

Perspectives on Coupled Oscillators: Geometry, Analysis and Computation

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Synchronization in Natural and Engineering Systems: Open Problems in Modeling, Analysis, and Control

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Outline

1 Recent progress

- Elastic and flow networks on the torus
- Cutset spaces
- Geometric graph theory on the n -torus
- Convexity, monotonicity, and contraction theory
- Multistability in phase-coupled oscillators
- Sync threshold: Approximate inverse via series methods
- Sync threshold: gap between necessary and sufficient conditions
- State-space oscillators

2 Open Problems

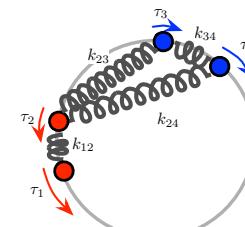
#1: Elastic and flow networks on the torus

$$\omega_i = \sum_{j=1}^n a_{ij} \sin(\theta_i - \theta_j)$$

Spring network

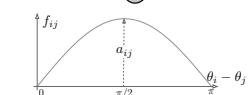
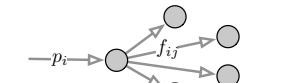
- $\omega_i = \tau_i$: torque at i
- $a_{ij} = k_{ij}$: spring stiffness i, j
- $\sin(\theta_i - \theta_j)$: modulation
- elastic energy

$$\mathcal{E} = \sum_{ij} (1 - \cos(\theta_i - \theta_j))$$

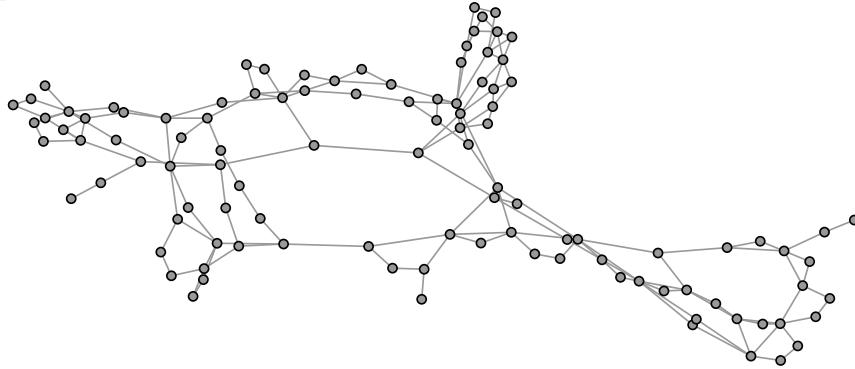


Power network

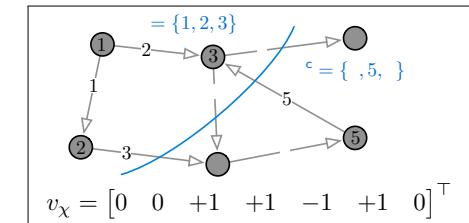
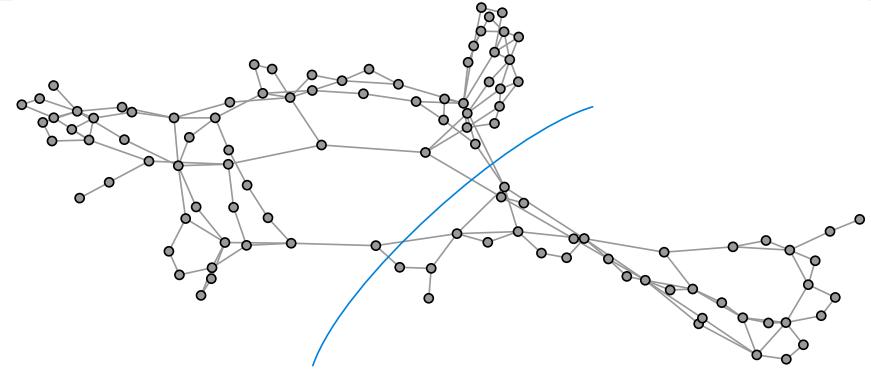
- $\omega_i = p_i$: injected power
- a_{ij} : max power flow i, j
- $\sin(\theta_i - \theta_j)$: modulation
- KCL flow conservation and Ohm's law



#2: Cutset spaces



#2: Cutset spaces



$$\underbrace{\mathbb{R}^m}_{\text{edge space}} = \underbrace{\text{Im}(B^\top)}_{\text{cutset space flow vectors}} \oplus \underbrace{\text{Ker}(BA)}_{\text{weighted cycle space cycle vectors}}$$

$$\mathcal{P} = B^\top L^\dagger BA \quad = \quad \text{cutset projection operator} \text{ — onto } \text{Im}(B^\top) \text{ parallel to } \text{Ker}(BA)$$

① if G unweighted, then \mathcal{P} is orthogonal and $\|\mathcal{P}\|_2 = 1$

② if G acyclic, then $\mathcal{P} = I_m$ and $\|\mathcal{P}\|_p = 1$

③ if G uniform complete or ring, then $\|\mathcal{P}\|_\infty = 2(n-1)/n \leq 2$

④ if θ is the minimal angle between the cutset space and the cycle space of G , then $\sin(\theta) = \|\mathcal{P}\|_2^{-1}$

⑤ if $R_{\text{eff}} \in \mathbb{R}^{n \times n}$ are effective resistances, then $\mathcal{P} = -\frac{1}{2} B^\top R_{\text{eff}} B A$

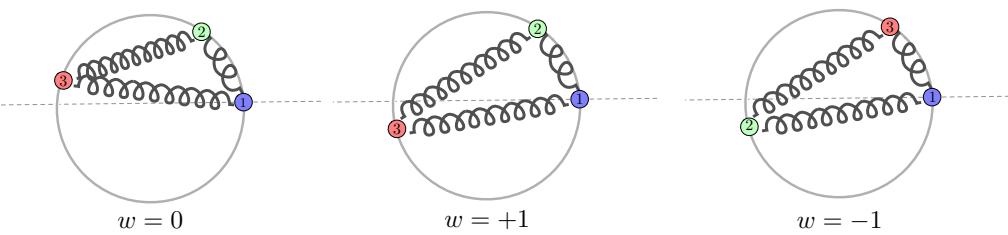
⑥ ...

#3: Winding numbers and partitions

Given a cycle $\sigma = (1, \dots, n_\sigma)$ and orientation

④ **winding number of $\theta \in \mathbb{T}^n$ along σ**

= number of times the **shortest-arc path wraps around torus**



② given basis $\sigma_1, \dots, \sigma_r$ for cycles, **winding vector of θ** is

$$w(\theta) = (w_{\sigma_1}(\theta), \dots, w_{\sigma_r}(\theta))$$

Theorem: Kirchhoff angle law on \mathbb{T}^n

winding number is at most $\pm \lfloor n_\sigma/2 \rfloor - 1$

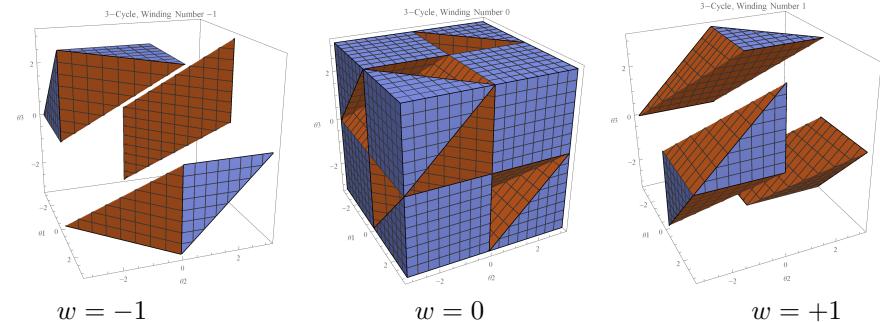


Theorem: Winding partition For each possible winding vector u , define

$$\text{WindingCell}(u) := \{\theta \in \mathbb{T}^n \mid w(\theta) = u\}$$

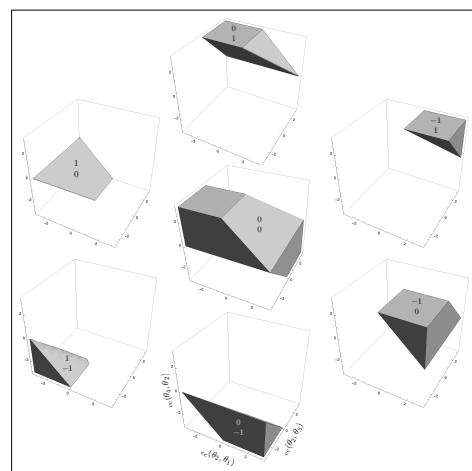
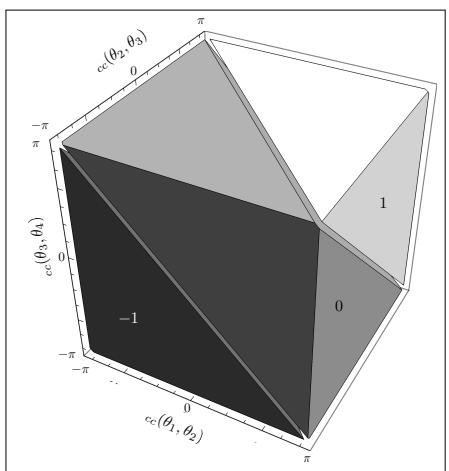
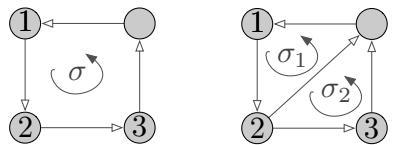
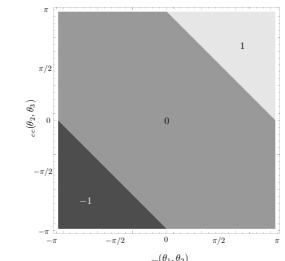
Then

$$\mathbb{T}^n = \cup_u \text{WindingCell}(u)$$

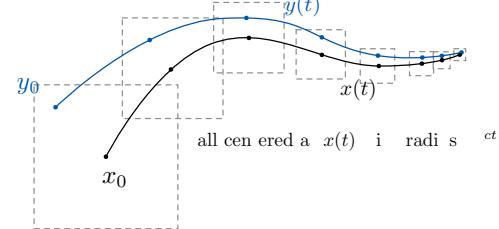


Theorem: Reduced cell is convex polytope

- each winding cell is connected and invariant under rotation
- **bijection:**
reduced winding cell \longleftrightarrow open convex polytope



#4: Analysis: Convexity, monotonicity, and contraction theory



- ① V is **strongly convex** with parameter m
- ② $-\text{grad } V$ is **m -strongly contracting**, that is

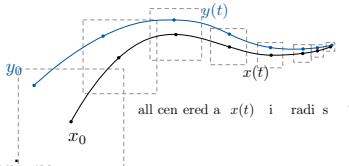
$$(-\text{grad } V(x) + \text{grad } V(y))^\top (x - y) \leq -m \|x - y\|_2^2$$

- ① F is a **monotone operator** (or a **coercive operator**) with parameter m ,
- ② $-F$ is **m -strongly contracting**

search for contraction properties
design engineering systems to be contracting

Highly ordered **transient** and **asymptotic** behavior:

- ① time-invariant F: unique globally exponential stable equilibrium
two natural Lyapunov functions
- ② periodic F: contracting system entrain to periodic inputs
- ③ accurate numerical integration and equilibrium computation
- ④ contractivity rate is natural measure/indicator of robust stability
input-to-state stability
finite input-state gain
contraction margin wrt unmodeled dynamics
input-to-state stability under delayed dynamics



#5: Multistable Sync = global partition + local contraction

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^n a_{ij} \sin(\theta_i - \theta_j)$$

in each winding cell

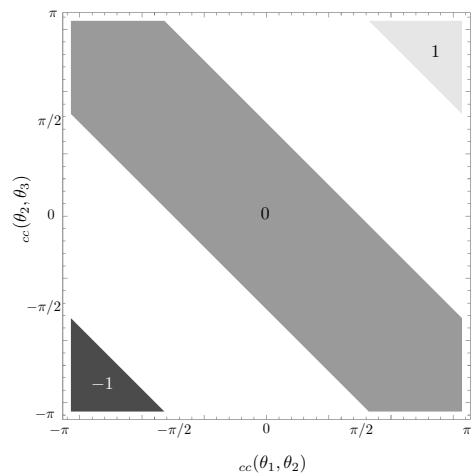
- ① $\dot{\theta} = -\text{grad } \mathcal{E}(\theta)$, where

$$\mathcal{E}(\theta) = \sum_{ij} (1 - \cos(\theta_i - \theta_j)) + \omega^\top \theta$$

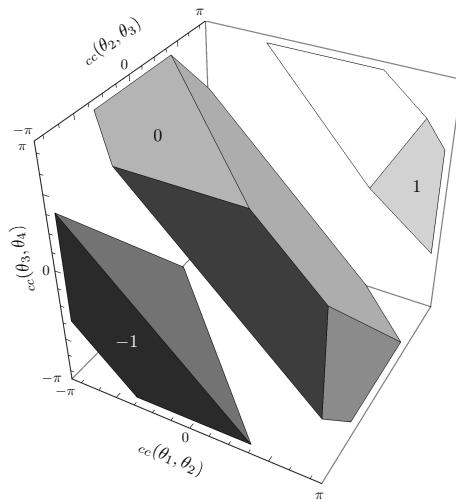
- ② Hessian $\mathcal{E}(\theta) = -\text{Cosine-Laplacian}(\theta) \preceq 0$
- ③ Hessian $\mathcal{E}(\theta) \preceq 0$ on the **cohesive subset** $|\theta_i - \theta_j| \leq \pi/2$
- ④ modulo the symmetry, the dynamics is strongly contracting

Theorem:

- ① each winding cell has at most one cohesive equilibrium
- ② contraction algorithm to decide/compute in each winding cell



(a)



()

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^n a_{ij} \sin(\theta_i - \theta_j + \phi_{ij})$$

same properties, by robustness of contracting dynamics

#6: Sync threshold: Approximate inverse via series methods

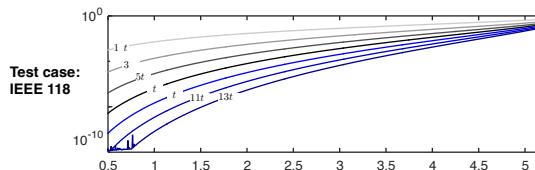
Projection onto to cutset space: $z = B^\top L^\dagger \omega$ and $x = B^\top \theta$

synchrony equilibrium equation is

$$z = \mathcal{P} \sin(x)$$

Given input z , unique solution is

$$\begin{aligned} x &= \sum_{i=0}^{\infty} A_{2i+1}(z), & A_1(z) &= z = B^\top L^\dagger \omega \\ A_3(z) &= \mathcal{P} \left(\frac{1}{3!} z^{o3} \right) \\ A_5(z) &= \mathcal{P} \left(\frac{3}{3!} A_3(z) \circ z^{o2} - \frac{1}{5!} z^{o5} \right) \dots \end{aligned}$$



State of the Art Empirical Results on IEEE Test Cases

Test Case	ratio of test prediction to numerical computation			
	$\ \cdot\ _2$	$\ \cdot\ _\infty$	$\alpha_\infty(\mathcal{P}) \approx 1$ approximate	numerical α_∞ (fmincon)
IEEE 9	16.5 %	73.7 %	92.1 %	85.1 % [†]
IEEE 14	8.3 %	59.4 %	83.1 %	81.3 % [†]
IEEE RTS 24	3.9 %	53.4 %	89.5 %	89.5 % [†]
IEEE 30	2.7 %	55.7 %	85.5 %	85.5 % [†]
IEEE 118	0.3 %	43.7 %	85.9 %	—*
IEEE 300	0.2 %	40.3 %	99.8 %	—*
Polish 2383	0.1 %	29.1 %	82.8 %	—*

[†] fmincon with 100 randomized initial conditions

* fmincon does not converge

#7: Sync threshold: gap between necessary and sufficient conditions

$$z = \mathcal{P} \sin(x)$$

given a norm, define

$$\alpha(\mathcal{P}) := \min \text{ amplification factor of } (\mathcal{P} \text{ diag[sinc}(x)]) < \|\mathcal{P}\|$$

Theorem: Sufficient Cohesive equilibrium angles exist if, in some norm,

$$\|B^\top L^\dagger \omega\| \leq \alpha(\mathcal{P})$$

Necessary Equilibrium angles do not exist if, in some norm

$$\|\mathcal{P}\| \leq \|B^\top L^\dagger \omega\|$$

Considering only first order term in expansion $\iff \alpha_\infty(\mathcal{P}) \approx 1$ (PNAS '13)

#8: State-space oscillators

Coupled networks of:

- ① Stuart-Landau oscillator
- ② FitzHugh–Nagumo neurons
- ③ Rössler chaotic oscillators
- ④ Lienard oscillators (Van Der Pol)
- ⑤ Biological Goodwin models
- ⑥ ...

semi-contraction theory

$$\dot{x}_i = f(t, x_i) - \sum_{j=1}^n a_{ij}(x_i - x_j), \quad i \in \{1, \dots, n\}$$

synchronization as function of

- ① growth rate of the internal dynamics
- ② strength of the diffusive coupling
- ③ heterogeneity of oscillators

Theorem: semi-contraction sufficient condition

If in some norm

$$\text{osLip}(f) < \lambda_2(L)$$

then

- ① semi-contraction rate $\lambda_2(L) - \text{osLip}(f)$,
- ② synchronization $\lim_{t \rightarrow \infty} \|x_i - x_j\| = 0 \quad \text{for every } i, j$

Our recent work

- ① S. Jafarpour, E. Y. Huang, and F. Bullo. [Synchronization of Kuramoto oscillators: Inverse Taylor expansions.](#)
SIAM Journal on Control and Optimization, 57(5):3388–3412, 2019.
doi:[10.1137/18M1216262](https://doi.org/10.1137/18M1216262)
- ② S. Jafarpour and F. Bullo. [Synchronization of Kuramoto oscillators via cutset projections.](#)
IEEE Transactions on Automatic Control, 64(7):2830–2844, 2019.
doi:[10.1109/TAC.2018.2876786](https://doi.org/10.1109/TAC.2018.2876786)
- ③ S. Jafarpour, E. Y. Huang, K. D. Smith, and F. Bullo. [Flow and elastic networks on the \$n\$ -torus: Geometry, analysis and computation.](#)
SIAM Review, 64(1):59–104, 2022.
doi:[10.1137/18M1242056](https://doi.org/10.1137/18M1242056)
- ④ S. Jafarpour, P. Cisneros-Velarde, and F. Bullo. [Weak and semi-contraction for network systems and diffusively-coupled oscillators.](#)
IEEE Transactions on Automatic Control, 67(3):1285–1300, 2022.
doi:[10.1109/TAC.2021.3073096](https://doi.org/10.1109/TAC.2021.3073096)
- ⑤ R. Delabays, S. Jafarpour, and F. Bullo. [Multistability and paradoxes in lossy oscillator networks.](#)
Submitted, February 2022.
URL: <https://arxiv.org/pdf/2202.02439.pdf>

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- ② Open Problems

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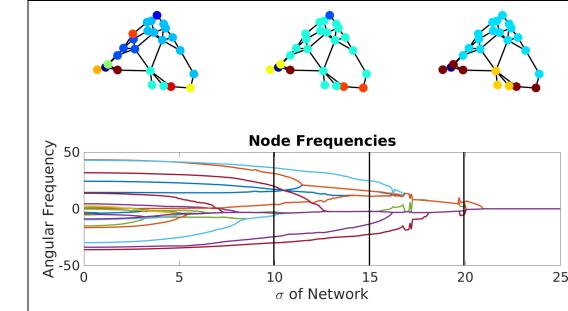
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Future research

- ① Fundamental theory of phased-coupled oscillators
- ② Fundamental theory of state-space-coupled oscillators
- ③ Applications in energy systems
- ④ Applications in machine learning and scientific computing

Fundamental theory of phased-coupled oscillators

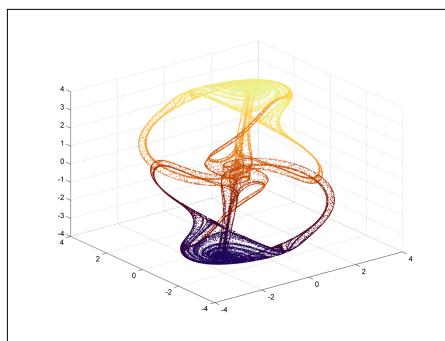
- ① outside cohesive set: signed graphs, symbolic dynamics, ...
- ② non-monotone phase couplings and higher-order dynamics
- ③ analysis and computation of cluster sync and bifurcation diagram



B. Gilg, *Critical Coupling and Synchronized Clusters in Arbitrary Networks of Kuramoto Oscillators*.
PhD thesis, Arizona State University, 2018

Fundamental theory of state-space-coupled oscillators

- ① sharpest sync conditions for benchmarks
- ② transverse contraction
- ③ fractal attractors via α -contraction theory



C. Wu, R. Pines, M. Margaliot, and J.-J. E. Slotine.
Generalization of the multiplicative and additive compounds of square matrices and contraction in the Hausdorff dimension.
IEEE Transactions on Automatic Control, 2022.
doi:10.1109/TAC.2022.3162547

Applications in energy systems

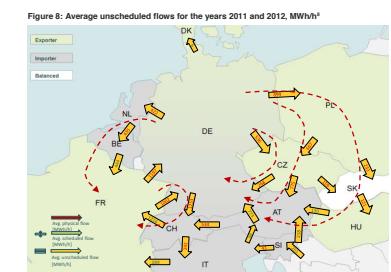
- ① understanding multi-stability in power flows
- ② thick torus conjecture for active/reactive power flow and for OPF
- ③ paradoxes in lossy networks

Practical observations:

sometimes undesirable power flows around loops
sometimes sizable difference between predicted and actual power flows



New York Independent System Operator, *Lake Erie Loop Flow Mitigation*, Technical Report, 2008



Source: THEMA Consulting Group, based on data from 16 TSOs
THEMA Consulting Group, *Loop-flows - Final advice*, Technical Report prepared for the European Commission, 2013

Applications in machine learning and scientific computing

- ① oscillator-based computing
 - ② nanotech allows construction of massively-parallel analog fast low-power devices
CMOS, spin torque nano-oscillators (spintronics), MEMS resonators, optomechanical crystal cavities, ...
 - ③ Example applications:
 - ① NP-complete computing
 - ② associative memory
 - ③ reservoir computing
-
- J. Von Neumann. [Non-linear capacitance or inductance switching, amplifying, and memory organs, December 1957.](#)
US Patent 2,815,488
 - M. H. Matheny et al. [Exotic states in a simple network of nanoelectromechanical oscillators.](#)
Science, 363(6431), 2019.
[doi:10.1126/science.aav7932](https://doi.org/10.1126/science.aav7932)
 - G. Csaba and W. Porod. [Coupled oscillators for computing: A review and perspective.](#)
Applied Physics Reviews, 7(1):011302, 2020.
[doi:10.1063/1.5120412](https://doi.org/10.1063/1.5120412)

Conclusions

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