SPHERE PACKINGS VI. TAME GRAPHS AND LINEAR PROGRAMS

THOMAS C. HALES

Abstract. This paper is the sixth and final part in a series of papers devoted to the proof of the Kepler conjecture, which asserts that no packing of congruent balls in three dimensions has density greater than the face-centered cubic packing. In a previous paper in this series, a continuous function f on a compact space is defined, certain points in the domain are conjectured to give the global maxima, and the relation between this conjecture and the Kepler conjecture is established. In this paper, we consider the set of all points in the domain for which the value of f is at least the conjectured maximum. To each such point, we attach a planar graph. It is proved that each such graph must be isomorphic to a *tame* graph, of which there are only finitely many up to isomorphism. Linear programming methods are then used to eliminate all possibilities, except for three special cases treated in earlier papers: pentahedral prisms, the face-centered cubic packing, and the hexagonal-close packing. The results of this paper rely on long computer calculations.

Introduction

This paper is the last in the series of paper devoted to the proof of the Kepler conjecture. The first several sections prove a result that asserts that "all contravening graphs are tame." A contravening graph is one that is attached to a potential counterexample to the Kepler conjecture. Contravening graphs by nature are elusive and are studied by indirect methods. In contrast, the defining properties of tame graphs lend themselves to direct examination. (By definition, tame graphs are planar graphs such that the degree of every vertex is at least 2 and at most 6, the length of every face is at least 3 and at most 8, and such that other similar explicit properties hold true.)

It is no coincidence that contravening graphs all turn out to be tame. The definition of tame graph has been tailored to suit the situation at hand. We set out to prove explicit properties of contravening graphs, and when we are satisfied with what we have proved, we brand a graph with these properties a tame graph.

The first section of this paper gives the definition of tame graph. The second section gives the classification of all tame graphs. There are several thousand such graphs. The classification was carried out by computer. This classification is one of the main uses of a computer in the proof of the Kepler conjecture. A detailed description of the algorithm that is used to find all tame graphs is presented in this section.

The third section of this paper gives a review of results from earlier parts of the paper that are relevant to the study of tame plane graphs. In the abridged version of the proof [Hal05a], the results cited in this section are treated as axioms. This section thus serves as a guide to the results that are proved in this volume, but not in the abridged version of the proof.

This section also contains a careful definition of what it means to be a contravening plane graph. The first approximation to the definition is that it is the combinatorial plane graph associated with the net of edges on the unit sphere bounding the standard regions of a contravening decomposition star. The precise definition is somewhat more subtle because we wish ensure that every face of a contravening plane graph is a simple polygon. To guarantee that this property holds, we simplify the net of edges on the unit sphere whenever necessary.

The fourth and fifth sections of this paper contain the proof that all contravening plane graphs are tame. These sections complete the first half of this paper.

The second half of this paper is about linear programming. Linear programs are used to prove that with the exception of three tame graphs (those attached to the face-centered cubic packing, the hexagonal-close-packing, and the pentahedral prism), a tame graph cannot be a contravening graph. This result reduces the proof of the Kepler conjecture to a close examination of three graphs. Pentahedral prism graphs are treated in Paper **??**. The face-centered cubic and hexagonal-close packing graphs are treated in Section **??** of Paper **??**. The linear programming results together with these earlier results complete the proof of the Kepler conjecture.

The sixth section of this paper describes how to attach a linear program to a tame plane graph. The output from this linear program is an upper bound on the score of all decomposition stars associated with the given tame plane graph. The seventh section of this paper shows how to use linear programs to eliminate what are called the *aggregate* tame plane graphs. The *aggregates* are those cases where the net of edges formed by the edges of standard regions was simplified to ensure that every face of a contravening plane graph is a polygon. By the end of this section, we have a proof that every standard region in a contravening decomposition star is bounded by a simple polygon.

The final section of this paper gives a long list of special strategies that are used when the output from the linear program in the sixth section does not give conclusive results. The general strategy is to partition the original linear program into a collection of refined linear programs with the property that the score is no greater than the maximum of the outputs from the linear programs in the collection. These branch and bound strategies are described in this final section. Linear programming shows that every decomposition star with a tame plane graph (other than the three mentioned above) has a score less than that of the decomposition stars attached to the face-centered cubic packing. This and earlier results imply the Kepler conjecture.

[TEMPLATE SECTION HEAD]

[PRINTER: Please insert Sections 18, 19, 20, 21, 22, 23, 24, and 25 here.]

References

[Hal97a] Thomas C. Hales, Sphere Packings I, Discrete and Computational Geometry, 17 (1997), 1-51.

[Hal97b] Thomas C. Hales, Sphere Packings II, Discrete and Computational Geometry, 18 (1997), 135-149.

[Hal03] Thomas C. Hales, Some algorithms arising in the proof of the Kepler Conjecture, Discrete and computational geometry, 489–507, Algorithms Combin., 25, Springer, Berlin, 2003.

[Hal05a] Thomas C. Hales, A Proof of the Kepler Conjecture (summary version), to appear in Annals of Mathematics.

[Hal05b] Thomas C. Hales, Packings,
http://www.math.pitt.edu/~thales/kepler98.html
(The source code, inequalities, and other computer data relating to the solution is also found at *http://xxx.lanl.gov/abs/math/9811078v1*.)

E-mail address: `hales@pitt.edu`

Math Department, University of Pittsburgh