

HISTORICAL OVERVIEW OF THE KEPLER CONJECTURE

THOMAS C. HALES

ABSTRACT. This paper is the first in a series of six papers devoted to the proof of the Kepler conjecture, which asserts that no packing of congruent balls in three dimensions has density greater than the face-centered cubic packing. After some preliminary comments about the face-centered cubic and hexagonal close packings, the history of the Kepler problem is described, including a discussion of various published bounds on the density of sphere packings. There is also a general historical discussion of various proof strategies that have been tried with this problem.

INTRODUCTION

The series of papers in this volume gives a proof of the Kepler conjecture, which asserts that the density of a packing of congruent spheres in three dimensions is never greater than $\pi/\sqrt{18} \approx 0.74048\dots$. This is the oldest problem in discrete geometry and is an important part of Hilbert's 18th problem. An example of a packing achieving this density is the face-centered cubic packing.

[TEMPLATE SECTION HEAD]

[PRINTER: Please insert Sections 1 and 2 here.]

REFERENCES

- [arXiv] <http://xxx.lanl.gov>.
- [Ben74] Bender, C., Bestimmung der grössten Anzahl gleich grosser Kugeln, welche sich auf eine Kugel von demselben Radius, wie die übrigen, auflegen lassen, *Archiv Math. Physik* 56 (1874), 302–306.
- [Bez90] A. Bezdek and W. Kuperberg, Maximum density space packing with congruent circular cylinders of infinite length, *Mathematica* 37 (1990), 74–80.
- [BKM91] A. Bezdek, W. Kuperberg, and E. Makai Jr., Maximum density space packing with parallel strings of balls, *DCG* 6 (1991), 227–283.
- [Bez94] A. Bezdek, A remark on the packing density in the 3-space in *Intuitive Geometry*, ed. K. Böröczky and G. Fejes Tóth, *Colloquia Math. Soc. János Bolyai* 63, North-Holland (1994), 17–22.
- [Bez97] K. Bezdek, Isoperimetric inequalities and the dodecahedral conjecture, *Internat. J. Math.* 8, no. 6 (1997), 759–780.
- [Bli19] H. F. Blichfeldt, Report on the theory of the geometry of numbers, *Bull. AMS*, 25 (1919), 449–453.

- [Bli29] H. F. Blichfeldt, The minimum value of quadratic forms and the closest packing of spheres, *Math. Annalen* 101 (1929), 605–608.
- [Bli35] H. F. Blichfeldt, The minimum values of positive quadratic forms in six, seven and eight variables, *Math. Zeit.* 39 (1935), 1–15.
- [Boe52] Boerdijk, A. H. Some remarks concerning close-packing of equal Spheres, *Philips Res. Rep.* 7 (1952), 303–313.
- [CK04] H. Cohn, A. Kumar, The densest lattice in twenty-four dimensions, math.MG/0408174, (2004).
- [CHMS94] J. H. Conway, T. C. Hales, D. J. Muder, and N. J. A. Sloane, On the Kepler conjecture, *Math. Intelligencer* 16, no. 2 (1994), 5.
- [CS95] J. H. Conway, N. J. A. Sloane, What are all the best sphere packings in low dimensions? *DCG* 13 (1995), 383–403.
- [CS93] J. H. Conway and N. J. A. Sloane, Sphere packings, lattices and groups, third edition, Springer-Verlag, New York, 1998.
- [Fej93] G. Fejes Tóth and W. Kuperberg, Recent results in the theory of packing and covering, in *New trends in discrete and computational geometry*, ed. J. Pach, Springer 1993, 251–279.
- [Fej95] G. Fejes Tóth, Review of [Hsi93], *Math. Review* 95g#52032, 1995.
- [Fej95b] G. Fejes Tóth, Densest packings of typical convex sets are not lattice-like, *DCG*, 14 (1995), 1–8.
- [Fej97] G. Fejes Tóth, Recent progress on packing and covering, *Advances in Discrete and Computational Geometry*, (South Hadley, MA, 1996), pp. 145–162. *Contemp. Math.* 223 (1999), AMS, Providence, RI, 1999. MR 99g:52036
- [Fej72] L. Fejes Tóth, *Lagerungen in der Ebene auf der Kugel und im Raum*, second edition, Springer-Verlag, Berlin New York, 1972.
- [Fej64] L. Fejes Tóth, *Regular figures*, Pergamon Press, Oxford London New York, 1964.
- [Fej42] L. Fejes Tóth, Über die dichteste Kugellagerung, *Math. Zeit.* 48 (1942 1943), 676–684.
- [Fej50] L. Fejes Tóth, Some packing and covering theorems, *Acta Scientiarum Mathematicarum (Szeged)* 12/A, 62–67.
- [Fej53] L. Fejes Tóth, *Lagerungen in der Ebene auf der Kugel und im Raum*, Springer, Berlin, first edition, 1953.
- [FH98] S. P. Ferguson and T. C. Hales, A formulation of the Kepler Conjecture, preprint 1998.
- [Fer97] S. P. Ferguson, Sphere Packings V, thesis, University of Michigan, 1997.
- [Gau31] C. F. Gauss, Untersuchungen über die Eigenscahften der positiven ternären quadratischen Formen von Ludwig August Seber, *Göttingische gelehrte Anzeigen*, 1831 Juli 9, also published in *J. reine angew. Math.* 20 (1840), 312–320, and *Werke*, vol. 2, Königliche Gesellschaft der Wissenschaften, Göttingen, 1876, 188–196.
- [Goo97] J. E. Goodman and J. O’Rourke, *Handbook of discrete and computational geometry*, CRC, Boca Raton and New York, 1997.
- [Gun75] S. Günther, *Ein stereometrisches Problem*, *Archiv der Math. Physik* 57 (1875), 209–215.
- [Hal92] T. C. Hales, The sphere packing problem, *J. Computational Applied Math.* 44 (1992), 41–76.
- [Hal93] T. C. Hales, Remarks on the density of sphere packings in three dimensions, *Combinatorica* 13 (1993), 181–187.
- [Hal94] T. C. Hales, The status of the Kepler conjecture, *Math. Intelligencer* 16, no. 3, (1994), 47–58.
- [Hal96] T. C. Hales, <http://www.pitt.edu/~thales/kepler98/holyoke.html>
- [Hal97a] T. C. Hales, Sphere Packings I, *Disc. Comp. Geom* 17:1-51 (1977).

- [Hal97b] T. C. Hales, Sphere Packings II, *Disc. Comp. Geom* 18:135–149 (1997).
- [Hal98a] T. C. Hales, <http://www.math.pitt.edu/~thales/kepler98/packings.html>.
The computer code is permanently archived at
<http://xxx.lanl.gov/abs/math.MG/9811078>.
- [Hal98b] T. C. Hales, Sphere Packings III, *math.MG/9811075*.
- [Hal98c] T. C. Hales, Sphere Packings IV, *math.MG/9811076*.
- [Hal98d] T. C. Hales, The Kepler Conjecture, *math.MG/9811078*.
- [Hal00] T. C. Hales, Cannonballs and Honeycombs, *Notices of the AMS*, Vol 47, No. 4.
- [Hal01] T. C. Hales, Sphere Packings in 3 Dimensions, *Arbeitstagung*, 2001, *math.MG/0205208*.
- [Hal03] Thomas C. Hales, Some algorithms arising in the proof of the Kepler Conjecture, *Discrete and Computational Geometry: The Goodman-Pollack Festschrift*, Jacob E. Goodman (Edt), Springer Verlag, July 2003.
- [Hil01] D. Hilbert, Mathematische Probleme, *Archiv Math. Physik* 1 (1901), 44–63, also in *Proc. Sym. Pure Math.* 28 (1976), 1–34.
- [Hop74] Hoppe R. *Bemerkung der Redaction*, *Math. Physik* 56 (1874), 307–312.
- [HPT95] R. Horst, P.M. Pardalos, N.V. Thoai, *Introduction to Global Optimization*, Kluwer, 1995.
- [Hsi93a] W.-Y. Hsiang, On the sphere packing problem and the proof of Kepler’s conjecture, *Internat. J. Math* 93 (1993), 739–831.
- [Hsi93b] W.-Y. Hsiang, On the sphere packing problem and the proof of Kepler’s conjecture, in *Differential geometry and topology* (Alghero, 1992), World Scientific, River Edge, NJ, 1993, 117–127.
- [Hsi93c] W.-Y. Hsiang, The geometry of spheres, in *Differential geometry* (Shanghai, 1991), World Scientific, River Edge, NJ, 1993, 92–107.
- [Hsi95] W.-Y. Hsiang, A rejoinder to T. C. Hales’s article “The status of the Kepler conjecture,” *Math. Intelligencer* 17, no. 1, (1995), 35–42.
- [Hsi02] W.-Y. Hsiang, Least Action Principle of Crystal Formation of Dense Packing Type and the Proof of Kepler’s Conjecture, World Scientific, 2002.
- [Kar66] R. Kargon, *Atomism in England from Hariot to Newton*, Oxford, 1966.
- [Kep66] J. Kepler, *The Six-cornered snowflake*, Oxford Clarendon Press, Oxford, 1966, forward by L. L. Whyte.
- [KZ73] A. Korkine and G. Zolotareff, Sur les formes quadratiques, *Math. Annalen* 6 (1873), 366–389.
- [KZ77] A. Korkine and G. Zolotareff, Sur les formes quadratiques positives, *Math. Annalen* 11 (1877), 242–292.
- [Lag73] J. L. Lagrange, Recherches d’arithmétique, *Nov. Mem. Acad. Roy. Sc. Bell Lettres Berlin* 1773, in *Œuvres*, vol. 3, 693–758.
- [Lee56] J. Leech, The Problem of the Thirteen Spheres, *The Mathematical Gazette*, Feb 1956, 22–23.
- [Lin86] J. H. Lindsey II, Sphere packing in R^3 , *Mathematika* 33 (1986), 137–147.
- [Mas66] B. J. Mason, On the shapes of snow crystals, in [Kep66].
- [McL98] S. McLaughlin, A proof of the dodecahedral conjecture, preprint, *math.MG/9811079*.
- [Mel97] J. B. M. Melissen, Packing and covering with circles, Ph.D. dissertation, Univ. Utrecht, Dec. 1997.
- [Mil76] J. Milnor, Hilbert’s problem 18: on crystallographic groups, fundamental domains, and on sphere packings, in *Mathematical developments arising from Hilbert problems*, *Proc. Symp. Pure Math.*, vol 28, 491–506, AMS, 1976.
- [MP93] W. Moser, J. Pach, Research problems in discrete geometry, DIMACS Technical Report, 93032, 1993.

- [Mud88] D. J. Muder, Putting the best face on a Voronoi polyhedron, *Proc. London Math. Soc.* (3) 56 (1988), 329–348.
- [Mud93] D. J. Muder A New Bound on the Local Density of Sphere Packings, *Discrete and Comp. Geom.* 10 (1993), 351–375.
- [Mud97] D. J. Muder, letter, in *Fermat’s enigma*, by S. Singh, Walker, New York, 1997.
- [Oes90] J. Oesterlé, Empilements de sphères, Séminaire Bourbaki, vol. 1989/90, Astérisque (1990), No. 189–190 exp. no. 727, 375–397.
- [PA95] J. Pach, P.K. Agarwal, *Combinatorial geometry*, John Wiley, New York 1995.
- [Plo00] K. Plofker, private communication, January 2000.
- [Ran47] R. A. Rankin, *Annals of Math.* 48 (1947), 228–229.
- [Rog58] C. A. Rogers, The packing of equal spheres, *Proc. London Math. Soc.* (3) 8 (1958), 609–620.
- [Rog64] C. A. Rogers, *Packing and covering*, Cambridge University Press, Cambridge, 1964.
- [SW53] K. Schütte and B.L. van der Waerden, Das Problem der dreizehn Kugeln, *Math. Annalen* 125, (1953), 325–334.
- [SM44] B. Segre and K. Mahler, On the densest packing of circles, *Amer. Math Monthly* (1944), 261–270.
- [Shi83] J. W. Shirley, *Thomas Harriot: a biography*, Oxford, 1983.
- [SHDC95] N. J. A. Sloane, R. H. Hardin, T. D. S. Duff, J. H. Conway, Minimal-energy clusters of hard spheres, *DCG* 14, no. 3, (1995), 237–259.
- [Szp02] G. G. Szpiro, *Kepler’s Conjecture*, Wiley, 2002.
- [Thu92] A. Thue, Om nogle geometrisk taltheoretiske Theoremer, *Forandlingerneved de Skandinaviske Naturforskere* 14 (1892), 352–353.
- [Thu10] A. Thue, Über die dichteste Zusammenstellung von kongruenten Kreisen in der Ebene, *Christinia Vid. Selsk. Skr.* 1 (1910), 1–9.
- [Why66] L. L. Whyte, forward to [Kep66].

E-mail address: `hales@pitt.edu`

MATH DEPARTMENT, UNIVERSITY OF PITTSBURGH