SPHERE PACKING IV. DETAILED BOUNDS

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Abstract. This paper is the fourth in a series of six papers devoted to the proof of the Kepler conjecture, which asserts that no packing of congruent balls in three dimensions has density greater than the face-centered cubic packing. In a previous paper in this series, a continuous function f on a compact space is defined, certain points in the domain are conjectured to give the global maxima, and the relation between this conjecture and the Kepler conjecture is established. The function f can be expressed as a sum of terms, indexed by regions on a unit sphere. In this paper, detailed estimates of the terms corresponding general regions are developed. These results form the technical heart of the proof of the Kepler conjecture, by giving detailed bounds on the function f. The results rely on long computer calculations.

Introduction

This paper contains the technical heart of the proof of the Kepler conjecture. Its primary purpose is to obtain good bounds on the score $\sigma_R(D)$ when R is an arbitrary standard region of a decomposition star D. This is particularly challenging, because we have no *a priori* restrictions on the combinatorial type of the standard region R. It is not known to be bounded by a simple polygon. It is not known to be simply connected. Moreover, there are multitudes of possible geometrical configurations of upright and flat quarters, each scored by a different rule. This paper will deal with these complexities and will bound the score $\sigma_R(D)$ in a way that depends on a simple numerical invariant $n(R)$ of R. When R is bounded by a simple polygon, the numerical invariant is simply the number of sides of the polygon. This bound on the score of a standard region represents the turning point of the proof, in the sense that it caps the complexity of a contravening decomposition star, and restrains the combinatorial possibilities. Later in the proof, it will be instrumental in the complete enumeration of the plane graphs attached to contravening stars.

The first section will prove a series of approximations for the score of upright quarters. The strategy is to limit the number of geometrical configurations of upright quarters by showing that a common upper bound (to the scoring function) can be found for quite disparate geometrical configurations of upright quarters. When a general upper bound can be found that is independent of the geometrical details of upright quarters, we say that the upright quarters can be *erased.* (A precise definition of what it means to erase an upright quarter appears below.) There are some upright quarters that cannot be treated in this manner; and this adds some complications to the proofs in this paper

The second section states the main result of the paper (Theorem **??**). An initial reduction reduces the proof to the case that the boundary of the given standard region is a polygon. A further argument is presented to reduce the proof to a convex polygon.

The third section completes the proof of the main theorem. This part of the proof relies on a new geometrical decomposition of the part of a V-cell over a standard region. The pieces in this decomposition are called *truncated corner cells*.

A final section in this paper collects miscellaneous further bounds that will be needed in later parts of the proof of the Kepler conjecture.

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References

[Hal97a] Thomas C. Hales, Sphere Packings I, Discrete and Computational Geometry, 17 (1997), 1-51.

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