the whole  $\lambda$ -plane, and for each fixed  $x \geq 0$ , the function  $\beta(x, \lambda)$  is entire in  $\lambda$ . Moreover, according to (2.3.43), (2.3.44),

$$\beta(x, \lambda_{ni}) = \beta_{ni}(x). \quad (2.3.45)$$

Let us show that for each fixed  $x \geq 0$ ,

$$|\beta(x, \lambda)| \leq \frac{C_x}{|\rho|} \exp(|\tau|x), \quad \lambda = \rho^2, \quad \tau := \operatorname{Im} \rho. \quad (2.3.46)$$

Indeed, using (2.2.6) and (2.1.87), we get

$$|\tilde{D}(x, \lambda, \lambda_{ki})| \leq \frac{C_x}{|\rho|} \exp(|\tau|x). \quad (2.3.47)$$

For definiteness, let  $\sigma := \operatorname{Re} \rho \geq 0$ . It follows from (2.3.43), (2.2.7) and (2.3.47) that

$$|\beta(x, \lambda)| \leq C_x \exp(|\tau|x) \left( \int_1^\infty \frac{|\hat{V}(\mu)|\theta}{|\rho - \theta| + 1} d\theta + \frac{1}{|\rho|} \right), \quad \mu = \theta^2. \quad (2.3.48)$$

In view of (2.2.4),

$$\begin{aligned} \left( \int_1^\infty \frac{|\hat{V}(\mu)|\theta}{|\rho - \theta| + 1} d\theta \right)^2 & \leq \left( \int_1^\infty |\hat{V}(\mu)|^2 \theta^4 d\theta \right) \left( \int_1^\infty \frac{d\theta}{\theta^2(|\rho - \theta| + 1)^2} \right) \\ & \leq C \int_1^\infty \frac{d\theta}{\theta^2(|\rho - \theta| + 1)^2}. \end{aligned} \quad (2.3.49)$$

Since

$$|\rho - \theta| = \sigma^2 + \tau^2 + \theta^2 - 2\sigma\theta, \quad ||\rho| - \theta| = \sigma^2 + \tau^2 + \theta^2 - 2|\rho|\theta,$$

we have

$$|\rho - \theta| \geq ||\rho| - \theta|. \quad (2.3.50)$$

By virtue of (2.3.49), (2.3.50) and (2.2.11),

$$\int_1^\infty \frac{|\hat{V}(\mu)|\theta}{|\rho - \theta| + 1} d\theta \leq \frac{C}{|\rho|}. \quad (2.3.51)$$

Using (2.3.48) and (2.3.51) we arrive at (2.3.46).

Furthermore, we construct the function  $\Gamma(x, \lambda)$  by the formula

$$\begin{aligned} \Gamma(x, \lambda) = & - \int_0^\infty \frac{\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle}{\lambda - \mu} \hat{V}(\mu) \beta(x, \mu) d\mu \\ & - \sum_{k=1}^m \frac{\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}_{k0}(x) \rangle}{\lambda - \lambda_{k0}} \alpha_{k0} \beta_{k0}(x) + \sum_{k=1}^{\tilde{m}} \frac{\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle}{\lambda - \lambda_{k1}} \alpha_{k1} \beta_{k1}(x). \end{aligned} \quad (2.3.52)$$

It follows from (2.1.70), (2.2.6), (2.3.43) and (2.3.52) that

$$\begin{aligned} \Gamma(x, \lambda) = & \tilde{M}(\lambda) \beta(x, \lambda) - \int_0^\infty \frac{\langle \tilde{S}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle}{\lambda - \mu} \hat{V}(\mu) \beta(x, \mu) d\mu \\ & - \sum_{k=1}^m \frac{\langle \tilde{S}(x, \lambda), \tilde{\varphi}_{k0}(x) \rangle}{\lambda - \lambda_{k0}} \alpha_{k0} \beta_{k0}(x) + \sum_{k=1}^{\tilde{m}} \frac{\langle \tilde{S}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle}{\lambda - \lambda_{k1}} \alpha_{k1} \beta_{k1}(x). \end{aligned} \quad (2.3.53)$$

Since  $\langle \tilde{S}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle|_{x=0} = -1$ , we infer from (1.6.1) that

$$\frac{\langle \tilde{S}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle}{\lambda - \mu} = -\frac{1}{\lambda - \mu} + \int_0^x \tilde{S}(t, \lambda) \tilde{\varphi}(t, \mu) dt.$$

Hence, (2.3.53) takes the form

$$\begin{aligned} \Gamma(x, \lambda) = & \tilde{M}(\lambda) \beta(x, \lambda) + \int_0^\infty \frac{\hat{V}(\mu) \beta(x, \mu)}{\lambda - \mu} d\mu \\ & + \sum_{k=1}^m \frac{\alpha_{k0} \beta_{k0}(x)}{\lambda - \lambda_{k0}} - \sum_{k=1}^{\tilde{m}} \frac{\alpha_{k1} \beta_{k1}(x)}{\lambda - \lambda_{k1}} + \Gamma_1(x, \lambda), \end{aligned} \quad (2.3.54)$$

where

$$\begin{aligned} \Gamma_1(x, \lambda) = & \int_0^\infty \left( \int_0^x \tilde{S}(t, \lambda) \tilde{\varphi}(t, \mu) dt \right) \hat{V}(\mu) \beta(x, \mu) d\mu \\ & - \sum_{k=1}^m \left( \int_0^x \tilde{S}(t, \lambda) \tilde{\varphi}_{k0}(t) dt \right) \alpha_{k0} \beta_{k0}(x) + \sum_{k=1}^{\tilde{m}} \left( \int_0^x \tilde{S}(t, \lambda) \tilde{\varphi}_{k1}(t) dt \right) \alpha_{k1} \beta_{k1}(x). \end{aligned}$$

The function  $\Gamma_1(x, \lambda)$  is entire in  $\lambda$  for each fixed  $x \geq 0$ . Using (2.3.35) we derive from (2.3.54):

$$\Gamma(x, \lambda) = M(\lambda) \beta(x, \lambda) + \Gamma_0(x, \lambda), \quad (2.3.55)$$

where

$$\Gamma_0(x, \lambda) = \Gamma_1(x, \lambda) + \Gamma_2(x, \lambda) + \Gamma_3(x, \lambda),$$



$$\Gamma_2(x, \lambda) = - \int_0^\infty \frac{\hat{V}(\mu)(\beta(x, \lambda) - \beta(x, \mu))}{\lambda - \mu} d\mu,$$

$$\Gamma_3(x, \lambda) = - \sum_{k=1}^m \frac{\alpha_{k0}}{\lambda - \lambda_{k0}} (\beta(x, \lambda) - \beta_{k0}(x)) + \sum_{k=1}^{\tilde{m}} \frac{\alpha_{k1}}{\lambda - \lambda_{k1}} (\beta(x, \lambda) - \beta_{k1}(x)).$$

In view of (2.3.45), the function  $\Gamma_0(x, \lambda)$  is entire in  $\lambda$  for each fixed  $x \geq 0$ . Using (2.3.55), (2.3.52), (2.3.46) and (2.1.88) we obtain the following properties of the function  $\Gamma(x, \lambda)$ .

(1) For each fixed  $x \geq 0$ , the function  $\Gamma(x, \lambda)$  is analytic in  $\Pi \setminus \Lambda'$  with respect to  $\lambda$  (with the simple poles  $\lambda_{k0}$ ,  $k = \overline{1, m}$ ), and continuous in  $\Pi_1 \setminus \Lambda'$ . Moreover,

$$\text{Res}_{\lambda=\lambda_{k0}} \Gamma(x, \lambda) = \alpha_{k0} \beta_{k0}(x), \quad k = \overline{1, m}. \quad (2.3.56)$$

(2) Denote

$$\Gamma^\pm(\lambda) = \lim_{z \rightarrow 0, \text{Re } z > 0} \Gamma(\lambda \pm iz), \quad \lambda > 0.$$

Then

$$\frac{1}{2\pi i} (\Gamma^-(\lambda) - \Gamma^+(\lambda)) = V(\lambda) \beta(x, \lambda), \quad \lambda > 0. \quad (2.3.57)$$

(3) For  $|\lambda| \rightarrow \infty$ ,

$$|\Gamma(x, \lambda)| \leq \frac{C_x}{|\rho|^2} \exp(-|\tau|x). \quad (2.3.58)$$

Now we construct the function  $B(x, \lambda)$  via the formula

$$B(x, \lambda) = \overline{\beta(x, \bar{\lambda})} \Gamma(x, \lambda). \quad (2.3.59)$$

By Cauchy's theorem

$$\frac{1}{2\pi i} \int_{\gamma_R^0} B(x, \lambda) d\lambda = 0,$$

where the contour  $\gamma_R^0$  is defined in Section 2.1 (see fig. 2.1.2). By virtue of (2.3.46), (2.3.58) and (2.3.59) we get

$$\lim_{R \rightarrow \infty} \frac{1}{2\pi i} \int_{|\lambda|=R} B(x, \lambda) d\lambda = 0,$$

and consequently

$$\frac{1}{2\pi i} \int_{\gamma} B(x, \lambda) d\lambda = 0, \quad (2.3.60)$$

where the contour  $\gamma$  is defined in Section 2.1 (see fig. 2.1.1). Moving the contour  $\gamma$  in (2.3.60) to the real axis and using the residue theorem and (2.3.56) we get for sufficiently small  $\varepsilon > 0$ ,

$$\sum_{n=1}^m \alpha_{n0} |\beta_{n0}(x)|^2 + \frac{1}{2\pi i} \int_{|\lambda|=\varepsilon} B(x, \lambda) d\lambda + \frac{1}{2\pi i} \int_{\gamma_\varepsilon''} B(x, \lambda) d\lambda = 0, \quad (2.3.61)$$

where  $\gamma_\varepsilon''$  is the two-sided cut along the arc  $\{\lambda : \lambda \geq \varepsilon\}$ . Since  $M(\lambda) = O(\rho^{-1})$  as  $|\lambda| \rightarrow 0$ , we get in view of (2.3.55) and (2.3.59) that for each fixed  $x \geq 0$ ,

$$B(x, \lambda) = O(\rho^{-1}) \quad \text{as } |\lambda| \rightarrow 0,$$

and consequently

$$\lim_{\varepsilon \rightarrow \infty} \frac{1}{2\pi i} \int_{|\lambda|=\varepsilon} B(x, \lambda) d\lambda = 0.$$

Together with (2.3.61), (2.3.57) and (2.3.59) this yields

$$\sum_{k=1}^m \alpha_{k0} |\beta_{k0}(x)|^2 + \int_0^\infty |\beta(x, \lambda)|^2 V(\lambda) d\lambda = 0.$$

Since  $\alpha_{k0} > 0$ ,  $V(\lambda) > 0$ , we get

$$\beta(x, \lambda) = 0, \quad \beta_{n0}(x) = 0, \quad \beta_{n1}(x) = \beta(x, \lambda_{n1}) = 0.$$

Thus,  $\beta(x) = 0$ , and Theorem 2.3.9 is proved.  $\square$

**Theorem 2.3.10.** *For a vector  $S = (\{V(\lambda)\}_{\lambda>0}, \{\lambda_k, \alpha_k\}_{k=\overline{1,m}}) \in \mathbf{W}'$  to be the spectral data for a certain  $L \in V'_N$ , it is necessary and sufficient that  $\varepsilon(x) \in W'_N$ , where  $\varepsilon(x)$  is defined via (2.3.40), and  $\psi(x)$  is the solution of (2.3.41). The function  $q(x)$  and the number  $h$  can be constructed via (2.2.19)-(2.2.20).*

Denote by  $\mathbf{W}''$  the set of functions  $M(\lambda)$  such that:

- (1)  $M(\lambda)$  is analytic in  $\Pi$  with the exception of an at most finite number of simple poles  $\Lambda' = \{\lambda_k\}_{k=1,m}$ ,  $\lambda_k = \rho_k^2 < 0$ , and  $\alpha_k := \operatorname{Res}_{\lambda=\lambda_k} M(\lambda) > 0$ ;
- (2)  $M(\lambda)$  is continuous in  $\Pi_1 \setminus \Lambda'$ ,  $M(\lambda) = O(\rho^{-1})$  for  $\rho \rightarrow 0$ , and

$$V(\lambda) := \frac{1}{2\pi i} (M^-(\lambda) - M^+(\lambda)) > 0 \text{ for } \lambda > 0;$$

- (3) there exists  $\tilde{L}$  such that (2.2.4) holds.

**Theorem 2.3.11.** *For a function  $M(\lambda) \in \mathbf{W}''$  to be the Weyl function for a certain  $L \in V'_N$ , it is necessary and sufficient that  $\varepsilon(x) \in W'_N$ , where  $\varepsilon(x)$  is defined via (2.3.40).*

We omit the proofs of Theorems 2.3.10-2.3.11, since they are similar to corresponding facts for Sturm-Liouville operators on a finite interval presented in Section 1.6 (see also the proof of Theorem 2.2.5).

**2.3.2. The nonselfadjoint case.** One can consider the inverse problem from the spectral data also for the nonselfadjoint case when  $q(x) \in L(0, \infty)$  is a complex-valued function, and  $h$  is a complex number. For simplicity we confine ourselves here to the case of a simple spectrum (see Definition 2.3.2). For nonselfadjoint differential operators with a simple spectrum we introduce the spectral data and study the inverse problem of recovering  $L$  from the given spectral data. We also consider a particular case when only the discrete spectrum is perturbed. Then the main equation of the inverse problem becomes a linear algebraic system, and the solvability condition is equivalent to the condition that the determinant of this system differs from zero.

**Definition 2.3.2.** We shall say that  $L$  has simple spectrum if  $\Delta(\rho)$  has a finite number of simple zeros in  $\Omega$ , and  $M(\lambda) = O(\rho^{-1})$ ,  $\rho \rightarrow 0$ . We shall write  $L \in V''_N$  if  $L$  has simple spectrum and  $q \in W_N$ .

Clearly,  $V'_N \subset V''_N$ , since the selfadjoint operators considered in Subsection 2.3.1 have simple spectrum.

Let  $L \in V''_N$ . Then  $\Lambda = \{\lambda_k\}_{k=\overline{1,m}}$ ,  $\lambda_k = \rho_k^2$  is a finite set, and the Weyl function  $M(\lambda)$  is analytic in  $\Pi \setminus \Lambda'$  and continuous in  $\Pi_1 \setminus \Lambda$ ;  $\Lambda' = \{\lambda_k\}_{k=\overline{1,r}}$  are the eigenvalues, and  $\Lambda'' = \{\lambda_k\}_{k=\overline{r+1,m}}$  are the spectral singularities of  $L$ . The next assertion is proved by the same arguments as in Lemma 2.3.1.

**Lemma 2.3.2.** *The following relation holds*

$$M(\lambda) = \int_0^\infty \frac{V(\mu)}{\lambda - \mu} d\mu + \sum_{k=1}^m \frac{\alpha_k}{\lambda - \lambda_k},$$

where

$$V(\lambda) := \frac{1}{2\pi i} (M^-(\lambda) - M^+(\lambda)), \quad M^\pm(\lambda) := \lim_{z \rightarrow 0, \operatorname{Re} z > 0} M(\lambda \pm iz),$$

$$\alpha_k := \begin{cases} e(0, \rho_k)(\Delta_1(\rho_k))^{-1}, & k = \overline{1, r}, \\ \frac{1}{2}e(0, \rho_k)(\Delta_1(\rho_k))^{-1}, & k = \overline{r+1, m}, \end{cases} \quad \Delta_1(\rho) := \frac{d}{d\lambda} \Delta(\rho).$$

**Definition 2.3.3.** The data  $S = (\{V(\lambda)\}_{\lambda > 0}, \{\lambda_k, \alpha_k\}_{k=\overline{1,m}})$  are called the spectral data of  $L$ .

Using Theorem 2.2.1 and Lemma 2.3.2 we obtain the following uniqueness theorem.

**Theorem 2.3.12.** *Let  $S$  and  $\tilde{S}$  be the spectral data for  $L$  and  $\tilde{L}$  respectively. If  $S = \tilde{S}$ , then  $L = \tilde{L}$ . Thus, the specification of the spectral data uniquely determines  $L$ .*

For  $L \in V''_N$  the main equation (2.3.41) of the inverse problem remains valid, and Theorem 2.3.8 also holds true. Moreover, the potential  $q$  and the coefficient  $h$  can be constructed by (2.3.40), (2.2.19)-(2.2.20).

**Perturbation of the discrete spectrum.** Let  $\tilde{L} \in V_N$ , and let  $\tilde{M}(\lambda)$  be the Weyl function for  $\tilde{L}$ . Consider the function

$$M(\lambda) = \tilde{M}(\lambda) + \sum_{\lambda_0 \in J} \frac{a_{\lambda_0}}{\lambda - \lambda_0}, \quad (2.3.62)$$

where  $J$  is a finite set of the  $\lambda$ -plane, and  $a_{\lambda_0}$ ,  $\lambda_0 \in J$  are complex numbers. In this case  $\hat{V}(\lambda) = 0$ . Then the main equation (2.2.12) becomes the linear algebraic system

$$\tilde{\varphi}(x, z_0) = \varphi(x, z_0) + \sum_{\lambda_0 \in J} \tilde{D}(x, z_0, \lambda_0) a_{\lambda_0} \varphi(x, \lambda_0), \quad z_0 \in J, \quad (2.3.63)$$

with the determinant  $\det(E + \tilde{G}(x))$ , where

$$\tilde{G}(x) = [\tilde{D}(x, z_0, \lambda_0) a_{\lambda_0}]_{z_0, \lambda_0 \in J}.$$

The solvability condition (Condition S in Theorem 2.2.5) takes here the form

$$\det(E + \tilde{G}(x)) \neq 0 \quad \text{for all } x \geq 0. \quad (2.3.64)$$

The potential  $q$  and the coefficient  $h$  can be constructed by the formulae

$$q(x) = \tilde{q}(x) + \varepsilon(x), \quad h = \tilde{h} - \sum_{\lambda_0 \in J} a_{\lambda_0}, \quad (2.3.65)$$

$$\varepsilon(x) = -2 \sum_{\lambda_0 \in J} a_{\lambda_0} \frac{d}{dx} (\tilde{\varphi}(x, \lambda_0) \varphi(x, \lambda_0)). \quad (2.3.66)$$

From Theorem 2.2.5 we have

**Theorem 2.3.13.** *Let  $\tilde{L} \in V_N$ . For a function  $M(\lambda)$  of the form (2.3.62) to be the Weyl function for a certain  $L \in V_N$  it is necessary and sufficient that (2.3.64) holds,  $\varepsilon(x) \in W_N$ , where  $\varepsilon(x)$  is defined via (2.3.66), and that  $\{\varphi(x, \lambda_0)\}_{\lambda_0 \in J}$  is a solution of (2.3.63). Under these conditions the potential  $q$  and the coefficient  $h$  are constructed by (2.3.65).*

**Example 2.3.1.** Let  $\tilde{q}(x) = 0$  and  $\tilde{h} = 0$ . Then  $\tilde{M}(\lambda) = \frac{1}{i\rho}$ . Consider the function

$$M(\lambda) = \tilde{M}(\lambda) + \frac{a}{\lambda - \lambda_0},$$

where  $a$  and  $\lambda_0$  are complex numbers. Then the main equation (2.3.63) becomes

$$\tilde{\varphi}(x, \lambda_0) = F(x) \varphi(x, \lambda_0),$$

where

$$\tilde{\varphi}(x, \lambda_0) = \cos \rho_0 x, \quad F(x) = 1 + a \int_0^x \cos^2 \rho_0 t \, dt, \quad \lambda_0 = \rho_0^2.$$

The solvability condition (2.3.64) takes the form

$$F(x) \neq 0 \text{ for all } x \geq 0, \quad (2.3.67)$$

and the function  $\varepsilon(x)$  can be found by the formula

$$\varepsilon(x) = \frac{2a\rho_0 \sin 2\rho_0 x}{F(x)} + \frac{2a^2 \cos^4 \rho_0 x}{F^2(x)}.$$

*Case 1.* Let  $\lambda_0 = 0$ . Then

$$F(x) = 1 + ax,$$

and (2.3.67) is equivalent to the condition

$$a \notin (-\infty, 0). \quad (2.3.68)$$

If (2.3.68) is fulfilled then  $M(\lambda)$  is the Weyl function for  $L$  of the form (2.2.1)-(2.2.2) with

$$q(x) = \frac{2a^2}{(1+ax)^2}, \quad h = -a,$$

$$\varphi(x, \lambda) = \cos \rho x - \frac{a}{1+ax} \cdot \frac{\sin \rho x}{\rho}, \quad e(x, \rho) = \exp(i\rho x) \left(1 - \frac{a}{i\rho(1+ax)}\right),$$

$$\Delta(\rho) = i\rho, \quad V(\lambda) = \frac{1}{\pi\rho}, \quad \varphi(x, 0) = \frac{1}{1+ax}.$$

If  $a < 0$ , then the solvability condition is not fulfilled, and the function  $M(\lambda)$  is not a Weyl function.

*Case 2.* Let  $\lambda_0 \neq 0$  be a real number, and let  $a > 0$ . Then  $F(x) \geq 1$ , and (2.3.67) is fulfilled. But in this case  $\varepsilon(x) \notin L(0, \infty)$ , i.e.  $\varepsilon(x) \notin W_N$  for any  $N \geq 0$ .

## 2.4. AN INVERSE PROBLEM FOR A WAVE EQUATION

In this section we consider an inverse problem for a wave equation with a focused source of disturbance. In applied problems the data are often functions of compact support localized within a relative small area of space. It is convenient to model such situations mathematically as problems with a focused source of disturbance (see [rom1]).

Consider the following boundary value problem  $B(q(x), h)$  :

$$u_{tt} = u_{xx} - q(x)u, \quad 0 \leq x \leq t, \quad (2.4.1)$$

$$u(x, x) = 1, \quad (u_x - hu)|_{x=0}, \quad (2.4.2)$$

where  $q(x)$  is a complex-valued locally integrable function (i.e. it is integrable on every finite interval), and  $h$  is a complex number. Denote  $r(t) := u(0, t)$ . The function  $r$  is called the trace of the solution. In this section we study the following inverse problem.

**Inverse Problem 2.4.1.** Given the trace  $r(t)$ ,  $t \geq 0$ , of the solution of  $B(q(x), h)$ , construct  $q(x)$ ,  $x \geq 0$ , and  $h$ .

We prove an uniqueness theorem for Inverse Problem 2.4.1 (Theorem 2.4.3), provide an algorithm for the solution of this inverse problem (Algorithm 2.4.1) and give necessary and sufficient conditions for its solvability (Theorem 2.4.4). Furthermore we show connections between Inverse Problem 2.4.1 and the inverse spectral problems considered in Sections 2.2-2.3.

**Remark 2.4.1.** Let us note here that the boundary value problem  $B(q(x), h)$  is equivalent to a Cauchy problem with a focused source of disturbance. For simplicity, we assume here that  $h = 0$ . We define  $u(x, t) = 0$  for  $0 < t < x$ , and  $u(x, t) = u(-x, t)$ ,  $q(x) = q(-x)$  for  $x < 0$ . Then, using symmetry, it follows that  $u(x, t)$  is a solution of the Goursat problem

$$u_{tt} = u_{xx} - q(x)u, \quad 0 \leq |x| \leq t,$$

$$u(x, |x|) = 1.$$

Moreover, it can be shown that this Goursat problem is equivalent to the Cauchy problem

$$u_{tt} = u_{xx} - q(x)u, \quad -\infty < x < \infty, \quad t > 0,$$

$$u|_{t=0} = 0, \quad u_t|_{t=0} = 2\delta(x),$$

where  $\delta(x)$  is the Dirac delta-function. Similarly, for  $h \neq 0$ , the boundary value problem (2.4.1)-(2.4.2) also corresponds to a problem with a focused source of disturbance.

Let us return to the boundary value problem (2.4.1)-(2.4.2). Denote

$$Q(x) = \int_0^x |q(t)| dt, \quad Q_*(x) = \int_0^x Q(t) dt, \quad d = \max(0, -h).$$

**Theorem 2.4.1.** *The boundary value problem (2.4.1)-(2.4.2) has a unique solution  $u(x, t)$ , and*

$$|u(x, t)| \leq \exp(d(t - x)) \exp\left(2Q_*\left(\frac{t + x}{2}\right)\right), \quad 0 \leq x \leq t. \quad (2.4.3)$$

*Proof.* We transform (2.4.1)-(2.4.2) by means of the replacement

$$\xi = t + x, \quad \eta = t - x, \quad v(\xi, \eta) = u\left(\frac{\xi - \eta}{2}, \frac{\xi + \eta}{2}\right)$$

to the boundary value problem

$$v_{\xi\eta}(\xi, \eta) = -\frac{1}{4}q\left(\frac{\xi - \eta}{2}\right)v(\xi, \eta), \quad 0 \leq \eta \leq \xi, \quad (2.4.4)$$

$$v(\xi, 0) = 1, \quad (v_{\xi}(\xi, \eta) - v_{\eta}(\xi, \eta) - hv(\xi, \eta))|_{\xi=\eta} = 0. \quad (2.4.5)$$

Since  $v_{\xi}(\xi, 0) = 0$ , integration of (2.4.4) with respect to  $\eta$  gives

$$v_{\xi}(\xi, \eta) = -\frac{1}{4} \int_0^{\eta} q\left(\frac{\xi - \alpha}{2}\right)v(\xi, \alpha) d\alpha. \quad (2.4.6)$$

In particular, we have

$$v_{\xi}(\xi, \eta)|_{\xi=\eta} = -\frac{1}{4} \int_0^{\eta} q\left(\frac{\eta - \alpha}{2}\right)v(\eta, \alpha) d\alpha. \quad (2.4.7)$$

It follows from (2.4.6) that

$$v(\xi, \eta) = v(\eta, \eta) - \frac{1}{4} \int_{\eta}^{\xi} \left( \int_0^{\eta} q\left(\frac{\beta - \alpha}{2}\right)v(\beta, \alpha) d\alpha \right) d\beta. \quad (2.4.8)$$

Let us calculate  $v(\eta, \eta)$ . Since

$$\frac{d}{d\eta}(v(\eta, \eta) \exp(h\eta)) = (v_{\xi}(\xi, \eta) + v_{\eta}(\xi, \eta) + hv(\xi, \eta))|_{\xi=\eta} \exp(h\eta),$$

we get by virtue of (2.4.5) and (2.4.7),

$$\frac{d}{d\eta}(v(\eta, \eta) \exp(h\eta)) = 2v_{\xi}(\xi, \eta)|_{\xi=\eta} \exp(h\eta) = -\frac{1}{2} \exp(h\eta) \int_0^{\eta} q\left(\frac{\eta - \alpha}{2}\right)v(\eta, \alpha) d\alpha.$$

This yields (with  $v(0, 0) = 1$ )

$$v(\eta, \eta) \exp(h\eta) - 1 = -\frac{1}{2} \int_0^{\eta} \exp(h\beta) \left( \int_0^{\beta} q\left(\frac{\beta - \alpha}{2}\right)v(\beta, \alpha) d\alpha \right) d\beta,$$

and consequently

$$v(\eta, \eta) = \exp(-h\eta) - \frac{1}{2} \int_0^{\eta} \exp(-h(\eta - \beta)) \left( \int_0^{\beta} q\left(\frac{\beta - \alpha}{2}\right)v(\beta, \alpha) d\alpha \right) d\beta. \quad (2.4.9)$$

Substituting (2.4.9) into (2.4.8) we deduce that the function  $v(\xi, \eta)$  satisfies the integral equation

$$\begin{aligned} v(\xi, \eta) = & \exp(-h\eta) - \frac{1}{2} \int_0^{\eta} \exp(-h(\eta - \beta)) \left( \int_0^{\beta} q\left(\frac{\beta - \alpha}{2}\right)v(\beta, \alpha) d\alpha \right) d\beta \\ & - \frac{1}{4} \int_{\eta}^{\xi} \left( \int_0^{\eta} q\left(\frac{\beta - \alpha}{2}\right)v(\beta, \alpha) d\alpha \right) d\beta. \end{aligned} \quad (2.4.10)$$

Conversely, if  $v(\xi, \eta)$  is a solution of (2.4.10) then one can verify that  $v(\xi, \eta)$  satisfies (2.4.4)-(2.4.5).

We solve the integral equation (2.4.10) by the method of successive approximations. The calculations are slightly different for  $h \geq 0$  and  $h < 0$ .

*Case 1.* Let  $h \geq 0$ . Denote

$$\begin{aligned} v_0(\xi, \eta) = \exp(-h\eta), \quad v_{k+1}(\xi, \eta) = & -\frac{1}{2} \int_0^\eta \exp(-h(\eta - \beta)) \left( \int_0^\beta q\left(\frac{\beta - \alpha}{2}\right) v_k(\beta, \alpha) d\alpha \right) d\beta \\ & - \frac{1}{4} \int_\eta^\xi \left( \int_0^\eta q\left(\frac{\beta - \alpha}{2}\right) v_k(\beta, \alpha) d\alpha \right) d\beta. \end{aligned} \quad (2.4.11)$$

Let us show by induction that

$$|v_k(\xi, \eta)| \leq \frac{1}{k!} \left( 2Q_*\left(\frac{\xi}{2}\right) \right)^k, \quad k \geq 0, \quad 0 \leq \eta \leq \xi. \quad (2.4.12)$$

Indeed, for  $k = 0$ , (2.4.12) is obvious. Suppose that (2.4.12) is valid for a certain  $k \geq 0$ . It follows from (2.4.11) that

$$|v_{k+1}(\xi, \eta)| \leq \frac{1}{2} \int_0^\xi \left( \int_0^\eta \left| q\left(\frac{\beta - \alpha}{2}\right) v_k(\beta, \alpha) \right| d\alpha \right) d\beta. \quad (2.4.13)$$

Substituting (2.4.12) into the right-hand side of (2.4.13) we obtain

$$\begin{aligned} |v_{k+1}(\xi, \eta)| & \leq \frac{1}{2k!} \int_0^\xi \left( 2Q_*\left(\frac{\beta}{2}\right) \right)^k \left( \int_0^\eta \left| q\left(\frac{\beta - \alpha}{2}\right) \right| d\alpha \right) d\beta \\ & \leq \frac{1}{k!} \int_0^\xi \left( 2Q_*\left(\frac{\beta}{2}\right) \right)^k \left( \int_0^{\beta/2} |q(s)| ds \right) d\beta = \frac{1}{k!} \int_0^\xi \left( 2Q_*\left(\frac{\beta}{2}\right) \right)^k Q\left(\frac{\beta}{2}\right) d\beta \\ & = \frac{1}{k!} \int_0^{\xi/2} (2Q_*(s))^k (2Q_*(s))' ds = \frac{1}{(k+1)!} \left( 2Q_*\left(\frac{\xi}{2}\right) \right)^{k+1}; \end{aligned}$$

hence (2.4.12) is valid.

It follows from (2.4.12) that the series

$$v(\xi, \eta) = \sum_{k=0}^{\infty} v_k(\xi, \eta)$$

converges absolutely and uniformly on compact sets  $0 \leq \eta \leq \xi \leq T$ , and

$$|v(\xi, \eta)| \leq \exp \left( 2Q_*\left(\frac{\xi}{2}\right) \right).$$

The function  $v(\xi, \eta)$  is the unique solution of the integral equation (2.4.10). Consequently, the function  $u(x, t) = v(t + x, t - x)$  is the unique solution of the boundary value problem (2.4.1)-(2.4.2), and (2.4.3) holds.

*Case 2.* Let  $h < 0$ . we transform (2.4.10) by means of the replacement

$$w(\xi, \eta) = v(\xi, \eta) \exp(h\eta)$$

to the integral equation

$$w(\xi, \eta) = 1 - \frac{1}{2} \int_0^\eta \left( \int_0^\beta q\left(\frac{\beta - \alpha}{2}\right) \exp(h(\beta - \alpha)) w(\beta, \alpha) d\alpha \right) d\beta - \frac{1}{4} \int_\eta^\xi \left( \int_0^\eta q\left(\frac{\beta - \alpha}{2}\right) \exp(h(\eta - \alpha)) w(\beta, \alpha) d\alpha \right) d\beta. \quad (2.4.14)$$

By the method of successive approximations we get similarly to Case 1 that the integral equation (2.4.14) has a unique solution, and that

$$|w(\xi, \eta)| \leq \exp\left(2Q_*\left(\frac{\xi}{2}\right)\right),$$

i.e. Theorem 2.4.1 is proved also for  $h < 0$ .  $\square$

**Remark 2.4.2.** It follows from the proof of Theorem 2.4.1 that the solution  $u(x, t)$  of (2.4.1)-(2.4.2) in the domain  $\Theta_T := \{(x, t) : 0 \leq x \leq t, 0 \leq x + t \leq 2T\}$

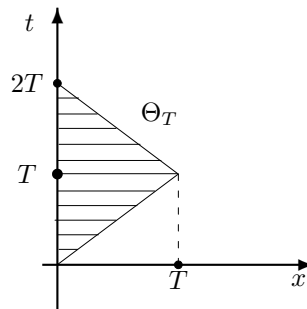


fig. 2.4.1.

is uniquely determined by the specification of  $h$  and  $q(x)$  for  $0 \leq x \leq T$ , i.e. if  $q(x) = \tilde{q}(x)$ ,  $x \in [0, T]$  and  $h = \tilde{h}$ , then  $u(x, t) = \tilde{u}(x, t)$  for  $(x, t) \in \Theta_T$ . Therefore, one can also consider the boundary value problem (2.4.1)-(2.4.2) in the domains  $\Theta_T$  and study the inverse problem of recovering  $q(x)$ ,  $0 \leq x \leq T$  and  $h$  from the given trace  $r(t)$ ,  $t \in [0, 2T]$ .

Denote by  $D_N$  ( $N \geq 0$ ) the set of functions  $f(x)$ ,  $x \geq 0$  such that for each fixed  $T > 0$ , the functions  $f^{(j)}(x)$ ,  $j = \overline{0, N-1}$  are absolutely continuous on  $[0, T]$ , i.e.  $f^{(j)}(x) \in L(0, T)$ ,  $j = \overline{0, N}$ . It follows from the proof of Theorem 2.4.1 that  $r(t) \in D_2$ ,  $r(0) = 1$ ,  $r'(0) = -h$ . Moreover, the function  $r''$  has the same smoothness properties as the potential  $q$ . For example, if  $q \in D_N$  then  $r \in D_{N+2}$ .

In order to solve Inverse Problem 2.4.1 we will use the Riemann formula for the solution of the Cauchy problem

$$\left. \begin{aligned} u_{tt} - p(t)u &= u_{xx} - q(x)u + p_1(x, t), \quad -\infty < x < \infty, \quad t > 0, \\ u|_{t=0} &= r(x), \quad u_t|_{t=0} = s(x). \end{aligned} \right\} \quad (2.4.15)$$

It is known (see, for example, [hel1, p.150]) that the solution of (2.4.15) has the form

$$u(x, t) = \frac{1}{2} \left( r(x+t) + r(x-t) \right) + \frac{1}{2} \int_{x-t}^{x+t} \left( s(\xi) R(\xi, 0, x, t) - r(\xi) R_2(\xi, 0, x, t) \right) d\xi +$$



$$\frac{1}{2} \int_0^t d\tau \int_{x+\tau-t}^{x+t-\tau} R(\xi, \tau, x, t) p_1(\xi, \tau) d\xi,$$

where  $R(\xi, \tau, x, t)$  is the Riemann function, and  $R_2 = \frac{\partial R}{\partial \tau}$ . Note that if  $q(x) \equiv \text{const}$ , then  $R(\xi, \tau, x, t) = R(\xi - x, \tau, t)$ . In particular, the solution of the Cauchy problem

$$\left. \begin{aligned} u_{tt} &= u_{xx} - q(x)u, & -\infty < t < \infty, & x > 0, \\ u|_{x=0} &= r(t), & u_x|_{x=0} &= hr(t), \end{aligned} \right\} \quad (2.4.16)$$

has the form

$$u(x, t) = \frac{1}{2} (r(t+x) + r(t-x)) - \frac{1}{2} \int_{t-x}^{t+x} r(\xi) (R_2(\xi - t, 0, x) - hR(\xi - t, 0, x)) d\xi.$$

The change of variables  $\tau = t - \xi$  leads to

$$u(x, t) = \frac{1}{2} (r(t+x) + r(t-x)) + \frac{1}{2} \int_{-x}^x r(t-\tau) G(x, \tau) d\tau, \quad (2.4.17)$$

where  $G(x, \tau) = -R_2(-\tau, 0, x) + hR(-\tau, 0, x)$ .

**Remark 2.4.3.** In (2.4.16) take  $r(t) = \cos \rho t$ . Obviously, the function  $u(x, t) = \varphi(x, \lambda) \cos \rho t$ , where  $\varphi(x, \lambda)$  was defined in Section 1.1, is a solution of problem (2.4.16). Therefore, (2.4.17) yields for  $t = 0$ ,

$$\varphi(x, \lambda) = \cos \rho x + \frac{1}{2} \int_{-x}^x G(x, \tau) \cos \rho \tau d\tau.$$

Since  $G(x, -\tau) = G(x, \tau)$ , we have

$$\varphi(x, \lambda) = \cos \rho x + \int_0^x G(x, \tau) \cos \rho \tau d\tau,$$

i.e. we obtained another derivation of the representation (1.3.11). Thus, the function  $G(x, t)$  is the kernel of the transformation operator. In particular, it was shown in Section 1.3 that

$$G(x, x) = h + \frac{1}{2} \int_0^x q(t) dt. \quad (2.4.18)$$

Let us go on to the solution of Inverse Problem 2.4.1. Let  $u(x, t)$  be the solution of the boundary value problem (2.4.1)-(2.4.2). We define  $u(x, t) = 0$  for  $0 \leq t < x$ , and  $u(x, t) = -u(x, -t)$ ,  $r(t) = -r(-t)$  for  $t < 0$ . Then  $u(x, t)$  is the solution of the Cauchy problem (2.4.16), and consequently (2.4.17) holds. But  $u(x, t) = 0$  for  $x > |t|$  (this is a connection between  $q(x)$  and  $r(t)$ ), and hence

$$\frac{1}{2} (r(t+x) + r(t-x)) + \frac{1}{2} \int_{-x}^x r(t-\tau) G(x, \tau) d\tau = 0, \quad |t| < x. \quad (2.4.19)$$

Denote  $a(t) = r'(t)$ . Differentiating (2.4.19) with respect to  $t$  and using the relations

$$r(0+) = 1, \quad r(0-) = -1, \quad (2.4.20)$$

we obtain

$$G(x, t) + F(x, t) + \int_0^x G(x, \tau)F(t, \tau) d\tau = 0, \quad 0 < t < x, \quad (2.4.21)$$

where

$$F(x, t) = \frac{1}{2} \left( a(t+x) + a(t-x) \right) \text{ with } a(t) = r'(t). \quad (2.4.22)$$

Equation (2.4.21) is called the Gelfand-Levitan equation.

**Theorem 2.4.2.** *For each fixed  $x > 0$ , equation (2.4.21) has a unique solution.*

*Proof.* Fix  $x_0 > 0$ . It is sufficient to prove that the homogeneous equation

$$g(t) + \int_0^{x_0} g(\tau)F(t, \tau) d\tau = 0, \quad 0 \leq t \leq x_0 \quad (2.4.23)$$

has only the trivial solution  $g(t) = 0$ .

Let  $g(t)$ ,  $0 \leq t \leq x_0$  be a solution of (2.4.23). Since  $a(t) = r'(t) \in D_1$ , it follows from (2.4.22) and (2.4.23) that  $g(t)$  is an absolutely continuous function on  $[0, x_0]$ . We define  $g(-t) = g(t)$  for  $t \in [0, x_0]$ , and furthermore  $g(t) = 0$  for  $|t| > x_0$ .

Let us show that

$$\int_{-x_0}^{x_0} r(t-\tau)g(\tau) d\tau = 0, \quad t \in [-x_0, x_0]. \quad (2.4.24)$$

Indeed, by virtue of (2.4.20) and (2.4.23), we have

$$\begin{aligned} \frac{d}{dt} \left( \int_{-x_0}^{x_0} r(t-\tau)g(\tau) d\tau \right) &= \frac{d}{dt} \left( \int_{-x_0}^t r(t-\tau)g(\tau) d\tau + \int_t^{x_0} r(t-\tau)g(\tau) d\tau \right) \\ &= r(+0)g(t) - r(-0)g(t) + \int_{-x_0}^{x_0} a(t-\tau)g(\tau) d\tau = 2 \left( g(t) + \int_0^{x_0} g(\tau)F(t, \tau) d\tau \right) = 0. \end{aligned}$$

Consequently,

$$\int_{-x_0}^{x_0} r(t-\tau)g(\tau) d\tau \equiv C_0.$$

Taking here  $t = 0$  and using that  $r(-\tau)g(\tau)$  is an odd function, we calculate  $C_0 = 0$ , i.e. (2.4.24) is valid.

Denote  $\Delta_0 = \{(x, t) : x - x_0 \leq t \leq x_0 - x, 0 \leq x \leq x_0\}$

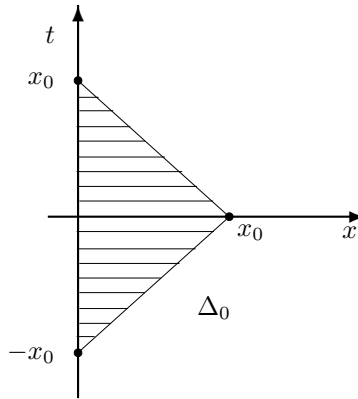


fig 2.4.2.

and consider the function

$$w(x, t) := \int_{-\infty}^{\infty} u(x, t - \tau)g(\tau) d\tau, \quad (x, t) \in \Delta_0, \quad (2.4.25)$$

where  $u(x, t)$  is the solution of the boundary value problem (2.4.1)-(2.4.2). Let us show that

$$w(x, t) = 0, \quad (x, t) \in \Delta_0. \quad (2.4.26)$$

Since  $u(x, t) = 0$  for  $x > |t|$ , (2.4.25) takes the form

$$w(x, t) = \int_{-\infty}^{t-x} u(x, t - \tau)g(\tau) d\tau + \int_{t+x}^{\infty} u(x, t - \tau)g(\tau) d\tau. \quad (2.4.27)$$

Differentiating (2.4.27) and using the relations

$$u(x, x) = 1, \quad u(x, -x) = -1, \quad (2.4.28)$$

we calculate

$$w_x(x, t) = g(t+x) - g(t-x) + \int_{-\infty}^{t-x} u_x(x, t - \tau)g(\tau) d\tau + \int_{t+x}^{\infty} u_x(x, t - \tau)g(\tau) d\tau, \quad (2.4.29)$$

$$w_t(x, t) = g(t+x) + g(t-x) + \int_{-\infty}^{t-x} u_t(x, t - \tau)g(\tau) d\tau + \int_{t+x}^{\infty} u_t(x, t - \tau)g(\tau) d\tau. \quad (2.4.30)$$

Since, in view of (2.4.28),

$$(u_x(x, t) \pm u_t(x, t))|_{t=\pm x} = \frac{d}{dx}u(x, \pm x) = 0,$$

it follows from (2.4.29)-(2.4.30) that

$$w_{xx}(x, t) - w_{tt}(x, t) - q(x)w(x, t) = \int_{-\infty}^{\infty} [u_{xx} - u_{tt} - q(x)u](x, t - \tau)g(\tau) d\tau,$$

and consequently

$$w_{tt}(x, t) = w_{xx}(x, t) - q(x)w(x, t), \quad (x, t) \in \Delta_0. \quad (2.4.31)$$

Furthermore, (2.4.25) and (2.4.29) yield

$$w(0, t) = \int_{-x_0}^{x_0} r(t - \tau)g(\tau) d\tau, \quad w_x(0, t) = h \int_{-x_0}^{x_0} r(t - \tau)g(\tau) d\tau = 0, \quad t \in [-x_0, x_0].$$

Therefore, according to (2.4.24), we have

$$w(0, t) = w_x(0, t) = 0, \quad t \in [-x_0, x_0]. \quad (2.4.32)$$

Since the Cauchy problem (2.4.31)-(2.4.32) has only the trivial solution, we arrive at (2.4.26).

Denote  $u_1(x, t) := u_t(x, t)$ . It follows from (2.4.30) that

$$\begin{aligned} w_t(x, 0) &= 2g(x) + \int_{-\infty}^{-x} u_1(x, \tau)g(\tau) d\tau + \int_x^{\infty} u_1(x, \tau)g(\tau) d\tau \\ &= 2\left(g(x) + \int_x^{\infty} u_1(x, \tau)g(\tau) d\tau\right) = 2\left(g(x) + \int_x^{x_0} u_1(x, \tau)g(\tau) d\tau\right). \end{aligned}$$

Taking (2.4.26) into account we get

$$g(x) + \int_x^{x_0} u_1(x, \tau) g(\tau) d\tau = 0, \quad 0 \leq x \leq x_0.$$

This integral equation has only the trivial solution  $g(x) = 0$ , and consequently Theorem 2.4.2 is proved.  $\square$

Let  $r$  and  $\tilde{r}$  be the traces for the boundary value problems  $B(q(x), h)$  and  $B(\tilde{q}(x), \tilde{h})$  respectively.

**Theorem 2.4.3.** *If  $r(t) = \tilde{r}(t)$ ,  $t \geq 0$ , then  $q(x) = \tilde{q}(x)$ ,  $x \geq 0$  and  $h = \tilde{h}$ . Thus, the specification of the trace  $r$  uniquely determines the potential  $q$  and the coefficient  $h$ .*

*Proof.* Since  $r(t) = \tilde{r}(t)$ ,  $t \geq 0$ , we have, by virtue of (2.4.22),

$$F(x, t) = \tilde{F}(x, t).$$

Therefore, Theorem 2.4.2 gives us

$$G(x, t) = \tilde{G}(x, t), \quad 0 \leq t \leq x. \quad (2.4.33)$$

By virtue of (2.4.18),

$$q(x) = 2 \frac{d}{dx} G(x, x), \quad h = G(0, 0) = -r'(0). \quad (2.4.34)$$

Together with (2.4.33) this yields  $q(x) = \tilde{q}(x)$ ,  $x \geq 0$  and  $h = \tilde{h}$ .  $\square$

The Gelfand-Levitan equation (2.4.21) and Theorem 2.4.2 yield finally the following algorithm for the solution of Inverse Problem 2.4.1.

**Algorithm 2.4.1.** Let  $r(t)$ ,  $t \geq 0$  be given. Then

- (1) Construct the function  $F(x, t)$  using (2.4.22).
- (2) Find the function  $G(x, t)$  by solving equation (2.4.21).
- (3) Calculate  $q(x)$  and  $h$  by the formulae (2.4.34).

**Remark 2.4.4.** We now explain briefly the connection of Inverse Problem 2.4.1 with the inverse spectral problem considered in Sections 2.2-2.3. Let  $q(x) \in L(0, \infty)$ , and let  $u(x, t)$  be the solution of (2.4.1)-(2.4.2). Then, according to (2.4.3),  $|u(x, t)| \leq C_1 \exp(C_2 t)$ . Denote

$$\begin{aligned} \Phi(x, \lambda) &= - \int_x^\infty u(x, t) \exp(i\rho t) dt, \quad x \geq 0, \quad \operatorname{Im} \rho \geq 0, \\ M(\lambda) &= - \int_0^\infty r(t) \exp(i\rho t) dt, \quad \operatorname{Im} \rho \geq 0. \end{aligned}$$

One can verify that  $\Phi(x, \lambda)$  and  $M(\lambda)$  are the Weyl solution and the Weyl function for the pair  $L = L(q(x), h)$  of the form (2.1.1)-(2.1.2). The inversion formula for the Laplace transform gives

$$r(t) = \frac{1}{2\pi} \int_{\gamma_1} M(\lambda) \exp(-i\rho t) d\rho = \frac{1}{2\pi i} \int_{\gamma_1} \frac{\sin \rho t}{\rho} (2\rho) M(\lambda) d\rho = \frac{1}{2\pi i} \int_{\gamma} \frac{\sin \rho t}{\rho} M(\lambda) d\lambda,$$

where  $\gamma$  is the contour defined in Section 2.1 (see fig. 2.1.1), and  $\gamma_1$  is the image of  $\gamma$  under the map  $\rho \rightarrow \lambda = \rho^2$ . Let  $\tilde{q}(x) = \tilde{h} = 0$ . Since then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{\sin \rho t}{\rho} \tilde{M}(\lambda) d\lambda = \tilde{r}(t) = 1,$$

we infer

$$r(t) = 1 + \frac{1}{2\pi i} \int_{\gamma} \frac{\sin \rho t}{\rho} \hat{M}(\lambda) d\lambda.$$

If (2.2.5) is valid we calculate

$$a(t) = \frac{1}{2\pi i} \int_{\gamma} \cos \rho t \hat{M}(\lambda) d\lambda,$$

and consequently

$$F(x, t) = \frac{1}{2\pi i} \int_{\gamma} \cos \rho x \cos \rho t \hat{M}(\lambda) d\lambda,$$

i.e. equation (2.4.21) coincides with equation (2.2.42).

Let us now formulate necessary and sufficient conditions for the solvability of Inverse Problem 2.4.1.

**Theorem 2.4.4.** *For a function  $r(t)$ ,  $t \geq 0$  to be the trace for a certain boundary value problem  $B(q(x), h)$  of the form (2.4.1)-(2.4.2) with  $q \in D_N$ , it is necessary and sufficient that  $r(t) \in D_{N+2}$ ,  $r(0) = 1$ , and that for each fixed  $x > 0$  the integral equation (2.4.21) is uniquely solvable.*

*Proof.* The necessity part of Theorem 2.4.4 was proved above, here we prove the sufficiency. For simplicity let  $N \geq 1$  (the case  $N = 0$  requires small modifications). Let a function  $r(t)$ ,  $t \geq 0$ , satisfying the hypothesis of Theorem 2.4.4, be given, and let  $G(x, t)$ ,  $0 \leq t \leq x$ , be the solution of (2.4.21). We define  $G(x, t) = G(x, -t)$ ,  $r(t) = -r(-t)$  for  $t < 0$ , and consider the function

$$u(x, t) := \frac{1}{2} (r(t+x) + r(t-x)) + \frac{1}{2} \int_{-x}^x r(t-\tau) G(x, \tau) d\tau, \quad -\infty < t < \infty, x \geq 0. \quad (2.4.35)$$

Furthermore, we construct  $q$  and  $h$  via (2.4.34) and consider the boundary value problem (2.4.1)-(2.4.2) with these  $q$  and  $h$ . Let  $\tilde{u}(x, t)$  be the solution of (2.4.1)-(2.4.2), and let  $\tilde{r}(t) := \tilde{u}(0, t)$ . Our goal is to prove that  $\tilde{u} = u$ ,  $\tilde{r} = r$ .

Differentiating (2.4.35) and taking (2.4.20) into account, we get

$$u_t(x, t) = \frac{1}{2} (a(t+x) + a(t-x)) + G(x, t) + \frac{1}{2} \int_{-x}^x a(t-\tau) G(x, \tau) d\tau, \quad (2.4.36)$$

$$\begin{aligned} u_x(x, t) &= \frac{1}{2} (a(t+x) - a(t-x)) + \frac{1}{2} (r(t-x) G(x, x) + r(t+x) G(x, -x)) \\ &\quad + \frac{1}{2} \int_{-x}^x r(t-\tau) G_x(x, \tau) d\tau. \end{aligned} \quad (2.4.37)$$

Since  $a(0+) = a(0-)$ , it follows from (2.4.36) that

$$u_{tt}(x, t) = \frac{1}{2} (a'(t+x) + a'(t-x)) + G_t(x, t) + \frac{1}{2} \int_{-x}^x a'(t-\tau) G(x, \tau) d\tau. \quad (2.4.38)$$

Integration by parts yields

$$\begin{aligned} u_{tt}(x, t) &= \frac{1}{2} \left( a'(t+x) + a'(t-x) \right) + G_t(x, t) - \frac{1}{2} \left( a(t-\tau)G(x, \tau) \Big|_{-x}^t + a(t-\tau)G(x, \tau) \Big|_t^x \right) \\ &\quad + \frac{1}{2} \int_{-x}^x a(t-\tau)G_t(x, \tau) d\tau = \frac{1}{2} \left( a'(t+x) + a'(t-x) \right) \\ &\quad + G_t(x, t) + \frac{1}{2} \left( a(t+x)G(x, -x) - a(t-x)G(x, x) \right) + \frac{1}{2} \int_{-x}^x r'(t-\tau)G_t(x, \tau) d\tau. \end{aligned}$$

Integrating by parts again and using (2.4.20) we calculate

$$\begin{aligned} u_{tt}(x, t) &= \frac{1}{2} \left( a'(t+x) + a'(t-x) \right) + G_t(x, t) + \frac{1}{2} \left( a(t+x)G(x, -x) - a(t-x)G(x, x) \right) \\ &\quad - \frac{1}{2} \left( r(t-\tau)G_t(x, \tau) \Big|_{-x}^t + r(t-\tau)G_t(x, \tau) \Big|_t^x \right) + \frac{1}{2} \int_{-x}^x r(t-\tau)G_{tt}(x, \tau) d\tau \\ &= \frac{1}{2} \left( a'(t+x) + a'(t-x) \right) + \frac{1}{2} \left( a(t+x)G(x, -x) - a(t-x)G(x, x) \right) \\ &\quad + \frac{1}{2} \left( r(t+x)G_t(x, -x) - r(t-x)G_t(x, x) \right) + \frac{1}{2} \int_{-x}^x r(t-\tau)G_{tt}(x, \tau) d\tau. \end{aligned} \quad (2.4.39)$$

Differentiating (2.4.37) we obtain

$$\begin{aligned} u_{xx}(x, t) &= \frac{1}{2} \left( a'(t+x) + a'(t-x) \right) + \frac{1}{2} \left( a(t+x)G(x, -x) - a(t-x)G(x, x) \right) \\ &\quad + \frac{1}{2} \left( r(t+x) \frac{d}{dx} G(x, -x) + r(t-x) \frac{d}{dx} G(x, x) \right) + \frac{1}{2} \left( r(t+x)G_x(x, -x) + r(t-x)G_x(x, x) \right) \\ &\quad + \frac{1}{2} \int_{-x}^x r(t-\tau)G_{xx}(x, \tau) d\tau. \end{aligned}$$

Together with (2.4.35), (2.4.39) and (2.4.34) this yields

$$u_{xx}(x, t) - q(x)u(x, t) - u_{tt}(x, t) = \frac{1}{2} \int_{-x}^x r(t-\tau)g(x, \tau) d\tau, \quad -\infty < t < \infty, \quad x \geq 0, \quad (2.4.40)$$

where

$$g(x, t) = G_{xx}(x, t) - G_{tt}(x, t) - q(x)G(x, t).$$

Let us show that

$$u(x, t) = 0, \quad x > |t|. \quad (2.4.41)$$

Indeed, it follows from (2.4.36) and (2.4.21) that  $u_t(x, t) = 0$  for  $x > |t|$ , and consequently  $u(x, t) \equiv C_0(x)$  for  $x > |t|$ . Taking  $t = 0$  in (2.4.35) we infer like above

$$C_0(x) = \frac{1}{2} \left( r(x) + r(-x) \right) + \frac{1}{2} \int_{-x}^x r(-\tau)G(x, \tau) d\tau = 0,$$

i.e. (2.4.41) holds.

It follows from (2.4.40) and (2.4.41) that

$$\frac{1}{2} \int_{-x}^x r(t-\tau)g(x, \tau) d\tau = 0, \quad x > |t|. \quad (2.4.42)$$

Differentiating (2.4.42) with respect to  $t$  and taking (2.4.20) into account we deduce

$$\frac{1}{2} \left( r(0+)g(x, t) - r(0-)g(x, t) \right) + \frac{1}{2} \int_{-x}^x a(t - \tau)g(x, \tau) d\tau = 0,$$

or

$$g(x, t) + \int_0^x F(t, \tau)g(x, \tau) d\tau = 0.$$

According to Theorem 2.4.2 this homogeneous equation has only the trivial solution  $g(x, t) = 0$ , i.e.

$$G_{tt} = G_{xx} - q(x)G, \quad 0 < |t| < x. \quad (2.4.43)$$

Furthermore, it follows from (2.4.38) for  $t = 0$  and (2.4.41) that

$$0 = \frac{1}{2} \left( a'(x) + a'(-x) \right) + G_t(x, 0) + \frac{1}{2} \int_{-x}^x a'(-\tau)G(x, \tau) d\tau.$$

Since  $a'(x) = -a'(-x)$ ,  $G(x, t) = G(x, -t)$ , we infer

$$\left. \frac{\partial G(x, t)}{\partial t} \right|_{t=0} = 0. \quad (2.4.44)$$

According to (2.4.34) the function  $G(x, t)$  satisfies also (2.4.18).

It follows from (2.4.40) and (2.4.43) that

$$u_{tt}(x, t) = u_{xx}(x, t) - q(x)u(x, t), \quad -\infty < t < \infty, x \geq 0.$$

Moreover, (2.4.35) and (2.4.37) imply (with  $h=G(0,0)$ )

$$u|_{x=0} = r(t), \quad u_x|_{x=0} = hr(t).$$

Let us show that

$$u(x, x) = 1, \quad x \geq 0. \quad (2.4.45)$$

Since the function  $G(x, t)$  satisfies (2.4.43), (2.4.44) and (2.4.18), we get according to (2.4.17),

$$\tilde{u}(x, t) = \frac{1}{2} \left( \tilde{r}(t+x) + \tilde{r}(t-x) \right) + \frac{1}{2} \int_{-x}^x \tilde{r}(t-\tau)G(x, \tau) d\tau. \quad (2.4.46)$$

Comparing (2.4.35) with (2.4.46) we get

$$\hat{u}(x, t) = \frac{1}{2} \left( \hat{r}(t+x) + \hat{r}(t-x) \right) + \frac{1}{2} \int_{-x}^x \hat{r}(t-\tau)G(x, \tau) d\tau,$$

where  $\hat{u} = u - \tilde{u}$ ,  $\hat{r} = r - \tilde{r}$ . Since the function  $\hat{r}(t)$  is continuous for  $-\infty < t < \infty$ , it follows that the function  $\hat{u}(x, t)$  is also continuous for  $-\infty < t < \infty$ ,  $x > 0$ . On the other hand, according to (2.4.41),  $\hat{u}(x, t) = 0$  for  $x > |t|$ , and consequently  $\hat{u}(x, x) = 0$ . By (2.4.2),  $\tilde{u}(x, x) = 1$ , and we arrive at (2.4.45).

Thus, the function  $u(x, t)$  is a solution of the boundary value problem (2.4.1)-(2.4.2). By virtue of Theorem 2.4.1 we obtain  $u(x, t) = \tilde{u}(x, t)$ , and consequently  $r(t) = \tilde{r}(t)$ . Theorem 2.4.4 is proved.  $\square$

## 2.5. THE GENERALIZED WEYL FUNCTION

Let us consider the differential equation and the linear form  $L = L(q(x), h)$  :

$$\ell y := -y'' + q(x)y = \lambda y, \quad x > 0,$$

$$U(y) := y'(0) - hy(0).$$

In this section we study the inverse spectral problem for  $L$  in the case when  $q(x)$  is a locally integrable complex-valued function, and  $h$  is a complex number. In this case we introduce the so-called generalized Weyl function as a main spectral characteristics.

For this purpose we define a space of generalized functions (distributions). Let  $D$  be the set of all integrable and bounded on the real line entire functions of exponential type with ordinary operations of addition and multiplication by complex numbers and with the following convergence:  $z_k(\rho)$  is said to converge to  $z(\rho)$  if the types  $\sigma_k$  of the functions  $z_k(\rho)$  are bounded ( $\sup \sigma_k < \infty$ ), and  $\|z_k(\rho) - z(\rho)\|_{L(-\infty, \infty)} \rightarrow 0$  as  $k \rightarrow \infty$ . The linear manifold  $D$  with this convergence is our space of test functions.

**Definition 2.5.1.** All linear and continuous functionals

$$R : D \rightarrow \mathbb{C}, \quad z(\rho) \mapsto R(z(\rho)) = (z(\rho), R),$$

are called generalized functions (GF). The set of these GF is denoted by  $D'$ . A sequence of GF  $R_k \in D'$  converges to  $R \in D'$ , if  $\lim(z(\rho), R_k) = (z(\rho), R)$ ,  $k \rightarrow \infty$  for any  $z(\rho) \in D$ . A GF  $R \in D'$  is called regular if it is determined by  $R(\rho) \in L_\infty$  via

$$(z(\rho), R) = \int_{-\infty}^{\infty} z(\rho) R(\rho) d\rho.$$

**Definition 2.5.2.** Let a function  $f(t)$  be locally integrable for  $t > 0$  (i.e. it is integrable on every finite segment  $[0, T]$ ). The GF  $L_f(\rho) \in D'$  defined by the equality

$$(z(\rho), L_f(\rho)) := \int_0^{\infty} f(t) \left( \int_{-\infty}^{\infty} z(\rho) \exp(i\rho t) d\rho \right) dt, \quad z(\rho) \in D, \quad (2.5.1)$$

is called the generalized Fourier-Laplace transform for the function  $f(t)$ .

Since  $z(\rho) \in D$ , we have

$$\int_{-\infty}^{\infty} |z(\rho)|^2 d\rho \leq \sup_{-\infty < \rho < \infty} |z(\rho)| \cdot \int_{-\infty}^{\infty} |z(\rho)| d\rho,$$

i.e.  $z(\rho) \in L_2(-\infty, \infty)$ . Therefore, by virtue of the Paley-Wiener theorem [zyg1], the function

$$B(t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} z(\rho) \exp(i\rho t) d\rho$$

is continuous and has compact support, i.e. there exists a  $d > 0$  such that  $B(t) = 0$  for  $|t| > d$ , and

$$z(\rho) = \int_{-d}^d B(t) \exp(-i\rho t) dt. \quad (2.5.2)$$

Consequently, the integral in (2.5.1) exists. We note that  $f(t) \in L(0, \infty)$  implies

$$(z(\rho), L_f(\rho)) := \int_{-\infty}^{\infty} z(\rho) \left( \int_0^{\infty} f(t) \exp(i\rho t) dt \right) d\rho,$$



i.e.  $L_f(\rho)$  is a regular GF (defined by  $\int_0^\infty f(t) \exp(i\rho t) dt$ ) and coincides with the ordinary Fourier-Laplace transform for the function  $f(t)$ . Since

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1 - \cos \rho x}{\rho^2} \exp(i\rho t) d\rho = \begin{cases} x - t, & t < x, \\ 0, & t > x, \end{cases}$$

the following inversion formula is valid:

$$\int_0^x (x - t) f(t) dt = \left( \frac{1}{\pi} \cdot \frac{1 - \cos \rho x}{\rho^2}, L_f(\rho) \right). \quad (2.5.3)$$

Let now  $u(x, t)$  be the solution of (2.4.1)-(2.4.2) with a locally integrable complex-valued function  $q(x)$ . Define  $u(x, t) = 0$  for  $0 < t < x$ , and denote (with  $\lambda = \rho^2$ )  $\Phi(x, \lambda) := -L_u(\rho)$ , i.e.

$$(z(\rho), \Phi(x, \lambda)) = - \int_x^\infty u(x, t) \left( \int_{-\infty}^\infty z(\rho) \exp(i\rho t) d\rho \right) dt. \quad (2.5.4)$$

For  $z(\rho) \in D$ ,  $\rho^2 z(\rho) \in L(-\infty, \infty)$ ,  $\nu = 1, 2$ , we put

$$(z(\rho), (i\rho)^\nu \Phi(x, \lambda)) := ((i\rho)^\nu z(\rho), \Phi(x, \lambda)),$$

$$(z(\rho), \Phi^{(\nu)}(x, \lambda)) := \frac{d^\nu}{dx^\nu} (z(\rho), \Phi(x, \lambda)).$$

**Theorem 2.5.1.** *The following relations hold*

$$\ell \Phi(x, \lambda) = \lambda \Phi(x, \lambda), \quad U(\Phi) = 1.$$

*Proof.* We calculate

$$\begin{aligned} (z(\rho), \ell \Phi(x, \lambda)) &= (z(\rho), -\Phi''(x, \lambda) + q(x) \Phi(x, \lambda)) \\ &= - \int_{-\infty}^\infty (i\rho) z(\rho) \exp(i\rho x) d\rho - u_x(x, x) \int_{-\infty}^\infty z(\rho) \exp(i\rho x) d\rho \\ &\quad + \int_x^\infty (u_{xx}(x, t) - q(x) u(x, t)) \left( \int_{-\infty}^\infty z(\rho) \exp(i\rho t) d\rho \right) dt, \\ (z(\rho), \lambda \Phi(x, \lambda)) &= - \int_{-\infty}^\infty (i\rho) z(\rho) \exp(i\rho x) d\rho - u_t(x, x) \int_{-\infty}^\infty z(\rho) \exp(i\rho x) d\rho \\ &\quad + \int_x^\infty u_{tt}(x, t) \left( \int_{-\infty}^\infty z(\rho) \exp(i\rho t) d\rho \right) dt. \end{aligned}$$

Using  $u_t(x, x) + u_x(x, x) = \frac{d}{dx} u(x, x) \equiv 0$ , we infer  $(z(\rho), \ell \Phi(x, \lambda) - \lambda \Phi(x, \lambda)) = 0$ . Furthermore, since

$$(z(\rho), \Phi'(x, \lambda)) = \int_{-\infty}^\infty z(\rho) \exp(i\rho x) d\rho - \int_x^\infty u_x(x, t) \left( \int_{-\infty}^\infty z(\rho) \exp(i\rho t) d\rho \right) dt,$$

we get

$$(z(\rho), U(\Phi)) = (z(\rho), \Phi'(0, \lambda) - h \Phi(0, \lambda)) = \int_{-\infty}^\infty z(\rho) d\rho.$$

□

**Definition 2.5.3.** The GF  $\Phi(x, \lambda)$  is called the *generalized Weyl solution*, and the GF  $M(\lambda) := \Phi(0, \lambda)$  is called the *generalized Weyl function (GWF)* for  $L(q(x), h)$ .

Note that if  $q(x) \in L(0, \infty)$ , then  $|u(x, t)| \leq C_1 \exp(C_2 t)$ , and  $\Phi(x, \lambda)$  and  $M(\lambda)$  coincide with the ordinary Weyl solution and Weyl function (see Remark 2.4.4).

The inverse problem considered here is formulated as follows:

**Inverse Problem 2.5.1.** Given the GWF  $M(\lambda)$ , construct the potential  $q(x)$  and the coefficient  $h$ .

Denote  $r(t) := u(0, t)$ . It follows from (2.5.4) that

$$(z(\rho), M(\lambda)) = - \int_0^\infty r(t) \left( \int_{-\infty}^\infty z(\rho) \exp(i\rho t) d\rho \right) dt,$$

i.e.  $M(\lambda) = -L_r(\rho)$ . In view of (2.5.3), we get by differentiation

$$r(t) = - \frac{d^2}{dt^2} \left( \frac{1}{\pi} \cdot \frac{1 - \cos \rho t}{\rho^2}, M(\lambda) \right), \quad (2.5.5)$$

and Inverse Problem 2.5.1 has been reduced to Inverse Problem 2.4.1 from the trace  $r$  considered in Section 2.4. Thus, the following theorems hold.

**Theorem 2.5.2.** Let  $M(\lambda)$  and  $\tilde{M}(\lambda)$  be the GWF's for  $L = L(q(x), h)$  and  $\tilde{L} = L(\tilde{q}(x), \tilde{h})$  respectively. If  $M(\lambda) = \tilde{M}(\lambda)$ , then  $L = \tilde{L}$ . Thus, the specification of the GWF uniquely determines the potential  $q$  and the coefficient  $h$ .

**Theorem 2.5.3.** Let  $\varphi(x, \lambda)$  be the solution of the differential equation  $\ell\varphi = \lambda\varphi$  under the initial conditions  $\varphi(0, \lambda) = 1$ ,  $\varphi'(0, \lambda) = h$ . Then the following representation holds

$$\varphi(x, \lambda) = \cos \rho x + \int_0^x G(x, t) \cos \rho t dt,$$

and the function  $G(x, t)$  satisfies the integral equation

$$G(x, t) + F(x, t) + \int_0^x G(x, \tau) F(t, \tau) d\tau = 0, \quad 0 < t < x, \quad (2.5.6)$$

where

$$F(x, t) = \frac{1}{2} (r'(t+x) + r'(t-x)).$$

The function  $r$  is defined via (2.5.5), and  $r \in D_2$ . If  $q \in D_N$  then  $r \in D_{N+2}$ . Moreover, for each fixed  $x > 0$ , the integral equation (2.5.6) is uniquely solvable.

**Theorem 2.5.4.** For a generalized function  $M \in D'$  to be the GWF for a certain  $L(q(x), h)$  with  $q \in D_N$ , it is necessary and sufficient that

- 1)  $r \in D_{N+2}$ ,  $r(0) = 1$ , where  $r$  is defined via (2.5.5);
- 2) for each  $x > 0$ , the integral equation (2.5.6) is uniquely solvable.

The potential  $q$  and the coefficient  $h$  can be constructed by the following algorithm.

**Algorithm 2.5.1.** Let the GWF  $M(\lambda)$  be given. Then

- (1) Construct the function  $r(t)$  by (2.5.5).

- (2) Find the function  $G(x, t)$  by solving the integral equation (2.5.6).  
 (3) Calculate  $q(x)$  and  $h$  by

$$q(x) = 2 \frac{dG(x, x)}{dx}, \quad h = G(0, 0).$$

Let us now prove an expansion theorem for the case of locally integrable complex-valued potentials  $q$ .

**Theorem 2.5.4.** *Let  $f(x) \in W_2$ . Then, uniformly on compact sets,*

$$f(x) = \frac{1}{\pi} \left( \varphi(x, \lambda) F(\lambda)(i\rho), M(\lambda) \right), \quad (2.5.7)$$

where

$$F(\lambda) = \int_0^\infty f(t) \varphi(t, \lambda) dt. \quad (2.5.8)$$

*Proof.* First we assume that  $q(x) \in L(0, \infty)$ . Let  $f(x) \in Q$ , where  $Q = \{f \in W_2 : U(f) = 0, \ell f \in L_2(0, \infty)\}$  (the general case when  $f \in W_2$  requires small modifications).

Let  $D^+ = \{z(\rho) \in D : \rho z(\rho) \in L_2(-\infty, \infty)\}$ . Clearly,  $z(\rho) \in D^+$  if and only if  $B(t) \in W_2^1[-d, d]$  in (2.5.2). For  $z(\rho) \in D^1$ , integration by parts in (2.5.2) yields

$$z(\rho) = \int_{-d}^d B(t) \exp(-i\rho t) dt = \frac{1}{i\rho} \int_{-d}^d B'(t) \exp(-i\rho t) dt.$$

Using (2.5.8) we calculate

$$F(\lambda) = \frac{1}{\lambda} \int_0^\infty f(t) \left( -\varphi''(t, \lambda) + q(t) \varphi(t, \lambda) \right) dt = \frac{1}{\lambda} \int_0^\infty \varphi(t, \lambda) \ell f(t) dt,$$

and consequently  $F(\lambda)(i\rho) \in D^+$ . According to Theorem 2.1.8 we have

$$f(x) = \frac{1}{2\pi i} \int_\gamma \varphi(x, \lambda) F(\lambda) M(\lambda) d\lambda = -\frac{1}{\pi} \int_{\gamma_1} \varphi(x, \lambda) F(\lambda)(i\rho) M(\lambda) d\rho, \quad (2.5.9)$$

where the contour  $\gamma_1$  in the  $\rho$ -plane is the image of  $\gamma$  under the mapping  $\rho \rightarrow \lambda = \rho^2$ . In view of Remark 2.4.4,

$$M(\lambda) = -\int_0^\infty r(t) \exp(i\rho t) dt, \quad |r(t)| \leq C_1 \exp(C_2 t). \quad (2.5.10)$$

Take  $b > C_2$ . Then, by virtue of Cauchy's theorem, (2.5.9)-(2.5.10) imply

$$\begin{aligned} f(x) &= \frac{1}{\pi} \int_{-\infty+ib}^{\infty+ib} \varphi(x, \lambda) F(\lambda)(i\rho) \left( -\int_0^\infty r(t) \exp(i\rho t) dt \right) d\rho \\ &= -\frac{1}{\pi} \int_0^\infty r(t) \left( \int_{-\infty+ib}^{\infty+ib} \varphi(x, \lambda) F(\lambda)(i\rho) \exp(i\rho t) d\rho \right) dt. \end{aligned}$$

Using Cauchy's theorem again we get

$$f(x) = -\frac{1}{\pi} \int_0^\infty r(t) \left( \int_{-\infty}^\infty \varphi(x, \lambda) F(\lambda)(i\rho) \exp(i\rho t) d\rho \right) dt = \frac{1}{\pi} \left( \varphi(x, \lambda) F(\lambda)(i\rho), M(\lambda) \right),$$

i.e. (2.5.7) is valid.

Let now  $q(x)$  be a locally integrable complex-valued function. Denote

$$q_R(x) = \begin{cases} q(x), & 0 \leq x \leq R, \\ 0, & x > R. \end{cases}$$

Let  $r_R(t)$  be the trace for the potential  $q_R$ . According to Remark 2.4.2,

$$r_R(t) = r(t) \quad \text{for } t \leq 2R. \quad (2.5.11)$$

Since  $q_R(x) \in L(0, \infty)$  we have by virtue of (2.5.7),

$$f(x) = -\frac{1}{\pi} \int_0^\infty r_R(t) \left( \int_{-\infty}^\infty \varphi(x, \lambda) F(\lambda)(i\rho) \exp(i\rho t) d\rho \right) dt.$$

Let  $x \in [0, T]$  for a certain  $T > 0$ . Then there exists a  $d > 0$  such that

$$f(x) = -\frac{1}{\pi} \int_0^d r_R(t) \left( \int_{-\infty}^\infty \varphi(x, \lambda) F(\lambda)(i\rho) \exp(i\rho t) d\rho \right) dt, \quad 0 \leq x \leq T.$$

For sufficiently large  $R$  ( $R > d/2$ ) we have in view of (2.5.11),

$$\begin{aligned} f(x) &= -\frac{1}{\pi} \int_0^d r(t) \left( \int_{-\infty}^\infty \varphi(x, \lambda) F(\lambda)(i\rho) \exp(i\rho t) d\rho \right) dt \\ &= -\frac{1}{\pi} \int_0^\infty r(t) \left( \int_{-\infty}^\infty \varphi(x, \lambda) F(\lambda)(i\rho) \exp(i\rho t) d\rho \right) dt \\ &= \frac{1}{\pi} \left( \varphi(x, \lambda) F(\lambda)(i\rho), M(\lambda) \right), \end{aligned}$$

i.e. (2.5.7) is valid, and Theorem 2.5.4 is proved.  $\square$

## 2.6. THE WEYL SEQUENCE

We consider in this section the differential equation and the linear form

$$\ell y := -y'' + q(x)y = \lambda y, \quad x > 0, \quad U_0(y) := y(0).$$

Let  $\lambda = \rho^2$ ,  $\operatorname{Im} \rho > 0$ . Then the Weyl solution  $\Phi_0(x, \lambda)$  and the Weyl function  $M_0(\lambda)$  are defined by the conditions

$$\Phi_0(0, \lambda) = 1, \quad \Phi_0(x, \lambda) = O(\exp(i\rho x)), \quad x \rightarrow \infty, \quad M_0(\lambda) = \Phi_0'(0, \lambda),$$

and (2.1.81) is valid. Let the function  $q$  be analytic at the point  $x = 0$  (in this case we shall write in the sequel  $q \in A$ ). It follows from (2.1.21) and (2.1.81) that for  $q \in A$  the following relations are valid

$$\Phi_0(x, \lambda) = \exp(i\rho x) \left( 1 + \sum_{k=1}^\infty \frac{b_k(x)}{(2i\rho)^k} \right), \quad |\rho| \rightarrow \infty, \quad (2.6.1)$$

$$M_0(\lambda) = i\rho + \sum_{k=1}^\infty \frac{M_k}{(2i\rho)^k}, \quad |\rho| \rightarrow \infty, \quad (2.6.2)$$

in the sense of asymptotic formulae. The sequence  $m := \{M_k\}_{k \geq 1}$  is called the *Weyl sequence* for the potential  $q$ .

Substituting (2.6.1)-(2.6.2) into the relations

$$-\Phi_0''(x, \lambda) + q(x)\Phi_0(x, \lambda) = \lambda\Phi_0(x, \lambda), \quad \Phi_0(0, \lambda) = 1, \quad \Phi_0'(0, \lambda) = M_0(\lambda),$$

we obtain the following recurrent formulae for calculating  $b_k(x)$  and  $M_k$ :

$$b_{k+1}'(x) = -b_k''(x) + q(x)b_k(x), \quad k \geq 0, \quad b_0(x) := 1, \quad (2.6.3)$$

$$b_k(0) = 0, \quad M_k = b_k'(0), \quad k \geq 1. \quad (2.6.4)$$

Hence

$$b_{k+1}(x) = b_k'(0) - b_k'(x) + \int_0^x q(t)b_k(t) dt, \quad k \geq 0.$$

In particular, this gives

$$b_1(x) = \int_0^x q(t) dt, \quad b_2(x) = q(0) - q(x) + \frac{1}{2} \left( \int_0^x q(t) dt \right)^2,$$

$$b_3(x) = q'(x) - q'(0) + (q(x) + q(0)) \int_0^x q(t) dt - \int_0^x q^2(t) dt + \frac{1}{3!} \left( \int_0^x q(t) dt \right)^3, \dots$$

Moreover, (2.6.3)-(2.6.4) imply

$$M_1 = q(0), \quad M_2 = -q'(0), \quad M_3 = q''(0) - q^2(0), \quad M_4 = -q'''(0) + 4q(0)q'(0),$$

$$M_5 = q^{(4)}(0) - 6q''(0)q(0) - 5(q'(0))^2 + 2q^3(0), \dots$$

We note that the Weyl sequence  $m = \{M_k\}_{k \geq 1}$  depends only on the Taylor coefficients  $Q = \{q^{(j)}(0)\}_{j \geq 0}$  of the potential  $q$  at the point  $x = 0$ , and this dependence is nonlinear.

In this section we study the following inverse problem

**Inverse Problem 2.6.1.** Given the Weyl sequence  $m := \{M_k\}_{k \geq 1}$ , construct  $q$ .

We prove the uniqueness theorem for Inverse problem 2.6.1, provide an algorithm for the solution of this inverse problem and give necessary and sufficient conditions for its solvability. We note that the specification of the Weyl sequence allows us to construct  $q$  in a neighbourhood of the original. However, if  $q(x)$  is analytic for  $x \geq 0$ , then one can obtain the global solution.

The following theorem gives us necessary and sufficient conditions on the Weyl sequence.

**Theorem 2.6.1.** 1) If  $q \in A$ , then there exists a  $\delta > 0$  such that

$$M_k = O\left(\left(\frac{k}{\delta}\right)^k\right) \quad \text{for } k \geq 0. \quad (2.6.5)$$

2) Let an arbitrary sequence  $\{M_k\}_{k \geq 1}$ , satisfying (2.6.5) for a certain  $\delta > 0$ , be given. Then there exists a unique function  $q \in A$  for which  $\{M_k\}_{k \geq 1}$  is the corresponding Weyl sequence.

First we prove auxiliary assertions. Denote

$$C_k^\nu = \binom{k}{\nu} = \frac{k!}{\nu!(k-\nu)!}, \quad 0 \leq \nu \leq k.$$

**Lemma 2.6.1.** *The following relation holds*

$$C_k^\nu \leq \frac{k^k}{\nu^\nu (k-\nu)^{k-\nu}}, \quad 0 \leq \nu \leq k, \quad (2.6.6)$$

where we put  $0^0 = 1$ .

*Proof.* By virtue of Stirling's formula [Hen1, p.42],

$$\sqrt{2\pi k} \left(\frac{k}{e}\right)^k \leq k! \leq 1.1 \sqrt{2\pi k} \left(\frac{k}{e}\right)^k, \quad k \geq 1. \quad (2.6.7)$$

For  $\nu = 0$  and  $\nu = k$ , (2.6.6) is obvious. Let  $1 \leq \nu \leq k-1$ . Then, using (2.6.7) we calculate

$$C_k^\nu = \frac{k!}{\nu! (k-\nu)!} \leq \frac{1.1}{\sqrt{2\pi}} \cdot \frac{k^k}{\nu^\nu (k-\nu)^{k-\nu}} \cdot \sqrt{\frac{k}{\nu(k-\nu)}}.$$

It is easy to check that

$$\frac{(1.1)^2}{2\pi} \cdot \frac{k}{\nu(k-\nu)} < 1, \quad 1 \leq \nu \leq k-1,$$

and consequently, (2.6.6) is valid.  $\square$

**Lemma 2.6.2.** *The following relations hold*

$$\frac{(k-s)^{k-s}}{k^k} \sum_{j=1}^{k-s} \frac{(j+s-2)^{j+s-2}}{j^j} \leq 1, \quad k-1 \geq s \geq 1, \quad (2.6.8)$$

$$\sum_{s=1}^{k-1} \frac{(k-s)^{k-s}}{k^k} \sum_{j=1}^{k-s} \frac{(j+s-2)^{j+s-2}}{j^j} \leq 3, \quad k \geq 2. \quad (2.6.9)$$

*Proof.* For  $s = 1$  and  $s = 2$ , (2.6.8) is obvious since

$$\frac{(j+s-2)^{j+s-2}}{j^j} \leq 1.$$

Let now  $s \geq 3$ . Since the function

$$f(x) = \frac{(x+a)^{x+a}}{x^x}$$

is monotonically increasing for  $x > 0$ , we get

$$\frac{(k-s)^{k-s}}{k^k} \sum_{j=1}^{k-s} \frac{(j+s-2)^{j+s-2}}{j^j} \leq \frac{(k-s)^{k-s}}{k^k} \frac{(k-2)^{k-2}}{(k-s)^{k-s}} (k-s) \leq \frac{(k-2)^{k-1}}{k^k} < 1,$$

i.e. (2.6.8) is valid. Furthermore,

$$\sum_{s=3}^{k-1} \frac{(k-s)^{k-s}}{k^k} \sum_{j=1}^{k-s} \frac{(j+s-2)^{j+s-2}}{j^j} \leq \frac{(k-2)^{k-2}}{k^k} \sum_{s=3}^{k-1} (k-s)$$

$$= \frac{(k-2)^{k-2}}{k^k} \cdot \frac{(k-3)(k-2)}{2} < \frac{1}{2},$$

and consequently (2.6.9) is valid.  $\square$

*Proof of Theorem 2.6.1.* 1) Let  $q \in A$ . We denote  $q_k = q^{(k)}(0)$ ,  $b_{k\nu} = b_k^{(\nu)}(0)$ . Differentiating  $k - \nu$  times the equality  $b'_{\nu+1}(x) = -b''_\nu(x) + q(x)b_\nu(x)$  and inserting then  $x = 0$  we get together with (2.6.4),

$$\left. \begin{aligned} b_{\nu+1, k-\nu+1} &= -b_{\nu, k-\nu+2} + \sum_{j=0}^{k-\nu} C_{k-\nu}^j q_{k-\nu-j} b_{\nu j}, \quad 0 \leq \nu \leq k, \\ b_{k+1, 0} &= 0, \quad b_{k+1, 1} = M_{k+1}, \quad b_{0, k} = \delta_{0k}, \quad k \geq 0, \end{aligned} \right\} \quad (2.6.10)$$

where  $\delta_{jk}$  is the Kronecker delta. Since  $q \in A$ , there exist  $\delta_0 > 0$  and  $C > 0$  such that

$$|q_k| \leq C \left( \frac{k}{\delta_0} \right)^k.$$

We denote  $a = 1 + C\delta_0^2$ . Let us show by induction with respect to  $\nu$  that

$$|b_{\nu+1, k-\nu+1}| \leq Ca^\nu \left( \frac{k}{\delta_0} \right)^k, \quad 0 \leq \nu \leq k. \quad (2.6.11)$$

For  $\nu = 0$  (2.6.11) is obvious by virtue of  $b_{1, k+1} = q_k$ . We assume that (2.6.11) holds for  $\nu = 1, \dots, s-1$ . Then, using (2.6.10) and Lemma 2.6.1 we obtain

$$\begin{aligned} |b_{s+1, k-s+1}| &\leq Ca^{s-1} \left( \frac{k}{\delta_0} \right)^k + \sum_{j=1}^{k-s} \frac{(k-s)^{k-s}}{j^j (k-s-j)^{k-s-j}} \cdot C \left( \frac{k-s-j}{\delta_0} \right)^{k-s-j} \cdot Ca^{s-1} \left( \frac{s+j-2}{\delta_0} \right)^{s+j-2} \\ &= Ca^{s-1} \left( \frac{k}{\delta_0} \right)^k \left( 1 + C\delta_0^2 \frac{(k-s)^{k-s}}{k^k} \sum_{j=1}^{k-s} \frac{(s+j-2)^{s+j-2}}{j^j} \right). \end{aligned}$$

By virtue of (2.6.8) this yields

$$|b_{s+1, k-s+1}| \leq Ca^{s-1} \left( \frac{k}{\delta_0} \right)^k (1 + C\delta_0^2) = Ca^s \left( \frac{k}{\delta_0} \right)^k,$$

i.e. (2.6.11) holds for  $\nu = s$ .

It follows from (2.6.11) for  $\nu = k$  that

$$|M_{k+1}| \leq Ca^k \left( \frac{k}{\delta_0} \right)^k, \quad k \geq 0,$$

and consequently (2.6.5) holds for  $\delta = \delta_0/a$ .

2) Let the sequence  $\{M_k\}_{k \geq 1}$ , satisfying (2.6.5) for a certain  $\delta > 0$ , be given. Solving (2.6.10) successively for  $k = 0, 1, 2, \dots$  with respect to  $q_k$  and  $\{b_{\nu+1, k-\nu+1}\}_{\nu=0, \overline{k}}$ , we calculate

$$b_{k-j+1, j+1} = (-1)^j M_{k+1} + \sum_{\nu=1}^{j-1} \sum_{\xi=1}^{\nu} (-1)^{j-\nu-1} C_{\nu}^{\xi} q_{\nu-\xi} b_{k-\nu, \xi}, \quad 0 \leq j \leq k, \quad (2.6.12)$$

$$q_k = (-1)^k M_{k+1} + \sum_{\nu=1}^{k-1} \sum_{\xi=1}^{\nu} (-1)^{k-\nu-1} C_{\nu}^{\xi} q_{\nu-\xi} b_{k-\nu,\xi}. \quad (2.6.13)$$

By virtue of (2.6.5) there exist  $C > 0$  and  $\delta_1 > 0$  such that

$$|M_{k+1}| \leq C \left( \frac{k}{\delta_1} \right)^k. \quad (2.6.14)$$

Let us show by induction with respect to  $k \geq 0$  that

$$|q_k| \leq C a_0^k \left( \frac{k}{\delta_1} \right)^k, \quad (2.6.15)$$

$$|b_{k-j+1,j+1}| \leq C a_0^k \left( \frac{k}{\delta_1} \right)^k, \quad 0 \leq j \leq k, \quad (2.6.16)$$

where  $a_0 = \max(2, \sqrt{6C}\delta_1)$ .

Since  $q_0 = b_{11} = M_1$ , (2.6.15)-(2.6.16) are obvious for  $k = 0$ . Suppose that (2.6.15)-(2.6.16) hold for  $k = 0, \dots, n-1$ . Then, using Lemma 2.6.1 and (2.6.13)-(2.6.14) for  $k = n$ , we obtain

$$\begin{aligned} |q_n| &\leq C \left( \frac{n}{\delta_1} \right)^n + \sum_{\nu=1}^{n-1} \sum_{\xi=1}^{\nu} \frac{\nu^{\nu}}{\xi^{\xi} (\nu - \xi)^{\nu - \xi}} \cdot C a_0^{\nu - \xi} \left( \frac{\nu - \xi}{\delta_1} \right)^{\nu - \xi} \cdot C a_0^{n - \nu + \xi - 2} \left( \frac{n - \nu + \xi - 2}{\delta_1} \right)^{n - \nu + \xi - 2} \\ &= C \left( \frac{n}{\delta_1} \right)^n a_0^n \left( \frac{1}{a_0^n} + \frac{C \delta_1^2}{a_0^2} \sum_{\nu=1}^{n-1} \frac{\nu^{\nu}}{n^n} \sum_{\xi=1}^{\nu} \frac{(n - \nu + \xi - 2)^{n - \nu + \xi - 2}}{\xi^{\xi}} \right). \end{aligned}$$

By virtue of (2.6.9),

$$\sum_{\nu=1}^{n-1} \frac{\nu^{\nu}}{n^n} \sum_{\xi=1}^{\nu} \frac{(n - \nu + \xi - 2)^{n - \nu + \xi - 2}}{\xi^{\xi}} = \sum_{s=1}^{n-1} \frac{(n - s)^{n-s}}{n^n} \sum_{\xi=1}^{n-s} \frac{(s + \xi - 2)^{s + \xi - 2}}{\xi^{\xi}} \leq 3,$$

and consequently

$$|q_n| \leq C \left( \frac{n}{\delta_1} \right)^n a_0^n \left( \frac{1}{a_0^n} + \frac{3C \delta_1^2}{a_0^2} \right) \leq C \left( \frac{n}{\delta_1} \right)^n a_0^n,$$

i.e. (2.6.15) holds for  $k = n$ . Similarly one can obtain that (2.6.16) also holds for  $k = n$ .

In particular, it follows from (2.6.15) that for  $k \geq 0$ ,

$$q_k = O\left(\left(\frac{k}{\delta_2}\right)^k\right), \quad \delta_2 = \frac{\delta_1}{a_0}.$$

We construct the function  $q \in A$  by

$$q(x) = \sum_{k=0}^{\infty} q_k \frac{x^k}{k!}.$$

It is easy to verify that  $\{M_k\}_{k \geq 1}$  is the Weyl sequence for this function. Notice that the uniqueness of the solution of the inverse problem is obvious. Theorem 2.6.1 is proved.  $\square$

**Remark 2.6.1.** The relations (2.6.12)-(2.6.13) give us an algorithm for the solution of Inverse Problem 2.6.1. Moreover, it follows from (2.6.12)-(2.6.13) that the specification of the Weyl sequence  $m := \{M_k\}_{k \geq 1}$  uniquely determines  $q$  in a neighbourhood of the origin.



### III. INVERSE SCATTERING ON THE LINE

In this chapter the inverse scattering problem for the Sturm-Liouville operator on the line is considered. In Section 3.1 we introduce the scattering data and study their properties. In Section 3.2, using the transformation operator method, we give a derivation of the so-called main equation and prove its unique solvability. In Section 3.3, using the main equation, we provide an algorithm for the solution of the inverse scattering problem along with necessary and sufficient conditions for its solvability. In Section 3.4 a class of reflectionless potentials, which is important for applications, is studied, and an explicit formula for constructing such potentials is given. We note that the inverse scattering problem for the Sturm-Liouville operator on the line was considered by many authors (see, for example, [mar1], [lev2], [fad1], [dei1]).

#### 3.1. SCATTERING DATA

**3.1.1.** Let us consider the differential equation

$$\ell y := -y'' + q(x)y = \lambda y, \quad -\infty < x < \infty. \quad (3.1.1)$$

Everywhere below in this chapter we will assume that the function  $q(x)$  is real, and that

$$\int_{-\infty}^{\infty} (1 + |x|)|q(x)| dx < \infty. \quad (3.1.2)$$

Let  $\lambda = \rho^2$ ,  $\rho = \sigma + i\tau$ , and let for definiteness  $\tau := \operatorname{Im} \rho \geq 0$ . Denote  $\Omega_+ = \{\rho : \operatorname{Im} \rho > 0\}$ ,

$$\begin{aligned} Q_0^+(x) &= \int_x^\infty |q(t)| dt, & Q_1^+(x) &= \int_x^\infty Q_0^+(t) dt = \int_x^\infty (t-x)|q(t)| dt, \\ Q_0^-(x) &= \int_{-\infty}^x |q(t)| dt, & Q_1^-(x) &= \int_{-\infty}^x Q_0^-(t) dt = \int_{-\infty}^x (t-x)|q(t)| dt. \end{aligned}$$

Clearly,

$$\lim_{x \rightarrow \pm\infty} Q_j^\pm(x) = 0.$$

The following theorem introduces the Jost solutions  $e(x, \rho)$  and  $g(x, \rho)$  with prescribed behavior in  $\pm\infty$ .

**Theorem 3.1.1.** *Equation (3.1.1) has unique solutions  $y = e(x, \rho)$  and  $y = g(x, \rho)$ , satisfying the integral equations*

$$\begin{aligned} e(x, \rho) &= \exp(i\rho x) + \int_x^\infty \frac{\sin \rho(t-x)}{\rho} q(t) e(t, \rho) dt, \\ g(x, \rho) &= \exp(-i\rho x) + \int_{-\infty}^x \frac{\sin \rho(x-t)}{\rho} q(t) g(t, \rho) dt. \end{aligned}$$

The functions  $e(x, \rho)$  and  $g(x, \rho)$  have the following properties:

1) For each fixed  $x$ , the functions  $e^{(\nu)}(x, \rho)$  and  $g^{(\nu)}(x, \rho)$  ( $\nu = 0, 1$ ) are analytic in  $\Omega_+$  and continuous in  $\overline{\Omega}_+$ .

2) For  $\nu = 0, 1$ ,

$$\left. \begin{aligned} e^{(\nu)}(x, \rho) &= (i\rho)^\nu \exp(i\rho x)(1 + o(1)), \quad x \rightarrow +\infty, \\ g^{(\nu)}(x, \rho) &= (-i\rho)^\nu \exp(-i\rho x)(1 + o(1)), \quad x \rightarrow -\infty, \end{aligned} \right\} \quad (3.1.3)$$

uniformly in  $\bar{\Omega}_+$ . Moreover, for  $\rho \in \bar{\Omega}_+$ ,

$$\left. \begin{aligned} |e(x, \rho) \exp(-i\rho x)| &\leq \exp(Q_1^+(x)), \\ |e(x, \rho) \exp(-i\rho x) - 1| &\leq Q_1^+(x) \exp(Q_1^+(x)), \\ |e'(x, \rho) \exp(-i\rho x) - i\rho| &\leq Q_0^+(x) \exp(Q_1^+(x)), \end{aligned} \right\} \quad (3.1.4)$$

$$\left. \begin{aligned} |g(x, \rho) \exp(i\rho x)| &\leq \exp(Q_1^-(x)), \\ |g(x, \rho) \exp(i\rho x) - 1| &\leq Q_1^-(x) \exp(Q_1^-(x)), \\ |g'(x, \rho) \exp(i\rho x) + i\rho| &\leq Q_0^-(x) \exp(Q_1^-(x)). \end{aligned} \right\} \quad (3.1.5)$$

3) For each fixed  $\rho \in \Omega_+$  and each real  $\alpha$ ,  $e(x, \rho) \in L_2(\alpha, \infty)$ ,  $g(x, \rho) \in L_2(-\infty, \alpha)$ . Moreover,  $e(x, \rho)$  and  $g(x, \rho)$  are the unique solutions of (3.1.1) (up to a multiplicative constant) having this property.

4) For  $|\rho| \rightarrow \infty$ ,  $\rho \in \bar{\Omega}_+$ ,  $\nu = 0, 1$ ,

$$\left. \begin{aligned} e^{(\nu)}(x, \rho) &= (i\rho)^\nu \exp(i\rho x) \left( 1 + \frac{\omega^+(x)}{i\rho} + o\left(\frac{1}{\rho}\right) \right), \quad \omega^+(x) = -\frac{1}{2} \int_x^\infty q(t) dt, \\ g^{(\nu)}(x, \rho) &= (-i\rho)^\nu \exp(-i\rho x) \left( 1 + \frac{\omega^-(x)}{i\rho} + o\left(\frac{1}{\rho}\right) \right), \quad \omega^-(x) = -\frac{1}{2} \int_{-\infty}^x q(t) dt, \end{aligned} \right\} \quad (3.1.6)$$

uniformly for  $x \geq \alpha$  and  $x \leq \alpha$  respectively.

5) For real  $\rho \neq 0$ , the functions  $\{e(x, \rho), e(x, -\rho)\}$  and  $\{g(x, \rho), g(x, -\rho)\}$  form fundamental systems of solutions for (3.1.1), and

$$\langle e(x, \rho), e(x, -\rho) \rangle = -\langle g(x, \rho), g(x, -\rho) \rangle \equiv -2i\rho, \quad (3.1.7)$$

where  $\langle y, z \rangle := yz' - y'z$ .

6) The functions  $e(x, \rho)$  and  $g(x, \rho)$  have the representations

$$\left. \begin{aligned} e(x, \rho) &= \exp(i\rho x) + \int_x^\infty A^+(x, t) \exp(i\rho t) dt, \\ g(x, \rho) &= \exp(-i\rho x) + \int_{-\infty}^x A^-(x, t) \exp(-i\rho t) dt, \end{aligned} \right\} \quad (3.1.8)$$

where  $A^\pm(x, t)$  are real continuous functions, and

$$A^+(x, x) = \frac{1}{2} \int_x^\infty q(t) dt, \quad A^-(x, x) = \frac{1}{2} \int_{-\infty}^x q(t) dt, \quad (3.1.9)$$

$$|A^\pm(x, t)| \leq \frac{1}{2} Q_0^\pm\left(\frac{x+t}{2}\right) \exp\left(Q_1^\pm(x) - Q_1^\pm\left(\frac{x+t}{2}\right)\right). \quad (3.1.10)$$

The functions  $A^\pm(x, t)$  have first derivatives  $A_1^\pm := \frac{\partial A^\pm}{\partial x}$ ,  $A_2^\pm := \frac{\partial A^\pm}{\partial t}$ ; the functions

$$A_i^\pm(x, t) \pm \frac{1}{4} q\left(\frac{x+t}{2}\right)$$

are absolutely continuous with respect to  $x$  and  $t$ , and

$$\left| A_i^\pm(x, t) \pm \frac{1}{4} q\left(\frac{x+t}{2}\right) \right| \leq \frac{1}{2} Q_0^\pm(x) Q_0^\pm\left(\frac{x+t}{2}\right) \exp(Q_1^\pm(x)), \quad i = 1, 2. \quad (3.1.11)$$

For the function  $e(x, \rho)$ , Theorem 3.1.1 was proved in Section 2.1 (see Theorems 2.1.1-2.1.3). For  $g(x, \rho)$  the arguments are the same. Moreover, all assertions of Theorem 3.1.1 for  $g(x, \rho)$  can be obtained from the corresponding assertions for  $e(x, \rho)$  by the replacement  $x \rightarrow -x$ .

In the next lemma we describe properties of the Jost solutions  $e_j(x, \rho)$  and  $g_j(x, \rho)$  related to the potentials  $q_j$ , which approximate  $q$ .

**Lemma 3.1.1.** *If  $(1 + |x|)|q(x)| \in L(a, \infty)$ ,  $a > -\infty$ , and*

$$\lim_{j \rightarrow \infty} \int_a^\infty (1 + |x|)|q_j(x) - q(x)| dx = 0, \quad (3.1.12)$$

then

$$\lim_{j \rightarrow \infty} \sup_{\rho \in \Omega_+} \sup_{x \geq a} |(e_j^{(\nu)}(x, \rho) - e^{(\nu)}(x, \rho)) \exp(-i\rho x)| = 0, \quad \nu = 0, 1. \quad (3.1.13)$$

If  $(1 + |x|)|q(x)| \in L(-\infty, a)$ ,  $a < \infty$ , and

$$\lim_{j \rightarrow \infty} \int_{-\infty}^a (1 + |x|)|q_j(x) - q(x)| dx = 0,$$

then

$$\lim_{j \rightarrow \infty} \sup_{\rho \in \Omega_+} \sup_{x \leq a} |(g_j^{(\nu)}(x, \rho) - g^{(\nu)}(x, \rho)) \exp(i\rho x)| = 0, \quad \nu = 0, 1. \quad (3.1.14)$$

Here  $e_j(x, \rho)$  and  $g_j(x, \rho)$  are the Jost solutions for the potentials  $q_j$ .

*Proof.* Denote

$$z_j(x, \rho) = e_j(x, \rho) \exp(-i\rho x), \quad z(x, \rho) = e(x, \rho) \exp(-i\rho x), \quad u_j(x, \rho) = |z_j(x, \rho) - z(x, \rho)|.$$

Then, it follows from (2.1.8) that

$$z_j(x, \rho) - z(x, \rho) = \frac{1}{2i\rho} \int_x^\infty (1 - \exp(2i\rho(t-x)))(q(t)z(t, \rho) - q_j(t)z_j(t, \rho)) dt.$$

From this, taking (2.1.29) into account, we infer

$$u_j(x, \rho) \leq \int_x^\infty (t-x)|(q(t) - q_j(t))z(t, \rho)| dt + \int_x^\infty (t-x)|q_j(t)|u_j(t, \rho) dt.$$

According to (3.1.4),

$$|z(x, \rho)| \leq \exp(Q_1^+(x)) \leq \exp(Q_1^+(a)), \quad x \geq a, \quad (3.1.15)$$

and consequently

$$u_j(x, \rho) \leq \exp(Q_1^+(a)) \int_a^\infty (t-a)|q(t) - q_j(t)| dt + \int_x^\infty (t-x)|q_j(t)| u_j(t, \rho) dt.$$

By virtue of Lemma 2.1.2 this yields

$$\begin{aligned} u_j(x, \rho) &\leq \exp(Q_1^+(a)) \int_a^\infty (t-a)|q(t) - q_j(t)| dt \exp\left(\int_x^\infty (t-x)|q_j(t)| dt\right) \\ &\leq \exp(Q_1^+(a)) \int_a^\infty (t-a)|q(t) - q_j(t)| dt \exp\left(\int_a^\infty (t-a)|q_j(t)| dt\right). \end{aligned}$$

Hence

$$u_j(x, \rho) \leq C_a \int_a^\infty (t-a)|q(t) - q_j(t)| dt. \quad (3.1.16)$$

In particular, (3.1.16) and (3.1.12) imply

$$\lim_{j \rightarrow \infty} \sup_{\rho \in \overline{\Omega}_+} \sup_{x \geq a} u_j(x, \rho) = 0,$$

and we arrive at (3.1.13) for  $\nu = 0$ .

Denote

$$v_j(x, \rho) = |(e'_j(x, \rho) - e'(x, \rho)) \exp(-i\rho x)|.$$

It follows from (2.1.20) that

$$v_j(x, \rho) \leq \int_x^\infty |q(t)z(t, \rho) - q_j(t)z_j(t, \rho)| dt,$$

and consequently,

$$v_j(x, \rho) \leq \int_a^\infty |(q(t) - q_j(t))z(t, \rho)| dt + \int_a^\infty |q_j(t)| u_j(t, \rho) dt. \quad (3.1.17)$$

By virtue of (3.1.15)-(3.1.17) we obtain

$$v_j(x, \rho) \leq C_a \left( \int_a^\infty |q(t) - q_j(t)| dt + \int_a^\infty (t-a)|q(t) - q_j(t)| dt \cdot \int_a^\infty |q_j(t)| dt \right).$$

Together with (3.1.12) this yields

$$\lim_{j \rightarrow \infty} \sup_{\rho \in \overline{\Omega}_+} \sup_{x \geq a} v_j(x, \rho) = 0,$$

and we arrive at (3.1.13) for  $\nu = 1$ . The relations (3.1.14) is proved analogously.  $\square$

**3.1.2.** For real  $\rho \neq 0$ , the functions  $\{e(x, \rho), e(x, -\rho)\}$  and  $\{g(x, \rho), g(x, -\rho)\}$  form fundamental systems of solutions for (3.1.1). Therefore, we have for real  $\rho \neq 0$ :

$$e(x, \rho) = a(\rho)g(x, -\rho) + b(\rho)g(x, \rho), \quad g(x, \rho) = c(\rho)e(x, \rho) + d(\rho)e(x, -\rho). \quad (3.1.18)$$

Let us study the properties of the coefficients  $a(\rho)$ ,  $b(\rho)$ ,  $c(\rho)$  and  $d(\rho)$ .

**Lemma 3.1.2.** For real  $\rho \neq 0$ , the following relations hold

$$c(\rho) = -b(-\rho), \quad d(\rho) = a(\rho), \quad (3.1.19)$$

$$\overline{a(\rho)} = a(-\rho), \quad \overline{b(\rho)} = b(-\rho), \quad (3.1.20)$$

$$|a(\rho)|^2 = 1 + |b(\rho)|^2, \quad (3.1.21)$$

$$a(\rho) = -\frac{1}{2i\rho} \langle e(x, \rho), g(x, \rho) \rangle, \quad b(\rho) = \frac{1}{2i\rho} \langle e(x, \rho), g(x, -\rho) \rangle. \quad (3.1.22)$$

*Proof.* Since  $\overline{e(x, \rho)} = e(x, -\rho)$ ,  $\overline{g(x, \rho)} = g(x, -\rho)$ , then (3.1.20) follows from (3.1.18). Using (3.1.18) we also calculate

$$\langle e(x, \rho), g(x, \rho) \rangle = \langle a(\rho)g(x, -\rho) + b(\rho)g(x, \rho), g(x, \rho) \rangle = -2i\rho a(\rho),$$

$$\langle e(x, \rho), g(x, -\rho) \rangle = \langle a(\rho)g(x, -\rho) + b(\rho)g(x, \rho), g(x, -\rho) \rangle = 2i\rho b(\rho),$$

$$\langle e(x, \rho), g(x, \rho) \rangle = \langle e(x, \rho), c(\rho)e(x, \rho) + d(\rho)e(x, -\rho) \rangle = 2i\rho d(\rho),$$

$$\langle e(x, -\rho), g(x, \rho) \rangle = \langle e(x, -\rho), c(\rho)e(x, \rho) + d(\rho)e(x, -\rho) \rangle = 2i\rho c(\rho),$$

i.e. (3.1.19) and (3.1.22) are valid. Furthermore,

$$\begin{aligned} -2i\rho &= \langle e(x, \rho), e(x, -\rho) \rangle = \langle a(\rho)g(x, -\rho) + b(\rho)g(x, \rho), a(-\rho)g(x, \rho) + b(-\rho)g(x, -\rho) \rangle = \\ &= a(\rho)a(-\rho)\langle g(x, -\rho), g(x, \rho) \rangle + b(\rho)b(-\rho)\langle g(x, \rho), g(x, -\rho) \rangle = -2i\rho(|a(\rho)|^2 - |b(\rho)|^2), \end{aligned}$$

and we arrive at (3.1.21).  $\square$

We note that (3.1.22) gives the analytic continuation for  $a(\rho)$  to  $\Omega_+$ . Hence, the function  $a(\rho)$  is analytic in  $\Omega_+$ , and  $\rho a(\rho)$  is continuous in  $\overline{\Omega}_+$ . The function  $\rho b(\rho)$  is continuous for real  $\rho$ . Moreover, it follows from (3.1.22) and (3.1.6) that

$$a(\rho) = 1 - \frac{1}{2i\rho} \int_{-\infty}^{\infty} q(t) dt + o\left(\frac{1}{\rho}\right), \quad b(\rho) = o\left(\frac{1}{\rho}\right), \quad |\rho| \rightarrow \infty \quad (3.1.23)$$

(in the domains of definition), and consequently the function  $\rho(a(\rho) - 1)$  is bounded in  $\overline{\Omega}_+$ .

Using (3.1.22) and (3.1.8) one can calculate more precisely

$$\left. \begin{aligned} a(\rho) &= 1 - \frac{1}{2i\rho} \int_{-\infty}^{\infty} q(t) dt + \frac{1}{2i\rho} \int_0^{\infty} A(t) \exp(i\rho t) dt, \\ b(\rho) &= \frac{1}{2i\rho} \int_{-\infty}^{\infty} B(t) \exp(i\rho t) dt, \end{aligned} \right\} \quad (3.1.24)$$

where  $A(t) \in L(0, \infty)$  and  $B(t) \in L(-\infty, \infty)$  are real functions.

Indeed,

$$\begin{aligned} 2i\rho a(\rho) &= g(0, \rho)e'(0, \rho) - e(0, \rho)g'(0, \rho) \\ &= \left(1 + \int_{-\infty}^0 A^-(0, t) \exp(-i\rho t) dt\right) \left(i\rho - A^+(0, 0) + \int_0^{\infty} A_1^+(0, t) \exp(i\rho t) dt\right) \\ &\quad + \left(1 + \int_0^{\infty} A^+(0, t) \exp(i\rho t) dt\right) \left(i\rho - A^-(0, 0) - \int_{-\infty}^0 A_1^-(0, t) \exp(-i\rho t) dt\right). \end{aligned}$$

Integration by parts yields

$$i\rho \int_{-\infty}^0 A^-(0, t) \exp(-i\rho t) dt = -A^-(0, 0) + \int_{-\infty}^0 A_2^-(0, t) \exp(-i\rho t) dt,$$

$$i\rho \int_0^\infty A^+(0, t) \exp(i\rho t) dt = -A^+(0, 0) - \int_0^\infty A_2^+(0, t) \exp(i\rho t) dt.$$

Furthermore,

$$\begin{aligned} & \int_{-\infty}^0 A^-(0, t) \exp(-i\rho t) dt \int_0^\infty A_1^+(0, s) \exp(i\rho s) ds \\ &= \int_{-\infty}^0 A^-(0, t) \left( \int_{-t}^\infty A_1^+(0, \xi + t) \exp(i\rho \xi) d\xi \right) dt \\ &= \int_0^\infty \left( \int_{-\xi}^0 A^-(0, t) A_1^+(0, \xi + t) dt \right) \exp(i\rho \xi) d\xi. \end{aligned}$$

Analogously,

$$\begin{aligned} & \int_{-\infty}^0 A_1^-(0, t) \exp(-i\rho t) dt \int_0^\infty A^+(0, s) \exp(i\rho s) ds \\ &= \int_0^\infty \left( \int_{-\xi}^0 A_1^-(0, t) A^+(0, \xi + t) dt \right) \exp(i\rho \xi) d\xi. \end{aligned}$$

Since

$$2(A^+(0, 0) + A^-(0, 0)) = \int_{-\infty}^\infty q(t) dt,$$

we arrive at (3.1.24) for  $a(\rho)$ , where

$$\begin{aligned} A(t) &= A_1^+(0, t) - A_1^-(0, -t) + A_2^-(0, -t) - A_2^+(0, t) - A^+(0, 0)A^-(0, -t) - A^-(0, 0)A^+(0, t) \\ &\quad + \int_{-t}^0 A^-(0, \xi) A_1^+(0, \xi + t) d\xi - \int_{-t}^0 A_1^-(0, \xi) A^+(0, \xi + t) d\xi. \end{aligned}$$

It follows from (3.1.10)-(3.1.11) that  $A(t) \in L(0, \infty)$ . For the function  $b(\rho)$  the arguments are similar.

Denote

$$e_0(x, \rho) = \frac{e(x, \rho)}{a(\rho)}, \quad g_0(x, \rho) = \frac{g(x, \rho)}{a(\rho)}, \quad (3.1.25)$$

$$s^+(\rho) = -\frac{b(-\rho)}{a(\rho)}, \quad s^-(\rho) = \frac{b(\rho)}{a(\rho)}. \quad (3.1.26)$$

The functions  $s^+(\rho)$  and  $s^-(\rho)$  are called the *reflection coefficients* (right and left, respectively). It follows from (3.1.18), (3.1.25) and (3.1.26) that

$$e_0(x, \rho) = g(x, -\rho) + s^-(\rho)g(x, \rho), \quad g_0(x, \rho) = e(x, -\rho) + s^+(\rho)e(x, \rho). \quad (3.1.27)$$

Using (3.1.25), (3.1.27) and (3.1.3) we get

$$e_0(x, \rho) \sim \exp(i\rho x) + s^-(\rho) \exp(-i\rho x) \quad (x \rightarrow -\infty), \quad e_0(x, \rho) \sim t(\rho) \exp(i\rho x) \quad (x \rightarrow \infty),$$

$$g_0(x, \rho) \sim t(\rho) \exp(i\rho x) \quad (x \rightarrow -\infty), \quad g_0(x, \rho) \sim \exp(-i\rho x) + s^+(\rho) \exp(i\rho x) \quad (x \rightarrow \infty),$$

where  $t(\rho) = (a(\rho))^{-1}$  is called the *transmission coefficient*.

We point out *the main properties of the functions*  $s^\pm(\rho)$ . By virtue of (3.1.20)-(3.1.22) and (3.1.26), the functions  $s^\pm(\rho)$  are continuous for real  $\rho \neq 0$ , and

$$\overline{s^\pm(\rho)} = s^\pm(-\rho).$$

Moreover, (3.1.21) implies

$$|s^\pm(\rho)|^2 = 1 - \frac{1}{|a(\rho)|^2},$$

and consequently,

$$|s^\pm(\rho)| < 1 \text{ for real } \rho \neq 0.$$

Furthermore, according to (3.1.23) and (3.1.26),

$$s^\pm(\rho) = o\left(\frac{1}{\rho}\right) \text{ as } |\rho| \rightarrow \infty.$$

Denote by  $R^\pm(x)$  the Fourier transform for  $s^\pm(\rho)$ :

$$R^\pm(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} s^\pm(\rho) \exp(\pm i\rho x) d\rho. \quad (3.1.28)$$

Then  $R^\pm(x) \in L_2(-\infty, \infty)$  are real, and

$$s^\pm(\rho) = \int_{-\infty}^{\infty} R^\pm(x) \exp(\mp i\rho x) dx. \quad (3.1.29)$$

It follows from (3.1.25) and (3.1.27) that

$$\rho e(x, \rho) = \rho a(\rho) \left( (s^-(\rho) + 1)g(x, \rho) + g(x, -\rho) - g(x, \rho) \right),$$

$$\rho g(x, \rho) = \rho a(\rho) \left( (s^+(\rho) + 1)e(x, \rho) + e(x, -\rho) - e(x, \rho) \right),$$

and consequently,

$$\lim_{\rho \rightarrow 0} \rho a(\rho) (s^\pm(\rho) + 1) = 0.$$

**3.1.3.** Let us now study the properties of the discrete spectrum.

**Definition 3.1.1.** The values of the parameter  $\lambda$ , for which equation (3.1.1) has nonzero solutions  $y(x) \in L_2(-\infty, \infty)$ , are called *eigenvalues* of (3.1.1), and the corresponding solutions are called *eigenfunctions*.

The properties of the eigenvalues are similar to the properties of the discrete spectrum of the Sturm-Liouville operator on the half-line (see Chapter 2).

**Theorem 3.1.2.** *There are no eigenvalues for  $\lambda \geq 0$ .*

*Proof.* Repeat the arguments in the proof of Theorems 2.1.6 and 2.3.6.  $\square$

Let  $\Lambda_+ := \{\lambda, \lambda = \rho^2, \rho \in \Omega_+ : a(\rho) = 0\}$  be the set of zeros of  $a(\rho)$  in the upper half-plane  $\Omega_+$ . Since the function  $a(\rho)$  is analytic in  $\Omega_+$  and, by virtue of (3.1.23),

$$a(\rho) = 1 + O\left(\frac{1}{\rho}\right), \quad |\rho| \rightarrow \infty, \operatorname{Im} \rho \geq 0,$$

we get that  $\Lambda_+$  is an at most countable bounded set.

**Theorem 3.1.3.** *The set of eigenvalues coincides with  $\Lambda_+$ . The eigenvalues  $\{\lambda_k\}$  are real and negative (i.e.  $\Lambda_+ \subset (-\infty, 0)$ ). For each eigenvalue  $\lambda_k = \rho_k^2$ , there exists only one (up to a multiplicative constant) eigenfunction, namely*

$$g(x, \rho_k) = d_k e(x, \rho_k), \quad d_k \neq 0. \quad (3.1.30)$$

The eigenfunctions  $e(x, \rho_k)$  and  $g(x, \rho_k)$  are real. Eigenfunctions related to different eigenvalues are orthogonal in  $L_2(-\infty, \infty)$ .

*Proof.* Let  $\lambda_k = \rho_k^2 \in \Lambda_+$ . By virtue of (3.1.22),

$$\langle e(x, \rho_k), g(x, \rho_k) \rangle = 0, \quad (3.1.31)$$

i.e. (3.1.30) is valid. According to Theorem 3.1.1,  $e(x, \rho_k) \in L_2(\alpha, \infty)$ ,  $g(x, \rho_k) \in L_2(-\infty, \alpha)$  for each real  $\alpha$ . Therefore, (3.1.30) implies

$$e(x, \rho_k), g(x, \rho_k) \in L_2(-\infty, \infty).$$

Thus,  $e(x, \rho_k)$  and  $g(x, \rho_k)$  are eigenfunctions, and  $\lambda_k = \rho_k^2$  is an eigenvalue.

Conversely, let  $\lambda_k = \rho_k^2$ ,  $\rho_k \in \Omega_+$  be an eigenvalue, and let  $y_k(x)$  be a corresponding eigenfunction. Since  $y_k(x) \in L_2(-\infty, \infty)$ , we have

$$y_k(x) = c_{k1}e(x, \rho_k), \quad y_k(x) = c_{k2}g(x, \rho_k), \quad c_{k1}, c_{k2} \neq 0,$$

and consequently, (3.1.31) holds. Using (3.1.22) we obtain  $a(\rho_k) = 0$ , i.e.  $\lambda_k \in \Lambda_+$ .

Let  $\lambda_n$  and  $\lambda_k$  ( $\lambda_n \neq \lambda_k$ ) be eigenvalues with eigenfunctions  $y_n(x) = e(x, \rho_n)$  and  $y_k(x) = e(x, \rho_k)$  respectively. Then integration by parts yields

$$\int_{-\infty}^{\infty} \ell y_n(x) y_k(x) dx = \int_{-\infty}^{\infty} y_n(x) \ell y_k(x) dx,$$

and hence

$$\lambda_n \int_{-\infty}^{\infty} y_n(x) y_k(x) dx = \lambda_k \int_{-\infty}^{\infty} y_n(x) y_k(x) dx$$

or

$$\int_{-\infty}^{\infty} y_n(x) y_k(x) dx = 0.$$

Furthermore, let  $\lambda^0 = u + iv$ ,  $v \neq 0$  be a non-real eigenvalue with an eigenfunction  $y^0(x) \neq 0$ . Since  $\overline{q(x)}$  is real, we get that  $\overline{\lambda^0} = u - iv$  is also the eigenvalue with the eigenfunction  $\overline{y^0(x)}$ . Since  $\lambda^0 \neq \overline{\lambda^0}$ , we derive as before

$$\|y^0\|_{L_2}^2 = \int_{-\infty}^{\infty} y^0(x) \overline{y^0(x)} dx = 0,$$

which is impossible. Thus, all eigenvalues  $\{\lambda_k\}$  are real, and consequently the eigenfunctions  $e(x, \rho_k)$  and  $g(x, \rho_k)$  are real too. Together with Theorem 3.1.2 this yields  $\Lambda_+ \subset (-\infty, 0)$ . Theorem 3.1.3 is proved.  $\square$

For  $\lambda_k = \rho_k^2 \in \Lambda_+$  we denote

$$\alpha_k^+ = \left( \int_{-\infty}^{\infty} e^2(x, \rho_k) dx \right)^{-1}, \quad \alpha_k^- = \left( \int_{-\infty}^{\infty} g^2(x, \rho_k) dx \right)^{-1}.$$

**Theorem 3.1.4.**  $\Lambda_+$  is a finite set, i.e. in  $\Omega_+$  the function  $a(\rho)$  has at most a finite number of zeros. All zeros of  $a(\rho)$  in  $\Omega_+$  are simple, i.e.  $a_1(\rho_k) \neq 0$ , where  $a_1(\rho) := \frac{d}{d\rho} a(\rho)$ . Moreover,

$$\alpha_k^+ = \frac{d_k}{ia_1(\rho_k)}, \quad \alpha_k^- = \frac{1}{id_k a_1(\rho_k)}, \quad (3.1.32)$$



where the numbers  $d_k$  are defined by (3.1.30).

*Proof.* 1) Let us show that

$$\left. \begin{aligned} -2\rho \int_{-A}^x e(t, \rho) g(t, \rho) dt &= \langle e(t, \rho), \dot{g}(t, \rho) \rangle \Big|_{-A}^x, \\ 2\rho \int_x^A e(t, \rho) g(t, \rho) dt &= \langle \dot{e}(t, \rho), g(t, \rho) \rangle \Big|_x^A, \end{aligned} \right\} \quad (3.1.33)$$

where in this subsection

$$\dot{e}(t, \rho) := \frac{d}{d\rho} e(t, \rho), \quad \dot{g}(t, \rho) := \frac{d}{d\rho} g(t, \rho).$$

Indeed,

$$\frac{d}{dx} \langle e(x, \rho), \dot{g}(x, \rho) \rangle = e(x, \rho) \dot{g}''(x, \rho) - e''(x, \rho) \dot{g}(x, \rho).$$

Since

$$-e''(x, \rho) + q(x)e(x, \rho) = \rho^2 e(x, \rho), \quad -\dot{g}''(x, \rho) + q(x)\dot{g}(x, \rho) = \rho^2 \dot{g}(x, \rho) + 2\rho g(x, \rho),$$

we get

$$\frac{d}{dx} \langle e(x, \rho), \dot{g}(x, \rho) \rangle = -2\rho e(x, \rho) g(x, \rho).$$

Similarly,

$$\frac{d}{dx} \langle \dot{e}(x, \rho), g(x, \rho) \rangle = 2\rho e(x, \rho) g(x, \rho),$$

and we arrive at (3.1.33).

It follows from (3.1.33) that

$$\begin{aligned} 2\rho \int_{-A}^A e(t, \rho) g(t, \rho) dt &= -\langle \dot{e}(x, \rho), g(x, \rho) \rangle - \langle e(x, \rho), \dot{g}(x, \rho) \rangle \\ &\quad + \langle \dot{e}(x, \rho), g(x, \rho) \rangle \Big|_{x=A} + \langle e(x, \rho), \dot{g}(x, \rho) \rangle \Big|_{x=-A}. \end{aligned}$$

On the other hand, differentiating (3.1.22) with respect to  $\rho$ , we obtain

$$2i\rho a_1(\rho) + 2ia(\rho) = -\langle \dot{e}(x, \rho), g(x, \rho) \rangle - \langle e(x, \rho), \dot{g}(x, \rho) \rangle.$$

For  $\rho = \rho_k$  this yields with the preceding formula

$$ia_1(\rho_k) = \int_{-A}^A e(t, \rho_k) g(t, \rho_k) dt + \delta_k(A), \quad (3.1.34)$$

where

$$\delta_k(A) = -\frac{1}{2\rho_k} \left( \langle \dot{e}(x, \rho_k), g(x, \rho_k) \rangle \Big|_{x=A} + \langle e(x, \rho_k), \dot{g}(x, \rho_k) \rangle \Big|_{x=-A} \right).$$

Since  $\rho_k = i\tau_k$ ,  $\tau_k > 0$ , we have by virtue of (3.1.4),

$$e(x, \rho_k), e'(x, \rho_k) = O(\exp(-\tau_k x)), \quad x \rightarrow +\infty.$$

According to (3.1.8),

$$\begin{aligned}\dot{e}(x, \rho_k) &= ix \exp(-\tau_k x) + \int_x^\infty it A^+(x, t) \exp(-\tau_k t) dt, \\ \dot{e}'(x, \rho_k) &= i \exp(-\tau_k x) - ix \tau_k \exp(-\tau_k x) - ix A^+(x, x) \exp(-\tau_k x) \\ &\quad + \int_x^\infty it A_1^+(x, t) \exp(-\tau_k t) dt.\end{aligned}$$

Hence

$$\dot{e}(x, \rho_k), \dot{e}'(x, \rho_k) = O(1), \quad x \rightarrow +\infty.$$

From this, using (3.1.30), we calculate

$$\begin{aligned}\langle \dot{e}(x, \rho_k), g(x, \rho_k) \rangle &= d_k \langle \dot{e}(x, \rho_k), e(x, \rho_k) \rangle = o(1) \quad \text{as } x \rightarrow +\infty, \\ \langle e(x, \rho_k), \dot{g}(x, \rho_k) \rangle &= \frac{1}{d_k} \langle g(x, \rho_k), \dot{g}(x, \rho_k) \rangle = o(1) \quad \text{as } x \rightarrow -\infty.\end{aligned}$$

Consequently,

$$\lim_{A \rightarrow +\infty} \delta_k(A) = 0.$$

Then (3.1.34) implies

$$ia_1(\rho_k) = \int_{-\infty}^\infty e(t, \rho_k) g(t, \rho_k) dt.$$

Using (3.1.30) again we obtain

$$ia_1(\rho_k) = d_k \int_{-\infty}^\infty e^2(t, \rho_k) dt = \frac{1}{d_k} \int_{-\infty}^\infty g^2(t, \rho_k) dt.$$

Hence  $a_1(\rho_k) \neq 0$ , and (3.1.32) is valid.

2) Suppose that  $\Lambda_+ = \{\lambda_k\}$  is an infinite set. Since  $\Lambda_+$  is bounded and  $\lambda_k = \rho_k^2 < 0$ , it follows that  $\rho_k = i\tau_k \rightarrow 0$ ,  $\tau_k > 0$ . By virtue of (3.1.4)-(3.1.5), there exists  $A > 0$  such that

$$\left. \begin{aligned} e(x, i\tau) &\geq \frac{1}{2} \exp(-\tau x) \quad \text{for } x \geq A, \tau \geq 0, \\ g(x, i\tau) &\geq \frac{1}{2} \exp(\tau x) \quad \text{for } x \leq -A, \tau \geq 0, \end{aligned} \right\} \quad (3.1.35)$$

and consequently

$$\left. \begin{aligned} \int_A^\infty e(x, \rho_k) e(x, \rho_n) dx &\geq \frac{\exp(-(\tau_k + \tau_n)A)}{4(\tau_k + \tau_n)} \geq \frac{\exp(-2AT)}{8T}, \\ \int_{-A}^{-A} g(x, \rho_k) g(x, \rho_n) dx &\geq \frac{\exp(-(\tau_k + \tau_n)A)}{4(\tau_k + \tau_n)} \geq \frac{\exp(-2AT)}{8T}, \end{aligned} \right\} \quad (3.1.36)$$

where  $T = \max_k \tau_k$ . Since the eigenfunctions  $e(x, \rho_k)$  and  $e(x, \rho_n)$  are orthogonal in  $L_2(-\infty, \infty)$  we get

$$\begin{aligned} 0 &= \int_{-\infty}^\infty e(x, \rho_k) e(x, \rho_n) dx = \int_A^\infty e(x, \rho_k) e(x, \rho_n) dx + \frac{1}{d_k d_n} \int_{-\infty}^{-A} g(x, \rho_k) g(x, \rho_n) dx \\ &\quad + \int_{-A}^A e^2(x, \rho_k) dx + \int_{-A}^A e(x, \rho_k) (e(x, \rho_n) - e(x, \rho_k)) dx. \end{aligned} \quad (3.1.37)$$

Take  $x_0 \leq -A$  such that  $e(x_0, 0) \neq 0$ . According to (3.1.30),

$$\frac{1}{d_k d_n} = \frac{e(x_0, \rho_k) e(x_0, \rho_n)}{g(x_0, \rho_k) g(x_0, \rho_n)}.$$

Since the functions  $e(x, \rho)$  and  $g(x, \rho)$  are continuous for  $\operatorname{Im} \rho \geq 0$ , we calculate with the help of (3.1.35),

$$\begin{aligned} \lim_{k, n \rightarrow \infty} g(x_0, \rho_k) g(x_0, \rho_n) &= g^2(x_0, 0) > 0, \\ \lim_{k, n \rightarrow \infty} e(x_0, \rho_k) e(x_0, \rho_n) &= e^2(x_0, 0) > 0. \end{aligned}$$

Therefore,

$$\lim_{k, n \rightarrow \infty} \frac{1}{d_k d_n} > 0.$$

Together with (3.1.36) this yields

$$\int_A^\infty e(x, \rho_k) e(x, \rho_n) dx + \frac{1}{d_k d_n} \int_{-A}^{-\infty} g(x, \rho_k) g(x, \rho_n) dx + \int_{-A}^A e^2(x, \rho_k) dx \geq C > 0 \quad (3.1.38)$$

for sufficiently large  $k$  and  $n$ . On the other hand, acting in the same way as in the proof of Theorem 2.3.4 one can easily verify that

$$\int_{-A}^A e(x, \rho_k) (e(x, \rho_n) - e(x, \rho_k)) dx \rightarrow 0 \quad \text{as } k, n \rightarrow \infty. \quad (3.1.39)$$

The relations (3.1.37)-(3.1.39) give us a contradiction.

This means that  $\Lambda_+$  is a finite set. □

Thus, the set of eigenvalues has the form

$$\Lambda_+ = \{\lambda_k\}_{k=\overline{1, N}}, \quad \lambda_k = \rho_k^2, \quad \rho_k = i\tau_k, \quad 0 < \tau_1 < \dots < \tau_m.$$

**Definition 3.1.2.** The data  $J^+ = \{s^+(\rho), \lambda_k, \alpha_k^+; \rho \in \mathbf{R}, k = \overline{1, N}\}$  are called the right scattering data, and the data  $J^- = \{s^-(\rho), \lambda_k, \alpha_k^-; \rho \in \mathbf{R}, k = \overline{1, N}\}$  are called the left scattering data.

*Example 3.1.1.* Let  $q(x) \equiv 0$ . Then

$$e(x, \rho) = \exp(i\rho x), \quad g(x, \rho) = \exp(-i\rho x), \quad a(\rho) = 1, \quad b(\rho) = 0, \quad s^\pm(\rho) = 0, \quad N = 0,$$

i.e. there are no eigenvalues at all.

**3.1.4.** In this subsection we study connections between the scattering data  $J^+$  and  $J^-$ . Consider the function

$$\gamma(\rho) = \frac{1}{a(\rho)} \prod_{k=1}^N \frac{\rho - i\tau_k}{\rho + i\tau_k}. \quad (3.1.40)$$

**Lemma 3.1.3.** (i<sub>1</sub>) The function  $\gamma(\rho)$  is analytic in  $\Omega_+$  and continuous in  $\overline{\Omega_+} \setminus \{0\}$ .  
 (i<sub>2</sub>)  $\gamma(\rho)$  has no zeros in  $\overline{\Omega_+} \setminus \{0\}$ .  
 (i<sub>3</sub>) For  $|\rho| \rightarrow \infty$ ,  $\rho \in \overline{\Omega_+}$ ,

$$\gamma(\rho) = 1 + O\left(\frac{1}{\rho}\right). \quad (3.1.41)$$

$$(i_4) \quad |\gamma(\rho)| \leq 1 \quad \text{for } \rho \in \overline{\Omega_+}.$$

*Proof.* The assertions  $(i_1) - (i_3)$  are obvious consequences of the preceding discussion, and only  $(i_4)$  needs to be proved. By virtue of (3.1.21),  $|a(\rho)| \geq 1$  for real  $\rho \neq 0$ , and consequently,

$$|\gamma(\rho)| \leq 1 \quad \text{for real } \rho \neq 0. \quad (3.1.42)$$

Suppose that the function  $\rho a(\rho)$  is analytic in the origin. Then, using (3.1.40) and (3.1.42) we deduce that the function  $\gamma(\rho)$  has a removable singularity in the origin, and  $\gamma(\rho)$  (after extending continuously to the origin) is continuous in  $\overline{\Omega_+}$ . Using (3.1.41), (3.1.42) and the maximum principle we arrive at  $(i_4)$ .

In the general case we cannot use these arguments for  $\gamma(\rho)$ . Therefore, we introduce the potentials

$$q_r(x) = \begin{cases} q(x), & |x| \leq r, \\ 0, & |x| > r, \end{cases} \quad r \geq 0,$$

and consider the corresponding Jost solutions  $e_r(x, \rho)$  and  $g_r(x, \rho)$ . Clearly,  $e_r(x, \rho) \equiv \exp(i\rho x)$  for  $x \geq r$  and  $g_r(x, \rho) \equiv \exp(-i\rho x)$  for  $x \leq -r$ . For each fixed  $x$ , the functions  $e_r^{(\nu)}(x, \rho)$  and  $g_r^{(\nu)}(x, \rho)$  ( $\nu = 0, 1$ ) are entire in  $\rho$ . Take

$$a_r(\rho) = -\frac{1}{2i\rho} \langle e_r(x, \rho), g_r(x, \rho) \rangle, \quad \gamma_r(\rho) = \frac{1}{a_r(\rho)} \prod_{k=1}^{N_r} \frac{\rho - i\tau_{kr}}{\rho + i\tau_{kr}},$$

where  $\rho_{kr} = i\tau_{kr}$ ,  $k = \overline{1, N_r}$  are zeros of  $a_r(\rho)$  in the upper half-plane  $\Omega_+$ . The function  $\rho a_r(\rho)$  is entire in  $\rho$ , and (see Lemma 3.1.2)  $|a_r(\rho)| \geq 1$  for real  $\rho$ . The function  $\gamma_r(\rho)$  is analytic in  $\overline{\Omega_+}$ , and

$$|\gamma_r(\rho)| \leq 1 \quad \text{for } \rho \in \overline{\Omega_+}. \quad (3.1.43)$$

By virtue of Lemma 3.1.1,

$$\lim_{r \rightarrow \infty} \sup_{\rho \in \overline{\Omega_+}} \sup_{x \geq a} |(e_r^{(\nu)}(x, \rho) - e^{(\nu)}(x, \rho)) \exp(-i\rho x)| = 0,$$

$$\lim_{r \rightarrow \infty} \sup_{\rho \in \overline{\Omega_+}} \sup_{x \leq a} |(g_r^{(\nu)}(x, \rho) - g^{(\nu)}(x, \rho)) \exp(i\rho x)| = 0,$$

for  $\nu = 0, 1$  and each real  $a$ . Therefore

$$\lim_{r \rightarrow \infty} \sup_{\rho \in \overline{\Omega_+}} |\rho(a_r(\rho) - a(\rho))| = 0,$$

i.e.

$$\lim_{r \rightarrow \infty} \rho a_r(\rho) = \rho a(\rho) \quad \text{uniformly in } \overline{\Omega_+}. \quad (3.1.44)$$

In particular, (3.1.44) yields that  $0 < \tau_{kr} \leq C$  for all  $k$  and  $r$ .

Let  $\delta_r$  be the infimum of distances between the zeros  $\{\rho_{kr}\}$  of  $a_r(\rho)$  in the upper half-plane  $\text{Im } \rho > 0$ . Let us show that

$$\delta^* := \inf_{r > 0} \delta_r > 0. \quad (3.1.45)$$

Indeed, suppose on the contrary that there exists a sequence  $r_k \rightarrow \infty$  such that  $\delta_{r_k} \rightarrow 0$ . Let  $\rho_k^{(1)} = i\tau_k^{(1)}$ ,  $\rho_k^{(2)} = i\tau_k^{(2)}$  ( $\tau_k^{(1)}, \tau_k^{(2)} \geq 0$ ) be zeros of  $a_{r_k}(\rho)$  such that  $\rho_k^{(1)} - \rho_k^{(2)} \rightarrow 0$  as  $k \rightarrow \infty$ . It follows from (3.1.4)-(3.1.5) that there exists  $A > 0$  such that

$$\left. \begin{aligned} e_r(x, i\tau) &\geq \frac{1}{2} \exp(-\tau x) \text{ for } x \geq A, \tau \geq 0, r \geq 0, \\ g_r(x, i\tau) &\geq \frac{1}{2} \exp(\tau x) \text{ for } x \leq -A, \tau \geq 0, r \geq 0. \end{aligned} \right\} \quad (3.1.46)$$

Since the functions  $e_{r_k}(x, \rho_k^{(1)})$  and  $e_{r_k}(x, \rho_k^{(2)})$  are orthogonal in  $L_2(-\infty, \infty)$ , we get

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} e_{r_k}(x, \rho_k^{(1)}) e_{r_k}(x, \rho_k^{(2)}) dx \\ &= \int_A^{\infty} e_{r_k}(x, \rho_k^{(1)}) e_{r_k}(x, \rho_k^{(2)}) dx + \frac{1}{d_k^{(1)} d_k^{(2)}} \int_{-\infty}^{-A} g_{r_k}(x, \rho_k^{(1)}) g_{r_k}(x, \rho_k^{(2)}) dx \\ &\quad + \int_{-A}^A e_{r_k}^2(x, \rho_k^{(1)}) dx + \int_{-A}^A e_{r_k}(x, \rho_k^{(1)}) (e_{r_k}(x, \rho_k^{(2)}) - e_{r_k}(x, \rho_k^{(1)})) dx, \end{aligned} \quad (3.1.47)$$

where the numbers  $d_k^{(j)}$  are defined by

$$g_{r_k}(x, \rho_k^{(j)}) = d_k^{(j)} e_{r_k}(x, \rho_k^{(j)}), \quad d_k^{(j)} \neq 0.$$

Take  $x_0 \leq -A$ . Then, by virtue of (3.1.46),

$$g_{r_k}(x_0, \rho_k^{(1)}) g_{r_k}(x_0, \rho_k^{(2)}) \geq C > 0,$$

and

$$\frac{1}{d_k^{(1)} d_k^{(2)}} = \frac{e_{r_k}(x_0, \rho_k^{(1)}) e_{r_k}(x_0, \rho_k^{(2)})}{g_{r_k}(x_0, \rho_k^{(1)}) g_{r_k}(x_0, \rho_k^{(2)})}.$$

Using Lemma 3.1.1 we get

$$\varliminf_{k \rightarrow \infty} e_{r_k}(x_0, \rho_k^{(1)}) e_{r_k}(x_0, \rho_k^{(2)}) \geq 0;$$

hence

$$\varliminf_{k \rightarrow \infty} \frac{1}{d_k^{(1)} d_k^{(2)}} \geq 0.$$

Then

$$\begin{aligned} &\int_A^{\infty} e_{r_k}(x, \rho_k^{(1)}) e_{r_k}(x, \rho_k^{(2)}) dx + \frac{1}{d_k^{(1)} d_k^{(2)}} \int_{-\infty}^{-A} g_{r_k}(x, \rho_k^{(1)}) g_{r_k}(x, \rho_k^{(2)}) dx \\ &\quad + \int_{-A}^A e_{r_k}^2(x, \rho_k^{(1)}) dx \geq C > 0. \end{aligned}$$

On the other hand,

$$\int_{-A}^A e_{r_k}(x, \rho_k^{(1)}) (e_{r_k}(x, \rho_k^{(2)}) - e_{r_k}(x, \rho_k^{(1)})) dx \rightarrow 0 \text{ as } k \rightarrow \infty.$$

From this and (3.1.47) we arrive at a contradiction, i.e. (3.1.45) is proved.

Let  $D_{\delta,R} := \{\rho \in \Omega_+ : \delta < |\rho| < R\}$ , where  $0 < \delta < \min(\delta^*, \tau_1)$ ,  $R > \tau_N$ . Using (3.1.44) one can show that

$$\lim_{r \rightarrow \infty} \gamma_r(\rho) = \gamma(\rho) \text{ uniformly in } \overline{D_{\delta,R}}. \quad (3.1.48)$$

It follows from (3.1.43) and (3.1.48) that  $|\gamma(\rho)| \leq 1$  for  $\rho \in \overline{D_{\delta,R}}$ . By virtue of arbitrariness of  $\delta$  and  $R$  we obtain  $|\gamma(\rho)| \leq 1$  for  $\rho \in \overline{\Omega_+}$ , i.e.  $(i_4)$  is proved.  $\square$

It follows from Lemma 3.1.3 that

$$(a(\rho))^{-1} = O(1) \text{ as } |\rho| \rightarrow 0, \rho \in \overline{\Omega_+}. \quad (3.1.49)$$

We also note that since the function  $\sigma a(\sigma)$  is continuous at the origin, it follows that for sufficiently small real  $\sigma$ ,

$$1 \leq |a(\sigma)| = \frac{1}{|\gamma(\sigma)|} \leq \frac{C}{|\sigma|}.$$

The properties of the function  $\gamma(\rho)$  obtained in Lemma 3.1.3 allow one to recover  $\gamma(\rho)$  in  $\Omega_+$  from its modulus  $|\gamma(\sigma)|$  given for real  $\sigma$ .

**Lemma 3.1.4.** *The following relation holds*

$$\gamma(\rho) = \exp\left(\frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\ln |\gamma(\xi)|}{\xi - \rho} d\xi\right), \quad \rho \in \Omega_+. \quad (3.1.50)$$

*Proof.* 1) The function  $\ln \gamma(\rho)$  is analytic in  $\Omega_+$  and  $\ln \gamma(\rho) = O(\rho^{-1})$  for  $|\rho| \rightarrow \infty$ ,  $\rho \in \overline{\Omega_+}$ . Consider the closed contour  $C_R$  (with counterclockwise circuit) which is the boundary of the domain  $D_R = \{\rho \in \Omega_+ : |\rho| < R\}$  (see fig. 3.1.1). By Cauchy's integral formula [con1, p.84],

$$\ln \gamma(\rho) = \frac{1}{2\pi i} \int_{C_R} \frac{\ln \gamma(\xi)}{\xi - \rho} d\xi, \quad \rho \in D_R.$$

Since

$$\lim_{R \rightarrow \infty} \frac{1}{2\pi i} \int_{\substack{|\xi|=R \\ \xi \in \overline{\Omega_+}}} \frac{\ln \gamma(\xi)}{\xi - \rho} d\xi = 0,$$

we obtain

$$\ln \gamma(\rho) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln \gamma(\xi)}{\xi - \rho} d\xi, \quad \rho \in \Omega_+. \quad (3.1.51)$$

2) Take a real  $\sigma$  and the closed contour  $C_{R,\delta}^\sigma$  (with counterclockwise circuit) consisting of the semicircles  $C_R^0 = \{\xi : \xi = R \exp(i\varphi), \varphi \in [0, \pi]\}$ ,  $\Gamma_\delta^\sigma = \{\xi : \xi - \sigma = \delta \exp(i\varphi), \varphi \in [0, \pi]\}$ ,  $\delta > 0$  and the intervals  $[-R, R] \setminus [\sigma - \delta, \sigma + \delta]$  (see fig.3.1.1).

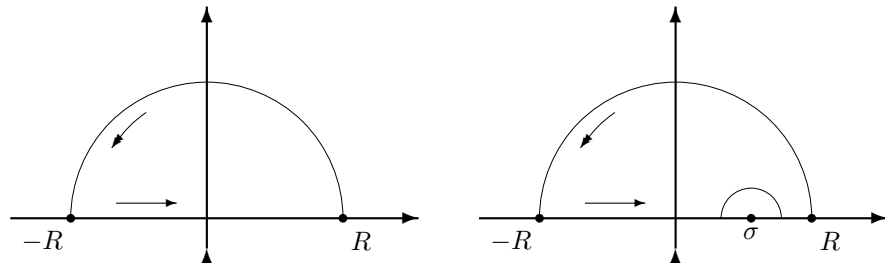


fig. 3.1.1.

By Cauchy's theorem,

$$\frac{1}{2\pi i} \int_{C_{R,\delta}^\sigma} \frac{\ln \gamma(\xi)}{\xi - \sigma} d\xi = 0.$$

Since

$$\lim_{R \rightarrow \infty} \frac{1}{2\pi i} \int_{C_R^0} \frac{\ln \gamma(\xi)}{\xi - \sigma} d\xi = 0, \quad \lim_{\delta \rightarrow 0} \frac{1}{2\pi i} \int_{\Gamma_\delta^\sigma} \frac{\ln \gamma(\xi)}{\xi - \sigma} d\xi = -\frac{1}{2} \ln \gamma(\sigma),$$

we get for real  $\sigma$ ,

$$\ln \gamma(\sigma) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\ln \gamma(\xi)}{\xi - \sigma} d\xi. \quad (3.1.52)$$

In (3.1.52) (and everywhere below where necessary) the integral is understood in the principal value sense.

3) Let  $\gamma(\sigma) = |\gamma(\sigma)| \exp(-i\beta(\sigma))$ . Separating in (3.1.52) real and imaginary parts we obtain

$$\beta(\sigma) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln |\gamma(\xi)|}{\xi - \sigma} d\xi, \quad \ln |\gamma(\sigma)| = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\beta(\xi)}{\xi - \sigma} d\xi.$$

Then, using (3.1.51) we calculate for  $\rho \in \Omega_+$ :

$$\begin{aligned} \ln \gamma(\rho) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln |\gamma(\xi)|}{\xi - \rho} d\xi - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\beta(\xi)}{\xi - \rho} d\xi \\ &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln |\gamma(\xi)|}{\xi - \rho} d\xi - \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \frac{d\xi}{(\xi - \rho)(s - \xi)} \right) \ln |\gamma(s)| ds. \end{aligned}$$

Since

$$\frac{1}{(\xi - \rho)(s - \xi)} = \frac{1}{s - \rho} \left( \frac{1}{\xi - \rho} - \frac{1}{\xi - s} \right),$$

it follows that for  $\rho \in \Omega_+$  and real  $s$ ,

$$\int_{-\infty}^{\infty} \frac{d\xi}{(\xi - \rho)(s - \xi)} = \frac{\pi i}{s - \rho}.$$

Consequently,

$$\ln \gamma(\rho) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\ln |\gamma(\xi)|}{\xi - \rho} d\xi, \quad \rho \in \Omega_+,$$

and we arrive at (3.1.50). □

It follows from (3.1.21) and (3.1.26) that for real  $\rho \neq 0$ ,

$$\frac{1}{|a(\rho)|^2} = 1 - |s^\pm(\rho)|^2.$$

By virtue of (3.1.40) this yields for real  $\rho \neq 0$ ,

$$|\gamma(\rho)| = \sqrt{1 - |s^\pm(\rho)|^2}.$$

Using (3.1.40) and (3.1.50) we obtain

$$a(\rho) = \prod_{k=1}^N \frac{\rho - i\tau_k}{\rho + i\tau_k} \exp \left( -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln(1 - |s^\pm(\xi)|^2)}{\xi - \rho} d\xi \right), \quad \rho \in \Omega_+. \quad (3.1.53)$$

We note that since the function  $\rho a(\rho)$  is continuous in  $\overline{\Omega_+}$ , it follows that

$$\frac{\rho^2}{1 - |s^\pm(\rho)|^2} = O(1) \text{ as } |\rho| \rightarrow 0.$$

Relation (3.1.53) allows one to establish connections between the scattering data  $J^+$  and  $J^-$ . More precisely, from the given data  $J^+$  one can uniquely reconstruct  $J^-$  (and vice versa) by the following algorithm.

**Algorithm 3.1.1.** Let  $J^+$  be given. Then

- 1) Construct the function  $a(\rho)$  by (3.1.53);
- 2) Calculate  $d_k$  and  $\alpha_k^-$ ,  $k = \overline{1, N}$  by (3.1.32);
- 3) Find  $b(\rho)$  and  $s^-(\rho)$  by (3.1.26).

## 3.2. THE MAIN EQUATION

**3.2.1.** The *inverse scattering problem* is formulated as follows:

Given the scattering data  $J^+$  (or  $J^-$ ), construct the potential  $q$ .

The central role for constructing the solution of the inverse scattering problem is played by the so-called main equation which is a linear integral equation of Fredholm type. In this section we give a derivation of the main equation and study its properties. In Section 3.3 we provide the solution of the inverse scattering problem along with necessary and sufficient conditions of its solvability.

**Theorem 3.2.1.** *For each fixed  $x$ , the functions  $A^\pm(x, t)$ , defined in (3.1.8), satisfy the integral equations*

$$F^+(x + y) + A^+(x, y) + \int_x^\infty A^+(x, t)F^+(t + y) dt = 0, \quad y > x, \quad (3.2.1)$$

$$F^-(x + y) + A^-(x, y) + \int_{-\infty}^x A^-(x, t)F^-(t + y) dt = 0, \quad y < x, \quad (3.2.2)$$

where

$$F^\pm(x) = R^\pm(x) + \sum_{k=1}^N \alpha_k^\pm \exp(\mp \tau_k x), \quad (3.2.3)$$

and the functions  $R^\pm(x)$  are defined by (3.1.28).

Equations (3.2.1) and (3.2.2) are called the *main equations* or *Gelfand-Levitan-Marchenko equations*.

*Proof.* By virtue of (3.1.18) and (3.1.19),

$$\left( \frac{1}{a(\rho)} - 1 \right) g(x, \rho) = s^+(\rho)e(x, \rho) + e(x, -\rho) - g(x, \rho). \quad (3.2.4)$$

Put  $A^+(x, t) = 0$  for  $t < x$ , and  $A^-(x, t) = 0$  for  $t > x$ . Then, using (3.1.8) and (3.1.29), we get

$$s^+(\rho)e(x, \rho) + e(x, -\rho) - g(x, \rho)$$



$$\begin{aligned}
&= \left( \int_{-\infty}^{\infty} R^+(y) \exp(i\rho y) dy \right) \left( \exp(i\rho x) + \int_{-\infty}^{\infty} A^+(x, t) \exp(i\rho t) dt \right) \\
&+ \int_{-\infty}^{\infty} (A^+(x, t) - A^-(x, t)) \exp(-i\rho t) dt = \int_{-\infty}^{\infty} H(x, y) \exp(-i\rho y) dy,
\end{aligned}$$

where

$$H(x, y) = A^+(x, y) - A^-(x, y) + R^+(x + y) + \int_x^{\infty} A^+(x, t) R^+(t + y) dt. \quad (3.2.5)$$

Thus, for each fixed  $x$ , the right-hand side in (3.2.4) is the Fourier transform of the function  $H(x, y)$ . Hence

$$H(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{1}{a(\rho)} - 1 \right) g(x, \rho) \exp(i\rho y) d\rho. \quad (3.2.6)$$

Fix  $x$  and  $y$  ( $y > x$ ) and consider the function

$$f(\rho) := \left( \frac{1}{a(\rho)} - 1 \right) g(x, \rho) \exp(i\rho y). \quad (3.2.7)$$

According to (3.1.6) and (3.1.23),

$$f(\rho) = \frac{c}{\rho} \exp(i\rho(y - x))(1 + o(1)), \quad |\rho| \rightarrow \infty, \quad \rho \in \overline{\Omega_+}. \quad (3.2.8)$$

Let  $C_{\delta, R}$  be a closed contour (with counterclockwise circuit) which is the boundary of the domain  $D_{\delta, R} = \{\rho \in \Omega_+ : \delta < |\rho| < R\}$ , where  $\delta < \tau_1 < \dots < \tau_N < R$ . Thus, all zeros  $\rho_k = i\tau_k$ ,  $k = \overline{1, N}$  of  $a(\rho)$  are contained in  $D_{\delta, R}$ . By the residue theorem,

$$\frac{1}{2\pi i} \int_{C_{\delta, R}} f(\rho) d\rho = \sum_{k=1}^N \operatorname{Res}_{\rho=\rho_k} f(\rho).$$

On the other hand, it follows from (3.2.7)-(3.2.8), (3.1.5) and (3.1.49) that

$$\lim_{R \rightarrow \infty} \frac{1}{2\pi i} \int_{\substack{|\rho|=R \\ \rho \in \Omega_+}} f(\rho) d\rho = 0, \quad \lim_{\delta \rightarrow 0} \frac{1}{2\pi i} \int_{\substack{|\rho|=\delta \\ \rho \in \Omega_+}} f(\rho) d\rho = 0.$$

Hence

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} f(\rho) d\rho = \sum_{k=1}^N \operatorname{Res}_{\rho=\rho_k} f(\rho).$$

From this and (3.2.6)-(3.2.7) it follows that

$$H(x, y) = i \sum_{k=1}^N \frac{g(x, i\tau_k) \exp(-\tau_k y)}{a_1(i\tau_k)}.$$

Using the fact that all eigenvalues are simple, (3.1.30), (3.1.8) and (3.1.32) we obtain

$$\begin{aligned}
H(x, y) &= i \sum_{k=1}^N \frac{d_k e(x, i\tau_k) \exp(-\tau_k y)}{a_1(i\tau_k)} \\
&= - \sum_{k=1}^N \alpha_k^+ \left( \exp(-\tau_k(x + y)) + \int_x^{\infty} A^+(x, t) \exp(-\tau_k(t + y)) dt \right). \quad (3.2.9)
\end{aligned}$$

Since  $A^-(x, y) = 0$  for  $y > x$ , (3.2.5) and (3.2.9) yield (3.2.1).  
Relation (3.2.2) is proved analogously.  $\square$

**Lemma 3.2.1.** *Let nonnegative functions  $v(x), u(x)$  ( $a \leq x \leq T \leq \infty$ ) be given such that  $v(x) \in L(a, T)$ ,  $u(x)v(x) \in L(a, T)$ , and let  $c_1 \geq 0$ . If*

$$u(x) \leq c_1 + \int_x^T v(t)u(t) dt, \quad (3.2.10)$$

then

$$u(x) \leq c_1 \exp \left( \int_x^T v(t) dt \right). \quad (3.2.11)$$

*Proof.* Denote

$$\xi(x) := c_1 + \int_x^T v(t)u(t) dt.$$

Then  $\xi(T) = c_1$ ,  $-\xi'(x) = v(x)u(x)$ , and (3.2.10) yields

$$0 \leq -\xi'(x) \leq v(x)\xi(x).$$

Let  $c_1 > 0$ . Then  $\xi(x) > 0$ , and

$$0 \leq -\frac{\xi'(x)}{\xi(x)} \leq v(x).$$

Integrating this inequality we obtain

$$\ln \frac{\xi(x)}{\xi(T)} \leq \int_x^T v(t) dt,$$

and consequently,

$$\xi(x) \leq c_1 \exp \left( \int_x^T v(t) dt \right).$$

According to (3.2.10)  $u(x) \leq \xi(x)$ , and we arrive at (3.2.11).

If  $c_1 = 0$ , then  $\xi(x) = 0$ . Indeed, suppose on the contrary that  $\xi(x) \neq 0$ . Then, there exists  $T_0 \leq T$  such that  $\xi(x) > 0$  for  $x < T_0$ , and  $\xi(x) \equiv 0$  for  $x \in [T_0, T]$ . Repeating the arguments we get for  $x < T_0$  and sufficiently small  $\varepsilon > 0$ ,

$$\ln \frac{\xi(x)}{\xi(T_0 - \varepsilon)} \leq \int_x^{T_0 - \varepsilon} v(t) dt \leq \int_x^{T_0} v(t) dt,$$

which is impossible. Thus,  $\xi(x) \equiv 0$ , and (3.2.11) becomes obvious.  $\square$

**Lemma 3.2.2.** *The functions  $F^\pm(x)$  are absolutely continuous and for each fixed  $a > -\infty$ ,*

$$\int_a^\infty |F^\pm(\pm x)| dx < \infty, \quad \int_a^\infty (1 + |x|) |F^{\pm'}(\pm x)| dx < \infty. \quad (3.2.12)$$

*Proof.* 1) According to (3.2.3) and (3.1.28),  $F^+(x) \in L_2(a, \infty)$  for each fixed  $a > -\infty$ . By continuity, (3.2.1) is also valid for  $y = x$ :

$$F^+(2x) + A^+(x, x) + \int_x^\infty A^+(x, t) F^+(t + x) dt = 0. \quad (3.2.13)$$

Rewrite (3.2.13) to the form

$$F^+(2x) + A^+(x, x) + 2 \int_x^\infty A^+(x, 2\xi - x) F^+(2\xi) d\xi = 0. \quad (3.2.14)$$

It follows from (3.2.14) and (3.1.10) that the function  $F^+(x)$  is continuous, and for  $x \geq a$ ,

$$|F^+(2x)| \leq \frac{1}{2} Q_0^+(x) + \exp(Q_1^+(a)) \int_x^\infty Q_0^+(\xi) |F^+(2\xi)| d\xi. \quad (3.2.15)$$

Fix  $r \geq a$ . Then for  $x \geq r$ , (3.2.15) yields

$$|F^+(2x)| \leq \frac{1}{2} Q_0^+(r) + \exp(Q_1^+(a)) \int_x^\infty Q_0^+(\xi) |F^+(2\xi)| d\xi.$$

Applying Lemma 3.2.1 we obtain

$$|F^+(2x)| \leq \frac{1}{2} Q_0^+(r) \exp(Q_1^+(a) \exp(Q_1^+(a))), \quad x \geq r \geq a,$$

and consequently

$$|F^+(2x)| \leq C_a Q_0^+(x), \quad x \geq a. \quad (3.2.16)$$

It follows from (3.2.16) that for each fixed  $a > -\infty$ ,

$$\int_a^\infty |F^+(x)| dx < \infty.$$

2) By virtue of (3.2.14), the function  $F^+(x)$  is absolutely continuous, and

$$\begin{aligned} & 2F^{+'}(2x) + \frac{d}{dx} A^+(x, x) - 2A^+(x, x) F^+(2x) \\ & + 2 \int_x^\infty (A_1^+(x, 2\xi - x) + A_2^+(x, 2\xi - x)) F^+(2\xi) d\xi = 0, \end{aligned}$$

where

$$A_1^+(x, t) = \frac{\partial A^+(x, t)}{\partial x}, \quad A_2^+(x, t) = \frac{\partial A^+(x, t)}{\partial t}.$$

Taking (3.1.9) into account we get

$$F^{+'}(2x) = \frac{1}{4} q(x) + P(x), \quad (3.2.17)$$

where

$$P(x) = - \int_x^\infty (A_1^+(x, 2\xi - x) + A_2^+(x, 2\xi - x)) F^+(2\xi) d\xi - \frac{1}{2} F^+(2x) \int_x^\infty q(t) dt.$$

It follows from (3.2.16) and (3.1.11) that

$$|P(x)| \leq C_a (Q_0^+(x))^2, \quad x \geq a. \quad (3.2.18)$$

Since

$$x Q_0^+(x) \leq \int_x^\infty t |q(t)| dt,$$

it follows from (3.2.17) and (3.2.18) that for each fixed  $a > -\infty$ ,

$$\int_a^\infty (1 + |x|)|F^{+'}(x)| dx < \infty,$$

and (3.2.12) is proved for the function  $F^+(x)$ . For  $F^-(x)$  the arguments are similar.  $\square$

**3.2.2.** Now we are going to study the solvability of the main equations (3.2.1) and (3.2.2). Let sets  $J^\pm = \{s^\pm(\rho), \lambda_k, \alpha_k^\pm; \rho \in \mathbf{R}, k = \overline{1, N}\}$  be given satisfying the following condition.

**Condition A.** For real  $\rho \neq 0$ , the functions  $s^\pm(\rho)$  are continuous,  $|s^\pm(\rho)| < 1$ ,  $\overline{s^\pm(\rho)} = s^\pm(-\rho)$  and  $s^\pm(\rho) = o\left(\frac{1}{\rho}\right)$  as  $|\rho| \rightarrow \infty$ . The real functions  $R^\pm(x)$ , defined by (3.1.28), are absolutely continuous,  $R^\pm(x) \in L_2(-\infty, \infty)$ , and for each fixed  $a > -\infty$ ,

$$\int_a^\infty |R^\pm(\pm x)| dx < \infty, \quad \int_a^\infty (1 + |x|)|R^{\pm'}(\pm x)| dx < \infty. \quad (3.2.19)$$

Moreover,  $\lambda_k = -\tau_k^2 < 0$ ,  $\alpha_k^\pm > 0$ ,  $k = \overline{1, N}$ .

**Theorem 3.2.2.** Let sets  $J^+$  ( $J^-$ ) be given satisfying Condition A. Then for each fixed  $x$ , the integral equation (3.2.1) ((3.2.2) respectively) has a unique solution  $A^+(x, y) \in L(x, \infty)$  ( $A^-(x, y) \in L(-\infty, x)$  respectively).

*Proof.* For definiteness we consider equation (3.2.1). For (3.2.2) the arguments are the same. It is easy to check that for each fixed  $x$ , the operator

$$(J_x f)(y) = \int_x^\infty F^+(t + y)f(t) dt, \quad y > x$$

is compact in  $L(x, \infty)$ . Therefore, it is sufficient to prove that the homogeneous equation

$$f(y) + \int_x^\infty F^+(t + y)f(t) dt = 0 \quad (3.2.20)$$

has only the zero solution. Let  $f(y) \in L(x, \infty)$  be a real function satisfying (3.2.20). It follows from (3.2.20) and Condition A that the functions  $F^+(y)$  and  $f(y)$  are bounded on the half-line  $y > x$ , and consequently  $f(y) \in L_2(x, \infty)$ . Using (3.2.3) and (3.1.28) we calculate

$$\begin{aligned} 0 &= \int_x^\infty f^2(y) dy + \int_x^\infty \int_x^\infty F^+(t + y)f(t)f(y) dt dy \\ &= \int_x^\infty f^2(y) dy + \sum_{k=1}^N \alpha_k^+ \left( \int_x^\infty f(y) \exp(-\tau_k y) dy \right)^2 + \frac{1}{2\pi} \int_{-\infty}^\infty s^+(\rho) \Phi^2(\rho) d\rho, \end{aligned}$$

where

$$\Phi(\rho) = \int_x^\infty f(y) \exp(i\rho y) dy.$$

According to Parseval's equality

$$\int_x^\infty f^2(y) dy = \frac{1}{2\pi} \int_{-\infty}^\infty |\Phi(\rho)|^2 d\rho,$$

and hence

$$\sum_{k=1}^N \alpha_k^+ \left( \int_x^\infty f(y) \exp(-\tau_k y) dy \right)^2 + \frac{1}{2\pi} \int_{-\infty}^\infty |\Phi(\rho)|^2 (1 - |s^+(\rho)| \exp(i(2\theta(\rho) + \eta(\rho)))) d\rho = 0,$$

where  $\theta(\rho) = \arg \Phi(\rho)$ ,  $\eta(\rho) = \arg(-s^+(\rho))$ . In this equality we take the real part:

$$\sum_{k=1}^N \alpha_k^+ \left( \int_x^\infty f(y) \exp(-\tau_k y) dy \right)^2 + \frac{1}{2\pi} \int_{-\infty}^\infty |\Phi(\rho)|^2 \left( 1 - |s^+(\rho)| \cos((2\theta(\rho) + \eta(\rho))) \right) d\rho = 0.$$

Since  $|s^+(\rho)| < 1$ , this is possible only if  $\Phi(\rho) \equiv 0$ . Then  $f(y) = 0$ , and Theorem 3.2.2 is proved.  $\square$

**Remark 3.2.1.** The main equations (3.2.1)-(3.2.2) can be rewritten in the form

$$\left. \begin{aligned} F^+(2x+y) + B^+(x,y) + \int_0^\infty B^+(x,t)F^+(2x+y+t) dt &= 0, & y > 0, \\ F^-(2x+y) + B^-(x,y) + \int_{-\infty}^0 B^-(x,t)F^-(2x+y+t) dt &= 0, & y < 0, \end{aligned} \right\} \quad (3.2.21)$$

where  $B^\pm(x,y) = A^\pm(x, x+y)$ .

### 3.3. THE INVERSE SCATTERING PROBLEM

**3.3.1.** In this section, using the main equations (3.2.1)-(3.2.2), we provide the solution of the inverse scattering problem of recovering the potential  $q$  from the given scattering data  $J^+$  (or  $J^-$ ). First we prove the uniqueness theorem.

**Theorem 3.3.1.** *The specification of the scattering data  $J^+$  (or  $J^-$ ) uniquely determines the potential  $q$ .*

*Proof.* Let  $J^+$  and  $\tilde{J}^+$  be the right scattering data for the potentials  $q$  and  $\tilde{q}$  respectively, and let  $J^+ = \tilde{J}^+$ . Then, it follows from (3.2.3) and (3.1.28) that  $F^+(x) = \tilde{F}^+(x)$ . By virtue of Theorems 3.2.1 and 3.2.2,  $A^+(x,y) = \tilde{A}^+(x,y)$ . Therefore, taking (3.1.9) into account, we get  $q = \tilde{q}$ . For  $J^-$  the arguments are the same.  $\square$

The solution of the inverse scattering problem can be constructed by the following algorithm.

**Algorithm 3.3.1.** Let the scattering data  $J^+$  (or  $J^-$ ) be given. Then

- 1) Calculate the function  $F^+(x)$  (or  $F^-(x)$ ) by (3.2.3) and (3.1.28).
- 2) Find  $A^+(x,y)$  (or  $A^-(x,y)$ ) by solving the main equation (3.2.1) (or (3.2.2) respectively).
- 3) Construct  $q(x) = -2\frac{d}{dx}A^+(x,x)$  (or  $q(x) = 2\frac{d}{dx}A^-(x,x)$ ).

Now we going to study necessary and sufficient conditions for the solvability of the inverse scattering problem. In other words we will describe conditions under which a set  $J^+$  (or  $J^-$ ) represents the scattering data for a certain potential  $q$ . First we prove the following auxiliary assertion.

**Lemma 3.3.1.** *Let sets  $J^\pm = \{s^\pm(\rho), \lambda_k, \alpha_k^\pm; \rho \in \mathbf{R}, k = \overline{1, N}\}$  be given satisfying Condition A, and let  $A^\pm(x,y)$  be the solutions of the integral equations (3.2.1)-(3.2.2). Construct the functions  $e(x, \rho)$  and  $g(x, \rho)$  by (3.1.8) and the functions  $q^\pm(x)$  by the formulae*

$$q^+(x) = -2\frac{d}{dx}A^+(x,x), \quad q^-(x) = 2\frac{d}{dx}A^-(x,x). \quad (3.3.1)$$

Then, for each fixed  $a > -\infty$ ,

$$\int_a^\infty (1 + |x|)|q^\pm(\pm x)| dx < \infty, \quad (3.3.2)$$

and

$$-e''(x, \rho) + q^+(x)e(x, \rho) = \rho^2 e(x, \rho), \quad -g''(x, \rho) + q^-(x)g(x, \rho) = \rho^2 g(x, \rho). \quad (3.3.3)$$

*Proof.* 1) It follows from (3.2.19) and (3.2.3) that

$$\int_a^\infty |F^\pm(\pm x)| dx < \infty, \quad \int_a^\infty (1 + |x|)|F^{\pm'}(\pm x)| dx < \infty. \quad (3.3.4)$$

We rewrite (3.2.1) in the form

$$F^+(y + 2x) + A^+(x, x + y) + \int_0^\infty A^+(x, x + t)F^+(t + y + 2x) dt = 0, \quad y > 0, \quad (3.3.5)$$

and for each fixed  $x$  consider the operator

$$(\mathcal{F}_x f)(y) = \int_0^\infty F^+(t + y + 2x)f(t) dt, \quad y \geq 0$$

in  $L(0, \infty)$ . It follows from Theorem 3.2.2 that there exists  $(E + \mathcal{F}_x)^{-1}$ , where  $E$  is the identity operator, and  $\|(E + \mathcal{F}_x)^{-1}\| < \infty$ . Using Lemma 1.5.1 (or an analog of Lemma 1.5.2 for an infinite interval) it is easy to verify that  $A^+(x, y)$ ,  $y \geq x$  and  $\|(E + \mathcal{F}_x)^{-1}\|$  are continuous functions. Since

$$\|\mathcal{F}_x\| = \sup_y \int_0^\infty |F^+(t + y + 2x)| dt = \sup_y \int_{y+2x}^\infty |F^+(\xi)| d\xi \leq \int_{2x}^\infty |F^+(\xi)| d\xi,$$

we get

$$\lim_{x \rightarrow \infty} \|\mathcal{F}_x\| = 0.$$

Therefore,

$$C_a^0 := \sup_{x \geq a} \|(E + \mathcal{F}_x)^{-1}\| < \infty.$$

Denote

$$\tau_0(x) = \int_x^\infty |F^{+'}(t)| dt, \quad \tau_1(x) = \int_x^\infty \tau_0(t) dt = \int_x^\infty (t - x)|F^{+'}(t)| dt.$$

Then

$$|F^+(x)| \leq \tau_0(x). \quad (3.3.6)$$

It follows from (3.3.5) that

$$A^+(x, x + y) = -(E + \mathcal{F}_x)^{-1}F^+(y + 2x),$$

consequently the preceding formulae yield,

$$\int_0^\infty |A^+(x, x + y)| dy \leq C_a^0 \int_0^\infty |F^+(y + 2x)| dy \leq C_a^0 \tau_1(2x), \quad x \geq a. \quad (3.3.7)$$

Using (3.3.5), (3.3.6) and (3.3.7) we calculate

$$\begin{aligned} |A^+(x, x+y)| &\leq \tau_0(y+2x) + \int_0^\infty |A^+(x, x+t)|\tau_0(t+y+2x) dt \\ &\leq (1 + C_a^0\tau_1(2x))\tau_0(y+2x). \end{aligned} \quad (3.3.8)$$

Applying Lemma 1.5.1 one can show that the function  $A^+(x, y)$  has the first derivatives

$$A_1^+(x, y) := \frac{\partial A^+(x, y)}{\partial x}, \quad A_2^+(x, y) := \frac{\partial A^+(x, y)}{\partial y},$$

and therefore, differentiating (3.2.1), we get

$$\left. \begin{aligned} F^{+'}(x+y) + A_1^+(x, y) - A^+(x, x)F^+(x+y) + \int_x^\infty A_1^+(x, t)F^+(t+y) dt &= 0, \\ F^{+'}(x+y) + A_2^+(x, y) + \int_x^\infty A^+(x, t)F^{+'}(t+y) dt &= 0. \end{aligned} \right\} \quad (3.3.9)$$

Denote  $A_0^+(x, x+y) := \frac{d}{dx}A^+(x, x+y)$ . Differentiating (3.3.5) with respect to  $x$  we obtain

$$\begin{aligned} &2F^{+'}(y+2x) + A_0^+(x, x+y) \\ &+ \int_0^\infty A_0^+(x, x+t)F^+(t+y+2x) dt + 2 \int_0^\infty A^+(x, x+t)F^{+'}(t+y+2x) dt = 0, \end{aligned} \quad (3.3.10)$$

and consequently

$$\begin{aligned} &\int_0^\infty |A_0^+(x, x+y)| dy \leq 2C_a^0 \left( \int_0^\infty |F^{+'}(y+2x)| dy \right. \\ &\left. + \int_0^\infty |A^+(x, x+t)| \left( \int_0^\infty |F^{+'}(t+y+2x)| dy \right) dt \right). \end{aligned}$$

By virtue of (3.3.8), this yields

$$\begin{aligned} \int_0^\infty |A_0^+(x, x+y)| dy &\leq 2C_a^0 \left( \tau_0(2x) + (1 + C_a^0\tau_1(2x)) \int_0^\infty \tau_0^2(t+2x) dt \right) \\ &\leq 2C_a^0\tau_0(2x) \left( 1 + (1 + C_a^0\tau_1(2x))\tau_1(2x) \right). \end{aligned} \quad (3.3.11)$$

It follows from (3.3.10) that

$$\begin{aligned} &|A_0^+(x, x+y) + 2F^{+'}(y+2x)| \leq \\ &\int_0^\infty |A_0^+(x, x+t)F^+(t+y+2x)| dt + 2 \int_0^\infty |A^+(x, x+t)F^{+'}(t+y+2x)| dt. \end{aligned}$$

Using (3.3.6), (3.3.11) and (3.3.8) we get

$$\begin{aligned} |A_0^+(x, x+y) + 2F^{+'}(y+2x)| &\leq 2C_a^0\tau_0(y+2x)\tau_0(2x)\tau_1(2x)(1 + C_a^0\tau_1(2x)) \\ &+ 2\tau_0(2x)\tau_0(y+2x)(1 + C_a^0\tau_1(2x)). \end{aligned}$$

For  $y = 0$  this implies

$$\left| 2\frac{d}{dx}A^+(x, x) + 4F^{+'}(2x) \right| \leq 2C_a\tau_0^2(2x), \quad x \geq a,$$

where  $C_a = 4(1 + C_a^0 \tau_1(2a))^2$ . Taking (3.3.1) into account we derive

$$\int_a^\infty (1 + |x|)|q^+(x)| dx \leq 4 \int_a^\infty (1 + |x|)|F^{+'}(2x)| dx + C_a \int_a^\infty (1 + |x|)\tau_0^2(2x) dx.$$

Since

$$x\tau_0(2x) \leq \int_{2x}^\infty t|F^{+'}(t)| dt \leq \int_0^\infty t|F^{+'}(t)| dt,$$

we arrive at (3.3.2) for  $q^+$ . For  $q^-$  the arguments are the same.

2) Let us now prove (3.3.3). For definiteness we will prove (3.3.3) for the function  $e(x, \rho)$ . First we additionally assume that the function  $F^{+'}(x)$  is absolutely continuous and  $F^{+''}(x) \in L(a, \infty)$  for each fixed  $a > -\infty$ . Differentiating the equality

$$J(x, y) := F^+(x + y) + A^+(x, y) + \int_x^\infty A^+(x, t)F^+(t + y) dt = 0, \quad y > x, \quad (3.3.12)$$

we calculate

$$J_{yy}(x, y) = F^{+''}(x + y) + A_{yy}^+(x, y) + \int_x^\infty A^+(x, t)F^{+''}(t + y) dt = 0, \quad (3.3.13)$$

$$\begin{aligned} J_{xx}(x, y) &= F^{+''}(x + y) + A_{xx}^+(x, y) - \frac{d}{dx}A^+(x, x)F^+(x + y) \\ &\quad - A^+(x, x)F^{+'}(x + y) - A_1^+(x, t)|_{t=x}F^+(x + y) + \int_x^\infty A_{xx}^+(x, t)F^+(t + y) dt = 0. \end{aligned} \quad (3.3.14)$$

Integration by parts in (3.3.13) yields

$$\begin{aligned} J_{yy}(x, y) &= F^{+''}(x + y) + A_{yy}^+(x, y) + \left( A^+(x, t)F^{+'}(t + y) - A_2^+(x, t)F^+(t + y) \right) \Big|_x^\infty \\ &\quad + \int_x^\infty A_{tt}^+(x, t)F^+(t + y) dt = 0. \end{aligned}$$

It follows from (3.3.8) and (3.3.9) that here the substitution at infinity is equal to zero, and consequently,

$$\begin{aligned} J_{yy}(x, y) &= F^{+''}(x + y) + A_{yy}^+(x, y) - A^+(x, x)F^{+'}(x + y) + A_2^+(x, x)F^+(x + y) \\ &\quad + \int_x^\infty A_{tt}^+(x, t)F^+(t + y) dt = 0. \end{aligned} \quad (3.3.15)$$

Using (3.3.14), (3.3.15), (3.3.12), (3.3.1) and the equality

$$J_{xx}(x, y) - J_{yy}(x, y) - q^+(x)J(x, y) = 0,$$

which follows from  $J(x, y) = 0$ , we obtain

$$f(x, y) + \int_x^\infty f(x, t)F^+(t + y) dt = 0, \quad y \geq x, \quad (3.3.16)$$

where

$$f(x, y) := A_{xx}^+(x, y) - A_{yy}^+(x, y) - q^+(x)A^+(x, y).$$

It is easy to verify that for each fixed  $x \geq a$ ,  $f(x, y) \in L(x, \infty)$ . According to Theorem 3.2.2, the homogeneous equation (3.3.16) has only the zero solution, i.e.

$$A_{xx}^+(x, y) - A_{yy}^+(x, y) - q^+(x)A^+(x, y) = 0, \quad y \geq x. \quad (3.3.17)$$



Differentiating (3.1.8) twice we get

$$\begin{aligned} e''(x, \rho) &= (i\rho)^2 \exp(i\rho x) - (i\rho)A^+(x, x) \exp(i\rho x) \\ &\quad - \left( \frac{d}{dx} A^+(x, x) + A_1^+(x, t)|_{t=x} \right) \exp(i\rho x) + \int_x^\infty A_{xx}^+(x, t) \exp(i\rho t) dt. \end{aligned} \quad (3.3.18)$$

On the other hand, integrating by parts twice we obtain

$$\begin{aligned} \rho^2 e(x, \rho) &= -(i\rho)^2 \exp(i\rho x) - (i\rho)^2 \int_x^\infty A^+(x, t) \exp(i\rho t) dt \\ &= -(i\rho)^2 \exp(i\rho x) + (i\rho)A^+(x, x) \exp(i\rho x) - A_2^+(x, t)|_{t=x} \exp(i\rho x) - \int_x^\infty A_{tt}^+(x, t) \exp(i\rho t) dt. \end{aligned}$$

Together with (3.1.8) and (3.3.18) this gives

$$\begin{aligned} e''(x, \rho) + \rho^2 e(x, \rho) - q^+(x)e(x, \rho) &= \left( -2 \frac{d}{dx} A^+(x, x) - q^+(x) \right) \exp(i\rho x) \\ &\quad + \int_x^\infty (A_{xx}^+(x, t) - A_{tt}^+(x, t) - q^+(x)A^+(x, t)) \exp(i\rho t) dt. \end{aligned}$$

Taking (3.3.1) and (3.3.17) into account we arrive at (3.3.3) for  $e(x, \rho)$ .

Let us now consider the general case when (3.3.4) holds. Denote by  $\tilde{e}(x, \rho)$  the Jost solution for the potential  $q^+$ . Our goal is to prove that  $\tilde{e}(x, \rho) \equiv e(x, \rho)$ . For this purpose we choose the functions  $F_j^+(x)$  such that  $F_j^+(x)$ ,  $F_j^{+'}(x)$  are absolutely continuous, and for each fixed  $a > -\infty$ ,  $F_j^{+''}(x) \in L(a, \infty)$  and

$$\lim_{j \rightarrow \infty} \int_a^\infty |F_j^+(x) - F^+(x)| dx = 0, \quad \lim_{j \rightarrow \infty} \int_a^\infty (1 + |x|) |F_j^{+'}(x) - F^{+'}(x)| dx = 0. \quad (3.3.19)$$

Denote

$$\tau_{0j}(x) = \int_x^\infty |F_j^{+'}(t) - F^{+'}(t)| dt, \quad \tau_{1j}(x) = \int_x^\infty \tau_{0j}(t) dt = \int_x^\infty (t - x) |F_j^{+'}(t) - F^{+'}(t)| dt.$$

Using (3.3.19) and Lemma 1.5.1 one can show that for sufficiently large  $j$ , the integral equation

$$F_j^+(x + y) + A_{(j)}^+(x, y) + \int_x^\infty A_{(j)}^+(x, t) F_j^+(t + y) dt = 0, \quad y > x,$$

has a unique solution  $A_{(j)}^+(x, y)$ , and

$$\int_x^\infty |A_{(j)}^+(x, y) - A^+(x, y)| dy \leq C_a \tau_{1j}(2x), \quad x \geq a. \quad (3.3.20)$$

Consequently,

$$\lim_{j \rightarrow \infty} \max_{x \geq a} \int_x^\infty |A_{(j)}^+(x, y) - A^+(x, y)| dy = 0. \quad (3.3.21)$$

Denote

$$e_j(x, \rho) = \exp(i\rho x) + \int_x^\infty A_{(j)}^+(x, t) \exp(i\rho t) dt, \quad (3.3.22)$$

$$q_j^+(x) = -2 \frac{d}{dx} A_{(j)}^+(x, x).$$

It was proved above that

$$-e_j''(x, \rho) + q_j^+(x)e_j(x, \rho) = \rho^2 e_j(x, \rho),$$

i.e.  $e_j(x, \rho)$  is the Jost solution for the potential  $q_j^+$ . Using (3.3.19)-(3.3.20) and acting in the same way as in the first part of the proof of Lemma 3.3.1 we get

$$\lim_{j \rightarrow \infty} \int_a^\infty (1 + |x|) |q_j^+(x) - q^+(x)| dx = 0.$$

By virtue of Lemma 3.1.1, this yields

$$\lim_{j \rightarrow \infty} \max_{\rho \in \Omega_+} \max_{x \geq a} |(e_j(x, \rho) - \tilde{e}(x, \rho)) \exp(-i\rho x)| = 0.$$

On the other hand, using (3.1.8), (3.3.21) and (3.3.22) we infer

$$\lim_{j \rightarrow \infty} \max_{\rho \in \Omega_+} \max_{x \geq a} |(e_j(x, \rho) - e(x, \rho)) \exp(-i\rho x)| = 0.$$

Consequently,  $e(x, \rho) \equiv \tilde{e}(x, \rho)$ , and (3.3.3) is proved for the function  $e(x, \rho)$ . For  $g(x, \rho)$  the arguments are similar. Therefore, Lemma 3.3.1 is proved.  $\square$

**3.3.2.** Let us now formulate necessary and sufficient conditions for the solvability of the inverse scattering problem.

**Theorem 3.3.1.** *For data  $J^+ = \{s^+(\rho), \lambda_k, \alpha_k^+; \rho \in \mathbf{R}, k = \overline{1, N}\}$  to be the right scattering data for a certain real potential  $q$  satisfying (3.1.2), it is necessary and sufficient that the following conditions hold:*

- 1)  $\lambda_k = -\tau_k^2$ ,  $0 < \tau_1 < \dots < \tau_N$ ;  $\alpha_k^+ > 0$ ,  $k = \overline{1, N}$ .
- 2) For real  $\rho \neq 0$ , the function  $s^+(\rho)$  is continuous,  $\overline{s^+(\rho)} = s^+(-\rho)$ ,  $|s^+(\rho)| < 1$ , and

$$s^+(\rho) = o\left(\frac{1}{\rho}\right) \text{ as } |\rho| \rightarrow \infty, \quad (3.3.23)$$

$$\frac{\rho^2}{1 - |s^+(\rho)|^2} = O(1) \text{ as } |\rho| \rightarrow 0. \quad (3.3.24)$$

- 3) The function  $\rho(a(\rho) - 1)$ , where  $a(\rho)$  is defined by

$$a(\rho) := \prod_{k=1}^N \frac{\rho - i\tau_k}{\rho + i\tau_k} \exp(B(\rho)), \quad B(\rho) := -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln(1 - |s^+(\xi)|^2)}{\xi - \rho} d\xi, \quad \rho \in \Omega_+, \quad (3.3.25)$$

is continuous and bounded in  $\overline{\Omega_+}$ , and

$$\frac{1}{a(\rho)} = O(1) \text{ as } |\rho| \rightarrow 0, \quad \rho \in \overline{\Omega_+}, \quad (3.3.26)$$

$$\lim_{\rho \rightarrow 0} \rho a(\rho) (s^+(\rho) + 1) = 0 \text{ for real } \rho. \quad (3.3.27)$$

- 4) The functions  $R^\pm(x)$ , defined by

$$R^\pm(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} s^\pm(\rho) \exp(\pm i\rho x) d\rho, \quad s^-(\rho) := -s^+(-\rho) \frac{a(-\rho)}{a(\rho)}, \quad (3.3.28)$$

are real and absolutely continuous,  $R^\pm(x) \in L_2(-\infty, \infty)$ , and for each fixed  $a > -\infty$ , (3.2.19) holds.

*Proof.* The necessity part of Theorem 3.3.1 was proved above. We prove here the sufficiency. Let a set  $J^+$  satisfying the hypothesis of Theorem 3.3.1 be given. According to (3.3.25),

$$B(\rho) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\theta(\xi)}{\xi - \rho} d\xi, \quad \rho \in \Omega_+, \quad \theta(\xi) := \ln \frac{1}{1 - |s^+(\xi)|^2}. \quad (3.3.29)$$

For real  $\xi \neq 0$ , the function  $\theta(\xi)$  is continuous and

$$\theta(\xi) = \theta(-\xi) \geq 0, \quad (3.3.30)$$

$$\theta(\xi) = o\left(\frac{1}{\xi^2}\right) \text{ as } \xi \rightarrow \infty,$$

$$\theta(\xi) = O\left(\ln \frac{1}{\xi}\right) \text{ as } \xi \rightarrow 0.$$

Furthermore, the function  $B(\rho)$  is analytic in  $\Omega_+$ , continuous in  $\overline{\Omega_+} \setminus \{0\}$  and for real  $\rho \neq 0$ ,

$$B(\rho) = \frac{1}{2} \theta(\rho) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\theta(\xi)}{\xi - \rho} d\xi, \quad (3.3.31)$$

where the integral in (3.3.31) is understood in the principal value sense:

$$\int_{-\infty}^{\infty} := \lim_{\varepsilon \rightarrow 0} \left( \int_{-\infty}^{\rho - \varepsilon} + \int_{\rho + \varepsilon}^{\infty} \right).$$

It follows from assumption 3) of the theorem, (3.3.25), (3.3.30) and (3.3.31) that

$$\overline{a(\rho)} = a(-\rho) \text{ for real } \rho \neq 0. \quad (3.3.32)$$

Moreover, for real  $\rho \neq 0$ , (3.3.25) and (3.3.31) imply

$$|a(\rho)|^2 = |\exp(B(\rho))|^2 = \exp(2\operatorname{Re} B(\rho)) = \exp(\theta(\rho)),$$

and consequently,

$$1 - |s^+(\rho)|^2 = \frac{1}{|a(\rho)|^2} \text{ for real } \rho \neq 0. \quad (3.3.33)$$

Furthermore, the function  $s^-(\rho)$ , defined by (3.3.28), is continuous for real  $\rho \neq 0$ , and by virtue of (3.3.32),

$$\overline{s^-(\rho)} = s^-(-\rho), \quad |s^-(\rho)| = |s^+(\rho)|.$$

Therefore, taking (3.3.23), (3.3.24) and (3.3.27) into account we obtain

$$s^-(\rho) = o\left(\frac{1}{\rho}\right) \text{ as } |\rho| \rightarrow \infty,$$

$$\frac{\rho^2}{1 - |s^-(\rho)|^2} = O(1) \text{ as } |\rho| \rightarrow 0,$$

$$\lim_{\rho \rightarrow 0} \rho a(\rho) (s^-(\rho) + 1) = 0 \quad \text{for real } \rho.$$

Let us show that

$$a_1^2(\rho_k) < 0, \quad k = \overline{1, N}, \quad (3.3.34)$$

where  $a_1(\rho) = \frac{d}{d\rho} a(\rho)$ ,  $\rho_k = i\tau_k$ . Indeed, it follows from (3.3.25) that

$$a_1(\rho_k) = \frac{d}{d\rho} \left( \prod_{j=1}^N \frac{\rho - i\tau_j}{\rho + i\tau_j} \right)_{|\rho=\rho_k} \exp(B(\rho_k)).$$

Using (3.3.30) we calculate

$$\begin{aligned} B(\rho_k) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\theta(\xi)}{\xi - i\tau_k} d\xi \\ &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\xi \theta(\xi)}{\xi^2 + \tau_k^2} d\xi + \frac{\tau_k}{2\pi} \int_{-\infty}^{\infty} \frac{\theta(\xi)}{\xi^2 + \tau_k^2} d\xi = \frac{\tau_k}{2\pi} \int_{-\infty}^{\infty} \frac{\theta(\xi)}{\xi^2 + \tau_k^2} d\xi. \end{aligned}$$

Since

$$\frac{d}{d\rho} \left( \prod_{j=1}^N \frac{\rho - i\tau_j}{\rho + i\tau_j} \right)_{|\rho=\rho_k} = \frac{1}{2i\tau_k} \prod_{\substack{j=1 \\ j \neq k}}^N \frac{\tau_k - \tau_j}{\tau_k + \tau_j},$$

the numbers  $a_1(\rho_k)$  are pure imaginary, and (3.3.34) is proved.

Denote

$$\alpha_k^- = -\frac{1}{a_1^2(\rho_k) \alpha_k^+}, \quad k = \overline{1, N}. \quad (3.3.35)$$

According to (3.3.34)-(3.3.35),

$$\alpha_k^- > 0, \quad k = \overline{1, N}.$$

Thus, we have the sets  $J^\pm = \{s^\pm(\rho), \lambda_k, \alpha_k^\pm; \rho \in \mathbf{R}, k = \overline{1, N}\}$  which satisfy Condition A. Therefore, Theorem 3.2.2 and Lemma 3.3.1 hold. Let  $A^\pm(x, y)$  be the solutions of (3.2.1)-(3.2.2). We construct the functions  $e(x, \rho)$  and  $g(x, \rho)$  by (3.1.8) and the functions  $q^\pm(x)$  by (3.3.1). Then (3.3.2)-(3.3.4) are valid.

**Lemma 3.3.2.** *The following relations hold*

$$\left. \begin{aligned} s^+(\rho) e(x, \rho) + e(x, -\rho) &= \frac{g(x, \rho)}{a(\rho)}, \\ s^-(\rho) g(x, \rho) + g(x, -\rho) &= \frac{e(x, \rho)}{a(\rho)}. \end{aligned} \right\} \quad (3.3.36)$$

*Proof.* 1) Denote

$$\begin{aligned} \Phi^+(x, y) &= R^+(x + y) + \int_x^\infty A^+(x, t) R^+(t + y) dt, \\ \Phi^-(x, y) &= R^-(x + y) + \int_{-\infty}^x A^-(x, t) R^-(t + y) dt. \end{aligned}$$

For each fixed  $x$ ,  $\Phi^\pm(x, y) \in L_2(-\infty, \infty)$ , and

$$\left. \begin{aligned} \int_{-\infty}^\infty \Phi^+(x, y) \exp(-i\rho y) dy &= s^+(\rho) e(x, \rho), \\ \int_{-\infty}^\infty \Phi^-(x, y) \exp(i\rho y) dy &= s^-(\rho) g(x, \rho). \end{aligned} \right\} \quad (3.3.37)$$

Indeed, using (3.1.8) and (3.3.28), we calculate

$$\begin{aligned}
s^+(\rho)e(x, \rho) &= \left( \exp(i\rho x) + \int_x^\infty A^+(x, t) \exp(i\rho t) dt \right) \int_{-\infty}^\infty R^+(\xi) \exp(-i\rho\xi) d\xi \\
&= \int_{-\infty}^\infty R^+(x+y) \exp(-i\rho y) dy + \int_x^\infty A^+(x, t) \left( \int_{-\infty}^\infty R^+(t+y) \exp(-i\rho y) dy \right) dt \\
&= \int_{-\infty}^\infty \left( R^+(x+y) + \int_x^\infty A^+(x, t) R^+(t+y) dt \right) \exp(-i\rho y) dy = \int_{-\infty}^\infty \Phi^+(x, y) \exp(-i\rho y) dy.
\end{aligned}$$

The second relation in (3.3.37) is proved similarly.

On the other hand, it follows from (3.2.1)-(3.2.2) that

$$\begin{aligned}
\Phi^+(x, y) &= -A^+(x, y) - \sum_{k=1}^N \alpha_k^+ \exp(-\tau_k y) e(x, i\tau_k), \quad y > x, \\
\Phi^-(x, y) &= -A^-(x, y) - \sum_{k=1}^N \alpha_k^- \exp(\tau_k y) g(x, i\tau_k), \quad y < x.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_{-\infty}^\infty \Phi^+(x, y) \exp(-i\rho y) dy = \int_{-\infty}^x \Phi^+(x, y) \exp(-i\rho y) dy \\
&- \int_x^\infty \left( A^+(x, y) + \sum_{k=1}^N \alpha_k^+ \exp(-\tau_k y) e(x, i\tau_k) \right) \exp(-i\rho y) dy.
\end{aligned}$$

According to (3.1.8),

$$\int_x^\infty A^+(x, y) \exp(-i\rho y) dy = e(x, -\rho) - \exp(-i\rho x),$$

and consequently,

$$\begin{aligned}
&\int_{-\infty}^\infty \Phi^+(x, y) \exp(-i\rho y) dy = \int_{-\infty}^x \Phi^+(x, y) \exp(-i\rho y) dy \\
&+ \exp(-i\rho x) - e(x, -\rho) - \sum_{k=1}^N \frac{\alpha_k^+}{\tau_k + i\rho} \exp(-i\rho x) \exp(-\tau_k x) e(x, i\tau_k).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{-\infty}^\infty \Phi^-(x, y) \exp(i\rho y) dy = \int_x^\infty \Phi^-(x, y) \exp(i\rho y) dy \\
&+ \exp(i\rho x) - g(x, -\rho) - \sum_{k=1}^N \frac{\alpha_k^-}{\tau_k + i\rho} \exp(i\rho x) \exp(\tau_k x) g(x, i\tau_k).
\end{aligned}$$

Comparing with (3.3.37) we deduce

$$\left. \begin{aligned} s^+(\rho)e(x, \rho) + e(x, -\rho) &= \frac{h^-(x, \rho)}{a(\rho)}, \\ s^-(\rho)g(x, \rho) + g(x, -\rho) &= \frac{h^+(x, \rho)}{a(\rho)}, \end{aligned} \right\} \quad (3.3.38)$$

where

$$\left. \begin{aligned} h^-(x, \rho) &:= \exp(-i\rho x) a(\rho) \left( 1 + \int_{-\infty}^x \Phi^+(x, y) \exp(i\rho(x-y)) dy \right. \\ &\quad \left. - \sum_{k=1}^N \frac{\alpha_k^+}{\tau_k + i\rho} \exp(-\tau_k x) e(x, i\tau_k) \right), \\ h^+(x, \rho) &:= \exp(i\rho x) a(\rho) \left( 1 + \int_x^{\infty} \Phi^-(x, y) \exp(i\rho(y-x)) dy \right. \\ &\quad \left. - \sum_{k=1}^N \frac{\alpha_k^-}{\tau_k + i\rho} \exp(\tau_k x) g(x, i\tau_k) \right). \end{aligned} \right\} \quad (3.3.39)$$

2) Let us study the properties of the functions  $h^\pm(x, \rho)$ . By virtue of (3.3.38),

$$\left. \begin{aligned} h^-(x, \rho) &= a(\rho) \left( s^+(\rho) e(x, \rho) + e(x, -\rho) \right), \\ h^+(x, \rho) &= a(\rho) \left( s^-(\rho) g(x, \rho) + g(x, -\rho) \right). \end{aligned} \right\} \quad (3.3.40)$$

In particular, this yields that the functions  $h^\pm(x, \rho)$  are continuous for real  $\rho \neq 0$ , and in view of (3.3.32),  $\overline{h^\pm(x, \rho)} = h^\pm(x, -\rho)$ . Since

$$\lim_{\rho \rightarrow 0} \rho a(\rho) (s^\pm(\rho) + 1) = 0,$$

it follows from (3.3.40) that

$$\lim_{\rho \rightarrow 0} \rho h^\pm(x, \rho) = 0, \quad (3.3.41)$$

and consequently the functions  $\rho h^\pm(x, \rho)$  are continuous for real  $\rho$ . By virtue of (3.3.39) the functions  $\rho h^\pm(x, \rho)$  are analytic in  $\Omega_+$ , continuous in  $\overline{\Omega}_+$ , and (3.3.41) is valid for  $\rho \in \overline{\Omega}_+$ . Taking (3.3.40) and (3.1.7) into account we get

$$\langle e(x, \rho), h^-(x, \rho) \rangle = \langle h^+(x, \rho), g(x, \rho) \rangle = -2i\rho a(\rho). \quad (3.3.42)$$

Since  $|s^\pm(\rho)| < 1$ , it follows from (3.3.38) that for real  $\rho \neq 0$ ,

$$\sup_{\rho \neq 0} \left| \frac{1}{a(\rho)} h^\pm(x, \rho) \right| < \infty. \quad (3.3.43)$$

Using (3.3.39) we get

$$h^+(x, i\tau_k) = ia_1(i\tau_k) \alpha_k^- g(x, i\tau_k), \quad h^-(x, i\tau_k) = ia_1(i\tau_k) \alpha_k^+ e(x, i\tau_k), \quad (3.3.44)$$

$$\lim_{\substack{|\rho| \rightarrow \infty \\ \text{Im } \rho \geq 0}} h^\pm(x, \rho) \exp(\mp i\rho x) = 1, \quad (3.3.45)$$

where  $a_1(\rho) = \frac{d}{d\rho} a(\rho)$ .

3) It follows from (3.3.38) that

$$\begin{aligned} s^+(\rho) e(x, \rho) + e(x, -\rho) &= \frac{h^-(x, \rho)}{a(\rho)}, \\ e(x, \rho) + s^+(-\rho) e(x, -\rho) &= \frac{h^-(x, -\rho)}{a(-\rho)}. \end{aligned}$$

Solving this linear algebraic system we obtain

$$e(x, \rho) \left( 1 - s^+(\rho) s^+(-\rho) \right) = \frac{h^-(x, -\rho)}{a(-\rho)} - s^+(-\rho) \frac{h^-(x, \rho)}{a(\rho)}.$$

By virtue of (3.3.32)-(3.3.33),

$$1 - s^+(\rho) s^+(-\rho) = 1 - |s^+(\rho)|^2 = \frac{1}{|a(\rho)|^2} = \frac{1}{a(\rho) a(-\rho)}.$$

Therefore,

$$\frac{e(x, \rho)}{a(\rho)} = s^-(\rho) h^-(x, \rho) + h^-(x, -\rho). \quad (3.3.46)$$

Using (3.3.46) and the second relation from (3.3.38) we calculate

$$h^-(x, \rho) g(x, -\rho) - h^-(x, -\rho) g(x, \rho) = G(\rho), \quad (3.3.47)$$

where

$$G(\rho) := \frac{1}{a(\rho)} \left( h^+(x, \rho) h^-(x, \rho) - e(x, \rho) g(x, \rho) \right). \quad (3.3.48)$$

According to (3.3.44) and (3.3.35),

$$h^+(x, i\tau_k) h^-(x, i\tau_k) - e(x, i\tau_k) g(x, i\tau_k) = 0, \quad k = \overline{1, N},$$

and hence (after removing singularities) the function  $G(\rho)$  is analytic in  $\Omega_+$  and continuous in  $\overline{\Omega_+} \setminus \{0\}$ . By virtue of (3.3.45),

$$\lim_{\substack{|\rho| \rightarrow \infty \\ \operatorname{Im} \rho \geq 0}} h^+(x, \rho) h^-(x, \rho) = 1.$$

Since

$$a(\rho) = 1 + O\left(\frac{1}{\rho}\right) \quad \text{as } |\rho| \rightarrow \infty, \quad \rho \in \overline{\Omega_+},$$

it follows from (3.3.48) that

$$\lim_{\substack{|\rho| \rightarrow \infty \\ \operatorname{Im} \rho \geq 0}} G(\rho) = 0.$$

By virtue of (3.3.47),

$$G(-\rho) = -G(\rho) \quad \text{for real } \rho \neq 0.$$

We continue  $G(\rho)$  to the lower half-plane by the formula

$$G(\rho) = -G(-\rho), \quad \operatorname{Im} \rho < 0. \quad (3.3.49)$$

Then, the function  $G(\rho)$  is analytic in  $\mathbf{C} \setminus \{0\}$ , and

$$\lim_{|\rho| \rightarrow \infty} G(\rho) = 0. \quad (3.3.50)$$

Furthermore, it follows from (3.3.48), (3.3.26), (3.3.41) and (3.3.49) that

$$\lim_{\rho \rightarrow 0} \rho^2 G(\rho) = 0,$$

i.e. the function  $\rho G(\rho)$  is entire in  $\rho$ . On the other hand, using (3.3.48), (3.3.41) and (3.3.43) we obtain that for real  $\rho$ ,

$$\lim_{\rho \rightarrow 0} \rho G(\rho) = 0,$$

and consequently, the function  $G(\rho)$  is entire in  $\rho$ . Together with (3.3.50), using Liouville's theorem, this yields  $G(\rho) \equiv 0$ , i.e.

$$h^+(x, \rho)h^-(x, \rho) = e(x, \rho)g(x, \rho), \quad \rho \in \overline{\Omega_+}, \quad (3.3.51)$$

$$h^-(x, \rho)g(x, -\rho) = h^-(x, -\rho)g(x, \rho) \quad \text{for real } \rho \neq 0. \quad (3.3.52)$$

4) Now we consider the function

$$p(x, \rho) := \frac{h^+(x, \rho)}{e(x, \rho)}.$$

Denote  $\mathcal{E} = \{x : e(x, 0)e(x, i\tau_1) \dots e(x, i\tau_N) = 0\}$ . Since  $e(x, \rho)$  is a solution of the differential equation (3.3.3) and for  $\rho \in \Omega_+$ ,

$$|e(x, \rho) \exp(-i\rho x) - 1| \leq \int_x^\infty |A^+(x, t)| dt \rightarrow 0 \quad \text{as } x \rightarrow \infty, \quad (3.3.53)$$

it follows that  $\mathcal{E}$  is a finite set. Take a fixed  $x \notin \mathcal{E}$ . Let  $\rho^* \in \overline{\Omega_+}$  be a zero of  $e(x, \rho)$ , i.e.  $e(x, \rho^*) = 0$ . Since  $x \notin \mathcal{E}$ , it follows that  $\rho^* \neq 0$ ,  $\rho^* \neq i\tau_k$ ,  $k = 1, N$ ; hence  $\rho^* a(\rho^*) \neq 0$ . In view of (3.3.42) this yields  $h^-(x, \rho^*) \neq 0$ . According to (3.3.51),  $h^+(x, \rho^*) = 0$ . Since all zeros of  $e(x, \rho)$  are simple (this fact is proved similarly to Theorem 2.3.3), we deduce that the function  $p(x, \rho)$  (after removing singularities) is analytic in  $\Omega_+$  and continuous in  $\overline{\Omega_+} \setminus \{0\}$ . It follows from (3.3.45) and (3.3.53) that

$$p(x, \rho) \rightarrow 1 \quad \text{as } |\rho| \rightarrow \infty, \quad \rho \in \overline{\Omega_+}.$$

By virtue of (3.3.51)-(3.3.52),

$$p(x, \rho) = p(x, -\rho) \quad \text{for real } \rho \neq 0.$$

We continue  $p(x, \rho)$  to the lower half-plane by the formula

$$p(x, \rho) = p(x, -\rho), \quad \text{Im } \rho < 0.$$

Then, the function  $p(x, \rho)$  is analytic in  $\mathbf{C} \setminus \{0\}$ , and

$$\lim_{|\rho| \rightarrow \infty} p(x, \rho) = 1. \quad (3.3.54)$$

Since  $e(x, 0) \neq 0$ , it follows from (3.3.41) that

$$\lim_{\rho \rightarrow 0} \rho p(x, \rho) = 0,$$

and consequently, the function  $p(x, \rho)$  is entire in  $\rho$ . Together with (3.3.54) this yields  $p(x, \rho) \equiv 1$ , i.e.

$$h^+(x, \rho) \equiv e(x, \rho). \quad (3.3.55)$$



Then, in view of (3.3.51),

$$h^-(x, \rho) \equiv g(x, \rho). \quad (3.3.56)$$

From this, using (3.3.38), we arrive at (3.3.36). Lemma 3.3.2 is proved.

Let us return to the *proof of Theorem 3.3.1*. It follows from (3.3.36) and (3.3.3) that

$$q^-(x) = q^+(x) := q(x). \quad (3.3.57)$$

Then (3.3.2) implies (3.1.2), and the functions  $e(x, \rho)$  and  $g(x, \rho)$  are the Jost solutions for the potential  $q$  defined by (3.3.57).

Denote by  $\tilde{J}^\pm = \{\tilde{s}^+(\rho), \tilde{\lambda}_k, \tilde{\alpha}_k^+; \rho \in \mathbf{R}, k = \overline{1, \tilde{N}}\}$  the scattering data for this potential  $q$ , and let

$$\tilde{a}(\rho) := -\frac{1}{2i\rho} \langle e(x, \rho), g(x, \rho) \rangle. \quad (3.3.58)$$

Using (3.3.42) and (3.3.55)-(3.3.56) we get

$$\langle e(x, \rho), g(x, \rho) \rangle = -2i\rho a(\rho).$$

Together with (3.3.58) this yields  $\tilde{a}(\rho) \equiv a(\rho)$ , and consequently,  $\tilde{N} = N$ ,  $\tilde{\lambda}_k = \lambda_k$ ,  $k = \overline{1, N}$ . Furthermore, it follows from (3.1.21) that

$$\tilde{s}^+(\rho)e(x, \rho) + e(x, -\rho) = \frac{g(x, \rho)}{a(\rho)},$$

$$\tilde{s}^-(\rho)g(x, \rho) + g(x, -\rho) = \frac{e(x, \rho)}{a(\rho)}.$$

Comparing with (3.3.26) we infer  $\tilde{s}^\pm(\rho) = s^\pm(\rho)$ ,  $\rho \in \mathbf{R}$ .

By virtue of (3.1.26)

$$\tilde{\alpha}_k^+ = \frac{d_k}{ia_1(\rho_k)}, \quad \tilde{\alpha}_k^- = \frac{1}{id_k a_1(\rho_k)}. \quad (3.3.59)$$

On the other hand, it follows from (3.3.55)-(3.3.56) and (3.3.44) that

$$e(x, i\tau_k) = ia_1(i\tau_k)\alpha_k^- g(x, i\tau_k),$$

$$g(x, i\tau_k) = ia_1(i\tau_k)\alpha_k^+ e(x, i\tau_k),$$

i.e.

$$d_k = ia_1(i\tau_k)\alpha_k^+ = \frac{1}{ia_1(i\tau_k)\alpha_k^-}.$$

Comparing with (3.3.59) we get  $\tilde{\alpha}^\pm = \alpha^\pm$ , and Theorem 3.3.1 is proved.  $\square$

**Remark 3.3.1.** There is a connection between the inverse scattering problem and the Riemann problem for analytic functions. Indeed, one can rewrite (3.1.27) in the form

$$Q^-(x, \rho) = Q^+(x, \rho)Q(\rho), \quad (3.3.60)$$

where

$$Q^-(x, \rho) = \begin{bmatrix} g(x, -\rho) & e(x, -\rho) \\ g'(x, -\rho) & e'(x, -\rho) \end{bmatrix}, \quad Q^+(x, \rho) = \begin{bmatrix} e(x, \rho) & g(x, \rho) \\ e'(x, \rho) & g'(x, \rho) \end{bmatrix},$$

$$Q(\rho) = \begin{bmatrix} \frac{1}{a(\rho)} & \frac{b(-\rho)}{a(\rho)} \\ -\frac{b(\rho)}{a(\rho)} & \frac{1}{a(\rho)} \end{bmatrix}.$$

For each fixed  $x$ , the matrix-functions  $Q^\pm(x, \rho)$  are analytic and bounded for  $\pm \operatorname{Im} \rho > 0$ . By virtue of (3.1.26) and (3.1.53), the matrix-function  $Q(\rho)$  can be uniquely reconstructed from the given scattering data  $J^+$  (or  $J^-$ ). Thus, the inverse scattering problem is reduced to the Riemann problem (3.3.60). We note that the theory of the solution of the Riemann problem can be found, for example, in [gak1]. Applying the Fourier transform to (3.3.60) as it was shown in Section 3.2, we arrive at the Gelfand-Levitan-Marchenko equations (3.2.1)-(3.2.2) or (3.2.21). We note that the use of the Riemann problem in the inverse problem theory has only methodical interest and does not constitute an independent method, since it can be considered as a particular case of the contour integral method (see [bea1], [yur1]).

### 3.4. REFLECTIONLESS POTENTIALS. MODIFICATION OF THE DISCRETE SPECTRUM.

**3.4.1.** A potential  $q$  satisfying (3.1.2) is called *reflectionless* if  $b(\rho) \equiv 0$ . By virtue of (3.1.26) and (3.1.53) we have in this case

$$s^\pm(\rho) \equiv 0, \quad a(\rho) = \prod_{k=1}^N \frac{\rho - i\tau_k}{\rho + i\tau_k}. \quad (3.4.1)$$

Theorem 3.3.1 allows one to prove the existence of reflectionless potentials and to describe all of them. Namely, the following theorem is valid.

**Theorem 3.4.1.** *Let arbitrary numbers  $\lambda_k = -\tau_k^2 < 0$ ,  $\alpha_k^+ > 0$ ,  $k = \overline{1, N}$  be given. Take  $s^+(\rho) \equiv 0$ ,  $\rho \in \mathbf{R}$ , and consider the data  $J^+ = \{s^+(\rho), \lambda_k, \alpha_k^+; \rho \in \mathbf{R}, k = \overline{1, N}\}$ . Then, there exists a unique reflectionless potential  $q$  satisfying (3.1.2) for which  $J^+$  are the right scattering data.*

Theorem 3.4.1 is an obvious corollary of Theorem 3.3.1 since for this set  $J^+$  all conditions of Theorem 3.3.1 are fulfilled and (3.4.1) holds.

For a reflectionless potential, the Gelfand-Levitan-Marchenko equation (3.2.1) takes the form

$$A^+(x, y) + \sum_{k=1}^N \alpha_k^+ \exp(-\tau_k(x+y)) + \sum_{k=1}^N \alpha_k^+ \exp(-\tau_k y) \int_x^\infty A^+(x, t) \exp(-\tau_k t) dt = 0. \quad (3.4.2)$$

We seek a solution of (3.4.2) in the form:

$$A^+(x, y) = \sum_{k=1}^N P_k(x) \exp(-\tau_k y).$$

Substituting this into (3.4.2), we obtain the following linear algebraic system with respect to  $P_k(x)$ :

$$P_k(x) + \sum_{j=1}^N \alpha_j^+ \frac{\exp(-(\tau_k + \tau_j)x)}{\tau_k + \tau_j} P_j(x) = -\alpha_k^+ \exp(-\tau_k x), \quad k = \overline{1, N}. \quad (3.4.3)$$

Solving (3.4.3) we infer  $P_k(x) = \Delta_k(x)/\Delta(x)$ , where

$$\Delta(x) = \det \left[ \delta_{kl} + \alpha_k^+ \frac{\exp(-(\tau_k + \tau_l)x)}{\tau_k + \tau_l} \right]_{k,l=\overline{1,N}}, \quad (3.4.4)$$

and  $\Delta_k(x)$  is the determinant obtained from  $\Delta(x)$  by means of the replacement of  $k$ -column by the right-hand side column. Then

$$q(x) = -2 \frac{d}{dx} A^+(x, x) = -2 \sum_{k=1}^N \frac{\Delta_k(x)}{\Delta(x)} \exp(-\tau_k x),$$

and consequently one can show that

$$q(x) = -2 \frac{d^2}{dx^2} \ln \Delta(x). \quad (3.4.5)$$

Thus, (3.4.4) and (3.4.5) allow one to calculate reflectionless potentials from the given numbers  $\{\lambda_k, \alpha_k^+\}_{k=\overline{1,N}}$ .

*Example 3.4.1.* Let  $N = 1$ ,  $\tau = \tau_1$ ,  $\alpha = \alpha_1^+$ , and

$$\Delta(x) = 1 + \frac{\alpha}{2\tau} \exp(-2\tau x).$$

Then (3.4.5) gives

$$q(x) = - \frac{4\tau\alpha}{\left(\exp(\tau x) + \frac{\alpha}{2\tau} \exp(-\tau x)\right)^2}.$$

Denote

$$\beta = -\frac{1}{2\tau} \ln \frac{2\tau}{\alpha}.$$

Then

$$q(x) = - \frac{2\tau^2}{\cosh^2(\tau(x - \beta))}.$$

**3.4.2.** If  $q = 0$ , then  $s^\pm(\rho) \equiv 0$ ,  $N = 0$ ,  $a(\rho) \equiv 1$ . Therefore, Theorem 3.4.1 shows that all reflectionless potentials can be constructed from the zero potential and given data  $\{\lambda_k, \alpha_k^+\}$ ,  $k = \overline{1, N}$ . Below we briefly consider a more general case of changing the discrete spectrum for an arbitrary potential  $q$ . Namely, the following theorem is valid.

**Theorem 3.4.2.** Let  $J^+ = \{s^+(\rho), \lambda_k, \alpha_k^+; \rho \in \mathbf{R}, k = \overline{1, N}\}$  be the right scattering data for a certain real potential  $q$  satisfying (3.1.2). Take arbitrary numbers  $\tilde{\lambda}_k = -\tilde{\tau}_k^2 < 0$ ,  $\tilde{\alpha}_k^+ > 0$ ,  $k = \overline{1, \tilde{N}}$ , and consider the set  $\tilde{J}^+ = \{s^+(\rho), \tilde{\lambda}_k, \tilde{\alpha}_k^+; \rho \in \mathbf{R}, k = \overline{1, \tilde{N}}\}$  with the same  $s^+(\rho)$  as in  $J^+$ . Then there exists a real potential  $\tilde{q}$  satisfying (3.1.2), for which  $\tilde{J}^+$  represents the right scattering data.

*Proof.* Let us check the conditions of Theorem 3.3.1 for  $\tilde{J}^+$ . According to (3.3.25) and (3.3.28) we construct the functions  $\tilde{a}(\rho)$  and  $\tilde{s}^-(\rho)$  by the formulae

$$\tilde{a}(\rho) := \prod_{k=1}^{\tilde{N}} \frac{\rho - i\tilde{\tau}_k}{\rho + i\tilde{\tau}_k} \exp\left(-\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln(1 - |s^+(\xi)|^2)}{\xi - \rho} d\xi\right), \quad \rho \in \Omega_+,$$

$$\tilde{s}^-(\rho) := -s^+(-\rho) \frac{\tilde{a}(-\rho)}{\tilde{a}(\rho)}. \quad (3.4.6)$$

Together with (3.1.53) this yields

$$\tilde{a}(\rho) = a(\rho) \prod_{k=1}^{\tilde{N}} \frac{\rho - i\tilde{\tau}_k}{\rho + i\tilde{\tau}_k} \prod_{k=1}^N \frac{\rho + i\tau_k}{\rho - i\tau_k}. \quad (3.4.7)$$

By virtue of (3.1.26),

$$s^-(\rho) = -s^+(-\rho) \frac{a(-\rho)}{a(\rho)}. \quad (3.4.8)$$

Using (3.4.6)-(3.4.8) we get

$$\tilde{s}^-(\rho) = s^-(\rho). \quad (3.4.9)$$

Since the scattering data  $J^+$  satisfy all conditions of Theorem 3.3.1 it follows from (3.4.7) and (3.4.9) that  $\tilde{J}^+$  also satisfy all conditions of Theorem 3.3.1. Then, by Theorem 3.3.1 there exists a real potential  $\tilde{q}$  satisfying (3.1.2) for which  $\tilde{J}^+$  are the right scattering data.  $\square$

## IV. APPLICATIONS OF THE INVERSE PROBLEM THEORY

In this chapter various applications of inverse spectral problems for Sturm-Liouville operators are considered. In Sections 4.1-4.2 we study the Korteweg-de Vries equation on the line and on the half-line by the inverse problem method. Section 4.3 is devoted to a synthesis problem of recovering parameters of a medium from incomplete spectral information. Section 4.4 deals with discontinuous inverse problems which are connected with discontinuous material properties. Inverse problems appearing in elasticity theory are considered in Section 4.5. In Section 4.6 boundary value problems with aftereffect are studied. Inverse problems for differential equations of the Orr-Sommerfeld type are investigated in Section 4.7. Section 4.8 is devoted to inverse problems for differential equations with turning points.

### 4.1. SOLUTION OF THE KORTEWEG-DE VRIES EQUATION ON THE LINE

Inverse spectral problems play an important role for integrating some nonlinear evolution equations in mathematical physics. In 1967, G.Gardner, G.Green, M.Kruskal and R.Miura [gar1] found a deep connection of the well-known (from XIX century) nonlinear Korteweg-de Vries (KdV) equation

$$q_t = 6qq_x - q_{xxx}$$

with the spectral theory of Sturm-Liouville operators. They could manage to solve globally the Cauchy problem for the KdV equation by means of reduction to the inverse spectral problem. These investigations created a new branch in mathematical physics (for further discussions see [abl1], [lax1], [tak1], [zak1]). In this section we provide the solution of the Cauchy problem for the KdV equation on the line. For this purpose we use ideas from [gar1], [lax1] and results of Chapter 3 on the inverse scattering problem for the Sturm-Liouville operator on the line.

**4.1.1.** Consider the Cauchy problem for the KdV equation on the line:

$$q_t = 6qq_x - q_{xxx}, \quad -\infty < x < \infty, \quad t > 0, \quad (4.1.1)$$

$$q|_{t=0} = q_0(x), \quad (4.1.2)$$

where  $q_0(x)$  is a real function satisfying (3.1.2). Denote by  $Q_0$  the set of real functions  $q(x, t)$ ,  $-\infty < x < \infty$ ,  $t \geq 0$ , such that for each fixed  $T > 0$ ,

$$\max_{0 \leq t \leq T} \int_{-\infty}^{\infty} (1 + |x|) |q(x, t)| dt < \infty.$$

Let  $Q_1$  be the set of functions  $q(x, t)$  such that  $q, \dot{q}, q', q'', q''' \in Q_0$ . Here and below, "dot" denotes derivatives with respect to  $t$ , and "prime" denotes derivatives with respect to  $x$ . We will seek the solution of the Cauchy problem (4.1.1)-(4.1.2) in the class  $Q_1$ . First we prove the following uniqueness theorem.

**Theorem 4.1.1.** *The Cauchy problem (4.1.1)-(4.1.2) has at most one solution.*

*Proof.* Let  $q, \tilde{q} \in Q_1$  be solutions of the Cauchy problem (4.1.1)-(4.1.2). Denote  $w := q - \tilde{q}$ . Then  $w \in Q_1$ ,  $w|_{t=0} = 0$ , and

$$w_t = 6(qw_x + w\tilde{q}_x) - w_{xxx}.$$

Multiplying this equality by  $w$  and integrating with respect to  $x$ , we get

$$\int_{-\infty}^{\infty} ww_t dx = 6 \int_{-\infty}^{\infty} w(qw_x + w\tilde{q}_x) dx - \int_{-\infty}^{\infty} ww_{xxx} dx.$$

Integration by parts yields

$$\int_{-\infty}^{\infty} ww_{xxx} dx = - \int_{-\infty}^{\infty} w_x w_{xx} dx = \int_{-\infty}^{\infty} w_x w_{xx} dx,$$

and consequently

$$\int_{-\infty}^{\infty} ww_{xxx} dx = 0.$$

Since

$$\int_{-\infty}^{\infty} qw w_x dx = \int_{-\infty}^{\infty} q \left( \frac{1}{2} w^2 \right)_x dx = - \frac{1}{2} \int_{-\infty}^{\infty} q_x w^2 dx,$$

it follows that

$$\int_{-\infty}^{\infty} ww_t dx = \int_{-\infty}^{\infty} \left( \tilde{q}_x - \frac{1}{2} q_x \right) w^2 dx.$$

Denote

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} w^2 dx, \quad m(t) = 12 \max_{x \in \mathbf{R}} \left| \tilde{q}_x - \frac{1}{2} q_x \right|.$$

Then

$$\frac{d}{dt} E(t) \leq m(t) E(t),$$

and consequently,

$$0 \leq E(t) \leq E(0) \exp \left( \int_0^t m(\xi) d\xi \right).$$

Since  $E(0) = 0$  we deduce  $E(t) \equiv 0$ , i.e.  $w \equiv 0$ . □

**4.1.2.** Our next goal is to construct the solution of the Cauchy problem (4.1.1)-(4.1.2) by reduction to the inverse scattering problem for the Sturm-Liouville equation on the line.

Let  $q(x, t)$  be the solution of (4.1.1)-(4.1.2). Consider the Sturm-Liouville equation

$$Ly := -y'' + q(x, t)y = \lambda y \tag{4.1.3}$$

with  $t$  as a parameter. Then the Jost-type of (4.1.3) solutions and the scattering data depend on  $t$ . Let us show that equation (4.1.1) is equivalent to the equation

$$\dot{L} = [A, L], \tag{4.1.4}$$

where in this section

$$Ay = -4y''' + 6qy' + 3q'y$$

is a linear differential operator, and  $[A, L] := AL - LA$ .

Indeed, since  $Ly = -y'' + qy$ , we have

$$\dot{L}y = \dot{q}y, \quad ALy = -4(-y'' + qy)''' + 6q(-y'' + qy)' + 3q'(-y'' + qy),$$

$$LAy = -(-4y''' + 6qy' + 3q'y)'' + q(-4y''' + 6qy' + 3q'y),$$

and consequently  $(AL - LA)y = (6qq' - q''')y$ .

Equation (4.1.4) is called the Lax equation or Lax representation, and the pair  $A, L$  is called the Lax pair, corresponding to (4.1.3).

**Lemma 4.1.1.** *Let  $q(x, t)$  be a solution of (4.1.1), and let  $y = y(x, t, \lambda)$  be a solution of (4.1.3). Then  $(L - \lambda)(\dot{y} - Ay) = 0$ , i.e. the function  $\dot{y} - Ay$  is also a solution of (4.1.3).*

Indeed, differentiating (4.1.3) with respect to  $t$ , we get  $\dot{L}y + (L - \lambda)\dot{y} = 0$ , or, in view of (4.1.4),  $(L - \lambda)\dot{y} = LAy - ALy = (L - \lambda)Ay$ .  $\square$

Let  $e(x, t, \rho)$  and  $g(x, t, \rho)$  be the Jost-type solutions of (4.1.3) introduced in Section 3.1. Denote  $e_{\pm} = e(x, t, \pm\rho)$ ,  $g_{\pm} = g(x, t, \pm\rho)$ .

**Lemma 4.1.2.** *The following relation holds*

$$\dot{e}_+ = Ae_+ - 4i\rho^3 e_+. \quad (4.1.5)$$

*Proof.* By Lemma 4.1.1, the function  $\dot{e}_+ - Ae_+$  is a solution of (4.1.3). Since the functions  $\{e_+, e_-\}$  form a fundamental system of solutions of (4.1.3), we have

$$\dot{e}_+ - Ae_+ = \beta_1 e_+ + \beta_2 e_-,$$

where  $\beta_k = \beta_k(t, \rho)$ ,  $k = 1, 2$  do not depend on  $x$ . As  $x \rightarrow +\infty$ ,

$$e_{\pm} \sim \exp(\pm i\rho x), \quad \dot{e}_+ \sim 0, \quad Ae_+ \sim 4i\rho^3 \exp(i\rho x),$$

consequently  $\beta_1 = -4i\rho^3$ ,  $\beta_2 = 0$ , and (4.1.5) is proved.  $\square$

**Lemma 4.1.3.** *The following relations hold*

$$\dot{a} = 0, \quad \dot{b} = -8i\rho^3 b, \quad \dot{s}^+ = 8i\rho^3 s^+, \quad (4.1.6)$$

$$\dot{\lambda}_j = 0, \quad \dot{\alpha}_j^+ = 8\kappa_j^3 \alpha_j^+. \quad (4.1.7)$$

*Proof.* According to (3.1.18),

$$e_+ = ag_+ + bg_-. \quad (4.1.8)$$

Differentiating (4.1.8) with respect to  $t$ , we get  $\dot{e}_+ = (\dot{a}g_+ + \dot{b}g_-) + (a\dot{g}_+ + b\dot{g}_-)$ . Using (4.1.5) and (4.1.8) we calculate

$$a(Ag_+ - 4i\rho^3 g_+) + b(Ag_- - 4i\rho^3 g_-) = (\dot{a}g_+ + \dot{b}g_-) + (a\dot{g}_+ + b\dot{g}_-). \quad (4.1.9)$$

Since  $g_{\pm} \sim \exp(\pm i\rho x)$ ,  $\dot{g}_{\pm} \sim 0$ ,  $Ag_{\pm} \sim \pm 4i\rho^3 \exp(\pm i\rho x)$  as  $x \rightarrow -\infty$ , then (4.1.9) yields  $-8i\rho^3 \exp(-i\rho x) \sim \dot{a} \exp(i\rho x) + \dot{b} \exp(-i\rho x)$ , i.e.  $\dot{a} = 0$ ,  $\dot{b} = -8i\rho^3 b$ . Consequently,  $\dot{s}^+ = 8i\rho^3 s^+$ , and (4.1.6) is proved.

The eigenvalues  $\lambda_j = \rho_j^2 = -\kappa_j^2$ ,  $\kappa_j > 0$ ,  $j = \overline{1, N}$  are the zeros of the function  $a = a(\rho, t)$ . Hence, according to  $\dot{a} = 0$ , we have  $\dot{\lambda}_j = 0$ . Denote

$$e_j = e(x, t, i\kappa_j), \quad g_j = g(x, t, i\kappa_j), \quad j = \overline{1, N}.$$

By Theorem 3.1.3,  $g_j = d_j e_j$ , where  $d_j = d_j(t)$  do not depend on  $x$ . Differentiating the relation  $g_j = d_j e_j$  with respect to  $t$  and using (4.1.5), we infer

$$\dot{g}_j = \dot{d}_j e_j + d_j \dot{e}_j = \dot{d}_j e_j + d_j A e_j - 4\kappa_j^3 d_j e_j$$

or

$$\dot{g}_j = \frac{\dot{d}_j}{d_j} g_j + A g_j - 4\kappa_j^3 g_j.$$

As  $x \rightarrow -\infty$ ,  $g_j \sim \exp(\kappa_j x)$ ,  $\dot{g}_j \sim 0$ ,  $A g_j \sim -4\kappa_j^3 \exp(\kappa_j x)$ , and consequently  $\dot{d}_j = 8\kappa_j^3 d_j$ , or, in view of (3.1.32),  $\dot{\alpha}_j^+ = 8\kappa_j^3 \alpha_j^+$ .  $\square$

Thus, it follows from Lemma 4.1.3 that we have proved the following theorem.

**Theorem 4.1.2.** *Let  $q(x, t)$  be the solution of the Cauchy problem (4.1.1)-(4.1.2), and let  $J^+(t) = \{s^+(t, \rho), \lambda_j(t), \alpha_j^+(t), j = \overline{1, N}\}$  be the scattering data for  $q(x, t)$ . Then*

$$\left. \begin{aligned} s^+(t, \rho) &= s^+(0, \rho) \exp(8i\rho^3 t), \\ \lambda_j(t) &= \lambda_j(0), \quad \alpha_j^+(t) = \alpha_j^+(0) \exp(8\kappa_j^3 t), \quad j = \overline{1, N} \quad (\lambda_j = -\kappa_j^2). \end{aligned} \right\} \quad (4.1.10)$$

The formulae (4.1.10) give us the evolution of the scattering data with respect to  $t$ , and we obtain the following algorithm for the solution of the Cauchy problem (4.1.1)-(4.1.2).

**Algorithm 4.1.1.** *Let the function  $q(x, 0) = q_0(x)$  be given. Then*

- 1) *Construct the scattering data  $\{s^+(0, \rho), \lambda_j(0), \alpha_j^+(0), j = \overline{1, N}\}$ .*
- 2) *Calculate  $\{s^+(t, \rho), \lambda_j(t), \alpha_j^+(t), j = \overline{1, N}\}$  by (4.1.10).*
- 3) *Find the function  $q(x, t)$  by solving the inverse scattering problem (see Section 3.3).*

Thus, if the Cauchy problem (4.1.1)-(4.1.2) has a solution, then it can be found by Algorithm 4.1.1. On the other hand, if  $q_0 \in Q_1$  then  $J^+(t)$ , constructed by (4.1.10), satisfies the conditions of Theorem 3.3.1 for each fixed  $t > 0$ , and consequently there exists  $q(x, t)$  for which  $J^+(t)$  are the scattering data. It can be shown (see [mar1]) that  $q(x, t)$  is the solution of (4.1.1)-(4.1.2).

We notice once more the main points for the solution of the Cauchy problem (4.1.1)-(4.1.2) by the inverse problem method:

- (1) The presence of the Lax representation (4.1.4).
- (2) The evolution of the scattering data with respect to  $t$ .
- (3) The solution of the inverse problem.

**4.1.3.** Among the solutions of the KdV equation (4.1.1) there are very important particular solutions of the form  $q(x, t) = f(x - ct)$ . Such solutions are called *solitons*. Substituting  $q(x, t) = f(x - ct)$  into (4.1.1), we get  $f''' + 6ff' + cf' = 0$ , or  $(f'' + 3f^2 + cf)' = 0$ . Clearly, the function

$$f(x) = -\frac{c}{2 \cosh^2\left(\frac{\sqrt{c}x}{2}\right)}$$

satisfies this equation. Hence, the function

$$q(x, t) = -\frac{c}{2 \cosh^2\left(\frac{\sqrt{c}(x-ct)}{2}\right)} \quad (4.1.11)$$



is a soliton.

It is interesting to note that solitons correspond to reflectionless potentials (see Section 3.4). Consider the Cauchy problem (4.1.1)-(4.1.2) in the case when  $q_0(x)$  is a reflectionless potential, i.e.  $s^+(0, \rho) = 0$ . Then, by virtue of (4.1.10),  $s^+(t, \rho) = 0$  for all  $t$ , i.e. the solution  $q(x, t)$  of the Cauchy problem (4.1.1)-(4.1.2) is a reflectionless potential for all  $t$ . Using (3.4.5) and (4.1.10) we derive

$$q(x, t) = -2 \frac{d^2}{dx^2} \Delta(x, t),$$

$$\Delta(x, t) = \det \left[ \delta_{kl} + \alpha_k^+(0) \exp(8\kappa_k^3 t) \frac{\exp(-(\kappa_k + \kappa_l)x)}{\kappa_k + \kappa_l} \right]_{k, l=1, \overline{N}}.$$

In particular, if  $N = 1$ , then  $\alpha_1^+(0) = 2\kappa_1$ , and

$$q(x, t) = -\frac{2\kappa_1^2}{\cosh^2(\kappa_1(x - 4\kappa_1^2 t))}.$$

Taking  $c = 4\kappa_1^2$ , we get (4.1.11).

## 4.2. KORTEWEG-DE VRIES EQUATION ON THE HALF-LINE. NONLINEAR REFLECTION

**4.2.1.** We consider the mixed problem for the KdV equation on the half-line:

$$q_t = 6qq_x - q_{xxx}, \quad x \geq 0, \quad t \geq 0, \quad (4.2.1)$$

$$q|_{t=0} = q_0(x), \quad q|_{x=0} = q_1(t), \quad q_{xx}|_{x=0} = q_2(t), \quad q_{xxx}|_{x=0} = q_3(t). \quad (4.2.2)$$

Here  $q_0(x)$  and  $q_j(t)$ ,  $j = 1, 2, 3$ , are continuous complex-valued functions, and  $q_0(x) \in L(0, \infty)$ ,  $q_0(0) = q_1(0)$ . In this section the mixed problem (4.2.1)-(4.2.2) is solved by the inverse problem method. For this purpose we use the results of Chapter 2 on recovering the Sturm-Liouville operator on the half-line from the Weyl function. We provide the evolution of the Weyl function with respect to  $t$ , and give an algorithm for the solution of the mixed problem (4.2.1)-(4.2.2) along with necessary and sufficient conditions for its solvability.

Denote by  $Q$  the set of functions  $q(x, t)$  such that the functions  $q, \dot{q}, q', q'', q'''$  are continuous in  $D = \{(x, t) : x \geq 0, t \geq 0\}$ , and integrable on the half-line  $x \geq 0$ , for each fixed  $t$ . We will seek the solution of (4.2.1)-(4.2.2) in the class  $Q$ .

Consider the Sturm-Liouville equation

$$Ly := -y'' + q(x, t)y = \lambda y. \quad (4.2.3)$$

where  $q(x, t) \in L(0, \infty)$  for each fixed  $t \geq 0$ . Let  $\Phi(x, t, \lambda)$  be the solution of (4.2.3) under the conditions  $\Phi(0, t, \lambda) = 1$ ,  $\Phi(x, t, \lambda) = O(\exp(i\rho x))$ ,  $x \rightarrow \infty$  (for each fixed  $t$ ). Denote  $M(t, \lambda) := \Phi'(0, t, \lambda)$ . Then,

$$\Phi(x, t, \lambda) = \frac{e(x, t, \rho)}{e(0, t, \rho)}, \quad M(t, \lambda) = \frac{e'(0, t, \rho)}{e(0, t, \rho)},$$

where  $e(x, t, \rho)$  is the Jost solution for  $q(x, t)$ . The function  $M(t, \lambda)$  is the Weyl function for equation (4.2.3) with respect to the linear form  $U(y) := y(0)$  (see Remark 2.1.4). From Chapter 2 we know that the specification of the Weyl function uniquely determines the potential  $q(x, t)$ . Our goal is to obtain the evolution of the Weyl function  $M(t, \lambda)$  with respect to  $t$  when  $q$  satisfies (4.2.1). For this purpose we consider two methods.

*Method 1.* Let us use the Lax equation (4.1.4), which is equivalent to the KdV equation. Let  $q(x, t)$  be a solution of the mixed problem (4.2.1)-(4.2.2). By Lemma 4.1.1, the function  $\dot{\Phi} - A\Phi$  is a solution of (4.2.3). Moreover, one can verify that  $\dot{\Phi} - A\Phi = O(\exp(i\rho x))$ . Hence  $\dot{\Phi} - A\Phi = \beta\Phi$ , where  $\beta = \beta(t, \lambda)$  does not depend on  $x$ . Taking  $x = 0$ , we calculate  $\beta = -(A\Phi)_{|x=0}$ , and consequently

$$\dot{\Phi} = A\Phi - (A\Phi)_{|x=0}\Phi.$$

Differentiating this relation with respect to  $x$  and taking  $x = 0$ , we get

$$\dot{M} = (A_1\Phi)_{|x=0} - M(A\Phi)_{|x=0}, \quad (4.2.4)$$

where  $A_1y := (Ay)'$ . If  $y$  is a solution of (4.2.3), then

$$Ay = -4y''' + 6qy' + 3q'y = F_{11}y + F_{12}y',$$

$$A_1y = -4y^{(4)} + 6qy'' + 9q'y' + 3q''y = F_{21}y + F_{22}y',$$

where  $F = F(x, t, \lambda)$  is defined by

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} = \begin{bmatrix} -q' & 4\lambda + 2q \\ (4\lambda + 2q)(q - \lambda) - q'' & q' \end{bmatrix}. \quad (4.2.5)$$

Hence

$$(A\Phi)_{|x=0} = F_{11}(0, t, \lambda) + F_{12}(0, t, \lambda)M(t, \lambda), \quad (A_1\Phi)_{|x=0} = F_{21}(0, t, \lambda) - F_{22}(0, t, \lambda)M(t, \lambda).$$

Substituting this into (4.2.4), we obtain the nonlinear ordinary differential equation

$$\dot{M}(t, \lambda) = F_{21}(0, t, \lambda) + (F_{22}(0, t, \lambda) - F_{11}(0, t, \lambda))M(t, \lambda) - F_{12}(0, t, \lambda)M^2(t, \lambda). \quad (4.2.6)$$

The functions  $F_{jk}(0, t, \lambda)$  depend on  $q_j(t)$ ,  $j = 1, 2, 3$ , only, i.e. they are known functions. Equation (4.2.6) gives us the evolution of the Weyl function  $M(t, \lambda)$  with respect to  $t$ . This equation corresponds to the equations (4.1.6)-(4.1.7). But here the evolution equation (4.2.6) is nonlinear. This happens because of nonlinear reflection at the point  $x = 0$ . That is the reason that the mixed problem on the half-space is more difficult than the Cauchy problem.

Here is the algorithm for the solution of the mixed problem (4.2.1)-(4.2.2).

**Algorithm 4.2.1.** (1) Using the functions  $q_0, q_1, q_2$  and  $q_3$ , construct the functions  $M(0, \lambda)$  and  $F_{jk}(0, t, \lambda)$ .

(2) Calculate  $M(t, \lambda)$  by solving equation (4.2.6) with the initial data  $M(0, \lambda)$ .

(3) Construct  $q(x, t)$  by solving the inverse problem.

**Remark 4.2.1.** Equation (4.2.6) is a (scalar) Riccati equation. Therefore it follows from Radon's lemma (see subsection 4.2.3 below) that the solution of (4.2.6) with given initial condition  $M(0, \lambda)$  has the form

$$M(t, \lambda) = \frac{X_2(t, \lambda)}{X_1(t, \lambda)},$$

where  $X_1(t, \lambda)$  and  $X_2(t, \lambda)$  is the unique solution of the initial value problem

$$\begin{bmatrix} \dot{X}_1(t, \lambda) \\ \dot{X}_2(t, \lambda) \end{bmatrix} = F(0, t, \lambda) \begin{bmatrix} X_1(t, \lambda) \\ X_2(t, \lambda) \end{bmatrix}, \quad \begin{bmatrix} X_1(0, \lambda) \\ X_2(0, \lambda) \end{bmatrix} = \begin{bmatrix} 1 \\ M(0, \lambda) \end{bmatrix}.$$

For convenience of the reader we present below in Subsection 4.2.3 a general version of Radon's lemma.

*Method 2.* Instead of the Lax equation we will use here another representation of the KdV equation. Denote

$$G = \begin{bmatrix} 0 & 1 \\ q - \lambda & 0 \end{bmatrix}, \quad U = \dot{G} - F' + GF - FG,$$

where  $F$  is defined by (4.2.5). Then, it follows by elementary calculations that

$$U = \begin{bmatrix} 0 & 0 \\ u & 0 \end{bmatrix},$$

where

$$u = q_t - 6qq_x + q_{xxx}.$$

Thus, the KdV equation is equivalent to the equation

$$U = 0. \tag{4.2.7}$$

We shall see that Equation (4.2.7) is sometimes more convenient for the solution of the mixed problem (4.2.1)-(4.2.2) than the Lax equation (4.1.4).

Let the matrix  $W(x, t, \lambda)$  and  $V(x, t, \lambda)$  be solutions of the Cauchy problems:

$$W' = GW, \quad W|_{x=0} = E, \quad \dot{V} = FV, \quad V|_{t=0} = E, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{4.2.8}$$

Denote by  $C(x, t, \lambda)$  and  $S(x, t, \lambda)$  the solutions of equation (4.2.3) under the initial conditions  $C|_{x=0} = S'|_{x=0} = 1$ ,  $S|_{x=0} = C'|_{x=0} = 0$ . Clearly,

$$W = \begin{bmatrix} C & S \\ C' & S' \end{bmatrix}.$$

**Lemma 4.2.1.** *Let  $q(x, t)$  be a solution of (4.2.1)-(4.2.2). Then*

$$\dot{W}(x, t, \lambda) = F(x, t, \lambda)W(x, t, \lambda) - W(x, t, \lambda)F(0, t, \lambda), \tag{4.2.9}$$

$$W(x, t, \lambda) = V(x, t, \lambda)W(x, 0, \lambda)V^{-1}(0, t, \lambda). \tag{4.2.10}$$

*Proof.* Using (4.2.8) we calculate

$$(\dot{W} - FW)' - G(\dot{W} - FW) = UW.$$

Since  $U = 0$ , it follows that the matrix  $\xi = \dot{W} - FW$  is a solution of the equation  $\xi' = G\xi$ . Moreover,  $\xi|_{x=0} = -F(0, t, \lambda)$ . Hence,  $\xi(x, t, \lambda) = -W(x, t, \lambda)F(0, t, \lambda)$ , i.e. (4.2.9) is valid.

Denote  $Y = W(x, t, \lambda)V(0, t, \lambda)$ ,  $Z = V(x, t, \lambda)W(x, 0, \lambda)$ . Then, using (4.2.8) and (4.2.9), we calculate  $\dot{Y} = F(x, t, \lambda)Y = \dot{Z}$ ,  $Y|_{t=0} = W(x, 0, \lambda) = Z|_{t=0}$ . By virtue of the uniqueness of the solution of the Cauchy problem, we conclude that  $Y \equiv Z$ , i.e. (4.2.10) is valid.  $\square$

Consider the matrices

$$\Psi = \begin{bmatrix} \Phi & S \\ \Phi' & S' \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 0 \\ M & 1 \end{bmatrix}, \quad N_1 := N^{-1} = \begin{bmatrix} 1 & 0 \\ -M & 1 \end{bmatrix}.$$

Since  $\Phi = C + MS$ , we get  $\Psi = WN$  or  $W = \Psi N_1$ . Then (4.2.9)-(4.2.10) take the form

$$\dot{\Psi}(x, t, \lambda) = F(x, t, \lambda)\Psi(x, t, \lambda) - \Psi(x, t, \lambda)D^0(t, \lambda), \quad (4.2.11)$$

$$\Psi(x, t, \lambda) = V(x, t, \lambda)\Psi(x, 0, \lambda)B(t, \lambda), \quad (4.2.12)$$

where

$$D^0(t, \lambda) = (\dot{N}_1(t, \lambda) + N_1(t, \lambda)F(0, t, \lambda))N(t, \lambda), \quad D^0 = \begin{bmatrix} d_{11}^0 & d_{12}^0 \\ d_{21}^0 & d_{22}^0 \end{bmatrix}, \quad (4.2.13)$$

$$B(t, \lambda) = N_1(0, \lambda)V^{-1}(0, t, \lambda)N(t, \lambda), \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}. \quad (4.2.14)$$

**Lemma 4.2.2.** *Let  $q(x, t)$  be a solution of (4.2.1)-(4.2.2). Then*

$$\dot{B}(t, \lambda) = -B(t, \lambda)D^0(t, \lambda), \quad B(0, \lambda) = E, \quad (4.2.15)$$

$$d_{21}^0(t, \lambda) \equiv 0, \quad b_{21}(t, \lambda) \equiv 0. \quad (4.2.16)$$

*Proof.* Differentiating (4.2.12) with respect to  $t$  and using the equality  $\dot{V} = FV$ , we get

$$\dot{\Psi} = \dot{V}(\Psi|_{t=0})B + V(\Psi|_{t=0})\dot{B} = F\Psi + V(\Psi|_{t=0})\dot{B}. \quad (4.2.17)$$

On the other hand, it follows from (4.2.11) and (4.2.12) that

$$\dot{\Psi} = F\Psi - V(\Psi|_{t=0})BD^0.$$

Together with (4.2.17) this gives  $\dot{B} = BD^0$ . Moreover, according to (4.2.12),  $B(0, \lambda) = E$ . Rewriting (4.2.11) in coordinates, we obtain

$$Sd_{21}^0 = -\dot{\Phi} + F_{11}\Phi + F_{12}\Phi' - \Phi d_{11}^0. \quad (4.2.18)$$

Since

$$-\dot{\Phi} + F_{11}\Phi + F_{12}\Phi' - \Phi d_{11}^0 = O(\exp(i\rho x)), \quad x \rightarrow \infty,$$

equality (4.2.18) is possible only if  $d_{21}^0 = 0$ . Then (4.2.15) yields

$$\dot{b}_{21} = -b_{21}d_{11}^0, \quad b_{21}|_{t=0} = 0,$$

and consequently  $b_{21} \equiv 0$ . □

Denote

$$R(t, \lambda) := V^{-1}(0, t, \lambda), \quad R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}.$$

Since  $R(V|_{x=0}) = E$ , we calculate  $\dot{R}(V|_{x=0}) + R(\dot{V}|_{x=0}) = 0$ . Thus, the matrix  $R(t, \lambda)$  is a solution of the Cauchy problem

$$\dot{R}(t, \lambda) = -R(t, \lambda)\tilde{F}(t, \lambda), \quad R(0, \lambda) = E, \quad (4.2.19)$$

where

$$\tilde{F} = \begin{bmatrix} \tilde{F}_{11} & \tilde{F}_{12} \\ \tilde{F}_{21} & \tilde{F}_{22} \end{bmatrix} = \begin{bmatrix} -q_2 & 4\lambda + 2q_1 \\ (4\lambda + 2q_1)(q_1 - \lambda) - q_3 & q_2 \end{bmatrix}.$$

It is important that (4.2.19) is a linear system.

**Lemma 4.2.3.** *Let  $q(x, t)$  be a solution of (4.2.1)-(4.2.2). Then*

$$\dot{M}(t, \lambda) = \tilde{F}_{21}(t, \lambda) + (\tilde{F}_{22}(t, \lambda) - \tilde{F}_{11}(t, \lambda))M(t, \lambda) - \tilde{F}_{12}(t, \lambda)M^2(t, \lambda), \quad (4.2.20)$$

$$M(t, \lambda) = -\frac{M^0(\lambda)R_{11}(t, \lambda) - R_{21}(t, \lambda)}{M^0(\lambda)R_{12}(t, \lambda) - R_{22}(t, \lambda)}. \quad (4.2.21)$$

where  $M^0(\lambda) = M(0, \lambda)$ .

*Proof.* Rewriting (4.2.13) in coordinates, we get

$$d_{21}^0(t, \lambda) = -\dot{M}(t, \lambda) + F_{21}(0, t, \lambda) + (F_{22}(0, t, \lambda) - F_{11}(0, t, \lambda))M(t, \lambda) - F_{12}(0, t, \lambda)M^2(t, \lambda). \quad (4.2.22)$$

By virtue of (4.2.16),  $d_{21}^0 \equiv 0$ , i.e. (4.2.20) holds. Rewriting (4.2.14) in coordinates, we have

$$b_{21}(t, \lambda) = -M(0, \lambda)(R_{11}(t, \lambda) + M(t, \lambda)R_{12}(t, \lambda)) + (R_{21}(t, \lambda) + M(t, \lambda)R_{22}(t, \lambda)).$$

By virtue of (4.2.16),  $b_{21} \equiv 0$ , i.e. (4.2.21) holds. □

Formula (4.2.21) gives us the solution of the nonlinear evolution equation (4.2.20). Thus, we obtain the following algorithm for the solution of the mixed problem (4.2.1)-(4.2.2) by the inverse problem method.

**Algorithm 4.2.2** (1) Using the functions  $q_0, q_1, q_2$  and  $q_3$ , construct the functions  $M^0(\lambda)$  and  $\tilde{F}_{jk}(t, \lambda)$ .

(2) Find  $R(t, \lambda)$  by solving the Cauchy problem (4.2.19).

(3) Calculate  $M(t, \lambda)$  by (4.2.21).

(4) Construct  $q(x, t)$  by solving the inverse problem (see Chapter 2).

**4.2.2.** In this subsection we study the existence of the solution of the mixed problem (4.2.1)-(4.2.2) and provide necessary and sufficient conditions for its solvability. The following theorem shows that the existence of the solution of (4.2.1)-(4.2.2) is equivalent to the solvability of the corresponding inverse problem.

**Theorem 4.2.1.** *Let for  $x \geq 0$ ,  $t \geq 0$  continuous complex-valued functions  $q_0(x)$ ,  $q_1(t)$ ,  $q_2(t)$ ,  $q_3(t)$  be given such that  $q_0(x) \in L(0, \infty)$ ,  $q_0(0) = q_1(0)$ . Construct the function  $M(t, \lambda)$  according to steps (1)-(3) of Algorithm 4.2.2. Assume that there exists a function  $q(x, t) \in Q$  for which  $M(t, \lambda)$  is the Weyl function. Then  $q(x, t)$  is the solution of the mixed problem (4.2.1)-(4.2.2).*

Thus, the problem (4.2.1)-(4.2.2) has a solution if and only if the solution of the corresponding inverse problem exists. Necessary and sufficient conditions for the solvability of the inverse problem for the Sturm-Liouville operator on the half-line from its Weyl function and an algorithm for the solution are given in Chapter 2.

*Proof.* We will use the notations of Subsection 4.2.1. Since

$$\begin{aligned} (\dot{W} - FW)' - G(\dot{W} - FW) &= UW, \\ (\dot{W} - FW)|_{x=0} &= -F(0, t, \lambda), \end{aligned}$$

we get

$$\dot{W}(x, t, \lambda) = F(x, t, \lambda)W(x, t, \lambda) - W(x, t, \lambda)\chi(x, t, \lambda), \quad (4.2.23)$$

where

$$\chi(x, t, \lambda) = F(0, t, \lambda) - \int_0^x W^{-1}(s, t, \lambda)U(s, t)W(s, t, \lambda) ds.$$

It follows from (4.2.23) and the relation  $W = \Psi N_1$  that

$$\dot{\Psi}(x, t, \lambda) = F(x, t, \lambda)\Psi(x, t, \lambda) - \Psi(x, t, \lambda)D(x, t, \lambda), \quad (4.2.24)$$

where

$$D(x, t, \lambda) = (\dot{N}_1(t, \lambda) - N_1(t, \lambda)\chi(x, t, \lambda))N(t, \lambda), \quad D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}. \quad (4.2.25)$$

Rewriting (4.2.24) and (4.2.25) in coordinates we obtain

$$d_{11}(x, t, \lambda) = d_{11}^0(t, \lambda) + \int_0^x u(s, t)\Phi(s, t, \lambda)S(s, t, \lambda) ds, \quad (4.2.26)$$

$$d_{21}(x, t, \lambda) = d_{21}^0(t, \lambda) - \int_0^x u(s, t)\Phi^2(s, t, \lambda) ds, \quad (4.2.27)$$

$$Sd_{21} = -\dot{\Phi} + F_{11}\Phi + F_{12}\Phi' - \Phi d_{11}. \quad (4.2.28)$$

Fix  $t \geq 0$  and  $\lambda$  ( $\lambda = \rho^2$ ,  $\text{Im } \rho > 0$ ). It follows from (4.2.26) that  $d_{11} = O(1)$ ,  $x \rightarrow \infty$ . From this and (4.2.28) we get  $d_{21} = O(\exp(2i\rho x))$ ,  $x \rightarrow \infty$ , and consequently  $d_{21} \rightarrow 0$  as  $x \rightarrow \infty$ . Then, by virtue of (4.2.27),

$$d_{21}^0(t, \lambda) = \int_0^\infty u(x, t)\Phi^2(x, t, \lambda) dx.$$

Together with (4.2.22) this yields

$$\begin{aligned} \dot{M}(t, \lambda) &= F_{21}(0, t, \lambda) + (F_{22}(0, t, \lambda) - F_{11}(0, t, \lambda))M(t, \lambda) - F_{12}(0, t, \lambda)M^2(t, \lambda) \\ &\quad - \int_0^\infty u(x, t)\Phi^2(x, t, \lambda) dx. \end{aligned} \quad (4.2.29)$$

Furthermore, it follows from (4.2.21) that

$$M(0, \lambda) = M^0(\lambda). \quad (4.2.30)$$

Since the specification of the Weyl function uniquely determines the potential we have from (4.2.30) that

$$q(x, 0) = q_0(x). \quad (4.2.31)$$

Differentiating (4.2.21) with respect to  $t$  we arrive at (4.2.20). Comparing (4.2.20) with (4.2.29) we obtain

$$\hat{F}_{21}(t, \lambda) + (\hat{F}_{22}(t, \lambda) - \hat{F}_{11}(t, \lambda))M(t, \lambda) - \hat{F}_{12}(t, \lambda)M^2(t, \lambda) - \int_0^\infty u(x, t)\Phi^2(x, t, \lambda) dx,$$

where  $\hat{F}_{jk}(t, \lambda) = F_{jk}(0, t, \lambda) - \tilde{F}_{jk}(t, \lambda)$ . Hence

$$\begin{aligned} & -\left(q_{xx}(0, t) - q_3(t)\right) + 2\lambda\left(q(0, t) - q_1(t)\right) + 2\left(q^2(0, t) - q_1^2(t)\right) + 2\left(q_x(0, t) - q_2(t)\right)M(t, \lambda) \\ & - 2\left(q(0, t) - q_1(t)\right)M^2(t, \lambda) - \int_0^\infty u(x, t)\Phi^2(x, t, \lambda) dx = 0. \end{aligned} \quad (4.2.32)$$

Since

$$\Phi(x, t, \lambda) = \frac{e(x, t, \rho)}{e(0, t, \rho)}, \quad M(t, \lambda) = \frac{e'(0, t, \rho)}{e(0, t, \rho)},$$

we have for a fixed  $t$ ,

$$M(t, \lambda) = (i\rho) + O\left(\frac{1}{\rho}\right), \quad \int_0^\infty u(x, t)\Phi^2(x, t, \lambda) dx = O\left(\frac{1}{\rho}\right), \quad |\rho| \rightarrow \infty.$$

Then, (4.2.32) implies

$$q(0, t) = q_1(t), \quad q_x(0, t) = q_2(t), \quad q_{xx}(0, t) = q_3(t), \quad (4.2.33)$$

$$\int_0^\infty u(x, t)\Phi^2(x, t, \lambda) dx = 0. \quad (4.2.34)$$

In particular, (4.2.34) yields

$$\int_0^\infty u(x, t)e^2(x, t, \rho) dx = 0. \quad (4.2.35)$$

According to (2.1.33),

$$e(x, t, \rho) = \exp(i\rho x) + \int_x^\infty A(x, t, \xi) \exp(i\rho\xi) d\xi,$$

and consequently,

$$e^2(x, t, \rho) = \exp(2i\rho x) + \int_x^\infty E(x, t, \xi) \exp(2i\rho\xi) d\xi, \quad (4.2.36)$$

where

$$E(x, t, \xi) = 4A(x, t, 2\xi - x) + \int_x^{2\xi - x} A(x, s)A(x, 2\xi - s) ds.$$

Substituting (4.2.36) into (4.2.35) we get

$$\int_0^\infty \left( u(x, t) + \int_0^x E(\xi, t, x) u(\xi, t) d\xi \right) \exp(2i\rho x) dx = 0.$$

Hence,  $u(x, t) = 0$ , i.e. the function  $q(x, t)$  satisfies the KdV equation (4.2.1). Together with (4.2.31) and (4.2.33) this yields that  $q(x, t)$  is a solution of the mixed problem (4.2.1)-(4.2.2). Theorem 4.2.1 is proved.  $\square$

**4.2.3.** In this subsection we formulate and prove Radon's lemma for the matrix Riccati equation.

Let us consider the following Cauchy problem for the Riccati equation:

$$\left. \begin{aligned} \dot{Z} &= Q_{21}(t) + Q_{22}(t)Z - ZQ_{11}(t) - ZQ_{12}(t)Z, \quad t \in [t_0, t_1], \\ Z(t_0) &= Z_0, \end{aligned} \right\} \quad (4.2.37)$$

where  $Z(t), Z_0, Q_{11}(t), Q_{12}(t), Q_{21}(t)$  and  $Q_{22}(t)$  are complex matrix of dimensions  $m \times n, m \times n, n \times n, n \times m, m \times n$ , and  $m \times m$  respectively, and  $Q_{jk}(t) \in C[t_0, t_1]$ .

Let the matrices  $(X(t), Y(t), t \in [t_0, t_1])$ , be the unique solution of the linear Cauchy problem

$$\left. \begin{aligned} \dot{X} &= Q_{11}(t)X + Q_{12}(t)Y, \quad X(t_0) = E, \\ \dot{Y} &= Q_{21}(t)X + Q_{22}(t)Y, \quad Y(t_0) = Z_0, \end{aligned} \right\} \quad (4.2.38)$$

where  $E$  is the identity  $n \times n$  matrix.

**Radon's Lemma.** *The Cauchy problem (4.2.37) has a unique solution  $Z(t) \in C_1[t_0, t_1]$ , if and only if  $\det X(t) \neq 0$  for all  $t \in [t_0, t_1]$ . Moreover,*

$$Z(t) = Y(t)(X(t))^{-1}. \quad (4.2.39)$$

*Proof.* 1) Let  $\det X(t) \neq 0$  for all  $t \in [t_0, t_1]$ . Define the matrix  $Z(t)$  via (4.2.39). Then, using (4.2.38), we get

$$\begin{aligned} Z(t_0) &= Y(t_0)(X(t_0))^{-1} = Z_0, \\ \dot{Z} &= \dot{Y}X^{-1} - YX^{-1}\dot{X}X^{-1} = (Q_{21}X + Q_{22}Y)X^{-1} - YX^{-1}(Q_{11}X + Q_{12}Y)X^{-1} \\ &= Q_{21} + Q_{22}Z - ZQ_{11} - ZQ_{12}Z. \end{aligned}$$

Thus,  $Z(t)$  is the solution of (4.2.37).

2) Let  $Z(t) \in C_1[t_0, t_1]$  be a solution of (4.2.37). Denote by  $\tilde{X}(t)$  the solution of the Cauchy problem

$$\dot{\tilde{X}} = (Q_{11}(t) + Q_{12}(t)Z(t))\tilde{X}, \quad \tilde{X}(t_0) = E. \quad (4.2.40)$$

Clearly,  $\det \tilde{X}(t) \neq 0$  for all  $t \in [t_0, t_1]$ . Take

$$\tilde{Y}(t) := Z(t)\tilde{X}(t).$$

Then, it follows from (4.2.37) and (4.2.40) that

$$\dot{\tilde{Y}} = \dot{Z}\tilde{X} + Z\dot{\tilde{X}} = Q_{21}\tilde{X} + Q_{22}\tilde{Y},$$



$$\dot{\tilde{X}} = Q_{11}\tilde{X} + Q_{12}\tilde{Y},$$

and  $\tilde{X}(t_0) = E$ ,  $\tilde{Y}(t_0) = Z_0$ . By virtue of the uniqueness theorem, this yields

$$\tilde{X}(t) \equiv X(t), \quad \tilde{Y}(t) \equiv Y(t),$$

and Radon's lemma is proved.  $\square$

### 4.3. CONSTRUCTING PARAMETERS OF A MEDIUM FROM INCOMPLETE SPECTRAL INFORMATION

In this section the inverse problem of synthesizing parameters of differential equations with singularities from incomplete spectral information. We establish properties of the spectral characteristics, obtain conditions for the solvability of such classes of inverse problems and provide algorithms for constructing the solution.

#### 4.3.1. Properties of spectral characteristics.

Let us consider the system

$$\frac{dy_1}{dx} = i\rho R(x) y_2, \quad \frac{dy_2}{dx} = i\rho \frac{1}{R(x)} y_1, \quad x \in [0, T] \quad (4.3.1)$$

with the initial conditions

$$y_1(0, \rho) = 1, \quad y_2(0, \rho) = -1.$$

Here  $\rho = k + i\tau$  is the spectral parameter, and  $R(x)$  is a real function.

For a wide class of problems describing the propagation of electromagnetic waves in a stratified medium, Maxwell's equations can be reduced to the canonical form (4.3.1), where  $x$  is the variable in the direction of stratification,  $y_1$  and  $y_2$  are the components of the electromagnetic field,  $R(x)$  is the wave resistance which describes the refractive properties of the medium and  $\rho$  is the wave number in a vacuum. System (4.3.1) also appears in radio engineering for the design of directional couplers for heterogeneous electronic lines, which constitute one of the important classes of radiophysical synthesis problems [mes1].

Some aspects of synthesis problems for system (4.3.1) with a positive  $R(x)$  were studied in [lit1], [mes1], [tik1]-[tik3], [zuy1] and other works. In this section we partially use these results. We study the inverse problem for system (4.3.1) from incomplete spectral information in the case when  $R(x)$  is nonnegative and can have zeros which are called turning points. More precisely, we shall consider two classes of functions  $R(x)$ . We shall say that  $R(x) \in B_0$  if  $R(x), R'(x)$  are absolutely continuous on  $[0, T]$ ,  $R''(x) \in L_2(0, T)$ ,  $R(x) > 0$ ,  $R(0) = 1$ ,  $R'(0) = 0$ . We also consider the more general case when  $R(x)$  has zeros  $0 < x_1 < \dots < x_p < T$ ,  $p \geq 0$  inside the interval  $(0, T)$ . We shall say that  $R(x) \in B_0^+$  if  $R(x)$  has the form

$$\frac{1}{R(x)} = \sum_{j=1}^p \frac{R_j}{(x - x_j)^2} + R_0(x), \quad 0 < x_1 < \dots < x_p < T, \quad R_j > 0, \quad 0 \leq j \leq p,$$

and  $R_0(x), R'_0(x)$  are absolutely continuous on  $[0, T]$ ,  $R''_0(x) \in L_2(0, T)$ ,  $R(x) > 0$  ( $x \neq x_j$ ),  $R(0) = 1$ ,  $R'(0) = 0$ . In particular, if here  $p = 0$  then  $R(x) \in B_0$ .

As the main spectral characteristics we introduce the amplitude reflection coefficient

$$r(\rho) = \frac{y_1(T, \rho) + R^0 y_2(T, \rho)}{y_1(T, \rho) - R^0 y_2(T, \rho)}, \quad R^0 := R(T);$$

the power reflection coefficient

$$\sigma(k) = |r(k)|^2, \quad k := Re \rho;$$

the transmission coefficients

$$f_1(\rho) = \frac{y_1(T, \rho) - R^0 y_2(T, \rho)}{2\sqrt{R^0}}, \quad f_2(\rho) = \frac{y_1(T, \rho) + R^0 y_2(T, \rho)}{2\sqrt{R^0}}$$

and the characteristic function

$$\Delta(\rho) = \frac{1}{\sqrt{R^0}} y_1(T, \rho).$$

Clearly,

$$r(\rho) = f_2(\rho)/f_1(\rho), \quad (4.3.2)$$

$$\Delta(\rho) = f_1(\rho) + f_2(\rho). \quad (4.3.3)$$

In many cases of practical interest, the phase is difficult or impossible to measure, while the amplitude is easily accessible to measurement. Such cases lead us to the so-called incomplete inverse problems where only a part of the spectral information is available. In this paper we study one of the incomplete inverse problems, namely, the inverse problem of recovering the wave resistance from the power reflection coefficient:

**Inverse Problem 4.3.1.** Given  $\sigma(k)$ , construct  $R(x)$ .

The lack of spectral information leads here to nonuniqueness of the solution of the inverse problem. Let us briefly describe a scheme of the solution of Inverse Problem 4.3.1.

Denote  $\alpha_j(k) = |f_j(k)|$ ,  $j = 1, 2$ . Since  $\alpha_1^2(k) - \alpha_2^2(k) \equiv 1$  (see (4.3.19) below), we get in view of (4.3.2),

$$\sigma(k) = 1 - (\alpha_1(k))^{-2}, \quad 0 \leq \sigma(k) < 1.$$

Firstly, from the given power reflection coefficient  $\sigma(k)$  we construct  $\alpha_j(k)$ . Then, using analytic properties of the transmission coefficients  $f_j(\rho)$  and, in particular, information about their zeros, we reconstruct the transmission coefficients from their amplitudes. Namely on this stage we are faced with nonuniqueness. Problems concerning the reconstruction of analytic functions from their moduli often appear in applications and have been studied in many works (see [hoe1], [hur1] and references therein). The last of our steps is to calculate the characteristic function  $\Delta(\rho)$  by (4.3.3) and to solve the inverse problem of recovering  $R(x)$  from  $\Delta(\rho)$ . Here we use the Gelfand-Levitan-Marchenko method (see [mar1], [lev2] and Chapters 1-2 of this book).

To realize this scheme, in Subsection 4.3.1 we study properties of the spectral characteristics. In Subsection 4.3.2 we solve the synthesis problem for  $R(x)$  from the characteristic function  $\Delta(\rho)$ , obtain necessary and sufficient conditions for its solvability and provide three algorithms for constructing the solution. In Subsection 4.3.3 we study the problem of recovering the transmission coefficients from their moduli. In Subsection 4.3.4, the so-called symmetrical case is considered, and in Subsection 4.3.5 we provide an algorithm for the

solution of Inverse Problem 1. We note that inverse problems for Sturm-Liouville equations with turning points have been studied in [FY1] and [FY3]. Some aspects of the turning point theory and a number of its applications are described in [ebe4], [mch1] and [was1].

We shall say that  $R(x) \in B_0^-$  if  $(R(x))^{-1} \in B_0^+$ . An investigation of the classes  $B_0^+$  and  $B_0^-$  is completely similar because the replacement  $R \rightarrow 1/R$  is equivalent to the replacement  $(y_1, y_2) \rightarrow (-y_2, -y_1)$ . Below it will be more convenient for us to consider the case  $R(x) \in B_0^-$ .

We transform (4.3.1) by means of the substitution

$$y_1(x, \rho) = \sqrt{R(x)}u(x, \rho), \quad y_2(x, \rho) = \frac{1}{\sqrt{R(x)}}v(x, \rho) \quad (4.3.4)$$

to the system

$$u' + h(x)u = ipv, \quad v' - h(x)v = ipu, \quad x \in [0, T] \quad (4.3.5)$$

with the initial conditions  $u(0, \rho) = 1, v(0, \rho) = -1$ , where

$$h(x) = \frac{R'(x)}{2R(x)}. \quad (4.3.6)$$

Hence the function  $u(x, \rho)$  satisfies the equation

$$-u'' + q(x)u = \lambda u, \quad \lambda = \rho^2 \quad (4.3.7)$$

and the initial conditions

$$u(0, \rho) = 1, \quad u'(0, \rho) = -i\rho,$$

where

$$q(x) = h^2(x) - h'(x) \quad (4.3.8)$$

or

$$q(x) = \frac{3}{4} \left( \frac{R'(x)}{R(x)} \right)^2 - \frac{1}{2} \frac{R''(x)}{R(x)}.$$

Similarly,

$$-v'' + g(x)v = \lambda v, \quad v(0, \rho) = -1, \quad v'(0, \rho) = i\rho,$$

where  $g(x) = h^2(x) + h'(x)$ .

In view of (4.3.4), the transmission coefficients, the amplitude reflection coefficient and the characteristic function take the form

$$\left. \begin{aligned} f_1(\rho) &= \frac{u(T, \rho) - v(T, \rho)}{2}, & f_2(\rho) &= \frac{u(T, \rho) + v(T, \rho)}{2}, \\ r(\rho) &= \frac{u(T, \rho) + v(T, \rho)}{u(T, \rho) - v(T, \rho)} = \frac{f_2(\rho)}{f_1(\rho)}, & \Delta(\rho) &= u(T, \rho). \end{aligned} \right\} \quad (4.3.9)$$

Since  $y_1(x, 0) \equiv 1$ , we have according to (4.3.4),

$$\sqrt{R(x)}u(x, 0) \equiv 1. \quad (4.3.10)$$

**Lemma 4.3.1.**  $R(x) \in B_0^-$  if and only if  $q(x) \in L_2(0, T)$ ,  $\Delta(0) \neq 0$ .

*Proof.* 1) Let  $R(x) \in B_0^-$ , i.e.

$$\left. \begin{aligned} R(x) &= \sum_{j=1}^p \frac{R_j}{(x-x_j)^2} + R_0(x), \quad 0 < x_1 < \dots < x_p < T, \quad R_j > 0, \\ R_0(x) &\in W_2^2(0, T), \quad R(x) > 0 \ (x \neq x_j), \quad R(0) = 1, \quad R'(0) = 0. \end{aligned} \right\} \quad (4.3.11)$$

Using (4.3.6) and (4.3.11) we get for  $x \rightarrow x_j$ ,

$$h(x) \sim -\frac{1}{x-x_j} + h_j^*(x), \quad h_j^*(x) \in W_2^1, \quad h_j^*(x_j) = 0,$$

and consequently  $q(x) \in L_2(0, T)$ . It follows from (4.3.10) that  $u(T, 0) \neq 0$ , i.e.  $\Delta(0) \neq 0$ .

2) Let now  $q(x) \in L_2(0, T)$ ,  $u(T, 0) \neq 0$ , and let  $0 < x_1 < \dots < x_p < T$  be zeros of  $u(x, 0)$ . It follows from (4.3.10) that

$$R(x) = (u(x, 0))^{-2}.$$

Hence  $R(x) > 0 \ (x \neq x_j)$ ,  $R(0) = 1$ ,  $R'(0) = 0$ , and

$$R(x) \sim \frac{R_j}{(x-x_j)^2}, \quad x \rightarrow x_j, \quad R_j > 0.$$

Denote

$$R_0(x) := R(x) - \sum_{j=1}^p \frac{R_j}{(x-x_j)^2}.$$

It is easy to see that  $R_0(x) \in W_2^2[0, T]$ . □

**Remark 4.3.1.** If  $R(x) \in B_0^-$ , then the functions  $y_1(x, \rho)$  and  $v(x, \rho)$  have singularities of order 1 at  $x = x_j$ , and there exist finite limits  $\lim_{x \rightarrow x_j} (x-x_j)v(x, \rho)$ ,  $\lim_{x \rightarrow x_j} (x-x_j)y_1(x, \rho)$ .

In the other words, we continue solutions in the neighbourhoods of the singular points with the help of generalized Bessel type solutions (see, [yur28]). It follows from (4.3.10) that

$$\lim_{x \rightarrow x_j} (x-x_j)\sqrt{R(x)} = R_j.$$

**Example 4.3.1.** Let  $R(x) = \frac{1}{\cos^2 x}$ . Then  $h(x) = \tan x$ ,  $q(x) = -1$ ,  $u(x, 0) = \frac{1}{\sqrt{R(x)}} = \cos x$ ,  $u(x, \rho) = \cos \mu x - i\rho \frac{\sin \mu x}{\mu}$ ,  $\mu^2 = \rho^2 + 1$ . If  $T \in \left(\frac{(2p-1)\pi}{2}, \frac{(2p+1)\pi}{2}\right)$ , then  $u(x, 0)$  has  $p$  zeros  $x_j = \frac{(2j-1)\pi}{2}$ ,  $j = \overline{1, p}$ .

In the sequel, we denote by  $AC[a, b]$  the set of absolutely continuous functions on the segment  $[a, b]$ .

**Theorem 4.3.1.** Let  $R(x) \in B_0^-$ . Then

(i) The characteristic function  $\Delta(\rho)$  is entire in  $\rho$ , and the following representation holds

$$\Delta(\rho) = e^{-i\rho T} + \int_{-T}^T \eta(t) e^{-i\rho t} dt, \quad \eta(t) \in AC[-T, T], \quad \eta'(t) \in L_2(-T, T), \quad \eta(-T) = 0, \quad (4.3.12)$$

where  $\eta(t)$  is a real function, and

$$\eta(T) = \frac{1}{2} \int_0^T q(t) dt = -\frac{h(T)}{2} + \frac{1}{2} \int_0^T h^2(t) dt.$$

(ii) For real  $\rho$ ,  $\delta(\rho)$  has no zeros. For  $\text{Im } \rho > 0$ ,  $\Delta(\rho)$  has at most a finite number of simple zeros of the form  $\rho_j = i\tau_j$ ,  $\tau_j > 0$ ,  $j = \overline{1, m}$ ,  $m \geq 0$ .

(iii) The function  $\delta(\rho)$  has no zeros for  $\text{Im } \rho \geq 0$  if and only if  $R(x) \in B_0$ .

*Proof.* 1) It is well-known (see [mar1] and Section 1.3 of this book) that  $u(x, \rho)$  has the form

$$u(x, \rho) = e^{-i\rho x} + \int_{-x}^x K(x, t) e^{-i\rho t} dt,$$

where  $K(x, t)$  is a real, absolutely continuous function,  $K_t(T, t) \in L_2(-T, T)$ ,  $K(x, -x) = 0$ ,  $K(x, x) = \frac{1}{2} \int_0^x q(t) dt$ . Moreover,

$$\overline{u(x, \rho)} = u(x, -\bar{\rho}), \quad (4.3.13)$$

$$\langle u(x, \rho), u(x, -\rho) \rangle \equiv 2i\rho, \quad (4.3.14)$$

where  $\langle y, z \rangle := yz' - y'z$ . Hence we arrive at (4.3.12), where  $\eta(t) = K(T, t)$ .

2) It follows from (4.3.13) that for real  $\rho$ ,  $\overline{\Delta(\rho)} = \Delta(-\rho)$ , and consequently, in view of (4.3.14), for real  $\rho \neq 0$ ,  $\Delta(\rho)$  has no zeros. By virtue of Lemma 4.3.1,  $\Delta(0) \neq 0$ . Furthermore, denote  $\Omega_+ = \{\rho : \text{Im } \rho > 0\}$ . Let  $\rho = k + i\tau \in \Omega_+$  be a zero of  $\Delta(\rho)$ . Since

$$-u'' + q(x)u = \rho^2 u, \quad -\bar{u}'' + q(x)\bar{u} = \bar{\rho}^2 \bar{u},$$

we get

$$(\rho^2 - \bar{\rho}^2) \int_0^T |u|^2 dx = \Big|_0^T \langle u, \bar{u} \rangle = -i(\rho + \bar{\rho}),$$

which is possible only if  $k := \text{Re } \rho = 0$ . Hence, all zeros of  $\Delta(\rho)$  in  $\Omega_+$  are pure imaginary, and by virtue of (4.3.12), the number of zeros in  $\overline{\Omega}_+$  is finite.

3) Let us show that in  $\Omega_+$  all zeros of  $\Delta(\rho)$  are simple. Suppose that for a certain  $\rho = i\tau$ ,  $\tau > 0$ ,  $\Delta(\rho) = \dot{\Delta}(\rho) = 0$ , where  $\dot{\Delta}(\rho) = \frac{d}{d\rho} \Delta(\rho)$ . Denote  $u_1 = \dot{u}$ . Then, by virtue of (4.3.7),

$$-u_1'' + q(x)u_1 = \rho^2 u_1 + 2\rho u, \quad u_1(0, \rho) = 0, \quad u_1'(0, \rho) = -i.$$

Consequently,

$$\begin{aligned} 2\rho \int_0^T u^2(x, \rho) dx &= \int_0^T u(x, \rho) (-u_1''(x, \rho) + q(x)u_1(x, \rho) - \rho^2 u_1(x, \rho)) dx \\ &= -\Big|_0^T \langle u, u_1 \rangle = -i, \end{aligned}$$

i.e.

$$2\tau \int_0^\tau |u(x, \rho)|^2 dx = -1,$$

which is impossible.

4) If  $R(x) \in B_0$ , then in view of (4.3.10)  $u(x, 0) > 0$ ,  $x \in [0, T]$ . Since  $u(0, i\tau) = 1$ ,  $\tau \geq 0$  and  $u(x, i\tau) > 0$ ,  $x \in [0, T]$  for large  $\tau$ , we get  $u(T, i\tau) > 0$  for all  $\tau \geq 0$ , i.e.  $\Delta(\rho)$  has no zeros in  $\overline{\Omega}_+$ . The inverse assertion is proved similarly.  $\square$

In the analogous manner one can prove the following theorem.

**Theorem 4.3.2.** Let  $R(x) \in B_0^-$ .

(i) The functions  $f_1(\rho), f_2(\rho)$  are entire in  $\rho$ , and have the form

$$f_1(\rho) = e^{-i\rho T} + \int_{-T}^T g_1(t) e^{-i\rho t} dt, \quad g_1(t) \in AC[-T, T], \quad g_1'(t) \in L_2(-T, T), \quad g_1(-T) = 0, \quad (4.3.15)$$

$$f_2(\rho) = \int_{-T}^T g_2(t) e^{-i\rho t} dt, \quad g_2(t) \in AC[-T, T], \quad g_2'(t) \in L_2(-T, T), \quad g_2(-T) = 0, \quad (4.3.16)$$

where  $g_j(t)$  are real, and

$$g_1(T) = \frac{1}{2} \int_0^T h^2(t) dt, \quad g_2(T) = -\frac{h}{2}, \quad h := h(T). \quad (4.3.17)$$

(ii)

$$f_1(\rho)f_1(-\rho) - f_2(\rho)f_2(-\rho) \equiv 1. \quad (4.3.18)$$

(iii) For real  $\rho$ ,  $f_1(\rho)$  has no zeros. In  $\overline{\Omega}_+$ ,  $f_1(\rho)$  has a finite number of simple zeros of the form  $\rho_j^* = i\tau_j^*$ ,  $\tau_j^* > 0$ ,  $j = \overline{1, m^*}$ ,  $m^* \geq 0$ .

(iv) If  $R(x) \in B_0$ , then  $f_1(\rho)$  has no zeros in  $\overline{\Omega}_+$ , i.e.  $m^* = 0$ .

We note that (4.3.18) follows from (4.3.14). Since  $\overline{f_j(\rho)} = f_j(-\bar{\rho})$ , we obtain from (4.3.18) that

$$\alpha_1^2(k) - \alpha_2^2(k) \equiv 1. \quad (4.3.19)$$

It follows from (4.3.9), (4.3.15), (4.3.16) and (4.3.19) that

$$\left. \begin{aligned} \alpha_1^2(k) &= 1 + \frac{h^2}{4k^2} + \frac{\omega(k)}{k^2}, & \alpha_2^2(k) &= \frac{h^2}{4k^2} + \frac{\omega(k)}{k^2}, \\ \sigma(k) &= \frac{h^2}{4k^2 + h^2} + \frac{\omega_1(k)}{k^2}, & |k| &\rightarrow \infty, \end{aligned} \right\} \quad (4.3.20)$$

where  $\omega(k), \omega_1(k) \in L_2(-\infty, \infty)$ .

**Lemma 4.3.2.** Denote  $g_j^*(t) = g_j(t - T)$ . Then

$$g_1^*(\xi) + \int_0^\xi g_1^*(t)g_1^*(t + 2T - \xi) dt = \int_0^\xi g_2^*(t)g_2^*(t + 2T - \xi) dt, \quad \xi \in [0, 2T]. \quad (4.3.21)$$

*Proof.* Indeed, using (4.3.15) and (4.3.16), we calculate

$$f_1(\rho)f_1(-\rho) = 1 + \int_{-2T}^{2T} G_1(\xi) e^{-i\rho\xi} d\xi, \quad f_2(\rho)f_2(-\rho) = \int_{-2T}^{2T} G_2(\xi) e^{-i\rho\xi} d\xi, \quad (4.3.22)$$

where  $G_j(-\xi) = G_j(\xi)$ ,

$$G_1(\xi) = g_1(T - \xi) + \int_{\xi-T}^T g_1(t)g_1(t - \xi)dt, \quad G_2(\xi) = \int_{\xi-T}^\xi g_2(t)g_2(t - \xi)dt, \quad \xi > 0.$$

Substituting (4.3.22) into (4.3.18) we arrive at (4.3.21).  $\square$

Denote

$$p(x) = \begin{cases} q(T-x) & \text{for } 0 \leq x \leq T, \\ 0 & \text{for } x > T, \end{cases}$$

and consider the equation

$$-y'' + p(x)y = \lambda y, \quad x > 0 \quad (4.3.23)$$

on the half-line. Let  $e(x, \rho)$  be the Jost solution of (4.3.23) such that  $e(x, \rho) \equiv e^{i\rho x}$  for  $x \geq T$ . Denote  $\delta(\rho) = e(0, \rho)$ . Clearly,

$$e(x, \rho) = e^{i\rho T} u(T-x, \rho), \quad 0 \leq x \leq T, \quad \delta(\rho) = e^{i\rho T} \Delta(\rho).$$

Then

$$e(x, \rho) = e^{i\rho x} + \int_x^{2T-x} G(x, t) e^{i\rho t} dt, \quad 0 \leq x \leq T, \quad e(x, \rho) \equiv e^{i\rho x}, \quad x \geq T, \quad (4.3.24)$$

$$\delta(\rho) = 1 + \int_0^{2T} \theta(t) e^{i\rho t} dt, \quad \theta(t) \in AC[0, 2T], \quad \theta'(t) \in L_2(0, 2T), \quad \theta(2T) = 0, \quad (4.3.25)$$

where  $G(x, t) = K(T-x, T-t)$ ,  $\theta(t) = \eta(T-t) = G(0, t)$ ,  $G(x, 2T-x) = 0$ ,

$$\theta(0) = \frac{1}{2} \int_0^T p(t) dt, \quad G(x, x) = \frac{1}{2} \int_x^T p(\xi) d\xi.$$

Denote

$$\alpha_j = \left( \int_0^\infty e^2(x, \rho_j) dx \right)^{-1} > 0, \quad \rho_j = i\tau_j, \quad \tau_j > 0, \quad j = \overline{1, m}.$$

**Lemma 4.3.3.**

$$\alpha_j = \frac{i\delta(-\rho_j)}{\dot{\delta}(\rho_j)}, \quad (4.3.26)$$

where  $\dot{\delta}(\rho) = \frac{d}{d\rho} \delta(\rho)$ .

*Proof.* Since  $-e'' + p(x)e = \rho^2 e$  and  $\delta(\rho_j) = 0$ , we have

$$(\rho^2 - \rho_j^2) \int_0^\infty e(x, \rho) e(x, \rho_j) dx = \left|_0^\infty \langle e(x, \rho), e(x, \rho_j) \rangle = -e'(0, \rho_j) \delta(\rho).$$

If yields

$$\alpha_j = -\frac{2\rho_j}{e'(0, \rho_j) \dot{\delta}(\rho_j)}. \quad (4.3.27)$$

Since  $\langle e(x, \rho), e(x, -\rho) \rangle \equiv 2i\rho$ , we get

$$\delta(-\rho_j) e'(0, \rho_j) = 2i\rho_j. \quad (4.3.28)$$

Together with (4.3.27) it gives (4.3.26).  $\square$

### 4.3.2. Synthesis of $R(x)$ from the characteristic function $\Delta(\rho)$ .

In this subsection we study the inverse problem of recovering the wave resistance  $R(x)$  from the given  $\Delta(\rho)$  in the class  $B_0^-$ . This inverse problem has a unique solution. We

provide physical realization conditions, i.e. necessary and sufficient conditions on a function  $\Delta(\rho)$  to be the characteristic function for a certain  $R(x) \in B_0^-$ . We also obtain three algorithms for the solution of the inverse problem.

Denote  $s(\rho) = \frac{\delta(-\rho)}{\delta(\rho)}$ . For real  $\rho$ ,  $s(\rho)$  is continuous and, by virtue of (4.3.25),  $s(\rho) = 1 + O(\frac{1}{\rho})$ ,  $|\rho| \rightarrow \infty$ . Consequently,  $1 - s(\rho) \in L_2(-\infty, \infty)$ . We introduce the Fourier transform

$$F_0(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 - s(\rho)) e^{i\rho t} d\rho, \quad (4.3.29)$$

$$1 - s(\rho) = \int_{-\infty}^{\infty} F_0(t) e^{-i\rho t} dt. \quad (4.3.30)$$

Substituting (4.3.25) and (4.3.30) into the relation  $\delta(\rho) s(\rho) = \delta(-\rho)$ , we obtain the connection between  $F_0(t)$  and  $\theta(t)$ :

$$\theta(t) + F_0(t) + \int_0^{2T} \theta(s) F_0(t+s) ds = 0, \quad t > 0, \quad (4.3.31)$$

where  $\theta(t) \equiv 0$  for  $t > 2T$ . Further, we consider the function

$$F(t) = F_0(t) + F_1(t), \quad (4.3.32)$$

where

$$F_1(t) = \sum_{j=1}^m \alpha_j e^{i\rho_j t}. \quad (4.3.33)$$

Since

$$F_1(t) + \int_0^{2T} \theta(s) F_1(t+s) ds = \sum_{j=1}^m \alpha_j e^{i\rho_j t} \delta(\rho_j) = 0,$$

we get by virtue of (4.3.31) that

$$\theta(t) + F(t) + \int_0^{2T} \theta(s) F(t+s) ds = 0, \quad t > 0. \quad (4.3.34)$$

Calculating the integral (4.3.29) by Jordan's lemma for  $t > 2T$  and using Lemma 4.3.3 we obtain

$$F_0(t) = i \sum_{j=1}^m \operatorname{Res}_{\rho=\rho_j} (1 - s(\rho)) e^{i\rho t} = -i \sum_{j=1}^m \frac{\delta(-\rho_j)}{\delta(\rho_j)} e^{i\rho_j t} = - \sum_{j=1}^m \alpha_j e^{i\rho_j t} = -F_1(t).$$

Thus,

$$F(t) \equiv 0, \quad t > 2T. \quad (4.3.35)$$

Taking (4.3.35) into account we rewrite (4.3.34) as follows

$$\theta(t) + F(t) + \int_t^{2T} \theta(s-t) F(s) ds = 0, \quad t \in (0, 2T). \quad (4.3.36)$$

In particular it yields  $F(t) \in AC[0, 2T]$ ,  $F'(t) \in L_2(0, 2T)$ ,  $F(2T) = 0$ .

We note that the function  $F(t)$  can be constructed from equation (4.3.36) which is usually better than to calculate  $F(t)$  directly by (4.3.32), (4.3.33), (4.3.29).



Acting in the same way as in [mar1] one can obtain that

$$G(x, t) + F(x, t) + \int_x^{2T-t} F(t+s)G(x, s) ds = 0, \quad 0 \leq x \leq T, \quad x < t < 2T - x. \quad (4.3.37)$$

Equation (4.3.37) is called the Gelfand-Levitan-Marchenko equation.

Now let us formulate physical realization conditions for the characteristic function  $\Delta(\rho)$ .

**Theorem 4.3.3.** *For a function  $\Delta(\rho)$  of the form (4.3.12) to be the characteristic function for a certain  $R(x) \in B_0^-$ , it is necessary and sufficient that all zeros of  $\Delta(\rho)$  in  $\overline{\Omega}_+$  are simple, have the form  $\rho_j = i\tau_j, \tau_j > 0, j = \overline{1, m}, m \geq 0$ , and*

$$\alpha_j := \frac{i\delta(-\rho_j)}{\dot{\delta}(\rho_j)} > 0.$$

*$R(x) \in B_0$  if and only if  $\Delta(\rho)$  has no zeros in  $\overline{\Omega}_+$ , i.e.  $m = 0$ . The specification of the characteristic function  $\Delta(\rho)$  uniquely determines  $R(x)$ .*

*Proof.* The necessity part of Theorem 4.3.3 was proved above. We prove the sufficiency. Put  $\delta(\rho) = e^{i\rho T}\Delta(\rho)$ . Then (4.3.25) holds, where  $\theta(t) = \eta(T - t)$ . Since for each fixed  $x \geq 0$ , the homogeneous integral equation

$$w(t) + \int_x^\infty F(t+s)w(s) ds = 0, \quad t > x \quad (4.3.38)$$

has only the trivial solution (see [mar1]), then equation (4.3.37) has a unique solution  $G(x, t)$ . The function  $G(x, t)$  is absolutely continuous,  $G(x, 2T - x) = 0$  and  $\frac{d}{dx}G(x, x) \in L_2(0, T)$ . We construct the function  $e(x, \rho)$  by (4.3.24).

Let us show that  $e(0, \rho) = \delta(\rho)$ . Indeed, it follows from (4.3.24) and (4.3.25) that

$$e(0, \rho) - \delta(\rho) = \int_0^{2T} (G(0, t) - \theta(t))e^{i\rho t} dt. \quad (4.3.39)$$

By virtue of (4.3.36) and (4.3.37),

$$(G(0, t) - \theta(t)) + \int_0^{2T-t} (G(0, s) - \theta(s))F(t+s) ds = 0.$$

In view of (4.3.35), it gives us that the function  $w(t) := G(0, t) - \theta(t)$  satisfies (4.3.38) for  $x = 0$ . Hence  $G(0, t) = \theta(t)$ , and according to (4.3.39),  $e(0, \rho) = \delta(\rho)$ .

Put  $p(x) := -2\frac{dG(x, x)}{dx}$ ,  $x \in [0, T]$ , and  $p(x) \equiv 0, x > T$ . It is easily shown that

$$-e''(x, \rho) + p(x)e(x, \rho) = \rho^2 e(x, \rho), \quad x > 0.$$

We construct  $R(x)$  by the formula

$$R(x) = \frac{1}{u^2(x, 0)}, \quad (4.3.40)$$

where  $u(x, \rho) = e^{-i\rho T}e(T - x, \rho)$ . Since  $\delta(0) \neq 0$ , we have  $u(T, 0) \neq 0$ . It follows from (4.3.40) that  $R(x) \in B_0^-$ , where  $0 < x_1 < \dots < x_p < T$  are zeros of  $u(x, 0)$ . In particular,

if  $\delta(\rho)$  has no zeros in  $\overline{\Omega}_+$ , then, by virtue of Theorem 1,  $R(x) \in B_0$ . The uniqueness of recovering  $R(x)$  from  $\Delta(\rho)$  was proved for example in [yur5].  $\square$

Theorem 4.3.3 gives us the following algorithm for constructing  $R(x)$  from the characteristic function  $\Delta(\rho)$ :

**Algorithm 4.3.1.** Let a function  $\Delta(\rho)$  satisfying the hypothesis of Theorem 4.3.3 be given. Then

- (1) Construct  $F(t)$ ,  $t \in (0, 2T)$  from the integral equation (4.3.36), where  $\theta(t) = \eta(T - t)$ , or directly from (4.3.26), (4.3.29), (4.3.32), (4.3.33).
- (2) Find  $G(x, t)$  from the integral equation (4.3.37).
- (3) Calculate  $R(x)$  by

$$R(x) = \frac{1}{e^{2(T-x)}}, \quad e(x) = 1 + \int_x^{2T-x} G(x, t) dt. \quad (4.3.41)$$

Next we provide two other algorithms for the synthesis of the wave resistance from the characteristic function, which sometimes may give advantage from the numerical point of view.

Let  $S(x, \lambda)$  be the solution of (4.3.23) under the conditions  $S(0, \lambda) = 0$ ,  $S'(0, \lambda) = 1$ . Then

$$S(x, \lambda) = \frac{\sin \rho x}{\rho} + \int_0^x Q(x, t) \frac{\sin \rho t}{\rho} dt, \quad (4.3.42)$$

where  $Q(x, t)$  is a real, absolutely continuous function,  $Q(x, 0) = 0$ ,  $Q(x, x) = \frac{1}{2} \int_0^x p(t) dt$ .

Since  $\langle e(x, \rho), S(x, \lambda) \rangle \equiv \delta(\rho)$  and  $\delta(\rho_j) = 0$  we get

$$e(x, \rho_j) = e'(0, \rho_j) S(x, \lambda_j), \quad \lambda_j = \rho_j^2, \quad (4.3.43)$$

and by virtue of (4.3.27) and (4.3.28),

$$\beta_j := \left( \int_0^\infty S^2(x, \lambda_j) dx \right)^{-1} = -\frac{2\rho_j e'(0, \rho_j)}{\dot{\delta}(\rho_j)} = -\frac{4i\rho_j^2}{\delta(-\rho_j)\dot{\delta}(\rho_j)} > 0. \quad (4.3.44)$$

Taking into account the relations

$$\dot{\delta}(\rho_j) = e^{i\rho_j T} \dot{\Delta}(\rho_j), \quad \delta(-\rho_j) = e^{-i\rho_j T} \Delta(-\rho_j),$$

we obtain from (4.3.44)

$$\beta_j = -\frac{4i\rho_j^2}{\Delta(-\rho_j)\dot{\Delta}(\rho_j)}.$$

Consider the function

$$a(x) = \sum_{j=1}^m \beta_j \frac{\cos \rho_j x}{\rho_j^2} + \frac{2}{\pi} \int_0^\infty \cos \rho x \left( \frac{1}{|\Delta(\rho)|^2} - 1 \right) d\rho. \quad (4.3.45)$$

By virtue of (4.3.12),

$$\rho \left( \frac{1}{|\Delta(\rho)|^2} - 1 \right) \in L_2(0, \infty),$$

and consequently  $a(x)$  is absolutely continuous, and  $a'(x) \in L_2$ . The kernel  $Q(x, t)$  from (4.3.42) satisfies the equation (see [lev2])

$$f(x, t) + Q(x, t) + \int_0^x Q(x, s)f(s, t)ds = 0, \quad x \geq 0, \quad 0 < t < x, \quad (4.3.46)$$

where

$$f(x, t) = \frac{1}{2}(a(x - t) - a(x + t)).$$

The function  $R(x)$  can be constructed by the following algorithm:

**Algorithm 4.3.2.** Let  $\Delta(\rho)$  be given. Then

- (1) Construct  $a(x)$  by (4.3.45).
- (2) Find  $Q(x, t)$  from the integral equation (4.3.46).
- (3) Calculate  $p(x) := 2\frac{d}{dx}Q(x, x)$ .
- (4) Construct

$$R(x) := \frac{1}{e^2(T - x)},$$

where  $e(x)$  is the solution of the Cauchy problem  $y'' = p(x)y, y(T) = 1, y'(T) = 0$ .

**Remark 4.3.2.** We also can construct  $R(x)$  using (4.3.6) and (4.3.8), namely:

$$R(x) = \exp(2 \int_0^x h(t)dt),$$

where  $h(x)$  is the solution of the equation

$$h(x) = \int_0^x h^2(t)dt + 2Q(T - x, T - x) - 2Q(T, T).$$

**Remark 4.3.3.** If  $R(x) \in B_0$ , then

$$a(x) = \frac{2}{\pi} \int_0^\infty \cos \rho x \left( \frac{1}{|\Delta(\rho)|^2} - 1 \right) d\rho,$$

and for constructing the solution of the inverse problem it is sufficient to specify  $|\Delta(k)|$  for  $k \geq 0$ .

Now we provide an algorithm for the solution of the inverse problem which uses discrete spectral characteristics.

Let  $X_k(x, \lambda)$ ,  $k = 1, 2$  be solutions of (4.3.7) under the conditions  $X_1(0, \lambda) = X_2'(0, \lambda) = 1, X_1'(0, \lambda) = X_2(0, \lambda) = 0$ . Denote  $\Delta_k(\lambda) = X_k(T, \lambda)$ . Clearly,

$$u(x, \rho) = X_1(x, \lambda) - i\rho X_2(x, \lambda), \quad \Delta(\rho) = \Delta_1(\lambda) - i\rho \Delta_2(\lambda),$$

and consequently

$$\Delta_1(\lambda) = \frac{\Delta(\rho) + \Delta(-\rho)}{2}, \quad \Delta_2(\lambda) = \frac{\Delta(\rho) - \Delta(-\rho)}{2i\rho}.$$

Let  $\{\mu_n\}_{n \geq 1}$  be the zeros of  $\Delta_2(\lambda)$ , and

$$\gamma_n := \int_0^T X_2^2(x, \mu_n)dx > 0.$$

It is easily shown that

$$\gamma_n = (\Delta_1(\mu_n))^{-1} \left( \frac{d}{d\lambda} \Delta_2(\lambda) \right)_{|\lambda=\mu_n},$$

and

$$\sqrt{\mu_n} = \frac{\pi n}{T} + \frac{1}{2\pi n} \int_0^T q(\xi) d\xi + \frac{\omega_n}{n}, \quad \gamma_n = \frac{T^3}{2n^2\pi^2} + \frac{\omega_{n1}}{n^3}, \quad \{\omega_n\}, \{\omega_{n1}\} \in \ell_2. \quad (4.3.47)$$

We introduce the function

$$A(x) = \frac{2}{T} \sum_{n=1}^{\infty} \left( \frac{T \cos \sqrt{\mu_n} x}{2\gamma_n \mu_n} - \cos \frac{\pi n x}{T} \right). \quad (4.3.48)$$

By virtue of (4.3.47),  $A(x)$  is absolutely continuous, and  $A'(x) \in L_2$ . It is known (see [mar1]) that

$$X_2(x, \lambda) = \frac{\sin \rho x}{\rho} + \int_0^x D(x, t) \frac{\sin \rho t}{\rho} dt,$$

where  $D(x, t)$  is a real, absolutely continuous function, and  $D(x, 0) = 0$ ,  $D(x, x) = \frac{1}{2} \int_0^x q(t) dt$ . The function  $D(x, t)$  satisfies the equation (see [lev2])

$$D(x, t) + B(x, t) + \int_0^x D(x, s) B(s, t) ds = 0, \quad 0 \leq x \leq T, \quad 0 < t < x, \quad (4.3.49)$$

where  $B(x, t) = \frac{1}{2}(A(x-t) - A(x+t))$ . The function  $R(x)$  can be constructed from the discrete data  $\{\mu_n, \gamma_n\}_{n \geq 1}$  by the the following algorithm.

**Algorithm 4.3.3.** Let  $\{\mu_n, \gamma_n\}_{n \geq 1}$  be given. Then

- (1) Construct  $A(x)$  by (4.3.48).
- (2) Find  $D(x, t)$  from the integral equation (4.3.49).
- (3) Calculate  $q(x) := 2 \frac{dD(x, x)}{dx}$ .
- (4) Construct  $R(x) := \frac{1}{u^2(x)}$ , where  $u(x)$  is the solution of the Cauchy problem

$$u'' = q(x)u, u(0) = 1, u'(0) = 0,$$

or by

$$R(x) = \exp(2 \int_0^x h(t) dt), \quad h(x) = \int_0^x h^2(t) dt - 2D(x, x).$$

### 4.3.3. Reconstruction of the transmission coefficients from their moduli.

**Lemma 4.3.4.** Suppose that a function  $\gamma(\rho)$  is regular in  $\overline{\Omega}_+$ , has no zeros in  $\Omega_+$ , and for  $|\rho| \rightarrow \infty$ ,  $\rho \in \overline{\Omega}_+$ ,  $\gamma(\rho) = 1 + O(\frac{1}{\rho})$ . Let  $\gamma(k) = |\gamma(k)|e^{-i\beta(k)}$ ,  $k = \text{Re } \rho$ . Then

$$\beta(k) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln |\gamma(\xi)|}{\xi - k} d\xi. \quad (4.3.50)$$

In (4.3.50) (and everywhere below, where necessary) the integral is understood in the principal value sense.

*Proof.* First we suppose that  $\gamma(k) \neq 0$  for real  $k$ . By Cauchy's theorem, taking into account the hypothesis of the lemma, we obtain

$$\frac{1}{2\pi i} \int_{C_{r,\epsilon}} \frac{\ln \gamma(\xi)}{\xi - k} d\xi = 0, \quad (4.3.51)$$

where  $C_{r,\epsilon}$  is the closed contour (with counterclockwise circuit) consisting of the semicircles  $C_r = \{\xi : \xi = re^{i\varphi}, \varphi \in [0, \pi]\}$ ,  $\Gamma_\epsilon = \{\xi : \xi - k = \epsilon e^{i\varphi}, \varphi \in [0, \pi]\}$  and the intervals  $\xi \in [-r, r] \setminus [k - \epsilon, k + \epsilon]$  of the real axis. Since

$$\lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{\Gamma_\epsilon} \frac{\ln \gamma(\xi)}{\xi - k} d\xi = -\frac{1}{2} \ln \gamma(k),$$

$$\lim_{r \rightarrow \infty} \frac{1}{2\pi i} \int_{C_r} \frac{\ln \gamma(\xi)}{\xi - k} d\xi = 0,$$

we get from (4.3.51) that

$$\ln \gamma(k) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\ln \gamma(\xi)}{\xi - k} d\xi.$$

Separating here real and imaginary parts, we arrive at (4.3.50).

Suppose now that for real  $\rho$ , the function  $\gamma(\rho)$  has one zero  $\rho_0 = 0$  of multiplicity  $s$  (the general case is treated in the same way). Denote

$$\tilde{\gamma}(\rho) = \gamma(\rho) \left( \frac{\rho + i\epsilon}{\rho} \right)^s, \quad \epsilon > 0; \quad \tilde{\gamma}(k) = |\tilde{\gamma}(k)| e^{-i\tilde{\beta}(k)}.$$

Then

$$\beta(k) = \tilde{\beta}(k) + s \operatorname{arctg} \frac{\epsilon}{k}. \quad (4.3.52)$$

For the function  $\tilde{\gamma}(\rho)$ , (4.3.50) has been proved. Hence, (4.3.52) takes the form

$$\beta(k) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln |\gamma(\xi)|}{\xi - k} d\xi + \frac{s}{2\pi} \int_{-\infty}^{\infty} \frac{\ln(1 + \frac{\epsilon^2}{\xi^2})}{\xi - k} d\xi + s \operatorname{arctg} \frac{\epsilon}{k}.$$

When  $\epsilon \rightarrow 0$ , it gives us (4.3.50). □

**Lemma 4.3.5.** Suppose that a function  $\gamma(\rho)$  is regular in  $\overline{\Omega}_+$ ,  $\gamma(-\bar{\rho}) = \overline{\gamma(\rho)}$ , and for  $|\rho| \rightarrow \infty, \rho \in \overline{\Omega}_+$ ,  $\gamma(\rho) = 1 + O(\frac{1}{\rho})$ . Let  $\gamma(k) = |\gamma(k)| e^{-i\beta(k)}$ , and let  $\rho_j = k_j + i\tau_j$ ,  $\tau_j > 0$ ,  $j = \overline{1, s}$  be zeros of  $\gamma(\rho)$  in  $\Omega_+$ . Then

$$\beta(k) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln |\gamma(\xi)|}{\xi - k} d\xi + 2 \sum_{j=1}^s \operatorname{arctg} \frac{\tau_j}{k - k_j}. \quad (4.3.53)$$

*Proof.* Since  $\gamma(-\bar{\rho}) = \overline{\gamma(\rho)}$ , the zeros of  $\gamma(\rho)$  in  $\Omega_+$  are symmetrical with respect to the imaginary axis. If  $\rho_j = i\tau_j, \tau_j > 0$ , then

$$\arg \frac{k + \rho_j}{k - \rho_j} = 2 \operatorname{arctg} \frac{\tau_j}{k}. \quad (4.3.54)$$

If  $\rho_j = k_j + i\tau_j$ ,  $k_j > 0, \tau_j > 0$ , then

$$\arg \frac{(k + \rho_j)(k - \bar{\rho}_j)}{(k - \rho_j)(k + \bar{\rho}_j)} = 2 \operatorname{arctg} \frac{\tau_j}{k - k_j} + 2 \operatorname{arctg} \frac{\tau_j}{k + k_j}. \quad (4.3.55)$$

Denote

$$\tilde{\gamma}(\rho) = \gamma(\rho) \prod_{j=1}^s \frac{\rho + \rho_j}{\rho - \rho_j}.$$

The function  $\tilde{\gamma}(\rho)$  satisfies the hypothesis of Lemma 4.3.4. Then using (4.3.50), (4.3.54) and (4.3.55) we arrive at (4.3.53).  $\square$

Using Lemma 5 one can construct the transmission coefficients from their moduli and information about their zeros in  $\Omega_+$ . For definiteness we confine ourselves to the case  $h \neq 0$ .

**Theorem 4.3.4.** *Let*

$$f_1(k) = \alpha_1(k) e^{-i\delta_1(k)}, \quad (4.3.56)$$

and let  $\rho_j^* = i\tau_j^*$ ,  $\tau_j^* > 0, j = \overline{1, m^*}$  be zeros of  $f_1(\rho)$  in  $\Omega_+$ . Then

$$\delta_1(k) = kT + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln \alpha_1(\xi)}{\xi - k} d\xi + 2 \sum_{j=1}^{m^*} \operatorname{arctg} \frac{\tau_j^*}{k}. \quad (4.3.57)$$

In particular, if  $R(x) \in B_0$ , then

$$\delta_1(k) = kT + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln \alpha_1(\xi)}{\xi - k} d\xi. \quad (4.3.58)$$

*Proof.* Denote  $\gamma(\rho) = e^{i\rho T} f_1(\rho)$ . It follows from (4.3.15) that

$$\gamma(\rho) = 1 + \int_{-T}^T g_1(t) e^{i\rho(T-t)} dt,$$

and consequently,  $\gamma(-\bar{\rho}) = \overline{\gamma(\rho)}$ , and for  $|\rho| \rightarrow \infty$ ,  $\rho \in \overline{\Omega}_+$ ,  $\gamma(\rho) = 1 + O\left(\frac{1}{\rho}\right)$ . Thus, the function  $\gamma(\rho)$  satisfies the hypothesis of Lemma 4.3.5. Using (4.3.53) and the relations  $|\gamma(k)| = \alpha_1(k)$ ,  $\delta_1(k) = \beta(k) + kT$ , we arrive at (4.3.57).  $\square$

Similarly we prove the following theorem.

**Theorem 4.3.5.** *Let*

$$f_2(k) = \alpha_2(k) e^{-i\delta_2(k)}, \quad (4.3.59)$$

and let  $\rho_j^0 = k_j^0 + i\tau_j^0$ ,  $\tau_j^0 > 0, j = \overline{1, m^0}$  be zeros of  $f_2(\rho)$  in  $\Omega_+$ . Then

$$\delta_2(k) = \frac{\pi}{2} \operatorname{sign} \left( \frac{k}{h} \right) + kT + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln \left| \frac{2\xi}{h} \alpha_2(\xi) \right|}{\xi - k} d\xi + 2 \sum_{j=1}^{m_0} \operatorname{arctg} \frac{\tau_j^0}{k - k_j^0}. \quad (4.3.60)$$

In particular, if  $f_2(\rho)$  has no zeros in  $\Omega_+$ , then

$$\delta_2(k) = \frac{\pi}{2} \operatorname{sign} \left( \frac{k}{h} \right) + kT + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln \left| \frac{2\xi}{h} \alpha_2(\xi) \right|}{\xi - k} d\xi. \quad (4.3.61)$$

Thus, the specification of  $\alpha_j(k)$  uniquely determines the transmission coefficients only when they have no zeros in  $\Omega_+$ . In particular, for the class  $B_0$ ,  $f_1(\rho)$  is uniquely determined by its modulus, and all possible transmission coefficients  $f_2(\rho)$  can be constructed by means of solving the integral equation (4.3.21).

#### 4.3.4. Symmetrical case.

The wave resistance is called symmetrical if  $R(T-x) = R(x)$ . For the symmetrical case the transmission coefficient  $f_2(k)$  is uniquely determined (up to the sign) from its modulus.

**Theorem 4.3.6.** *For the wave resistance to be symmetrical it is necessary and sufficient that  $\operatorname{Re} f_2(k) = 0$ . Moreover,  $f_2(k) = -f_2(-k)$ ,  $g_2(t) = -g_2(-t)$ , and (see (4.3.16))*

$$f_2(k) = -2i \int_0^T g_2(t) \sin kt \, dt.$$

*Proof.* According to (4.3.9) and (4.3.5),

$$f_2(\rho) = \frac{1}{2i\rho} (u'(T, \rho) + i\rho u(T, \rho) + hu(T, \rho)).$$

Since  $u(x, \rho) = X_1(x, \lambda) - i\rho X_2(x, \lambda)$ , we calculate

$$\operatorname{Re} f_2(k) = \frac{1}{2} (X_1(T, \lambda) - X_2'(T, \lambda) - hX_2(T, \lambda)). \quad (4.3.62)$$

If  $R(x) = R(T-x)$ , then it follows from (4.3.6) and (4.3.8) that  $h(x) = -h(T-x)$ ,  $q(x) = q(T-x)$ , and consequently  $h = 0$ ,  $X_1(T, \lambda) \equiv X_2'(T, \lambda)$  (see [yur2]), i.e.  $\operatorname{Re} f_2(k) = 0$ .

Conversely, if  $\operatorname{Re} f_2(k) = 0$ , then (4.3.62) gives us

$$X_1(T, \lambda) - X_2'(T, \lambda) - hX_2(T, \lambda) \equiv 0. \quad (4.3.63)$$

Since for  $k \rightarrow \infty$ ,

$$X_2(T, k^2) = \frac{\sin kT}{k} + O\left(\frac{1}{k^2}\right), \quad X_1(T, \lambda) - X_2'(T, \lambda) = O\left(\frac{1}{k^2}\right),$$

we obtain from (4.3.63) that  $h = 0$ ,  $X_1(T, \lambda) \equiv X_2'(T, \lambda)$ . Consequently,  $q(x) = q(T-x)$  (see [yur2]). Similarly one can prove that  $g(x) = g(T-x)$ . Hence  $h(x) = -h(T-x)$ , and  $R(x) = R(T-x)$ .

Furthermore, it follows from (4.3.16) that

$$\operatorname{Re} f_2(k) = \int_{-T}^T g_2(t) \cos kt \, dt, \quad \operatorname{Im} f_2(k) = - \int_{-T}^T g_2(t) \sin kt \, dt.$$

For the symmetrical case  $\operatorname{Re} f_2(k) = 0$ , and consequently  $g_2(t) = -g_2(-t)$ . Then

$$f_2(k) = i \operatorname{Im} f_2(k) = -2i \int_0^T g_2(t) \sin kt \, dt, \quad f_2(-k) = -f_2(k).$$

□

Thus,  $f_2(k)$  can be constructed (up to the sign) by the formulas

$$\operatorname{Re} f_2(k) = 0, \quad |\operatorname{Im} f_2(k)| = \alpha_2(k), \quad f_2(k) = -f_2(-k).$$

#### 4.3.5. Synthesis of the wave resistance from the power reflection coefficient.

In this subsection, using results obtained above, we provide a procedure for constructing  $R(x)$  from the given power reflection coefficient  $\sigma(k)$ . For definiteness we confine ourselves to the case  $h \neq 0$ . Let  $\sigma(k)$  ( $0 \leq \sigma(k) < 1$ ,  $\sigma(k) = \sigma(-k)$ ) be given. Our *scheme of calculation* is:

*Step 1.* Calculate  $\alpha_1(k)$  and  $\alpha_2(k)$  by the formula

$$\alpha_1^2(k) = 1 + \alpha_2^2(k) = \frac{1}{1 - \sigma(k)}. \quad (4.3.64)$$

*Step 2.* Construct  $f_1(k)$  by (4.3.56), (4.3.57) or for  $R(x) \in B_0$  by (4.3.56), (4.3.58). Find  $g_1(t)$  from the relation

$$\int_{-T}^T g_1(t) e^{-ikt} dt = f_1(k) - e^{ikT}.$$

*Step 3.* Construct  $f_2(k)$  by (4.3.59), (4.3.60) or, if  $f_2(\rho)$  has no zeros in  $\Omega_+$ , by (4.3.59), (4.3.61). Find  $g_2(t)$  from the relation

$$\int_{-T}^T g_2(t) e^{-ikt} dt = f_2(k).$$

We note that  $g_2(t)$  can be constructed also from the integral equation (4.3.21). In this case  $f_2(k)$  is calculated by (4.3.16).

*Step 4.* Calculate  $\eta(t) = g_1(t) + g_2(t)$  and  $\Delta(\rho)$  by (4.3.12).

*Step 5.* Construct  $R(x)$  using one of the Algorithms 4.3.1 - 4.3.3.

**Remark 4.3.4.** In some concrete algorithms it is not necessary to make all calculations mentioned above. For example, let  $R(x) \in B_0$ , and let us use Algorithm 4.3.2. Then it is not necessary to calculate  $g_1(t), g_2(t)$  and  $\eta(t)$ , since we need only  $|\Delta(k)|$ .

Now we consider a concrete algorithm which realizes this scheme. For simplicity, we consider the case  $R(x) \in B_0$ .

For  $t \in [0, T]$  we consider the functions

$$\varphi_j(t) = g_j(t) + g_j(-t), \quad \psi_j(t) = g_j(t) - g_j(-t), \quad j = 1, 2, \quad (4.3.65)$$

$$\varphi(t) = \eta(t) + \eta(-t), \quad \psi(t) = \eta(t) - \eta(-t). \quad (4.3.66)$$

Since  $\eta(t) = g_1(t) + g_2(t)$ , we get

$$\psi(t) = \varphi_1(t) + \varphi_2(t), \quad \psi(t) = \psi_1(t) + \psi_2(t). \quad (4.3.67)$$

Solving (4.3.65) and (4.3.66) with respect to  $g_j(t)$  and  $\eta(t)$  we obtain

$$g_j(t) = \begin{cases} \frac{1}{2}(\varphi_j(t) + \psi_j(t)) & , \quad t > 0, \\ \frac{1}{2}(\varphi_j(-t) - \psi_j(-t)) & , \quad t < 0, \end{cases} \quad \eta(t) = \begin{cases} \frac{1}{2}(\varphi(t) + \psi(t)) & , \quad t > 0 \\ \frac{1}{2}(\varphi(-t) + \psi(-t)) & , \quad t < 0 \end{cases} \quad (4.3.68)$$



It follows from (4.3.17) and (4.3.65) that

$$\varphi_1(T) = \psi_1(T) = -w_1, \quad \varphi_2(T) = \psi_2(T) = -\frac{h}{2}, \quad \psi_1(0) = \psi_2(0) = 0, \quad (4.3.69)$$

where

$$w_1 = -\frac{1}{2} \int_0^T h^2(\xi) d\xi.$$

By virtue of Lemma 4.3.3,

$$\left. \begin{aligned} f_1(k) &= (\cos kT + C_1(k)) - i(\sin kT + S_1(k)), \\ f_2(k) &= C_2(k) - iS_2(k), \\ \Delta(k) &= ((\cos kT + C(k)) - i(\sin kT + S(k))), \end{aligned} \right\} \quad (4.3.70)$$

where

$$C_j(k) = \int_0^T \varphi_j(t) \cos kt dt, \quad S_j(k) = \int_0^T \psi_j(t) \sin kt dt, \quad (4.3.71)$$

$$C(k) = \int_0^T \varphi(t) \cos kt dt, \quad S(k) = \int_0^T \psi(t) \sin kt dt. \quad (4.3.72)$$

Using (4.3.69) and (4.3.71) we obtain the following asymptotic formulae for  $C_j(k)$  and  $S_j(k)$  as  $k \rightarrow \infty$ :

$$\left. \begin{aligned} C_1(k) &= -w_1 \frac{\sin kT}{k} + \frac{\omega(k)}{k}, \quad C_2(k) = -\frac{h}{2} \frac{\sin kT}{k} + \frac{\omega(k)}{k}, \\ S_1(k) &= w_1 \frac{\cos kT}{k} + \frac{\omega(k)}{k}, \quad S_2(k) = \frac{h}{2} \frac{\cos kT}{k} + \frac{\omega(k)}{k}. \end{aligned} \right\} \quad (4.3.73)$$

Here and below, one and the same symbol  $\omega(k)$  denotes various functions from  $L_2(-\infty, \infty)$ .

Comparing (4.3.70) with the relations  $f_j(k) = \alpha_j(k) e^{-i\delta_j(k)}$ , we derive

$$C_1(k) = \alpha_1(k) \cos \delta_1(k) - \cos kT, \quad S_1(k) = \alpha_1(k) \sin \delta_1(k) - \sin kT, \quad (4.3.74)$$

$$C_2(k) = \alpha_2(k) \cos \delta_2(k), \quad S_2(k) = \alpha_2(k) \sin \delta_2(k). \quad (4.3.75)$$

For calculating the arguments of the transmission coefficients we will use (4.3.58) and (4.3.61), i.e.

$$\delta_1(k) = kT + \tilde{\delta}_1(k), \quad \delta_2(k) = \frac{\pi}{2}w + kT + \tilde{\delta}_2(k), \quad k > 0, \quad (4.3.76)$$

where  $w = \text{sign } h$ ,

$$\tilde{\delta}_1(k) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln \alpha_1(\xi)}{\xi - k} d\xi, \quad (4.3.77)$$

$$\tilde{\delta}_2(k) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln \tilde{\alpha}_2(\xi)}{\xi - k} d\xi, \quad \tilde{\alpha}_2(\xi) := \left| \frac{2\xi}{h} \alpha_2(\xi) \right|. \quad (4.3.78)$$

It follows from (4.3.73)-(4.3.75) and (4.3.20) that for  $k \rightarrow +\infty$ ,

$$\delta_1(k) = kT + \frac{w_1}{k} + \frac{\omega(k)}{k}, \quad \delta_2(k) = \frac{\pi}{2}w + kT + \omega(k). \quad (4.3.79)$$

Substituting (4.3.76) into (4.3.74) and (4.3.75), we calculate for  $k > 0$ ,

$$\left. \begin{aligned} C_1(k) &= \cos kT(\alpha_1(k) \cos \tilde{\delta}_1(k) - 1) - \sin kT(\alpha_1(k) \sin \tilde{\delta}_1(k)), \\ S_1(k) &= \sin kT(\alpha_1(k) \cos \tilde{\delta}_1(k) - 1) + \cos kT(\alpha_1(k) \sin \tilde{\delta}_1(k)), \end{aligned} \right\} \quad (4.3.80)$$

$$\left. \begin{aligned} C_2(k) &= -\alpha_2(k)w(\sin kT \cos \tilde{\delta}_2(k) + \cos kT \sin \tilde{\delta}_2(k)), \\ S_2(k) &= \alpha_2(k)w(\cos kT \cos \tilde{\delta}_2(k) - \sin kT \sin \tilde{\delta}_2(k)). \end{aligned} \right\} \quad (4.3.81)$$

Furthermore, consider the functions

$$\left. \begin{aligned} \varphi_1^*(t) &= \varphi_1(t) + w_1, \quad \psi_1^*(t) = \psi_1(t) + w_1 \frac{t}{T}, \\ \varphi_2^*(t) &= \varphi_2(t) + \frac{h}{2}, \quad \psi_2^*(t) = \psi_2(t) + \frac{h}{2} \cdot \frac{t}{T}. \end{aligned} \right\} \quad (4.3.82)$$

Then in view of (4.3.69),

$$\varphi_j^*(T) = \psi_j^*(T) = \psi_j^*(0) = 0, \quad j = 1, 2. \quad (4.3.83)$$

Denote

$$C_j^*(k) = \int_0^T \varphi_j^*(t) \cos kt \, dt, \quad S_j^*(k) = \int_0^T \psi_j^*(t) \sin kt \, dt. \quad (4.3.84)$$

Integrating the integrals in (4.3.84) by parts and taking (4.3.83) into account, we obtain

$$C_j^*(k) = - \int_0^T \varphi_j^{*'}(t) \frac{\sin kt}{k} \, dt, \quad S_j^*(k) = \int_0^T \psi_j^{*'}(t) \frac{\cos kt}{k} \, dt. \quad (4.3.85)$$

Clearly,

$$C_1(k) = C_1^*(k) - w_1 \frac{\sin kT}{k}, \quad S_1(k) = S_1^*(k) + w_1 \left( \frac{\cos kT}{k} - \frac{\sin kT}{Tk^2} \right), \quad (4.3.86)$$

$$C_2(k) = C_2^*(k) - \frac{h}{2} \frac{\sin kT}{k}, \quad S_2(k) = S_2^*(k) + \frac{h}{2} \left( \frac{\cos kT}{k} - \frac{\sin kT}{Tk^2} \right). \quad (4.3.87)$$

Now let the power reflection coefficient  $\sigma(k)$  ( $0 \leq \sigma(k) < 1$ ,  $\sigma(k) = \sigma(-k)$ ) be given for  $|k| \leq B$ , and put

$$\sigma(k) = \frac{h^2}{4k^2 + h^2}, \quad |k| > B, \quad (4.3.88)$$

where  $B > |h|$  is chosen sufficiently large, such that (see (4.3.20))  $\sigma(k)$  is sufficiently accurate for  $|k| > B$ .

Using (4.3.64) we calculate  $\alpha_1(k)$  and  $\alpha_2(k)$ . Then

$$\alpha_1^2(k) - \alpha_2^2(k) = 1, \quad \alpha_j(k) = \alpha_j(-k) > 0,$$

and

$$\alpha_1^2(k) = 1 + \frac{h^2}{4k^2}, \quad \alpha_2^2(k) = \frac{h^2}{4k^2}, \quad |k| > B. \quad (4.3.89)$$

In order to calculate  $\tilde{\delta}_1(k)$  we use (4.3.77). First let  $|k| > B + \chi$ ,  $\chi > 0$ . In view of (4.3.89), equality (4.3.77) takes the form

$$\tilde{\delta}_1(k) = \frac{1}{\pi} \int_{-B}^B \frac{\ln \alpha_1(\xi)}{\xi - k} d\xi + \frac{1}{2\pi} \int_{|\xi| > B} \frac{\ln(1 + \frac{h^2}{4\xi^2})}{\xi - k} d\xi.$$

Since

$$\begin{aligned} \frac{1}{2\pi} \int_{|\xi|>B} \frac{\ln\left(1 + \frac{h^2}{4\xi^2}\right)}{\xi - k} d\xi &= \frac{1}{2\pi} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \left(\frac{h^2}{4}\right)^j \int_{|\xi|>B} \frac{d\xi}{\xi^{2j}(\xi - k)} \\ &= \frac{1}{2\pi} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \left(\frac{h^2}{4}\right)^j \int_{|\xi|>B} \left\{ \frac{1}{k^{2j}} \left(\frac{1}{\xi - k} - \frac{1}{\xi}\right) - \sum_{s=1}^{2j-1} \frac{1}{k^{2j-s}\xi^{s+1}} \right\} d\xi, \end{aligned}$$

we get

$$\begin{aligned} \tilde{\delta}_1(k) &= \frac{1}{\pi} \int_{-B}^B \frac{\ln \alpha_1(\xi)}{\xi - k} d\xi + \frac{1}{2\pi} \ln\left(1 + \frac{2B}{k - B}\right) \ln\left(1 + \frac{h^2}{4k^2}\right) \\ &+ \frac{1}{\pi} \sum_{j=1}^{\infty} \frac{(-1)^j}{j} \left(\frac{h^2}{4}\right)^j \sum_{\mu=0}^{j-1} \frac{1}{(2j - 2\mu - 1)B^{2j-2\mu-1}k^{2\mu+1}}. \end{aligned} \quad (4.3.90)$$

Using the relation

$$\frac{1}{\pi} \int_{-B}^B \frac{\ln \alpha_1(\xi)}{\xi - k} d\xi = -\frac{1}{k\pi} \int_{-B}^B \ln \alpha_1(\xi) d\xi + \frac{1}{\pi} \int_{-B}^B \frac{\xi \ln \alpha_1(\xi)}{k(\xi - k)} d\xi,$$

we separate in (4.3.90) the terms of order  $1/k$ , i.e.

$$\tilde{\delta}_1(k) = \frac{w_1}{k} + \tilde{\delta}_1^*(k), \quad (4.3.91)$$

where

$$\begin{aligned} \tilde{\delta}_1^*(k) &= \frac{1}{\pi} \int_{-B}^B \frac{\xi \ln \alpha_1(\xi)}{k(\xi - k)} d\xi + \frac{1}{2\pi} \ln\left(1 + \frac{2B}{k - B}\right) \ln\left(1 + \frac{h^2}{4k^2}\right) \\ &+ \frac{1}{\pi} \sum_{j=1}^{\infty} \frac{(-1)^j}{j} \left(\frac{h^2}{4}\right)^j \sum_{\mu=1}^{j-1} \frac{1}{(2j - 2\mu - 1)B^{2j-2\mu-1}k^{2\mu+1}}, \quad |k| > B + \chi, \end{aligned} \quad (4.3.92)$$

$$w_1 = -\frac{1}{\pi} \int_{-B}^B \ln \alpha_1(\xi) d\xi + \frac{1}{\pi} \sum_{j=1}^{\infty} \frac{(-1)^j}{j} \left(\frac{h^2}{4B^2}\right)^j \cdot \frac{B}{2j - 1} < 0. \quad (4.3.93)$$

It is easy to show that

$$|\tilde{\delta}_1^*(k)| \leq \frac{\Omega_1^*}{k^2},$$

where

$$\Omega_1^* = \frac{B + \chi}{\pi\chi} \int_0^B \xi \ln \alpha_1(\xi) d\xi + \frac{h^2}{8\pi} \ln\left(1 + \frac{2B}{\chi}\right) + \frac{h^2}{2\pi}.$$

Now let  $|k| \leq B + \chi$ . In this case we rewrite (4.3.77) in the form

$$\tilde{\delta}_1(k) = \frac{1}{\pi} \int_0^\infty \frac{\ln \alpha_1(\xi + k) - \ln \alpha_1(\xi - k)}{\xi} d\xi. \quad (4.3.94)$$

Take  $r \geq 2B + \chi$ . Since

$$\left| \ln\left(1 + \frac{h^2}{4\xi^2}\right) - \frac{h^2}{4\xi^2} \right| \leq \frac{h^4}{32\xi^4},$$

we get

$$\frac{1}{\pi} \int_r^\infty \frac{\ln \alpha_1(\xi + k) - \ln \alpha_1(\xi - k)}{\xi} d\xi = \frac{h^2}{8\pi} \int_r^\infty \frac{1}{\xi} \left( \frac{1}{(\xi + k)^2} - \frac{1}{(\xi - k)^2} \right) d\xi + \epsilon(k),$$

where

$$|\epsilon(k)| \leq \frac{h^4}{96\pi r(r-B-\chi)^3}.$$

Calculating this integral and substituting into (4.3.94), we obtain

$$\begin{aligned} \tilde{\delta}_1(k) = \frac{1}{\pi} \int_0^r \frac{\ln \alpha_1(\xi+k) - \ln \alpha_1(\xi-k)}{\xi} d\xi + \frac{h^2}{8\pi} \left( \frac{1}{k^2} \ln \frac{r+k}{r-k} - \frac{1}{k} \left( \frac{1}{R+k} + \frac{1}{R-k} \right) \right) \\ + \epsilon(k), \quad \tilde{\delta}_1(0) = 0, \quad |k| \leq B + \chi, \quad r \geq 2B + \chi. \end{aligned} \quad (4.3.95)$$

In order to calculate  $\delta_2(k)$  we use (4.3.78). Let  $|k| > B + \chi$ ,  $\chi > 0$ . According to (4.3.89) we have  $\tilde{\alpha}_2(k) = 1$  for  $|k| > B$ . Hence

$$\tilde{\delta}_2(k) = \frac{1}{\pi} \int_{-B}^B \frac{\ln \tilde{\alpha}_2(\xi)}{\xi - k} d\xi, \quad |\xi| > B + \chi. \quad (4.3.96)$$

For  $|k| \leq B + \chi$  it is more convenient to use another formula:

$$\tilde{\delta}_2(k) = \frac{1}{\pi} \int_0^r \frac{\ln \tilde{\alpha}_2(\xi+k) - \ln \tilde{\alpha}_2(\xi-k)}{\xi} d\xi, \quad |k| \leq B + \chi, \quad r = 2B + \chi. \quad (4.3.97)$$

Let us calculate  $\psi_j^*(t), \varphi_j^*(t)$ . By virtue of (4.3.84),

$$\begin{aligned} \psi_j^*(t) &= \frac{2}{T} \sum_{n=1}^{\infty} S_j^* \left( \frac{n\pi}{T} \right) \sin \frac{n\pi}{T} t, \\ \varphi_j^*(t) &= \frac{1}{T} C_j^*(0) + \frac{2}{T} \sum_{n=1}^{\infty} C_j^* \left( \frac{n\pi}{T} \right) \cos \frac{n\pi}{T} t. \end{aligned} \quad (4.3.98)$$

According to (4.3.85), the series in (4.3.98) converge absolutely and uniformly. By virtue of (4.3.80), (4.3.81), (4.3.86), (4.3.87), the coefficients  $S_j^* \left( \frac{n\pi}{T} \right)$  and  $C_j^* \left( \frac{n\pi}{T} \right)$  can be calculated via the formulae

$$\left. \begin{aligned} S_1^*(k) &= (-1)^n (\alpha_1(k) \sin \tilde{\delta}_1(k) - \frac{w_1}{k}), & k = \frac{n\pi}{T}, \quad n \geq 1, \\ S_2^*(k) &= (-1)^n (\alpha_2(k) w \cos \tilde{\delta}_2(k) - \frac{h}{2k}), & k = \frac{n\pi}{T}, \quad n \geq 1, \\ C_1^*(k) &= (-1)^n (\alpha_1(k) \cos \tilde{\delta}_1(k) - 1) + \delta_{n0} w_1 T, & k = \frac{n\pi}{T}, \quad n \geq 0, \\ C_2^*(k) &= (-1)^{n+1} w \sin \tilde{\delta}_2(k) + \delta_{n0} \frac{hT}{2}, & k = \frac{n\pi}{T}, \quad n \geq 0, \end{aligned} \right\} \quad (4.3.99)$$

where  $\delta_{nj}$  is the Kronecker delta.

Using the formulae obtained above we arrive at the following numerical algorithm for the solution of Inverse Problem 4.3.1.

**Algorithm 4.3.4.** Let the power reflection coefficient  $\sigma(k)$  be given for  $|k| \leq B$ , and take  $\sigma(k)$  according to (88) for  $|k| > B$ . Then

- (1) Construct  $\alpha_1(k)$  and  $\alpha_2(k)$  by (4.3.64).
- (2) Find the number  $w_1$  by (4.3.93).
- (3) Calculate  $\tilde{\delta}_1(k), \tilde{\delta}_2(k)$  by (4.3.91), (4.3.92), (4.3.96) for  $|k| > B + \chi$ , and by (4.3.95), (4.3.97) for  $|k| \leq B + \chi$ .

- (4) Construct  $\varphi_j^*(t), \psi_j^*(t)$  by (4.3.98), where  $S_j^*\left(\frac{n\pi}{T}\right), C_j^*\left(\frac{n\pi}{T}\right)$  is defined by (4.3.99).
- (5) Find  $\varphi_j(t), \psi_j(t)$  using (4.3.82).
- (6) Calculate  $\eta(t), t \in [-T, T]$  by (4.3.67), (4.3.68).
- (7) Find  $F(t), 0 < t < 2T$  from the integral equation (4.3.36), where  $\theta(t) = \eta(T - t)$ .
- (8) Calculate  $G(x, t)$  from the integral equation (4.3.37).
- (9) Construct  $R(x), x \in [0, T]$  via (4.3.41).

**Remark 4.3.5.** This algorithm is one of several possible numerical algorithms for the solution of Inverse Problem 4.3.1. Using the obtained results one can construct various algorithms for synthesizing  $R(x)$  from spectral characteristics.

#### 4.4. DISCONTINUOUS INVERSE PROBLEMS

This section deals with Sturm-Liouville boundary value problems in which the eigenfunctions have a discontinuity in an interior point. More precisely, we consider the boundary value problem  $L$  for the equation:

$$\ell y := -y'' + q(x)y = \lambda y \quad (4.4.1)$$

on the interval  $0 < x < T$  with the boundary conditions

$$U(y) := y'(0) - hy(0) = 0, \quad V(y) := y'(T) + Hy(T) = 0, \quad (4.4.2)$$

and with the jump conditions

$$y(a+0) = a_1 y(a-0), \quad y'(a+0) = a_1^{-1} y'(a-0) + a_2 y(a-0). \quad (4.4.3)$$

Here  $\lambda$  is the spectral parameter;  $q(x), h, H, a_1$  and  $a_2$  are real;  $a \in (0, T)$ ,  $a_1 > 0$ ,  $q(x) \in L_2(0, T)$ . Attention will be focused on the inverse problem of recovering  $L$  from its spectral characteristics.

Boundary value problems with discontinuities inside the interval often appear in mathematics, mechanics, physics, geophysics and other branches of natural sciences. As a rule, such problems are connected with discontinuous material properties. The inverse problem of reconstructing the material properties of a medium from data collected outside of the medium is of central importance in disciplines ranging from engineering to the geosciences. For example, discontinuous inverse problems appear in electronics for constructing parameters of heterogeneous electronic lines with desirable technical characteristics ([lit1], [mes1]). After reducing the corresponding mathematical model we come to the boundary value problem  $L$  where  $q(x)$  must be constructed from the given spectral information which describes desirable amplitude and phase characteristics. Spectral information can be used to reconstruct the permittivity and conductivity profiles of a one-dimensional discontinuous medium ([kru1], [she1]). Boundary value problems with discontinuities in an interior point also appear in geophysical models for oscillations of the Earth ([And1], [lap1]). Here the main discontinuity is caused by reflection of the shear waves at the base of the crust. Further, it is known that inverse spectral problems play an important role for investigating some

nonlinear evolution equations of mathematical physics. Discontinuous inverse problems help to study the blow-up behavior of solutions for such nonlinear equations. We also note that inverse problems considered here appear in mathematics for investigating spectral properties of some classes of differential, integrodifferential and integral operators.

Direct and inverse spectral problems for differential operators without discontinuities have been studied in Chapter 1. The presence of discontinuities produces essential qualitative modifications in the investigation of the operators. Some aspects of direct and inverse problems for discontinuous boundary value problems in various formulations have been considered in [she1], [hal1], [akt2], [ebe4] and other works. In particular, it was shown in [hal1] that if  $q(x)$  is known a priori on  $[0, T/2]$ , then  $q(x)$  is uniquely determined on  $[T/2, T]$  by the eigenvalues. In [she1] the discontinuous inverse problem is considered on the half-line.

Here in Subsection 4.4.1 we study properties of the spectral characteristics of the boundary value problem  $L$ . Subsections 4.4.2-4.4.3 are devoted to the inverse problem of recovering  $L$  from its spectral characteristics. In Subsection 4.4.2 the uniqueness theorems are proved, and in Subsection 4.4.3 we provide necessary and sufficient conditions for the solvability of the inverse problem and also obtain a procedure for the solution of the inverse problem.

In order to study the inverse problem for the boundary value problem  $L$  we use the method of spectral mappings described in Section 1.6. We will omit the proofs (or provide short versions of the proofs) for assertions which are similar to those given in Section 1.6.

**4.4.1. Properties of the spectral characteristics.** Let  $y(x)$  and  $z(x)$  be continuously differentiable functions on  $[0, a]$  and  $[a, T]$ . Denote  $\langle y, z \rangle := yz' - y'z$ . If  $y(x)$  and  $z(x)$  satisfy the jump conditions (4.4.3), then

$$\langle y, z \rangle|_{x=a+0} = \langle y, z \rangle|_{x=a-0}, \quad (4.4.4)$$

i.e. the function  $\langle y, z \rangle$  is continuous on  $[0, T]$ . If  $y(x, \lambda)$  and  $z(x, \mu)$  are solutions of the equations  $\ell y = \lambda y$  and  $\ell z = \mu z$  respectively, then

$$\frac{d}{dx} \langle y, z \rangle = (\lambda - \mu)yz. \quad (4.4.5)$$

Let  $\varphi(x, \lambda), \psi(x, \lambda), C(x, \lambda), S(x, \lambda)$  be solutions of (4.4.1) under the initial conditions  $C(0, \lambda) = \varphi(0, \lambda) = S'(0, \lambda) = \psi(T, \lambda) = 1, C'(0, \lambda) = S(0, \lambda) = 0, \varphi'(0, \lambda) = h, \psi'(T, \lambda) = -H$ , and under the jump conditions (4.4.3). Then  $U(\varphi) = V(\psi) = 0$ . Denote

$$\Delta(\lambda) = \langle \varphi(x, \lambda), \psi(x, \lambda) \rangle.$$

By virtue of (4.4.4) and Liouville's formula for the Wronskian [cod1, p.83],  $\Delta(\lambda)$  does not depend on  $x$ . The function  $\Delta(\lambda)$  is called the characteristic function of  $L$ . Clearly,

$$\Delta(\lambda) = -V(\varphi) = U(\psi). \quad (4.4.6)$$

**Theorem 4.4.1.** 1) The eigenvalues  $\{\lambda_n\}_{n \geq 0}$  of the boundary value problem  $L$  coincide with zeros of the characteristic function  $\Delta(\lambda)$ . The functions  $\varphi(x, \lambda_n)$  and  $\psi(x, \lambda_n)$  are eigenfunctions, and

$$\psi(x, \lambda_n) = \beta_n \varphi(x, \lambda_n), \quad \beta_n \neq 0. \quad (4.4.7)$$

2) Denote

$$\alpha_n = \int_0^T \varphi^2(x, \lambda_n) dx. \quad (4.4.8)$$

Then

$$\beta_n \alpha_n = \Delta_1(\lambda_n), \quad (4.4.9)$$

where  $\Delta_1(\lambda) = \frac{d}{d\lambda} \Delta(\lambda)$ . The data  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  are called the spectral data of  $L$ .

3) The eigenvalues  $\{\lambda_n\}$  and the eigenfunctions  $\varphi(x, \lambda_n), \psi(x, \lambda_n)$  are real. All zeros of  $\Delta(\lambda)$  are simple, i.e.  $\Delta_1(\lambda_n) \neq 0$ . Eigenfunctions related to different eigenvalues are orthogonal in  $L_2(0, T)$ .

We omit the proof of Theorem 4.4.1, since the arguments here are the same as for the classical Sturm-Liouville problem (see Section 1.1).

Let  $C_0(x, \lambda)$  and  $S_0(x, \lambda)$  be smooth solutions of (4.4.1) on the interval  $[0, T]$  under the initial conditions  $C_0(0, \lambda) = S'_0(0, \lambda) = 1, C'_0(0, \lambda) = S_0(0, \lambda) = 0$ . Then, using the jump condition (4.4.3) we get

$$C(x, \lambda) = C_0(x, \lambda), \quad S(x, \lambda) = S_0(x, \lambda), \quad x < a, \quad (4.4.10)$$

$$C(x, \lambda) = A_1 C_0(x, \lambda) + B_1 S_0(x, \lambda), \quad S(x, \lambda) = A_2 C_0(x, \lambda) + B_2 S_0(x, \lambda), \quad x > a, \quad (4.4.11)$$

where

$$\begin{cases} A_1 = a_1 C_0(a, \lambda) S'_0(a, \lambda) - a_1^{-1} C'_0(a, \lambda) S_0(a, \lambda) - a_2 C_0(a, \lambda) S_0(a, \lambda), \\ B_1 = (a_1^{-1} - a_1) C_0(a, \lambda) C'_0(a, \lambda) + a_2 C_0^2(a, \lambda), \\ A_2 = (a_1 - a_1^{-1}) S_0(a, \lambda) S'_0(a, \lambda) - a_2 S_0^2(a, \lambda), \\ B_2 = a_1^{-1} C_0(a, \lambda) S'_0(a, \lambda) - a_1 C'_0(a, \lambda) S_0(a, \lambda) + a_2 C_0(a, \lambda) S_0(a, \lambda). \end{cases} \quad (4.4.12)$$

Let  $\lambda = \rho^2$ ,  $\rho = \sigma + i\tau$ . It was shown in Section 1.1 that the function  $C_0(x, \lambda)$  satisfies the following integral equation:

$$C_0(x, \lambda) = \cos \rho x + \int_0^x \frac{\sin \rho(x-t)}{\rho} q(t) C_0(t, \lambda) dt, \quad (4.4.13)$$

and for  $|\rho| \rightarrow \infty$ ,

$$C_0(x, \lambda) = \cos \rho x + O\left(\frac{1}{\rho} \exp(|\tau|x)\right).$$

Then (4.4.13) implies

$$C_0(x, \lambda) = \cos \rho x + \frac{\sin \rho x}{2\rho} \int_0^x q(t) dt + \frac{1}{2\rho} \int_0^x q(t) \sin \rho(x-2t) dt + O\left(\frac{1}{\rho^2} \exp(|\tau|x)\right), \quad (4.4.14)$$

$$C'_0(x, \lambda) = -\rho \sin \rho x + \frac{\cos \rho x}{2} \int_0^x q(t) dt + \frac{1}{2} \int_0^x q(t) \cos \rho(x-2t) dt + O\left(\frac{1}{\rho} \exp(|\tau|x)\right), \quad (4.4.15)$$

Analogously,

$$S_0(x, \lambda) = \frac{\sin \rho x}{\rho} - \frac{\cos \rho x}{2\rho^2} \int_0^x q(t) dt + \frac{1}{2\rho^2} \int_0^x q(t) \cos \rho(x-2t) dt + O\left(\frac{1}{\rho^3} \exp(|\tau|x)\right), \quad (4.4.16)$$

$$S'_0(x, \lambda) = \cos \rho x + \frac{\sin \rho x}{2\rho} \int_0^x q(t) dt - \frac{1}{2\rho} \int_0^x q(t) \sin \rho(x-2t) dt + O\left(\frac{1}{\rho^2} \exp(|\tau|x)\right). \quad (4.4.17)$$

By virtue of (4.4.12) and (4.4.14)-(4.4.17),

$$A_1 = b_1 + b_2 \cos 2\rho a + \left(b_2 \int_0^a q(t) dt - \frac{a_2}{2}\right) \frac{\sin 2\rho a}{\rho} + O\left(\frac{1}{\rho^2}\right),$$

$$B_1 = b_2 \left( \rho \sin 2\rho a - \cos 2\rho a \int_0^a q(t) dt - \int_0^a q(t) \cos 2\rho(a-t) dt \right) + \frac{a_2}{2} \left( 1 + \cos 2\rho a \right) + O\left(\frac{1}{\rho}\right),$$

$$A_2 = b_2 \frac{\sin 2\rho a}{\rho} + O\left(\frac{1}{\rho^2}\right), \quad B_2 = b_1 + b_2 \cos 2\rho a + O\left(\frac{1}{\rho}\right),$$

where  $b_1 = (a_1 + a_1^{-1})/2$ ,  $b_2 = (a_1 - a_1^{-1})/2$ . Since  $\varphi(x, \lambda) = C(x, \lambda) + hS(x, \lambda)$ , we calculate using (4.4.10)-(4.4.12), (4.4.14)-(4.4.17):

$$\varphi(x, \lambda) = \cos \rho x + \left(h + \frac{1}{2} \int_0^x q(t) dt\right) \frac{\sin \rho x}{\rho} + o\left(\frac{1}{\rho} \exp(|\tau|x)\right), \quad x < a, \quad (4.4.18)$$

$$\varphi(x, \lambda) = (b_1 \cos \rho x + b_2 \cos \rho(2a - x)) + f_1(x) \frac{\sin \rho x}{\rho} + f_2(x) \frac{\sin \rho(2a - x)}{\rho} + o\left(\frac{1}{\rho} \exp(|\tau|x)\right), \quad x > a, \quad (4.4.19)$$

$$\varphi'(x, \lambda) = -\rho \sin \rho x + \left(h + \frac{1}{2} \int_0^x q(t) dt\right) \cos \rho x + o(\exp(|\tau|x)), \quad x < a, \quad (4.4.20)$$

$$\varphi'(x, \lambda) = \rho(-b_1 \sin \rho x + b_2 \sin \rho(2a - x)) + f_1(x) \cos \rho x - f_2(x) \cos \rho(2a - x) + o(\exp(|\tau|x)), \quad x > a, \quad (4.4.21)$$

where

$$f_1(x) = b_1 \left( h + \frac{1}{2} \int_0^x q(t) dt \right) + \frac{a_2}{2}, \quad f_2(x) = b_2 \left( h - \frac{1}{2} \int_0^x q(t) dt + \int_0^a q(t) dt \right) - \frac{a_2}{2}.$$

In particular, (4.4.18)-(4.4.21) yield

$$\varphi^{(\nu)}(x, \lambda) = O(|\rho|^\nu \exp(|\tau|x)), \quad 0 \leq x \leq \pi. \quad (4.4.22)$$

Similarly,

$$\psi^{(\nu)}(x, \lambda) = O(|\rho|^\nu \exp(|\tau|(T - x))), \quad 0 \leq x \leq \pi. \quad (4.4.23)$$

It follows from (4.4.6), (4.4.19) and (4.4.21) that

$$\Delta(\lambda) = \rho(b_1 \sin \rho T - b_2 \sin \rho(2a - T)) - \omega_1 \cos \rho T - \omega_2 \cos \rho(2a - T) + o(\exp(|\tau|T)), \quad (4.4.24)$$

where

$$\omega_1 = b_1 \left( H + h + \frac{1}{2} \int_0^T q(t) dt \right) + \frac{a_2}{2}, \quad \omega_2 = b_2 \left( H - h + \frac{1}{2} \int_0^T q(t) dt - \int_0^a q(t) dt \right) - \frac{a_2}{2}. \quad (4.4.25)$$

Using (4.4.24), by the well-known methods (see, for example, [bel1]) one can obtain the following properties of the characteristic function  $\Delta(\lambda)$  and the eigenvalues  $\lambda_n = \rho_n^2$  of the boundary value problem  $L$ :

1) For  $|\rho| \rightarrow \infty$ ,  $\Delta(\lambda) = O(|\rho| \exp(|\tau|T))$ .

2) There exist  $h > 0$ ,  $C_h > 0$  such that  $|\Delta(\lambda)| \geq C_h |\rho| \exp(|\tau|T)$  for  $|Im \rho| \geq h$ . Hence, the eigenvalues  $\{\lambda_n\}$  lie in the domain  $|Im \rho| < h$ .



3) The number  $N_a$  of zeros of  $\Delta(\lambda)$  in the rectangle  $R_a = \{\rho : |\tau| \leq h, \sigma \in [a, a+1]\}$  is bounded with respect to  $a$ .

4) Denote  $G_\delta = \{\rho : |\rho - \rho_n| \geq \delta\}$ . Then

$$|\Delta(\lambda)| \geq C_\delta |\rho| \exp(|\tau|T), \quad \rho \in G_\delta. \quad (4.4.26)$$

5) There exist numbers  $R_N \rightarrow \infty$  such that for sufficiently small  $\delta > 0$ , the circles  $|\rho| = R_N$  lie in  $G_\delta$  for all  $N$ .

6) Let  $\lambda_n^0 = (\rho_n^0)^2$  be zeros of the function

$$\Delta_0(\lambda) = \rho(b_1 \sin \rho T - b_2 \sin \rho(2a - T)). \quad (4.4.27)$$

Then

$$\rho_n = \rho_n^0 + o(1), \quad n \rightarrow \infty. \quad (4.4.28)$$

Substituting (4.4.18), (4.4.19) and (4.4.28) into (4.4.8) we get

$$\alpha_n = \alpha_n^0 + o(1), \quad n \rightarrow \infty, \quad (4.4.29)$$

where

$$\alpha_n^0 = \frac{a}{2} + \left( \frac{b_1^2 + b_2^2}{2} + b_1 b_2 \cos 2\rho_n^0 a \right) (T - a). \quad (4.4.30)$$

It follows from (4.4.29) and (4.4.30) that

$$|\alpha_n| \asymp C, \quad (4.4.31)$$

which means that  $\alpha_n = O(1)$  and  $(\alpha_n)^{-1} = O(1)$ . By virtue of (4.4.7),

$$\beta_n = \psi(0, \lambda_n) = \frac{1}{\varphi(T, \lambda_n)}. \quad (4.4.32)$$

Taking (4.4.22) and (4.4.23) into account, we deduce  $|\beta_n| \asymp C$ . Together with (4.4.9) and (4.4.31) this yields

$$|\Delta_1(\lambda_n)| \asymp C. \quad (4.4.33)$$

We need more precise asymptotics for  $\rho_n$  and  $\alpha_n$  than (4.4.28) and (4.4.29). Substituting (4.4.24) into the relation  $\Delta(\lambda_n) = 0$ , we get

$$b_1 \sin \rho_n T - b_2 \sin \rho_n(2a - T) = O\left(\frac{1}{\rho_n^0}\right). \quad (4.4.34)$$

According to (4.4.28),  $\rho_n = \rho_n^0 + \varepsilon_n$ ,  $\varepsilon_n \rightarrow 0$ . Since

$$\Delta^0(\lambda_n^0) = \rho_n^0(b_1 \sin \rho_n^0 T - b_2 \sin \rho_n^0(2a - T)) = 0,$$

it follows from (4.4.34) that

$$\varepsilon_n(b_1 T \cos \rho_n^0 T - b_2(2a - T) \cos \rho_n^0(2a - T)) = O\left(\frac{1}{\rho_n^0}\right) + O(\varepsilon_n^2).$$

In view of (4.4.27),

$$\Delta_1^0(\lambda_n^0) := \left( \frac{d}{d\lambda} \Delta^0(\lambda) \right)_{|\lambda=\lambda_n^0} = 2^{-1} (b_1 T \cos \rho_n^0 T - b_2(2a - T) \cos \rho_n^0(2a - T)).$$

Hence

$$\varepsilon_n \Delta_1^0(\lambda_n^0) = O\left(\frac{1}{\rho_n^0}\right) + O(\varepsilon_n^2).$$

By virtue of (4.4.33),  $|\Delta_1^0(\lambda_n^0)| \asymp C$ , and we conclude that

$$\varepsilon_n = O\left(\frac{1}{\rho_n^0}\right). \quad (4.4.35)$$

Substituting (4.4.24) into the relation  $\Delta(\lambda_n) = 0$  again and using (4.4.35) we arrive at

$$\rho_n = \rho_n^0 + \frac{\theta_n}{\rho_n^0} + \frac{\kappa_n}{\rho_n^0}, \quad (4.4.36)$$

where  $\kappa_n = o(1)$ , and

$$\theta_n = (\omega_1 \cos \rho_n^0 T + \omega_2 \cos \rho_n^0 (2a - T))(2\Delta_1^0(\lambda_n^0))^{-1}.$$

Here  $\omega_1$  and  $\omega_2$  are defined by (4.4.25). At last, using (4.4.8), (4.4.18), (4.4.19) and (4.4.36), one can calculate

$$\alpha_n = \alpha_n^0 + \frac{\theta_{n1}}{\rho_n^0} + \frac{\kappa_{n1}}{\rho_n^0}, \quad (4.4.37)$$

where  $\kappa_{n1} = o(1)$ , and

$$\theta_{n1} = b_0 \sin 2\rho_n^0 a + \frac{b_1^2}{4} \sin 2\rho_n^0 T - \frac{b_2^2}{4} \sin 2\rho_n^0 (2a - T) + \frac{b_1 b_2}{2} \sin 2\rho_n^0 (T - a),$$

$$b_0 = -2a\theta_n - \frac{b_1^2}{4} + \frac{b_2^2}{4} + (T - a) \left( b_1 b_2 (2h + \int_0^a q(t) dt) + \frac{a_2}{2} (b_2 - b_1) \right).$$

Furthermore, one can show that  $\{\kappa_n\}, \{\kappa_{n1}\} \in \ell_2$ . If  $q(x)$  is a smooth function one can obtain more precise asymptotics for the spectral data.

**4.4.2. Formulation of the inverse problem. Uniqueness theorems.** Let  $\Phi(x, \lambda)$  be the solution of (4.4.1) under the conditions  $U(\Phi) = 1$ ,  $V(\Phi) = 0$  and under the jump conditions (4.4.3). We set  $M(\lambda) := \Phi(0, \lambda)$ . The functions  $\Phi(x, \lambda)$  and  $M(\lambda)$  are called the Weyl solution and the Weyl function for the boundary value problem  $L$ , respectively. Clearly,

$$\Phi(x, \lambda) = \frac{\psi(x, \lambda)}{\Delta(\lambda)} = S(x, \lambda) + M(\lambda)\varphi(x, \lambda), \quad (4.4.38)$$

$$\langle \varphi(x, \lambda), \Phi(x, \lambda) \rangle \equiv 1, \quad (4.4.39)$$

$$M(\lambda) = \frac{\delta(\lambda)}{\Delta(\lambda)}, \quad (4.4.40)$$

where  $\delta(\lambda) = \psi(0, \lambda) = V(S)$  is the characteristic function of the boundary value problem  $L_1$  for equation (4.4.1) with the boundary conditions  $U(y) = 0$ ,  $y(T) = 0$  and with the jump conditions (4.4.3). Let  $\{\mu_n\}_{n \geq 0}$  be zeros of  $\delta(\lambda)$ , i.e. the eigenvalues of  $L_1$ .

In this section we study the following inverse problems of recovering  $L$  from its spectral characteristics:

- (i) from the Weyl function  $M(\lambda)$ ;
- (ii) from the spectral data  $\{\lambda_n, \alpha_n\}_{n \geq 0}$ ;

(iii) from two spectra  $\{\lambda_n, \mu_n\}_{n \geq 0}$ .

These inverse problems are a generalization of the inverse problems for Sturm-Liouville equations without discontinuities (see Section 1.4).

First, let us prove the uniqueness theorems for the solutions of the problems (i)-(iii). For this purpose we agree that together with  $L$  we consider a boundary value problem  $\tilde{L}$  of the same form but with different coefficients  $\tilde{q}(x)$ ,  $\tilde{h}$ ,  $\tilde{H}$ ,  $\tilde{a}$ ,  $\tilde{a}_k$ . If a certain symbol  $\gamma$  denotes an object related to  $L$ , then  $\tilde{\gamma}$  will denote the analogous object related to  $\tilde{L}$ , and  $\hat{\gamma} = \gamma - \tilde{\gamma}$ .

**Theorem 4.4.2.** *If  $M(\lambda) = \tilde{M}(\lambda)$ , then  $L = \tilde{L}$ . Thus, the specification of the Weyl function uniquely determines the operator.*

*Proof.* It follows from (4.4.23), (4.4.26) and (4.4.38) that

$$|\Phi^{(\nu)}(x, \lambda)| \leq C_\delta |\rho|^{\nu-1} \exp(-|\tau|x), \quad \rho \in G_\delta. \quad (4.4.41)$$

Let us define the matrix  $P(x, \lambda) = [P_{jk}(x, \lambda)]_{j,k=1,2}$  by the formula

$$P(x, \lambda) \begin{bmatrix} \tilde{\varphi}(x, \lambda) & \tilde{\Phi}(x, \lambda) \\ \tilde{\varphi}'(x, \lambda) & \tilde{\Phi}'(x, \lambda) \end{bmatrix} = \begin{bmatrix} \varphi(x, \lambda) & \Phi(x, \lambda) \\ \varphi'(x, \lambda) & \Phi'(x, \lambda) \end{bmatrix}.$$

Taking (4.4.39) into account we calculate

$$\left. \begin{aligned} P_{j1}(x, \lambda) &= \varphi^{(j-1)}(x, \lambda) \tilde{\Phi}'(x, \lambda) - \Phi^{(j-1)}(x, \lambda) \tilde{\varphi}'(x, \lambda), \\ P_{j2}(x, \lambda) &= \Phi^{(j-1)}(x, \lambda) \tilde{\varphi}(x, \lambda) - \varphi^{(j-1)}(x, \lambda) \tilde{\Phi}(x, \lambda), \end{aligned} \right\} \quad (4.4.42)$$

$$\left. \begin{aligned} \varphi(x, \lambda) &= P_{11}(x, \lambda) \tilde{\varphi}(x, \lambda) + P_{12}(x, \lambda) \tilde{\varphi}'(x, \lambda), \\ \Phi(x, \lambda) &= P_{11}(x, \lambda) \tilde{\Phi}(x, \lambda) + P_{12}(x, \lambda) \tilde{\Phi}'(x, \lambda). \end{aligned} \right\} \quad (4.4.43)$$

According to (4.4.38) and (4.4.42), for each fixed  $x$ , the functions  $P_{jk}(x, \lambda)$  are meromorphic in  $\lambda$  with simple poles in the points  $\lambda_n$  and  $\tilde{\lambda}_n$ . Denote  $G_\delta^0 = G_\delta \cap \tilde{G}_\delta$ . By virtue of (4.4.22), (4.4.41) and (4.4.42) we get

$$|P_{12}(x, \lambda)| \leq C_\delta |\rho|^{-1}, \quad |P_{11}(x, \lambda)| \leq C_\delta, \quad \rho \in G_\delta^0. \quad (4.4.44)$$

It follows from (4.4.38) and (4.4.42) that

$$P_{11}(x, \lambda) = \varphi(x, \lambda) \tilde{S}'(x, \lambda) - S(x, \lambda) \tilde{\varphi}'(x, \lambda) + (\tilde{M}(\lambda) - M(\lambda)) \varphi(x, \lambda) \tilde{\varphi}'(x, \lambda),$$

$$P_{12}(x, \lambda) = S(x, \lambda) \tilde{\varphi}(x, \lambda) - \varphi(x, \lambda) \tilde{S}(x, \lambda) + (M(\lambda) - \tilde{M}(\lambda)) \varphi(x, \lambda) \tilde{\varphi}(x, \lambda).$$

Thus, if  $M(\lambda) \equiv \tilde{M}(\lambda)$ , then for each fixed  $x$ , the functions  $P_{1k}(x, \lambda)$  are entire in  $\lambda$ . Together with (4.4.44) this yields  $P_{12}(x, \lambda) \equiv 0$ ,  $P_{11}(x, \lambda) \equiv A(x)$ . Using (4.4.43) we derive

$$\varphi(x, \lambda) \equiv A(x) \tilde{\varphi}(x, \lambda), \quad \Phi(x, \lambda) \equiv A(x) \tilde{\Phi}(x, \lambda). \quad (4.4.45)$$

It follows from (4.4.18) and (4.4.19) that for  $|\rho| \rightarrow \infty$ ,  $\arg \rho \in [\varepsilon, \pi - \varepsilon]$ ,  $\varepsilon > 0$ ,

$$\varphi(x, \lambda) = 2^{-1} b \exp(-i\rho x) (1 + O(\rho^{-1})),$$

where  $b = 1$  for  $x < a$ , and  $b = b_1$  for  $x > a$ . Similarly, one can calculate

$$\Phi(x, \lambda) = (i\rho b)^{-1} \exp(i\rho x)(1 + O(\rho^{-1})).$$

Together with (4.4.39) and (4.4.45) this gives  $b_1 = \tilde{b}_1$ ,  $A(x) \equiv 1$ , i.e.  $\varphi(x, \lambda) \equiv \tilde{\varphi}(x, \lambda)$ ,  $\Phi(x, \lambda) \equiv \tilde{\Phi}(x, \lambda)$  for all  $x$  and  $\lambda$ . Consequently,  $L = \tilde{L}$ .  $\square$

**Theorem 4.4.3.** *If  $\lambda_n = \tilde{\lambda}_n$ ,  $\alpha_n = \tilde{\alpha}_n$ ,  $n \geq 0$ , then  $L = \tilde{L}$ . Thus, the specification of the spectral data  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  uniquely determines the operator.*

*Proof.* It follows from (4.4.40) that the Weyl function  $M(\lambda)$  is meromorphic with simple poles  $\lambda_n$ . Using (4.4.40), (4.4.32) and (4.4.9) we calculate

$$\operatorname{Res}_{\lambda=\lambda_n} M(\lambda) = \frac{\delta(\lambda_n)}{\Delta_1(\lambda_n)} = \frac{\beta_n}{\Delta_1(\lambda_n)} = \frac{1}{\alpha_n}. \quad (4.4.46)$$

Let  $\Gamma = \{\lambda = u + iv : u = (2h^2)^{-2}v^2 - h^2\}$  be the image of the set  $Im \rho = \pm h$  under the mapping  $\lambda = \rho^2$ . Denote  $\Gamma_n = \Gamma \cap \{\lambda : |\lambda| \leq r_n\}$ , and consider the closed contours  $\Gamma_{n0} = \Gamma_n \cup \{\lambda : |\lambda| = r_n, \lambda \notin \operatorname{int} \Gamma\}$ ,  $\Gamma_{n1} = \Gamma_n \cup \{\lambda : |\lambda| = r_n, \lambda \in \operatorname{int} \Gamma\}$  (with counterclockwise circuit). Since the Weyl function  $M(\lambda)$  is regular for  $\lambda \in \operatorname{int} \Gamma_{n0}$ , we get by Cauchy's theorem that

$$M(\lambda) = \frac{1}{2\pi i} \int_{\Gamma_{n0}} \frac{M(\mu)}{\mu - \lambda} d\mu, \quad \lambda \in \operatorname{int} \Gamma_{n0}.$$

Furthermore, since  $\delta(\lambda) = \psi(0, \lambda)$ , estimate (4.4.23) gives

$$\delta(\lambda) = O(\exp(|\tau|T)). \quad (4.4.47)$$

Using (4.4.40), (4.4.26) and (4.4.47) we obtain

$$|M(\lambda)| \leq C_\delta |\rho|^{-1}, \quad \rho \in G_\delta. \quad (4.4.48)$$

Hence

$$\lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{|\mu|=r_n} \frac{M(\mu)}{\mu - \lambda} d\mu = 0,$$

i.e.

$$M(\lambda) = \lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{\Gamma_{n1}} \frac{M(\mu)}{\lambda - \mu} d\mu.$$

Calculating this integral by the residue theorem and taking (4.4.46) into account we arrive at

$$M(\lambda) = \sum_{k=0}^{\infty} \frac{1}{\alpha_k(\lambda - \lambda_k)}, \quad (4.4.49)$$

where the series converges "with brackets":  $\sum_{k=0}^{\infty} = \lim_{n \rightarrow \infty} \sum_{|\lambda_k| < r_n}$ . Moreover, it is obvious that the series converges absolutely. Under the hypothesis of the theorem we get, in view of (4.4.49), that  $M(\lambda) \equiv \tilde{M}(\lambda)$ , and consequently by Theorem 4.4.2,  $L = \tilde{L}$ .  $\square$

**Theorem 4.4.4.** *If  $\lambda_n = \tilde{\lambda}_n$ ,  $\mu_n = \tilde{\mu}_n$ ,  $n \geq 0$ , then  $q(x) = \tilde{q}(x)$  a.e. on  $(0, T)$ ,  $h = \tilde{h}$ ,  $H = \tilde{H}$ ,  $a = \tilde{a}$ ,  $a_1 = \tilde{a}_1$  and  $a_2 = \tilde{a}_2$ .*

*Proof.* It follows from (4.4.24) that the function  $\Delta(\lambda)$  is entire in  $\lambda$  of order  $1/2$ , and consequently  $\Delta(\lambda)$  is uniquely determined up to a multiplicative constant by its zeros:

$$\Delta(\lambda) = C \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right) \quad (4.4.50)$$

(the case when  $\Delta(0) = 0$  requires minor modifications). According to (4.4.27),

$$\Delta_0(\lambda) = \Omega_0 \lambda \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_n^0}\right), \quad \Omega_0 = T b_1 - (2a - T) b_2.$$

Then

$$\frac{\Delta(\lambda)}{\Delta_0(\lambda)} = C \frac{\lambda_0 - \lambda}{\lambda_0 \Omega_0 \lambda} \prod_{n=1}^{\infty} \frac{\lambda_n^0}{\lambda_n} \prod_{n=1}^{\infty} \left(1 + \frac{\lambda_n - \lambda_n^0}{\lambda_n^0 - \lambda}\right).$$

Since

$$\lim_{\lambda \rightarrow -\infty} \frac{\Delta(\lambda)}{\Delta_0(\lambda)} = 1, \quad \lim_{\lambda \rightarrow -\infty} \prod_{n=1}^{\infty} \left(1 + \frac{\lambda_n - \lambda_n^0}{\lambda_n^0 - \lambda}\right) = 1,$$

we get

$$C = -\lambda_0 \Omega_0 \prod_{n=1}^{\infty} \frac{\lambda_n}{\lambda_n^0}.$$

Substituting into (4.4.50) we arrive at

$$\Delta(\lambda) = \Omega_0 (\lambda - \lambda_0) \prod_{n=1}^{\infty} \frac{\lambda_n - \lambda}{\lambda_n^0}. \quad (4.4.51)$$

It follows from (4.4.51) that the specification of the spectrum  $\{\lambda_n\}_{n \geq 0}$  uniquely determines the characteristic function  $\Delta(\lambda)$ . Analogously, the function  $\delta(\lambda)$  is uniquely determined by its zeros  $\{\mu_n\}_{n \geq 0}$ . Let now  $\lambda_n = \tilde{\lambda}_n$ ,  $\mu_n = \tilde{\mu}_n$ ,  $n \geq 0$ . Then  $\Delta(\lambda) \equiv \tilde{\Delta}(\lambda)$ ,  $\delta(\lambda) \equiv \tilde{\delta}(\lambda)$ , and consequently by (4.4.40),  $M(\lambda) \equiv \tilde{M}(\lambda)$ . From this, in view of Theorem 4.4.2, we obtain  $q(x) = \tilde{q}(x)$  a.e. on  $(0, T)$ ,  $h = \tilde{h}$ ,  $H = \tilde{H}$ ,  $a = \tilde{a}$ ,  $a_1 = \tilde{a}_1$  and  $a_2 = \tilde{a}_2$ .  $\square$

**4.4.3 Solution of the inverse problem.** For definiteness we consider the inverse problem of recovering  $L$  from the spectral data  $\{\lambda_n, \alpha_n\}_{n \geq 0}$ . Let boundary value problems  $L$  and  $\tilde{L}$  be such that

$$a = \tilde{a}, \quad \sum_{n=0}^{\infty} \xi_n |\rho_n| < \infty, \quad (4.4.52)$$

where  $\xi_n := |\rho_n - \tilde{\rho}_n| + |\alpha_n - \tilde{\alpha}_n|$ . Denote

$$\begin{aligned} \lambda_{n0} &= \lambda_n, \quad \lambda_{n1} = \tilde{\lambda}_n, \quad \alpha_{n0} = \alpha_n, \quad \alpha_{n1} = \tilde{\alpha}_n, \quad \varphi_{ni}(x) = \varphi(x, \lambda_{ni}), \quad \tilde{\varphi}_{ni}(x) = \tilde{\varphi}(x, \lambda_{ni}), \\ \tilde{Q}_{kj}(x, \lambda) &= \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{kj}(x) \rangle}{\alpha_{kj}(\lambda - \lambda_{kj})} = \frac{1}{\alpha_{kj}} \int_0^x \tilde{\varphi}(t, \lambda) \tilde{\varphi}_{kj}(t) dt, \quad \tilde{Q}_{ni,kj}(x, \lambda) = \tilde{Q}_{kj}(x, \lambda_{ni}). \end{aligned}$$

Here we lean on (4.4.4) and (4.4.5). It follows from (4.4.18)-(4.4.21) and (4.4.31) that

$$|\varphi_{nj}^{(\nu)}(x)| \leq C(|\rho_n^0| + 1)^\nu, \quad |\tilde{\varphi}_{nj}^{(\nu)}(x)| \leq C(|\rho_n^0| + 1)^\nu, \quad (4.4.53)$$

$$|\tilde{Q}_{ni,kj}(x)| \leq \frac{C}{|\rho_n^0 - \rho_k^0| + 1}, \quad |\tilde{Q}_{ni,kj}^{(\nu+1)}(x)| \leq C(|\rho_n^0| + |\rho_k^0| + 1)^\nu. \quad (4.4.54)$$

**Lemma 4.4.1.** *The following relation holds*

$$\tilde{\varphi}(x, \lambda) = \varphi(x, \lambda) + \sum_{k=0}^{\infty} (\tilde{Q}_{k0}(x, \lambda)\varphi_{k0}(x) - \tilde{Q}_{k1}(x, \lambda)\varphi_{k1}(x)). \quad (4.4.55)$$

*Proof.* By virtue of (4.4.52) we have

$$a = \tilde{a}, \quad a_1 = \tilde{a}_1. \quad (4.4.56)$$

Then it follows from (4.4.18)-(4.4.21) that

$$|\varphi^{(\nu)}(x, \lambda) - \tilde{\varphi}^{(\nu)}(x, \lambda)| \leq C|\rho|^{\nu-1} \exp(|\tau|x). \quad (4.4.57)$$

Similarly,

$$|\psi^{(\nu)}(x, \lambda) - \tilde{\psi}^{(\nu)}(x, \lambda)| \leq C|\rho|^{\nu-1} \exp(|\tau|(T-x)). \quad (4.4.58)$$

Denote  $G_\delta^0 = G_\delta \cap \tilde{G}_\delta$ . In view of (4.4.38), (4.4.23), (4.4.26) and (4.4.58) we obtain

$$|\Phi^{(\nu)}(x, \lambda) - \tilde{\Phi}^{(\nu)}(x, \lambda)| \leq C_\delta |\rho|^{\nu-2} \exp(-|\tau|x), \quad \rho \in G_\delta^0. \quad (4.4.59)$$

Let  $P(x, \lambda)$  be the matrix defined in Subsection 4.4.2. Since for each fixed  $x$ , the functions  $P_{1k}(x, \lambda)$  are meromorphic in  $\lambda$  with simple poles  $\lambda_n$  and  $\tilde{\lambda}_n$ , we get by Cauchy's theorem

$$P_{1k}(x, \lambda) - \delta_{1k} = \frac{1}{2\pi i} \int_{\Gamma_{n0}} \frac{P_{1k}(x, \xi) - \delta_{1k}}{\xi - \lambda} d\xi, \quad k = 1, 2, \quad (4.4.60)$$

where  $\lambda \in \text{int } \Gamma_{n0}$ , and  $\delta_{jk}$  is the Kronecker delta.

Further, (4.4.39) and (4.4.42) imply

$$P_{11}(x, \lambda) = 1 + (\varphi(x, \lambda) - \tilde{\varphi}(x, \lambda))\tilde{\Phi}'(x, \lambda) - (\Phi(x, \lambda) - \tilde{\Phi}(x, \lambda))\tilde{\varphi}'(x, \lambda). \quad (4.4.61)$$

Using (4.4.22), (4.4.23), (4.4.41), (4.4.42), (4.4.57), (4.4.59) and (4.4.61) we infer

$$|P_{1k}(x, \lambda) - \delta_{1k}| \leq C_\delta |\rho|^{-1}, \quad \rho \in G_\delta^0. \quad (4.4.62)$$

By virtue of (4.4.62),

$$\lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{|\xi|=r_n} \frac{P_{1k}(x, \xi) - \delta_{1k}}{\xi - \lambda} d\xi = 0,$$

and consequently, (4.4.60) yields

$$P_{1k}(x, \lambda) - \delta_{1k} = \lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{\Gamma_{n1}} \frac{P_{1k}(x, \xi)}{\xi - \lambda} d\xi.$$

Substituting into (4.4.43) we obtain

$$\varphi(x, \lambda) = \tilde{\varphi}(x, \lambda) + \lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{\Gamma_{n1}} \frac{\tilde{\varphi}(x, \lambda)P_{11}(x, \xi) + \tilde{\varphi}'(x, \lambda)P_{12}(x, \xi)}{\lambda - \xi} d\xi.$$

Taking (4.4.42) into account we calculate

$$\varphi(x, \lambda) = \tilde{\varphi}(x, \lambda) + \lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{\Gamma_{n1}} (\tilde{\varphi}(x, \lambda)(\varphi(x, \xi)\tilde{\Phi}'(x, \xi) - \Phi(x, \xi)\tilde{\varphi}'(x, \xi)) +$$

$$\tilde{\varphi}'(x, \lambda)(\Phi(x, \xi)\tilde{\varphi}(x, \xi) - \varphi(x, \xi)\tilde{\Phi}(x, \xi))\frac{d\xi}{\lambda - \xi},$$

or, in view of (4.4.38),

$$\tilde{\varphi}(x, \lambda) = \varphi(x, \lambda) + \lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{\Gamma_{n1}} \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \xi) \rangle}{\lambda - \xi} \hat{M}(\xi) \varphi(x, \xi) d\xi. \quad (4.4.63)$$

It follows from (4.4.46) that

$$\operatorname{Res}_{\xi=\lambda_{kj}} \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \xi) \rangle}{\lambda - \xi} \hat{M}(\xi) \varphi(x, \xi) = \tilde{Q}_{kj}(x, \lambda) \varphi_{kj}(x).$$

Calculating the integral in (4.4.63) by the residue theorem we arrive at (4.4.55).  $\square$

Analogously one can obtain the following relation

$$\tilde{\Phi}(x, \lambda) = \Phi(x, \lambda) + \sum_{k=0}^{\infty} (\tilde{D}_{k0}(x, \lambda) \varphi_{k0}(x) - \tilde{D}_{k1}(x, \lambda) \varphi_{k1}(x)), \quad (4.4.64)$$

where

$$\tilde{D}_{kj}(x, \lambda) := \frac{\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}_{kj}(x) \rangle}{\alpha_{kj}(\lambda - \lambda_{kj})}.$$

It follows from (4.4.55) that

$$\tilde{\varphi}_{ni}(x) = \varphi_{ni}(x) + \sum_{k=0}^{\infty} (\tilde{Q}_{ni,k0}(x) \varphi_{k0}(x) - \tilde{Q}_{ni,k1}(x) \varphi_{k1}(x)). \quad (4.4.65)$$

Let  $V$  be a set of indices  $u = (n, i)$ ,  $n \geq 0$ ,  $i = 0, 1$ . For each fixed  $x$ , we define the vectors

$$\psi(x) = [\psi_u(x)]_{u \in V} = \begin{bmatrix} \psi_{n0}(x) \\ \psi_{n1}(x) \end{bmatrix}_{n \geq 0}, \quad \tilde{\psi}(x) = [\tilde{\psi}_u(x)]_{u \in V} = \begin{bmatrix} \tilde{\psi}_{n0}(x) \\ \tilde{\psi}_{n1}(x) \end{bmatrix}_{n \geq 0}$$

by the formulas

$$\begin{aligned} \psi_{n0}(x) &= (\varphi_{n0}(x) - \varphi_{n1}(x)) \xi_n^{-1}, & \psi_{n1}(x) &= \varphi_{n1}(x), \\ \tilde{\psi}_{n0}(x) &= (\tilde{\varphi}_{n0}(x) - \tilde{\varphi}_{n1}(x)) \xi_n^{-1}, & \tilde{\psi}_{n1}(x) &= \tilde{\varphi}_{n1}(x) \end{aligned}$$

(if  $\xi_n = 0$  for a certain  $n$ , then we put  $\psi_{n0}(x) = \tilde{\psi}_{n0}(x) = 0$ ). Similarly we define the block matrix

$$\tilde{H}(x) = [\tilde{H}_{u,v}(x)]_{u,v \in V} = \begin{bmatrix} \tilde{H}_{n0,k0}(x) & \tilde{H}_{n0,k1}(x) \\ \tilde{H}_{n1,k0}(x) & \tilde{H}_{n1,k1}(x) \end{bmatrix}_{n,k \geq 0},$$

where  $u = (n, i)$ ,  $v = (k, j)$ ,

$$\tilde{H}_{n0,k0}(x) = (\tilde{Q}_{n0,k0}(x) - \tilde{Q}_{n1,k0}(x)) \xi_k \xi_n^{-1}, \quad \tilde{H}_{n1,k1}(x) = \tilde{Q}_{n1,k0}(x) - \tilde{Q}_{n1,k1}(x),$$

$$\tilde{H}_{n1,k0}(x) = \tilde{Q}_{n1,k0}(x) \xi_k, \quad \tilde{H}_{n0,k1}(x) = (\tilde{Q}_{n0,k0}(x) - \tilde{Q}_{n1,k0}(x) - \tilde{Q}_{n0,k1}(x) + \tilde{Q}_{n1,k1}(x)) \xi_n^{-1}.$$

It follows from (4.4.18)-(4.4.21), (4.4.31), (4.4.36), (4.4.33), (4.4.53), (4.4.54) and Schwarz's lemma that

$$|\psi_{nj}^{(\nu)}(x)| \leq C(|\rho_n^0| + 1)^\nu, \quad |\tilde{\psi}_{nj}^{(\nu)}(x)| \leq C(|\rho_n^0| + 1)^\nu, \quad (4.4.66)$$

$$|\tilde{H}_{ni,kj}(x)| \leq \frac{C\xi_k}{|\rho_n^0 - \rho_k^0| + 1}, \quad |\tilde{H}_{ni,kj}^{(\nu+1)}(x)| \leq C(|\rho_n^0| + |\rho_k^0| + 1)^\nu \xi_k. \quad (4.4.67)$$

Let us consider the Banach space  $m$  of bounded sequences  $\alpha = [\alpha_u]_{u \in V}$  with the norm  $\|\alpha\|_m = \sup_{u \in V} |\alpha_u|$ . It follows from (4.4.67) that for each fixed  $x$ , the operator  $E + \tilde{H}(x)$  (here  $E$  is the identity operator), acting from  $m$  to  $m$ , is a linear bounded operator, and

$$\|\tilde{H}(x)\| \leq C \sup_n \sum_k \frac{\xi_k}{|\rho_n^0 - \rho_k^0| + 1} < \infty.$$

Taking into account our notations, we can rewrite (4.4.65) in the form

$$\tilde{\psi}_{ni}(x) = \psi_{ni}(x) + \sum_{k=0}^{\infty} (\tilde{H}_{ni,k0}(x)\psi_{k0}(x) + \tilde{H}_{ni,k1}(x)\psi_{k1}(x)),$$

or

$$\tilde{\psi}(x) = (E + \tilde{H}(x))\psi(x). \quad (4.4.68)$$

Thus, for each fixed  $x$ , the vector  $\psi(x) \in m$  is a solution of equation (4.4.68) in the Banach space  $m$ . Equation (4.4.68) is called the main equation of the inverse problem.

Now let us denote

$$\varepsilon_0(x) = \sum_{k=0}^{\infty} \left( \frac{1}{\alpha_{k0}} \tilde{\varphi}_{k0}(x) \varphi_{k0}(x) - \frac{1}{\alpha_{k1}} \tilde{\varphi}_{k1}(x) \varphi_{k1}(x) \right), \quad \varepsilon(x) = -2\varepsilon'_0(x). \quad (4.4.69)$$

By virtue of (4.4.52), the series converges absolutely and uniformly on  $[0, a]$  and  $[a, T]$ ;  $\varepsilon_0(x)$  is absolutely continuous on  $[0, a]$  and  $[a, T]$ , and  $\varepsilon(x) \in L_2(0, T)$ .

**Lemma 4.4.2** *The following relations hold*

$$q(x) = \tilde{q}(x) + \varepsilon(x), \quad (4.4.70)$$

$$a_2 = \tilde{a}_2 + (a_1^{-1} - a_1^3)\varepsilon_0(a - 0), \quad (4.4.71)$$

$$h = \tilde{h} - \varepsilon_0(0), \quad H = \tilde{H} + \varepsilon_0(T). \quad (4.4.72)$$

*Proof.* Differentiating (4.4.55) twice with respect to  $x$  and using (4.4.5) we get

$$\tilde{\varphi}'(x, \lambda) - \varepsilon_0(x)\tilde{\varphi}(x, \lambda) = \varphi'(x, \lambda) + \sum_{k=0}^{\infty} (\tilde{Q}_{k0}(x, \lambda)\varphi'_{k0}(x) - \tilde{Q}_{k1}(x, \lambda)\varphi'_{k1}(x)), \quad (4.4.73)$$

$$\tilde{\varphi}''(x, \lambda) = \varphi''(x, \lambda) + \sum_{k=0}^{\infty} (\tilde{Q}_{k0}(x, \lambda)\varphi''_{k0}(x) - \tilde{Q}_{k1}(x, \lambda)\varphi''_{k1}(x)) +$$

$$2\tilde{\varphi}(x, \lambda) \sum_{k=0}^{\infty} \left( \frac{1}{\alpha_{k0}} \tilde{\varphi}_{k0}(x) \varphi'_{k0}(x) - \frac{1}{\alpha_{k1}} \tilde{\varphi}_{k1}(x) \varphi'_{k1}(x) \right) +$$

$$\sum_{k=0}^{\infty} \left( \frac{1}{\alpha_{k0}} (\tilde{\varphi}(x, \lambda) \tilde{\varphi}_{k0}(x))' \varphi_{k0}(x) - \frac{1}{\alpha_{k1}} (\tilde{\varphi}(x, \lambda) \tilde{\varphi}_{k1}(x))' \varphi_{k1}(x) \right).$$

We replace here the second derivatives, using equation (4.4.1), and then we replace  $\varphi(x, \lambda)$ , using (4.4.55). After canceling terms with  $\tilde{\varphi}'(x, \lambda)$  we arrive at (4.4.70).



By virtue of (4.4.3), it follows from (4.4.69) that

$$\varepsilon_0(a+0) = a_1^2 \varepsilon_0(a-0). \quad (4.4.74)$$

In (4.4.73) we set  $x = a+0$ . Since the functions  $\tilde{Q}_{kj}(x, \lambda)$  are continuous with respect to  $x \in [0, T]$ , we calculate using (4.4.3), (4.4.56) and (4.4.74):

$$\begin{aligned} a_1^{-1} \tilde{\varphi}'(a-0, \lambda) + \tilde{a}_2 \tilde{\varphi}(a-0, \lambda) - a_1^3 \varepsilon_0(a-0) \tilde{\varphi}(a-0, \lambda) &= a_1^{-1} \varphi'(a-0, \lambda) + a_2 \varphi(a-0, \lambda) + \\ &+ a_1^{-1} \sum_{k=0}^{\infty} (\tilde{Q}_{k0}(a, \lambda) \varphi'_{k0}(a-0) - \tilde{Q}_{k1}(a, \lambda) \varphi'_{k1}(a-0)) + \\ &+ a_2 \sum_{k=0}^{\infty} (\tilde{Q}_{k0}(a, \lambda) \varphi_{k0}(a-0) - \tilde{Q}_{k1}(a, \lambda) \varphi_{k1}(a-0)). \end{aligned}$$

Replacing here  $\varphi(a-0, \lambda)$  and  $\varphi'(a-0, \lambda)$  from (4.4.55) and (4.4.73) respectively, we get after simple calculations

$$(\tilde{a}_2 - a_1^3 \varepsilon_0(a-0) + a_1^{-1} \varepsilon_0(a-0) - a_2) \tilde{\varphi}(a-0, \lambda) = 0,$$

i.e. (4.4.71) holds. Denote  $h_0 = -h$ ,  $h_T = H$ ,  $U_0 = U$ ,  $U_T = V$ . In (4.4.55) and (4.4.73) we put  $x = 0$  and  $x = T$ . Then

$$\begin{aligned} \tilde{\varphi}'(d, \lambda) + (h_d - \varepsilon_0(d)) \tilde{\varphi}(d, \lambda) &= U_d(\varphi(x, \lambda)) + \\ &+ \sum_{k=0}^{\infty} (\tilde{Q}_{k0}(d, \lambda) U_d(\varphi_{k0}) - \tilde{Q}_{k0}(d, \lambda) U_d(\varphi_{k0})), \quad d = 0, T. \end{aligned} \quad (4.4.75)$$

Let  $d = 0$ . Since  $U_0(\varphi(x, \lambda)) = 0$ ,  $\tilde{\varphi}(0, \lambda) = 1$ ,  $\tilde{\varphi}'(0, \lambda) = -\tilde{h}_0$ , we get  $h_0 - \tilde{h}_0 - \varepsilon_0(0) = 0$ , i.e.  $h = \tilde{h} - \varepsilon_0(0)$ . Let  $d = T$ . Since

$$U_T(\varphi(x, \lambda)) = -\Delta(\lambda), \quad U_T(\varphi_{k0}) = 0, \quad U_T(\varphi_{k1}) = -\Delta(\lambda_{k1}),$$

$$\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle = \tilde{\varphi}(T, \mu) \tilde{\Delta}(\lambda) - \tilde{\varphi}(T, \lambda) \tilde{\Delta}(\mu),$$

it follows from (4.4.75) that

$$\tilde{\varphi}'(T, \lambda) + (h_T - \varepsilon_0(T)) \tilde{\varphi}(T, \lambda) = -\Delta(\lambda) + \sum_{k=0}^{\infty} \frac{\tilde{\varphi}_{k1}(T) \tilde{\Delta}(\lambda)}{\alpha_{k1}(\lambda - \lambda_{k1})} \Delta(\lambda_{k1}).$$

For  $\lambda = \lambda_{n1}$  this yields

$$\tilde{\varphi}'_{n1}(T) + (h_T - \varepsilon_0(T)) \tilde{\varphi}_{n1}(T) = \Delta(\lambda_{n1}) \left( -1 + (\alpha_{n1})^{-1} \tilde{\varphi}_{n1}(T) \tilde{\Delta}_1(\lambda_{n1}) \right).$$

By virtue of (4.4.7) and (4.4.9),

$$(\alpha_{n1})^{-1} \tilde{\varphi}_{n1}(T) \tilde{\Delta}_1(\lambda_{n1}) = 1,$$

i.e.

$$\tilde{\varphi}'_{n1}(T) + (h_T - \varepsilon_0(T)) \tilde{\varphi}_{n1}(T) = 0.$$

On the other hand,

$$\tilde{\varphi}'_{n1}(T) + \tilde{h}_T \tilde{\varphi}_{n1}(T) = -\tilde{\Delta}(\lambda_{n1}) = 0.$$

Then  $(h_T - \varepsilon_0(T) - \tilde{h}_T)\tilde{\varphi}_{n1}(T) = 0$ , i.e.  $h_T - \tilde{h}_T = \varepsilon_0(T)$ .  $\square$

Now let us formulate necessary and sufficient conditions for the solvability of the inverse problem.

**Theorem 4.4.5.** *For real numbers  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  to be the spectral data for a certain boundary value problem  $L$  of the form (4.4.1)-(4.4.3), it is necessary and sufficient that  $\alpha_n > 0$ ,  $\lambda_n \neq \lambda_m$  ( $n \neq m$ ), and there exists  $\tilde{L}$  such that (4.4.52) holds. The boundary value problem  $L$  can be constructed by (4.4.56), (4.4.70)-(4.4.72).*

*Proof.* We give only a short sketch of the proof of Theorem 4.4.5, since it is similar to the proof of Theorem 1.6.2 for the classical Sturm-Liouville boundary value problems.

The necessity part of Theorem 4.4.5 was proved above. We prove the sufficiency. Let numbers  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  satisfying the hypothesis of Theorem 4.4.5 be given. We construct  $\tilde{\psi}(x)$  and  $\tilde{H}(x)$ , and consider the main equation (4.4.68).

**Lemma 4.4.3** *For each fixed  $x \in [0, T]$ , the operator  $E + \tilde{H}(x)$ , acting from  $m$  to  $m$ , has a bounded inverse operator, and the main equation (4.4.68) has a unique solution  $\psi(x) \in m$ .*

We omit the proof of Lemma 4.4.3, since the arguments here are the same as in the proof of Lemma 1.6.6.

Let  $\psi(x) = [\psi_u(x)]_{u \in V}$  be the solution of the main equation (4.4.68). Then the functions  $\psi_{ni}^{(\nu)}(x)$ ,  $\nu = 0, 1$  are absolutely continuous on  $[0, a]$  and  $[a, T]$ , and (4.4.66) holds. We define the functions  $\varphi_{ni}(x)$  by the formulae

$$\varphi_{n1}(x) = \psi_{n1}(x), \quad \varphi_{n0}(x) = \psi_{n0}(x)\xi_n + \psi_{n1}(x).$$

Then (4.4.65) and (4.4.53) are valid. Furthermore, we construct the functions  $\varphi(x, \lambda)$  and  $\Phi(x, \lambda)$  via (4.4.55) and (4.4.64), and the boundary value problem  $L$  via (4.4.56), (4.4.69)-(4.4.72). Clearly,  $\varphi_{ni}(x) = \varphi(x, \lambda_{ni})$ .

Using (4.4.55), (4.4.64) and (4.4.65) we get analogously to the proof of Lemma 1.6.9,

$$\ell\varphi_{ni}(x) = \lambda_{ni}\varphi_{ni}(x), \quad \ell\varphi(x, \lambda) = \lambda\varphi(x, \lambda), \quad \ell\Phi(x, \lambda) = \lambda\Phi(x, \lambda).$$

Furthermore, from (4.4.55) by differentiation we get (4.4.73). For  $\lambda = \lambda_{ni}$ , (4.4.73) implies

$$\tilde{\varphi}'_{ni}(x) = \varphi'_{ni}(x) + \varepsilon_0(x)\tilde{\varphi}_{ni}(x) + \sum_{k=0}^{\infty} (\tilde{Q}_{ni,k0}(x)\varphi'_{k0}(x) - \tilde{Q}_{ni,k1}(x)\varphi'_{k1}(x)). \quad (4.4.76)$$

Let us show that the functions  $\varphi(x, \lambda)$  and  $\Phi(x, \lambda)$  satisfy the jump conditions (4.4.3). For this purpose we put  $x = a + 0$  in (4.4.65). Using the jump conditions for  $\tilde{\varphi}_{ni}(x)$ , we get

$$a_1\tilde{\varphi}_{ni}(a - 0) = \varphi_{ni}(a + 0) + \sum_{k=0}^{\infty} (\tilde{Q}_{ni,k0}(a)\varphi_{k0}(a + 0) - \tilde{Q}_{ni,k1}(a)\varphi_{k1}(a + 0)).$$

Comparing with (4.4.65) for  $x = a - 0$ , we infer

$$g_{ni} + \sum_{k=0}^{\infty} (\tilde{Q}_{ni,k0}(a)g_{k0} - \tilde{Q}_{ni,k1}(a)g_{k1}) = 0, \quad g_{ni} := \varphi_{ni}(a + 0) - a_1\varphi_{ni}(a - 0).$$

According to Lemma 4.4.3 this yields  $g_{ni} = 0$ , i.e.

$$\varphi_{ni}(a+0) = a_1 \varphi_{ni}(a-0). \quad (4.4.77)$$

Taking  $x = a+0$  in (4.4.76) and using (4.4.77) and the jump conditions for  $\tilde{\varphi}_{ni}(x)$ , we get

$$a_1^{-1} \tilde{\varphi}'_{ni}(a-0) + \tilde{a}_2 \tilde{\varphi}_{ni}(a-0) = \varphi'_{ni}(a+0) + a_1^3 \varepsilon_0(a-0) \tilde{\varphi}_{ni}(a+0) + \sum_{k=0}^{\infty} (\tilde{Q}_{ni,k0}(a) \varphi'_{k0}(a+0) - \tilde{Q}_{ni,k1}(a) \varphi'_{k1}(a+0)).$$

Replacing here  $\tilde{\varphi}_{ni}(a-0)$  and  $\tilde{\varphi}'_{ni}(a-0)$  from (4.4.65) and (4.4.76) and taking (4.4.71) into account, we obtain

$$G_{ni} + \sum_{k=0}^{\infty} (\tilde{Q}_{ni,k0}(a) G_{k0} - \tilde{Q}_{ni,k1}(a) G_{k1}) = 0, \quad G_{ni} := \varphi'_{ni}(a+0) - a_1^{-1} \varphi'_{ni}(a-0) - a_2 \varphi_{ni}(a-0).$$

By Lemma 4.4.3 this yields  $G_{ni} = 0$ , i.e.

$$\varphi'_{ni}(a+0) = a_1^{-1} \varphi'_{ni}(a-0) + a_2 \varphi_{ni}(a-0). \quad (4.4.78)$$

It follows from (4.4.55), (4.4.71), (4.4.73), (4.4.77), (4.4.78) and the jump conditions for  $\tilde{\varphi}(x, \lambda)$  that

$$\varphi(a+0, \lambda) = a_1 \varphi(a-0, \lambda), \quad \varphi'(a+0, \lambda) = a_1^{-1} \varphi'(a-0, \lambda) + a_2 \varphi(a-0, \lambda).$$

Similarly one can calculate

$$\Phi(a+0, \lambda) = a_1 \Phi(a-0, \lambda), \quad \Phi'(a+0, \lambda) = a_1^{-1} \Phi'(a-0, \lambda) + a_2 \Phi(a-0, \lambda).$$

Thus, the functions  $\varphi(x, \lambda)$  and  $\Phi(x, \lambda)$  satisfy the jump conditions (4.4.3).

Let us now show that

$$\varphi(0, \lambda) = 1, \quad \varphi'(0, \lambda) = h, \quad U(\Phi) = 1, \quad V(\Phi) = 0. \quad (4.4.79)$$

Indeed, using (4.4.55) and (4.4.64) and acting in the same way as in the proof of Lemma 4.4.2, we get

$$\tilde{U}_d(\tilde{\varphi}(x, \lambda)) = U_d(\varphi(x, \lambda)) + \sum_{k=0}^{\infty} (\tilde{Q}_{k0}(d, \lambda) U_d(\varphi_{k0}) - \tilde{Q}_{k1}(d, \lambda) U_d(\varphi_{k1})), \quad (4.4.80)$$

$$\tilde{U}_d(\tilde{\Phi}(x, \lambda)) = U_d(\Phi(x, \lambda)) + \sum_{k=0}^{\infty} (\tilde{D}_{k0}(d, \lambda) U_d(\varphi_{k0}) - \tilde{D}_{k1}(d, \lambda) U_d(\varphi_{k1})), \quad (4.4.81)$$

where  $d = 0, T$ ;  $U_0 = U$ ,  $U_T = V$ . Since  $\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{kj}(x) \rangle|_{x=0} = 0$ , it follows from (4.4.55) and (4.4.80) that  $\varphi(0, \lambda) = 1$ ,  $U_0(\varphi) = 0$ , and consequently  $\varphi'(0, \lambda) = h$ . In (4.4.81) we put  $d = 0$ . Since  $U_0(\varphi) = 0$  we get  $U_0(\Phi) = 1$ .

Denote  $\Delta(\lambda) := -V(\varphi)$ . It follows from (4.4.80) and (4.4.81) for  $d = T$  that

$$\tilde{\Delta}(\lambda) = \Delta(\lambda) + \sum_{k=0}^{\infty} (\tilde{Q}_{k0}(T, \lambda) \Delta(\lambda_{k0}) - \tilde{Q}_{k1}(T, \lambda) \Delta(\lambda_{k1})), \quad (4.4.82)$$

$$0 = V(\Phi) - \sum_{k=0}^{\infty} \left( \tilde{D}_{k0}(T, \lambda) \Delta(\lambda_{k0}) - \tilde{D}_{k1}(T, \lambda) \Delta(\lambda_{k1}) \right). \quad (4.4.83)$$

In (4.4.82) we set  $\lambda = \lambda_{n1}$  :

$$\Delta(\lambda_{n1}) + \sum_{k=0}^{\infty} \left( \tilde{Q}_{n1,k0}(T) \Delta(\lambda_{k0}) - \tilde{Q}_{n1,k1}(T) \Delta(\lambda_{k1}) \right) = 0.$$

Since  $\tilde{Q}_{n1,k1}(T) = \delta_{nk}$  we get

$$\sum_{k=0}^{\infty} \tilde{Q}_{n1,k0}(T) \Delta(\lambda_{k0}) = 0.$$

Then, by virtue of Lemma 4.4.3,  $\Delta(\lambda_{k0}) = 0$ ,  $k \geq 0$ . Substituting this into (4.4.83) and using the relation  $\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle_{|x=T} = 0$ , we obtain  $V(\Phi) = 0$ , and (4.4.79) is proved.

We additionally proved that  $\Delta(\lambda_n) = 0$ , i.e. the numbers  $\{\lambda_n\}_{n \geq 0}$  are eigenvalues of  $L$ . It follows from (4.4.64) for  $x = 0$  that

$$M(\lambda) = \tilde{M}(\lambda) + \sum_{k=0}^{\infty} \left( \frac{1}{\alpha_{k0}(\lambda - \lambda_{k0})} - \frac{1}{\alpha_{k1}(\lambda - \lambda_{k1})} \right).$$

But according to (4.4.49),

$$\tilde{M}(\lambda) = \sum_{k=0}^{\infty} \frac{1}{\alpha_{k1}(\lambda - \lambda_{k1})}.$$

Hence

$$M(\lambda) = \sum_{k=0}^{\infty} \frac{1}{\alpha_k(\lambda - \lambda_k)}.$$

Thus, the numbers  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  are the spectral data for the constructed boundary value problem  $L$ . Theorem 4.4.5 is proved.  $\square$

**Example 4.4.1.** Take  $\tilde{L}$  such that  $\tilde{q}(x) = 0$ ,  $\tilde{h} = \tilde{H} = \tilde{a}_2 = 0$ . Let  $\{\tilde{\lambda}_n, \tilde{\alpha}_n\}_{n \geq 0}$  be the spectral data of  $\tilde{L}$ . Clearly,

$$\tilde{\lambda}_0 = 0, \quad \tilde{\alpha}_0 = a + (T - a)a_1, \quad \tilde{\varphi}_{00}(x) = 1 \ (x < a), \quad \tilde{\varphi}_{00}(x) = a_1 \ (x > a).$$

Let  $\lambda_n = \tilde{\lambda}_n$  ( $n \geq 0$ ),  $\alpha_n = \tilde{\alpha}_n$  ( $n \geq 1$ ), and let  $\alpha_0 > 0$  be an arbitrary positive number. Denote  $A := \frac{1}{\alpha_0} - \frac{1}{\tilde{\alpha}_0}$ . Then, (4.4.65) and (4.4.69) yield

$$\tilde{\varphi}_{00}(x) = \varphi_{00}(x) \left( 1 + A \int_0^x \tilde{\varphi}_{00}^2(t) dt \right), \quad \varepsilon_0(x) = A \tilde{\varphi}_{00}(x) \varphi_{00}(x),$$

and consequently,

$$\varphi_{00}(x) = \begin{cases} (1 + Ax)^{-1}, & x < a, \\ a_1(B + Aa_1^2x)^{-1}, & x > a, \end{cases} \quad \varepsilon_0(x) = \begin{cases} A(1 + Ax)^{-1}, & x < a, \\ Aa_1^2(B + Aa_1^2x)^{-1}, & x > a, \end{cases}$$

where  $B = 1 + Aa - Aa_1^2a$ . Using (4.4.70)-(4.4.72) we calculate

$$q(x) = \begin{cases} 2A^2(1 + Ax)^{-2}, & x < a, \\ 2A^2a_1^4(B + Aa_1^2x)^{-2}, & x > a, \end{cases}$$

$$h = -A, \quad H = Aa_1^2(B + Aa_1^2T)^{-1}, \quad a_2 = (a_1^{-1} - a_1^3)A(1 + Aa)^{-1}.$$

**Remarks.** 1) Analogous results are also valid for boundary value problems having an arbitrary number of discontinuities.

2) In Theorem 4.4.5, (4.4.52) is a condition on the asymptotic behaviour of the spectral data. It follows from the proof that Theorem 4.4.5 is also valid if instead of (4.4.52) we take  $\{\xi_n \rho_n\} \in \ell_2$ .

3) In order to choose a model boundary value problem  $\tilde{L}$ , one can use the asymptotics of the spectral data. We note that in applications a model boundary value problem often is given a priori. It describes a known object, and its perturbations are studied.

4) For boundary value problems without discontinuities (i.e.  $a_1 = 1, a_2 = 0$ ), Theorem 4.4.5 gives the following assertion as a corollary:

For real numbers  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  to be the spectral data for a certain boundary value problem  $L$  of the form (4.4.1)-(4.4.2), it is necessary and sufficient that  $\alpha_n > 0$ ,  $\lambda_n \neq \lambda_m$  ( $n \neq m$ ), and

$$\rho_n = \frac{\pi n}{T} + \frac{\omega}{n} + \frac{\kappa_n}{n}, \quad \alpha_n = \frac{T}{2} + \frac{\kappa_{n1}}{n}, \quad \{\kappa_n\}, \{\kappa_{n1}\} \in \ell_2.$$

5) This method also works for nonselfadjoint boundary value problems and for more general jump conditions.

## 4.5. INVERSE PROBLEMS IN ELASTICITY THEORY

The problem of determining the dimensions of the transverse cross-sections of a beam from the given frequencies of its natural vibrations is examined. Frequency spectra are indicated that determine the dimensions of the transverse cross-sections of the beam uniquely, an effective procedure is presented for solving the inverse problem, and a uniqueness theorem is proved.

**4.5.1.** Consider the differential equation describing beam vibrations in the form

$$(h^\mu(x)y'')'' = \lambda h(x)y, \quad 0 \leq x \leq T. \quad (4.5.1)$$

Here  $h(x)$  is a function characterizing the beam transverse section, and  $\mu = 1, 2, 3$  is a fixed number. We will assume that the function  $h(x)$  is absolutely continuous in the segment  $[0, T]$ , and  $h(x) > 0$ ,  $h(0) = 1$ . The inverse problem for (4.5.1) in the case  $\mu = 2$  (similar transverse sections) was investigated in [ain1] in determining small changes in the beam transverse sections from given small changes in a finite number of its natural vibration frequencies.

Let  $\{\lambda_{kj}\}_{k \geq 1}$ ,  $j = 1, 2$  be the eigenvalues of the boundary value problems  $Q_j$  for (4.5.1) with the boundary conditions

$$y(0) = y^{(j)}(0) = y(T) = y'(T) = 0.$$

The inverse problem is formulated as follows.

**Inverse Problem 4.5.1.** Find the function  $h(x)$ ,  $x \in [0, T]$  from the given spectra  $\{\lambda_{kj}\}_{k \geq 1, j=1,2}$ .

Let us show that this inverse problem can be reduced to the inverse problem of recovering the differential equation (4.5.1) from the corresponding Weyl function. Let  $\Phi(x, \lambda)$  be a solution of (4.5.1) under the conditions

$$\Phi(0, \lambda) = \Phi(T, \lambda) = \Phi'(T, \lambda) = 0, \quad \Phi'(0, \lambda) = 1.$$

We set  $M(\lambda) := \Phi''(0, \lambda)$ . The function  $M(\lambda)$  is called the Weyl function for (4.5.1). Let the functions  $C_\nu(x, \lambda)$ ,  $\nu = \overline{0, 3}$  be solutions of (4.5.1) under the initial conditions  $C_\nu^{(\mu)}(0, \lambda) = \delta_{\nu\mu}$ ,  $\nu, \mu = \overline{0, 3}$ . Denote

$$\Delta_j(\lambda) = C_{3-j}(T, \lambda)C_3'(T, \lambda) - C_3(T, \lambda)C_{3-j}'(T, \lambda), \quad j = 1, 2.$$

Then

$$\Phi(x, \lambda) = \frac{1}{\Delta_1(\lambda)} \det[C_\nu(x, \lambda), C_\nu(T, \lambda), C_\nu'(T, \lambda)]_{\nu=1,2,3},$$

and hence

$$M(\lambda) = -\frac{\Delta_2(\lambda)}{\Delta_1(\lambda)}.$$

Denote

$$\gamma(x) = \int_0^x (h(t))^{(1-\mu)/4} dt, \quad \tau = \gamma(T).$$

Let  $\lambda = \rho^4$ . By the well-known method (see, for example [nail, Ch.1]) one can obtain the following asymptotic formulae

$$\lambda_{kj} = \left(\frac{k\pi}{\tau}\right)^4 \left(1 + \frac{A_{j1}}{k} + O\left(\frac{1}{k^2}\right)\right), \quad k \rightarrow \infty, \quad (4.5.2)$$

$$\Delta_j(\lambda) = O(\rho^{j-5} \exp(C|\rho|)), \quad |\lambda| \rightarrow \infty. \quad (4.5.3)$$

For definiteness, let  $\rho \in S := \{\rho : \arg \rho \in [0, \pi/4]\}$ . Then, for  $|\lambda| \rightarrow \infty$ ,  $\rho \in S$ ,  $\arg \lambda = \varphi \neq 0$ ,

$$\Delta_j(\lambda) = \rho^{j-5} A_{j2} \exp(\rho(1-i)\tau) \left(1 + O\left(\frac{1}{\rho}\right)\right), \quad (4.5.4)$$

$$\Phi^{(\nu)}(x, \lambda) = \rho^{\nu-1} \sum_{\xi=1}^2 (R_\xi \gamma'(x))^\nu g_\xi(x) \exp(\rho R_\xi \gamma(x)) \left(1 + O\left(\frac{1}{\rho}\right)\right), \quad x \in [0, T], \quad (4.5.5)$$

where  $R_1 = -1$ ,  $R_2 = i$ ; the functions  $g_\xi(x)$  are absolutely continuous, and  $g_\xi(x) \neq 0$ ,  $g_1(0) = -g_2(0) = (-1-i)^{-1}$ . The numbers  $A_{js}$  in (4.5.2) and (4.5.4) depend on  $\tau$ . In particular, (4.5.5) yields

$$M(\lambda) = \rho(-1+i) \left(1 + O\left(\frac{1}{\rho}\right)\right).$$

**Lemma 4.5.1.** *The Weyl function  $M(\lambda)$  is uniquely defined by giving the spectra  $\{\lambda_{kj}\}_{k \geq 1, j=1,2}$ .*

*Proof.* The eigenvalues  $\{\lambda_{kj}\}_{k \geq 1, j=1,2}$  of the boundary value problems  $Q_j$  coincide with the zeros of the entire functions  $\Delta_j(\lambda)$ . Indeed, let  $\lambda^0$  be an eigenvalue and  $\psi(x)$  an eigenfunction of the boundary value problem  $Q_j$ . Then

$$\psi(x) = \sum_{\mu=0}^3 \beta_\mu C_\mu(x, \lambda^0),$$

where

$$\sum_{\mu=0}^3 \beta_{\mu} C_{\mu}^{(k)}(0, \lambda^0) = 0, \quad \sum_{\mu=0}^3 \beta_{\mu} C_{\mu}^{(s)}(T, \lambda^0), \quad k = 0, j \quad s = 0, 1.$$

Since  $\psi \neq 0$ , this linear homogeneous algebraic system has non-zero solutions and, therefore, its determinant equals zero, i.e.  $\Delta_j(\lambda^0) = 0$ . Repeating all arguments in reverse order, we obtain that if  $\Delta_j(\lambda^0) = 0$ , then  $\lambda^0$  is an eigenvalue of the boundary value problem  $Q_j$ .

It follows from (4.5.3) that the order of the entire functions  $\Delta_j(\lambda)$  is less than 1, and therefore, according to Hadamard's factorization theorem [con1, p.289]

$$\Delta_j(\lambda) = B_j \prod_{k=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_{kj}}\right), \quad (4.5.6)$$

with some constants  $B_j$ . Let us consider a positive function  $\tilde{h}(x) \in AC[0, T]$  such that  $\tilde{h}(0) = 1$ . As before we agree that if a certain symbol  $\alpha$  denotes an object related to  $h$ , then the corresponding symbol  $\tilde{\alpha}$  with tilde denotes the analogous object related to  $\tilde{h}$ , and  $\hat{\alpha} := \alpha - \tilde{\alpha}$ .

Let  $\tau = \tilde{\tau}$ . We have from (4.5.6)

$$\frac{\Delta_j(\lambda)}{\tilde{\Delta}_j(\lambda)} = \frac{B_j}{\tilde{B}_j} \prod_{k=1}^{\infty} \frac{\tilde{\lambda}_{kj}}{\lambda_{kj}} \prod_{k=1}^{\infty} \left(1 - \frac{\tilde{\lambda}_{kj} - \lambda_{kj}}{\tilde{\lambda}_{kj} - \lambda}\right).$$

By virtue of (4.5.2) and (4.5.4), we have for  $|\lambda| \rightarrow \infty$ ,  $\arg \lambda = \varphi \neq 0$ ,  $\rho \in S$ ,

$$\frac{\Delta_j(\lambda)}{\tilde{\Delta}_j(\lambda)} \rightarrow 1, \quad \prod_{k=1}^{\infty} \left(1 - \frac{\tilde{\lambda}_{kj} - \lambda_{kj}}{\tilde{\lambda}_{kj} - \lambda}\right) \rightarrow 1.$$

This yields

$$\frac{B_j}{\tilde{B}_j} \prod_{k=1}^{\infty} \frac{\tilde{\lambda}_{kj}}{\lambda_{kj}} = 1,$$

and therefore,

$$M(\lambda) = \tilde{B} \prod_{k=1}^{\infty} \frac{\tilde{\lambda}_{k1}}{\tilde{\lambda}_{k2}} \cdot \frac{\lambda_{k2} - \lambda}{\lambda_{k1} - \lambda}, \quad \tilde{B} = -\frac{\tilde{B}_2}{\tilde{B}_1}.$$

Hence the specification of the spectra  $\{\lambda_{kj}\}_{k \geq 1, j=1,2}$  uniquely determines the Weyl function  $M(\lambda)$ .  $\square$

Thus, Inverse Problem 4.5.1 is reduced to the following inverse problem.

**Inverse Problem 4.5.2.** Given the Weyl function  $M(\lambda)$ , find  $h(x)$ ,  $x \in [0, T]$ .

**4.5.2.** We shall solve Inverse Problem 4.5.2 by the method of standard models. First we prove several auxiliary assertions.

**Lemma 4.5.2.** Let  $p(x) := h^{\mu}(x)$ . The following relationship holds

$$\int_0^T \left( \hat{h}(x) \lambda \Phi(x, \lambda) \tilde{\Phi}(x, \lambda) - \hat{p}(x) \Phi''(x, \lambda) \tilde{\Phi}''(x, \lambda) \right) dx = \hat{M}(\lambda). \quad (4.5.7)$$

*Proof.* Denote

$$\ell_{\lambda} y = (p(x) y'')'' - \lambda h(x) y, \quad L(y, z) = (p(x) y'')' z - p(x) y'' z' + p(x) y' z'' - y(p(x) z'')'.$$

Then

$$\int_0^T \ell_\lambda y(x) z(x) dx = L(y(x), z(x)) \Big|_0^T + \int_0^T y(x) \ell_\lambda z(x) dx. \quad (4.5.8)$$

Using (4.5.8) and the equality  $\ell_\lambda \Phi(x, \lambda) = \tilde{\ell}_\lambda \tilde{\Phi}(x, \lambda) = 0$ , we obtain

$$\begin{aligned} \int_0^T (\ell_\lambda - \tilde{\ell}_\lambda) \Phi(x, \lambda) \tilde{\Phi}(x, \lambda) dx &= - \int_0^T \tilde{\ell}_\lambda \Phi(x, \lambda) \tilde{\Phi}(x, \lambda) dx \\ &= -\tilde{L}(\Phi(x, \lambda), \tilde{\Phi}(x, \lambda)) \Big|_0^T - \int_0^T \Phi(x, \lambda) \tilde{\ell}_\lambda \tilde{\Phi}(x, \lambda) dx \\ &= \Phi'(0, \lambda) \tilde{\Phi}''(0, \lambda) - \Phi''(0, \lambda) \tilde{\Phi}'(0, \lambda) = -\hat{M}(\lambda). \end{aligned}$$

On the other hand, integrating by parts we have

$$\begin{aligned} \int_0^T (\ell_\lambda - \tilde{\ell}_\lambda) \Phi(x, \lambda) \tilde{\Phi}(x, \lambda) dx &= \int_0^T ((\hat{p}(x) \Phi''(x, \lambda))'' - \lambda \hat{h}(x) \Phi(x, \lambda)) \tilde{\Phi}(x, \lambda) dx \\ &= ((\hat{p}(x) \Phi''(x, \lambda))' \tilde{\Phi}(x, \lambda) - \hat{p}(x) \Phi''(x, \lambda) \tilde{\Phi}'(x, \lambda)) \Big|_0^T \\ &\quad + \int_0^T (\hat{p}(x) \Phi''(x, \lambda) \tilde{\Phi}''(x, \lambda) - \lambda \hat{h}(x) \Phi(x, \lambda) \tilde{\Phi}(x, \lambda)) dx. \end{aligned}$$

Since the substitution vanishes, we hence obtain (4.5.7).  $\square$

**Lemma 4.5.3.** *Consider the integral*

$$J(z) = \int_0^T f(x) H(x, z) dx, \quad (4.5.9)$$

where

$$f(x) \in C[0, T], \quad f(x) \sim f_\alpha \frac{x^\alpha}{\alpha!} \quad (x \rightarrow +0), \quad H(x, z) = \exp(-za(x)) \left(1 + \frac{\xi(x, z)}{z}\right),$$

$$a(x) \in C^1[0, T], \quad 0 < a(x_1) < a(x_2) \quad (0 < x_1 < x_2),$$

$$a^{(\nu)}(x) \sim a_0 x^{1-\nu} \quad (x \rightarrow +0, \nu = 0, 1), \quad a'(x) > 0,$$

the function  $\xi(x, z)$  is continuous and bounded for  $x \in [0, T]$ ,

$$z \in Q := \left\{ z : \arg z \in \left[ -\frac{\pi}{2} + \delta_0, \frac{\pi}{2} - \delta_0 \right], \delta_0 > 0 \right\}.$$

Then as  $z \rightarrow \infty$ ,  $z \in Q$ ,

$$J(z) \sim f_\alpha (a_0 z)^{-\alpha-1}. \quad (4.5.10)$$

*Proof.* The function  $t = a(x)$  has the inverse  $x = b(t)$ , where  $b(t) \in C^1[0, T_1]$ ,  $T_1 = a(T)$ ,  $b(t) > 0$  for  $t > 0$ , and  $b^{(\nu)}(t) \sim (a_0)^{-1} t^{1-\nu}$ ,  $\nu = 0, 1$  as  $t \rightarrow +0$ . Let us make the change of variable  $t = a(x)$  in the integral in (4.5.9). We obtain

$$J(z) = \int_0^T g(t) \exp(-zt) (1 + z^{-1} \xi(b(t), z)) dt, \quad (4.5.11)$$

where  $g(t) = f(b(t))b'(t)$ . It is clear that for  $t \rightarrow +0$ ,

$$g(t) \sim (a_0)^{-1-\alpha} f_\alpha \frac{t^\alpha}{\alpha!}.$$



Applying Lemma 1.7.1 to (4.5.11), we obtain (4.5.10).  $\square$

Denote

$$A_\alpha = \frac{1}{(R_1 - R_2)^2} \sum_{\xi, s=1}^2 \frac{(-1)^{\xi+s} (1 - \mu R_\xi^2 R_s^2)}{(R_\xi + R_s)^{\alpha+1}}, \quad \alpha \geq 1.$$

Let us show that  $A_\alpha \neq 0$  for all  $\alpha \geq 1$ . For definiteness, we put  $\arg \rho \in (0, \frac{\pi}{4})$ , i.e.  $R_1 = -1$ ,  $R_2 = i$ . Then

$$A_\alpha = -\frac{1}{(R_1 - R_2)^2} \cdot \frac{1}{(-2)^{\alpha+1}} a_\alpha,$$

where

$$a_\alpha = (\mu - 1)(1 + i^{\alpha+1}) + 2(\mu + 1)(1 + i)^{\alpha+1}.$$

Since  $|1 + i^{\alpha+1}| \leq 2$  and  $|1 + i|^{\alpha+1} = (\sqrt{2})^{\alpha+1}$ , it follows that  $a_\alpha \neq 0$  for all  $\alpha \geq 1$ . Hence  $A_\alpha \neq 0$  for all  $\alpha \geq 1$ .  $\square$

**Lemma 4.5.3.** *As  $x \rightarrow 0$  let  $\hat{h}(x) \sim \hat{h}_\alpha (\alpha!)^{-1} x^\alpha$ . Then as  $|\lambda| \rightarrow \infty$ ,  $\arg \lambda = \varphi \neq 0$ , there exists a finite limit  $I_\alpha = \lim \rho^{\alpha-1} \hat{M}(\lambda)$ , and*

$$A_\alpha \hat{h}_\alpha = I_\alpha. \quad (4.5.12)$$

*Proof.* Since  $p(x) = h^\mu(x)$ , then by virtue of the conditions of the lemma we have

$$\hat{p}(x) \sim \mu \hat{h}_\alpha \frac{x^\alpha}{\alpha!}, \quad x \rightarrow +0.$$

Using the asymptotic formulae (4.5.5) and Lemma 4.5.3 we find as  $\lambda \rightarrow \infty$ ,  $\arg \lambda = \varphi \neq 0$ ,  $\rho \in S$ :

$$\begin{aligned} \int_0^T \hat{h}(x) \lambda \Phi(x, \lambda) \tilde{\Phi}(x, \lambda) dx &\sim \frac{1}{\rho^{\alpha-1}} \cdot \frac{\hat{h}_\alpha}{(R_1 - R_2)^2} \sum_{\xi, s=1}^2 \frac{(-1)^{\xi+s}}{(R_\xi + R_s)^{\alpha+1}}, \\ \int_0^T \hat{p}(x) \Phi''(x, \lambda) \tilde{\Phi}''(x, \lambda) dx &\sim \frac{1}{\rho^{\alpha-1}} \cdot \frac{\mu \hat{h}_\alpha}{(R_1 - R_2)^2} \sum_{\xi, s=1}^2 \frac{(-1)^{\xi+s} R_\xi^2 R_s^2}{(R_\xi + R_s)^{\alpha+1}}. \end{aligned}$$

Substituting the expressions obtained in (4.5.7), we obtain the assertion of the lemma.  $\square$

Let  $A$  be the set of analytic functions on  $[0, T]$ . From the facts presented above we have the following theorem.

**Theorem 4.5.1.** *Inverse Problem 4.5.2 has a unique solution in the class  $h(x) \in A$ . This solution can be found according to the following algorithm:*

1) We calculate  $h_\alpha = h^{(\alpha)}(0)$ ,  $\alpha \geq 0$ ,  $h_0 = 1$ . For this purpose we successively perform the following operations for  $\alpha = 1, 2, \dots$ : we construct the function  $\tilde{h}(x) \in A$ ,  $\tilde{h}(x) > 0$  such that  $\tilde{h}^{(\nu)}(0) = h_\nu$ ,  $\nu = 0, \alpha - 1$ , and arbitrary in the rest, and we calculate  $h_\alpha$  by (4.5.12).

We construct the function  $h(x)$  by the formula

$$h(x) = \sum_{\alpha=0}^{\infty} h_\alpha \frac{x^\alpha}{\alpha!}, \quad 0 < x < R,$$

where

$$R = \left( \lim_{\alpha \rightarrow \infty} \sqrt[\alpha]{\frac{|h_\alpha|}{\alpha!}} \right)^{-1}.$$

If  $R < T$ , then for  $R < x < T$  the function  $h(x)$  is constructed by analytic continuation.

We note that the inverse problem in the class of piecewise-analytic function can be solved in an analogous manner.

#### 4.6. BOUNDARY VALUE PROBLEMS WITH AFTEREFFECT

In this section, the perturbation of the Sturm-Liouville operator by a Volterra integral operator is considered. The presence of an "aftereffect" in a mathematical model produces qualitative changes in the study of the inverse problem. The main results of the section are expressed by Theorems 4.6.1-4.6.3. We use here several methods which were described in Chapter 1. In order to prove the uniqueness theorem (Theorem 4.6.1) the transformation operator method is applied. With the help of the method of standard models we get an algorithm for the solution of the inverse problem (Theorem 4.6.2), and the Borg method is used for the local solvability of the inverse problem and for the study the stability of the solution (Theorem 4.6.3).

**4.6.1.** Let  $\{\lambda_n\}_{n \geq 1}$  be the eigenvalues of a boundary value problem  $L = L(q, M)$  of the form

$$\ell y(x) \equiv -y''(x) + q(x)y(x) + \int_0^x M(x-t)y(t) dt = \lambda y(x) = \rho^2 y(x), \quad (4.6.1)$$

$$y(0) = y(\pi) = 0. \quad (4.6.2)$$

Considered the following problem:

**Inverse Problem 4.6.1.** Given the function  $q(x)$  and the spectrum  $\{\lambda_n\}_{n \geq 1}$ , find the function  $M(x)$ .

Put

$$M_0(x) = (\pi - x)M(x), \quad M_1(x) = \int_0^x M(t) dt, \quad Q(x) = M_0(x) - M_1(x).$$

We shall assume that  $q(x), Q(x) \in L_2(0, \pi)$ ,  $M_k(x) \in L(0, \pi)$ ,  $k = 0, 1$ .

Let  $S(x, \lambda)$  be the solution of (4.6.1) under the initial conditions  $S(0, \lambda) = 0$ ,  $S'(0, \lambda) = 1$ .

**Lemma 4.6.1.** The representation

$$S(x, \lambda) = \frac{\sin \rho x}{\rho} + \int_0^x K(x, t) \frac{\sin \rho t}{\rho} dt, \quad (4.6.3)$$

holds, where  $K(x, t)$  it is a continuous function, and  $K(x, 0) = 0$ .

*Proof.* The function  $S(x, \lambda)$  is the solution of the integral equation

$$S(x, \lambda) = \frac{\sin \rho x}{\rho} + \int_0^x \frac{\sin \rho(x-\tau)}{\rho} \left( q(\tau)S(\tau, \lambda) + \int_0^\tau M(\tau-s)S(s, \lambda) ds \right) d\tau. \quad (4.6.4)$$

Since

$$\int_0^x \frac{\sin \rho(x-\tau)}{\rho} f(\tau) d\tau = \int_0^x \left( \int_0^t f(\tau) \cos \rho(t-\tau) d\tau \right) dt,$$

then (4.6.4) is transformable to the form

$$S(x, \lambda) = \frac{\sin \rho x}{\rho} + \int_0^x \int_0^t \left( q(\tau) S(\tau, \lambda) + \int_0^\tau M(\tau-s) S(s, \lambda) ds \right) \cos \rho(t-\tau) d\tau dt. \quad (4.6.5)$$

Apply the method of successive approximations to solve the equation (4.6.5):

$$S_0(x, \lambda) = \frac{\sin \rho x}{\rho},$$

$$S_{n+1}(x, \lambda) = \int_0^x \int_0^t \left( q(\tau) S_n(\tau, \lambda) + \int_0^\tau M(\tau-s) S_n(s, \lambda) ds \right) \cos \rho(t-\tau) d\tau dt.$$

Transform  $S_1(x, \lambda)$  :

$$\begin{aligned} S_1(x, \lambda) &= \frac{1}{2\rho} \int_0^x \sin \rho t \left( \int_0^t q(\tau) d\tau \right) dt + \frac{1}{2\rho} \int_0^x \left( \int_0^t q(\tau) \sin \rho(2\tau-t) d\tau \right) dt \\ &+ \frac{1}{2\rho} \int_0^x \int_0^t \int_0^\tau M(\tau-s) \sin \rho(s+t-\tau) ds d\tau dt + \frac{1}{2\rho} \int_0^x \int_0^t \int_0^\tau M(\tau-s) \sin \rho(s-t+\tau) ds d\tau dt. \end{aligned}$$

Carrying out the change of variables  $\xi = 2\tau - t$ ,  $\xi = s + t - \tau$ ,  $\xi = s - t + \tau$ , respectively, in the last three integrals and reversing the order of integration, we obtain

$$S_1(x, \lambda) = \int_0^x K_1(x, \xi) \frac{\sin \rho \xi}{\rho} d\xi,$$

where

$$\begin{aligned} K_1(x, \xi) &= \frac{1}{2} \int_0^\xi q(\tau) d\tau + \frac{1}{4} \int_\xi^x \left( q\left(\frac{\xi+t}{2}\right) - q\left(\frac{t-\xi}{2}\right) \right) dt \\ &+ \frac{1}{2} \int_\xi^x \left( \xi M(t-\xi) + \int_{\frac{\xi+t}{2}}^t M(2\tau-\xi-t) d\tau - \int_{\frac{t-\xi}{2}}^{t-\xi} M(2\tau+\xi-t) d\tau \right) dt. \end{aligned} \quad (4.6.6)$$

Clearly  $K_1(x, 0) = 0$ . In an analogous manner we compute

$$S_n(x, \lambda) = \int_0^x K_n(x, \xi) \frac{\sin \rho \xi}{\rho} d\xi,$$

where the functions  $K_n(x, \xi)$  are determined by the recurrence formula

$$\begin{aligned} K_{n+1}(x, \xi) &= \frac{1}{2} \int_\xi^x \left( \int_{t-\xi}^t q(\tau) K_n(\tau, \xi + \tau - t) d\tau + \int_{\frac{\xi+t}{2}}^t q(\tau) K_n(\tau, \xi + t - \tau) d\tau \right. \\ &- \left. \int_{\frac{t-\xi}{2}}^{t-\xi} q(\tau) K_n(\tau, -\xi + t - \tau) d\tau + \int_{t-\xi}^t \left( \int_{\xi-t+\tau}^\tau M(\tau-s) K_n(s, \xi - t + \tau) ds \right) d\tau \right. \\ &+ \left. \int_{\frac{t+\xi}{2}}^\xi \left( \int_{\xi+t-\tau}^\tau M(\tau-s) K_n(s, \xi + t - \tau) ds \right) d\tau \right. \\ &- \left. \int_{\frac{t-\xi}{2}}^{t-\xi} \left( \int_{-\xi+t-\tau}^\tau M(\tau-s) K_n(s, -\xi + t - \tau) ds \right) d\tau \right) dt. \end{aligned} \quad (4.6.7)$$

Clearly,  $K_n(x, 0) = 0$ . From (4.6.6) and (4.6.7) we obtain by induction the estimate

$$|K_n(x, \xi)| \leq \frac{1}{n!}(Cx)^n, \quad 0 \leq \xi \leq x \leq \pi.$$

Thus,

$$S(x, \lambda) = \sum_{n=0}^{\infty} S_n(x, \lambda) = \frac{\sin \rho x}{\rho} + \int_0^x K(x, \xi) \frac{\sin \rho \xi}{\rho} d\xi,$$

where

$$K(x, \xi) = \sum_{n=1}^{\infty} K_n(x, \xi), \quad (4.6.8)$$

and the series (4.6.8) converges absolutely and uniformly for  $0 \leq \xi \leq x \leq \pi$ . Lemma 4.6.1 is proved.  $\square$

Denote  $\Delta(\lambda) = S(\pi, \lambda)$ . The eigenvalues  $\{\lambda_n\}_{n \geq 1}$  of the boundary value problem  $L$  coincide with the zeros of the function  $\Delta(\lambda)$ , and like in the proof of Theorem 1.1.3 we get for  $n \rightarrow \infty$ ,

$$\rho_n = \sqrt{\lambda_n} = n + \frac{A_1}{n} + \frac{\kappa_n}{n}, \quad \{\kappa_n\} \in \ell_2, \quad A_1 = \frac{1}{2\pi} \int_0^\pi q(t) dt. \quad (4.6.9)$$

The following assertion can be proved like Theorem 1.1.4.

**Lemma 4.6.2.** *The function  $\Delta(\lambda)$  is uniquely determined by its zeros. Moreover,*

$$\Delta(\lambda) = \pi \prod_{n=1}^{\infty} \frac{\lambda_n - \lambda}{n^2}. \quad (4.6.10)$$

We shall now prove the uniqueness theorem for the solution of Inverse Problem 4.6.1. Let  $\{\tilde{\lambda}_n\}_{n \geq 1}$  be the eigenvalues of the boundary value problem  $\tilde{L} = L(q, \tilde{M})$ .

**Theorem 4.6.1.** *If  $\lambda_n = \tilde{\lambda}_n$ ,  $n \geq 1$ , then  $M(x) = \tilde{M}(x)$ ,  $x \in (0, \pi)$ .*

*Proof.* Let the function  $S^*(x, \lambda)$  be the solution of the equation

$$\ell^* z(x) := -z''(x) + q(x)z(x) + \int_x^\pi M(t-x)z(t) dt = \lambda z(x) \quad (4.6.11)$$

under the conditions  $S^*(\pi, \lambda) = 0$ ,  $S^{*'}(\pi, \lambda) = -1$ . Put  $\Delta^*(\lambda) = S^*(0, \lambda)$ , and denote  $\hat{M}(x) := M(x) - \tilde{M}(x)$ . Then

$$\begin{aligned} & \int_0^\pi S^*(x, \lambda) \int_0^x \hat{M}(x-t) \tilde{S}(t, \lambda) dt dx \\ &= \int_0^\pi S^*(x, \lambda) \ell \tilde{S}(x, \lambda) dx - \int_0^\pi S^*(x, \lambda) \tilde{\ell} \tilde{S}(x, \lambda) dx \\ &= \int_0^\pi \ell^* S^*(x, \lambda) \tilde{S}(x, \lambda) dx - \int_0^\pi S^*(x, \lambda) \tilde{\ell} \tilde{S}(x, \lambda) dx \\ &+ \left( \tilde{S}(x, \lambda) S^{*'}(x, \lambda) - \tilde{S}'(x, \lambda) S^*(x, \lambda) \right) \Big|_0^\pi = \Delta^*(\lambda) - \tilde{\Delta}(\lambda). \end{aligned}$$

For  $\tilde{\ell} = \ell$  we have  $\Delta^*(\lambda) \equiv \Delta(\lambda)$ , and consequently

$$\int_0^\pi S^*(x, \lambda) \int_0^x \hat{M}(x-t) \tilde{S}(t, \lambda) dt dx = \hat{\Delta}(\lambda). \quad (4.6.12)$$

Transform (4.6.12) into

$$\int_0^\pi \hat{M}(x) \left( \int_x^\pi S^*(t, \lambda) \tilde{S}(t-x, \lambda) dt \right) dx = \hat{\Delta}(\lambda). \quad (4.6.13)$$

Denote  $w(x, \lambda) = S^*(\pi-x, \lambda)$ ,  $\hat{N}(x) = \hat{M}(\pi-x)$ ,

$$\varphi(x, \lambda) = \int_0^x w(t, \lambda) \tilde{S}(t-x, \lambda) dt. \quad (4.6.14)$$

Then (4.6.13) takes the form

$$\int_0^\pi \hat{N}(x) \varphi(x, \lambda) dx = \hat{\Delta}(\lambda). \quad (4.6.15)$$

**Lemma 4.6.3.** *The representation*

$$\varphi(x, \lambda) = \frac{1}{2\rho^2} \left( -x \cos \rho x + \int_0^x V(x, t) \cos \rho t dt \right) \quad (4.6.16)$$

holds, where  $V(x, t)$  is a continuous function.

*Proof.* Since  $w(x, \lambda) = S^*(\pi-x, \lambda)$ , the function  $w(x, \lambda)$  is the solution of the Cauchy problem

$$\begin{aligned} -w''(x, \lambda) + q(\pi-x)w(x, \lambda) + \int_0^x M(x-t)w(t, \lambda) dt &= \lambda w(x, \lambda), \\ w(0, \lambda) &= 0, \quad w'(0, \lambda) = 1. \end{aligned}$$

Therefore, by Lemma 4.6.1, the representation

$$w(x, \lambda) = \frac{\sin \rho x}{\rho} + \int_0^x K^0(x, t) \frac{\sin \rho t}{\rho} dt \quad (4.6.17)$$

holds, where  $K^0(x, t)$  is a continuous function. Substituting (4.6.3) and (4.6.17) into (4.6.14), we obtain

$$\varphi(x, \lambda) = \varphi_1(x, \lambda) + \varphi_2(x, \lambda) + \varphi_3(x, \lambda) + \varphi_4(x, \lambda),$$

where

$$\begin{aligned} \varphi_1(x, \lambda) &= \frac{1}{\rho^2} \int_0^x \sin \rho t \sin \rho(x-t) dt, \\ \varphi_2(x, \lambda) &= \frac{1}{\rho^2} \int_0^x \sin \rho(x-t) \left( \int_0^t K^0(t, \xi) \sin \rho \xi d\xi \right) dt, \\ \varphi_3(x, \lambda) &= \frac{1}{\rho^2} \int_0^x \sin \rho t \left( \int_0^{x-t} \tilde{K}(x-t, \eta) \sin \rho \eta d\eta \right) dt, \\ \varphi_4(x, \lambda) &= \frac{1}{\rho^2} \int_0^x \int_0^t K^0(t, \xi) \sin \rho \xi \int_0^{x-t} \tilde{K}(x-t, \eta) \sin \rho \eta d\eta d\xi dt. \end{aligned}$$

For  $\varphi_1(x, \lambda)$  we have

$$\varphi_1(x, \lambda) = \frac{1}{2\rho^2} \int_0^x (\cos \rho(x-2t) - \cos \rho x) dt = \frac{1}{2\rho^2} \left( -x \cos \rho x + \int_0^x \cos \rho t dt \right).$$

Transform  $\varphi_2(x, \lambda)$  :

$$\begin{aligned} \varphi_2(x, \lambda) &= \frac{1}{2\rho^2} \int_0^x \int_0^t K^0(t, \xi) (\cos \rho(x-t-\xi) - \cos \rho(x-t+\xi)) d\xi dt \\ &= \frac{1}{2\rho^2} \int_0^x \left( \int_{x-2t}^{x-t} K^0(t, x-t-s) \cos \rho s ds - \int_{x-t}^x K^0(t, s+t-x) \cos \rho s ds \right) dt. \end{aligned}$$

Changing the order of integration, we obtain

$$\varphi_2(x, \lambda) = \frac{1}{2\rho^2} \int_0^x V_2(x, t) \cos \rho t dt,$$

where

$$V_2(x, t) = \int_{\frac{x-t}{2}}^{x-t} K^0(s, x-t-s) ds + \int_{\frac{x+t}{2}}^x K^0(s, x+t-s) ds - \int_{x-t}^x K^0(s, t+s-x) ds.$$

The integrals  $\varphi_3(x, \lambda)$  and  $\varphi_4(x, \lambda)$  are transformable in an analogous manner. Lemma 4.6.3 is proved.  $\square$

Let us return to proving Theorem 4.6.1. Since  $\lambda_n = \tilde{\lambda}_n$ ,  $n \geq 1$ , we have by Lemma 4.6.2

$$\Delta(\lambda) \equiv \tilde{\Delta}(\lambda).$$

Then, substituting (4.6.16) into (4.6.15), we obtain

$$\int_0^\pi \cos \rho x \left( -x \hat{N}(x) + \int_x^\pi V(t, x) \hat{N}(t) dt \right) dx \equiv 0,$$

and consequently,

$$-x \hat{N}(x) + \int_x^\pi V(t, x) \hat{N}(t) dt = 0. \quad (4.6.18)$$

For each fixed  $\varepsilon > 0$ , (4.6.18) is a homogeneous Volterra integral equation of the second kind in the interval  $(\varepsilon, \pi)$ . Consequently,  $\hat{N}(x) = 0$  a.e. in  $(\varepsilon, \pi)$  and, since  $\varepsilon$  is arbitrary, this holds in the whole interval  $(0, \pi)$ . Thus,  $M(x) = \tilde{M}(x)$  a.e. in  $(0, \pi)$ .  $\square$

**4.6.2.** Relation (4.6.15) also makes it possible to obtain an algorithm for solving Inverse Problem 4.6.1 in the case when  $M(x) \in PA[0, \pi]$  (i.e.  $M$  is piecewise analytic in  $[0, \pi]$ ). Indeed, consider the boundary value problems  $L(q, M)$  and  $L(q, \tilde{M})$ , and assume that  $q(x) \in L_2(0, \pi)$ ;  $M(x), \tilde{M}(x) \in PA[0, \pi]$ . Let for some fixed  $a > 0$

$$\hat{N}(x) = 0, \quad x \in (a, \pi); \quad \hat{N}(x) \sim \hat{N}_\alpha^a(\alpha!)^{-1}(a-x)^\alpha, \quad x \rightarrow a-0. \quad (4.6.19)$$

It follows from (4.6.16) that as  $|\rho| \rightarrow \infty$ ,  $\arg \rho \in [\delta, \pi - \delta]$ ,  $x \in [\varepsilon, \pi]$ ,  $\delta > 0$ ,  $\varepsilon > 0$ , the asymptotic formula

$$\varphi(x, \lambda) = -\frac{x}{4\rho^2} \exp(-i\rho x) \left( 1 + O\left(\frac{1}{\rho}\right) \right) \quad (4.6.20)$$

holds. Furthermore, we infer from (4.6.16) that

$$|\varphi(x, \lambda)| < C|\rho|^{-2} \exp(-i\rho x), \quad x \in [0, \pi], \quad \operatorname{Im} \rho \geq 0. \quad (4.6.21)$$

With (4.6.21) we obtain the estimate

$$\left| \int_0^\varepsilon \hat{N}(x) \varphi(x, \lambda) dx \right| < C|\rho|^{-2} \exp(-i\rho \varepsilon), \quad \operatorname{Im} \rho \geq 0. \quad (4.6.22)$$

Using (4.6.19), (4.6.20) and Lemma 1.7.1, we get for  $|\rho| \rightarrow \infty$ ,

$$\int_\varepsilon^a \hat{N}(x) \varphi(x, \lambda) dx = \frac{a}{4(-i\rho)^{\alpha+3}} \exp(-i\rho a) (\hat{N}_\alpha^a + o(1)). \quad (4.6.23)$$

Since  $\hat{N}(x) = 0$  for  $x \in (a, \pi)$ , it follows from (4.6.15), (4.6.22) and (4.6.23) that for  $|\rho| \rightarrow \infty$ ,  $\arg \rho \in [\delta, \pi - \delta]$ ,

$$\hat{\Delta}(\lambda) = \frac{a}{4} (-i\rho)^{-\alpha-3} \exp(-i\rho a) (\hat{N}_\alpha^a + o(1)),$$

and consequently

$$\hat{N}_\alpha^a = \frac{4}{a} \lim \hat{\Delta}(\lambda) (-i\rho)^{\alpha+3} \exp(i\rho a), \quad |\rho| \rightarrow \infty, \quad \arg \rho \in [\delta, \pi - \delta]. \quad (4.6.24)$$

Thus we have proved the following theorem.

**Theorem 4.6.2.** *Let  $\{\lambda_n\}_{n \geq 1}$  be the eigenvalues of  $L(q, M)$ , where  $q(x) \in L_2(0, \pi)$ ,  $M(x) \in PA[0, \pi]$ . Then the solution of Inverse Problem 4.6.1 can be found by the following algorithm:*

- 1) From  $\{\lambda_n\}_{n \geq 1}$  construct the function  $\Delta(\lambda)$  by (4.6.10).
- 2) Take  $a = \pi$ .
- 3) For  $\alpha = 0, 1, 2, \dots$  carry out successively the following operations: construct a function  $\tilde{M}(x) \in PA[0, \pi]$  such that  $\hat{N}(x) = 0$ ,  $x \in (a, \pi)$ ;  $\hat{N}^{(k)}(a - 0) = 0$ ,  $k = \overline{0, \alpha - 1}$  and find  $N_\alpha^a = (-1)^\alpha N^{(\alpha)}(a - 0)$  from (4.6.24).
- 4) Construct  $N(x)$  for  $x \in (a^+, a)$  by the formula

$$N(x) = \sum_{\alpha=0}^{\infty} N_\alpha^a \frac{(a-x)^\alpha}{\alpha!}.$$

- 5) If  $a^+ > 0$  set  $a := a^+$  and go to step 3.

**4.6.3.** We shall now investigate local solvability of Inverse Problem 4.6.1 and its stability. First let us prove an auxiliary assertion.

**Lemma 4.6.4.** *Consider in a Banach space  $B$  the nonlinear equation*

$$r = f + \sum_{j=2}^{\infty} \psi_j(r), \quad (4.6.25)$$

where

$$\|\psi_j(r)\| \leq (C\|r\|)^j, \quad \|\psi_j(r) - \psi_j(r^*)\| \leq \|r - r^*\| (C \max(\|r\|, \|r^*\|))^{j-1}.$$

There exists  $\delta > 0$  such that if  $\|f\| < \delta$ , then in the ball  $\|r\| < 2\delta$  equation (4.6.25) has a unique solution  $r \in B$ , for which  $\|r\| \leq 2\|f\|$ .

*Proof.* Assume that  $C \geq 1$ . Put

$$\psi(r) = \sum_{j=2}^{\infty} \psi_j(r), \quad C_0 = 2C^2, \quad \delta = \frac{1}{4C_0}.$$

If  $\|r\|, \|r^*\| \leq (2C_0)^{-1}$ , then

$$\left. \begin{aligned} \|\psi(r)\| &\leq \sum_{j=2}^{\infty} (C\|r\|)^j \leq C_0\|r\|^2 \leq \frac{1}{2}\|r\|, \\ \|\psi(r) - \psi(r^*)\| &\leq \|r - r^*\| \sum_{j+2}^{\infty} (C(2C_0)^{-1})^{j-1} \leq \frac{1}{2}\|r - r^*\|. \end{aligned} \right\} \quad (4.6.26)$$

Let  $\|f\| \leq \delta$ ; construct

$$r_0 = f, \quad r_{k+1} = f + \psi(r_k), \quad k \geq 0.$$

By induction, using (4.6.26), we obtain the estimates

$$\|r_k\| \leq 2\|f\|, \quad \|r_{k+1} - r_k\| \leq \frac{1}{2^{k+1}}\|f\|, \quad k \geq 0.$$

Consequently, the series

$$r = r_0 + \sum_{k=0}^{\infty} (r_{k+1} - r_k)$$

converges to the solution of (4.6.25), and  $\|r\| \leq 2\|f\|$ . Lemma 4.6.4 is proved.  $\square$

**Theorem 4.6.3.** For the boundary value problem  $L = L(q, M)$  with the spectrum  $\{\lambda_n\}_{n \geq 1}$  there exists  $\delta > 0$  (which depends on  $L$ ) such that if numbers  $\{\tilde{\lambda}_n\}_{n \geq 1}$  satisfy the condition

$$\Lambda := \left( \sum_{n=1}^{\infty} |\lambda_n - \tilde{\lambda}_n|^2 \right)^{1/2} < \delta,$$

then there exists a unique  $\tilde{L} = L(q, \tilde{M})$  for which the numbers  $\{\tilde{\lambda}_n\}_{n \geq 1}$  are the eigenvalues, and, furthermore,

$$\|Q(x) - \tilde{Q}(x)\|_{L_2(0, \pi)} \leq C\Lambda,$$

$$\|M_k(x) - \tilde{M}_k(x)\|_{L(0, \pi)} \leq C\Lambda, \quad k = 0, 1.$$

Here and below,  $C$  denotes various constants depending on  $L$ .

*Proof.* For brevity, we confine ourselves to the case when all eigenvalues are simple. The Cauchy problem  $\ell y(x) - \lambda y(x) + f(x) = 0$ ,  $y(0) = y'(0) = 0$  has a unique solution

$$y(x) = \int_0^x g(x, t, \lambda) f(t) dt,$$

where  $g(x, t, \lambda)$  is the Green function satisfying the relations

$$-g_{xx}(x, t, \lambda) + q(x)g(x, t, \lambda) - \lambda g(x, t, \lambda) + \int_t^x M(x - \tau)g(\tau, t, \lambda) d\tau = 0, \quad x > t,$$



$$g(t, t, \lambda) = 0, \quad g_x(x, t, \lambda)|_{x=t} = 1.$$

Denote

$$\begin{aligned} G(x, t, \lambda) &= g_t(x, t, \lambda), \quad y_n(x) = S(x, \tilde{\lambda}_n), \quad \varepsilon_n = n^2 \Delta(\tilde{\lambda}_n), \\ v_n(x, t) &= \begin{cases} w'(\pi - x - t, \tilde{\lambda}_n), & 0 < t < \pi - x \\ 0, & \pi - x < t < \pi, \end{cases} \\ G_n(x, t, s) &= \begin{cases} G(x, s + t, \tilde{\lambda}_n), & s + t \leq x \\ 0, & s + t > x, \end{cases} \\ \varphi_n(x) &= \int_0^x w(t, \tilde{\lambda}_n) S(x - t, \tilde{\lambda}_n) dt, \quad \xi_n(x) = \int_0^\pi v_n(x, t) y_n(t) dt, \\ \psi_n(x) &= \frac{n}{x} \varphi'_n(x), \quad \psi_{n0}(x) = \frac{n}{2\tilde{\rho}_n} \sin \tilde{\rho}_n x, \quad \eta_n(x) = \frac{n}{\pi - x} \overline{\xi_n(x)}. \end{aligned}$$

Let here  $W_2^1$  be the space of functions  $f(x)$  absolutely continuous on  $[0, \pi]$  and such that  $f'(x) \in L_2(0, \pi)$ , with the norm  $\|f\|_{W_2^1} = \|f\|_{L_2(0, \pi)} + \|f'\|_{L_2(0, \pi)}$ , and let  $W_{20}^1 = \{f(x) : f(x) \in W_2^1, f(0) = f(\pi) = 0\}$ .

**Lemma 4.6.5.** *The functions  $\{\psi_n(x)\}_{n \geq 1}$  constitute a Riesz basis in  $L_2(0, \pi)$ , and the biorthogonal basis  $\{\psi_n^*(x)\}_{n \geq 1}$  possesses the following properties:*

- 1)  $\psi_n^*(x) \in W_{20}^1$ ;
- 2)  $|\psi_n(x)| \leq C, n \geq 1, x \in [0, \pi]$ ;
- 3) for any  $\{\theta_n\} \in \ell_2$

$$\theta(x) := \sum_{n=1}^{\infty} \frac{\theta_n}{n} \psi_n^*(x) \in W_{20}^1, \quad \|\theta(x)\|_{W_{20}^1} \leq C \left( \sum_{n=1}^{\infty} \|\theta_n\|^2 \right)^{1/2}.$$

*Proof.* We shall use the well-known results for the inverse Sturm-Liouville problem (see Chapter 1). Since  $\Lambda < \infty$ , it follows from (4.6.9) that

$$\tilde{\rho}_n = \sqrt{\tilde{\lambda}_n} = n + \frac{A_1}{n} + \frac{\tilde{\kappa}_n}{n}, \quad \{\tilde{\kappa}_n\} \in \ell_2. \quad (4.6.27)$$

Consequently, there exists a function  $\tilde{q}(x)$  (not unique) such that the numbers  $\{\tilde{\lambda}_n\}_{n \geq 1}$  are the eigenvalues of the Sturm-Liouville boundary value problem

$$y'' + \tilde{q}(x)y = \lambda y, \quad y(0) = y(\pi) = 0. \quad (4.6.28)$$

Let  $\tilde{s}_n(x)$  be the eigenfunctions of (4.6.28) normalized by the condition  $\tilde{s}'(0) = \frac{n}{2}$ . The functions  $\{\tilde{s}_n(x)\}_{n \geq 1}$  constitute a Riesz basis in  $L_2(0, \pi)$ , and

$$\int_0^\pi \tilde{s}_n(x) \tilde{s}_m(x) dx = \delta_{nm} \tilde{\alpha}_n. \quad (4.6.29)$$

Using Lemma 4.6.1, we obtain

$$\tilde{s}_n(x) = \psi_{n0}(x) + \int_0^x \tilde{K}(x, t) \psi_{n0}(t) dt, \quad \tilde{K}(x, 0) = 0. \quad (4.6.30)$$

In particular, it follows from (4.6.30), (4.6.29) and (4.6.27) that

$$\tilde{s}_n(x) = \frac{1}{2} \sin nx + O\left(\frac{1}{n}\right), \quad \tilde{\alpha}_n = \int_0^\pi \tilde{s}_n^2(x) dx = \frac{\pi}{8} + O\left(\frac{1}{n}\right), \quad n \rightarrow \infty.$$

Due to (4.6.30), the functions  $\{\psi_{n0}(x)\}_{n \geq 1}$  constitute a Riesz basis in  $L_2(0, \pi)$ . Denote

$$\psi_{n0}^{**}(x) = \tilde{s}_n(x) + \int_x^\pi \tilde{K}(t, x) \tilde{s}_n(t) dt. \quad (4.6.31)$$

It follows from (4.6.29)-(4.6.31) that

$$\begin{aligned} \int_0^\pi \psi_{n0}(x) \psi_{m0}^{**}(x) dx &= \int_0^\pi \psi_{n0} \left( \tilde{s}_m(x) + \int_x^\pi \tilde{K}(t, x) \tilde{s}_m(t) dt \right) dx \\ &= \int_0^\pi \tilde{s}_m(x) \left( \psi_{n0}(x) + \int_0^x \tilde{K}(x, t) \psi_{n0}(t) dt \right) dx = \int_0^\pi \tilde{s}_n(x) \tilde{s}_m(x) dx = \delta_{nm} \tilde{\alpha}_n. \end{aligned} \quad (4.6.32)$$

Furthermore, we compute

$$\psi_n(x) = \frac{n}{x} \int_0^x w(t, \tilde{\lambda}_n) S'(x - t, \tilde{\lambda}_n) dt. \quad (4.6.33)$$

Since

$$S'(x, \lambda) = \cos \rho x + \int_0^x K^1(x, t) \cos \rho t dt, \quad (4.6.34)$$

we obtain, substituting (4.6.34) and (4.6.17) into (4.6.33) like in the proof of Lemma 4.6.3,

$$\psi_n(x) = \psi_{n0}(x) + \int_0^x V_0(x, t) \psi_{n0}(t) dt, \quad (4.6.35)$$

where  $V_0(x, t)$  is a continuous function with  $V_0(x, 0) = 0$ . Solving the integral equation (4.6.35), we find

$$\psi_{n0}(x) = \psi_n(x) + \int_0^x V_1(x, t) \psi_n(t) dt, \quad V_1(x, 0) = 0. \quad (4.6.36)$$

Consider the functions

$$\psi_n^{**}(x) = \psi_{n0}^{**}(x) + \int_x^\pi V_1(t, x) \psi_{n0}^{**}(t) dt. \quad (4.6.37)$$

It follows from (4.6.32), (4.6.36) and (4.6.37) that

$$\int_0^\pi \psi_n(x) \psi_m^{**}(x) dx = \delta_{nm} \tilde{\alpha}_n. \quad (4.6.38)$$

By virtue of (4.6.35) and (4.6.38), the functions  $\{\psi_n(x)\}_{n \geq 1}$  constitute a Riesz basis in  $L_2(0, \pi)$ , and the biorthogonal basis  $\{\psi_n^*(x)\}_{n \geq 1}$  has the form  $\psi_n^*(x) = \tilde{\alpha}_n^{-1} \psi_{n0}^{**}(x)$ . Substituting (4.6.31) into (4.6.37), we have

$$\psi_n^{**}(x) = \tilde{s}_n(x) + \int_x^\pi V_1^0(t, x) \tilde{s}_n(t) dt, \quad V_1^0(t, 0) = 0.$$

Hence we obtain the required properties of the biorthogonal basis.  $\square$

Since  $\eta_n(x) = \psi_n(\pi - x)$ , Lemma 4.6.5 implies

**Corollary 4.6.1.** *The function  $\{\eta_n(x)\}_{n \geq 1}$  constitute a Riesz basis in  $L_2(0, \pi)$ , and the biorthogonal basis  $\{\chi_n(x)\}_{n \geq 1}$  has the properties:*

- 1)  $\chi_n^*(x) \in W_{20}^1$ ;
- 2)  $|\chi_n(x)| \leq C$ ,  $n \geq 1$ ,  $x \in [0, \pi]$ ;

3) for any  $\{\theta_n\} \in \ell_2$

$$\theta(x) = \sum_{n=1}^{\infty} \frac{\theta_n}{n} \chi_n(x) \in W_{20}^1, \quad \|\theta(x)\|_{W_2^1} \leq C \left( \sum_{n=1}^{\infty} \|\theta_n\|^2 \right)^{1/2}.$$

Let us return to proving Theorem 4.6.3. Put

$$\varepsilon(x) = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{n} \chi_n(x). \quad (4.6.39)$$

Using Lemma 4.6.1, the relations  $\Delta(\lambda) = S(\pi, \lambda)$ ,  $\Delta(\lambda_n) = 0$ , and formulae (4.6.9), (4.6.27), we obtain the estimate

$$|\varepsilon_n| = n^2 |\Delta(\tilde{\lambda}_n) - \Delta(\lambda_n)| \leq C |\lambda_n - \tilde{\lambda}_n|.$$

Now by Corollary 4.6.1 we have

$$\varepsilon(x) \in W_{20}^1, \quad \|\varepsilon(x)\|_{W_2^1} \leq C\Lambda.$$

Consider in  $W_{20}^1$  the nonlinear equation

$$r = \varepsilon + \sum_{j=2}^{\infty} \psi_j(r), \quad (4.6.40)$$

where  $\varepsilon(x)$  is defined by (4.6.39), and the operators  $z_j = \psi_j(r)$  act from  $W_{20}^1$  to  $W_{20}^1$  according to the formula

$$z_j(x) = - \sum_{n=1}^{\infty} \left( \underbrace{\int_0^\pi \dots \int_0^\pi}_{j} r(t_1) \dots r(t_j) B_{nj}(t_1, \dots, t_j) dt_1 \dots dt_j \right) \chi_n(x),$$

where

$$\begin{aligned} r(x) \in W_{20}^1, \quad B_{nj}(t_1, \dots, t_j) &= \frac{n}{(\pi - t_1) \dots (\pi - t_j)} \\ &\times \underbrace{\int_0^\pi \dots \int_0^\pi}_{j} v_n(t_1, s_1) G_n(s_1, t_2, s_2) \dots G_n(s_{j-1}, t_j, s_j) y_n(s_j) ds_1 \dots ds_j \end{aligned}$$

and

$$\|\psi_j(r)\|_{W_2^1} \leq (C\|r\|_{W_2^1})^j,$$

$$\|\psi_j(r) - \psi_j(r^*)\|_{W_2^1} \leq \|r - r^*\|_{W_2^1} (C \max(\|r\|_{W_2^1}, \|r^*\|_{W_2^1}))^{j-1}.$$

By Lemma 4.6.4, there exists  $\delta > 0$  such that for  $\Lambda < \delta$  equation (4.6.40) has a solution  $r(x) \in W_{20}^1$ ,  $\|r(x)\|_{W_{20}^1} \leq C\Lambda$ . Put  $\tilde{M}(x) = M(x) - ((\pi - x)^{-1}r(x))'$ , and consider the boundary value problem  $\tilde{L} = L(q, \tilde{M})$ . Clearly

$$\tilde{Q}(x) = Q(x) - r'(x) \in L_2(0, \pi), \quad \|Q(x) - \tilde{Q}(x)\|_{L_2(0, \pi)} \leq C\Lambda.$$

Since

$$\hat{M}_1(x) = -\frac{1}{\pi - x} \int_x^\pi \hat{Q}(t) dt, \quad \hat{M}_0(x) = \hat{Q}(x) + \hat{M}_1(x),$$

we have

$$\|M_k(x) - \tilde{M}_k(x)\|_{L_2(0,\pi)} \leq C\Lambda, \quad k = 0, 1.$$

It remains to show that the numbers  $\{\tilde{\lambda}\}_{n \geq 1}$  are the eigenvalues of the problem  $\tilde{L}$ . To do this, consider the functions  $\tilde{y}_n(x)$  which are solutions of the integral equations

$$\tilde{y}_n(x) = y_n(x) + \int_0^\pi \hat{M}_1(t) \left( \int_0^\pi G_n(x, t, s) \tilde{y}_n(s) ds \right) dt \quad (4.6.41)$$

or, which is the same,

$$\tilde{y}_n(x) = y_n(x) + \int_0^x \hat{M}_1(t) \left( \int_0^{x-t} G(x, s+t, \tilde{\lambda}_n) \tilde{y}_n(s) ds \right) dt. \quad (4.6.42)$$

After integration by parts, (4.6.42) takes the form

$$\tilde{y}_n(x) = y_n(x) - \int_0^x \hat{M}_1(t) \int_t^x g(x, s, \tilde{\lambda}_n) \tilde{y}_n'(s-t) ds dt.$$

Reverse the integration order:

$$\tilde{y}_n(x) = y_n(x) - \int_0^x g(x, t, \tilde{\lambda}_n) \int_0^t \hat{M}_1(s) \tilde{y}_n'(t-s) ds dt.$$

Integrate by parts:

$$\tilde{y}_n(x) = y_n(x) - \int_0^x g(x, t, \tilde{\lambda}_n) \int_0^t \hat{M}(t-s) \tilde{y}_n(s) ds dt. \quad (4.6.43)$$

It follows from (4.6.43) that

$$\ell(\tilde{y}_n(x) - y_n(x)) = \int_0^x \hat{M}(t-s) \tilde{y}_n(s) ds = (\ell - \tilde{\ell}) \tilde{y}_n(x),$$

and consequently,

$$\tilde{\ell} \tilde{y}_n(x) - \tilde{\lambda}_n \tilde{y}_n(x), \quad \tilde{y}_n(0) = (0), \quad \tilde{y}_n'(0) = 1.$$

Since the solution of the Cauchy problem is unique, we have  $\tilde{y}_n(x) = \tilde{S}(x, \tilde{\lambda}_n)$ .

Write (4.6.13) in the form

$$\int_0^\pi \hat{M}(x) \left( \int_0^{\pi-x} w(\pi-x-t, \lambda) \tilde{S}(t, \lambda) dt \right) dx = \hat{\Delta}(\lambda).$$

Integrating by parts, we obtain for  $\lambda = \tilde{\lambda}_n$

$$\int_0^\pi \hat{M}_1(x) \left( \int_0^\pi v_n(x, t) \tilde{y}_n(t) dt \right) dx = \hat{\Delta}(\tilde{\lambda}_n). \quad (4.6.44)$$

Solving (4.6.41) by the method of successive approximations, we have

$$\tilde{y}_n(x) = y_n(x) + Y_n(x), \quad (4.6.45)$$

where

$$Y_n(x) = \sum_{j=1}^{\infty} \underbrace{\int_0^\pi \dots \int_0^\pi}_{j} \hat{M}_1(t_1) \dots \hat{M}_1(t_j) \left( \underbrace{\int_0^\pi \dots \int_0^\pi}_{j} G_n(x, t_1, s_1) \right)$$

$$\times G_n(s_1, t_2, s_2) \dots G_n(s_{j-1}, t_j, s_j) y_n(s_j) ds_1 \dots ds_j) dt_1 \dots dt_j.$$

Furthermore, multiplying (4.6.40) by  $\overline{\eta_n(x)}$  and integrating from 0 to  $\pi$ , we obtain

$$\int_0^\pi r(x) \overline{\eta_n(x)} dx + \sum_{j=2}^\infty \underbrace{\int_0^\pi \dots \int_0^\pi}_{j} r(t_1) \dots r(t_j) B_{nj}(t_1) \dots t_j) dt_1 \dots dt_j = \frac{\varepsilon_n}{n}. \quad (4.6.46)$$

Since  $r(x) = (\pi - x) \hat{M}_1(x)$ ,  $\overline{\eta_n(x)} = n(\pi - x)^{-1} \xi_n(x)$ , we can transform (4.6.46) to the form

$$\begin{aligned} \int_0^\pi \hat{M}_1(x) \xi_n(x) dx + \sum_{j=2}^\infty \underbrace{\int_0^\pi \dots \int_0^\pi}_{j} \hat{M}_1(t_1) \dots \hat{M}_1(t_j) \left( \underbrace{\int_0^\pi \dots \int_0^\pi}_{j} v_n(t_1, s_1) \right. \\ \left. \times G_n(s_1, t_2, s_2) \dots G_n(s_{j-1}, t_j, s_j) y_n(s_j) ds_1 \dots ds_j \right) dt_1 \dots dt_j = \frac{\varepsilon_n}{n^2}. \end{aligned}$$

Hence, taking (4.6.45) into account, we obtain

$$\int_0^\pi \hat{M}_1(x) \left( \int_0^\pi v_n(x, t) \tilde{y}_n(t) dt \right) dx = \Delta(\tilde{\lambda}_n). \quad (4.6.47)$$

Comparing (4.6.44) with (4.6.47), we find that  $\tilde{\Delta}(\tilde{\lambda}_n) = 0$ . Hence the number  $\{\tilde{\lambda}_n\}_{n \geq 1}$  are the eigenvalues of the boundary value problem  $\tilde{L}$ . Theorem 4.6.3 is proved.  $\square$

## 4.7. DIFFERENTIAL OPERATORS OF THE ORR-SOMMERFELD TYPE

In this section we study the inverse problem of recovering differential operators of the Orr-Sommerfeld type (or Kamke type) from the Weyl matrix. Properties of the Weyl matrix are investigated, and an uniqueness theorem for the solution of the inverse problem is proved.

**4.7.1.** Let us consider the differential equation

$$Ly = z\ell y \quad (4.7.1)$$

where

$$Ly = y^{(n)} + \sum_{k=0}^{n-2} p_k(x) y^{(k)}, \quad \ell y = y^{(m)} + \sum_{j=0}^{m-1} q_j(x) y^{(j)}, \quad n > m \geq 1, \quad x \in (0, T).$$

Here  $p_k$  and  $q_j$  are complex-valued integrable functions.

Differential equations of the form (4.7.1) play an important role in various areas of mathematics as well as in applications. The Orr-Sommerfeld equation from the theory of hydrodynamic stability is a typical example for equation (4.7.1). Spectral properties of boundary value problems for the Orr-Sommerfeld equation and the more general equation (4.7.1) have been studied in many works (see [ebe5], [tre1], [shk1] and the references given therein). However at present inverse spectral problems for such classes of operators have not been studied yet because of their complexity.

In this paper we study the inverse problem of recovering the coefficients of equation (4.7.1) from the so-called Weyl matrix. We introduce and investigate the Weyl matrix and

prove an uniqueness theorem for the solution of the inverse problem. We note that for the case of  $m = 0$ , i.e. for the equation  $Ly = zy$ , inverse problems of recovering  $L$  from various spectral characteristics have been studied in [bea1], [bea2], [kha3], [lei1], [sak1], [yur1], [yur16], [yur22] and other works. In particular, for such operators the theory of the solution of the inverse problem by means of the Weyl matrix (and its applications) has been constructed in [yur1], [yur12], [yur15], [yur16], [yur22]. The Weyl matrix introduced in this paper is a generalization of the Weyl matrix for the equation  $Ly = zy$  introduced in [yur16].

**4.7.2.** Let the functions  $\Phi_k(x, z)$ ,  $k = \overline{1, n}$ , be solutions of equation (4.7.1) under the conditions  $\Phi_k^{(j-1)}(0, z) = \delta_{kj}$ ,  $j = \overline{1, k}$ ,  $\Phi_k^{(s-1)}(T, z) = 0$ ,  $s = \overline{1, n-k}$ . Here  $\delta_{kj}$  is the Kronecker delta. Denote  $M_{kj}(z) = \Phi_k^{(j-1)}(0, z)$ ,  $j = \overline{k+1, n}$ . The functions  $\Phi_k(x, z)$  are called the Weyl solutions, and the functions  $M_{kj}(z)$  are called the Weyl functions. The matrix  $M(z) = [M_{kj}(z)]_{k,j=\overline{1,n}}$ ,  $M_{kj}(z) = \delta_{kj}$ ,  $j = \overline{1, k}$  is called the Weyl matrix for equation (4.7.1). The inverse problem studied in this section is formulated as follows:

*Given the Weyl matrix, construct  $L$  and  $\ell$ .*

Let us consider the fundamental system of solutions of equation (4.7.1)  $C_k(x, z)$ ,  $k = \overline{1, n}$ , which satisfies the conditions  $C_k^{(j-1)}(0, z) = \delta_{kj}$ ,  $j = \overline{1, n}$ . Then

$$\Phi_k(x, z) = C_k(x, z) + \sum_{j=k+1}^n M_{kj}(z) C_j(x, z), \quad (4.7.2)$$

$$\det[\Phi_k^{(j-1)}(x, z)]_{k,j=\overline{1,n}} = \det[C_k^{(j-1)}(x, z)]_{k,j=\overline{1,n}} = \begin{cases} 1, & m < n-1, \\ \exp(zx), & m = n-1. \end{cases} \quad (4.7.3)$$

Let us denote

$$\Delta_{kj}(z) = (-1)^{k+j} \det[C_\nu^{(\xi-1)}(T, z)]_{\xi=\overline{1, n-k}; \nu=\overline{k, n}, \nu \neq j}.$$

The function  $\Delta_{kj}(z)$  is entire in  $z$  of order less than 1, and its zeros coincide with the eigenvalues of the boundary-value problem for equation (4.7.1) under the conditions  $y^{(s-1)}(0) = y^{(p-1)}(T) = 0$ ;  $s = \overline{1, k-1}$ ,  $j$ ;  $p = \overline{1, n-k}$ . The function  $\Delta_{kj}(z)$  is uniquely determined by its zeros.

Using the boundary conditions on the Weyl solutions we calculate

$$\Phi_k(x, z) = (\Delta_{kk}(z))^{-1} \det[C_j(x, z), C_j(T, z), \dots, C_j^{(n-k-1)}(T, z)]_{j=\overline{k, n}}, \quad (4.7.4)$$

and consequently

$$M_{kj}(z) = (\Delta_{kk}(z))^{-1} \Delta_{kj}(z). \quad (4.7.5)$$

Let  $h_k(x)$ ,  $k = \overline{1, m}$ , be the solutions of the differential equation  $\ell y = 0$  under the initial conditions  $h_k^{(\nu-1)}(0) = \delta_{k\nu}$ ,  $\nu = \overline{1, m}$ . Denote

$$B_s(x) = \det[h_k(x), h_k(T), h'_k(T), \dots, h_k^{(m-s-1)}(T)]_{k=\overline{s, m}}, \quad s = \overline{1, m-1}, \quad B_m(x) = h_m(x),$$

$$\alpha_s = \det[h_k^{(\nu)}(T)]_{k=\overline{s, m}; \nu=\overline{0, m-s}}, \quad s = \overline{1, m}, \quad \alpha_{m+1} = 1.$$

Clearly,

$$\alpha_1 = \exp\left(-\int_0^T q_{m-1}(t) dt\right).$$

For definiteness, we shall assume that  $\alpha_s \neq 0$ ,  $s = \overline{2, m}$ . Other cases require separate calculations. We shall also suppose the coefficients  $p_k$  and  $q_j$  to be sufficiently smooth such that the asymptotic formulae (4.7.12)-(4.7.13) hold (see [ebe5], [tre1]).

By virtue of (4.7.5), the Weyl functions  $M_{kj}(z)$  are meromorphic. It follows from above that the specification of the Weyl matrix  $M(z)$  is equivalent to the specification of the spectra of the corresponding boundary-value problems for equation (4.7.1) or corresponds to the specification of poles and residues of the Weyl functions. Therefore the inverse problem of recovering equation (4.7.1) from the Weyl matrix is a generalization of the well-known inverse problems for the Sturm-Liouville operators from discrete spectral characteristics (see, for example, [lev2] and Chapter 1).

Let  $b_s(x)$ ,  $s = \overline{1, m}$  be the solutions of the equation  $\ell y = 0$  with the conditions  $b_s^{(\nu-1)}(0) = \delta_{s\nu}$ ,  $\nu = \overline{1, s}$ ;  $b_s^{(\xi-1)}(T) = 0$ ,  $\xi = \overline{1, m-s}$ . In other words, the functions  $b_s(x)$  are the Weyl solutions for the equation  $\ell y = 0$ . As in (4.7.3) and (4.7.4), we see that

$$b_s(x) = (\alpha_{s+1})^{-1} B_s(x)$$

and

$$\det[b_s^{(\nu-1)}(x)]_{s,\nu=\overline{1,m}} = \det[h_s^{(\nu-1)}(x)]_{s,\nu=\overline{1,m}} = \exp\left(-\int_0^x q_{m-1}(t) dt\right).$$

Together with  $L$  and  $\ell$  we consider operators  $\tilde{L}$  and  $\tilde{\ell}$  of the same form but with different coefficients. We agree that if a symbol  $\alpha$  denotes an object related to  $L$  and  $\ell$ , then  $\tilde{\alpha}$  will denote the analogous object related to  $\tilde{L}$  and  $\tilde{\ell}$ .

The main result of this section is the following uniqueness theorem for the solution of the inverse problem by means of the Weyl matrix.

**Theorem 4.7.1.** *If  $M(z) = \tilde{M}(z)$  then  $L = \tilde{L}$  and  $\ell = \tilde{\ell}$ .*

*Proof.* Let

$$C(x, z) = [C_k^{(j-1)}(x, z)]_{j,k=\overline{1,n}}, \quad \Phi(x, z) = [\Phi_k^{(j-1)}(x, z)]_{j,k=\overline{1,n}}.$$

Then (4.7.2) takes the form

$$\Phi(x, z) = C(x, z)M^t(z), \quad (4.7.6)$$

where  $t$  denotes transposition. We define the matrix  $Q(x, z) = [Q_{jk}(x, z)]_{j,k=\overline{1,n}}$  from the relation  $Q(x, z)\tilde{\Phi}(x, z) = \Phi(x, z)$ . Let  $r := n - m$ .

**Lemma 4.7.1.** *The following relations hold*

$$Q'_{jk}(x, z) = Q_{j+1,k}(x, z) - Q_{j,k-1}(x, z) + (-1)^{n-k+1} Q_{j,n}(x, z) \pi_{k-1}(x, z), \quad j, k = \overline{1, n-1}, \quad r \geq 1, \quad (4.7.7)$$

$$Q'_{jn}(x, z) = Q_{j+1,n}(x, z) - Q_{j,n-1}(x, z) - z Q_{j,n}(x, z), \quad j = \overline{1, n-1}, \quad r = 1, \quad (4.7.8)$$

$$Q'_{jn}(x, z) = Q_{j+1,n}(x, z) - Q_{j,n-1}(x, z), \quad j = \overline{1, n-1}, \quad r > 1, \quad (4.7.9)$$

where  $\pi_s(x, z) = \tilde{p}_s(x) - z\tilde{q}_s(x)$ .

*Proof of Lemma 1.* Let us denote  $W(x, z) = \det \Phi(x, z)$ . In virtue of (4.7.3),  $W(x, z) \neq 0$ , and consequently one can write  $Q(x, z) = \Phi(x, z)(\tilde{\Phi}(x, z))^{-1}$  or in the coordinates

$$Q_{jk}(x, z) = (\tilde{W}(x, z))^{-1} \det[\tilde{\Phi}_s(x, z), \tilde{\Phi}'_s(x, z) \dots \tilde{\Phi}_s^{(k-2)}(x, z), \Phi_s^{(j-1)}(x, z), \tilde{\Phi}_s^{(k)}(x, z), \tilde{\Phi}_s^{(k+1)}(x, z) \dots \tilde{\Phi}_s^{(n-2)}(x, z), \tilde{\Phi}_s^{(n-1)}(x, z)]_{s=\overline{1,n}}. \quad (4.7.10)$$

Let  $j, k = \overline{1, n-1}$ . Differentiating (4.7.10) with respect to  $x$  we get

$$Q'_{jk}(x, z) = -z\delta_{r1}Q_{jk}(x, z) + Q_{j+1,k}(x, z) - Q_{j,k-1}(x, z) + (\tilde{W}(x, z))^{-1} \det[\tilde{\Phi}_s(x, z), \\ \tilde{\Phi}'_s(x, z) \dots \tilde{\Phi}_s^{(k-2)}(x, z), \Phi_s^{(j-1)}(x, z), \tilde{\Phi}_s^{(k)}(x, z), \tilde{\Phi}_s^{(k+1)}(x, z) \dots \tilde{\Phi}_s^{(n-2)}(x, z), \tilde{\Phi}_s^{(n)}(x, z)]_{s=\overline{1, n}}.$$

We replace here  $\tilde{\Phi}_s^{(n)}(x, z)$  with the help of equation (4.7.1), and after some simple calculations we arrive at (4.7.7).

The relations (4.7.8) and (4.7.9) are obtained analogously.  $\square$

**Lemma 4.7.2.** *If  $Q_{jn}(x, z) \equiv 0$  for  $j = \overline{1, n-1}$ , then  $Q_{jk}(x, z) \equiv 0$  for  $j < k$ .*

Lemma 4.7.2 can be proved by induction with respect to  $k$  using the formulae (4.7.7)-(4.7.9).

Let us now study the asymptotic behavior of the Weyl solutions  $\Phi_k(x, z)$  for  $z \rightarrow \infty$ . Let  $z = \rho^r$ . We can divide  $\rho$ -plane into the sectors

$$S_\nu = \{\rho : \arg \rho \in (\nu\pi/r, (\nu+1)\pi/r), \nu = \overline{0, 2r-1}\},$$

in each of which the roots  $R_k$  of the equation  $R^r = 1$  can be numbered so that

$$\operatorname{Re}(\rho R_1) < \operatorname{Re}(\rho R_2) < \dots < \operatorname{Re}(\rho R_r), \quad \rho \in S_\nu. \quad (4.7.11)$$

It is known (see, for example, [ebe5], [tre1] and the references therein) that in each sector  $S_\nu$  with the property (4.7.11) there exists a fundamental system of solutions  $\{Y_s(x, \rho)\}_{s=\overline{1, n}}$  of equation (4.7.1) of the form

$$Y_s^{(\nu)}(x, \rho) = (\rho R_s)^\nu \exp(\rho R_s x)(a(x) + O(\rho^{-1})), \quad s = \overline{1, r}, |\rho| \rightarrow \infty, \nu = \overline{0, n-1}, \quad (4.7.12)$$

$$Y_{s+r}^{(\nu)}(x, \rho) = h_s^{(\nu)}(x) + O(z^{-1}), \quad s = \overline{1, m}, |\rho| \rightarrow \infty, \nu = \overline{0, n-1}, \quad (4.7.13)$$

where

$$a(x) = \exp\left(\frac{1}{r} \int_0^x q_{m-1}(t) dt\right).$$

Denote

$$\delta_k(\rho) = \det[Y_s(0, \rho), \dots, Y_s^{(k-1)}(0, \rho), Y_s(T, \rho), \dots, Y_s^{(n-k-1)}(T, \rho)]_{s=\overline{1, n}}, \quad k = \overline{1, n}.$$

For definiteness, let  $r = 2\mu$ . The case when  $r$  is odd is considered similarly. Using (4.7.12)-(4.7.13) we obtain for  $|\rho| \rightarrow \infty$ ,  $\rho \in S$ ,  $\arg z = \text{const} \neq 0, \pi$ ,

$$\delta_s(\rho) = (-1)^{m(r-s)} \alpha_1 \omega_{s+1, r} \omega_{1s} (R_{s+1} \dots R_r)^m a^{r-s}(T) \rho^{\sigma_s}.$$

$$\exp(\rho(R_{s+1} + \dots + R_r)T)(1 + O(\rho^{-1})), \quad s = \overline{1, \mu}, \quad (4.7.14)$$

$$\delta_{\mu+s}(\rho) = (-1)^{\mu(m+s)} \alpha_{s+1} \omega_{\mu+1, r} \omega_{1\mu} (R_{\mu+1} \dots R_r)^{m-s} (R_1 \dots R_\mu)^s a^\mu(T) \rho^{\sigma_{s+\mu}}.$$

$$\exp(\rho(R_{\mu+1} + \dots + R_r)T)(1 + O(\rho^{-1})), \quad s = \overline{0, m}, \quad (4.7.15)$$

$$\delta_{\mu+m+s}(\rho) = \omega_{\mu+s+1, r} \omega_{1, \mu+s} (R_1 \dots R_{\mu+s})^m a^{\mu-s}(T) \rho^{\sigma_{s+\mu+m}}.$$

$$\exp(\rho(R_{\mu+s+1} + \dots + R_r)T)(1 + O(\rho^{-1})), \quad s = \overline{0, \mu}, \quad (4.7.16)$$



where

$$\begin{aligned}\omega_{jk} &= \det[R_\xi^\nu]_{\xi=\overline{j,k}; \nu=\overline{o,k-j}}, \\ \sigma_s &= (s(s-1) + (n-s-1)(n-s) - (m-1)m)/2, \quad s = \overline{1, \mu}, \\ \sigma_{s+\mu} &= ((\mu+s-1)(\mu+s) - s(s-1) + (n-\mu-s-1)(n-\mu-s) - (m-s)(m-s-1))/2, \quad s = \overline{0, m}, \\ \sigma_{s+\mu+m} &= ((\mu-s-1)(\mu-s) + (m+\mu+s-1)(m+\mu+s) - (m-1)m)/2, \quad s = \overline{0, \mu}.\end{aligned}$$

Moreover

$$|\delta_s(\rho)| \leq C|\rho|^\beta \exp(\alpha|\rho|), \quad |\rho| \rightarrow \infty, \quad \rho \in \bar{S}, \quad s = \overline{1, n},$$

for some  $\alpha > 0$  and  $\beta > 0$ .

Using the boundary conditions on the Weyl solutions we calculate

$$\begin{aligned}\Phi_k(x, z) &= (\delta_k(\rho))^{-1} \det[Y_s(0, \rho), \dots, Y_s^{(k-2)}(0, \rho), \\ &\quad Y_s(x, \rho), Y_s(T, \rho), \dots, Y_s^{(n-k-1)}(T, \rho)]_{s=\overline{1, n}}.\end{aligned}\tag{4.7.17}$$

Substituting the asymptotic formulas (4.7.12)-(4.7.16) into (4.7.17) we obtain for  $|\rho| \rightarrow \infty$ ,  $\rho \in S$ ,  $\arg z = \text{const} \neq 0, \pi$ ,

$$\Phi_s^{(\nu)}(x, z) = \omega_{1,s-1}(\omega_{1s})^{-1} a(x) \rho^{1-s} (\rho R_s)^\nu \exp(\rho R_s x) (1 + O(\rho^{-1})), \quad s = \overline{1, \mu}, \tag{4.7.18}$$

$$\Phi_{\mu+s}^{(\nu)}(x, z) = (-1)^\mu (R_1 \cdots R_\mu)^{-1} b_s^{(\nu)}(x) \rho^{-\mu} (1 + O(\rho^{-1})), \quad s = \overline{1, m}, \tag{4.7.19}$$

$$\begin{aligned}\Phi_{\mu+m+s}^{(\nu)}(x, z) &= \omega_{1,\mu+s-1}(\omega_{1,\mu+s})^{-1} R_{\mu+s}^{-m} a(x) \rho^{1-s-m-\mu} (\rho R_{\mu+s})^\nu \exp(\rho R_{\mu+s} x) \times \\ &\quad (1 + O(\rho^{-1})), \quad s = \overline{1, \mu},\end{aligned}\tag{4.7.20}$$

where  $\omega_{10} = 1$ .

Now we transform the matrix  $Q(x, z)$ . For this we use (4.7.6). Since  $M(z) = \tilde{M}(z)$  by the assumption of Theorem 4.7.1, we get

$$Q(x, z) = \Phi(x, z)(\tilde{\Phi}(x, z))^{-1} = C(x, z)M^t(z)(\tilde{M}^t(z))^{-1}(\tilde{C}(x, z))^{-1} = C(x, z)(\tilde{C}(x, z))^{-1}.$$

Using (4.7.3) we conclude that for each fixed  $x$  the functions  $Q_{jk}(x, z)$  are entire in  $z$  of order  $1/r$ . Substituting (4.7.18)-(4.7.20) into (4.7.10) we calculate for  $z \rightarrow \infty$ ,  $\arg z = \text{const} \neq 0, \pi$ ,

$$Q_{jn}(x, z) = O(\rho^{-1}), \quad j = \overline{1, n-1}.$$

Then the theorems of Phragmen-Lindelöf and Liouville give us

$$Q_{jn}(x, z) \equiv 0, \quad j = \overline{1, n-1}.$$

By virtue of Lemma 4.7.2, this yields

$$Q_{jk}(x, z) \equiv 0, \quad j < k.$$

Since  $Q(x, z)\tilde{\Phi}(x, z) = \Phi(x, z)$ , we obtain

$$Q_{11}(x, z)\tilde{\Phi}_k(x, z) \equiv \Phi_k(x, z), \quad k = \overline{1, n}.$$

This implies

$$\Phi_k^{(\nu)}(x, z) = Q_{11}(x, z)\tilde{\Phi}_k^{(\nu)}(x, z) + \sum_{s=0}^{\nu-1} C_\nu^s Q_{11}^{(\nu-s)}(x, z)\tilde{\Phi}_k^{(s)}(x, z),$$

and hence  $Q_{11}^n(x, z)\tilde{W}(x, z) \equiv W(x, z)$ . In view of (4.7.3),  $\tilde{W}(x, z) \equiv W(x, z)$ , i.e.  $Q_{11}^n(x, z) \equiv 1$ . Since  $Q_{11}(0, z) \equiv 1$ , we have  $Q_{11}(x, z) \equiv 1$ . Consequently,  $\tilde{\Phi}_k(x, z) \equiv \Phi_k(x, z)$ ,  $k = \overline{1, n}$ ,  $L = \tilde{L}$ ,  $\ell = \tilde{\ell}$ . Theorem 4.7.1 is proved.  $\square$

Using this method one can also obtain a constructive procedure for the solution of this inverse problem along with necessary and sufficient conditions of its solvability.

## 4.8. DIFFERENTIAL EQUATIONS WITH TURNING POINTS

**4.8.1.** We consider in this section boundary value problems  $L$  of the form

$$\ell y := -y'' + q(x)y = \lambda R^2(x)y, \quad x \in [0, 1], \quad (4.8.1)$$

$$U(y) := y'(0) - hy(0) = 0, \quad (4.8.2)$$

$$V(y) := y'(1) + h_1y(1) = 0. \quad (4.8.3)$$

Here  $\lambda = \rho^2$  is the spectral parameter;  $R^2$  and  $q$  are real functions, and  $h, h_1$  are real numbers. We suppose that

$$R^2(x) = \prod_{\nu=1}^m (x - x_\nu)^{\ell_\nu} R_0(x),$$

where  $0 < x_1 < x_2 < \dots < x_m < 1$ ,  $\ell_\nu \in \mathbf{N}$ ,  $R_0(x) > 0$  for  $x \in I := [0, 1]$ , and  $R_0$  is twice continuously differentiable on  $I$ . In the other words,  $R^2$  has in  $I$   $m$  zeros  $x_\nu$ ,  $\nu = \overline{1, m}$  of order  $\ell_\nu$ . The zeros  $x_\nu$  of  $R^2$  are called turning points. We also assume that  $q$  is bounded and integrable on  $I$ .

In this section we study the following three inverse problems of recovering  $L$  from its spectral characteristics, namely

- (i) from the Weyl function,
- (ii) from two spectra and
- (iii) from the so-called spectral data.

Differential equations with turning points play an important role in various areas of mathematics as well as in applications (see [ebe1]–[ebe4], [gol1], [mch1], [was1] and the references therein). For example, turning points connected with physical situations in which zeros correspond to the limit of motion of a wave mechanical particle bound by a potential field. Turning points appear also in elasticity, optics, geophysics and other branches of natural sciences. Moreover, a wide class of differential equations with Bessel-type singularities and their perturbations can be reduced to differential equations having turning points. Inverse problems for equations with turning points also help to study blow-up behavior of solutions for nonlinear evolution equations in mathematical physics [Con1].

In order to study the inverse problem in this section we use the method of spectral mappings described in Section 1.6 for the classical Sturm-Liouville equation without turning points. The presence of turning points in the differential equation produces essential qualitative modifications in the method. An important role in this investigation is played by the special fundamental system of solutions for equation (4.8.1) constructed in [ebe4]. This fundamental system of solutions gives us an opportunity to obtain the asymptotic behavior

of the so-called Weyl solutions and the Weyl function for the boundary value problem L and to solve the corresponding inverse problems. In Subsection 4.8.2 we prove uniqueness theorems, and in Subsection 4.8.3 we provide a constructive procedure for the solution of the inverse problem.

Let  $\varepsilon > 0$  be fixed, sufficiently small and let  $D_{0\varepsilon} = [0, x_1 - \varepsilon]$ ,  $D_{\nu\varepsilon} = [x_\nu + \varepsilon, x_{\nu+1} - \varepsilon]$  for  $1 \leq \nu \leq m-1$ ,  $D_{m\varepsilon} = [x_m + \varepsilon, 1]$ ,  $D_\varepsilon = \bigcup_{\nu=0}^m D_{\nu\varepsilon}$ , and  $I_{\nu\varepsilon} = D_{\nu-1,\varepsilon} \cup [x_\nu - \varepsilon, x_\nu + \varepsilon] \cup D_{\nu\varepsilon}$ .

We distinguish four different types of turning points: For  $1 \leq \nu \leq m$

$$T_\nu = \begin{cases} I, & \text{if } \ell_\nu \text{ is even and } R^2(x)(x - x_\nu)^{-\ell_\nu} < 0 \text{ in } I_{\nu\varepsilon}, \\ II, & \text{if } \ell_\nu \text{ is even and } R^2(x)(x - x_\nu)^{-\ell_\nu} > 0 \text{ in } I_{\nu\varepsilon}, \\ III, & \text{if } \ell_\nu \text{ is odd and } R^2(x)(x - x_\nu)^{-\ell_\nu} < 0 \text{ in } I_{\nu\varepsilon}, \\ IV, & \text{if } \ell_\nu \text{ is odd and } R^2(x)(x - x_\nu)^{-\ell_\nu} > 0 \text{ in } I_{\nu\varepsilon}, \end{cases}$$

is called type of  $x_\nu$ . Further we set for  $1 \leq \nu \leq m$

$$\mu_\nu = \frac{1}{2 + \ell_\nu},$$

$$\theta_\nu = \begin{cases} 1 & \text{if } \mu_\nu > \frac{1}{4}, \\ 1 - \delta_0 \text{ (with } \delta_0 > 0 \text{ arbitrary small)} & \text{if } \mu_\nu = \frac{1}{4}, \\ 4\mu_\nu & \text{if } \mu_\nu < \frac{1}{4}, \end{cases}$$

and  $0 < \theta_0 = \min\{\theta_\nu | 1 \leq \nu \leq m\}$ . We also denote

$$I_+ = \{x : R^2(x) > 0\}, \quad I_- = \{x : R^2(x) < 0\},$$

$$\xi(x) = \begin{cases} 0, & \text{for } x \in I_+ \\ 1, & \text{for } x \in I_-, \end{cases}$$

$$\gamma_\nu = \begin{cases} 2 \sin \frac{\pi \mu_\nu}{2}, & \text{for } T_\nu = III, IV, \\ \sin \pi \mu_\nu, & \text{for } T_\nu = I, II, \end{cases}$$

$$K_\pm(x) = \left( \prod_{x_\nu \in (0, x)} \gamma_\nu^{-1} \right) \exp(\pm i \frac{\pi}{4} (\xi(x) - \xi(0))),$$

$$K_\pm^*(x) = \left( \prod_{x_\nu \in (0, x)} \gamma_\nu \right) \exp(\pm i \frac{\pi}{4} (\xi(x) + \xi(0))),$$

$$R_+^2(x) = \max(0, R^2(x)), \quad R_-^2(x) = \max(0, -R^2(x)).$$

Clearly,

$$K_\pm(x) K_\pm^*(x) = \exp(\pm i \frac{\pi}{2} \xi(x)) = \begin{cases} 1, & \text{if } x \in I_+ \\ \pm i, & \text{if } x \in I_-. \end{cases}$$

Let

$$S_k = \{\rho : \arg \rho \in [\frac{k\pi}{4}, \frac{(k+1)\pi}{4}]\},$$

$$\sigma_s^\delta = \{\rho : \arg \rho \in [\frac{s\pi}{2} - \delta, \frac{s\pi}{2} + \delta]\}, \quad \delta > 0,$$

$$\sigma^\delta = \bigcup_s \sigma_s^\delta, \quad S_k^\delta = S_k \setminus \sigma^\delta, \quad S^\delta = \bigcup_{k=-2}^1 S_k^\delta.$$

Below it is sufficient to consider the sectors  $S_k$  and  $S_k^\delta$  for  $k = -2, -1, 0, 1$  only.

It is shown in [ebe4] that for each fixed sector  $S_k$  ( $k = -2, -1, 0, 1$ ) there exists a fundamental system of solutions of (1)  $\{z_1(x, \rho), z_2(x, \rho)\}$ ,  $x \in I$ ,  $\rho \in S_k$  such that the functions

$$(x, \rho) \mapsto z_j^{(s)}(x, \rho) \quad (j = 1, 2; s = 0, 1)$$

are continuous for  $x \in I$ ,  $\rho \in S_k$  and holomorphic for each fixed  $x \in I$  with respect to  $\rho \in S_k$ ; moreover for  $|\rho| \rightarrow \infty$ ,  $\rho \in S_k$ ,  $x \in D_\varepsilon$ ,  $j = 1, 2$ ,

$$\begin{aligned} z_1^{(j)}(x, \rho) &= (\pm i\rho)^j |R(x)|^{j-1/2} (\exp(\mp i \frac{\pi}{2} \xi(x)))^j \exp(\rho \int_0^x |R_-(t)| dt) \\ &\quad \times \exp(\pm i\rho \int_0^x |R_+(t)| dt) K_\pm(x) \kappa(x, \rho), \end{aligned} \quad (4.8.4)$$

$$\begin{aligned} z_2^{(j)}(x, \rho) &= (\mp i\rho)^j |R(x)|^{j-1/2} (\exp(\mp i \frac{\pi}{2} \xi(x)))^j \exp(-\rho \int_0^x |R_-(t)| dt) \\ &\quad \times \exp(\mp i\rho \int_0^x |R_+(t)| dt) K_\pm^*(x) \kappa(x, \rho), \end{aligned} \quad (4.8.5)$$

$$\begin{vmatrix} z_1(x, \rho) & z_2(x, \rho) \\ z_1'(x, \rho) & z_2'(x, \rho) \end{vmatrix} = \mp(2i\rho)[1]. \quad (4.8.6)$$

Here and in the sequel:

- (i) the upper or lower signs in formulas correspond to the sectors  $S_{-2}, S_{-1}$  or  $S_0, S_1$ , respectively;
- (ii)  $[1] \stackrel{\text{def}}{=} 1 + O(\frac{1}{\rho^{\theta_0}})$  uniformly in  $x \in D_\varepsilon$ ,
- (iii) one and the same symbol  $\kappa(x, \rho)$  denotes various functions such that:
  - (1) uniformly in  $x \in D_\varepsilon$ ,  $\kappa(x, \rho) = O(1)$  as  $|\rho| \rightarrow \infty$ ,  $\rho \in S_k$ ,
  - (2) for each fixed  $\delta > 0$ ,  $\kappa(x, \rho) = [1]$  as  $|\rho| \rightarrow \infty$ ,  $\rho \in S_k^\delta$ .

**4.8.2.** Let  $\varphi(x, \lambda)$  be the solution of (4.8.1) under the initial conditions  $\varphi(0, \lambda) = 1$ ,  $\varphi'(0, \lambda) = h$  and denote

$$\Delta(\lambda) = \varphi'(1, \lambda) + h_1 \varphi(1, \lambda). \quad (4.8.7)$$

The function  $\lambda \mapsto \Delta(\lambda)$  is entire in  $\lambda$ , and its zeros coincide with the eigenvalues  $\{\lambda_n\}_{n \geq 0}$  of the boundary value problem  $L$ . The functions  $x \mapsto \varphi(x, \lambda_n)$  are eigenfunctions of  $L$ .

Let  $\Phi(x, \lambda)$  be the solution of (4.8.1) under the boundary conditions  $U(\Phi) = 1$ ,  $V(\Phi) = 0$ . We set  $M(\lambda) = \Phi(0, \lambda)$ . The functions  $x \mapsto \Phi(x, \lambda)$  and  $\lambda \mapsto M(\lambda)$  are called the *Weyl solution* and the *Weyl function* for the boundary value problem (4.8.1)-(4.8.3), respectively. Clearly

$$\langle \varphi(x, \lambda), \Phi(x, \lambda) \rangle \equiv 1 \quad (4.8.8)$$

for all  $x$  and  $\lambda$ , where  $\langle y, z \rangle := yz' - y'z$ .

The inverse problem studied in this section is formulated as follows. Suppose that the function  $R^2$  is known a priori. Our goal is to find  $q(x), h$  and  $h_1$  from the given Weyl function  $M$ .

In order to formulate and prove the uniqueness theorem for the solution of this inverse problem we agree that together with  $L = L(R^2(x), q(x), h, h_1)$  we consider a boundary value problem  $\tilde{L} = L(R^2(x), \tilde{q}(x), \tilde{h}, \tilde{h}_1)$  of the same form (4.8.1)-(4.8.3) but with different coefficients. If a certain symbol denotes an object related to  $L$ , then the corresponding symbol with tilde will denote the analogous object related to  $\tilde{L}$ .

**Theorem 4.8.1.** *If  $M = \tilde{M}$  then  $q(x) = \tilde{q}(x)$  for  $x \in I$ ,  $h = \tilde{h}$  and  $h_1 = \tilde{h}_1$ .*

Thus, the specification of the Weyl function  $M$  uniquely determines  $L$ .

*Proof.* Let us define the matrix  $P(x, \lambda) = [P_{jk}(x, \lambda)]_{j,k=1,2}$  by the formula

$$P(x, \lambda) \begin{bmatrix} \tilde{\varphi}(x, \lambda) & \tilde{\Phi}(x, \lambda) \\ \tilde{\varphi}'(x, \lambda) & \tilde{\Phi}'(x, \lambda) \end{bmatrix} = \begin{bmatrix} \varphi(x, \lambda) & \Phi(x, \lambda) \\ \varphi'(x, \lambda) & \Phi'(x, \lambda) \end{bmatrix}. \quad (4.8.9)$$

Using (4.8.8) we calculate

$$\begin{cases} P_{11}(x, \lambda) = \varphi(x, \lambda)\tilde{\Phi}'(x, \lambda) - \Phi(x, \lambda)\tilde{\varphi}'(x, \lambda) \\ P_{12}(x, \lambda) = \Phi(x, \lambda)\tilde{\varphi}(x, \lambda) - \varphi(x, \lambda)\tilde{\Phi}(x, \lambda) \end{cases} \quad (4.8.10)$$

Let us now study the asymptotic behavior of  $\varphi(x, \lambda)$ ,  $\Phi(x, \lambda)$  and  $P_{1j}(x, \lambda)$  as  $|\lambda| \rightarrow \infty$ . For this purpose we use the above-mentioned fundamental system of solutions  $\{z_1(x, \rho), z_2(x, \rho)\}$  in each fixed sector  $S_k$ ,  $k = -2, -1, 0, 1$ .

From the initial conditions on  $\varphi(x, \lambda)$  we calculate

$$\varphi(x, \lambda) = \frac{1}{w(\lambda)}(U(z_2(x, \rho))z_1(x, \rho) - U(z_1(x, \rho))z_2(x, \rho)), \quad (4.8.11)$$

where

$$w(\lambda) = \begin{vmatrix} z_1(0, \rho) & z_2(0, \rho) \\ z_1'(0, \rho) & z_2'(0, \rho) \end{vmatrix}.$$

Using (4.8.4)-(4.8.6) we get for  $|\rho| \rightarrow \infty, \rho \in S_k$

$$\begin{cases} U(z_1(x, \rho)) = (\pm i \rho)|R(0)|^{1/2} \exp(\mp i \frac{\pi}{2} \xi(0)) \kappa(\rho), \\ U(z_2(x, \rho)) = (\mp i \rho)|R(0)|^{1/2} \kappa(\rho), \\ w(\lambda) = \mp 2i \rho [1]; \end{cases} \quad (4.8.12)$$

here and in the sequel  $\kappa(\rho) = O(1)$  for  $\rho \in S_k$  and  $\kappa(\rho) = [1]$  for  $\rho \in S_k^\delta$  as  $|\rho| \rightarrow \infty$ .

Substituting (4.8.4), (4.8.5) and (4.8.12) into (4.8.11) we conclude that for  $|\rho| \rightarrow \infty, \rho \in S_k, x \in D_\varepsilon, j = 0, 1$ ,

$$\begin{aligned} \varphi^{(j)}(x, \lambda) &= \frac{1}{2}(\pm i \rho)^j |R(0)|^{1/2} |R(x)|^{j-1/2} (\exp(\mp i \frac{\pi}{2} \xi(x)))^j \\ &\times \exp(\rho \int_0^x |R_-(t)| dt) \exp(\pm i \rho \int_0^x |R_+(t)| dt) K_\pm(x) \kappa(x, \rho). \end{aligned} \quad (4.8.13)$$

Consequently, taking (4.8.7) into account, we have

$$\Delta(\lambda) = \frac{1}{2}(\pm i \rho) |R(0)R(1)|^{1/2} \exp(\rho \int_0^1 |R_-(t)| dt)$$

$$\times \exp(\pm i\rho \int_0^1 |R_+(t)|dt) K_{\pm}(1) \kappa(\rho), \quad \rho \in S_k, \quad |\rho| \rightarrow \infty. \quad (4.8.14)$$

From (4.8.13) and (4.8.14) we obtain the estimates:

$$\begin{aligned} |\varphi^{(j)}(x, \lambda)| &\leq C|\rho|^j \exp(\rho \int_0^x |R_-(t)|dt) \\ &\times \exp(\pm i\rho \int_0^x |R_+(t)|dt), \quad \rho \in S_k, \quad x \in D_{\varepsilon}, \end{aligned} \quad (4.8.15)$$

and

$$|\Delta(\lambda)| \leq C|\rho \exp(\rho \int_0^1 |R_-(t)|dt) \exp(\pm i\rho \int_0^1 |R_+(t)|dt)|, \quad \rho \in S_k. \quad (4.8.16)$$

Here and below one and the same symbol  $C$  denotes various positive constants in estimates. Moreover, it can be shown that

$$|\Delta(\lambda)| \geq C_{\delta} |\rho \exp(\rho \int_0^1 |R_-(t)|dt) \exp(\pm i\rho \int_0^1 |R_+(t)|dt)|, \quad \lambda \in G_{\delta}, \quad (4.8.17)$$

where

$$G_{\delta} = \{\lambda : |\lambda - \lambda_n| \geq \delta, \quad n \geq 0\}, \quad \delta > 0.$$

It follows from (4.8.15) and (4.8.16) that the functions  $\varphi^{(j)}(x, \cdot)$  and  $\Delta$  are entire functions of order  $1/2$ .

Let us go on to the Weyl solution  $\Phi(x, \lambda)$ . Applying the boundary conditions to  $\Phi(x, \lambda)$  we calculate

$$\Phi(x, \lambda) = \frac{1}{\Delta_0(\rho)} (V(z_2(x, \rho))z_1(x, \rho) - V(z_1(x, \rho))z_2(x, \rho)) \quad (4.8.18)$$

where

$$\Delta_0(\rho) = \begin{vmatrix} U(z_1(x, \rho)) & U(z_2(x, \rho)) \\ V(z_1(x, \rho)) & V(z_2(x, \rho)) \end{vmatrix}.$$

Taking (4.8.4)-(4.8.6) into account we obtain from (4.8.18) that for  $|\rho| \rightarrow \infty$ ,  $\rho \in S^{\delta}$ ,  $x \in D_{\varepsilon}$ ,  $j = 0, 1$ ,

$$\begin{aligned} \Phi^{(j)}(x, \lambda) &= (\mp i\rho)^{j-1} |R(0)|^{-1/2} |R(x)|^{j-1/2} (\exp(\mp i\frac{\pi}{2}\xi(x)))^j \\ &\times \exp(-\rho \int_0^x |R_-(t)|dt) \exp(\mp i\rho \int_0^x |R_+(t)|dt) K_{\pm}^*(x)[1] \end{aligned} \quad (4.8.19)$$

It follows from the boundary conditions on  $\varphi(x, \lambda)$  and  $\Phi(x, \lambda)$  that

$$\Phi(x, \lambda) = S(x, \lambda) + M(\lambda)\varphi(x, \lambda), \quad (4.8.20)$$

where  $S(x, \lambda)$  is the solution of (4.8.1) satisfying the conditions  $S(0, \lambda) = 0$ ,  $S'(0, \lambda) = 1$ . Using the fundamental system of solutions  $\{z_2(x, \rho), z_1(x, \rho)\}$  we get

$$S(x, \lambda) = \frac{1}{w(\lambda)} (z_1(0, \rho)z_2(x, \rho) - z_2(0, \rho)z_1(x, \rho)).$$

By virtue of (4.8.4)-(4.8.6), we infer that for  $\rho \in S_k$ ,  $x \in D_{\varepsilon}$ ,  $j = 0, 1$ ,

$$S^{(j)}(x, \lambda) = \frac{1}{2} (\pm i\rho)^{j-1} |R(0)|^{-1/2} |R(x)|^{j-1/2} (\exp(\mp i\frac{\pi}{2}\xi(x)))^j \exp(\pm i\frac{\pi}{2}\xi(0))$$

$$\times \exp(\pm i \frac{\pi}{2} \xi(0)) \exp(\rho \int_0^x |R_-(t)| dt) \exp(\pm i \rho \int_0^x |R_+(t)| dt) K_{\pm}(x) \kappa(x, \rho) \quad (4.8.21)$$

and

$$|S^{(j)}(x, \lambda)| \leq C |\rho|^{j-1} \exp(\rho \int_0^x |R_-(t)| dt) \exp(\pm i \rho \int_0^x |R_+(t)| dt). \quad (4.8.22)$$

It follows from (4.8.22) that the functions  $S^{(j)}(x, \cdot)$  are entire of order  $1/2$ . Using (4.8.10), (4.8.13) and (4.8.19) we get

$$P_{11}(x, \lambda) = [1], \quad P_{12}(x, \lambda) = O(\frac{1}{\rho^{\theta_0}}), \quad |\rho| \rightarrow \infty, \quad \rho \in S^\delta, \quad x \in D_\varepsilon. \quad (4.8.23)$$

Furthermore, substituting (4.8.20) into (4.8.10), we calculate

$$P_{11}(x, \lambda) = \varphi(x, \lambda) \tilde{S}'(x, \lambda) - S(x, \lambda) \tilde{\varphi}'(x, \lambda) + (\tilde{M}(\lambda) - M(\lambda)) \varphi(x, \lambda) \tilde{\varphi}'(x, \lambda),$$

$$P_{12}(x, \lambda) = S(x, \lambda) \tilde{\varphi}(x, \lambda) - \varphi(x, \lambda) \tilde{S}(x, \lambda) + (M(\lambda) - \tilde{M}(\lambda)) \varphi(x, \lambda) \tilde{\varphi}(x, \lambda).$$

By the hypothesis of Theorem 4.8.1,  $M(\lambda) \equiv \tilde{M}(\lambda)$ , and consequently the functions  $P_{11}(x, \lambda)$  and  $P_{12}(x, \lambda)$  are entire in  $\lambda$  of order  $1/2$ . It follows from this and (4.8.23) that  $P_{11}(x, \lambda) \equiv 1$ ,  $P_{12}(x, \lambda) \equiv 0$ . Substituting into (4.8.9) we obtain  $\varphi(x, \lambda) \equiv \tilde{\varphi}(x, \lambda)$ ,  $\Phi(x, \lambda) \equiv \tilde{\Phi}(x, \lambda)$  for all  $x$  and  $\lambda$ . Consequently,  $q(x) = \tilde{q}(x)$  for  $x \in I$ ,  $h = \tilde{h}$  and  $h_1 = \tilde{h}_1$ . Theorem 4.8.1 is proved.  $\square$

Let  $\{\lambda_n^1\}_{n \geq 0}$  be the sequence of eigenvalues of the boundary value problem  $L_1$  for equation (4.8.1) with the boundary conditions  $y(0) = V(y) = 0$ . Now we consider the inverse problem of recovering  $q(x)$ ,  $h$  and  $h_1$  from the given two spectra  $\{\lambda_n\}_{n \geq 0}$  of  $L$  and  $\{\lambda_n^1\}_{n \geq 0}$  of  $L_1$ .

**Theorem 4.8.2.** If  $\lambda_n = \tilde{\lambda}_n$ ,  $\lambda_n^1 = \tilde{\lambda}_n^1$  for all  $n \geq 0$ , then  $q(x) = \tilde{q}(x)$  for  $x \in I$ ,  $h = \tilde{h}$  and  $h_1 = \tilde{h}_1$ .

*Proof.* The function  $\Delta$  is entire of order  $1/2$ , and  $\Delta(\lambda)$  is uniquely determined by its zeros  $\{\lambda_n\}_{n \geq 0}$  via the formula

$$\Delta(\lambda) = A \prod_{n=0}^{\infty} (1 - \frac{\lambda}{\lambda_n}) \quad (\text{if } \lambda_n \neq 0 \forall n),$$

where the constant  $A$  is uniquely determined with the help of the asymptotic formula (4.8.14), (the case when  $\Delta(0) = 0$  requires minor modifications).

The eigenvalues  $\{\lambda_n^1\}_{n \geq 0}$  of the boundary value problem  $L_1$  coincide with zeros of the entire function  $\Delta_1(\lambda) = S'(1, \lambda) + h_1 S(1, \lambda)$  of order  $1/2$ . It follows from (4.8.21) that

$$\begin{aligned} \Delta_1(\lambda) &= \frac{1}{2} |R(0)|^{-1/2} |R(1)|^{1/2} \exp(\pm i \frac{\pi}{2} \xi(0)) \exp(\rho \int_0^1 |R_-(t)| dt) \\ &\quad \times \exp(\pm i \rho \int_0^1 |R_+(t)| dt) K_{\pm}(1) \kappa(\rho). \end{aligned} \quad (4.8.24)$$

The function  $\Delta_1$  is uniquely determined by its zeros  $\{\lambda_n^1\}_{n \geq 0}$  via the formula

$$\Delta_1(\lambda) = A_1 \prod_{n=0}^{\infty} (1 - \frac{\lambda}{\lambda_n^1}),$$

where the constant  $A_1$  is uniquely determined with the help of (4.8.24).

Since  $V(\Phi) = 0$ , it follows from (4.8.20) that

$$M(\lambda) = -\frac{\Delta_1(\lambda)}{\Delta(\lambda)}. \quad (4.8.25)$$

By hypothesis of Theorem 4.8.2,  $\lambda_n = \tilde{\lambda}_n$ ,  $\lambda_n^1 = \tilde{\lambda}_n^1$  for all  $n \geq 0$ , and consequently  $\Delta(\lambda) \equiv \tilde{\Delta}(\lambda)$ ,  $\Delta_1(\lambda) \equiv \tilde{\Delta}_1(\lambda)$ . From this, in view of (4.8.25), we get  $M(\lambda) \equiv \tilde{M}(\lambda)$ . Using Theorem 4.8.1 we conclude that  $q(x) = \tilde{q}(x)$  for  $x \in I$ ,  $h = \tilde{h}$  and  $h_1 = \tilde{h}_1$ ; hence Theorem 4.8.2 is proved.  $\square$

We consider now the inverse problem of recovering  $L$  from the so-called spectral data. For simplicity, we confine ourselves to the case when all zeros of  $\Delta(\lambda)$  are simple.

Denote

$$\alpha_n = \int_0^1 R^2(x) \varphi^2(x, \lambda_n) dx.$$

The data  $\{\lambda_n, \alpha_n\}_{n \geq 0}$  are called the *spectral data* of the boundary value problem  $L$ .

**Theorem 4.8.3.** *If  $\lambda_n = \tilde{\lambda}_n$ ,  $\alpha_n = \tilde{\alpha}_n$  for all  $n \geq 0$ , then  $q(x) = \tilde{q}(x)$  for  $x \in I$ ,  $h = \tilde{h}$  and  $h_1 = \tilde{h}_1$ .*

*Proof.* Since

$$\begin{aligned} -\Phi''(x, \lambda) + q(x)\Phi(x, \lambda) &= \lambda R^2(x)\Phi(x, \lambda), \\ -\varphi''(x, \lambda_n) + q(x)\varphi(x, \lambda_n) &= \lambda_n R^2(x)\varphi(x, \lambda_n), \end{aligned}$$

we get

$$(\Phi(x, \lambda)\varphi'(x, \lambda_n) - \Phi'(x, \lambda)\varphi(x, \lambda_n))' = (\lambda - \lambda_n)R^2(x)\Phi(x, \lambda)\varphi(x, \lambda_n)$$

and hence

$$(\lambda - \lambda_n) \int_0^1 R^2(x)\Phi(x, \lambda)\varphi(x, \lambda_n)dx = \left|_0^1 (\Phi(x, \lambda)\varphi'(x, \lambda_n) - \Phi'(x, \lambda)\varphi(x, \lambda_n)) = 1.\right.$$

In view of (4.8.20) and (4.8.25), we have

$$(\lambda - \lambda_n) \int_0^1 R^2(x)S(x, \lambda)\varphi(x, \lambda_n)dx - (\lambda - \lambda_n) \frac{\Delta_1(\lambda)}{\Delta(\lambda)} \int_0^1 R^2(x)\varphi(x, \lambda)\varphi(x, \lambda_n)dx = 1.$$

Let  $\lambda \rightarrow \lambda_n$ , then

$$\frac{\Delta_1(\lambda_n)}{\dot{\Delta}(\lambda_n)} \int_0^1 R^2(x)\varphi^2(x, \lambda_n)dx = -1,$$

where  $\dot{\Delta}(\lambda) = \frac{d}{d\lambda}\Delta(\lambda)$ . Hence

$$\alpha_n = -\frac{\dot{\Delta}(\lambda_n)}{\Delta_1(\lambda_n)}. \quad (4.8.26)$$

It follows from (4.8.25) that the Weyl function  $M$  is meromorphic with simple poles in the points  $\lambda_n, n \geq 0$ . Using (4.8.26) we calculate

$$\text{Res}_{\lambda=\lambda_n} M(\lambda) = \frac{1}{\alpha_n}. \quad (4.8.27)$$



Furthermore, by virtue of (4.8.19), we derive

$$M(\lambda) = \frac{1}{(\mp i\rho)|R(0)|} \cdot \exp(\pm i\frac{\pi}{2}\xi(0)), \quad |\rho| \rightarrow \infty, \quad \rho \in S^\delta,$$

and consequently

$$|M(\lambda)| \leq \frac{C}{|\rho|}, \quad |\rho| \rightarrow \infty, \quad \rho \in S^\delta. \quad (4.8.28)$$

Let us consider the function  $\lambda \mapsto N(\lambda) \stackrel{\text{def}}{=} M(\lambda) - \tilde{M}(\lambda)$ . By hypothesis of Theorem 4.8.3 and (4.8.27), we have

$$\text{Res}_{\lambda=\lambda_n} N(\lambda) = \frac{1}{\alpha_n} - \frac{1}{\tilde{\alpha}_n} = 0.$$

Thus, the function  $N$  is an entire function of order  $1/2$ . On the other hand, in view of (4.8.28),

$$|N(\lambda)| \leq \frac{C}{|\rho|}, \quad \rho \in S^\delta, \quad |\rho| \rightarrow \infty.$$

Consequently,  $N(\lambda) \equiv 0$ , i.e.  $M(\lambda) \equiv \tilde{M}(\lambda)$ . Using Theorem 4.8.1 we obtain  $q(x) = \tilde{q}(x)$  for  $x \in I$ ,  $h = \tilde{h}$  and  $h_1 = \tilde{h}_1$ . Theorem 4.8.3 is proved.  $\square$

**Remark 4.8.1.** Theorem 4.8.2 and 4.8.3 are generalizations of results by Borg and Marchenko (see Section 1.4) for the Sturm-Liouville equation without turning points where  $R^2(x) \equiv 1$ .

**Remark 4.8.2.** We can also apply this method to investigate the inverse problems for equation (4.8.1) with turning points in  $0$  or (and)  $1$ .

**4.8.3.** Let us now go on to constructing the solution of the inverse problem. The central role for solving the inverse problem is played by the so-called *main equation* of the inverse problem which connects the spectral characteristics with the corresponding solutions of the differential equation. We give a derivation of the main equation, which is a linear equation in the Banach space  $m$  of bounded sequences. Moreover we prove the unique solvability of the main equation. For simplicity, we confine ourselves to the most important particular case when  $m = 1$  and  $T_1 = IV$ , i.e. the weight-function changes sign exactly once. The general case can be treated analogously.

For deriving the main equation of the inverse problem we need more precise asymptotics for the solutions of equation (4.8.1). For definiteness, everywhere below  $\rho \in S_0 \cup S_1$  (the case  $\rho \in S_{-1} \cup S_{-2}$  is considered in the same way). It has been shown in [ebe4] that for  $|\rho| \rightarrow \infty$ ,  $j = 0, 1$  the following asymptotic formulas are valid

$$\left. \begin{aligned} z_1^{(j)}(x, \rho) &= \rho^j |R(x)|^{j-1/2} \exp(\rho \int_0^x |R(t)| dt) [1], \quad x < x_1, \\ z_1^{(j)}(x, \rho) &= (-i\rho)^j |R(x)|^{j-1/2} \exp(\rho \int_0^{x_1} |R(t)| dt) \frac{1}{2} \csc \frac{\pi \mu_1}{2} \exp(i \frac{\pi}{4}) \\ &\times (\exp(-i\rho \int_{x_1}^x |R(t)| dt) [1] + (-1)^{j+1} i \exp(i\rho \int_{x_1}^x |R(t)| dt) [1]), \quad x > x_1, \end{aligned} \right\} \quad (4.8.29)$$

$$\left. \begin{aligned} z_2^{(j)}(x, \rho) &= (-\rho)^j |R(x)|^{j-1/2} (-i \exp(-\rho \int_0^x |R(t)| dt) [1] \\ &+ (-1)^j \exp(\rho \int_0^x |R(t)| dt) \exp(-2\rho \int_0^{x_1} |R(t)| dt) [1]), \quad x < x_1, \\ z_2^{(j)}(x, \rho) &= (i\rho)^j |R(x)|^{j-1/2} 2 \sin \frac{\pi\mu_1}{2} \exp(-i\frac{\pi}{4}) \exp(-\rho \int_0^{x_1} |R(t)| dt) \\ &\times \exp(i\rho \int_{x_1}^x |R(t)| dt) [1], \quad x > x_1. \end{aligned} \right\} \quad (4.8.30)$$

Here, as above,  $[1] = 1 + O(\frac{1}{\rho^{\theta_0}})$  uniformly in  $x \in D_\varepsilon$ . It follows from (4.8.11), in view of (4.8.6), (4.8.29)-(4.8.30), that

$$\left. \begin{aligned} \varphi^{(j)}(x, \lambda) &= \frac{1}{2} \rho^j |R(0)|^{1/2} |R(x)|^{j-1/2} (\exp(\rho \int_0^x |R(t)| dt) [1] \\ &+ (-1)^j \exp(-\rho \int_0^x |R(t)| dt) [1]), \quad x < x_1, \\ \varphi^{(j)}(x, \lambda) &= \frac{1}{2} (i\rho)^j |R(0)|^{1/2} |R(x)|^{j-1/2} (A_1(\rho) \exp(i\rho \int_{x_1}^x |R(t)| dt) [1] \\ &+ (-1)^j A_2(\rho) \exp(-i\rho \int_{x_1}^x |R(t)| dt) [1]), \quad x > x_1, \end{aligned} \right\} \quad (4.8.31)$$

where

$$\left. \begin{aligned} A_2(\rho) &= \frac{1}{2} \csc \frac{\pi\mu_1}{2} \exp(i\frac{\pi}{4}) (\exp(\rho \int_0^{x_1} |R(t)| dt) [1] \\ &- i \exp(-\rho \int_0^{x_1} |R(t)| dt) [1]), \\ A_1(\rho) &= -i A_2(\rho) + 2 \sin \frac{\pi\mu_1}{2} \exp(i\frac{\pi}{4}) \exp(-\rho \int_0^{x_1} |R(t)| dt) [1]. \end{aligned} \right\} \quad (4.8.32)$$

**Remark 4.8.3.** Let  $\xi = \int_{x_1}^x |R(t)| dt$ . It follows from results of [ebe4] that (4.8.29)-(4.8.31) are also valid uniformly for  $|\rho\xi| \geq 1$  with  $[1] = 1 + O(\frac{1}{(\rho\xi)^{\theta_0}})$ ; moreover for  $|\rho\xi| \leq 1$  we have the estimates

$$\left. \begin{aligned} |\varphi(x, \lambda)| &\leq C |R(x)|^{-1/2} |\exp(\rho \int_0^x |R(t)| dt)|, \quad x \leq x_1 \\ |\varphi(x, \lambda)| &\leq C |R(x)|^{-1/2} |\exp(\rho \int_0^{x_1} |R(t)| dt)|, \quad x \geq x_1. \end{aligned} \right\} \quad (4.8.33)$$

**Lemma 4.8.1.** (i) The spectrum  $\{\lambda_k\}$  of the boundary value problem  $L$  consists of two sequences of eigenvalues:  $\{\lambda_k\} = \{\lambda_k^+\} \cup \{\lambda_k^-\}$ ,  $k \in \mathbf{N}$ , such that

$$\rho_k^\pm = \sqrt{\lambda_k^\pm} = \rho_{k,0}^\pm + O(\frac{1}{k^{\theta_0}}), \quad k \rightarrow +\infty, \quad (4.8.34)$$

where

$$\rho_{k,0}^+ = (\int_{x_1}^1 |R(t)| dt)^{-1} (k + \frac{1}{4})\pi, \quad \rho_{k,0}^- = (\int_0^{x_1} |R(t)| dt)^{-1} (k + \frac{1}{4})\pi i.$$

(ii) Denote  $\alpha_k^\pm = \int_0^1 R^2(x) \varphi^2(x, \lambda_k^\pm) dx$ , i.e.  $\{\alpha_k\} = \{\alpha_k^+\} \cup \{\alpha_k^-\}$ . Then

$$\left. \begin{aligned} \alpha_k^+ &= \frac{1}{2} |R(0)| \int_{x_1}^1 |R(t)| dt (\frac{1}{2} (\csc \frac{\pi\mu_1}{2})^2 \omega_k^2 (1 + O(\frac{1}{k^\sigma})), \quad k \rightarrow +\infty, \\ \alpha_k^- &= -\frac{1}{2} |R(0)| \int_0^{x_1} |R(t)| dt (1 + O(\frac{1}{k^\sigma})), \quad k \rightarrow +\infty, \end{aligned} \right\} \quad (4.8.35)$$

where

$$\omega_k = \exp(\rho_{k,0}^+ \int_0^{x_1} |R(t)|dt), \quad \sigma = \min(1 - \delta_0, \theta_0).$$

*Proof.* Substituting (4.8.31) into (4.8.7) we calculate

$$\begin{aligned} \Delta(\lambda) &= \frac{1}{2}(i\rho)|R(0)R(1)|^{1/2}(A_1(\rho) \exp(i\rho \int_{x_1}^x |R(t)|dt)[1] \\ &\quad - A_2(\rho) \exp(-i\rho \int_{x_1}^x |R(t)|dt)[1]). \end{aligned} \quad (4.8.36)$$

Let  $\rho \in S_0$ . It follows from (4.8.32) that

$$\begin{aligned} A_1(\rho) &= \frac{1}{2} \csc \frac{\pi\mu_1}{2} \exp(-i\frac{\pi}{4}) \exp(\rho \int_0^{x_1} |R(t)|dt), \\ A_2(\rho) &= \frac{1}{2} \csc \frac{\pi\mu_1}{2} \exp(i\frac{\pi}{4}) \exp(\rho \int_0^{x_1} |R(t)|dt)[1]. \end{aligned}$$

Hence, by virtue of (4.8.36), the equation  $\Delta(\lambda) = 0$  can be rewritten in the form

$$\exp(2i\rho \int_{x_1}^1 |R(t)|dt) = i[1].$$

This equation has a countable set of roots  $\rho_k^+$  such that  $\rho_k^+ = \rho_{k,0}^+ + O(\frac{1}{k^{\theta_0}})$ . Analogously, if  $\rho \in S_1$  then the equation  $\Delta(\lambda) = 0$  can be transformed to

$$\exp(2\rho \int_0^{x_1} |R(t)|dt) = i[1];$$

therefore this equation has a countable set of roots  $\rho_k^-$  such that  $\rho_k^- = \rho_{k,0}^- + O(\frac{1}{k^{\theta_0}})$ .

Let us now consider  $\alpha_k^-$ . Put  $\alpha_k^- = \alpha_{k,1}^- + \alpha_{k,2}^-$ , where

$$\alpha_{k,1}^- = \int_0^{x_1} R^2(x) \varphi^2(x, \lambda_k^-) dx, \quad \alpha_{k,2}^- = \int_{x_1}^1 R^2(x) \varphi^2(x, \lambda_k^-) dx.$$

Denote  $I_{k,1} = \{x \in [0, x_1] : |\xi \rho_k^-| \geq 1\}$ ,  $I_{k,2} = \{x \in [0, x_1] : |\xi \rho_k^-| \leq 1\}$ , where  $\xi = \int_{x_1}^x |R(t)|dt$ . According to (4.8.31),

$$\begin{aligned} \int_{I_{k,1}} R^2(x) \varphi^2(x, \lambda_k^-) dx &= -\frac{1}{4} |R(0)| \int_{I_{k,1}} |R(x)| (\exp(\rho \int_0^x |R(t)|dt)[1] \\ &\quad + \exp(-\rho \int_0^x |R(t)|dt)[1])^2_{/\rho=\rho_k^-} dx. \end{aligned}$$

The change of variables gives

$$\begin{aligned} \int_{I_{k,1}} R^2(x) \varphi^2(x, \lambda_k^-) dx &= -\frac{1}{4} |R(0)| \int_{1/|\rho|}^{\xi_1} (\exp(\rho(\xi_1 - \xi))(1 + O((\rho\xi)^{-\theta_0})) \\ &\quad + \exp(-\rho(\xi_1 - \xi))(1 + O((\rho\xi)^{-\theta_0}))^2 d\xi|_{\rho=\rho_k^-}, \end{aligned}$$

where  $\xi_1 = \int_0^{x_1} |R(t)|dt$ . Hence

$$\int_{I_{k,1}} R^2(x) \varphi^2(x, \lambda_k^-) dx = -\frac{1}{2} |R(0)| \int_0^{x_1} |R(x)| dx (1 + O(\frac{1}{k^\sigma})).$$

Furthermore, it follows from (4.8.33) that

$$|\int_{I_{k,2}} R^2(x) \varphi^2(x, \lambda_k^-) dx| \leq \int_{I_{k,2}} |R(x) \exp(2\rho \int_0^x |R(t)|dt)|_{\rho=\rho_k^-} dx.$$

In view of (4.8.34), the exponential is bounded, and consequently,

$$|\int_{I_{k,2}} R^2(x) \varphi^2(x, \lambda_k^-) dx| \leq \int_{I_{k,2}} |R(x)| dx = \int_0^{1/|\rho|} d\xi_{|\rho=\rho_k^-} = O(\frac{1}{k}).$$

Thus, we arrive at

$$\alpha_{k,1}^- = -\frac{1}{2} |R(0)| \int_0^{x_1} |R(x)| dx (1 + O(\frac{1}{k^\sigma})). \quad (4.8.37)$$

Let us now estimate  $\alpha_{k,2}^-$ . Denote  $J_{k,1} = \{x \in [x_1, 1] : |\xi \rho_k^-| \geq 1\}$ ,  $J_{k,2} = \{x \in [x_1, 1] : |\xi \rho_k^-| \leq 1\}$ . In the same way as above one can show that

$$|\int_{J_{k,2}} R^2(x) \varphi^2(x, \lambda_k^-) dx| = O(\frac{1}{k}).$$

Then

$$\alpha_{k,2}^- = \int_{J_{k,1}} R^2(x) \varphi^2(x, \lambda_k^-) dx + O(\frac{1}{k}).$$

Taking (4.8.31) into account, we get

$$\begin{aligned} \alpha_{k,2}^- &= -\frac{|R(0)|}{4} \int_{J_{k,1}} (A_1(\rho) \exp(i\rho \int_{x_1}^x |R(t)|dt)[1] \\ &\quad + A_2(\rho) \exp(-i\rho \int_{x_1}^x |R(t)|dt)[1])_{\rho=\rho_k^-}^2 dx + O(\frac{1}{k}). \end{aligned}$$

The integral seems to be unbounded as  $k \rightarrow +\infty$ . But it follows from (4.8.36) and (4.8.32) that

$$\frac{A_2(\rho)}{A_1(\rho)_{|\rho=\rho_k^-}} = \exp(2i\rho_k^- \int_{x_1}^1 |R(t)|dt)[1], \quad (4.8.38)$$

$$A_1(\rho_k^-) = 2 \exp(i\frac{\pi}{4}) \sin \frac{\pi\mu_1}{2} \exp(-\rho_k^- \int_0^{x_1} |R(t)|dt)[1]. \quad (4.8.39)$$

Then

$$\begin{aligned} \alpha_{k,2}^- &= -\frac{|R(0)|}{4} A_1(\rho_k^-) \int_{J_{k,1}} |R(x)| (\exp(i\rho \int_{x_1}^x |R(t)|dt)[1] \\ &\quad + \exp(2i\rho \int_{x_1}^1 |R(t)|dt) \exp(-i\rho \int_{x_1}^x |R(t)|dt)[1])_{\rho=\rho_k^-}^2 dx + O(\frac{1}{k}). \end{aligned}$$

After changing variables  $x \mapsto \xi = \int_{x_1}^x |R(t)|dt$  and integrating we arrive at the estimate

$$\alpha_{k,2}^- = O(\frac{1}{k^\sigma}).$$

Combining this with (4.8.37), we obtain (4.8.35) for  $\alpha_k^-$ . For  $\alpha_k^+$  the proof is analogous; hence Lemma 4.8.1 is proved.  $\square$

Now we go on to the derivation of the main equation of the inverse problem. We assume that the spectral data  $\{\lambda_k, \alpha_k\}_{k \geq 0}$  of  $L$  are given. Let  $\tilde{L} = L(R^2(x), \tilde{q}(x), \tilde{h}, \tilde{h}_1)$  be a certain known model boundary value problem with the same weight-function  $R^2(x)$  and with the spectral data  $\{\tilde{\lambda}_k, \tilde{\alpha}_k\}_{k \geq 0}$ .

**Lemma 4.8.2.** *For each fixed  $x \neq x_1$ , and  $k \rightarrow +\infty$  the following estimates hold:*

(i) *If  $x < x_1$  then*

$$\varphi(x, \lambda_k^-) = O(1), \quad \varphi(x, \tilde{\lambda}_k^-) = O(1), \quad (4.8.40)$$

$$\varphi(x, \lambda_k^-) - \varphi(x, \tilde{\lambda}_k^-) = O(\rho_k^- - \tilde{\rho}_k^-), \quad (4.8.41)$$

$$\left. \begin{aligned} \varphi(x, \lambda_k^+) &= O(\exp(\rho_{k,0}^+ \int_0^x |R(t)| dt)), \\ \varphi(x, \tilde{\lambda}_k^+) &= O(\exp(\rho_{k,0}^+ \int_0^x |R(t)| dt)), \end{aligned} \right\} \quad (4.8.42)$$

$$\varphi(x, \lambda_k^+) - \varphi(x, \tilde{\lambda}_k^+) = O(|\rho_k^+ - \tilde{\rho}_k^+| \exp(\rho_{k,0}^+ \int_0^x |R(t)| dt)). \quad (4.8.43)$$

(ii) *If  $x > x_1$  then*

$$\varphi(x, \lambda_k^-) = O(\exp(i\rho_{k,0}^- \int_{x_1}^x |R(t)| dt)), \quad (4.8.44)$$

$$\varphi(x, \tilde{\lambda}_k^-) = O(k^{-\theta_0} \exp(-i\rho_{k,0}^- \int_{x_1}^x |R(t)| dt)), \quad (4.8.45)$$

$$\varphi(x, \lambda_k^+) = O(\omega_k), \quad \varphi(x, \tilde{\lambda}_k^+) = O(\omega_k), \quad (4.8.46)$$

$$\varphi(x, \lambda_k^+) - \varphi(x, \tilde{\lambda}_k^+) = O(|\rho_k^+ - \tilde{\rho}_k^+| \omega_k). \quad (4.8.47)$$

We mark the difference in the estimates (4.8.44) and (4.8.46) for the functions  $\varphi(x, \lambda_k^-)$  and  $\varphi(x, \tilde{\lambda}_k^-)$ ; the real part of the exponent in (4.8.44) is negative but the real part of the exponent in (4.8.46) is positive.

*Proof.* Let  $x < x_1$ . The estimates (4.8.40) and (4.8.42) follow from (4.8.31) and the asymptotics of  $\lambda_k^+$  and  $\lambda_k^-$ . Using (4.8.40), (4.8.42) and Schwarz's lemma we obtain (4.8.41) and (4.8.43).

Let now  $x > x_1$ . For  $\lambda = \lambda_k^-$  and  $\lambda = \tilde{\lambda}_k^-$  we rewrite (4.8.31) as follows

$$\begin{aligned} \varphi(x, \lambda) &= \frac{1}{2} |R(0)|^{1/2} |R(x)|^{-1/2} A_1(\rho) (\exp(i\rho \int_{x_1}^x |R(t)| dt) [1] \\ &\quad + \frac{A_2(\rho)}{A_1(\rho)} \exp(-i\rho \int_{x_1}^x |R(t)| dt) [1]). \end{aligned} \quad (4.8.48)$$

From this, using (4.8.38) and (4.8.39), we obtain (4.8.44). Furthermore, it is easy to show that

$$\frac{A_2(\rho)}{A_1(\rho)} \Big|_{\rho=\tilde{\rho}_k^-} = O\left(\frac{1}{k^{\theta_0}}\right), \quad A_1(\tilde{\rho}_k^-) = O(1).$$

Substituting into (4.8.48) we arrive at (4.8.45). The estimates (4.8.46) and (4.8.47) are proved in the same way as (4.8.42) and (4.8.43). Lemma 4.8.2 is proved.  $\square$

**Lemma 4.8.3.** *For each fixed  $x \in I$ , the following relations hold*

$$\tilde{\varphi}(x, \lambda) = \varphi(x, \lambda) + \sum_k \left( \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \lambda_k) \rangle}{\alpha_k(\lambda - \lambda_k)} \varphi(x, \lambda_k) - \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \tilde{\lambda}_k) \rangle}{\tilde{\alpha}_k(\lambda - \tilde{\lambda}_k)} \varphi(x, \tilde{\lambda}_k) \right), \quad (4.8.49)$$

$$\begin{aligned} & \frac{\langle \varphi(x, \lambda), \varphi(x, \mu) \rangle}{\lambda - \mu} - \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle}{\lambda - \mu} \\ & + \sum_k \left( \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \lambda_k) \rangle \langle \varphi(x, \lambda_k), \varphi(x, \mu) \rangle}{\alpha_k(\lambda - \lambda_k)(\lambda_k - \mu)} - \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \tilde{\lambda}_k) \rangle \langle \varphi(x, \tilde{\lambda}_k), \varphi(x, \mu) \rangle}{\tilde{\alpha}_k(\lambda - \tilde{\lambda}_k)(\tilde{\lambda}_k - \mu)} \right) = 0. \end{aligned} \quad (4.8.50)$$

We omit the proof of Lemma 4.8.3 since the arguments here are the same as in Lemma 1.6.3.

Denote  $\lambda_{k0} = \lambda_k$ ,  $\lambda_{k1} = \tilde{\lambda}_k$ ,  $\alpha_{k0} = \alpha_k$ ,  $\alpha_{k1} = \tilde{\alpha}_k$ ,  $\varphi_{kj}(x) = \varphi(x, \lambda_{kj})$ ,  $\tilde{\varphi}_{kj}(x) = \tilde{\varphi}(x, \lambda_{kj})$ ,

$$\tilde{P}_{ni,kj}(x) = \frac{\langle \tilde{\varphi}_{ni}(x), \tilde{\varphi}_{kj}(x) \rangle}{\alpha_{kj}(\lambda_{nj} - \lambda_{kj})} = \frac{1}{\alpha_{kj}} \int_0^x R^2(t) \tilde{\varphi}_{ni}(t) \tilde{\varphi}_{kj}(t) dt,$$

$$P_{ni,kj}(x) = \frac{\langle \varphi_{ni}(x), \varphi_{kj}(x) \rangle}{\alpha_{kj}(\lambda_{ni} - \lambda_{kj})} = \frac{1}{\alpha_{kj}} \int_0^x R^2(t) \varphi_{ni}(t) \varphi_{kj}(t) dt.$$

It follows from (4.8.49) and (4.8.50) that

$$\tilde{\varphi}_{ni}(x) = \varphi_{ni}(x) + \sum_k (\tilde{P}_{ni,k0}(x) \varphi_{k0}(x) - \tilde{P}_{ni,k1}(x) \varphi_{k1}(x)), \quad (4.8.51)$$

$$P_{ni,\ell j}(x) - \tilde{P}_{ni,\ell j}(x) + \sum_k (\tilde{P}_{ni,k0}(x) P_{k0,\ell j}(x) - \tilde{P}_{ni,k1}(x) P_{k1,\ell j}(x)) = 0. \quad (4.8.52)$$

Let  $x < x_1$ . Denote

$$\gamma_k(x) = \begin{cases} \exp(\rho_{k,0}^+ \int_0^x |R(t)| dt), & \text{for } \lambda_k = \lambda_k^+ \\ 1, & \text{for } \lambda_k = \lambda_k^-, \end{cases}$$

$$\gamma_k = \gamma_k(x_1), \quad \xi_k = |\rho_k - \tilde{\rho}_k| + \frac{|\alpha_k - \tilde{\alpha}_k|}{\gamma_k^2}.$$

Clearly,  $\xi_k = O(k^{-\sigma})$ . It follows from (4.8.40)-(4.8.43) that

$$\left. \begin{aligned} |\varphi_{kj}(x)| &\leq C\gamma_k(x), \quad |\varphi_{k0}(x) - \varphi_{k1}(x)| \leq C\xi_k\gamma_k(x), \\ |P_{ni,kj}(x)| &\leq \frac{C\gamma_n(x)\gamma_k(x)}{(|n-k|+1)\gamma_k^2} \end{aligned} \right\}. \quad (4.8.53)$$

Let  $V$  be a set of indices  $v = (k, j)$ ,  $k \in \mathbf{N}$ ,  $j = 0, 1$ . Define the vector  $\psi(x) = [\psi_v(x)]_{v \in V} := [\psi_{00}, \psi_{01}, \psi_{10}, \psi_{11}, \dots]^T$  and similarly the block matrix

$$H(x) = [H_{u,v}(x)]_{u,v \in V} = \begin{bmatrix} H_{n0,k0}(x) & H_{n0,k1}(x) \\ H_{n1,k0}(x) & H_{n1,k1}(x) \end{bmatrix}_{n,k \in \mathbf{N}},$$

$u = (n, i)$ ,  $v = (k, j)$ , where

$$\psi_{k0}(x) = (\varphi_{k0}(x) - \varphi_{k1}(x))(\xi_k\gamma_k(x))^{-1}, \quad \psi_{k1}(x) = \varphi_{k1}(x)(\gamma_k(x))^{-1},$$

$$\begin{aligned}
H_{n0,k0}(x) &= (P_{n0,k0}(x) - P_{n1,k0}(x))\xi_k\gamma_k(x)(\xi_n\gamma_n(x))^{-1}, \\
H_{n1,k0}(x) &= P_{n1,k0}(x)\xi_k\gamma_k(x)(\gamma_n(x))^{-1}, \quad H_{n1,k1}(x) = (P_{n1,k0}(x) - P_{n1,k1}(x))\gamma_k(x)(\gamma_n(x))^{-1}, \\
H_{n0,k1}(x) &= (P_{n0,k0}(x) - P_{n1,k0}(x) - P_{n0,k1}(x) + P_{n1,k1}(x))\gamma_k(x)(\xi_n\gamma_n(x))^{-1}.
\end{aligned}$$

Analogously we define  $\tilde{\psi}(x)$ ,  $\tilde{H}(x)$ . It follows from (4.8.53) and Schwarz's lemma that for each fixed  $x \in (0, x_1)$ ,

$$|\psi_{ni}(x)| \leq C, \quad |H_{ni,kj}(x)| \leq \frac{C\xi_k}{(|n-k|+1)}. \quad (4.8.54)$$

Similarly,

$$|\tilde{\psi}_{ni}(x)| \leq C, \quad |\tilde{H}_{ni,kj}(x)| \leq \frac{C\xi_k}{(|n-k|+1)}. \quad (4.8.55)$$

Let us consider the Banach space  $m$  of bounded sequences  $\alpha = [\alpha_v]_{v \in V}$  with the norm  $\|\alpha\|_m = \sup_{v \in V} |\alpha_v|$ . It follows from (4.8.54) and (4.8.55) that for each fixed  $x \in (0, x_1)$  the operators  $E + \tilde{H}(x)$  and  $E - H(x)$  (here  $E$  is the identity operator), acting from  $m$  to  $m$ , are linear bounded operators, and

$$\|H(x)\|, \|\tilde{H}(x)\| \leq C \sup_n \sum_k \frac{\xi_k}{(|n-k|+1)} < \infty.$$

**Theorem 4.8.4.** *For each fixed  $x \in (0, x_1)$ , the vector  $\psi(x) \in m$  is the solution of the equation*

$$\tilde{\psi}(x) = (E + \tilde{H}(x))\psi(x) \quad (4.8.56)$$

*in the Banach space  $m$ . The operator  $E + \tilde{H}(x)$  has a bounded inverse operator, i.e. equation (4.8.56) is uniquely solvable.*

*Proof.* Indeed, taking into account our notations, we can rewrite (4.8.51) and (4.8.52) in the form

$$\tilde{\psi}(x) = (E + \tilde{H}(x))\psi(x), \quad (E + \tilde{H}(x))(E - H(x)) = E.$$

Interchanging places for  $L$  and  $\tilde{L}$  we obtain analogously

$$\psi(x) = (E - H(x))\tilde{\psi}(x), \quad (E - H(x))(E + \tilde{H}(x)) = E.$$

Hence the operator  $(E + \tilde{H}(x))^{-1}$  exists, and it is a linear bounded operator.  $\square$

Equation (4.8.56) is called the main equation of the inverse problem. Solving (4.8.56) we find the vector  $\psi(x)$ , and consequently, the functions  $\varphi_{ni}(x)$ . Since  $\varphi_{ni}(x) = \varphi(x, \lambda_{ni})$  are the solutions of (4.8.1), we can construct the function  $q(x)$  for  $x \in (0, x_1)$  and the coefficient  $h$ . Thus, the inverse problem has been solved for the interval  $x \in (0, x_1)$ . For  $x > x_1$  we can act analogously, starting from (4.8.51)-(4.8.52) and using (4.8.44)-(4.8.47) instead of (4.8.40)-(4.8.43), and construct  $q(x)$  for  $x \in (x_1, 1)$  and the coefficient  $h_1$ .

**Remark 4.8.4.** To construct  $q(x)$  for  $x \in (x_1, 1)$  we can act also in another way. Suppose that, using the main equation of the inverse problem, we have constructed  $q(x)$  for  $x \in (0, x_1)$  and  $h$ . Consequently, the solutions  $\varphi(x, \lambda)$  and  $S(x, \lambda)$  are known for  $x \in [0, x_1]$ . By virtue of (4.8.20), the solution  $\Phi(x, \lambda)$  is known for  $x \in [0, x_1]$  too. Let

$\psi(x, \lambda)$  be the solution of (4.8.1) under the conditions  $\psi(1, \lambda) = 1$ ,  $\psi'(1, \lambda) = -h_1$ . Clearly,  $\Phi(x, \lambda) = -\frac{\psi(x, \lambda)}{\Delta(\lambda)}$ . Thus, using  $q(x)$  for  $x \in [0, x_1]$ , we can construct the functions

$$\delta_j(\lambda) = \psi^{(j-1)}(x_1, \lambda), \quad j = 1, 2.$$

The functions  $\delta_j(\lambda)$  are characteristic functions of the boundary value problems  $Q_j$  for equation (4.8.1) on  $x \in (x_1, 1)$  with the conditions  $y^{(j-1)}(x_1) = y'(1) + h_1 y(1) = 0$ . Thus, we can reduce our problem to the inverse problem of recovering  $q(x)$ ,  $x \in (x_1, 1)$  from two spectra of boundary value problems  $Q_j$  on the interval  $(x_1, 1)$ . In this problem the weight-function  $R^2(x)$  does not change sign. We can treat this inverse problem by the same method as above. In this case the main equation will be simpler. We note that the case when the weight-function does not change sign were studied in more general cases in [Yur22] and other papers.



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