A FORMULATION OF THE KEPLER CONJECTURE

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Abstract. This paper is the second in a series of six papers devoted to the proof of the Kepler conjecture, which asserts that no packing of congruent balls in three dimensions has density greater than the face-centered cubic packing. The top level structure of the proof is described. A compact topological space is described. Each point of this space can be described as a finite cluster of balls with additional combinatorial markings. A continuous function on this compact space is defined. It is proved that the Kepler conjecture will follow if the value of this function is never greater than a given explicit constant.

Introduction

The following papers give a proof of the Kepler conjecture, which asserts that no packing of congruent balls in three dimensional Euclidean space has density exceeding that of the face-centered cubic packing.

A historical overview of the Kepler conjecture is found in the first paper in this series. Since the history of this problem is treated there, this paper does not go into the details of the extensive literature on this problem. We mention that Hilbert included the Kepler conjecture as part of his eighteenth problem [Hil01]. L. Fejes Tóth was the first to formulate a plausible strategy for a proof [Fej72]. He also suggested that computers might play a role in the solution of this problem. The historical account also discusses the development of some of the key concepts of this paper.

An expository account of the proof is contained in [Hal00]. A general reference on sphere packings is [CS98]. A general discussion of the computer algorithms that are used in the proof can be found in [Hal03]. Some speculations on the structure of a second-generation proof can be found in [Hal01]. Details of computer calculations can be found on the internet at [Hal05b].

The first section of this paper gives the top level structure of the proof of the Kepler conjecture. The next two sections describe the fundamental decompositions of space that are needed in the proof. The first decomposition, which is called the Q-system, is a collection of simplices that do not overlap. This decomposition was originally inspired by the Delaunay decomposition of space. The other decomposition, which is called the V-cell decomposition, is closely related to the Voronoi decomposition of space. In the following section, these two decompositions of space are combined into geometrical objects called *decomposition stars*. The decomposition star is the fundamental geometrical object in the proof of the Kepler conjecture.

The final section of this paper, which was coauthored with Samuel P. Ferguson, describes a particular nonlinear function on the set of all decomposition stars, called the scoring function. The Kepler conjecture reduces to an optimization problem involving this nonlinear function on the set of all decomposition stars. This is an optimization problem in a finite number of variables. The subsequent papers (Papers **??** – **??**) solve that optimization problem.

The choice of the particular scoring function to use was arrived at jointly with Samuel P. Ferguson. He has contributed to this project in many important ways, including the results in Section **??**.

Some history of the proof and this paper is as follows. The original proof, as envisioned in 1994 and accomplished in 1998, was divided into a five-step program. As a result, the original papers were called "Sphere Packings I," "Sphere Packings II," and so forth. The first two papers in the series were published in an earlier volume of DCG. As it turned out, the fourth step "Sphere Packings IV" is considerably more difficult than the other steps in the program. It became clear that a single paper would not suffice, and the fourth step of the proof was divided into two parts "Sphere Packings IV" and "Kepler Conjecture (Sphere Packings VI)." Samuel Ferguson's thesis "Sphere Packings V" solved one of the five major steps in the proof. (Although "Sphere Packings IV" and "Sphere Packings VI" belonged together, because of the numbering scheme, Ferguson's theses "Sphere Packings V" was inserted between these two papers.)

The proof that is contained in this volume is a rewritten version of the proof. For historical reasons, the papers in this volume have retained the original titles, but because of extensive revisions over the past several years, the proof is no longer arranged according to the five steps of the 1994 program.

In addition to the $5+1$ papers corresponding to the five steps of the original program, there is the current paper. It has the following origin. In 1996, it became clear that progress on the problem required some adjustments in the main nonlinear optimization problem of "Sphere Packings I" and "II." As the original 1996 manuscript put it, "There are infinitely many scoring schemes that should lead to a proof of the Kepler conjecture. The problem is to formulate the scheme that makes the Kepler conjecture as accessible as possible" [Hal96]. The original purpose of this paper was to make some useful improvements in the scoring function from "Sphere Packings I" and "II" and to make the changes in such a way that the main results of those papers would still hold true.

Over the past years, this paper has grown considerably in scope to the point that it is now lays the foundation for all of the papers in the series. In fact, all of the foundational material from "Sphere Packings I," and "II," and the 1998 preprint series has been collected together in this article. The scoring function is no longer the same as the one presented in "Sphere Packings I," and "II." This paper adapts the relevant material from these earlier papers to the current scoring function. This paper has expanded to the point that it is now possible to understand the entire proof of the Kepler conjecture without reading "I" and "II."

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References

[CS98] J. H. Conway and N. J. A. Sloane, Sphere packings, lattices and groups, third edition, Springer-Verlag, New York, 1998.

[Fej72] L. Fejes Tóth, *Lagerungen in der Ebene auf der Kugel und im Raum*, second edition, Springer-Verlag, Berlin New York, 1972.

[Hal96] Thomas C. Hales, A reformulation of the Kepler Conjecture, unpublished manuscript, Nov. 1996.

[Hal97a] Thomas C. Hales, Sphere Packings I, Discrete and Computational Geometry, 17 (1997), 1-51.

[Hal97b] Thomas C. Hales, Sphere Packings II, Discrete and Computational Geometry, 18 (1997), 135-149.

[Hal00] Thomas C. Hales, Cannonballs and Honeycombs, Notices Amer. Math. Soc. 47 (2000), no. 4, 440–449.

[Hal01] Thomas C. Hales, Sphere Packings in 3 Dimensions, Arbeitstagung, 2001.

[Hal03] Thomas C. Hales, Some algorithms arising in the proof of the Kepler Conjecture, Discrete and computational geometry, 489–507, Algorithms Combin., 25, Springer, Berlin, 2003.

[Hal05a] Thomas C. Hales, A Proof of the Kepler Conjecture (summary version), to appear in Annals of Mathematics.

[Hal05b] Thomas C. Hales, Packings,
http://www.math.pitt.edu/~thales/kepler98.html
(The source code, inequalities, and other computer data relating to the solution is also found at *http://xxx.lanl.gov/abs/math/9811078v1*.)

[Hil01] D. Hilbert, Mathematische Probleme, *Archiv Math. Physik* 1 (1901), 44–63, also in *Proc. Sym. Pure Math.* 28 (1976), 1–34.

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