SPHERE PACKING III. EXTREMAL CASES

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ABSTRACT. This paper is the third in a series of six papers devoted to the proof of the Kepler conjecture, which asserts that no packing of congruent balls in three dimensions has density greater than the face-centered cubic packing. In the previous paper in this series, a continuous function f on a compact space is defined, certain points in the domain are conjectured to give the global maxima, and the relation between this conjecture and the Kepler conjecture is established. This paper shows that those points are indeed local maxima. Various approximations to f are developed, that will be used in subsequent papers to bound the value of the function f. The function f can be expressed as a sum of terms, indexed by regions on a unit sphere. Detailed estimates of the terms corresponding to triangular and quadrilateral regions are developed.

INTRODUCTION

This paper is the third in a series of six papers devoted to the proof of the Kepler conjecture, which asserts that no packing of congruent balls in three dimensions has density greater than the face-centered cubic packing. In the previous paper in this series, a continuous function f on a compact space is defined, certain points in the domain are conjectured to give the global maxima, and the relation between this conjecture and the Kepler conjecture is established. This paper shows that those points are indeed local maxima. Various approximations to f are developed, that will be used in subsequent papers to bound the value of the function f. The function f can be expressed as a sum of terms, indexed by regions on a unit sphere. Detailed estimates of the terms corresponding to triangular and quadrilateral regions are developed.

This paper has three objectives. The first is dealing with the two types of decomposition stars that attain the optimal Kepler conjecture bound. The second is obtaining general upper bounds on the score of decomposition star by truncation. The third is obtaining various upper bounds on the score associated to individual triangular and quadrilateral regions of a general decomposition star.

The first section contains a proof that the decomposition stars attached to the face-centered cubic and hexagonal-close packings give local maxima to the scoring function on the space of all decomposition stars. The proof describes precisely determined neighborhoods of these critical points. These special decomposition stars are shown to yield the global maximum of the scoring function on these restricted neighborhoods.

The second section gives an approximation to a decomposition star that provides an upper bound approximation to the scoring function σ. In the simplest cases, the approximation to the decomposition star is obtained by truncating the decomposition star at distance $t_0 = 1.255$ from the origin. More generally, we define a collection of simplices (that do not overlap any simplices in the Q-system), and define a somewhat different truncation for each type of simplex in the collection. For want of a more suggestive term, these simplices are said to form the $\mathcal{S}$-system.

When truncation at t_0 cuts too deeply, we reclaim a scrap of volume that lies outside the ball of radius t_0 but still inside the V-cell. This scrap is called a *crown*. These scraps are studied in that same section.

In a final section, we develop a series of bounds on the score function in triangular and quadrilateral regions, for use in later papers.

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References

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