homework_10

March 25, 2022

1 1

1. Let xy be an edge of G, with $x \in X$ and $y \in Y$, and let $G' = G - x - y$.
2. Each set $S \subseteq X - \{x\}$ loses at most one neighbor when y is deleted. Therefore, $|N_{G'}(S)| \geq |N_G(S)| - 1 \geq |S|$.
3. G' satisfies Hall's Condition and has a matching that saturates $X - \{x\}$.
4. Thus, Every edge xy of G belongs to some matching that saturates X.

2 2

induction proof

1. Let A be a square matrix with rows and columns summing to k. From induction on k, it yields A as a sum of k permutation matrices. For $k = 1, A$ is a permutation matrix.
2. For $k > 1$, form a bipartite graph G with vertices $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, and the number of edges joining x_i and y_j is $a_{i,j}$.
3. The graph G has a perfect matching since it is bipartite and regular. (the Marriage Theorem)
4. Let $b_{i,j} = 1$ if $x_i y_j$ belongs to this matching. Also, let $b_{i,j} = 0$ if $x_i y_j$ don't belongs to this matching. Therefore, the matrix B is a permutation matrix. Each row and column of B has exactly one 1 .
5. Thus $A' = B - A$ is a non-negative integer matrix. In A' matrix, rows and columns sum to $k - 1$.
6. Therefore, A' yields $k - 1$ additional permutation matrices that with B sum to A.
7. Therefore, for any square matrix A of non-negative integers, A can be expressed as the sum of k permutation matrices iff all row sums and column sums of A equal k.

3 3

1. Each vertex of G covers at most $\Delta(G)$ edges.
2. Since all edges must be covered in a vertex cover, this yields $\beta(G) \geq e(G)/\Delta(G)$.
3. $\alpha'(G) = \beta(G)$ when G is bipartite (the König-Egerváry Theorem). Thus $\alpha'(G) \geq e(G)/\Delta(G)$.
4. Every subgraph of $K_{n,n}$ with more than $(k-1)n$ edges has a matching of size at least k.
5. Such a graph G is a simple bipartite graph with partite sets of size n.
6. Thus $\Delta(G) \leq n$, and we compute $\alpha'(G) \geq e(G)/\Delta(G) > (k-1)n/n = k - 1$.
7. Thus G has a matching of size k.

4 4

1. Let S be an independent set of size $\alpha(G)$.
2. Since $V(G)-S$ is a vertex cover, summing the vertex degrees in $V(G)-S$ provides an upper bound on $e(G)$.
3. Thus $e(G) \leq (n(G)-\alpha(G))\Delta(G)$.

If G is regular, set $e(G) = n(G)\Delta(G)/2$. From above proved inequality, $\alpha(G) \leq n(G)/2$.