Distributed Abstract Optimization via Constraints Consensus

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Workshop on Network Induced Constraints in Control
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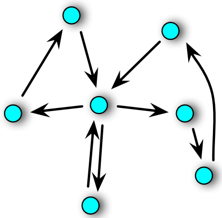
Joint work with Giuseppe Notarstefano, University of Lecce

Preliminary #1: Consensus algorithms



simplest distributed algorithm = linear averaging

each node contains a value x_i and repeatedly executes:

$$x_i^+ := \text{average}(x_i, \{x_j, \text{ for all in-neighbor nodes } j\})$$

each node's value converges to common value
(for strongly connected and aperiodic digraphs)

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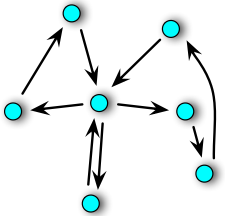

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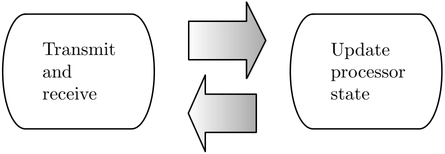
Preliminary #1: Distributed algorithms on networks



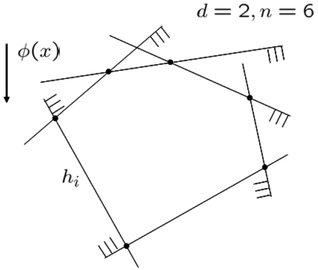
Distributed algorithm for a network of processors consists of

1. W, the **processor state set**
2. $\mathbb{A}$, the **communication alphabet**
3. $\text{stf} : W \times \mathbb{A}^n \to W$, the **state-transition map**
4. $\text{msg} : W \to \mathbb{A}$, the **message-generation map** (often identity map)

Preliminary #2: Optimization problems

Standard LP in d variables with n constraints

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & a_i^T x \leq b_i \qquad i \in \{1, \ldots, n\} \end{aligned}$$



cost function = direction
linear inequalities = halfspace constraints

solution uniquely determined by precisely d constraints
(For special cases, use lexicographic minimum solution)

Howto setup a "distributed optimization problem" from this LP?

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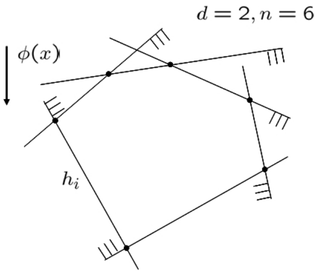

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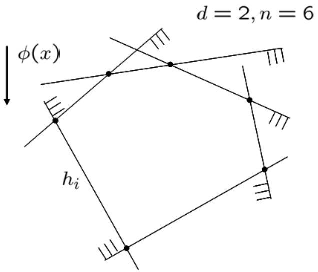

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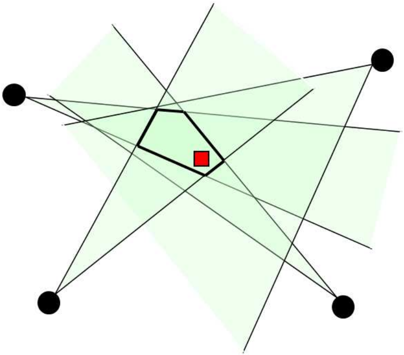

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2. **set-membership localization** = intersection of n convex sets

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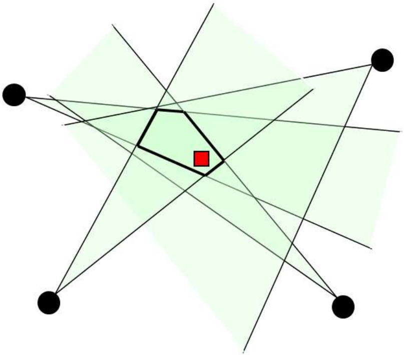

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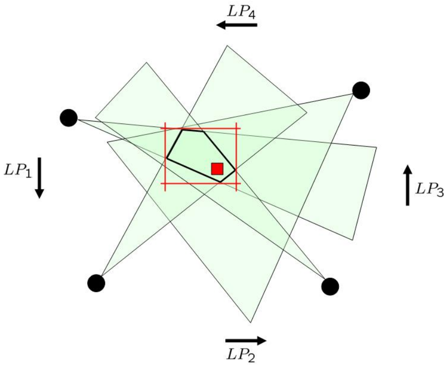

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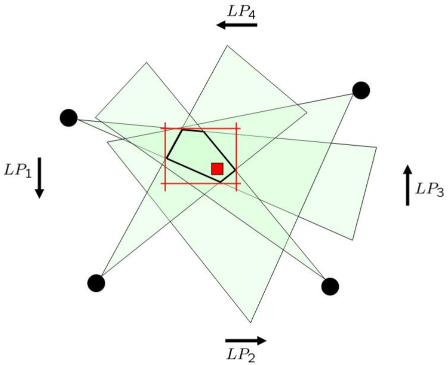

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Problem statement: Distributed optimization

A distributed LP

Assume

1. {direction, n halfspace constraints} is feasible LP in d variables
2. G is directed graph with n nodes, strongly connected
3. memory of node i contains {direction, ith halfspace constraint}

Design distributed algorithm so each node computes global LP solution

Dimensionality assumption

$$d \ll n$$

- network with many nodes (order n) and finite memory (order d)
- network with bounded node degree, also
- for $d \sim n$, see "Parallel Computation" by Bertsekas & Tsitsiklis

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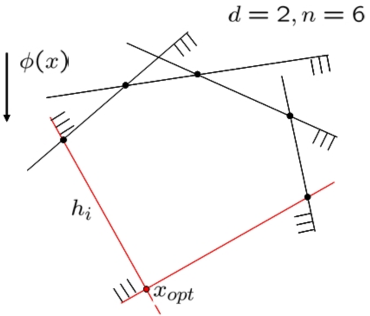
Simple observations



- each node knows some local constraints
- each node can solve "local LP" & compute "local active constraints"
- **achieve consensus upon "global active constraints"**

Solution: first attempt

processor state: a set of constraints $\mathcal{C}_i$ — initialized $\mathcal{C}_i := \{(a_i, b_i)\}$

message generation: transmit the set of constraints $\mathcal{C}_i$

state update rule:

1. collect all constraints

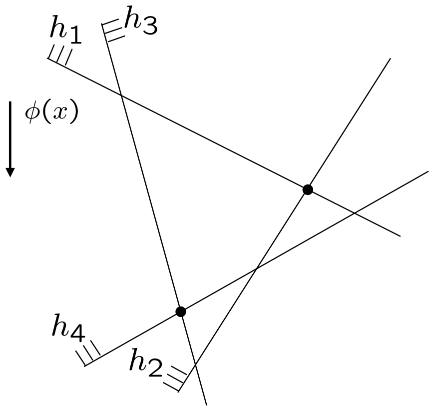
$$\mathcal{C}_{\text{tmp}} := \mathcal{C}_i \cup \Big(\cup_{\text{for all in-neighbor } j} \mathcal{C}_j \Big)$$

2. solve local LP

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && a_k^T x \leq b_k \qquad \text{for all } (a_k, b_k) \in \mathcal{C}_{\text{tmp}} \end{aligned}$$

3. store $\mathcal{C}_i :=$ active constraints in solution of local LP

But constraints need to be re-examined!



Note: h_2 is a global active constraints, but not local:

1. $\{h_1, h_2\}$ is a basis for $\{h_1, h_2, h_3, h_4\}$, but
2. $\{h_3, h_4\}$ is a basis for $\{h_2, h_3, h_4\}$

Solution: Constraints Consensus

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message generation: transmit the set of constraints $\mathcal{C}_i$

state update rule:

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$$\mathcal{C}_{\text{tmp}} := \mathcal{C}_i \cup \big(\cup_{\text{for all in-neighbor } j} \mathcal{C}_j \big) \cup \{(a_i, b_i)\}$$

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3. store

$$\mathcal{C}_i := \begin{cases} \text{active constraints} & \text{if local LP is bounded} \\ \emptyset & \text{if local LP is unbounded} \end{cases}$$

Formal properties of constraints consensus

(Assume one node has bounded solution at initial time)

Monotonicity: the **LP value** at each node is monotonically non-decreasing

Finite time: the **LP value** at each node converges in finite time

Consensus: the **LP values** at all node are equal in finite time

LP solution: after convergence, the **LP constraints set** at each node is an active constraint set for global LP

Uniqueness: if global LP has unique set of active constraints, then the LP constraint set at each node converges that unique set

Time Complexity: unknown, conjectured to be $O(n)$

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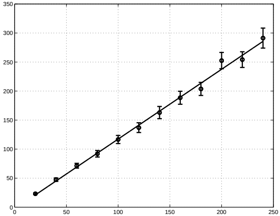
Linear time complexity via Monte Carlo analysis

Nominal problem: $d = 4$, graph $=$ line graph, random LP $=$ hyperplanes with normal vectors uniformly distributed on the unit sphere, and at unit distance from the origin.

Monte Carlo probability estimation: With 99% confidence, there is 99% probability that a nominal problem with $n \in \{40, 60, 80\}$ is solved via constraints consensus in time bounded by $4(n-1)$.



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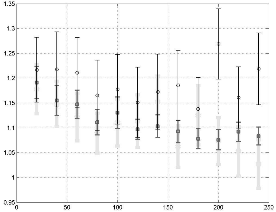
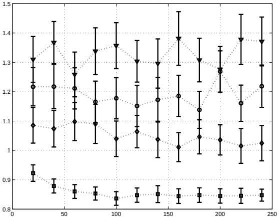


line/Erdős-Rènyi/random-geom graphs



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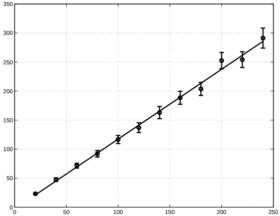

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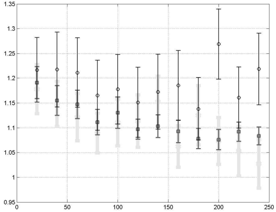

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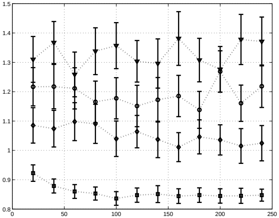

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End of story ... almost

1. we only considered distributed LPs!
2. what about more general optimization problems?
3. how to generalize constraints consensus?
4. what about formation control?

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Abstract Optimization

Abstract optimization problem is (H, ω)

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- $\omega(G)$ is the *value function*
  (minimum value attainable by cost function subject to $G \subset H$)

Axioms

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$$\omega(G) < \omega(G \cup \{h\}) \quad \Longrightarrow \quad \omega(F) < \omega(F \cup \{h\})$$

abstract framework that captures the main features of LP

rich lit: Matousek, Sharir, Welzl, Gärtner, Agarwal, ...

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Ex #1: Distributed training of Support Vector Machines

Max Margin Problem

Separable training set = a separable set $\{(x_i, \ell_i)\}$ of n examples $x_i \in \mathbb{R}^k$ and labels $\ell_i \in \{-1, +1\}$. Find $(t_+, t_-) \in \mathbb{R}^2$ and $w \in \mathbb{R}^k$

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}\|w\|^2 - (t_+ - t_-) \\ \text{subject to} \quad & w \cdot x_i \geq t_+ \quad \text{if } \ell_i = +1 \\ & w \cdot x_i \leq t_- \quad \text{if } \ell_i = -1 \end{aligned}$$

Balcázar et al, TCS '08: Max Margin satisfies axioms

Distributed Max Margin Problem

1. a separable training set $\{(x_i, \ell_i)\}$
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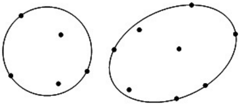

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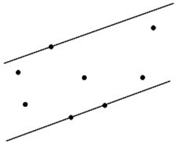

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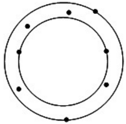

Application to motion coordination in robotic networks

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2. from distributed shape consensus, easy to design formation control

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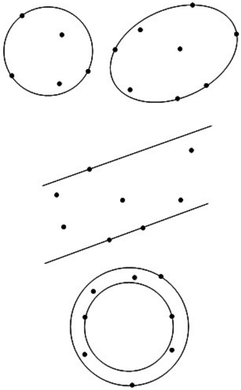

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End of story

1. distributed abstract optimization
2. consensus constraints: correctness and time complexity
3. applications to target tracking & formation control

References:

- B. Gärtner. A subexponential algorithm for abstract optimization problems. *SIAM J Computing*, 24(5):1018–1035, 1995
- P. K. Agarwal and M. Sharir. Efficient algorithms for geometric optimization. *ACM Computing Surveys*, 30(4):412–458, 1998
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- G. Notarstefano and F. Bullo. Distributed abstract optimization via constraints consensus: theory and applications, October 2009. Available at `http://arxiv.org/abs/0910.5816`