Homework 6

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12/2/2020

Part A.i

Compute the covariance matrix of Z

$$Var(Z_1) = \frac{1}{2}Var(X_1 + X_2) = \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2\right)$$
$$Var(Z_2) = \frac{1}{2}Var(X_1 - X_2) = \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2\right)$$

$$Cov(Z_1, Z_2) = \mathbb{E}(Z_1 Z_2) - \mathbb{E}Z_1 \mathbb{E}Z_2 = \mathbb{E}(Z_1 Z_2)$$

and

$$\mathbb{E}(Z_1 Z_2) = \frac{1}{2}\mathbb{E}\left(X_1^2 - X_2^2\right) = \frac{1}{2}\left(\sigma_1^2 - \sigma_2^2\right)$$

Thus,

$$\Sigma = \frac{1}{2}\begin{pmatrix} \sigma_1^2 + \sigma_2^2 & \sigma_1^2 - \sigma_2^2 \\ \sigma_1^2 - \sigma_2^2 & \sigma_1^2 + \sigma_2^2 \end{pmatrix}$$

Compute Λ

The precision matrix is:

$$\Lambda = \frac{1}{2\sigma_1^2\sigma_2^2}\begin{pmatrix} \sigma_1^2 + \sigma_2^2 & \sigma_2^2 - \sigma_1^2 \\ \sigma_2^2 - \sigma_1^2 & \sigma_1^2 + \sigma_2^2 \end{pmatrix}$$

Compute eigenvalues and eigenvectors of Σ

$$
\begin{aligned}
0 &= det \begin{pmatrix} \frac{1}{2} & a+b-\lambda & b-a \\ & b-a & a+b-\lambda \end{pmatrix} \\
&= (\frac{a+b}{2} - \lambda)^2 - (\frac{b-a}{2})^2 \\
&= ab + \lambda^2 - \lambda(a+b) = (a-\lambda)(b-\lambda)
\end{aligned}
$$

Thus, the eigenvalues of Σ are σ_1^2 and σ_2^2.

So we now have the two eigenvalues, σ_1^2 and σ_2^2

We then seek the corresponding eigenvectors.

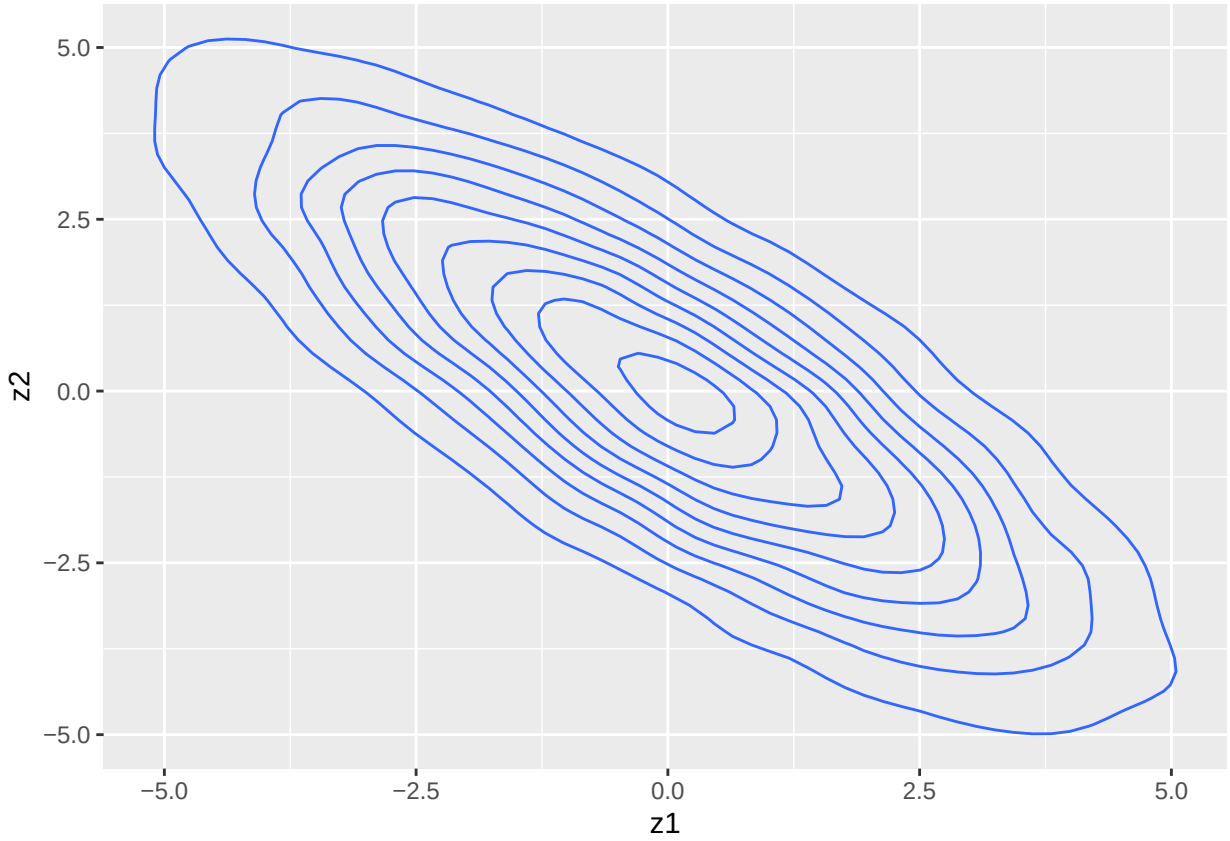
We see that these are $(1, 1)$ and $(1, -1)$.

Level curves of the pdf

```
s1 <- 1
s2 <- 10
```

```
x1 <- rnorm(n = 10000, mean = 0, sd = sqrt(s1))
x2 <- rnorm(n = 10000, mean = 0, sd = sqrt(s2))
z1 <- (x1 + x2) / sqrt(2)
z2 <- (x1 - x2) / sqrt(2)
```

```
library(ggplot2)
library(magrittr)
tibble::tibble(z1 = z1, z2 = z2) %>%
  ggplot() + geom_density_2d(aes(x = z1, y = z2))
```



Part A.ii

Find the minimizer

Level curves

Part A.iii

Find the minimizer

Level curves