Chapter 11

Oscillation theory and the spectra of eigenvalues

The basic problems of the Sturm-Liouville theory are two: (1) to establish the existence of eigenvalues and eigenfunctions and describe them qualitatively and, to some extent, quantitatively and (2) to prove that an "arbitrary" function can be expressed as an infinite series of eigenfunctions. In this chapter we take up the first of these problems. The principal tool for this is the oscillation theory. It has extensions to other, related eigenvalue problems that are of interest in applications, and we shall take up two of these as well.

11.1 The Prüfer substitution

For a fixed value of the parameter λ the Sturm-Liouville differential equation takes the form

$$\frac{d}{dx}\left(P\left(x\right)\frac{du}{dx}\right)+Q\left(x\right)u=0. \tag{11.1}$$

We temporarily put aside the Sturm-Liouville problem and investigate this equation, under the assumptions

$$P\in C^{1}\text{ and }P>0,\ Q\in C\text{ on the interval }[a,b]. \tag{11.2}$$

Equations for which some solutions have two or more zeros in the open interval (a,b) will be called *oscillatory*[1]. Our first results will concern conditions

[1] One zero will not do: *any* equation of the form (11.1) has a solution with one zero.

guaranteeing that equation (11.1) is oscillatory and estimating the number of zeros that its solutions have. This is facilitated by a change of variables.

Define, in place of u and u' variables r and θ through the equations

$$u = r\sin\theta, \;\; Pu' = r\cos\theta. \tag{11.3}$$

This substitution, which is called the *Prüfer* substitution after its discoverer, represents a legitimate change of variables provided r is never zero. Since we shall always assume that equation (11.1) is solved with initial data

$$u\,(a) = u_0,\;\; u'\,(a) = u_0',\;\; u_0, u_0' \text{ not both zero,} \tag{11.4}$$

and since $r = \sqrt{u^2 + P^2u'^2}$, it is indeed never zero on the interval $[a, b]$. Differentiating the equations (11.3) and using the differential equation, we find the system

$$r' = \left(\frac{1}{P} - Q\right) r\sin\theta\cos\theta,\;\; \theta' = \frac{1}{P}\left(\cos\theta\right)^2 + Q\left(\sin\theta\right)^2. \tag{11.5}$$

This system is to be solved with initial data r_0, θ_0 such that

$$u_0 = r_0\sin\theta_0,\;\; P\,(a)\,u_0' = r_0\cos\theta_0. \tag{11.6}$$

For $\theta_0 \in [0, 2\pi)$ there is a unique solution of these equations for r_0 and θ_0. Likewise, given a solution of the system (11.5), one can easily check that equation (11.1) is satisfied. There is therefore a complete equivalence of these two systems; we'll use whichever is the more convenient in a given context.

The advantage of the Prüfer system for the study of the zeros of u are that (1) $u = 0$ whenever θ is a multiple of π and (2) the equation for θ is independent of r, i.e., is a first-order equation for θ alone. It is further true that for the study of zeros it is immaterial whether we consider u or $-u$, and for that reason we may assume that $u_0 \geq 0$. It is easy to verify that $\theta(a)$ can then be restricted to either of the half-open intervals $[0, \pi)$ and $(0, \pi]$; we choose the first of these[2] and can learn about the zeros of u by studying the first-order initial-value problem

$$\theta' = f\,(x, \theta) = \frac{1}{P}\left(\cos\theta\right)^2 + Q\left(\sin\theta\right)^2,\;\; \theta\,(a) = \gamma \in [0, \pi). \tag{11.7}$$

[2] We'll choose the second in connection with the right-hand boundary condition, to be considered later.

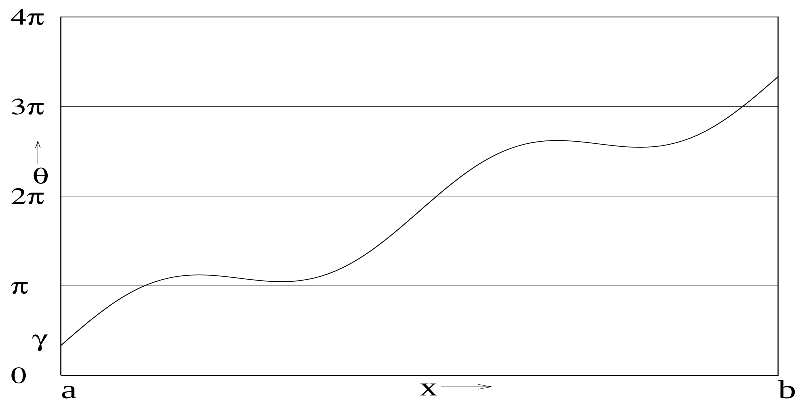

Figure 11.1: The behavior of the θ variable of the Prüfer substitution is illustrated here. It is not in general monotone-increasing, but once it reaches the value $n\pi$ at a certain point x_n, it remains greater than $n\pi$ for $x > x_n$.

The r-equation can be solved by quadratures if the solution to this equation is assumed known.

Since the solutions of equation (11.1) exist on the entire interval $[a, b]$, the same is true of equation (11.7) and it is not difficult to establish this directly (see the next problem set).

The solution θ of equation (11.7) need not be uniformly increasing on $[a, b]$, but it has the following similar property regarding points where θ is an integer multiple of π.

Proposition 11.1.1 *There is at most one value of $x \in [a, b]$ such that $\theta(x) = m\pi$, where m is an integer, say $x = x_m$. If $x < x_m$ we have $\theta(x) < m\pi$ and if $x > x_m$ we have $\theta(x) > m\pi$.*

Proof: Observe that if $x_m < b$ exists then

$$\theta'(x_m) = 1/P(x_m) > 0,$$

so that $\theta > m\pi$ at least on some sufficiently small interval to the right of x_m. Suppose for some $c > x_m$ that $\theta(c) \leq m\pi$. Consider the set $S = \{x \in (x_m, c] | \, \theta(x) \leq m\pi\}$. S has a greatest lower bound $x_* > x_m$ satisfying the conditions that $\theta - m\pi > 0$ on (x_m, x_*) and $\theta(x_*) - m\pi = 0$. It follows that $\theta'(x_*) \leq 0$. But the inequality above holds also at x_*: $\theta'(x_*) > 0$. This contradiction shows that there is no such point c and therefore that $\theta > m\pi$ for $x > x_m$. If there were another value $\tilde{x}$ where $\theta = m\pi$ then necessarily

$\tilde{x} < x_m$ and , on applying the reasoning above to $\tilde{x}$ we would arrive at the contradiction $\theta(x_m) > m\pi$. $\square$

This result is illustrated in Figure 11.1

11.2 Comparison theorems

In this section we present comparison theorems under assumptions appropriate to equations (11.7) and (11.1) above.

Theorem 11.2.1 *Suppose F and G are defined and continuous in a region $\Omega \subset R^2$ and $F(x,y) \geq G(x,y)$ there, and suppose further that each satisfies a Lipschitz condition with respect to y on Ω, with Lipschitz constant L. If there are functions $y(x)$ and $z(x)$ such that*

$$y' = F(x,y) \quad \text{and} \quad z' = G(x,z) \ \text{ on } (a,b), \tag{11.8}$$

and if, further, $y(a) \geq z(a)$, then $y(x) \geq z(x)$ on $[a,b]$.

Proof: Let $g(x) = z(x) - y(x)$. Then $g(a) \leq 0$ and the object is to show that $g \leq 0$ on $[a,b]$. Suppose to the contrary that for some $c \in (a,b)$ $g(c) > 0$. Consider the set

$$S = \{x \in [a,c] \mid g(x) \leq 0\}.$$

This set is bounded above by c and is not empty, so it has a least upper bound x_1. We must have $g(x_1) = 0$, and $g > 0$ on $(x_1, c]$. We'll consider the two equations on this interval.

The equivalent integral-equation formulations of equations (11.8) give

$$z(x) - y(x) = \int_{x_1}^{x} \{G(s, z(s)) - F(s, y(s))\}\, ds.$$

Rewrite the integrand:

$$G(s,z) - F(s,y) = (G(s,z) - G(s,y)) + (G(s,y) - F(s,y)).$$

The last terms on the right make a negative (or at least non-positive) contribution, so

$$z(x) - y(x) \leq \int_{x_1}^{x} \{G(s, z(s)) - G(s, y(s))\}\, ds.$$

The integrand is dominated by $L\left|z(s)-y(s)\right|$ where L is the Lipschitz constant. On the interval $(x_1,c]$ $g(s)=z(s)-y(s)>0$ so the absolute-value sign is not needed and the preceding equation reads

$$g(x) \le L\int_{x_1}^{x} g(s)\,ds.$$

It now follows from Gronwall's lemma (Lemma 1.4.1) that $g(x) \le 0$ on $(x_1,c]$, which is a contradiction. Consequently there is no such point c. □

This theorem can be strengthened by "running it backwards:"

Corollary 11.2.1 *Suppose that for some $x_* > a$ we have $y(x_*) = z(x_*)$. Then $y(x) = z(x)$ on $[a, x_*]$.*

Proof: Let $x = x_* - \xi$ so that for $0 \le \xi \le x_* - a$ we have $x \in [a, x_*]$. Set $y(x) = \eta(\xi)$ and $z(x) = \zeta(\xi)$. The differential equations become

$$\eta' = \Phi(\xi,\eta)\,,\;\; \zeta' = \Psi(\xi,\zeta)$$

with initial data $\eta(0) = \zeta(0)\,,$ $\Phi(\xi,\eta) = -F(x_* - \xi, \eta)$ and $\Psi(\xi,\zeta) = -G(x_* - \xi,\zeta)\,.$ We now have $\Psi \ge \Phi$ and conclude on applying the theorem to these initial-value problems that $\zeta \ge \eta$ on $[0, x_* - a]$, i.e., that $y(x) \le z(x)$ on $[a, x_*]$. But we know from the theorem that $y(x) \ge z(x)$ on this interval. □

We can apply these results to the θ equation of the Prüfer system. Consider the initial-value problems

$$\hat\theta' = \hat Q\left(\sin\hat\theta\right)^2 + \frac{1}{\hat P}\left(\cos\hat\theta\right)^2 \equiv G\left(x,\hat\theta\right),\;\; \hat\theta(a) = \hat\theta_0, \tag{11.9}$$

$$\theta' = Q(\sin\theta)^2 + \frac{1}{P}(\cos\theta)^2 \equiv F(x,\theta)\,,\;\; \theta(a) = \theta_0. \tag{11.10}$$

Here the region $\Omega = [a,b] \times R$ and the coefficients all satisfy the conditions (11.2).

Proposition 11.2.1 *Suppose that in equations (11.9) and (11.10)*

$$Q \ge \hat Q \text{ and } P \le \hat P.$$

If $\theta_0 \ge \hat\theta_0$, then $\theta(x) \ge \hat\theta(x)$ on $[a,b]$. Moreover, if for some $x_* > a$ we have $\theta(x_*) = \hat\theta(x_*)$, then $\theta(x) = \hat\theta(x)$ on $[a, x_*]$.

Proof: The assumptions imply that $F \geq G$, so the conclusion follows from Theorem 11.2.1 above and its corollary. $\square$

This result can be strengthened in a manner that will be useful below.

Corollary 11.2.2 *Suppose that one of the assumptions of Proposition 11.2.1 is strengthened: $Q > \hat{Q}$ on (a,b). Then $\theta(x) > \hat{\theta}(x)$ on $(a,b]$.*

Proof: Suppose, to the contrary, that $\theta(x_*) = \hat{\theta}(x_*)$ for some $x_* \in (a,b]$. By the Proposition $\theta = \hat{\theta}$ on $[a, x_*]$. It follows that

$$\left(Q - \hat{Q}\right)(\sin\theta)^2 + \left(\frac{1}{P} - \frac{1}{\hat{P}}\right)(\cos\theta)^2 = 0$$

on that interval. Each of the two terms is non-negative so each must vanish separately. Since $Q > \hat{Q}$ on (a,b) this is only possible if $\sin\theta = 0$ on (a, x_*), and hence by continuity on the interval $[a, x_*]$, and therefore θ is constant there. Equation (11.10) then leads to a contradiction. $\square$

Remark: An alternative hypothesis leading to the same conclusion is that $P < \hat{P}$ and Q does not vanish identically on any open subinterval of $[a,b]$.

Proposition 11.2.1 provides a comparison theorem for solutions of equation (11.1).

Theorem 11.2.2 *(Sturm Comparison Theorem) Let $\hat{u}$ be a nontrivial solution of the equation*

$$\frac{d}{dx}\left(\hat{P}\frac{d\hat{u}}{dx}\right) + \hat{Q}\hat{u} = 0$$

vanishing at points x_1 and x_2 in $[a,b]$. Let u be any solution of equation (11.1) on this interval, and suppose that $Q \geq \hat{Q}$ and $P \leq \hat{P}$ on $[x_1, x_2]$. Then u vanishes at least once on this interval. If $\hat{u}$ has k zeros on this interval then u has at least $k-1$ zeros there.

Proof: Consider the corresponding Prüfer equations (11.10) and (11.9). Since $\sin\hat{\theta}(x_1) = 0$, we may assign the value $\hat{\theta}(x_1) = 0$ without loss of generality. Similarly, at x_2, $\sin\hat{\theta} = 0$ and we have $\hat{\theta}(x_2) = m\pi$ where the integer m is at least equal to 1, by Proposition 11.1.1. For θ, the Prüfer variable corresponding to u, we may choose $\theta(x_1) \in [0, \pi)$. The conditions of Proposition 11.2.1 are satisfied and we infer that $\theta(x_2) \geq m\pi$. Therefore θ takes on the value π somewhere on the interval, and at that point $u = 0$. If $\hat{u}$ has k

zeros, including those at the endpoints, then u has at least $k-1$ zeros on $[x_1, x_2]$. $\square$

This comparison theorem provides the key to showing that equation (11.1) is oscillatory and to estimating the number of zeros of its solutions. For the comparison equation choose the constant coefficients

$$\hat{P} = P_M \text{ and } \hat{Q} = Q_m$$

where P_M is the maximum value of P on $[a, b]$ and Q_m is the minimum value of Q. The comparison equation is therefore

$$\hat{u}'' + \frac{Q_m}{P_M}\hat{u} = 0.$$

If Q_m/P_M is positive, this equation has the solution $\hat{u} = \sin(\kappa(x-a))$ on $[a, b]$, where $\kappa = \sqrt{Q_m/P_M}$. If $\kappa(b-a) > (k-1)\pi$, this has at least k zeros on $[a, b]$ and consequently any solution u of equation (11.1) must have at least $k-1$ zeros.

We need to be able to choose Q to be large. In the Sturm-Liouville theory the means for doing this is the presence of the parameter λ. We now turn to this.

11.3 The Sturm-Liouville Theorem

Recall the Sturm-Liouville system:

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + \{\lambda\rho(x) - q(x)\}u = 0, \quad a < x < b, \tag{11.11}$$

$$A[u] \equiv \alpha u(a) + \alpha' u'(a) = 0, \quad B[u] \equiv \beta u(b) + \beta' u'(b) = 0. \tag{11.12}$$

The conditions on the coefficients are now familiar and are given precisely in statement (10.12), but we remind the reader of two of these:

$$p(x) > 0 \text{ for each } x \text{ in the closed interval } [a, b] \tag{11.13}$$

and

$$\rho(x) > 0 \text{ for each } x \text{ at least in the open interval } (a, b). \tag{11.14}$$

Our objective is to prove the following

Theorem 11.3.1 *(Sturm-Liouville) The boundary-value problem (11.11), (11.12) has an infinite sequence of eigenvalues* $\{\lambda_n\}_0^\infty$ *with* $\lambda_n < \lambda_{n+1}$ *and* $\lambda_n \to +\infty$ *as* $n \to \infty$. *The eigenfunction* u_n *associated to* λ_n *has precisely* n *zeros in the open interval* (a, b).

We shall obtain this result through consideration of the θ-equation of the corresponding Prüfer system,

$$\theta' = (\lambda\rho(x) - q(x))(\sin\theta)^2 + \frac{1}{p(x)}(\cos\theta)^2, \quad \theta(a) = \gamma, \tag{11.15}$$

where $\gamma \in [0, \pi)$ is chosen so that the right-hand boundary condition $A[u] = 0$ is satisfied, i.e., γ is the unique number in $[0, \pi)$ such that

$$\alpha \sin\gamma + \alpha' (P(a))^{-1} \cos\gamma = 0. \tag{11.16}$$

The solution $\theta = \theta(x, \lambda)$ is a continuous function of λ for each $x \in [a, b]$, by the discussion in §6.3.

Lemma 11.3.1 *The solution* $\theta(x, \lambda)$ *of the initial-vlaue problem (11.15) is, for each fixed* $x \in (a, b]$, *a continuous, monotonically increasing function of* λ.

Proof: The functions P, Q of the preceding sections take the values $P = p$ and $Q = \lambda\rho - q$, respectively, and a choice of comparison equation has coefficients $\hat{P} = p$ and $\hat{Q} = \hat{\lambda}\rho - q$. Consequently, if $\lambda > \hat{\lambda}$ then $Q > \hat{Q}$ on (a, b), and Corollary 11.2.2 implies that $\theta(x, \lambda) > \theta\left(x, \hat{\lambda}\right)$ for any $x > a$. In other words, for any $x > a$, θ is a strictly increasing function of λ. $\square$

We denote by $x_n(\lambda)$ the nth zero of the solution u of equation (11.11) in (a, b), if it exists. The next lemma guarantees that it does exist and describes its behavior as function of λ.

Lemma 11.3.2 *Let* $u(x, \lambda)$ *be any solution of equation (11.11). For any* $n \geq 1$ *and sufficiently large* λ, $x_n(\lambda)$ *is defined and* $x_n(\lambda) \to a$ *as* $\lambda \to \infty$.

Proof: We first show that x_n exists if λ is large enough. Take for a comparison equation one with coefficients

$$\hat{P} = p_M, \quad \hat{Q} = \lambda\rho_m - q_M$$

where p_M, q_M denote the maximum values of p, q on $[a, b]$, and ρ_m the minimum value of ρ. Assume provisionally that $\rho_m > 0$. Then for a given positive value of λ the conditions of the Sturm Comparison Theorem hold. The comparison equation is

$$\hat{u}'' + \kappa^2 \hat{u} = 0, \quad \kappa^2 = \frac{\lambda \rho_m - q_M}{p_M},$$

which has a solution $\hat{u} = \sin(\kappa[x - a])$. If λ is large enough, κ^2 is not only positive but as large as we please. Choose λ so that $\kappa[b-a] > (n+1)\pi$. Then $\hat{u}$ has at least $n+1$ zeros in (a, b) and, by the Sturm Comparison Theorem, u has at least n zeros in this interval. This shows that $x_n(\lambda)$ exists for large enough λ. It also establishes the final statement that $x_n(\lambda) \to a$, for the k-th zero of the comparison equation in (a, b) is $\hat{x}_k = a + k\pi/\kappa$. For any k this tends to a as λ tends to infinity. But $x_n(\lambda) \leq \hat{x}_{n+1}(\lambda)$, again by the Sturm Comparison Theorem.

If ρ vanishes at one or both endpoints, the analysis above must be modified. Choose ϵ, δ to be arbitrarily small positive numbers and consider the comparison equation on the interval $[a_1, b_1]$, where $a_1 = a + \epsilon$ and $b_1 = b - \delta$; if ϵ, δ are chosen small enough, this is an interval slightly smaller than $[a, b]$. On this modified interval the minimum $\rho_m > 0$. Repeating the analysis above we find that $\hat{u}$ has $n+1$ zeros in (a_1, b_1) if λ is large enough, and that any solution of equation (11.11) has at least n zeros on this interval and therefore also on (a, b). This shows that $x_n(\lambda)$ exists. It also tends to a_1 as $\lambda \to \infty$, so for sufficiently large λ, $x_n(\lambda) - a_1 < \epsilon$ and therefore $x_n(\lambda) - a < 2\epsilon$. Since ϵ is arbitrary, this shows that $x_n \to a$. $\square$

Remark: We emphasize the phrase *any solution* in the statement of this lemma: irrespective of initial conditions, and for any $n \geq 1$, there exists a number Λ_n such that, if $\lambda > \Lambda_n$, $x_n(\lambda)$ exists and therefore $\theta(b, \lambda) > n\pi$.

We now obtain the decisive result, the asymptotic behavior of θ as $\lambda \to \pm\infty$.

Proposition 11.3.1 *For any $x > a$, $\theta(x, \lambda) \to +\infty$ as $\lambda \to +\infty$ and $\theta(x, \lambda) \to 0$ as $\lambda \to -\infty$.*

Proof: The first statement can be inferred from the preceding lemma, as follows. Since for any fixed n we have $x_n(\lambda) \to a$, for any $x > a$ we have $x_n(\lambda) < x$ for all sufficiently large λ. This means that $\theta(x, \lambda) > n\pi$, from which the first statement follows.

To prove the second statement we need to show that, given $\epsilon > 0$, there is a number Λ, depending in general on ϵ and on the choice of $x \in (a, b]$, such that $\theta(x, \lambda) < \epsilon$ provided $\lambda < \Lambda$. We proceed in stages.

1. Consider equation 11.15 for θ. The terms that are not multiplied by λ are bounded by $K = |q|_M + 1/p_m$, where the subscripts M and m denote the maximum and minimum values, respectively, that these functions take on the interval $[a, b]$. Therefore we have

$$\theta' \leq \lambda\rho(x)\,(sin\theta)^2 + K. \tag{11.17}$$

2. We first establish the result on the open interval (a, b) and return to a consideration of the right-hand point b later. We pick a point x_1 in this open interval.

3. The initial value $\theta(a, \lambda) = \gamma \in [0, \pi)$. For negative values of λ, $\theta' \leq K$ by equation (11.17) above. From this one infers via the mean-value theorem that $\theta(x, \lambda) \leq \gamma + K(x - a)$ and therefore, for sufficiently small values of $x - a$, $\theta < \pi$. In particular, given the value of ϵ, we can choose $a_1 \in (a, x_1)$ such that

$$\theta(a_1, \lambda) < \gamma_1 = \gamma + \epsilon < \pi - \epsilon.$$

   For this we may first have to rechoose ϵ to be smaller than the originally prescribed value: this is permitted. We choose it so that $\gamma + 2\epsilon < \pi$. Since the estimate K for θ' is independent of λ, so also is the choice of a_1, provided only that $\lambda \leq 0$.

4. The straight line $\theta = s(x)$ connecting the points a_1, γ_1 and x_1, ϵ in the $x - \theta$ plane has slope

$$m = -\left(\gamma_1 - \epsilon\right) / \left(x_1 - a_1\right),$$

   negative unless $\gamma = \gamma_1 - \epsilon = 0$. Let ρ_1 be the minimum value of ρ on $[a_1, x_1]$ and choose

$$\Lambda_1 = \left(m - K\right) / \rho_1 \left(\sin \epsilon\right)^2;$$

   we shall suppose $\lambda < \Lambda_1$. At $x = a_1, \theta(x) < s(x)$. If for some value of x in the interval $[a_1, x_1]$ $\theta(x) > s(x)$ then there must be a first such value x_*, at which $\theta(x_*) = s(x_*)$ and

$$\theta'(x_*) \geq m.$$

But then $\theta(x_*) \in [\epsilon, \pi - \epsilon]$ so that $\sin\theta(x_*) \geq \sin\epsilon$ and

$$m < \lambda\rho(x_*)\left(\sin\theta(x_*)\right)^2 + K \leq \lambda\rho_1\left(\sin\epsilon\right)^2 + K < m$$

if $\lambda < \Lambda_1$ as given above. This contradiction shows that indeed $\theta(x_1, \lambda) < \epsilon$ if $\lambda < \Lambda_1$.

5. We have thus far excluded the right-hand endpoint b from consideration because it is possible that $\rho(b) = 0$. Choose $b_1 < b$ so that $\theta(b) \leq \theta(b_1) + \epsilon$. This can be done, independently of λ for $\lambda < 0$, because of the estimate $\theta'(x, \lambda) < K$. Then applying the reasoning above at b_1, we infer that $\theta(b_1, \lambda) < \epsilon$ – and therefore $\theta(b, \lambda) < 2\epsilon$ – provided $\lambda < \Lambda$. This completes the proof. $\square$

Proof of Theorem 11.3.1: The initial condition on the function θ ensures that the left-hand boundary condition is satisfied. The corresponding solution u of equation (11.11) will be an eigenfunction if the second boundary condition is also satisfied. This will be so if $\theta(b, \lambda) = \delta + n\pi$ for $n = 0, 1, \ldots$, provided δ is such that

$$\beta \sin\delta + \beta'\left(p\left(b\right)\right)^{-1}\cos\delta = 0. \tag{11.18}$$

There is a unique value $\delta \in (0, \pi]$ satisfying this equation. Choosing this value, we now ask: can we find λ such that $\theta(b, \lambda) = \delta$? The answer is yes, because of Proposition 11.3.1; call this value λ_0 and the corresponding eigenfunction u_0. Since $\delta \leq \pi$, $\theta < \pi$ in (a, b) and therefore u does not vanish in this interval. Can we find λ such that $\theta(b, \lambda) = \delta + \pi$? Yes again, by virtue of Proposition 11.3.1, and the corresponding eigenfunction u_1 has a single zero in (a, b) since θ takes the value π there. Continuing in this way, we see that Proposition 11.3.1, in conjunction with Proposition 11.1.1, proves the Sturm-Liouville theorem. $\square$

PROBLEM SET 11.3.1

1. Show that for any real k, the equation $u'' - k^2u = 0$ is non-oscillatory, i.e., has at most one zero on any interval.

2. Verify that any solution of the Prüfer system (11.5) provides a solution of equation (11.1) through the relations (11.3).

3. Let u be a solution of equation (11.1) that has zeros at points x_1 and x_2 in (a, b). Suppose v is any linearly independent solution. Show that v has a zero in the interval (x_1, x_2).

4. Let P be as described in the conditions (11.2) and suppose $Q = 1/P$. Solve equation (11.1) in quadratures (i.e., find a formula requiring only the integration of known functions).

5. Show that the function f defined by equation (11.7) satisfies a Lipschitz condition in the region $\Omega = [a, b] \times R$ and estimate the Lipschitz constant. Argue that equation (11.3) has a solution on $[a, b]$ without referring to equation (11.1).

6. In Theorem 11.2.1 it is assumed that each of the functions F and G satisfies a Lipschitz condition.

   (a) Show that it suffices if one them satisfies a Lipschitz condition.

   (b) Show that some condition stronger than continuity must be imposed on at least one of the functions by producing a counterexample otherwise (Hint: consider equations with non-unique solutions like $y' = \sqrt{y}$).

7. Prove the remark following Corollary 11.2.2.

8. Suppose that $q > 0$ in equation (11.11), and that the boundary conditions are $u\,(a) = u\,(b) = 0$. Show that all eigenvalues are positive.

9. Suppose that ρ is a positive, decreasing function, q a positive, strictly increasing function on (a, b), i.e., if $x_2 > x_1$ then

$$\rho(x_2) \le \rho(x_1),\ q(x_2) > q(x_1),\ \rho > 0, q > 0 \text{ on } (a, b).$$

   Let u be an eigenfunction of the Sturm-Liouville problem (11.11), (11.12) with consecutive zeros at a_1 and a_2, and suppose that $u > 0$ on the interval (a_1, a_2). Show that u has a single maximum on that interval (Hint: if it has two, then there are three points of the interval at which $u' = 0$; examine the sign of pu''/u at these points).

10. For the eigenvalues $\{\lambda_k\}_0^\infty$ of the Sturm-Liouville problem, show that there exists a number $h > 0$ such that $\lambda_{k+1} - \lambda_k \ge h$ for all $k = 0, 1, 2, \ldots$ (Hint: assume the contrary and use the fact that the continuous function $\theta(b, \lambda)$ satisfies the condition $\theta(b, \lambda_{k+1}) - \theta(b, \lambda_k) = \pi$).

11. Consider the eigenvalue problem

$$u'' + \lambda u = 0,\ 0 < x < \pi;\ \ u(0) = 0,\ u(\pi) + u'(\pi) = 0.$$

Describe the eigenfunctions and give a formula determining the eigenvalues.

12. Consider the problem

$$u^{iv} + \lambda u = 0,\ 0 < x < \pi;\ \ u = u'' = 0 \text{ at } 0 \text{ and } \pi.$$

Show that all eigenvalues are real. What are the signs of the eigenvalues? What are they?

11.4 The nature of the spectrum when ρ changes sign

One can envision numerous modifications of the conditions leading to the basic theorem (11.3.1) above. We address one of these in this section, namely the relaxation of the condition that the coefficient ρ be positive on the entire interval (a, b). The object of this section is to prove the following:

Theorem 11.4.1 *Consider the boundary-value problem consisting of equations (11.11) and (11.12) under the condition that the continuous function ρ take different signs in different subintervals of the interval (a, b). Then there is a sequence of eigenvalues $\left\{\lambda_k^+\right\}_{k=0}^{\infty}$ for which $\lambda_k^+ \to +\infty$ as $k \to \infty$, and a sequence of eigenvalues $\left\{\lambda_k^-\right\}_{k=0}^{\infty}$ for which $\lambda_k^- \to -\infty$ as $k \to \infty$.*

This guarantees not only that there are negative eigenvalues in the case when ρ changes sign but also that there are infinitely many of them. No statement is made regarding the numbers of interior zeros of the eigenfunctions. There are important applications where ρ indeed changes sign on the interval. An extended example follows.

Example 11.4.1 A paradigmatic problem in hydrodynamics is that of the flow of an inviscid liquid between coaxial cylinders. Any flow for which the velocity field, expressed in cylindrical coordinates r, ϕ, z, has the structure

$$U_r = 0, U_\phi = r\Omega(r), U_z = 0, \tag{11.19}$$

represents a time-independent solution of the underlying fluid-dynamical equations (the so-called Euler equations). Here the function Ω is quite arbitrary, apart from smoothness conditions (we shall assume that it is C^1 in the interval (R_1, R_2), where R_1 and R_2 are the radii of the inner and outer bounding cylnders). It is important to distinguish among different choices of this function Ω on the basis of their *stability*. A criterion was proposed by Lord Rayleigh. Rayleigh's criterion is that the flow determined by the function $\Omega(r)$ is stable to small, axially symmetric perturbations of the velocity field provided that

$$\frac{d}{dr}\left(r^2\Omega(r)\right)^2 \geq 0 \tag{11.20}$$

for all values of the radial coordinate r in the interval (R_1, R_2); the implication is further that the flow is unstable if this criterion is violated at any point of the interval.

The limitation to axially symmetric perturbations implies that the perturbations u_r, u_ϕ, u_z to the time-independent velocity field (11.19) are independent of the angle coordinate ϕ. Thus a conclusion of stability under such conditions leaves open the possibility of instability to more general perturbations (i.e. to those that are not axially symmetric), while a conclusion of instability is decisive, since the existence of any motion that departs from the flow (11.19) shows that that flow cannot be stable. Our discussion is limited to the case of axially symmetric perturbations.

By considering the representative perturbation of the radial velocity in the form

$$u_r = u(r)e^{i(\sigma t + kz)} \tag{11.21}$$

one derives[3] for the equation governing u and σ the eigenvalue problem

$$\left(DD_* - k^2\right)u + \frac{k^2}{\sigma^2}\Phi(r)u = 0;\; u(R_1) = u(R_2) = 0, \tag{11.22}$$

where

$$D = d/dr, D_* = D + 1/r \text{ and } \Phi(r) = \frac{1}{r^3}\frac{d}{dr}\left(r^2\Omega(r)\right)^2.$$

Thus Rayleigh's criterion for stability is that $\Phi(r) \geq 0$ for each value of r in the interval R_1, R_2. This problem can be rewritten in the form

$$\frac{d}{dr}\left(p(r)\frac{dv}{dr}\right) + (\lambda\rho - q)\, v = 0, \tag{11.23}$$

[3]See the monograph Hydrodynamic Stability by Drazin and Reid for a derivation of the basic equation (11.22).

$$v\left(R_1\right)=0, \quad v\left(R_2\right)=0 \tag{11.24}$$

where

$$v(r)=ru(r),\ p(r)=1/r,\ \lambda=k^2/\sigma^2, \rho(r)=\frac{1}{r}\Phi(r) \text{ and } q(r)=k^2/r.$$

This is exactly in the form of the Sturm-Liouville problem consisting of equations (11.11) and boundary conditions (11.12), with the modification that the function ρ is not necessarily positive on the interval (R_1, R_2).

In the case when ρ *is* positive on this interval, i.e., in the case when Rayleigh's criterion is satisfied, we indeed find that the time-independent solution (11.19) is stable, as predicted. This can be shown in the following way. Refer to Problem (8) of the preceding problem set; this establishes that any eigenvalue λ is necessarily positive. Since $\lambda = k^2/\sigma^2$, this implies that σ is real, that is, that the time dependence $\exp i\sigma t$ of the perturbation is oscillatory, and the perturbation is therefore bounded in time and, in particular, remains small if it is initially small.

However, if there should be a subinterval of (R_1, R_2) where $\rho(r) < 0$, the conclusion above cannot be drawn. Indeed, in that case, according to Theorem 11.4.1 above, there are negative eigenvalues λ. The reasoning leading to the structure of the perturbation in the form (11.21) then shows that there are values of σ in the form $\sigma = \pm is$ with s real and positive. The choice $\sigma = -is$ then leads to a time dependence $\exp(st)$, which increases without bound, so an initially small perturbation does not remain small. This verifies that the flow (11.19) is unstable if it violates Rayleigh's criterion. □

To fix ideas, we restrict consideration to the special case of Theorem (11.4.1) when ρ suffers only one sign change, at x_*, and assume that

$$\rho(x)>0 \text{ on } (a, x_*),\ \rho(x_*)=0,\ \rho(x)<0 \text{ on } (x_*, b). \tag{11.25}$$

Before turning to the proof of Theorem (11.4.1), we note the following variant of Lemma (11.3.2) above:

Lemma 11.4.1 *Let $u(x,\lambda)$ be any solution of equation (11.11) with ρ as described by the conditions (11.25) above. Given any integer $n \geq 1$, u has n zeros on the interval (x_*, b) provided λ is negative and sufficiently large in absolute value.*

The proof of this is an obvious modification of the proof of Lemma (11.3.2); we recall the remark there that this conclusion is independent of the initial values for u at x_*.

Proof of Theorem (11.4.1):
We begin the analysis with equation (11.15) above. The function $u(x)$ related to the solution $\theta(x)$ of this equation via equation (11.3) is an eigenfunction of the system (11.11), (11.12) if $\theta(a) = \gamma$ and $\theta(b) = \delta + n\pi$, where n is a non-negative integer and, as in equations (11.16) and (11.18), γ and δ are the unique numbers, in the intervals $[0, \pi)$ and $(0, \pi]$ respectively, assuring that the boundary data (11.12) are satisfied.

Since on the interval (a, x_*) ρ is positive, Proposition (11.3.1) also holds on this interval. Consider then a solution of equation (11.15) satisfying the initial condition $\theta(a, \lambda) = \gamma$, ensuring that the left-hand boundary condition of the Sturm-Liouville system is satisfied. The value of $\theta(x_*, \lambda)$ can be controlled by choosing λ. In particular, it can take on any value in the interval $(0, \infty)$ for appropriate choice of λ. Choose a provisional value λ_0 such that $\theta(x_*, \lambda_0) = \pi$. By Proposition (11.1.1) the solution $u(x, \lambda_0)$ does not vanish in the open interval (a, x_*), but $u(x_*, \lambda_0) = 0$, and $\theta(x, \lambda_0) > \pi$ for $x > x_*$. Consider in particular $\theta(b, \lambda_0)$. Choose the least integer N such that

$$N \geq \left(\theta(b, \lambda_0) - \delta\right)/\pi;$$

then $N \geq 1$ and $\theta(b, \lambda_0)$ lies in an interval of the form $((N-1)\pi + \delta, N\pi + \delta]$. If it should happen that $\theta(b, \lambda_0) = N\pi + \delta$, then the corresponding function $u_0(x)$ would be an eigenfunction and λ_0 an eigenvalue. The eigenfunction u_0 would have no zeros on the interval (a, x_*); on the interval (x_*, b) it would have $N - 1$ zeros.

In the general case $\theta(b, \lambda_0) < N\pi + \delta$. Let λ decrease. Since $\theta(x, \lambda)$ is continuous and increases without bound as λ decreases for each fixed $x \in (x_*, b]$, it follows that there is a first value of λ, say λ_0^- for which $\theta(b, \lambda) = N\pi + \delta$.[4] This is an eigenvalue and the corresponding function $u(x, \lambda_0^-)$ is an eigenfunction. Since by making λ more negative we can infer that $\theta(b, \lambda)$ tends to ∞, it must take on the successive values $(N+1)\pi+\delta, (N+2)\pi+\delta, \ldots$.

[4]Here it is important that we may rely on Lemma 11.4.1 above, rather than on Lemma 11.3.1 and Propostion 11.3.1, which are valid when the value of θ is fixed at the left-hand endpoint; in the present case, as λ decreases, so also does the value $\theta(x_*, \lambda)$ at the left-hand endpoint.

This shows that there is an infinite sequence of eigenvalues $\left\{\lambda_k^-\right\}_{k=0}^{\infty}$ with $\lambda_k^- \to -\infty$ as $l \to \infty$.

Now consider u on the interval (a, x_*). Recall that for $\lambda = \lambda_0\ \theta(x_*, \lambda_0) = \pi$ and $\theta(b, \lambda_0)$ lies in the interval $((N-1)\pi + \delta, N\pi + \delta]$. Now let λ increase. As λ increases, the number of zeros of u on the interval (a, x_*) increases without bound and therefore $\theta(x_*, \lambda)$ increases without bound. For some value of λ we must have $\theta(x_*, \lambda) = (N+1)\pi$ and therefore $\theta(b, \lambda) > (N+1)\pi$, by Proposition (11.1.1). There must therefore be a somewhat smaller value of λ, say λ_0^+ such that $\theta(b, \lambda_0^+) = N\pi + \delta$. The corresponding solution u of equation (11.11) is then an eigenfunction and λ_0^+ an eigenvalue. Increasing λ further we find that $\theta(b, \lambda)$ must successively take on the values $(N+2)\pi + \delta, (N+3)\pi + \delta, \ldots$ as λ takes on values $\lambda_1^+, \lambda_2^+, \ldots$. This provides a sequence of eigenvalues tending to $+\infty$. $\square$

The proof of Theorem (11.4.1) shows that there is a sequence of eigenvalues ($\{\lambda_k^-\}$) with the property that the corresponding eigenfunctions have no zeros in the interval (a, x_*), all zeros being confined to the interval (x_*, b) where ρ is negative. It leaves open the possibility that there may be further eigenfunctions belonging to negative eigenvalues that *do* have a zero or zeros in the interval (a, x_*). A similar remark may be made regarding the zeros of the eigenfunctions belonging to the sequence $\{\lambda_k^+\}$ of eigenvalues tending to $+\infty$. Thus the qualitative desription of the eigenfunctions is less complete than in the case when $\rho > 0$ in (a, b)

PROBLEM SET 11.4.1

1. Consider the boundary-value problem

$$u'' + \lambda\rho(x)u = 0,\ -a < x < a;\ u(-a) = u(a) = 0$$

   in the case when ρ is an odd function of x: $\rho(-x) = -\rho(x)$. Show directly (i.e., without referring to Theorem 11.4.1) that to each positive eigenvalue with eigenfunction $u(x)$ there is a negative eigenvalue and corresponding eigenfunction. Can $\lambda = 0$ be an eigenvalue?

2. In the preceding problem let $a = \pi$ and take for ρ the step function

$$\rho(x) = \begin{cases} +1 \text{ if } -\pi < x \le 0 \\ -1 \text{ if } 0 < x < \pi \end{cases}$$

Since ρ is discontinuous at $x = 0$ so is u'' in general so we generalize the notion of a solution to functions u that are C^2 at points $x \neq 0$ but need be only C^1 at $x = 0$. On this understanding find the relation determining the eigenvalues for this problem.

11.5 The periodic case

The standard Sturm-Lioville problem has separated endpoint conditions. This is a commonly occurring case, but it does rule out certain physically interesting possibilities. We explore one of these here, that when the problem is posed on the entire real axis and the coefficients are periodic:

$$p(t+1) = p(t), \quad \rho(t+1) = \rho(t) \text{ and } q(t+1) = q(t). \tag{11.26}$$

Here we have (without loss of generality) taken the period to be one, and have denoted the independent variable by t rather than x, in keeping with custom when in typical applications the independent variable represents time rather than space. The coefficients are defined on the entire real axis once they are given in the fundamental interval $[0,1]$, and are assumed to satisfy the conditions (10.12) on that interval. In particular, we revert to the assumption that ρ is positive at least on the open interval $(0,1)$.

Suppose that equation (11.11) is to be solved on the interval $[0,1]$ with the (non-separated) endpoint conditions

$$u(0) = u(1) \text{ and } u'(0) = u'(1). \tag{11.27}$$

It is easy to see that any such solution, when continued to the real t axis, will be periodic with period one (see Problem 6 of Exercises 10.3.1). As usual, $u \equiv 0$ is a solution for any choice of the parameter λ, but we seek nontrivial solutions, i.e., eigenfunctions; we shall find that these eigenfunctions will again require that the λ be chosen to be one of a sequence of eigenvalues. Similarly, consider equation (11.11) on $[0,1]$ with the further, nonseparated boundary conditions:

$$u(0) = -u(1) \text{ and } u'(0) = -u'(1), \tag{11.28}$$

In this case one finds that $u(t+1) = -u(t)$, implying that $u(t+2) = u(t)$, i.e., the eigenfunctions in this case represent functions having twice the period of the coefficients.

Definition 11.5.1 *We'll refer to the problems consisting of equation (11.11) together with either of the boundary conditions (11.27) or (11.28) collectively as* the periodic problem

We begin by defining, independently of any boundary data, a standard basis of solutions $\phi(t,\lambda)$ and $\psi(t,\lambda)$ of equation (11.11), satisfying the initial conditions

$$\phi(0,\lambda) = 1, \phi'(0,\lambda) = 0;\ \psi(0,\lambda) = 0, \psi'(0,\lambda) = 1. \tag{11.29}$$

Any solution u of equation (11.11) can be written

$$u(t,\lambda) = a\phi(t,\lambda) + b\psi(t,\lambda) \tag{11.30}$$

for some choice of constants a and b. Imposing on this solution the boundary conditions (11.27) leads to the system of equations

$$\begin{pmatrix} \phi(1,\lambda) & \psi(1,\lambda) \\ \phi'(1,\lambda) & \psi'(1,\lambda) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Thus the matrix

$$A(\lambda) = \begin{pmatrix} \phi(1,\lambda) & \psi(1,\lambda) \\ \phi'(1,\lambda) & \psi'(1,\lambda) \end{pmatrix}$$

must possess the eigenvalue 1. Similarly, in order for the boundary conditions (11.28) to be satisfied, the matrix $A(\lambda)$ must possess the eigenvalue -1.

The eigenvalues of the matrix $A(\lambda)$ are the values of μ such that the determinant

$$\det\,(A - \mu I) = \begin{vmatrix} \phi(1,\lambda) - \mu & \psi(1,\lambda) \\ \phi'(1,\lambda) & \psi'(1,\lambda) - \mu \end{vmatrix}$$

vanishes. This determinant may be expressed as

$$\det\,(A - \mu I) = \mu^2 - f(\lambda)\mu + W(1,\lambda)$$

where

$$f(\lambda) = \phi(1,\lambda) + \psi'(1,\lambda), \tag{11.31}$$

and $W(1,\lambda)$ is precisely the Wronskian $W(t,\lambda)$ of the solutions ϕ and ψ evaluated at $t = 1$. For the self-adjoint form of the equation (11.11) the expression (2.17) for the Wronskian takes the form

$$p(t)W(t,\lambda) = p(0)W(0,\lambda).$$

Noting that $W(0,\lambda) = 1$ and that $p(1) = p(0)$, we now find that $W(1,\lambda) = 1$, and the characteristic equation for the eigenvalues μ of the matrix $A(\lambda)$ takes the simple form

$$\mu^2 + f(\lambda)\mu + 1 = 0. \tag{11.32}$$

Denote the two roots of this equation by μ_+ and μ_-; they are given by the formula

$$\mu_\pm = -(1/2)f(\lambda) \pm \sqrt{f(\lambda)^2 - 4}. \tag{11.33}$$

If $f(\lambda) < -2$ or $f(\lambda) > +2$, these roots are real, one greater than one in absolute value and the other less (Note that $\mu_+\mu_- = 1$). Otherwise they form a complex-conjugate pair of modulus one.

There is a closely related stability problem in which these roots play an important role.

Example 11.5.1 *Stability of the origin* Consider the initial-value problem for equation (11.11) on the interval $(0,\infty)$. The solution $u \equiv 0$ is valid for any value of λ. We may inquire into its stability, in the following way. Can we choose the numbers a and b in equation (11.30) so that the size of the solution increases without bound as $t \to \infty$, or is it the case that the solution remains bounded for any choice of a and b? In the first case we say the origin is unstable, in the second stable (compare the definitions in §8).

Consider first the case when, for the given value of λ, one of the roots (11.33) satisfies the condition $|\mu| > 1$, i.e., $|f(\lambda)| > 2$. Then this root is real and the corresponding corresponding eigenvector of coefficients, $(a,b) = \xi$ (say), of the matrix $A(\lambda)$ may likewise be chosen to be real. Choose this eigenvector as the coefficient vector. After one period, the coefficient vector ξ has become $A(\lambda)\xi$. On the next interval, from time $t = 1$ to time $t = 2$, the equation is the same as on the preceding interval, so the initial-value vector has become $A(\lambda) \cdot A(\lambda)\xi = A(\lambda)^2\xi$. In this way, at a time $t = n$, the coefficient vector ξ has grown to $A(\lambda)^n\xi = \mu^n\xi$, and is clearly growing without bound. A criterion for instability is therefore that $|f(\lambda)| > 2$.

On the other hand, when $|f(\lambda)| < 2$, the roots $\mu_\pm$ are complex and both of modulus one. Any real coefficient vector ξ may be written $\xi = c_+z_+ + c_-z_-$ where $z_\pm$ are the (complex) eigenvectors of the matrix A and $c_\pm$ complex coefficients. Since

$$A^n\xi = c_+\mu_+^n z_+ + c_-\mu_-^n z_-,$$

it is easy to check that

$$\|A^n\xi\| \le |c_+|\|z_+\| + |c_-|\|z_-\|,$$

which is bounded independently of n. This is therefore the stable case. □

This example shows that the two cases that we have identified above as leading to periodic solutions, of period one if $f(\lambda) = +2$ or period two if $f(\lambda) = -2$, lie on the borderline between stability and instability. Since in applications the discovery of this borderline is a frequently sought goal, the study of these cases acquires a special significance.

They are also precisely the two cases when equation (11.32) has a double root. When $f(\lambda) = \pm 2$ and $\mu = \pm 1$ is therefore a double root of the characteristic equation, there are two possibilities for the matrix $A(\lambda)$. It may have a unique eigenvector, in which case there is likewise a unique eigenfunction of the boundary-value problem consisting of equation (11.11) and the conditions (11.27). Alternatively, it may have a pair of linearly independent eigenvectors. In the latter case it is easy to see that $A(\lambda)$ carries any vector (a, b) to itself, i.e., $A(\lambda) = I$. This means that

$$\phi(1, \lambda) = \pm, 1, \psi(1, \lambda) = 0, \phi'(1, \lambda) = 0, \psi'(1, \lambda) = \pm 1, \tag{11.34}$$

the plus signs holding in case $f(\lambda) = +2$, the minus sign if $f(\lambda) = -2$. Given the initial conditions on ϕ and ψ, these provide a number of boundary-value problems on the interval $(0, 1)$. For example, the problem consisting of equation (11.11) on this interval together with the separated endpoint conditions

$$u(0) = 0 \text{ and } u(1) = 0 \tag{11.35}$$

is satisfied by the function ψ on this interval, up to a constant factor. Alternatively we may consider the problem solved by the function ϕ in this special case: it satisfies equation (11.11) and the separated endpoint conditions

$$u'(0) = 0 \text{ and } u'(1) = 0. \tag{11.36}$$

The same considerations apply to the other borderline case, when $f(\lambda) = -2$ and $\mu = -1$ is a double root. We therefore are able to learn useful information regarding the function $f(\lambda)$ by considering standard Sturm-Liouville problems with separated endpoint conditions, and we now turn to this.

We derive a series of properties of the function f based on the regular Sturm-Liouville problem (11.11), (11.35) on the interval $(0, 1)$. Once these properties have been established, the analysis of the periodic problem will be straightforward. They are the following:

Lemma 11.5.1 *Denote by $\{\mu_j\}_{j=0}^{\infty}$ the increasing sequence of eigenvalues of the problem (11.11), (11.35), and let the function $f(\lambda)$ be given by equation (11.31). Then*

1. *$f(\mu_{2k}) \leq -2$ and $f(\mu_{2k+1}) \geq 2$, $k = 0, 1, 2, \ldots$*
2. *There exists a number $\mu_* < \mu_0$ such that $f(\mu_*) \geq 2$; for $\lambda < \mu_*$, $f(\lambda) > 2$.*
3. *If $f(\lambda) = \pm 2$ and $\lambda \neq \mu_n$ for any $n = 0, 1, 2, \ldots$, then the periodic problem has only a single linearly independent eigenfunction at λ. For that value of λ the derivative $f_\lambda = df/d\lambda$ satisfies*
   - *(a) $f_\lambda(\lambda) < 0$ if $\lambda < \mu_0$,*
   - *(b) $f_\lambda(\lambda) > 0$ if $\mu_{2k} < \lambda < \mu_{2k+1}$, and $f_\lambda(\lambda) < 0$ if $\mu_{2k+1} < \lambda < \mu_{2k+2}$.*

   On λ intervals for which $\lambda \neq \mu_n$ and $|f(\lambda)| < 2$, f is monotone, increasing or decreasing according as ψ_1 is negative or positive.
4. *Suppose $f(\mu_{2k}) = -2$. Then*
   - *(a) If $f_\lambda(\mu_{2k}) \neq 0$, the periodic problem has only a single linearly independent eigenfunction; in this case $f_\lambda(\mu_{2k}) < 0$.*
   - *(b) If $f_\lambda(\mu_{2k}) = 0$, the periodic problem has two linearly independent eigenfunctions; in this case $f_{\lambda\lambda}(\mu_{2k}) > 0$.*
5. *Suppose $f(\mu_{2k+1}) = +2$. Then*
   - *(a) If $f_\lambda(\mu_{2k+1}) \neq 0$, the periodic problem has only a single linerly independent eigenfunction; in this case $f_\lambda(\mu_{2k+1} > 0$.*
   - *(b) If $f_\lambda(\mu_{2k}) = 0$, the periodic problem has two linearly independent eigenfunctions; in this case $f_{\lambda\lambda}(\mu_{2k+1}) < 0$.*

Proof:

1. The eigenfunctions of the problem (11.11), (11.35) may be chosen to have $u'(0) = 1$, and therefore are precisely the functions $\psi(t, \mu_j)$. Since the Wronskian evaluated at $t = 1$ is

$$\phi(1, \mu_j)\psi'(1, \mu_j) - \phi'(1, \mu_j)\psi(1, \mu_j) = \phi(1, \mu_j)\psi'(1, \mu_j) = 1,$$

we find for f the expression

$$f(\mu_j) = \psi'(1, \mu_j) + 1/\psi'(1, \mu_j),$$

where we note that $\psi'(1, \mu_j) \neq 0$ since otherwise $\psi(t, \mu_j)$ would vanish identically on $(0, 1)$. The function $g(x) = x + 1/x$ satisfies $g(x) \geq 2$ on $(0, \infty)$ and $g(x) \leq -2$ on $(-\infty, 0)$. If j is even, $\psi(t, \mu_j)$ has an even number of zeros in the open interval $(0, 1)$. Since $\psi'(0, \mu_j) > 0$, this implies that $\psi'(1, \mu_j) < 0$. Thus $f(\mu_j) < -2$ if j is even, and similarly $f(\mu_j) > 2$ if j is odd. This proves the inequalities in item (1) in the list of statements.

2. For the statement regarding μ_* consider the problem consisting of equation (11.11) with the separated endpoint conditions (11.36), and let μ_* be the least eigenvalue of this problem. The corresponding eigenfunction, which is $\phi(t, \mu_*)$, has no zeros on $(0, 1)$ and therefore remains positive there. To see that $\mu_* < \mu_0$, consider Theorem 11.2.2 applied with $\hat{P}(t) = P(t) = p(t)$, $\hat{Q}(t) = \mu_* \rho(t) - q(t)$ and $Q(t) = \mu_0 \rho(t) - q(t)$. The solution $\psi(t, \mu_0)$ of the 'unhatted' equation vanishes at $t = 0$ and $t = 1$. If it were the case that $\mu* \geq \mu_0$, it would follow that any solution of the 'hatted' equation would vanish at least once on $(0, 1)$. But the eigenfunction of the problem (11.11), (11.36) belonging to the least eigenvalue does not vanish on this interval. Therefore $\mu_* < \mu_0$. Again making use of the Wronskian expression as above, one finds that the expression for $f(\mu_*)$ now takes the form

$$f(\mu_*) = \phi(1, \mu_*) + 1/\phi(1, \mu_*);$$

since $\phi(1, \mu_*) > 0$, it follows that $f(\mu_*) \geq 2$. The conclusion that $f(\lambda) > 2$ for $\lambda < \mu_*$ will be drawn below, with the aid of equation (11.42).

3. If $f(\lambda) = \pm 2$ then the periodic problem has a nontrivial solution. If there were two linearly independent solutions, then $\psi(t, \lambda)$ would be a solution of the system (11.11), (11.35), implying that $\lambda = \mu_j$ for some $j = 0, 1, 2, \ldots$, whereas here we assume the opposite.

   To address the inequalities regarding the derivative f_λ, we first observe that this derivative exists. This will be so if the derivatives $\phi_\lambda(t, \lambda)$ and $\psi'_\lambda(t, \lambda)$ exist. This in turn follows from Theorem 6.4.2 in Chapter 6; it further follows that the derivatives $\phi_\lambda, \psi_\lambda$ satisfy the linear differential equation (the *variational* equation) obtained by differentiating formally with respect to λ in equation (11.11). Therefore if we provisionally put $v(t) = \phi_\lambda(t, \lambda)$, we find for v the initial-value problem

$$\frac{d}{dt}\left(p(t)\frac{dv}{dt}\right) + (\lambda\rho(t) - q(t))\, v = -\rho(t)\phi(t, \lambda), \quad v(0) = 0, \; v'(0) = 0. \tag{11.37}$$

Here we have observed that $\phi(0, \lambda)$ and $\phi'(0, \lambda)$ are both independent of λ; this implies that $v(0) = v'(0) = 0$. Exactly the same equation and initial conditions are obtained for the derivative ψ_λ except that ψ rather than ϕ appears on the right-hand side.

The solutions of these equations are provided by the variation-of-parameters formula, conveniently given by equation (10.27) of Chapter 10. In that equation the denominator $p(\xi)W(\xi)$ is constant, and therefore equal to its value at the left hand endpoint; in the present case that is $p(0)$. Thus the solution of equation (11.37) is

$$\phi_\lambda(t, \lambda) = \frac{-1}{p(0))}\int_0^t (\phi(s, \lambda)\psi(t, \lambda) - \phi(t, \lambda)\psi(s, \lambda))\, \rho(s)\phi(s, \lambda)\, ds. \tag{11.38}$$

The solution for ψ_λ is given by the same formula as that above with the exception that the last term in the integrand is $\psi(s, \lambda)$ instead of $\phi(s, \lambda)$. Further, we require not ψ_λ but ψ'_λ (or $\partial^2\psi(t, \lambda)/\partial t\partial\lambda)$). This results in the formula

$$\psi'_\lambda = \frac{-1}{p(0))}\int_0^t (\phi(s, \lambda)\psi'(t, \lambda) - \phi'(t, \lambda)\psi(s, \lambda))\, \rho(s)\psi(s, \lambda)\, ds. \tag{11.39}$$

Adding these expressions and evaluating them at $t = 1$ gives for f_λ the formula

$$f_\lambda(\lambda) = (1/p(0))\int_0^1 B(s, \lambda)\rho(s)\, ds, \tag{11.40}$$

where

$$B(s, \lambda) = \phi'_1\psi^2 + (\phi_1 - \psi'_1)\,\phi\psi - \psi_1\phi^2, \tag{11.41}$$

and we have used the following notation:

$$\phi_1 = \phi(1, \lambda), \phi'_1 = \phi'(1, \lambda)$$

and the same for ψ_1 and ψ_1'; ϕ, written without indices, is short for $\phi(s,\lambda)$, and likewise for ψ.

Multiplying the function B by $-\psi_1$ and completing the square results in the formula

$$-\psi_1 B = \left[\psi_1\phi - \frac{1}{2}\left(\phi_1 - \psi_1'\right)\psi\right]^2 - \frac{1}{4}\left[\left(\phi_1 - \psi_1'\right)^2 + 4\psi_1\phi_1'\right]\psi^2.$$

Finally, observing that in the last term above we can replace the term $\psi_1\phi_1'$ by the term $\phi_1\psi_1'-1$ through the Wronskian relation $\phi_1\psi_1'-\phi_1'\psi_1 = 1$, we find

$$-\psi_1 B = \left[\psi_1\phi - \frac{1}{2}\left(\phi_1 - \psi_1'\right)\psi\right]^2 - \frac{1}{4}\left[f(\lambda)^2 - 4\right]\psi^2. \tag{11.42}$$

The last assertion of statement (3) follows from this formula. The assumption that $\lambda \neq \mu_j$ implies that $\psi_1 \neq 0$ so the further assumption that $f(\lambda) = \pm 2$ implies that $B(s,\lambda)$ has the sign of $-\psi_1$ on the entire interval $0 < s < 1$. Therefore,

(a) if $\lambda < \mu_0$, the function ψ does not vanish on $(0,1]$. Since $\psi(0,\lambda) = 0$ and $\psi'(0,\lambda) = 1$, ψ_1 is positive on this interval and f_λ is therefore negative.

(b) Consider $\psi_1 = \psi(1,\lambda)$ as a function of λ. At $\lambda = \mu_0$ ψ_1 has its first zero, and is negative in the interval $\mu_0 < \lambda < \mu_1$; then positive again in the interval $\mu_1 < \lambda < \mu_2$; and so on, as stated.

We can now verify the assertion in statement (2) that $f(\lambda) > 2$ if $\lambda < \mu_*$. Suppose first that $f(\mu_*) = 2$. Then equation (11.42) shows that $f_\lambda(\mu_*) < 0$, so that $f > 2$ at least on some interval to the left of μ_*. If $f(\lambda) < 2$ for some value of $\lambda < \mu_*$ there would have to be a nearest value of $\lambda < \mu_*$ where $f = 2$. Necessarily $f_\lambda > 0$ at this point, but equation (11.42) guarantees that $f_\lambda < 0$ there. Thus there is no such point and $f(\lambda) > 2$ for $\lambda < \mu_*$. If $f(\mu_*) > 2$ then there is a point λ_0 with $\mu_* < \lambda_0 < \mu_0$ where $f(\lambda_0) = 2$. Applying the preceding reasoning to λ_0 we infer that $f > 2$ if $\lambda < \lambda_0$ and a fortiori for $\lambda < \mu_*$.

4. If $f(\mu_{2k}) = -2$, then the periodic problem is satisfied with $\lambda = \mu_{2k}$. This means that $\psi_1 = 0$. Then the expression (11.41) becomes

$$B = \phi_1'\psi^2 + \left(\phi_1 - \psi_1'\right)\phi\psi.$$

The Wronskian relation then gives $\phi_1 = 1/\psi_1'$ and, since $f(\mu_{2k}) = \psi_1' + 1/\psi_1' = -2$, we must have $\phi_1 = \psi_1' = -1$. The second term above in the expression for B therefore vanishes, and we see from the formula (11.40) that $f_\lambda = 0$ if and only if $\phi_1' = 0$.

(a) Assume first that this is *not* so, i.e., that $f_\lambda(\mu_{2k}) \neq 0$. Then $\phi_1' \neq 0$. It follows that there is only one linearly independent eigenfunction of the periodic problem. That $f_\lambda(\mu_{2k})$ is in fact negative follows from the facts that f is a continuous function of λ and is negative for values of $\lambda = \mu_{2k} - \epsilon$ for sufficiently small $\epsilon > 0$.

(b) Next suppose that both $f(\mu_{2k})$ and $f_\lambda(\mu_{2k})$ vanish. By the remark above that in that case also $\phi_1' = 0$, we infer that the conditions (11.34) hold and there are indeed two linearly independent solutions of the periodic problem. We need to take the second derivative of f – and therefore with respect to ϕ and ψ – with respect to λ. This can be justified by applying Theorem 6.4.2 to the variational equation for ϕ_λ above.[5] We can find a formula for $f_{\lambda\lambda}(\mu_{2k})$ by differentiating the expression (11.40) for f_λ, obtaining

$$f_{\lambda\lambda}(\lambda) = \int_0^1 \rho(s) B_\lambda(s, \lambda)\, ds.$$

The expression for B_λ obtained from equation (11.41) is lengthy in general but simplifies when we put $\lambda = \mu_{2k}$ and employ the conditions (11.34):

$$B_\lambda(s, \mu_{2k}) = \phi_{1,\lambda}'\psi^2 + \left(\phi_{1,\lambda} - \psi_{1,\lambda}'\right)\phi\psi - \psi_{1,\lambda}\phi^2.$$

The coefficients in this formula can be evaluated by differentiating the expressions (11.38) and (11.39) as well as corresponding expressions for ϕ' and ψ. Carrying these out and again applying the conditions (11.34), we find (for example)

$$\phi_{1,\lambda} = -\int_0^1 \rho(u)\phi(u, \lambda)\psi(u, \lambda)\, du \tag{11.43}$$

[5]In the present case, one can in fact show that the solutions $\phi(t, \lambda)$ and $\psi(t, \lambda)$ are, for each fixed $t \in [0, 1]$, analytic functions of λ in the entire complex λ plane.

and three similar equations for the other derivatives appearing in the expression for $B_\lambda(\mu_2 k)$, with the result that that expression takes the form

$$B_\lambda = \psi^2 \int_0^1 \rho\phi^2 \, du - 2\psi\phi \int_0^1 \rho\phi\psi \, ds + \phi^2 \int_0^1 \rho\psi^2.$$

This expression is positive for all choices of ψ and ϕ provided the discriminant is negative, i.e., provided

$$\left(\int_0^1 \rho\phi\psi \, du \right)^2 < \int_0^1 \rho\phi^2 \, du \int_0^1 \rho\psi^2 \, du.$$

This is the Cauchy-Schwartz[6] inequality which holds generally and verifies that indeed $f_{\lambda\lambda}(\mu_{2k}) > 0$.

5. The proof of this last case is virtually identical with the preceding one and is therefore omitted.

We are now able to describe the eigenvalues and eigenfunctions of the periodic problem. We'll denote those of the problem (11.11), (11.27) by $\{\lambda_j\}_0^\infty, \{v_j\}_0^\infty$ and those of the problem (11.11), (11.28) by $\{\nu_j\}_1^\infty, \{w_j\}_1^\infty$. We contnue to denote by $\{\mu_j\}_0^\infty, \{\mu_j\}_0^\infty$ the eigenvalues and eigenfunctions of the related Sturm-Liouville problem (11.11), (11.35).

Consider the first two statements of Lemma 11.5.1. These show that there is a number λ_0 in the interval $\mu_* \geq \lambda_0 < \mu_0$ where $f = +2$; this is an eigenvalue of the problem (11.11), (11.27). It has only a single, linearly independent eigenfunction because $\lambda < \mu_0$, whereas, if there were two linearly independent eigenfunctions it would necessarily be the case that $\lambda_0 = \mu_j$ for some $j = 0, 1, \ldots$.

Allowing λ to increase, we see that, at the value μ_0, $f \leq -2$. If $f(\mu_0) < -2$ then there is a value $\lambda_1 < \mu_0$ where $f = -2$ and, since $f(\mu_1) \geq +2$, a subsequent value λ_2 at which $f = -2$ again. If $f(\mu_0) = -2$, there are two possibilities. One is that $f_\lambda(\mu_0) \neq 0$. In this case it must be negative since it is continuous function of λ which is negative on the interval (λ_0, μ_0) according to equations (11.40) and (11.42). Therefore $f < -2$ for values of λ to the right of μ_0. It is therefore the case that $f(\lambda) = -2$ again for

[6] See Theorem 12.2.1 below; equality rather than strict inequality may hold, but only if the functions ϕ and ψ are linearly dependent, which is excluded in the present case.

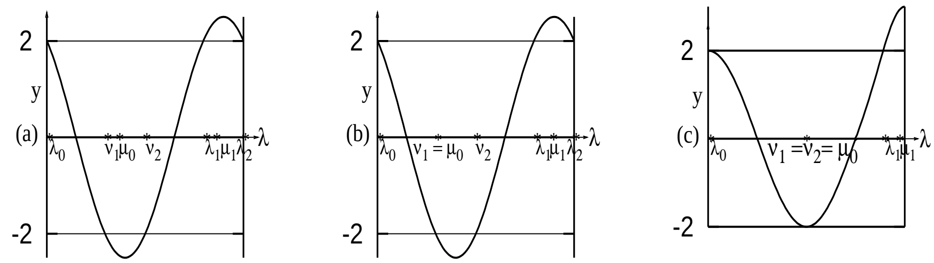

Figure 11.2: The curve $y = f(\lambda)$. These three diagrams illustrate the three possible dispositions of the eigenvalues of the periodic problem with respect to those of the auxiliary Sturm-Liouville problem. In frame (a), $\nu_1 < \mu_0 < \nu_2$; in frame (b) $\nu_1 = \mu_0 < \nu_2$ and $f_\lambda(\mu_0) < 0$; in frame (c) $\nu_1 = \nu_2 = \mu_0$ and f takes a local minimum value of -2 at this point.

a value of λ in the interval (μ_0, μ_1). This gives a pair of values $\nu_1 = \mu_0$ and $\nu_2 \in (\mu_0, \mu_1)$. According to Lemma 11.5.1 there is a single linearly independent eigenfunction corresponding to each of these. Finally, it may be that $f(\mu_0) = -2$ and $f_\lambda(\mu_0) = 0$. In this case, there are two linearly independent eigenfunctions and $\nu_1 = \nu_2 = \mu_0$. Since then $f_{\lambda\lambda}(\mu_0) > 0$, for values of λ just to the right of μ_0, $f > -2$. On an interval where $-2 < f(\lambda) < +2$, f is monotone increasing according to equations (11.40) and (11.42). It must equal or exceed this value at or near the value $\lambda = \mu_1$, where the preceding analysis may be repeated. We have now completed most of the following

Theorem 11.5.1 *The least eigenvalue λ_0 of the problem (11.11), (11.27) has a single, linearly independent eigenfunction v_0 associated with it, which does not vanish on $[0,1]$. The remaining eigenvalues $\{\lambda_j\}_1^\infty$ of this problem and the eigenvalues $\{\nu_j\}_1^\infty$ of the problem (11.11), (11.28) satisfy the following relations:*

$$\nu_{2j+1} \le \mu_{2j} \le \nu_{2j+2} < \lambda_{2j+1} \le \mu_{2j+1} \le \lambda_{2j+2} < \nu_{2j+3} \le \mu_{2j+2} \le \nu_{2j+4} < \cdots$$

for $j = 0, 1, 2, \ldots$. Here the sequence $\{\mu_j\}$ is the sequence of eigenvalues of the regular Sturm-Liouville problem (11.11), (11.35). The eigenfunctions w_{2j+1} and w_{2j+2} have precisely $2j$ zeros in the interval $[0,1)$ and the eigenfunctions v_{2j+1}, v_{2j+2} have precisely $2j+1$ zeros in the interval $[0,1)$.

Proof: Only the remarks on the numbers of zeros remain to be established. Consider the function v_0. At the endpoints, v_0 does not vanish: if it did, it would be a solution of the problem (11.11), (11.35) and it would follow that λ_0 is one of the eigenvalues $\mu_0, \mu_1, \ldots$ of that problem. But $\lambda_0 < \mu_0$, so $v_0(0) = v_0(1) \neq 0$. Moreover, it cannot vanish in the interior. If it did, it would have to vanish at least twice, since the boundary condition $v_0(0) = v_0(1)$ implies that the number of zeros is even. But then the eigenfunction u_0 of the problem (11.11), (11.35) would vanish at least once in the interior, by the Sturm comparison theorem, since $\mu_0 > \lambda_0$. Hence v_0 does not vanish on $[0, 1]$.

Next consider the functions w_1 and w_2 belonging to the eigenvalues ν_1 and ν_2. Suppose first that $\nu_1 < \mu_0 < \nu_2$. Then neither w_1 nor w_2 can vanish at the endpoints by the same reasoning as that above for v_0. But since $w_1(1) = -w_1(0)$, this function has an odd number of zeros in $(0, 1)$, and likewise for w_2. If w_1 had three or more zeros, then u_1 would have at least two by the Sturm comparison theorem, and this is not so; therefore w_1 has exactly one zero on $(0, 1)$ and likewise for w_2. Suppose next that $\nu_1 = \mu_0$ but $f_\lambda(\mu_0) \neq 0$: then $\mu_0 < \nu_2$. Then w_2 has one zero in $(0, 1)$ by the preceding reasoning, and $w_1 = u_0$ vanishes at the endpoints but not in the interior. Finally, if $\nu_1 = \nu_2 = \mu_0$ then the functions ψ and ϕ satisfy the boundary conditions (11.35) and (11.36) respectively. The first of these vanishes at the endpoints only, the second cannot vanish at the endpoints and therefore has a single zero in the interior. Therefore, in each case, the eigenfunction has a single zero on $[0, 1)$.

Continue this reasoning to the functions v_1 and v_2 belonging to the eigenvalues λ_1 and λ_2. In the case when $\lambda_1 < \mu_1 < \lambda_2$, the eigenfunction v_1 must vanish at least once on $(0, 1)$ since $\lambda_1 > \mu_0$, and since it must vanish an even number of times and cannot vanish at the endpoints, it has at least two zeros there. It cannot have more than two for if it did u_2 would have at least three. Thus v_1 has exactly two zeros in $(0, 1)$ and likewise for v_2. Again, if $\lambda_1 < \mu_1 = \lambda_2$, the reasoning above applies to λ_1, and we have $v_2 = u_1$, so v_2 has three zeros in $[0, 1]$, two at the endpoints and one in the interior. Finally, if $\lambda_1 = \lambda_2 = \mu_1$, then ψ and ϕ satisfy the boundary conditions (11.35) and (11.36) respectively. The latter function has two zeros in the interior by now-familiar reasoning, the former one in the interior and two at the endpoints. Therefore, in all cases, there are two zeros on $[0, 1)$. $\square$

Stating the number of zeros on the interval $[0, 1)$ is legally correct but is done principally to keep the statement of the theorem reasonably short: the proof gives a more precise description of the disposition of the zeros.

PROBLEM SET 11.5.1

1. Prove that any solution u of equation (11.11) with the boundary conditions (11.28) satisfies the condition $u(t+1) = -u(t)$ for all real t.
2. Suppose $f(\lambda) = 2$. Show that the conditions (11.34) are both necessary and sufficient for the problem (11.11), (11.27) to have two linearly independent solutions.
3. Let $p(t) = \rho(t) \equiv 1$, $q(t) \equiv 0$. Find the function $f(\lambda)$ explicitly and verify the conclusions of Theorem 11.5.1 in this case.
4. Derive equation (11.43) and the corresponding equations giving $\phi'_{1,\lambda}$, $\psi_{1,\lambda}$ and $\psi'_{1,\lambda}$ evaluated at $\lambda = \mu_{2k}$.
5. Show that the eigenfunctions $\{v_j\}$ of the problem (11.11), (11.27) are orthogonal when the eigenvalues are distinct, and may be chosen to be orthogonal in all cases, and likewise for the problem (11.11), (11.28).
6. Need the eigenfunctions of the problem (11.11), (11.28) be orthogonal to those of the problem (11.11), (11.27)? Use Problem 3 above as a test case.