TUTORIAL

Should you hedge with implied or realized? Volatility Arbitrage

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Sep 2022

How to use

- While we re-visit relevant first principles, the tutorial is <u>not a substitute</u> for the core lectures, eg Understanding Volatility, Black-Scholes. There may be no opportunity to re-explain each concept or formula. If you have not reviewed the core lecture(s) and <u>Exercises/Solutions</u> – the difficulty to follow the material can be expected.

- The tutorial/Python lab is not a full lecture with a set program of content. Frequent changes between slides/computation to be expected – the flow might be 'punctuated'.

- The tutorial is delivered 'from the desk' and typically includes a computation (Excel, Python, R etc) – not built from the first principles. The teaching is by presenting an example – each example, case is inevitably limited in scope (eg, we will not cover the entire delta-hedging)

Learning outcomes

- Revisit delta-hedging and gain/loss in a hedged option position
- Case 1: Volatility Arbitrage – ‘the market is wrong’ and we have a better forecast for volatility. Outcome relies on (Rlzd – Imp).
- Case 2: Sell Side business – we **will not** revisit but, the inverse applies (Implied – Realized).
- EXCEL walkthrough Delta-replication with Gamma Payoff

These slides will be released AFTER tutorial.

We refer to Understanding Volatility Lecture SOLUTIONS – recommend to download from CQF Portal, print and review on paper.

Sell Call to Open

Continuous hedging means sliding up and down the CDF: **buying OR selling**

– common sense: no need to have 100% exposure to the falling price

– however, falling stock = high realized volatility (☹ hedger).

The more price drops the less shares we need to hold. At 0% or 100% hedged we are not much sensitive to change in BS Delta
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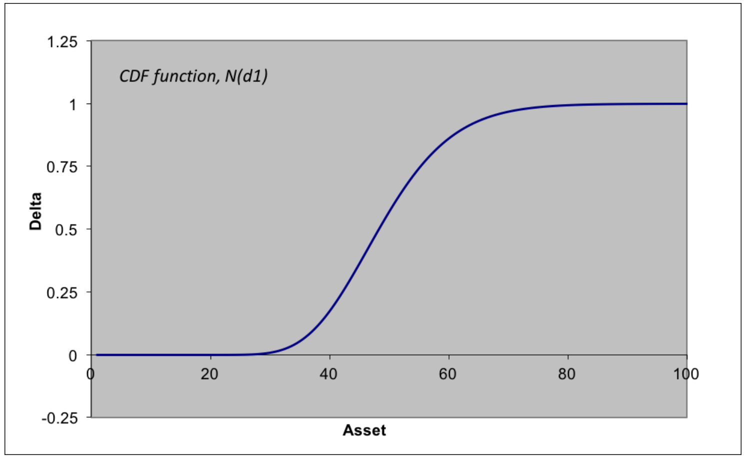
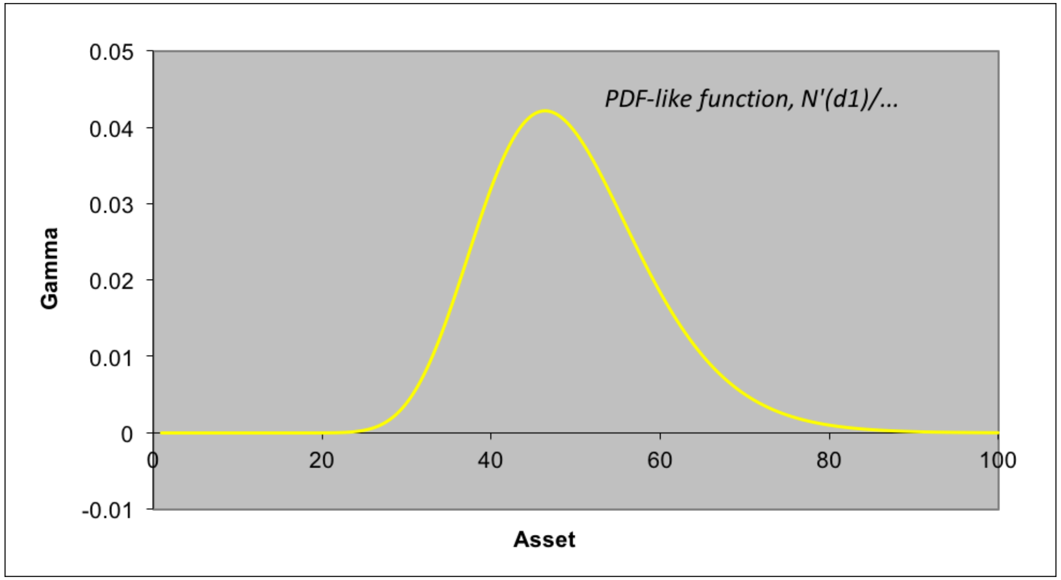


Not selling fast enough means extra losses (right-to-left)

(left-to-right) Not buying fast enough means missed profits

“Short Gamma”

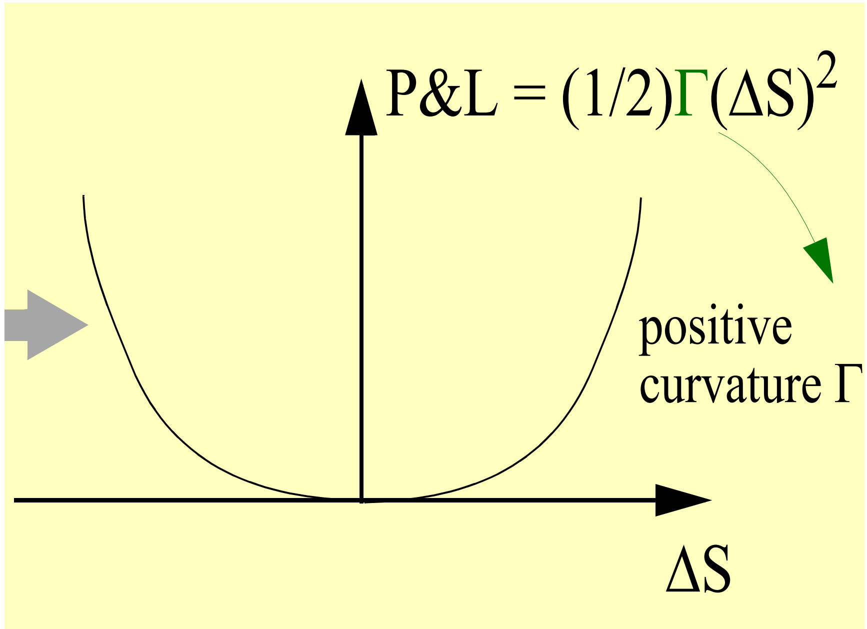


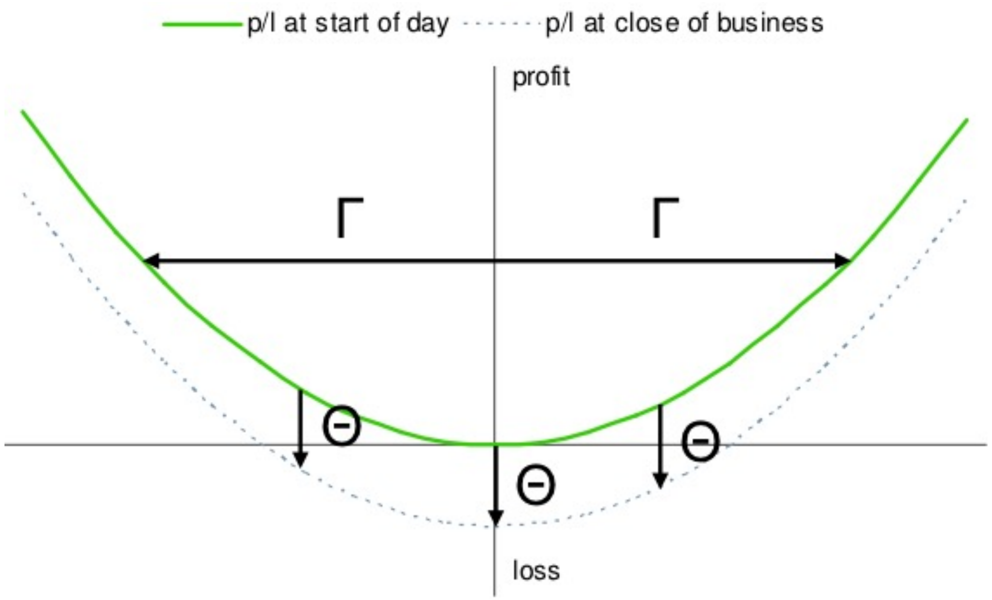
“Fast enough” equates to Gamma.

Traders professional language to say **short gamma**, as in “not enough gamma” or failure to anticipate that Gamma increases.

Convex Payoff

d(P&L) gains from curvature and loses from time decay.





Convex Payoff – applies even if hedged

$$V(S+\Delta S,\, t+\Delta t) = V(S,t) + \left.\frac{\partial V}{\partial t}\right|_t \Delta t + \left.\frac{\partial V}{\partial S}\right|_S \Delta S + \frac{1}{2}\left.\frac{\partial^2 V}{\partial S^2}\right|_S (\Delta S)^2$$

$$d(V - \Delta S) = \frac{1}{2}\Gamma S^2 \sigma^2 \Delta t + \Theta \Delta t$$

LHS is naturally hedged position in the option (long option, short stock), albeit an inverted position to the sell-side market maker. Matches our Case 1 Vol arb.

RHS matches Black-Scholes if equated to zero,

– gain from curvature (see next slide too) $\frac{1}{2}\Gamma(\Delta S)^2$

– time decay $\underbrace{\Theta\Delta t}$

$$\left(\frac{\Delta S}{S}\right)^2 = r^2 \approx \sigma^2 \Delta t \qquad \text{prop to variance}$$

Consider P&L from a hedged position in an option, where option price has already been decomposed into Greeks:

$$\begin{aligned}
\mathrm{P\&L}_{\Delta t, \Delta S} &= \cancel{\Delta(\Delta S)} + \frac{1}{2}\Gamma(\Delta S)^2 + \underbrace{\Theta \Delta t} - \cancel{\Delta(\Delta S)} \\
& \qquad \text{the Black-Sholes } \Theta_i = -\frac{1}{2}\sigma_i^2 S^2 \Gamma_i \\
&= \frac{1}{2}\Gamma(\Delta S)^2 - \frac{1}{2}\Gamma_i S^2 \sigma_i^2 (\Delta t) \\
&= \frac{1}{2}\Gamma S^2 \left[\left(\frac{\Delta S}{S}\right)^2 - \sigma_i^2 \Delta t\right].
\end{aligned}$$

MtM is the **difference between Realized vs Implied**!

Reminder of Case 1: Volatility Arb (see Understanding Volatility core lecture)

We buy V_i, and **replicate a short position** in the better-valued option V_a.

- Selling as the stock goes up will lose us some money (deep ITM we short 100 shares). Buying back occurs as the stock goes down – natural short covering and de-risking.

  See Excel Column J Cashflow from Replication.

  Higher realized volatility is good for us. $\sigma_a > \sigma_i$

- Our shorting will be compensated by $(S_T - K)^+$,

  V_i option will deliver that at time T.

Hedge with the actual volatility, σ_a

So you believe an option at $\sigma_i = 20\%$ is mispriced... **how can you profit from this?**

Buy an option and delta-replicate: cash from *buying* V^i and *selling* Δ^a quantity of the stock:

$$-V^i + \Delta^a S$$

By continually selling stock we replicate a short position in a correctly priced option V^a.

Eventually, we shall earn a pile of money equal to option premium $\$V^a$... at the market's expense!

P&L (MtM) mathematical result

Let's do the maths **on the mark-to-market basis**, by which we mean to consider P&L over each time step.

'Today' at time t:

Option	V^i
Stock	$-\Delta^a\, S$
Cash	$-V^i + \Delta^a\, S$

'Tomorrow' at time $t + dt$:

Option	$V^i + dV^i$
Stock	$-\Delta^a\, S - \Delta^a\, dS$
Cash	$(-V^i + \Delta^a\, S)(1 + r\, dt)$

Therefore we have made marked to market,

$$dV^i - \Delta^a \, dS - (V^i - \Delta^a \, S) \, rdt \; \dagger \qquad (1)$$

Because the option would be correctly valued at V^a then we have

$$dV^a - \Delta^a \, dS - (V^a - \Delta^a \, S) \, rdt = 0 \; \ddagger$$

This is profit from time t to $t + dt$ is

$$\begin{aligned} \dagger - \ddagger &= dV^i - dV^a + r(V^a - \Delta^a \, S) \, dt - r(V^i - \Delta^a \, S) \, dt \\ &= dV^i - dV^a - r(V^i - V^a) \, dt \\ &= e^{rt} \, d\left(e^{-rt}(V^i - V^a)\right) \qquad \textit{by Integrating Factor } e^{-rt} \\ d(e^{-rt}V) &= e^{-rt}dV - re^{-rt}V dt = e^{-rt}(dV - rV dt) \end{aligned}$$

PV-ing that increment of profit to t_0 gives

$$e^{-r(t-t_0)}\underbrace{e^{rt}\,d\left(e^{-rt}(V^i - V^a)\right)} = e^{rt_0}\,d\left(e^{-rt}(V^i - V^a)\right)$$

And **the total profit** from t_0 to expiration comes from summation (integration in continuous time)

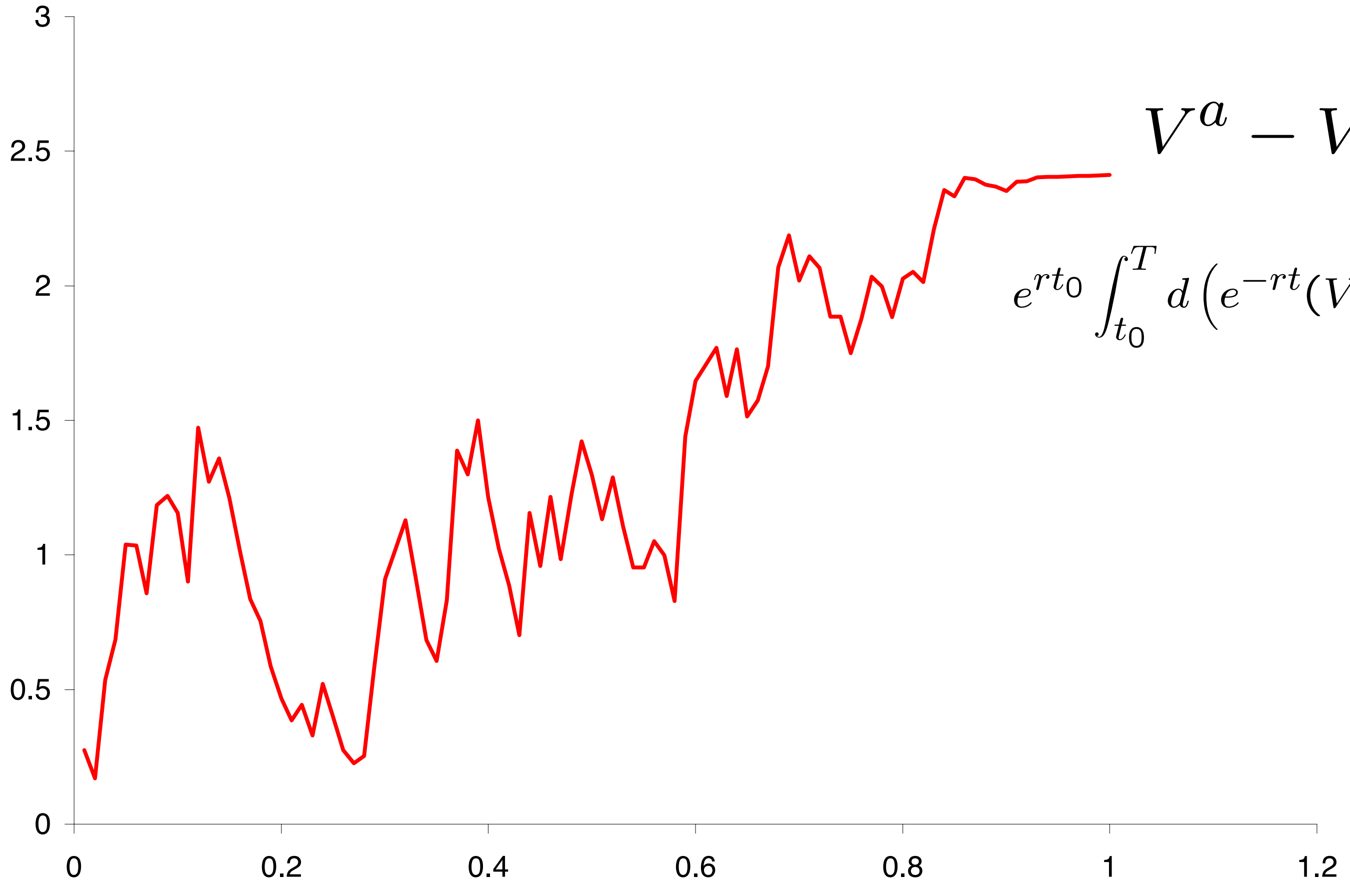
$$e^{rt_0}\int_{t_0}^{T} d\left(e^{-rt}(V^i - V^a)\right) = V^a - V^i$$

The total profit is a known quantity.

Tutorial Notes 1 (integrating factor working)

Tutorial Notes 2 (total profit working)

P&L from hedging with the actual

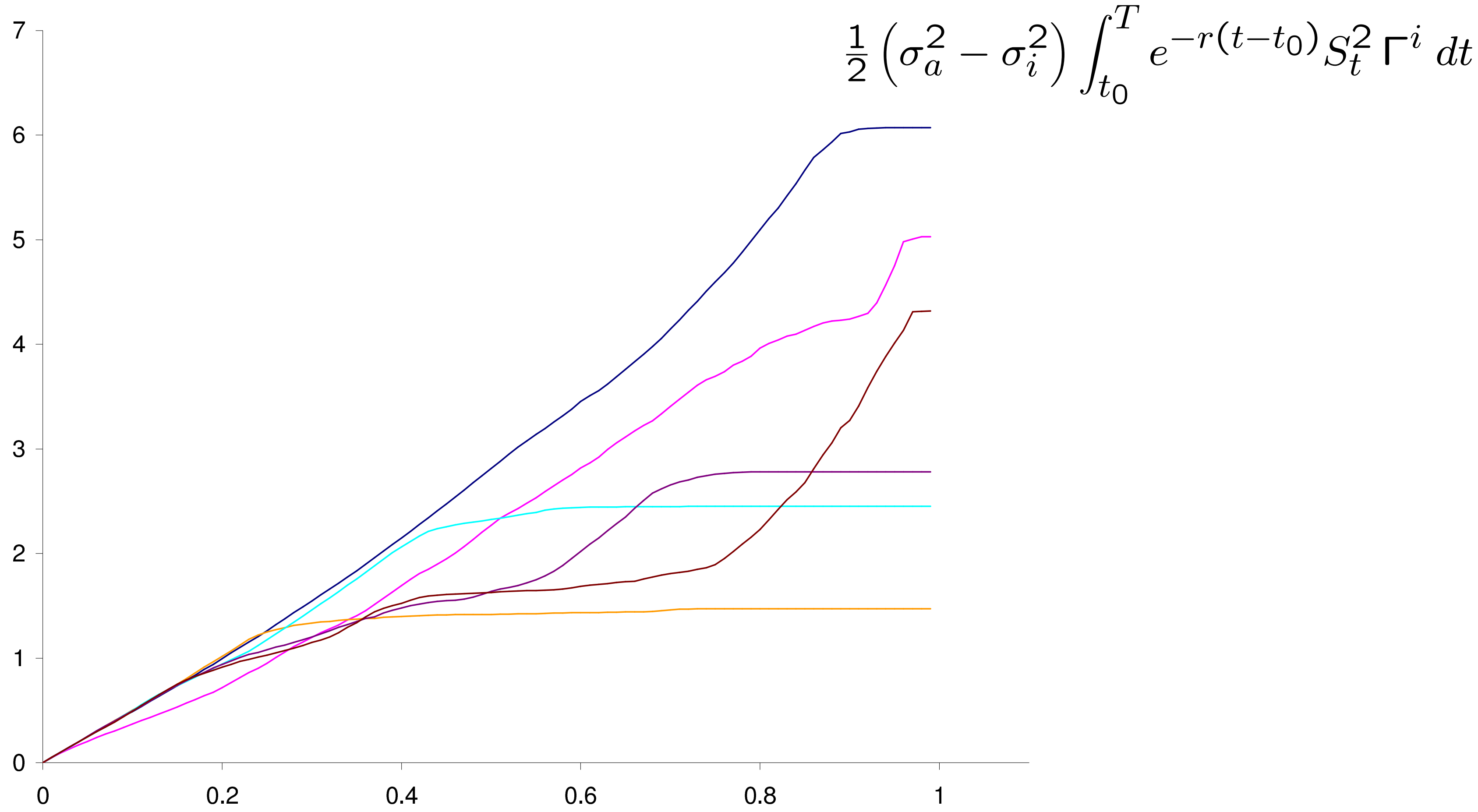


$$V^a - V^i$$

$$e^{rt_0}\int_{t_0}^{T} d\left(e^{-rt}(V^i - V^a)\right) = V^a - V^i$$

How the guaranteed profit is achieved is random. MtM P&L is affected by asset price and Gamma.

P&L from hedging with the implied

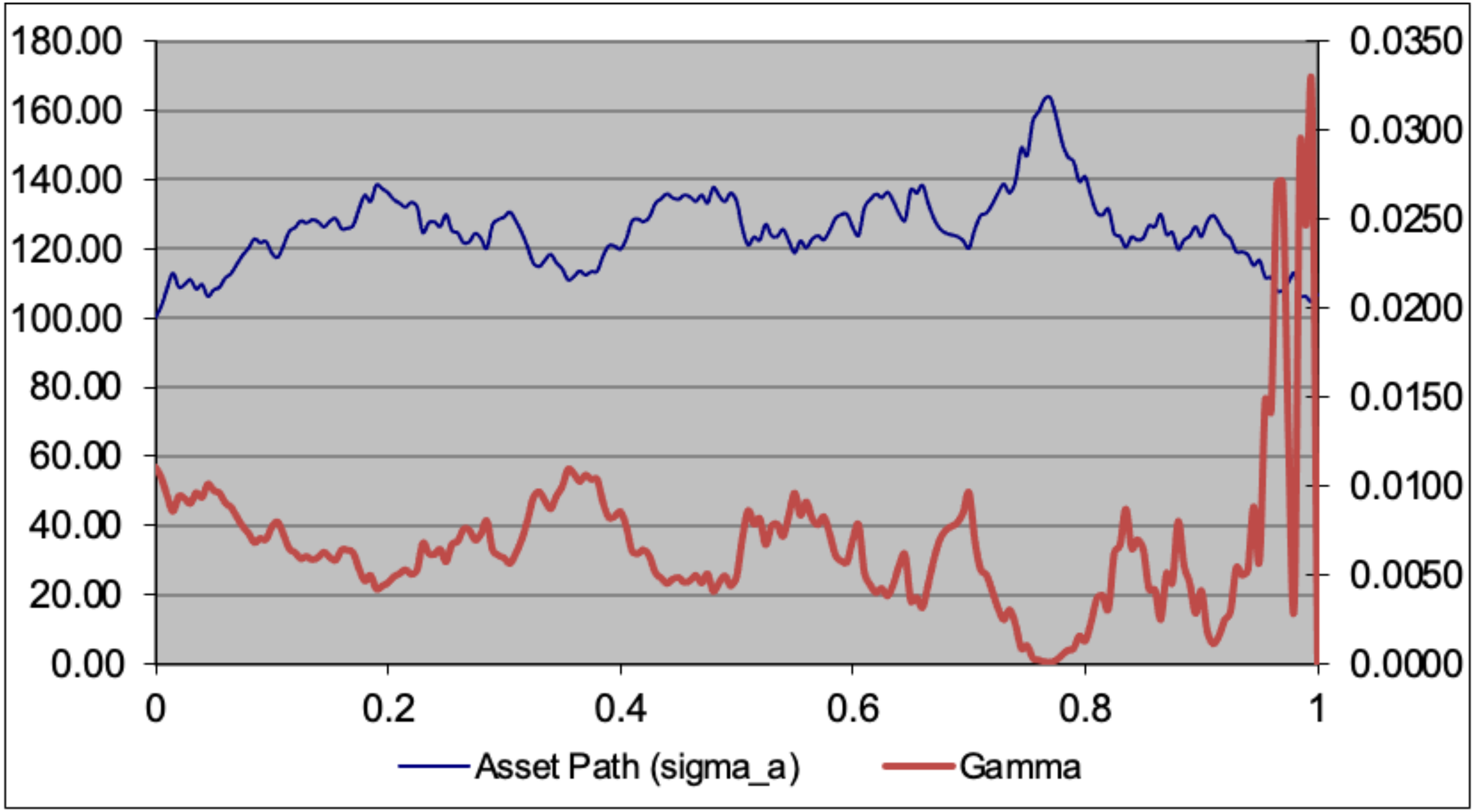


Using implied volatility σ_i as a prediction, while asset evolves according to its own actual volatility σ_a. **End result is uncertain.**

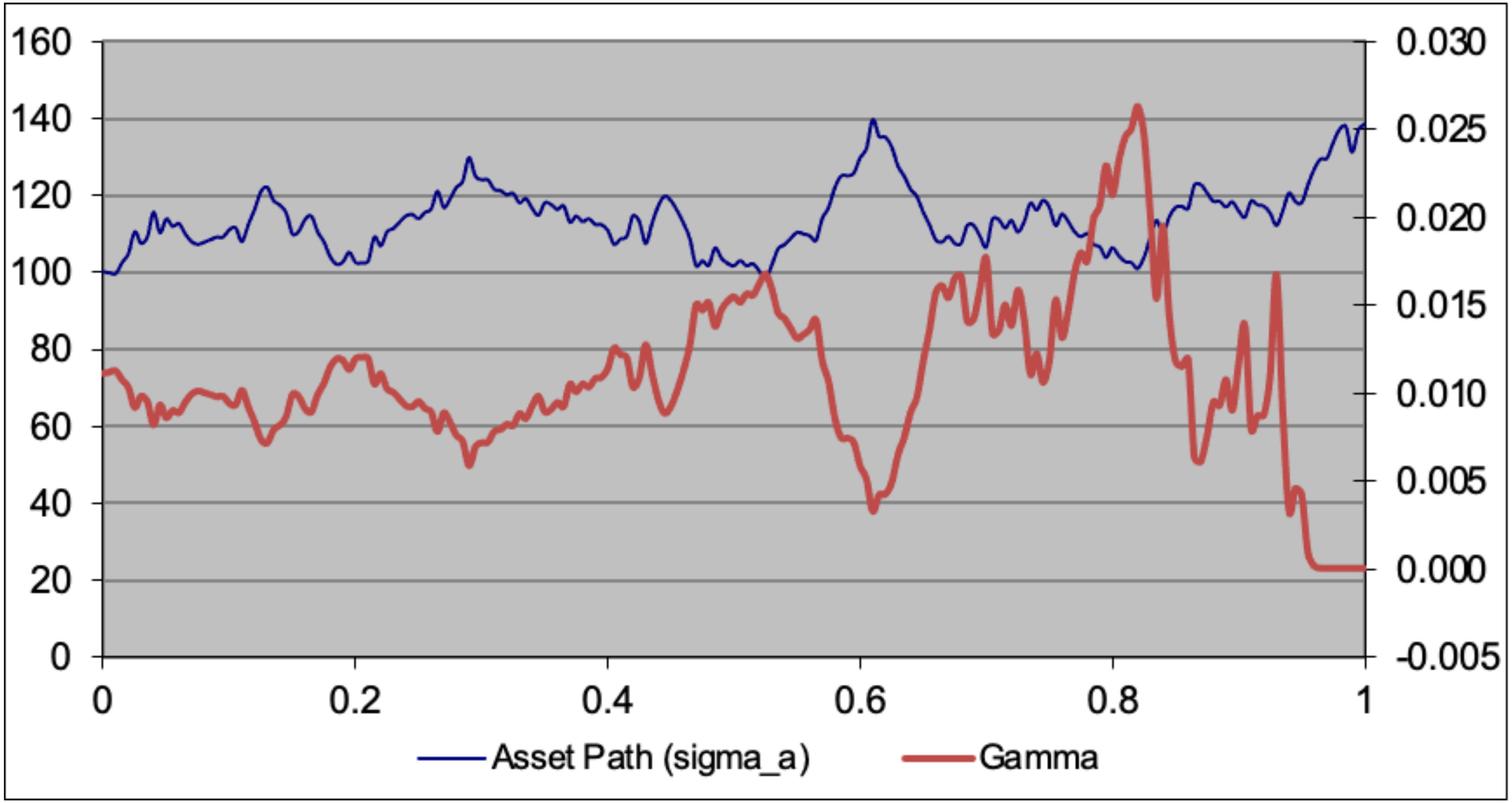
$$V^a - V^i$$

The following slides present modelling results from Excel implementation of Case 1: Volatility Arbitrage.

However, remember we have proven mathematically that the total arb profit is a known quantity.

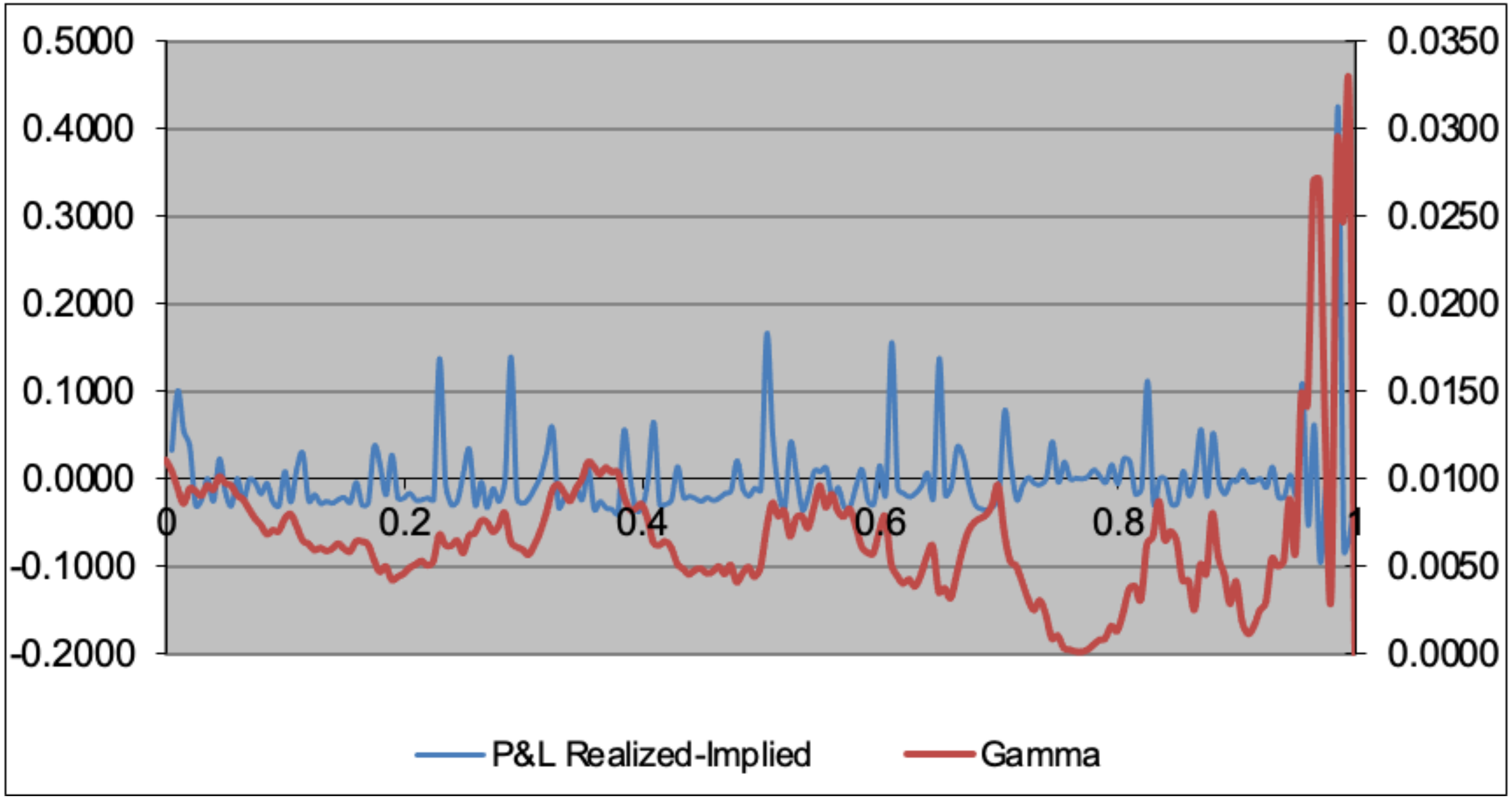


Scenario 1: asset raises, and call is deep ITM, Black-Scholes Gamma goes to near-zero.



Scenario 2: deep ITM, ATM and near-expiry situations for Black-Scholes Gamma

$$\text{Total P\&L} \approx \sum_t^T \frac{1}{2} \Gamma_t S_t^2 \left[r_t^2 - \sigma_{t,imp}^2 \Delta t \right]$$



However, it is difficult to trace relationship over dt.

The formula suggests Vol Arb needs a high Dollar Gamma to win…
Is that a good guess?

Summary I. Should You Hedge With Implied…

Q: Do you hedge with implied volatility or actual volatility?

A: Actual, but there are strong assumptions on what we know about it, eg timescale, relationship to real variables (*S, t)*.

Q: If have a good estimate of volatility, can I enter volatility arbitrage.

A: Only in the situation when you have a good reason (forecast) that Realized will turn out above Implied. "Market is wrong".

Q: What about the market-makers?

A: The party that sells an option benefits from Implied – Realized.

Implied vs Realized

S&P500 Index Level vs Volatility (short-term std dev)

Summary II. Trading in Implied Volatility

Q: What have we learned about Implied vs Realized? (Bloomberg plots)

A: Implied volatility trends usually **above** Realized volatility. Options are sold at the IV price level (eg, 25% input to Black-Scholes formulae, when realized can be 18%), expensive!

Q: When can we enter the volatility arbitrage of Case 1 (collect from the market the premium equal to V_a dollar value.

A: For the arbitrage opportunity to exist we need Realized volatility to turn out (in future, and over the time) above the Implied, at least 2-3%.

That is not the frequent scenario, we learned.

Summary III. Volatility Arb

“It sure is the hell of a lot easier to just be first.” to quote from *Margin Call* movie.

Q: Are there other patterns about Implied Volatility?

A: Peaks in implied vol are followed by the periods of increased realized volatility – possible to run delta-replication (volatility arb) but one has to be quick to enter.

What is considered a peak depends on historical calibration:

- VIX at 35 happens once per few years (can occur twice in sequence).
- VIX at 85 is GFC 2008 or Pandemic 2020 event (supposed to be one-in-hundred years event but actually occurs far more often, even more often than 20 years).

Update September 2022

S&P 500 daily return (high frequency if considering 30 years) – top.

VIX (30-day **forward volatility**) – bottom.

Source: Broadgate Advisers (Bernard Renaud), Bloomberg data.