Math454_A07

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1 1

For $n(T) = 2$, the only such subgraph is T itself. Suppose $n(T) > 2$. Observe that every pendant edge must appear in the subgraph to give the leaves odd degree. Let x be an endpoint of a longest path P, with neighbor u. If u has another leaf neighbor y, add ux and uy to the unique such subgraph found in $T - \{x, y\}$. Otherwise, $d(u) = 2$, since P is a longest path. In this case, add the isolated edge ux to the unique such subgraph found in $T - \{u, x\}$.

2 2

2.1 a

A u, v-path of length $d(u, v)$ and a v, w-path of length $d(v, w)$ together form a u, w-walk of length $l = d(u, v) + d(v, w)$. Every u, w-walk contains a u, w-path among its edges, so there is a u, w path of length at most l.

Hence the shortest u, w-path has length at most l.

2.2 b

Let u, v be two vertices such that $d(u, v) = d$. Let w be a vertex in the center of G; it has eccentricity r. Thus $d(u, w) \leq r$ and $d(w, v) \leq r$. By part (a), $d = d(u, v) \leq d(u, w) + d(w, v) \leq 2r$

2.3 c

Consider the lollypop, a cycle C_{2r} with handle P_{d-r} attached at vertex x, with $r \leq d \leq 2r$. If $d = r$, this is just the cycle, which has radius = diameter = r. By adding a path to x of length less or equal to r, the eccentricity of x remain unchanged (the farthest vertex is still on the other side of the cycle). However, the end of the handle is $d - r$ father away from the antipode of x, and so is $d - r + r = d$ away from x. Since anything closer in on the handle and anything closer in the cycle both have smaller eccentricities than these two vertices, d is the new diameter.

3 3

A single vertex x has at most k neighbors. Each of these has at most k other incident edges, and hence there are at most $k(k-1)$ vertices at distance 2 from x. Assuming that new vertices always get generated, the tree of paths from x has at most $k(k-1)^{i-1}$ vertices at distance i from x. Hence $n(G) \leq 1 + \sum_{i=1}^{d} k(k-1)^{i-1} = 1 + k\frac{(k-1)^d - 1}{k-1-1}$.

4 4

By symmetry each edge of K_n on $\{1, 2, \ldots, n\}$ is on the same number of vertices. Since a spanning tree contains $n-1$ edges, the number of pairs (T, ε) with T a spanning tree and ε an edge of T is, by Cayley's formula, $n^{n-2}(n-1)$. Since there are $n(n-1)/2$ edges in K_n, the number of spanning trees containing any specific edge is $\frac{n^{n-2}(n-1)}{n(n-1)/2} = 2n^{n-3}$ Hence deleting that edge we get $n^{n-2} - 2n^{n-3} = n^{n-3}(n-2)$ spanning trees.