CQF - Exam 1 - Optimal Portfolio Allocation

Python imports

In [ ]:
```
import pandas as pd
import numpy as np
import math
from scipy.stats import norm
import matplotlib.pyplot as plt
%matplotlib inline
```

Question 1

Consider mininum variance portfolio with a target return m

$$\underset{w}{argmin} \frac{1}{2} w^{'} \sum w$$

$s.t.$

$$w^{'} \mathbf{1} = 1$$

$$\mu_{\pi} = w^{'} \mu = m$$

- Formulate the Lagrangian and give its partial derivatives

We form the Lagrange function with two Lagrange multipliers λ and γ:

$$L(w, \lambda, \gamma) = \frac{1}{2} w^{'} \sum w + \lambda(m - w^{'} \mu) + \gamma(1 - w^{'} \mathbf{1})$$

with partial derivatives:

$$\frac{\partial L}{\partial w} = \sum w - \lambda \mu - \gamma$$

$$\frac{\partial L}{\partial \lambda} = m - w^{'} \mu$$

$$\frac{\partial L}{\partial \gamma} = 1 - w^{'} \mathbf{1}$$

- Write down the anlytical solution for optimal allocations w^{*} (derivation not required)

$$w^* = \frac{1}{(AC - B^2)} \sum^{-1} [(A\mu - B\mathbf{1})m + (C\mathbf{1} - B\mu)]$$

where:

$$\begin{cases} A = \mathbf{1}' \sum^{-1} \mathbf{1} \\ B = \mathbf{1}' \sum^{-1} \mu \\ C = \mu' \sum^{-1} \mu \end{cases} \tag{1}$$

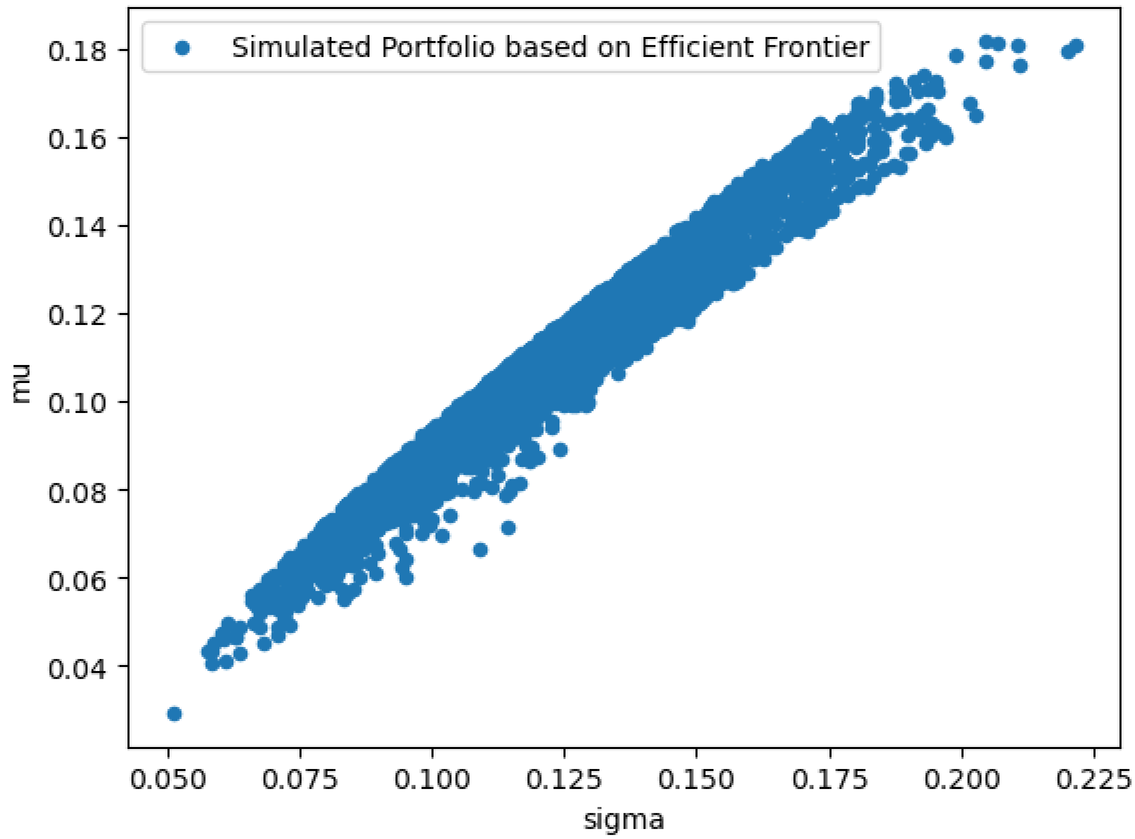
- Inverse optmization: generate above 700 random allocation sets (vectors) 4x1, those will not be optimal allocations.

  Standardize each set to satisfy $w'\mathbf{1} = 1$ For each vector of allocations compute $\mu_\pi = w'\mu$ and $\sigma_\pi = \sqrt{w' \sum w}$

In [ ]:
```
mu = np.array([0.02, 0.07, 0.15, 0.20])

sigma = np.array([0.05, 0.12, 0.17, 0.25])

R = np.matrix([
        [1.0, 0.3, 0.3, 0.3],
        [0.3, 1.0, 0.6, 0.6],
        [0.3, 0.6, 1.0, 0.6 ],
        [0.3, 0.6, 0.6, 1.0],
        ])

def get_covariance_matrix(sigma, R):
    S = np.diag(sigma)
    return S*R*S
```

In [ ]:
```
def gen_w(size):
    w = np.array([np.random.uniform() for i in range(0,size,1)])
    return w / w.sum()

n_assets = 4
cases = 5000
W = [gen_w(n_assets) for i in range(0,cases,1)]
cov = get_covariance_matrix(sigma=sigma, R=R)

data = [[np.dot(mu, w),np.sqrt(np.dot(np.dot(w,cov),w).sum())] for w in W]
df_simul = pd.DataFrame(data=data,columns=['mu','sigma'])
df_simul.plot.scatter(x='sigma', y='mu', label='Simulated Portfolio based on Effici
```

Out[ ]:
```
<AxesSubplot:xlabel='sigma', ylabel='mu'>
```



The plot above shows the efficient frontier based on different weighted portfolios simulation.

Question 2

Consider optmization for a tangency portfolio (maximum Sharpe Ratio).

- Formulate optimization expression

$$\min_{w} \frac{1}{2} w' \sum w$$

subject to:

$$r + w^{'}(\mu - r\mathbf{1}) = m$$

- Formulate Lagrangian function and give its partial derivatives only

As residual of wealth not invested in risky assets will be invested in risk free, the budget constraint has been removed

$$L(x, \lambda) = \frac{1}{2} w^{'} \sum w + \lambda(m - r - w^{'}(\mu - r\mathbf{1}))$$

$$\frac{\partial L}{\partial w} = \sum w - \lambda(\mu - r\mathbf{1})$$

- For the range of tangency portfolios given by $r_f = 50bps, 100bps, 150bps, 175bps$ optimal compute allocations (ready formula) and σ_π.

  Plot the efficient frontier in the presence of a risk-free aset for $r_f = 100bps, 175bps$.

The tangencial portfolio alocation w_t is given by:

$$w_t = \frac{\sum^{-1}(\mu - r\mathbf{1})}{B - Ar}$$

where its return and standard deviation:

$$m_t = \frac{C - Br}{B - Ar}$$

$$\sigma_t = \sqrt{\frac{C - 2Br + Ar^2}{(B - Ar)^2}}$$

where:

$$\begin{cases} A = \mathbf{1}' \sum^{-1} \mathbf{1} \\ B = \mathbf{1}' \sum^{-1} \mu \\ C = \mu' \sum^{-1} \mu \end{cases} \tag{2}$$

In [ ]:
```
def calculate_tangency_portfolio(mu, sigma, R, r_f):

    m_cov = get_covariance_matrix(sigma=sigma, R=R)
    v_ones = np.ones(len(m_cov))
    m_cov_inv = np.linalg.inv(m_cov)
    A = m_cov_inv.sum()
    B = np.dot(np.dot(v_ones, m_cov_inv), mu).sum()
    C = np.dot(np.dot(mu, m_cov_inv), mu).sum()

    den = (B-A*r_f)
    w_t = np.dot(m_cov_inv,mu - r_f) / den
    m_t = (C - B*r_f) / den
    sigma_t = np.sqrt((C - 2*B*r_f + A*(r_f**2))/(den**2))

    return w_t, m_t, sigma_t
```

In [ ]:
```
mu = np.array([0.02, 0.07, 0.15, 0.20])

sigma = np.array([0.05, 0.12, 0.17, 0.25])

R = np.matrix([
        [1.0, 0.3, 0.3, 0.3],
```

```
        [0.3, 1.0, 0.6, 0.6],
        [0.3, 0.6, 1.0, 0.6 ],
        [0.3, 0.6, 0.6, 1.0],
        ])

df_pt = pd.DataFrame(data=[0.005, 0.01, 0.015, 0.0175], columns=['r_f'])
df_pt['w_t'], df_pt['m_t'], df_pt['sigma_t'] = zip(*df_pt.r_f.apply(lambda x: calcu
df_pt[['r_f', 'w_t', 'm_t', 'sigma_t']]
```

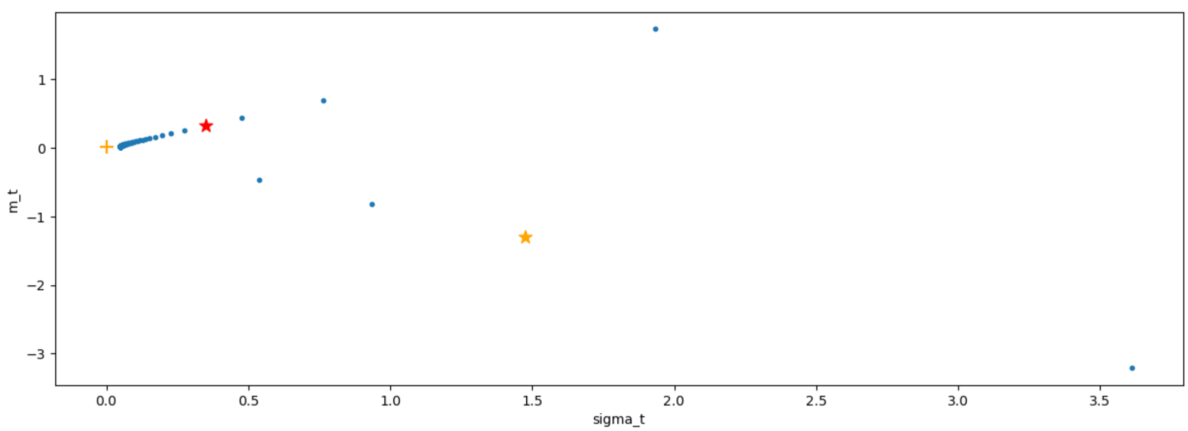
Out[ ]:

	r_f	w_t	m_t	sigma_t
0	0.0050	[[[[0.0168352 -0.22936698 0.81434026 0.39...	0.186070	0.196511
1	0.0100	[[[[-0.74593711 -0.51056937 1.49024934 0.76...	0.326130	0.350665
2	0.0150	[[[[-8.64485405 -3.42257114 8.48965087 4.57...	1.776525	1.972392
3	0.0175	[[[[8.10350247 2.75185052 -6.3514309 -3.50...	-1.298799	1.473515

In [ ]:

```
rs = np.linspace(-5,0.02, 3000)
mus = np.zeros(len(rs))
sigmas = np.zeros(len(rs))
for i in range(0, len(rs), 1):
    w, m, s = calculate_tangency_portfolio(mu=mu, sigma=sigma, R=R, r_f=rs[i])
    mus[i] = m
    sigmas[i] = s

plt.subplots(figsize=[15,5])
plt.scatter(sigmas, mus, marker='.')

df_100bps = df_pt[df_pt['r_f'] == 0.01]
df_175bps = df_pt[df_pt['r_f'] == 0.0175]

plt.scatter(df_100bps.iloc[0].sigma_t, df_100bps.iloc[0].m_t, marker='*', color='re
plt.scatter(0, 0.01, marker='+', color='red', s=100)
plt.scatter(df_175bps.iloc[0].sigma_t, df_175bps.iloc[0].m_t, marker='*', color='or
plt.scatter(0, 0.0175, marker='+', color='orange', s=100)

plt.xlabel('sigma_t')
plt.ylabel('m_t')
```

Out[ ]: Text(0, 0.5, 'm_t')



Question 3

Implement the multi-step binomial method as described in Binomial Method lecture with the following variables and parameters: sotck $S = 100$, interset rate $r = 0.05$ (continuously compounded) for a call option s with strike $E = 100$, and maturity $T = 1$.

- Use any suitable parametrisation for up and down moves uS, vS.
- Compute the options value for a range of volatilities $[0.05, \ldots, 0.8]$ and plot the result. Set trees to have a minimum four time steps.
- Now, compute and plot the value of one options, $\sigma_{imp} - 0.2$ as you increase the number of time steps $NTS = 4, 5, \ldots, 50$.

In [ ]:

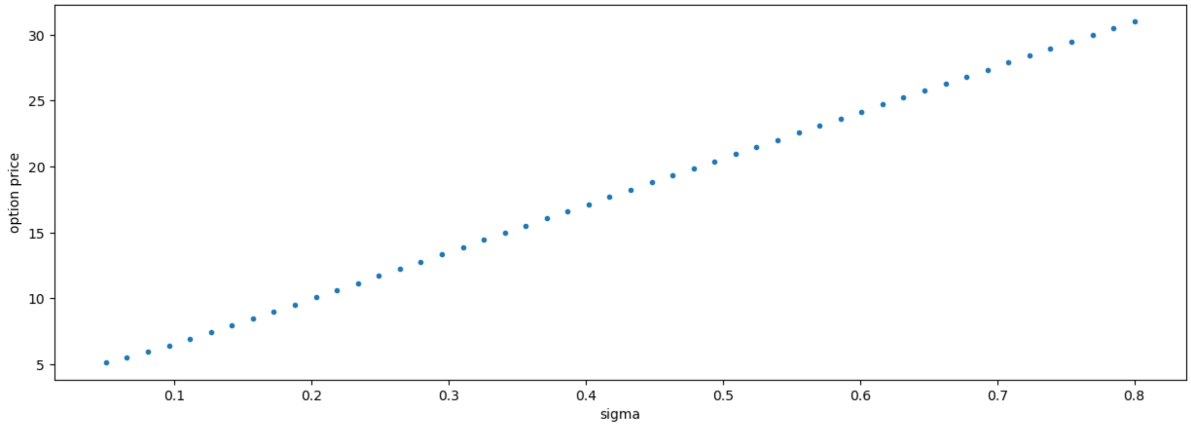
```
S = 100
r = 0.05
K = 100
T = 1
N = 4

def combinat(n, i):
    return math.factorial(n) / (math.factorial(n-i)*math.factorial(i))

def opt_binomial(S0, K, T, r, sigma, N, CP):
    dt = T / N
    u = np.exp(sigma * np.sqrt(dt))
    d = np.exp(-sigma * np.sqrt(dt))
    p = (np.exp(r*dt)-d)/(u-d)
    value = 0
    for i in range(N+1):
        prob = combinat(N, i)*(p**i)*((1-p)**(N-i))
        ST = S0*(u**i)*(d**(N-i))
        if CP == 'C':
            value += max(ST-K, 0) * prob
        else:
            value += max(K-ST, 0) * prob

    return value*np.exp(-r*T)
```

In [ ]:

```
sigmas = np.linspace(0.05, 0.80)

prices = []
for sigma in sigmas:
    prices.append(opt_binomial(S0=S, K=K, T=T, r=r, sigma=sigma, N=N, CP='C'))

plt.subplots(figsize=[15,5])
plt.scatter(sigmas, prices, marker='.')
```

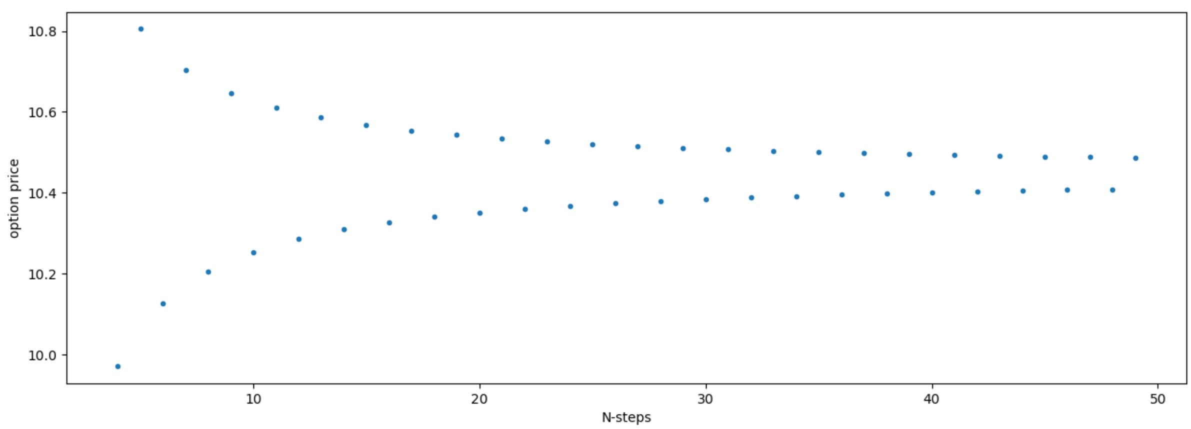
```
plt.xlabel('sigma')
plt.ylabel('option price')
```

Out[ ]: Text(0, 0.5, 'option price')



In [ ]:
```
ns = list(range(4, 50, 1))

prices = []
for n in ns:
    prices.append(opt_binomial(S0=S, K=K, T=T, r=r, sigma=0.2, N=n, CP='C'))

plt.subplots(figsize=[15,5])
plt.scatter(ns, prices, marker='.')

plt.xlabel('N-steps')
plt.ylabel('option price')
```

Out[ ]: Text(0, 0.5, 'option price')



Question 4

Use the ready formula for Expected Shortfall in order to compute the standardised value of Expected Shortfall for $N(0, 1)$.

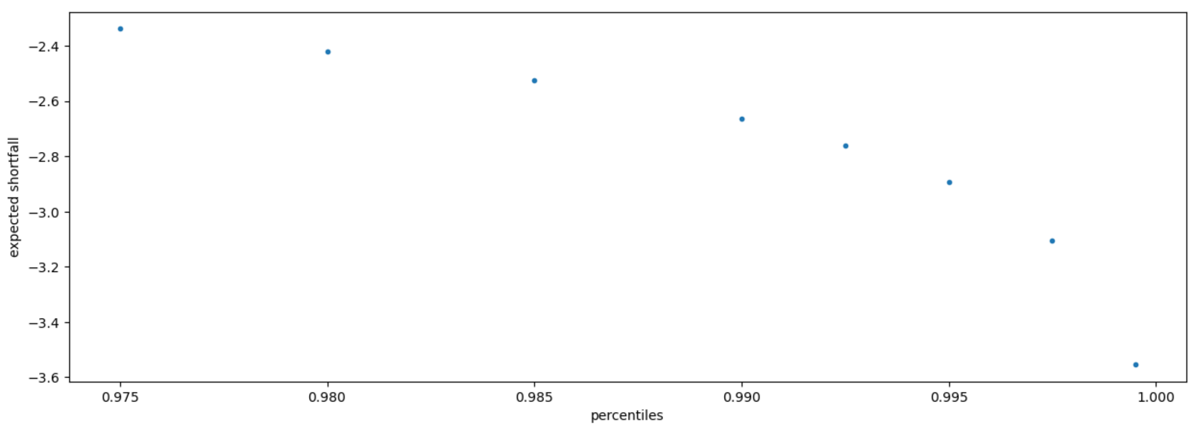
- Compute for the following range of percentiles $[99.95; 99.75; 99.5; 99.25; 99; 98.5; 98; 97.5]$

- The formula to use, and $1-c$ refers to $1-99.95$ and so on,

$$ES_c(X) = \mu - \sigma\frac{\phi(\Phi^{-1}(1-c))}{1-c}$$

In [ ]:
```
alpha = 1-0.995

def ExpectedShortfall(mu, sigma, percentil):
    alpha = 1.0 - percentil
    return mu - (sigma*(norm.pdf(norm.ppf(alpha)))) / alpha

mu = 0
sigma = 1
percentiles = [0.9995, 0.9975, 0.995, 0.9925, 0.99, 0.985, 0.98, 0.975]
ess = []
for percentile in percentiles:
    es = ExpectedShortfall(mu=mu, sigma=sigma, percentil=percentile)
    ess.append(es)

plt.subplots(figsize=[15,5])
plt.scatter(percentiles, ess, marker='.')

plt.xlabel('percentiles')
plt.ylabel('expected shortfall')
```

Out[ ]:
```
Text(0, 0.5, 'expected shortfall')
```



Question 5

In [ ]:
```
df = pd.read_csv('Data_SP500.csv', skiprows=1, names=['date', 'close'])
df['log_return'] = (np.log(df.close) - np.log(df.close.shift(1)))
df['log_return_10d'] = (np.log(df.close.shift(-10)) - np.log(df.close))
df['sigma'] = df.log_return.rolling(21).std()
df['sigma_10d'] = df.sigma * np.sqrt(10)
df['var10d'] = df.sigma_10d * norm.ppf(0.01)
df['breaches'] = df.log_return_10d < df.var10d
result = df[['log_return_10d','var10d','breaches']].dropna()
breaches = result[result.breaches]
```

In [ ]:
```
print(rf'Breaches: {len(breaches)} - {len(breaches)/len(result):.2%}')
```

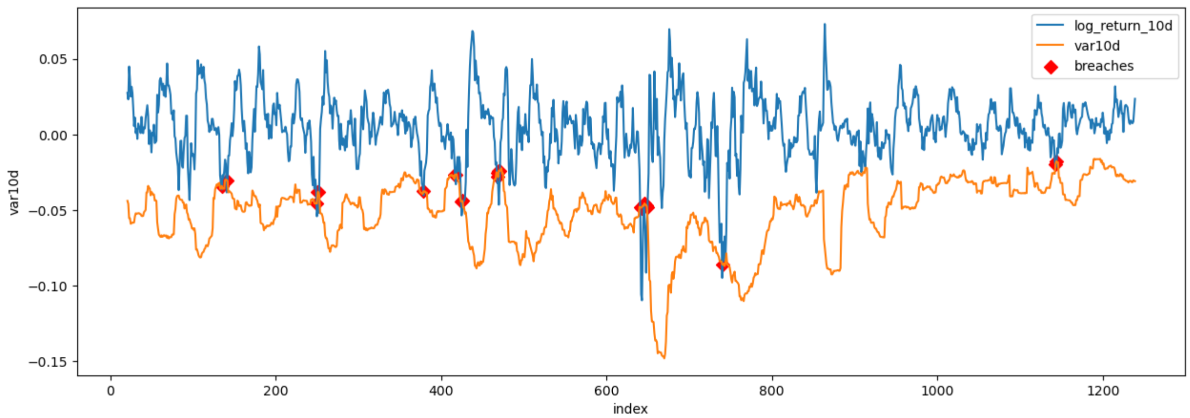
```
Breaches: 25 - 2.05%
```

In [ ]:
```
consecutives_breaches = breaches.index[1:] - breaches.index[:-1]
total_consecutive_breaches = consecutives_breaches.value_counts()[1]
print(rf'Consecutive Breaches: {total_consecutive_breaches} - {total_consecutive_br
```

```
Consecutive Breaches: 14 - 56.00%
```

In [ ]:
```
ax = df[['log_return_10d','var10d']].dropna().plot(figsize=[15,5])
breaches.reset_index().plot.scatter(ax = ax, x='index', y='var10d', color='red', ma
```

Out[ ]:
```
<AxesSubplot:xlabel='index', ylabel='var10d'>
```



Question 6

In [ ]:
```
ewma_lambda = 0.72

df_index = 21
std_ewma = []
sigma_0 = df.log_return.std()
std_ewma.append(sigma_0)

for i in range(1, len(df.index) - df_index, 1):
    std_ewma.append( np.sqrt( ewma_lambda*std_ewma[i-1]**2 + (1-ewma_lambda)*(df.il

df['ewma_sigma'] = pd.Series(data=std_ewma,index=df.index[21:])
df['ewma_sigma10d'] = df.ewma_sigma * np.sqrt(10)

df['ewma_var10d'] = df.ewma_sigma10d * norm.ppf(0.01)
df['ewma_breaches'] = df.log_return_10d < df.ewma_var10d
result = df[['log_return_10d','ewma_var10d','ewma_breaches']].dropna()
ewma_breaches = result[result.ewma_breaches]
```

In [ ]:
```
print(rf'Breaches (EWMA): {len(ewma_breaches)} - {len(ewma_breaches)/len(result):.2
```

```
Breaches (EWMA): 32 - 2.63%
```

In [ ]:
```
consecutives_breaches = ewma_breaches.index[1:] - ewma_breaches.index[:-1]
total_consecutive_breaches = consecutives_breaches.value_counts()[1]
```

```
print(rf'Consecutive Breaches (EWMA): {total_consecutive_breaches} - {total_consecu
```

```
Consecutive Breaches (EWMA): 17 - 53.12%
```

In [ ]:
```
ax = df[['log_return_10d','ewma_var10d']].dropna().plot(figsize=[15,5])
ewma_breaches.reset_index().plot.scatter(ax = ax, x='index', y='ewma_var10d', color
```

Out[ ]:
```
<AxesSubplot:xlabel='index', ylabel='ewma_var10d'>
```