homework_14

April 22, 2022

1 1

By Menger's Theorem, for each x, y there are $\kappa'(G)$ pairwise edge-disjoint x, y paths. Since G is 3 -regular, these paths cannot share internal vertices. Hence for each x, y there are $\kappa'(G)$ pairwise internally disjoint x, y-paths. This implies that $\kappa(G) \geq \kappa'(G)$, and it always holds that $\kappa(G) \leq \kappa'(G)$.

2 2

If every edge of a graph G appears in at most one cycle, then every block of G is an edge, a cycle, or an isolated vertex. A block B with at least three vertices is 2 -connected and has a cycle C. Hence, $B = C$.

Every 2-connected graph has an ear decomposition. If B is not a cycle, then adding the next ear completes two cycles sharing a path.

$\chi(G)$ equal the largest chromatic number of its blocks. Therefore, the blocks are edges or cycles and have chromatic number at most 3 .

3 3

The vertices of G are the points of intersection of a family of lines; the edges are the segments on the lines joining two points of intersection.

If $H \subseteq G$, the vertex of H with largest x-coordinate has degree at most 2 in H.

On each line through that vertex it has at most one neighbor with smaller x-coordinate and none with larger x-coordinate.

According to the Szekeres-Wilf Theorem, $\chi(G) \leq 1 + \max_{H \subseteq G} \delta(H) \leq 3$

4 4

From left to right:

If G is m-colorable, then $\chi\left(G \square K_m\right) = \max\left\{\chi(G), \chi\left(K_m\right)\right\} = m$.

Since $\chi(H) \geq n(H)/\alpha(H)$ for every graph H, and $n\left(G \square K_m\right) = n(G)m$, $\alpha\left(G \square K_m\right) \geq n(G)m/m = n(G)$.

From right to left:

Since an independent set has at most one vertex in each copy of K_m, $\alpha\left(G \square K_m\right) \geq n(G)$ yields $\alpha\left(G \square K_m\right)=n(G)$,

The vertices of a maximum independent set S have the form (v, i), where $v \in V(G)$ and $i \in[m]$. By the definition of cartesian product, adding 1 (modulo m) to the second coordinate in each vertex of S yields another independent set of size $n(G)$. Doing this m times yields m pairwise disjoint independent sets covering all the vertices of $G \square K_m$.

Hence, $G \square K_m$ is m-colorable. Since G is a subgraph of $G \square K_m$, also G is m-colorable.