Math 454_HW13

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1 1

If v has no such neighbor, then for every $u \in N(v)$, the graph $G - v - u$ is disconnected. Choose $u \in N(v)$ such that $G - v - u$ has as small a component as possible;

Let H be the smallest component of $G - v - u$. Since G is 2-connected, v and u have neighbors in every component of $G - v - u$.

Let x be a neighbor of v in H. If $G - v - x$ is disconnected, then it has a component that is a proper subgraph of H. Contradicts the choice of u

Thus, $G - v - x$ is connected.

2 2

Suppose that G is 2-connected. $G - xy$ is 2connected iff x and y lie on a cycle in $G - xy$.

If $G - xy$ is not 2 -connected, then there is a vertex v whose deletion separates x and y. Therefore, all x, y-paths in $G - xy$ pass through v and $G - xy$ has no cycle containing x and y.

x, y, as pair of vertices, lies on a cycle if $G - xy$ is 2 -connected. If a cycle in G has a chord x, y, then this argument shows that $G - xy$ is still 2-connected, and hence G is not minimally 2-connected.

If no cycle has a chord, then for any edge xy, the graph $G - xy$ has no cycle containing x and y.

Thus, $G - xy$ is not 2 -connected.

3 3

Let G be a X, Y bipartite graph. Construct H with $V(H) = V(G) \cup \{a, b\}, E(H) = E(G) \cup T$, where $T = \{ax, by : \forall x \in X, \forall y \in Y\}$

$\lambda(a, b) = \alpha'(G)$. Choose any maximum matching which we denote $\{x_i, y_i\}$. Then $a \to x_j \to y_j \to b$ is internally disjoint from $a \to x_k \to y_k \to b$ for all $i \neq j$ in our matching. Thus $\lambda_{G'}(a, b) \geq \alpha'(G)$. If $\lambda_{G'}(x, y) > \alpha'(G)$, we can see that this would provide a matching that would be greater than maximum.

In order to disconnect a, b in G', we need to delete vertices so that no edges from X to Y exist, or else that would be a path. Thus a disconnecting set of a, b is a vertex cover of G. Therefore $\kappa_{G'}(a, b) \geq \beta(G)$.

We have $\alpha'(G) = \lambda_{G'}(a,b) = \kappa_{G'}(a,b) \geq \beta(G)$. We know that $\alpha'(G) \leq \beta(G)$ because distinct vertices must be used to cover the edges of a matching.

Thus, we have $\alpha' = \beta$ when G bipartite.