HW6_Baoshu_Feng

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1 Prove or disprove, for each $n \geq 1$:

1.1 Every n-vertex simple graph has two vertices with the same degree.

Assume that a finite graph G has n vertices. Then each vertex has a degree between $n-1$ and 0 . But if any vertex has degree 0 , then no vertex can have degree $n-1$, so it's not possible for the degrees of the graph's vertices to include both 0 and $n-1$. Thus, the n vertices of the graph can only have $n-1$ different degrees, so by the pigeonhole principle at least two vertices must have the same degree.

1.2 Every n-vertex simple digraph has two vertices with the same in-degree or two vertices with the same out-degree.

Counterexample. For n = 4,

If the digraph is simple, the possible outdegrees are $0,1,2,3$. If they are all different, we must have one of each. Call the vertices v_0, v_1, v_2, v_3 where od $(v_i) = i$. The number of edges is the sum of the outdegrees, $0+1+2+3=6$. Since $6 = \left(\begin{array}{l}4 \\ 2\end{array}\right)$, every pair of vertices is joined by an edge; that is, we have an oriented complete graph aka a tournament. Therefore we have outdegree + indegree = 3 at each vertex; that is, v_0 has outdegree 0 and indegree $3, v_1$ has outdegree 1 and indegree 2, etc. And now the digraph is completely determined: we have an edge from v_i to v_j if $j < i$.

2 Prove that every graph G has an orientation such that $|d^+(v) - d^-(v)| \leq 1$ at every vertex v.

As a base case, note that the result is trivial for a 1-vertex tree. For the inductive step, let T be a tree with $|V(T)| \geq 2$ and assume the result holds true for every tree with fewer than $|V(T)|$ vertices. Now choose a leaf vertex $v \in V(T)$ and let $T' = T - v$. By induction we may choose an orientation of T' satisfying

$$\left|\deg^+(v) - \deg^-(v)\right| \leq 1$$

. Let u be the unique neighbour of v in T. Now, we construct an orientation of T by taking the orientation of the edges of T' and then choosing an orientation of uv. Since u satisfies rule

$$\left|\deg^+(v) - \deg^-(v)\right| \leq 1$$

in T' we may choose an orientation of uv so that u also satisfies

$$\left|\deg^+(v) - \deg^-(v)\right| \leq 1$$

in T. Since v automatically satisfies

$$\left|\deg^+(v) - \deg^-(v)\right| \leq 1$$

in T this solves the problem.

3 We already proved that if G is acyclic and $e(G) = n(G) - 1$, then G is connected. Reprove this by adding edges to G.

Let suppose that G is a not connected acyclic graph and $|E(G)| \geq |V(G)| - 1$. This will lead to contradiction, G is acyclic if there is an eadge between v1 and v2 then for every thing v2 is connected to there is no path from that vertex to v1 since acyclic dissallow cycle and the same happens for all vertices and since the G is not connected there exists a vertex with incoming and outgoing degree equals to o, then $|E(G)| < |V(G)| - 1$. which proof that if g is acyclic and $|E(G)| \geq |V(G)| - 1$ it implies that G is connected By contradiction.

4 A saturated hydrocarbon is a molecule formed from hydrogen and carbon atoms, where each hydrogen atom is part of one bond (between atoms), each carbon atom is part of four bonds (between atoms), and there is no cycle structure within the molecule. Prove that $a = 2b + 2$ for any saturated hydrocarbon with a hydrogen atoms and b carbon atoms.

Let us consider a graph G of hydrocarbons with k carbon atoms and m hydrogen atoms. Therefore, total vertices in the graph are $k+m$. Now, consider the following property. For any positive integer n, any tree with n vertices has $n-1$ edges. So, a graph G has $k+m-1$ edges. Therefore, by using the concept of degree, total degree of a graph G is $2(k+m-1)$. Now, let us consider the definition of total degree of the graph. The total degree of G is the sum of the degrees of all the vertices of G. Thus, the total degree is calculated as follows $\deg(G) = \deg(\text{ carbon atoms }) + \deg(\text{ hydrogen atoms }) = 4k + m$. Thus, G has a total degree of $4k + m$.

From above,

$$\begin{aligned}
&2(k+m-1) = 4k + m \\
&2k + 2m - 2 = 4k + m \\
&m = 2k + 2
\end{aligned}$$

Therefore, we can say that a saturated hydrocarbon molecule with k carbon atoms and a maximum number of hydrogen atoms has $2k + 2$ hydrogen atoms.

5 Suppose that T is a tree with $n(T) \geq 2$ in which every vertex adjacent to a leaf has degree at least 3. Prove that T must have a pair of leaves with a common neighbor.

Suppose no two leaves have a common neighbor, then the graph obtained by removing all the leaves has degree at least 2. Now removing a leaf creates a new tree, and so the new graph obtained by removing all the leaves (one by one) should be a tree. However, if every vertex has degree at least 2, we saw in class that the graph contains a cycle using a maximal path argument. This is a contradiction.