Kelvinlets Calibration

Wednesday, November 14, 2018 5:49 PM

This document explains where the calibration factors for the Kelvinlets come from. This is well covered in the Regularized Kelvinlets paper at https://graphics.pixar.com/library/Kelvinlets/ for the single scale Kelvinlet, but this document covers how the factors are derived for the biscale Kelvinlets that the sculpting-and-simulations-sample code uses. First we'll cover how to calibrate for a single scale, then cover the calibration for biscale.

First, what is meant by calibration: To make Kelvinlets useful for sculpting, the equations need to be "calibrated". That is, the force vector and force matrix for a Kelvinlet must be multiplied by a factor so that the deformation at the point at the sculpting tool's tip exactly follow the movement of the sculpting tool.

The Kelvinlets paper at https://graphics.pixar.com/library/Kelvinlets/ suggests these constraints, which are easy to satisfy due to the linearity of the regularized Kelvinlets:

- For translation, the displacement at the tool's tip $\vec{u}$=(0,0,0) should be exactly the translation of the tool.
- For twist, the vorticity of the vector field at $\vec{u}$=(0,0,0) should be exactly the vorticity of the tool tip.
- For scale, the gradient at $\vec{u}$=(0,0,0) should be exactly the vector (s, s, s), where s is the scale factor.

It's important to realize that the stiffness value used to compute the calibration factor is distinct from the material stiffness value used later when evaluating the Kelvinlet. When they are the same, then the stiffness cancels out, and the Kelvinlet deformation at r=(0,0,0) matches the move tool's movement (in other words, the material has no stiffness). But when the stiffness value of the Kelvinlet is different than the one used for calibration, the material's movement is affected by its stiffness.

The stiffness value is Young's modulus, which is the ratio of stress to strain (the ratio of how much force is applied to how much the material deforms). It is convenient to have the calibration term use a default stiffness value of 1.0, although it could technically be any value, as long as the ratio is correct.

Translation Calibration

Wednesday, November 14, 2018 5:49 PM

To set the calibration for translation, the Kelvinlets calibration adjusts the force vector f to satisfy the constraint that the displacement at point $r = (0,0,0)$ is equal to the translation f. We start with the translation Kelvinlet, multiplied by the force vector:

$$u(r) = K_t(r)\, f = \left[\frac{a-b}{r_e} I \; + \frac{b}{r_e^3} r\, r^T + \frac{a}{2}\frac{e^2}{r_e^3}\; I\right] f,$$

e = radius,

$$r_e = \sqrt{\|r\| * \|r\| + e * e}$$

At $r = (0,0,0)$, the second term cancels, and $r_e = e$. We then have the following, which we simplify:

$$u(0,0,0) = \left[\frac{a-b}{e} I \;\; + \frac{a}{2}\frac{e^2}{e^3} I\right] f$$

$$u(0,0,0) = \left[\left(\frac{2(a-b)}{2\, e} + \frac{a}{2}\frac{e^2}{e^3}\right) I\right] f$$

$$u(0,0,0) = \left[\left(\frac{3a \; - 2b}{2\, e}\right) I\right] f$$

To satisfy the constraint that the displacement at $r = (0,0,0)$ is exactly the translation f, we set the calibration c to be the inverse of that term:

$$c \; = \frac{2}{3a \; - \; 2b}$$

The calibrated Kelvinlet equation becomes:

$$u(r) = \; c \;\; e \;\; K_t(r)\, f$$

which is equation 7 in the Kelvinlets paper.

Note that the calibration term $c\; e$ is linear in terms of f, and can be multiplied into f. That's why the Kelvinlets paper distinguishes the force vector from the translation: the force vector is the translation multiplied by the calibration factor.

Twist Calibration

Wednesday, December 5, 2018 8:05 AM

The Kelvinlets twist (rotation) deformation is defined in equation 15 in the paper at https://graphics.pixar.com/library/Kelvinlets/ as:

$$u(\vec{r}) = K_{twist}(\vec{r})\,[\,\vec{q}\,]_x = -a\left(\frac{1}{r_e^3} + \frac{3\,e^2}{2\,r_e^5}\right)\vec{q} \times \vec{r}$$

And the gradient of that equation at point (0,0,0) is equation 4 in the Kelvinlets supplemental material:

$$\nabla t_e(0) = -\frac{5}{2}\frac{a}{e^3}[\,\vec{q}\,]_x$$

$[\,\vec{q}\,]_x$ is the skew symmetric (cross product) matrix of the angular velocity vector q.

We want to calibrate the twist Kelvinlet such that its vorticity at the point (0,0,0) is equal to the vorticity of the tool tip. In other words, we want the gradient $\nabla t_e(0)$ to be $[\,\vec{q}\,]_x$. This is necessary or the center of the tool will rotate with a different speed than your hand. In fact, without calibration, the twist Kelvinlet above will rotate in the opposite direction of your hand (because of the negative sign).

Therefore, we need to multiply $\nabla t_e(0)$ by $-\frac{2}{5}\frac{e^3}{a}$. Because multiplicative factors are the same in equations and their derivatives, we can just multiply the original $u(\vec{r})$ by this term.

This means that our calibrated twist equation is:

$u(\vec{r}) = c\,e^3\,K_{twist}(\vec{r})\,[\,\vec{q}\,]_x$, where $c = -\frac{2}{5\,a}$

Scale Calibration

Sunday, January 27, 2019 6:05 PM

The Kelvinlets twist (rotation) deformation is defined in equation 15 in the paper at https://graphics.pixar.com/library/Kelvinlets/ as:

$$u(\vec{r}) = K_{scale}(\vec{r})\, s = (2b - a)\left(\frac{1}{r_e^3} + \frac{3\, e^2}{2\, r_e^5}\right)\vec{r} s$$

And the gradient of that equation at point (0,0,0) is equation 5 in the Kelvinlets supplemental material:

$$\nabla s(0) = s(2b - a)\frac{5}{2}\frac{1}{e^3} I$$

s is the uniform scale factor, and I is the identity matrix.

We want to calibrate the scale Kelvinlet so that its scale at point (0,0,0) is exactly s. Therefore, we need to multiply $\nabla s(0)$ with $\frac{2e^3}{5(2b-a)}$. Then the terms in $\nabla s(0)$ cancel out such that the result is s.

This means our calibrated scale equation is:

$$u(\vec{r}) = c\, e^3\, K_{scale}(\vec{r})\, s, \text{where } c\ = \frac{2}{5(2b-a)}$$

Translation Calibration, Biscale

Wednesday, December 5, 2018 7:30 AM

For tighter falloff, you can use biscale Kelvinlets (two Kelvinlets with slightly different radii that are subtracted from each other). This works because of the linearity of the equations.

Biscale subtracts two Kelvinlets of different radii e_1 and e_2, where $e_1 < e_2$, which gives us:

$$u(\vec{r}) = K_{e_1}(\vec{r}) - K_{e_2}(\vec{r})$$

We want to satisfy the calibration constraint that the translation at $\vec{r} = (0,0,0)$ is equal to f. At (0,0,0), the above simplifies to:

$$u(0,0,0) = \left[\left(\frac{3a - 2b}{2\, e_1}\right) I\right] f - \left[\left(\frac{3a - 2b}{2\, e_2}\right) I\right] f$$

$$u(0,0,0) = \left[\left(\frac{3a - 2b}{2}\right)\left(\frac{1}{e_1} - \frac{1}{e_2}\right) I\right] f$$

We multiply everything with the inverse of the first term above to satisfy the constraint:

$$u(0,0,0) = \left[\frac{2}{3a - 2b}\left(\frac{1}{e_1} - \frac{1}{e_2}\right)^{-1}\right]\left[\left(\frac{3a - 2b}{2}\right)\left(\frac{1}{e_1} - \frac{1}{e_2}\right) I\right] f = f$$

So for the calibrated biscale translation equation we use:

$$u(\vec{r}) = c\left(\frac{1}{e_1} - \frac{1}{e_2}\right)^{-1}\left(K_{e_1}(\vec{r}) - K_{e_2}(\vec{r}) f\right), \text{ where } c = \frac{2}{3a - 2b}$$

Twist Calibration, Biscale

Sunday, January 27, 2019 6:04 PM

Biscale subtracts two Kelvinlets of different radii e_1 and e_2, where $e_1 < e_2$, which gives us:

$$u(\vec{r}) = K_{e_1}(\vec{r}) - K_{e_2}(\vec{r})$$

And the gradient of the twist Kelvinlet is

$$\nabla t_e(0) = -\frac{5}{2}\frac{a}{e^3}[\vec{q}]_x$$

So the gradient of the biscale twist Kelvinlet is

$$-\frac{5}{2}\frac{a}{e_1^3}[\vec{q}]_x - -\frac{5}{2}\frac{a}{e_2^3}[\vec{q}]_x$$

$$-\frac{5}{2}a\left(\frac{1}{e_1^3}-\frac{1}{e_2^3}\right)[\vec{q}]_x$$

We want to calibrate so that this gradient is equal to $[\vec{q}]_x$. So we multiply by the inverse of everything but $[\vec{q}]_x$ which leads us to:

$$-\frac{2}{5a}\left(\frac{1}{e_1^3}-\frac{1}{e_2^3}\right)^{-1} -\frac{5}{2}a\left(\frac{1}{e_1^3}-\frac{1}{e_2^3}\right)[\vec{q}]_x$$

So the calibrated biscale twist Kelvinlet is:

$$u(\vec{r}) = c\left(\frac{1}{e_1^3}-\frac{1}{e_2^3}\right)^{-1}\left(K_{twist\ e_1}(\vec{r}) - K_{twist\ e_2}(\vec{r})\right)[\vec{q}]_x, \text{ where } c = -\frac{2}{5\,a}$$

Scale Calibration, Biscale

Sunday, January 27, 2019 6:05 PM

Biscale subtracts two Kelvinlets of different radii e_1 and e_2, where $e_1 < e_2$, which gives us:

$$u(\vec{r}) = K_{e_1}(\vec{r}) - K_{e_2}(\vec{r})$$

And the gradient of the scale Kelvinlet is

$$\nabla s(0) = s(2b-a)\frac{5}{2}\frac{1}{e^3}I$$

So the gradient of the biscale scale Kelvinlet is

$$s(2b-a)\frac{5}{2}\frac{1}{e_1^3}I - s(2b-a)\frac{5}{2}\frac{1}{e_2^3}I$$

$$\frac{5(2b-a)}{2}\left(\frac{1}{e_1^3} - \frac{1}{e_2^3}\right)sI$$

We want to calibrate so that this gradient is equal to s. So we multiply by the inverse of everything but s which leads us to:

$$\frac{2}{5(2b-a)}\left(\frac{1}{e_1^3} - \frac{1}{e_2^3}\right)^{-1}\frac{5(2b-a)}{2}\left(\frac{1}{e_1^3} - \frac{1}{e_2^3}\right)sI$$

So the calibrated biscale twist Kelvinlet is:

$$u(\vec{r}) = c\left(\frac{1}{e_1^3} - \frac{1}{e_2^3}\right)^{-1}\left(K_{scale\ e_1}(\vec{r}) - K_{scale\ e_2}(\vec{r})\right)s, \text{ where } c = \frac{2}{5*(2b-a)}$$