HW4_Baoshu_Feng

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#1

In a 26-vertex graph which decomposes into a 4-regular bipartite graph with partites X and Y both of order 13, and two 9-regular graphs of order 13 (one on each partite of X and Y). The latter isn't possible because a graph cannot have an odd number odd-degree vertices.

#2

Proof. Suppose for the sake of contradiction that G is a k-regular bipartite graph ($k \geq 2$) with a cut edge ab. When ab is removed from G, the component of G containing the edge ab splits into two new components; call them A and B, with $a \in A$ and $b \in B$. Both of these components are nontrivial, since their vertices have degrees at least $k - 1 \geq 1$.

Now, the component A is bipartite (since it is a subgraph of a bipartite graph), so there is a bipartition $V(A) = X \cup Y$ of A with each edge of A running between X and Y. Without loss of generality, say $a \in Y$. Then every vertex of X has degree k. By contrast every vertex of Y has degree k, except for the vertex a, which has degree $k - 1$.

the number of edges in A:

$$\begin{aligned}
|E(A)| = k|X| &= k(|Y| - 1) + (k - 1) \\
k|X| &= k|Y| - 1 \\
1 &= k(|Y| - |X|)
\end{aligned}$$

From the above, it follows that $k = 1$, contradicting the fact that $k \geq 2$.

#3

a)

TRUE. Let G be a graph with $n \geq 2$ vertices and m edges. Let v be a vertex in G of maximum degree, $\Delta(G)$. Then, using the degree-sum formula, we have

$$\text{avg degree in } G = \frac{1}{n} \sum_{u \in V(G)} d_G(u) = \frac{2m}{n}$$

$$
\begin{aligned}
\text{avg degree in } G - v &= \frac{1}{n-1}\left(\sum_{u \in V(G)-v} d_{G-v}(u)\right) \\
&= \frac{1}{n-1}\left(\sum_{u \in V(G)} d_G(u) - 2\Delta(G)\right) \\
&= \frac{2m - 2\Delta(G)}{n-1} \\
&\leq \frac{2m - 2\frac{2m}{n}}{n-1} \text{ since } \Delta(G) \geq \text{ avg degree of } G = \frac{2m}{n} \\
&= \left(\frac{2m}{n}\right)\left(\frac{n-2}{n-1}\right) \\
&< \frac{2m}{n}.
\end{aligned}
$$

b)

FALSE. Consider the path on 3 vertices, P_3. The average degree is $\frac{1}{3}(1+1+2) = \frac{4}{3}$ but for a leaf $v \in V(P_3)$, the average degree of $P_3 - v = P_2$ is $\frac{1}{2}(1+1) = 1$. When a vertex of minimum degree is removed, the average degree can be reduced. The average degree in G is k, and

$$
\text{avg degree in } G - v = \frac{1}{n-1}(nk - 2k) = \left(\frac{n-2}{n-1}\right) k < k = \text{ avg degree in } G
$$

#4

Proof. Consider the set of bits which are not shared amongst all of the vertices of the cycle C. The cycle sits in the subcube generated by varying only those bits which vary in C. There are at most r of them since the farthest a vertex can be from any other vertex in C half the length of the cycle. So C sits inside some copy of Q_r. On the other hand,

$$
(0,0,0) - (1,0,0) - (1,1,0) - (1,1,1) - (1,0,1) - (0,0,1) - (0,1,1) - (0,1,0) - (0,0,0)
$$

is a cycle of length $8 = 2 * 4$ in a hypercube of dimension 3 .

#5

Proof. There is a bijection between simple graphs on $[n-1]$ and even simple graphs on $[n]$ given by

$$
\begin{aligned}
\{ \text{ simple graphs on } [n-1]\} &\to \{ \text{ simple even graphs on } [n]\} \\
G &\mapsto G + v_n + \{v_n v_i \mid d(v_i) \text{ is odd }\}
\end{aligned}
$$

The inverse of this map is

$$
\begin{aligned}
\{ \text{ simple even graphs on } [n]\} &\to \{ \text{ simple graphs on } [n-1]\} \\
G &\mapsto G - v_n.
\end{aligned}
$$

Since there are $2\binom{n-1}{2}$ simple graphs on $[n-1]$, there are also $2\left(\begin{array}{c} n-1 \\ 2 \end{array}\right)$ simple even graphs on $[n]$.