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#1

Proof: Let G be a 2-connected graph that is not a cycle. We know that G has at least 3 vertices because it is 2-connected. Pick any two vertices in G, then we know that G contains a cycle through these 2 vertices, so we know in particular that G contains a cycle C. Let C be a shortest cycle in G.

Now since G is not a cycle, it must contain edges e that are not edges of C. If some such edge e joins two vertices x and y in C (such an edge is called a chord of C), then e together with either one of the (x, y)-paths on C is a cycle that is shorter than C, contradicting our choice of C. Therefore all edges of G that are not edges of C must be incident to at least one vertex not on C.

Let x be a vertex not on C. Since G is connected, there is a path P from x to some vertex of C. Suppose P is the shortest such path; then the last vertex y of P is on C but no other vertices of P are on C. Let z be the neighbour of y on P (it is possible $z=x$). Now since G is 2-connected, $G-y$ is connected, so there is a path Q from z to a vertex of C that doesn't contain y. Again we may assume only the last vertex u of Q is on C. We know at least one of the (u, y) paths on C contains another vertex w different from u and y, since C has length at least 3 . Then C together with the path yzQ is a subdivision of H, with vertices y, z, u and w.

#2

##i

$\left(\begin{array}{l}n \\ 2\end{array}\right)-\left(\begin{array}{l}k \\ 2\end{array}\right)$. Achieved by the graph obtained by deleting from K_n all edges with both endpoints in a chosen k-element subset of $V\left(K_n\right)$.

##ii

$\left(\begin{array}{c}n-k+1 \\ 2\end{array}\right)$. Let $H_1, \ldots, H_k$ be the k components of G with $n_i:=n\left(H_i\right)$ for $i=1, \ldots, k$. We may assume (after rearranging, if necessary) that $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k \geqslant 1$. Then $e(G)=e\left(H_1\right)+e\left(H_2\right)+\cdots+e\left(H_k\right) \leqslant\left(\begin{array}{c}n_1 \\ 2\end{array}\right)+\left(\begin{array}{c}n_2 \\ 2\end{array}\right)+\cdots+\left(\begin{array}{c}n_k \\ 2\end{array}\right)=\left(n_1^2+n_2^2+\cdots+n_k^2-n\right) / 2$. This number of edges corresponds to that of a disjoint union of k complete graphs $\left(K_{n_1}+K_{n_2}+\cdots+K_{n_k}\right)$. Now for $n_i>1$ we have $e\left(K_{n_1+1}+K_{n_2}+\cdots+K_{n_{i-1}}+K_{n_i-1}+K_{n_{i+1}}+\cdots+K_{n_k}\right)-e\left(K_{n_1}+K_{n_2}+\cdots+K_{n_k}\right)=\left(n_1-n_i\right)+1 \geqslant 1$, i.e., we gain at least an edge by pushing a vertex from the i-th component to the first one. Thus if $n_2=\cdots=n_k=1$ (so that $n_1=n-k+1$), we have the maximum value of $e(G)$, i.e., $e(G) \leqslant\binom{n-k+1}{2}$. This bound is achieved by $K_{n-k+1}+K_1+\cdots+K_1$.

##iii

$\left(\begin{array}{c} n-1 \\ 2 \end{array} \right)$. A disconnected graph G has $k \geqslant 2$ components and hence by Part ii) can have a maximum of $\left(\begin{array}{c} n-k+1 \\ 2 \end{array} \right)$ edges. This number is maximized for $k = 2$, which implies that $n(G) \leqslant \left(\begin{array}{c} n-1 \\ 2 \end{array} \right)$. This bound is achieved by $K_{n-1} + K_1$.

#3

Assume that G isn't connected, then it has at least 2 connected components. So by Pigeonhole Principle there exists a connected component, say $\boldsymbol{H}$ with at most $\frac{n}{2}$ vertices. So now take $v \in V(H)$. Then by the assumption has a degree of at least $\frac{n}{2}$. But these edges are all in H_1, which is impossible as H_1 has at most $\frac{n}{2}$ vertices.

#4

Disprove. Assume that G isn't connected, then it has at least 2 connected components. So by Pigeonhole Principle there exists a connected component, say H with at most $\frac{n}{2}$ vertices. So now take $v \in V(H)$. Then by the assumption has a degree of at least $\frac{n}{2}$. But these edges are all in $\boldsymbol{H}_1$, which is impossible as $\boldsymbol{H}_1$ has at most $\frac{n}{2}$ vertices.

#5

Proof. Transitivity of order. A sting of inequalities produced by our criteria is equivalent to a walk in G; since G is acyclic, every walk is a u, v-path. So our criteria for the ordering is equivalent to requiring that whenever v_i appears before v_j in any path, i must be less than j. If there is no such order, then there must be some pair of vertices u and v for which there is both a u, v-path and a v, u-path. However, this would produce a cycle (follow the u, v path until it intersects with the v, u-path and then follow the v, u-path back). So there must be some ordering which agrees with all paths.