Math_454_A11

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1 q1

If G has a maximal matching of size k, then the $2k$ endpoints of these edges form a set of vertices covering the edges, because any uncovered edge could be added to the matching. Hence $\beta(G) \leq 2\alpha'(G)$.

Let G be a graph and let B be a minimum vertex cover. So $|B| = \beta(G)$. Since B is incident on every edge any matching must have each edge incident on some vertex of B.

To be spercific,

A graph consisting of k disjoint triangles has $\alpha' = k$ and $\beta = 2k$. the inequality is proved.

Also, for the graph K_{2k+1}, since we cannot omit two vertices from a vertex cover of K_{2k+1}. Every disjoint union of cliques of odd order satisfies $\beta(G) = 2\alpha'(G)$.

2 q2

Necessity. Assume G is a 3-regular simple graph with a 1-factor.

Let M be the edges of this perfect matching and let $H = G - M$. Then G is 2-regular and therefore a union of disjoint cycles. Orient each cycle in some (consistent) direction. Now construct $|M|$ copies of P_4 each with a matching edge as the middle edge by attaching the out-edges at each end-vertex on the cycles of the 2-factor. This decomposes the graph into P_4 's.

Sufficiency. Assume G is a 3-regular graph that decomposes into P_4 's.

Since vertices interior to a P_4 have degree 2 , no vertex can be a middle vertex of more than one P_4. Let k be the number of P_4 's in the decomposition. Then $e(G) = 3n(G)/2 = 3k$. Thus, $k = n/2$. That is, an elementary edge count implies that there are $n/2$ middle edges and so every vertex must be an interior vertex on some P_4.

Thus, a 3-regular simple graph has a 1-factor if and only if it decomposes into copies of P_4.

3 q4

Let S be a minimum vertex cut, which means $|S| = \kappa(G)$.

Since $\kappa(G) \leq \kappa'(G)$, to provide an edge cut of size $|S|$.

Let H_1 and H_2 be two components of $G - S$. Since S is a minimum vertex cut, each $v \in S$ has a neighbor in H_1 and a neighbor in H_2.

Since $\Delta(G) \leq 3, v$ cannot have two neighbors in H_1 and two in H_2. For each such v, delete the edge to a member of $\{H_1, H_2\}$ in which v has only one neighbor.

These $\kappa(G)$ edges break all paths from H_1 to H_2 except in the case drawn below, where a path can come into S via v_1 and leave via v_2. Choose the edge to H_1 for each v_i.

Thus, $\kappa'(G) = \kappa(G)$

4 q5

To verify Tutte's 1 -factor condition. When $|S| = \varnothing$, the only component of $G - S$ has even order. When $1 \leq |S| \leq r - 1$, there is only one component of $G - S$. For $|S| \geq r$, we prove that $G - S$ has at most $|S|$ components.

Each component H of $G - S$ sends edges to at least r distinct vertices in S, since $\kappa(G) = r$. For each such H, choose edges to r distinct vertices in S. Given $v \in S$, we have chosen at most one edge from v to each component of $G - S$. If $G - S$ has more than $|S|$ components, then we have chosen more than $r|S|$ edges to S.

Accoring to the pigeonhole principle, some $x \in S$ appears in more than r of these edges. Since we chose at most one edge from x to each component of $G - S$, the chosen edges containing x have endpoints in distinct components of $G - S$, which creates the forbidden induced $K_{1,r+1}$.

Therefore, it is not enough to assume that G is r edge-connected or that G is $r - 1$-connected. Thus, G has a 1-factor.