Demand and Auction Estimation

Part 1 (Demand Estimation)

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Ph.D. Training Module
SeNSS Workshop
University of Essex
February 28-29, 2024

- **Topics:** Approximate outline of the main topics

Readings

Main Sources

- **Readings:** There is no single text for this workshop. These lecture notes draw heavily from several sources. The primary ones are listed below.

    - Berry, S. T. (1994). Estimating discrete-choice models of product differentiation. *RAND Journal of Economics*, 25(2):242–262.

    - Berry, S., Levinsohn, J., and Pakes, A. (1995). Automobile prices in market equilibrium.
      Econometrica, 63(4):841–890.

    - Nevo, A. (2001). Measuring market power in the ready-to-eat cereal industry. *Econometrica*, 69(2):307–342.

    - Nevo, A. (2000b). A practitioner's guide to estimation of random-coefficients logit models of demand.
      Journal of Economics and Management Strategy, 9(4):513–548.

  Other useful material to consult includes Ackerberg et al. (2007), Cameron and Trivedi (2005) (Chapter 6), Train (2003) (Chapters 3 & 9), Hausman et al. (1994), and Reiss and Wolak (2007). And most recently, Conlon and Gortmaker (2020).

These lecture notes are based on several sources and draw heavily from the following articles/chapters: Cameron and Trivedi (2005, Chap. 6); Deaton and Muellbauer (1980b, Chap. 3 & 5); Hausman et al. (1994); Bokhari and Fournier (2013); Bokhari and Mariuzzo (2018); Berry (1994); Berry et al. (1995); Ackerberg et al. (2007); Nevo (2000b, 2001).

In addition to these primary sources, I have also benefitted from presentations/lecture notes on the same topics by other researchers who have generously put their slides on the internet. These sources include (1) Matthew Shum (Lecture notes: Demand in differentiated-product markets); (2) Matthijs Wildenbeest (Structural Econometric Modeling in Industrial Organization); (3) Eric Rasmusen (The BLP Method of Demand Curve Estimation in Industrial Organization); (4) John Asker and Allan Collard-Wexler (Demand Systems for Empirical Work in IO); (5) Jonathan Levin (Differentiated Products Demand Systems); (6) Ariel Pakes (NBERMetrics); and (7) Aviv Nevo (NBER Methods Lecture – Estimation of Static Discrete Choice Models Using Market Level Data).

Finally, I am also in debt to my colleague Franco Mariuzzo for providing significant feedback on these notes. All errors are mine.

PRELIMINARIES

- Demand systems often form the bedrock upon which empirical work in industrial organization rests
- A fundamental issue is to measure market power, which is measured by the price-cost margin

$$L \equiv \frac{p - mc}{p} \qquad \text{(L = Lerner Index)} \tag{1}$$

- Lerner Index is a measure of a firm's market power (the index ranges from a high of 1 to a low of 0, where for a perfectly competitive firm with $p = mc$, the value of the Lerner index is zero)
- But cost is often not observed – the "new empirical industrial organization" (NEIO) literature is motivated by this data problem
- General idea – measure the demand side and back out the price cost margins
- How?

- Consider the monopolist's maximization problem

$$\max_p pq(p) - c(q(p)) \tag{2}$$

FOC imply

$$q(p) + p\frac{\partial q(p)}{\partial p} = \frac{\partial c(q(p))}{\partial q}\frac{\partial q(p)}{\partial p} = mc(q(p))\frac{\partial q(p)}{\partial p} \tag{3}$$

At the optimal price

$$(p^* - mc(q(p^*))) = -\left.\frac{q(p)}{\partial q(p)/\partial p}\right|_{p=p^*} \tag{4}$$

or equivalently

$$L = \frac{p^* - mc(q(p^*))}{p^*} = -\frac{1}{\eta(p^*)} \tag{5}$$

where $\eta(p^*) = \frac{p}{q(p)} \left.\frac{\partial q(p)}{\partial p}\right|_{p=p^*}$ is the price elasticity of demand

WHY DEMAND ESTIMATION?
SINGLE PRODUCT MONOPOLIST

- Inferring costs:

$$L \equiv \frac{p^* - mc(q(p^*))}{p^*} = -\frac{1}{\eta(p^*)}$$

- If the monopolist is pricing optimally, then estimate/knowledge of elasticity η allows us to infer marginal cost mc
- Similarly, if there was a cost shock, and if we have inferred the marginal cost mc, then we can figure out its impact on price (assuming the firm still behaves optimally) from the Lerner condition

$$p = mc + \frac{1}{(\partial q(p)/\partial p)} q(p)$$

- Price is equal to marginal cost plus a markup
- The markup depends on the curvature of the demand curve (if demand is perfectly elastic, as in the case of the perfect competition, then $p = mc$)
- Thus, if we can estimate demand elasticity, we can back out the markups
- The idea extends to oligopoly as well

- Topology of Various Approaches
  - single vs multi-products
  - product or characteristics space
  - representative vs heterogeneous agents
- Common Problems
  - endogeneity
  - multicollinearity
  - the dimensionality problem
  - unobserved heterogeneity among consumers
- Depending on the context and the question, a researcher needs to be careful about choosing the appropriate estimation methodology, as there are tradeoffs between how well different methods deal with these issues and how relevant any given problem is within a context

- When there are differentiated products, we want to estimate the system of demand equations and infer the markups using the full cross-elasticity matrix

$$
\begin{aligned}
q_1 &= q_1(p_1, p_2, \dots p_j, \dots, p_J, \boldsymbol{X}_1; \xi_1, \boldsymbol{\theta}_1) \\
q_2 &= q_2(p_1, p_2, \dots p_j, \dots, p_J, \boldsymbol{X}_2; \xi_2, \boldsymbol{\theta}_2) \\
&\vdots \\
q_j &= q_j(p_1, p_2, \dots p_j, \dots, p_J, \boldsymbol{X}_3; \xi_j, \boldsymbol{\theta}_j)
\end{aligned}
$$

where $j = 1, \dots, j, \dots, J$ represent the J different related products and $\boldsymbol{\theta}_j$ are the paraments in the j-th demand function $q_j(\cdot)$ that need to be estimated

- Elasticity matrix is represented by

$$
\boldsymbol{\eta} = \begin{bmatrix} \eta_{11} & \eta_{12} & \dots & \eta_{1j} \\ \eta_{21} & \eta_{22} & \dots & \eta_{2j} \\ & & \vdots & \\ \eta_{J1} & \eta_{J2} & \dots & \eta_{JJ} \end{bmatrix} \qquad \text{where} \qquad \eta_{ji} = \frac{\partial q_j(\cdot)}{\partial p_i} \frac{p_i}{q_j(\cdot)}
$$

- Example ...

- Example ...
- Say there are just three related products ... $J = 3$ and demand is specified in log-log form (aka Cobb-Douglas)

$$lnq_1 = \alpha_{10} + \beta_{11} lnp_1 + \beta_{12} lnp_2 + \beta_{13} lnp_3 + \gamma_{14} X_1 + \eta_1$$
$$lnq_2 = \alpha_{20} + \beta_{21} lnp_1 + \beta_{22} lnp_2 + \beta_{23} lnp_3 + \gamma_{24} X_2 + \eta_2$$
$$lnq_3 = \alpha_{30} + \beta_{31} lnp_1 + \beta_{32} lnp_2 + \beta_{33} lnp_3 + \gamma_{34} X_3 + \eta_3$$

then the elasticity matrix is constructed from the β parameters

$$\boldsymbol{\eta} = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix} \qquad \text{where} \qquad \eta_{ji} = \frac{\partial q_j(\cdot)}{\partial p_i} \frac{p_i}{q_j(\cdot)} = \frac{\partial lnq_j}{\partial lnp_i} = \beta_{ji}$$

- Note that with just three products, the elasticity matrix in the example above requires estimating at least nine parameters from the demand **system** above

Preliminaries
Single vs. Multiproduct Systems

- Should we be measuring demand for aggregate product type (drugs) or individual brands? Prices move together
- Most products have substitutes or complements and it is often necessary to explicitly account for the substitution possibilities to adequately answer the research question at hand
- In the context of multi-products, the researcher also has to face the problem of dimensionality and multicollinearity
  - Consider a system of demand equations

$$\mathbf{q} = D(\mathbf{p}, \mathbf{z}; \boldsymbol{\theta}, \boldsymbol{\xi}) \tag{6}$$

    where $\mathbf{q}$ is a $J \times 1$ vector of quantities, $\mathbf{p}$ is a vector of prices, $\mathbf{z}$ is a vector of exogenous variables that shift demand, $\boldsymbol{\theta}$ are the parameters to be estimated, and $\boldsymbol{\xi}$ are the error terms
  - In a system with J products, even with some simple and restrictive forms, the number of parameters to estimate is large
  - If $D(\cdot)$ is linear so that $D(\mathbf{p}) = \mathbf{A}\mathbf{p}$ where $\mathbf{A}$ is a $J \times J$ matrix of slope coefficients, then there are J^2 parameters to estimate (plus additional ones due to the exogenous variables $\mathbf{z}$)
  - Restrictions ...

- Imposing the symmetry of the Slutsky matrix or adding up restrictions (Engle and Cournout aggregation) reduces the number of parameters to be estimated
- However, the essential problem, that the number of parameters increases in the square of the number of products, remains
  - Slutsky equation: $\frac{\partial q_j}{\partial p_i} = \frac{\partial h_j}{\partial p_i} - q_i \frac{\partial q_j}{\partial y}$
  - Engle aggregation: $\sum_j s_j \eta_{jy} = 1$
  - Cournot aggregation: $\sum_j s_j \eta_{ji} = -s_i$) where η_{ji}
    - q_j and h_j are the Marshallian and Hicksian demand functions respectively for product j, and y is the income or total expenditure
    - η_{ji} is the cross price elasticity of product j with respect to price of i, η_{jy} is the income elasticity of product j and s_i, s_j are the expenditure shares

- *If the research question allows*, avoid the problem of estimating too many parameters by working with a more restrictive form
- Consider the constant elasticity of substitution (CES) utility function

$$u(\mathbf{q};\rho) = u(q_1, q_2, \ldots, q_J;\rho) = \left(\sum_i^J q_i^\rho\right)^{1/\rho} \tag{7}$$

where ρ is the parameter of interest that measures the elasticity of substitution
- The demand for a representative consumer is then given by

$$q_j(\mathbf{p}, I;\rho) = \frac{p_j^{1/(1-\rho)}}{\sum_i^J p_i^{\rho/(1-\rho)}} I \qquad j = 1, \ldots, J. \tag{8}$$

- Need to estimate only one parameter ... not J^2 – problem solved!
- But now the cross elasticity between products i and j is the same as between k and j for all combinations of i, j, k,

$$\frac{\partial q_i}{\partial p_j}\frac{p_j}{q_i} = \frac{\partial q_k}{\partial p_j}\frac{p_j}{q_k} \qquad \forall i, j, k. \tag{9}$$

- An alternative to the single parameter of the CES utility function is the logit demand (Anderson, de Palma, and Thisse, 1992)

$$u(\mathbf{q};\boldsymbol{\delta}) = \sum_{j}^{J} \delta_j q_j - \sum_{j}^{J} q_j \ln q_j \,. \tag{10}$$

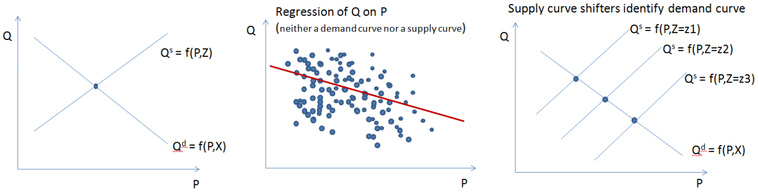
- Elasticities in this model depend on market shares (given by J number of parameters δ_j) but not on the similarities among the products
- What if products j and k are more alike (coke,pepsi) and product i is somewhat more different (fanta)?
- Will discuss logit properties further (later)

- Demand models often suffer from the endogeneity problem
  - Endogeneity means when in an econometric equation, a right-hand side is correlated with the error term
  - In demand models, this is because the prices on the right-hand side are typically correlated with the error term
  - A consequence of that is that it violates one of the classical assumptions of the OLS regression theory and hence leads to biased estimates of the demand parameters
- The Problem – Consider an equation such as

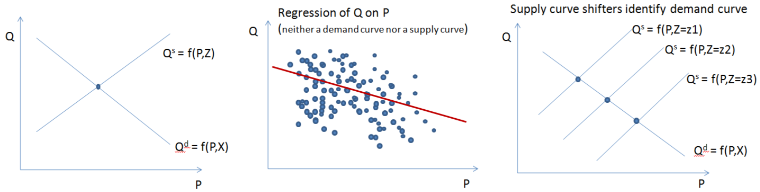
$$Y_i = \beta_1 + \beta_2 X_{2i} + u_i$$

where the interest is in knowing the value of β_2.

  - If $(E[X_{2i}, u_i] \neq 0)$ then simple regression based methods will produce biased estimates such that $E(\widehat{\beta_2}) \neq \beta_2$.
  - This is because $E[X_{2i}, u_i] \neq 0$ (crucial assumption in OLS) due to
    - measurement error of X_2
    - omitted variable(s) X_3 correlated with both Y and X_2
    - simultaneity – i.e., where X_2 and Y are jointly determined

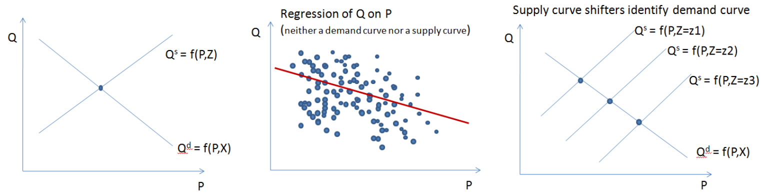


- In typical demand analysis with n products
  - Quantity demanded is a function of own price, price of related products and other demand shifters, $Q_i^d = f(p_1, p_2, \ldots, p_i, \ldots, p_n, X_i)$.
  - The supply curve Q_i^s is also a function of its own price and marginal cost $Q_i^s = f(p_i, C_i)$.
  - The *observed* price and quantity (or shares) are jointly determined via market clearing (demand equals supply, $Q_i^d = Q_i^s$).
  - Regression of quantity on prices (even after holding other variables constant) will result in neither the estimates of the demand curve nor of the supply curve.
  - Demand curve can be *identified* via variables that shift the supply curve (e.g. cost of production).



- The Cure – For each endogenous variable such as X_2, find a variable (instrument) Z such that
  - it is relevant (i.e., $E[X_{2i}, Z_i] \neq 0$)
  - it is valid (i.e., $E[Y_i, Z_i] = 0$)
- The IV procedure – In two easy steps
  - Regress X_{2i} on Z_i and obtain predicted values of X_2 (say $\widehat{X}_2$)
  - Regress Y on $\widehat{X}_2$ – coefficient on X_2 is now an unbiased estimate of β_2

Demand Models
Endogeneity and Identification



- Instruments
  - To estimate demand curves, we need at least n relevant and valid instruments $(Z_1, Z_2, \ldots, Z_n)$.
  - C_i enter the supply function and hence are relevant (i.e., $E[p_i, C_i] \neq 0$).
  - C_i do not enter the demand function and hence are valid (i.e., $E[Q_i, C_i] = 0$).
  - Good News: Can use the (marginal) costs C_i of the products as instruments for the prices.
  - Bad News: Data on marginal costs by product line is often not available.
- Need some different types of instruments to estimate demand curves.

- Prices are often endogenous ...
- Consider a simple linear demand/supply model for a single homogenous product over T markets, where aggregate demand/supply relations are given by

$$\begin{aligned} q_t^d &= \beta_{10} + \gamma_{12} p_t + \beta_{11} x_{1t} + \xi_{1t}, \\ p_t &= \beta_{20} + \gamma_{22} q_t^s + \beta_{22} x_{2t} + \xi_{2t}, \\ q_t^s &= q_t^d \end{aligned} \tag{11}$$

- error terms are such that*

$$\begin{aligned} E(\xi_{1t}|\mathbf{x}_t) = 0, E(\xi_{2t}|\mathbf{x}_t) &= 0, \\ E(\xi_{1t}^2|\mathbf{x}_t) = \sigma_1^2, E(\xi_{2t}^2|\mathbf{x}_t) &= \sigma_2^2 \\ E(\xi_{1t}\mathbf{x}_t) = 0, E(\xi_{2t}\mathbf{x}_t) &= 0, \\ \text{and } E(\xi_{1t}\xi_{2t}|\mathbf{x}_t) &= 0 \end{aligned} \tag{12}$$

where $\mathbf{x}_t = [1 \; x_{1t} \; x_{2t}]$

*Since we have already made the stronger assumption that $E(\xi_{1t}|\mathbf{x}_t) = 0$, technically we do not need to explicitly assume that $E(\xi_{1t}\mathbf{x}_t) = 0$, since the latter is implied by the former assumption of zero conditional mean due to the law of iterated expectations. Nonetheless, I include it just to be clear.

- Prices are often endogenous ...
- solve for the reduced form equilibrium values of q^* and p^* – dropping subscript t, we get

$$\begin{aligned} q^* &= \frac{\beta_{10} + \beta_{20}\gamma_{12}}{1 - \gamma_{12}\gamma_{22}} + \frac{\beta_{11}}{1 - \gamma_{12}\gamma_{22}} x_1 + \frac{\gamma_{12}\beta_{22}}{1 - \gamma_{12}\gamma_{22}} x_2 + \frac{\xi_1 + \gamma_{12}\xi_2}{1 - \gamma_{12}\gamma_{22}} \\ p^* &= \frac{\beta_{20} + \beta_{10}\gamma_{22}}{1 - \gamma_{12}\gamma_{22}} + \frac{\beta_{11}\gamma_{22}}{1 - \gamma_{12}\gamma_{22}} x_1 + \frac{\beta_{22}}{1 - \gamma_{12}\gamma_{22}} x_2 + \frac{\gamma_{22}\xi_1 + \xi_2}{1 - \gamma_{12}\gamma_{22}} \end{aligned} \tag{13}$$

- p^* is a function of ξ_1 (and ξ_2) and hence an OLS estimation of the demand equation above (regress q on p, x_1) will result in an inconsistent estimate of γ_{12} and other parameters

- Prices are often endogenous ...
- Useful to explicitly compute the conditional covariance between p and ξ_1
- Note that conditional on $\mathbf{x}_t$,

$$\begin{aligned} p^* - E(p^*) &= \frac{\gamma_{22}\xi_1 + \xi_2}{1 - \gamma_{12}\gamma_{22}} \\ \text{and } \xi_1 - E(\xi_1) &= \xi_1 \end{aligned} \tag{14}$$

  Thus

$$\mathrm{cov}(p, \xi_1) = \frac{\gamma_{22}}{1 - \gamma_{12}\gamma_{22}}\sigma_1^2 + \frac{\mathrm{E}(\xi_1\xi_2)}{1 - \gamma_{12}\gamma_{22}} \tag{15}$$

- Even if the error terms across the two equations were uncorrelated ($\mathrm{E}(\xi_{1t}\xi_{2t}|\mathbf{x}_t = 0)$), the covariance between p and ξ_1 would still not be zero
- On the other hand, if γ_{22} is zero, q does not appear in the supply equation, i.e., it is a triangular system of equations and OLS estimation is fine as long as $\mathrm{E}(\xi_{1t}\xi_{2t}|\mathbf{x}_t) = 0$
- For completeness – complete system of equations, i.e., the number of equations are equal to the number of endogenous variables – we also require that $\gamma_{12} \neq 1/\gamma_{22}$.

- we can re-write the system in (11) in matrix notation

$$\mathbf{y}_t' = \begin{bmatrix} q_t & p_t \end{bmatrix} \mathbf{x}_t = \begin{bmatrix} 1 & x_{1t} & x_{2t} \end{bmatrix} \boldsymbol{\xi}_t' = \begin{bmatrix} \xi_{1t} & \xi_{2t} \end{bmatrix}$$
$$\boldsymbol{\Gamma} = \begin{bmatrix} 1 & -\gamma_{22} \\ -\gamma_{12} & 1 \end{bmatrix} \text{ and, } \mathbf{B} = \begin{bmatrix} \beta_{10} & \beta_{20} \\ \beta_{11} & 0 \\ 0 & \beta_{22} \end{bmatrix} \tag{16}$$

then, the system of equations above can be written as

$$\mathbf{y}_t'\boldsymbol{\Gamma} - \mathbf{x}_t\mathbf{B} = \boldsymbol{\xi}_t' \tag{17}$$

so that the reduced form equation is

$$\mathbf{y}_t' = \mathbf{x}_t\boldsymbol{\Pi} + \mathbf{v}_t' \qquad \text{where } \boldsymbol{\Pi} = \mathbf{B}\boldsymbol{\Gamma}^{-1} \text{ and, } \mathbf{v}_t' = \boldsymbol{\xi}_t'\boldsymbol{\Gamma}^{-1} \tag{18}$$

Note that in the equation above we are taking the inverse of the $\boldsymbol{\Gamma}$ – but the inverse exists if the determinant ($\det(\boldsymbol{\Gamma}) = 1 - \gamma_{12}\gamma_{22}$) is not zero, which goes back to the condition $\gamma_{12} \neq 1/\gamma_{22}$ mentioned above

- The moment restrictions in (12) (in general we do not need to impose $E(\xi_{1t}\xi_{2t}|\mathbf{x}_t) = 0$) are

$$\begin{aligned} &\mathrm{E}(\boldsymbol{\xi}_t|\mathbf{x}_t) = \mathbf{0}, && \mathrm{E}(\boldsymbol{\xi}_t\boldsymbol{\xi}_t'|\mathbf{x}_t) = \boldsymbol{\Sigma} \\ &\mathrm{E}(\mathbf{v}_t|\mathbf{x}_t) = \mathbf{0}, && \mathrm{E}(\mathbf{v}_t\mathbf{v}_t'|\mathbf{x}_t) = \boldsymbol{\Omega} \\ &\text{where } \boldsymbol{\Omega} = (\boldsymbol{\Gamma}^{-1})'\boldsymbol{\Sigma}\boldsymbol{\Gamma}^{-1}. \end{aligned} \tag{19}$$

- Estimation can proceed with IV/2SLS (or 3SLS for joint estimation), where the demand equation is estimated using x_{2t} as the instrument, and supply equation is estimated using x_{1t} as the instrument
- If either $\beta_{22} = 0$ or if data on x_{2t} is not available, demand equation cannot be identified/estimated consistently (vice versa for supply equation)
- Since the x's are exogenous variables, they can serve as instruments
  - x_{2t} are cost shifters – they affect production costs; Correlated with p_t but not with ξ_{1t}, hence use as instruments in demand function
  - x_{1t} are demand shifters – affect willingness-to-pay, but not a firm's production costs; Correlated with q_t but not with ξ_{2t}, hence use as instruments in supply function

- Product Space
  - Consumers have preferences over products
  - Usual utility maximization problem
  - Leads to demand at the product level
  - In that sense, demand analysis in product space is more natural (or at least more familiar)
- Characteristics Space
  - Views products as bundles of characteristics
  - Consumers have preferences over those characteristics
  - Each individual's demand for a given product is just a function of the characteristics of the product
- We can think of a set of products (Toyota Minivan, Lexus SUV, etc.) or we can think of them as a collection of various properties (horsepower, size, color, etc.)
- In general, demand systems in characteristic space are approximations to product space demand systems and hence, we can either model consumers as having preferences over products, or over characteristics (note that not all of the characteristics need to be observed and may form part of the error term)

- Considerations
  - Dimensionality of Products
  - Dimensionality of Characteristics
  - New Goods
  - Cross elasticities

- Considerations
  - Dimensionality of Products
    - For large number of products (say $J = 50$), the product space approach leads to the dimensionality problem mentioned earlier, and may require grouping/nesting these products. By contrast, if we can reduce J products to just a few K characteristics, and the preferences over those characteristics are, say normally distributed, then we have to estimate K means and $K(K+1)/2$ covariances. *If* there were no unobserved characteristics, then $K(1 + (K+1)/2)$ parameters would suffice to analyze own and cross-price elasticities for all J goods.
  - Dimensionality of Characteristics
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- Considerations
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    - If there are too many characteristics (K is large), then the the problem of too many parameters re-appears as in the product space case, and we need data on each of these characteristics. A solution is to model some of them as unobserved characteristics – but this leads to the endogeneity problem if the unobserved characteristics (think product quality) are correlated with the price, which they usually are.
  - New Goods
  - Cross elasticities

- Considerations
  - Dimensionality of Products
  - Dimensionality of Characteristics
  - New Goods
    - If we are interested in the counterfactual exercise to assess the welfare impact of a new introduction in an ex-ante period (say a new proposed generic drug or a me-too drug), it is difficult to do so in the product space (we can do it using ex-post data though), but it is easier to do this exercise using the characteristic space approach. This is because if we have estimated the demand system using the characteristic approach, and we know the proposed characteristics of the new good, we can, in principle, analyze what the demand for the new good would be. Note however that if the new good is totally different from products already in the market, i.e., have very different (and new) properties, characteristics space approaches may not help either (e.g., could we have predicted the demand for laptops based on the characteristics of desktop computers, or for a new drug which proposes treatment of a formerly un-treatable disease?)
  - Cross elasticities

- Considerations
    - Dimensionality of Products
    - Dimensionality of Characteristics
    - New Goods
    - Cross elasticities
        - Most of the characteristics space estimation, at least on aggregate data, does not easily lend to analyzing products that are used in bundles or as complements. This is an ongoing area of research.

- Consider the demand function of single product j in market t for a representative consumer, given by

$$q_{jt} = \gamma_j + \alpha_j p_{jt} + \mathbf{x}_{jt}\beta_j + \xi_{jt} \tag{22}$$

  where $\mathbf{x}_{jt}$ is a vector of product characteristics and ξ_{jt} are the unobserved components of demand

  - Interest is in estimating α_j and demand elasticity
  - Even though product specific intercepts γ_j have been included in the model, they are demand shifters, and as such do not change the sensitivity to price depending on the level of income or other demographic characteristics such as family size
  - Micro studies often show that the price coefficient depends on an important way on income/wealth, i.e., lower-income people care more about price
  - Consequently, if the income distribution varies across the markets, we should expect the price coefficient to vary across these markets, and we need to find a way to allow for it

Preliminaries
Representative or Heterogenous Consumer

- Consider the demand function of single product j in market t for a representative consumer, given by

$$q_{jt} = \gamma_j + \alpha_j p_{jt} + \mathbf{x}_{jt}\beta_j + \xi_{jt} \tag{22}$$

  where $\mathbf{x}_{jt}$ is a vector of product characteristics and ξ_{jt} are the unobserved components of demand
  - One could make γ_j to be a function of income, but they are still demand shifters and do not change the sensitivity to price. Similarly, other demographic differences may be important to model as well
  - One could potentially include some ad-hoc interaction terms between average values of demographic variables in market t with price (and other product characteristics) but may not represent demand derived from a consumer's utility maximization problem

- To make it a heterogenous agent model, it is more typical to build a micro model where the parameters that enter the utility function of a consumer – say γ_j and α_j – vary over individuals and are perhaps functions of their demographics

  In that case, the demand equations to be estimated would end up looking something like

$$q_{jt} = \int \gamma_{ij} dG(\gamma_{ij}) + \int \alpha_{ij} p_{jt} dF(\alpha_{ij}) + \mathbf{x}_{jt}\beta_j + \xi_{jt} \tag{23}$$

  - where γ_{ij} and α_{ij} are person and product specific random intercepts and slope coefficients, with known or assumed distribution functions $\gamma_{ij} \sim G(\gamma|\tau)$ and $\alpha_{ij} \sim F(\alpha|\theta)$, and where θ and τ are parameters to be estimated and are functions of demographic variables
  - This is called a random coefficients model

- Depending on the context and the question, a researcher needs to be careful about choosing the appropriate estimation methodology
- Earlier empirical work focused on specifying representative consumer demand systems such that they allowed for various substitution patterns, and were consistent with economic theory
  - Linear Expenditure model (Stone, 1954)
  - the Rotterdam model (Theil, 1965; and Barten 1966)
  - or the more flexible ones such as the Translog model (Christensen, Jorgenson, and Lau, 1975) and the Almost Ideal Demand System (AIDS – Deaton and Muellbauer, 1980a)
- We will focus on the AIDS model but within the context of multistage budgeting as well as variants of the logit models — logit, nested logit, random coefficients logit – based on works by Berry (1994) and Berry et al. (1995) (henceforth BLP)

Estimation in Product Space
(AIDS Model only)

- Several demand models in product space can be linked to consumer theory — linear, linear expenditure model, constant elasticity of substitution (CES), Cobb-Douglas, Rotterdam model, Translog model, etc., with varying theoretical properties
- A popular demand system, introduced by Deaton and Muellbauer (1980a,b), is the "Almost Ideal Demand System" (AIDS) — it has several desirable theoretical properties (not discussed in detail here but see the appendix)[†]
  - aggregates over consumers and allows for non-linear Engle curves
  - has a flexible substitution pattern and provides a first-order approximation to any other demand system
  - we can impose and test restrictions on parameters (symmetry, homogeneity)
  - can be linearized via the Stone price index (but that has some consequences on the estimation of elasticities ...)

[†] Aggregation and non-linear Engle curves properties are related to the Gorman polar form of the expenditure functions. Further, AIDS modes are often estimated in the context of separability and multistage budgeting by a consumer. I have skipped the details but discuss them in more detail in the Appendix.

- The model starts by specifying a representative consumer's expenditure function, given by[‡]

$$\ln(y) = \ln(e(\mathbf{p}, u_0)) = (1 - u_0)\ln(a(\mathbf{p})) + u_0 \ln(b(\mathbf{p})) \tag{24}$$

where y is the total expenditure, $\mathbf{p}$ is the vector of prices of relevant goods and u_0 is the utility of the representative consumer, and

$$\begin{aligned} \ln a(\mathbf{p}) &= \alpha_0 + \sum_j \alpha_j \ln p_j + \frac{1}{2}\sum_j \sum_k \gamma^*_{jk} \ln p_j \ln p_k \\ \ln b(\mathbf{p}) &= \ln a(\mathbf{p}) + \beta_0 \prod_j p_j^{\beta_j} \end{aligned} \tag{25}$$

- The expenditure function will be linearly homogenous in $\mathbf{p}$ as long as $\sum_j \alpha_j = 1, \sum_j \gamma^*_{kj} = \sum_k \gamma^*_{kj} = \sum_j \beta_j = 0$

[‡]Recall that an expenditure function $e(\mathbf{p}, u_0)$ indicates the minimum amount of money necessary to purchase as many units of goods at the given prices $\mathbf{p}$ to obtain utility level u_0

- Microeconomic theory tells us that if take partial derivatives of the expenditure function wrt prices, we will obtain the Hicksian (compensated) demand functions – and if we further replace the utility with indirect utility, we will obtain the observable demand curves (Marshallian or uncompensated demand functions)
- Thus, for a set of J products, the demand for an good j is given by

$$q_j(\mathbf{p}, y) = \frac{y}{p_j}\left(\alpha_j + \sum_k \gamma_{jk} \ln p_k + \beta_j \ln(y/P)\right)$$

and where P is a translog price index defined by (26)

$$\ln P = \alpha_0 + \sum_k \alpha_k \ln p_k + \frac{1}{2} \sum_i \sum_k \gamma_{ki} \ln p_k \ln p_i$$

where $\gamma_{jk} = \frac{1}{2}(\gamma^*_{jk} + \gamma^*_{kj})$

- The demand system given above is estimated in expenditure share form $s_j = q_j p_j / y$, and hence the system of equations to be estimated are given by

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$$
\begin{aligned}
s_j &= \alpha_j + \sum_k \gamma_{jk} \ln p_k + \beta_j \ln(y/P) + u_j \\
\ln P &= \alpha_0 + \sum_k \alpha_k \ln p_k + \frac{1}{2} \sum_i \sum_k \gamma_{ki} \ln p_k \ln p_i
\end{aligned} \tag{27}
$$

  - Note that I have added in an econometric error term u_j – also, demographic differences can be added in by modeling them as functions of α_j
  - The restrictions on the parameter of the cost function impose restriction on the parameters of the AIDS demand system (27) given by

$$
\begin{aligned}
&\sum_{j=1}^{J} \alpha_j = 1 \qquad \sum_{j=1}^{J} \gamma_{jk} = 0 \qquad \sum_{j=1}^{J} \beta_j = 0 \\
&\sum_k \gamma_{jk} = 0 \qquad \gamma_{jk} = \gamma_{kj}
\end{aligned} \tag{28}
$$

  - Provided the restrictions above hold (or are imposed), (27) represents a system of demand functions which add up to total expenditure $(\sum s_j = 1)$, are homogeneous of degree zero in prices and total expenditure taken together, and satisfy Slutsky symmetry and give nonlinear Engle curves

- The system of equations (27) is non-linear: estimation of parameters in the share equation requires that we know the value of the price index – but that can't be computed until we have the parameters – so need to use non-linear estimation methods
- A popular simplification is to linearize via the Stone price index which does not use these parameters (called LA-AIDS)

$$\ln P = \sum_j s_j \ln p_j \tag{29}$$

- We can now estimate the system of equations as $\ln P$ can be computed from the data before estimation – but now the problem is that we will introduce a simultaneity bias (endogeneity) even if prices were exogenous as the share s_j appears on both sides of the equation
    - To deal with this endogeneity, in panel settings s_j is often replaced by (i) a lagged value $s_{j,t-1}$, (ii) first period average value $\bar{s}_{j0}$ (aka Laspeyres price index) (iii) sample average value $\bar{s}_j$, (iv) other ... all such choices impact how elasticity is computed

- Under LA-AIDS (and with first-period average values in the Stone price index, i.e., Laspeyres index), the own and cross-price elasticity (Marshallian) for product j wrt to price of k can be computed as

$$\eta_{jk} \equiv \frac{\ln q_j}{\ln p_k} = \frac{1}{s_j}(-\beta_j \bar{s}_{k0} + \gamma_{jk}) - \delta_{jk} \tag{30}$$

  where δ_{jk} is equal to 1 if $j = k$ and zero otherwise

- The expenditure elasticity of product j, denoted e_j, and compensated (Hicksian) elasticities h_{ik} are then given by

$$\begin{aligned} e_j &= 1 + \frac{\beta_j}{s_j} \\ h_{jk} &= \eta_{jk} + s_k e_j \end{aligned} \tag{31}$$

- Prices are likely to be endogenous in most applications
- Earlier we discussed how endogeneity can arise in the context of a competitive single-product demand-supply model, where due to the simultaneity, the price and the error term in the demand equation are correlated (see equation (15))
- The endogeneity concern arises in a variety of differentiated products pricing models as well
- Let the demand for the i^{th} product be given by $q_i = D_i(\mathbf{p}, \mathbf{z}_i; \xi_i)$, where ξ_i is the error term and consists of unobserved product characteristics, and $\mathbf{z}_i$ is the vector of exogenous demand shifters (say the observed product characteristics)
- If there are L firms, and the lth firm produces a subset $\mathfrak{L}_l$ of the products, then it maximizes its joint profit over these products as

$$\Pi_l = \sum_{r \in \mathfrak{L}_l} (p_r - c_r) q_r(\mathbf{p}, \mathbf{z}_r, \xi_r), \tag{32}$$

where c_r is the constant marginal cost of the r^{th} product

- Nash-Bertrand price competition, price p_i of any product i produced by firm l satisfies the first-order condition

$$q_i(\mathbf{p}, \mathbf{z}_i; \xi_i) + \sum_{r \in \mathfrak{L}_l} (p_r - c_r) \frac{\partial q_r(\mathbf{p}, \mathbf{z}_r; \xi_r)}{\partial p_i} = 0 \tag{33}$$

- The equilibrium price for product i would be a function of its marginal cost and a markup term, and in matrix form (for all equilibrium prices) is given by

$$\mathbf{p} = \mathbf{c} + \boldsymbol{\Omega}^{-1} \boldsymbol{q}(\boldsymbol{p}, \boldsymbol{z}; \boldsymbol{\xi}), \tag{34}$$

- where
  - $\boldsymbol{\Omega}$ is defined such that $\Omega_{ri} = -O_{ri} \frac{\partial q_r(\mathbf{p}, \mathbf{z}_r; \xi_r)}{\partial p_i}$
  - $\boldsymbol{O}$ is 1/0 joint ownership matrix with ones in the leading diagonals and in r, i position if these products are produced by the same firm and zeros everywhere else
- The markup term is a function of the same error terms, and hence generally, prices will be endogenous so that OLS/SUR estimation will lead to biased estimates of the demand parameters

- The usual starting place for demand-side instruments is to use cost shifters (terms that affect $\mathbf{c}$, such as cost of raw materials) that are uncorrelated with demand shocks
- These can work well for homogenous products, but in the case of differentiated products, we would need cost shifters that vary by individual brands, which are often very difficult to obtain
- Two types of instruments that have grown in popularity (use with caution as may or may not be valid in your application)
  - Berry (1994)/Berry et al. (1995) (BLP)
  - Hausman et al. (1994)

- Berry (1994) builds on Bresnahan's (1981) assumption that the location of products in a characteristics space is determined prior to the revelation of the consumer's valuation of the unobserved product characteristics
- BLP use this assumption to generate a set of instrumental variables: they use the observed product characteristics (excluding price and any other endogenous characteristics of the product), the sums of the values of the same characteristics of other products offered by that firm, and the sums of the values of the same characteristics of products offered by other firms

- Consider the case when there are two firms, X and Y and each is producing three products A,B,C and D,E,F respectively
- Suppose further that each of these products has two observable characters, S (say, package size, which is the number of pills in a box) and T (number of times a pill must be taken during a day for a standard diagnosis)
- Then for the price of A, which is produced by firm X, there are 6 potential instruments:
  - S_{AX} and T_{AX} – the values of S and T of product A
  - $S_{BX} + S_{CX}$ and $T_{BX} + T_{CX}$ – the sum of S and T over the firms two other products B and C
  - $S_{DY} + S_{EY} + S_{FY}$ and $T_{DY} + T_{EY} + T_{FY}$ – the sum of S and T over the competitor's products D,E, and F
- Similar instruments can be constructed for prices of other products

- Main advantage of this approach (if valid) is that it gives instruments that vary by brands
- Problems arise if the assumption that the unobserved characteristics are uncorrelated with observed characteristics is not valid
  - for instance, if the observed characteristics are changing over time, and the change in observed characteristics is for the same unobserved factors that determine the price
- Another potential issue arises if brand dummies are included in the estimation, since then it must be the case that there is variation in products offered in different markets, else there will be no variation between the instruments in these markets

AIDS Model

Endogeneity and Instruments (à la Hausman)

- A second set of instruments is due to Hausman et al. (1994) and has been used in several papers
- Hausman uses the panel nature of data and the assumption that prices in different areas (cities) are correlated via **common cost shocks**, to use prices from other areas as instruments for prices in a given city and there are **no common demand side shocks** across the two cities
- The identifying assumption is that after controlling for brand-specific intercepts and demographics, the city-specific valuations of a product are independent across cities but may be correlated within a city over time
- Given this assumption, the prices of the brand in other cities are valid instruments so that prices of brand j in two cities will be correlated due to the common marginal cost, but due to the independence assumption will be uncorrelated with the market-specific valuation of the product

- A second set of instruments is due to Hausman et al. (1994) and has been used in several papers – **common cost shocks** and **no common demand side shocks** across cities
- The reduced form price of a product i in two cities, $a = 1$ and $a = 2$ at time period t, will be given by

$$\begin{aligned} \ln p_{i1t} &= \pi_1 \ln c_{it} + \mathbf{x}_{i1t}\boldsymbol{\pi}_2 + v_{i1t} \\ \ln p_{i2t} &= \pi_1 \ln c_{it} + \mathbf{x}_{i2t}\boldsymbol{\pi}_2 + v_{i2t}, \end{aligned} \tag{35}$$

- where
  - c_{it} is the common cost component of the price in two different cities
  - $\mathbf{x}_{iat}$ are brand level demand shifters (demographics, time trends) as well city-specific brand differentials (intercepts by brands and cities) due to differences in transportation costs or local wages
  - In general, the error terms v_{iat} will be correlated with the error term in equation (27) (or φ_{iat} in equation (36) in a later example), and hence OLS/SUR will give inconsistent estimates
  - If however, v_{i1t} is uncorrelated with v_{i2t}, then city two's prices will be uncorrelated with the error term in equation (27) (or φ_{i1t} in equation (36)), and hence the instrument will be valid

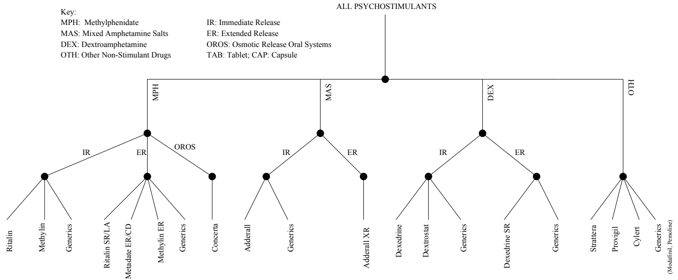
  - Further, since the prices in the two cities are driven by the same underlying common costs c_{it}, they will be correlated to each other and hence relevant
- Hausman instruments also rely on no correlation between v_{i1t} and v_{i2t} – this assumption may be invalid if the terms are related due to common demand side shocks across the two cities
  - Example: a national campaign will increase the unobserved valuation of product i in both cities, thus violating the independence assumption

- We will estimate such a demand system shortly (using SAS and/or STATA)
  - However, AIDS modeling is often done in the context of **multistage budgeting** along with **separability** of preferences (related but distinct concepts)
  - Separability refers to the case when a consumer's preferences for products of one group are independent of product-specific consumption of products from other groups
  - Multistage budgeting refers to when a consumer (or household) can allocate their total expenditure on different goods in sequential stages, represented as a utility tree, where in the first stage, the total current expenditure is allocated to broad groups of products (food, housing, entertainment) followed by the allocation of expenditures within each broad group (e.g., meats, vegetables, etc. within the food group)

- Typical applications involve a three (or four) stage system where
  - The top level corresponds to the overall demand for the product (e.g., beer, pharmaceutical drugs, RTE cereals, etc.)
  - The middle level consists of the demand for different market segments (e.g., in the demand for beer example, the middle segment consists of four groups of beer – premium beer, light beer, imported beer, and non-premium beer, while in the RTE cereal example, the middle segments are family, kids, and adult cereals)
  - The bottom-level segment involves a flexible brand demand system corresponding to the competition between the different brands within each segment
- For each of these stages a flexible parametric functional form is assumed
  - The choice of functional form is driven by the need for flexibility, but also requires that the conditions for multistage budgeting are met
  - Note – all stages are not necessarily modeled via AIDS and may include cobb-douglas and linear models at different levels
- Examples
  - Bokhari and Fournier (2013) – a 4-level system for ADHD drugs
  - Hausman et al. (1994) – a 3-level system for Beers

AIDS Model
Example w/ Multistage Budgeting

- We will use a four-level system example from Bokhari and Fournier (2013)
    - The top level consists of the aggregate demand for drugs used in the treatment of ADHD
    - The second level segments by the types of molecules used in different drugs (four different groups of molecules)
    - The third level further segments the market by the form of the drug, i.e., if it is 4hr, 8hr or a 12hr effect drug
    - The the bottom level, different brands, and generics are considered within each molecule-form segment of the market



- A typical application has the AIDS model at the lowest level
- The demand for product i in segment fm, which consists of I_{fm} number of products, in area a at period t is given by

Level 1 (Bottom):

$$s_{iat_{fm}} = \alpha_{i_{fm}} + \beta_{i_{fm}} ln(\frac{R_{fmat}}{P_{fmat}}) + \sum_{j=1}^{I_{fm}} \gamma_{ij_{fm}} \ln P_{jat_{fm}} + \mathbf{x}_{iat_{fm}} \boldsymbol{\lambda}_{i_{fm}} + \varphi_{iat_{fm}} \tag{36}$$

- where
  - $s_{iat_{fm}}$ is the revenue share of product i
  - $\ln P_{jat_{fm}}$ is the (log) price of product j (also in segment f-m)
  - R_{fmat} is the total expenditure on the segment
  - P_{fmat} is a price index for the segment
  - $\mathbf{x}_{iat_{fm}}$ are other exogenous variables which may be varying by product, market, or segment and may include terms like demographic variables, time trends, area fixed effects, or any observable product characteristics if they vary by markets
- Estimate a system of such equations for each segment, either jointly (all equations from all segments together) or on a segment-by-segment basis – e.g., estimate the system for MPH-IR, MPH-ER, MAS-IR, etc.

- The demand for product i in segment fm, which consists of I_{fm} number of products, in area a at period t is given by

Level 1 (Bottom):

$$s_{iat_{fm}} = \alpha_{i_{fm}} + \beta_{i_{fm}} ln(\frac{R_{fmat}}{P_{fmat}}) + \sum_{j=1}^{I_{fm}} \gamma_{ij_{fm}} \ln P_{jat_{fm}} + \mathbf{x}_{iat_{fm}} \boldsymbol{\lambda}_{i_{fm}} + \varphi_{iat_{fm}} \tag{36}$$

- To impose the restrictions, we require (for each segment)

$$\begin{aligned} &\sum_{i=1}^{I_{fm}} \alpha_{i_{fm}} = 1 \qquad \sum_{i=1}^{I_{fm}} \gamma_{ik_{fm}} = 0 \qquad \sum_{i=1}^{I_{fm}} \beta_{i_{fm}} = 0 \\ &\sum_{k} \gamma_{ik_{fm}} = 0 \qquad \gamma_{ik_{fm}} = \gamma_{ki_{fm}} \end{aligned} \tag{37}$$

where the last share equation per segment is not estimated as the shares must add up to one (recall that the revenue shares are shares relative to total spending in this segment and not total spending on all drugs)

- The demand for product i in segment fm, which consists of I_{fm} number of products, in area a at period t is given by

Level 1 (Bottom):

$$s_{iat_{fm}} = \alpha_{i_{fm}} + \beta_{i_{fm}} ln(\frac{R_{fmat}}{P_{fmat}}) + \sum_{j=1}^{I_{fm}} \gamma_{ij_{fm}} \ln P_{jat_{fm}} + \mathbf{x}_{iat_{fm}} \boldsymbol{\lambda}_{i_{fm}} + \varphi_{iat_{fm}} \tag{36}$$

- Price Index: Deaton and Muellbaur's exact price index P_{fmat} is given by

$$lnP_{fmat} = \alpha_{0_{fm}} + \sum_{i}^{I_{fm}} \alpha_{i_{fm}} \ln P_{iat_{fm}} + \frac{1}{2} \sum_{i}^{I_{fm}} \sum_{k}^{I_{fm}} \gamma_{ki_{fm}} \ln P_{kat_{fm}} \ln P_{iat_{fm}} \tag{38}$$

This index involves the same parameters that need to be estimated, and hence AIDS estimation requires non-linear estimation methods

- Alternatively, use **Stone price index**

$$\ln P_{fmat} = \sum_{i}^{I_{fm}} s_{iat_{fm}} \ln P_{iat_{fm}} \tag{39}$$

which makes the estimation linear – but now equation (36) involves shares on both the left-hand side and right hand side of the equation

- The demand for product i in segment fm, which consists of I_{fm} number of products, in area a at period t is given by

Level 1 (Bottom):

$$s_{iat_{fm}} = \alpha_{i_{fm}} + \beta_{i_{fm}} ln(\frac{R_{fmat}}{P_{fmat}}) + \sum_{j=1}^{I_{fm}} \gamma_{ij_{fm}} \ln P_{jat_{fm}} + \mathbf{x}_{iat_{fm}} \boldsymbol{\lambda}_{i_{fm}} + \varphi_{iat_{fm}} \tag{36}$$

  - Alternatively, use **Stone price index**

$$\ln P_{fmat} = \sum_{i}^{I_{fm}} s_{iat_{fm}} \ln P_{iat_{fm}} \tag{39}$$

which makes the estimation linear – but now equation (36) involves shares on both the left-hand side and right hand side of the equation

- In the price index, replace observed shares with average shares
- In (39), Hausman and colleagues replace $s_{iat_{fm}}$ with $\bar{s}_{ia_{fm}}$ – area specific average value of $s_{iat_{fm}}$, thus the value is different for each city but the same for all periods (data is from many periods and a few cities)
- In (39), B&F replace $s_{iat_{fm}}$ with $\bar{s}_{it_{fm}}$ – period specific average value of $s_{iat_{fm}}$, thus the value is different for each period but the same for all areas (data is from many counties and a few periods)

- At the next level up (the middle level, or level 2), demand captures the allocation between segments and can again be modeled using the AIDS specification, in which case the demand specified by the equation (36) is used with both expenditure shares and prices aggregated to a segment level
- Level 2 is aggregation up from level 1
- Prices are aggregated using either equations (38) or (39) (exact or Stone price index)
- If the latter (Stone price index) is used, then use $s_{iat_{fm}}$ for the purpose of creating a price index for the upper level rather than $\bar{s}_{at_{fm}}$ or $\bar{s}_{it_{fm}}$

- An alternative for level 2 is the log-log equation used by Hausman, Leonard, and Zona (1994) and Hausman (1996) and is given by

  Level 2 (Middle):

$$\ln(q_{[fm]at}) = A_{[fm]} + B_{[fm]}\ln(R_{at}) + \sum_{n=1}^{FM}\Gamma_{[fm]n}\ln P_{nat} + \mathbf{x}_{[fm]at}\boldsymbol{\lambda}_{[fm]} + \xi_{[fm]at} \tag{40}$$

- where (suppressing subscripts at for areas and periods)
  - $q_{[fm]}$ is the aggregate quantity of the $[fm]$ bottom level segment, i.e., the total quantity of RTE cereals for the family, kids or the adults segments in market at (city and quarter)
  - $P_{[fm]}$ is the price of each of these $[fm]$ segments, written as $\ln P_n$ in the equation above, where n is an indexing number for the lower level $[fm]$ segment
  - The segment level prices are the price indexes from the lower level equations and are computed using equations (38) or (39) as discussed earlier
  - The variable R_{at} is the total expenditure by market on all related products – e.g., it is the sum of total sales of RTE cereals over the the three segments, kids, family, and adults
  - And $\mathbf{x}_{[fm]at}$ are the exogenous variables that are segment-specific characteristics – if they are different for each market – or just demographic variables by markets

- Note that the lower level of the demand system is AIDS, which satisfies the generalized Gorman polar form,
- In order to be consistent with exact two-stage budgeting, the preferences of the second level should be additively separable (i.e., overall utility from ready-to-eat cereal or all ADHD drugs should be additively separable in the sub-utilities from the various subsegments)
- Neither the second-level AIDS nor the log-log system satisfies this requirement[§]
- For exact multistage budgeting to hold to the next level of aggregation (see appendix) these preferences should be of generalized Gorman polar form

[§] Deaton and Muellbauer also discuss approximate – instead of exact – two-stage budgeting, and show that if one uses the Rotterdam model, approximate two-stage budgeting implies that higher stages also have Rotterdam functional form

- B&F have two middle-level segments that differentiate drugs by forms within molecules (level 2) and by molecules among all ADHD drugs (level 3)

Level 2 (Middle):

$$u_{fat_m} = a_{f_m} + b_{f_m} \ln(\frac{R_{mat}}{P_{mat}}) + \sum_{h=1}^{F_m} g_{fh_m} \ln P_{hat_m} + \mathbf{x}_{fat_m} \boldsymbol{\lambda}_{f m} + \mu_{fat_m}$$

Level 3 (Middle):

$$\ln(q_{mat}) = A_m + B_m \ln(R_{at}) + \sum_{n=1}^{M} \Gamma_{mn} \ln P_{nat} + \mathbf{x}_{mat} \boldsymbol{\lambda}_m + \xi_{mat} \tag{41}$$

- where (suppressing subscripts at for exposition)
  - u_{f_m} is revenue share of form f within molecule m
  - P_{h_m} is the price of the form (i.e., the price indexes from level 1 segments) given by $- ln(P_{f_m}) = \sum_{j=1}^{I_{f_m}} s_{i_{f_m}} \ln(P_{j_{f_m}})$
  - The terms $\frac{R_m}{P_m}$ are the total expenditures from all forms within molecule m, and a price index for molecule m where the later is computed (using Stone index form) as

$$\ln(P_m) = \sum_{h=1}^{F_m} u_{f_m} \ln(P_{h_m}) \tag{42}$$

- For level 2, one needs to estimate as many equations as there are forms per molecule (F_m), and repeat the process for each molecule
  - For instance, if there are four molecules, and each admits up to three forms, then a total of four sets of system equations, with each set consisting of three equations need to be estimated
  - Again, depending on the data, the estimations can be joint for all segments, or segment by segment, and restrictions can be imposed within each segment much like the lower levels
- Level 3 is an aggregation from level 2
  - Thus, $\ln q_m$ is the aggregate quantity for segment m and is the the sum of quantities over all forms within this molecule
  - Similarly, $\ln P_n$ is the price of molecule n used earlier in level 2 and is given by (42)
  - Total number of equations to be estimated equals the number of upper level segments, e.g., the total number of molecules and the rest is the same as discussed earlier in the context of middle-level equation (40)

- The top level is the demand for the entire set of subsegments (RTE cereal, beer, ADHD drugs etc.) and is typically specified as

$$\textbf{Level 4 (Top):} \quad \ln q_{at} = A + B\ln(Y_{at}) + G\ln P_{at} + \mathbf{x}_{at}\boldsymbol{\lambda} + \zeta_{at} \tag{43}$$

- where
  - q_{at} is the total quantity
  - Y_{at} is the real income
  - $\mathbf{x}_{at}$ are the demand shifters
  - and P_{at} is the overall price index for these products, given by share weighted sum of (log) prices at the previous level and given by (again suppressing subscripts at),

$$\ln(P) = \sum_{m=1}^{M} v_m \ln(P_m) \tag{44}$$

  - and where v_m is the revenue share and P_m is the price index for molecule m computed earlier in (42). Note that this form does satisfy additive separability, which is required for exact two-stage budgeting.

- Note that every time we move up one level up, the price index from the lower level is the 'price' at the higher level – and the 'price' at the higher level is constructed as share weighted average (NOT average fixed share)
- Note that this form does satisfy additive separability, which is required for exact two-stage budgeting

AIDS Model
Example w/ Multistage Budgeting

- Multi-budgeting process allows estimation of the conditional demand functions (conditional on expenditures on the segment) at the lower levels and the cross-price elasticities are limited to within the segment
- From these conditional demand estimates, and estimates of the upper level equations, it is possible to derive the unconditional cross-price elasticities across the full range of products in different segments
- Conditional on segment expenditure R_{fm} (in market at), price elasticity of a product is

$$\begin{aligned} \frac{\partial \ln q_{i_{fm}}}{\partial \ln p_{k_{f'm'}}} = & \frac{1}{s_{i_{fm}}}\Big\{\Big(- \beta_{i_{fm}}\bar{s}_{k_{f'm'}} + \gamma_{ij_{f'm'}}\Big) \cdot 1[f'=f, m'=m]\Big\} \\ & - 1[i=k, f'=f, m'=m], \end{aligned} \tag{45}$$

- where
  - $1[\cdot]$ is the indicator function
  - elasticities conditional on R_{fm} are zero across products in different f-m segments
  - the subscript at has been suppressed in the equation above but is present on all quantities, shares, prices etc. and $\bar{s}_{k_{f'm'}}$ is either $\bar{s}_{kt_{f'm'}}$ or $\bar{s}_{ka_{f'm'}}$ depending on whichever one was used in the Stone price index in level 1 share equations
  - elasticities can be computed in each market or at the average value of shares

- Elasticity at level 2 with respect to the *price index* for the segment and conditional on segment revenue R_m in market *at* (where the market subscripts have been suppressed), has a similar formula as for the bottom level (since both are in AIDS form) and is given by

$$\frac{\partial \ln q_{f_m}}{\partial \ln p_{f'_{m'}}} = \frac{1}{u_{f_m}} \Big\{ \Big(-b_{f_m} \bar{u}_{f'_{m'}} + g_{fh_{m'}} \Big) \cdot 1[m' = m] \Big\} - 1[f' = f, m' = m], \tag{46}$$

- Conditional cross price elasticity of forms in different level 3 segments (i.e., for forms in different molecules) is zero
- Price elasticities at level 3 (for example, at the molecule level), are just the Γ_{mn} parameters in level 3 equation,
- Elasticity with respect to price for the aggregate product is the value of the parameter G in top-level equation

- Given all the parameters, unconditional elasticities can be computed as

$$
\begin{aligned}
\frac{\partial \ln q_{i_{fm}}}{\partial \ln p_{k_{f'_{m'}}}} &= \left(1 + \frac{\beta_{i_{fm}}}{s_{i_{fm}}}\right) \bar{s}_{k_{f'_{m'}}} \left[\frac{g_{ff'_{m'}}}{u_{fm}} + \bar{u}_{f'_{m'}}\right] \cdot 1[m = m'] \\
&+ \left(1 + \frac{\beta_{i_{fm}}}{s_{i_{fm}}}\right) \bar{s}_{k_{f'_{m'}}} \left[\frac{b_{fm}\bar{u}_{f'_{m'}}}{u_{fm}} + \bar{u}_{f'_{m'}}\right] \Gamma_{mm'} \\
&+ \frac{1}{s_{i_{fm}}} \left\{\gamma_{ik_{f'_{m'}}} - \beta_{i_{fm}} \bar{s}_{k_{f'_{m'}}}\right\} \cdot 1[f' = f, m' = m] \\
&- 1[i = k, f' = f, m' = m]
\end{aligned} \tag{47}
$$

- Please see the file `model-estimate-AIDS-ver01.sas` on how to estimate all the segments on the simulated data along with computing all the elasticities
- The above file produces, as output, two HTML files: one with all the regression coefficients (both SUR and 3SLS) and a second file with all the elasticity measures (SUR and 3SLS) for conditional and unconditional elasticities
- An example of an unconditional elasticities matrix is given on the next slide (an 11 by 11 from the full 17 by 17 matrix)

AIDS Model
Example w/ Multistage Budgeting

	Unconditional Marshallian Elasticities										
	Ritalin	**Methylin**	**Generics (MPH-IR)**	**Ritalin SRLA**	**Metadate ERCD**	**MethylinER**	**Generics (MPH-ER)**	**Concerta**	**Aderall**	**Generic (MAS-IR)**	**Aderall XR**
Ritalin	-1.263	0.139	0.403	-0.258	-0.260	-0.086	-0.273	0.948	0.007	0.006	0.010
Methylin	0.258	-0.347	-0.277	-0.131	-0.132	-0.044	-0.139	0.482	0.004	0.003	0.005
Generics (MPH-IR)	0.242	-0.135	-0.611	-0.180	-0.182	-0.060	-0.191	0.663	0.005	0.004	0.007
Ritalin SRLA	-0.333	-0.275	-0.718	-2.130	-0.077	0.478	0.061	2.404	0.007	0.005	0.009
Metadate ERCD	-0.250	-0.207	-0.539	0.046	-1.316	-0.065	0.083	1.805	0.005	0.004	0.007
MethylinER	-0.292	-0.241	-0.630	1.493	-0.259	-1.945	-0.752	2.108	0.006	0.005	0.008
Generics (MPH-ER)	-0.249	-0.205	-0.536	0.182	0.082	-0.215	-1.293	1.794	0.005	0.004	0.007
Concerta	0.091	0.075	0.196	0.185	0.187	0.062	0.196	-1.528	0.006	0.005	0.008
Aderall	0.002	0.001	0.004	0.001	0.001	0.000	0.001	0.013	-1.419	0.189	0.149
Generic (MAS-IR)	0.001	0.001	0.002	0.001	0.001	0.000	0.001	0.006	0.631	-1.176	0.066
Aderall XR	0.001	0.001	0.003	0.001	0.001	0.000	0.001	0.010	0.069	0.053	-0.990

- We will now use a simulated data set to estimate the AIDS model for the segment MPH-IR with four drugs
  - We will do so using SAS and STATA and where we will use the Stone price index with fixed weights based on first period (Laspeyres price index)
  - We will also estimate the elasticities at the sample mean, which are typically not easy to estimate unless we use software with proper matrix language — we will do this second part in SAS only
  - We will then use a canned routine `aidsills` (a package) within STATA, which makes estimation a lot easier and computes elasticity matrices for us at the sample mean ... but does not give us the flexibility to set our price index ... and hence the regression estimates as well as the elasticities will be different
- Please download the SAS/STATA datasets “simulateddrugs01.sas7bdat”, “`simulateddrugs01.dat`”, the read-me file “`readme-data-simulateddrugs01.pdf`” and all the *.sas and *.do files provided in the ‘training’ folder

- MPH-IR segment: four drugs, 780 counties large counties from the US, 4 years (2000-2003)
- Relevant variables (re-name as appropriate)
  - revenues: r4-r7
  - segment expenditure: y2
  - log prices: lpo4-lpo7
  - shares: s4-s7
  - average shares across all counties in base year: so4-so7
  - Stone price index for the segment: lpoi2
  - Hausman style price instruments: lpoz4-lpoz7; lpzi2
  - Other exogenous variables: t1,t2, lnkids,lnmds, lncaiddrugs, lnmcaidenrollees and many more available (see readme-data-simulateddrugs01.pdf)
- If estimating only this segment, re-name the variables so numbers go from 1-4

- Retain, rename and/or create new variables (SAS code)

```
/***************************** STEP 1 *********************************

data segment122 ;
   set training.simulateddrugs01
   (keep = fips year s4-s7 so4-so7 lpo1-lpo17 lpoz1-lpoz17
           lpoi2 lpoii2  lpzi2 lpzii2 y2 t1 t2
           lnpop lnkids lnmds lncaiddrugs lnmcaidenrollees pcpi
           msac stabr cenregc cenregn cendivc cendivn cntyst poptot
           msapmsa99c
           r4-r7 po4-po7 poz1-poz17
);
   if year < 2000 then delete;
   rename
   y2     = y
   lpoi2  = lpi      lpoii2 = lpii
   lpzi2  = lpzi     lpzii2 = lpzii

   so4    = s01      so5    = s02
   so6    = s03      so7    = s04
   s4    = s1        s5    = s2
   s6    = s3        s7    = s4
   r4    = r1        r5    = r2
   r6    = r3        r7    = r4
   po4    = p1       po5    = p2
   po6    = p3       po7    = p4
  ;

   lp1    = lpo4;   lp2    = lpo5;
   lp3    = lpo6;   lp4    = lpo7;
   lpz1   = lpoz4;  lpz2   = lpoz5;
   lpz3 =   lpoz6;  lpz4 =   lpoz7;

attrib _all_ label=' ';
run; quit;
```

Load data

rename variables

```
data segment122;
   set segment122;
   ly = log(y);
   lypi = ly - lpi;
   lypzi = ly - lpzi;
   t1ly = t1*ly;
   t2ly = t2*ly;
run; quit;
```

create new vars as needed

- Use `proc model` procedure to estimate (SAS code)

```
proc model data=segment122 plots = none print;
    /*Omit Last Share Equation  for adding up restrictions */
    s1 = a1 + b1*lypi +  c11*t1 + c12*t2 + c13*lnkids + c14*lnmds + c15*lncaiddrugs + c16*lnmcaidenrollees +
         g11*lp1 + g12*lp2 + g13*lp3 + g14*lp4 ;

    s2 = a2 + b2*lypi + c21*t1  + c22*t2 + c23*lnkids + c24*lnmds + c25*lncaiddrugs + c26*lnmcaidenrollees +
         g21*lp1 + g22*lp2 + g23*lp3 + g24*lp4 ;

    s3 = a3 + b3*lypi + c31*t1 + c32*t2 + c33*lnkids + c34*lnmds + c35*lncaiddrugs + c36*lnmcaidenrollees  +
         g31*lp1 + g32*lp2 + g33*lp3 + g34*lp4 ;
    fit s1 s2 s3 / sur 3sls hausman  converge = .00001;
    /* Homogenity restrictions  */
    restrict g11 + g12 + g13 + g14  = 0;
    restrict g21 + g22 + g23 + g24  = 0;
    restrict g31 + g32 + g33 + g34  = 0;
   /*Symmetry restrictions */
    restrict g12 = g21;
    restrict g13 = g31;
    restrict g23 = g32;
    /* Save estimated coefficients and covariances on prices and expenditures in work.mlf2results*/
    /* This is so thata we can later on print the coefficients or call them up in a different */
    /* procedure to estimate elasticities */
    estimate b1, g11, g12, g13, g14,
             b2, g21, g22, g23, g24,
             b3, g31, g32, g33, g34,
            'b4' -(b1+b2+b3),
            'g41' g14,
            'g42' g24,
            'g43' g34,
            'g44' -(g14 + g24 + g34),
     /outest=mlf2results outcov;
     /* Save estimated coefficients and covariances on exogenous variables in work.mlf2resultsB*/
     estimate c11, c12, c13, c14, c15, c16,
              c21, c22, c23, c24, c25, c26,
              c31, c32, c33, c34, c35, c36,
             'c41' - (c11 + c21 + c31),
             'c42' - (c12 + c22 + c32),
             'c43' - (c13 + c23 + c33),
             'c44' - (c14 + c24 + c34),
             'c45' - (c15 + c25 + c35),
             'c46' - (c16 + c26 + c36),
     /outest=mlf2resultsb outcov ;
   endogenous s1-s3 lp1-lp4 lypi ;
   instruments  lpz1 lpz2 lpz3 lpz4 lpzi
                 t1 t2 lnkids lnmds lncaiddrugs lnmcaidenrollees
                 ly t1ly t2ly;
   run; quit ;
```

- SAS's `proc model` will produce SUR and 3SLS estimates of the parameters but will not directly provide elasticities
- We can compute elasticities within the same `proc model` via the `estimate` command and it will also provide the standard errors but it is cumbersome to do so here
- Instead we can use various data steps to compute mean values of variables and then load the estimates in IML to compute elasticity matrices

- Use `proc iml` to compute elasticities (and display all results)

```
proc iml;
use coeffs;
read all into coeffs[colname = varnames];
read all var{b1 b2 b3 b4} into b;
b = b`;                          /*b is a column vector */
read all var{g11 g12 g13 g14} into g1;
read all var{g21 g22 g23 g24} into g2;
read all var{g31 g32 g33 g34} into g3;
read all var{g41 g42 g43 g44} into g4;
g = g1//g2//g3//g4;
close coeffs;

use &bnames;
read all var{description} into key;
close &bnames;

use &atshares;
read all var{s01 s02 s03 s04} into s0;
read all var{s1 s2 s3 s4} into s;
close &atshares;


brands  = {'Brand1', 'Brand2', 'Brand3', 'Brand4'};
brandsp = brands`;

one = i(4);


s0b = s0@b;             /*Note: b is a column vector and s is a row vector*/
sp = s`;
emij = (g - s0b)#(1/sp) - one;           /*Marshallian Elasticity */
eta = 1 + b/sp;                          /*Expenditure Elasticity */
etap = eta`;                             /*Expenditure Elasticity as a row*/
ehij = emij + s@eta;                    /*Compensated (Hicksian) Elasticity*/
haij = ehij#(1/s);                      /*Hicks-Allen measures of elasticity */
```

Load parameters

s is the vector of sample mean values of s1, s2, s3, s4

So in the mean share in base year in store index

Elasticity calculations in matrix form

- Note: additional code to clean print the parameters and elasticities omitted (see the SAS file) – results follow

- 3SLS estimates

Segment MPH-ER (M1F2) -- 3SLS Estimates				
	s1 (Ritalin SR/LA	s2 (Metadate ER/CD)	s3 (Methylin ER)	s4 (Generics)
ln(R/P)	0.058	-.030	0.005	-.033
	0.003	0.003	0.002	0.004
lnp1	-.274	0.036	0.160	0.079
	0.019	0.013	0.013	0.014
lnp2	0.036	-.072	-.012	0.048
	0.013	0.017	0.010	0.015
lnp3	0.160	-.012	-.088	-.060
	0.013	0.010	0.013	0.011
lnp4	0.079	0.048	-.060	-.068
	0.014	0.015	0.011	0.019
lnkids	-.012	0.012	0.004	-.004
	0.002	0.002	0.001	0.002
lnmds	0.009	-.007	-.005	0.003
	0.001	0.001	0.001	0.001
lncaiddrugs	0.007	-.001	-.004	-.001
	0.001	0.001	0.001	0.001
lncaidenrollees	-.004	0.001	0.001	0.001
	0.001	0.001	0.001	0.001
time	-.004	0.004	0.001	-.002
	0.000	0.000	0.000	0.000
timeSq	0.001	-.000	-.000	0.000
	0.000	0.000	0.000	0.000

- 3SLS elasticities

Elasticities computed at the following Shares (S_i)			
Brand1	Brand2	Brand3	Brand4
0.29381	0.29671	0.09805	0.31143

Expenditure Elasticities (ETA_i) - Per '3SLS'					
	Brand1	Brand2	Brand3	Brand4	SUM_i [ETA_i*S_i] = 1
ETA_i	1.19769	0.89930	1.05016	0.89364	1.00000

Conditional Marshallian Price Elasticities (Em_ij) - Per '3SLS'				
	Brand1	Brand2	Brand3	Brand4
Brand1	-1.99197	0.06255	0.52413	0.20761
Brand2	0.14961	-1.21174	-0.03033	0.19317
Brand3	1.61400	-0.13656	-1.90424	-0.62337
Brand4	0.28519	0.18571	-0.18090	-1.18365

Conditional Compensated Price Elasticities (Eh_ij) - Per '3SLS'				
	Brand1	Brand2	Brand3	Brand4
Brand1	-1.64008	0.41791	0.64156	0.58061
Brand2	0.41384	-0.94492	0.05784	0.47324
Brand3	1.92255	0.17503	-1.80127	-0.29631
Brand4	0.54776	0.45086	-0.09329	-0.90533

Conditional Hicks-Allen Price Elasticities (Eh_ij/S_j) - Per '3SLS'				
	Brand1	Brand2	Brand3	Brand4
Brand1	-5.58202	1.40850	6.54345	1.86431
Brand2	1.40850	-3.18468	0.58992	1.51955
Brand3	6.54345	0.58992	-18.37182	-0.95145
Brand4	1.86431	1.51955	-0.95145	-2.90698

- Retain, rename and/or create new variables (STATA code)

```
use simulateddrugs01, clear
```

Load

```
keep fips year s4-s7 so4-so7 lpo1-lpo17 lpoz1-lpoz17    ///
           lpoi2 lpoii2  lpzi2 lpzii2 y2 t1 t2          ///
           lnpop lnkids lnmds lncaiddrugs lnmcaidenrollees pcpi  ///
           msac stabr cenregc cenregn cendivc cendivn cntyst poptot  ///
           msapmsa99c  ///
           r4-r7 po4-po7 poz1-poz17
```

Retain

```
drop if year < 2000

rename  y2      y
rename  lpoi2   lpi
rename  lpoii2  lpii
rename  lpzi2   lpzi
rename  lpzii2  lpzii
```

re-name

- Only part of the code shown (the rest is like SAS code in terms of renaming and creating new variables)

```
gen lpz1 = lpoz4
gen  lpz2 = lpoz5
gen  lpz3 = lpoz6
gen  lpz4 = lpoz7

gen ly = ln(y)
gen lypi = ly - lpi
gen lypzi = ly - lpzi
gen t1ly = t1*ly
gen t2ly = t2*ly
```

- Estimate via `reg3` command (STATA code)

```
global eq1  "(s1 lypi t1 t2 lnkids lnmds lncaiddrugs lnmcaidenrollees lp1 lp2 lp3 lp4)"
global eq2  "(s2 lypi t1 t2 lnkids lnmds lncaiddrugs lnmcaidenrollees lp1 lp2 lp3 lp4)"
global eq3  "(s3 lypi t1 t2 lnkids lnmds lncaiddrugs lnmcaidenrollees lp1 lp2 lp3 lp4)"
global eq4  "(s4 lypi t1 t2 lnkids lnmds lncaiddrugs lnmcaidenrollees lp1 lp2 lp3 lp4)"
global enlist "(lp1 lp2 lp3 lp4 lypi)"
global exlist "(lpz1 lpz2 lpz3 lpz4 lpzi ly t1ly t2ly)"


/*symmetry*/
constraint 1 [s1]lp2 = [s2]lp1
constraint 2 [s1]lp3 = [s3]lp1
constraint 3 [s2]lp3 = [s3]lp2

/* Homogenity restrictions  */
constraint 4 [s1]lp1 + [s1]lp2 + [s1]lp3 + [s1]lp4 = 0
constraint 5 [s2]lp1 + [s2]lp2 + [s2]lp3 + [s2]lp4 = 0
constraint 6 [s3]lp1 + [s3]lp2 + [s3]lp3 + [s3]lp4 = 0


reg3 $eq1 $eq2 $eq3, endog($enlist) exog($exlist) constr(1 2 3 4 5 6)  3sls
```

define equs & var list.

Impose constraints

3SLS or SUR via reg3 command.

- This will give a nice compact output of all the regression coefficients (and these should be the same as what we obtained in SAS)
- However, it will not give elasticity estimate ... for that, you can use either the `nlcom` command to program in each elasticity, or use STATA's matrix language to compute all of them together

- 3SLS estimates (Same as SAS estimates)

	Coef.	Std. Err.	z	P>z	[95% Conf.	Interval]
s1						
lypi	.0580837	.0030064	19.32	0.000	.0521913	.0639761
t1	-.0035769	.0000443	-80.79	0.000	-.0036637	-.0034901
t2	.0005384	6.38e-06	84.42	0.000	.0005259	.0005509
lnkids	-.012246	.0015231	-8.04	0.000	-.0152311	-.0092608
lnmds	.0085688	.0009405	9.11	0.000	.0067256	.0104121
lncaiddrugs	.0067264	.0010055	6.69	0.000	.0047557	.0086971
lnmcaidenrollees	-.0038225	.0012463	-3.07	0.002	-.0062652	-.0013798
lp1	-.2743889	.0192285	-14.27	0.000	-.3120761	-.2367018
lp2	.0356113	.0128959	2.76	0.006	.0103359	.0608868
lp3	.1596903	.0126453	12.63	0.000	.1349061	.1844745
lp4	.0790873	.0140347	5.64	0.000	.0515798	.1065948
_cons	.3801477	.0267833	14.19	0.000	.3276535	.432642
s2						
lypi	-.0298788	.0032582	-9.17	0.000	-.0362648	-.0234929
t1	.0038591	.0000481	80.16	0.000	.0037648	.0039535
t2	-.0004539	6.93e-06	-65.45	0.000	-.0004675	-.0004403
lnkids	.0124577	.0016562	7.52	0.000	.0092116	.0157038
lnmds	-.0069539	.0010213	-6.81	0.000	-.0089556	-.0049523
lncaiddrugs	-.0011513	.0010929	-1.05	0.292	-.0032934	.0009908
lnmcaidenrollees	.0009349	.0013552	0.69	0.490	-.0017213	.0035911
lp1	.0356113	.0128959	2.76	0.006	.0103359	.0608868
lp2	-.0716907	.0170384	-4.21	0.000	-.1050853	-.038296
lp3	-.0119295	.0101917	-1.17	0.242	-.0319048	.0080458
lp4	.0480089	.0148076	3.24	0.001	.0189864	.0770313
_cons	-.2742804	.028849	-9.51	0.000	-.3308235	-.2177374
s3						
lypi	.004918	.0023827	2.06	0.039	.0002479	.0095881
t1	.001447	.000035	41.40	0.000	.0013785	.0015155
t2	-.0001945	5.03e-06	-38.64	0.000	-.0002044	-.0001847
lnkids	.0035347	.0012011	2.94	0.003	.0011806	.0058889
lnmds	-.004987	.0007445	-6.70	0.000	-.0064462	-.0035279
lncaiddrugs	-.0044576	.0007933	-5.62	0.000	-.0060125	-.0029027
lnmcaidenrollees	.0014438	.0009831	1.47	0.142	-.0004831	.0033707
lp1	.1596903	.0126453	12.63	0.000	.1349061	.1844745
lp2	-.0119295	.0101917	-1.17	0.242	-.0319048	.0080458
lp3	-.0881739	.0126504	-6.97	0.000	-.1129683	-.0633795
lp4	-.0595869	.0107937	-5.52	0.000	-.0807422	-.0384316
_cons	-.1548779	.0211749	-7.31	0.000	-.1963799	-.1133759

AIDS Model
Estimation example (with code)

- There are user-written packages in STATA that estimate AIDS models
  - A big advantage is that they also provide elasticity estimates along with standard errors
  - A potential disadvantage is that they do not allow as much flexibility as you may want in terms of how certain issues should be dealt
  - If you are going to use such a package, read the documentation carefully to be sure that any restrictions they impose are ok in your specific case
  - The biggest limitation of such packages is they do not allow for multilevel budgeting/nesting and so you need to do some programming yourself
- The package `aidsills` (where ills stands for iterated least squares) provides lots of good options for estimating the AIDS model
  - Importantly, it allows for the endogeneity of prices and the expenditure function (for endogeneity, it uses the control function approach)
  - It provides elasticity matrices at the sample mean
  - However, it does not use the Stone price index, and hence the estimates can be somewhat different

- `aidsills` (STATA's user-written package)

```
/* (ssc?) install aidsills if you don't have it */

aidsills s1 s2 s3 s4 , prices (p1 p2 p3 p4) exp(y) ///
        intercept(t1 t2 lnkids lnmds lncaiddrugs lnmcaidenrollees) ///
        ivprices(lpz1 lpz2 lpz3 lpz4) ///
        ivexpenditure(lpzi) ///
        homogeneity symmetry

/*post estimation elasticitiy command */
aidsills_elas
```

→ Variables should be provided w/out logging them!!

- `aidsills` estimates

Price coefficients in eqn 1

	Coefficient	Std. err.	z	P>|z|	[95% conf.	interval]
s1						
gamma_lnp1	-.3640854	.0635678	-5.73	0.000	-.4886761	-.2394948
gamma_lnp2	.043454	.0369639	1.18	0.240	-.028994	.115902
gamma_lnp3	.1715655	.0308277	5.57	0.000	.1111443	.2319868
gamma_lnp4	.1490659	.0501505	2.97	0.003	.0507727	.2473591
beta_lnx	.1080618	.0329923	3.28	0.001	.0433981	.1727255
rho_vp1	.9069537	.0275587	32.91	0.000	.8529397	.9609677
rho_vp2	-.243946	.0209375	-11.65	0.000	-.2849828	-.2029092
rho_vp3	-.2320032	.0189291	-12.26	0.000	-.2691035	-.1949029
rho_vp4	-.2708364	.0237526	-11.40	0.000	-.3173905	-.2242822
rho_vy	-.0485012	.0330555	-1.47	0.142	-.1132887	.0162864
alpha_t1	-.0033301	.0001678	-19.84	0.000	-.0036591	-.0030012
alpha_t2	.0004979	.0000273	18.23	0.000	.0004444	.0005514
alpha_lnkids	-.0220879	.0066957	-3.30	0.001	-.0352111	-.0089646
alpha_lnmds	.0043943	.002946	1.49	0.136	-.0013797	.0101683
alpha_lncaiddrugs	.0072543	.0011148	6.51	0.000	.0050694	.0094392
alpha_lnmcaidenrollees	-.0024294	.0016223	-1.50	0.134	-.0056091	.0007503
alpha_cons	.0273761	.232217	0.12	0.906	-.4277609	.4825131
s2						
gamma_lnp1	.043454	.0386761	1.12	0.261	-.0323498	.1192579
gamma_lnp2	-.0654016	.018754	-3.49	0.000	-.1021588	-.0286445
gamma_lnp3	-.0089523	.0189301	-0.47	0.636	-.0460547	.0281501
gamma_lnp4	.0308999	.0264069	1.17	0.242	-.0208567	.0826566
beta_lnx	-.0135093	.0348652	-0.39	0.698	-.0818438	.0548252
rho_vp1	-.2914822	.0305356	-9.55	0.000	-.3513308	-.2316336
rho_vp2	.5722547	.022957	24.93	0.000	.5272598	.6172496
rho_vp3	-.0683751	.0202533	-3.38	0.001	-.1080709	-.0286793
rho_vp4	-.2046372	.0259258	-7.89	0.000	-.2554508	-.1538236
rho_vy	-.0156898	.0349308	-0.45	0.653	-.0841529	.0527733
alpha_t1	.0039348	.0001736	22.67	0.000	.0035946	.0042749
alpha_t2	-.0004663	.0000282	-16.51	0.000	-.0005217	-.000411
alpha_lnkids	.0093663	.0069308	1.35	0.177	-.0042177	.0229503
alpha_lnmds	-.0082764	.0030442	-2.72	0.007	-.0142429	-.0023099
alpha_lncaiddrugs	-.0010141	.0011484	-0.88	0.377	-.0032649	.0012368

control fn coefficients

Price coeffs in eqn 2.

Compare these w/ nonlinear 3SLS estimates. e.g. -.36 vs -.27

-.065 vs -.072

- `aidsills` elasticities

UNCOMPENSATED CROSS-PRICE ELASTICITIES

	p1 b/se	p2 b/se	p3 b/se	p4 b/se
s1	-2.013***	-0.003	0.461***	0.187***
	(0.077)	(0.066)	(0.061)	(0.064)
s2	0.118	-1.202***	-0.015	0.144**
	(0.079)	(0.069)	(0.064)	(0.068)
s3	1.562***	0.028	-1.755***	-0.546***
	(0.171)	(0.149)	(0.139)	(0.147)
s4	0.345***	0.186***	-0.180***	-1.139***
	(0.081)	(0.071)	(0.066)	(0.069)

* p<0.1, ** p<0.05, *** p<0.01

COMPENSATED CROSS-PRICE ELASTICITIES

	p1 b/se	p2 b/se	p3 b/se	p4 b/se
s1	-1.613***	0.403***	0.596***	0.614***
	(0.070)	(0.056)	(0.059)	(0.058)
s2	0.397***	-0.919***	0.079	0.442***
	(0.073)	(0.058)	(0.061)	(0.061)
s3	1.770***	0.239*	-1.685***	-0.324**
	(0.159)	(0.126)	(0.133)	(0.133)
s4	0.575***	0.420***	-0.102	-0.893***
	(0.075)	(0.060)	(0.063)	(0.063)

* p<0.1, ** p<0.05, *** p<0.01

- The appendix provides details about estimating all of the other segments on the same simulated data and for all 4-levels
- Importantly, it shows how to estimate cross-elasticities between products that may be in different nests (referred to as unconditional elasticities)
- You should go over them your self and we will return to them only if there is additional time at the end
  - (there is also an accompanying SAS code available for estimating the full 4-level system on the simulated data – see file `model-estimate-AIDS-ver01.sas`)

Discrete Choice Models

- Consumer chooses a single product from a finite set of goods
- Each product is defined as a bundle of attributes (including price, which is a special attribute), and consumers have preferences over these attributes
- Consumers can have different relative preferences, which gives rise to the random coefficients models, and they choose the product that maximizes their utility subject to the usual constraints – when we impose constraints that preferences/marginal utilities are the same, we obtain the logit model
- This leads to different choices by different consumers
- Aggregate demand is then derived as the sum over individuals and depends on the entire distribution of consumer preferences

- Indirect utility for individual n for product j in market t is given by

$$u_{njt} = U(\mathbf{x}_{jt}, \xi_{jt}, y_{nt} - p_{jt}, \boldsymbol{d}_{nt}, \boldsymbol{\nu}_{nt}, \epsilon_{njt}; \boldsymbol{\theta}_n), \qquad \text{for } j = 0, 1, 2, \ldots, J \tag{48}$$

  - 'outside good' is numbered 0 (when the consumer does not purchase any of the observed products)
  - price of the outside good is often considered to be exogenous
  - vector $\mathbf{x}_{jt}$ and random term ξ_{jt} are the observed and unobserved (to the econometrician, but not to the consumer) product characteristics and do not vary over consumers
  - product characteristics, multiplied by the parameters $\boldsymbol{\theta_n}$ determine the level of utility for consumer n

- Indirect utility for individual n for product j in market t is given by

$$u_{njt} = U(\mathbf{x}_{jt}, \xi_{jt}, y_{nt} - p_{jt}, \boldsymbol{d}_{nt}, \boldsymbol{\nu}_{nt}, \epsilon_{njt}; \boldsymbol{\theta}_n), \qquad \text{for } j = 0, 1, 2, \ldots, J \tag{48}$$

  - vectors $\boldsymbol{d}_{nt}$ and $\boldsymbol{\nu}_{nt}$ are vectors of observed and unobserved sources of differences in consumer tastes
  - they do not enter the utility function directly, but rather enter into the model by changing the value of the parameters of interest for each consumer
  - $\boldsymbol{d}_{nt}$ may be a vector of observed demographics (income, family size, etc.), that affect the parameters (marginal valuations) of product characteristics by individual and change the value of $\boldsymbol{\theta}$ for each attribute of the product by individual n
  - for each product attribute (including price) there is an additional randomness to the marginal valuation by individuals and is captured by $\boldsymbol{\nu}_{nt}$
  - accounts for other unobserved person-specific characteristics that affect their marginal valuation for an observed product characteristic – e.g., the number of dogs a family owns affects their marginal valuation of the size of a car

- Indirect utility for individual n for product j in market t is given by

$$u_{njt} = U(\mathbf{x}_{jt}, \xi_{jt}, y_{nt} - p_{jt}, \boldsymbol{d}_{nt}, \boldsymbol{\nu}_{nt}, \epsilon_{njt}; \boldsymbol{\theta}_n), \qquad \text{for } j = 0, 1, 2, \dots, J \tag{48}$$

  - if $\mathbf{x}_{jt}$ is a $k-1$ vector of observed characteristics, then $\boldsymbol{\nu}_{nt}$ is a vector of length k
  - the coefficients $\boldsymbol{\theta}_n$ depend on $\boldsymbol{d}_{nt}$ and $\boldsymbol{\nu}_{nt}$
  - ϵ_{njt} is a mean-zero stochastic term that enters directly into the utility of product j for consumer n
  - for each consumer, $\boldsymbol{\epsilon}_{nt} = (\epsilon_{n0t}, \epsilon_{n1t}, \dots, \epsilon_{nJt})$ is a vector of error terms with the length of the vector equal to the number of products
  - y_{nt} is the consumer's income but is often subsumed into either $\boldsymbol{\nu}$ or in $\boldsymbol{d}$, so that utility is modeled explicitly depending on prices, i.e., $u_{njt} = U(\mathbf{x}_{jt}, \xi_{jt}, p_{jt}, \boldsymbol{d}_{nt}, \boldsymbol{\nu}_{nt}, \epsilon_{njt}; \boldsymbol{\theta}_n)$
  - utility of the outside good is denoted as $u_{n0t} = U(\mathbf{x}_{0t}, \xi_{0t}, \boldsymbol{d}_{nt}, \boldsymbol{\nu}_{nt}, \epsilon_{n0t}; \boldsymbol{\theta})$ and is normalized to zero

- Consumer n will choose product j when $u_{njt} \geq u_{nlt}$ for all $l = 0, 1, \ldots, J$ and $l \neq j$
- Differences in consumer choices arise only due to differences in the marginal valuations $\boldsymbol{\theta}_n$ (which are themselves functions of $\boldsymbol{d}_{nt}$ and $\boldsymbol{\nu}_{nt}$), and the idiosyncratic terms ϵ_{njt}, a consumer can be described as a tuple $(\boldsymbol{d}, \boldsymbol{\nu}, \boldsymbol{\epsilon})$
- The set $\mathbb{A}_{jt}$ defines characteristics of the individuals that choose brand j in market t

$$\mathbb{A}_{jt}(\boldsymbol{\theta}) = \{(\boldsymbol{d}_{nt}, \boldsymbol{\nu}_{nt}, \epsilon_{n0t}, \epsilon_{n1t}, \ldots, \epsilon_{nJt}) \mid u_{njt} > u_{nlt} \qquad \forall\, l = 0, 1, 2 \ldots J,\ l \neq j\} \tag{49}$$

- Market share of product j is just the probability weighted sum of individuals in the set $\mathbb{A}_{jt}$
- Let $F(\boldsymbol{d}, \boldsymbol{\nu}, \boldsymbol{\epsilon})$ be the population joint distribution function, then the market share of product j is the integral of this distribution over the mass of individuals in the region $\mathbb{A}_{jt}$,

$$s_{jt}(\mathbf{x}, \mathbf{p}; \boldsymbol{\theta}) = \int_{\mathbb{A}_{jt}} dF(\boldsymbol{d}, \boldsymbol{\nu}, \boldsymbol{\epsilon}). \tag{50}$$

If the size of the market is M (total number of consumers) then the aggregate demand for the jth product is $Ms_{jt}(\mathbf{x}, \mathbf{p}; \boldsymbol{\theta})$

Logit Demand Model

Utility function and market shares

- Let the indirect utility for consumer n for product j in market t be given by

$$\begin{aligned} & u_{njt} = \alpha_n(y_n - p_{jt}) + \mathbf{x}_{jt}\boldsymbol{\beta}_n + \xi_{jt} + \epsilon_{njt}, \text{ where} \\ & n = 1, \ldots, N, \qquad j = 0, 1, \ldots, J, \qquad t = 1, 2, \ldots, T, \text{ and} \\ & \boldsymbol{\beta}_n = \boldsymbol{\beta}, \qquad \alpha_n = \alpha, \qquad \text{for all } N \end{aligned} \tag{51}$$

- where
  - $\mathbf{x}_{jt}$ is a $k-1$ dimensional vector of observable characteristics (which may vary by market)
  - ξ_{jt} is a *scalar* that summarizes the unobservable (to the econometrician) product characteristics
  - neither of these terms varies over consumers
  - also, no variation in tastes across consumers, and the terms $\boldsymbol{d}_{nt}$ and $\boldsymbol{\nu}_{nt}$ do not enter this model (in BLP/Random coefficient models, $\boldsymbol{\beta}_n$ and α_n vary across individuals and in some applications we make them functions of $\boldsymbol{d}_n$ and $\boldsymbol{\nu}_n$ mentioned earlier – as in Nevo (2001, 2000a))
  - outside option (product 0) is normalized by assuming that the price and other characteristics are zero for this option so that

$$u_{n0t} = \alpha y_n + \epsilon_{n0t} \tag{52}$$

- Utility function in (51) can be written more compactly as just

$$u_{njt} = \alpha y_n + \delta_{jt} + \epsilon_{njt}, \tag{53}$$

where $\delta_{jt} \equiv \alpha(-p_{jt}) + \mathbf{x}_{jt}\boldsymbol{\beta} + \xi_{jt}$ is the **mean utility** for product j in market t

- Since income is common to all options, and consumers only differ in the terms ϵ, the set of individuals choosing product j is given by

$$\mathbb{A}_{jt}(\alpha, \beta) = \{(\epsilon_{n0t}, \epsilon_{n1t}, \dots \epsilon_{nJt}) | u_{njt} > u_{nlt} \qquad \forall\, l = 0, 1, 2 \dots J,\ l \neq j\} \tag{54}$$

- Assume ϵ_{njt} are independently and identically distributed (iid) and follow a Type-1 extreme value distribution, given by

$$f(\epsilon) = \exp(-\epsilon)\exp(-\exp(-\epsilon)) \text{ and } \qquad F(\epsilon) = \exp(-\exp(-\epsilon)), \tag{55}$$

where $f(\epsilon)$ and $F(\epsilon)$ are the PDF and CDF of the random variable ϵ

- If ϵ_{njt} are iid Type-1 extreme value distribution, then market share of product j (and the probability that individual n chooses product j) is

$$s_{jt}(\boldsymbol{\delta}_t) = \int_{\mathbb{A}_{jt}} dF(\boldsymbol{\epsilon}) = \frac{\exp(\delta_{jt})}{\sum_{j=0}^{J} \exp(\delta_{jt})}. \tag{56}$$

- Since $\delta_{0t} = 0$ (so that $(\exp(\delta_{0t}) = \exp(0) = 1)$, the share equation becomes

$$\begin{aligned} s_{jt} &= \frac{\exp(\delta_{jt})}{1 + \sum_{j=1}^{J} \exp(\delta_{jt})} \\ s_{0t} &= \frac{1}{1 + \sum_{j=1}^{J} \exp(\delta_{jt})} = 1 - \sum_{j=1}^{J} s_{jt}. \end{aligned} \tag{57}$$

- Since $s_{jt}/s_{0t} = \exp(\delta_{jt})$, and hence

$$\ln(s_{jt}) - \ln(s_{0t}) = \delta_{jt} = \alpha(-p_{jt}) + \mathbf{x}_{jt}\boldsymbol{\beta} + \xi_{jt} \tag{58}$$

can be estimated using linear regression methods

- Since $s_{jt}/s_{0t} = \exp(\delta_{jt})$, and hence

$$\ln(s_{jt}) - \ln(s_{0t}) = \delta_{jt} = \alpha(-p_{jt}) + \mathbf{x}_{jt}\boldsymbol{\beta} + \xi_{jt} \quad (58)$$

  can be estimated using linear regression methods
- Instead of estimating J^2 number of parameters, we only have to estimate a handful
- Own and cross-price elasticities depend on only one parameter α
- The closed (logit) form for the shares is due to both, the extreme value distribution, and the iid assumption
- The independence part of iid, causes serious limitations on the substitution patterns

- The logit model suffers from the property known as the Independence of Irrelevant Alternatives (IIA)
- The (logit) probability that individual n chooses product j is given by (see (56))

$$\Pr(j) = \frac{\exp(\delta_j)}{\sum_{j=0}^{J} \exp(\delta_j)} \tag{56}$$

The relative probabilities of options j and k are thus

$$\frac{\Pr(j)}{\Pr(k)} = \frac{\exp(\delta_j)}{\exp(\delta_k)} = \exp(\delta_j - \delta_k) \tag{59}$$

- Ratio does not depend on characteristics of any other alternative other than those of j and k
- Thus the relative odds of choosing j over k are the same no matter what other alternatives are available or what are the attributes of other alternatives (the values of $\delta' s$)

- IIA leads to substitution patterns that may be unrealistic
- Blue Bus/Red Bus Example
  - A traveler can commute to work either by car (c) or by blue bus (bb)
  - Suppose further that it turns out (for simplicity) that $Pr(bb) = Pr(c) = .5$
  - Say a new type of bus is introduced that is identical in all other respects to the existing blue bus (fare, route, smell, time it takes to get to work, etc.,) except that it is red (rb)
  - We expect the new probabilities of the travel model would be $Pr(bb) = Pr(rb) = .25$ and $Pr(c) = .5$
  - logit model would predict that the substitution from the two old modes of travel (blue bus or car) to the new mode of travel (red bus) are such that they would depend on the ratio of old probabilities
  - Since the old probabilities were equal, new probabilities for each of the new modes would be $Pr(bb) = Pr(rb) = Pr(c) = 1/3$

- IIA has implications for own and cross elasticities estimated via logit specification for the aggregate demand
- Price elasticities from the model are

$$\eta_{jkt} = \frac{\partial s_{jt}}{\partial p_{kt}} \frac{p_{kt}}{s_{jt}} = \begin{cases} -\alpha p_{jt}(1 - s_{jt}) & \text{if } j = k, \\ \alpha p_{kt} s_{kt} & \text{otherwise} \end{cases} \tag{60}$$

- Cross elasticity
  - cross price elasticity between product j and k depends *only* on the prices and shares of product k
  - let Coca Cola = product j; Pepsi Cola = product k; and Orange Cola = product l
  - if the price of Pepsi Cola increases by 1%, then ceteris paribus, the market shares of Coca-Cola and Orange Cola will increase by the same proportion even though Coca Colas and Pepsi Cola are more like each other (blue bus/red bus) compared to Orange Cola (car)

- IIA has implications for own and cross elasticities estimated via logit specification for the aggregate demand
- Price elasticities from the model are

$$\eta_{jkt} = \frac{\partial s_{jt}}{\partial p_{kt}} \frac{p_{kt}}{s_{jt}} = \begin{cases} -\alpha p_{jt}(1 - s_{jt}) & \text{if } j = k, \\ \alpha p_{kt} s_{kt} & \text{otherwise} \end{cases} \tag{60}$$

- Own elasticity
    - often market shares (when there are many differentiated products) are small
    - own elasticity will be roughly proportional to the price of the product ($\eta_{jjt} \approx -\alpha p_{jt}$ because $(1 - s_{jt}) \approx 1$)
    - if price increases, sensitivity to prices also increases – but people who buy more expensive products may in fact be less price sensitive compared to those who buy less expensive products
    - if as the price increases, so does elasticity, it implies that the markups for cheaper-priced products will be larger than those with higher priced products (price-cost margin inversely related to own elasticities) – markups are higher for cheaper-priced generics compared to the blockbuster patented?

- If we compute a logit model on the same simulated data, the elasticity matrix (from 2SLS) at the sample average value of prices and shares for the first 11 drugs look as follows

Logit price elasticities -- IV estimates [Entry (j,k) --> DlnQj/DlnPk]											
	Ritalin	Methylin	Generics (MPH-IR)	Ritalin SRLA	Metadate ERCD	MethylinER	Generics (MPH-ER)	Concerta	Aderall	Generic (MAS-IR)	Aderall XR
Ritalin	-2.579	0.014	0.038	0.012	0.012	0.004	0.013	0.133	0.082	0.016	0.052
Methylin	0.018	-1.831	0.038	0.012	0.012	0.004	0.013	0.133	0.082	0.016	0.052
Generics (MPH-IR)	0.018	0.014	-1.836	0.012	0.012	0.004	0.013	0.133	0.082	0.016	0.052
Ritalin SRLA	0.018	0.014	0.038	-3.364	0.012	0.004	0.013	0.133	0.082	0.016	0.052
Metadate ERCD	0.018	0.014	0.038	0.012	-2.927	0.004	0.013	0.133	0.082	0.016	0.052
MethylinER	0.018	0.014	0.038	0.012	0.012	-2.896	0.013	0.133	0.082	0.016	0.052
Generics (MPH-ER)	0.018	0.014	0.038	0.012	0.012	0.004	-2.650	0.133	0.082	0.016	0.052
Concerta	0.018	0.014	0.038	0.012	0.012	0.004	0.013	-5.007	0.082	0.016	0.052
Aderall	0.018	0.014	0.038	0.012	0.012	0.004	0.013	0.133	-1.253	0.016	0.052
Generic (MAS-IR)	0.018	0.014	0.038	0.012	0.012	0.004	0.013	0.133	0.082	-1.290	0.052
Aderall XR	0.018	0.014	0.038	0.012	0.012	0.004	0.013	0.133	0.082	0.016	-2.181

- Notice something odd in the columns?
  - look again at the formula for cross-price elasticity between drugs j and k – $\eta_{jk} = \alpha p_{kt} s_{kt}$ ... the formula does not depend on j

Logit Demand Model
Estimation Details

- Despite the earlier noted shortcomings, logit may be ok in some situations – even if not, it's easy to estimate and can be a starting point for more elaborate models
- If we have aggregate sales data (quantities and prices), along with product characteristics, equation (58) can be estimated by defining the dependent variable y_{jt} as
  $y_{jt} = \ln(s_{jt}) - \ln(s_{0t})$
- To start, we need to estimate the share of the outside good – done by first defining the (potential) size of the market
- Examples
    - Bresnahan et al (1997) define it as the total number of office-based employees
    - BLP define it as the total number of households
    - Nevo (2001) defines the potential size of the market as one bowl of cereal per day per person
    - In the example of ADHD drugs considered earlier, one could define it as a 12-hr day-long coverage of a standard dose of ADHD drug – 3×30mg strength of Ritalin IR (a 30mg pill covers about 4hrs of a day) which can be multiplied by a base line candidate population, say 10% of all school-aged children (current ADHD prevalence rates of whom only 69% are given any ADHD drugs), and a smaller proportion of the older population

Logit Demand Model
Estimation Details

- Thus, first define the potential size of the market M_t
- Next, based on the observed values of $q_{1t}, \ldots, q_{Jt}$, define the shares of the 'inside' goods $s_{1t}, \ldots s_{Jt}$ relative to the market size as

$$s_{jt} = q_{jt}/M_t \qquad j = 1, \ldots, J \text{ for all } t = 1, \ldots, T. \tag{61}$$

- Then, the share of the outside good per market is just

$$s_{0t} = 1 - \sum_{j=1}^{J} s_{jt} \qquad \forall t \tag{62}$$

- With these definitions in place, can estimate the equation (58) (reproduced below)

$$\ln(s_{jt}) - \ln(s_{0t}) = \delta_{jt} \equiv \alpha(-p_{jt}) + \mathbf{x}_{jt}\boldsymbol{\beta} + \xi_{jt}, \tag{58}$$

via linear regression methods — in fact can estimate the equation with data from just one market

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via linear regression methods — in fact can estimate the equation with data from just one market

  - let $\mathbf{y}'_t = (y_{1t}, y_{2t}, \ldots, y_{Jt})$ be a row vector (for market t) given by $\mathbf{y}'_t = ([\ln s_{1t} - \ln s_{0t}], [\ln s_{2t} - \ln s_{0t}], \ldots, [\ln s_{Jt} - \ln s_{0t}])$ so that $\mathbf{y}_t$ is a column vector of length J
  - let $\mathbf{p}'_t = (p_{1t}, \ldots, p_{Jt})$ and $\boldsymbol{\xi}'_t = (\xi_{1t}, \ldots, \xi_{Jt})$ be row vectors with J entries for the t^{th} market
  - since $\mathbf{x}_{jt}$ is a row vector of observable characteristics of product j in market t, i.e., $\mathbf{x}_{jt} = (x_{1jt}, x_{2jt}, \ldots, x_{Kjt})$, thus let $\mathbf{X}'_t = (\mathbf{x}'_{1t}, \mathbf{x}'_{2t}, \ldots, \mathbf{x}'_{jt}, \ldots, \mathbf{x}'_{Jt})$ so that $\mathbf{X}_t$ is a $J \times K$ matrix, such that each row is itself a k dimensional vector of observable product characteristics

Then (58) can be written in 'long' form and even estimated with observations from one market t

$$\mathbf{y}_t = (\ln \mathbf{s}_{jt} - \ln s_{0t}) \qquad = \alpha(-\mathbf{p}_t) + \mathbf{X}_t\boldsymbol{\beta} + \boldsymbol{\xi}_t \equiv \boldsymbol{\delta}_t$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_J \end{bmatrix}_t = \begin{bmatrix} \ln s_1 - \ln s_0 \\ \ln s_2 - \ln s_0 \\ \vdots \\ \ln s_J - \ln s_0 \end{bmatrix}_t = \alpha \begin{bmatrix} -p_1 \\ -p_2 \\ \vdots \\ -p_J \end{bmatrix}_t + \begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1K} \\ x_{21} & x_{22} & \ldots & x_{2K} \\ \vdots & & & \\ x_{J1} & x_{J2} & \ldots & x_{JK} \end{bmatrix}_t \begin{bmatrix} \beta_1 \\ \beta_2 \\ \ldots \\ \beta_k \end{bmatrix} + \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_J \end{bmatrix} \tag{63}$$

- Data from multiple markets can be vertically 'stacked'

$$\mathbf{y} = \alpha(-\mathbf{p}) + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\xi} \equiv \boldsymbol{\delta}$$

$$\begin{bmatrix} \begin{pmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{J1} \end{pmatrix} \\ \vdots \\ \begin{pmatrix} y_{1t} \\ y_{2t} \\ \vdots \\ y_{Jt} \end{pmatrix} \\ \vdots \\ \begin{pmatrix} y_{1T} \\ y_{2T} \\ \vdots \\ y_{JT} \end{pmatrix} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} \ln s_{11} - \ln s_{01} \\ \ln s_{21} - \ln s_{01} \\ \vdots \\ \ln s_{J1} - \ln s_{01} \end{pmatrix} \\ \vdots \\ \begin{pmatrix} \ln s_{1t} - \ln s_{0t} \\ \ln s_{2t} - \ln s_{0t} \\ \vdots \\ \ln s_{Jt} - \ln s_{01} \end{pmatrix} \\ \vdots \\ \begin{pmatrix} \ln s_{1T} - \ln s_{0T} \\ \ln s_{2T} - \ln s_{0T} \\ \vdots \\ \ln s_{JT} - \ln s_{0T} \end{pmatrix} \end{bmatrix} = \alpha \begin{bmatrix} \begin{pmatrix} -p_{11} \\ -p_{21} \\ \vdots \\ -p_{J1} \end{pmatrix} \\ \vdots \\ \begin{pmatrix} -p_{1t} \\ -p_{2t} \\ \vdots \\ -p_{Jt} \end{pmatrix} \\ \vdots \\ \begin{pmatrix} -p_{1T} \\ -p_{2T} \\ \vdots \\ -p_{JT} \end{pmatrix} \end{bmatrix} + \begin{bmatrix} \begin{pmatrix} x_{111}\, x_{121} \cdots x_{1K1} \\ x_{211}\, x_{221} \cdots x_{2K1} \\ \vdots \\ x_{J11}\, x_{J21} \cdots x_{JK1} \end{pmatrix} \\ \vdots \\ \begin{pmatrix} x_{11t}\, x_{12t} \cdots x_{1Kt} \\ x_{21t}\, x_{22t} \cdots x_{2Kt} \\ \vdots \\ x_{J1t}\, x_{J2t} \cdots x_{JKt} \end{pmatrix} \\ \vdots \\ \begin{pmatrix} x_{11T}\, x_{12T} \cdots x_{1KT} \\ x_{21T}\, x_{22T} \cdots x_{2KT} \\ \vdots \\ x_{J1T}\, x_{J2T} \cdots x_{JKT} \end{pmatrix} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdots \\ \beta_k \end{bmatrix} + \begin{bmatrix} \begin{pmatrix} \xi_{11} \\ \xi_{21} \\ \vdots \\ \xi_{J1} \end{pmatrix} \\ \vdots \\ \begin{pmatrix} \xi_{1t} \\ \xi_{2t} \\ \vdots \\ \xi_{Jt} \end{pmatrix} \\ \vdots \\ \begin{pmatrix} \xi_{1t} \\ \xi_{2t} \\ \vdots \\ \xi_{Jt} \end{pmatrix} \end{bmatrix}$$

- As discussed earlier, very likely that $\text{cov}(p_{jt}, \xi_{jt}) \neq 0$
  - As before, one needs to find instruments that are correlated with price but not with any of the unobserved product characteristics
  - See the earlier discussion on various instruments (Hausman, BLP, etc.)
- Regardless of the instruments used, a first approach to consistent estimation would be to estimate a fixed effects model with dummies for products (and markets)
  - Requires that data be available from multiple markets
  - Thus, with data available from multiple markets, one can estimate via OLS

$$\ln(s_{jt}) - \ln(s_{0t}) = \delta_{jt} = \alpha(-p_{jt}) + \mathbf{x}_{jt}\boldsymbol{\beta} + \xi_j + \xi_t + \Delta\xi_{jt} \tag{64}$$

  where ξ_j is the brand fixed effect and ξ_t is the market fixed effect
  - Identifying assumption for OLS estimation is

$$\mathrm{E}(\Delta\xi_{jt}p_{jt}|\mathbf{x}_{jt}) = 0 \tag{65}$$

- Thus, with data available from multiple markets, one can estimate via OLS

$$\ln(s_{jt}) - \ln(s_{0t}) = \delta_{jt} = \alpha(-p_{jt}) + \mathbf{x}_{jt}\boldsymbol{\beta} + \xi_j + \xi_t + \Delta\xi_{jt} \tag{64}$$

- A brand-specific dummy variable captures all the observed characteristics of the product that do not vary across markets, as well as the product-specific mean of the unobserved characteristics, i.e., $\mathbf{x}_j\boldsymbol{\beta}$, where, note the missing market subscript of t from the vector $\mathbf{x}$
- Thus, the correlation between prices and brand-specific mean of unobserved quality is fully accounted for and does not require an instrument
- Once brand-specific dummy variables are included in the regression, the error term now is just the market-specific deviation from the mean of the unobserved characteristics, and may still require the use of instruments if the condition in equation (65) is not true

- Thus, with data available from multiple markets, one can estimate via OLS

$$\ln(s_{jt}) - \ln(s_{0t}) = \delta_{jt} = \alpha(-p_{jt}) + \mathbf{x}_{jt}\boldsymbol{\beta} + \xi_j + \xi_t + \Delta\xi_{jt} \tag{64}$$

- Similarly, if the mean unobserved quality – where the mean is now across all brands – is different by markets, then it too is fully accounted for by the market dummies
- If the subscript t for the markets is in the context of time periods, then this could be because of the unobserved quality of all products are improving over time (think computer quality over time)
- If the subscript t is in the cross-sectional setting, then this may or may not make much sense, since adding such dummies to the equation, the researcher is effectively arguing that the unobserved quality components of all brands in, Hooker, OK, are higher than those in Boring, OR
    - This may be true if the products under study require some additional local input for providing the product (radio channels with local DJs and ads), or if shipping from long-distance affects the quality of all products (fresh food), but not if they are centrally produced (RTE cereals) and shipping does not impact quality

- Two objections to the use of brand dummies
- Use of brand dummies increases the number of parameters to be estimated by J (rather than by J^2) – may not be too serious an issue if the number of markets is large
- A potentially more serious difficulty is that the coefficients $\boldsymbol{\beta}$ cannot be identified if observed characteristics do not vary by markets

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  - Nevo (2001) points out that in fact they can be recovered using minimum distance procedure by regressing the estimated brand dummy variables on the observed characteristics
  - Let $\mathbf{b}_t$ be the $J \times 1$ vector of brand dummies and let $\mathbf{X}_t$ be the $J \times K$ matrix of observed product characteristics and $\boldsymbol{\xi}_t$ be the $J \times 1$ vector of unobserved product qualities, neither of which varies by markets
  - Let also $\hat{\boldsymbol{b}}$ be the estimated values of coefficients ($J \times 1$) of the brand dummies and $\hat{\boldsymbol{V}}_{\boldsymbol{b}}^{-1}$ their estimated $J \times J$ variance-covariance matrix, both of which are available from initially estimating equation (64)

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  - Then, the estimates of $\boldsymbol{\beta}$ and $\boldsymbol{\xi}$ in equation

$$\mathbf{b}_t = \mathbf{X}_t \boldsymbol{\beta} + \boldsymbol{\xi}_t, \tag{66}$$

can be recovered via the GLS estimator

$$\widehat{\boldsymbol{\beta}} = (\mathbf{X}_t' \hat{\mathbf{V}}_{\mathbf{b}}^{-1} \mathbf{X}_t)^{-1} \mathbf{X}_t' \hat{\mathbf{V}}_{\mathbf{b}}^{-1} \hat{\mathbf{b}}_t, \text{ and } \boldsymbol{\xi}_t = \hat{\mathbf{b}}_t - \mathbf{X}_t \widehat{\boldsymbol{\beta}} \tag{67}$$

where the latter is just the calculated value of the residual term from the regression above

Logit Demand Model
Estimation Example

- Simulated dataset for the same 17 drugs also available as in the 'long' format – long here means that within each county, the data – shares, prices, other characteristics, etc. – are set as 17 rows as opposed to 17 different columns per variable – download `simulateddrugs02.sas7bdat` and `simulateddrugs02.dat`
- The accompanying SAS program `estimate-LOGIT-ver01.sas` shows how to estimate the model in SAS using OLS/2SLS – it also computes the elasticity matrix at the sample mean — you can do the same in STATA (I will not do that here as I will shortly introduce a special package `mergersim` for STATA that does all that plus more)

Standard Logit

Estimation Example

- Partial code (define outside good share and shares)

```
data foo2;
/*****************
outside good, potential market and shares:

Potential market is defined via 30mg * 3 of mph per day.
 a ADHD kid gets 30mg per day.
 --> 30/1000 grams per day.
 --> 30*30/1000 grams per month.
--> (30*30*12)/1000 grams per year.
--> .9*12 grams per year.
--> 10.8 grams per year.

mso -> potential market 15% of all children + 5% of other adults
consuming at 3 times the DDD amount  OR 3 times the current consumption
rate per child.

*****************/
set foo;
  qpc = qq/(.07*.75*kids5t19);     ** quantity per child;
  mso = ((kids5t19)*.15 + (poptot - (kids5t19 + kids0t4))*.05)*(3*qpc);
  q0 = mso - qq ;                  ** outside good;
  s0 = q0/mso;                     ** share of outside good;
  lns0 =   log(s0);                ** log of share of outside good;
  sqi = qi/mso;                    ** re-compute shares relative to potential market ;

 /* or use
 qpc = qq/(.1*.60*kids5t19);     ** quantity per child;
 mso = ((kids5t19)*.1 + (poptot - (kids5t19 + kids0t4))*0.01)*(qpc);
  q0 = mso - qq ;                  ** outside good;
  s0 = q0/mso;                     ** share of outside good;
  lns0 =   log(s0);                ** log of share of outside good;
  sqi = qi/mso;                    ** re-compute shares relative to potential market ;
*/

  if sqi ~ = 0 then lnsqi =   log(sqi);  ** log of shares;
  if sqi ~ = 0 then  lnsqim0 = lnsqi - lns0;
  if sqi = 0 then lnsqi = .;
```

- Partial code (define outside good share and shares)

```
proc syslin data = logits  2sls ;

ENDOGENOUS  lnsqim0 poi ;
INSTRUMENTS pzi t1 t2
  lnkids lnmds lncaiddrugs lnmcaidenrollees
  drug1-drug16 d1-d48;

 model lnsqim0 = poi t1 t2
  lnkids lnmds lncaiddrugs lnmcaidenrollees
  drug1-drug16 d1-d48;

 run; quit;
```

Brand dummies & state dummies

- Partial output

Parameter Estimates						
Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|	Variable Label
Intercept	1	0.502753	0.485966	1.03	0.3009	Intercept
poi	1	-5.21234	0.211303	-24.67	<.0001	poi: Price ($/DDD gms) (Constant 2000 Dollars) -- price of 4,
t1	1	-0.00368	0.000186	-19.79	<.0001	Time (t1): 1 is year 1999
t2	1	0.000364	0.000025	14.35	<.0001	Time Sq (t2): 1 is year 1999
lnkids	1	0.034712	0.006464	5.37	<.0001	
lnmds	1	0.019831	0.004072	4.87	<.0001	
lncaiddrugs	1	0.064888	0.013569	4.78	<.0001	Log (Total Amount Reimbured for Drugs by Medicaid)
lnmcaidenrollees	1	0.039136	0.033232	1.18	0.2389	Log(Medicaid Enrollees in State (Census Estimates))
DRUG1	1	-3.91399	0.107768	-36.32	<.0001	
DRUG2	1	-4.53114	0.138050	-32.82	<.0001	
DRUG3	1	-3.52148	0.136902	-25.72	<.0001	

- Partial output (2SLS vs OLS)

Logit price elasticities -- IV estimates [Entry (j,k) --> DlnQj/DlnPk]

	Ritalin	Methylin	Generics (MPH-IR)	Ritalin SRLA	Metadate ERCD	MethylinER	Generics (MPH-ER)	Concerta	Aderall	Generic (MAS-IR)	Aderall XR
Ritalin	-2.579	0.014	0.038	0.012	0.012	0.004	0.013	0.133	0.082	0.016	0.052
Methylin	0.018	-1.831	0.038	0.012	0.012	0.004	0.013	0.133	0.082	0.016	0.052
Generics (MPH-IR)	0.018	0.014	-1.836	0.012	0.012	0.004	0.013	0.133	0.082	0.016	0.052
Ritalin SRLA	0.018	0.014	0.038	-3.364	0.012	0.004	0.013	0.133	0.082	0.016	0.052
Metadate ERCD	0.018	0.014	0.038	0.012	-2.927	0.004	0.013	0.133	0.082	0.016	0.052
MethylinER	0.018	0.014	0.038	0.012	0.012	-2.896	0.013	0.133	0.082	0.016	0.052
Generics (MPH-ER)	0.018	0.014	0.038	0.012	0.012	0.004	-2.650	0.133	0.082	0.016	0.052
Concerta	0.018	0.014	0.038	0.012	0.012	0.004	0.013	-5.007	0.082	0.016	0.052
Aderall	0.018	0.014	0.038	0.012	0.012	0.004	0.013	0.133	-1.253	0.016	0.052
Generic (MAS-IR)	0.018	0.014	0.038	0.012	0.012	0.004	0.013	0.133	0.082	-1.290	0.052
Aderall XR	0.018	0.014	0.038	0.012	0.012	0.004	0.013	0.133	0.082	0.016	-2.181
Dexederine	0.018	0.014	0.038	0.012	0.012	0.004	0.013	0.133	0.082	0.016	0.052
Dextrostat	0.018	0.014	0.038	0.012	0.012	0.004	0.013	0.133	0.082	0.016	0.052
Generics (DEX-IR)	0.018	0.014	0.038	0.012	0.012	0.004	0.013	0.133	0.082	0.016	0.052
Dexederine SR	0.018	0.014	0.038	0.012	0.012	0.004	0.013	0.133	0.082	0.016	0.052
Generics (DEX-ER)	0.018	0.014	0.038	0.012	0.012	0.004	0.013	0.133	0.082	0.016	0.052
Offlabels	0.018	0.014	0.038	0.012	0.012	0.004	0.013	0.133	0.082	0.016	0.052

Logit price elasticities -- OLS estimates [Entry (j,k) --> DlnQj/DlnPk]

	Ritalin	Methylin	Generics (MPH-IR)	Ritalin SRLA	Metadate ERCD	MethylinER	Generics (MPH-ER)	Concerta	Aderall	Generic (MAS-IR)	Aderall XR
Ritalin	-0.309	0.002	0.005	0.001	0.001	0.000	0.002	0.016	0.010	0.002	0.006
Methylin	0.002	-0.219	0.005	0.001	0.001	0.000	0.002	0.016	0.010	0.002	0.006
Generics (MPH-IR)	0.002	0.002	-0.220	0.001	0.001	0.000	0.002	0.016	0.010	0.002	0.006
Ritalin SRLA	0.002	0.002	0.005	-0.403	0.001	0.000	0.002	0.016	0.010	0.002	0.006
Metadate ERCD	0.002	0.002	0.005	0.001	-0.350	0.000	0.002	0.016	0.010	0.002	0.006
MethylinER	0.002	0.002	0.005	0.001	0.001	-0.347	0.002	0.016	0.010	0.002	0.006
Generics (MPH-ER)	0.002	0.002	0.005	0.001	0.001	0.000	-0.317	0.016	0.010	0.002	0.006
Concerta	0.002	0.002	0.005	0.001	0.001	0.000	0.002	-0.599	0.010	0.002	0.006
Aderall	0.002	0.002	0.005	0.001	0.001	0.000	0.002	0.016	-0.150	0.002	0.006
Generic (MAS-IR)	0.002	0.002	0.005	0.001	0.001	0.000	0.002	0.016	0.010	-0.154	0.006
Aderall XR	0.002	0.002	0.005	0.001	0.001	0.000	0.002	0.016	0.010	0.002	-0.261

- The IIA problem in logit arose from the iid structure of the error terms
- Particularly, while consumers have different rankings of the products, these differences arise only due to the iid shocks to the error term ϵ_{njt}
- One solution to this problem is to make the random shocks to the utility correlated across products by generating correlations through the error term
- An example is the nested logit model in which products are grouped and ϵ_{njt} is decomposed into an iid shock plus a group specific component which results in a correlation between products in the same group
- The basic idea is to relax the IIA by grouping products (similar to the grouping idea in multilevel budgeting/AIDS we saw earlier), but within each group, we have a standard logit model, and products in different groups have less in common and are not good substitutes

- Let the utility for consumer n for product j in group g be

$$u_{njt} = \delta_{jt} + \zeta_{ngt}(\sigma) + (1-\sigma)\epsilon_{njt}, \tag{68}$$

- where
  - $\delta_{jt} = \alpha(-p_{jt}) + \mathbf{x}_{jt}\boldsymbol{\beta} + \xi_{jt}$ is the mean utility for product j common to all consumers (as before)
  - ϵ_{njt} is (still) the person-specific iid random shock with extreme value distribution
  - but ζ_{ngt} is the person-specific shock that is common to all products in group g
  - The distribution of the group-specific random variable ζ_{ngt} depends on the parameter σ so that $\zeta_{ngt}(\sigma) + (1-\sigma)\epsilon_{njt}$ is extreme value
  - If σ approaches zero, the model is reduced to that of the simple logit case discussed earlier while if it approached one, only the nests matter
- Gives a closed form that can be estimated using linear estimation methods

$$\ln(s_{jt}) - \ln(s_{0t}) = \alpha(-p_{jt}) + \mathbf{x}_{jt}\boldsymbol{\beta} + \sigma\ln(s_{jt}/s_{gt}) + \xi_{jt} \tag{69}$$

- The additional term $\ln(s_{jt}/s_{gt})$ is the share of product j in group g
- All previous issues (define outside good, use of dummies, instruments etc.) apply here as well
- One difference from the previous case is that even if prices are exogenous, the term $\ln(s_{jt}/s_{gt})$ is endogenous and we need some instrumental variable for it

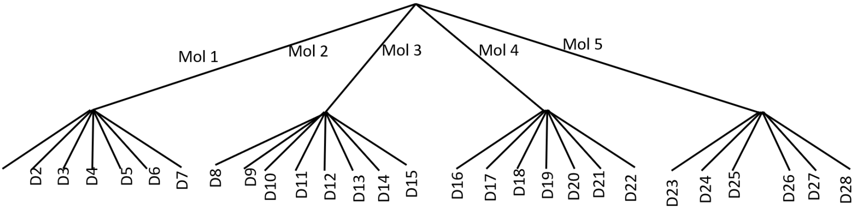
Nested Logit
Utility function and market shares

- A significant refinement over the model comes from the nested logit variant – groups of products that are close substitutes are placed in nests – and consumers choose the nest and then the specific product

$$\ln(s_{jt}/s_{0t}) = \alpha(-p_{jt}) + \mathbf{x}_{jt}\boldsymbol{\beta} + \sigma \ln(s_{jt}/s_{gt}) + \xi_{jt}$$

- The additional term $\ln(s_{jt}/s_{gt})$ is the share of product j in group g (and the term is endogenous as is often the price variable)

- The figure shows a nesting choice of 28 across four molecules, where the patient/doctor first chooses a molecule and then the brand

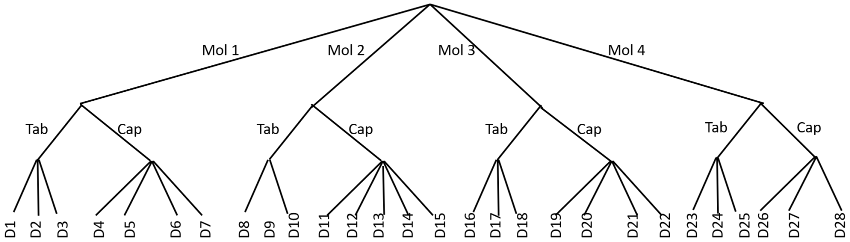


Nested Logit
Utility function and market shares

- We can refine this further to second-level nesting (or more, but becomes difficult)

$$\ln(s_{jt}/s_{0t}) = \alpha(-p_{jt}) + \mathbf{x}_{jt}\boldsymbol{\beta} + \sigma_1 \ln(s_{jt}/s_{hgt}) + \sigma_2 \ln(s_{ht}/s_{gt}) + \xi_{jt}$$

- The additional terms $\ln(s_{jt}/s_{hgt})$ and $\ln(s_{ht}/s_{gt})$ are the shares of product j in subgroup h of group g and of group h in group g
- The figure shows a nesting choice of 28 across four molecules and two formulations, where the patient/doctor first chooses a molecule, then a formulation and then the brand



- Such a model can be estimated and used in merger simulations with STATA's user written command `mergersim` by Björnerstedt and Verboven (2014)
- In a nutshell
  - Easy to use add-in for STATA
  - Estimates a logit, nested logit, or double nested logit (OLS or IV) using standard STATA commands for linear regressions with or without fixed effects
  - By declaring product id and firm id variables, initialization of the program automatically creates ownership matrix $\boldsymbol{\Theta_0}$ in the background, and using estimates from the logit model and observed shares and prices, creates the markup $\boldsymbol{\Omega_0}$ matrix
  - Post estimation gives estimates of marginal costs and allows for mergers between any number of firms – also allows for the computation of minimum required efficiencies per product for price not to increase after the merger
- Example/demo follows ...

Nested Logit

Estimation Example – mergersim tool

- Sample sales data by the authors from the European car market
  - Markets: Countries-year combination – Belgium, France, Germany, Italy, UK and years 1970-1999
  - Products: 351 (for instance Alpha Romeo 33 is a distinct product from Alpha Romeo 75); brands 38
  - Firms: 26
  - Nests: upper nest is *segment* – subcompact, compact, intermediate, standard, and luxury and lower nest is *domestic* which takes values 1/0 if a firm is domestic or foreign in a given market (for instance, Fiat is domestic in Italy and foreign in other countries)
  - Price is measured in 1,000 Euro (1999 values) and quantity is new car registrations
  - The data set includes several other product characteristics such horsepower, fuel efficiency, height, width

- The program runs in four parts
  - Step 1: Initialize the program (`mergersim int`) – this entails declaring variables firm and product id, price, and quantity variables, variables that capture the nest, and variables for potential market size (so shares can be computed)
  - Step 2: Estimate the logit or nested logit model using standard STATA commands (including IV-based commands) – the previous step has already created all the variables necessary for estimating the model
  - Step 3: Compute pre-merger variables (`mergersim market`) – this step computes the mean gross valuation of each product $\delta_{jt} \equiv \mathbf{x_{jt}}\boldsymbol{\beta} + \boldsymbol{\xi}_{jt}$, own and cross elasticities, and marginal costs
  - Step 4: Simulate a merger (`mergersim simulate`) – performs a merger simulation where a user specifies which firms are merging and outputs results
- Selected inputs (code snippets) and outputs follow

- Step 1 – (`mergersim int`)
  - Set the size of the potential market to 1/4 of the population and run step 1 initialization
  - population variable is `pop`, and market size variable is `MSIZE`
  - price variable is `price`, quantity is `qu`, and firm id is `firm`
  - nesting variables are `segment` and `domestic`
  - the product id is `co`, and is declared as part of STATA's panel declaration command (`xtset`) along with the other dimension being `yearcountry`

Nested Logit
Estimation Example – mergersim tool

```
. egen yearcountry=group(year country), label

. xtset co yearcountry
       panel variable:  co (unbalanced)
        time variable:  yearcountry, 1 to 150, but with gaps
                delta:  1 unit

. gen MSIZE=pop/4

.
. mergersim init, nests(segment domestic) price(price) quantity(qu) marketsize(MSIZE) firm(firm)
--------------------------------------------------------------------------------
MERGERSIM: Merger Simulation Program
Version 1.0, Revision: 218
--------------------------------------------------------------------------------
Unit demand two-level nested logit

                 Depvar             Price              Group shares
                 ----------------------------------------------------------------
                 M_ls               price              M_lsjh M_lshg
--------------------------------------------------------------------------------
Variables generated: M_ls M_lsjh M_lshg
--------------------------------------------------------------------------------
```

Nested Logit

Estimation Example – mergersim tool

- Step 2 – (`estimate parameters of nested logit model`)
  - In this simple example, we used a fixed effects linear model via `xtreg` (where the fixed effects are over the product ids) but should be run using `ivreg` or `xtivreg`

```
. xtreg M_ls price M_lsjh M_lshg horsepower fuel width height domestic year country2-c
> ountry5, fe

Fixed-effects (within) regression               Number of obs      =     11,483
Group variable: co                              Number of groups   =        351

R-sq:                                           Obs per group:
     within  = 0.8948                                         min =          1
     between = 0.7576                                         avg =       32.7
     overall = 0.8427                                         max =        146

                                                F(13,11119)        =    7271.50
corr(u_i, Xb)  = -0.0147                        Prob > F           =     0.0000
```

M_ls	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
price	-.0468375	.0013002	-36.02	0.000	-.0493861	-.0442888
M_lsjh	.9047371	.0041489	218.07	0.000	.8966045	.9128696
M_lshg	.5677968	.0085109	66.71	0.000	.551114	.5844796
horsepower	.0038279	.0005921	6.46	0.000	.0026672	.0049886
fuel	-.0270919	.004539	-5.97	0.000	-.0359892	-.0181946
width	.0103757	.0016768	6.19	0.000	.0070889	.0136625
height	.0004322	.0022161	0.20	0.845	-.0039117	.0047761
domestic	.5230743	.0124205	42.11	0.000	.4987279	.5474206
year	.0017336	.0012022	1.44	0.149	-.000623	.0040902

- Step 3 – (`back out marginal cost etc.`) (here we do so using only 1998 data) – **output part 1**

```
. mergersim market if year == 1998

Supply: Bertrand competition
Demand: Unit demand two-level nested logit

Demand estimate
xtreg M_ls price M_lsjh M_lshg horsepower fuel width height domestic year country2-cou
> ntry5, fe
Dependent variable: M_ls


Parameters

alpha = -0.047
sigma1 = 0.905
sigma2 = 0.568

Own- and Cross-Price Elasticities:  unweighted market averages
```

variable	mean	sd	min	max
M_ejj	-7.488	3.761	-30.454	-1.710
M_ejk	0.766	1.276	0.003	10.908
M_ejl	0.068	0.120	0.000	0.768
M_ejm	0.001	0.002	0.000	0.011

```
Observations: 449
```

Nested Logit

Estimation Example – mergersim tool

- Step 3 – (`back out marginal cost etc.`) (here we do so using only 1998 data) – **output part 2**

```
Pre-merger Market Conditions
Unweighted averages by firm
```

firm code	price	Marginal costs	Pre-merger Lerner
BMW	20.194	17.499	0.146
Fiat	15.277	10.553	0.372
Ford	14.557	11.923	0.207
Honda	20.094	17.941	0.128
Hyundai	12.915	10.849	0.179
Kia	10.814	8.772	0.207
Mazda	14.651	12.557	0.156
Mercedes	25.598	21.569	0.162
Mitsubishi	15.955	13.825	0.145
Nissan	15.438	13.259	0.159
GM	21.054	18.633	0.135
PSA	16.243	13.533	0.194
Renault	15.518	12.837	0.203
Suzuki	9.289	7.226	0.234
Toyota	14.560	12.430	0.172
VW	18.990	16.388	0.181
Volvo	23.167	20.912	0.099
Daewoo	13.871	11.789	0.170

```
Variables generated: M_costs M_delta
```

- Step 4 – (`mergersim simulate`) Simulate a merger between GM (seller=15) and VW (buyer=26) and looks at effects in Germany 1998 – **output part 1**

```
. mergersim simulate if year == 1998 & country == 3, seller(15) buyer(26) detail
```

firm code	Pre-merger	Post-merger	Relative change
BMW	17.946	18.002	0.003
Fiat	15.338	15.341	0.000
Ford	13.093	13.362	0.023
Honda	15.778	15.780	0.000
Hyundai	12.912	12.912	0.000
Kia	11.276	11.276	0.000
Mazda	14.229	14.231	0.000
Mercedes	20.114	20.155	0.003
Mitsubishi	15.832	15.834	0.000
Nissan	15.101	15.103	0.000
GM	19.921	21.054	0.076
PSA	16.397	16.399	0.000
Renault	15.292	15.295	0.000
Suzuki	9.225	9.225	0.000
Toyota	13.019	13.020	0.000
VW	17.182	17.739	0.036
Volvo	22.149	22.154	0.000
Daewoo	13.483	13.484	0.000

- Step 4 – (`mergersim simulate`) Simulate a merger between GM (seller=15) and VW (buyer=26) and looks at effects in Germany 1998 – **output part 2**

```
Market shares by quantity
Unweighted averages by firm
```

firm code	Pre-merger	Post-merger	Difference
BMW	0.074	0.079	0.005
Fiat	0.043	0.045	0.003
Ford	0.095	0.132	0.037
Honda	0.012	0.012	0.001
Hyundai	0.006	0.006	0.000
Kia	0.003	0.003	0.000
Mazda	0.025	0.027	0.002
Mercedes	0.100	0.116	0.017
Mitsubishi	0.015	0.017	0.001
Nissan	0.025	0.027	0.002
GM	0.166	0.108	-0.058
PSA	0.034	0.037	0.003
Renault	0.051	0.054	0.003
Suzuki	0.006	0.006	0.000
Toyota	0.027	0.029	0.002
VW	0.300	0.280	-0.020
Volvo	0.012	0.013	0.001
Daewoo	0.006	0.007	0.001

- Step 4 – (`mergersim simulate`) Simulate a merger between GM (seller=15) and VW (buyer=26) and looks at effects in Germany 1998 – **output part 3**

	Pre-merger	Post-merger
HHS:	1501	1972
C4:	66.07	71.50
C8:	86.21	88.01

	Change
Consumer surplus:	-1,839,750
Producer surplus:	1,303,353

- The `mergersim` tool also allows a user to explore the effects of
  - efficiencies (by changing marginal costs)
  - remedies such as divestitures (via adjusting ownership matrix for other products by the merging parties)
  - conduct parameter (allows for partial collusion pre-merger)
- The tool also allows for calibration where users can set the values of $\alpha, \sigma_1, \sigma_2$ and computation of minimum required efficiencies so that prices do not increase
- As such can be used as an initial or additional screen

- Say we have an additional set of exogenous variables $\mathbf{z}_t$ that are correlated with $\mathbf{x}_t$ but not with the error terms so that $\mathrm{E}[u_t|\mathbf{z}_t] = 0$
- Then, $\mathrm{E}[(y_t - \mathbf{x}_t\boldsymbol{\beta})|\mathbf{z}_t] = 0$, and as before, we can multiply $\mathbf{z}_t$ with the residual terms to get K unconditional population moment conditions

$$\mathrm{E}[\mathbf{z}_t'(y_t - \mathbf{x}_t\boldsymbol{\beta})] = \mathbf{0} \tag{70}$$

- Then the MM estimator solves the sample moment conditions given by

$$\frac{1}{T}\sum_{t=1}^{T}\mathbf{z}_t'(y_t - \mathbf{x}_t\boldsymbol{\beta}) = \mathbf{0} \tag{71}$$

- If $\dim(\mathbf{z}) = K$, then this yields the MM estimator which is just the IV estimator

$$\widehat{\boldsymbol{\beta}}_{MM} = (\sum_t \mathbf{z}_t'\mathbf{x}_t)^{-1}\sum_t \mathbf{z}_t' y_t = (\mathbf{Z}'\mathbf{X})^{-1}\mathbf{Z}'\mathbf{y} \tag{72}$$

- If however, $\dim(\mathbf{z}) > K$, (more potential instruments than the original number of regressors) then there is no unique solution – more moment conditions than the number of parameters to be estimated
- We can use the GMM estimator which chooses $\widehat{\boldsymbol{\beta}}$ so as to make the vector $T^{-1}\sum_{t=1}^{T}\mathbf{z}_t'(y_t - \mathbf{x}_t\boldsymbol{\beta})$ as small as possible using quadratic loss
- Thus find $\widehat{\boldsymbol{\beta}}_{\text{GMM}}$ which minimizes the function

$$Q(\boldsymbol{\beta}) = \left[\frac{1}{T}\sum_t \mathbf{z}_t'(y_t - \mathbf{x}_t\boldsymbol{\beta})\right]' \boldsymbol{\Phi} \left[\frac{1}{T}\sum_t \mathbf{z}_t'(y_t - \mathbf{x}_t\boldsymbol{\beta})\right] \tag{73}$$

where $\boldsymbol{\Phi}$ is a $\dim(\mathbf{z}) \times \dim(\mathbf{z})$ weighting matrix

- In matrix notation define $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$ (where $\mathbf{y}$ and $\mathbf{u}$ are $T \times 1$, $\mathbf{X}$ is $T \times K$ and $\boldsymbol{\beta}$ is $K \times 1$ as before), and let $\mathbf{Z}$ be $T \times R$ matrix, then $\sum_{t=1}^{T}\mathbf{z}_t'(y_t - \mathbf{x}_t\boldsymbol{\beta}) = \mathbf{Z}'\mathbf{u}$ and (73) becomes

$$Q(\boldsymbol{\beta}) = \left[\frac{1}{T}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'\mathbf{Z}\right] \boldsymbol{\Phi} \left[\frac{1}{T}\mathbf{Z}'(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})\right] \tag{74}$$

where $\boldsymbol{\Phi}$ is a $R \times R$ full rank symmetric weighting matrix

- First order conditions, $\partial Q(\boldsymbol{\beta})/\partial\boldsymbol{\beta} = \mathbf{0}$ for the linear IV case are

$$\frac{\partial Q(\boldsymbol{\beta})}{\partial\boldsymbol{\beta}} = -2\left[\frac{1}{T}\mathbf{X}'\mathbf{Z}\right]\boldsymbol{\Phi}\left[\frac{1}{T}\mathbf{Z}'(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})\right] = \mathbf{0} \tag{75}$$

- Then the GMM linear IV estimator and its variance are

$$\begin{aligned}\widehat{\boldsymbol{\beta}}_{\text{GMM}} &= \left(\mathbf{X}'\mathbf{Z}\boldsymbol{\Phi}\mathbf{Z}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{Z}\boldsymbol{\Phi}\mathbf{Z}'\mathbf{y} \\ \mathrm{V}(\widehat{\boldsymbol{\beta}})_{\text{GMM}} &= T\left(\mathbf{X}'\mathbf{Z}\boldsymbol{\Phi}\mathbf{Z}'\mathbf{Z}\right)^{-1}\left(\mathbf{X}'\mathbf{Z}\boldsymbol{\Phi}\widehat{\mathbf{S}}\boldsymbol{\Phi}\mathbf{Z}'\mathbf{X}\right)\left(\mathbf{X}'\mathbf{Z}\boldsymbol{\Phi}\mathbf{Z}'\mathbf{X}\right)^{-1}\end{aligned} \tag{76}$$

where $\widehat{\mathbf{S}}$ is a consistent estimate of

$$\mathbf{S} = \text{plim}\frac{1}{T}\sum_i\sum_j\left[\mathbf{z}_i' u_i u_j \mathbf{z}_j\right] \tag{77}$$

- Different choices of the weighting matrix $\mathbf{\Phi}$ lead to different estimators
- If the model is just identified ($R = K$) and the matrix $\mathbf{X}'\mathbf{Z}$ is invertible, then the choice of the weighting matrix $\mathbf{\Phi}$ does not matter as the GMM estimator is just the IV estimator:

$$\begin{aligned}\widehat{\boldsymbol{\beta}}_{\text{GMM}} &= \left(\mathbf{X}'\mathbf{Z}\mathbf{\Phi}\mathbf{Z}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{Z}\mathbf{\Phi}\mathbf{Z}'\mathbf{y} \\ &= (\mathbf{Z}'\mathbf{X})^{-1}\mathbf{\Phi}^{-1}(\mathbf{X}'\mathbf{Z})^{-1}(\mathbf{X}'\mathbf{Z})\mathbf{\Phi}\mathbf{Z}'\mathbf{y} \\ &= (\mathbf{Z}'\mathbf{X})^{-1}\mathbf{Z}'\mathbf{y} = \widehat{\boldsymbol{\beta}}_{\text{IV}}\end{aligned} \tag{78}$$

- If $R > K$, and the errors are homoscedastic, then $\mathbf{\Phi} = (T^{-1}\mathbf{Z}'\mathbf{Z})^{-1}$ and $\widehat{\mathbf{S}}^{-1} = \left[s^2T^{-1}\mathbf{Z}'\mathbf{Z}\right]$ leads to the usual 2SLS estimator

$$\begin{aligned}\widehat{\boldsymbol{\beta}}_{\text{GMM}} &= \left(\mathbf{X}'\mathbf{P}_{\mathbf{z}}\mathbf{X}\right)^{-1}\left(\mathbf{X}'\mathbf{P}_{\mathbf{z}}\mathbf{y}\right) = \widehat{\boldsymbol{\beta}}_{\text{2SLS}} \\ \mathrm{V}(\widehat{\boldsymbol{\beta}}_{\text{GMM}}) &= s^2\left(\mathbf{X}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X}\right)^{-1} \\ \text{where } \mathbf{P}_{\mathbf{z}} &= \mathbf{Z}(\mathbf{Z}\mathbf{Z}')^{-1}\mathbf{Z}' \text{ and } s^2 = (T-K)^{-1}\sum_t \hat{u}_t^2\end{aligned} \tag{79}$$

Generalized Method of Moments
Brief Review

- Alternatively, if errors are heteroscedastic, then instead we can use

$$\mathrm{V}(\widehat{\boldsymbol{\beta}}_{\text{GMM}}) = T\left(\mathbf{X}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X}\right)^{-1}\left(\mathbf{X}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\widehat{\mathbf{S}}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X}\right)\left(\mathbf{X}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X}\right.$$
$$\text{and } \widehat{\mathbf{S}} = T^{-1}\sum_t \hat{u}_t^2 \mathbf{z}_t \mathbf{z}'_t. \tag{80}$$

- The **optimal** weighting matrix (optimal in the sense of efficiency/smallest variance) is one which is proportional to the inverse of $\mathbf{S}$
- The optimal GMM two-step estimator (for the linear IV case) is when $\boldsymbol{\Phi} = \widehat{\mathbf{S}}^{-1}$

$$\widehat{\boldsymbol{\beta}}_{\text{OGMM}} = \left(\mathbf{X}'\mathbf{Z}\widehat{\mathbf{S}}^{-1}\mathbf{Z}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{Z}\widehat{\mathbf{S}}^{-1}\mathbf{Z}'\mathbf{y} \tag{81}$$

  - Step 1: Use 2SLS as the first step to estimate $\widehat{\boldsymbol{\beta}}$ and then compute residuals as in the heteroscedastic case above
  - Step 2: Construct the $\widehat{\mathbf{S}}^{-1}$ and then use it in (81) to compute the estimator

- Variance is given by

$$\mathrm{V}(\widehat{\boldsymbol{\beta}}_{\text{OGMM}}) = T\left(\mathbf{X}'\mathbf{Z}\widehat{\mathbf{S}}^{-1}\mathbf{Z}'\mathbf{X}\right)^{-1} \tag{82}$$

- This approach extends easily to the general case with other moment conditions
- Let $\boldsymbol{\theta}$ be a $q \times 1$ vector of parameters and $\mathbf{h}(\mathbf{w}, \boldsymbol{\theta})$ be an $r \times 1$ vector function such that at the true value of the parameter $\boldsymbol{\theta}_0$, there are r moment conditions ($r > q$) give by

$$\mathrm{E}\left[\mathbf{h}(\mathbf{w}_t, \boldsymbol{\theta}_0)\right] = \mathbf{0} \tag{83}$$

  - where the expectations are not zero if $\boldsymbol{\theta} \neq \boldsymbol{\theta}_0$
  - the vector $\mathbf{w}_t$ includes all observable variables, including $\mathbf{y}_t$, $\mathbf{x}_t$ and, $\mathbf{z}_t$
- Then the GMM objective function (equivalent of (73)) is

$$Q(\boldsymbol{\beta}) = \left[\frac{1}{T}\sum_t \mathbf{h}(\mathbf{w}_t, \boldsymbol{\theta})\right]' \boldsymbol{\Phi} \left[\frac{1}{T}\sum_t \mathbf{h}(\mathbf{w}_t, \boldsymbol{\theta})\right] \tag{84}$$

  and the corresponding first-order conditions are

$$\frac{\partial Q(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} = \left[\frac{1}{T}\sum_t^T \frac{\partial \mathbf{h}_t(\widehat{\boldsymbol{\theta}})'}{\partial \boldsymbol{\theta}}\right] \boldsymbol{\Phi} \left[\frac{1}{T}\sum_t^T \mathbf{h}_t(\widehat{\boldsymbol{\theta}})\right] = \mathbf{0} \tag{85}$$

  where $\mathbf{h}_t(\boldsymbol{\theta}) = \mathbf{h}(\mathbf{w}_t\boldsymbol{\theta})$
- Note that If $\mathbf{h}_t(\boldsymbol{\theta}) = \mathbf{z}_t'(y_t - \mathbf{x}_t\boldsymbol{\beta}) = \mathbf{z}_t' u_t$ then $\partial \mathbf{h}/\partial \boldsymbol{\beta}' = -\mathbf{z}_t'\mathbf{x}_t$ and the earlier results of linear IV follows

- GMM also extends to non-linear models, where the error term u_t may or may not be additively separable
- For instance, $u_t = y_t - g(\mathbf{x}_t; \boldsymbol{\theta})$ where $g(\cdot)$ is some nonlinear function but the error term is additively separable, or non-separable so that $u_t = g(y_t, \mathbf{x}_t; \boldsymbol{\theta})$
- If $\mathrm{E}(u_t|\mathbf{x}_t) \neq 0$ but we have instruments available so that $\mathrm{E}(u_t|\mathbf{z}_t) = 0$, then the moment conditions are $\mathrm{E}(\mathbf{z}_t' u_t) = \mathbf{0}$
- The GMM estimator minimizes the objective function

$$Q(\boldsymbol{\beta}) = \left[\frac{1}{T}\mathbf{u}'\mathbf{Z}\right] \boldsymbol{\Phi} \left[\frac{1}{T}\mathbf{Z}'\mathbf{u}\right] \tag{86}$$

- Unlike the linear case, the first-order conditions do not give closed forms for the estimators

- Earlier saw that standard logit can be estimated as a linear equation when the dependent variable is defined as $y_{jt} \equiv \ln s_{jt} - \ln s_{0t}$ and the equation is given as $y_{jt} = \alpha(-p_{jt}) + \mathbf{x}_{jt}\boldsymbol{\beta} + \xi_{jt}$
- When the price is correlated with the unobserved heterogeneity term ξ_{jt}, so that $\mathrm{E}(p, \xi) \neq 0$ and we have a set of instruments such that $\mathrm{E}(Z\xi) = 0$, then we can use the GMM/IV methods described in the earlier section to estimate the parameters of the equation
- The linear equation arose out of Berry's (1994) inversion trick
- Useful to work through this again for extending the method to random coefficients model

- Let the observed shares be given by $\mathbf{s}$ so that $\mathbf{s}_t = (s_{0t}, s_{1t}, \ldots, s_{Jt})$ where, as before, $s_{0t} = 1 - \sum_{j=1}^{J} s_{jt}$
- Let also $\boldsymbol{\theta}_1 \equiv \begin{bmatrix}\alpha & \boldsymbol{\beta}'\end{bmatrix}'$) and let model predicted market shares in equation (57) be given by $\tilde{\mathbf{s}}$ so that $\tilde{\mathbf{s}}_t = (\tilde{s}_{0t}, \tilde{s}_{1t}, \ldots, \tilde{s}_{Jt})$
- Given a value of $\boldsymbol{\theta}_1$, can compute the model predicted shares as

$$\tilde{s}_{jt} = \frac{\exp(\delta_{jt})}{1 + \sum_{j=1}^{J} \exp(\delta_{jt})} \tag{57}$$

- Thus, may want to use NLS methods to find $\boldsymbol{\theta}_1$ to minimize the distance between predicted and observed market shares

$$\min_{\boldsymbol{\theta}_1} \sum_{j=1}^{J} \left[s_{jt} - \tilde{s}_{jt}(\alpha, \boldsymbol{\beta}, \xi_{1t}, \xi_{2t}, \ldots, \xi_{Jt})\right]^2 \tag{87}$$

- The econometric error terms $\boldsymbol{\xi}_t$ – unobserved product qualities – enter the predicted market share and are not additively separable. Hence, non-linear least squares methods will not give consistent estimates *even if* prices were not endogenous

- Assume that we have a set of M instruments given by matrix $\mathbf{Z}$ with dimensions $JT \times M$ (the jt^{th} row is given by $\mathbf{z}_{jt} = (z_{jt}^{(1)}, z_{jt}^{(2)}, \ldots, z_{jt}^{(M)})$) which are uncorrelated with error terms in the utility model ξ_{jt}
- Then the M moment conditions are given by $\mathrm{E}(\mathbf{z}'_{jt}\xi_{jt}) = \mathbf{0}$
- The key insight comes from the fact that the error terms enter the mean utility linearly ($\delta_{jt} = \alpha(-p_{jt}) + \mathbf{x}_{jt}\boldsymbol{\beta} + \xi_{jt}$), and that they only enter the mean utility and hence one can separate out the ξ_{jt} terms to compute the moment conditions above

$$\frac{1}{J}\sum_{j} z_{jt}^{(m)}\xi_{jt} = \frac{1}{J}\sum_{j} z_{jt}^{(m)}(\delta_{jt} - \mathbf{x}_{jt}\boldsymbol{\beta} + \alpha p_{jt}) \tag{88}$$

- Thus want to estimate the parameters $\alpha, \boldsymbol{\beta}$ that minimize the sample moment conditions (or rather their weighted sum of squares)
- But since we cannot observe δ_{jt} we cannot proceed as is
- Berry (1994) suggests a two-step approach: first obtain an estimate of δ_{jt}, – call it $\widehat{\delta}_{jt}$ – and insert it into the moment conditions above, and second, search for values of $\alpha, \boldsymbol{\beta}$ that minimize the weighted sum of squares of these moment conditions

(1) Figure out the values of δ_{jt}

(A) If we normalize $\delta_{0t} = 0$ and equate the observed shares to the model predicted shares, then we have J non-linear equations per market – see logit share equation (57) – in J unknowns

$$\begin{aligned} s_{1t} &= \tilde{s}_{1t}(\delta_{1t}, \ldots, \delta_{Jt}) \\ s_{2t} &= \tilde{s}_{2t}(\delta_{1t}, \ldots, \delta_{Jt}) \\ &\vdots \\ s_{Jt} &= \tilde{s}_{jt}(\delta_{1t}, \ldots, \delta_{Jt}) \end{aligned} \tag{89}$$

(B) If we can invert this system, we can solve for $\delta_{1t}, \delta_{2t}, \ldots, \delta_{jt}$ as a function of observed shares $s_{1t}, s_{2t}, \ldots, s_{jt}$.

(C) Thus, we now have $\widehat{\delta}_{jt} \equiv \tilde{s}_{jt}^{-1}(s_{1t}, s_{2t}, \ldots, s_{Jt})$, J numbers per market which we can use to carry out step 2 (in the simple logit case, $\widehat{\delta}_{jt} = \ln(s_{jt}) - \ln(s_{0t})$)

(2) With the estimated values of δ_{jt}, use GMM to estimate parameters (in this case, α and $\boldsymbol{\beta}$) so as to minimize (88).

(A) Recall that δ_j is the mean utility of product j defined linearly as $\delta_{jt} = \alpha(-p_{jt}) + \mathbf{x}_{jt}\boldsymbol{\beta} + \xi_{jt}$ for all j,

$$\begin{aligned} \delta_{1t} &= \alpha(-p_{1t}) + \mathbf{x}_{1t}\boldsymbol{\beta} + \xi_{1t} \\ \delta_{2t} &= \alpha(-p_{2t}) + \mathbf{x}_{2t}\boldsymbol{\beta} + \xi_{2t} \\ &\vdots \\ \delta_{Jt} &= \alpha(-p_{Jt}) + \mathbf{x}_{Jt}\boldsymbol{\beta} + \xi_{Jt} \end{aligned} \tag{90}$$

(B) We can now use the estimated values of $\widehat{\delta}_j$ to calculate the sample moments

$$\frac{1}{J}\sum_j z_{jt}^{(m)}\xi_{jt} = \frac{1}{J}\sum_j z_{jt}^{(m)}(\widehat{\delta}_{jt} - \mathbf{x}_{jt}\boldsymbol{\beta} + \alpha p_{jt}) \tag{91}$$

minimize these to calculate the values of α, $\boldsymbol{\beta}$

- In step (1a) above, we equated observed market shares to model predicted market shares
  - In the case of logits, the model predicted market shares take the closed-form (57) given by $\tilde{s}_{jt} = \exp(\delta_{jt}) / \left[1 + \sum_{j=1}^{J} \exp(\delta_{jt})\right]$
  - In other cases, there will be no closed form available to compute the model-predicted market shares and we will need to resort to numerical simulation methods to estimate the model-predicted shares
  - In fact, these may be functions of additional parameters (call them $\boldsymbol{\theta}_2$) – thus, equations (89) will be of the form

$$s_{jt} = \tilde{s}_{jt}(\delta_{1t}, \ldots, \delta_{Jt}, \boldsymbol{\theta}_2) \tag{92}$$

- In steps (1b/1c), we 'inverted' these equations to solve for $\widehat{\delta}_{jt}$
  - In the case of logit, an analytical solution was available since $\delta_{jt} = \ln s_{jt} - \ln s_{0t}$
  - More generally, these equations are nonlinear and need to be solved numerically
  - Berry/BLP suggest a contraction mapping (and prove that it converges) for $\boldsymbol{\delta}_t$ given by

$$\boldsymbol{\delta}_t^{h+1} = \boldsymbol{\delta}_t^{h} + \left[\ln(\mathbf{s}_t) - \ln(\tilde{\mathbf{s}}_t(\boldsymbol{\delta}_t^{h}; \boldsymbol{\theta}_2))\right] \tag{93}$$

where $\mathbf{s}_t(\cdot)$ is the observed market share, $\tilde{\mathbf{s}}_t(\cdot)$ is the model predicted market share at mean utility $\boldsymbol{\delta}_t^{h}$ at iteration h and $||\boldsymbol{\delta}_t^{h+1} - \boldsymbol{\delta}_t^{h}||$ is below some tolerance level

- To sum up, Berry's (1994) two-step GMM approach with a matrix of instruments $\mathbf{Z}$ is as follows:
  (1) Compute $\widehat{\delta}_{jt}$
  - Without loss of generality, subsume p_{jt} within $\mathbf{x}_{jt}$ as just another column (a special attribute of product J), and rather than introduce new (unnecessary) notation, redefine $\mathbf{x}_{jt} = \begin{bmatrix} -p_{jt} & \mathbf{x}_{jt} \end{bmatrix}$ – similarly, redefine matrix $\mathbf{X}$ to be inclusive of the price vector so that $\mathbf{X} = \begin{bmatrix} \mathbf{p} & \mathbf{X} \end{bmatrix}$. Also, let $\mathbf{s}_t$ be the vector of observed shares and $\boldsymbol{\theta}_1 = \begin{bmatrix} \alpha & \boldsymbol{\beta}' \end{bmatrix}'$
  - Conveniently, $\widehat{\delta}_{jt} = \ln(s_{jt}) - \ln(s_{0t})$ (in the case of simple logit) and $\widehat{\boldsymbol{\delta}} = \ln(\mathbf{s}) - \ln(\mathbf{s}_0)$
  - Then $\xi_{jt}(\boldsymbol{\theta}_1) = \widehat{\delta}_{jt}(\mathbf{s}_t) - \mathbf{x}_{jt}\boldsymbol{\theta}_1$ – and in matrix notation, $\boldsymbol{\xi}(\boldsymbol{\theta}_1) = \widehat{\boldsymbol{\delta}} - \mathbf{X}\boldsymbol{\theta}_1$

  (2) Define the moment conditions as $\mathrm{E}(\mathbf{Z}'\boldsymbol{\xi}(\boldsymbol{\theta}_1)) = \mathbf{0}$
  - Next, $\min_{\boldsymbol{\theta}_1} \boldsymbol{\xi}(\boldsymbol{\theta}_1)'\mathbf{Z}\boldsymbol{\Phi}\mathbf{Z}'\boldsymbol{\xi}(\boldsymbol{\theta}_1)$ where $\boldsymbol{\Phi} = (\mathrm{E}[\mathbf{Z}'\boldsymbol{\xi}\boldsymbol{\xi}'\mathbf{Z}])^{-1}$
  - In the case of logit, we have an analytical solution – see equation (81) in the GMM section, and replace $\mathbf{y}$ in that equation with $\widehat{\boldsymbol{\delta}}$: $\widehat{\boldsymbol{\theta}}_1 = (\mathbf{X}'\mathbf{Z}\boldsymbol{\Phi}\mathbf{Z}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Z}\boldsymbol{\Phi}\mathbf{Z}'\widehat{\boldsymbol{\delta}}$
  - Since we don't know $\boldsymbol{\Phi}$, we start with $\boldsymbol{\Phi} = \mathbf{I}$ or $\boldsymbol{\Phi} = (\mathbf{Z}'\mathbf{Z})^{-1}$, get an initial estimate of $\boldsymbol{\theta}_1$, use this to get residuals, and then recompute $\boldsymbol{\Phi} = (\mathrm{E}[\mathbf{Z}'\boldsymbol{\xi}\boldsymbol{\xi}'\mathbf{Z}])^{-1}$ to get the new estimates of $\boldsymbol{\theta}_1$
- We will use this 2 step approach explicitly in the next model

- Let the utility be given by

$$u_{njt} = \alpha_n(y_n - p_{jt}) + \mathbf{x}_{jt}\boldsymbol{\beta}_n + \xi_{jt} + \epsilon_{njt}, \text{ where} \\ n = 1, \ldots, N, \qquad j = 0 \ldots, J, \qquad t = 1 \ldots, T \tag{94}$$

- where

$$\begin{aligned} \begin{bmatrix} \alpha_n \\ \boldsymbol{\beta}_n \end{bmatrix} &= \underbrace{\begin{bmatrix} \alpha \\ \boldsymbol{\beta} \end{bmatrix}}_{\boldsymbol{\theta}_1} + \underbrace{\boldsymbol{\Pi} \boldsymbol{d}_n + \boldsymbol{\Sigma} \boldsymbol{\nu}_n}_{\boldsymbol{\theta}_2 = \{\boldsymbol{\Pi}, \boldsymbol{\Sigma}\}} \\ &= \begin{bmatrix} \alpha \\ \boldsymbol{\beta} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Pi}_\alpha \\ \boldsymbol{\Pi}_{\boldsymbol{\beta}} \end{bmatrix} \boldsymbol{d}_n + \begin{bmatrix} \boldsymbol{\Sigma}_\alpha \\ \boldsymbol{\Sigma}_{\boldsymbol{\beta}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\nu}_{n\alpha} & \boldsymbol{\nu}_{\boldsymbol{n\beta}} \end{bmatrix} \end{aligned} \tag{95}$$

- and where

$$\boldsymbol{d}_n \sim F_{\boldsymbol{d}}(\boldsymbol{d}) \qquad \boldsymbol{\nu}_n \sim F_{\boldsymbol{\nu}}(\boldsymbol{\nu}) \tag{96}$$

  - note that the person-specific coefficients are equal to the <u>mean</u> value of the parameters $\boldsymbol{\theta}_1 = \begin{bmatrix} \alpha & \boldsymbol{\beta}' \end{bmatrix}'$, plus deviation from the mean due to a second set of parameters $\boldsymbol{\theta}_2 = \{\boldsymbol{\Pi}, \boldsymbol{\Sigma}\}$ and given by $\boldsymbol{\Pi}\boldsymbol{d}_n + \boldsymbol{\Sigma}\boldsymbol{\nu}_n$
  - each consumer is assumed to have a fixed set of coefficients $\{\alpha_n, \boldsymbol{\beta_n}\}$
  - we do not impose the restriction that taste parameters $\{\alpha, \boldsymbol{\beta}\}$ – the marginal utilities of product characteristics – are the same for all consumers
  - the person-specific coefficients are modeled as a function of underlying common parameters $\{\boldsymbol{\Pi} \text{ and } \boldsymbol{\Sigma}\}$ that are multiplied to the person-specific characteristics $(\boldsymbol{d}_n, \boldsymbol{\nu}_n)$, each of which is random draws from an underlying mean zero population with distribution functions $F_{\boldsymbol{d}}(\boldsymbol{d})$ and $F_{\boldsymbol{\nu}}(\boldsymbol{\nu})$

Random Coefficients Logit

Heterogenous Tastes

- Let π_{ab} and σ_{ef} be the terms of $\mathbf{\Pi}$ and $\mathbf{\Sigma}$ respectively and let ($\boldsymbol{d}_n = (d_{1n}, \ldots, d_{5n})'$) be the five demographics of the n^{th} person recorded as deviation from the population mean values – then

$$\begin{aligned} \alpha_n = \alpha \quad & +\pi_{11} d_{1n} + \pi_{12} d_{2n} + \ldots + \pi_{15} d_{5n} \\ & +\sigma_{11} v_{1n} + \sigma_{12} v_{2n} + \ldots + \sigma_{14} v_{4n} \\ \beta_{kn} = \beta_k \quad & +\pi_{k1} d_{1n} + \pi_{k2} d_{2n} + \ldots + \pi_{k5} d_{5n} \\ & +\sigma_{k1} v_{1n} + \sigma_{k2} v_{2n} + \ldots + \sigma_{k4} v_{4n} \end{aligned} \tag{97}$$

- If there are D person specific observed characteristics ($\boldsymbol{d}_n = (d_{1n}, \ldots, d_{Dn})'$) and $k-1$ product characteristics, then $\mathbf{\Pi}$ is a $k \times D$ and Σ is a $k \times k$ matrix of parameters, i.e.,

$$\underbrace{\begin{bmatrix} \alpha_n \\ \boldsymbol{\beta}_n \end{bmatrix}}_{k \times 1} = \underbrace{\begin{bmatrix} \alpha \\ \boldsymbol{\beta} \end{bmatrix}}_{k \times 1} + \underbrace{\mathbf{\Pi} \boldsymbol{d}_n}_{k \times D \text{ by } D \times 1} + \underbrace{\mathbf{\Sigma} \boldsymbol{\nu}_n}_{k \times k \text{ by } k \times 1} \tag{98}$$

  - suppose there are three observed product characteristics (so $k-1=3$)
  - five observed person-specific characteristics so that $\begin{bmatrix} \alpha & \boldsymbol{\beta}' \end{bmatrix}'$ is a 4×1 vector (the additional dimension is for price) and $\boldsymbol{d}_n$ is a 5×1 vector
  - $\boldsymbol{\nu}_n$ is also a 4×1 vector – these are the person specific random error terms that provide part of the deviation from the mean values of $\begin{bmatrix} \alpha & \boldsymbol{\beta}' \end{bmatrix}'$
  - Then $\mathbf{\Pi}$ is 4×5 matrix (20 parameters) and $\mathbf{\Sigma}$ is a 4×4 matrix (16 parameters) and so the total number of parameters affecting the utility function are 4 + 20 + 16 = 40

- If we insert (95) back into (94) and simplify, then the utility function can be decomposed into three parts (or four, if we count $\alpha_n y_n$ term, but it drops out later on)

$$
\begin{aligned}
&u_{njt} = \alpha_n y_n + \delta_{jt} + \mu_{njt} + \epsilon_{njt} \\
&\text{where,} \\
&\delta_{jt} = \delta(\mathbf{x}_{jt}, p_{jt}, \xi_{jt}; \boldsymbol{\theta}_1) = \alpha(-p_{jt}) + \mathbf{x}_{jt}\boldsymbol{\beta} + \xi_{jt} \\
&\mu_{njt} = \mu(\mathbf{x}_{jt}, p_{jt}, \boldsymbol{d}_n, \boldsymbol{\nu}_n; \boldsymbol{\theta}_2) = (-p_{jt}, \mathbf{x}_{jt})(\boldsymbol{\Pi}\boldsymbol{d}_n + \boldsymbol{\Sigma}\boldsymbol{\nu}_n)
\end{aligned}
\tag{99}
$$

- Note the following
  - except for the μ_{njt} term, which arises due to multiplication of $(\boldsymbol{\Pi}\boldsymbol{d}_n + \boldsymbol{\Sigma}\boldsymbol{\nu}_n)$ with the observed product characteristics, the rest of the form is the same as in the logit case
  - as before, $\alpha_n y_n$ will drop out of the model, δ_{jt} is the mean utility of product j and is common to all consumers
  - $\mu_{njt} + \epsilon_{njt}$ is the mean-zero heteroscedastic error term that captures the deviation from the mean utility
  - it is this last composite error term $\mu_{njt} + \epsilon_{njt}$, that allows us to break away from the IIA property

- Utility can be written as

$$
\begin{aligned}
&u_{njt} = \alpha_n y_n + \delta_{jt} + \mu_{njt} + \epsilon_{njt} \\
&\text{where,} \\
&\delta_{jt} = \delta(\mathbf{x}_{jt}, p_{jt}, \xi_{jt}; \boldsymbol{\theta}_1) = \alpha(-p_{jt}) + \mathbf{x}_{jt}\boldsymbol{\beta} + \xi_{jt} \\
&\mu_{njt} = \mu(\mathbf{x}_{jt}, p_{jt}, \boldsymbol{d}_n, \boldsymbol{\nu}_n; \boldsymbol{\theta}_2) = (-p_{jt}, \mathbf{x}_{jt})(\boldsymbol{\Pi}\boldsymbol{d}_n + \boldsymbol{\Sigma}\boldsymbol{\nu}_n)
\end{aligned}
\tag{99}
$$

- Recall that in the logit model, the IIA property was arising due to the independence of the error terms ϵ_{njt}
  - One way around this problem is to allow these error terms to be correlated across different brands – and in principle, one can allow a completely unrestricted variance-covariance matrix for the shocks ϵ_{njt} – leads to the dimensionality problem (all pair-wise covariances between products and variances of each of the J products)
  - The nested logit took a restricted version of this by imposing some structure on the error terms so that all products within a group have a correlation between them but not with those in other groups
- In the current context, we retain the iid extreme value distribution assumption on ϵ_{njt}, but the correlation among the choices is generated via the μ_{njt} component of the composite error term $\mu_{njt} + \epsilon_{njt}$
  - Correlation between the utility of different products is a function of both product and consumer attributes so that products with similar characteristics will have similar rankings and consumers with similar demographics will also have similar rankings of products ($\mu_{njt} = (-p_{jt}, \mathbf{x}_{jt})(\boldsymbol{\Pi}\boldsymbol{d}_n + \boldsymbol{\Sigma}\boldsymbol{\nu}_n)$)
  - Rather than estimate a large number of parameters of a completely unrestricted variance-covariance matrix for ϵ_{njt}, we need to estimate relatively fewer parameters $\boldsymbol{\theta}_1 = (\alpha,\ \boldsymbol{\beta})'$, $\boldsymbol{\theta}_2 = \{\boldsymbol{\Pi}, \boldsymbol{\Sigma}\}$

- Utility of product j for two different consumers differs only by $\mu_{njt} + \epsilon_{njt}$ (see (99) – $u_{njt} = \alpha_n y_n + \delta_{jt} + \mu_{njt} + \epsilon_{njt}$)
  - the δ_j term is the same for all consumers and $\alpha_n y_n$ is the same for all choices
  - hence the fact that one consumer chooses product j while another chooses product i must only be because the two consumers differ in their product-specific idiosyncratic error terms $\mu_{njt} + \epsilon_{njt}$
- Hence, we can describe each consumer as a tuple of demographic and product-specific shocks $(\boldsymbol{d}_n, \boldsymbol{\nu}_n, \epsilon_{n0t}, \epsilon_{n1t}, \ldots, \epsilon_{nJt})$, which implicity defines the set of individual attributes that choose product j given by

$$\mathbb{A}_{jt}(\mathbf{x}_t, \mathbf{p}_t, \boldsymbol{\delta}_t(\mathbf{x}_t, \mathbf{p}_t; \boldsymbol{\theta}_1); \boldsymbol{\theta}_2) = \{(\boldsymbol{d}_{nt}, \boldsymbol{\nu}_{nt}, \epsilon_{n0t}, \epsilon_{n1t}, \ldots, \epsilon_{nJt}) \mid u_{njt} > u_{nlt} \\ \forall\, l = 0, 1, 2 \ldots J,\ l \neq j\}. \tag{100}$$

- The market share of product j is the integral of the joint distribution of $(\boldsymbol{d}, \boldsymbol{\nu}, \boldsymbol{\epsilon})$ over the mass of individuals in the region A_{jt},

$$s_{jt} = \int_{\mathbb{A}_{jt}} dF(\boldsymbol{d}, \boldsymbol{\nu}, \boldsymbol{\epsilon}) = \int_{\mathbb{A}_{jt}} dF_{\boldsymbol{d}}(\boldsymbol{d}) dF_{\boldsymbol{\nu}}(\boldsymbol{\nu}) dF_{\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}) \tag{101}$$

  - where the second part follows only if we assume that the three random variables for a given consumer are independently distributed
  - note also that set $\mathbb{A}_{jt}$ is only defined via the parameters $\boldsymbol{\theta}_2 = \{\boldsymbol{\Pi}, \boldsymbol{\Sigma}\}$, since they were part of the μ_{njt} term, and not over the parameters $\boldsymbol{\theta}_1$

- Unlike the logit case, the integral does not have a closed form
- If we continue to assume that ϵ_{njt} has iid extreme value distribution, then the probability that a given individual $\tilde{n}$ – with endowed values of $\tilde{\boldsymbol{d}}_n$ and $\tilde{\boldsymbol{\nu}}_n$, or equivalently with a given value of $\tilde{\mu}_{njt}$ – chooses product j, continues to have a closed logit form like the equation 5.6 and in this case is given by

$$s_{njt} = \frac{\exp(\delta_{jt} + \tilde{\mu}_{njt})}{\sum_{j=0}^{J} \exp(\delta_{jt} + \tilde{\mu}_{njt})} \tag{102}$$

- Since $\mu_{njt} = \mu(\mathbf{x}_{jt}, p_{jt}, \boldsymbol{d}_n, \boldsymbol{\nu}_n; \boldsymbol{\theta}_2)$, we can integrate individual probability over the distribution of $\boldsymbol{d}_n$ and $\boldsymbol{\nu}_n$ to recover market share of product j

$$\begin{aligned} s_{jt} &= \int_{\mathbb{A}_{jt}} s_{njt} dF_{\boldsymbol{d}}(\boldsymbol{d}) dF_{\boldsymbol{\nu}}(\boldsymbol{\nu}) \\ &= \int_{\mathbb{A}_{jt}} \left\{ \frac{\exp(\delta_{jt} + \mu_{njt})}{\sum_{j=0}^{J} \exp(\delta_{jt} + \mu_{njt})} \right\} dF_{\boldsymbol{d}}(\boldsymbol{d}) dF_{\boldsymbol{\nu}}(\boldsymbol{\nu}) \end{aligned} \tag{103}$$

- Price elasticities of market shares are given by

$$\eta_{jkt} = \frac{\partial s_{jt}}{\partial p_{kt}} \frac{p_{kt}}{s_{jt}} = \begin{cases} -\frac{p_{jt}}{s_{jt}} \int_{\mathbb{A}_{jt}} \alpha_n s_{njt}(1 - s_{njt}) dF_{\boldsymbol{d}}(\boldsymbol{d}) dF_{\boldsymbol{\nu}}(\boldsymbol{\nu}) & \text{if } j = k, \\ \frac{p_{kt}}{s_{jt}} \int_{\mathbb{A}_{jt}} \alpha_n s_{njt} s_{nkt} dF_{\boldsymbol{d}}(\boldsymbol{d}) dF_{\boldsymbol{\nu}}(\boldsymbol{\nu}) & \text{otherwise} \end{cases} \tag{104}$$

  where $s_{njt} = \frac{\exp(\delta_{jt} + \tilde{\mu}_{njt})}{\sum_{j=0}^{J} \exp(\delta_{jt} + \tilde{\mu}_{njt})}$
- The main advantage of this model is that estimation requires estimation of a handful of parameters (rather than the square of the number of parameters), elasticities do not exhibit the problems noted earlier for the logit (own or cross-elasticities) and allows us to model consumer heterogeneity rather than rely on a representative consumer
- Compare to the earlier elasticities from the logit model

$$\eta_{jkt} = \frac{\partial s_{jt}}{\partial p_{kt}} \frac{p_{kt}}{s_{jt}} = \begin{cases} -\alpha p_{jt}(1 - s_{jt}) & \text{if } j = k, \\ \alpha p_{kt} s_{kt} & \text{otherwise} \end{cases} \tag{60}$$

- Nothing comes for free ... now we must integrate the expression numerically

- Let x be some arbitrary random variable[¶] with a probability distribution $f(x) = dF(x)/dx \rightarrow dF(x) = f(x)dx$
  - then note that the integral – $\int x \cdot f(x)dx$ – is just the expected value of x, i.e., $\mathrm{E}[x] = \int x \cdot dF(x)$
  - the sample analog would be the weighted average of x given by $\bar{x} = \sum_n x_n Pr(x_n)$
  - further, if all values are equally possible, then it is just the simple sample average $\bar{x} = (1/N) \sum_n x_n$
- The idea carries over to any function $g(x)$ defined over x such that
  - $\mathrm{E}[g(x)] = \int g(x) \cdot dF(x)$
  - and the sample analog would be $\overline{g(x)} = \sum_n g(x_n) Pr(x_n)$
- Thus, if we wanted to numerically evaluate the integral of $g(x)$ with a known distribution of x (i.e., evaluate $\int g(x) \cdot dF(x)$), all we need to do is
  - take lots of draws of x from this known distribution
  - evaluate $g(x)$ at each of these points
  - and then just take a simple average of all these values of $g(x)$
  - we will get a pretty good value of the integral by this method if we have taken enough *good* draws of the random variable x

[¶]This x has nothing to do with the earlier characteristic vector $\mathbf{x}_{jt}$

- Consider the case where x is distributed between 0 and 3 such that the probabilities of draws are
  - $\Pr(0 \leq x < 1) = .45$,
  - $\Pr(1 \leq x < 2) = .10$, and
  - $\Pr(2 \leq x < 3) = .45$
- If we drew 100 random numbers from this distribution, we would expect about 45 of them to be between 0 and 1, another 10 observations between 1 and 2, and 45 observations between 2 and 3
  - If that were the case, we could safely evaluate $g(x)$ at each of these 100 random draws and take their average to compute $\mathrm{E}[g(x)] = \int g(x) \cdot dF(x)$
  - If on the other hand, we find that the drawing sequence (algorithm) is such that for the first 100 draws, we have 1/3 of observations from each of the three regions, then with just 100 draws, average values of $g(x)$ will give a very poor (if not outright wrong) approximation to the integral in question
- There is a large literature on drawing from different types of random distributions, for a good review of basic techniques, see chapter 9 in Train

- To compute the integral in (103), we need to know the distribution functions $F_{\boldsymbol{d}}(\boldsymbol{d})$ and $F_{\boldsymbol{\nu}}(\boldsymbol{\nu})$ and draw from these distributions
- Drawing from $F_{\boldsymbol{d}}(\boldsymbol{d})$
  - note that $\boldsymbol{d}_n$ is the vector of demographics for consumer n (income, family size, age, gender, etc.)
  - one way to proceed is to make use of other data sources, such as the census data, to construct a non-parametric distribution. We can then take random draws from this distribution to compute the integral above
  - in practice one can directly draw N number of consumers – where N is a reasonably large number – from each of the t markets and record their demographic information
  - thus, let us assume that $\boldsymbol{d}_n$ is a 5×1 vector of demographics, and that we have obtained N_s random draws from each market and recorded the values of these demographics
- Drawing from $F_{\boldsymbol{\nu}}(\boldsymbol{\nu})$
  - recall that if $\mathbf{x}_{jt}$ is a vector of three observed characteristics $(k - 1 = 3)$ for product j, then for each person, $\boldsymbol{\nu}_n$ is a 4×1 (or more generally $k \times 1$) vector of random error terms that provide part of the deviation from the mean values of $\begin{bmatrix}\alpha & \boldsymbol{\beta}'\end{bmatrix}'$
  - researchers often specify $F_{\boldsymbol{\nu}}(\boldsymbol{\nu})$ as standard multivariate normal and take N draws per market to obtain $\boldsymbol{\nu}_n$
  - let us again assume that with the help of a good random number generator, we have taken N_s such draws per market and have recorded a series of 4×1 vectors for each person

- Given the values of the parameters $\boldsymbol{\theta}_2 = \{\boldsymbol{\Pi}, \boldsymbol{\Sigma}\}$, a value of mean utility δ_{jt} and N_s random values of $\boldsymbol{d}_n$ and $\boldsymbol{\nu}_n$, the predicted market share of good j can be computed using the smooth simulator as the average value of s_{njt} over the N_s observations,

$$\begin{aligned}\tilde{s}_{jt} &= \int_{A_{jt}} s_{njt} dF_{\boldsymbol{d}}(\boldsymbol{d}) dF_{\boldsymbol{\nu}}(\boldsymbol{\nu}) \\ &= \frac{1}{N_s} \sum_{n}^{N_s} s_{njt} = \frac{1}{N_s} \sum_{n}^{N_s} \left\{ \frac{\exp(\delta_{jt} + \mu_{njt})}{\sum_{j=0}^{J} \exp(\delta_{jt} + \mu_{njt})} \right\} \end{aligned} \tag{105}$$

where $\mu_{njt} = (-p_{jt}, \mathbf{x}_{jt})(\boldsymbol{\Pi} \boldsymbol{d}_n + \boldsymbol{\Sigma} \boldsymbol{\nu}_n)$

- Recall from earlier example (5 demographics and 3+1 product characteristics), there were 40 parameters to estimate
- Data may not allow estimation of such a rich set of parameters
  - BLP does not use individual demographics to create variation in person specific coefficients
  - equivalently, the $k \times d$ matrix $\mathbf{\Pi}$ consists of zeros and the variation in $\begin{bmatrix}\alpha_n & \boldsymbol{\beta}_n'\end{bmatrix}'$ is only due to $\mathbf{\Sigma}\boldsymbol{\nu}_n$
  - Nevo sets only some of the terms of $\mathbf{\Pi}$ to zero and estimates the other coefficients
  - Often researchers set $\mathbf{\Sigma}$ as a diagonal matrix and estimate only the leading terms of this matrix
  - this is not as restrictive as it may appear at first pass

- To understand the logic of choosing parameters that are set to zero, and the implications, consider a very simple example where there is only one observed characteristic of each product, plus price, so that $\begin{bmatrix}\alpha_n & \boldsymbol{\beta}_n'\end{bmatrix}'$ is just a 2×1 column vector instead of $k \times 1$
  - just to be clear, in what follows in the next couple of paragraphs, think of $\boldsymbol{\beta}_n$ and $\boldsymbol{\beta}$ as just 1×1 scalars even though I continue to write them in bold font for vectors
  - Further, suppose that all the elements of $\mathbf{\Pi}$ are zero (again, only to simplify the algebra as the main idea carries through with or without $\mathbf{\Pi}$ in the utility function)
- Then sans the $\mathbf{\Pi}\boldsymbol{d}_n$ term

$$\begin{bmatrix}\alpha_n \\ \boldsymbol{\beta}_n\end{bmatrix} = \begin{bmatrix}\alpha \\ \boldsymbol{\beta}\end{bmatrix} + \mathbf{\Sigma}\boldsymbol{\nu}_n = \begin{bmatrix}\alpha \\ \boldsymbol{\beta}\end{bmatrix} + \begin{bmatrix}\sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22}\end{bmatrix}\begin{bmatrix}\nu_{1n} \\ \nu_{2n}\end{bmatrix} \tag{106}$$

- Since $\boldsymbol{\nu}_n$ is a mean zero error term, then

$$\begin{aligned}
\alpha_n &= \alpha + \sigma_{11}\nu_{1n} + \sigma_{12}\nu_{2n} \\
\boldsymbol{\beta}_n &= \boldsymbol{\beta} + \sigma_{21}\nu_{1n} + \sigma_{22}\nu_{2n} \\
\mathrm{E}[\alpha_n] &= \alpha \qquad \mathrm{E}[\boldsymbol{\beta}_n] = \boldsymbol{\beta} \\
\mathrm{Var}[\alpha_n] &= \sigma_{11}^2\mathrm{Var}[\nu_{1n}] + 2\sigma_{11}\sigma_{12}\mathrm{Cov}[\nu_{1n}, \nu_{2n}] + \sigma_{12}^2\mathrm{Var}[\nu_{2n}] \\
\mathrm{Var}[\boldsymbol{\beta}_{\boldsymbol{n}}] &= \sigma_{21}^2\mathrm{Var}[\nu_{1n}] + 2\sigma_{21}\sigma_{22}\mathrm{Cov}[\nu_{1n}, \nu_{2n}] + \sigma_{22}^2\mathrm{Var}[\nu_{2n}]
\end{aligned} \tag{107}$$

- Since $\boldsymbol{\nu}_n$ is a mean zero error term, then

$$\begin{aligned}
\alpha_n &= \alpha + \sigma_{11}\nu_{1n} + \sigma_{12}\nu_{2n} \\
\boldsymbol{\beta}_n &= \boldsymbol{\beta} + \sigma_{21}\nu_{1n} + \sigma_{22}\nu_{2n} \\
\mathrm{E}[\alpha_n] &= \alpha \qquad \mathrm{E}[\boldsymbol{\beta}_n] = \boldsymbol{\beta} \\
\mathrm{Var}[\alpha_n] &= \sigma_{11}^2\mathrm{Var}[\nu_{1n}] + 2\sigma_{11}\sigma_{12}\mathrm{Cov}[\nu_{1n}, \nu_{2n}] + \sigma_{12}^2\mathrm{Var}[\nu_{2n}] \\
\mathrm{Var}[\boldsymbol{\beta}_{\boldsymbol{n}}] &= \sigma_{21}^2\mathrm{Var}[\nu_{1n}] + 2\sigma_{21}\sigma_{22}\mathrm{Cov}[\nu_{1n}, \nu_{2n}] + \sigma_{22}^2\mathrm{Var}[\nu_{2n}]
\end{aligned} \tag{107}$$

- Implications of setting the off-diagonal terms in $\mathbf{\Sigma}$ to zero: if $\sigma_{12} = \sigma_{21} = 0$, then
  - α_n is a deviation from the mean value of α and the deviation is determined only by a random shock ν_{1n} multiplied by a coefficient σ_{11}
  - the shock to the marginal utility of the second characteristic ν_{2n}, does not affect the deviation from the mean for the first characteristics, i.e., the marginal (dis)utility of price
  - put another way, the unobserved heterogeneity has been modeled such that if the price and speed of a computer are the only two characteristics in consideration, and a given person gets a positive shock to the marginal utility of speed (they get more utility from the speed of computer relative to another person), it does not imply that they also get a higher (dis)utility from the price of the computer due to the higher utility from speed
  - the (dis)utility from price is equal to α plus a person specific deviation only for price $\sigma_{11}\nu_{1n}$
  - similarly, variances of α_n and $\boldsymbol{\beta}_n$ depend on the variances of the shocks of these characteristics (e.g. $\mathrm{Var}[\alpha_n] = \sigma_{11}^2\mathrm{Var}[\nu_{1n}]$) but not on the *covariance* of the shocks, even if $\mathrm{Cov}[\nu_{1n}, \nu_{2n}] \neq 0$, since $\sigma_{12} = \sigma_{21} = 0$

- Next, consider the covariance between α_n and $\boldsymbol{\beta}_n$
- Covariance between the two random variables is defined as $\text{Cov}(\alpha_n, \boldsymbol{\beta}_n) = \text{E}\left[\{\alpha_n - \text{E}(\alpha_n)\}\{\boldsymbol{\beta}_n - \text{E}(\boldsymbol{\beta}_n)\}\right]$ hence

$$\begin{aligned}\text{Cov}(\alpha_n, \boldsymbol{\beta}_n) &= \text{E}(\alpha_n \boldsymbol{\beta}_n) - \alpha\boldsymbol{\beta} \\ &= \sigma_{11}\sigma_{21}\text{Var}(\nu_{1n}) + \sigma_{12}\sigma_{22}\text{Var}(\nu_{2n}) \\ &+ \sigma_{11}\sigma_{22}\text{Cov}(\nu_{1n}, \nu_{2n}) + \sigma_{12}\sigma_{21}\text{Cov}(\nu_{1n}, \nu_{2n}) \\ &= \sigma_{11}\sigma_{22}\text{Cov}(\nu_{1n}, \nu_{2n}).\end{aligned} \tag{108}$$

  - the first line is due to the definition of a covariance and the observation that $\text{E}[\alpha_n] = \alpha$ and $\text{E}[\boldsymbol{\beta}_n] = \boldsymbol{\beta}$
  - the second line follows from substituting values of α_n and $\boldsymbol{\beta}_n$ from equation (107), taking the expectations, setting $\text{E}[\boldsymbol{\nu}_n] = \mathbf{0}$ and simplifying
  - the last line is if we set $\sigma_{12} = \sigma_{21} = 0$ and shows that even after setting the off-diagonals in $\boldsymbol{\Sigma}$ equal to zero, the covariance between the marginal utilities is not necessarily zero – *unless we now further assume that the mean zero error terms $\boldsymbol{\nu}_n$ are not correlated across the characteristics*

- Common to assume that $\boldsymbol{\nu}_n$ are drawn from multivariate standard normal or log-normal, i.e., covariances between the error terms are zero as well
  - In the special case where the terms of $\mathbf{\Pi}$ are also zero – as in the foregoing discussion – this implies that covariance between marginal utilities will also be zero
- However, if the terms of $\mathbf{\Pi}$ are not all zero, they will still invoke correlations between the marginal utilities of different characteristics
  - as equation (97), reproduced below for this special case of two characteristics and five demographics shows

$$\begin{aligned}\alpha_n = \alpha \quad & +\pi_{11}d_{1n} + \pi_{12}d_{2n} + \ldots + \pi_{15}d_{5n} \\ & \qquad +\sigma_{11}\nu_{1n} + \sigma_{12}\nu_{2n} \\ \boldsymbol{\beta}_n = \boldsymbol{\beta} \quad & +\pi_{21}d_{1n} + \pi_{22}d_{2n} + \ldots + \pi_{25}d_{5n} \\ & \qquad +\sigma_{21}\nu_{1n} + \sigma_{22}\nu_{2n}\end{aligned} \tag{97}$$

  - in this case, the covariance between α_n and $\boldsymbol{\beta_n}$ will be invoked via the π terms and the covariances between the demographic variables, even if we set $\sigma_{12} = \sigma_{21} = 0$ and choose the distribution of $\boldsymbol{\nu}_n$ to be multivariate standard normal
- Thus as mentioned earlier, if we use demographic data and don't set the $\mathbf{\Pi}$ to zero (at least not all terms) then setting the off diagonals of $\mathbf{\Sigma}$ to zero and drawing $\boldsymbol{\nu}_n$ from multivariate standard normal is not so restrictive

- The essential idea of estimation remains the same as that of a two-step estimation outlined in the section on logits
- Briefly,
  - estimate mean utility δ_{jt} and then use it in the second step to estimate the moment functions and find parameters that minimize the value
  - this requires first estimating model predicted market shares via (103), equating them to observed market shares, and then inverting the relation and using a contraction mapping to compute δ_{jt}
- We consider each of these along the way and following Nevo (2001), combine everything in a 5-step algorithm

(-1) For each market t, draw N_s random values for $(\boldsymbol{\nu}_n, \boldsymbol{d}_n)$ from the distributions $F_{\boldsymbol{\nu}}(\boldsymbol{\nu})$ and $F_{\boldsymbol{d}}(\boldsymbol{d})$

- the distribution $F_{\boldsymbol{d}}(\boldsymbol{d})$ can be estimated using census data
- for $F_{\boldsymbol{\nu}}(\boldsymbol{\nu})$ we can use zero mean multivariate normal with a pre-specified covariance matrix

(0) Select arbitrary initial values of δ_{jt} and $\boldsymbol{\theta}_2 = \{\boldsymbol{\Pi}, \boldsymbol{\Sigma}\}$ and for $\boldsymbol{\theta}_1$

- for $\boldsymbol{\theta}_1 = \begin{bmatrix} \alpha & \boldsymbol{\beta}' \end{bmatrix}'$ use initial values from simple logit estimation

(1) Use random draws and the initial parameter values to estimate the model predicted market shares $\tilde{s}_{jt}$ of each product in each market

- use (105) to compute these shares

(2) Obtain $\widehat{\delta}_{jt}$

(A) Keep $\boldsymbol{\theta}_2 = \{\mathbf{\Pi}, \mathbf{\Sigma}\}$ fixed and change values of δ_{jt} until predicted shares $\tilde{s}_{jt}$ in step above, equal the observed shares – this is the inversion step where we want to find $\boldsymbol{\delta}_t$ such that $s_{jt} = \tilde{s}_{jt}(\delta_{1t}, \ldots, \delta_{Jt}, \boldsymbol{\theta}_2)$ in each market

(B) This can be done using the contraction mapping $\boldsymbol{\delta}_t^{h+1} = \boldsymbol{\delta}_t^h + [ln(\mathbf{s}_t) - ln(\tilde{\mathbf{s}}_t)]$

(C) Note carefully that mean utility is a function of observed market shares and parameters $\boldsymbol{\theta}_2$ thus, $\delta_{jt} = \delta_{jt}(\mathbf{s}_t, \boldsymbol{\theta}_2)$

(3) Define error term as $\xi_{jt} = \widehat{\delta}_{jt}(\mathbf{s}_t, \boldsymbol{\theta}_2) + \alpha p_{jt} - \mathbf{x}_{jt}\boldsymbol{\beta}$ and calculate the value of the moment condition, i.e., the GMM objective function

- (A) As before, subsume p_{jt} within $\mathbf{x}_{jt}$ as just another column of $\mathbf{x}_{jt}$ and redefine $\mathbf{x}_{jt} = \begin{bmatrix} -p_{jt} & \mathbf{x}_{jt} \end{bmatrix}$; similarly, redefine matrix $\mathbf{X}$ to be inclusive of the price vector so that $\mathbf{X} = \begin{bmatrix} -\mathbf{p} & \mathbf{X} \end{bmatrix}$
- (B) Thus $\xi_{jt}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = \widehat{\delta}_{jt}(\mathbf{s}_t, \boldsymbol{\theta}_2) - \mathbf{x}_{jt}\boldsymbol{\theta}_1$.
  In matrix notation $\boldsymbol{\xi} = \widehat{\boldsymbol{\delta}}(\mathbf{s}, \boldsymbol{\theta}_2) - \mathbf{X}\boldsymbol{\theta}_1$
- (C) Then the objective function to be minimized is
  $\Big(\boldsymbol{\xi}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)'\mathbf{Z}\Big)\boldsymbol{\Phi}\Big(\mathbf{Z}'\boldsymbol{\xi}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)\Big)$,
  where $\boldsymbol{\Phi}$ is the GMM weighting matrix
- (D) Initially set the weighting matrix as $\boldsymbol{\Phi} = (\mathbf{Z}'\mathbf{Z})^{-1}$

(4) Search for better values of $\boldsymbol{\theta}_1 = \begin{bmatrix}\alpha & \boldsymbol{\beta}'\end{bmatrix}'$ and $\boldsymbol{\theta}_2 = \{\boldsymbol{\Pi}, \boldsymbol{\Sigma}\}$ and the GMM weighting matrix $\boldsymbol{\Phi}$ that minimize the objective function as follows:

(A) Note that while $\boldsymbol{\xi}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ is a function of both sets of parameters $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$, it actually partitions into two components: $\xi_{jt}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = \widehat{\delta}_{jt}(\mathbf{s}_t, \boldsymbol{\theta}_2) - \mathbf{x}_{jt}\boldsymbol{\theta}_1$
– this is important because we can help the search algorithm by solving for $\boldsymbol{\theta}_1$, conditional on $\boldsymbol{\theta}_2$ analytically – how? in the GMM objective function given above $[(\boldsymbol{\xi}'\mathbf{Z})\boldsymbol{\Phi}(\mathbf{Z}'\boldsymbol{\xi})]$, set $\boldsymbol{\xi} = \widehat{\boldsymbol{\delta}}(\boldsymbol{\theta}_2) - \mathbf{X}\boldsymbol{\theta}_1$
– now consider the first-order condition with respect to $\boldsymbol{\theta}_1$ and solve for $\boldsymbol{\theta}_1$. See equations 5.31 and 5.32 for FOC and its solution for the GMM estimator
– this implies that if we have some fixed values of $\boldsymbol{\theta}_2$, then $\boldsymbol{\theta}_1$ can be solved for analytically as $\boldsymbol{\theta}_1 = (\mathbf{X}'\mathbf{Z}\boldsymbol{\Phi}\mathbf{Z}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Z}\boldsymbol{\Phi}\mathbf{Z}'\widehat{\boldsymbol{\delta}}(\boldsymbol{\theta}_2)$

(B) Thus, first solve (search) for $\boldsymbol{\theta}_1$ as $\widehat{\boldsymbol{\theta}}_1 = (\mathbf{X}'\mathbf{Z}\boldsymbol{\Phi}\mathbf{Z}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Z}\boldsymbol{\Phi}\mathbf{Z}'\widehat{\boldsymbol{\delta}}(\boldsymbol{\theta}_2)$

(C) Use new $\boldsymbol{\theta}_1 = \begin{bmatrix}\alpha & \boldsymbol{\beta}'\end{bmatrix}'$ to re-compute error term $\boldsymbol{\xi}$ (see 3b above)

(D) Next, update the weighting matrix $\boldsymbol{\Phi}$ as $\boldsymbol{\Phi} = (\mathbf{Z}'\boldsymbol{\xi}\boldsymbol{\xi}'\mathbf{Z})^{-1}$

(E) Take the new value of $\boldsymbol{\Phi}$ and update the GMM objective function, $(\boldsymbol{\xi}'\mathbf{Z})\boldsymbol{\Phi}(\mathbf{Z}'\boldsymbol{\xi})$

(F) Finally, update $\boldsymbol{\theta}_2 = \{\boldsymbol{\Pi}, \boldsymbol{\Sigma}\}$ – do a non-linear search over $\{\boldsymbol{\Pi}, \boldsymbol{\Sigma}\}$ to minimize the objective function

(5) Return to step (1) above with all new shiny parameter values (keep the original draws) and iterate
   – Note that you can skip the updating of the weighting matrix $\mathbf{\Phi}$ in step 4e from now on

- Brand Dummies
  - In the section on logits, we discussed adding in the brand dummies to the vector $\mathbf{x}_{jt}$ and recovering the $\boldsymbol{\beta}$ coefficients for the brand characteristics
  - The same can be done here as well but will need to have two separate versions of data matrix $\mathbf{X}$ (call them $\mathbf{X}_1$ and $\mathbf{X}_2$)
  - Observe that $\mathbf{X}$ (defined to be inclusive of the price vector) enters the utility function twice:
    - in the linear part of the estimation as mean utility $\boldsymbol{\delta}(\mathbf{X};\boldsymbol{\theta}_1) = \mathbf{X}\boldsymbol{\theta}_1 + \boldsymbol{\xi}$ – this is from $\delta_{jt} = \delta(\mathbf{x}_{jt}, p_{jt}, \xi_{jt}; \boldsymbol{\theta}_1) = \alpha(-p_{jt}) + \mathbf{x}_{jt}\boldsymbol{\beta} + \xi_{jt}$
    - and in the non-linear part of the estimation as an individual deviation from the mean utility $\boldsymbol{\mu}_n(\mathbf{X};\boldsymbol{\theta}_2, \boldsymbol{d}_n, \boldsymbol{\nu}_n) = \mathbf{X}(\boldsymbol{\Pi}\boldsymbol{d}_n + \boldsymbol{\Sigma}\boldsymbol{\nu}_n)$ – this follows from $\mu_{njt} = (-p_{jt}, \mathbf{x}_{jt})(\boldsymbol{\Pi}\boldsymbol{d}_n + \boldsymbol{\Sigma}\boldsymbol{\nu}_n)$ – and allows for random coefficients on product characteristics
    - In practice we may not want to allow random coefficients on all characteristics, in which case the data matrix $\mathbf{X}$ appearing in $\boldsymbol{\mu}_n$ can be a subset of the one appearing the linear part $\boldsymbol{\delta}$
  - Thus, we can write the two components as
    $\boldsymbol{\delta}(\mathbf{X}_1;\boldsymbol{\theta}_1) = \mathbf{X}_1\boldsymbol{\theta}_\mathbf{1} + \boldsymbol{\xi}$ and,
    $\boldsymbol{\mu}_n(\mathbf{X}_2;\boldsymbol{\theta}_2, \boldsymbol{d}_n, \boldsymbol{\nu}_n) = \mathbf{X}_2(\boldsymbol{\Pi}\boldsymbol{d}_n + \boldsymbol{\Sigma}\boldsymbol{\nu}_n)$

- Brand Dummies
  - Thus, we can write the two components as
    $\boldsymbol{\delta}(\mathbf{X}_1;\boldsymbol{\theta}_1) = \mathbf{X}_1\boldsymbol{\theta}_\mathbf{1} + \boldsymbol{\xi}$ and,
    $\boldsymbol{\mu}_n(\mathbf{X}_2;\boldsymbol{\theta}_2,\boldsymbol{d}_n,\boldsymbol{\nu}_n) = \mathbf{X}_2(\boldsymbol{\Pi}\boldsymbol{d}_n + \boldsymbol{\Sigma}\boldsymbol{\nu}_n)$
  - $\mathbf{X}_1$ includes all variables that are common to all individuals (price, promotional activities, and brand characteristics <u>or brand dummies</u> instead of brand characteristics)
  - $\mathbf{X}_2$ contains variables that can have random coefficients (price and product characteristics but not brand dummies)
  - Note that if we use $\mathbf{X}_1$ and $\mathbf{X}_2$, then the estimator $\widehat{\boldsymbol{\theta}}_1$ in step 4a/4b above will be
    $\widehat{\boldsymbol{\theta}}_1 = (\mathbf{X}_1'\mathbf{Z}\boldsymbol{\Phi}\mathbf{Z}'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{Z}\boldsymbol{\Phi}\mathbf{Z}'\widehat{\boldsymbol{\delta}}(\boldsymbol{\theta}_2)$

- Additional Instruments
  - The instruments matrix $\mathbf{Z}$ consists of all exogenous variables
  - If the brand characteristics (excluding price) are exogenous, then the brand characteristics plus the instrument(s) for the price variable consist of the matrix $\mathbf{Z}$, or alternatively, if we use brand dummies, then the brand dummies and the price instrument(s) form the matrix $\mathbf{Z}$
  - However, note that if we have only one additional instrument for price, it will not be enough for the identification of the model parameters
    - The brand characteristics (or brand dummies) plus the one additional instrument for price will give *exactly* as many moment conditions as the number of components of the parameter vector $\boldsymbol{\theta}_1$
    - These would be enough in the linear logit case
    - However, in the random coefficients case, we have to estimate additional $k \times D + k \times k$ parameters of $\boldsymbol{\theta}_2 = \{\mathbf{\Pi}, \mathbf{\Sigma}\}$
    - This is not possible unless we have an additional $k \times D + k \times k$ moment conditions
    - In practice, researchers often set some of the terms of the $\mathbf{\Pi}$ matrix to zero and also set the parameter matrix $\mathbf{\Sigma}$ to be diagonal (see earlier discussions)
    - This reduces the need for additional moment conditions from $kD + k^2$ to $g + k$ where g is the number of non-zero terms in $\mathbf{\Pi}$

- Additional Instruments
    - These may be relatively easier to overcome (these instruments should also not be nearly collinear else will give rise to redundant moment conditions)
    - If one is using BLP-style instruments for price (and product characteristics are exogenous) then recall that, in general, one gets more than one instrument for price by using sums of the values of characteristics of other products offered by a firm, and the sums of the values of the same characteristics of products offered by other firms
    - Alternatively, if using Hausman-style instruments, the price of the product from more than one market needs to be used (for instance, Nevo (2001) uses data from 20 quarters and multiple cities and constructs 20 additional instruments from other cities matching one from each quarter)
    - An additional set of instruments could be the average value (average over n individuals) of the product characteristics interacted with the person-specific demographics to account for the parameters in the $\mathbf{\Pi}$ matrix and similarly the average value of the person specific shocks $\boldsymbol{\nu}$ interacted with product characteristics

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APPENDIX

- Separability and Aggregation
- Merger Simulation

- Separability
  - The main method we will look at in the products space approach is one which solves the dimensionality problem by dividing the products into small sub-groups and then allow some relatively flexible substitution patterns between the products within a group
  - Useful if we could break down the overall consumer decision problem into separate parts, some of which could be estimated separately
  - This is the issue of **separability**
  - What assumptions do we need on an individual consumer's utility function to treat and analyze demand for some products separately from the demand for other products?

- Aggregation
  - A related problem is that of **aggregation**, which considers the relationship between individual consumers' behavior and aggregate consumer behavior (which is the sum of individual behavior over all individuals)
  - When working with aggregate data, one can ask whether there are assumptions on preferences such that aggregate demand is generated by a "representative consumer" with "rationalizable" preferences
  - There is no reason why aggregate data, or any data that is an average over many people should conform to a theory of consumer behavior that focuses on individual people or households

- Preferences ($\succeq$)are homothetic if $t\mathbf{q}_1 \succeq t\mathbf{q}_2 \Leftrightarrow \mathbf{q}_1 \succeq \mathbf{q}_2$ for any $t > 0$
  - the consumer is indifferent between bundles $t\mathbf{q}_1$ and $t\mathbf{q}_2$ whenever they are indifferent between bundles $\mathbf{q}_1$ and $\mathbf{q}_2$
  - there is only one indifference curve and any indifference curve is a radial blowup of another and all indifference sets are related by proportional expansion along rays
  - marginal rates of substitution are unaffected by equal proportional changes in all quantities, so that income expansion paths are straight lines through the origin
  - preferences are homothetic if and only if they are of the form

$$u(\mathbf{q}) = F(f(\mathbf{q})) \text{ where } f(t\mathbf{q}) = tf(\mathbf{q}), \tag{109}$$

    and $F(\cdot)$ is a monotone increasing function
  - the utility function must admit a function that is homogenous of degree one (the $f(\cdot)$) and since utility functions are only defined up to monotonic transformations, then we may as well write the utility function to be just $u(\mathbf{q}) = f(\mathbf{q})$ where the latter is, as before, homogeneous of degree one

- Consider the consumer's expenditure minimization problem $\min \mathbf{p} \cdot \mathbf{q}$ s.t. $u(\mathbf{q}) = f(\mathbf{q}) = u$.
  - Since the function is homogenous of degree one, doubling $\mathbf{q}$ will double the target utility, but doubling $\mathbf{q}$ means doubling the expenditure
  - This means that if $e(\mathbf{p}, u) = \mathbf{q}^* \cdot \mathbf{p}$ is the minimum expenditure for target utility u, then for a target utility of tu, the minimum expenditure is $e(\mathbf{p}, tu) = t\mathbf{q}^* \cdot \mathbf{p} = te(\mathbf{p}, u)$
  - Now if the initial target utility is equal to 1, then by letting $t = u$, we can write $e(\mathbf{p}, u) = ue(\mathbf{p}, 1)$ and hence, for homothetic utility preferences, the expenditure function is of the form
  $$e(\mathbf{p}, u) = ub(\mathbf{p}), \tag{110}$$
  where $b(\mathbf{p})$ is some linearly homogenous and concave function of prices

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$$e(\mathbf{p}, u) = ub(\mathbf{p}), \tag{110}$$
    where $b(\mathbf{p})$ is some linearly homogenous and concave function of prices
- This implies the following forms for indirect utility, Hicksian and Marshallian demand curves ($V(\mathbf{p}, y)$, $h(\mathbf{p}, u)$ and $q(\mathbf{p}, y)$ respectively)

$$V(\mathbf{p}, y) = \frac{y}{b(\mathbf{p})}, \qquad h_j(\mathbf{p}, u) = u\frac{\partial b(\mathbf{p})}{\partial p_j}, \qquad q_j(\mathbf{p}, y) = yq_j(\mathbf{p}), \tag{111}$$

where $y = \sum_j p_j q_j$ is the total expenditure

- Example: cobb-douglas utility function given by $u(\mathbf{q}) = q_1^{\beta_1} q_2^{\beta_2} \ldots, q_J^{\beta_J}$ where the associated demand functions are of the form

$$q_j = y \frac{1}{p_j} \frac{\beta_i}{\sum_j^J \beta_j}$$

- Implications for demand estimation
  - demand for each good is proportional to expenditure (income), or alternatively, the Engel curve for each good is a straight line going through the origin
  - expenditure elasticity of good j is always one

$$\eta_j = \frac{\partial ln q_j}{\partial \ln y} = 1 \qquad \forall j = 1, \ldots, J.$$

  - known as the expenditure proportionality, which is equivalent to the requirement that budget shares ($w_j = \frac{p_j q_j}{y}$) of all commodities are independent of the level of total expenditure (income) so that a consumer always spends a constant proportion of their income on a product, even though income may be varying across different consumers
  - all expenditure elasticities are equal to one – a result that is contradicted by most empirical studies
  - demand for each good is independent of prices of other products implying that cross-price elasticities are zero

- A less restrictive form is that of **quasi-homotheticity**
- In this formulation, a fixed expenditure element ($a(\mathbf{p})$) is added to the expenditure function in equation (110) so that it is now given by

$$e(\mathbf{p}, u) = a(\mathbf{p}) + ub(\mathbf{p}) \tag{112}$$

- This form is called the **Gorman Polar Form**
- The term $a(\mathbf{p})$ represents the subsistence level of expenditure when $u = 0$ and $b(\mathbf{p})$ is the marginal cost of utility
- The associated indirect utility and demand functions (per the usual derivations) take the forms

$$\begin{aligned} V(\mathbf{p}, y) &= \frac{y - a(\mathbf{p})}{b(\mathbf{p})} \quad \text{and} \quad q_j(\mathbf{p}, y) = a_j(\mathbf{p}) + \frac{b_j(\mathbf{p})}{b(\mathbf{p})}\left[y - a(\mathbf{p})\right] \\ \text{where } a_j(\mathbf{p}) &= \frac{\partial a(\mathbf{p})}{\partial p_j} \quad \text{and} \quad b_j(\mathbf{p}) = \frac{\partial b(\mathbf{p})}{\partial p_j} \end{aligned} \tag{113}$$

- $a(\mathbf{p})$ is interpreted as the subsistence spending amount and $b(\mathbf{p})$ is a price index that deflates income/expenditure over and above the subsistence level

- Some authors write it in an alternative form
  - we can define $A(\mathbf{p}) = \frac{1}{b(\mathbf{p})}$ and $B(\mathbf{p}) = -\frac{a(\mathbf{p})}{b(\mathbf{p})}$
  - and define $\alpha_j(\mathbf{p}) = a_j(\mathbf{p}) + b_j(\mathbf{p})B(\mathbf{p}) = a_j(\mathbf{p}) - \beta_j(\mathbf{p})a(\mathbf{p})$ and $\beta_j(\mathbf{p}) = b_j(\mathbf{p})A(\mathbf{p}) = \frac{b_j(\mathbf{p})}{b(\mathbf{p})}$
- Then (112) and (113) can be expressed as

$$\begin{aligned}
& e(\mathbf{p}, u) = a(\mathbf{p}) + ub(\mathbf{p}) \\
& V(\mathbf{p}, y) = A(\mathbf{p})y + B(\mathbf{p}) \\
& q_j(\mathbf{p}, y) = \alpha_j(\mathbf{p}) + \beta_j(\mathbf{p})y \\
& \text{where, } A(\mathbf{p}) = \frac{1}{b(\mathbf{p})} \qquad B(\mathbf{p}) = -\frac{a(\mathbf{p})}{b(\mathbf{p})} \\
& \text{and, } \alpha_j(\mathbf{p}) = \frac{\partial a(\mathbf{p})}{\partial p_j} - \beta_j(\mathbf{p})a(\mathbf{p}) \qquad \beta_j(\mathbf{p}) = \frac{1}{b(\mathbf{p})}\frac{\partial b(\mathbf{p})}{\partial p_j}
\end{aligned} \tag{114}$$

- The budget share equations in this case are given by a weighted average of two terms

$$w_j = \left(\frac{a}{y}\right)\left(\frac{p_j a_j}{a}\right) + \left(1 - \frac{a}{y}\right)\left(\frac{p_j b_j}{b}\right), \tag{115}$$

- Implications
  - if $a = y$ (subsistence level is equal to the entire income) the budget share of good j is equal to just $\frac{p_j a_j}{a}$, and if expenditure is much larger than the subsistence level (so $a/y \approx 0$) then the share is given by $\frac{p_j b_j}{b}$
  - In aggregate, the expenditure patterns are a weighted average of value shares appropriate to very rich and very poor consumers
  - Engle curves are still linear but they do not go through the origin anymore
  - although homotheticity implies unitary income elasticities for all commodities, quasi-homotheticity implies elasticities that only tend to unity as total expenditure increases
  - significant generalization/improvement over the previous case, but still restrictive as it is unlikely to be true for narrowly defined commodities
  - even for broad commodities such as food, household budget studies tend to give nonlinear Engel curves (we will get to that further below)

- Example: Stone-Geary utility/linear expenditure system (LES) – $u(\mathbf{q}) = \prod_j^J (q_j - \alpha_j)^{\beta_j}$ or equivalently as $u(\mathbf{q}) = \sum_j^J \beta_j \ln(q_j - \alpha_j)$ with $\sum_j^J \beta_j = 1$
  - implied expenditure, indirect utility and demand functions are

$$e(\mathbf{p}, u) = \sum_i^J p_j \alpha_j + u \prod_j^J p_j^{\beta_j}, \qquad V(\mathbf{p}, y) = \frac{y - \sum_j^J p_j \alpha_j}{\prod_j^J p_j^{\beta_j}},$$

$$\text{and} \qquad q_j(\mathbf{p}, y) = \alpha_j + \beta_j \frac{y - \sum_j^J p_j \alpha_j}{p_j}$$

  - expenditure on good j is

$$p_j q_j = p_j \alpha_j + \beta_j (y - \sum_j^J p_j \alpha_j)$$

    and is called the **linear expenditure system** (LES) (expenditure is linear in prices and income) which is easy to estimate, and has been very popular in empirical studies for this reason

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  and is called the **linear expenditure system** (LES) (expenditure is linear in prices and income) which is easy to estimate, and has been very popular in empirical studies for this reason
  - characterized by the marginal budget share and subsistence level parameters, requiring estimation of $2J$ parameters
  - compare that to the more general case of estimating $J^2 + J$ parameters (own and cross-price elasticities and income/expenditure elasticities), or, if adding up, homogeneity, and symmetry restrictions are imposed, there are $(2J - 1)(J/2 + 1)$ parameters to be estimated
  - nonetheless, if concavity of the expenditure function is allowed, then by construction, all cross-price elasticities are positive and hence the system cannot be used if some of the products are complements
  - also there is an approximate proportionality between own-price and expenditure elasticities

- Aggregate demand data raises the problem as to whether the aggregate demand function is consistent with consumer theory
- Certain conditions are necessary under which we can treat the aggregate demand estimations as resulting from the behavior of a single utility maximizing consumer (exact aggregation)
- As you can guess by now, they have to do with quasi-homotheticity and Gorman Polar Form

- Suppose there are N consumers (or households) that face the same prices but differ only in the incomes or expenditures on different products so that the demand for good j for the n^{th} individual is of form

$$q_{jn} = g_{jn}(\mathbf{p}, y_n). \tag{116}$$

Then the average demand $\bar{q}_j$ – aggregated by adding up *quantities* over all individuals and dividing by N – is given by some function f_j as

$$\bar{q}_j = f_j(\mathbf{p}, y_1, y_2, \ldots y_N) = \frac{1}{N}\sum_n^N g_{jn}(\mathbf{p}, y_n) \tag{117}$$

exact aggregation is possible if we can write (117) in the form

$$\bar{q}_j = g_j(\mathbf{p}, \bar{y}) \text{ where } \bar{y} = \frac{1}{N}\sum_n^N y_n \tag{118}$$

- An implication is that the general function in (116) must be linear in y_n, that is, for some function α_{jn} and β_j of $\mathbf{p}$ alone, be of form

$$q_{jn}(\mathbf{p}, y_n) = \alpha_{jn}(\mathbf{p}) + \beta_j(\mathbf{p})y_n \tag{119}$$

- Thus, if the aggregate (average) demand is a function of prices and average income, as in (118), then the underlying individual demand must be of the form given by (119)
- But this is the same demand function from quasi-homothetic preferences as in (114) with a subscript n for the n^{th} consumer, and α_j and y both vary over consumers, but importantly, β_j does not vary over consumers (i.e, person-specific $\alpha(\mathbf{p})$ but identical $\beta(\mathbf{p})$)

- Conversely, if the n^{th} consumer has quasi-homothetic preferences with demand given by (119), then the average demand – aggregated via adding up *quantities* over all individuals and dividing by N – is

$$\begin{aligned}\bar{q}_j &= \frac{1}{N}\sum_n^N q_{jn}(\mathbf{p}, y_n) \\ &= \alpha_j(\mathbf{p}) + \beta_j(\mathbf{p})\bar{y}, \text{ where} \qquad (120) \\ \alpha_j(p) &= \frac{1}{N}\sum_n^N \alpha_{jn}(\mathbf{p}), \text{ and } \bar{y} = \frac{1}{N}\sum_n^N y_n.\end{aligned}$$

- Thus, for exact linear aggregation, underlying individual demand must be from quasi-homothetic preferences and if the consumer has a demand corresponding to quasi-homothetic preferences, then aggregate demand must be of a similar form
- (119) is necessary and sufficient for (118)
- Note that the forms above are arising only due to aggregation requirements, and have nothing to do with requiring aggregate utility maximization

- Suppose now that individuals maximize utility and the individuals demand function is of form (119)
- Gorman showed that the quasi-homothetic demand of the form above is generated by a consumer with the expenditure function given by

$$e_n(\mathbf{p}, u_n) = a_n(\mathbf{p}) + u_n b(\mathbf{p}), \tag{121}$$

  i.e., expenditure is of (Gorman) polar form with subscript n in the equation (114)
- Deaton and Muellbaur show that it is a 'if and only if' condition
- Similarly, the average of the expenditure functions in (121) is

$$\bar{e}(\mathbf{p}, u_n) = \bar{a}(\mathbf{p}) + ub(\mathbf{p}), \tag{122}$$

  and corresponds to the expenditure function for the average demand function in (120)
- If individuals maximize utility, and preferences are such that they satisfy the exact aggregation condition, then the average demand function will be consistent with utility maximization
- Moral of the story ... if we want exact aggregation and want to think of the aggregate demand as arising from a utility maximization of a aggregate consumer, then we have to work with quasi-homothetic utility functions

- Aggregation given earlier leads to the linear Engel curves.
- Muellbauer (1975,1976) introduced exact nonlinear aggregation by starting with budget shares rather than with quantities, so that aggregation is over the budget shares of different consumers
- For n consumers, the average budget share of good j is given by

$$\bar{w}_j = \frac{p_j \sum_n q_{jn}(\mathbf{p}, y_n)}{\sum_n y_n} = \sum_n \left(\frac{y_n}{\sum_n y_n} \right) w_{jn}. \tag{123}$$

  defined as a weighted average of individual shares w_{jn} with weights given by the share of each individual in total expenditure on good j.
- Turns out that such a representative consumer (and the assumed cost function) exists only if the preferences are such that the expenditure function of each individual has the form (called **Generalized Gorman Polar Form**)

$$e_n(\mathbf{p}, u_n) = \theta_n(u_n, a(\mathbf{p}), b(\mathbf{p})) + \phi_n(\mathbf{p}) \tag{126}$$

  where $a(\mathbf{p}), b(\mathbf{p})$ and $\phi(\mathbf{p})$ are homogenous of degree 1 in prices, $\theta_n(\ )$ is homogenous in $a(\mathbf{p})$ and $b(\mathbf{p})$ and, $\sum_n \phi_n(\mathbf{p}) = 0$

- Deaton and Muellbauer consider a special case, in which the representative consumer's expenditure level (income) y_0 is assumed to depend on the distribution of individual expenditures (incomes), $y_1, \ldots, y_n$ but not on prices, which leads to particularly useful class of demand equations
- For a representative consumer the expenditure function takes the form

$$e(\mathbf{p}, u_0) = [a(\mathrm{p})^{\alpha}(1 - u_0) + b(\mathbf{p})^{\alpha} u_0]^{1/\alpha} \tag{132}$$

  and the corresponding budget share equations are said to have the **price independent generalized linear** form (PIGL).
- As $\alpha \to 0$, the representative expenditure function becomes

$$\ln(e(\mathbf{p}, u_0)) = (1 - u_0)\ln(a(\mathbf{p})) + u_0 \ln(b(\mathbf{p})) \tag{133}$$

- These give the nonlinear Engel curves as

$$w_j = \begin{cases} \gamma_j + \eta_j (y/k)^{-\alpha} & \text{PIGL} \\ \gamma_j^* + \eta_j^* \ln(y/k) & \text{PIGLOG} \end{cases} \tag{134}$$

  where γ's and η's are functions of prices only, k varies over individuals (or households) and can be used to capture demographic effects

- PIGL/PIGLOG family generates exact nonlinear aggregation over individuals or households with nonlinear Engel curves
- Merits of representation of market demand as if they were the outcome of decisions by a rational representative consumer has made for extensive application of this class of models
- A specific application comes from a second-order Taylor series expansion of equation (133) so that the first and second derivatives of the expenditure function with respect to prices and utility can be set equal to those of any arbitrary expenditure function at any point (a flexible functional form)
- Deaton and Muellbauer suggest functional forms for $a(\mathbf{p})$ and $b(\mathbf{p})$ in (133) which result in a flexible system they call the 'almost ideal demand system', where

$$\begin{aligned} \ln a(p) &= \alpha_0 + \sum_j \alpha_j \ln p_j + \frac{1}{2}\sum_j\sum_k \gamma^*_{jk} \ln p_j \ln p_k \\ \ln b(p) &= \ln a(p) + \beta_0 \prod_j p_j^{\beta_j} \end{aligned} \tag{135}$$

- AIDS expenditure function is given by

$$\ln e(\mathbf{p}, u) = \alpha_0 + \sum_j \alpha_j \ln p_j + \frac{1}{2}\sum_j\sum_k \gamma^*_{jk} \ln p_j \ln p_k + u\beta_0 \prod_j p_j^{\beta_j} \tag{136}$$

- The expenditure function will be linearly homogenous in $\mathbf{p}$ as long as $\sum_j \alpha_j = 1, \sum_j \gamma^*_{kj} = \sum_k \gamma^*_{kj} = \sum_j \beta_j = 0$

- AIDS demand functions in budget share form are

$$w_j = \alpha_j + \sum_k \gamma_{jk} \ln p_k + \beta_j \ln(y/P)$$

where P is a price index defined by (27)

$$lnP = \alpha_0 + \sum_k \alpha_k \ln p_k + \frac{1}{2} \sum_i \sum_k \gamma_{ki} \ln p_k \ln p_i$$

where $\gamma_{jk} = \frac{1}{2}(\gamma^*_{jk} + \gamma^*_{kj})$

- The restrictions on the parameter of the cost function impose restriction on the parameters of the AIDS demand system (27) given by

$$\sum_{j=1}^{J} \alpha_j = 1 \qquad \sum_{j=1}^{J} \gamma_{jk} = 0 \qquad \sum_{j=1}^{J} \beta_j = 0$$
$$\sum_k \gamma_{jk} = 0 \qquad \gamma_{jk} = \gamma_{kj} \tag{137}$$

- Provided the restrictions above hold (or are imposed), (27) represents a system of demand functions which add up to total expenditure ($\sum w_j = 1$), are homogeneous of degree zero in prices and total expenditure taken together, and satisfy Slutsky symmetry and give nonlinear Engle curves.

Separability & Multi-Stage Budgeting

Related but distinct

- Separability refers to the case when a consumer's preferences for products of one group are independent of product-specific consumption of products from other groups
- Multi-stage budgeting refers to when a consumer (or household) can allocate their total expenditure on different goods in sequential stages, represented as a utility tree, where in the first stage, the total current expenditure is allocated to broad groups of products (food, housing, entertainment) followed by the allocation of expenditures within each broad group (e.g., meats, vegetables, etc. within the food group)

- **Separability** Preferences for products of one group are independent of product-specific consumption of products from other groups
- Thus,
$$u(q_1, \ldots, q_j) = f[v_1(\mathbf{q_{(1)}}), \ldots, v_k(\mathbf{q_{(k)}}), \ldots v_K(\mathbf{q_{(K)}})], \tag{138}$$
where $(q_1, \ldots, q_j) = (\mathbf{q_{(1)}}, \mathbf{q_{(2)}}, \ldots, \mathbf{q_{(k)}})$ i.e., the set $\{\mathbf{q_{(j)}}\}$ is a partition of $(q_1, \ldots, q_j)$ and there are $K < J$ partitions and $f(\cdot)$ is an increasing function of sub-utility functions $v_1, \ldots, v_k$ defined over the partitions
- The groups could be broad categories such as food, shelter, etc. or within a class of related products, it could be subgroups such as the type of food (meat, vegetables, etc.)
- This does not remove the dimensionality problem but does lessen it. For example, for a linear demand system, the total number of parameters reduces from $J^2 + J$ (additional J parameters are for income) to $J^2/K + K^2$ number of parameters (for $J = 20$ products and $K = 10$ subgroups, we go from a total of 420 parameters to 140 parameters)

- The implied subgroup demand functions – conditional demand functions – for all products j in group G are of the form

$$q_j = g(y_g, \mathbf{p_g}), \tag{139}$$

where $y_g = \sum_{i \in G} p_i q_i$ is the total expenditure on products in group G and $\mathbf{p_g}$ is the vector of prices of these products

- Note that they do not include the prices of products not in Group G
- Let $s_{ij} = \partial q_i^h / \partial p_j$ be the terms of the Slutsky matrix (i.e., partials of the Hicksian demand function with respect to prices), then for any two product $i \in G$ and $j \in H$ where $H \neq G$,

$$\begin{gathered} s_{ij} = \mu_{GH} \frac{\partial q_i}{\partial y} \frac{\partial q_j}{\partial y} = \lambda_{GH} \frac{\partial q_i}{\partial y_g} \frac{\partial q_j}{\partial y_h} \\ \text{where } \lambda_{GH} = \mu_{GH} \frac{\partial y_g}{\partial y} \frac{\partial y_h}{\partial y} \end{gathered} \tag{140}$$

- μ_{GH} summarizes the interrelation between groups
- λ_{GH} is the compensated derivative of expenditure on group G with respect to a proportional change in all prices in group H (i.e., $\lambda_{GH} = \sum_{j \in H} p_j \frac{\partial y_g}{\partial p_j}\Big|_{u=const}$)
- If there are K total groups, then we can write a $K \times K$ matrix from the $\lambda's$ that is interpretable as the Slutsky substitution matrix of the group aggregates
- Weak separability results in a two-tier structure of substitution matrices: there are K completely general intragroup Slutsky matrices with no restrictions on substitutions within each group, but between groups substitution is limited by (140)

- When the marginal rate of substitution between any two goods belonging to the same group is independent of the consumption of goods within the other groups, it is consider as **weak** separability of preferences
- If the marginal rate of substitution between any two goods belonging to two different groups is independent of the consumption of any good in any third group, this separability is called strong separability or block additivity.‖
- Strong form is when

$$u(q_1, \ldots, q_j) = f[v_1(\mathbf{q_{(1)}}) + \ldots + v_k(\mathbf{q_{(k)}}) + \ldots + v_K(\mathbf{q_{(K)}})], \tag{141}$$

and $f'(\cdot) > 0$. In turn, the equivalent form of (140) is given by

$$s_{ij} = \mu \frac{\partial q_i}{\partial x} \frac{\partial q_j}{\partial x} \tag{142}$$

where note that μ is independent of groups to which i and j belong

‖Note that some authors refer to this form as just 'additive' separability (without the use of the word block), but technically that is the case when there is only one good in each group.

- **Multi-stage Budgeting:** Consumers can allocate total expenditures in stages, starting with the top-level group and then to any subgroups or sub-subgroups within them
- At each stage, information appropriate for that stage only is required, i.e., the allocation decision is a function of only that group's total expenditure and price indexes for the subgroups and not of prices or price indexes of products in the other groups
  - If the first stage consists of broad categories (food, housing, entertainment) then the consumer decides how much of the budget to allocate to each of these categories depending on three price indexes and not individual prices of types of food items etc
  - Within the food category, the consumer decides how much to spend on different food items (or subgroups) based on the total amount allocated for food and prices of individual food items (or price indexes if there are further subgroups with the food group)
  - Similarly, allocations are done within other groups (housing, entertainment)
  - The process repeats at a third level if there are subgroups (for instance, within the foods group, there may be subgroups of meat, vegetables, etc., and then within any of these subgroups, there are individual items)
- Thus the consumer can allocate the expenditures to the subgroups in sequential stages
- However, all these sequential allocations must equal those that would occur if the consumer's utility maximization problem was done in one complete information step

- Because expenditure allocation to any good within a group can be written as a function only of the total group expenditure and the prices of goods within that group, the demand for any good belonging to the group must also be expressed as a function only of total expenditures on the group and the prices of goods within the group
- Thus

$$u(q_1, \ldots, q_m, q_v, q_d, \ldots, q_j) = f[v_1(\mathbf{q_{(1)}}), \ldots, v_F(q_m, q_v, q_d), \ldots, v_K(\mathbf{q_{(K)}})] \quad (143)$$

implies

$$q_j = g(y_F, p_m, p_v, p_d) \qquad j \in \{m, v, d\} \quad (144)$$

where y_F is the total expenditure on the food items

- In fact, the converse is also true: the existence of subgroup demand functions implies weak separability

- Weak separability and multi-stage budgeting are closely related concepts but are not the same nor does one imply the other
- Weak separability is necessary and sufficient for the last stage of multi-stage budgeting
- While weak separability is necessary and sufficient for the last stage of multi-stage budgeting, and one can proceed with group-specific demand functions at the bottom level (as above)
- Allocation of total budget to different groups at higher stages requires further restrictions on preferences, or on stronger notions of separability and on composite commodity theorem
- To be able to do upper-level allocation, there must be an aggregate quantity and price index for each group which can be calculated without knowing the choices within the group
- A useful set of requirements is that (1) the overall utility is separably additive in the sub-utilities, and that (2) the indirect utility functions for each group are of the generalized Gorman polar form

Appendix – Merger Simulation

(1) How to back out marginal costs
(2) Compute new prices under a merger
(3) Allow cost efficiency for merging firms
(4) Example with Linear demand

- Commonly used demand systems
  - Linear/Log-linear
  - Almost Ideal Demand System (AIDS)
  - Logit/Nested Logit
  - Random Coefficients Logit
- Pros and cons – models differ in flexibility for own- and cross-price elasticities, requirements on data, and difficulty of estimation
  - Linear and AIDS – flexible and can give negative cross-elasticities (complements), but difficult to estimate if too many products (the ‘dimensionality curse’)
  - Logit – easy to estimate, suffers from the ‘independence of irrelevant alternatives’ (IIA) problem, and if shares are small, own elasticity is proportional to price
  - Random coefficients and its variants difficult to estimate under strict time restrictions
- Endogeneity – models must account for simultaneity of price and quantity

Models of Competition

- What are the strategic variables?
  - Prices, quantities, quality, advertising
- How do firms set their values?
  - Cooperatively or non-cooperatively
  - Simultaneously or sequentially
- What is the equilibrium concept?
  - Typically Nash equilibrium
- We will focus on differentiated products Bertrand competition where
  - Firms move simultaneously to set prices
  - Outcome is via Bertrand-Nash equilibrium

- Costs can be obtained from independent sources (e.g. firms accounts, industry reports)
- Can also be backed out from demand model when combined with a model of competition such as Bertrand-Nash equilibrium
- Intuition from a monopolist's problem ...

$$\max_p pq(p) - TC(q(p))$$

where FOC's give

$$\frac{p^* - c(q(p^*))}{p^*} = -\frac{1}{\eta(p^*)}$$

- The equation can be rewritten as **price is equal to marginal cost plus a markup**

$$p = c + \frac{1}{(\partial q(p)/\partial p)} q(p)$$

- Let there be J differentiated products and F firms and where the f-th firm produces a subset $\mathfrak{F}_f$ of the J products
- Let the demand for the j-th product be given by

$$q_j = q_j(\mathbf{p})$$

  where $\mathbf{p}$ is a vector of all related prices (could be any of the demand functions we discussed earlier)
- The the f-th firm maximizes its joint profit over products that it produces

$$\Pi_f = \sum_{k \in \mathfrak{F}_f} (p_k - c_k) q_k(\mathbf{p})$$

  where c_k is the marginal cost of the k-th product, typically assumed constant over the relevant range, and the sum is over all the products owned by firm f

- For firm f, the first order conditions for profit maximization (Nash-Bertand competition) are

$$q_j(\mathbf{p}) + \sum_{k \in \mathfrak{F}_f} (p_k - c_k) \frac{\partial q_k(\mathbf{p})}{\partial p_j} = 0 \qquad \text{for all } j \in \mathfrak{F}_f$$

- Let $\boldsymbol{\Theta}$ be a 1/0 joint "ownership" so that terms θ_{jk} (row j column k) equal 1 if products j and k belong to the same firm and 0 otherwise (and 1 on the leading diagonal)
- Then we can re-write the FOC equations above for each firm f as

$$q_j(\mathbf{p}) + \sum_{k=1}^{J} \theta_{jk} (p_k - c_k) \frac{\partial q_k(\mathbf{p})}{\partial p_j} = 0 \qquad \text{for all } j \in \mathfrak{F}_f$$

which will give us a total of J such equations

- Example: firm 1 owns products 1,2, firm 2 owns products 3,4 and firms 3 and 4 own products 5 and 6 respectively

$$
\begin{aligned}
q_1 + \theta_{11}(p_1 - c_1)\frac{\partial q_1}{\partial p_1} + \ldots + \theta_{61}(p_6 - c_6)\frac{\partial q_6}{\partial p_1} &= 0 \\
q_2 + \theta_{12}(p_1 - c_1)\frac{\partial q_1}{\partial p_2} + \ldots + \theta_{62}(p_6 - c_6)\frac{\partial q_6}{\partial p_2} &= 0 \\
&\vdots \\
q_6 + \theta_{16}(p_1 - c_1)\frac{\partial q_1}{\partial p_6} + \ldots + \theta_{66}(p_6 - c_6)\frac{\partial q_6}{\partial p_6} &= 0
\end{aligned}
$$

where note that only those terms survive where $\theta_{jk} \neq 0$

Rewrite in matrix notation as

$$
\mathbf{q} - \boldsymbol{\Omega}(\mathbf{p} - \mathbf{c}) = \mathbf{0} \qquad \text{where} \qquad \Omega_{jk} = -\theta_{jk}\frac{\partial q_k(\mathbf{p})}{\partial p_j}
$$

- Equivalently, given a demand system $q_j = D_j(\mathbf{p})$, if the matrix of slope coefficients $\frac{\partial q_j(\mathbf{p})}{\partial p_k}$ (row j column k) is given by $\mathbf{B}$, then

$$\boldsymbol{\Omega} = -\boldsymbol{\Theta} \cdot \mathbf{B}'$$

(note: the symbol $\cdot$ is *element by element multiplication* and not the usual matrix multiplication)

- The quantity equation above can be rewritten as the price markup equation

$$\mathbf{p} = \mathbf{c} + \boldsymbol{\Omega}^{-1}\mathbf{q}(\mathbf{p})$$

(compare this to the monopolist's equation on **slide 197** – same/similar)

- This price equation, along with a demand system equations $q_j = D_j(\mathbf{p})$ jointly determines equilibrium prices and quantities and are at the heart of merger simulation

- Given estimates of demand functions, information about ownership, and observed prices and quantities, we can back out markups and marginal costs

$$\mathbf{p} = \mathbf{c} + \mathbf{\Omega}^{-1}\mathbf{q}(\mathbf{p})$$
$$\Rightarrow$$
$$\mathbf{c} = \mathbf{p} - \mathbf{\Omega}^{-1}\mathbf{q}(\mathbf{p})$$

- For merger simulations we change the ownership matrix $\mathbf{\Theta}$ and re-solve for prices using the equations $\mathbf{p} = \mathbf{c} + \mathbf{\Omega}^{-1}\mathbf{q}(\mathbf{p})$ and $q_j = D_j(\mathbf{p})$

MERGER SIMULATION
ALGORITHM

- Step 0a: Estimate the demand system $q_j = D_j(\mathbf{p})$ and obtain $\mathbf{B}$ the matrix of slope coefficients (or use previous studies); i.e. $B_{jk} = \frac{\partial q_j(\mathbf{p})}{\partial p_k}$
- Step 0b: Construct $\boldsymbol{\Omega}_0 = -\boldsymbol{\Theta}_0 \cdot \mathbf{B}'$ using pre-merger ownership matrix $\boldsymbol{\Theta}_\mathbf{0}$
- Step 1: Given data on price and quantity back out estimates of marginal cost $\widehat{\mathbf{c}} = \mathbf{p}_0 - \boldsymbol{\Omega}_\mathbf{0}^{-\mathbf{1}}\mathbf{q}_\mathbf{0}(\mathbf{p}_\mathbf{0})$ (unless available from outside)
- Step 2: Construct the new ownership matrix $\boldsymbol{\Theta}_\mathbf{1}$ (optionally, adjust mc of merging parties as necessary)
- Step 3: Compute the new equilibrium price $\mathbf{p}_\mathbf{1}^*$ using the equation $\mathbf{p}_\mathbf{1}^* = \widehat{\mathbf{c}} + \boldsymbol{\Omega}_\mathbf{1}^{-\mathbf{1}}\mathbf{q}(\mathbf{p}_\mathbf{1}^*)$
    - If the demand system is linear we get a closed form solution for price and quantity
    - If not linear, will need to search for new price equilibrium using numerical methods
    - Given type of demand model, can iteratively search for $\mathbf{p}_\mathbf{1}^*$ such that $|\mathbf{p}^{(h+1)} - \mathbf{p}^{(h)}| < \epsilon$ and where $\mathbf{p}^{(h+1)} = \widehat{\mathbf{c}} + \boldsymbol{\Omega}_\mathbf{1}^{-\mathbf{1}}(\boldsymbol{p}^{(\boldsymbol{h})})\boldsymbol{q}(\boldsymbol{p}^{(\boldsymbol{h})})$ and h is the iteration loop

- Data requirements can be high
  - Sales data including product characteristics, cost data and/or data on inputs that affect cost (additional supply side estimation)
  - Expertise in demand estimation
- Sensibility and sensitivity checks
  - Do elasticities, margins, marginal costs seem reasonable? Do they match some known outside information?
  - How much do they change with demand specification?
  - Do the assumptions made for the model make sense?
- Proceed with caution
  - They can provide reasonable predictions but require great care
  - Predictions are sensitive to modelling assumptions
  - Perhaps use it as internal screen that complements other qualitative work

- Suppose demand functions are linear, and the demand for jth product is given by

$$q_j = a_j + \sum_{k=1}^{J} b_{jk} p_j$$

and marginal cost for each product is mc_j

- We can write the demand equation in matrix notation as

$$\mathbf{q} = \mathbf{a} + \mathbf{B}\mathbf{p}$$

where for instance vector $\mathbf{a}$ and matrix $\mathbf{B}$ are given by

$$\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_j \\ \vdots \\ a_J \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_{11} & \dots & b_{1k} & \dots & b_{1J} \\ \vdots & & \vdots & & \vdots \\ b_{j1} & \dots & b_{jk} & \dots & b_{jJ} \\ \vdots & & \vdots & & \vdots \\ b_{J1} & \dots & b_{Jk} & \dots & b_{JJ} \end{bmatrix}$$

- Suppose there are 6 independent firms and 6 products
  - Demand functions are linear and previously estimated to be

$$q_j = 10 - 2p_j + 0.3 \sum_{k \neq j}^{5} q_k$$

  - In a typical market, say price and quantity are **observed** to be 4.8 and 7.6 respectively for all the products
  - $\mathbf{p}' = (4.8, 4.8, 4.8, 4.8, 4.8, 4.8)$ and $\mathbf{q}' = (7.6, 7.6, 7.6, 7.6, 7.6, 7.6)$
- Using the equations above we can back out the marginal cost and compute markups and price-cost margins

$$\mathbf{B}' = \begin{bmatrix} -2 & .3 & .3 & .3 & .3 & .3 \\ .3 & -2 & .3 & .3 & .3 & .3 \\ .3 & .3 & -2 & .3 & .3 & .3 \\ .3 & .3 & .3 & -2 & .3 & .3 \\ .3 & .3 & .3 & .3 & -2 & .3 \\ .3 & .3 & .3 & .3 & .3 & -2 \end{bmatrix} \qquad \boldsymbol{\Theta} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- Let $\mathbf{a}$ be column vector of intercept terms (all equal to 10 in this example), so $\mathbf{a}' = (10, 10, 10, 10, 10, 10)$
- Then from $\mathbf{p} = \mathbf{c} + \mathbf{\Omega}^{-1}\mathbf{q}(\mathbf{p})$ and $\mathbf{\Omega} = -\mathbf{\Theta} \cdot \mathbf{B}'$, it follows that estimated marginal cost $\widehat{\mathbf{c}}$ can be computed as

$$\mathbf{c} = \mathbf{p} \quad - \quad \mathbf{\Omega}^{-1} \quad \mathbf{q}(\mathbf{p})$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix} = \begin{bmatrix} 4.8 \\ 4.8 \\ 4.8 \\ 4.8 \\ 4.8 \\ 4.8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 7.6 \\ 7.6 \\ 7.6 \\ 7.6 \\ 7.6 \\ 7.6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Thus we have backed out the marginal costs (all equal to 1 in this example) with *price cost margins* being 100(4.8-1)/4.8 = 79.16% for each product

- Equipped with marginal costs and demand parameters, we can now simulate new equilibrium prices and quantities
- For the moment, let's continue with our linear demand system
- We start by determining/solving for Nash-equilibrium given the set of J demand equations $\mathbf{q} = \mathbf{a} + \mathbf{B}\mathbf{p}$ and the set of J price equations $\mathbf{p} = \mathbf{c} + \mathbf{\Omega^{-1}}\mathbf{q}(\mathbf{p})$ derived from the first order conditions specific to this linear demand system

- The set of 2J equations $\mathbf{q} = \mathbf{a} + \mathbf{B}\mathbf{p}$ and $\mathbf{p} = \mathbf{c} + \boldsymbol{\Omega}^{-1}\mathbf{q}(\mathbf{p})$ jointly determine equilibrium price and quantity vectors in any market
  - Write the 2 matrix form equations as

$$\mathbf{q} = \mathbf{a} + \mathbf{B}\mathbf{p} \qquad \text{and} \qquad \mathbf{q} = \boldsymbol{\Theta} \cdot \mathbf{B}'(\mathbf{p} - \mathbf{c})$$

  - They can be stacked with the endogenous variables p,q on the LHS as

$$\begin{bmatrix} (\boldsymbol{\Theta} \cdot \mathbf{B}') & \mathbf{I} \\ -\mathbf{B} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} (\boldsymbol{\Theta} \cdot \mathbf{B}') & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{a} \end{bmatrix}$$

    where $\mathbf{I}$ are $\mathbf{0}$ are $J \times J$ identity and zero matrices respectively, and hence

$$\begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} (\boldsymbol{\Theta} \cdot \mathbf{B}') & \mathbf{I} \\ -\mathbf{B} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} (\boldsymbol{\Theta} \cdot \mathbf{B}') & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{a} \end{bmatrix}$$

- The set of equations

$$\begin{bmatrix}\mathbf{p}\\ \mathbf{q}\end{bmatrix} = \begin{bmatrix}(\boldsymbol{\Theta}\cdot\mathbf{B}') & \mathbf{I}\\ -\mathbf{B} & \mathbf{I}\end{bmatrix}^{-1}\begin{bmatrix}(\boldsymbol{\Theta}\cdot\mathbf{B}') & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{c}\\ \mathbf{a}\end{bmatrix}$$

  can be easily solved using any matrix based software (Matlab, R, Mathematica, SAS, STAT, etc. ... and can even be programmed in Excel)

- Thus given the demand parameters of a linear demand system, marginal costs and the ownership matrix, we get a unique Nash equilibrium solution in prices and quantities
  - Let $\boldsymbol{\Theta}$ and $\mathbf{B}$ be as specified for the linear demand system for six products owned by six separate firms, and let $c' = (1,1,1,1,1,1)$
  - Then

$$\mathbf{p}^* = \begin{bmatrix}4.8\\4.8\\4.8\\4.8\\4.8\\4.8\end{bmatrix} \qquad \mathbf{q}^* = \begin{bmatrix}7.6\\7.6\\7.6\\7.6\\7.6\\7.6\end{bmatrix}$$

- Suppose firms 1 and 2 merge, firms 3 and 4 merge and firms 5 and 6 merge – then all we need to do is change the owenership matrix $\mathbf{\Theta}$ to reflect the new post merger ownership and resolve the system of equations using the new ownership matrix
- Let the pre merger and post merger ownership matrices be given by $\mathbf{\Theta}_0$ and $\mathbf{\Theta}_1$ respectively (i.e., for time 0 and 1)

$$\mathbf{\Theta}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \rightarrowtail \qquad \mathbf{\Theta}_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

- Now solve for $\mathbf{p}$ and $\mathbf{q}$ using $\mathbf{\Theta}_1$

$$\begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} (\mathbf{\Theta}_1 \cdot \mathbf{B}') & \mathbf{I} \\ -\mathbf{B} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} (\mathbf{\Theta}_1 \cdot \mathbf{B}') & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{a} \end{bmatrix}$$

- Suppose firms 1 and 2 merge, firms 3 and 4 merge and firms 5 and 6 merge
- The old and new equilibria are as follows

	Pre-merger values				Post-merger values				
Product	p	q	(p-c)/p	π	p	q	(p-c)/p	π	% Δp
1	4.8	7.6	79.2%	28.88	5.32	7.34	81.2%	31.70	10.80%
2	4.8	7.6	79.2%	28.88	5.32	7.34	81.2%	31.70	10.80%
3	4.8	7.6	79.2%	28.88	5.32	7.34	81.2%	31.70	10.80%
4	4.8	7.6	79.2%	28.88	5.32	7.34	81.2%	31.70	10.80%
5	4.8	7.6	79.2%	28.88	5.32	7.34	81.2%	31.70	10.80%
6	4.8	7.6	79.2%	28.88	5.32	7.34	81.2%	31.70	10.80%

- Overall prices increase by 10.8% for each product and total output falls, which would reduce consumer surplus
- What if there was an efficiency defence – say 25% reduction in costs?

- Suppose there is a merger specific efficiency defence – that marginal costs would reduce by 25% – then in addition to changing the ownership matrix, we can multiply mc by 0.75 and resolve

- Let the pre merger and post merger ownership matrices be given by $\boldsymbol{\Theta}_0$ and $\boldsymbol{\Theta}_1$ respectively (i.e., for time 0 and 1)

$$\boldsymbol{\Theta}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrowtail \boldsymbol{\Theta}_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

- Now solve for $\mathbf{p}$ and $\mathbf{q}$ using $\boldsymbol{\Theta}_1$

$$\begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} (\boldsymbol{\Theta}_1 \cdot \mathbf{B}') & \mathbf{I} \\ -\mathbf{B} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} (\boldsymbol{\Theta}_1 \cdot \mathbf{B}') & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{0.75 \cdot c} \\ \mathbf{a} \end{bmatrix}$$

Merger Simulation
Linear Demand Example

- Suppose firms 1 and 2 merge, firms 3 and 4 merge and firms 5 and 6 merge and costs reduce by 25% due to mergers

- The the old and new equilibria are as follows

	Pre-merger values				**Post-merger values**				
Product	p	q	(p-c)/p	π	p	q	(p-c)/p	π	% Δp
1	4.8	7.6	79.2%	28.88	5.13	7.44	85.4%	32.54	6.77%
2	4.8	7.6	79.2%	28.88	5.13	7.44	85.4%	32.54	6.77%
3	4.8	7.6	79.2%	28.88	5.13	7.44	85.4%	32.54	6.77%
4	4.8	7.6	79.2%	28.88	5.13	7.44	85.4%	32.54	6.77%
5	4.8	7.6	79.2%	28.88	5.13	7.44	85.4%	32.54	6.77%
6	4.8	7.6	79.2%	28.88	5.13	7.44	85.4%	32.54	6.77%

- Overall prices still increase by 6.77% and output is reduced so merger does not improve consumer surplus

- Can also compute change in total profits and compare to the change in total CS for welfare criteria

- Thus, we can modify the ownership matrix and/or the vector of estimated (or known) marginal costs to simulate unilateral effects

- In the previous analysis, the demand curves were linear and hence the solutions, the Nash-Bertrand equilibrium, was easy to compute no matter how large the system of equations (dictated by J)

- More generally, the most appropriate demand system may not be linear but the overall process stays the same