Kron Reduction of Graphs with Applications to Electrical Networks

Florian Dörfler and Francesco Bullo

Center for Control,
Dynamical Systems & Computation
University of California at Santa Barbara
http://motion.me.ucsb.edu

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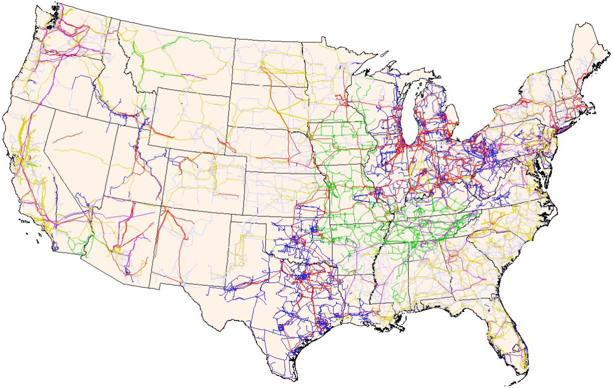
Motivation: the current power grid is . . .



". . . the greatest engineering achievement of the 20th century."

[National Academy of Engineering '10]

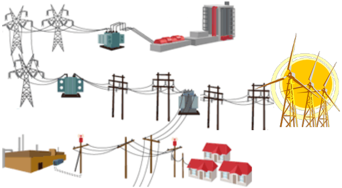
". . . the largest and most complex machine engineered by humankind."

[P. Kundur '94, V. Vittal '03, . . .]

Motivation: the envisioned power grid

Energy is one of the top three national priorities

Expected developments in "**smart grid**":



1. large number of distributed power sources
2. increasing adoption of renewables
3. sophisticated cyber-coordination layer

☹ **challenges:** increasingly complex networks & stochastic disturbances

☺ **opportunity:** some smart grid keywords:
control/sensing/optimization $\oplus$ *distributed/coordinated/decentralized*

Today: "reducing the complexity by means of circuit and graph theory"

Motivation: the envisioned power grid

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Energy is one of the top three national priorities

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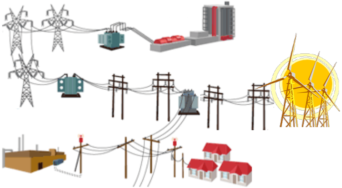

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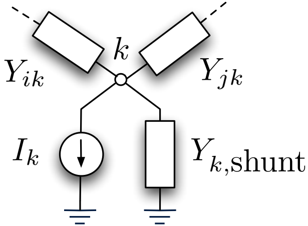
Today: "reducing the complexity by means of circuit and graph theory"

Kron reduction of a resistive circuit

1. Nodal analysis by Kirchhoff's and Ohm's laws:

$$\boxed{I = Y \cdot V}$$



$I \in \mathbb{C}^n$	nodal current injections
$V \in \mathbb{C}^n$	nodal voltages/potentials
$Y \in \mathbb{C}^{n \times n}$	nodal conductance matrix

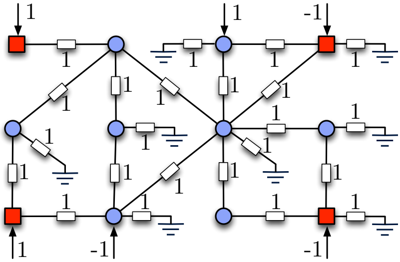
$$Y = Y^T = \begin{bmatrix} \vdots & \ddots & \vdots & \iddots & \vdots \\ -Y_{i1} & \cdots & \sum_{k=1, k \neq i}^{n} Y_{ik} + Y_{k,\text{shunt}} & \cdots & -Y_{in} \\ \vdots & \iddots & \vdots & \ddots & \vdots \end{bmatrix}$$

$$= \{\text{ weighted Laplacian matrix } \} + \mathrm{diag}\big(Y_{k,\text{shunt}}\big) = \text{ “loopy Laplacian”}$$

Kron reduction of a resistive circuit

2. Partition circuit equations via boundary nodes & interior nodes:

$$\begin{bmatrix} I_{\text{boundary}} \\ \hline I_{\text{interior}} \end{bmatrix} = \left[\begin{array}{c|c} Y_{\text{boundary}} & Y_{\text{bound-int}} \\ \hline Y_{\text{bound-int}}^T & Y_{\text{interior}} \end{array}\right] \begin{bmatrix} V_{\text{boundary}} \\ \hline V_{\text{interior}} \end{bmatrix}$$

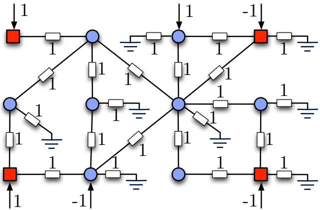


Boundary nodes ■ arise as natural terminals in applications.

Kron reduction of a resistive circuit

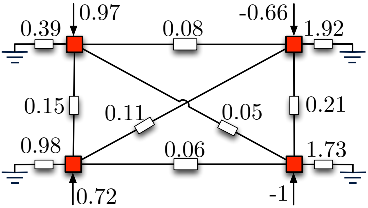
3. **Kron reduction**: eliminate interior nodes ● via Schur complement:

$$Y_{\text{red}} = Y / Y_{\text{interior}} = Y_{\text{boundary}} - Y_{\text{bound-int}} \cdot Y_{\text{interior}}^{-1} \cdot Y_{\text{bound-int}}^{T}$$



original circuit

$$I = Y \cdot V$$



"equivalent" reduced circuit

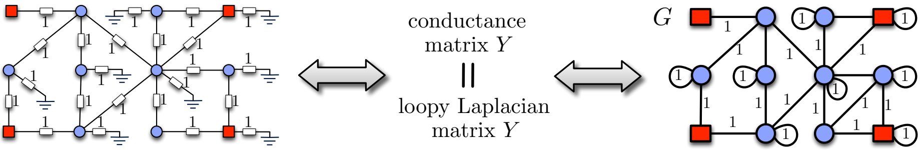
$$I_{\text{red}} = Y_{\text{red}} \cdot V_{\text{boundary}}$$

G. Kron, "Tensor Analysis of Networks," *Wiley*, 1939.

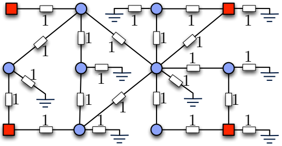
Kron reduction of graphs

Consider either of the following three equivalent setups:

1. a connected electrical network with conductance matrix Y, terminals ■ , interior nodes ● , & possibly shunt conductances
2. a symmetric and irreducible loopy Laplacian matrix Y with partition (■ ,●), & possibly diagonally dominance
3. an undirected, connected, & weighted graph with boundary nodes ■ , interior nodes ● , & possibly self-loops



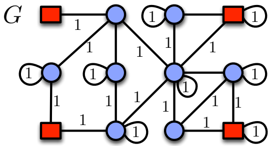
Kron reduction via Schur complement: $\boxed{Y_{\text{red}} = Y / Y_{\text{interior}}}$



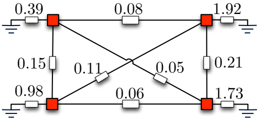
conductance matrix Y

$||$

loopy Laplacian matrix Y



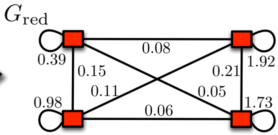
Kron reduction | Kron reduction | Kron reduction



transfer conductance matrix Y_{red}

$||$

Kron-reduced loopy Laplacian Y_{red}



Kron reduction of graphs

Kron reduction via Schur complement: $\boxed{Y_{\mathbf{red}} = Y / Y_{\mathbf{interior}}}$

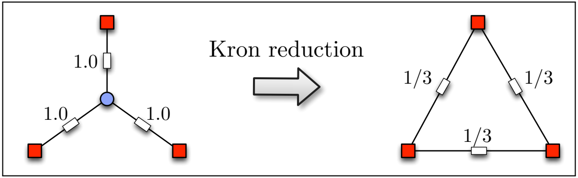
- Relation of spectrum and algebraic properties of Q and Q_{red}?
- How about the graph topologies and the effective resistances?
- What is the effect of a perturbation in the original graph on the reduced graph, its spectrum, and its effective resistance?
- Finally, why is this graph reduction process of practical importance and in which application areas?

Kron reduction of graphs: applications

Purpose: construct low-dimensional equivalent circuits / graphs / models

Simplest non-trivial case: star-Δ transformation

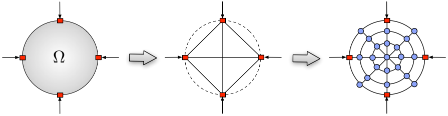
[A. E. Kennelly 1899, A. Rosen 1924]



- **Engineering applications:** smart grid monitoring, circuit theory, model reduction for power and water networks, power electronics, large-scale integration chips, electrical impedance tomography, data-mining, ...
- **Mathematics applications:** sparse matrix algorithms, finite-element methods, sparse multi-grid solvers, Markov chain reduction, stochastic complementation, applied linear algebra & matrix analysis, Dirichlet-to-Neumann map, ...
- **Physics applications:** knot theory, Yang-Baxter equations and applications, high-energy physics, statistical mechanics, vortices in fluids, entanglement of polymers & DNA, ... [F. Dörfler & F. Bullo '11, J.H.H. Perk & H. Au-Yang '06]

Kron reduction of graphs: applications

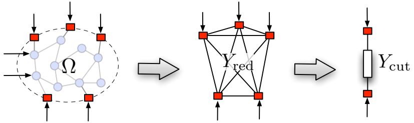
Electrical impedance tomography



to reconstruct spatial conductivity

[E. Curtis and J. Morrow '94 & '00]

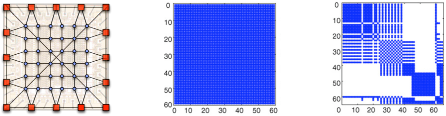
Smart grid monitoring



through cut-set variables

[I. Dobson '11]

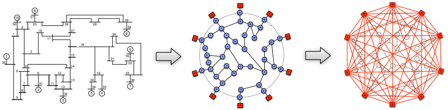
Representation of **integration chips**



for sparse computation

[J. Rommes and W. H. A. Schilders '09]

Reduced power network modeling



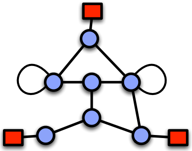
for stability analysis and control

[F. Dörfler and F. Bullo '09]

Kron reduction of graphs: properties

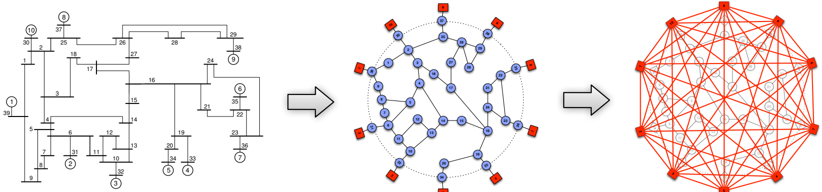
Kron reduction of a graph with

- boundary ■ , interior ● , non-neg self-loops ⟲
- loopy Laplacian matrix Y
- Schur complement: $Y_{\mathbf{red}} = Y / Y_{\mathbf{interior}}$



Properties of Kron reduction:

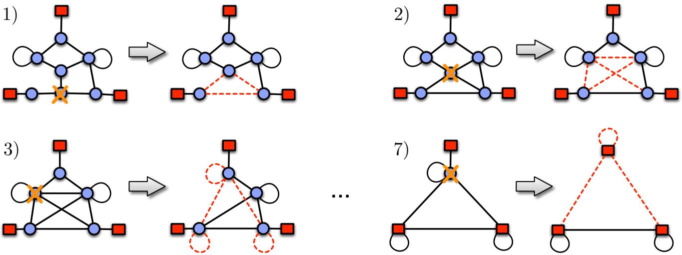
1. **Well-posedness:** set of loopy Laplacian matrices is closed



Kron reduction of graphs: properties

2. **Iterative 1-dim Kron reduction**: $\mathbf{Y}_{\text{red}}^{k+1} = \mathbf{Y}_{\text{red}}^{k} / \bullet$

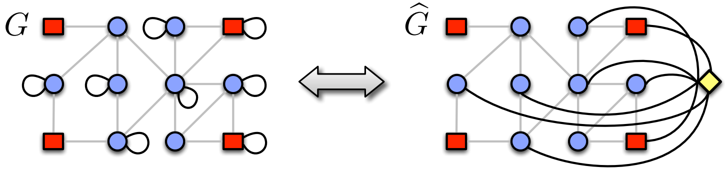
$\Rightarrow$ topological evolution of the corresponding graph



$\Rightarrow$ **Equivalence**: the following diagram commutes:

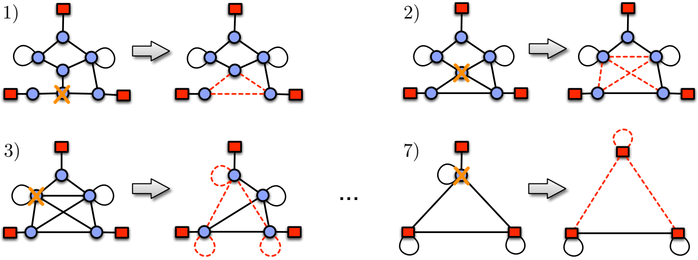
$$\begin{array}{ccc} & \text{Kron reduction of } \bullet & \\ Y = Y_{\text{red}}^{0} & \xrightarrow{\hspace{6em}} & Y_{\text{red}} = Y_{\text{red}}^{n-|\blacksquare|} \\ \downarrow & & \uparrow \\ Y_{\text{red}}^{1} \longrightarrow Y_{\text{red}}^{2} & \longrightarrow \cdots \longrightarrow & Y_{\text{red}}^{n-|\blacksquare|+1} \\ & \text{iterative Kron reduction} & \end{array}$$

3. **Augmentation**: replace self-loops ↻ by edge to grounded node ◊



$\Rightarrow$ **Equivalence**: the following diagram commutes:

$$
\begin{tikzcd}
Y \arrow[r, "\text{augment}"] \arrow[d, swap, "\text{Kron reduction of } \bullet"] & \widehat{Y} \arrow[d, "\text{Kron reduction of } \bullet"] \\
Y_{\text{red}} \arrow[r, swap, "\text{augment}"] & \widehat{Y}_{\text{red}}
\end{tikzcd}
$$



4. **Topological properties:**
   - interior network connected ⇒ reduced network complete
   - at least one node in interior network features a self-loop ⟲
     ⇒ all nodes in reduced network feature self-loops ⟲
5. **Algebraic properties:** self-loops in interior network
   - decrease mutual coupling in reduced network
   - increase self-loops in reduced network

6. **Spectral properties**:
   - interlacing property: $\lambda_i(Y) \leq \lambda_i(Y_{\text{red}}) \leq \lambda_{i+n-|\blacksquare|}(Y)$

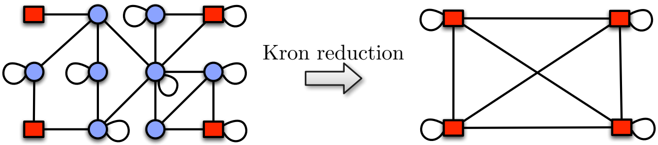
   $\Rightarrow$ algebraic connectivity λ_2 is non-decreasing

   - effect of self-loops $\circlearrowleft$ on loop-less Laplacian matrices:

$$\lambda_2(L_{\text{red}}) + \max\{\circlearrowleft\} \geq \lambda_2(L) + \min\{\circlearrowleft\}$$

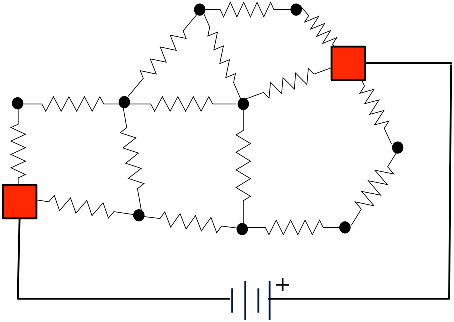
   $\Rightarrow$ self-loops weaken the algebraic connectivity λ_2

Example: all mutual edges have unit weight



without self-loops: $\lambda_2(L) = 0.39 \leq 0.69 = \lambda_2(L_{\text{red}})$
with unit self-loops: $\lambda_2(L) = 0.39 \geq 0.29 = \lambda_2(L_{\text{red}})$

7. **Effective resistance** R_{ij} :



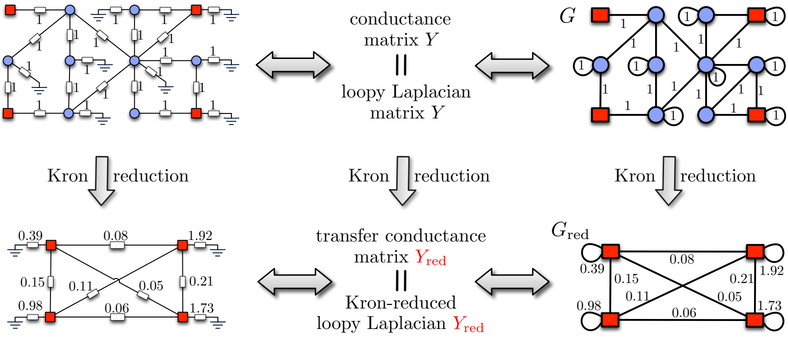
7. **Effective resistance** R_{ij} :

- **Equivalence and invariance** of R_{ij} among ■ nodes:

$$\begin{tikzcd}
Y \arrow[rr, "\text{augment}"] \arrow[dr] \arrow[dd, "\text{Kron reduction of } \bigcirc"'] & & \widehat{Y} \arrow[dl] \arrow[dd, "\text{Kron reduction of } \bigcirc"] \\
& \begin{array}{c} R_{ij} \\ i,j \in \alpha \end{array} & \\
Y_{\text{red}} \arrow[rr, "\text{augment}"'] \arrow[ur] & & \widehat{Y}_{\text{red}} \arrow[ul]
\end{tikzcd}$$

- no self-loops: R_{ij} among ■ uniform $\Leftrightarrow$ $\frac{1}{R_{ij}} = \frac{■}{2} |Y_{\text{red}}(i,j)|$
- self-loops: R_{ij} among ■ & ◇ uniform $\Leftrightarrow$ $\frac{1}{R_{ij}} = \frac{■}{2} |Y_{\text{red}}(i,j)| + \max\{⟲\}$

Conclusions



- Kron reduction is important in various applications
- Analysis of Kron reduction via algebraic graph theory
- Open problem: directed & complex-weighted graphs