# Appendices to Dense Sphere Packings

a blueprint for formal proofs

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# Contents

# Abstract

These are unpublished appendices for ”Dense Sphere Packings” (DSP). These are notes that were useful in the formalization of some lemmas in the proof of the Kepler conjecture but were considered too technical for inclusion in DSP. It also includes material related to some changes that were made to the formalization that occurred after the publication of DSP.

These appendices have not been edited into a form that is suitable for publication. They were primarily written for the benefit of the members of the Flyspeck team who carried out the formalization.

# 1

## Appendix on the main estimate

This appendix gives further details about the proof of the main estimate. It contains numerous improvements over the published text of *Dense Sphere Packings*. The verification of the main estimate can be viewed as a combinatorial proof that $S_{\text{init}} \Rightarrow S_{\text{term}}$.

### 1.1 statement of results

This appendix largely replaces [1][Section 7.4]. For completeness, we repeat a few definitions and results.

**Definition 1.1** $(h_0, \tau)$ `[CUFCNHB]` [$\rho_0$ ⇜⇝ `rho_fun`] [$\rho_0$ ⇜⇝ `rho`] [$\tau$ ⇜⇝ `tau_fun`] [$\mathrm{sol}_0$ ⇜⇝ `sol0`] [$h_0$ ⇜⇝ `h0`] [azim ⇜⇝ `azim_in_fan`] Let $(V, E, F)$ be a nonreflexive local fan. Recall that $h_0 = 1.26$ and $L(h) = (h_0 - h)/(h_0 - 1)$, when $h \le h_0$. Set

$$\rho_0(y) = 1 + \frac{\mathrm{sol}_0}{\pi} \cdot \frac{y-2}{2h_0 - 2} = 1 + \frac{\mathrm{sol}_0}{\pi}(1 - L(y/2))$$

$$\begin{aligned}\tau(V, E, F) &= \sum_{x \in F} \rho_0(\|\mathrm{node}(x)\|)\mathrm{azim}(x) + (\pi + \mathrm{sol}_0)\,(2 - k(F))\\ &= \sum_{\mathbf{v} \in V} \rho_0(\|\mathbf{v}\|)\angle(\mathbf{v}) + (\pi + \mathrm{sol}_0)\,(2 - \mathrm{card}(V)),\end{aligned}$$

where $\mathrm{sol}_0 = 3\arccos(1/3) - \pi \approx 0.551$ is the solid angle of a spherical equilateral triangle of side $\pi/3$, and $k(F)$ is the cardinality of $F$.

**Definition 1.2** ($\tau_{tri}, \tau_3, , \mathrm{dih}_i$) [$\tau_{tri}$ ⇝ `taum`] We define additional functions in the case $k = 3$. If $\mathrm{card}(V) = 3$, then write $V = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Set

$$\tau_3(V) = \sum_{i=1}^{3} \rho_0(\|\mathbf{v}_i\|)\, \mathrm{dih}_V\{\mathbf{0}, \mathbf{v}_i\}\{\mathbf{v}_{i+1}, \mathbf{v}_{i+2}\} - (\pi + \mathrm{sol}_0).$$

Let

$$\tau_{tri}(y_1, y_2, y_3, y_4, y_5, y_6) = \sum_{i=1}^{3} \rho_0(y_i)\, \mathrm{dih}_i(y_1, \ldots, y_6) - (\pi + \mathrm{sol}_0)\,, \tag{1.3}$$

where

$$\begin{aligned} \mathrm{dih}_1(y_1, y_2, y_3, y_4, y_5, y_6) &= \mathrm{dih}(y_1, y_2, y_3, y_4, y_5, y_6), \\ \mathrm{dih}_2(y_1, y_2, y_3, y_4, y_5, y_6) &= \mathrm{dih}(y_2, y_3, y_1, y_5, y_6, y_4), \quad \text{and} \\ \mathrm{dih}_3(y_1, y_2, y_3, y_4, y_5, y_6) &= \mathrm{dih}(y_3, y_1, y_2, y_6, y_4, y_5). \end{aligned} \tag{1.4}$$

**Definition 1.5** (standard, protracted, diagonal) `[KRACSCQ]` `[standard` ⇝ `standard (deprecated)]` `[protracted` ⇝ `protracted (deprecated)]` `[diagonal` ⇝ `diagonal0 (deprecated)]` Let $(V, E)$ be a fan. We write $\|\varepsilon\|$ for $\|\mathbf{v} - \mathbf{w}\|$, when $\varepsilon = \{\mathbf{v}, \mathbf{w}\} \subset V$. We say that $\varepsilon$ is *standard* if

$$2 \le \|\varepsilon\| \le 2h_0.$$

We say that $\varepsilon$ is *protracted* if

$$2h_0 \le \|\varepsilon\| \le \sqrt{8}.$$

If $\mathbf{v}, \mathbf{w} \in V$ are distinct, and $\varepsilon = \{\mathbf{v}, \mathbf{w}\}$ is not an edge in $E$, then we call $\varepsilon$ a *diagonal* of the fan.

**Theorem 1.6** (main estimate) `[JEJTVGB]` *[*(ANNULUS) ⇝ *`ball_annulus`]* *[`main estimate` ⇝ `main_estimate (deprecated)`]* *[*(DIAGONAL) ⇝ *`diagonal1 (deprecated)`]* *[`*` ⇝ `check:quad_std_cs, pent_std_cs, hex_st_cs, pent_diag_cs, pent_pro_cs,` ] Let* $(V, E, F)$ *be a nonreflexive local fan ([1][Definition 7.2]). We make the following additional assumptions on* $(V, E, F)$.

1. (PACKING) *$V$ is a packing.*
2. (ANNULUS) *$V \subset \mathcal{B}$.*
3. (DIAGONAL) *If $\{\mathbf{v}, \mathbf{w}\}$ is a diagonal, then*

$$\|\mathbf{v} - \mathbf{w}\| \ge 2h_0.$$

4. (CARD) *Let* $k = \mathrm{card}(E) = \mathrm{card}(F)$. *Then* $3 \le k \le 6$.

*In this context, we have the following conclusions.*

1. *Assume $k \geq 3$. If every edge of $E$ is standard, then*

$$\tau(V,E,F) \geq \begin{cases} 0.0, & \text{if } k = 3, \\ 0.206, & \text{if } k = 4, \\ 0.4819, & \text{if } k = 5, \\ 0.712, & \text{if } k = 6. \end{cases}$$

2. *Assume $k = 5$. Assume that every edge of $E$ is standard. Assume that every diagonal $\varepsilon$ of the fan satisfies $\|\varepsilon\| \geq \sqrt{8}$. Then*

$$\tau(V,E,F) \geq 0.616. \qquad \text{(check:pent\_diag\_cs)}$$

3. *Assume $k = 5$. Assume there exists some protracted edge in $E$ and that the other four edges are standard. Then*

$$\tau(V,E,F) \geq 0.616. \qquad \text{(check:pent\_pro\_cs)}$$

4. *Assume that $k = 4$. Assume that there exists some protracted edge in $E$ and that the other three edges are standard. Assume that both diagonals $\varepsilon$ of the fan satisfy $\|\varepsilon\| \geq \sqrt{8}$. Then*

$$\tau(V,E,F) \geq 0.477. \qquad \text{(check:quad\_pro\_cs)}$$

5. *Assume $k = 4$. Assume that every edge of $E$ is standard. Assume that both diagonals $\varepsilon$ of the fan satisfy $\|\varepsilon\| \geq 3$. Then*

$$\tau(V,E,F) \geq 0.467. \qquad \text{(check:quad\_diag\_cs)}$$

## 1.2 definitions

**Definition 1.7** (torsor, adjacent) `[XCZLSVS]` `[torsor ⇝ torsor]` Let $k > 1$ be an integer. A *torsor* is a set $I$ with a given simply transitive action of $\mathbb{Z}/k\mathbb{Z}$ on $I$. We write the application of $j \in \mathbb{Z}/k\mathbb{Z}$ to $i \in I$ as $j+i$ or $i+j$. We also write $j+i$ for the application of the image of $j \in \mathbb{Z}$ in $\mathbb{Z}/k\mathbb{Z}$ to $i \in I$. Note that each choice of base point $i_0 \in I$ gives a bijection $i \mapsto i + i_0$ between $\mathbb{Z}/k\mathbb{Z}$ and $I$. We say that $i$ and $j$ are *adjacent* if $i = j+1$ or $j = i+1$. When $i$ and $j$ are adjacent, we call $\{i, j\}$ an edge of $I$. A *diagonal* is a pair $\{i, j\} \subset I$ such that $\{i, j\}$ is not a singleton and not an edge.

**Definition 1.8** `[TEQQCLX]` The *opposite* $I'$ of a torsor $I$ is the torsor with the same underlying set and the action is composed with the group automorphism $\mathbb{Z}/k\mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$, sending $i \mapsto -i$. An *isomorphism of torsors* is a bijection that respects the action. Two torsors are *equivalent* if they are isomorphic or if one is isomorphic to the opposite of the other.

**Example 1.9** *If $H = (D, e, n, f)$ is a hypermap with face $F$, then $F$ is a torsor under the action $x \mapsto fx$. If $H$ is isomorphic to $\mathrm{Dih}_{2k}$ and has vertex set $V$, then $V$ is a torsor under the action of $x \mapsto \rho x$, for some $\rho$ similar to $\rho_{(V,E,F)}$. We may mostly restrict our attention to these instances of torsors.*

Recall that $c_{\mathrm{stab}} = 3.01$.

**Definition 1.10** (stable constraint system) `[ZGFHNKX]` `[constraint system` $\leftrightsquigarrow$ `constraint_system]` `[stable constraint system` $\leftrightsquigarrow$ `stable_system]` `[`$(k, a, b, d, I, J, +1)$ $\leftrightsquigarrow$ `(k_sy,a_sy,b_sy,d_sy,I_sy,J_sy,f_sy)]` `[tri-stable` $\leftrightsquigarrow$ `tri_stable]` `[`$(a, b, d, k, I, J, +1)$ $\leftrightsquigarrow$ $(a\_ts, b\_ts, d\_ts, k\_ts, I\_ts, J\_ts, f\_ts)$`]` `[substandard (stable)` $\leftrightsquigarrow$ `augmented_constraint_system1]` `[substandard (tri-stable)` $\leftrightsquigarrow$ `augmented_constraint_system2]` A *stable constraint system* $s$ (or *SCS* for short) consists of the following data subject to conditions listed below.

1. a natural number $k \in \{3, 4, 5, 6\}$,
2. a $\mathbb{Z}/k\mathbb{Z}$-torsor $I$,
3. a real number $d$,
4. real constants $a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij}$, for $i, j \in I$.
5. a set of edges $J \subset I\{\{i, 1 + i\} \ : \ i \in I\}$,
6. subsets $I_{lo}, I_{hi}, I_{str} \subset I$.

The data is subject to the following conditions.

1. $d < 0.9$,
2. $a_{ij} = a_{ji}, b_{ij} = b_{ji}, \alpha_{ij} = \alpha_{ji}, \beta_{ij} = \beta_{ji}$, for $i, j \in I$.
3. $a_{ij} \le \alpha_{ij} \le \beta_{ij} \le b_{ij}$, for $i, j \in I$,
4. $$0 = a_{ii} \text{ and } 2 \le a_{ij} \text{ for all } i, j \in I \text{ such that } i \ne j.$$
5. Also,
$$\begin{cases} b_{i,i+1} < 4, & \text{if } k = 3 \\ b_{i,i+1} \le c_{\mathrm{stab}}, & \text{if } k > 3. \end{cases}$$
6. If $\{i, j\} \in J$, then $[a_{ij}, b_{ij}] = [\sqrt{8}, c_{\mathrm{stab}}]$.
7. $m+k \le 6$, where $m$ is the number of edges $\{i, i+1\} \subset I$ such that $b_{i,i+1} > 2h_0$ or $a_{i,i+1} > 2$. In particular, $\mathrm{card}(J) + k \le 6$.

For each SCS $s$, we write $k(s)$, $d(s)$, $I(s)$, $a_{ij}(s)$, and so forth for the associated parameters. We simply write $k$, $d$, and so forth when there is a single SCS $s$ in a given context.

**Definition 1.11** (unadorned, basic) [SDJTENL] We say that a SCS $s$ is *unadorned* if the following additional properties hold (with established notation):

1. For all $i, j \in I(s)$, $a_{ij}(s) = \alpha_{ij}(s)$ and $b_{ij}(s) = \beta_{ij}(s)$.
2. $I_{lo}(s) = I_{hi} = I_{str}(s) = \emptyset$.

We say the SCS $s$ is *basic* if it is unadorned and $J(s) = \emptyset$.

Intuitively, we think of the SCS as involving both *hard* and *soft* constraints. The soft constraints are those that involve the adornments $\alpha_{ij}, \beta_{ij}, I_{lo}, I_{hi}, I_{str}$.

**Example 1.12** *We may always transform a SCS $s$ into another $s'$ that is unadorned by setting $\alpha_{ij}(s') = a_{ij}(s)$, $\beta_{ij}(s') = b_{ij}(s)$, $I_{lo}(s') = I_{hi}(s') = I_{str}(s') = \emptyset$, and keeping the rest of the data the same.*

**Definition 1.13** (ear) [HVFQIBQ] [$a_{ij}$ ⇝ a_ear0] [$b_{ij}$ ⇝ b_ear0] [ear ⇝ ear_sy] We have an unadorned SCS $s$ given by $k = 3$, $d = 0.11$, $J$ a singleton, and

$$[a_{ij}, b_{ij}] = \begin{cases} [0,0], & \text{if } i = j, \\ [\sqrt{8}, c_{\mathrm{stab}}], & \text{if } \{i,j\} \in J, \\ [2, 2h_0], & \text{otherwise.} \end{cases}$$

We call $s$ an *ear* (by analogy with an ear in a triangulation of a polygon, which is a triangle that has two of its edges in common with the polygon).

Next we associate a set $\mathcal{B}_s$ with each SCS $s$.

**Definition 1.14** ($\mathcal{B}_s$) [KTFVGXF] For every SCS $s$, and every function $\mathbf{v} : I(s) \to \mathbb{R}^3$, let $V_{\mathbf{v}} \subset \mathbb{R}^3$ be the image of $\mathbf{v}$. Let $E_{\mathbf{v}}$ be the image of $i \mapsto \{\mathbf{v}_i, \mathbf{v}_{i+1}\}$. Let $F_{\mathbf{v}}$ be the image of $i \mapsto (\mathbf{v}_i, \mathbf{v}_{i+1})$. Let $\mathcal{B}_s$ be the set of all functions $\mathbf{v}$ that have the following properties.

1. $V_{\mathbf{v}} \subset \mathcal{B}$.
2. $a_{ij}(s) \le \|\mathbf{v}_i - \mathbf{v}_j\| \le b_{ij}(s)$, for all $i, j \in I(s)$.
3. if $k(s) > 3$, then $(V_{\mathbf{v}}, E_{\mathbf{v}}, F_{\mathbf{v}})$ is a nonreflexive local fan.

Note that $\mathcal{B}_s$ does not depend on the data $\alpha_{ij}, \beta_{ij}, J, I_{str}, I_{lo}, I_{hi}$, and $d$. The set $J$ is used to make a small correction $d(s, \mathbf{v})$ to the constant $d(s)$.

**Definition 1.15** ($d(s, \mathbf{v})$) [TPLCZFL] [$\sigma$ ⇝ sigma_sy] Let $s$ be a SCS. Set $\sigma(s) = 1$ when $s$ is an ear; $\sigma = -1$, otherwise. Write

$$d(s, \mathbf{v}) = d(s) + 0.1\,\sigma(s) \sum_{\{i,j\} \in J(s)} (c_{\mathrm{stab}} - \|\mathbf{v}_i - \mathbf{v}_j\|). \tag{1.16}$$

This correction to $d(s)$ makes it a bit easier to prove inequalities when $\sigma(s) = -1$, at the cost of slightly more difficult inequalities for ears.

When $k(s) = 3$ and $\mathbf{v} \in \mathcal{B}_s$, the set $V_{\mathbf{v}} = \{\mathbf{v}_i \ : \ i \in I(s)\}$ may degenerate to planar configurations, because the local fan constraint in Definition 1.14 is not imposed in this case. Nevertheless, by the constraint $b_{ij}(s) < 4$, the tetrahedron $\{\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ has well-defined dihedral angles $\mathrm{dih}_V\{\mathbf{0}, \mathbf{v}_i\}\{\mathbf{v}_{i+1}, \mathbf{v}_{i+2}\}$, so that $\tau_3(V_{\mathbf{v}})$ is defined.

**Definition 1.17** ($\tau^*$) `[BGCEUKP]` Let $s$ be a SCS. Define

$$\tau^* : \{(s, \mathbf{v}) \ : \ \mathbf{v} \in \mathcal{B}_s\} \to \mathbb{R}$$

by

$$\tau^*(s, \mathbf{v}) = \begin{cases} \tau(V_{\mathbf{v}}, E_{\mathbf{v}}, F_{\mathbf{v}}) - d(s, \mathbf{v}), & \text{if } k(s) > 3 \\ \tau_3(V_{\mathbf{v}}) - d(s, \mathbf{v}), & \text{if } k(s) = 3. \end{cases}$$

**Definition 1.18** ($S_{\text{init}}$) `[XOSFOMP]` The constants in the conclusions of the main estimate (Theorem 1.6) can be packaged into unadorned SCSs. For example, the standard main estimate for $k = 6$ gives the SCS data: $d = 0.712$, $J = \emptyset$, $I$ an indexing set of cardinality six, and

$$a_{ij} = \begin{cases} 0, & i = j, \\ 2, & j = i \pm 1, \\ 2h_0, & \text{otherwise}, \end{cases} \qquad b_{ij} = \begin{cases} 0, & i = j, \\ 2h_0, & j = i \pm 1, \\ 4h_0^+, & \textit{otherwise}, \end{cases}$$

where $h_0^+$ is any constant greater than $h_0$. The upper bound $4h_0$ on any diagonal comes from the triangle inequality: $\|\mathbf{v}_i - \mathbf{v}_j\| \le \|\mathbf{v}_i\| + \|\mathbf{v}_j\| \le 4h_0$. We write $S_{\text{init}}$ for the set of SCSs $s$, for all cases of the main estimate.

**Lemma 1.19** `[ZITHLQN][formal proof by Hoang Le Truong].` *The main estimate holds if and only if for every $s \in S_{init}$ and for every $\mathbf{v} \in \mathcal{B}_s$, we have $\tau^*(s, \mathbf{v}) \ge 0$.*

*Proof* This follows by expanding the definition of $S_{\text{init}}$ and $\mathcal{B}_s$. Note that the set $J(s)$ is empty for $s \in S_{\text{init}}$, so

$$d(s, \mathbf{v}) = d(s), \text{ for all } s \in S_{\text{init}}.$$

□

## 1.3 minimization

**Definition 1.20** (index, $\iota$, $\mathcal{M}_s$) `[FNUEPJW]` Let $s$ be a SCS. Let

$$\mathcal{M}_s \subset \mathcal{B}_s'' \subset \mathcal{B}_s' \subset \mathcal{B}_s$$

be defined as follows. Let $\mathcal{B}_s'$ be the set of all $\mathbf{v} \in \mathcal{B}_s$ such that

1. $\tau^*(s, \mathbf{v})$ is equal to the minimum of $\tau^*(s, *)$ over $\mathcal{B}_s$.
2. $\tau^*(s, \mathbf{v}) < 0$.

Define the *index* $\iota(s, \mathbf{v})$ of $\mathbf{v} \in \mathcal{B}_s$ to be the number of edges $\{i, j\}$ of $I(s)$ for which $a_{ij}(s) = \|\mathbf{v}_i - \mathbf{v}_j\|$. Let $\iota(s)$ be the minimum of the index of $\mathbf{v}$ as $\mathbf{v}$ runs over $\mathcal{B}_s'$. We let $\mathcal{B}_s''$ be the set of $\mathbf{v} \in \mathcal{B}_s'$ that attain the smallest possible index $\iota(s)$, and let $\mathcal{M}_s \subset \mathcal{B}_s''$ be the subset of all $\mathbf{v}$ satisfying the additional soft constraints.

1. If $i \in I_{str}$, then $\mathbf{v}_i$ is straight.
2. If $i \in I_{lo}$, then $\|\mathbf{v}_i\| = 2$.
3. If $i \in I_{hi}$, then $\|\mathbf{v}_i\| = 2h_0$.
4. $\alpha_{ij}(s) \leq \|\mathbf{v}_i - \mathbf{v}_j\| \leq \beta_{ij}(s)$ for all $i, j$.

Note that if $s$ is unadorned, then $\mathcal{M}_s = \mathcal{B}_s''$.

**Lemma 1.21** `[GKFMJLC]` *$s$ is a SCS for every $s \in S_{init}$.*

*Proof* This is a direct result of definitions. □

**Lemma 1.22** `[UXCKFPE][old:XWITCCN][formal proof by Hoang Le Truong].` *Let $s$ be a SCS, and assume that*

$$\tau^*(s, \mathbf{v}) < 0.$$

*for some $\mathbf{v} \in \mathcal{B}_s$. Then $\mathcal{B}_s'$ is nonempty.*

*Proof* Let $s$ be a SCS. Then by DSP [1], $\mathcal{B}_s$ is compact (as a subset of $\mathcal{B}^k \subset \mathbb{R}^{3k}$).

The function

$$\mathbf{v} \mapsto \tau^*(s, \mathbf{v})$$

is a continuous function on $\mathcal{B}_s$. Moreover, if $\mathcal{B}_s$ is nonempty, then the function attains a minimum. This follows from Lemma HDPLYGY and Lemma PCRTTID. The set of minima is then nonempty. □

**Lemma 1.23** `[SGTRNAF][old:AYQJTMD][formal proof by Hoang Le Truong].` *Let $s$ be an unadorned SCS, and assume that*

$$\tau^*(s, \mathbf{v}) < 0.$$

*for some $\mathbf{v} \in \mathcal{B}_s$. Then $\mathcal{M}_s$ is nonempty.*

*Proof* By Lemma 1.22, the set $\mathcal{B}'_s$ of minimizers is nonempty. The subset $\mathcal{B}''_s$ on which the index is as small as possible is then also nonempty. By assumption, $s$ is unadorned, and $\mathcal{B}''_s = \mathcal{M}_s$. □

**Lemma 1.24** `[EAPGLEJ][formal proof by Hoang Le Truong].` *The main estimate holds if and only if $\mathcal{M}_s = \emptyset$ for all $s \in S_{init}$.*

*Proof* By Lemma 1.23 and Lemma 1.19. □

**Lemma 1.25** `[JKQEWGV][formal proof by Hoang Le Truong].` *Let $s$ be a SCS. Let $\mathbf{v} \in \mathcal{B}_s$. Suppose that $\tau^*(s, \mathbf{v}) < 0$ and $k(s) > 3$. Then* $\mathrm{sol}(V_{\mathbf{v}}, E_{\mathbf{v}}, F_{\mathbf{v}}) < \pi$. *Furthermore, the local fan is not circular, and the local fan can be lunar only when the pole has acute interior angle.*

*Proof* By the definition of SCS, we have $d(s) < 0.9$. The proof of Lemma GBYCPXS extends readily to this context. The solid angle of a lune is less than $\pi$ if and only if the pole has acute interior angle. □

## 1.4 operations

This section describes some operations on SCSs. The first of these, a restriction, is a *hardening* of some of the soft constraints.

**Definition 1.26** (restriction) `[PFEOBSC]` Let $s$ be a SCS. We say that $s'$ is a *restriction* of $s$ of the *first type* if $I(s) = I(s')$ and

$$b_{ij}(s') = \beta_{ij}(s), \quad i, j \in I(s).$$

and all other parameters $k$, $d$, $J$, $a$, $\alpha$, $\beta$, etc. are the same for $s$ and $s'$.

We say that $s'$ is a *restriction* of $s$ of the *second type* if

1. $I(s) = I(s')$,
2. $\alpha_{ij}(s) = \beta_{ij}(s)$ for all $i, j \in I(s)$,
3. $a_{ij}(s') = b_{ij}(s') = \alpha_{ij}(s)$, and
4. all other parameters $k$, $d$, etc. are the same for $s$ and $s'$.

We say that $s'$ is a restriction of $s$ if it is a restriction of the first or second type.

**Definition 1.27** (subdivision) `[YYKMEWW]` Let $s$ be a SCS, and let $p, q \in I(s)$, with $p \neq q$. Let $c \in \mathbb{R}$. Define constants $\beta_0 = \min(b_{pq}(s), c)$ and $\alpha_0 = \max(a_{pq}(s), c)$. Define $s_1$ to be the same as $s$ except that

$$\beta_{pq}(s_1) = \beta_0, \quad b_{pq}(s_1) = c.$$

Define $s_2$ to be the same as $s$ except that

$$a_{pq}(s_2) = c, \quad \alpha_{pq}(s_2) = \alpha_0.$$

Define the subdivision of $s$ to be the following list of one or two SCSs, according to the case.

$$\begin{cases} c \leq a_{pq}(s), & [s] \\ a_{pq}(s) < c \leq \alpha_{pq}(s), & [s_2] \\ \alpha_{pq}(s) < c < \beta_{pq}(s), & [s_1; s_2] \\ \beta_{pq}(s) \leq c < b_{pq}(s), & [s_1] \\ b_{pq}(s) \leq c, & [s] \end{cases}$$

The subdivision thus corresponds to splitting an interval $[a_{pq}, b_{pq}]$ into

$$[a_{pq}, c] \cup [c, b_{pq}].$$

**Definition 1.28** `[LCTBALA]` We say that a SCS $s$ *transfers* to a SCS $s'$ if

1. If $s$ is an ear, then $s = s'$.
2. $s'$ is unadorned.
3. $I(s) = I(s')$.
4. $d(s) \leq d(s')$.
5. For all $i, j$, we have $a_{ij}(s') \leq a_{ij}(s) \leq b_{ij}(s) \leq b_{ij}(s')$.
6. $J(s') \subset J(s)$.

**Definition 1.29** (equivalent SCS) If $s$ is a SCS with torsor $I(s)$, and if $I'$ is any equivalent torsor, then we can use the bijection between $I(s)$ and $I'$ to obtain a SCS $s'$ with $I' = I(s')$. An SCS $s'$ related in this way to $s$ is said to be *equivalent* to $s$. An *equi-transfer* of $s$ to $s'$ is a transfer from $s$ to a SCS that is equivalent to $s'$.

**Definition 1.30** (torsor slice) `[ZTBHGMO]` Let $I$ be a $\mathbb{Z}/k\mathbb{Z}$-torsor, with action given by $(j, i) \mapsto j + i$, for $i \in I$. Let $p, q \in I$ be non-adjacent. Set

$$I[p, q] = \{p, 1 + p, 2 + p, \ldots, q\} \subset I.$$

Note that the cardinality of $I[p, q]$ is

$$m = 1 + \min\{j \in \mathbb{N} \ : \ j + p = q\}.$$

We make $I[p,q]$ into a $\mathbb{Z}/m\mathbb{Z}$-torsor with action $(j,i) \mapsto j +' i$, given by the iterates of

$$1 +' i = \begin{cases} 1 + i & \text{if } i \neq q, \\ p & \text{if } i = q. \end{cases}$$

The $\mathbb{Z}/m\mathbb{Z}$-torsor $I[p,q]$ is called the *slice* of $I$ along $(p,q)$.

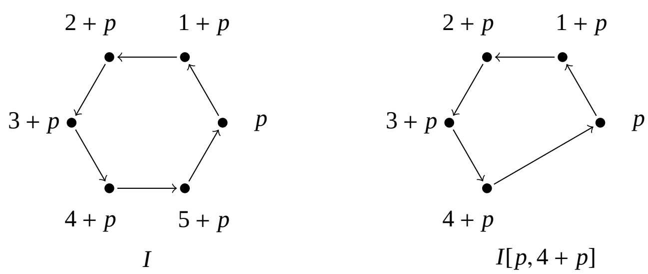


Figure 1.1 `[WKUYEXM]` Given $p, q \in I$, the slice $I[p,q]$ follows the cyclic order through $I$ from $p$ to $q$, then returns directly from $q$ to $p$.

**Definition 1.31** (SCS slice) `[CJMHFAT]` Let $s$ be a SCS and let $\{p,q\} \subset I(s)$ be a diagonal. (In particular, $k(s) > 3$.) We say that a pair $\{s', s''\}$ is a *slice* along the diagonal $\{p,q\}$ of $s$, if the following conditions hold.

1. $s'$ and $s''$ are unadorned SCSs.
2. $I(s') = I[p,q]$ and $I(s'') = I[q,p]$.
3. $d(s) \leq d(s') + d(s'')$.
4. $J(t) \subset J(s) \cup \{\{p,q\}\}$, for $t = s', s''$.
5. $\{p,q\} \in J(s')$ iff $\{p,q\} \in J(s'')$.
6. $\{p,q\} \in J(s')$ iff $s'$ or $s''$ is an ear.
7. $$a_{ij}(t) = a_{ij}(s), \text{ and } b_{ij}(t) = b_{ij}(s),$$
   for $t = s', s''$ and all $i, j \in I(t)$ except the given diagonal $\{i, j\} = \{p,q\}$.
8. $a_{pq}(t) = \alpha_{pq}(s), \quad b_{pq}(t) = \beta_{pq}(s), \quad t = s', s''$.
9. $\beta_{pq}(s) < 4$.
10. $k(s) = 4$ or $\beta_{pq}(s) \leq c_{\text{stab}}$.

The word *slice* is used for related operations on the indexing set, the local fan, and the SCS. A slice of a SCS is used in parallel with the slice a fan of cardinality $k(s)$ into two smaller fans with cardinalities $k(s')$ and $k(s'')$. All of the edge length constraints are to be preserved under slicing.

If $\{s', s''\}$ is an slice of a diagonal $\{p, q\}$, then we can use the inclusions of sets $I(s') \subset I(s)$ and $I(s'') \subset I(s)$ to restrict an element $\mathbf{v} : I(s) \to \mathcal{B}$ to $\mathbf{v}' : I(s') \to \mathcal{B}$ and $\mathbf{v}'' : I(s'') \to \mathcal{B}$.

**Lemma 1.32** `[QKNVMLB] [formal proof by Hoang Le Truong].` *Let $s$ be a SCS with diagonal $\{p, q\}$ and slice $\{s', s''\}$ along $\{p, q\}$. Let $\mathbf{v} \in \mathcal{M}_s$ and let $\mathbf{v}'$ and $\mathbf{v}''$ be constructed from $\mathbf{v}$ as above. Then $\mathbf{v}' \in \mathcal{B}_{s'}$ and $\mathbf{v}'' \in \mathcal{B}_{s''}$. Moreover,*

$$d(s, \mathbf{v}) \le d(s', \mathbf{v}') + d(s'', \mathbf{v}'') \tag{1.33}$$

*and*

$$\tau^*(s, \mathbf{v}) \ge \tau^*(s, \mathbf{v}') + \tau^*(s, \mathbf{v}''). \tag{1.34}$$

*Proof* See Lemma MTUWLUN. □

## 1.5 propagation

The proof of the main lemma consists in showing that the nonemptiness of $\mathcal{M}_s$ propagates in an orderly way under the operations of restriction, slicing, equivalence, subdivision, and deformation.

**Definition 1.35** ($\Rightarrow$) `[AZGJNZO] [⇒ ⇝ scs_arrow...]` Let **SCS** be the set of SCSs and let $P(\mathbf{SCS})$ be the powerset of **SCS**. We define a binary relation ($\Rightarrow$) on $P(\mathbf{SCS})$. When $S, T \subset \mathbf{SCS}$, we write $S \Rightarrow T$ to mean either that $\mathcal{M}_s = \emptyset$ for all $s \in S$, or that there exists $t \in T$ such that $\mathcal{M}_t \ne \emptyset$.

**Lemma 1.36** `[FZIOTEF] [formal proof by Hales].` *The relation* ($\Rightarrow$) *is reflexive and transitive.*

*Proof* Clearly, $\{s\} \Rightarrow \{s\}$, and this implies reflexivity. Transitivity is a simple matter. Assume $S_1 \Rightarrow S_2$ and $S_2 \Rightarrow S_3$. Assume $s_1 \in S_1$, with $\mathcal{M}_{s_1} \ne \emptyset$. Then select $s_2 \in S_2$ such that $\mathcal{M}_{s_2} \ne \emptyset$, and $s_3 \in S_3$ such that $\mathcal{M}_{s_3} \ne \emptyset$. Then $\{s_1\} \Rightarrow \{s_3\}$. So $S_1 \Rightarrow S_3$. □

**Lemma 1.37** (restriction) `[EQTTNZI] [formal proof by Hoang Le Truong].` *Let $s$ be a SCS, and let $t$ be a restriction of $s$. When $t$ has the first type, assume also that for all $\{i, j\} \in J$, we have $\beta_{ij}(s) = b_{ij}(s)$. Assume further for this type that $J(s) = \emptyset$ or $k(s) > 3$. When $t$ has the second type, assume also that $J(s) = \emptyset$ and $m(t) + k(t) \le 6$, with $m$ as in the definition of SCS. Then $\{s\} \Rightarrow \{t\}$.*

*Proof* $\mathcal{B}_t \subset \mathcal{B}_s$. Assume $\mathbf{w} \in \mathcal{M}_s$. The assumptions according to the type give $\mathbf{w} \in \mathcal{B}_t$.

Assume the restriction has the first type. Then a global minimizer $\mathbf{w}$ also minimizes on a subset, and the index does not change. Hence $\mathbf{w} \in \mathcal{M}_t$ and $\{s\} \Rightarrow \{t\}$.

Assume that the restriction has the second type. Then $\alpha_{ij}(s) = \beta_{ij}(s)$. We have $\mathcal{B}_t \subset \mathcal{B}_s$. For every $\mathbf{v} \in \mathcal{B}'_t$, the index of $\mathbf{v}$ is $k$: every edge is fixed at its lower bound. Hence $\mathcal{B}'_t = \mathcal{B}''_t$, and $\mathbf{w} \in \mathcal{M}_t$. □

**Lemma 1.38** (subdivision) `[UAGHHBM]` `[formal proof by Hoang Le Truong]`. *Let $s$ be a SCS and let Let $\{s_1, s_2\}$ be a subdivision of $s$ along $\{i, j\} \subset I(s)$. Then* $\{s\} \Rightarrow \{s_1, s_2\}$.

*Proof* Let $c$ denote the constant used for the subdivision. We have $\mathcal{B}_s = \mathcal{B}_{s_1} \cup \mathcal{B}_{s_2}$. If $\mathbf{w} \in \mathcal{M}_s$, then $\mathbf{w} \in \mathcal{B}'_{s_1} \cup \mathcal{B}'_{s_2}$.

Assume first that there exists $\mathbf{w} \in \mathcal{M}_s$ such that $c < \|\mathbf{w}_i - \mathbf{w}_j\|$. We claim that $\iota(s) \le \iota(s_2)$. In fact,

$$\begin{aligned}
\iota(s) &= \min_{\mathbf{v}\in\mathcal{B}'_s} \iota(s, \mathbf{v}) \\
&\le \min_{\mathbf{v}\in\mathcal{B}'_{s_2}} \iota(s, \mathbf{v}) \\
&\le \min_{\mathbf{v}\in\mathcal{B}'_{s_2}} \iota(s_2, \mathbf{v}) \\
&= \iota(s_2).
\end{aligned}$$

We have $\mathbf{w} \in \mathcal{B}'_{s_2}$ and $\mathcal{B}'_{s_2} = \mathcal{B}'_s \cap \mathcal{B}_{s_2}$. Also,

$$\iota(s_2) \ge \iota(s) = \iota(s, \mathbf{w}) = \iota(s_2, \mathbf{w}) \ge \iota(s_2)$$

so that $\mathbf{w} \in \mathcal{B}''_{s_2}$ and $\mathbf{w} \in \mathcal{M}_{s_2}$.

Finally, assume that every $\mathbf{w} \in \mathcal{M}_s$ has $\|\mathbf{w}_i - \mathbf{w}_j\| \le c$. This means $\mathcal{M}_s \subset \mathcal{B}_{s_1}$. Let $\mathbf{w} \in \mathcal{M}_s$. We claim that $\iota(s) = \iota(s_1)$. In fact,

$$\begin{aligned}
\iota(s) &= \min_{\mathbf{v}\in\mathcal{B}'_s} \iota(s, \mathbf{v}) \\
&\le \min_{\mathbf{v}\in\mathcal{B}'_{s_1}} \iota(s, \mathbf{v}) \\
&= \min_{\mathbf{v}\in\mathcal{B}'_{s_1}} \iota(s_1, \mathbf{v}) \\
&= \iota(s_1).
\end{aligned}$$

Also,

$$\iota(s) = \iota(s, \mathbf{w}) = \iota(s_1, \mathbf{w}) \ge \iota(s_1).$$

We have $\mathbf{w} \in \mathcal{M}_s \subset \mathcal{B}'_s \cap \mathcal{B}_{s_1} = \mathcal{B}'_{s_1}$. Since $\iota(s) = \iota(s_1)$, we also have $\mathbf{w} \in \mathcal{B}''_{s_1}$. So $\mathbf{w} \in \mathcal{M}_{s_1}$. □

**Lemma 1.39** (transfer, equivalence) `[YXIONXL]` `[formal proof by Hoang Le Truong].` *Let $s$ be a SCS and let $t$ be a transfer or equivalent of $s$. Then* $\{s\} \Rightarrow \{t\}$.

*Proof* We treat transfer first. Assume that $\mathbf{w} \in \mathcal{M}_s$. We have $\mathbf{w} \in \mathcal{B}_s \subset \mathcal{B}_t$, and $d(t, \mathbf{w}) \geq d(s, \mathbf{w})$, as well as $\tau^*(t, \mathbf{w}) \leq \tau^*(t, \mathbf{w}) < 0$. By Lemma 1.22, there exists a global minimizer of smallest index $\mathbf{v} \in \mathcal{B}''_t \neq \emptyset$. It satisfies $\tau^*(t, \mathbf{v}) \leq \tau^*(t, \mathbf{w}) < 0$. By the definition of transfer, $t$ is unadorned, so that $\mathcal{B}''_t = \mathcal{M}_t$. (Note that $\mathcal{M}_s$ and $\mathcal{M}_t$ might not be directly related.) This shows that $\mathcal{M}_t \neq \emptyset$. Hence $\{s\} \Rightarrow \{t\}$. Similarly, if $t$ is equivalent to $s$, then we again have $\{s\} \Rightarrow \{t\}$. □

**Lemma 1.40** (slice) `[LKGRQUI]` `[formal proof by Hoang Le Truong].` *Let $\{s', s''\}$ be a slice of a SCS $s$ along a diagonal $\{p, q\}$. Then $\{s\} \Rightarrow \{s', s''\}$.*

*Proof* Assume that $\mathbf{w} \in \mathcal{M}_s$. Let $\mathbf{w}' \in \mathcal{B}_{s'}$ and $\mathbf{w}'' \in \mathcal{B}_{s''}$ be obtained by restriction of parameters. From Lemma 1.32, we have $\tau^*(s', \mathbf{w}') < 0$ or $\tau^*(s'', \mathbf{w}'') < 0$. To be concrete, say $\tau^*(s', \mathbf{w}') < 0$. A global minimizer $\mathbf{v}' \in \mathcal{B}_{s'}$ of smallest index then also satisfies $\tau^*(s', \mathbf{v}') < 0$. By the definition of slice, $s'$ is unadorned, so that $\mathcal{B}''_{s'} = \mathcal{M}_{s'}$ and $\mathbf{v}' \in \mathcal{M}_{s'}$. Hence $\{s\} \Rightarrow \{s', s''\}$. □

## 1.6 deformation

This section proves some deformation results and implements them as arrows $\{s\} \Rightarrow T$.

**Lemma 1.41** `[HXHYTIJ]` `[formal proof by Hoang Le Truong].` *Let $s$ be a SCS, and let $\mathbf{v} \in \mathcal{B}''_s$. For Let $\mathbf{w} : I(s) \to \mathbb{R}^3$. Then one of the following holds:*

1. $\mathbf{w} \notin \mathcal{B}_s$.
2. $\tau^*(s, \mathbf{w}) > \tau^*(s, \mathbf{v})$.
3. $\tau^*(s, \mathbf{w}) = \tau^*(s, \mathbf{v})$ *and the index of* $\mathbf{w}$ *is at least that of* $\mathbf{v}$.

*Proof* If the first two conclusions fail, then $\mathbf{w} \in \mathcal{B}'_s$. Since $\mathbf{v} \in \mathcal{B}''_s$, it must minimize the index over $\mathcal{B}'_s$. Hence the third conclusion holds. □

**Lemma 1.42** `[ODXLSTCv2]` `[formal proof by Hoang Le Truong].` *Let $s$ be a SCS and let $\mathbf{w} \in \mathcal{M}_s$. Fix $\ell \in I(s)$. Assume that $\mathbf{w}_\ell$ is not the pole of a lunar local fan $(V_{\mathbf{w}}, E_{\mathbf{w}}, F_{\mathbf{w}})$. Assume that $4h_0 < b_{\ell i}(s)$ for every diagonal $\{\ell, i\}$ at $\ell$. Then one of the following hard constraints hold at index $\ell$.*

1. $\|\mathbf{w}_\ell - \mathbf{w}_i\|$ *attains its lower bound $a_{\ell i}(s)$, for some $i \neq \ell$.*

2. $\|\mathbf{w}_\ell\|$ *attains its lower bound* 2.
3. *There exists $i$ adjacent to $\ell$ such that* $\{\ell, i\} \in J(s)$.

*Proof* For a contradiction, assume that none of the enumerated constraints hold. By Lemma 1.25, the fan is not circular. The hypotheses allow us to use Lemmas MHAEYJN and ZLZTHIC to deform $\mathbf{w}$ at $\mathbf{w}_\ell$.

The function $\tau^*$ is decreasing along the curve of the form [1][Equation 7.70] such that $\mathbf{w}_\ell(t) = (1-t)\mathbf{w}_\ell$. That is, we push the point $\mathbf{w}_\ell$ radially towards the origin. Explicitly,

$$\tau^*(s, w) = c_1 + c_2\|\mathbf{w}_\ell(t)\|$$

for some $c_2 > 0$ and $c_1$. We have $\mathbf{w}(t) \in \mathcal{B}_s$ for all $t$ positive and sufficiently small. This contradicts the minimality properties of $\mathcal{M}_s$. □

**Lemma 1.43** [IMJXPHRv2] *Let $s$ be a SCS, $\mathbf{w} \in \mathcal{M}_s$, and $\mathbf{w}_\ell$ straight. Assume that $\mathbf{w}_\ell$ is not the pole of a lunar local fan $(V_\mathbf{w}, E_\mathbf{w}, F_\mathbf{w})$. Assume that $4h_0 < b_{\ell i}(s)$ for every diagonal $\{\ell, i\}$ at $\ell$. Assume that $\mathbf{w}_\ell \in \mathrm{aff}_+(\mathbf{0}, \{\mathbf{w}_{\ell+1}, \mathbf{w}_{\ell-1}\})$. Then one of the following hard constraints holds at $\ell$.*

1. $\|\mathbf{w}_\ell - \mathbf{w}_{\ell+1}\|$ *attains its lower bound* $a_{\ell,\ell+1}(s)$, *and* $\|\mathbf{w}_\ell - \mathbf{w}_{\ell-1}\|$ *attains its lower bound* $a_{\ell,\ell-1}(s)$.
2. $\|\mathbf{w}_\ell\|$ *attains its lower bound* 2.
3. *There exists $i$ adjacent to $\ell$ such that* $\{\ell, i\} \in J(s)$.
4. *Some diagonal* $\{\ell, i\} \subset I(s)$ *at $\ell$ satisfies* $\|\mathbf{w}_\ell - \mathbf{w}_i\| = a_{\ell i}(s)$.

*Proof* For a contradiction, assume that none of the enumerated constraints hold. By Lemma 1.25, the fan is not circular. The hypotheses allow us to use Lemmas MHAEYJN and ZLZTHIC to deform $\mathbf{w}$ at $\mathbf{w}_\ell$.

The set $\{\mathbf{0}, \mathbf{w}_{\ell-1}, \mathbf{w}_\ell, \mathbf{w}_{\ell+1}\}$ lies in a plane $A$. By the previous lemma one of the norm constraints is satisfied, say

$$\|\mathbf{w}_\ell - \mathbf{w}_{\ell+1}\| = a_{\ell,\ell+1}(s).$$

We consider a deformation of $\mathbf{w}$ that moves $\mathbf{w}_\ell(t)$ along a circle through $\mathbf{w}_\ell$ with center $\mathbf{w}_{\ell+1}$ in the plane $A$. Parameterize the curve so that as $t$ increases, the norm $\|\mathbf{w}_\ell(t)\|$ decreases. The function $\tau^*(s, \mathbf{w})$ is decreasing in $t$. Explicitly, the function again depends linearly on $\|\mathbf{w}_\ell(t)\|$, because the angle at $\ell$ remains straight. This contradicts the minimality of $\mathcal{M}_s$. The result ensues. □

**Lemma 1.44** [NUXCOEAv2] *Let $s$ be a SCS and let $\mathbf{w} \in \mathcal{M}_s$. Fix $\ell \in I_{str}(s)$ Assume $j$ is an index adjacent to $\ell$ such that $\|\mathbf{w}_\ell - \mathbf{w}_j\| = a_{\ell j}(s)$. Assume that $\mathbf{w}_\ell$ is not the pole of a lunar local fan $(V_\mathbf{w}, E_\mathbf{w}, F_\mathbf{w})$. Assume that $4h_0 < b_{\ell i}(s)$ for every diagonal $\{\ell, i\}$ at $\ell$. Then one of the following hard conditions holds at index $\ell$.*

1. *We have* $\|\mathbf{w}_\ell - \mathbf{w}_i\| = a_{\ell k}(s)$ *for both choices of* $i \in I(s)$ *adjacent to* $\ell$.
2. *There exists* $i$ *adjacent to* $\ell$ *such that* $\{\ell, i\} \in J(s)$.
3. *Some diagonal* $\{\ell, i\} \subset I(s)$ *at* $\ell$ *satisfies* $\|\mathbf{w}_\ell - \mathbf{w}_i\| = a_{\ell i}(s)$.

*Proof* The lemma is a special case of the previous lemma, unless $\|\mathbf{w}_\ell\| = 2$, which we assume. Let $i \neq j$ be the other index adjacent to $\ell$. Assume that the three enumerated parts of the conclusion fail.

We consider a deformation of $\mathbf{w}$ that only moves $\mathbf{w}_\ell$. We take the motion of $\mathbf{w}_\ell$ to be in a circular arc with center $\mathbf{0}$ through the point $\mathbf{w}_\ell$ and in the fixed plane determined by $\{\mathbf{0}, \mathbf{w}_\ell, \mathbf{w}_j, \mathbf{w}_i\}$. The function $\tau^*(s, *)$ is constant along this curve. We orient the curve to be increasing in $\|\mathbf{w}_\ell - \mathbf{w}_j\|$. For sufficiently, small $t$, we find that $\mathbf{w}(t) \in \mathcal{B}'_s$ has smaller index than $\mathbf{w}$. This is contrary to the minimizing properties of $\mathbf{w} \in \mathcal{M}_s$.

As in the proofs of the previous lemmas, the fan is not circular, and the constraints on generic and local fans allow us to use Lemmas MHAEYJN and ZLZTHIC, showing that fan conditions are preserved. □

In the preceding three lemmas, we specifically allow the deformations $\mathbf{v}(t)$ to occur within a lunar fan, moving a single node that is not a pole of the lunar fan, as given by Lemma MHAEYJN. We consider some further deformations on generic $\mathbf{v}$.

**Lemma 1.45** (482) `[CUXVZOZ]` *Let* $s$ *be a basic SCS, and let* $p \in I(s)$. *Let* $\mathbf{v} \in \mathcal{M}_s$ *be generic. We make the following assumptions.*

1. *For all diagonals except* $\{p-1, p+1\}$, *we have* $a_{i,j}(s) < \|\mathbf{v}_i - \mathbf{v}_j\|$.
2. $3 \le \|\mathbf{v}_{p-1} - \mathbf{v}_{p+1}\|$.
3. $\angle \mathbf{v}_p < pi$.
4. $\pi/2 < \angle \mathbf{v}_p$ *or* $\angle \mathbf{v}_{p+1} < \pi$.
5. *The constraints along* $\{p, p+1\}$ *are standard.*
6. $2 \le \|\mathbf{v}_p - \mathbf{v}_{p-1}\| \le c_{stab}$.

*Then* $\|\mathbf{v}_p - \mathbf{v}_{p+1}\| = 2$.

*Proof* We construct a deformation using Lemma 3.1. Specifically, we take the base triangle to be $\{\mathbf{0}, \mathbf{v}_{p-1}, \mathbf{v}_{p+1}\}$ and variable point to be $\mathbf{v}_p$ with variable edge $\{\mathbf{v}_p, \mathbf{v}_{p+1}\}$. `LEMMA_4828966562` states that the condition $\|\mathbf{v}_p - \mathbf{v}_{p+1}\| > 2$ is not a local minimum. Because of the condition $\pi/2 < \angle \mathbf{v}_p < \pi$, decreasing $\|\mathbf{v}_p - \mathbf{v}_{p+1}\|$ along this deformation will preserve the nonreflexivity condition of a nonreflexive local fan. □

**Lemma 1.46** (482 bis) `[CJBDXXN]` *Let* $s$ *be a basic SCS, and let* $p \in I(s)$. *Let* $\mathbf{v} \in \mathcal{M}_s$ *be generic. Assume that for all diagonals except* $\{p-1, p+1\}$, *we* *have $a_{i,j}(s) < \|\mathbf{v}_i - \mathbf{v}_j\|$. Assume $3 \le \|\mathbf{v}_{p-1} - \mathbf{v}_{p+1}\|$. Assume $\pi/2 < \angle\mathbf{v}_p < \pi$ or $\angle\mathbf{v}_{p-1} < \pi$. Assume that the constraints along $\{p-1, p\}$ are standard. Assume $2 \le \|\mathbf{v}_p - \mathbf{v}_{p+1}\| \le c_{stab}$. Then $\|\mathbf{v}_p - \mathbf{v}_{p-1}\| = 2$.*

*Proof* We construct a deformation using Lemma 3.1 as in the previous lemma, now using $\{\mathbf{v}_p, \mathbf{v}_{p-1}\}$ as the variable edge. □

Abbreviate $\mathrm{xrr}_V(\mathbf{u}, \mathbf{v}) = \mathrm{xrr}(\|\mathbf{u}\|, \|\mathbf{v}\|, \|\mathbf{u} - \mathbf{v}\|)$.

**Lemma 1.47** *684 pent*`[IUNBUIG]` *Let $s$ be a basic SCS with $k(s) = 5$. Let $p \in I(s)$ and $\mathbf{v} \in \mathcal{M}_s$ be generic. Assume that for all diagonals, we have $a_{i,j}(s) \le c_{stab} < \|\mathbf{v}_i - \mathbf{v}_j\|$ and $4h_0 < b_{ij}(s)$. Assume that $\angle\mathbf{v}_{p+2} < \pi$ and $\angle\mathbf{v}_{p+3} < \pi$. Assume that $2 \le a_{p+2,p+3} \le b_{p+2,p+3} \le cstab$. Assume that $\|\mathbf{v}_i - \mathbf{v}_j\| = 2$ for every edge other than $\{p+2, p+3\}$. Assume that $2 \le \|\mathbf{v}_i - \mathbf{v}_j\|$ and $\mathrm{xrr}_V(\mathbf{v}_i, \mathbf{v}_j) \le 15.53$, for $\{i, j\} = \{p, p+2\}, \{p, p+3\}$. Then $\|\mathbf{v}_{p+2} - \mathbf{v}_{p+3}\| = a_{p+2,p+3}$.*

We only need to use the special case $(p+2, p+3) = (0, 1)$.

*Proof* We construct a deformation using Lemma 3.1. We use $\{\mathbf{0}, \mathbf{v}_p, \mathbf{v}_{p+3}\}$ as the base triangle with constructed point $\mathbf{v}_{p+2}$ and variable edge $\{p+2, p+3\}$. We construct a second time on the base triangle $\{\mathbf{0}, \mathbf{v}_p, \mathbf{v}_{p+2}(t)\}$ with constructed point $\mathbf{v}_{p+1}$. `LEMMA_6843920790` asserts that $\|\mathbf{v}_{p+2} - \mathbf{v}_{p+3}\| > a_{p+2,p+3}$ is incompatible with the minimality of $\mathbf{v}$. The conditions $\angle\mathbf{v}_{p+2} < \pi$ and $\angle\mathbf{v}_{p+3} < \pi$ are used to maintain nonreflexivity.

To use `LEMMA_6843920790`, we must check the precondition that $\{\mathbf{0}, \mathbf{v}_p, \mathbf{v}_{p+2}, \mathbf{v}_{p+3}\}$ is not coplanar. We leave this as an exercise based on the triangle inequality? □

**Lemma 1.48** (1834) `[NEHXMWH]` *Let $s$ be a basic SCS, with $k \le 4$, and let $p \in I(s)$. Let $\mathbf{v} \in \mathcal{M}_s$ be generic. Assume that for all diagonals we have $a_{i,j}(s) \le c_{stab} < \|\mathbf{v}_i - \mathbf{v}_j\|$ and $4h_0 < b_{i,j}(s)$. Assume $\angle\mathbf{v}_i < \pi$ for all $i \ne p+2$. Then $\|\mathbf{v}_p - \mathbf{v}_{p+1}\| = a_{p,p+1}$ or $b_{p,p+1}$.*

*Proof* As in previous lemmas, use Lemma 3.1 to construct a deformation. Use $\{\mathbf{0}, \mathbf{v}_{p+1}, \mathbf{v}_{p-1}\}$ as the base triangle with contructed point $\mathbf{v}_p$, where $\|\mathbf{v}_{p+1} - \mathbf{v}_{p+2}\|$ varies. Continue as before with `LEMMA_1834976363`. □

**Lemma 1.49** (1834 sym) `[BZQNDMM]` *Let $s$ be a basic SCS, with $k \le 4$, and let $p \in I(s)$. Let $\mathbf{v} \in \mathcal{M}_s$ be generic. Assume that for all diagonals we have $a_{i,j}(s) \le c_{stab} < \|\mathbf{v}_i - \mathbf{v}_j\|$ and $4h_0 < b_{i,j}(s)$. Assume $\angle\mathbf{v}_i < \pi$ for all $i \ne p-1$. Then $\|\mathbf{v}_p - \mathbf{v}_{p+1}\| = a_{p,p+1}$ or $b_{p,p+1}$.*

*Proof* As in previous lemmas, use Lemma 3.1 to construct a deformation. Use $\{\mathbf{0}, \mathbf{v}_p, \mathbf{v}_{p+2}\}$ as the base triangle with contructed point $\mathbf{v}_{p+1}$, where $\|\mathbf{v}_p - \mathbf{v}_{p+1}\|$ varies. Continue as before with `LEMMA_1834976363`. □

**Lemma 1.50** (684) `[JOTSWIX]` *Let $s$ be a basic SCS, with $k(s) = 4$, and let $p \in I(s)$. Let $\mathbf{v} \in \mathcal{M}_s$ be generic. Assume that for all diagonals we have $a_{i,j}(s) \le c_{stab} < \|\mathbf{v}_i - \mathbf{v}_j\|$ and $4h_0 < b_{i,j}(s)$. Assume $\angle \mathbf{v}_i < \pi$ for all $i \ne p + 2$. Assume $\mathrm{xrr}_V(\mathbf{v}_{p-1}, \mathbf{v}_{p+1}) \le 15.53$. Assume $a_{p,p+1}(s) < c_{stab}$. Then $\|\mathbf{v}_p - \mathbf{v}_{p+1}\| < c_{stab}$ or $\|\mathbf{v}_p - \mathbf{v}_{p-1}\| < c_{stab}$.*

*Proof* As in previous lemmas, use Lemma 3.1 to construct a deformation. Use $\{\mathbf{0}, \mathbf{v}_{p-1}, \mathbf{v}_{p+1}\}$ as the base triangle with contructed point $\mathbf{v}_p$, where $\|\mathbf{v}_{p-1} - \mathbf{v}_p\|$ varies. Continue as before with `LEMMA_6843920790`. ◻

# 2

# Appendix on checking completeness

In this section we prove the implication $S_{\text{init}} \Rightarrow S_{\text{term}}$.

## 2.1 definitions

We let

$$M(s) = \{(i, i+1)\ :\ i \in I(s), \quad (2 < a_{i,i+1}(s) \text{ or } 2h_0 < b_{i,i+1}(s))\}.$$

By the definition of SCS, the cardinality $m(s)$ of $M(s)$ satisfies $k(s) + m(s) \le 6$.

The stable constraints systems we discuss here are all unadorned with $J = I_{str} = I_{lo} = I_{hi} = \emptyset$; that is, basic stable constraint systems. Little further mention will be made of this sets.

We say that $\mathbf{v} \in \mathcal{B}_s$ is generic if its local fan $(V_{\mathbf{v}}, E_{\mathbf{v}}, F_{\mathbf{v}})$ is generic. We say that $\mathbf{v} \in \mathcal{B}_s$ is lunar if its local fan $(V_{\mathbf{v}}, E_{\mathbf{v}}, F_{\mathbf{v}})$ is generic.

## 2.2 init list

We name stable constraint systems $s$ with a three-character identifier. The first digit is $k(s)$. The second is the letter $I$ (initial), $T$ (terminal), or $M$ (intermediate). The third character is a distinguishing digit.

We let std denote the range $[2, 2h_0]$ of a standard edge, pro denote the range $[2h_0, \sqrt{8}]$ of a prolonged edge, and $\text{pro}^+$ denote the range $[2h_0, c_{\text{stab}}]$. Let $\text{pro}_8$ denote the range $[\sqrt{8}, c_{\text{stab}}]$. If $r$ is a real number, we write $r+$ for the interval $[r, 6]$, and by abuse of notation $r$ for the interval $[r, r]$.

We have the given table (2.2) of SCSs in the initial list. In each case we give the value of $k$ and $d$, the range of the diagonals, the default range on edges, and the range of one distinguished edge if there is one.

| **name** | interval |
|---|---|
| std | $[2, 2h_0]$ |
| pro | $[2h_0, \sqrt{8}]$ |
| $\mathrm{pro}_8$ | $[\sqrt{8}, c_{\mathrm{stab}}]$ |
| $\mathrm{pro}^+$ | $[2h_0, c_{\mathrm{stab}}]$ |
| pro | $[2h_0, \sqrt{8}]$ |

Table 2.1 *intervals*

| **name** | k | d | diag | edge | dist.edge |
|---|---|---|---|---|---|
| 6I1 | 6 | 0.712 | $2h_0+$ | std | * |
| 5I1 | 5 | 0.4819 | $2h_0+$ | std | * |
| 5I2 | 5 | 0.616 | $\sqrt{8}+$ | std | * |
| 5I3 | 5 | 0.616 | $2h_0+$ | std | pro |
| 4I1 | 4 | 0.206 | $2h_0+$ | std | * |
| 4I2 | 4 | 0.467 | 3+ | std | * |
| 4I3 | 4 | 0.477 | $\sqrt{8}+$ | std | pro |
| 3I1 | 3 | 0 | * | std | * |

Table 2.2 *Initial SCS*

We have the given table (2.2) of SCSs in the terminal list. Two entries in the diagonal column separated by a semicolon indicates that there are two diagonals (in a quadrilateral) with different length constraints. In 3T7, there are three different edge ranges.

| **name** | k | d | diag | edge | dist.edge |
|---|---|---|---|---|---|
| 6T1 | 6 | 0.712 | $c_{\mathrm{stab}}+$ | 2 | * |
| 5T1 | 5 | 0.616 | $c_{\mathrm{stab}}+$ | 2 | * |
| 4T1 | 4 | 0.467 | 3+ | 2 | * |
| 4T2 | 4 | 0.467 | 3 | std | * |
| 4T3 | 4 | 0.513 | $c_{\mathrm{stab}}+$ | 2 | $c_{\mathrm{stab}}$ |
| 4T4 | 4 | 0.477 | $\sqrt{8}+$; $\mathrm{pro}_8$ | std | pro |
| 4T5 | 4 | 0.513 | $c_{\mathrm{stab}}$; $c_{\mathrm{stab}}+$ | std | $\mathrm{pro}^+$ |
| 3T1 | 3 | 0.11 | * | std | $\mathrm{pro}_8$ |
| 3T2 | 3 | 0 | * | 2 | * |
| 3T3 | 3 | 0.476 | * | $\mathrm{pro}^+$ | * |
| 3T4 | 3 | 0.2759 | * | $\mathrm{pro}^+$ | std |
| 3T5 | 3 | 0.103 | * | std | pro |
| 3T6 | 3 | 0.4348 | * | $\mathrm{pro}_8$ | std |
| 3T7 | 3 | 0.2565 | * | 2 | $c_{\mathrm{stab}}$; $[c_{\mathrm{stab}}, 3.62]$ |

Table 2.3 *terminal SCS*

We define basic SCSs with the following parameters. In 4M4, 4M5, 4M7, and 4M8, there are two standard edges and two with range $\mathrm{pro}^{+}$, differing according to whether the standard edges are adjacent or opposite. In particular,

| **name** | k | d | diag | edge | dist.edge |
|---|---|---|---|---|---|
| 6M1 | 6 | 0.712 | $2h_0+$ | std | * |
| 5M1 | 5 | 0.616 | $2h_0+$ | std | $\mathrm{pro}^{+}$ |
| 5M2 | 5 | 0.616 | $c_{\mathrm{stab}}+$ | std | $[2, c_{\mathrm{stab}}]$ |
| 4M2 | 4 | 0.3789 | $2h_0+$ | std | $\mathrm{pro}^{+}$ |
| 4M3’ | 4 | 0.513 | $\sqrt{8}+$ | std | $\mathrm{pro}_8$ |
| 4M4’ | 4 | 0.513 | $2h_0+$ | std (2 adj) | $\mathrm{pro}^{+}$(2 adj) |
| 4M5’ | 4 | 0.513 | $2h_0+$ | std (2 opp) | $\mathrm{pro}^{+}$(2 opp) |
| 4M6’ | 4 | 0.513 | $c_{\mathrm{stab}}+$ | std | $\mathrm{pro}^{+}$ |
| 4M7 | 4 | 0.513 | $c_{\mathrm{stab}}+$ | std (2 adj) | $\mathrm{pro}^{+}$(2 adj) |
| 4M8 | 4 | 0.513 | $c_{\mathrm{stab}}+$ | std (2 opp) | $\mathrm{pro}^{+}$(2 opp) |
| 3M1 | 3 | 0.103 | * | std | $\mathrm{pro}^{+}$ |

## 2.3 preliminary lemmas

**Lemma 2.1** `[FEKTYIY]`. `[formal proof by Hoang Le Truong].` *Let $s$ be an SCS with $k(s) > 3$, and let $\mathbf{v} \in \mathcal{M}_s$. Then $\{\mathbf{0}\} \cup V_{\mathbf{v}}$ is not coplanar.*

*Proof* If $\{\mathbf{0}\} \cup V_{\mathbf{v}}$ is coplanar, then every interior angle in the local fan $(V_{\mathbf{v}}, E_{\mathbf{v}}, F_{\mathbf{v}})$ is 0 or $\pi$. However, the fan conditions prevent angle 0. Hence every interior angle is $\pi$. This gives $\mathrm{sol}(V_{\mathbf{v}}, E_{\mathbf{v}}, F_{\mathbf{v}}) = 2\pi$, which is contrary to the estimate of Lemma 1.25 (JKQEWGV). □

**Lemma 2.2** `[AURSIPD]` `[formal proof by Hoang Le Truong].` *Let $s$ be an SCS with $k(s) > 3$, and let $\mathbf{v} \in \mathcal{M}_s$ be generic. Then $3 + \ell \le k$, where $\ell$ is the number of straight interior angles of $(V_{\mathbf{v}}, E_{\mathbf{v}}, F_{\mathbf{v}})$.*

*Proof* Otherwise $2 + \ell \ge k$, and we can find two distinct $i_1, i_2$ such that the interior angle is straight at $\mathbf{v}_i$ for $i \ne i_1, i_2$. By the straightness of $i_1 + 1, \ldots$ we have that

$$\{\mathbf{0}, \mathbf{v}_{i_1}, \mathbf{v}_{i_1+1}, \ldots, \mathbf{v}_{i_2}\}$$

lies in the plane $A = \mathrm{aff}\{\mathbf{0}, \mathbf{v}_{i_1}, \mathbf{v}_{i_2}\}$. Similarly,

$$\{\mathbf{0}, \mathbf{v}_{i_2}, \mathbf{v}_{i_2+1}, \ldots, \mathbf{v}_{i_1}\}$$

lies in $A$. Hence $V_{\mathbf{v}}$ is coplanar, which is contrary to Lemma 2.1. □

**Lemma 2.3** (polar test) `[PPBTYDQ]` `[formal proof by Hoang Le Truong].` *Let* $\mathbf{u}$, $\mathbf{p}$, *and* $\mathbf{v} \in \mathbb{R}^3$ *be vectors such that* $\{\mathbf{0}, \mathbf{p}, \mathbf{u}\}$ *is not collinear and* $\{\mathbf{0}, \mathbf{p}, \mathbf{v}\}$ *is not collinear. If*

$$\mathrm{arc}_V(\mathbf{0}, \mathbf{u}, \mathbf{p}) + \mathrm{arc}_V(\mathbf{0}, \mathbf{p}, \mathbf{v}) < \pi$$

*then* $\mathbf{0}$ *is not in the convex hull of* $\{\mathbf{u}, \mathbf{v}\}$.

*Proof* Assume for a contradiction that $\mathbf{0}$ does lie in the convex hull. Let $\mathbf{e} = \mathbf{u} \times \mathbf{p}$. By the orientation chosen for $\mathbf{e}$, and by (DIHV_ARCV), we have

$$\mathrm{azim}(\mathbf{0}, \mathbf{e}, \mathbf{u}, \mathbf{p}) = \mathrm{dih}_V(\mathbf{0}, \mathbf{e}, \mathbf{u}, \mathbf{p}) = \mathrm{arc}_V(\mathbf{0}, \mathbf{u}, \mathbf{p})$$
$$\mathrm{azim}(\mathbf{0}, \mathbf{e}, \mathbf{p}, \mathbf{v}) = \mathrm{dih}_V(\mathbf{0}, \mathbf{e}, \mathbf{p}, \mathbf{v}) = \mathrm{arc}_V(\mathbf{0}, \mathbf{p}, \mathbf{v}).$$

Hence the assumption of the lemma and additivity of azimuth angles gives

$$\mathrm{azim}(\mathbf{0}, \mathbf{e}, \mathbf{u}, \mathbf{v}) = \mathrm{azim}(\mathbf{0}, \mathbf{e}, \mathbf{u}, \mathbf{p}) + \mathrm{azim}(\mathbf{0}, \mathbf{e}, \mathbf{p}, \mathbf{v}) < \pi.$$

If $\mathbf{0}$ does lie in the convex hull, then $\{\mathbf{0}, \mathbf{u}, \mathbf{v}, \mathbf{e}\}$ is coplanar, which implies that $\mathrm{azim}(\mathbf{0}, \mathbf{e}, \mathbf{u}, \mathbf{v}) \in \{0, \pi\}$. We have eliminated the possibility $\pi$. If the azimuth angle is 0, then $\mathbf{u}$ and $\mathbf{v}$ lie in the same half plane $\mathbf{u} \in \mathrm{aff}_+^0(\{\mathbf{0}, \mathbf{e}\}, \{\mathbf{v}\})$. Hence $\mathbf{0}$ is not in the convex hull. □

**Lemma 2.4** `[OIQKKEP]` *Let* $\mathbf{u}, \mathbf{v} \in \mathcal{B}$. *Assume that* $\|\mathbf{u} - \mathbf{v}\| \le c < 4$, *for some* $c$. *Then*

$$\mathrm{arc}_V(\mathbf{0}, \mathbf{u}, \mathbf{v}) \le \mathrm{arc}(2, 2, c).$$

*Proof*

$$\mathrm{arc}_V(\mathbf{0}, \mathbf{u}, \mathbf{v}) = \mathrm{arc}(\|\mathbf{u}\|, \|\mathbf{v}\|, \|\mathbf{u} - \mathbf{v}\|).$$

The function arc is decreasing in the first two arguments and increasing in the third. □

**Lemma 2.5** `[BGSAKHJ]` `[formal proof by Hoang Le Truong].` *Let* $(V, E, F)$ *be a lunar fan with pole* $(\mathbf{v}, \mathbf{w})$. *Then* $\mathbf{0}$ *lies in the convex hull of* $\{\mathbf{v}, \mathbf{w}\}$.

*Proof* This follows directly from the definition of a lunar fan. □

**Lemma 2.6** `[MXQTIED]` `[formal proof by Hoang Le Truong].` *Let* $s$ *be a basic SCS, and let* $\mathbf{v} \in \mathcal{M}_s$. *Let* $s'$ *be a basic SCS such that* $M(s') = M(s)$, $k(s') = k(s)$, $I(s') = I(s)$, $\mathbf{v} \in \mathcal{B}_{s'}$, $d(s') = d(s)$, $a_{ii}(s') = 0$, $a_{ij}(s') = a_{ij}(s)$ *for all edges* $(i, j)$, *and*

$$a_{ij}(s) \le a_{ij}(s') \le \|\mathbf{v}_i - \mathbf{v}_j\| \le b_{ij}(s') \le b_{ij}(s).$$

*Then* $\mathbf{v} \in \mathcal{M}_{s'}$.

*Proof* Clearly $\mathbf{v} \in \mathcal{B}_{s'}$. The domain shrinks and $\mathbf{v}$ is a global minimizer, so $\mathbf{v} \in \mathcal{B}'_{s'}$. By the hypotheses the constants $a_{ij}$ are the same on edges, so the index remains the same, and $\mathbf{v} \in \mathcal{B}''_{s'}$. The SCS is unadorned, so $\mathbf{v} \in \mathcal{M}_{s'}$. □

**Lemma 2.7** `[XWNHLMD]` `[formal proof by Hoang Le Truong].` *Let $s$ and $s'$ be basic SCSs with $d(s') \geq d(s)$, $k(s) = k(s')$, and $I(s) = I(s')$. Suppose that $\mathbf{v} \in \mathcal{M}_s \cap \mathcal{B}_{s'}$. Then $\{s\} \Rightarrow \{s'\}$.*

*Proof* The condition $\mathbf{v} \in \mathcal{M}_s$ gives $\tau^*(s', \mathbf{v}) \leq \tau^*(s, \mathbf{v}) < 0$. Hence $\mathcal{B}'_s$ is nonempty. The SCS is basic, so $\mathcal{B}''_{s'} = \mathcal{M}_{s'}$ is also nonempty. □

**Lemma 2.8** `[SYNQIWN]` `[formal proof by Hales].` *Let $s$ be a basic SCS and let $\mathbf{v} \in \mathcal{B}_s$. Let $i \in I(s)$. Assume that at least two the following four lengths are equal to 2:*

$$\|\mathbf{v}_i\|, \quad \|\mathbf{v}_i - \mathbf{v}_{i+1}\|, \quad \|\mathbf{v}_{i+2}\|, \quad \|\mathbf{v}_{i+2} - \mathbf{v}_{i+1}\|$$

*Assume that $c_{stab} \leq \|\mathbf{v}_i - \mathbf{v}_{i+2}\|$. Then the interior angle at $i+1$ is greater than $\pi/2$.*

*Proof* This is a *computer calculation*[1] [2]. □

**Lemma 2.9** (lunar prep) `[AXJRPNC]` `[formal proof by Hoang Le Truong].` *Let $s$ be a basic SCS and let $\mathbf{v} \in \mathcal{M}_s$. Assume that $b_{ij}(s) \leq c_{stab}$ for all edges $\{i, j\}$. Assume that $(V_{\mathbf{v}}, E_{\mathbf{v}}, F_{\mathbf{v}})$ is a lune with pole $(\mathbf{v}_p, \mathbf{v}_q)$. Then $k(s) = 6$ and $q = p + 3$.*

*Proof* We first consider the case that $k = 6$ and $q = p$, $p + 1$, or $p + 2$. Then $M(s) = \emptyset$ and the polar criterion gives the result, when applied to

$$2\,\mathrm{arc}(2, 2, 2h_0) < \pi.$$

Assume for a contradiction that $k < 6$. In particular, the floor of $k/2$ is at most 2. We have $\|\mathbf{v}_i - \mathbf{v}_j\| < 4$ for every edge, and this implies that $(p, q)$ is not an edge. We have that $k > 3$; for otherwise $(p, q)$ is an edge. Without loss of generality we may assume that $q = p + 2$.

We start a special case by assuming that $M(s)$ meets $\{(p, p+1), (p+1, p+2)\}$ in a set of cardinality at most one. Then by the monotonicity of arc, we apply the polar test

$$\mathrm{arc}(\mathbf{0}, \mathbf{v}_p, \mathbf{p}) + \mathrm{arc}(\mathbf{0}, \mathbf{p}, \mathbf{v}_q) \leq \mathrm{arc}(2, 2, 2h_0) + \mathrm{arc}(2, 2, c_{\mathrm{stab}}) < \pi,$$

to see that $(p, q)$ is not a pole.

In the final case we assume that $M(s)$ meets $\{(p, p+1), (p+1, p+2)\}$ in a set

[1] `[1117202051 4559601669 4559601669b]`

of cardinality two. By the inequality $m + k \le 6$, we see that $m = 2$ and $k = 4$. This implies that $M(s)$ is disjoint from $\{(p+2, p+3), (p+3, p)\}$. By the polar test,

$$\mathrm{arc}_V(\mathbf{0}, \mathbf{v}_p, \mathbf{v}_q) \le 2\,\mathrm{arc}(2, 2, 2h_0) < \pi,$$

and again $(p, q)$ is not a pole. □

A weak version of the following lemma has been formalized as `RRCWNS_WEAK`.

**Lemma 2.10** (genericity) `[RRCWNSJ]` `[formal proof by Hoang Le Truong]`. *Let $s$ be a basic SCS and $\mathbf{v} \in \mathcal{M}_s$. Assume that $4h_0 < b_{ij}(s)$ for every diagonal $(i, j)$. Assume that $a_{ij}(s) \le c_{stab} < \|\mathbf{v}_i - \mathbf{v}_j\|$ for every diagonal $(i, j)$. Assume that $b_{ij}(s) \le c_{stab}$ for every edge $(i, j)$. Then $(V_{\mathbf{v}}, E_{\mathbf{v}}, F_{\mathbf{v}})$ is generic.*

*Proof* An earlier lemma shows that $(V_{\mathbf{v}}, E_{\mathbf{v}}, F_{\mathbf{v}})$ is not circular.

Assume for a contradiction that $(V_{\mathbf{v}}, E_{\mathbf{v}}, F_{\mathbf{v}})$ is lunar with pole $(\mathbf{v}_p, \mathbf{v}_q)$. The previous lemma gives $k(s) = 6$ and $q = p + 3$. By the definition of SCS, we have $m = 0$. That is, $b_{ij}(s) \le 2h_0$ and $a_{ij} = 2$ for all edges.

By the deformation ODX applied at $p + 1$, we have

$$\|\mathbf{v}_{p+1}\| = 2, \text{ or } \|\mathbf{v}_{p+1} - \mathbf{v}_p\| = 2, \text{ or } \|\mathbf{v}_{p+1} - \mathbf{v}_{p+2}\| = 2.$$

Furthermore, if the third possibility holds: $\|\mathbf{v}_{p+1} - \mathbf{v}_{p+2}\| = 2$, then the deformation IMJ at $p + 1$ gives that one of the first two possibilities hold.

By similar reasoning at vertex $p - 1$, we have

$$\|\mathbf{v}_{p-1}\| = 2, \text{ or } \|\mathbf{v}_p - \mathbf{v}_{p-1}\| = 2.$$

By Lemma 2.8, the interior angle at $p$ is greater than $\pi/2$. This is contrary to Lemma 1.25 (JKQEWGV). □

**Lemma 2.11** (exists-2) `[JCYFMRP]` *Let $s$ be a basic SCS, with $3 < k(s)$, and let $\mathbf{v} \in \mathcal{M}_s$ be generic. Assume that $4h_0 < b_{ij}(s)$ for all diagonals, and that $m(s) \le 1$. Assume that $a_{ij}(s) = 2$ for all edges $\{i, j\}$. Assume that for all diagonals $\{i, j\}$*

$$a_{ij}(s) \le c_{stab} < \|\mathbf{v}_i - \mathbf{v}_j\| \text{ and } 4h_0 < b_{ij}(s).$$

*Then there exists an edge $\{i, j\}$ such that $\|\mathbf{v}_i - \mathbf{v}_j\| = 2$.*

*Proof* For a contradiction, assume $\|\mathbf{v}_i - \mathbf{v}_j\| > 2$ for all edges. Apply the deformation ODX at every vertex $\ell$. Then $\|\mathbf{v}_\ell\| = 2$ for all $\ell$. Since $m(s) \le 1$, we have some index $p$ such that for every edge $(i, i+1)$ other than $(p, p+1)$, we have $b_{ij}(s) \le 2h_0$.

By Lemma 2.8, the interior angle is obtuse at every $i$, such that $i \ne p, p+1$.

We claim that each of the vertices $\mathbf{v}_i$ is straight $i \neq p, p+1$. For otherwise, we may apply the obtuse case of deformation 482 to conclude that some edge has length 2.

Thus, we have constructed $k-2$ straight angles. This is contrary to Lemma 2.2. □

**Lemma 2.12** (propagate-min-std) `[TFITSKC]` *Let $s$ be a basic SCS, with $3 < k(s)$, and let $\mathbf{v} \in \mathcal{M}_s$ be generic. Let $p \in I(s)$. Assume that $(p, p+1)$ is a standard edge. Assume that $\|\mathbf{v}_{p-1} - \mathbf{v}_p\| = 2$. Assume that for all diagonals $\{i, j\}$*

$$a_{ij}(s) \le c_{stab} < \|\mathbf{v}_i - \mathbf{v}_j\| \text{ and } 4h_0 < b_{ij}(s).$$

*Assume*

$$2 < b_{p-1,p}(s), \quad \le a_{p+1,p+2}(s) < b_{p+1,p+2}(s)$$

*Then $\|\mathbf{v}_p - \mathbf{v}_{p+1}\| = 2$.*

*Proof* Assume for a contradiction that $\|\mathbf{v}_p - \mathbf{v}_{p+1}\| > 2$.

We claim that $\mathbf{v}_p$ is not straight. Otherwise, deformation NUX applied at $\mathbf{v}_p$ gives $\|\mathbf{v}_p - \mathbf{v}_{p+1}\| = 2$.

We claim that $\|\mathbf{v}_{p+1}\| > 2$. Otherwise, by Lemma 2.8, the interior angle at $\mathbf{v}_p$ is obtuse, and the obtuse case of deformation 482 gives the $\|\mathbf{v}_p - \mathbf{v}_{p+1}\| = 2$.

By deformation ODX applied at $p+1$, we have $\|\mathbf{v}_{p+1} - \mathbf{v}_{p+2}\| = a_{p+1,p+2}$.

We claim that $\mathbf{v}_{p+1}$ is not straight. Otherwise, deformation NUX applied at $\mathbf{v}_{p+1}$ gives $\|\mathbf{v}_p - \mathbf{v}_{p+1}\| = 2$.

Finally, we may apply the deformation 482 in the non-obtuse case at $p+1$ to obtain $\|\mathbf{v}_p - \mathbf{v}_{p+1}\| = 2$. □

**Lemma 2.13** (propagate-min-std2) `[CQAOQLR]` *Let $s$ be a basic SCS, with $3 < k(s)$, and let $\mathbf{v} \in \mathcal{M}_s$ be generic. Assume that for all diagonals $\{i, j\}$, we have $a_{ij}(s) \le c_{stab} < \|\mathbf{v}_i - \mathbf{v}_j\|$ and $4h_0 < b_{ij}(s)$. Assume*

$$2 < b_{p,p+1}, \quad 2 < b_{p,p-1}, \quad a_{p+1,p+2} < b_{p+1,p+2}, \quad a_{p-2,p_1} < b_{p-2,p-1}.$$

*Assume*

$$a_{ij} < b_{ij}$$

*for all edges. Then for every $p$, such that $(p-1, p)$ and $(p, p+1)$ are both standard, either both of $\|\mathbf{v}_p - \mathbf{v}_{p+1}\|$ and $\|\mathbf{v}_p - \mathbf{v}_{p-1}\|$ equal 2, or neither equals 2.*

*Proof* This follows from Lemma 2.12 and the reversal symmetry of stable constraint systems. □

**Lemma 2.14** (propagate-min-std3) `[JLXFDMJ]` *Let $s$ be a basic SCS, and let $\mathbf{v} \in \mathcal{M}_s$ be generic. Assume that $m(s) \le 1$. Assume that for all diagonals $\{i, j\}$, we have $a_{ij}(s) \le c_{stab} < \|\mathbf{v}_i - \mathbf{v}_j\|$ and $4h_0 < b_{ij}(s)$. Assume*

$$a_{ij} < b_{ij}$$

*for all edges. Assume that $\|\mathbf{v}_i - \mathbf{v}_{i+1}\| = 2$ for some edge $\{i, i+1\}$. Then $\|\mathbf{v}_i - \mathbf{v}_j\| = 2$ at every edge not in $M(s)$.*

*Proof* Let $\{j, j+1\}$ be any standard edge. We may create a chain avoiding $M(s)$ of the form

$$\{j, j+1\}, \{j+1, j+2\}, \ldots, \{i-1, i\},$$

or

$$\{i+1, i+2\}, \{i+2, i+3\}, \ldots \{j, j+1\}.$$

Induction along the chain using Lemma 2.13 gives the result. □

If $s$ is a SCS, with diagonal $\{i, j\}$, let $D(s, \{i, j\})$ be the SCS which is identical to $s$, except that $b_{ij}(D(s, \{i, j\})) = c_{\mathrm{stab}}$. In particular, we have a SCS $D(6I1, \{i, j\})$.

**Lemma 2.15** `[YRTAFYH]` *If $s$ is a basic SCS, $\{i, j\}$ is a diagonal, and $a_{ij}(s) \le c_{stab}$, then $D(s, \{i, j\})$ is a basic SCS.*

**Lemma 2.16** (equivalent diagonal) `[WKEIDFT]` *Let $s$ be a basic SCS with $k(s) > 3$. Assume that the constants $a_{ij}(s)$ and $b_{ij}(s)$ are independent of the subscript for all edges $\{i, j\}$. Assume that the constants $a_{ij}(s)$ and $b_{ij}(s)$ are independent of the subscript for all diagonals. Assume $a_{ij}(s) \le$ stab for all diagonals. Let $\{p, q\}$ and $\{p', q'\}$ be diagonals with $p' + q = p + q'$. Then*

$$\{D(s, \{p, q\})\} \Rightarrow \{D(s, \{p', q'\})\}.$$

*Proof* This follows from the equivalence of the two SCSs. □

## 2.4 hexagons

In this subsection, we treat the initial SCS 6I1.

**Lemma 2.17** `[PEDSLGV]` *Let $\mathbf{v} \in \mathcal{M}_{6I1}$. If for some diagonal $\{i, j\}$, we have $\|\mathbf{v}_i - \mathbf{v}_j\| \le c_{stab}$, then $\mathbf{v} \in \mathcal{M}_{D(6I1,\{i,j\})}$. If for every diagonal $\{i, j\}$, we have $c_{stab} < \|\mathbf{v}_i - \mathbf{v}_j\|$, then $\mathbf{v} \in \mathcal{M}_{6M1}$.*

*Proof* This is a direct consequence of Lemma 2.6. □

**Lemma 2.18** (hex slice) `[AQICLXA]` *We have*

$$D(6I1, \{0, 2\}) \Rightarrow \{5M1, 3M1\}.$$

*Proof* Slice $D(6I1, \{0, 2\})$ along the diagonal $\{0, 2\}$. □

**Lemma 2.19** (hex slice2) `[FUNOUYH]` *We have*

$$D(6I1, \{0, 3\}) \Rightarrow \{4M2\}.$$

*Proof* Slice $D(6I1, \{0, 3\})$ along the diagonal $\{0, 3\}$. The two halves are equivalent. □

**Lemma 2.20** (hex slice 3) `[OEHDBEN]` $\{6I1\} \Rightarrow \{6T1, 5M1, 4M2, 3M1\}$.

*Proof* Let $\mathbf{v} \in \mathcal{M}_{6I1}$.

Assume first that there is a diagonal $\{i, i+2\}$ such that $\|\mathbf{v}_i - \mathbf{v}_{i+2}\| \leq c_{\text{stab}}$. Then by the previous lemmas,

$$\{6I1\} \Rightarrow \{D(6I1, \{i, j\})\} \Rightarrow \{D(6I1, \{0, 2\})\} \Rightarrow \{5M1, 3M1\}.$$

Assume next that there is a diagonal $\{i, i+3\}$ such that $\|\mathbf{v}_i - \mathbf{v}_{i+1}\| \leq c_{\text{stab}}$. Then by the previous lemmas,

$$\{6I1\} \Rightarrow \{D(6I1, \{i, j\})\} \Rightarrow \{D(6I1, \{0, 3\})\} \Rightarrow \{4M2\}.$$

In a hexagon, every diagonal has the form $\{i, i+2\}$ or $\{i, i+3\}$. We may now assume that for every diagonal, $c_{\text{stab}} < \|\mathbf{v}_i - \mathbf{v}_j\|$. By Lemma 2.10, $\mathbf{v}$ is generic. By Lemma 2.11, there exists an edge $\{i, j\}$ such that $\|\mathbf{v}_i - \mathbf{v}_j\| = 2$. By Lemma 2.14 [propagate-min-std3], all edges have length 2. Hence $\mathbf{v} \in \mathcal{B}_{6T1}$ and $\{6I1\} \Rightarrow \{6T1\}$. □

In the section on pentagons, we make a further analysis of 5M1; and in the section on quadrilaterals, we analyze 4M2. This completes our analysis of hexagons.

## 2.5 pentagons

In this section, we treat the pentagonal cases: 5I1, 5I2, 5I3, 5M1.

**Lemma 2.21** [OTMTOTJ]

$$\begin{aligned}
\{5I1\} &\Rightarrow \{D(5I1,\{0,2\}), 5M2\}\\
\{5I2\} &\Rightarrow \{D(5I2,\{0,2\}), 5M2\}\\
\{5I3\} &\Rightarrow \{D(5M1,\{0,2\}), D(5M1,\{0,3\}), D(5M1,\{2,4\}), 5M2\}\\
\{5M1\} &\Rightarrow \{D(5M1,\{0,2\}), D(5M1,\{0,3\}), D(5M1,\{2,4\}), 5M2\}
\end{aligned}$$

*Proof* Let $s \in \{5I1, 5I2, 5I3, 5M1\}$. Let $\mathbf{v} \in \mathcal{M}_s$. If $\|\mathbf{v}_i - \mathbf{v}_j\| > c_{\mathrm{stab}}$ for every diagonal $\{i, j\}$, then $s$ transfers to 5M2, and $\{s\} \Rightarrow \{5M2\}$.

For every diagonal, we have

$$\{D(5I3,\{i,j\})\} \Rightarrow \{D(5M1,\{i,j\})\}. \tag{2.22}$$

Now, let $\{i, j\}$ be a diagonal such that $\|\mathbf{v}_i - \mathbf{v}_j\| \leq c_{\mathrm{stab}}$. Then $\{s\} \Rightarrow \{D(s,\{i,j\})\}$ and by equivalence $\{D(s,\{i,j\})\}$ and the arrow (2.22), we may pass to one of the SCSs enumerated in the lemma. □

**Lemma 2.23** [HIJQAHA]

$$\{5M2\} \Rightarrow \{3T1, 3T4, 4M6, 4M7, 4M8, 5T1, D(5I2,\{0,2\}), D(5M1,\{0,2\}), D(5M1,\{0,3\}), D(5M1,\{2,4\})\}$$

*Proof* Let $\mathbf{v} \in \mathcal{M}_{5M2}$. Assume first that $\|\mathbf{v}_i - \mathbf{v}_j\| \leq c_{\mathrm{stab}}$ for some diagonal $\{i, j\}$. We consider subcases depending on the length of edges $\|\mathbf{v}_i - \mathbf{v}_j\|$. If $\|\mathbf{v}_0 - \mathbf{v}_1\| \leq 2h_0$, then

$$\{5M2\} \Rightarrow \{D(5I2,\{i,j\})\},$$

and otherwise if $2h_0 < \|\mathbf{v}_0 - \mathbf{v}_1\| \leq \sqrt{8}$, then

$$\{5M2\} \Rightarrow \{D(5I3,\{i,j\})\},$$

These SCSs transfer to one of the terms on the right-hand side in the lemma. In the final case $\sqrt{8} \leq \|\mathbf{v}_0 - \mathbf{v}_1\| \leq c_{\mathrm{stab}}$. We can take the diagonal to be $\{0, 2\}$, $\{0, 3\}$, or $\{2, 4\}$. For these three diagonals, we have respectively,

$$\{5M2\} \Rightarrow \{D(5M2,\{i,j\})\} \Rightarrow \{4M6, 3T4\}, \quad \text{or } \{3T1, 4M7\}, \quad \text{or } \{3T1, 4M8\}.$$

We now assume $\|\mathbf{v}_i - \mathbf{v}_j\| > c_{\mathrm{stab}}$ for all diagonals. By Lemma 2.10, $\mathbf{v}$ is generic. By Lemma 2.11, there exists an edge $\{i, j\}$ such that $\|\mathbf{v}_i - \mathbf{v}_j\| = 2$. By Lemma 2.14, all standard edges of $\mathbf{v}$ have length 2.

Index the vertices so that $\{0, 1\}$ is the non-standard edge. If either $\mathbf{v}_0$ or $\mathbf{v}_1$ is straight, then the deformation NUX implies that the edge $\{0, 1\}$ is as short as possible, length 2. If neither of the two vertices is straight, then we may apply deformation 684, to again conclude the edge $\{0, 1\}$ is as short as possible. In these cases $\{5M2\} \Rightarrow \{5T1\}$. □

**Lemma 2.24** [CNICGSF]

$$\{D(5I1,\{0,2\})\} \Rightarrow \{4M2, 3M1\}$$
$$\{D(5I2,\{0,2\})\} \Rightarrow \{4M3, 3T1\}$$
$$\{D(5M1,\{0,2\})\} \Rightarrow \{4M2, 3T4\}$$
$$\{D(5M1,\{0,3\})\} \Rightarrow \{4M4, 3M1\}$$
$$\{D(5M1,\{2,4\})\} \Rightarrow \{4M5, 3M1\}$$

*Proof* In each case, we slice the given SCS along the given diagonal $\{i, j\}$ to produce a quadrilateral and a triangle. The resulting pieces (up to equivalence and transfer) are shown on the right. □

## 2.6 quadrilaterals

Next we turn to the analysis of quadrilaterals.

**Lemma 2.25** [GSXRFWM] *Let $s$ be an SCS with $k(s) = 4$. Let $\mathbf{v} \in \mathcal{M}_s$. Then $\mathbf{v}$ is generic.*

*Proof* Assume $k = 4$. Recall $m(s) + k(s) \leq 6$ so that $m(s) \leq 2$. There is no pole, because of the polar test

$$\mathrm{arc}(2, 2, 2h_0) + \mathrm{arc}(2, 2, c_{\mathrm{stab}}) < \pi.$$

□

**Lemma 2.26** [ARDBZYE]

$$\{4I2\} \Rightarrow \{4T1, 4T2\}.$$

*Proof* Let $\mathbf{v} \in M_{4I2}$. By Lemma 2.25, $\mathbf{v}$ is generic.

We consider the case when both diagonals of $\mathbf{v}$ are greater than $c_{\mathrm{stab}}$. By Lemma 2.11, there exists an edge $\{i, j\}$ such that $\|\mathbf{v}_i - \mathbf{v}_j\| = 2$. By Lemma 2.14, all edges have length 2. This gives $\{4I2\} \Rightarrow \{4T1\}$.

We turn to the case when some diagonal, say $\{1, 3\}$ has length in the range $[3, c_{\mathrm{stab}}]$. Without loss of generality, we may assume that $\{1, 3\}$ is the shorter of the two diagonals. By Lemma `ear_acute`, a quadrilateral with diagonals at least 3 does not have any straight $\mathbf{v}_i$. If diagonal $\{0, 2\}$ has length 3 then since it is the longest, both diagonals are 3, and $\{4I2\} \Rightarrow \{4T2\}$. If the diagonal $\{0, 2\}$ has length greater than 3, we may triangulate the quadrilateral along the diagonal $\{1, 3\}$ and apply non-obtuse deformation 482 at any edge to show that it has length 2. Hence, $\{4I2\} \Rightarrow \{4T1\}$. □

**Lemma 2.27** `[FYSSVEV]`

$$\{4I1\} \Rightarrow \{4I2, D(4I1, \{0, 2\})\}$$

*Proof* Let $\mathbf{v} \in \mathcal{M}_{4I1}$. If $\mathbf{v}$ has some diagonal $\{0, 2\}$ that is less than $c_{\text{stab}}$, then

$$\{4I1\} \Rightarrow \{D(4I1, \{i, j\})\} \Rightarrow \{D(4I1, \{0, 2\})\}.$$

Otherwise, with long diagonals, we have a transfer

$$\{4I1\} \Rightarrow \{4I2\}.$$

□

**Lemma 2.28** `[AUEAHEH]`

$$\{D(4I1, \{0, 2\})\} \Rightarrow \{3M1\}.$$

*Proof* Slice along the diagonal $\{0, 2\}$. □

**Lemma 2.29** `[ZNLLLDL]`

$$\{D(4I3, \{0, 2\})\} \Rightarrow \{4T4\}.$$

*Proof* In fact, 4T4 is equivalent to $D(413, \{0, 2\})$. □

*Remark* 2.30 The terminal case 4T4 is treated by slicing along the diagonal $\{0, 2\}$ and introducing a nontrivial $J$.

**Lemma 2.31** `[VQFYMZY]`

$$\{4I3\} \Rightarrow \{4T4, 4M6\}.$$

*Proof* Let $\mathbf{v} \in \mathcal{M}_{4I3}$. If there is a diagonal $\{i, j\}$ of length at most $c_{\text{stab}}$, and we have

$$\{4I3\} \Rightarrow \{D(4I3, \{i, j\})\} \Rightarrow \{D(4I3, \{0, 2\})\} \Rightarrow \{4T4\}.$$

If both diagonals are greater than $c_{\text{stab}}$, there is an equi-transfer to 4M6. □

**Lemma 2.32** `[BNAWVNH]` $\{4M2\} \Rightarrow \{3M1, 3T4, 4M6\}$.

*Proof* If there exists $\mathbf{v} \in \mathcal{M}_{4M2}$ and a diagonal $\{i, j\}$ such that $\|\mathbf{v}_i - \mathbf{v}_j\| \leq c_{\text{stab}}$, then slice along the diagonal $\{i, j\}$ to obtain $\{4M2\} \Rightarrow \{3T5, 3T4\}$. Otherwise, $\{4M2\} \Rightarrow \{4M6\}$. □

**Lemma 2.33** `[RAWZDIB]` $\{4M3\} \Rightarrow \{3T1, 3T6, 4M6\}$.

*Proof* If there exists $\mathbf{v} \in \mathcal{M}_{4M3}$ and a diagonal $\{i, j\}$ such that $\|\mathbf{v}_i - \mathbf{v}_j\| \leq c_{\text{stab}}$, slice along the diagonal $\{i, j\}$ to obtain $\{4M3\} \Rightarrow \{3M1, 3T6\}$. Otherwise, $\{4M3\} \Rightarrow \{4M6\}$. □

**Lemma 2.34** `[MFKLVDK]` $\{4M4\} \Rightarrow \{3T4, 3M1, 3T3, 4M7\}$.

*Proof* If there exists $\mathbf{v} \in \mathcal{M}_{4M4}$ and a diagonal $\{i, j\}$ such that $\|\mathbf{v}_i - \mathbf{v}_j\| \le c_{\text{stab}}$, slice along the diagonal $\{i, j\}$ to obtain $\{4M4\} \Rightarrow \{3T4, 3M1, 3T3\}$. Otherwise, $\{4M4\} \Rightarrow \{4M7\}$. □

**Lemma 2.35** `[RYPDIXT]` $\{4M5\} \Rightarrow \{3T4, 4M8\}$.

*Proof* If there exists $\mathbf{v} \in \mathcal{M}_{4M5}$ and a diagonal $\{i, j\}$ such that $\|\mathbf{v}_i - \mathbf{v}_j\| \le c_{\text{stab}}$, slice along the diagonal $\{i, j\}$ to obtain $\{4M5\} \Rightarrow \{3T4\}$. Otherwise, $\{4M5\} \Rightarrow \{4M8\}$. □

The quadrilateral cases that remain are 4M6, 4M7, and 4M8.

**Lemma 2.36** `[WGDHPPI]` *Let $s$ be an SCS with $k(s) = 4$. Let $\mathbf{v} \in \mathcal{M}_s$. Then the number of straight vertices $\mathbf{v}_i$ is at most* 1.

*Proof* Lemma 2.25 implies that $\mathbf{v}$ is generic. Lemma 2.2 gives the result. □

**Lemma 2.37** `[ASSWPOW]` *Let $s$ be a basic SCS with $k(s) = 4$. Let $\mathbf{v} \in \mathcal{M}_s$. Let $p \in I(s)$. Assume that the constraints at $(p-1, p)$ and $(p, p+1)$ are both standard. Then* $\mathrm{xrr}_V(\mathbf{v}_{p-1}, \mathbf{v}_{p+1}) \le 15.53$.

*Remark* 2.38 The constant 15.53 appears in several of the computer calculations for nonlinear inequalities.

*Proof* This follows from the triangle inequality.

$$\mathrm{arc}_V(\mathbf{0}, \mathbf{v}_{\ell-1}, \mathbf{v}_{\ell+1}) \le 2\,\mathrm{arc}(2, 2, 2h_0) < \mathrm{arc}(2, 2, \sqrt{15.53}).$$

□

*Remark* 2.39 (Running assumptions) The following lemmas will have several assumptions in common. We list them here as conditions on $(s, v)$.

1. $s$ is an SCS
2. $k = 4$
3. $\mathbf{v} \in \mathcal{M}_s$.
4. For all diagonals, $a_{ij} \le c_{\text{stab}} < \|\mathbf{v}_i - \mathbf{v}_j\|$ and $4h_0 < b_{ij}$.
5. The constraint on each edge is standard or pro$^+$. (In particular $m$ is the number of edges with pro$^+$ constraints.)

**Lemma 2.40** `[YEBWJNG]` *Let $(s, v)$ satisfy the running assumptions. If the constraint on $(p, p+1)$ is standard and if $\mathbf{v}$ is not straight at $p$ and $p+1$, then* $\|\mathbf{v}_p - \mathbf{v}_{p+1}\| = 2$.

*Proof* This is a direct consequence of deformation 482. □

**Lemma 2.41** `[TUAPYYU]` *Let* $(s, v)$ *satisfy the running assumptions. Assume that* $\mathbf{v}$ *is not straight at* $p$ *and* $p+1$. *Then* $\|\mathbf{v}_p - \mathbf{v}_{p+1}\| = a_{p,p+1}$ *or* $b_{p,p+1}$.

*Proof* Either $\mathbf{v}_{p-1}$ or $\mathbf{v}_{p+2}$ is not straight. Use deformation 1834 or deformation (1834 sym), as appropriate. ◻

**Lemma 2.42** `[WKZZEEH]` *Let* $(s, v)$ *satisfy the running assumptions. Assume that* $\mathbf{v}$ *is not straight at* $p$, $p-1$, *and* $p+1$. *Then* $\|\mathbf{v}_p - \mathbf{v}_{p-1}\| < c_{stab}$ *or* $\|\mathbf{v}_p - \mathbf{v}_{p+1}\| < c_{stab}$.

*Proof* This is a restatement of deformation 684. ◻

**Lemma 2.43** `[PWEIWBZ]` *Let* $(s, v)$ *satisfy the running assumptions. Then* $\|\mathbf{v}_p - \mathbf{v}_{p+1}\| = 2$ *for every standard edge.*

*Proof* Assume $(p, p+1)$ standard. If $\mathbf{v}$ is not straight at $p$ and $p+1$, then YEBWJNG gives the result. We may now assume without loss of generality that $\mathbf{v}$ is straight at $p$ or $p+1$. Shifting indices, we may assume that $\mathbf{v}$ is straight at $p$ and that $(p, q)$ is standard, where $q = p \pm 1$. For a contradiction, we may assume that $\|\mathbf{v}_p - \mathbf{v}_q\|$ is not minimal. In fact, by NUX, we may assume that no edge at $p$ is minimal. By IMJ, $\|\mathbf{v}_p\| = 2$.

We review the conditions at $(i, j) = (p+1, p+2)$ and $(p+2, p+3)$, the two edges away from the straight angle $p$. By Lemma `TUAPYYU`, $\mathbf{v}$ is extremal along edges $(p+1, p+2)$ and $(p+2, p+3)$. Hence along these two edges $\|\mathbf{v}_i - \mathbf{v}_j\|$ is 2, $2h_0$, or $c_{\text{stab}}$. By Lemma `TUAPYYU`, one of the two edges $\|\mathbf{v}_i - \mathbf{v}_j\| \leq 2h_0$. By Lemma `YEBWJNG`, if the constraint at $(i, j)$ is standard, then $\|\mathbf{v}_i - \mathbf{v}_j\| = 2$.

If $(p, j)$ is a standard edge at $p$, with $j = p \pm 2$, then $(j, p+2)$ is pro$^+$. Otherwise, by earlier remarks $\|\mathbf{v}_j - \mathbf{v}_{p+2}\| = 2$, which propagates to make $\|\mathbf{v}_p - \mathbf{v}_j\| = 2$ by `TFITSKC`, which is contrary to what we have.

We consider four subcases:

1. $(p-1, p)$ and $(p, p+1)$ are both standard. Also, $\|\mathbf{v}_{p+1} - \mathbf{v}_{p+1}\| \leq 2h_0$.
2. $(p-1, p)$ and $(p, p+1)$ are both standard. Also, $\|\mathbf{v}_{p+2} - \mathbf{v}_{p-1}\| \leq 2h_0$.
3. $(p, p+1)$ is standard, $(p-1, p)$ is pro$^+$.
4. $(p, p-1)$ is standard, $(p, p+1)$ is pro$^+$.

If $(p, j)$ is a standard edge at $p$, with $j = p \pm 1$, then the angle at $j$ is acute. Otherwise, deformation 482 obtuse makes $(p, j)$ minimal. We use a computer calculation to contradict this fact.

In the first two cases, by *computer calculation*[2] [2] we may use `LEMMA_PWE1` and `LEMMA_PWE2`, respectively. Note that both edges at $p+2$ are pro$^+$.

[2] `[6184614449, 1348932091, 1348932091 delta, 5557288534, 5557288534 delta]`

In the third case, the edge $(p+1, p+2)$ is pro$^+$, and the edge $(p+2, p+3)$ is standard (because $m \leq 2$) of length 2. Use `LEMMA_PWE3`, which is based on a *computer calculation*[3] [2].

The fourth case is symmetrical with the third, and uses `LEMMA_PWE4`. □

**Lemma 2.44** `[VASYYAU]` *Let* $(s, v)$ *satisfy the running assumptions. Let* $\mathbf{v}$ *be straight at* $p$*. Then both edges at* $p$ *are minimal.*

*Proof* If there exists a standard constraint at some edge at $p$, then by PWEIBZ, it is minimal. By NUX, the other edge at $p$ is also minimal. In this case we are done.

If, on the other hand, there does not exist a standard constraint at some edge at $p$, then since $m \leq 2$, both edges at $p+2$ are standard. The edges are $\mathbf{v}$-minimal and hence have length 2. The two edges at $p$ are pro$^+$ and hence have length $\geq 2h_0$. Also, by ODX, IMJ, since some edge at $p$ is not minimal, we have $\|\mathbf{v}_p\| = 2$.

This violates the triangle inequality:

$$\begin{aligned}
\mathrm{arc}(2, \|\mathbf{v}_{p-1}\|, 2) + \mathrm{arc}(2, \|\mathbf{v}_{p+1}\|, 2) &< \mathrm{arc}(2, \|\mathbf{v}_{p-1}\|, 2h_0) + \mathrm{arc}(2, \|\mathbf{v}_{p+1}\|, 2h_0)\\
&\leq \mathrm{arc}_V(\mathbf{0}, \{\mathbf{v}_{p-1}, \mathbf{v}_p\}) + \mathrm{arc}_V(\mathbf{0}, \{\mathbf{v}_{p+1}, \mathbf{v}_p\})\\
&= \mathrm{arc}_V(\mathbf{0}, \{\mathbf{v}_{p-1}, \mathbf{v}_{p+1}\})\\
&\leq \mathrm{arc}_V(\mathbf{0}, \{\mathbf{v}_{p-1}, \mathbf{v}_{p+2}\}) + \mathrm{arc}_V(\mathbf{0}, \{\mathbf{v}_{p+2}, \mathbf{v}_{p+1}\})\\
&\leq \mathrm{arc}(2, \|\mathbf{v}_{p-1}\|, 2) + \mathrm{arc}(2, \|\mathbf{v}_{p+1}\|, 2).
\end{aligned}$$

□

**Lemma 2.45** `[NWDGKXH]`

$$\{4M6\} \Rightarrow \{4T3, 4T5\}.$$

*Proof* Let $\mathbf{v} \in \mathcal{M}_{4M6}$. By the definition of 4M6, the diagonals are at least $c_{\mathrm{stab}}$. If some diagonal has $\|\mathbf{v}_i - \mathbf{v}_j\| = c_{\mathrm{stab}}$, then

$$\{4M6\} \Rightarrow \{4T5\}.$$

Otherwise, by the preceding lemmas, all edges are extremal, and all standard edges have length 2.

By Lemma `quad_nonexist_849`, the nonstandard edge does not have $\|\mathbf{v}_i - \mathbf{v}_j\| = 2h_0$. Since the non-standard edge is extremal, this forces $\|\mathbf{v}_i - \mathbf{v}_j\| = c_{\mathrm{stab}}$. In this case $\{4M6\} \Rightarrow \{4T3\}$. □

[3] `[2073661826, 8405387449, 9368433105, 5550839403, 5550839403 delta]`

**Lemma 2.46** [EFLYGAU] *Let* $\mathbf{v} \in \mathcal{M}_{4M7}$. *If both diagonals of* $\mathbf{v}$ *are greater than* $c_{stab}$, *then*

$$\{4M7\} \Rightarrow \{4M6\}.$$

*Proof* By preceding lemmas, the nonstandard edges of $\mathbf{v}$ is extremal.

If one of those edges is minimal $2h_0$, then $\mathbf{v}$ is in the domain of $\mathcal{M}_{4M6}$ and it transfers.

We prove in fact, that if both standard edges are maximal, then we obtain a contradiction. Suppose for a contradiction, the two nonstandard edges are adjacent and have length $c_{\mathrm{stab}}$. By Lemma VASYYAU, neither nonstandard edge is adjacent to a vertex that is straight. By WKZZEEH, one of the nonstandard edges is minimal. earlier lemma has ruled out adjacent edges of length $c_{\mathrm{stab}}$. ◻

**Lemma 2.47** [YOBIMPP] *Let* $\mathbf{v} \in \mathcal{M}_{4M7}$. *If some diagonal has length* $c_{stab}$, *then*

$$\{4M7\} \Rightarrow \{3M1, 3T3, 3T4\}$$

*Proof* Let $\{i, i+2\}$ be the diagonal such that $\|\mathbf{v}_i - \mathbf{v}_{i+2}\| = c_{\mathrm{stab}}$.

Assume first that the diagonal is $\{0, 2\}$. Slice along the diagonal to obtain 3M1 and 3T3. Then $\{4M7\} \Rightarrow \{3M1, 3T3\}$.

In the remaining case, the diagonal is $\{1, 3\}$. Slice along the diagonal to obtain two equivalent pieces that transfer to 3T4. ◻

**Lemma 2.48** [BJTDWPS] *Let* $\mathbf{v} \in \mathcal{M}_{4M8}$. *If both diagonals of* $\mathbf{v}$ *are greater than* $c_{stab}$, *then*

$$\{4M8\} \Rightarrow \{4M6, 3T7\}.$$

*Proof* By preceding lemmas, the non-standard edges of $\mathbf{v}$ is extremal. If one of those edges is minimal $2h_0$, then $\mathbf{v}$ is in the domain of $\mathcal{M}_{4M6}$ and it transfers.

Otherwise, both non-standard edges have length $c_{\mathrm{stab}}$. Every non-standard edge is adjacent to a standard edge. By Lemma PWEI WBZ, these standard edges are minimal; that is, they have length 2. By Lemma quad_diag_362, the shortest diagonal has length at most 3.62. Slice along that diagonal, to obtain two pieces like 3T7. Then $\{4M7\} \Rightarrow \{3T7\}$. ◻

**Lemma 2.49** [MIQMCSN] *Let* $\mathbf{v} \in \mathcal{M}_{4M8}$. *If some diagonal has length precisely* $c_{stab}$, *then*

$$\{4M8\} \Rightarrow \{3T4\}$$

*Proof* Slice along the diagonal to obtain two equivalent pieces that transfer to 3T4. (Compare the proof of RYPDIXT.) ◻

This completes the analysis of the cases 4M6, 4M7, and 4M8. All quad cases have been resolved into terminal cases.

## 2.7 triangles

**Lemma 2.50** `[LFLACKU]` `[formal proof by TCH].` $\{3I1\} \Rightarrow \{3T2, 3T5\}$

*Proof* By deformation 1834, we may assume that each edge $\{i, j\}$ is extremal, hence of length 2 or $2h_0$. If all edges have length 2, then

$$\{3I1\} \Rightarrow \{3T2\}.$$

Otherwise,

$$\{3I1\} \Rightarrow \{3T5\}.$$

□

**Lemma 2.51** `[BKOSSGE]` `[formal proof by TCH].` $\{3M1\} \Rightarrow \{3T1, 3T5\}$.

**Lemma 2.52** `[OCBICBY]` `[formal proof by T. Hales].`

$$\mathcal{M}_s = \emptyset,$$

*for all* $s \in S_{term}$.

*Proof* When $s \in S_{\text{term}}$ with $k(s) \le 4$, by a *computer calculation*[4] [2], we show that $\tau^*(s, \mathbf{v}) \ge 0$ for all $\mathbf{v} \in \mathcal{B}_s$. This implies for such $s$ that $\mathcal{M}_s = \emptyset$. The two cases $5T1$ and $6T1$ remain. They are treated in [1][Section 7.4.5],where it is shown that $\mathcal{M}_s = \emptyset$ for $s = 5T1$ and $6T1$. This shows that $\mathcal{M}_s = \emptyset$ for all $s \in S_{\text{term}}$. □

## 2.8 unsorted lemmas

We dump a few unsorted lemmas at the end of this appendix.

**Lemma 2.53** `[TECOXBM]` `[formal proof by Hoang Le Truong].` *Let $s$ be a SCS, and let $\mathbf{v} \in \mathcal{B}_s$. Let $\mathbf{u}, \mathbf{w} \in V_{\mathbf{v}}$ satisfy $2 \le \|\mathbf{u} - \mathbf{w}\| \le c_{stab}$ where $\{\mathbf{u}, \mathbf{w}\} \notin E_{\mathbf{v}}$. Then $\mathbf{u}$ and $\mathbf{w}$ are nonparallel. Moreover, $C^0\{\mathbf{u}, \mathbf{w}\} \subset W^0_{dart}(x)$ for all $x \in F$.*

*Proof* This is a repetition of [1][Lemma TECOXBM]. □

[4] `[various inequalities]`

**Definition 2.54** ($XR$) `[LIHVJNG]` We let

$$XR(y_1, y_2, y_6) = 8(1 - \frac{y_1^2 + y_2^2 - y_6^2}{2y_1y_2}).$$

**Lemma 2.55** `[KPIDBQH]` `[formal proof by Hales].`

$$\mathrm{dih}_y(y_1, y_2, \ldots, y_6) = \mathrm{dih}_x(4, 4, 4, XR(y_2, y_3, y_4), XR(y_1, y_2, y_6), XR(y_1, y_3, y_5)).$$

*Proof* Compute both sides. □

**Lemma 2.56** `[DRNDRDV]` `[formal proof by Hales].`

$$\frac{\partial XR(y_1, y_2, y_6)}{\partial y_6} = \frac{8y_6}{y_1y_2}.$$

**Lemma 2.57** `[TBRMXRZ]` `[formal proof by Hales].` *Let* $h(x) = g(f(x))$ *and* $f(x) = y$. *Assume* $f'(x) > 0$. *Then under appropriate differentiabilty conditions:*

*1.* $g'(y) \sim h'(x)$.
*2.* $h'(x) = 0$ *implies that* $h''(x) \sim g''(y)$.

*Proof*

$$h'(x) = g'(y) f'(x) \sim g'(y).$$

Also, if $h'(x) = 0$, then $g'(y) = 0$ and

$$h''(x) = g''(y)(f'(x))^2 + g'(y) f''(x) = g''(y)(f'(x))^2 \sim g''(y).$$

□

# 3

## Appendix on explicit deformations

This is an appendix to Section 7.2 of Dense Sphere Packings. This appendix gives the explicit construction of particular deformations.

The first lemma constructs a simplex $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subset \mathbb{R}^3$ on a given base triangle $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\}$. Our intention is to define a new constant to equal the right-hand side of Equation 3.2. The variable $x_5$ will run over an interval to define a continuous deformation of a local fan $(V, E, F)$.

**Lemma 3.1** `[PQCSXWG]` *Let* $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\} \subset \mathbb{R}^3$. *Assume that* $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\}$ *is not collinear. Let* $x_1, \ldots, x_6 \in \mathbb{R}^3$ *be given with* $x_i > 0$ *and*

$$x_1 = \|\mathbf{v}_1 - \mathbf{v}_0\|^2, \quad x_2 = \|\mathbf{v}_2 - \mathbf{v}_0\|^2, \quad x_6 = \|\mathbf{v}_1 - \mathbf{v}_2\|^2.$$

*Assume that* $\Delta(x_1, \ldots, x_6) > 0$ *(that is,* `delta_x x1 x2 ...`*). Then there exists* $\mathbf{v}_3$ *such that*

- $x_3 = \|\mathbf{v}_3 - \mathbf{v}_0\|^2, \quad x_5 = \|\mathbf{v}_3 - \mathbf{v}_1\|^2, \quad x_4 = \|\mathbf{v}_3 - \mathbf{v}_2\|^2$, *and*
- $(\mathbf{v}_1 - \mathbf{v}_0) \cdot (\mathbf{v}_2 - \mathbf{v}_0) \times (\mathbf{v}_3 - \mathbf{v}_0) > 0.$

*Explicitly, the following vector works:*

$$\mathbf{v}_3 = \mathbf{v}_0 + \frac{2\sqrt{\Delta}(\mathbf{v}_1 - \mathbf{v}_0) \times (\mathbf{v}_2 - \mathbf{v}_0) + \Delta_5(\mathbf{v}_1 - \mathbf{v}_0) + \Delta_4(\mathbf{v}_2 - \mathbf{v}_0)}{\upsilon(x_1, x_2, x_6)}, \tag{3.2}$$

*with subscripts on* $\Delta$ *indicating partial derivatives and* $\upsilon =$ `ups_x`*, the upsilon function. Moreover, fixing* $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2$ *and fixing all the variables* $x_i$ *except* $x_5$*, the vector* $\mathbf{v}_3 \in \mathbb{R}^3$ *in Equation 3.2 depends continuously on* $x_5$ *on the domain*

$$\{x_5 \ : \ x_5 > 0, \quad \Delta(x_1, \ldots, x_6) > 0\}.$$

*Remark* 3.3 There is a symmetry to the data fixing $\mathbf{v}_0$, $x_3$, $x_6$, $\upsilon(x_1, x_2, x_6)$ and swapping $\mathbf{v}_1 \leftrightarrow \mathbf{v}_2$, $x_1 \leftrightarrow x_2$, $x_4 \leftrightarrow x_5$, $\Delta_5 \leftrightarrow \Delta_4$. Under this symmetry, the vector $\mathbf{v}_3$ is given by the same formula, except that $\sqrt{\Delta}$ is replaced with $-\sqrt{\Delta}$, and the sign of the triple product is reversed.

*Proof* It can be shown by direct computation that the vector $\mathbf{v}_3$ works. This proof gives details about how $\mathbf{v}_3$ is found.

Without loss of generality, we may move $\mathbf{v}_0$ to the origin. Explicitly, we let $\mathbf{w}_i = \mathbf{v}_i - \mathbf{v}_0$. We will construct a unique $\mathbf{w}_3$ (for $\mathbf{w}_0 = 0$) and then set $\mathbf{v}_3 = \mathbf{v}_0 + \mathbf{w}_3$.

Note that the non-collinearity condition on $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\}$ gives $\mathbf{w}_1 \times \mathbf{w}_2 \neq 0$. We may write $\mathbf{w}_3$ in terms of the basis $\mathbf{w}_1 \times \mathbf{w}_2$, $\mathbf{w}_1$, and $\mathbf{w}_2$:

$$\mathbf{w}_3 = \alpha(\mathbf{w}_1 \times \mathbf{w}_2) + \beta\mathbf{w}_1 + \gamma\mathbf{w}_2.$$

We compute the norms of $\mathbf{w}_3$, $\mathbf{w}_3 - \mathbf{w}_1$ and $\mathbf{w}_3 - \mathbf{w}_2$ in this basis:

$$\begin{aligned}
\|\mathbf{w}_3\|^2 &= x_3 = \alpha^2\|\mathbf{w}_1 \times \mathbf{w}_2\|^2 + \|\beta\mathbf{w}_1 + \gamma\mathbf{w}_2\|^2\\
\|\mathbf{w}_3 - \mathbf{w}_1\|^2 &= x_5 = \alpha^2\|\mathbf{w}_1 \times \mathbf{w}_2\|^2 + \|(\beta - 1)\mathbf{w}_1 + \gamma\mathbf{w}_2\|^2\\
\|\mathbf{w}_3 - \mathbf{w}_2\|^2 &= x_4 = \alpha^2\|\mathbf{w}_1 \times \mathbf{w}_2\|^2 + \|\beta\mathbf{w}_1 + (\gamma - 1)\mathbf{w}_2\|^2.
\end{aligned}$$

We eliminate $\alpha^2$ from the equations and write the resulting equations for $\beta$ and $\gamma$ as a linear system:

$$\begin{pmatrix} \|\mathbf{w}_1\|^2 & \mathbf{w}_1 \cdot \mathbf{w}_2 \\ \mathbf{w}_1 \cdot \mathbf{w}_2 & \|\mathbf{w}_2\| \end{pmatrix}\begin{pmatrix}\beta\\ \gamma\end{pmatrix} = \begin{pmatrix}(x_1 + x_3 - x_5)/2\\ (x_2 + x_3 - x_4)/2\end{pmatrix}.$$

The determinant of the system is

$$\|\mathbf{w}_1\|^2\|\mathbf{w}_2\|^2 - (\mathbf{w}_1 \cdot \mathbf{w}_2)^2 = \|\mathbf{w}_1 \times \mathbf{w}_2\|^2 > 0.$$

By the law of sines or cosines (Lemma 2.59), this determinant can also be written in terms of $x_i$:

$$x_1x_2 - ((x_6 - x_1 - x_2)/2)^2 = \upsilon(x_1, x_2, x_6)/4 > 0.$$

Thus, there are unique solutions $\beta$ and $\gamma$ as functions of the variables $x_i$. Explicitly,

$$\beta = \Delta_5/\upsilon(x_1, x_2, x_6), \quad \gamma = \Delta_4/\upsilon(x_1, x_2, x_6),$$

where $\Delta_i = \Delta_i(x_1, \ldots, x_6)$ is the $i$th partial derivative of $\Delta$.

With $\beta$ and $\gamma$ in hand, we solve for $\alpha$ using the equation:

$$x_3 - \|\beta\mathbf{w}_1 + \gamma\mathbf{w}_2\|^2 = \alpha^2\|\mathbf{w}_1 \times \mathbf{w}_2\|^2.$$

The left-hand side of this equation, expressed in terms of the variables $x_i$ is precisely $\Delta/\upsilon(x_1, x_2, x_6) > 0$. Hence

$$\alpha = \pm 2\frac{\sqrt{\Delta(x_1, \ldots, x_6)}}{\upsilon(x_1, x_2, x_6)}.$$

The triple product is

$$((\mathbf{v}_1 - \mathbf{v}_0) \times (\mathbf{v}_2 - \mathbf{v}_0)) \cdot (\mathbf{v}_3 - \mathbf{v}_0) = (\mathbf{w}_1 \times \mathbf{w}_2) \cdot \mathbf{w}_3 = \alpha |\mathbf{w}_1 \times \mathbf{w}_2|^2.$$

This is positive exactly when $\alpha$ is chosen to be positive. This shows the existence of a unique vector $\mathbf{w}_3$ subject to the given conditions.

Continuity follows from the continuous dependence of $\alpha$, $\beta$, and $\gamma$ on $x_5$. In fact, $\beta$ and $\gamma$ are polynomials in $x_5$ and $\alpha$ requires the extraction of a square root of a positive polynomial in $x_5$. □

We will need a second lemma for deformations that occur within a plane. Our intention is to define a new constant to equal the right-hand side of Equation 3.5. It will be used in two contexts. Sometimes, the variable $x_3$ will run over an interval to define a continuous deformation of a local fan $(V, E, F)$. At other times, the vector $\mathbf{v}_2$ will vary in a continuous deformation constructed in the previous lemma, and $\mathbf{v}_3$ will be carried continuously along in the plane of $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2(t)\}$ by the construction of this lemma.

**Lemma 3.4** `[EYYPQDW] [formal proof by Hoang Le Truong].` *Let* $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\} \subset \mathbb{R}^3$*. Assume that* $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\}$ *is not collinear. Let* $x_1, x_2, x_3, x_5, x_6 \in \mathbb{R}^3$ *be given with* $x_i > 0$ *and*

$$x_1 = \|\mathbf{v}_1 - \mathbf{v}_0\|^2, \quad x_2 = \|\mathbf{v}_2 - \mathbf{v}_0\|^2, \quad x_6 = \|\mathbf{v}_1 - \mathbf{v}_2\|^2.$$

*Assume that* $\upsilon(x_1, x_3, x_5) > 0$*. Let* $\sigma \in \{\pm 1\}$ *be a choice of sign. Then there exists* $\mathbf{v}_3$ *such that*

- $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *is coplanar.*
- $x_3 = \|\mathbf{v}_3 - \mathbf{v}_0\|^2, \quad x_5 = \|\mathbf{v}_3 - \mathbf{v}_1\|^2$*, and*
- $(\mathbf{v}_3 - \mathbf{v}_0) \times (\mathbf{v}_1 - \mathbf{v}_0)$ *is a positive scalar times* $\sigma(\mathbf{v}_1 - \mathbf{v}_0) \times (\mathbf{v}_2 - \mathbf{v}_0)$.

*Explicitly, the following vector works:*

$$\mathbf{v}_3 = \mathbf{v}_0 + \frac{x_1 + x_3 - x_5}{2x_1}(\mathbf{v}_1 - \mathbf{v}_0) + \frac{\sigma}{x_1}\sqrt{\frac{\upsilon(x_1, x_3, x_5)}{\upsilon(x_1, x_2, x_6)}}(\mathbf{v}_1 - \mathbf{v}_0) \times ((\mathbf{v}_1 - \mathbf{v}_0) \times (\mathbf{v}_2 - \mathbf{v}_0)). \tag{3.5}$$

*Moreover, fixing* $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2$ *and fixing all the variables* $x_i$ *except* $x_3$*, the vector* $\mathbf{v}_3 \in \mathbb{R}^3$ *depends continuously on* $x_3$ *on the domain*

$$\{x_3 \ :\ x_3 > 0, \quad \upsilon(x_1, x_3, x_5) > 0\}.$$

*Moreover, fixing* $\mathbf{v}_0, \mathbf{v}_1$ *and fixing all the variables* $x_i$*, the vector* $\mathbf{v}_3 \in \mathbb{R}^3$ *depends continuously on* $\mathbf{v}_2$ *on the domain*

$$\{\mathbf{v}_2 \in \mathbb{R}^3 \ :\ \mathbf{v}_2 \ \text{ is not on the line through } \ \mathbf{v}_0 \ \text{ and } \ \mathbf{v}_1\}.$$

*Remark* 3.6 There is a symmetry in the data $\mathbf{v}_0 \leftrightarrow \mathbf{v}_1$, $x_3 \leftrightarrow x_5$, $x_2 \leftrightarrow x_6$ fixing $\mathbf{v}_2$ and $x_1$. The symmetry preserves the solution $\mathbf{v}_3$.

*Proof* Without loss of generality, we may move $\mathbf{v}_0$ to the origin. Explicitly, we let $\mathbf{w}_i = \mathbf{v}_i - \mathbf{v}_0$. We will construct a unique $\mathbf{w}_3$ (for $\mathbf{w}_0 = 0$) and then set $\mathbf{v}_3 = \mathbf{v}_0 + \mathbf{w}_3$.

Let $\mathbf{n} = \mathbf{w}_1 \times (\mathbf{w}_1 \times \mathbf{w}_2)$. By the non-collinearity assumption and the positivity of $x_i$, we have $\mathbf{n} \neq 0$. In fact, $\mathbf{w}_1$ are $\mathbf{n}$ are orthogonal and span $\operatorname{aff}\{\mathbf{0}, \mathbf{w}_1, \mathbf{w}_2\}$. The norm of $\mathbf{n}$ is computed as in the previous lemma by the law of cosiines:

$$\|\mathbf{n}\|^2 = \|\mathbf{w}_1\|^2 \|\mathbf{w}_1 \times \mathbf{w}_2\|^2 = x_1 \upsilon(x_1, x_2, x_6)/4 > 0.$$

We solve for $\mathbf{w}_3$ as a combination of $\mathbf{w}_1$ and $\mathbf{n}$:

$$\mathbf{w}_3 = \alpha \mathbf{w}_1 + \beta \mathbf{n}.$$

The norms of $\mathbf{w}_3$ and $\mathbf{w}_3 - \mathbf{w}_1$ are computed as

$$\begin{aligned} \|\mathbf{w}_3\|^2 &= x_3 = \alpha^2 \|\mathbf{w}_1\|^2 + \beta^2 \|\mathbf{n}\| \\ \|\mathbf{w}_3 - \mathbf{w}_1\|^2 &= x_5 = \alpha^2 \|\mathbf{w}_1\|^2 + \beta^2 \|\mathbf{n}\| \end{aligned}$$

Eliminate $\beta$ and solve the resulting linear equation uniquely for $\alpha$:

$$\alpha = (x_1 + x_3 - x_5)/(2x_1).$$

The right-hand side of $\beta^2 \|\mathbf{n}\|^2 = x_3 - \alpha^2 x_1$, expressed in terms of the variables $x_i$ is $\upsilon(x_1, x_3, x_5)/(4x_1)$. Hence,

$$\beta = \pm \frac{1}{x_1} \sqrt{\frac{\upsilon(x_1, x_3, x_5)}{\upsilon(x_1, x_2, x_6)}}.$$

To compute the sign of $\beta$, we examine the cross-product condition.

$$\begin{aligned} \mathbf{w}_3 \times \mathbf{w}_1 &= (\alpha \mathbf{w}_1 + \beta \mathbf{n}) \times \mathbf{w}_1 \\ &= \beta(\mathbf{w}_1 \times (\mathbf{w}_1 \times \mathbf{w}_2)) \times \mathbf{w}_1 \\ &= \beta(\mathbf{w}_1 \times \mathbf{w}_2)(\mathbf{w}_1 \cdot \mathbf{w}_1) \\ &= \beta x_1 (\mathbf{w}_1 \times \mathbf{w}_2). \end{aligned}$$

This is a positive multiple of $\sigma \mathbf{w}_1 \times \mathbf{w}_2$ when $\beta$ has sign $\sigma$. This establishes the unique existence of $\mathbf{w}_3$.

Continuity follows from the explicit formulas for $\alpha$ and $\beta$. □

# 4

# Appendix on deformations of local fans

This is an appendix to Section 7.2 of Dense Sphere Packings. This appendix gives further details about the proof of the wedge property in Lemma 7.25 (ZLZTHIC).

**Lemma 4.1** `[WEECNNS]` `[formal proof by J. Harrison].` *The function* $\mathrm{dih}_V$ *is continuous on the set*

$$\{(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)\ :\ \textit{not collinear}\ \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}\ \textit{and not collinear}\ \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}\}.$$

**Lemma 4.2** `[XBJRPHC]` *The function* azim *is continuous on the set*

$$\{(\mathbf{v}1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)\ :\ \textit{not collinear}\ \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}\ \textit{and not collinear}\ \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}\mathbf{v}_4 \notin \mathrm{aff}^0_+(\{\mathbf{v}_1, \mathbf{v}_2\}, \mathbf{v}_3)\}.$$

The following lemma rewrites the generic blade condition as a union of two sets that are manifestly open.

**Lemma 4.3** (generic_alt) `[IHWVUIZ]` `[formal proof by T. Hales].` *Assume that* $\{0, \mathbf{v}, \mathbf{w}\}$ *is not collinear and that* $\mathbf{u} \neq \mathbf{0}$. *Then the generic blade condition holds*

$$\mathrm{aff}_+(\mathbf{0}, \{\mathbf{v}, \mathbf{w}\}) \cap \mathrm{aff}^0_-(\{\mathbf{0}\}, \{\mathbf{u}\}) = \varnothing$$

*iff* $\{\mathbf{0}, \mathbf{u}, \mathbf{v}, \mathbf{w}\}$ *is not coplanar or* $-\mathbf{u} \in W^0(\mathbf{0}, \mathbf{v} \times \mathbf{w}, \mathbf{w}, \mathbf{v})$.

**Lemma 4.4** `[NHCXLRV]` *Let* $(V, E, F)$ *be a generic convex local fan. Let* $(\varphi, I, V)$ *be a deformation of the fan. Let* $\mathbf{v}, \mathbf{w} \in V$. *Assume that* $\mathbf{v} \in W^0_{dart}(\mathbf{w}, \rho\mathbf{w})$. *Then for sufficiently small* $t$, *we have*

$$\mathbf{v}(t) \in W^0_{dart}(\mathbf{w}(t), \rho\mathbf{w}(t)).$$

*Proof* *We claim that* $(\mathbf{v}, \mathbf{w})$ *is not a pole of the fan.* In fact, by `WEDGE_ALT`, if a pole, then $\mathbf{v} \notin W^0_{\mathrm{dart}}(\mathbf{w}, \rho\mathbf{w})$.

The condition $\mathbf{v} \in W^0_{\text{dart}}(\mathbf{w}, \rho\mathbf{w})$ gives

$$0 < \text{azim}(\mathbf{0}, \mathbf{w}, \rho\mathbf{w}, \mathbf{v}) < \text{azim}(\mathbf{0}, \mathbf{w}, \rho\mathbf{w}, \rho^{-1}\mathbf{w}).$$

Pick $c$ such that

$$0 < \text{azim}(\mathbf{0}, \mathbf{w}, \rho\mathbf{w}, \mathbf{v}) < c < \text{azim}(\mathbf{0}, \mathbf{w}, \rho\mathbf{w}, \rho^{-1}\mathbf{w}). \tag{4.5}$$

The pole claim gives that the domain conditions for the continuity of azim are met. Therefore Equation (4.5) holds under a small deformation. This gives the conclusion. □

**Lemma 4.6** `[WNWSHJT]` *Let $(V, E, F)$ be generic fan. Let $(\varphi, I, V)$ be a deformation of the fan. Let $\mathbf{u} \in V$ have interior angle $\angle(\mathbf{u}) < \pi$. Then for sufficiently small $t$, we have $\angle(\mathbf{u}(t)) < \pi$.*

*Proof* By `INTERIOR_ANGLE1_POS`, we have $0 < \angle(\mathbf{u}) < \pi$. The continuity domain conditions for azim are met at $(\mathbf{0}, \mathbf{u}, \rho\mathbf{u}, \rho^{-1}\mathbf{u}$. Hence

$$0 < \text{azim}(\mathbf{u}(t)) < \pi$$

for $t$ sufficiently small. □

The following case is not needed, because it is a special case of `MHAEYJN`.

**Lemma 4.7** `[CREUXCQ]` *Let $(V, E, F)$ be a generic fan. with pole $(\mathbf{v}, \mathbf{w})$. Assume that the interior angle at the pole is less than $\pi$. Let $\mathbf{u} \neq \mathbf{v}, \mathbf{w}$. Let $(\varphi, I, V)$ be a deformation of the fan that moves a single $\mathbf{u} \in V \setminus \{\mathbf{v}, \mathbf{w}\}$ and such that $\mathbf{u} \in V$, we have $\mathbf{u}(t) \in \text{aff}\{\mathbf{0}, \mathbf{v}, \mathbf{w}, \mathbf{u}\}$. Fix $\mathbf{u}' \in V$ such that $\angle(\mathbf{u}') = \pi$. Then for sufficiently small $t$, we have $\angle(\mathbf{u}'(t)) \leq \pi$.*

*Proof* Let $\mathbf{u}_1 = \mathbf{u}'$, $\mathbf{u}_2 = \rho\mathbf{u}'$, and $\mathbf{u}_0 = \rho^{-1}\mathbf{u}'$. Let $S = \{\mathbf{0}, \mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{v}, \mathbf{w}\}$.

We consider the case $\mathbf{u} \notin S$. Then the set $S$ is fixed under the deformation, so that

$$\angle(\mathbf{u}_1(t)) = \angle(\mathbf{u}'(t)) = \angle(\mathbf{u}') = \pi.$$

In the remaining case $\mathbf{u} \in S$. By the lunar geometry lemma, $S \subset A$, where $A = \text{aff}\{\mathbf{0}, \mathbf{v}, \mathbf{w}, \mathbf{u}\}$, a plane. The deformation moves $\mathbf{u}$ within this plane, so $S(t) \subset A$. This implies that $S(t)$ is coplanar, which gives

$$\angle(\mathbf{u}'(t)) = \text{azim}(\mathbf{0}, \mathbf{u}_0(t), \mathbf{u}_1(t), \mathbf{u}_2(t)) \in \{\mathbf{0}, \pi\}.$$

We claim the the domain conditions for the continuity of azim are met. Hence by continuity, $\angle(\mathbf{u}'(t)) = \pi$ for small $\pi$. □

In the next series of lemmas we introduce an assumption `ECAU` given as follows.

*Remark* 4.8 (ECAU) Let $\mathbf{u}_0, \ldots, \mathbf{u}_r \in \mathbb{R}^3$, and define $\mathbf{e} = \mathbf{u}_0 \times \mathbf{u}_1$. We make the following assumptions.

1. (C) The set $\{\mathbf{0}, \mathbf{u}_i, \mathbf{u}_j\}$ is not collinear for $i \neq j$.
2. (A) For all $i \geq 1$, we have $\mathbf{u}_i \in \mathrm{aff}_+^0(\{\mathbf{0}, \mathbf{u}_0\}, \mathbf{u}_1)$.
3. (U) The set $U = \{\mathbf{u}_0, \ldots, \mathbf{u}_r\}$ is cyclic with respect to $(\mathbf{0}, \mathbf{e})$, with cycle

$$\sigma \mathbf{u}_i = \mathbf{u}_{i+1}, \text{ if } i < r.$$

We let $A$ be the plane $\mathrm{aff}\{\mathbf{0}, \mathbf{u}_0, \mathbf{u}_1\}$.

**Lemma 4.9** `[VUYCADE][formal proof by Hales].` *Let* $\mathbf{u}_1, \ldots, \mathbf{u}_r$ *with cross product* $\mathbf{e}$ *satisfy the conditions* ECAU*. Then for all* $i \leq r$*, we have constants* $t_0$ *and* $t_1$*, with* $t_1 > 0$ *such that*

$$\mathbf{u}_i = t_0 \mathbf{u}_0 + t_1 \mathbf{u}_1, \quad \mathbf{e} \cdot \mathbf{u}_i = 0, \quad \mathrm{azim}(\mathbf{0}, \mathbf{e}, \mathbf{u}_0, \mathbf{u}_i) < \pi.$$

*Also,* $\{\mathbf{0}, \mathbf{e}, \mathbf{u}_i\}$ *is not collinear.*

*Proof* Condition A gives $\mathbf{u}_i = t_0 \mathbf{u}_0 + t_1 \mathbf{u}_1$, with $t_1 \geq 0$. We get $t_1 > 0$ from C. Taking the dot product of both sides of this identity gives $\mathbf{e} \cdot \mathbf{u}_i = 0$. We have that $\{\mathbf{0}, \mathbf{e}, \mathbf{u}_i\}$ is not collinear because $\mathbf{e}$ and $\mathbf{u}_i$ are orthogonal and nonzero. The condition $\mathrm{azim}(\mathbf{0}, \mathbf{e}, \mathbf{u}_0, \mathbf{u}_i) < \pi$ is equivalent to $\mathbf{e} \cdot (\mathbf{u}_0 \times \mathbf{u}_i) \neq 0$. The left-hand-side of this equation evaluates to $\|\mathbf{e}\| t_1 > 0$. □

**Lemma 4.10** `[YBTASCZ][formal proof by Hales].` *Let* $\mathbf{u}_1, \ldots, \mathbf{u}_r$ *with cross product* $\mathbf{e}$ *satisfy the conditions* ECAU*. For all* $i < j \leq r$*, we have*

$$\mathrm{azim}(\mathbf{0}, \mathbf{e}, \mathbf{u}_0, \mathbf{u}_i) < \mathrm{azim}(\mathbf{0}, \mathbf{e}, \mathbf{u}_0, \mathbf{u}_j).$$

*Furthermore,* $\mathrm{azim}(\mathbf{0}, \mathbf{e}, \mathbf{u}_i, \mathbf{u}_j) < \pi$.

*Proof* Let $j \leq r$. We prove the result by induction on $i$.

When $i = 0$, we have

$$0 = \mathrm{azim}(\mathbf{0}, \mathbf{e}, \mathbf{u}_0, \mathbf{u}_0) \leq \mathrm{azim}(\mathbf{0}, \mathbf{e}, \mathbf{u}_0, \mathbf{u}_j).$$

We do not have equality because of A and C.

Assume for $i$ and assume for a contradiction $i + 1$ fails. That is,

$$\mathrm{azim}(\mathbf{0}, \mathbf{e}, \mathbf{u}_0, \mathbf{u}_i) < \mathrm{azim}(\mathbf{0}, \mathbf{e}, \mathbf{u}_0, \mathbf{u}_j) \leq \mathrm{azim}(\mathbf{0}, \mathbf{e}, \mathbf{u}_0, \mathbf{u}_{i+1}).$$

By AC, the inequality is strict. By U, this contradicts $\sigma \mathbf{u}_i = \mathbf{u}_{i+1}$. This proves the main statement.

The further claim follows from the addition law

$$\mathrm{azim}(\mathbf{0}, \mathbf{e}, \mathbf{u}_0, \mathbf{u}_i) + \mathrm{azim}(\mathbf{0}, \mathbf{e}, \mathbf{u}_i, \mathbf{u}_j) = \mathrm{azim}(\mathbf{0}, \mathbf{e}, \mathbf{u}_0, \mathbf{u}_j).$$

□

**Lemma 4.11** `[KCZXLLE]` `[formal proof by Hales].` *Let* $\mathbf{u}_1, \ldots, \mathbf{u}_r$ *with cross product* $\mathbf{e}$ *satisfy the conditions* `ECAU`. *Assume* $j < k \le r$. *If* $i < j$ *or* $k < i$ *then* $\mathrm{azim}(\mathbf{0}, \mathbf{u}_i, \mathbf{u}_j, \mathbf{u}_k) = \mathbf{0}$.

*Proof* By $A$, the set $\{\mathbf{0}, \mathbf{u}_i, \mathbf{u}_j, \mathbf{u}_k\}$ is coplanar. This implies that the azimuth angle is 0 or $\pi$. By `LDURDPN`, using that $\{\mathbf{0}, \mathbf{u}_i, \mathbf{u}_j, \mathbf{u}_k\}$ is coplanar, we get that $\mathrm{azim}(\mathbf{0}, \mathbf{u}_i, \mathbf{u}_j, \mathbf{u}_k) \ne \pi$ if and only iff

$$\mathrm{aff}\{\mathbf{0}, \mathbf{u}_i\} \cap \mathrm{conv}^0\{\mathbf{u}_j, \mathbf{u}_k\} = \emptyset. \tag{4.12}$$

Since $\mathrm{azim}(\mathbf{0}, \mathbf{e}, \mathbf{u}_j, \mathbf{u}_k) < \pi$, we can relate the wedge to a lune:

$$\mathrm{conv}^0\{\mathbf{u}_j, \mathbf{u}_k\} \subset \mathrm{aff}^0_+\{\mathbf{0}, \mathbf{e}\}\{\mathbf{u}_j, \mathbf{u}_k\} = W^0(\mathbf{0}, \mathbf{e}, \mathbf{u}_j, \mathbf{u}_k) \subset \mathrm{aff}^0_+(\{\mathbf{0}, \mathbf{e}, \mathbf{u}_0\}, \mathbf{u}_1)$$

and by the $t_1 > 0$ calculation above, we also have

$$\mathrm{aff}\{\mathbf{0}, \mathbf{u}_i\} \cap \mathrm{aff}^0_+(\{\mathbf{0}, \mathbf{e}, \mathbf{u}_0\}, \mathbf{u}_1) \subset \mathrm{aff}^0_+\{\mathbf{0}\}\{\mathbf{u}_i\}.$$

Hence the intersection (4.12) is contained in $W^0(\mathbf{0}, \mathbf{e}, \mathbf{u}_j, \mathbf{u}_k) \cap \mathrm{aff}^0_+\{\mathbf{0}\}\{\mathbf{u}_i\}$. A point $\mathbf{p}$ in the intersection on the right gives

$$0 < \mathrm{azim}(\mathbf{0}, \mathbf{e}, \mathbf{u}_j, \mathbf{p}) < \mathrm{azim}(\mathbf{0}, \mathbf{e}, \mathbf{u}_j, \mathbf{u}_k) \text{ and } \mathrm{azim}(\mathbf{0}, \mathbf{u}_j, \mathbf{p}) = \mathrm{azim}(\mathbf{0}, \mathbf{u}_j, \mathbf{u}_i).$$

However, when $i < j$ or $k < i$, these inequalities fail. □

**Lemma 4.13** `[FSQKWKK]` `[formal proof by JH].` *If* $\mathrm{azim}(\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \le \pi$ *then*

$$\mathrm{azim}(\mathbf{0}, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_1) \le \pi.$$

*Proof* $\mathrm{azim}(\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \le \pi$ iff $0 \le \sin(\mathrm{azim}(\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3))$ iff $0 \le (\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3$. The final condition is invariant under even permutations of the subscripts. □

Recall that if we have $\mathrm{azim}(\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \le \pi$ and appropriate noncollinearity constraints, then $\mathrm{azim}(\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \mathrm{dih}_V(\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

**Lemma 4.14** `[MKIFWJT]` *Let* $\mathbf{v}, \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in \mathbb{R}^3$. *Assume that none of the sets* $\{\mathbf{0}, \mathbf{v}, \mathbf{w}_i\}$ *are collinear and that* $\mathrm{azim}(\mathbf{0}, \mathbf{v}, \mathbf{w}_1, \mathbf{w}_2) + \mathrm{azim}(\mathbf{0}, \mathbf{v}, \mathbf{w}_2, \mathbf{w}_3) < 2\pi$. *Then*

$$\mathrm{azim}(\mathbf{0}, \mathbf{v}, \mathbf{w}_1, \mathbf{w}_2) + \mathrm{azim}(\mathbf{0}, \mathbf{v}, \mathbf{w}_2, \mathbf{w}_3) = \mathrm{azim}(\mathbf{0}, \mathbf{v}, \mathbf{w}_1, \mathbf{w}_3).$$

*Proof* This is (Fan.sum3_azim_fan). □

**Lemma 4.15** (deform-wedge) `[XIVPHKS]` `[formal proof by John Harrison].` *Let* $\mathbf{w}_0, \ldots, \mathbf{w}_n \in \mathbb{R}^3$, *for some* $n \ge 1$. *Assume that* $i \ne j$ *implies* $\{\mathbf{0}, \mathbf{w}_i, \mathbf{w}_j\}$ *is not collinear. Let* $r = n - 1$. *Set*

$$d(i, j, k) = \mathrm{dih}_V(\mathbf{0}, \mathbf{w}_i, \mathbf{w}_j, \mathbf{w}_k) \quad \textit{and} \quad a(i, j, k) = \mathrm{azim}(\mathbf{0}, \mathbf{w}_i, \mathbf{w}_j, \mathbf{w}_k).$$

*Introduce abbreviations for wedges* $W(i) = W(\mathbf{0}, \mathbf{w}_i, \mathbf{w}_{i+1}, \mathbf{w}_{i-1})$. *Let* $\epsilon \in \mathbb{R}$ *satisfy* $0 < 2\epsilon < a(i, i+1, i-1)$*, for* $i = 1, \ldots, r$. *Assume* $a(i, i+1, i-1) \leq \pi$ *for* $i = 1, \ldots, r$. *Assume that* $d(p, q, q+1) < \epsilon$ *for all sets of three distinct elements* $\{p, q, q+1\} \subset \{0, \ldots, r\}$. *Assume that* $d(p, p+1, q) < \epsilon$ *for all sets of three distinct elements* $\{p, p+1, q\} \subset \{0, \ldots, r\}$ *with* $q > p+1$. *Assume that* $d(p+1, p, q) < \epsilon$ *for all sets of three distinct elements* $\{p+1, p, q\} \subset \{0, \ldots, r\}$ *with* $q < p$.

*Then for all* $k$ *we have the statement* $S_k$*: for all* $j \leq r - k$ *we have* $\mathbf{w}_j \in W(j+k)$ *and* $\mathbf{w}_{j+k} \in W(j)$.

*Proof* We prove $S_k$ by induction on $k$. The cases $k = 0, 1$ are trivially satisfied. Assume that $S_k$, for some $k \geq 1$. We show $S_{k+1}$. Fix $j \leq r - (k+1)$.

By induction, $\mathbf{w}_j \in W(j+k)$, giving

$$a(j+k, j+k+1, j) \leq a(j+k, j+k+1, j+k-1) \leq \pi.$$

By Lemma 4.13, $a(j, j+k, j+k+1) \leq \pi$, and converting to dihedral, $a(j, j+k, j+k+1) < \epsilon$.

By induction, $\mathbf{w}_{j+k} \in W(j)$, giving

$$a(j, j+1, j+k) \leq a(j, j+1, j-1) \leq \pi,$$

and converting to dihedral, $a(j, j+1, j+k) < \epsilon$.

Since $2\epsilon < 2\pi$, we can add the angles (Lemma 4.14),

$$a(j, j+1, j+k) + a(j, j+k, j+k+1) = a(j, j+1, j+k+1) < 2\epsilon < a(j, j+1, j-1).$$

This says that $\mathbf{w}_{j+k+1} \in W(j)$.

The proof that $\mathbf{w}_j \in W(j+k+1)$ is similar.

By induction, $\mathbf{w}_{j+k+1} \in W(j+1)$, giving

$$a(j+1, j+k+1, j) \leq a(j+1, j+2, j) \leq \pi.$$

By Lemma 4.13, $a(j+k+1, j, j+1) \leq \pi$, and converting to dihedral $a(j+k+1, j, j+1) < \epsilon$.

By induction, $\mathbf{w}_{j+1} \in W(j+k+1)$, giving

$$a(j+k+1, j+1, j+k) \leq a(j+k+1, j+k+2, j+k) \leq \pi,$$

and converting to dihedral $a(j+k+1, j+1, j+k) < \epsilon$.

Since $2\epsilon < 2\pi$, we can add angles,

$$a(j+k+1, j, j+1) + a(j+k+1, j+1, j+k) = a(j+k+1, j, j+k) < 2\epsilon < a(j+k+1, j+k+2, j+k).$$

This says that $\mathbf{w}_j \in W(j+k+1)$. □

We apply this lemma to prove the wedge property in ZLZTHIC (Dense Sphere Packings Lemma 7.25). We return to the context and notation of that lemma.

**Lemma 4.16** `[ITNZZRD]` *Let $(\varphi, V, I)$ be a deformation of a generic local fan $(V, E, F)$ over an interval $I$. Assume that the azimuth angle of $\mathbf{v}_i(t)$ is at most $\pi$ for all $t \in I$, whenever $\mathbf{v}_i$ is straight. Assume that $(V(t), E(t), F(t))$ is a generic for all $t \in I$. (Or assume that $(V, E, F)$ is a local fan and a deformation that moves a single nonpolar element of $V$.) Then for all sufficiently small $t \in I$, we have that $(V(t), E(t), F(t))$ satisfies the wedge property of local fans. That is, $V(t) \subset W_{dart}(x(t))$ for all $x(t) \in F(t)$.*

*Proof* As in the proof in Dense Sphere Packings, by continuity, the proof of the wedge property reduces to the case $\mathbf{w} \in W_{\text{dart}}(\mathbf{u}, \rho\mathbf{u}) \setminus W^0_{\text{dart}}(\mathbf{u}, \rho\mathbf{u})$. We may prove the more symmetrical statement

$$\mathbf{u}(t) \in W_{\text{dart}}(\mathbf{w}(t), \rho\mathbf{w}(t)) \text{ and } \mathbf{w}(t) \in W_{\text{dart}}(\mathbf{u}(t), \rho\mathbf{u}(t)).$$

Exchanging $\mathbf{u}$ and $\mathbf{w}$ as needed, we have a sequence $\mathbf{u} = \mathbf{v}_0, \ldots, \mathbf{v}_r = \mathbf{w}$, with $\mathbf{v}_i = \rho^i\mathbf{v}$, where all of the terms $\mathbf{v}_i$ $(i = 1, \ldots, r-1)$ are straight. We use Lemmas 7.15 and 7.19 to check the properties `ECAU`.

The interior angles of the fan are positive. Pick $\epsilon$ such that $0 < 2\epsilon <$ azim$(\mathbf{0}, \mathbf{v}_i, \mathbf{v}_{i+1}, \mathbf{v}_{i-1})$ for $i = 0, \ldots, r$. By the continuity of azim, there is some $\epsilon > 0$ such that $2\epsilon <$ azim$(\mathbf{0}, \mathbf{v}_i(t), \mathbf{v}_{i+1}(t), \mathbf{v}_{i-1}(t))$ for $i = 0, \ldots, r$ when $|t| < \epsilon$.

When $t = 0$, the straight conditions give

$$\text{dih}(\mathbf{0}, \mathbf{v}_p, \mathbf{v}_q, \mathbf{v}_{q+1}) = 0,$$

for distinct triples $\{p, q, q + 1\} \subset \{0, \ldots, r\}$, and

$$\text{dih}(\mathbf{0}, \mathbf{v}_p, \mathbf{v}_{p+1}, \mathbf{v}_q) = 0,$$

for distinct triples $\{p, p + 1, q\} \subset \{0, \ldots, r\}$ with $p + 1 < q$. (This follows from the lemma above, which shows that azim $= 0$.) By the continuity of $\text{dih}_V$, shrinking $\epsilon$ as needed, we may assume the corresponding dihedral angles are less than $\epsilon$, when evaluated at $\mathbf{v}_i(t)$, for $|t| < \epsilon$.

All of the conditions of Lemma 4.15 are satisfied for vectors $\mathbf{v}_i(t)$ for any $|t| < \epsilon$. Hence for all $i$ and all $j$ such that $i, j \le r$, we have

$$\mathbf{v}_i(t) \in W_{\text{dart}}(\mathbf{v}_j(t), \mathbf{v}_{j+1}(t)) \text{ and } \mathbf{v}_j(t) \in W_{\text{dart}}(\mathbf{v}_i(t), \mathbf{v}_{i+1}(t)).$$

If we take $i = 0$ and $j = r$, this is the desired wedge condition. □

# 5

# Appendix on the proof of BGMIFTE

This is an appendix to Section 7.3 of Dense Sphere Packings.

This page contains some notes on the verification that the polar $(V', E', F')$ is a local fan. This is an expanded version of the last paragraph of the proof of Lemma 7.34 (BGMIFTE). We check the intersection property of fans, the dihedral property of local fans, and the face property of local fans.

Since we are treating the last paragraph of the proof, we may assume that all the other parts of that lemma dealing with arcs and azimuth angles have been established.

We place ourselves in the context of Lemma BGMIFTE, adopting the notation from that lemma. In particular, $\mathbf{v}_i = \rho^i \mathbf{v}$ and $\mathbf{w}_i = \mathbf{v}_i \times \mathbf{v}_{i+1}$. We take the indices modulo $k = \mathrm{card}(V)$, so that $\mathbf{v}_{i+k} = \mathbf{v}_i$. By earlier parts of the proof, we have

$$\mathbf{w}_j \in W^0(\mathbf{0}, \mathbf{w}_i, \mathbf{w}_{i+1}, \mathbf{w}_{i-1}), \quad j \neq i-1, i, i+1 \tag{5.1}$$

and

$$\{\mathbf{0}, \mathbf{w}_i, \mathbf{w}_j\} \text{ is not collinear, when } i \neq j \tag{5.2}$$

and

$$\mathrm{arc}_V(\mathbf{0}, \{\mathbf{w}_i, \mathbf{w}_{i+1}\}) = \pi - \angle(\mathbf{v}_{i+1}) > 0 \tag{5.3}$$

and

$$\angle'(\mathbf{w}_{i+1}) = \pi - \mathrm{arc}_V(\mathbf{0}, \{\mathbf{v}_{i+1}, \mathbf{v}_{i+2}\}) \in (0, \pi). \tag{5.4}$$

We verify the fan intersection property of $(V', E', F')$. Remark GMLWKPK gives some hints about verifying the intersection property, and notes that it comes down to two cases:

1. $\varepsilon \cap \varepsilon' = \emptyset$ implies $C(\varepsilon) \cap C(\varepsilon') = \emptyset$.
2. $\varepsilon \cap \varepsilon' = \{\mathbf{v}\}$ implies $C(\varepsilon) \cap C(\varepsilon') = C\{v\}$.

If $\varepsilon$ and $\varepsilon'$ are both singletons then the intersection property follows from (5.2). Without loss of generality assume that $\varepsilon$ is not a singleton. By the definition of $E'$, we have $\varepsilon = \{\mathbf{w}_i, \mathbf{w}_{i+1}\}$, for some $i$. We may partition $C(\varepsilon)$ as

$$C(\varepsilon) = C^0(\varepsilon) \cup C^0(\mathbf{w}_i) \cup C^0(\mathbf{w}_{i+1}) \cup \{\mathbf{0}\}.$$

Since we know the intersection property for singletons, we are reduced to showing the following.

1. if $\varepsilon' = \{\mathbf{w}_j, \mathbf{w}_{j+1}\} \neq \varepsilon$ is also an edge, then $C^0(\varepsilon) \cap C^0(\varepsilon') = \emptyset$.
2. for every $\mathbf{w}_j \in V'$, we have $C^0(\varepsilon) \cap C^0(\mathbf{w}_j) = \emptyset$.

Consider the first of these two enumerated cases. Exhanging $i$ with $j$ if necessary, we may assume that $j \neq i, i+1$. Set $\alpha(\mathbf{p}) = \operatorname{azim}(\mathbf{0}, \mathbf{w}_i, \mathbf{w}_{i+1}, \mathbf{p})$. For every point $\mathbf{p}$ in $C^0(\varepsilon)$ we have $\alpha(\mathbf{p}) = 0$ and $C^0(\varepsilon) \subset A := \operatorname{aff}\{\mathbf{0}, \mathbf{w}_i, \mathbf{w}_{i+1}\}$. We separate this from $C^0(\varepsilon')$ by showing that $C^0(\varepsilon') \subset A^0_+ := \operatorname{aff}^0_+(\{\mathbf{0}, \mathbf{w}_i, \mathbf{w}_{i+1}\}, \{\mathbf{w}_{i-1}\})$, and using the disjointness of this open half-space $A^0_+$ from its bounding plane $A$.

If $j = i - 1$, then by (5.4), for every $\mathbf{q} \in C^0(\varepsilon')$, we have

$$\alpha(\mathbf{q}) = \alpha(\mathbf{w}_j) = \alpha(\mathbf{w}_{i-1}) \in (0, \pi).$$

The values of $\alpha$ separate $C^0(\varepsilon)$ from $C^0(\varepsilon')$. Note also that this gives

$$\mathbf{w}_{i-1} \in A^0_+ \tag{5.5}$$

If $\ell \neq i, i+1$, then by (5.1) and (5.5), we have $\mathbf{w}_\ell \in A^0_+$. From this, we obtain $\varepsilon' \subset A^0_+$ and from the conic structure of the halfspace $A^0_+$, it follows that $C^0(\varepsilon') \subset A^0_+$.

The second enumerated case is similar, if $j \in \{i, i+1\}$, then the empty intersection property $C^0\{\mathbf{w}_i, \mathbf{w}_{i+1}\} \cap C^0(\mathbf{w}_j) = \emptyset$ follows from the strict inequality in the definition of $C^0$ and the linear independence of $\mathbf{w}_i$ and $\mathbf{w}_{i+1}$. For example,

$$t_0 \mathbf{w}_i + t_1 \mathbf{w}_{i+1} = s \mathbf{w}_i,$$

has no solution in positive real numbers $t_0, t_1, s$. Otherwise, if $j \notin \{i, i+1\}$, we separate $C^0(\varepsilon)$ from $C^0(\mathbf{w}_j)$ by the disjointness of $A^0_+$ and $A$, as in the first case.

This completes the proof of the intersection property of fans for the polar. Next we verify the dihedral property of local fans.

For this, we review how a hypermap is attached to the fan $(V', E', F')$. We have sets $(V', E', F')$ defined to be

$$V' = \{\mathbf{w}_i = \mathbf{v}_i \times \mathbf{v}_{i+1} \ : \ i\},$$

$$E' = \{\{\mathbf{w}_i, \mathbf{w}_{i+1}\} \ : \ i\},$$

$$F' = \{(\mathbf{w}_i, \mathbf{w}_{i+1}) \ : \ i\}.$$

The set of darts of the hypermap consists of all orderings of edges:

$$D = \{(\mathbf{w}_i, \mathbf{w}_{i+1}) \ : \ i\} \cup \{(\mathbf{w}_{i+1}, \mathbf{w}_i) \ : \ i\} = F' \cup F'' \text{ say .}$$

The set $E'(\mathbf{w}_i)$, with overloaded notation, is defined as the set

$$\{\mathbf{w} \in V' \ : \ \{\mathbf{w}, \mathbf{w}_i\} \in E'\},$$

which in this situation reduces to

$$E'(\mathbf{w}_i) = \{\mathbf{w}_{i-1}, \mathbf{w}_{i+1}\}.$$

The permutation $\sigma(\mathbf{w}_i)$ of $E'(\mathbf{w}_i)$ is defined as the azimuth cycle on this set. Since the set has only two elements, the permutation is forced to swap $\mathbf{w}_{i-1}$ and $\mathbf{w}_{i+1}$. The hypermap is a tuple $(D, e, n, f)$, where the permutations of $D$ are generally defined as follows.

$$\begin{aligned}
n(\mathbf{v}, \mathbf{w}) &= (\mathbf{v}, \sigma(\mathbf{v}, \mathbf{w})),\\
f(\mathbf{v}, \mathbf{w}) &= (\mathbf{w}, \sigma(\mathbf{w})^{-1}\mathbf{v}),\\
e(\mathbf{v}, \mathbf{w}) &= (\mathbf{w}, \mathbf{v}).
\end{aligned}$$

In the present context, this reduces to

$$\begin{aligned}
n(\mathbf{w}_i, \mathbf{w}_{i\pm1}) &= (\mathbf{w}_i, \mathbf{w}_{i\mp1}),\\
f(\mathbf{w}_i, \mathbf{w}_{i\pm1}) &= (\mathbf{w}_{i\pm1}, \mathbf{w}_{i\pm2}),\\
e(\mathbf{w}_i, \mathbf{w}_{i\pm1}) &= (\mathbf{w}_{i\pm1}, \mathbf{w}_i).
\end{aligned}$$

To prove the dihedral property, by Lemma QQYVCFM, it is enough to prove the following properties

1. the hypermap is connected.
2. the number of darts is $2k$.
3. the orders of $f, n, e$ are $k$, 2, 2, respectively.

We have

$$f^j(\mathbf{w}_i, \mathbf{w}_{i\pm1}) = (\mathbf{w}_{i\pm j}, \mathbf{w}_{i\pm(j+1)}).$$

Note that the orbit of $f$ on $(\mathbf{w}_i, \mathbf{w}_{i\pm1})$ is $F'$ or $F'' \subset D$. (This proves in particular the face property of local fans: $F'$ is a face of the hypermap.) and $n$ exchanges darts in $F'$ and $F''$. Hence the hypermap is connected.

The sets $F'$ and $F''$ are disjoint and each contain $k$ darts. Hence the number of darts is $2k$.

The smallest positive $j$ such that $f^j(\mathbf{w}_i, \mathbf{w}_{i\pm1}) = (\mathbf{w}_i, \mathbf{w}_{i\pm1})$ is $k$. Hence $f$

has order $k$. The orders of $e$ and $n$ are 2 by inspection. This completes the verification of the dihedral property of local fans.

While proving the dihedral property, the face property fell out as well.

This completes the verification of properties intersection, fan, and dihedral.

# 6

# Appendix on saturation

**Lemma 6.1** [CPNKNXN] *Let $V \subset \mathbb{R}^3$ be any packing. Then there exists a saturated packing $V_{sat}$ that contains $V$.*

*Proof* If $V$ is any packing and $r$ is a real number, let $m(V, r)$ be the maximum over the following set of natural numbers:

$$\{\mathrm{card}(V' \cap B(\mathbf{0}, r)) \ : \ V \subset V' \text{ and } V' \text{ is a packing }\}.$$

By (the proof of) Lemma 6.2 of DSP (KIUMVTC) and Lemma (WQZISRI), there is a one-to-one map from $V' \cap B(\mathbf{0}, r)$ to $\mathbb{Z}^3 \cap B(\mathbf{0}, 2r+1)$, which gives an upper bound on the cardinality of $V' \cap B(\mathbf{0}, r)$ depending only on $r$. Thus the maximum exists.

Skolemizing variables, we have a function $W$ of $V$ and $r$ with the property that for any packing $V$ and and any real number $r$, we have that $W(V, r)$ is a packing with $V \subset W(V, r)$ that realizes the maximum cardinality $m(V, r)$.

We define a packing by recursion. Let $V_0 = V$ and let $V_{n+1} = W(V_n, n)$. Set $V_{sat} = \bigcup_n V_n$. We will show that $V_{sat}$ is a saturated packing that contains $V$. It will take a few steps to reach the conclusion.

We show that $V \subset V_{sat}$. In fact, $V = V_0 \subset \bigcup_n V_n = V_{sat}$.

It is easily checked that if $V' \subset V''$ is an inclusion of packings, then for all $r$, we have $V' \subset W(V'', r)$. From this, it follows that if $V' \subset V_n$ then $V' \subset V_{n+1}$. By induction, it now follows that if $n \le m$, then $V_n \subset V_m$.

We show that $V_{sat}$ is a packing. If $\mathbf{v}, \mathbf{w} \in V_{sat}$, then $\mathbf{v}$, $\mathbf{w} \in V_m$ for some $m$. The set $V_m$ is a packing, so that if $\mathbf{v} \ne \mathbf{w}$, we have $\|\mathbf{v} - \mathbf{w}\| \ge 2$. Hence $V_{sat}$ is a packing.

We claim that $V_{sat}$ is saturated. Otherwise, there exists $\mathbf{w} \in \mathbb{R}^3$, with $\mathbf{w} \notin V_{sat}$ such that $\|\mathbf{v} - \mathbf{w}\| \ge 2$ for all $\mathbf{v} \in V_{sat}$. Choose $n$ such that $\mathbf{w} \in B(\mathbf{0}, n)$. Let $V' = V_{n+1} \cup \{\mathbf{w}\}$. Then $V'$ is a packing containing $V_n$ that achieves a larger cardinality of intersection with $B(\mathbf{0}, n)$ than does $V_{n+1}$. This contradicts the maximality of $V_{n+1}$. $\square$

# 7

# Appendix on OXLZLEZ

This appendix was written in Nov 2012 to give details of the formalization of Lemma `OXLZLEZ`.

## 7.1 formulas for $\gamma$ and dih

The dihedral angle of a cell is given by a formula dihX, which is defined in terms of dihV. Various lemmas related dihV to the function giving the dihedral angle of a simplex as a function of its edge lengths.

The function $\gamma$ can also be expressed as a function of edge lengths. In the following lemmas, we fix a saturated packing $V$ and take cells with respect to $V$.

**Lemma 7.1** `[YJBIAOE]` *Let $X$ be a* 4*-cell. Let $y_1, \ldots, y_6$ be its edge lengths. Then $\gamma(X, L)$ is given by*

`gamma4fgcy y1 y2 y3 y4 y5 y6 lmfun`

*defined formally in the module Sphere.*

**Lemma 7.2** `[XKYBPAI]` *Let $X$ be a* 3*-cell. Let $y_4, y_5, y_6$ be the lengths of the three edges in $E(X)$. Then $\gamma(X, L)$ is given by*

`gamma3f y4 y5 y6 sqrt2 lmfun`

*defined formally in the module Sphere.*

**Lemma 7.3** `[KKHWUHM]` *Let $X$ be a* 2*-cell. Let $y$ be the length of the unique edge in $E(X)$. Let $\alpha$ be the dihedral angle along that edge. Then $\gamma(X, L) = \alpha \cdot t$, where $t$ is*

`gamma2_x_div_azim (h0cut y) (y * y)`

*defined formally in the module Functional_equation.*

**Lemma 7.4** [OWEWPJG] *Let $X$ be a 1-cell. Let $\mathbf{v}$ be the unique element of $V(X)$. Let $s = \mathrm{sol}(X, \mathbf{v})$. Then $\gamma(X, L) = st$, where $t$ is*

$$\frac{8\pi\sqrt{2}}{3} - 8m_1$$

**Lemma 7.5** [KPJNKIL] *Let $X$ be a 0-cell. Then $\gamma(X, L)$ is equal to the volume of $X$.*

## 7.2 leaf and cell

**Definition 7.6** (leaf) [NIPHFIE] Let $V$ be a saturated packing. A *leaf* of $V$ is an element $\underline{\mathbf{u}} = [\mathbf{u}_0; \mathbf{u}_1; \mathbf{u}_2] \in \underline{V}(2)$ such that $h(\underline{\mathbf{u}}) < \sqrt{2}$. The *stem* of the leaf is $\{\mathbf{u}_0, \mathbf{u}_1\}$.

**Lemma 7.7** [GBEWYFX] *Let $V$ be a saturated packing, and let $\underline{\mathbf{u}} = [\mathbf{u}_0; \mathbf{u}_1; \mathbf{u}_2]$ be a leaf of $V$. Then $S = \{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2\}$ is not collinear.*

*Proof* Part 1 of MHFTTZN states that $S$ has affine dimension 2, hence the set is not collinear. □

**Lemma 7.8** [NWVRFMF] *Let $V$ be a saturated packing, and let $\underline{\mathbf{u}}$ be a leaf of $V$. Let $\mathbf{p} \in \mathbb{R}^3$ be such that $\{\mathbf{p}\}$ is a facet of $\Omega(V, \underline{\mathbf{u}})$. Then there exists $\underline{\mathbf{v}} \in \underline{V}(3)$ such that $d_2\underline{\mathbf{v}} = \underline{\mathbf{u}}$ and $\omega_3(\mathbf{v}) = \mathbf{p}$.*

*Proof* This follows directly from Lemma IDBEZAL and $\underline{\mathbf{u}} \in \underline{V}(2)$. □

**Lemma 7.9** [YBZFUPO] *Let $V$ be a saturated packing with leaf $\underline{\mathbf{u}}$. Then there exist distinct $\mathbf{p}_1$ and $\mathbf{p}_2$ such that $\Omega(V, \underline{\mathbf{u}})$ is the convex hull of $\{\mathbf{p}_1, \mathbf{p}_2\}$ and such that $F$ is a facet of $\Omega(V, \underline{\mathbf{u}})$ if and only if $F \in \{\{\mathbf{p}_1\}, \{\mathbf{p}_2\}\}$.*

*Proof* By the definition of $\underline{V}(2)$, we have that $\underline{\mathbf{u}} \in \underline{V}(2)$ implies that the affine dimension of $\Omega(V, \underline{\mathbf{u}})$ is one. This is a bounded polyhedron of dimension one, hence a segment given as a convex hull of distinct points $\mathbf{p}_1$ and $\mathbf{p}_2$. The facets of a segment are its extreme points as given. □

**Lemma 7.10** [ZASUVOR] *Let $V$ be a saturated packing with leaf $[\mathbf{u}_0; \mathbf{u}_1; \mathbf{u}_2]$. Then $[\mathbf{u}_1; \mathbf{u}_0; \mathbf{u}_2]$ is also a leaf with the same stem.*

*Proof* The stem is clearly the same, and the circumradius does not change upon reordering of elements. Let $\underline{\mathbf{v}} \in \underline{V}(3)$ be an element constructed in the previous lemma such that $d_2\underline{\mathbf{v}} = \underline{\mathbf{u}}$. Let $\underline{\mathbf{v}}'$ be obtained by transposing the first two elements. By YNHYJIT, we have $\underline{\mathbf{v}}' \in \underline{V}(3)$. Then $d_2\underline{\mathbf{v}}' = [\mathbf{u}_1; \mathbf{u}_0; \mathbf{u}_2] \in \underline{V}(2)$. The result ensues. □

**Lemma 7.11** [FUZBZGI] *Let $V$ be a saturated packing with leaf $\underline{\mathbf{u}}$. let $\mathbf{q}$ be the circumcenter of $\underline{\mathbf{u}}$. Then $\mathbf{q} \in \Omega(V, \underline{\mathbf{u}})$, but is not an extreme point of $\Omega(V, \underline{\mathbf{u}})$.*

*Proof* The third part of Lemma MHFTTZN gives that $\mathbf{q} \in \Omega(V, \underline{\mathbf{u}})$.

Assume for a contradiction that $\mathbf{q}$ is an extreme point, then Lemma 7.8 gives $\underline{\mathbf{v}} \in \underline{V}(3)$ such that $d_2\underline{\mathbf{v}} = \underline{\mathbf{u}}$ and $\omega_3(\mathbf{v}) = \mathbf{q}$. The set $\Omega(V, \underline{\mathbf{v}})$ is convex of affine dimension 0, and is therefore a singleton $\omega_3(\underline{\mathbf{v}})$. By Lemma MHFTTZN applied to $\underline{\mathbf{v}}$, we have that $\mathbf{q} = \omega_3(\underline{\mathbf{v}})$ is the circumcenter of $\underline{\mathbf{v}}$. This contradicts the strict inequality of Lemma XYOFCGX. □

**Definition 7.12** ($\chi$) [MSBKFLD] For any list $\underline{\mathbf{u}} = [\mathbf{u}_0; \mathbf{u}_1; \mathbf{u}_2]$ of elements in $\mathbb{R}^3$, define $\chi(\underline{\mathbf{u}}, \mathbf{p}) = ((\mathbf{u}_1 - \mathbf{u}_0) \times (\mathbf{u}_2 - \mathbf{u}_0)) \cdot (\mathbf{p} - \mathbf{u}_0)$.

**Lemma 7.13** [JDHAWAY] *Let $V$ be a saturated packing with leaf $\underline{\mathbf{u}}$. Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be the distinct points constructed in Lemma 7.9. Then $\chi(\underline{\mathbf{u}}, \mathbf{p}_i)$ is not zero, and $\chi(\underline{\mathbf{u}}, \mathbf{p}_1)$ and $\chi(\underline{\mathbf{u}}, \mathbf{p}_2)$ have opposite signs.*

*Proof* Let $\mathbf{q}$ be the circumcenter of $\underline{\mathbf{u}}$. If $\chi(\underline{\mathbf{u}}, \mathbf{p}_i) = 0$, then $\mathbf{p}_i$ lies in the affine hull of $\underline{\mathbf{u}}$. By MHFTTZN, this implies that $\mathbf{q} = \mathbf{p}_i$, which is impossible by the previous lemma. Hence $\chi(\underline{\mathbf{u}}, \mathbf{p}_i) \neq 0$.

By the previous lemma, the circumcenter $\mathbf{q}$ of $\underline{\mathbf{u}}$ has the form $\mathbf{q} = \mathbf{p}_1 t_1 + \mathbf{p}_2 t_2$, for some $t_i$ such that $t_1 + t_2 = 1$ and $t_i > 0$. Since $\mathbf{q}$ lies in the affine hull of $\underline{\mathbf{u}}$, we have

$$0 = \chi(\underline{\mathbf{u}}, \mathbf{q}) = t_1\chi(\underline{\mathbf{u}}, \mathbf{p}_1) + t_2\chi(\underline{\mathbf{u}}, \mathbf{p}_2)$$

Since $t_i > 0$, this implies that $\chi(\underline{\mathbf{u}}, \mathbf{p}_1)$ and $\chi(\underline{\mathbf{u}}, \mathbf{p}_2)$ have opposite signs. □

*Remark* 7.14 [LITLFSC] Recall from Lemma JBDNJJB that $\chi(\underline{\mathbf{u}}, \mathbf{p})$ has the same sign as

$$\sin(\mathrm{azim}(\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{p})).$$

Also, $\chi([\mathbf{u}_0; \mathbf{u}_1; \mathbf{u}_2], \mathbf{p}) = -\chi([\mathbf{u}_1; \mathbf{u}_0; \mathbf{u}_2], \mathbf{p})$.

**Definition 7.15** ($\mathbf{p}_\mathbf{e}$, $\hat{\underline{\mathbf{u}}}$, $k_\mathbf{e}$, $c$) [AQEQEDX] Let $V$ be a saturated packing with leaf $\underline{\mathbf{u}}$. We define $\mathbf{p}_\mathbf{e}(\underline{\mathbf{u}})$, $\hat{\underline{\mathbf{u}}}$, and $k_\mathbf{e}$ as functions of $\underline{\mathbf{u}}$ as follows. By the previous lemma, there exists a unique extreme point $\mathbf{p}_\mathbf{e}$ of $\Omega(V, \underline{\mathbf{u}})$ such that $\chi(\underline{\mathbf{u}}, \mathbf{p}_\mathbf{e}) > 0$. By an earlier lemma, there exists $\hat{\underline{\mathbf{u}}} \in \underline{V}(3)$ (choose one) where $d_2(\hat{\underline{\mathbf{u}}}) = \underline{\mathbf{u}}$ and $\omega_3(\hat{\underline{\mathbf{u}}}) = \mathbf{p}_\mathbf{e}$. Finally, let $k_\mathbf{e} \in \{3, 4\}$ be given by

$$k_\mathbf{e} = \begin{cases} 4, & h(\hat{\underline{\mathbf{u}}}) < \sqrt{2} \\ 3, & \text{otherwise.} \end{cases}$$

We abbreviate $c(\underline{\mathbf{u}}) = \mathrm{cell}(\hat{\underline{\mathbf{u}}}, k_{\mathbf{e}}(\underline{\mathbf{u}}))$.

This construction uses a leaf $\underline{\mathbf{u}} = [\mathbf{u}_0; \mathbf{u}_1; \mathbf{u}_2]$ to select an extreme point of $\Omega(V, \underline{\mathbf{u}})$ and an associated cell. By earlier lemmas, the other extreme point of $\Omega(V, \underline{\mathbf{u}})$ is determined by the other leaf $[\mathbf{u}_1; \mathbf{u}_0; \mathbf{u}_2]$ with the same stem. It reverses the sign of $\chi$. Note that $[\mathbf{u}_1; \mathbf{u}_0; \mathbf{u}_2]$ is the unique nontrivial rearrangement of $\underline{\mathbf{u}}$ with the same stem. If $\underline{\mathbf{u}}$ is a leaf, we write

$$A^0_+(\underline{\mathbf{u}}) = \mathrm{aff}^0_+(\{\mathbf{u}_0; \mathbf{u}_1\}, \mathbf{u}_2), \text{ and } A(\underline{\mathbf{u}}) = \mathrm{aff}\{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2\}.$$

**Lemma 7.16** `[NUNRRDS]` *Let $V$ be a saturated packing with leaf $\underline{\mathbf{u}}$. Then $c(\underline{\mathbf{u}})$ meets $A^0_+(\underline{\mathbf{u}})$. Furthermore, for every $\mathbf{q} \in c(\underline{\mathbf{u}})$, we have $\chi(\underline{\mathbf{u}}, \mathbf{q}) \geq 0$.*

*Proof* The cell is given as a convex hull of $\{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{p}\}$, for some point $\mathbf{p}$. Hence the cell contains $\mathbf{u}_2$ which lies in $A_+(\underline{\mathbf{u}})$. The convex hull clearly lies in a half space bounded by $A(\underline{\mathbf{u}})$.

We have $\chi(\underline{\mathbf{u}}, \mathbf{p}) > 0$. Since $\mathbf{q}$ is in the convex hull of four points with $\chi \geq 0$, we also have $\chi(\underline{\mathbf{u}}, \mathbf{q}) \geq 0$. □

## 7.3 planarity

**Lemma 7.17** `[RIJRIED]` *Let $V$ be a saturated packing. Let $X$ be a cell with an edge. Then $X$ is not coplanar.*

*Proof* By definition, the vertex set is empty, if the cell is a null set. □

**Lemma 7.18** `[ZWVCBMN]` *Assume that $S = \{\mathbf{u}_0, \ldots, \mathbf{u}_3\} \subset \mathbb{R}^3$ is not coplanar. Then the convex hull of $S$ has positive measure.*

*Proof* The volume of a tetrahedron has the form $\sqrt{\Delta}/12$, and the condition of planarity is $\Delta = 0$. □

**Lemma 7.19** `[ASVAYEW]` *Let $V$ be a saturated packing and let $X$ be a nonempty 3 or 4-cell. Then $X$ is not coplanar.*

*Proof* Write the cell as $\mathrm{cell}(\underline{\mathbf{u}}, k)$. A nonempty 3 or 4-cell has the form of a convex hull of four points, $\{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{p}\}$, where $\mathbf{p} = \mathbf{u}_3$ in the case of a 4-cell ($h(\underline{\mathbf{u}}) < \sqrt{2}$) or $\mathbf{p} = \xi$ in the case of a 3-cell ($h_2 < \sqrt{2} \leq h(\underline{\mathbf{u}})$). In the case of a 4-cell the points are not coplanar by Lemma MHFTTZN.

In the case of a 3-cell, we have that $A(\underline{\mathbf{u}})$ has dimension 2 by Lemma MHFTTZN. It is enough to show that $\xi$ is not in this affine hull. The point $\xi$ is equidistant from $\mathbf{u}_0, \mathbf{u}_1$ and $\mathbf{u}_2$. If $\xi$ is in the affine hull, then it is the circumcenter, and we arrive at a contradiction

$$\sqrt{2} = \|\xi - \mathbf{u}_0\| = h(d_2\underline{\mathbf{u}}) < \sqrt{2}.$$

□

## 7.4 classification of cells

**Lemma 7.20** `[CFFONNL]` *Let $V$ be a saturated packing with leaf $\underline{\mathbf{u}} = [\mathbf{u}_0; \mathbf{u}_1; \mathbf{u}_2]$. Let $X$ be a cell of $V$, and let $\{\mathbf{u}_0, \mathbf{u}_1\} \in E(X)$ be an edge. Suppose that*

$$X \cap A^0_+(\underline{\mathbf{u}}) \neq \emptyset.$$

*Then there exists a 2-rearrangement $\underline{\mathbf{u}}'$ of $\underline{\mathbf{u}}$ such that*

$$X = c(\underline{\mathbf{u}}').$$

*Proof* Since $X$ has an edge, it is at least a 2-cell. The cell $X$ is not a subset of $A(\underline{\mathbf{u}})$. Pick $\mathbf{q} \in X \setminus A$, and choose a 2-rearrangement $\underline{\mathbf{u}}'$ of $\underline{\mathbf{u}}$ such that $\chi(\underline{\mathbf{u}}', \mathbf{q}) > 0$. The cell $X$ contains the convex hull of four points

$$\mathbf{u}_0, \mathbf{u}_1, \mathbf{p}, \mathbf{q}$$

where $\mathbf{p} \in A^0_+$ and $\chi(\underline{\mathbf{u}}', \mathbf{q}) > 0$.

Let $\underline{\mathbf{v}} = \underline{\hat{\mathbf{u}}}' \in \underline{V}(3)$ and $k = k_{\mathbf{e}}(\underline{\mathbf{u}}')$. Let $X' = \mathrm{cell}(\underline{\mathbf{v}}, k)$. The cell $X'$ also contains the convex hull of four points

$$\mathbf{u}_0, \mathbf{u}_1, \mathbf{p}', \mathbf{q}'$$

where $\mathbf{p}' \in A^0_+$ and $\chi(\underline{\mathbf{u}}', \mathbf{q}') > 0$. Any two such convex hulls meet in a set of positive measure. So $X \cap X'$ has positive measure. Hence by Lemma AJRIPQN, we have $X = X'$. This gives the result. □

**Lemma 7.21** `[FUEIMOV]` *Suppose that $V$ is a saturated packing with leaves $\underline{\mathbf{u}}$ and $\underline{\mathbf{u}}'$ with the same stem. Let $k = k_{\mathbf{e}}(\underline{\mathbf{u}})$ and $k' = k_{\mathbf{e}}(\underline{\mathbf{u}}')$. Assume that $\underline{\mathbf{u}} \neq \underline{\mathbf{u}}'$. and $c(\underline{\mathbf{u}}) = c(\underline{\mathbf{u}}')$. Then $k = k' = 4$, $\underline{\hat{\mathbf{u}}} = [\mathbf{v}_0; \mathbf{v}_1; \mathbf{v}_2; \mathbf{v}_3]$, and $\underline{\hat{\mathbf{u}}}' = [\mathbf{v}_1; \mathbf{v}_0; \mathbf{v}_3; \mathbf{v}_2]$.*

*Proof* If the cells are equal, then $k = k' \in \{3, 4\}$ by the definiton of $k_{\mathbf{e}}$ and Lemma AJRIPQN.

We consider two cases depending on the value of $k$. Suppose that $k = 4$. The definition of $k_{\mathbf{e}}$ gives $h(\underline{\hat{\mathbf{u}}}) < \sqrt{2}$. The set $V \cap \mathrm{cell}(\underline{\mathbf{v}}, 4)$ determines the parameter $\underline{\mathbf{v}}$ up to rearrangement. The stem is fixed, giving only two possibilities $[\mathbf{u}_0; \mathbf{u}_1; \ldots]$ and $[\mathbf{u}_1; \mathbf{u}_0; \ldots]$ for the first two elements. The last two entries must also be equal or transpositions. To preserve the sign of $\chi$, the permutation must be even. The only nontrivial possibility is the one given in the lemma.

Suppose that $k = 3$. The cell is the convex hull of its extreme points, three of which are elements of the vertex set $V(X)$. The first two entries of $\underline{\mathbf{u}}$ are determined by the stem, up to transposition. The third entry is fixed by membership in $V(X)$. The fourth entry must be the common extreme point $\xi$. If $\underline{\mathbf{u}} = [\mathbf{u}_0; \mathbf{u}_1; \mathbf{u}_2]$ and $\underline{\mathbf{u}}' = [\mathbf{u}_1; \mathbf{u}_0; \mathbf{u}_2]$, then

$$0 < \chi(\underline{\mathbf{u}}, \xi) = -\chi(\underline{\mathbf{u}}', \xi).$$

This gives incompatible sign constraints. □

**Lemma 7.22** `[YSAKKTX]` *Let $V$ be a saturaed packing. Let $X$ be a cell with edge $\{\mathbf{u}_0, \mathbf{u}_1\}$. Then $X = \mathrm{cell}(\underline{\mathbf{u}}, k)$, for some $\underline{\mathbf{u}}$ of the form $[\mathbf{u}_0; \mathbf{u}_1; \ldots]$.*

*Proof* This follows from RVFXZBU. □

**Lemma 7.23** `[RBUTTCS]` *Let $V$ be a saturated packing. Let $X$ be a 4-cell with edge $\{\mathbf{u}_0, \mathbf{u}_1\}$. Then there exists a leaf $\underline{\mathbf{u}}$ with stem $\{\mathbf{u}_0, \mathbf{u}_1\}$ such that $X = c(\underline{\mathbf{u}})$.*

*Proof* By YSAKKTX, we may write $X = \mathrm{cell}(\underline{\mathbf{v}}, 4)$, where $\underline{\mathbf{v}}$ has the form $\underline{\mathbf{v}} = [\mathbf{u}_0; \mathbf{u}_1; \mathbf{v}_2; \mathbf{v}_3]$, for some $\mathbf{v}_2$ and $\mathbf{v}_3$. Set $\underline{\mathbf{u}} = d_2\underline{\mathbf{v}}$. We have $h(\underline{\mathbf{u}}) < \sqrt{2}$, so that $\underline{\mathbf{u}}$ is a leaf. The cell meets $A_+^0(\underline{\mathbf{u}})$ at $\mathbf{v}_2$. Lemma 7.20 gives that $X = c(\underline{\mathbf{u}}')$ for some 2-rearrangement of $\underline{\mathbf{u}}$. □

**Lemma 7.24** `[FCHKUGT]` *Let $V$ be a saturated packing, and let $\underline{\mathbf{u}} = [\mathbf{u}_0; \mathbf{u}_1; \mathbf{u}_2]$ and $\underline{\mathbf{u}}' = [\mathbf{u}_0; \mathbf{u}_1; \mathbf{u}_2']$ be two leaves with the same first two entries, such that $A_+^0(\underline{\mathbf{u}}) = A_+^0(\underline{\mathbf{u}}')$. Then $\underline{\mathbf{u}} = \underline{\mathbf{u}}'$.*

*Proof* We have that $c(\underline{\mathbf{u}})$ and $c(\underline{\mathbf{u}}')$ both meet $A_+^0$. Hence, there is a 2-rearrangement $\underline{\mathbf{u}}''$ of $\underline{\mathbf{u}}'$ such that $c(\underline{\mathbf{u}}) = c(\underline{\mathbf{u}}'')$. By FUEIMOV, $k = k' = 4$, and $\underline{\mathbf{u}}''$ is a rearrangement of $\underline{\mathbf{u}}$. If $\underline{\mathbf{u}} \neq \underline{\mathbf{u}}''$, then $\underline{\mathbf{u}}'' = [\mathbf{u}_1; \mathbf{u}_0; \mathbf{u}_3]$, where $c(\underline{\mathbf{u}}) = \mathrm{conv}\{\mathbf{u}_0, \ldots, \mathbf{u}_3\}$. But this final possibility is impossible, since $\{\mathbf{u}_0, \ldots, \mathbf{u}_3\}$ is not coplanar. □

**Lemma 7.25** `[BDXKHTW]` *Let $V$ be a saturated packing. Let $X$ be a cell of $V$ with edge $e = \{\mathbf{u}_0, \mathbf{u}_1\}$. Let $\underline{\mathbf{u}} = [\mathbf{u}_0; \mathbf{u}_1; \mathbf{u}_2]$ and $\underline{\mathbf{u}}' = [\mathbf{u}_0; \mathbf{u}_1; \mathbf{u}_2']$ be distinct leaves with stem $e$. Assume that the intersection of $X$ with $\mathrm{wedge}^0(\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_2')$ is nonempty. Then $X$ is a subset of $\mathrm{wedge}(\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_2')$.*

*Proof* We claim that $X$ is not coplanar. Otherwise it is a null set, and by definition a null set has no vertices or edges.

Since the leaves are distinct, by FCHKUGT, we have $A_+^0(\underline{\mathbf{u}}) \neq A_+^0(\underline{\mathbf{u}}')$. This implies that the wedges are nondegenerate.

For a contradiction, pick $\mathbf{p} \in X \cap \mathrm{wedge}^0(\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_2')$ and $\mathbf{q} \in X \cap \mathrm{wedge}^0(\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2', \mathbf{u}_2)$. We may assume that $\{\mathbf{u}_0, \mathbf{u}_1, \mathbf{p}, \mathbf{q}\}$ is not coplanar. The sets wedge$^0$ are open and disjoint. By the connectedness of the unit interval, the path $t \mapsto (1-t)\mathbf{p} + t\mathbf{q}$ crosses $A_+^0(\underline{\mathbf{u}})$ or $A_+^0(\underline{\mathbf{u}}')$ at $\mathbf{q}_t$, for some $0 < t < 1$. We then have that $\{\mathbf{u}_0, \mathbf{u}_1, \mathbf{q}_t\}$ is not collinear. This point of crossing, by an earlier lemma gives $X = c(\underline{\mathbf{u}}'')$ for some 2-rearrangement of $\underline{\mathbf{u}}$ or $\underline{\mathbf{u}}'$, and say $A_+^0(\underline{\mathbf{u}}'') = A_+^0(\underline{\mathbf{u}})$. By an earlier lemma,

$$0 \le (1-t)\chi(\underline{\mathbf{u}}'', \mathbf{p}) + t\chi(\underline{\mathbf{u}}'', \mathbf{q}) = \chi(\underline{\mathbf{u}}'', \mathbf{q}_t) = 0.$$

This forces $\chi(\underline{\mathbf{u}}'', \mathbf{p}) = \chi(\underline{\mathbf{u}}'', \mathbf{q}) = 0$, which is contrary to our assumption that $\{\mathbf{p}, \mathbf{q}, \mathbf{u}_0, \mathbf{u}_1\}$ is not coplanar. □

**Lemma 7.26** `[EWYBJUA]` *Let $V$ be a saturated packing. Let $X$ be any cell with edge $\{\mathbf{u}_0, \mathbf{u}_1\}$. Let $\underline{\mathbf{u}} = [\mathbf{u}_0; \mathbf{u}_1; \mathbf{u}_2]$ and $\underline{\mathbf{u}}' = [\mathbf{u}_0; \mathbf{u}_1; \mathbf{u}_2']$ be distinct leaves of $V$. Then*

$$X \subset \mathrm{wedge}(\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_2') \text{ or } X \subset \mathrm{wedge}(\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2', \mathbf{u}_2).$$

*Proof* The cell $X$ is not a null set, so there exists a point in the intersection of $X$ with

$$\mathrm{wedge}^0(\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_2') \text{ or } X \subset \mathrm{wedge}^0(\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2', \mathbf{u}_2).$$

Then apply the previous lemma. □

## 7.5 angle sums revisited

In the following lemma, the hypothesis (7.28) always holds by preceding lemmas, but we include it to make it independent of these lemmas. The lemma is a variant of Lemma GRUTOTI.

**Lemma 7.27** `[REUHADY]` *Let $V$ be a saturated packing. Assume that $\mathbf{u}_0, \mathbf{u}_1 \in V$ satisfy $\|\mathbf{u}_0 - \mathbf{u}_1\| < 2\sqrt{2}$. Set $\varepsilon = \{\mathbf{u}_0, \mathbf{u}_1\}$. Let $[\mathbf{u}_0; \mathbf{u}_1; \mathbf{w}]$ and $[\mathbf{u}_0; \mathbf{u}_1; \mathbf{w}']$ be two leaves (not necessarily distinct) such that for every cell $X$ with edge $\{\mathbf{u}_0, \mathbf{u}_1\}$ we have*

$$X \subset \mathrm{wedge}(\mathbf{u}_0, \mathbf{u}_1, \mathbf{w}, \mathbf{w}') \quad \text{or} \quad X \subset \mathrm{wedge}(\mathbf{u}_0, \mathbf{u}_1, \mathbf{w}', \mathbf{w}). \tag{7.28}$$

*Then*

$$\sum_{X \in \mathcal{X}} \mathrm{dih}(X, \varepsilon) = \mathrm{azim}(\mathbf{u}_0, \mathbf{u}_1, \mathbf{w}, \mathbf{w}').$$

*The sum runs over the set $\mathcal{X}$ of cells $X$ such that $\varepsilon \in E(X)$ and $X \subset$* $\mathrm{wedge}(\mathbf{u}_0, \mathbf{u}_1, \mathbf{w}, \mathbf{w}')$.

*Proof* Consider the sets

$$C = B(\mathbf{u}_0, r) \cap \mathrm{rcone}^0(\mathbf{u}_0, \mathbf{u}_1, a), \text{ and } C' = C \cap \mathrm{wedge}(\mathbf{u}_0, \mathbf{u}_1, \mathbf{w}, \mathbf{w}'),$$

where $r$ and $a$ are small positive real numbers. From the definition of $k$-cells, it follows that we can choose $r$ and $a$ sufficiently small so that if $X$ is a $k$ cell that meets $C'$ in a set of positive measure, then $k \geq 2$ and there exists $\underline{\mathbf{u}} \in \underline{V}(3)$ such that $X = \mathrm{cell}(\underline{\mathbf{u}}, k)$ and $d_1\underline{\mathbf{u}} = [\mathbf{u}_0; \mathbf{u}_1]$. Moreover,

$$C' \cap X = C \cap X = C \cap A, \quad A = \mathrm{aff}_+(\{\mathbf{u}_0, \mathbf{u}_1\}, \{\mathbf{v}, \mathbf{w}\}),$$

where $A$ is the lune of [1][Definition MVIADQK] and $\mathbf{v}$, $\mathbf{w}$ are chosen as in [1][Definition RSDYMHV]. By [1][Lemma FMSWMVO] and Definition RSDYMHV, the volume of this intersection is

$$\mathrm{vol}(C \cap A) = \mathrm{vol}(C)\,\mathrm{dih}_V(\{\mathbf{u}_0, \mathbf{u}_1\}, \{\mathbf{v}, \mathbf{w}\})/(2\pi) = \mathrm{vol}(C)\,\mathrm{dih}(X, \varepsilon)/(2\pi).$$

The set of cells meeting $C'$ in a set of positive measure gives a partition of $C'$ into finitely many measurable sets. This gives

$$\begin{aligned}\mathrm{vol}(C)\mathrm{azim}(\mathbf{u}_0, \mathbf{u}_1, \mathbf{w}, \mathbf{w}')/(2\pi) &= \mathrm{vol}(C') \\ &= \sum_{X \in \mathcal{X}} \mathrm{vol}(C \cap X) \\ &= \mathrm{vol}(C) \sum_{X \in \mathcal{X}} \mathrm{dih}(X, \varepsilon)/(2\pi).\end{aligned}$$

The calculation of volumes in [1][Chapter 3] gives $\mathrm{vol}(C) > 0$. The conclusion follows by canceling $\mathrm{vol}(C)$ from both sides of the equation. □

## 7.6 linear programs

Let $V$ be a saturated packing and let $\{\mathbf{u}_0, \mathbf{u}_1\}$ be a critical edge (of some cell). We may order the set

$$\mathrm{Leaf} = \{\mathbf{v} \in V \;:\; [\mathbf{u}_0; \mathbf{u}_1; \mathbf{v}] \text{ is a leaf }\}$$

(and the corresponding leaves) by arbitrarily fixing one element $\mathbf{v}_0 \in L$ and then ordering them by increasing azimuth angle $\mathrm{azim}(\mathbf{u}_0, \mathbf{u}_1, \mathbf{v}_0, \mathbf{v})$, for $\mathbf{v} \in \mathrm{Leaf}$. This partitions $\mathbb{R}^3$ into finitely many wedges delimited by the leaves. Each cell with edge $\{\mathbf{u}_0, \mathbf{u}_1\}$ lies in one of these wedges.

If $h(\underline{\hat{\mathbf{u}}}) < \sqrt{2}$, then $c(\underline{\mathbf{u}}) = c(\underline{\mathbf{u}}')$ is a 4-cell in the wedge, and this is the only cell (of positive measure or in fact the only cell at all) contained in the wedge.

If $h(\underline{\hat{\mathbf{u}}}) \geq \sqrt{2}$, then $c(\underline{\mathbf{u}})$ and $c(\underline{\mathbf{u}}')$ are distinct 3-cells in the wedge, and these are the only 3 cells in the wedge. The wedge may also include a finite number of 2-cells, but it has no 4-cells. The 2-cells within a given wedge are combined into a total $\gamma$ and total azimuth angle.

We have a number of nonlinear inequalities bounding the value of

$$\gamma(X, L)\mathrm{wt}(X) + \beta(\{\mathbf{u}_0, \mathbf{u}_1\}, X)$$

as a function of its azimuth angle. These inequalities are based on the partition of cells according to wedges. Based on these inequalities, we run linear programs giving lower bounds for $\sum \gamma(X, L)$, subject to the constraint that the azimuth angles sum to $2\pi$. In every case, we find that $\Gamma \geq 0$ for each cluster. We run a separate linear program depending on the number of leaves. A generic case handles the case of five or more leaves. The following subsection gives details.

## 7.7 cell cluster inequality

This section shows how to deduce Lemma OXLZLEZ from a collection of nonlinear inequalities. The nonlinear inequalities are specified in an abbreviated way according to the table at the end of this section. Some collections of nonlinear inequalities have been merged into a single inequality. Once the source of a nonlinear inequality has been cited, we assume that the reader has become familiar with it, and it is not repeatedly cited.

*Remark* 7.29 (preparation) `[XSBYGIQ]`

1. If a leaf separates two 3-cells, erase it, scoring it at 0.008azim. This is justified by a *computer calculation*[1] [2].
2. Score all 2-cells at 0.008azim by a *computer calculation*[2] [2].
3. Score all 3-cells with $\eta^+$ at 0.008azim, but keep the leaf, by *computer calculation*[3] [2].

**Definition 7.30** (small leaf, subcritical, supercritical) `[HYTORSD]` A *critical* edge has length $2h_- \leq y \leq 2h_+$. A *subcritical* edge has length $2 \leq y < 2h_-$. A *supercritical* edge has length $2h_+ < y \leq \sqrt{8}$. A *small leaf* is a triangle along the critical edge $y_1$ with one critical edge $y_1$ and two subcritical edges $y_2$, $y_6$. A $\beta$-cell is one with two critical edges $y_1, y_4$ and the remaining four edges subcritical. Set $\check{\gamma} = \gamma(X)\mathrm{wt} + \beta(X)$. A 23-cell is used to refer to the collection of 2 and 3-cells that lie between two consecutive nonerased leaves. The term *cell* will now refer to a 4-cell or a 23-cell, rather than a Marchal cell.

[1] `[cell3]`
[2] `[grki]`
[3] `[cell3]`

A cell along a fixed stem $[\mathbf{u}_0; \mathbf{u}_1]$ is call a quarter, if it is a 4-cell with a critical edge along the stem and five subcritical edges. We write QU for a quarter. We write QX as an abbreviation for any 4-cell that is not a quarter, and QY for any 23-cell. Write $\epsilon = 0.0057$. When we speak of a leaf, we restrict our usage to the two leaves along the common stem. We write $\eta^+$ for the condition that the circumradius of a leaf $> 1.34$ and $\eta^-$ for the circumradius $\leq 1.34$.

*Remark* 7.31 (Running Assumptions) In the following lemmas, we work in the context of OXLZLEZ. That is, we have a number of cells sharing a critical edge. For a contradiction, we assume that we are working with a counterexample. That is, we assume as a hypothesis in each lemma that $\sum \check{\gamma}_i < 0$. We have running assumptions `bool_model`, `bool_prep`, `real_model`, and $\sum \check{\gamma}_i < 0$.

**Lemma 7.32** `[CHQSQEY]` *Assume the running assumptions. There are at least three* 4*-cells (in every counterexample).*

*Proof* We consider cases, according to the number of 4-cells. If there are no quarters, then $\check{\gamma}_i \geq 0$ for all $i$ by *computer calculation*[4] [2]. We may in fact assume that there is a quarter with $\check{\gamma} < 0$.

If there is one 4-cell (a quarter), then dih $< 2\pi$ by *computer calculation*[5] [2] so there is another cell, a 23-cell. Then $\check{\gamma}_4+\check{\gamma}_3 > 0$ by *computer calculation*[6] [2] If there are two 4-cells. The total angle of the two cells is at most $2(2.8) < 2\pi$, so there is at least one 23-cell, and each 4-cell is flanked by at least one 23-cell. Then use $\check{\gamma}_4 + \check{\gamma}_3 > 0$. □

**Lemma 7.33** `[MTMLSRF]` *Assume the running assumptions. There exists a quarter with* $\check{\gamma} < 0$ *and flanked on both sides by another* 4*-cell.*

*Proof* Otherwise, we again have $\check{\gamma}_4 + \check{\gamma}_3 > 0$. □

**Lemma 7.34** `[LXDEYBO]` *Assume the running assumptions. There are at most four* 4*-cells.*

*Proof* Otherwise, by a *computer calculation*[7] [2]

$$\sum_{(5)} \check{\gamma} \geq 5a_5 + b_5\alpha \geq 5a_5 + b_5(2\pi) > 0.$$

□

**Lemma 7.35** `[UNPNFVW]` *Assume the running assumptions. The* 4*-cells are all contiguous. That is, there is at most one* 23*-cell.*

4 `[gamma_qx]`
5 `[azim_c4]` ,
6 `[quqy]` .
7 `[ztg4]`

*Proof* The number of 4-cells is three or four. By Lemma 7.33, there is a contiguous block of three or four 4-cells. Thus, if they are not all contiguous, there is a block of three and a block of one, with 23-cells interspersed. That is, there would be at least two 23-cells, each with angle at least 0.606. Then by a *computer calculation*[8] [2]

$$\sum \check{\gamma} \geq 2(0.606)(0.008) + 4a_5 + (2\pi - 2(0.606))b_5 > 0.$$

□

**Lemma 7.36** `[DHCVTVE]` *Assume the running assumptions. The configuration must be one of the four possibilities $C_{n,k}$, where $n$ is the number of leaves, $k$ is the number of* 4*-cells, and $n-k$ is the number of* 23*-cells:*

$$C_{3,3} \quad C_{4,4} \quad C_{4,3} \quad C_{5,4}.$$

*Proof* Since there are three or four 4-cells and zero or one 23-cell, there are four cases as given. □

**Lemma 7.37** `[PMZTATI]` *Assume the running assumptions. Let* QX *be a cell with one small leaf and the other not small. Then $\check{\gamma} > \epsilon$ or* QX *is a $\beta$-cell.*

*Proof* This is a *computer calculation*[9] [2]. □

**Lemma 7.38** `[RJSZKQX]` *Assume the running assumptions. Let* QU *be a cell with (at least) one leaf $\eta^+$. Then $\check{\gamma} > \epsilon$.*

*Proof* This is a *computer calculation*[10] [2]. □

**Lemma 7.39** `[IXPFBKA]` *Assume the running assumptions. If $\eta^-$, then the leaf is small.*

*Proof* This is a *computer calculation*[11] [2]. □

*Remark* 7.40 (neutralize) `[XCLCXWG]` $\check{\gamma}_{\rm QU} > -\epsilon$ and $\check{\gamma}_{\rm QX} \geq 0$ by a *computer calculation*[12] [2] If QX has one leaf that is small and another that is not small, it *neutralizes* a quarter in the sense that it gives $\epsilon$ against the $-\epsilon$ of the quarter. Notice that this condition propagates, forcing one cell after another to have small leaves, if it doesn't neutralize. If a 4-cell is next to a 23-cell, then we have by a *computer calculation*[13] [2],

$$\check{\gamma}_4 + \check{\gamma}_3 > \epsilon,$$

8 `[gckb ztg4 cell3 grki]`
9 `[gamma8]`
10 `[fhvb2]`
11 `[jsp]`
12 `[gamma_qu gamma_gx]` .
13 `[gamma10_gamma11 qu_qy]`

and again we can neutralize a quarter. Call this 234-neutralization.

**Lemma 7.41** `[IPVICGW]` *Assume the running assumptions. All leaves along the stem are small.*

*Proof* We consider many cases, breaking it down intially according to the cases $C_{n,k}$.

$C_{3,3}$: If there are at least two quarters, every leaf lies along a quarter and is small. If there is one quarter, we can neutralize it, unless all leaves are small.

$C_{4,4}$, $C_{4,3}$: If there are at least three quarters (or two that are nonadjacent), then every leaf lies along a quarter and is small. If there are two adjacent or one quarter, we can neutralize them unless every leaf is small.

$C_{5,4}$: If there are four quarters, then every leaf is along a quarter. If there is one quarter, we can neutralize it, if it is not small. There remain two cases: two or three quarters.

$C_{5,4}$: two or three quarters. There are various combinatorial placements of the quarters among the four 4-cells. If two of them are not adjacent to the 23-cell, then we can treat it with neutralization arguments. The one case that is not easily neutralized is an arragnement with a single contiguous block of quarters, occupying the two slots that are not adjacent to the 23-cell, and possibly one other slot. Furthermore, there is exactly one leaf (along the 23-cell) that is not small. That nonsmall leaf has $\eta^+$. We break this case into subcases.

*Assume* $\mathrm{azim}_{23} > 1.074$. By a *computer calculation*[14] [2], we have

$$\sum \check{\gamma} > 0.008\mathrm{azim}_{23} + (4a_5 + b_5(2\pi - \mathrm{azim}_{23})) > 0.$$

*Assume* $\mathrm{azim}_{23} \le 1.074$ *and assume* $\eta^-$ *on the small leaf along the* 23*-cell*. By a *computer calculation*[15] [2], the 23-cell can be included in the estimate,

$$\sum \check{\gamma} > 5a_5 + b_5(2\pi) > 0.$$

*Assume* $\mathrm{azim}_{23} \le 1.074$ *and assume* $\eta^+$ *on the small leaf along the* 23*-cell*. In this case we can split the 4-cells into two groups of two, each with a quarter paired with a neutralizing 4-cell by a *computer calculation*[16] [2]. □

**Lemma 7.42** `[RSIWAMP]` *Assume the running assumptions. There are at most four leaves. That is,* $C_{5,4}$ *does not occur.*

*Proof* Otherwise, we have reduced to the case of five leaves, four 4-cells, and one 23-cell, and all leaves small. Let $A$ and $B$ be the two leaves along the 23-cell. We consider various cases.

[14] `[cell3 grki ztg4]`
[15] `[pem ztg4]`
[16] `[gamma10_gamma11 fhbv2]`

*Assume* $\mathrm{azim}_{23} > 1.074$. This case is the same as in the previous lemma. Without generality we now assume that $\mathrm{azim}_{23} \le 1.074$.

*Assume* $\eta_A^-$ *or* $\eta_B^-$. In this case, by a *computer calculation*[17] [2], we have

$$\sum_{(5)} \check{\gamma} > 5a_5 + b_5(2\pi) > 0.$$

*Assume* $\eta_A^+$ *and* $\eta_B^+$. If further, some 4-cell is not a quarter, then by a *computer calculation*[18] [2],

$$\sum_{(5)} \check{\gamma} > 0.606(0.008) + 0.21849 + 3(0.161517) - (2\pi - 0.606)0.119482 > 0.$$

And if all 4-cells are quarters, then we can easily neutralizes the two pairs of 4-cells. ◻

**Lemma 7.43** `[BKLETJQ]` *Assume the running assumptions. If* QX *is adjacent to a* 23-*cell, then* $\check{\gamma}_4 + \check{\gamma}_3 > \epsilon$.

*Proof* This is a *computer calculation*[19] [2]. ◻

**Lemma 7.44** `[UTEOITF]` *Assume the running assumptions. There is no* 23-*cell. That is,* $C_{4,3}$ *does not occur.*

*Proof* Again, there are several cases. Label the three consecutive 4-cells as $A, B, C$. All leaves are small, $B$ is a quarter with $\check{\gamma}_B < 0$ and its two leaves have $\eta^-$. If any leaf has $\eta^+$, then we can easily neutralize. Assume now that $\eta^-$ for all leaves. If any 4-cell is not a quarter, we can neutralize.

Assume that all 4-cells are quarters.

*Assume* $\mathrm{azim}_{23} \ge 2.089$. By a *computer calculation*[20] [2],

$$\sum \check{\gamma} > \mathrm{azim}_{23}(0.008) + 3(0.161517) + (2\pi - \mathrm{azim}_{23})(-0.119482) > 0.$$

*Assume* $\mathrm{azim}_{23} \le 1.946$. By a *computer calculation*[21] [2],

$$\sum \check{\gamma} > \mathrm{azim}_{23}(0.008) + 3(-0.0659) + (2\pi - \mathrm{azim}_{23})0.42 > 0.$$

*Assume* $1.946 \le \mathrm{azim}_{23} \le 2.089$. By a *computer calculation*[22] [2],

$$\sum \check{\gamma} > 3\epsilon - 3\epsilon \ge 0.$$

◻

[17] `[pem tew]`
[18] `[gaz9 gaz6 gckb]`
[19] `[gamma10_gamma11]`
[20] `[gaz6]`
[21] `[azim1]`
[22] `[txq]`

**Lemma 7.45** `[LUIKGMH]` *Assume the running assumptions. The case $C_{3,3}$ does not occur.*

*Proof* By a *computer calculation*[23] [2], we have azim $< 2.8$. If QX and $\breve{\gamma} < \epsilon$, then azim $< 2.3$. If QU and $\breve{\gamma} < 0$, then azim $< 1.65$.

If there are at least two negative quarters, then the total angle is

$$2\pi \leq 1.65 + 1.65 + 2.8 < 2\pi.$$

If there is one negative quarter, it will be neutralized unless both others have angle $< 2.3$. Then

$$2\pi \leq 1.65 + 2(2.3) < 2\pi.$$

□

**Lemma 7.46** `[GRHIDFA]` *Assume the running assumptions. The case $C_{4,4}$ does not occur.*

*Proof* Let $k$ be the number of quarters. We have seen that all faces are small.

If $k \leq 2$, then by a *computer calculation*[24] [2],

$$k\mathrm{QU} + (4-k)\mathrm{QX}: \quad \sum \breve{\gamma} > k1.61517 + (4-k)0.213849 - 2\pi(0.119482) > 0.$$

if $k = 4$, then by a *computer calculation*[25] [2],

$$\sum \breve{\gamma} > -4(0.0659) + 0.42(2\pi) > 0.$$

Finally assume that $k = 3$. If the fourth cell is a $\beta$-cell, we are done by a *computer calculation*[26] [2] Thus, the fourth edge of the fourth cell is supercritical. This gives by a *computer calculation*[27] [2],

$$3\mathrm{QU} + \mathrm{QX}: \quad 3(-0.0142852) + (0.00457511) + 2\pi(0.00609451) > 0.$$

□

## 7.8 table of inequalities

Here is a table of the inequalities that were used. The first column gives the name referenced in the text. Then it is indicated if the inequality is in the module *Ineq* or *Module* and the name of the inequality in the computer development. The final column gives a shorthand form of the inequality.

[23] `[azim1 azim_c4 g_qxd]`
[24] `[gaz6 gaz9]`
[25] `[azim1]`
[26] `[ox3q1h]`.
[27] `[gaz4 azim2]`

| | | | |
|---|---|---|---|
| **general bounds** | | | |
| jsp | ineq | JSPEVYT | $\eta^+$ |
| gckb | ineq | GCKBQEA | azim $> 0.606$ |
| azim_c4 | ineq | BIXPCGW 6652007036 a2 | QU $\vee$ QX $\Rightarrow$ azim $< 2.8$ |
| ox3q1h | merge | ox3q1h_merge | 3QU + 1$B \Rightarrow \sum_4 \check{\gamma} > 0.$ |
| **quarters** | | | |
| gamma_qu | ineq | BIXPCGW 9455898160 | QU $\Rightarrow \check{\gamma} > -\epsilon$ |
| fhbv2 | ineq | FHBVYXZv2 a | QU $\eta^+ \Rightarrow \check{\gamma} > \epsilon$ |
| quqy | merge | g_quqya_g_quqyb | QU + QY $\Rightarrow \check{\gamma}_4 + \check{\gamma}_3 > 0$ |
| ztg4 | merge | ztg4 | QU $\vee$ QX $\Rightarrow (a_5, b_5)$ |
| azim1 | ineq | QITNPEA 5653573305 | QU $\Rightarrow (\check{\gamma}$, azim) |
| gaz4 | ineq | QITNPEA 6206775865 | QU $\Rightarrow (\check{\gamma}$, azim) |
| gaz6 | ineq | QITNPEA 3848804089 | QU $\Rightarrow (\check{\gamma}$, azim) |
| **nonquarter** 4-**cells** | | | |
| gamma_qx | merge | gamma_qx | QX $\Rightarrow \check{\gamma} > 0$ |
| g_qxd | merge | gamma_qxd | QXD $\Rightarrow \check{\gamma} > \epsilon$ |
| gamma10_gamma11 | merge | gamma10_gamma11 | QX$^{ss}$ + QY $\Rightarrow \check{\gamma}_4 + \check{\gamma}_3 > \epsilon$ |
| gamma8 | merge | QITNPEA_9063653052_weak | QX$^{1s} \Rightarrow \check{\gamma} > \epsilon$ |
| gaz9 | ineq | QITNPEA 2134082733 | QX$^{ss} \Rightarrow (\check{\gamma}$, azim) |
| azim2 | ineq | QITNPEA 9939613598 | QX$^{ss,super} \Rightarrow (\check{\gamma}$, azim) |
| 23-**cells** | | | |
| cell3 | merge | cell3_008_from_ineq | $\check{\gamma}_3 > 0.008$azim |
| grki | ineq | GRKIBMP a | $\check{\gamma}_2 > 0.008$azim |
| pem | ineq | PEMKWKU | QY $\eta^+$ $\eta^-$ azim$^- \Rightarrow (a_5, b_5)$ |
| tew | ineq | TEWNSCJ | QY, $2\eta^- \Rightarrow (a_5, b_5)$ |
| txq | ineq | TXQTPVC, IXPOTPA | QY $2\eta^-$azim[] $\Rightarrow \check{\gamma} > 3\epsilon$ |

Table 7.1 *inequalities*

# 8

# Appendix on connecting with Bauer-Nipkow work in Isabelle

This section builds on the HOL-Light files hypermap.hl and import-tame-classificaton.hl.

For simplicity of typesetting, we use a dash in this text to represent an undescore in HOL Light.

## 8.1 basic definitions

(This section has been formalized 2/2014.)

In the Bauer-Nipkow work, there is a type for Isabelle graphs, which we abbreviate to lgraph. There is a predicate planegraph that expresses lgraph planarity. There is also a slightly broader class planegraph-relaxed.

**Lemma 8.1** `[DPZGBYF] [formal proof by tchales].` *If g is planegraph, then it is also planegraph-relaxed.*

There is a function, fgraph, that maps an lgraph to the list of list representation of the lgraph. For example, the list of list representation of a planar graph consisting of a square triangulated into four squares with common vertex 0 is

$$[[0; 1; 2]; [0; 2; 3]; [0; 3; 4]; [0; 4; 1]; [4; 3; 2; 1]].$$

We refer to these entries [0; 1; 2], [0; 2; 3], etc. as the faces of the lgraph.

There is a function, hypermap-of-list, that maps the list of list representation of the planar graph into the corresponding hypermap. The darts of the hypermap are consecutive pairs of elements in the list of list representation. For example, the example above gives the dart set

$$\{(0,1),(1,2),(2,0),\ (0,2),(2,3),(3,0),\ (0,3),(3,4),(4,0),\ (0,4),(4,1),(1,0),\ (4,3),(3,2),(2,1),(1,4)\}.$$

The face map follows each pair around the face in which it occurs $f(0,1) = (1,2)$, etc. The edge map reverses an ordered pair $e(0,1) = (1,0)$, etc. The set of integers that appear in this list of list is in bijection with the set of vertices of the lgraph. In fact, we can take this is as the definition of the set of vertices of the lgraph. When we refer to the hypermap of $g$, we refer to this construction.

There is a function, finals, that returns a sublist of the list of list representaton of the lgraph.

In the Isabelle development, there is a function facesAt.

Since our aim is relate hypermaps to the constructions in Isabelle, which are based on lists, we give a series of list-based definitions that run in parallel with hypermap-based definitions.

We make a table of notions about hypermaps and their corresponding list-based notion. This table serves as a dictionary translating between the different data structures.

| **hypermap** | **lists** | **theorem** |
|---|---|---|
| loop | loop-list | |
| is-contour | contour-list | contour-list-is-contour |
| is-normal | normal-list | normal-list-normal |
| hyp'm, hyp'p, hyp'q | l'm, l'p, l'q | |
| hyp'y, hyp'z | l'y, l'z | |
| final-quotient-face | final-list | |
| quotient | quotient-list | |
| - | final-dart-list | |
| - | split-normal-list | |
| transform | transform-list | |
| hyp'S | s-list | |
| s-flagged | s-flag-list | |
| empty-flagged | flag-list | |
| is-marked | marked-list | |
| iso | iso-list | |
| hyp-iso | isop-list | |
| f-map, n-map, e-map | f-list, n-list, e-list | |
| dih2k | dih2k-list | |
| atom-choice | find-atom | |
| atom | atom-list | |
| head-of-atom | TAIL | |
| tail-of-atom | HD | |
| is-split-condition | split-condition-list | |
| index | indexf | index-indexf |
| loop-of-face | loop-of-list | loop-of-face-list |

We also describe the general correspondence of variable names.

| **hypermap** | **lists** | **function** |
|---|---|---|
| H (hypermap) | L (list of lists) | hypermap-of-list |
| NF (normal family) | N | loop-family-of-list |
| L (loop) | r | loop-of-list |
| x (dart) | x | |

We also have conversions from lists to hypermaps. The constant loop-of-list converts a list to a loop, and loop-family-of-list converts a list of lists to a family of loops.

There is a function match-core-list that is true when there is a correspondence between faces of an lgraph g and those of the quotient of L by N, such that the final faces of g are included in those of $N$. A function match-list also measures the corespondence. The function loop-choice generates a pair (r,x) to be used as part of a marked-list, from a lgraph g.

## 8.2 properties of planegraph

**Definition 8.2** (good-list, good-list-nodes, good-graph) Recall that a list of lists $L$ is a good-list if it has three properties:

1. The list of darts of $L$ has no duplicates.
2. Every member of $L$ is non-nil.
3. If $(x, y)$ is a dart of $L$, then so is $(y, x)$.

Recall that a good list $L$ is a good-list-nodes if the number of nodes in its hypermap is equal to the number of vertices of $L$. We say that an lgraph is is good-graph if it has the properties:

1. Its fgraph $L$ is a good-list and a good-list-nodes.
2. all faces of the fgraph are final.
3. In each face, each vertex occur at most once: (all uniq L).
4. The vertex set coincides with the set of elements of the fgraph.
5. The facesAt a vertex $v$ is the same as the list of faces that contain $v$.

(We don't need to formalize JUJUWAT, CGGZYRC, EAHHATZ, ETDLJXT, and HWDMZDM because we will get them as a corollary of the correspondence between restricted hypermaps and plane-graphs.)

**Lemma 8.3** `[JUJUWAT]` *If g is planegraph-relaxed, then its fgraph is a good-list.*

**Lemma 8.4** `[CGGZYRC]` *Let g be a planegraph. Let F be a face of g that is not final. Let $(x, y)$ be a dart of (the hypermap of) g on face F. Then $(y, x)$ is a dart on a face that is final.*

**Lemma 8.5** `[EAHHATZ]` *If g is a Plane-graph, then it is a good-graph.*

**Lemma 8.6** `[ETDLJXT]` *If g is planegraph-relaxed, then the vertices of g are* $0, \ldots, n-1$*, where n is the number of nodes of g.*

**Lemma 8.7** `[HWDMZDM]` *Let g be planegraph-relaxed, and v in the vertex setof g. Then $facesAt\ g\ v$ is the set of faces of g that v is a member of.*

In general we want to have the properties of good-list, good-list-nodes, and all-uniq. The following lemmas show how these properties progagate. The following properties can be proved by structural induction for planegraph-relaxed. (We no longer need to use planegraph-relaxed; planegraph will be sufficient for our purposes.)

**Lemma 8.8** `[PMBRINH]` *If L is good-list-nodes and normal-list-L-N, then good-list-nodes holds of quotient-list-L-N.*

*Proof* It is enough to show that all of the darts with a given first coordinate form a single node of the quotient. This is clear, because by assumption all the darts of $L$ with a given first coordinate for a single node for $L$. □

**Lemma 8.9** `[SNVACWG][formal proof by tch 3/2014].` *If L is a good-list and good-list-nodes, its hypermap is restricted, and normal-list-L-N, then good-list holds of quotient-list-L-N.*

*Proof* We examine each of the defining properties of a good-list.

(UNIQ) If a dart $(a, b)$ occurs more than once then there are two darts $(x_1, y_1)$ and $(x_2, y_2)$ in $N$, that run from nodes $a$ to $b$. This is impossible by the no-double-joins property of restricted hypermaps.

(ALL-NON-NIL) It is a property of normal that its members are not nil. The same is then true of the quotient.

(SYM). Let $(a, b)$ be a dart in the quotient. Then there is a dart $(a, b)$ in $N$. In fact, we see that $(a, b)$ is the last member of its atom in $N$. The family $N$ meets the node $b$, and $(b, a)$ is a dart at the node $b$. By properties of normal, all darts at the node, including $(b, a)$, are on $N$. We have $n^{-1}(b, a) = f(a, b)$. By unicity, $(b, a)$ is followed by an $f$-step in $N$. This implies that $(b, a)$ is a dart in the quotient. □

**Lemma 8.10** `[LYNVPSU][formal proof by tch 3/3/2014].` $(L, N, r, s)$ *be a marked list. Then the quotient of L by N is all-uniq.*

*Proof* This is a defining property of a marked hypermap. □

**Lemma 8.11** `[LEBHIJR]` *Let L be a good-list and good-list-nodes. Let* $(L, N, r, x)$ *be a marked list. Let* $(N', r')$ *be the transform. Then the set of elements of* $L//N'$ *is the union of the set of elements of* $L//N$*, together with the set of nodes that visit* $N'$ *but not* $N$*.*

**Lemma 8.12** `[HOJODCM]` *Let L be a good-list and good-list-nodes. Let* $(L, N, r, x)$ *be a marked list with transform* $(N', r')$*. Let* $r''$ *be the offshoot of the transform. Let* $V$ *be the set of nodes that visit* $N'$ *but not* $N$*. Let* $s$ *be a loop in* $N'$*, and* $v$ *a node of* $N'$*. Then we have the following rules describing when* $v$ *is visited by* $s$*.*

1. *If* $s \neq r', r''$*, then* $v$ *is visited by* $s$ *iff* $v$ *is visited by* $s$ *in* $N$*.*
2. *If* $v \in V$*, then* $v$ *is visited by* $s$ *iff* $s = r'$ *or* $r''$*. Now assume* $v \notin V$*.*
3. $v$ *is visited by* $r'$ *iff* $v$ *is at the node of* $l'z$*,* $l'y$ *or between them.*
4. $v$ *is visited by* $r''$ *iff* $v$ *is at the node of* $l'y$*,* $l'z$ *or between them.*

## 8.3 main result

This result is imported from Isabelle.

**Theorem 8.13** (Import Tame Classification) *Let g be a final planegraph that is tame. Then there exists a y in the archive such that the fgraph of g is fgraph congruent to y.*

The main result to be proved is the following lemma.

**Lemma 8.14** (LSKOKJE) *Let H be a restricted hypermap. Then there exists a good-graph g that is a PlaneGraphs, such that H is isomorphic to the hypermap of g.*

## 8.4 isomorphism

We develop the properties of iso-list, which is the list-based version of isomorphism. It is a particularly rigid notion, because it does not allow the rearrangement of the elements of a list. However, it is sufficient for our purposes.

We often need to consider $L$ together with a normal-list $N$. The definition of iso-list carries with it a second argument that checks that $N$ is sent to $N'$ under the isomorphism.

**Lemma 8.15** `[GNBEVVU]` `[formal proof by tchales].` *If $L$ is a good-list and $L'$ is an iso-list, then the hypermaps of $L$ and $L'$ are isomorphic.*

*Proof* This follows directly from the theorem hypermap-of-list-map. □

**Lemma 8.16** (iso-list-refl, iso-list-sym, iso-list-trans) `[formal proof by tchales].` *The relation iso-list is reflexive, symmetric, and transitive.*

The properties of a list $L$ that are invariant under an injective map remain true when $L$ is replaced with an iso-list. This gives the following results.

**Lemma 8.17** `[PEUTLZH]` `[formal proof by tchales, 2/2014].` *Let $L$ be a good-list, and $L'$ an iso-list. Then $L'$ is a good-list.*

**Lemma 8.18** `[OISRWOF]` `[formal proof by tchales, 2/2014].` *Let $L$ be good-list and good-list-nodes, and $L'$ an iso-list. Then $L'$ is a good-list-nodes.*

**Lemma 8.19** `[UEYETNI]` `[formal proof by tchales, 2/2014].` *Let $L$ be all-uniq, and $L'$ an iso-list. Then $L'$ is a all-uniq.*

**Lemma 8.20** `[XKDZKWV]` `[formal proof by tch, 2/25/2014].` *Let $L$ be a good-list with normal-list $N$. If $(L', N')$ is iso-list, then $N'$ is a normal-list of $L'$.*

**Lemma 8.21** `[DAKEFCC]` `[formal proof by tch 3/10/2014].` *If $H$ and $K$ are isomorphic hypermaps, and if $H$ is restricted, then $K$ is restricted.*

**Lemma 8.22** `[MEEIXJO]` *Let $L$ be a good-list with marked-list $(L, N, r, x)$. If $(L', N', r', x')$ is iso-list, then $(L', N', r', x')$ is a marked-list.*

## 8.5 representing hypermap as lists

It is convenient to work with lists of list $L$ rather than hypermaps $H$. This means we should give a list version of notions such as loop and normal family. We have loop-list as the list version of a loop and norm-list as the list version of a normal family of loops.

**Lemma 8.23** `[RXOKSKC]` `[formal proof by HLT, 2/26/2014].` *Let $H$ be a restricted hypermap. Then there exists a list $L$ such that $H$ is its hypermap. In fact, there exists such an $L$ that is good-list, good-list-nodes, all-uniq.*

*Proof* We may convert a hypermap to list $L$ as follows. From each face $F$ of

$H$ pick a dart $x$ and form the list $[\mathrm{node}(x), \mathrm{node}(f x), \mathrm{node}(f^2 x), \ldots]$. The list of all these lists for each face is defined to be $L$.

A restricted hypermap is simple, hence the elements of the list are uniq. This gives the all-uniq property.

By construction the elements of these lists are the nodes. This gives the good-list-nodes property.

We check that it is a good list. For the uniqueness property, note that we show below that the map between darts sets is one-to-one. This means that each dart appears in a face once.

For the non-nil property of good-lists: each list contains $\mathrm{node}(x)$ as the first element. Hence it is not the nil-list.

For the symmetry property of good-lists: if $(\mathrm{node}(x), \mathrm{node}(f x))$ is a dart, then

$$(\mathrm{node}(n f x), \mathrm{node}(f n f x)) = (\mathrm{node}(f x), \mathrm{node}(x))$$

is also a dart.

Finally, we check that $H$ is isomorphic to the hypermap of $L$. The darts of the hypermap of $L$ are formed by consecutive pairs $(\mathrm{node}(f^i x), \mathrm{node}(f^{i+1} x))$.

We claim the map between dart sets, $y \mapsto (\mathrm{node}(y), \mathrm{node}(f y))$, is one-to-one. This follows from the no-double-joins property of restricted hypermaps. Specifically, if $(\mathrm{node}(y), \mathrm{node}(f y)) = (\mathrm{node}(z), \mathrm{node}(f z))$, then the edges $\{y, ey\}$ and $\{z, ez\}$ run between the same nodes, forcing $y = z$.

The face maps are compatible on the two hypermaps by construction. Finally, the edge map is compatible because

$$(\mathrm{node}(ey), \mathrm{node}(f(ey))) = (\mathrm{node}(f y), \mathrm{node}(y)) = e - list(\mathrm{node}(y), \mathrm{node}(f y)).$$

□

**Lemma 8.24** `[JXBJOAB] [formal proof by tchales, 2/2014].` *If $L$ is any list of lists, then there is a iso-list $L'$ whose elements are natural numbers.*

*Proof* Pick any injective map from the elements of $L$ to $\mathbb{N}$ and take the resulting iso-list. □

For the correspondence with Isabelle, we will need to use lists $L$ whose elements are natural numbers and whose elements are ordered in a particular way. By taking a permutation on the natural numbers we can obtain any convenient ordering. A particular ordering is given by lemma SHXWKXQ. We refer to the formal statement in the HOL Light files. What follows is an informal summary.

**Lemma 8.25** `[SHXWKXQ].` *For any $L$ (good-list, good-list-nodes, all-uniq)* *and normal-list $N$. By passing to an iso-list, we can arrange that the elements appearing in $N$ are the natural numbers $\{0, 1, \ldots, n-1\}$, where $n$ is the number of vertices visited by $N$. Furthermore, if $x$ is any dart, and $F$ is the face in $L$ containing $x$ listed so that $x$ appears at the head of the list. Let $F'$ be the filtered list of $F$ containing all the darts of $F$ that are not in $N$. Then we may also assume that $F' = [n+1; n+2; n+k]$, where $k$ is the length of $F'$.*

The following lemma gives a reduction of the main result LSKOKJE.

**Lemma 8.26** (JCAJYDU) `[formal proof by HLT, 2/28/2014].` *It is enough to prove LSKOKJE in the case when $H$ is the hypermap of $L$, where $L$ is a good-list, good-list-nodes, and having elements in the natural numbers.*

## 8.6 translating notions between hypermaps and lists

We skip the following proofs. In each case the definitions involved in the correspondence are direct translations of one another. Thus, the proofs are a matter of comparing defintions.

**Lemma 8.27** `[GZLJIGN][formal proof by tch 3/9/2014].` *Under the correspondence between $L$ and its hypermap, the constants $l'm$, $l'p$, $l'q$, $l'y$, $l'z$ correspond respectively to hyp'm, hyp'p, hyp'q, hyp'y, hyp'z.*

**Lemma 8.28** `[EVNAPDQ][formal proof by tch 2/28/2014].` *Let $L$ be a restricted, good-list and good-list-nodes. Let $N$ be a normal list of $L$. Then the loop family is a normal family of the hypermap of $L$.*

**Lemma 8.29** `[ABKCJWD][formal proof by tch 3/8/2014].` *Let $L$ be a good-list. Let $N$ be a normal list of $L$. Then the hypermap of the quotient-list is isomorphic to the quotient of the data on the hypermap side.*

*Proof* Their respective definitions are direct translations of one another. □

**Lemma 8.30** `[ODWAFRG][formal proof by tch 3/8/2014].` *Let $L$ be a good-list and $(L, N, r, x)$ a marked-list. Then the corresponding data in the hypermap is is-marked.*

There are also specific lemmas about hypermaps that we wish to translate back into theorems about lists. Specifically, we have lemmas HQYMRTX1-list, HQYMRTX2-list, HQYMRTX3-list, QRDYXYJ-list, AQIUNPP1-list, and AQIUNPP2-list that are list versions of lemmas given in *Dense Sphere Packings*. We omit the proofs here, because the proofs are direct translations from hypermap language to list language. They have been deduced form the hypermap versions of the lemams using transform-assumption-v2.

**Lemma 8.31** (transform-assumption-v2) `[AQ]` *Let L be good-list, good-list-nodes, with marked list* $(L, N, r, x)$ *and transform* $(N', r')$. *Then marked-list* $(L, N', r', x)$.

## 8.7 dihedral initialization

The fgraph of Seed p is the list of lists $[[0; 1; ...; n+2]; [n+2; n+1; ...; 0]]$.

**Lemma 8.32** (good-list-seed, all-uniq-seed) `[FOEGZEQ] [formal proof by tch].` *Seed p is a good-list and all-uniq.*

**Lemma 8.33** `[ENWCUED] [formal proof by tch].` *Seed p is dih2k-list.*

**Lemma 8.34** `[TAGYMWJ] [formal proof by tch].` *Seed p is a good-list-nodes. More generally, for any L that is dih2k-list, it is good-list-nodes.*

**Lemma 8.35** `[formal proof by tch, 3/14/2014].` *If L is dih2k-list, then it is all uniq.*

**Lemma 8.36** `[formal proof by tch, 3/14/2014].` *If L is dih2k-list, then it is good-list.*

The following is not needed:

**Lemma 8.37** `[DLWOJBB]` *If L is dih2k-list, then its hypermap is dih2k.*

**Lemma 8.38** `[KUKASGD] [formal proof by tch 3/14/2014].` *Any two dih2k-list's L and L′ of the same face size are iso-lists.*

**Lemma 8.39** `[UNHQYQM] [formal proof by tch, 3/14/2014].` *If L and L′ are iso-lists and L is dih2k-list, then so is L′.*

This is a version of AUQTZYZ for lists. It can be deduced from the hypermap version.

**Lemma 8.40** (AUQTZYZ-list) *Let L be good-list with restricted hypermap. Let* $f$ *be a face of L. Then there exists* $f'$ *such that* $[f; f']$ *is normal and the quotient is* $[f, rev\ f]$ *(which is dih2k-list).*

*Proof* Pick $f'$ to be the complementary path (a concatenation of complementary nodes) of $f$. □

The following is a more precise version of the correspondence of a restricted hypermap with the seed. We will apply this lemma to a face with the largest size of $L$.

**Lemma 8.41** `[UYOUIXG]` *Let $L$ be a good-list with restricted hypermap. Let $f$ be any face of $L$. Then there exists data $(L', N')$ and $f'$ such that $(L, [f; f'])$ is an iso-list of $(L', N')$ and such that $N'$ is a normal-list of $L'$ and match-quotient-list between the seed and $(L', N')$.*

*Proof* This is obtained by renaming the elements of $L$ to agree exactly with those of the seed. □

## 8.8 termination

We will obtain termination after a finite sequence of iterations of subdivFace. Each subdivFace will increase the number of final faces by one. The number of final faces is no more than the number of faces in the original. Finally, if every face is final, then every loop in $N$ is also final, and the quotient list of $L$ by $N$ is equal to $L$.

This is not needed. We will count final faces rather than darts.

**Lemma 8.42** `[ADACDYF]` *Let $(L, N, r, x)$ be a marked-list in which $r$ is not final. Let $(N', r')$ be the transform of the marked-list. Then the number of darts in the quotient of $L$ by $N$ is less than the number of darts in the quotient of $L$ by $N'$.*

*Proof* The number of darts in a quotient hypermap is equal to the number of quotient darts (atoms) in the normal family $N$. The normal family $N'$ replaces a loop $r$ with $r_1$ and $r_2$ and keeps the other loops the same. The atoms of $r$ at $y$ and $z$ are split into two atoms to create $r_1$ and $r_2$. Hence the number increases. □

**Lemma 8.43** `[ZBHENEI]` *The number of final faces in a quotient is no more than the number of faces in the original $L$.*

*Proof* Each final face in the quotient is a face in the original $L$, up to rotation. □

The following upper bound on the number of elements that are final in an lgraph, combined with a monotonicity result for $g$, will give termination of the the procedure.

**Lemma 8.44** `[XZAJELF]` *If we have a match-quotient-list $(g, L, N)$, then the number of final faces in $g$ is at most the number of faces in $L$.*

*Proof* The set of elements that are final in $g$ injects into the set of darts in the quotient. Then apply ZBHENEI. □

We can continue to subdivFace until every face is final. Note that match-quotient-list allows for there to be loops that are final in N, whose corresponding face in $g$ is not final. In this case, the iteration of subdivFace has no effect other than to make the face final. In this case, no transform of $N$ occurs.

The following lemma shows that when we terminate, the fgraph of $g$ will equal $L$.

**Lemma 8.45** `[XWCNBMA]` *If* $(L, N, r, x)$ *is a marked-list, and if every path of* $N$ *is a final-list, then* $L$ *is equal to its quotient-list by* $N$.

This has been proved at the level of hypermaps in STKBEPH. We can deduce the lemma from that.

*Proof* This is Example 4.51 (maximal normal family), translated into lists. If $f$ is a loop in the family, it is canonically true. Hence its darts are singletons and the darts in the loop form a face. Thus, the set of darts visited by the family $N$ is a union of faces.

By the third, property of normal family, the set of darts visited by the family $N$ is a union of nodes. Hence the set of such darts is a connected component.

$L$ is restricted, hence connected. Our conditions force $N$ to be a connected set of faces, hence all of $L$ □

Thus, a series of transforms leads in a finite number of steps to a quotient-list that equals $L$.

## 8.9 finals and nonFinals

This section shows that showDups and hideDups are inverse operations, enum-of-VertexList and indexToVertexList are inverse operations, mk-triple and dest-triple are inverse operations. It analyzes containsDuplicateEdge and gives an alternative to subdivFace.

We also analyze enumerator and simplify formulas for splitFace.

## 8.10 enumeration lists

## 8.11 index calculus of higher transforms

We establish a common notation for the following lemmas.

Let $\iota(y, z, s)$ be the index of the dart $z$ in the list $s$ relative to the base point $y$ (where both $y$ and $z$ are members of $s$).

**Lemma 8.46** `[QYHXIVZ]` `[formal proof by tchales, 2/2014]. [formal note: indexf-refl, indexf1, next-el-indexf, indexf-rotn, indexf-rotn, indexf-add-right, indexf-add-left, indexf-add-sum, indexf-antisym, size-betwn, indexl-betwn, indexf-betwn-eq, next-eln-indexf, indexf-n, betwn-cases, indexf-add-betwn].` *The index satisfies the following properties. Assume that $a, b, c$ are members of $s$, and that $s$ is uniq. Let $n$ be the length of $s$.*

- $\iota(a, a, s) = 0$.
- *$\iota$ is invariant under rotations of $s$.*
- $\iota(a, b, s) + \iota(b, c, s) = \iota(a, c, s)$ *if* $\iota(b, c, s) \leq \iota(a, c, s)$.
- $\iota(a, b, s) + \iota(b, c, s) = \iota(a, c, s)$ *if* $\iota(a, b, s) \leq \iota(a, c, s)$.
- $\iota(a, b, s) + \iota(b, c, s) = \iota(a, c, s)$ *if* $\iota(a, b, s) + \iota(b, c, s) < n$.
- $\iota(a, b, s) + \iota(b, a, s) = n$, *if* $(a = b)$.
- $n^i a = b$, *where* $i = \iota(a, b, s)$ *and $n$ is the next element map.*
- $\iota(a, b, s) = 1$ *iff $b$ is the next element after $a$ in $s$.*
- *$b$ is between $a$ and $c$ on $s$ iff* $a \neq c$ *and* $0 < \iota(a, b, s) < \iota(a, c, s)$.
- *We have one of the following: $b = a$, $b = c$, $b$ is between $a$ and $c$, or $b$ is between $c$ and $a$.*
- *if $b$ is not between $c$ and $a$ on $s$, then* $\iota(a, b, s) + \iota(b, c, s) = \iota(a, c, s)$.

**Lemma 8.47** `[FWDDPHY]` `[formal proof by tchales, 2/2014]. [formal note: indexf-prev].` *Assume that $b$ follows $a$ in $s$ and that $a \neq b$. Then* $\iota(b, a, s) + 1$ *is the length of $s$.*

*Proof* This is a special case of the preceeding lemma, since $\iota(a, b, s) = 1$. □

*Remark* 8.48 (F, T, w, $N_i$, $r_i$, $y_i$, $z_i$) Let $(L, N, r, x)$ be a marked list. Let $F$ be the face of $x$ in $L$. We let $w = fx$ be the dart following $x$ in $F$. It will serve as a base point.

Let $T = T_{L,x}$ be the transform. Set $(N_i, r_i) = T^i(N, r)$. In particular, $(N_0, r_0) = (N, r)$. Let $y_i = l'y(L, r_i, x)$ and $z_i = l'z(L, N_i, r_i, x)$. By construction $y_i$ and $z_i$ are members of $r_i$. Also, let $z_i^-$ be the predecessor of $z_i$ in $r_i$ and $y_i^+$ the successor of $y_i$ in $r_i$.

**Lemma 8.49** `[DANGEYJ]` `[formal proof by HLT 3/2/2014].` *Let* $(L, N, r, x)$ *be a marked list with notation as established above. Then every dart of $N_i$ is a dart of $N_{i+1}$.*

*Proof* The darts of $N_i$ in each face are the same in each face, except the face which is split. Its darts are divided between the two new faces. □

**Lemma 8.50** `[PWSSRAT]` `[formal proof by HLT 3/5/2014]`. *Let* $(L, N, r, x)$ *be a marked list with notation as established above. Then a dart* $d$ *of* $r_i$ *appears in* $r_{i+1}$ *if and only if* $\iota(z_i, d, r_i) \le \iota(z_i, y_i, r_i)$.

(In particular, $z_i$ appears in $r_{i+1}$.)

*Proof* This follows from the explicit ordering of darts on faces and the transform. □

**Lemma 8.51** `[OHCGKFU]` `[formal proof by HLT 3/2014]`. *Let* $(L, N, r, x)$ *be a marked list with notation as established above. Assume that* $d$ *is a dart in both* $r_i$ *and* $r_{i+1}$. *Then*

$$\iota(z_i, d, r_i) = \iota(z_i, d, r_{i+1}).$$

*Proof* This follows from the explicit ordering of darts on faces and the transform. □

**Lemma 8.52** `[PPLHULJ]` `[formal proof by HLT 3/2014]`. $(L, N, r, x)$ *be a marked list with notation as established above. Then* $y_i$ *and* $z_i$ *belong to* $F$, *and* $\iota(w, y_i, F) < \iota(w, z_i, F)$.

*Proof* This follows from the face map power applied to $y_i$ to obtain $z_i$, an the result that $z_i$ is not in the s-list (HQYMRTX1). □

**Lemma 8.53** `[NCVIBWU]` `[formal proof by HLT 3/2014]`. $(L, N, r, x)$ *be a marked list with notation as established above. If* $j \le \iota(w, z_i, F)$, *then* $f^j w$ *belongs to* $r_{i+1}$ *and* $j = \iota(w, f^j w, r_{i+1})$.

*Proof* This follows from the explicit ordering of darts on faces and the transform. □

**Lemma 8.54** `[QCDVKEA]` $(L, N, r, x)$ *be a marked list with notation as established above. Then* $\iota(w, z_i, F) \le \iota(w, y_{i+1}, F)$.

*Proof* By construction, $\iota(w, y_{i+1}, F)$ is the largest index $j$ such that the indexing on $f$ and $r_{i+1}$ agree up to $j$. By the previous lemma, the indexing agrees at least up to $\iota(w, z_i, F)$. □

By PWSSRAT, $y_0$ belongs to $r_i$ for all $i$.

**Lemma 8.55** `[PBFLHET]` $(L, N, r, x)$ *be a marked list with notation as established above. Let* $d$ *be a dart in* $r_i$ *such that* $\iota(z_i, d, r_i) \le \iota(z_i, y_0, r_i)$. *Then* $d$ *belongs to* $r$. *Moreover,* $\iota(z_i, d, r_i) = \iota(z_i, d, r)$.

*Proof* By assumption $d$ belongs to $r_i$. Fix such a $d$. By PWSSRAT, and the monotonicity of the parameters PPLHULJ and QCDVKEA, by induction, $d$ belongs to $r_{i-j}$, for $j = 0, \ldots, i$. In particular, $d$ belongs to $r_0 = r$.

Taking $j = 0$, we get $d = z_i$ belongs to $r$.

The index calculation is also backwards induction on indices $r_{i-j}$, using OHCGKFU. □

**Lemma 8.56** [PNXVWFS] $(L, N, r, x)$ *be a marked list with notation as established above. For all* $i$*, we have* $y_i$ *and* $z_i$ *belong to* $r$.

*Proof* Apply the previous lemma to $d = y_{i+1}$ and $d = z_i$, which belong to $r_i$ by construction. □

**Lemma 8.57** [DIOWAAS] $(L, N, r, x)$ *be a marked list with notation as established above. Then* $\iota(w, y_i, r) < \iota(w, z_i, r)$.

*Proof* This is equivalent to $\iota(z_i, w, r) \le \iota(z_i, y_i, r)$, or to $\iota(z_i, w, r_i) \le \iota(z_i, y_i, r_i)$. For this it is enough to show that $w$ is not between $y_i$ and $z_i$ on $r_i$. This is clear, because the segment between $y_i$ and $z_i$ does not meet $r_{i+1}$, but $w$ belongs to $r_{i+1}$. □

**Lemma 8.58** [RYIUUVK] $(L, N, r, x)$ *be a marked list with notation as established above. Then* $\iota(w, z_i, r) \le \iota(w, y_{i+1}, r)$.

*Proof* The dart $y_{i+1}$ belongs to $r_{i+1}$. We claim that it lies on the segment of $r_{i+1}$ between $z_i$ and $w$. Otherwise, it is between $w$ and $z_i$. But from $w$ to $z_i$, the list $r_{i+1}$ coincides with $f$, by NCVIBWU. This is impossible, since $y_{i+1}$ comes after $z_i$ on $f$.

On the segment of $r_{i+1}$ between $z_i$ and $w$, the ordering is the same as on $r$. So $\iota(z_i, y_{i+1}, r_{i+1}) < \iota(z_i, w, r_{i+1})$ gives $\iota(z_i, y_{i+1}, r) < \iota(z_i, w, r)$ and hence $\iota(w, z_i, r) \le \iota(w, y_{i+1}, r)$. □

**Lemma 8.59** [KBWPBHQ] $(L, N, r, x)$ *be a marked list with notation as established above. The set* $\{y_i\}$ *consists of darts* $y$ *such that* $y$ *on* $F$ *and* $N$ *such that* $y$ *is followed by an* $n^{-1}$*-step on* $r$*. The set* $\{z_i\}$ *consists of darts* $z$ *on* $F$ *and* $N$ *such that* $z$ *is preceded by an* $n^{-1}$*-step on* $r$.

*Proof* We have seen that the darts $y_i$ and $z_i$ lie on $F$ and $r$. By construction the darts $y_i$ are followed by an $n^{-1}$-step, and the darts $z_i$ are preceded by an $n^{-1}$ step.

Conversely, let $y$ be any dart on $F$ and $N$ that is followed by an $n^{-1}$ step on $N$. If $y$ is between $y_i$ and $z_i$, then these darts are not in $N_i$, hence not in $N$. This is contrary to the assumption.

If $y$ is between $z_i$ and $y_{i+1}$, then $y$ is reached from $z_i$ by $f$-steps on $r_i$. So they are also $f$-steps on $r$. □

## 8.12 match $g$ $L$ and $N$

## 8.13 tame hypermaps

The formalization of this section is complete.

**Lemma 8.60** `[OXAXUCS]` `[formal proof by tch].` *If a hypermap $H$ is isomorphic to one that has property tame $k$, for $k \in \{9a, 10, 11a, 11b, 12o, 13a\}$, then $H$ has that property as well.*

Each of the tame properties $I = \{9a, 10, 11a, 11b, 12o, 13a\}$ has a corresponding definition for lgraphs, say $I'$. This correspondence is defined in such a way that the following holds.

**Lemma 8.61** `[WMLNYMD]` `[formal proof by tch].` *Let $g$ be a good graph. Let $H$ be a tame hypermap that is isomorphic to the hypermap of $g$. Then $g$ is tame.*

Let $H$ be any tame hypermap. It is restricted, so there exists a final plane-graph $g$ and an isomorphism between $H$ and the hypermap of $g$. It follows that the hypermap of $g$ has all of the tameness properties $I$. Hence $g$ itself has all of the tameness properties $I'$.

By the Bauer-Nipkow formalization on tame graphs, $g$ is fgraph congruent to an fgraph $y$ in the archive.

**Lemma 8.62** `[XRFJNDO]` `[formal proof by Solovyev].` `[formal note: See tame/good_list_archive.hl].` *Every member of the archive is a good-list.*

*Proof* This is by direct enumeration of the archive. □

**Lemma 8.63** `[ELLLNYZ]` `[formal proof by tch].` *Let $x$ and $y$ be two good-lists that are fgraph congruent. Then their hypermaps are isomorphic, or the opposite hypermap of $x$ is isomorphic to the hypermap of $y$.*

Putting these results together we have that $H$ isomorphic to the hypermap of $y$ or the opposite of $H$ is isomorphic to the hypermap of $y$.

The main linear programming result, formalized by Solovyev, shows that if $H$ is isomorphic to the hypermap of $y$, then it is not contravening. We need the opposite as well.

**Lemma 8.64** `[ASFUTBF][formal proof by tch].` *If the opposite of $H$ is contravening, then $H$ is also contravening.*

*Proof* If $H$ is contravening, this means there is a finite packing $V$ which is contravening and whose associated hypermap $\mathrm{hyp}(V, E_{std}(V))$ is $H$. The finite packing $-V$ obtained by negating all the coordinates is also contravening. Its hypermap is isomorphic to the opposite of $H$. □

## 8.14 notes on AQ

Here we discuss the proof of AQ12 and AQ13. Specifically, we discuss the proof of the second part of those statements (not involving the contour list).

Let $N' =$ ntrans $L\ N\ r\ x$ 1, and $r' =$ rtrans $L\ N\ r\ x$ 1. Write AQ12' for the goal $AQ(L, N', r', x)$ and AQ12 for the assumption $AQ(L, N, r, x)$. Similarly, for AQ13' and AQ13. We write $S$ =s-list L r x, and $S'$ =s-list L r' x. Note that every member of $S$ is a member of $S'$.

*Proof* We start with the proof of AQ12'. We may assume (final-list L r') and not(final-list L r). Otherwise, there is nothing to prove.

We let u = LAST p, where p is a part of s in N'. We assume that not(final-list L s), for otherwise there is nothing to prove. Our goal is to prove that e-list u is a member of final-dart-list L N'.

(A) We may assume that e-list u is not a member of r', because otherwise e-list u is in the a final face r', and we are done. Note that every member of $S$ is in r'. Thus, we may also assume that e-list u is not a member of $S'$. Since not(final-list L s) and final-list L r', we have $s \neq r'$; so that u is not a member of r' and that u is not a member of $S'$.

We have a slightly weaker statement: (A') e-list u is not a member $S'$, and u is not a member of $S'$.

We will see that if we give a proof of AQ12' based on (A'), then exactly the same proof can be used for AQ13'. The following condition also holds:

(B) Every member of final-dart-list L N is a member of final-dart-list L N'.

*Case 1. u is a member of N' but not N.* Comparing N' with N, we see that u is a new dart that is added at new nodes of degree 2 on the quotient hypermap. (These are the darts at the nodes between l'y and l'z along the face of $x$.) By (A'), u is not a member of $S'$. This implies that u is not in r'. It follows that u lies at a node of degree 2 on the quotient hypermap. Hence e-list u is in $S'$, which is contrary to (A').

*Case 2. u is a member of r.* Recall that u has the form LAST p, for p a part of s. If p is also a part of r, then using that r is not final, and that e-list u does not depend on N, N', we see that by AQ13, e-list u is a member of $S$ or a final-dart-list L N. By (A'), we know that e-list u is not a member of $S'$. So it is a member of final-dart-list N. The result follows by (B).

If u has the form LAST p, where p is a part of s, but not a part of r, then there are only two possbilities for u. Either it is the dart l'y, or the dart node-list (l'z). The first case u = l'y is incompatible with (A'), since l'y is a member of $S'$. The second case u =node-list (l'z), gives e-list u $= f^{-1} l'z$, which is also a member of $S'$. This is also incompatible with (A').

*Case 3. u is a member of N but not r.* For such u, we have that u is final-dart-list for N iff it is in the final-dart-list for N'. Recall that s is not-final.

In view of (A'), the assumption AQ13 tells us that since s is a face of both N and N', and since not(final-list s), we have that e-list u is a member of the final-dart-list L N. By (B), e-list u is a member of final-dart-list L N' as well. This completes the proof. □

*Proof* Now we turn to the proof of AQ13'. We may assume that not(final-list L r') and not(final-list L r). Otherwise, there is nothing to prove.

We let u = LAST p, where p is a part of s in N'. We assume that not(final-list L s), for otherwise there is nothing to prove. To avoid a trivial case, we may also assume that u is not a member of $S'$.

Our goal is to prove that e-list u is a member of $S'$ or final-dart-list L N'. In other words, assuming that e-list u is not a member of $S'$, we wish to prove it is a member of final-dart-list L N'. In this form, we have assumption (A'), as stated in the previous proof. Assumption (B) also holds.

We have written the proof of AQ12' in such a way that the three claims from the previous proof work word-for-word to give the proof of AQ13' as well. □

# References


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[2] Thomas C. Hales, Alexey Solovyev, and Hoang Le Truong et al. The Flyspeck Project, 2014. `http://code.google.com/p/flyspeck`.