Homework 8_Baoshu Feng

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0.1 q1

If G has two minimum-weight spanning trees, then let e be the lightest edge of the symmetric difference. Distinct edge weight appears in only one of the two trees, T and T'. Since $e \in E(T) - E\left(T'\right)$, there exists $e' \in E\left(T'\right) - E(T)$ such that $T' + e - e'$ is a spanning tree. By the choice of $e, w\left(e'\right) > w(e)$. Now $w\left(T' + e - e'\right) < w\left(T'\right)$, contradicting the assumption that T' is an MST. Hence that G has only one minimum-weight spanning tree and there cannot be two MSTs.

1 q2

2 q3

Let T be a minimum spanning tree in G. If $e \in E(T)$, then $T - e$ has two components with vertex sets U and U'. The subgraph $C - e$ is a path with endpoints in U and U'; hence it contains an edge e' joining U and U'. Since $w\left(e'\right) \leq w(e)$ by hypothesis, $T - e + e'$ is a tree as cheap as T that avoids e. there is a minimum spanning tree not containing e.

Given a weighted graph, iteratively deleting a heaviest non-cut-edge produces a minimum spanning tree. A non-cut-edge is an edge on a cycle. A heaviest such edge is a heaviest edge on that cycle. From above proof, some minimum spanning tree avoids it, so deleting it does not change the minimum weight of a spanning tree. When no cycles remain, we have a connected acyclic subgraph. It is the only remaining spanning tree and has the minimum weight among spanning trees of the original graph. Hence, deleting a heaviest non-cut-edge until none remain produces a minimum-weight spanning tree.