Optimal Delta Hedging for Options

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First version: September 9, 2015
This version: April 25, 2016

ABSTRACT

The "practitioner Black-Scholes delta" for hedging options is a delta calculated from the Black-Scholes-Merton model (or one of its extensions) with the volatility parameter set equal to the implied volatility. As has been pointed out by a number of researchers, this delta does not minimize the variance of changes in the value of a trader's position. This is because there is a non-zero correlation between movements in the price of the underlying asset and implied volatility movements. The minimum variance delta takes account of both price changes and the expected change in implied volatility conditional on a price change. This paper determines empirically a model for the minimum variance delta. We test the model using data on options on the S&P 500 and show that it is an improvement over stochastic volatility models, even when the latter are calibrated afresh each day for each option maturity. We present results for options on the S&P 100, the Dow Jones, individual stocks, and commodity and interest-rate ETFs.

Key words: Options, delta, vega, stochastic volatility, minimum variance

*We thank Peter Carr, Peter Christoffersen, Tom Coleman, Emanuel Derman, Bruno Dupire, Andrew Lesniewski, Andrei Lyashenko, Tom McCurdy, Massimo Morini, Michael Pykhtin, Lorenzo Ravagli, and seminar participants at University of Toronto, the Fields Institute, the Global Risk Institute, Bloomberg, FinCAD, and the 2015 RiskMinds International conference for helpful comments. We are grateful to the Global Risk Institute in Financial Services for funding. Earlier versions of the paper were circulated with titles "Optimal Delta Hedging" and Optimal Delta Hedging for Equity Options"

Optimal Delta Hedging for Options

I. Introduction

The textbook approach to managing the risk in a portfolio of options involves specifying a valuation model and then calculating partial derivatives of the option prices with respect to the underlying stochastic variables. The most popular valuation models are those based on the assumptions made by Black and Scholes (1973) and Merton (1973). When hedge parameters are calculated from these models, the usual market practice is to set each option's volatility parameter equal to its implied volatility. This is sometimes referred to as using the "practitioner Black-Scholes model." The "practitioner Black-Scholes delta" for example is the partial derivative of the option price with respect to the underlying asset price with other variables, including the implied volatility, kept constant.

Delta is by far the most important hedge parameter and fortunately it is the one that can be most easily adjusted by trading the underlying asset. Ever since the birth of exchange-traded options markets in 1973, delta hedging has played a major role in the management of portfolios of options. Option traders adjust delta frequently, making it close to zero, by trading the underlying asset.

Even though the Black-Scholes-Merton model assumes volatility is constant, market participants usually calculate a "practitioner Black-Scholes vega" to measure their volatility exposure and for hedging this exposure. This vega is the partial derivative of the option price with respect to implied volatility with all other variables, including the asset price, kept constant.[1] This approach, although not based on an internally consistent model, has the advantage of simplicity and can lead to effective hedging. This is because the price of an option at any given time is, to a good approximation, a deterministic function of the underlying asset price and the implied

[1] In a portfolio of options dependent on a particular asset, the options typically have different implied volatilities. The usual practice when vega is calculated is to calculate the portfolio vega as the sum of vegas of the individual options. This is equivalent to considering the impact of a parallel shift in the volatility surface.

volatility.[2] A Taylor series expansion shows that changes in the price are hedged if the impact of changes in these two variables are hedged. However, vega is less easy to adjust than delta because this requires trades in more complex products such as options or volatility swaps.

As is well known, there is a negative relationship between an equity price and its volatility. Christie (1982) showed this is true when volatility estimates are based on equity return data. Other authors have shown that it is true when implied volatility estimates are used. One model explaining the negative relationship in terms of leverage was proposed by Geske (1979). In this model, the value of the assets of a company has constant volatility. As the equity price moves up (down), leverage decreases (increases) and as a result volatility decreases (increases). An alternative hypothesis, known as the volatility feedback effect, is considered by, for example, French *et al* (1987), Campbell and Hentschel (1992) and Bollerslev *et al* (2006). In this, the causality is the other way round. When there is an increase (decrease) in volatility, the required rate of return increases (decreases) causing the stock price to decline (increase).

A number of researchers have recognized that the negative relationship between an equity price and its volatility means that the Black-Scholes delta does not give the position in the underlying equity that minimizes the variance of the hedger's position. The minimum variance (MV) delta hedge takes account of the impact of both a change in the underlying equity price and the expected change in volatility conditional on the change in the underlying equity price. Given that delta hedging is relatively straightforward, it is important that traders get as much mileage as possible from it. Switching from the practitioner Black-Scholes delta to the minimum variance delta is therefore a desirable objective. Indeed it has two advantages. First, it lowers the variance of daily changes in the value of the hedged position. Second, it lowers the residual vega exposure because part of vega exposure is handled by the position that is taken in the underlying asset.

A number of stochastic volatility models have been suggested in the literature. These include Hull and White (1987, 1988), Heston (1993), and Hagan *et al* (2002). A natural assumption might be that using a stochastic volatility model automatically improves delta. In fact, this is not the case if delta is calculated in the usual way, as the partial derivative of the option price with respect to the asset price. To calculate the MV delta, it is necessary to use the model to

[2] This is exactly true if we ignore uncertainties relating to interest rates and dividends.

determine the expected change in the option price arising from both the change in the underlying asset and the associated expected change in its volatility.

A number of researchers have implemented stochastic volatility models and used the models' assumptions to convert the usual delta to an MV delta. They have found that this produces an improvement in delta hedging performance, particularly for out-of-the-money options. The researchers include Bakshi *et al* (1997) who implemented three different stochastic volatility models using data on call options on the S&P 500 between June 1988 and May 1991[3]; Bakshi *et al* (2000), who looked at short and long-term options on the S&P 500 between September 1993 and August 1995; Alexander and Nogueira (2007), who looked at call options on the S&P 500 during a six month period in 2004; Alexander *et al* (2009), who consider the hedging performance of six different models using put and call options on the S&P 500 trading in 2007; and Poulsen *et al* (2009) who looked at data on S&P 500 options, Eurostoxx index options, and options on the U.S. dollar euro exchange rate during the 2004 to 2008 period. Bartlett (2006) shows how a minimum variance hedge can be used in conjunction with the SABR stochastic volatility model proposed by Hagen *et al* (2002).

This paper is different from the research just mentioned in that it is not based on a stochastic volatility model. It is similar in spirit to papers such as Crépey (2004), Vähämaa (2004) and Alexander *et al* (2012) which consider different ways in which the Black-Scholes delta can be adjusted to reflect the volatility smile. These authors note that minimum variance delta is the Black-Scholes delta plus the Black-Scholes vega times the partial derivative of the expected implied volatility with respect to the asset price. Improving delta therefore requires an assumption about the partial derivative of the expected implied volatility with respect to the asset price. Crépey (2004) and Vähämaa (2004) test setting the partial derivative equal to (or close to) the (negative) slope of the volatility smile.[4] Alexander *et al* (2012) build on the research of Derman (1999) and test eight different models for the partial derivative, including a number of regime-switching models.

[3] They also looked at puts on the S&P 500, but did not report the results as they were similar to calls.
[4] As discussed by Derman *et al* (1995) and Coleman *et al* (2001), this assumption corresponds to the local volatility model of Dupire (1994). We explain this later.

This paper extends previous research by determining empirically a model for the partial derivative of the expected implied volatility with respect to asset price. We look at data on a wide range of assets over a long period of time. Specifically, we use data on options on three stock indices, a number of individual stocks, and ETFs based on commodity and bond prices over a twelve-year period. Moneyness is measured by the Black-Scholes delta. We show that to a good approximation the partial derivative of an option's expected implied volatility with respect to the asset price is a quadratic function of the delta of the option divided by the product of the asset price and the square root of the time to maturity. This leads to a simple model where the MV delta is calculated from the practitioner Black-Scholes delta, the practitioner Black-Scholes vega, the asset price, and the time to maturity. We show that the hedging gain is better than that obtained using a stochastic volatility model or a local volatility model. The results have practical relevance to traders, many of whom still base their decision making on output from the practitioner Black-Scholes model. The results also lend support to the volatility feedback effect explanation for the negative relation between equity prices and implied volatility, mentioned earlier.

The structure of the rest of the papers is as follows. We first discuss the nature of the data that we use. Second, we develop the theory that allows us to parameterize the evolution of the implied volatilities of options. The theory is implemented and tested out-of-sample using options on the S&P 500, which are very actively traded. The results are compared with those from a stochastic volatility and a local volatility model. Based on the results for the S&P 500 we then carry out tests for options on other indices and for options on individual stocks and ETFs.

II. Data

We used data from OptionMetrics. This is a convenient data source for our research. It provides daily prices for the underlying asset, closing bid and offer quotes for options, and hedge parameters based on the practitioner Black-Scholes model. We chose to consider options on the S&P 500, S&P 100, the Dow Jones Industrial Average of 30 stocks (DJIA), the individual stocks underlying the DJIA and five ETFs. The assets underlying three of the ETFs are commodities, gold (GLD), silver (SLV) and oil (USO). The assets underlying the other two ETFs were the

Barclays U.S. 20+ year Treasury Bond Index (TLT) and the Barclays U.S. 7-10 year Treasury Bond Index (IEF). The options on the S&P 500 and the DJIA are European. Both European and American options on the S&P 100 are included in our data set. Options on individual stocks and those on ETFs are American. The period covered by the data we used is January 2, 2004 to August 31, 2015 except for the commodity ETFs where data was first available in 2008.[5]

Only option quotes for which the bid price, offer price, implied volatility, delta, gamma, vega, and theta were available were retained. The option data set was sorted to produce observations for the same option on two successive trading days. For every pair of observations the data was normalized so that the underlying price on the first of the two days was one. Options with remaining lives less than 14 days were removed from the data set. Call options for which the practitioner Black-Scholes delta was less than 0.05 or greater than 0.95, and put options for which the practitioner Black-Scholes delta was less than –0.95 or greater than –0.05 were removed from the data set. For options on individual stocks, in addition to the filters used for options on the indices, days on which stock splits occurred were removed.

Table 1 shows the number of price quotations for options of different maturities and moneyness after all filtering. The degree of moneyness is measured by the option delta.[6] The option deltas in Table 1 are rounded to the nearest tenth. Table 2 shows the volume of trading by option delta. The trading volume for puts on the S&P 500 is much greater than that for calls.[7] Puts and calls trade in approximately equal volumes for other indices. Calls trade more actively than puts for the individual stocks. Trading tends to be concentrated in close-to-the-money and out-of-the-money options. One notable feature of Table 2 is that the trading of close-to-the-money call options is particularly popular. Although not evident from Table 2, the majority of trading is in options with maturities less than 91 days.

[5] This is a much longer period than that used by other researchers except Alexander *et al* (2012).

[6] This measure of moneyness is commonly used by traders. An "at-the-money option" is considered to be a call with $\delta_{BS} = 0.5$ or put with $\delta_{BS} = -0.5$. An advantage of this measure over K/S (where K is the strike price and S is the equity price) is that it takes account of the life of the option. A two-week call option where $K/S = .1$ is more out of the money that a two-year call option with the same value of K/S.

[7] The bid-offer spread for puts on the S&P 500 is smaller than that for calls except in the case of deep in-the-money options where the spreads are about the same.

III. Background Theory

Define ΔS as the change in an asset price over one trading day and Δf as the change in the price of an option on the asset during this period. The minimum variance delta, δ_{MV}, is the value that minimizes the variance of [8]

$$\Delta f - \delta_{\mathrm{MV}} \Delta S \tag{1}$$

The dependence of the price of an option on uncertainties associated with interest rates and dividends is usually very small. The price of a European-style option, *f*, on an asset whose price is *S* can therefore be approximated as

$$f = f_{\mathrm{BS}}\left(S, \sigma_{\mathrm{imp}}\right)$$

where f_{BS} is the Black-Scholes-Merton pricing function and σ_{imp} is the implied volatility.

The MV delta, δ_{MV}, takes into account both changes in *S* and expected changes in σ_{imp} as a result of changes in *S*. This leads to

$$\delta_{\mathrm{MV}} = \frac{\partial f_{\mathrm{BS}}}{\partial S} + \frac{\partial f_{\mathrm{BS}}}{\partial \sigma_{\mathrm{imp}}} \frac{\partial E\left(\sigma_{\mathrm{imp}}\right)}{\partial S} = \delta_{\mathrm{BS}} + \nu_{\mathrm{BS}} \frac{\partial E\left(\sigma_{\mathrm{imp}}\right)}{\partial S} \tag{2}$$

where ν_{BS} is the practitioner Black-Scholes vega and $E(\sigma_{imp})$ is the expected value of the implied volatility as a function of *S*. Other authors, in particular Alexander *et al* (2012), have explored the effectiveness of various estimates $\partial E(\sigma_{imp})/\partial S$ in determining the minimum variance delta. In what follows we estimate this function empirically and then conduct out of sample tests of the effectiveness of the estimated function. The tests are carried out on European and American stock index options, options on individual stocks, options on commodity ETFs, and options on interest rate ETFs.

When presenting our results, we shall define the effectiveness of a hedge as the percentage reduction in the sum of the squared residuals resulting from the hedge. We denote the Gain from

[8] An early application of this type of hedging analysis to futures markets is Ederington (1979)

an MV hedge as the percentage increase in the effectiveness of an MV hedge over the effectiveness of the Black-Scholes hedge. Thus:

$$\text{Gain} = 1 - \frac{SSE\left[\Delta f - \delta_{\text{MV}} \Delta S\right]}{SSE\left[\Delta f - \delta \Delta S\right]} \tag{3}$$

where SSE denotes sum of squared errors.

IV. Exploratory Analysis of S&P 500 Options

In this section we explore the characteristics of the MV delta for options on the S&P 500 with the objective of determining the functional form of the MV delta. Once we have a candidate functional form, we test it out of sample for both options on the S&P 500 and options on other assets.

We start our exploration of MV hedging for the S&P 500 options with an implementation of equation (1).

$$\Delta f = \delta_{\text{MV}} \Delta S + \varepsilon \tag{4}$$

Because the mean of ΔS and Δf are both close to zero, minimizing the variance of ε in this equation, and other similar equations that we will test, is functionally equivalent to minimizing the sum of squared values. Several other variations on the model were tried such as using non-normalized data, replacing Δf with $\Delta f - \theta_{\text{BS}}\Delta t$, where θ_{BS} is the practitioner Black-Scholes theta[9] and Δt is one trading day, or including an intercept. None of the variations had a material effect on the results we present. The results that we report are for the model in (4), or similar models.

We estimated equation (4) for call and put options divided into nine different buckets according to the value of δ_{BS} rounded to the nearest tenth, and seven different option maturities. For each

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[9] The practitioner Black-Scholes theta is the partial derivative with respect to the passage of time with the volatility set equal to the implied volatility) and Δt is one day. If the asset price and its implied volatility do not change, the option price can be expected to decline by about $\theta_{\text{BS}}\Delta t$ in one day.

delta and each maturity bucket the value of δ_{MV} is estimated. The differences between the estimated δ_{MV} and the average δ_{BS} for the buckets are shown in Table 3. In all cases $\delta_{MV} - \delta_{BS}$ is negative. This means that on average, traders of S&P 500 index options should under-hedge call options and over-hedge put options relative to δ_{BS}.[10] Our results are consistent with those of other researchers who find that the MV delta is less than the practitioner Black-Scholes delta.

We now present arguments that δ_{MV} for an option depends only on the moneyness of the option, as measured by δ_{BS}. First, it is reasonable to assume that the option pricing model is scale invariant.[11] This means that the option price depends on the asset price only through its dependence on δ_{BS}. Second, the results in Table 3 show that the MV delta is not particularly sensitive to the option maturity except for very short- and very long-term put options. This means that the option price can be assumed to depend on time only through its dependence on δ_{BS}. Third, the dependence of the option price on dividend yields and interest rates is small. From equation (2), this means that

$$\nu_{\mathrm{BS}} \frac{\partial E(\sigma_{\mathrm{imp}})}{\partial S}$$

is dependent on δ_{BS}, but can otherwise be assumed to be independent of S and T.

The practitioner Black-Scholes vega, ν_{BS}, of a European option is, to a reasonable approximation, given by[12]

$$\nu_{\mathrm{BS}} = S\sqrt{T}G\left(\delta_{\mathrm{BS}}\right)$$

[10] A call has a positive delta and the MV delta, δ_{MV}, is less positive than δ_{BS}; a put has a negative delta and δ_{MV} is more negative than δ_{BS}.

[11] A scale invariant model is one where the distribution of S_t / S_0 is independent of S_0. See for example Alexander and Nogueira (2007).

[12] For European options, $\nu_{\mathrm{BS}} = S\sqrt{T}N'\left(d_1\right)e^{-qT}$ where $d_1 = \left[\ln(S/K) + (r - q + \sigma^2/2)T\right] \big/ \sigma\sqrt{T}$, K is the strike price, T is the time to maturity, r is the risk-free rate, q is the dividend yield, and N is the cumulative normal distribution function. However, $\delta_{\mathrm{BS}} = N(d_1)e^{-qT}$ so that $d_1 = N^{-1}(\delta_{\mathrm{BS}}e^{qT})$. As a result, $\nu_{\mathrm{BS}} = S\sqrt{T}N'\left(N^{-1}(\delta_{\mathrm{BS}}e^{qT})\right)e^{-qT}$. Given that q is small (less than 3%), this shows that $\nu_{\mathrm{BS}} \big/ \left(S\sqrt{T}\right)$ is approximately dependent only on δ_{BS}.

for some function G where T is the time to the option maturity. It follows that the expected rate of change of the implied volatility with respect to changes in the stock price must have the form

$$\frac{\partial E\left(\sigma_{\text{imp}}\right)}{\partial S}=\frac{H\left(\delta_{\text{BS}}\right)}{S\sqrt{T}} \tag{5}$$

The results in Table 3 suggest that a single quadratic function for H is likely to provide a good fit across all delta buckets and maturities. Equation (2) then becomes

$$\delta_{\text{MV}}=\delta_{\text{BS}}+\nu_{\text{BS}}\frac{a+b\delta_{\text{BS}}+c\delta_{\text{BS}}^{2}}{S\sqrt{T}}$$

and equation (4) becomes

$$\Delta f=\delta_{\text{BS}}\Delta S+\frac{\nu_{\text{BS}}}{\sqrt{T}}\frac{\Delta S}{S}\left(a+b\delta_{\text{BS}}+c\delta_{\text{BS}}^{2}\right)+\varepsilon \tag{6}$$

From equation (5), the expected volatility surface change for a proportional change in S is given by

$$E\left(\Delta\sigma_{\text{imp}}\right)=\left(\frac{a+b\delta_{\text{BS}}+c\delta_{\text{BS}}^{2}}{\sqrt{T}}\right)\frac{\Delta S}{S} \tag{7}$$

For a given value of δ_{BS} the response of the implied volatility to a percentage change in the stock price is inversely proportional to the square root of the time to maturity. Similar "square root of time rules" have been mentioned elsewhere.[13]

To this point our work has been largely descriptive, motivated by a desire to produce a simple model of how the volatility surface for S&P 500 options evolves as a result of stock price changes. Our simple model is that for a particular moneyness and a particular stock price change, the expected size of the change in the implied volatility is inversely proportional to the square-root of the option life. For a particular option maturity and a particular stock price change, the

[13] See for example Hull (2015, p439) which explains that some traders choose to incorporate the square root of time rule in estimates of the volatility smile and Daglish *et al* (2007) which provides some empirical support for the square root of time rule.

expected size of the change in the implied volatility is the option vega multiplied by a quadratic function of our measure of moneyness, δ_{BS}. The same model applies across the range of deltas considered.

V. Out of Sample Tests of S&P 500 Options

We now consider whether our simple model of the evolution of the volatility surface can be used to improve hedging performance. That is, can we estimate the MV delta using historical data and then use that estimate to reduce the variance of the hedging error in the future. In carrying out this test we will use a moving window where parameters are estimated over a 36-month period and then used to determine MV hedges during the following month. The first month for which MV hedges are estimated is January 2007 and the last is August 2015. We tested moving windows of length between 12- and 60-months but did not find that any one of these was materially better than the others.[14]

The only element of our simple model that is unknown is the quadratic function of moneyness in equation (6), our H function. We estimate the model parameters, a, b and c, using equation (6) fitted to all options in each estimation period. The estimation is done separately for puts and calls. The estimated coefficients, a, b and c, are shown in Figure 1. Usually, the parameters of the best fit quadratic model change slowly through time, but during the credit crisis of 2008 some extreme changes were observed.

The Gain (equation (3)) resulting from using our model to hedge for the next month is then calculated using the estimated parameters. On average the Gain is about 17% for calls and 11% for puts. We also calculated the Gain for each delta bucket. The average Gain achieved for each delta bucket and the standard error of the estimate are shown in Table 4. This shows that for call options the Gain is largest for out-of-the-money options (a Gain of about 32% for the highest

[14] In all our reported results we consider one day changes in option prices and implied volatilities when estimating the MV hedge parameters. Slightly better results occur if the observation period is increased to several trading days.

strike options) and smallest for in-the-money options. For put options the Gains are more uniformly distributed across option strikes.

Figures 2 and 3 show the value of $\delta_{MV} - \delta_{BS}$, and the expected change in the implied volatility, that is estimated for calls and puts based on the average three-year calibration results for the period January 2012 to July 2015.[15] (This period was chosen to avoid using data from the credit crisis.) Put-call parity shows that $\delta_{MV} - \delta_{BS}$ and the expected change in implied volatility should be the same for a call and a put when they have the same strike price and time to maturity.[16] In Figures 2 and 3, calls and puts with the put δ_{BS} equal to the call δ_{BS} minus one are compared. This means that calls and puts with approximately the same strike prices are being compared.[17] In the case of Figure 2 the comparison is across a range of maturities.

We confined our hedging effectiveness test to options with maturities greater than 13 days. This eliminates very short term options. Including the very short maturity options worsens our results due to the large gammas of short-term options that are close to the money, but does not eliminate them.

MV hedging works better for calls than puts and better for out-of-the-money options than in-the-money options. To understand why this is the case we directly estimate the relationship between implied volatility changes and stock price changes by estimating α in

$$\Delta\sigma_{\text{imp}} = \frac{\alpha}{\sqrt{T}} \frac{\Delta S}{S} + \varepsilon \tag{8}$$

The estimation is done separately for puts and calls for every delta bucket using all options observed between 2004 and 2015. The R^2 for each delta bucket is shown in Figure 4.

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[15] As explained in the previous section both $\delta_{MV} - \delta_{BS}$ and the rate of change of implied volatility with respected to the proportional change in the stock price are dependent only on δ_{BS} to a good approximation.

[16] Define $\delta_{MV,c}$ and $\delta_{MV,p}$ as the minimum variance call and put deltas, and $\delta_{BS,c}$ and $\delta_{BS,p}$ as the Black-Scholes call and put deltas. From the put call parity equation and equation (2) it follows that $\delta_{MV,c} - \delta_{MV,p}$ and $\delta_{BS,c} - \delta_{BS,p}$ both equal e^{-qT} where q is the dividend yield and T is the time to maturity. Subtracting the two equations shows that $\delta_{MV,c} - \delta_{BS,c} = \delta_{MV,p} - \delta_{BS,p}$.

[17] Define q as the dividend yield and T as the time to maturity. Calls and puts have the same strike price if the put delta equals the call delta minus e^{-qT}. They have only approximately the same strike price when the put delta equals the call delta minus one.

The average R^2 from the estimation in equation (8) for calls with δ_{BS} in the 0.1 bucket is about 0.60. That is, the change in the implied volatility due to changes in the stock price explains about 60% of the total variation in implied volatilities. As δ_{BS} increases the average R^2 declines due to increased idiosyncratic noise in the implied volatility data. This explains the effectiveness of MV hedging for out-of-the money calls and the declining effectiveness of MV hedging for in-the-money calls.

The results for put options are somewhat different. The fraction of total variance in implied volatilities explained by changes in the stock price is much smaller than that observed for call options. There is much more idiosyncratic variation in the implied volatilities of put options. As a result, MV hedging of puts is much less effective than for calls.

A Put-Call Parity Test

We used our quadratic model to test how well put-call parity has held over the period covered by our data. We first used the put-call parity relationship to turn all call prices in our data set into synthetic put prices. We estimated a, b, and c in our quadratic form using the actual put prices, and $\hat{a}$, $\hat{b}$, and $\hat{c}$ using the synthetic put prices for each of our three-year calibration periods. We then calculated the root mean square error of the difference between the estimated put parameters and the put parameters calculated from the synthetic put data under the assumption that put-call parity holds:

$$RMSE = \sqrt{\frac{(a-\hat{a})^2 + (b-\hat{b})^2 + (c-\hat{c})^2}{3}} \tag{9}$$

The results are shown in Figure 5. These results suggest that put-call parity was seriously violated before December 2008 but that thereafter it was approximately true. (The first observation of the post-December 2008 period is December 2011.)

VI. Comparison with Alternative Models

In the previous section we tested an empirical model to determine the minimum variance delta hedge. The results show that a reasonable improvement in hedging accuracy can be achieved in this way. However, as mentioned earlier, other researchers have calculated minimum variance deltas from stochastic volatility models and local volatility models. In this section we compare the performance of our empirical model with these two categories of models.

Stochastic Volatility Model

The stochastic volatility model we use is a particular version of the SABR model discussed by Hagen *et al.* (2002):[18]

$$\begin{aligned} dF &= \sigma F dz \\ d\sigma &= \xi \sigma dw \end{aligned} \tag{10}$$

where F is the futures stock price when the numeraire is the zero coupon bond with Maturity T. The dz and dw are Wiener processes with constant correlation ρ and ξ is a constant volatility of volatility parameter. In this model the expected change in the volatility given a particular change in the futures price is

$$E\left(d\sigma \middle| dF\right) = \xi\rho\frac{dF}{F}$$

Hagan *et al* (2002) and Rebonato *et al* (2009) show that under the model defined by equation (10), a good approximation to the implied volatility for an option with strike price K and time to maturity T is

$$\sigma_{\text{imp}}\left(S_0, \sigma_0\right) = \sigma_0\left(\frac{y}{\chi(y)}\right)\left\{1 + \left[\frac{\rho\sigma_0\xi}{4} + \frac{2-3\rho^2}{24}\xi^2\right]T\right\} \tag{11}$$

[18] As pointed out by Poulsen *et al* (2009), similar results are obtained for different stochastic volatility models. In the general SABR model $dF = \sigma F^{\beta} dz$. Setting β=1 ensures scale invariance which is a reasonable property for equities and equity indices. The model we choose is equivalent to a version of the model in Hull and White (1987).

where

$$y = \frac{\xi}{\sigma_0} \ln \frac{F_0}{K}$$
$$\chi(y) = \ln\left(\frac{\sqrt{1-2\rho y + y^2} + y - \rho}{1-\rho} \right)$$

and F_0 and σ_0 are the initial values of the futures stock price and the stochastic volatility respectively.

Define $f_{\mathrm{BS}}(F,\sigma)$ as the value of an option given by the Black-Scholes-Merton assumptions when the futures stock price is F and the volatility is σ. If f is the value of an option, an estimate of the minimum variance delta given by the model is then

$$\delta_{\mathrm{SV}} = \frac{E\left(\Delta f \mid \Delta F\right)}{\Delta F} = \frac{f_{\mathrm{BS}}\left(F_0 + \Delta F, \sigma_{\mathrm{imp}}\left(F_0 + \Delta F, \sigma_0 + \xi\rho\Delta F/F\right)\right) - f_{\mathrm{BS}}\left(F_0, \sigma_{\mathrm{imp}}\left(F_0, \sigma_0\right)\right)}{\Delta F} \quad (12)$$

The procedure for implementing this model is as follows. On each trading day the implied volatilities of all options with a particular maturity are determined.[19] The parameters for the stochastic volatility model (σ_0, ξ, and ρ) that are to be used for that particular option maturity are chosen to minimize the sum of squared differences between the market implied volatilities and the model implied volatilities given by equation (11).[20] Once the model parameters are determined for the particular maturity, the minimum variance delta is then determined for each option with that maturity using equation (12). This procedure is repeated for every maturity observed on each trading day.

To align the tests of the stochastic volatility model with the tests of our empirical model we calibrated the model for every option maturity every day from the start of 2007 to August 2015. Puts and calls were considered separately. Since there are about 13 different maturities observed

[19] In practice the SABR model is used as a model for the behavior of all options with a particular maturity. When calibrated to all options of all maturities we find that it provides poor results. This is not surprising as the model is not designed to fit the term structure of implied volatilities.
[20] For a particular maturity to be included in our sample on any day we require that there be options with more than 10 different strike prices and that the root mean square error in fitting the implied volatilities be smaller than 1%.

on each trading day we are estimating about 78 model parameters on each trading day. In total, about 29,000 optimizations are carried out and about 87,000 model parameters are estimated. The estimated parameters are reasonable and provide a good fit to the observed implied volatilities. The average initial volatility, σ_0, is about 19% which is approximately equal to the average at-the-money option implied volatility, the average volatility of the volatility, ξ, is about 1.2, and the average correlation is about –0.85 while the root mean square error in fitting the implied volatility is about 0.32%.

Table 5 compares the Gain from the SABR model with the Gain from the empirical model developed in this paper. The results are aggregated by Black-Scholes delta rounded to the nearest tenth. The table shows, the stochastic volatility model is materially worse at reducing hedging variance than is our empirical model. To put this in perspective recall that in our empirical approach we estimate only the three coefficients of the quadratic function (equation (6)) and update the estimate once month, a total of 104 calibrations and 312 parameters estimated compared to the nearly one hundred thousand parameters of the SABR model. Overall the SABR model performs less well than our empirical model. Its performance is better than the empirical model only for very-deep-in-the-model options. For options which trade actively, the empirical model is clearly better.

Local Volatility Model

The slope of the volatility smile plays a key role determining the partial derivative of the expected implied volatility with respect to the asset price for the local volatility model. To understand this, suppose that the current futures price of an asset for a contract with maturity T is F_0. Define $c\left(F_0, K, r, T \middle| \Omega\right)$ and $p\left(F_0, K, r, T \middle| \Omega\right)$ as the prices of European call and European put futures options with strike price K and maturity T. The variable Ω denotes the stochastic process followed by the futures price when the numeraire is a zero-coupon bond price with maturity T. The risk-free rate for maturity T, r, is the yield on the numeraire zero coupon bond.

The call option is equivalent to a European put option with maturity T to sell the strike price for the futures price. If we use the futures price, F, as the numeraire, the equivalent put option can be seen to be F_0 times a put option to sell K/F for 1. It follows that[21]

$$c\left(F_0, K, r, T \mid \Omega\right) = F_0 p\left(\frac{K}{F_0}, 1, r, T \mid \Omega\right) \tag{13}$$

A position in F_0 options to sell K/F for 1 is equivalent to a position in one option to sell K for F. Hence equation (13) becomes

$$c\left(F_0, K, r, T \mid \Omega\right) = p\left(K, F_0, r, T \mid \Omega\right) \tag{14}$$

Now one particular assumption about Ω is that it is geometric Brownian motion with volatility σ. We denote this by "GBM, σ". It follows that

$$c\left(F_0, K, r, T \mid \text{GBM}, \sigma\right) = p\left(K, F_0, r, T \mid \text{GBM}, \sigma\right) \tag{15}$$

The implied volatility for the call is the value of σ that equates the right hand side of equation (15) with the right hand side of equation (14). Equations (14) and (15) imply that this value of σ also equates the left hand side of equation (15) with the left hand side of equation (14). It follows that the implied volatility of the call option in equation (14) is the same as the implied volatility of the put option in equation (14). Because the implied volatility of a European call is the same as the implied volatility of a European put, it follows that

$$\sigma_{\text{imp}}\left(F_0, K, r, T \mid \Omega\right) = \sigma_{\text{imp}}\left(K, F_0, r, T \mid \Omega\right) \tag{16}$$

where $\sigma_{\text{imp}}\left(F_0, K, r, T \mid \Omega\right)$ is the implied volatility of a European option with parameters, F_0, K, r, and T. Note that we have not yet made any assumptions about Ω. Equations (14) and (16) are identities true for all stochastic processes Ω.

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[21] One way to understand equation (13) is to suppose that we are dealing with a foreign currency option that can be regarded as a call to buy one unit of currency B for K units of currency A or a put to sell K units currency A for one unit of currency B. As a first step, the call option is valued in currency A and the put option is valued in currency B.

If a local volatility model of the form

$$dF = \sigma(F,t)Fdz$$

is assumed, the implied volatility must be a deterministic function of F. Similarly to Coleman (2001), we can then differentiate equation (16) with respect to F_0 to obtain[22]

$$\frac{\partial \sigma_{\text{imp}}\left(F_0, K, r, T \middle| \Omega\right)}{\partial F_0} = \frac{\partial \sigma_{\text{imp}}\left(K, F_0, r, T \middle| \Omega\right)}{\partial F_0} \tag{17}$$

The left hand side is the partial derivative of the implied volatility with respect to the futures price for the option under consideration. The right hand side is the partial derivative of the implied volatility with respect to the strike price for an imaginary option where today's futures price is the strike price, K, of the option we are considering and the strike price is today's futures price. For an at-the-money option where $K = F_0$, equation (17) shows that the partial derivative of the implied volatility with respect to the futures price equals the slope of the volatility smile.[23]

We now apply this result to the S&P 500 spot options we are considering. Because the futures contract has maturity T, a call (put) option on spot with maturity T and strike K is the same as a call (put) futures option with maturity T and strike K. Suppose that the current value of the index is S_0 and the dividend yield for maturity T is q. The futures option is at-the-money when $F_0 = K$. Because $F_0 = S_0e^{(r-q)T}$ the spot option is at the money when $K = S_0e^{(r-q)T}$. The partial derivative of the expected implied volatility with respect to S_0 equals $e^{(r-q)T}$ times the partial derivative with respect to F_0. It follows that, for an option where $K = S_0e^{(r-q)T}$, the partial derivative of the implied volatility with respect to S_0 equals the slope of the volatility smile times $e^{(r-q)T}$. We assume this result is approximately true for other options which are not at the money. This is

[22] The subsequent analysis is valid if the weaker result

$$\frac{\partial E\left[\sigma_{\text{imp}}\left(F_0, K, r, T \middle| \Omega\right)\right]}{\partial F_0} = \frac{\partial E\left[\sigma_{\text{imp}}\left(K, F_0, r, T \middle| \Omega\right)\right]}{\partial F_0}$$

where E denotes expected value holds. This result may be a reasonable approximation for some stochastic volatility models. The analysis given is also valid for American-style futures options.

[23] Interestingly, for American futures options, a similar argument shows that the partial derivative of an at-the-money call equals the slope of the volatility smile for puts at the at-the-money point and vice versa.

equivalent to the assumption that (a) the volatility smile is linear and (b) the volatility smile exhibits parallel shifts. These two assumptions are approximately, but not exactly, true.

We find that a quadratic gives an excellent fit to the implied volatility smile for a particular maturity. We therefore determined the slope of smile model for each maturity on each day by fitting a quadratic function to the smile and using it to determine the slope of the smile when $K = S_0e^{(r-q)T}$. The results are shown in Table 5.[24] The results are clearly worse than for the empirical model. The slope-of-smile results for out-of-the money call options are better than those for the stochastic volatility model, but they are worse for at-the-money and in-the-money options. For put options the results are generally bad, presumably because of the idiosyncratic variation in observed prices mentioned earlier.

VII. Results for Other Stock Indices

We now return to a consideration of the empirical model and test how well it works for other stock indices. Specifically, we consider European (ticker XEO) and American (ticker OEX) options on the S&P 100, and European options on the Dow Jones Industrial Index (ticker DJX). We carry out out-of-sample tests similar to those done on the S&P 500. The two contracts on the S&P 100 are the same except for exercise terms. They therefore allow us to explore the degree to which hedging differs for American options.

The out of sample test was based on estimating the three parameters of the quadratic function in equation (6) using options of all strikes and maturities. The model parameters were estimated using a 36-month estimation period and the three estimated parameters were then used to delta hedge for a one-month testing period. The Gain (equation (3)) resulting from using our model to hedge in the test period is then calculated. The average Gain achieved for puts and calls in each delta bucket is shown in Table 6.

The results for call options for all indices are essentially the same as those found for options on the S&P 500. It is tempting to think that the results for the American style (OEX) call options are

[24] We experimented with other implementations of the slope-of-smile model but did not obtain better results.

the same as those for the European style options because American style call options are almost never exercised early and hence are effectively European. However, for more than 80% of the sample tested the S&P 100 dividend yield is more than 1.5% higher than the interest rate.[25] In these circumstances the probability of early exercise is high. As a result, it appears that the Americanness of the option does not affect the hedging effectiveness of our rule of thumb.

The results for put options are a bit more complicated. The results for XEO options were similar to those for options on the S&P 500 and the results for DJX options are similar but weaker. The weaker DJX results may be caused by the fact that there are only 30 stocks in the index which means that there will be more idiosyncratic variation in the implied volatilities.

Our results for in-the-money American (OEX) options are different from our results for all other assets in that the Gain for put options is greater than the Gain for call options. One reason could be that, as previously mentioned, interest rates are much lower than dividend yields for most of the period considered so that the American puts are effectively European. Overall, the conclusion that can be drawn from Table 6 is that our rule of thumb for hedging works at least as well for American options as for European options.

VIII. Results for Single Stocks and ETFs

We repeated the out-of-sample hedging tests based on the quadratic model in equation (6) for each of the thirty individual stocks underlying the DJX and each of the five ETFs. The average hedging variance reduction found in these tests is reported in Table 7.

The average hedging gain for call options on single stocks are similar to but rather smaller than those for options on the Dow Jones Industrial Average. For put options the results are very poor. MV hedging contributes nothing or has a negative effect for puts. To understand why this is the case we carried out the regression in equation (8) for puts and calls for every delta bucket for each of the 30 stocks. The average R^2 across the thirty stocks is shown in Figure 6.

[25] The relevant interest rates were almost invariably at least 1.5% lower than the dividend yield between January 2009 and August 2015.

The behavior of the implied volatilities for call options is consistent with our theoretical development. The average change in the implied volatility as a result of a 1% increase in the stock price for call options is similar to the predicted expected change shown in Figure 3. The R^2 exhibits the same pattern observed in Figure 4 for options on the S&P 500 but is somewhat smaller than that for the index options indicating that the idiosyncratic noise is larger for individual stocks. The increased idiosyncratic noise reduces the MV hedging effectiveness by inserting a wedge between parameters estimated in one period and the parameters that would produce the most effective MV hedge in the following period. The results for puts are quite different. The fraction of the variance of changes in the implied volatility explained by stock price changes, R^2, is essentially zero. As a result we can expect no improvement from MV hedging which is what we see.

The results for the ETFs are divided into results for options on commodities (gold, silver, and oil) and options on interest-rate products (20+ year Treasury Bonds and 7 to 10 year Treasury bonds). The results for options on commodities are similar to those for individual stocks while the results for interest-rate products are similar but weaker. As in most of the stock index results, MV hedging provides a much bigger Gain for call options than put options and the gain is greatest for out-of-the-money options. The negative correlation between price and implied volatility for commodities and interest rate products cannot be explained by a leverage and therefore lends support for the volatility feedback hypothesis mentioned earlier.

IX. Conclusions

Delta is by far the most important Greek letter. It plays a key role in the management of portfolios of options. Option traders take steps to ensure that they are close to delta neutral at least once a day and derivatives dealers usually specify delta limits for their traders. This paper has investigated empirically the difference between the practitioner Black-Scholes delta and the minimum variance delta. The negative relation between price and volatility for equities means that the minimum variance delta is always less than the practitioner Black-Scholes delta. Traders should under-hedge equity call options and over-hedge equity put options relative to the practitioner Black-Scholes delta.

The main contribution of this paper is to show that a good estimate of the minimum variance delta can be obtained from the practitioner Black-Scholes delta and an empirical estimate of the historical relationship between implied volatilities and asset prices. We show that the expected movement in implied volatility for an option on a stock index can be approximated as a quadratic function in the option's delta divided by the square root of time. This leads to a formula for converting the practitioner Black-Scholes delta to the minimum variance delta. When the formula is tested out of sample, we obtain good results for both European and American call options on stock indices. However, the reduction in the variance of the hedged position is greater for call options than for put options. For options on the S&P 500 we find that our model gives better results that either a stochastic volatility model or a model based on the slope of the smile.

Call options on individual stocks and ETFs exhibit the same general behavior as call options on stock indices, but the effectiveness of MV hedging is greatly reduced because there is more noise in the relationship between volatility changes and price changes. For nearly all the assets we considered, the results for put options are much worse than those for call options. In the case of put options on individual stocks and ETFs, the results are particularly disappointing in that virtually none of the variation in changes in implied volatility is explained by changes in stock prices. The relatively poor performance of MV hedging for put options is a puzzle because (a) in the case of the European options considered put-call parity means that puts and calls can be regarded as substitutes for each other and (b) that in the case of American options puts are less likely to be exercised early than call options for most of our sample period. It appears that the reason for the discrepancy between calls and puts is a result of a very high level of idiosyncratic noise in the prices of put options.

The most striking result is the ubiquity of the negative relation between asset price and implied volatilities for call option prices. When asset prices rise, implied volatilities decline resulting in an MV delta that is less than the Black-Scholes delta. For options on equities and equity indices this might be explained by a leverage argument. As equity prices rise the firm becomes less levered and equity volatility declines. However, this argument does not seem to apply to commodity or bond prices. For these assets it seems likely that we have to rely on the volatility feedback effect in which an increase in volatility raises the required rate of return resulting in a stock price decline.

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Table 1

The number of price quotations for options on the S&P 500 (SPX), European options on the S&P 100 (XEO), American options on the S&P 100 (OEX), options on the Dow Jones Industrial Average (DJX), the thirty stocks underlying DJX and five ETFs. Numbers reported for the thirty stocks are averages per stock. Numbers reported for the five ETFs are averages per ETF.

Results are reported by bucket based on the practitioners' Black-Scholes delta rounded to the nearest tenth.

Call Options (Thousands of quotations per underlying asset)

δ_{BS}	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	Total
SPX	153.2	109.6	101.1	102.5	112.4	130.3	159.7	204.7	280.5	1,354.0
XEO	46.1	36.4	35.1	35.5	38.8	44.6	54.8	68.5	90.4	450.1
OEX	37.8	28.9	27.8	28.4	31.4	36.0	44.0	60.5	123.8	418.5
DJX	78.1	56.7	52.5	53.2	57.2	65.3	78.9	93.6	90.0	625.4
30 Stocks	29.2	18.0	15.3	14.5	14.6	15.6	17.8	22.4	34.1	181.5
5 ETFs	89.0	50.7	40.3	36.3	35.0	35.1	36.7	41.4	60.0	424.6

Put Options (Thousands of quotations per underlying asset)

δ_{BS}	–0.9	–0.8	–0.7	–0.6	–0.5	–0.4	–0.3	–0.2	–0.1	Total
SPX	151.1	100.1	95.7	99.5	110.1	127.9	156.0	207.4	381.3	1,429.2
XEO	38.6	30.8	32.3	34.2	37.4	43.4	52.9	72.9	131.0	473.6
OEX	76.9	34.5	29.8	30.4	32.2	36.9	45.3	61.9	111.3	459.4
DJX	82.9	50.2	47.8	49.9	56.1	63.5	76.7	101.7	162.7	691.4
30 Stocks	25.8	18.6	15.4	14.6	14.9	16.2	18.9	24.6	45.2	194.3
5 ETFs	93.5	60.0	45.4	40.2	39.0	39.2	41.4	47.9	77.8	484.4

Table 2

Volume of trading for options on the S&P 500 (SPX), European options on the S&P 100 (XEO), American options on the S&P 100 (OEX), options on the Dow Jones Industrial Average (DJX), the thirty stocks underlying DJX, and five ETFs. Numbers reported for the thirty stocks are averages per stock. Numbers reported for the five ETFs are averages per ETF.

Results are reported by bucket based on the practitioners' Black-Scholes delta rounded to the nearest tenth.

Call Options (Millions of contracts per underlying asset)

δ_{BS}	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	Total
SPX	75.8	60.4	53.8	53.4	110.0	41.5	12.7	5.9	3.8	417.2
XEO	0.7	0.3	0.2	0.5	1.4	0.6	0.1	0.1	0.1	4.1
OEX	4.6	3.5	2.9	2.7	2.8	1.8	0.7	0.4	0.4	19.7
DJX	1.4	1.9	2.5	3.0	4.1	2.2	1.0	0.8	0.9	17.9
30 Stocks	5.4	7.8	8.9	8.8	7.9	5.5	3.7	2.6	3.4	54.0
5 ETFs	8.7	9.4	9.2	8.8	8.8	4.8	2.4	1.5	1.1	54.8

Put Options (Millions of contracts per underlying asset)

δ_{BS}	–0.9	–0.8	–0.7	–0.6	–0.5	–0.4	–0.3	–0.2	–0.1	Total
SPX	2.0	4.0	8.3	23.4	111.0	103.9	104.4	123.6	176.4	656.9
XEO	0.0	0.1	0.2	0.4	1.3	0.9	0.4	0.7	1.1	5.0
OEX	0.2	0.3	0.5	1.1	2.8	3.4	3.4	4.0	6.4	22.0
DJX	0.3	0.5	0.7	1.5	3.3	3.5	3.0	2.7	2.5	17.9
30 Stocks	0.8	1.1	1.6	2.6	4.4	5.9	6.6	6.4	5.3	34.8
5 ETFs	0.5	0.8	1.3	2.5	6.0	7.5	7.8	8.0	7.4	41.8

Table 3

The excess of the MV delta, δ_{MV}, over the practitioner Black-Scholes delta, δ_{BS}, for all options on the S&P 500 observed from 2004 to 2015. In this exploratory analysis options are divided into ten buckets according to their practitioner Black-Scholes deltas rounded to the nearest tenth. Options are also bucketed by option maturity.

The average standard error of the estimated values is about 0.0007.

Call Options: $\delta_{MV} - \delta_{BS}$

	Option delta								
Option Life (days)	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
14 to 30	–0.036	–0.044	–0.050	–0.058	–0.063	–0.065	–0.069	–0.064	–0.036
31 to 60	–0.042	–0.055	–0.062	–0.068	–0.072	–0.072	–0.073	–0.067	–0.046
61 to 91	–0.038	–0.050	–0.061	–0.065	–0.067	–0.068	–0.072	–0.063	–0.045
92 to 122	–0.041	–0.053	–0.056	–0.066	–0.073	–0.070	–0.073	–0.067	–0.055
123 to 182	–0.040	–0.056	–0.066	–0.071	–0.077	–0.083	–0.072	–0.065	–0.055
183 to 365	–0.037	–0.053	–0.062	–0.064	–0.069	–0.070	–0.066	–0.064	–0.056
More than 365	–0.037	–0.049	–0.054	–0.057	–0.057	–0.057	–0.054	–0.047	–0.030
All Maturities	–0.039	–0.052	–0.059	–0.064	–0.067	–0.068	–0.067	–0.062	–0.046

Put Options: $\delta_{MV} - \delta_{BS}$

	Option delta								
Option Life (days)	–0.9	–0.8	–0.7	–0.6	–0.5	–0.4	–0.3	–0.2	–0.1
14 to 30	–0.056	–0.036	–0.031	–0.031	–0.036	–0.037	–0.041	–0.034	–0.013
31 to 60	–0.046	–0.043	–0.044	–0.046	–0.048	–0.049	–0.051	–0.045	–0.024
61 to 91	–0.037	–0.041	–0.045	–0.047	–0.047	–0.050	–0.052	–0.048	–0.026
92 to 122	–0.036	–0.036	–0.035	–0.045	–0.052	–0.046	–0.050	–0.046	–0.029
123 to 182	–0.052	–0.061	–0.055	–0.054	–0.060	–0.060	–0.054	–0.048	–0.031
183 to 365	–0.049	–0.063	–0.062	–0.058	–0.059	–0.057	–0.056	–0.053	–0.033
More than 365	–0.061	–0.078	–0.091	–0.087	–0.081	–0.078	–0.071	–0.061	–0.037
All Maturities	–0.048	–0.054	–0.057	–0.057	–0.058	–0.057	–0.056	–0.049	–0.027

Table 4

The out-of-sample average hedging Gain (equation (3)) for options on the S&P 500 from MV delta hedging when the model parameters, *a*, *b* and *c* in equation (6) are estimated using all options traded in a 36 month window and then applied to determine the hedge in the next month. Results are reported for buckets based on rounding δ_{BS} to the nearest tenth. The column headed 'S.E.' is the standard error of the estimate.

Call Options			Put Options		
δ_{BS}	Gain	S.E.	δ_{BS}	Gain	S.E.
0.1	32.1%	3.2%	–0.9	10.5%	1.9%
0.2	25.1%	3.1%	–0.8	9.6%	2.5%
0.3	20.9%	2.9%	–0.7	9.9%	2.6%
0.4	17.9%	2.7%	–0.6	10.6%	2.7%
0.5	15.0%	2.7%	–0.5	11.1%	2.8%
0.6	13.4%	2.7%	–0.4	12.1%	3.0%
0.7	11.9%	2.6%	–0.3	13.4%	3.2%
0.8	9.5%	2.7%	–0.2	13.8%	3.5%
0.9	2.7%	2.8%	–0.1	9.1%	4.0%

Table 5

The out-of-sample average hedging Gain (equation (3)) for options on the S&P 500 from MV delta hedging between January 2007 and August 2015. For the columns headed SABR model, the SABR model is calibrated daily and applied to determine the hedge for the next day. For columns headed Local Vol, the partial derivative of the expected implied volatility with respect to the asset price is calculated from the slope of the smile. For columns headed Empirical model, the model parameters, *a*, *b* and *c* in equation (6) are estimated using all options traded in a 36 month window and then applied to determine the hedge on every day in the next month. Results are reported for buckets based on rounding δ_{BS} to the nearest tenth.

Calls				Puts			
Delta	SABR model	Local Vol	Empirical model	Delta	SABR model	Local Vol	Empirical model
0.1	23.1%	32.8%	32.1%	–0.9	11.9%	–3.9%	10.5%
0.2	13.7%	20.0%	25.1%	–0.8	11.7%	–0.2%	9.6%
0.3	6.8%	10.3%	20.9%	–0.7	9.3%	–3.4%	9.9%
0.4	3.4%	3.9%	17.9%	–0.6	5.3%	–8.2%	10.6%
0.5	0.4%	–0.9%	15.0%	–0.5	–0.1%	–12.3%	11.1%
0.6	2.4%	–0.7%	13.4%	–0.4	–5.5%	–14.9%	12.1%
0.7	5.4%	1.2%	11.9%	–0.3	–8.5%	–14.4%	13.4%
0.8	8.1%	3.1%	9.5%	–0.2	–11.1%	–14.8%	13.8%
0.9	4.8%	–1.7%	2.7%	–0.1	–16.3%	–22.7%	9.1%

Table 6

The average out-of-sample hedging Gain (equation (3)) from MV delta hedging when the model parameters, *a*, *b* and *c* in equation (6) are estimated using options with all strikes and maturities observed in a 36 month window and then applied to determine the hedge in the next month. Results are reported for each delta bucket for European (XEO) and American (OEX) options on the S&P 100 and for European options on the Dow Jones Industrial Index (DJX).

Call Options

δ_{BS}	XEO	OEX	DJX
0.1	32.2%	31.7%	27.0%
0.2	26.3%	24.0%	20.2%
0.3	22.2%	21.0%	17.3%
0.4	18.4%	17.8%	15.2%
0.5	16.0%	15.5%	14.1%
0.6	14.5%	14.1%	13.4%
0.7	12.6%	13.3%	12.4%
0.8	10.3%	11.2%	8.2%
0.9	2.5%	1.4%	1.2%

Put Options

δ_{BS}	XEO	OEX	DJX
–0.9	4.1%	21.5%	1.2%
–0.8	1.8%	16.6%	1.0%
–0.7	4.1%	14.5%	2.0%
–0.6	4.0%	12.4%	2.8%
–0.5	5.8%	12.4%	3.4%
–0.4	7.0%	11.6%	3.9%
–0.3	8.7%	12.1%	5.0%
–0.2	10.8%	13.6%	6.1%
–0.1	10.0%	9.7%	5.5%

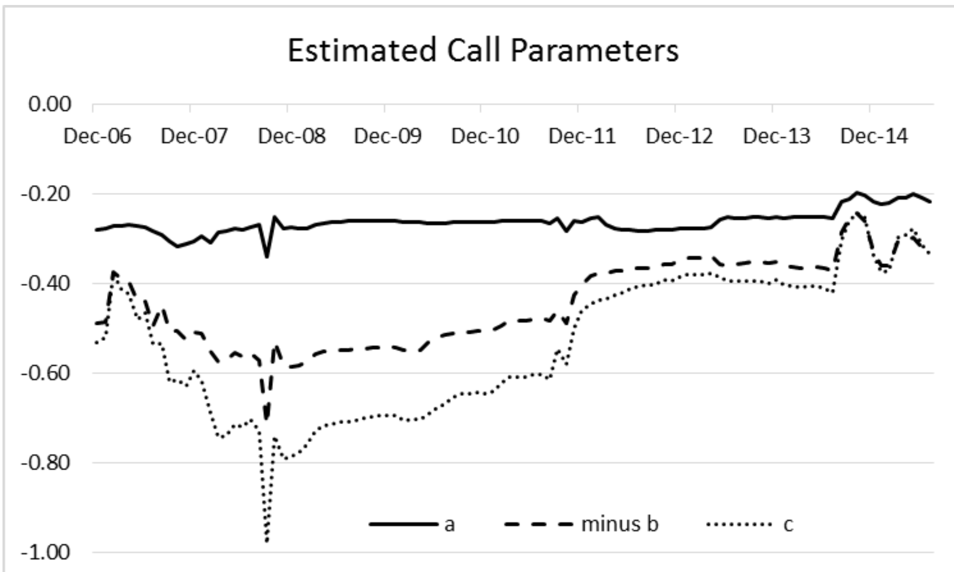
Table 7

The average out-of-sample hedging Gain (equation (3)) observed across the 30 stocks underlying the DJX, the three commodities (gold, silver and oil), and two interest-rate products (20+ year bonds and 7 to 10 year bonds). The model parameters, *a*, *b* and *c* in equation (6) are estimated using options with all strikes and maturities observed in a 36 month window and then applied to determine the hedge in the next month. Gain results are reported for each delta bucket in the test month based on rounding δ_{BS} to the nearest tenth.

	Call Options					Put Options		
δ_{BS}	Stocks	Commodities	Int. Rates		δ_{BS}	Stocks	Commodities	Int. Rates
0.1	20.7%	23.3%	9.1%		–0.9	1.1%	3.0%	–0.4%
0.2	12.7%	15.4%	6.2%		–0.8	1.5%	3.7%	–1.6%
0.3	8.5%	7.9%	6.2%		–0.7	1.5%	1.6%	–2.5%
0.4	5.4%	2.6%	4.8%		–0.6	0.1%	–0.6%	–2.7%
0.5	1.9%	–1.5%	4.4%		–0.5	–2.4%	–3.4%	–2.4%
0.6	–0.5%	–4.6%	3.5%		–0.4	–4.4%	–6.1%	–1.9%
0.7	–1.1%	–5.2%	2.5%		–0.3	–4.6%	–6.7%	–2.1%
0.8	–0.4%	–3.6%	1.7%		–0.2	–2.4%	–6.3%	–2.8%
0.9	0.8%	–0.2%	–0.4%		–0.1	3.1%	–4.0%	3.7%

Figure 1

The estimated parameters for the quadratic in equation (6) puts and calls on the S&P 500 observed between 2004 and 2015. The estimations use overlapping 36-month periods. For call options the negative of the *b* parameter is plotted so that the same scale can be used for both charts.



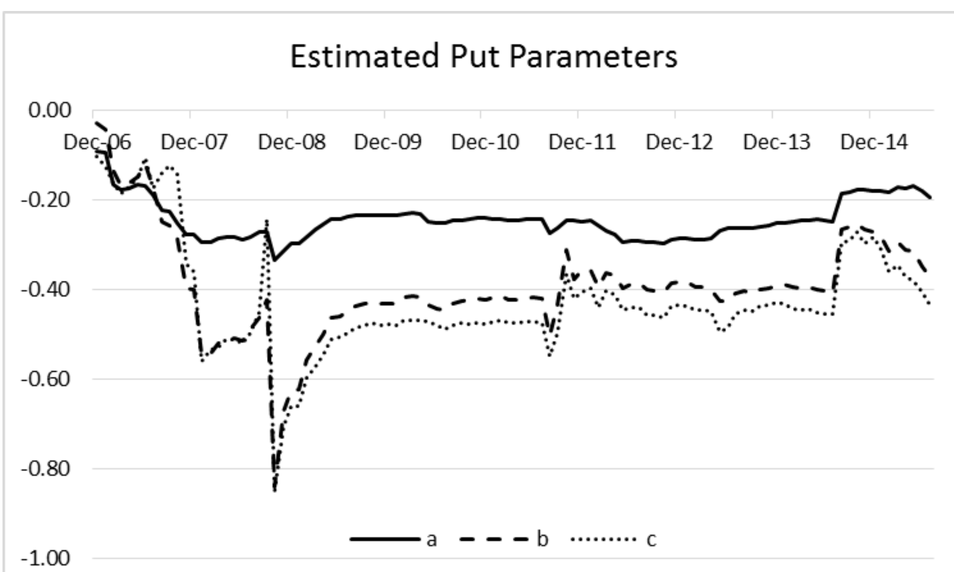


Figure 2

The difference between the MV delta and the Black–Scholes delta, $\delta_{MV} - \delta_{BS}$, for options on the S&P 500 calculated using the quadratic approximation in equation (6). The chart is based on the average of the three year calibration parameters estimated for the periods between January 2012 and July 2015.

The horizontal axis is ordered so that high strike prices are on the right hand end and low strike prices are on the left.

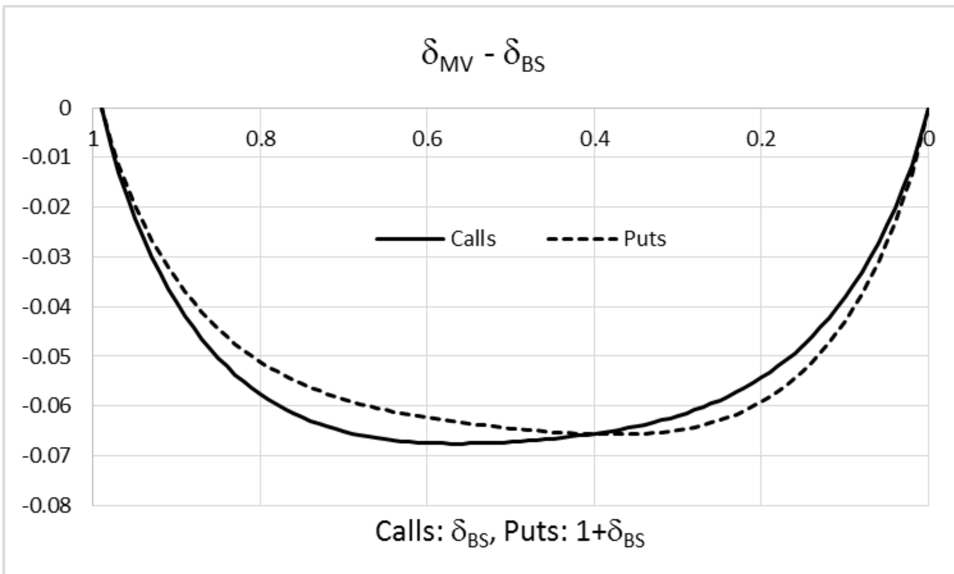


Figure 3

The change in the expected implied volatility, $\Delta E(\sigma_{imp})$, as a result of a 1% increase in the stock price for one-year options on the S&P 500 based on equation (7). For other option maturities, the results shown should be divided by the square-root of the option life measured in years. The chart is based on the average of the three year calibration parameters estimated for the periods between January 2012 and July 2015.

The horizontal axis is the Black-Scholes delta, δ_{BS}, ordered so that high strike prices are on the right hand end and low strike prices are on the left.

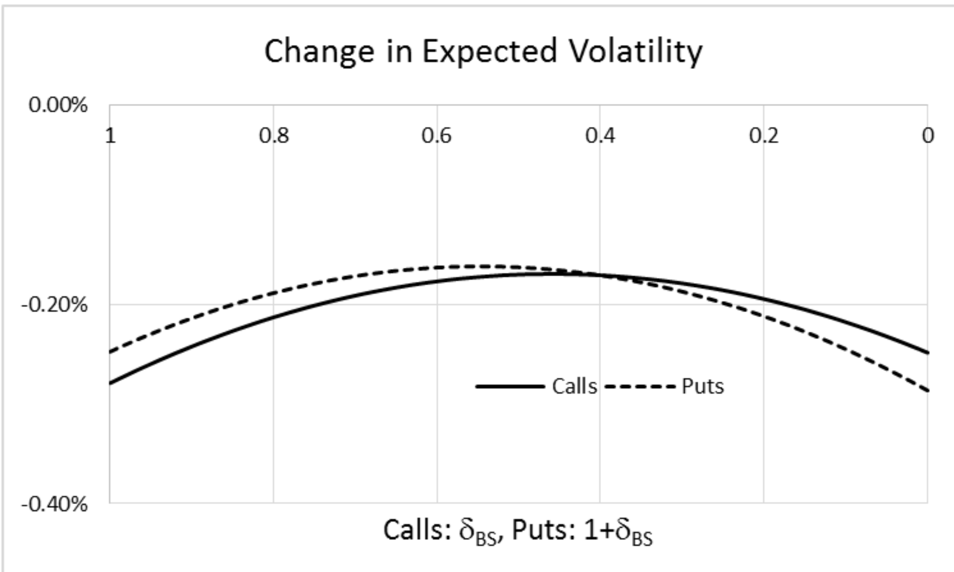


Figure 4

The average R^2, the fraction of total variance of changes in implied volatility explained by changes in the index, for all options on the S&P 500 observed between 2004 and 2015.

The horizontal axis is ordered so that high strike prices are on the right hand end and low strike prices are on the left.

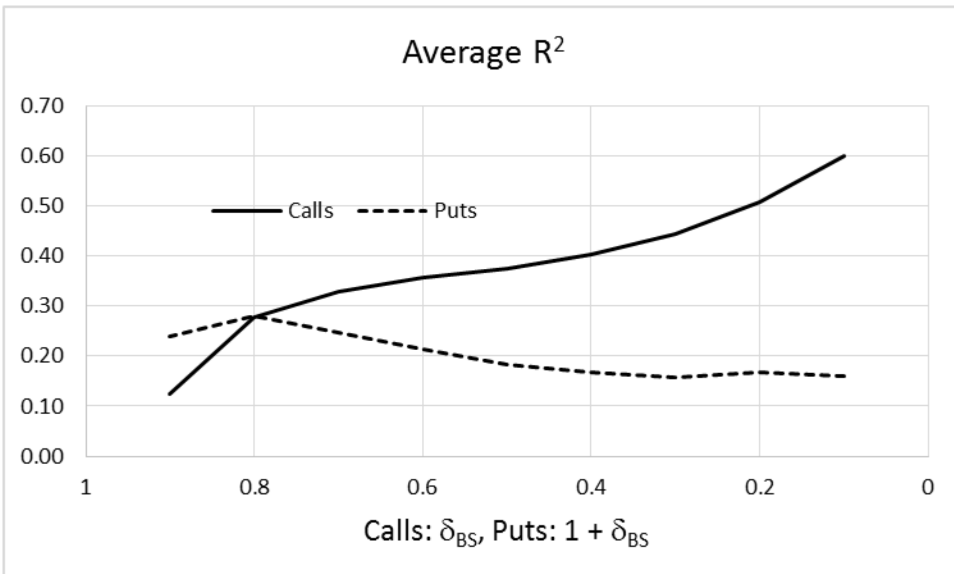


Figure 5

The root mean squared difference between estimated parameters for the quadratic in equation (6) for put options, and the parameter values for put option prices that are calculated from call option prices under the assumption that put-call parity holds (equation (9)). The estimation uses overlapping 36-month periods based on options on the S&P 500.

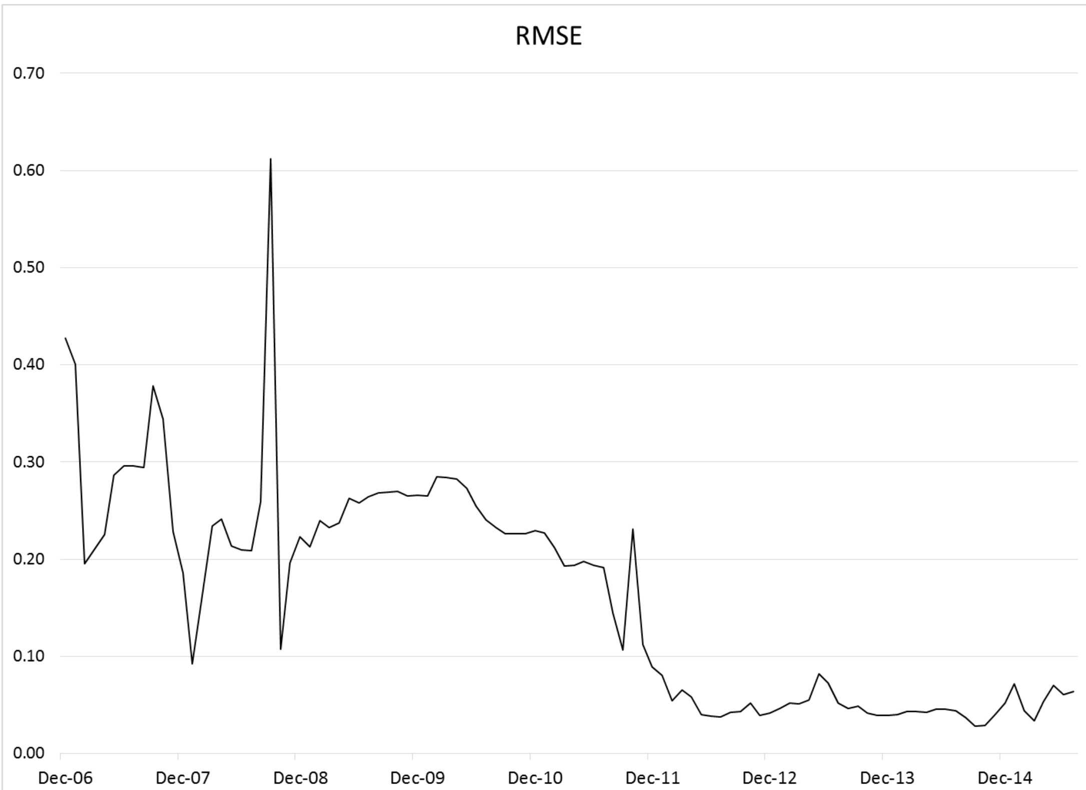


Figure 6

The average R^2, the fraction of total variance of changes in implied volatility explained by changes in the stock price. The results shown are averaged across the stocks underlying the DJIA.

The horizontal axis is ordered so that high strike prices are on the right hand end and low strike prices are on the left.