Problem 1

$H = X(X^TX)^{-1}X^T$ so $\hat{Y} = X\hat{\beta} = HY$

$\mathbb{E}(SSE) = \mathbb{E}[Y^T(I-H)Y] = \mathbb{E}[(X\beta+\varepsilon)^T(I-H)(X\beta+\varepsilon)]$

$= \sigma^2\,\mathbb{E}[\text{trace}(I-H)] = \sigma^2(n-p)$

Problem 2

① $\hat{Y} = \beta_0 + \beta_1X_1 + \beta_2X_2 + \beta_3X_3 + \beta_4X_4 + \beta_5X_5 + \beta_6X_6$

$= 10.8 + .61X_1 - .073X_2 + .32X_3 + .082X_4 + .038X_5 - .22X_6$

② $\sum y_i = \sum \hat{y}_i = 1939$

③ If the model includes the intercept and we let

$Q(\beta_0, \beta_1, \dots, \beta_p) = \sum(y_i - \hat{y}_i)^2 = \sum(y_i - \beta_0 - \beta_1X_1 - \dots - \beta_pX_p)^2$

then minimizing least squares guarantees

$\frac{dQ}{d\beta_0} = 0 = 2\sum(y_i - \hat{y}_i) \longrightarrow \sum e_i = 0$

④ Classic method gives $\hat{\beta}_3 = .43$

Page 63 method also gives $\hat{\beta}_4 = .43$ since:

Step 1: $\hat{Y} = 20 + .69X_4$

residual is part of Y not explained by X_4

Step 2: $\hat{X}_3 = 9.6 + .72X_4$

residual is part of X_3 not explained by X_4

Step 3: gives $\hat{\beta}_3 = .43$, which is the same since we are measuring the effect of X_3 without X_4 on Y without X_4

problem 3

(a) $\hat{F} = \hat{\beta}_0 + \hat{\beta}_1 P_1 = -22.3 + 1.26 P_1$ (Model I)

$\hat{F} = \hat{\beta}_0 + \hat{\beta}_2 P_2 = -1.8 + 1.0 P_2$ (Model II)

$\hat{F} = \hat{\beta}_0 + \hat{\beta}_1 P_1 + \hat{\beta}_2 P_2 = -14.5 + .44 P_1 + .67 P_2$ (Model III)

(b) Model I: $|t^*| = 1.93 < 2.086 = t_{\alpha/2, n-2} \Rightarrow$ cannot reject

Model II: $|t^*| = 0.24 < 2.086 = t_{\alpha/2, n-2} \Rightarrow$ cannot reject

Model III: $|t^*| = 1.57 < 2.093 = t_{\alpha/2, n-3} \Rightarrow$ cannot reject

(c) Model II, which has predictor P_2, is better since it has a smaller SSE, or equivalently a larger R^2

(d) Model III has the lowest SSE / highest R^2 and predicts

$\hat{F} = -14.9 + .44(78) + .67(85) = 80.7$

problem 4

Note $\sum (x_i - \bar{x})^2 = \sum (x_i^2 - 2\bar{x}x_i + \bar{x}^2) = \sum x_i^2 - 2n\bar{x}^2 + n\bar{x}^2 = \sum x_i^2 - n\bar{x}^2$

and $\sum (y_i - \bar{y})(x_i - \bar{x}) = \sum (x_i y_i - \bar{x} y_i - \bar{y} x_i + \bar{x}\bar{y})$

$= \sum x_i y_i - n\bar{x}\bar{y} - n\bar{x}\bar{y} + n\bar{x}\bar{y}$

$= \sum x_i y_i - n\bar{x}\bar{y}$.

$\therefore \hat{\beta} = \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = (X^T X)^{-1} X^T Y$ for $X = \begin{bmatrix} 1 & x_{11} & \cdots & x_{1p} \\ 1 & x_{21} & \cdots & x_{2p} \\ \vdots & & & \vdots \\ 1 & x_{n1} & & x_{np} \end{bmatrix}$

$= \begin{pmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$

$= \dfrac{1}{n\sum x_i^2 - (\sum x_i)^2} \begin{pmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{pmatrix} \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$

$$= \frac{1/n}{\sum(x_i-\bar{x})^2}\begin{pmatrix}\sum x_i^2 & -n\bar{x}\\ -n\bar{x} & n\end{pmatrix}\begin{pmatrix} n\bar{y} \\ \sum x_i y_i\end{pmatrix}$$

so $\hat{\beta}_1 = \frac{\sum x_i y_i - n\bar{x}\bar{y}}{\sum(x_i-\bar{x})^2} = \frac{\sum(y_i-\bar{y})(x_i-\bar{x})}{\sum(x_i-\bar{x})^2}$ and

$$\bar{y}-\hat{\beta}_1\bar{x} = \frac{\bar{y}\sum(x_i-\bar{x})^2 - \bar{x}\left(\sum x_i y_i - n\bar{x}\bar{y}\right)}{\sum(x_i-\bar{x})^2}$$

$$= \frac{\bar{y}\sum x_i^2 - n\bar{y}\bar{x}^2 - \bar{x}\sum x_i y_i + n\bar{x}^2\bar{y}}{\sum(x_i-\bar{x})^2}$$

$$= \frac{\bar{y}\sum x_i^2 - \bar{x}\sum x_i y_i}{\sum(x_i-\bar{x})^2}$$

which we see equals $\hat{\beta}_0$, the first element in the $\hat{\beta}$ matrix.

Now, $\mathrm{var}(\hat{\beta}) = \mathrm{var}((X^TX)^{-1}X^Ty) = (X^TX)^{-1}X^T\,\mathrm{var}(Y)\,X(X^TX)^{-1}$

$= \sigma^2(X^TX)^{-1}$ for $\mathrm{var}(Y)=\sigma^2$

For SLR:

$$\begin{pmatrix}\mathrm{var}(\hat{\beta}_0) & \mathrm{cov}(\hat{\beta}_0,\hat{\beta}_1)\\ \mathrm{cov}(\hat{\beta}_0,\hat{\beta}_1) & \mathrm{var}(\hat{\beta}_1)\end{pmatrix} = \frac{\sigma^2/n}{\sum(x_i-\bar{x})^2}\begin{pmatrix}\sum x_i^2 & -n\bar{x}\\ -n\bar{x} & n\end{pmatrix}$$

So $\mathrm{var}(\hat{\beta}_1) = \frac{\sigma^2}{\sum(x_i-\bar{x})^2}$ and

$$\mathrm{var}(\hat{\beta}_0) = \frac{\sigma^2}{n}\cdot\frac{\sum x_i^2}{\sum(x_i-\bar{x})^2} = \frac{\sigma^2}{n}\cdot\frac{\sum x_i^2}{\sum x_i^2 - n\bar{x}^2} = \frac{\sigma^2}{n}\cdot\left(1+\frac{n\bar{x}}{\sum x_i^2 - n\bar{x}^2}\right)$$

$$= \sigma^2\left(\frac{1}{n}+\frac{\bar{x}}{\sum x_i^2 - n\bar{x}^2}\right)$$

Problem 5

$$F^* = \left[\frac{SSE(R) - SSE(F)}{df_R - df_F}\right] \Big/ \left[\frac{SSE(F)}{df_F}\right]$$

$$= \left[\frac{SSE(R) - SSE(F)}{3}\right] \Big/ \left[\frac{SSE(F)}{n-4-1}\right]$$

$$= 20.5$$

$F_{\alpha,3,n-5} = 2.71$ and we reject since $F^* > F_\alpha$