EINDHOVEN UNIVERSITY OF TECHNOLOGY

Coherent De-Dispersion of Radio Pulsar Signals using Dataflow on FPGA

Graduation project

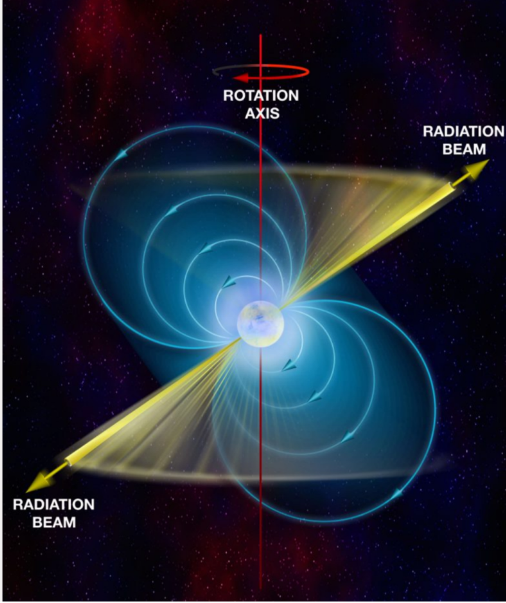
Frank Boerman

Electrical Engineering

Radio Pulsars

"rapidly rotating heavily magnetised neutron stars"
slowly losing rotation speed $\rightarrow$ energy loss $\dot{E}$

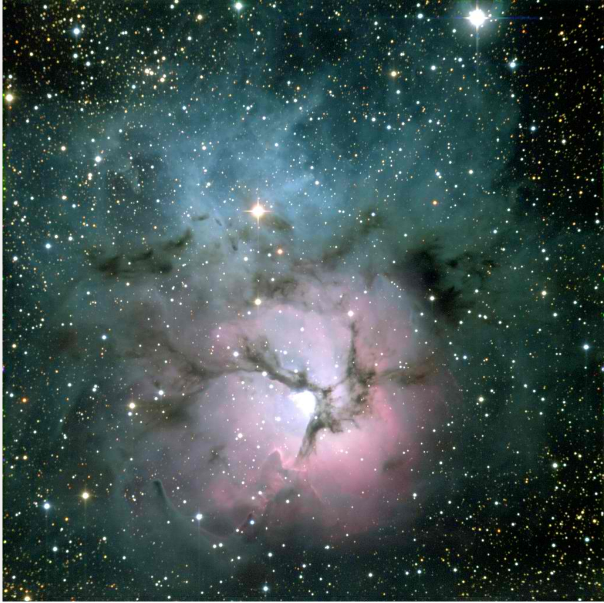
small fraction $\rightarrow$ radio emission reaching Earth
Rotating radiation beam $\rightarrow$ "blinking" pulse effect



The InterStellar Medium (ISM)

consist of cold ionised plasma
Four distinct effects:

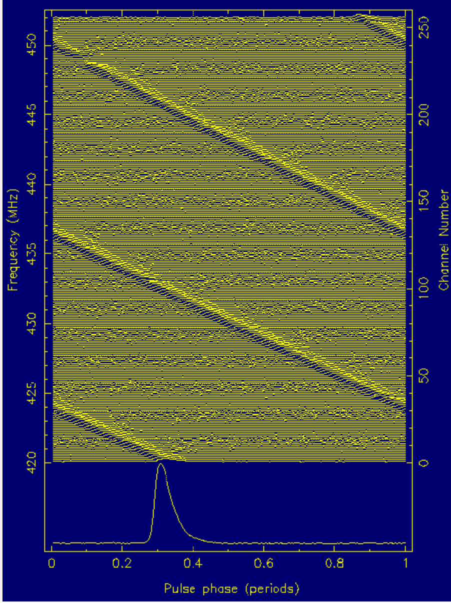
- Faraday Rotation
- Scintillation
- Scattering
- **Frequency Dispersion**



TU/e

Frequency Dispersion

"the effect in which signals with different wavelength have different propagation speeds through a non-vacuum medium"

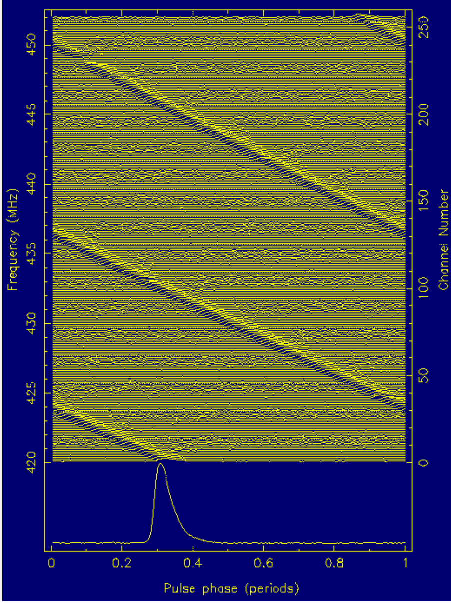


TU/e

Frequency Dispersion

"the effect in which signals with different wavelength have different propagation speeds through a non-vacuum medium"

different frequencies within the signal will slowly shift in time relative to each other



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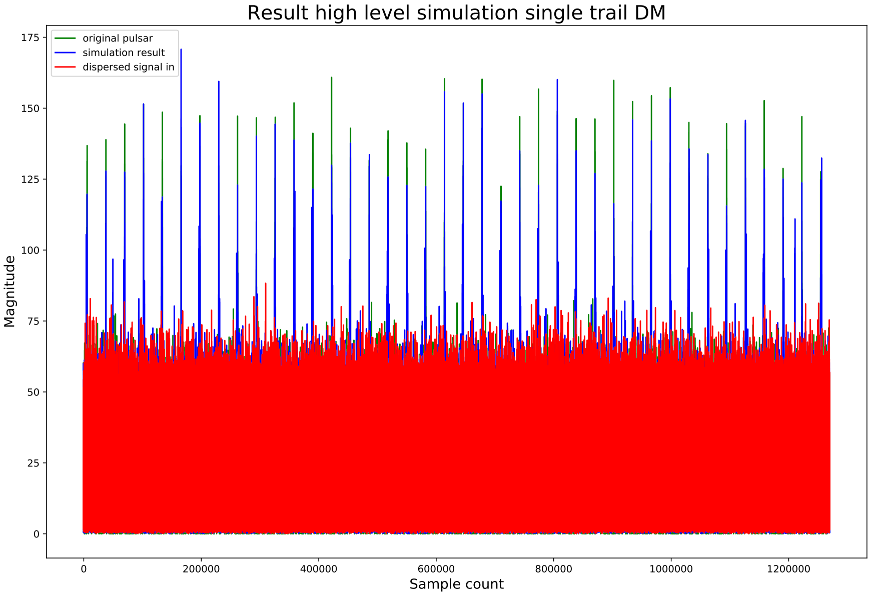
with **Dispersion Measure**:

$$DM = \int_0^d \eta_e(l)dl \tag{3}$$

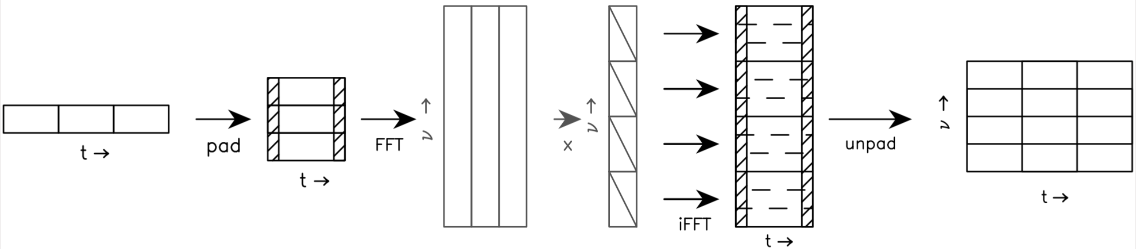
(De)Dispersion visualized

Green:
pulsar signal
Red:
telescope data
-> no longer peaks!
Blue:
example output
-> recovered peaks



CDMT (Bassa et al, 2016)

coherent dispersion measure trials



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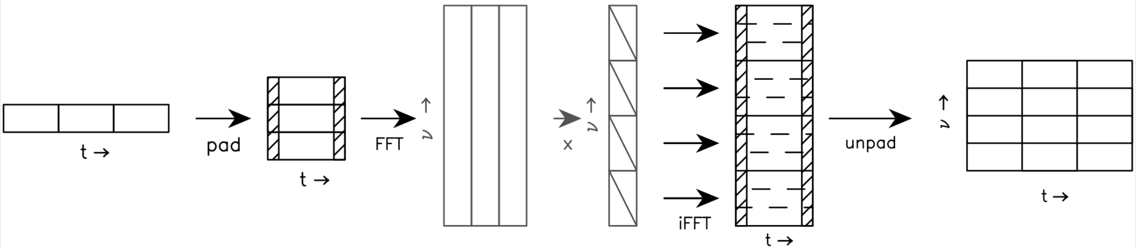

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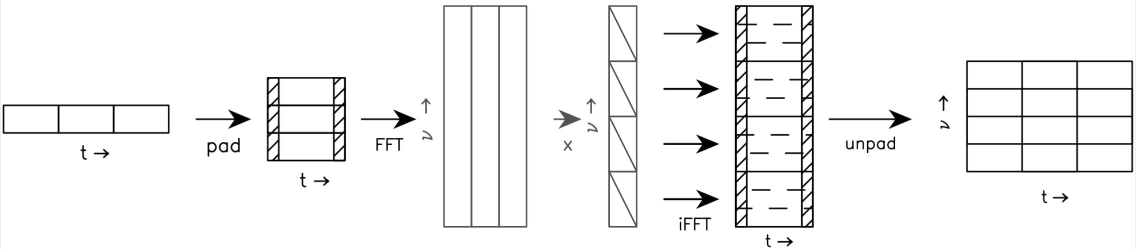

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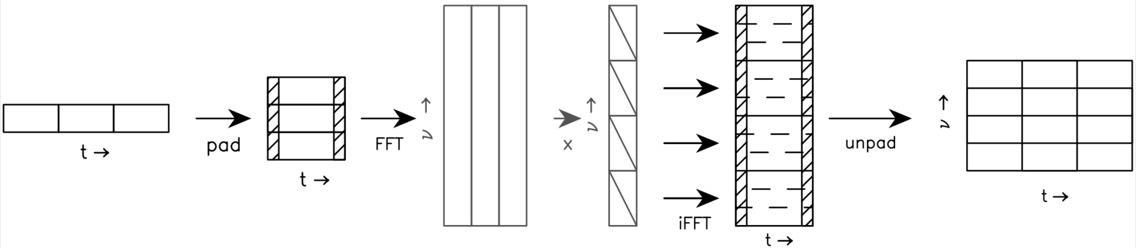

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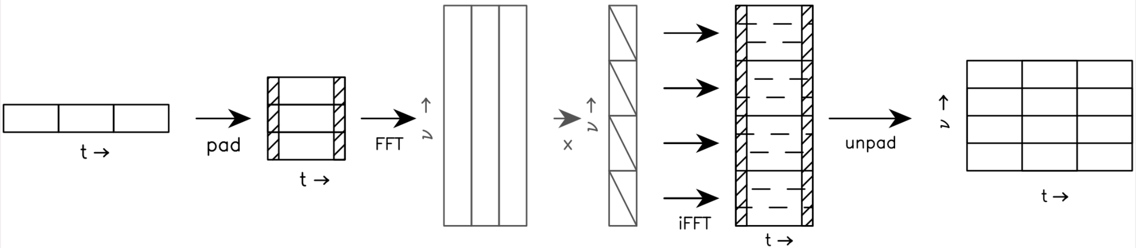

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Not real time -> large intermediate storage needed

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Can we process faster to real time?

Some terminology

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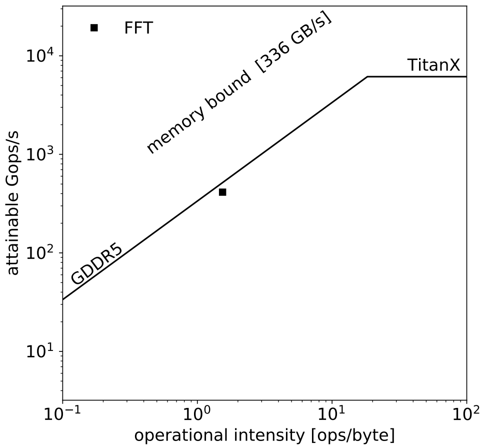
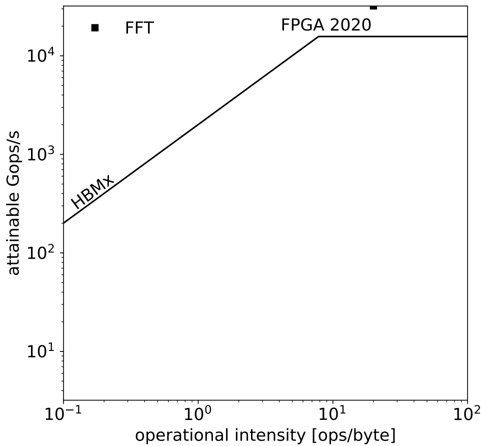
- approximately every three stages full read and write
- **intermediate results are saved many times!**

RoofLines

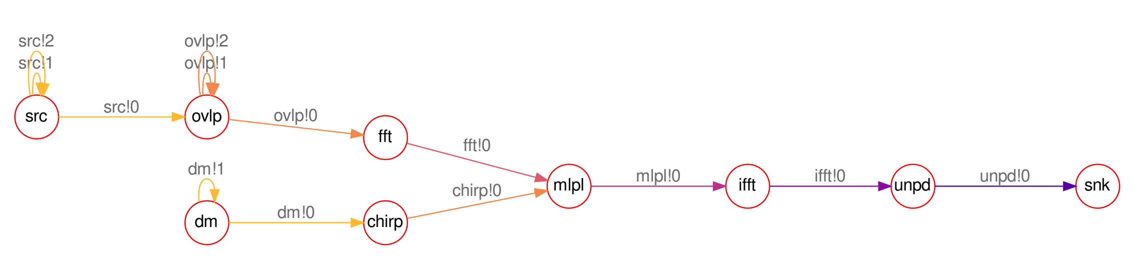
GPU



FPGA



De-Dispersion flow (*cdmt*)



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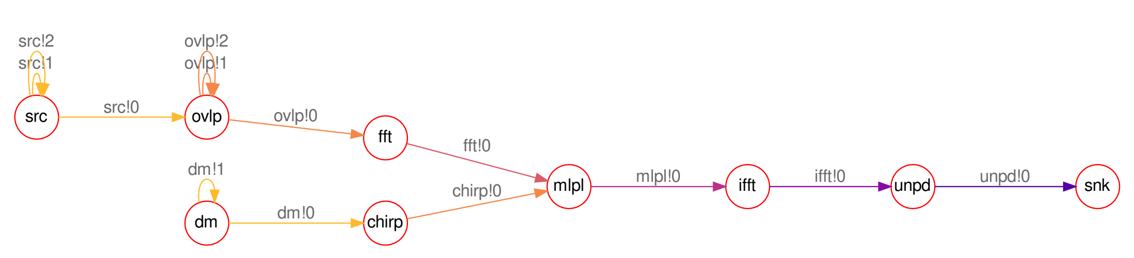

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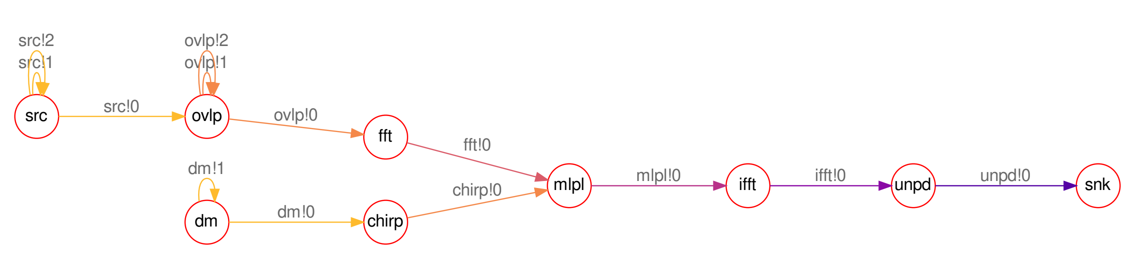
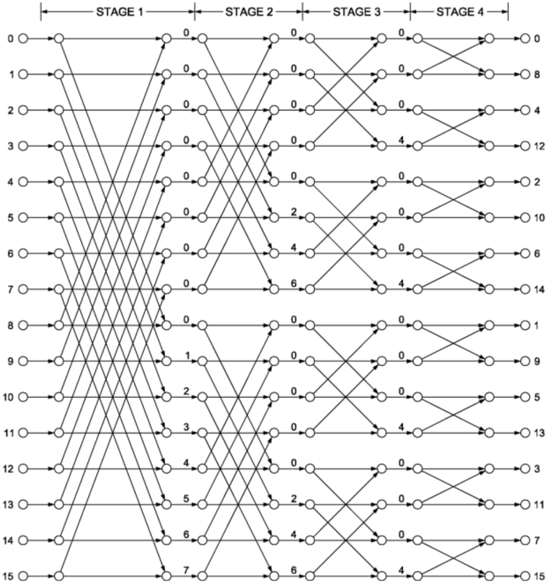


Dominant complexity -> **Crossing time-frequency boundary**

For FPGA: the whole trail fully on chip!

The DFT

Important:
N number of samples in single FFT
across $S = \log_2(N)$ stages



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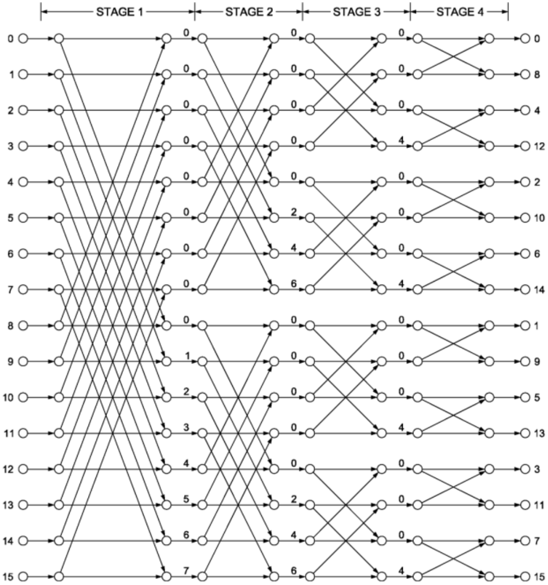
"butterfly" addition and subtraction
"twiddle" rotating factor

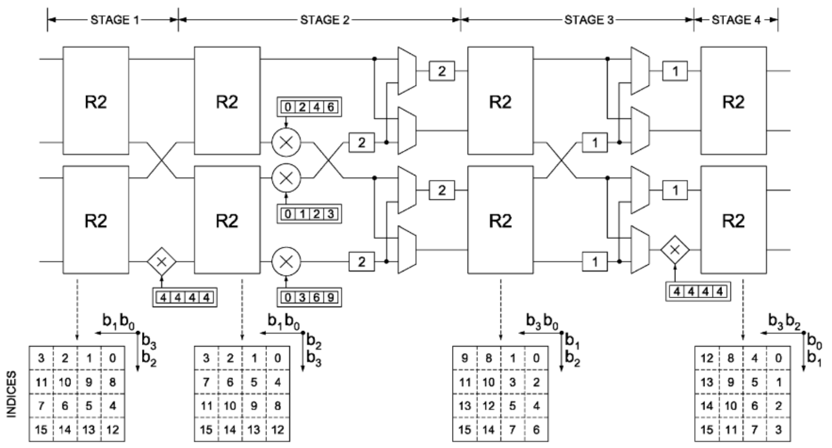
optimized algorithm is called
Fast **F**ourier **T**ransform (**FFT**)



FFT: Desired Properties

- High FFT size N
- High throughput, high parallel samples P
- Small buffers (memory hardware is expensive!)
- easily scalable

DFT Architecture (Garrido et al, 2013)



LOFAR use case: how many trails can we fit?

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- Throughput of $143 \times 10^6 \cdot 16 = 2.29 \times 10^9$
- $\dfrac{2.29 \times 10^9}{78.128 \times 10^6} = 29$ trails fit realtime!

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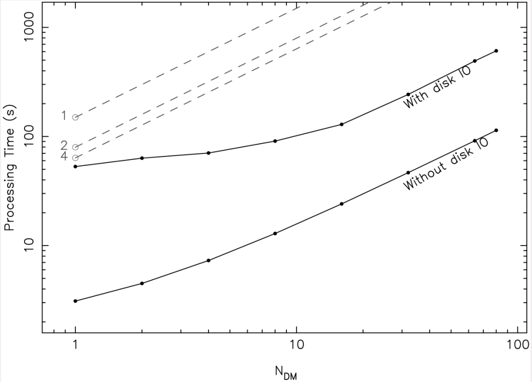
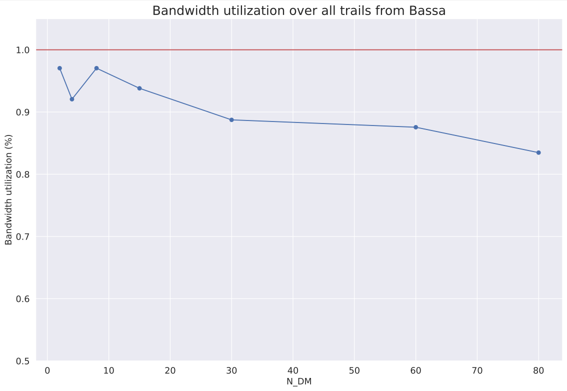
cdmt benchmark

lets look in detail of the whole dispersion





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- Grand total: $4 + 1 + 6 * (16 + 16) = 197$ bytes

Numerical Comparison (cont.)

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FPGA per polarization pair:

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- Output: 1 byte *Stokes parameter*
- Grand total: $4 + 1 = 5$ bytes
- $\frac{197}{5} \approx 40$ times improvement possible!

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model complex systems as independent actors with communication in between

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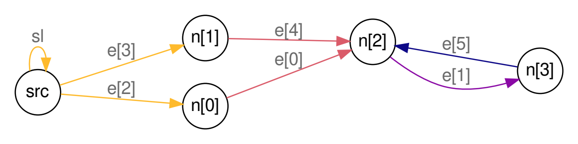
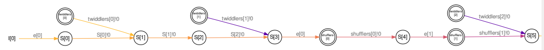


Implementation FFT

Conversion to dataflow step by step

Dataflow implementation

FFT:



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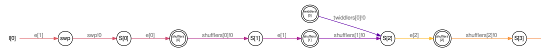

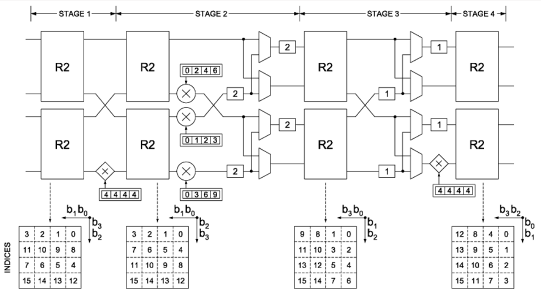

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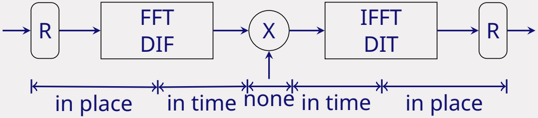
Requires every action to be compatible!

Input/output ordering

Property of FFT: *bit reversed order*
indices of output samples are bit reversed
example: 0010 (2) -> 0100 (4)



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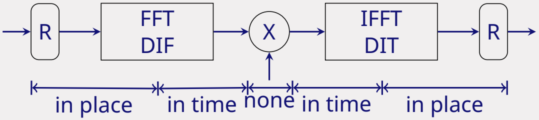

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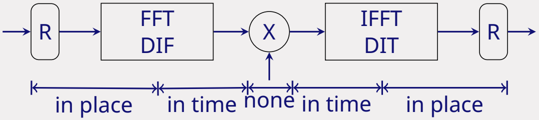

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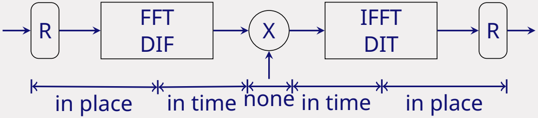

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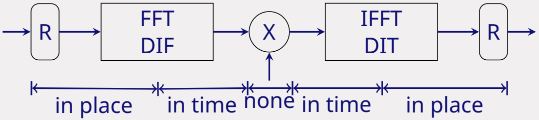

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Using both saves a reorder!

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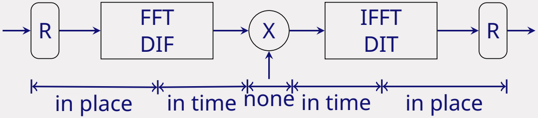

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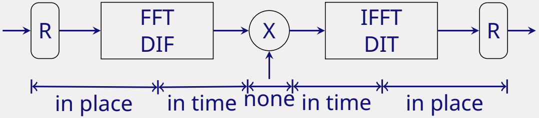

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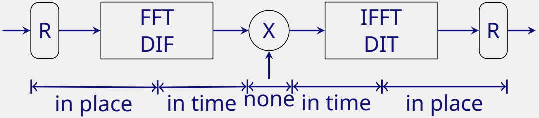

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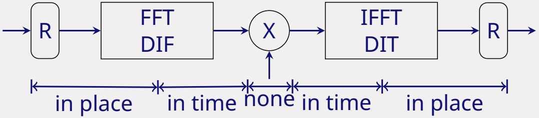

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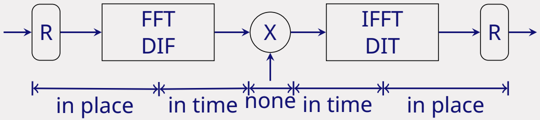

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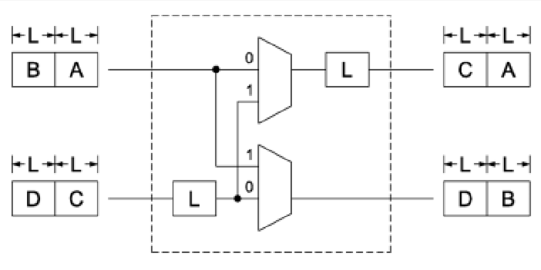
The first only changes samples over input ports (vertical), the second also reorders in time (horizontal)

Shuffler blocks

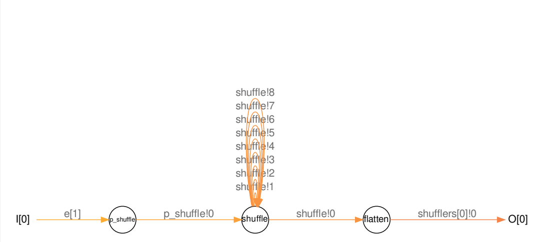


Continues the reordering of previous step
reorders samples across time and place, bufferlength L

there are always $\log_2(N) - \log_2(P)$ shuffles

Shuffler blocks (cont.)

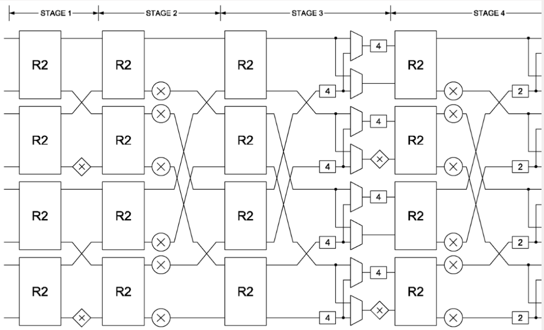
example of $P = 8$



Buffers per edge(P) -> take apart the SIMD!

Multiplexor -> Finite State Machine (thus *Cyclo Static Dataflow*)

Connection Pattern



Depending on P: implemented as single vector in token

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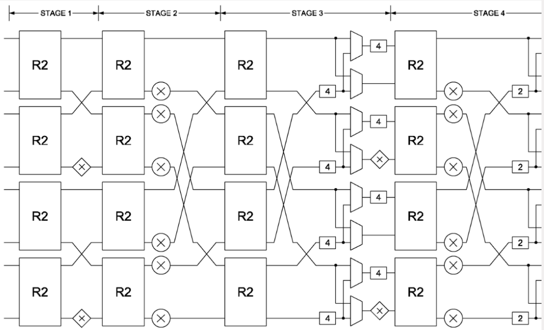


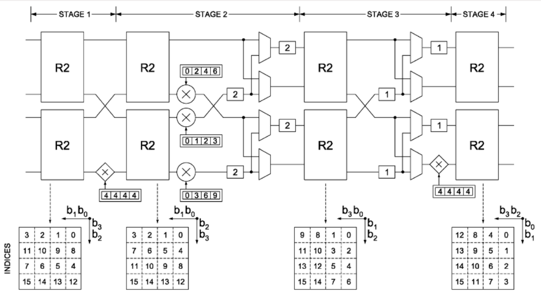
Depending on P: implemented as single vector in token

In dataflow:
Instead of wiring -> fixed reorder of samples within the vector

hardcoded precomputed ordering per stage

Butterfly (R2)

R2 block defined on pairs of 2 samples



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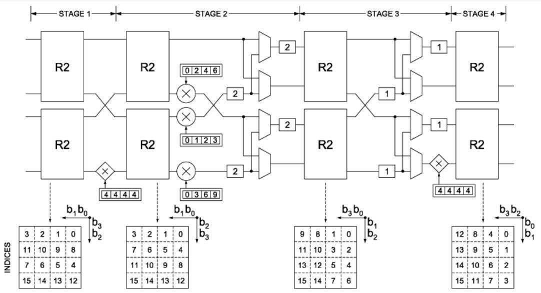


R2 block defined on pairs of 2 samples

in dataflow split input vector in pairs and apply:

$$
\begin{aligned}
O_0 &= I_0 + I_1 \\
O_1 &= I_0 - I_1
\end{aligned} \tag{5}
$$

Twiddle rotation in Architecture

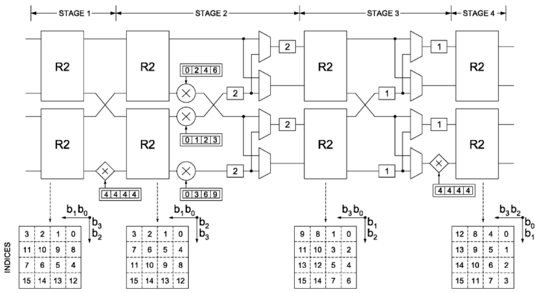
Twiddle factors not discussed at all in original architecture!

Rotations expressed as integer ϕ

$$W_N^{nk} = W_N^{\phi} = e^{\dfrac{-j \cdot 2\pi \cdot \phi}{N}} \tag{6}$$

Trivial rotations: $\{0, N/4, N/2, 3N/4\}$

Trivial: $\rightarrow \{0, 4, 8, 12\}$



Realizing non-trivial rotations

example of $N = 64$ $P = 16$ for integer

0	0	0	0	0	16	8	24	0	8	4	12	0	24	12	36
0	2	1	3	0	18	9	27	0	10	5	15	0	26	13	39
0	4	2	6	0	20	10	30	0	12	6	18	0	28	14	42
0	6	3	9	0	22	11	33	0	14	7	21	0	30	15	45

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0	6	3	9	0	22	11	33	0	14	7	21	0	30	15	45

Based on a pattern:

$$(0 \cdot \frac{N}{P} \cdot \{0, 2, 1, 3\}, 2 \cdot \frac{N}{P} \cdot \{0, 2, 1, 3\},$$
$$1 \cdot \frac{N}{P} \cdot \{0, 2, 1, 3\}, 3 \cdot \frac{N}{P} \cdot \{0, 2, 1, 3\})$$
$$+ \alpha_{row}$$

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Nodes:

- $base_\alpha$

Realized as:

- CORDIC: hardware

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- $R_{n\alpha}$ (with $n \in \{2, 3\}$)

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- $R_{N//P}$

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- CORDIC: hardware
- Rotation matrix using goniometric properties
- Rotation matrix precomputed

Rotator Nodes

CORDIC: hardware to approximate any angle, used for base step

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$R_{N//P}$ precomputed rotation matrix (exact rotation!)

$$R(\alpha) \cdot V_C = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \tag{7}$$

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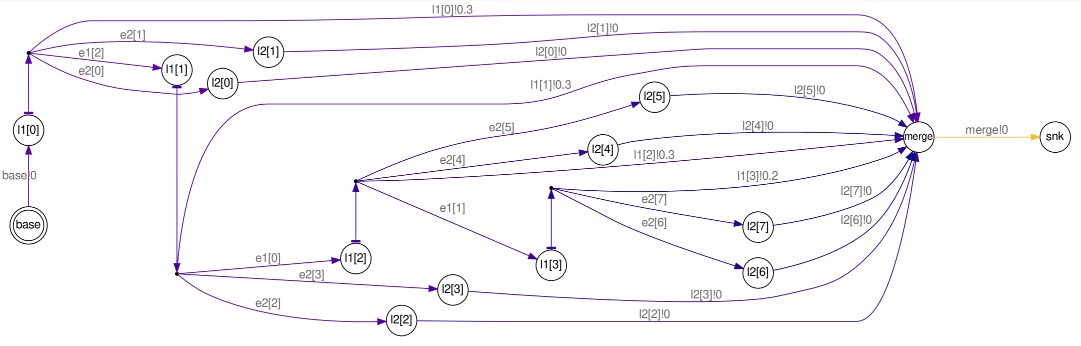
$R_{n\alpha}$ make use of goniometric identity in rotation matrix:

$$\begin{aligned} \begin{bmatrix} x'_{2\alpha} \\ y'_{2\alpha} \end{bmatrix} &= \begin{bmatrix} \cos(2\alpha_0) & -\sin(2\alpha_0) \\ \sin(2\alpha_0) & \cos(2\alpha_0) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x^2 - y^2 \\ 2 \cdot x \cdot y \end{bmatrix} \\ \begin{bmatrix} x'_{3\alpha} \\ y'_{3\alpha} \end{bmatrix} &= \begin{bmatrix} \cos(3\alpha_0) & -\sin(3\alpha_0) \\ \sin(3\alpha_0) & \cos(3\alpha_0) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \cdot x^3 - 3 \cdot x \\ 3 \cdot y - 4 \cdot y^3 \end{bmatrix} \end{aligned} \tag{8}$$

Twiddle generator Dataflow implementation $s = 1$

Everything together produces this work of art:



the Inverse Fourier Transform

the DFT definition:

$$X[k] = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{i2\pi}{N}kn} \tag{9}$$

inverse DFT (IFFT):

$$x_n = \frac{1}{N} \cdot \sum_{k=0}^{N-1} X_k \cdot e^{\frac{i2\pi}{N}kn} \tag{10}$$

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rewriting in eachother:

$$\mathcal{F}^{-1}(\vec{x}) = \frac{1}{N} \cdot swap(\mathcal{F}(swap(\vec{x}))) \tag{11}$$

with $swap : a + bi \to b + ai$
or in math:

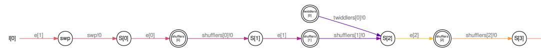
$$swap(x_n) = ix_n^* \tag{12}$$

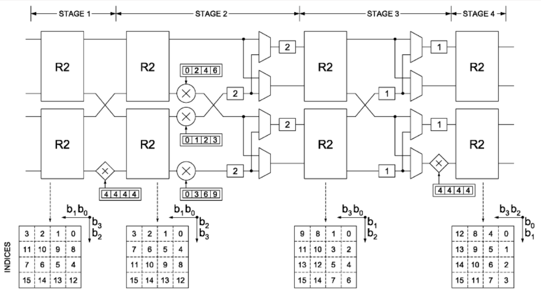
can now express the inverse as the original with simple swap

Dataflow Graph final result (again)

FFT:

IFFT:





Library created to generate design, fully scalable through *N* and *P*

Dataflow Graph final result (again)

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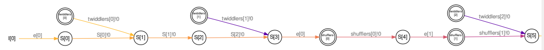
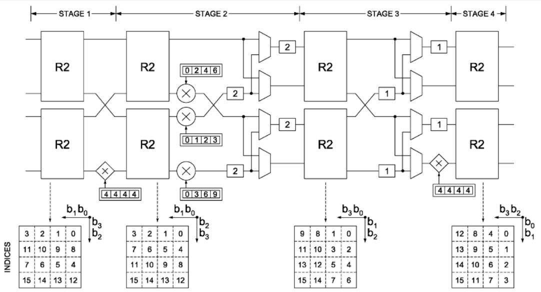
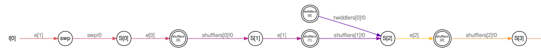




IFFT:



Library created to generate design, fully scalable through *N* and *P*
All arithmetic internally done in fixed point with *Z* bits per sample part

Testing Library

Test the FFT with its main properties, from industry standard FFTW library:

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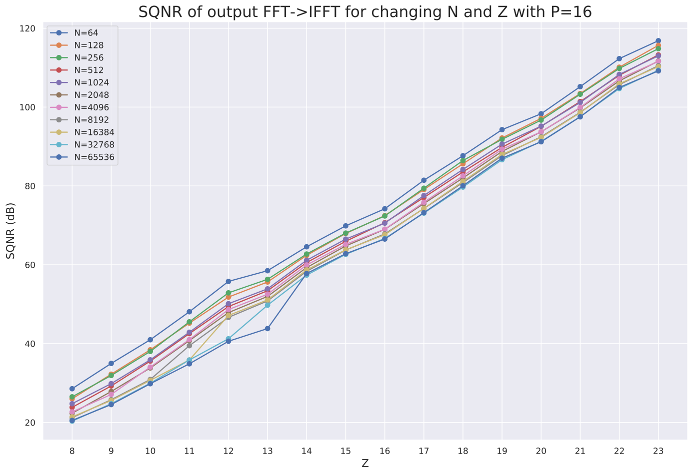
Test the FFT with its main properties, from industry standard FFTW library:

- Unit pulse input $\rightarrow$ "flat" output: tests butterfly without twiddles
- Linearity: $\mathcal{F}(\vec{V_A} + \vec{V_B}) = \mathcal{F}(\vec{V_A}) + \mathcal{F}(\vec{V_B})$
- Time shift of D samples: $\mathcal{F}(\vec{V_I^D}) = \left[e^{\left(\dfrac{-2j\pi D}{N} n\right)} \right] \cdot \mathcal{F}\left(\vec{V_I}\right)$
- Alternative $+2^Z$, -2^Z to test for overflow
- White noise: general data test

All simulation only

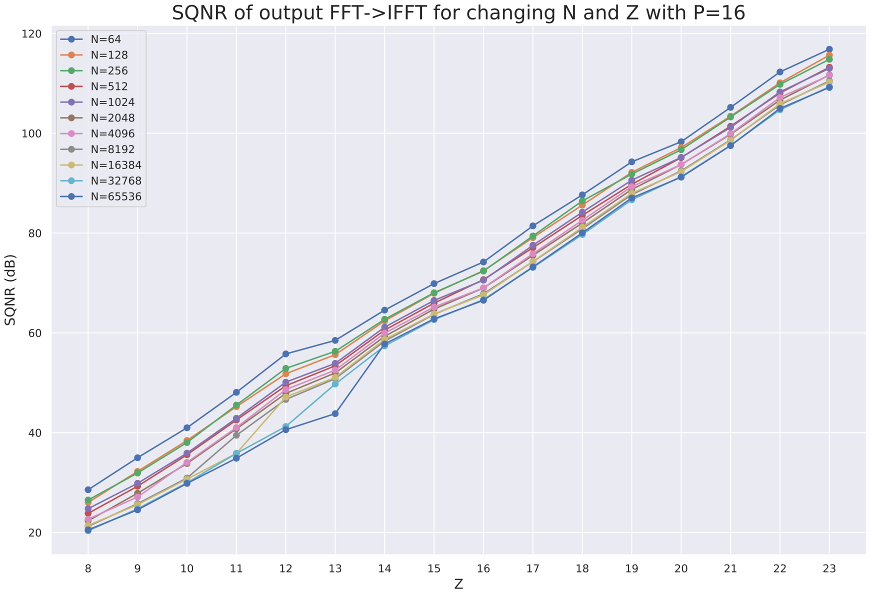
FFT -> IFFT



SQNR of output FFT->IFFT for changing N and Z with P=16
N=64
N=128
N=256
N=512
N=1024
N=2048
N=4096
N=8192
N=16384
N=32768
N=65536
SQNR (dB)
120
100
80
60
40
20
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
Z

FFT -> IFFT

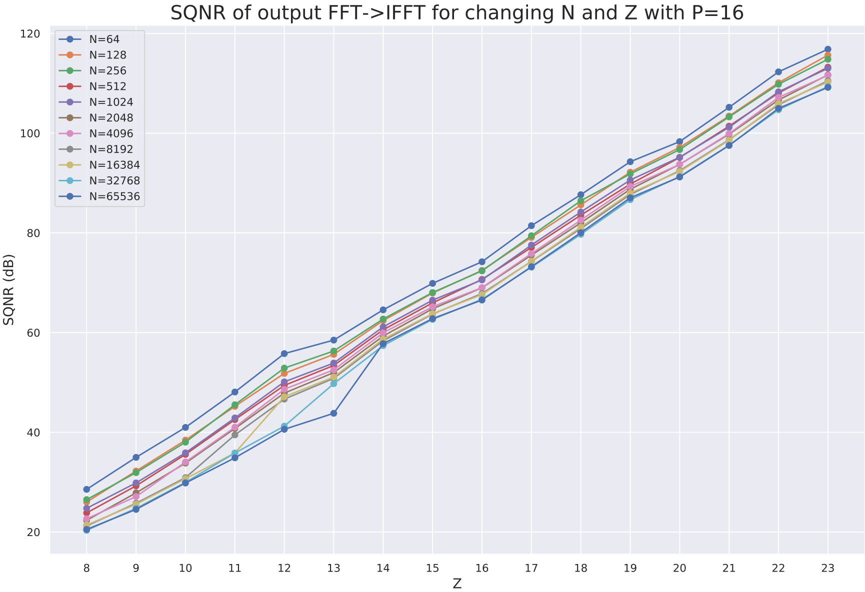
What do we see?



FFT -> IFFT

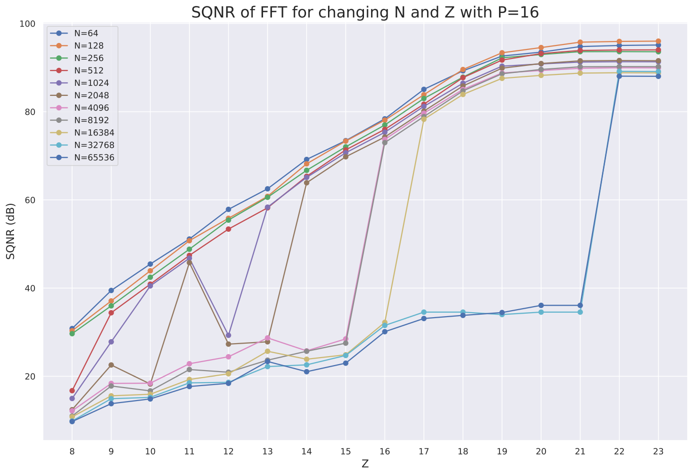
What do we see?

Higher bits → higher accuracy



TU/e

FFT only



SQNR of FFT for changing N and Z with P=16
SQNR (dB)
Z
N=64
N=128
N=256
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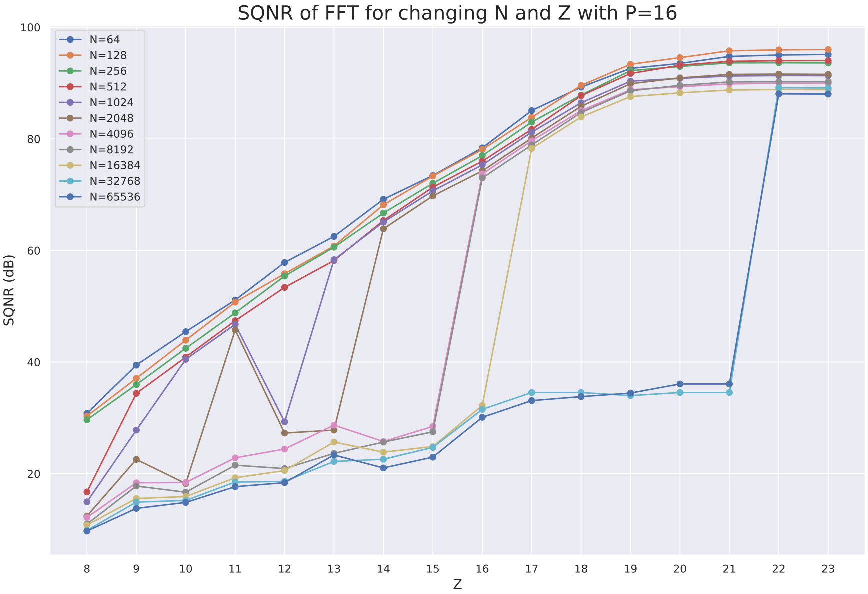
N=65536
100
80
60
40
20
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23

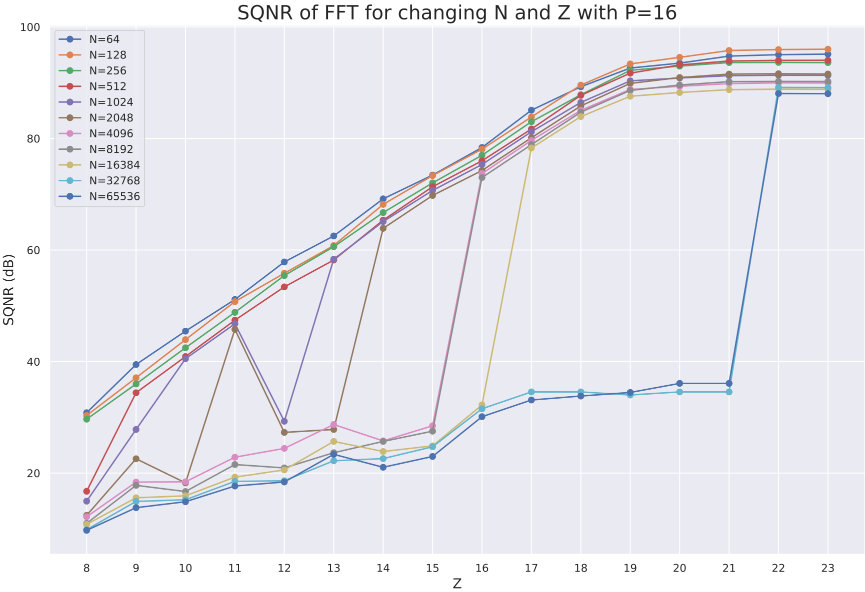
FFT

What do we see?



FFT

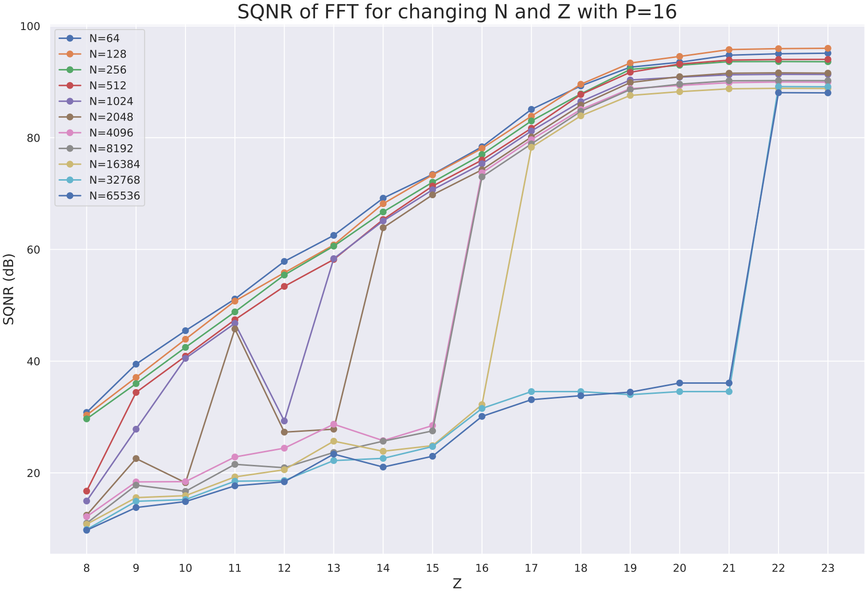
What do we see?
Higher bits → higher accuracy



FFT

What do we see?
Higher bits → higher accuracy

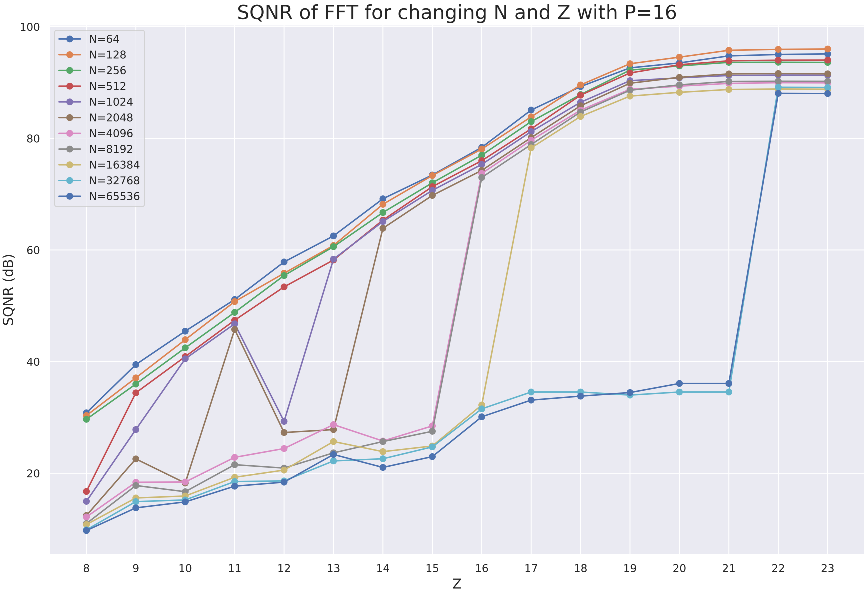
clear cut off point!
Why?



FFT

What do we see?
Higher bits $\rightarrow$
higher accuracy

clear cut off point!
Why?
possibly:
high N means:
very small twiddle



Future work

Hardware synthesis step from StaccatoLab (not done yet in library)

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Hardware synthesis step from StaccatoLab (not done yet in library)

Make twiddle fabrication more accurate (more smooth FFT SQNR graph)

In Conclusion

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analysis presented:
Pulsar search -> signal with heavy frequency dispersion

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cdmt: Frequency De-Dispersion through phase only filter

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FFT redesigned with dataflow for FPGA

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Design tested and verified in simulation

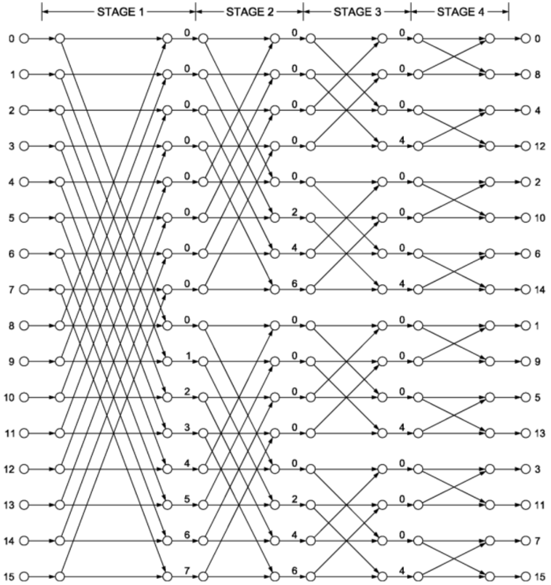
Questions?

Extra slides

The DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot W_N^{nk}, k = 0, 1, ..., N-1 \tag{13}$$

$$W_N^{nk} = W_N^{\phi} = e^{\dfrac{-j \cdot 2\pi \cdot \phi}{N}} \tag{14}$$



Radix 2 $\rightarrow$ **Radix** 2^2

Split the twiddle (rotation on complex plane):

$$W_N^{nk} = W_N^{\phi} = e^{\dfrac{-j \cdot 2\pi \cdot \phi}{N}} \tag{15}$$

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into trivial and non trivial part:

$$Ae^{-j\dfrac{2\pi}{N}\phi'} \pm Be^{-j\dfrac{2\pi}{N}(\phi' + N/4)} = [A \pm (-j)B] \cdot e^{-j\dfrac{2\pi}{N}\phi'} \tag{16}$$

Radix 2 $\rightarrow$ Radix 2^2

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Trivial: $\{0, N/4, N/2, 3N/4\} \rightarrow \{0, \pi/2, \pi, 1.5\pi\}$
Trivial is already in R2, what about non trivial?

What is Fixed point arithmetic?

Simple example:
Floating point number one third: 0.33333333333333333333...

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do math on integers $\rightarrow$ much easier on hardware!

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shift result back with $\div 2^{10}$
make sure intermediate results keep fitting! $\rightarrow$ scale back intermediate

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Input data: complex samples with 2 bytes (8 bits each)

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limiting scope of project, relative simple scaling:

- twiddle multipliers immediate scale back intermediate result
- for butterfly addition $\rightarrow$ *orthogonal scaling*

the Inverse Fourier Transform and Orthogonal Scaling

making use of: $\frac{1}{N} = \frac{1}{\sqrt{N}} \cdot \frac{1}{\sqrt{N}}$

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$$x_n = \frac{1}{\sqrt{N}} \cdot \sum_{k=0}^{N-1} X_k \cdot e^{\frac{i2\pi}{N}kn} \qquad (18)$$

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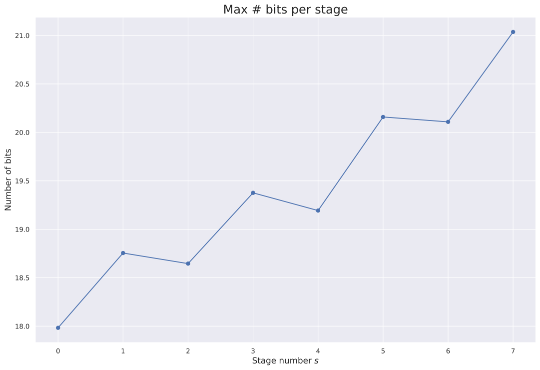
Example with N=256 and Z=18



Connection Pattern

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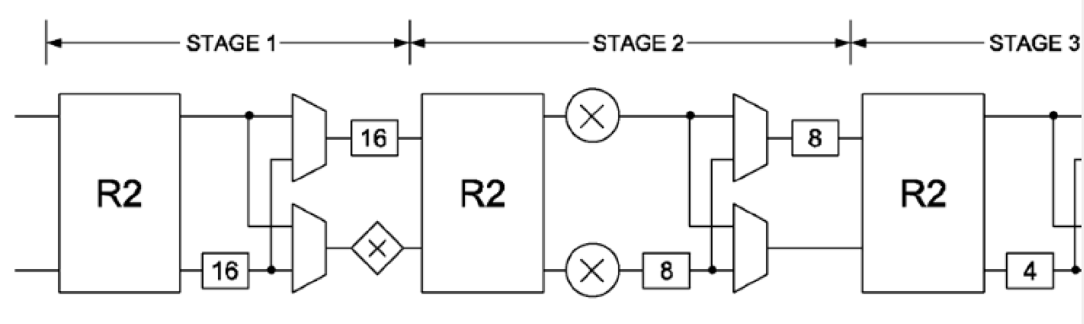
Depending on P: implemented as single vector in token
P=2 $\rightarrow$ simple straight



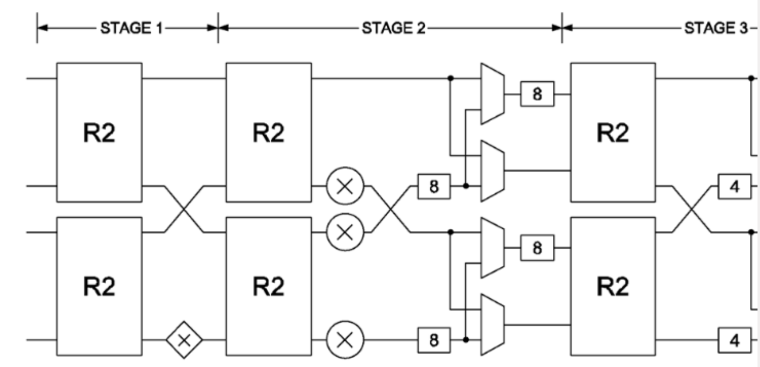
Connection Pattern (cont.)

Depending on P: implemented as single vector in token

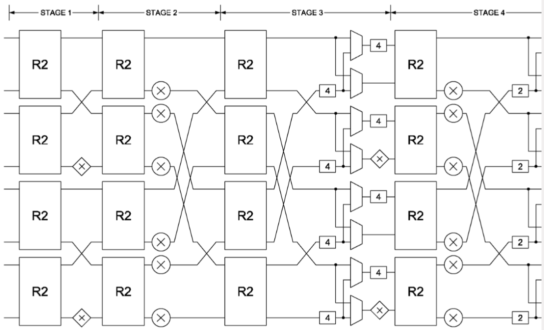
Connection Pattern (cont.)

Depending on P: implemented as single vector in token

P=4 $\rightarrow$ crossing pattern

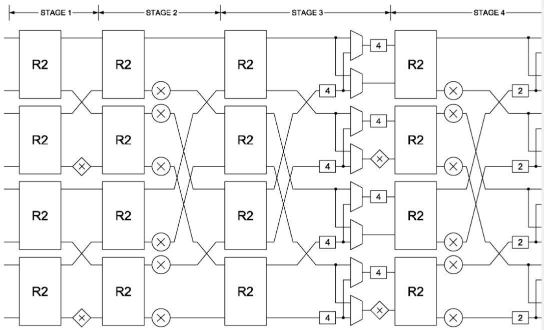


Connection Pattern (cont.)



Depending on P: implemented as single vector in token

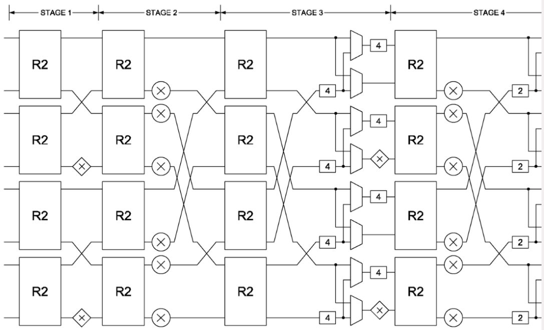
Connection Pattern (cont.)



Depending on P: implemented as single vector in token

P=8 $\rightarrow$ first repeat of pattern

Connection Pattern (cont.)



Depending on P: implemented as single vector in token

P=8 → first repeat of pattern

→ then stretched interleaving

Rotating factors

s=1, N=64, P=16
each row: non trivial rotating integers ϕ for one clock cycle

0	0	0	0	0	16	8	24	0	8	4	12	0	24	12	36
0	2	1	3	0	18	9	27	0	10	5	15	0	26	13	39
0	4	2	6	0	20	10	30	0	12	6	18	0	28	14	42
0	6	3	9	0	22	11	33	0	14	7	21	0	30	15	45

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0	4	2	6	0	20	10	30	0	12	6	18	0	28	14	42
0	6	3	9	0	22	11	33	0	14	7	21	0	30	15	45

P columns by $\frac{N}{P} = \frac{64}{16} = 4$ rows

How to generate?

Taking first row:

0	0	0	0	0	16	8	24	0	8	4	12	0	24	12	36
0	2	1	3	0	18	9	27	0	10	5	15	0	26	13	39
0	4	2	6	0	20	10	30	0	12	6	18	0	28	14	42
0	6	3	9	0	22	11	33	0	14	7	21	0	30	15	45

$$0 \cdot \frac{N}{P} \cdot \{0, 2, 1, 3\}, 2 \cdot \frac{N}{P} \cdot \{0, 2, 1, 3\},$$
$$1 \cdot \frac{N}{P} \cdot \{0, 2, 1, 3\}, 3 \cdot \frac{N}{P} \cdot \{0, 2, 1, 3\}$$

How to generate?

Taking first row:
4 groups of 4, consisting of:

0	0	0	0	0	16	8	24	0	8	4	12	0	24	12	36
0	2	1	3	0	18	9	27	0	10	5	15	0	26	13	39
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0	2	1	3	0	18	9	27	0	10	5	15	0	26	13	39
0	4	2	6	0	20	10	30	0	12	6	18	0	28	14	42
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Taking first row:
4 groups of 4, consisting of:
$base \cdot \{0, 2, 1, 3\}$

How to generate?

0	0	0	0	0	16	8	24	0	8	4	12	0	24	12	36
0	2	1	3	0	18	9	27	0	10	5	15	0	26	13	39
0	4	2	6	0	20	10	30	0	12	6	18	0	28	14	42
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Taking first row:
4 groups of 4, consisting of:
$base \cdot \{0, 2, 1, 3\}$
with base multiples of $\frac{N}{P}$ $(= 4$ here)

How to generate?

0	0	0	0	0	16	8	24	0	8	4	12	0	24	12	36
0	2	1	3	0	18	9	27	0	10	5	15	0	26	13	39
0	4	2	6	0	20	10	30	0	12	6	18	0	28	14	42
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in order of (again!) $\frac{N}{P} \cdot \{0, 2, 1, 3\}$

How to generate?

0	0	0	0	0	16	8	24	0	8	4	12	0	24	12	36
0	2	1	3	0	18	9	27	0	10	5	15	0	26	13	39
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with base multiples of $\frac{N}{P}$ $(= 4$ here)

in order of (again!) $\frac{N}{P} \cdot \{0, 2, 1, 3\}$

each row increases base with 1

Hardware needed

Nodes:

0	0	0	0	0	16	8	24	0	8	4	12	0	24	12	36
0	2	1	3	0	18	9	27	0	10	5	15	0	26	13	39
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0	2	1	3	0	18	9	27	0	10	5	15	0	26	13	39
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Nodes:

- $base_\alpha$

Realized as:

- CORDIC: hardware

Hardware needed

0	0	0	0	0	16	8	24	0	8	4	12	0	24	12	36
0	2	1	3	0	18	9	27	0	10	5	15	0	26	13	39
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Nodes:

- $base_\alpha$
- $R_{n\alpha}$ (with $n \in \{2,3\}$)

Realized as:

- CORDIC: hardware
- Rotation matrix using goniometric properties

Hardware needed

0	0	0	0	0	16	8	24	0	8	4	12	0	24	12	36
0	2	1	3	0	18	9	27	0	10	5	15	0	26	13	39
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Nodes:

- $base_\alpha$
- $R_{n\alpha}$ (with $n \in \{2, 3\}$)
- $R_{N//P}$

Realized as:

- CORDIC: hardware
- Rotation matrix using goniometric properties
- Rotation matrix precomputed

Hardware needed

0	0	0	0	0	16	8	24	0	8	4	12	0	24	12	36
0	2	1	3	0	18	9	27	0	10	5	15	0	26	13	39
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This is all done in single sample token mode!

$\rightarrow$ relative simple connection pattern

Dataflow implementation

after first stage only one of the four blocks

0	0	0	0	0	16	8	24	0	8	4	12	0	24	12	36
0	2	1	3	0	18	9	27	0	10	5	15	0	26	13	39
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after first stage only one of the four blocks

with base increasing with $\left\lfloor \frac{s-1}{2} \right\rfloor$

Dataflow implementation

0	0	0	0	0	16	8	24	0	8	4	12	0	24	12	36
0	2	1	3	0	18	9	27	0	10	5	15	0	26	13	39
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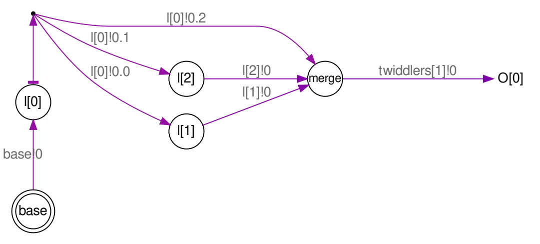
after first stage only one of the four blocks

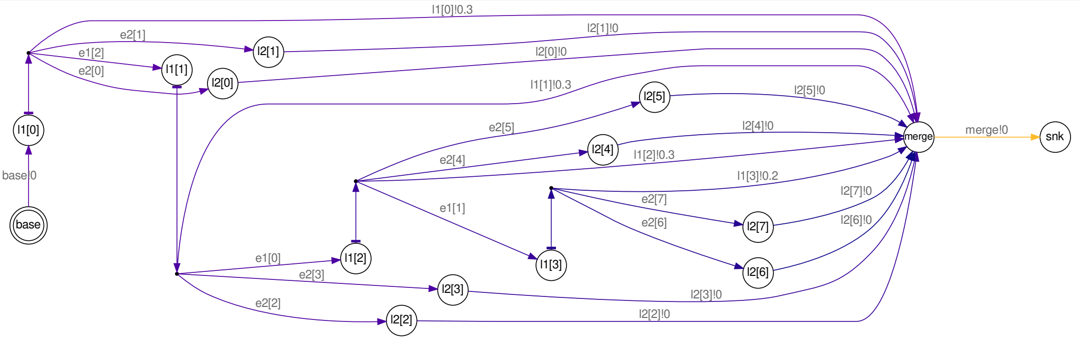
with base increasing with $\left\lfloor \frac{s-1}{2} \right\rfloor$

Graph form:



Dataflow implementation $s = 1$

Everything together produces this work of art:



CORDIC

$$
\begin{aligned}
x_{i+1} &= x_i - d_i \cdot 2^{-i} \cdot y_i \\
y_{i+1} &= y_i + d_i \cdot 2^{-i} \cdot x_i \\
\omega_{i+1} &= \omega_i - d_i \cdot \tanh(2^{-i}) \\
d_i &= -sign(\omega_i)
\end{aligned} \tag{19}
$$

Here the states are described as:
x_i and y_i for the real and imaginary part
ω_i as the remaining angle
d_i as the decision variable. i is the iteration number with $i \in \{0, 1, ..., n-1\}$.

Scaled Rotation Matrix

$$\begin{aligned}
\begin{bmatrix} x'_{2\alpha} \\ y'_{2\alpha} \end{bmatrix} &= \begin{bmatrix} \cos(2\alpha_0) & -\sin(2\alpha_0) \\ \sin(2\alpha_0) & \cos(2\alpha_0) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x^2 - y^2 \\ 2 \cdot x \cdot y \end{bmatrix} \\
\begin{bmatrix} x'_{3\alpha} \\ y'_{3\alpha} \end{bmatrix} &= \begin{bmatrix} \cos(3\alpha_0) & -\sin(3\alpha_0) \\ \sin(3\alpha_0) & \cos(3\alpha_0) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \cdot x^3 - 3 \cdot x \\ 3 \cdot y - 4 \cdot y^3 \end{bmatrix}
\end{aligned} \tag{20}$$

For the 3α rotation matrix

$$\left(\begin{bmatrix} x & -y \\ y & x \end{bmatrix} \cdot \begin{bmatrix} (x^2 - y^2) \gg Z \\ (2 \cdot x \cdot y) \gg Z \end{bmatrix} \right) \gg Z \tag{21}$$

Measure unit

Measured in Signal-to-quantization-noise ratio (SQNR):

$$SQNR(dB) = 10 \cdot \log_{10}\left(\frac{E\{\mid X_{ID} \mid^2\}}{E\{\mid X_Q - X_{ID} \mid^2\}}\right) \tag{22}$$

Measure unit

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$$SQNR(dB) = 10 \cdot \log_{10}\left(\frac{E\{\mid X_{ID} \mid^2\}}{E\{\mid X_Q - X_{ID} \mid^2\}}\right) \tag{22}$$

Misleading name: not only quantization errors!
But Garrido uses it