Contraction Theory in Systems and Control

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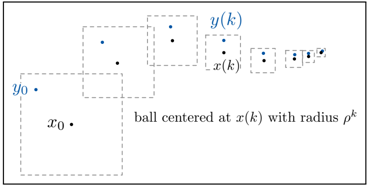

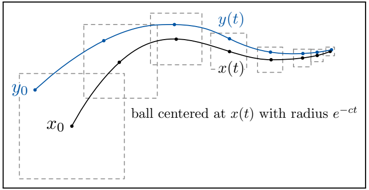

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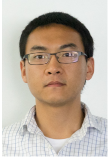

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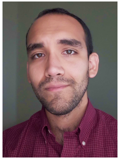

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contractivity = robust computationally-friendly stability

fixed point theory + Lyapunov stability theory + geometry of metric spaces

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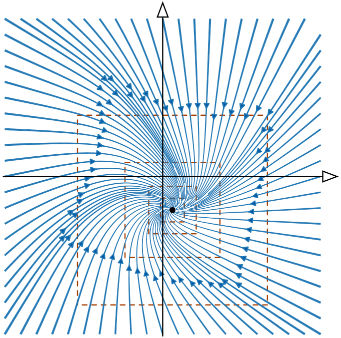
fixed point theory + Lyapunov stability theory + geometry of metric spaces

highly-ordered transient and asymptotic behavior:

1. unique globally exponential stable equilibrium
   & two natural Lyapunov functions
2. robustness properties
   - bounded input, bounded output (iss)
   - finite input-state gain
   - robustness margin wrt unmodeled dynamics
   - robustness margin wrt delayed dynamics
3. periodic input, periodic output
4. modularity and interconnection properties
5. accurate numerical integration and equilibrium point computation



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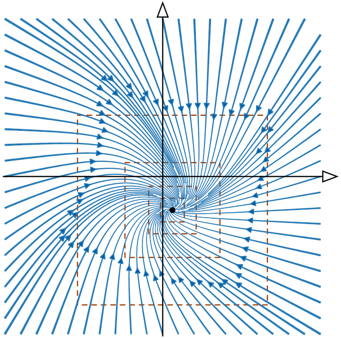

search for contraction properties

design engineering systems to be contracting

Contraction theory: historical notes

- **Origins**

  S. Banach. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae*, 3(1):133–181, 1922.

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  S. M. Lozinskii. Error estimate for numerical integration of ordinary differential equations. I. *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, 5:52–90, 1958. URL **http://mi.mathnet.ru/eng/ivm2980**. (in Russian)

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- **Systems and control**:

  W. Lohmiller and J.-J. E. Slotine. On contraction analysis for non-linear systems. *Automatica*, 34(6): 683–696, 1998.

- **Incomplete list of contributors who influenced me**
  Aminzare, Arcak, Chung, Coogan, Di Bernardo, Manchester, Margaliot, Pavlov, Pham, Proskurnikov, Russo, Sepulchre, Slotine, Sontag, ...

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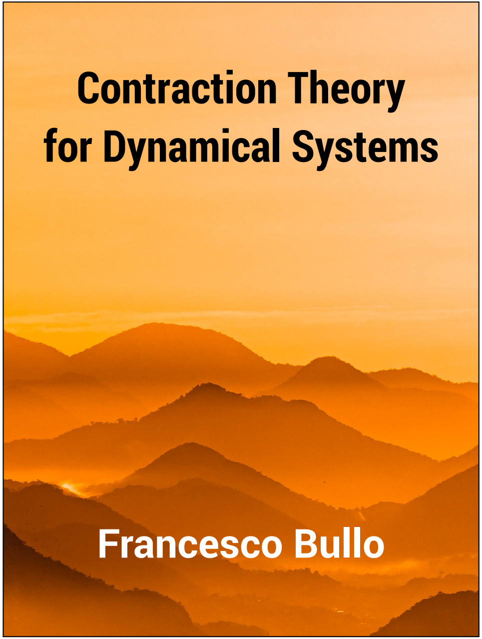
- **Surveys**:

  Z. Aminzare and E. D. Sontag. Contraction methods for nonlinear systems: A brief introduction and some open problems. In *IEEE Conf. on Decision and Control*, pages 3835–3847, Dec. 2014.

  M. Di Bernardo, D. Fiore, G. Russo, and F. Scafuti. Convergence, consensus and synchronization of complex networks via contraction theory. In *Complex Systems and Networks*. Springer, 2016.

  H. Tsukamoto, S.-J. Chung, and J.-J. E. Slotine. Contraction theory for nonlinear stability analysis and learning-based control: A tutorial overview. *Annual Reviews in Control*, 52:135–169, 2021.

  P. Giesl, S. Hafstein, and C. Kawan. Review on contraction analysis and computation of contraction metrics. *Journal of Computational Dynamics*, 10(1):1–47, 2023.



Contraction Theory for Dynamical Systems, Francesco Bullo, KDP, 1.1 edition, 2023, ISBN 979-8836646806

1. Textbook with exercises and answers. Format: textbook, slides, and paperbook
2. Content:
   Fixed point theory
   Theory of contracting dynamics on vector spaces
   Applications to nonlinear and interconnected systems
3. Self-Published and Print-on-Demand at:
   https://www.amazon.com/dp/B0B4K1BTF4
4. PDF Freely available at
   https://fbullo.github.io/ctds
5. 10h minicourse on youtube:
   https://youtu.be/RvR47ZbqJjc
6. Future version to include: systems on Riemannian manifolds, homogeneous spaces, and solid cones

   "Continuous improvement is better than delayed perfection"
   Mark Twain

Outline

1. Contractivity of dynamical systems
   - From discrete-time to continuous-time dynamics
   - Table of infinitesimal contractivity conditions
   - Application to recurrent neural networks
   - Connection with convex optimization
2. From closed to open, interconnected and optimal systems
3. From nominal to uncertain, local and weakly contracting systems
4. Contractivity on Riemannian manifolds
   - A contractivity conjecture
5. Conclusions and Future Research

Linear algebra: induced norms

Vector norm	Induced matrix norm	Induced matrix log norm
$\|x\|_1 = \sum_{i=1}^n \|x_i\|$	$\|A\|_1 = \max_{j\in\{1,\dots,n\}} \sum_{i=1}^n \|a_{ij}\|$	$\mu_1(A) = \max_{j\in\{1,\dots,n\}} \left(a_{jj} + \sum_{i=1,i\neq j}^n \|a_{ij}\|\right)$ $=$ max column "absolute sum" of A
$\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$	$\|A\|_2 = \sqrt{\lambda_{\mathsf{max}}(A^\top A)}$	$\mu_2(A) = \lambda_{\mathsf{max}}\left(\frac{A+A^\top}{2}\right)$
$\|x\|_\infty = \max_{i\in\{1,\dots,n\}} \|x_i\|$	$\|A\|_\infty = \max_{i\in\{1,\dots,n\}} \sum_{j=1}^n \|a_{ij}\|$	$\mu_\infty(A) = \max_{i\in\{1,\dots,n\}} \left(a_{ii} + \sum_{j=1,j\neq i}^n \|a_{ij}\|\right)$ $=$ max row "absolute sum" of A

Discrete-time dynamics and Lipschitz constants

$$x_{k+1} = \mathsf{F}(x_k) \qquad \text{on } \mathbb{R}^n \text{ with norm } \|\cdot\| \text{ and induced norm } \|\cdot\|$$

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Lipschitz constant

$$\begin{aligned}\mathsf{Lip}(\mathsf{F}) &= \inf\{\ell > 0 \;\text{ such that }\; \|\mathsf{F}(x) - \mathsf{F}(y)\| \leq \ell \|x - y\| \quad \text{for all } x, y\} \\ &= \sup_x \|\mathsf{J}_{\mathsf{F}}(x)\|\end{aligned}$$

For **scalar map** f, $\quad \mathsf{Lip}(f) = \sup_x |f'(x)|$

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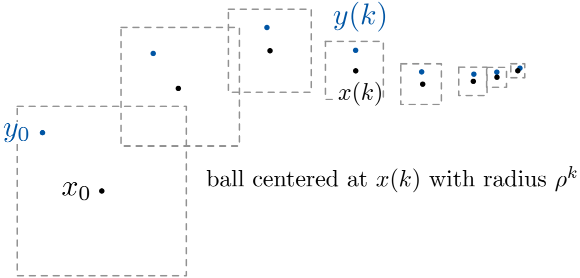
For **affine map** $\mathsf{F}_A(x) = Ax + a$

$$\begin{aligned}\|x\|_{2,P} &= (x^\top P x)^{1/2} & \mathsf{Lip}_{2,P}(\mathsf{F}_A) &= \|A\|_{2,P} \leq \ell & \iff & \quad A^\top P A \preceq \ell^2 P \\ \|x\|_{\infty,\eta} &= \max_i |x_i|/\eta_i & \mathsf{Lip}_{\infty,\eta}(\mathsf{F}_A) &= \|A\|_{\infty,\eta} \leq \ell & \iff & \quad \eta^\top |A| \leq \ell \eta^\top\end{aligned}$$

Banach contraction theorem for discrete-time dynamics:

If $\rho := \mathrm{Lip}(\mathsf{F}) < 1$, then

1. F is **contracting** $=$ distance between trajectories decreases exp fast (ρ^k)
2. F has a unique, glob exp stable equilibrium x^*



From discrete to continuous time

The **induced log norm** of $A \in \mathbb{R}^{n\times n}$ wrt to $\|\cdot\|$:

$$\mu(A) := \lim_{h\to 0^+} \frac{\|I_n + hA\| - 1}{h}$$

subadditivity:	$\mu(A+B) \leq \mu(A) + \mu(B)$	
scaling:	$\mu(bA) = b\mu(A),$	$\forall b \geq 0$

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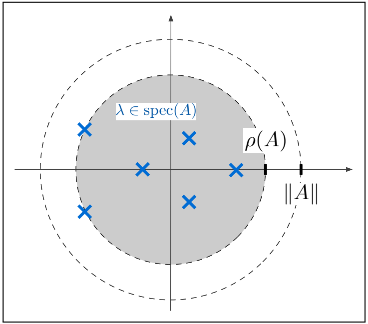

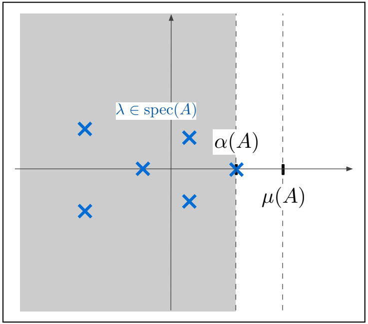

Example induced log norms

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Continuous-time dynamics and one-sided Lipschitz constants

$\dot{x} = \mathsf{F}(x)$ on $\mathbb{R}^n$ with norm $\|\cdot\|$ and induced log norm $\mu(\cdot)$

Continuous-time dynamics and one-sided Lipschitz constants

$$\dot{x} = \mathsf{F}(x) \qquad \text{on } \mathbb{R}^n \text{ with norm } \|\cdot\| \text{ and induced log norm } \mu(\cdot)$$

One-sided Lipschitz constant

$$\begin{aligned} \mathsf{osLip}(\mathsf{F}) &= \inf\{b \in \mathbb{R} \text{ such that } \langle\!\langle \mathsf{F}(x) - \mathsf{F}(y), x - y \rangle\!\rangle \leq b\|x-y\|^2 \quad \text{for all } x, y\} \\ &= \sup_x \mu(\mathsf{J}_{\mathsf{F}}(x)) \end{aligned}$$

For **scalar map** f, $\quad \mathsf{osLip}(f) = \sup_x f'(x)$

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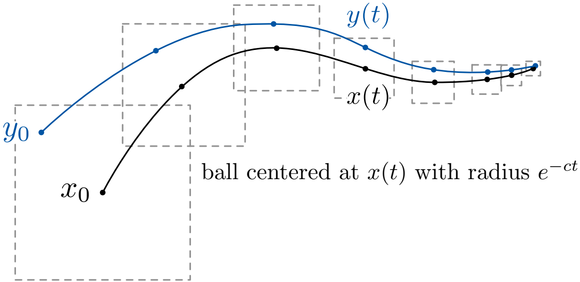

From inner products to weak pairings

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|_2^2 \;=\; \dot{x}^\top x \;=\; \langle\!\langle \dot{x}, x \rangle\!\rangle$$

$$\implies \qquad \tfrac{1}{2}D^+\|x(t)\|^2 \;=:\; [\![\dot{x}, x]\!]$$

- D^+ is upper-right Dini derivative

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- D^+ is upper-right Dini derivative
- **weak pairing** $[\![\cdot,\cdot]\!] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ exists for each norm, i.e.,

$$[\![y, x]\!]_1 \;:=\; \|x\|_1 \operatorname{sign}(x)^\top y \qquad \text{(sign pairing)}$$

$$[\![y, x]\!]_\infty \;:=\; \max_{i \in \mathcal{A}_\infty(x)} x_i y_i \qquad \text{for } \mathcal{A}_\infty(x) = \{i \ \text{ s.t. } |x_i| = \|x\|_\infty\} \qquad \text{(max pairing)}$$

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theory of weak pairings: computational properties and applications to monotone operators

Log norm bounds	Demidovich conditions	One-sided Lipschitz conditions
$\mu_{2,P}(\mathsf{J}_{\mathsf{F}}(x)) \leq -c$	$P\mathsf{J}_{\mathsf{F}}(x) + \mathsf{J}_{\mathsf{F}}(x)^\top P \preceq -2cP$	$(x-y)^\top P\big(\mathsf{F}(x) - \mathsf{F}(y)\big) \leq -c\\|x-y\\|_{P^{1/2}}^2$
$\mu_1(\mathsf{J}_{\mathsf{F}}(x)) \leq -c$	$\operatorname{sign}(v)^\top \mathsf{J}_{\mathsf{F}}(x)v \leq -c\\|v\\|_1$	$\operatorname{sign}(x-y)^\top(\mathsf{F}(x) - \mathsf{F}(y)) \leq -c\\|x-y\\|_1$
$\mu_\infty(\mathsf{J}_{\mathsf{F}}(x)) \leq -c$	$\max\limits_{i\in\mathcal{A}_\infty(v)} v_i\,(\mathsf{J}_{\mathsf{F}}(x)v)_i \leq -c\\|v\\|_\infty^2$	$\max\limits_{i\in\mathcal{A}_\infty(x-y)} (x_i-y_i)(\mathsf{F}_i(x)-\mathsf{F}_i(y)) \leq -c\\|x-y\\|_\infty^2$

Each row = three equivalent statements. To be understood for all $x, y \in \mathbb{R}^n$ and all $v \in \mathbb{R}^n$.

One sided Lipschitz conditions

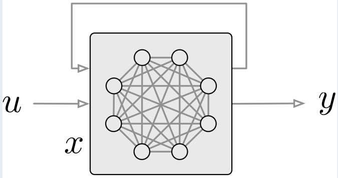
1. **simple sufficient condition** for uniqueness of continuous ODEs in: A. F. Filippov. *Differential Equations with Discontinuous Righthand Sides*. Kluwer, 1988. ISBN 902772699X (Chapter 1, page 5, citing Krasnosel'skii and Krein 1955)
2. **one-sided Lipschitz maps** in: E. Hairer, S. P. Nørsett, and G. Wanner. *Solving Ordinary Differential Equations I. Nonstiff Problems*. Springer, 1993. doi (Section I.10)

3. **uniformly decreasing maps** in: L. Chua and D. Green. A qualitative analysis of the behavior of dynamic nonlinear networks: Stability of autonomous networks. *IEEE Transactions on Circuits and Systems*, 23(6):355–379, 1976. doi
4. **maps with negative nonlinear measure** in: H. Qiao, J. Peng, and Z.-B. Xu. Nonlinear measures: A new approach to exponential stability analysis for Hopfield-type neural networks. *IEEE Transactions on Neural Networks*, 12(2): 360–370, 2001. doi
5. **dissipative Lipschitz maps** in: T. Caraballo and P. E. Kloeden. The persistence of synchronization under environmental noise. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 461 (2059):2257–2267, 2005. doi
6. **maps with negative lub log Lipschitz constant** in: G. Söderlind. The logarithmic norm. History and modern theory. *BIT Numerical Mathematics*, 46(3):631–652, 2006. doi
7. **QUAD maps** in: W. Lu and T. Chen. New approach to synchronization analysis of linearly coupled ordinary differential systems. *Physica D: Nonlinear Phenomena*, 213(2):214–230, 2006. doi
8. **incremental quadratically stable maps** in: L. D'Alto and M. Corless. Incremental quadratic stability. *Numerical Algebra, Control and Optimization*, 3:175–201, 2013. doi

Advantages of non-Euclidean approaches

1. *well suited for certain class of systems*
   ℓ_1 for monotone flow systems
2. *computational advantages*
   ℓ_1/ℓ_∞ constraints lead to LPs, whereas ℓ_2 constraints leads to LMIs
3. *robustness to structural perturbations*
   ℓ_1/ℓ_∞ contractions are connectively robust (i.e., edge removal)
4. *adversarial input-output analysis*
   ℓ_∞ better suited for the analysis of adversarial examples than ℓ_2
5. *asynchronous distributed computation*
   ℓ_∞ contractions converge under fully asynchronous distributed execution

Application: ℓ_∞-contracting neural networks



$$\dot{x} = -x + \Phi(Ax + Bu + b) \qquad \text{(*recurrent NN*)}$$

$$x = \Phi(Ax + Bu + b) \qquad \text{(*implicit NN*)}$$

$$x_{k+1} = (1-\alpha)x_k + \alpha\Phi(Ax_k + Bu + b) \qquad \text{(*forward Euler*)}$$

If

$$\mu_\infty(A) < 1 \qquad \left(\text{i.e., } a_{ii} + \sum_j |a_{ij}| < 1 \text{ for all } i\right)$$

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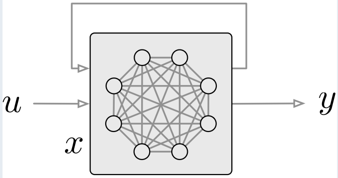

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If

$$\mu_\infty(A) < 1 \qquad \Big(\text{i.e., } a_{ii} + \sum_j |a_{ij}| < 1 \text{ for all } i\Big)$$

- recurrent NN is contracting with rate $1 - \mu_\infty(A)_+$
- implicit NN is well posed
- forward Euler is contracting with factor $1 - \dfrac{1-\mu_\infty(A)_+}{1-\min_i(a_{ii})_-}$ at $\alpha = \dfrac{1}{1-\min_i(a_{ii})_-}$

Detour: convexity and fixed point theory

For differentiable $V : \mathbb{R}^n \to \mathbb{R}$, equivalent statements:

1. V is **strongly convex** with parameter m
2. $-\operatorname{grad} V$ is m-**strongly infinitesimally contracting** with respect to $\|\cdot\|_2$

Forward Euler theorem for contracting dynamics

Given arbitrary norm $\|\cdot\|$, equivalent statements

1. $\dot{x} = \mathsf{F}(x)$ is infinitesimally contracting
2. there exists $\alpha > 0$ such that $x_{k+1} = x_k + \alpha \mathsf{F}(x_k)$ is contracting

Forward Euler theorem for contracting dynamics

Given arbitrary norm $\|\cdot\|$, equivalent statements

1. $\dot{x} = \mathsf{F}(x)$ is infinitesimally contracting
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Given *contraction rate* c and *Lipschitz constant* ℓ, define *condition number* $\kappa = \dfrac{\ell}{c} \geq 1$

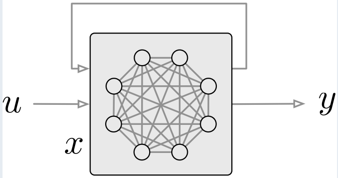
1. $\mathsf{Id} + \alpha \mathsf{F}$ is contracting for

$$0 < \alpha < \frac{1}{c\kappa(1+\kappa)}$$

2. the optimal step size minimizing and minimum contraction factor:

$$\alpha^* = \frac{1}{c}\Big(\frac{1}{2\kappa^2} - \frac{3}{8\kappa^3} + \mathcal{O}\Big(\frac{1}{\kappa^4}\Big)\Big)$$
$$\ell^* = 1 - \frac{1}{4\kappa^2} + \frac{1}{8\kappa^3} + \mathcal{O}\Big(\frac{1}{\kappa^4}\Big)$$

Application: ℓ_∞-contracting neural networks



$$\dot{x} = -x + \Phi(Ax + Bu + b) \qquad \text{(*recurrent NN*)}$$

$$x = \Phi(Ax + Bu + b) \qquad \text{(*implicit NN*)}$$

$$x_{k+1} = (1-\alpha)x_k + \alpha\Phi(Ax_k + Bu + b) \qquad \text{(*forward Euler*)}$$

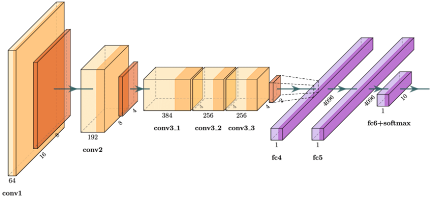
If

$$\mu_\infty(A) < 1 \qquad \Big(\text{i.e., } a_{ii} + \sum\nolimits_j |a_{ij}| < 1 \text{ for all } i\Big)$$

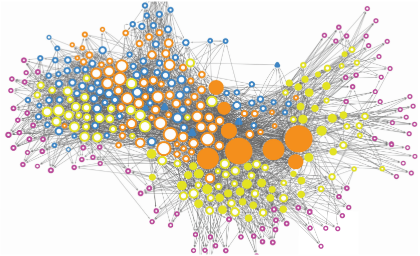
- recurrent NN is contracting with rate $1 - \mu_\infty(A)_+$
- implicit NN is well posed
- forward Euler is contracting with factor $1 - \dfrac{1-\mu_\infty(A)_+}{1-\min_i(a_{ii})_-}$ at $\alpha^* = \dfrac{1}{1-\min_i(a_{ii})_-}$

Motivation: ℓ_∞-contracting neural networks

While most ML architectures are feedforward,
biological neural networks are recurrent and recent interest for implicit ML architectures



artificial neural network AlexNet '12



C. elegans connectome '17

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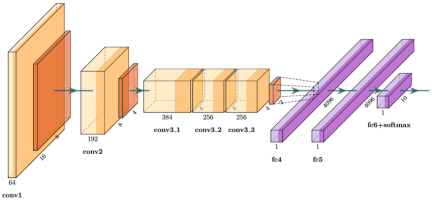

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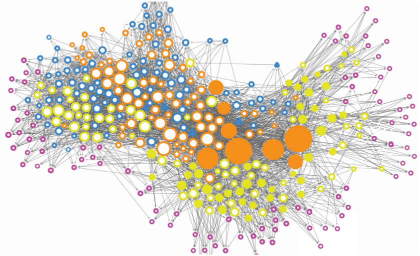

C. elegans connectome '17

For recurrent NN, ℓ_∞-contractivity characterizes the synaptic weights to ensure:

- reproducible & robust behavior
- highly-ordered transient+asymptotic dynamic behavior
- efficient computational methods

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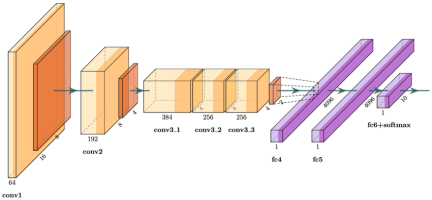

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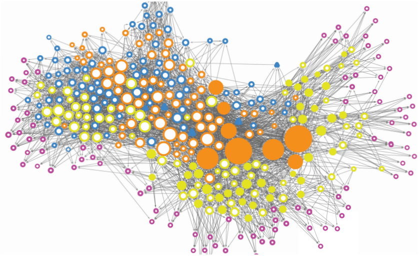

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For recurrent NN, ℓ_∞-contractivity characterizes the synaptic weights to ensure:

- reproducible & robust behavior
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- efficient computational methods

A. Krizhevsky, I. Sutskever, and G. E. Hinton. Imagenet classification with deep convolutional neural networks. *Advances in Neural Information Processing Systems*, 25, 2012

G. Yan, P. E. Vértes, E. K. Towlson, Y. L. Chew, D. S. Walker, W. R. Schafer, and A.-L. Barabási. Network control principles predict neuron function in the Caenorhabditis elegans connectome. *Nature*, 550(7677):519–523, 2017.

Outline

1. Contractivity of dynamical systems
   - From discrete-time to continuous-time dynamics
   - Table of infinitesimal contractivity conditions
   - Application to recurrent neural networks
   - Connection with convex optimization
2. From closed to open, interconnected and optimal systems
3. From nominal to uncertain, local and weakly contracting systems
4. Contractivity on Riemannian manifolds
   - A contractivity conjecture
5. Conclusions and Future Research

Given normed spaces $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{U}, \|\cdot\|_{\mathcal{U}})$, consider

$$\dot{x} = \mathsf{F}(x, u(t)), \qquad x_0 \in \mathcal{X}, \qquad u(t) \in \mathcal{U}$$

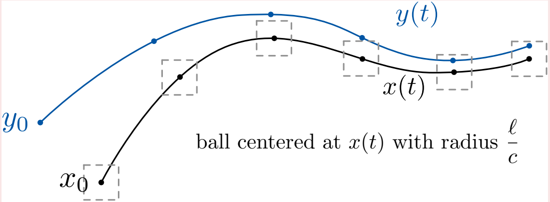
Assume:

- **contractivity wrt x**: $\mathsf{osLip}_x(\mathsf{F}) \leq -c < 0$, uniformly in u
- **Lipschitz wrt u**: $\mathsf{Lip}_u(\mathsf{F}) \leq \ell$, uniformly in x

Then

1. each soltns: $x(t)$ with input u_x and $y(t)$ with input u_y

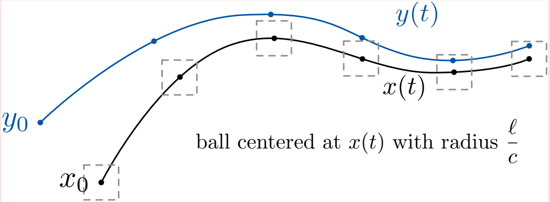
$$D^+\|x(t)-y(t)\|_{\mathcal{X}} \;\leq\; -c\|x(t)-y(t)\|_{\mathcal{X}} \;+\; \ell\|u_x(t)-u_y(t)\|_{\mathcal{U}}$$



Then

1. each soltns: $x(t)$ with input u_x and $y(t)$ with input u_y

$$D^+\|x(t) - y(t)\|_{\mathcal{X}} \;\leq\; -c\|x(t) - y(t)\|_{\mathcal{X}} \;+\; \ell\|u_x(t) - u_y(t)\|_{\mathcal{U}}$$



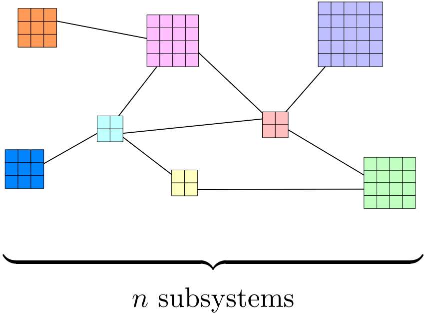
2. F is **incrementally ISS**, that is, for all x_0, y_0

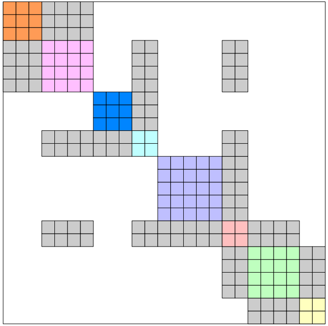
$$\|x(t) - y(t)\|_{\mathcal{X}} \;\leq\; e^{-ct}\|x_0 - y_0\|_{\mathcal{X}} \;+\; \frac{\ell(1 - e^{-ct})}{c} \sup_{\tau\in[0,t]} \|u_x(\tau) - u_y(\tau)\|_{\mathcal{U}}$$

Consider n interconnected subsystems

$$\dot{x}_i = \mathsf{F}_i(x_i, x_{-i}), \qquad \text{for } i \in \{1, \dots, n\}$$

with state $x_i \in \mathbb{R}^{N_i}$
with states of connected subsystems $x_{-i} \in \mathbb{R}^{N-N_i}$, and
consider n *local norms* $\|\cdot\|_i$ on $\mathbb{R}^{N_i}$





Assume for each node i:

- **contractivity wrt x_i**: $\mathsf{osLip}_{x_i}(\mathsf{F}_i) \leq -c_i < 0$, uniformly in x_{-i}
- **Lipschitz wrt x_j**: $\mathsf{Lip}_{x_j}(\mathsf{F}_i) \leq \ell_{ij}$, uniformly in x_{-j}

Assume for each node i:

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- **Lipschitz wrt x_j**: $\quad \mathsf{Lip}_{x_j}(\mathsf{F}_i) \leq \ell_{ij}, \quad$ uniformly in x_{-j}

Network contraction theorem

If the Lipschitz constants matrix $\begin{bmatrix} -c_1 & \dots & \ell_{1n} \\ \vdots & & \vdots \\ \ell_{n1} & \dots & -c_n \end{bmatrix}$ is **Hurwitz**

$\implies$ the **interconnected system** is infinitesimally contracting

History: interconnection of stable systems, method of vector Lyapunov functions, connective stability via M-matrix theory – Matrosov and Bellman 1962, Ström, Siljak, Russo/DiBernardo/Sontag, ...

$$\begin{bmatrix} -c_1 & \dots & \ell_{1n} \\ \vdots & & \vdots \\ \ell_{n1} & \dots & -c_n \end{bmatrix} \text{ is } \textbf{Metzler}$$

Hurwitzness depends upon both topology and edge weights

- Hurwitz iff there exists a positive ξ such that $M\xi < \mathbb{0}_n$ (power method)
- Hurwitz iff Lyapunov diagonally stable

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- for $n = 2$, Hurwitz if and only if **small gain condition**

$$\text{cycle gain} := \frac{\ell_{12}}{c_1}\frac{\ell_{21}}{c_2} < 1$$

and, for $n \geq 3$, **network small-gain theorem for Metzler matrices**

X. Duan, S. Jafarpour, and F. Bullo. Graph-theoretic stability conditions for Metzler matrices and monotone systems. *SIAM Journal on Control and Optimization*, 59(5):3447–3471, 2021.

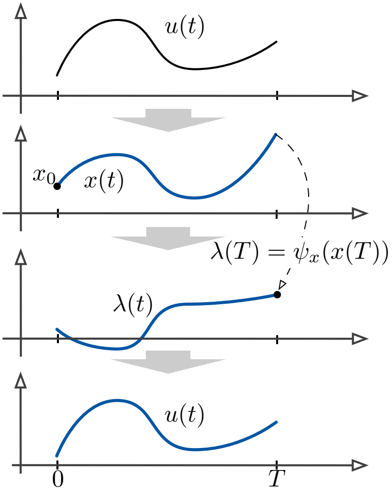
#3: From closed to systems with optimal controls

For $\dot{x} = \mathsf{F}(x, u)$, compute $u : [0, T] \to \mathbb{R}^k$ to minimize $\quad \psi(x(T)) + \int_0^T \phi(x, u)dt$

#3: From closed to systems with optimal controls

For $\dot{x} = \mathsf{F}(x,u)$, compute $u : [0,T] \to \mathbb{R}^k$ to minimize $\qquad \psi(x(T)) + \int_0^T \phi(x,u)dt$

Pontryagin Minimum Principle: $\qquad u = \mathcal{FBS}[u]$



$\mathcal{F} : \quad \dot{x} = \mathsf{F}(x,u)$

$\mathcal{B} : \quad \dot{\lambda} = -\mathsf{J}_{\mathsf{F}}^{\top}(x,u)\lambda - \phi_x(x,u)$

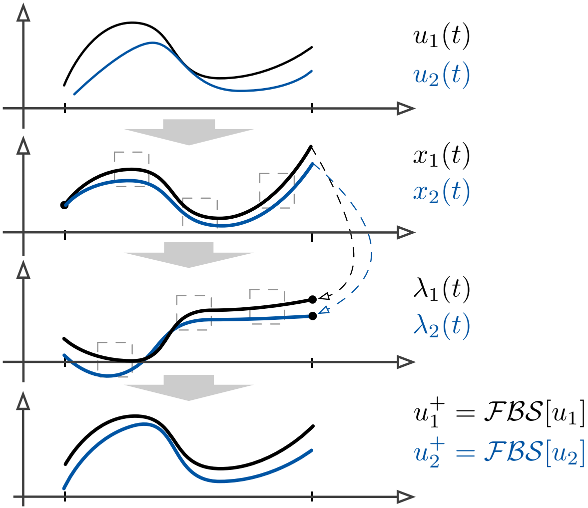
$\mathcal{S} : \quad u = \mathrm{argmin}_{\tilde{u}} \; \underbrace{\lambda^{\top}\mathsf{F}(x,\tilde{u}) + \phi(x,\tilde{u})}_{\mathcal{H}(x,\tilde{u},\lambda)}$

To compute a solution to:

$$u = \mathcal{FBS}[u]$$

adopt

$$u^{+} = \mathcal{FBS}[u]$$



To compute a solution to:

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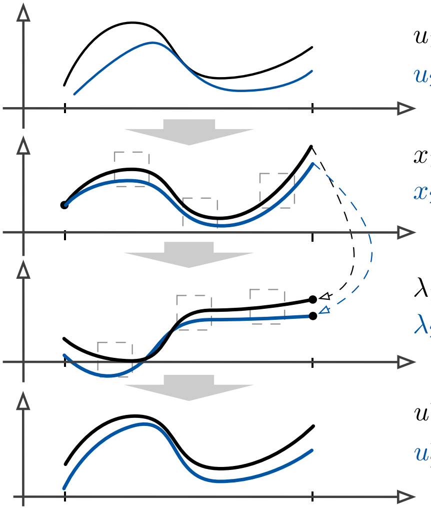

If $\mathsf{osLip}_x(\mathsf{F}) = -c$ and all other maps are Lipschitz,

1. $\mathsf{osLip}_\lambda(\mathsf{Adjoint}(\mathsf{F})) \;=\; \mathsf{osLip}_x(\mathsf{F})$
2. $\mathsf{Lip}(\mathcal{FBS}) = \text{constant} \times \dfrac{1 - \mathrm{e}^{-cT}}{c}$

$\mathcal{FBS}$ contracting for short T or large c

Summary

contractivity = robust computationally-friendly stability

fixed point theory + Lyapunov stability theory + geometry of metric spaces

From closed to open, interconnected and optimal systems:

1. iISS
2. network small gain theorems
3. numerical optimal control

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contractivity = robust computationally-friendly stability

fixed point theory + Lyapunov stability theory + geometry of metric spaces

From closed to open, interconnected and optimal systems:

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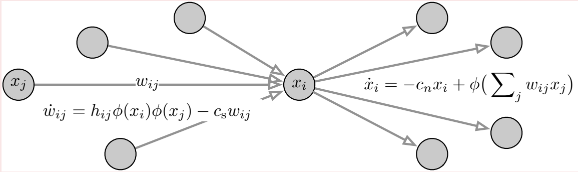
Applications coupled neural-synaptic dynamics and ML via optimal control



Outline

From nominal to uncertain systems

Given a norm $\|\cdot\|$, consider

$$\dot{x} = \mathsf{F}(x) + \Delta(x)$$

Assume:

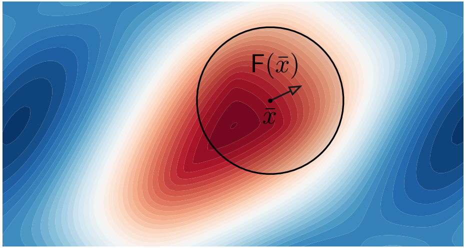
- **contractivity**: $\quad \mathrm{osLip}(\mathsf{F}) \leq -c < 0$
- **bounded disturbance**: $\quad \mathrm{osLip}(\Delta) \leq d < c$

Then

1. $\mathsf{F} + \Delta$ is strongly contracting with rate $c - d$
2. the unique equilibria x^*_{F} of F and $x^*_{\mathsf{F}+\Delta}$ of $\mathsf{F} + \Delta$ satisfy

$$\|x^*_{\mathsf{F}} - x^*_{\mathsf{F}+\Delta}\| \leq \frac{\|\Delta(x^*_{\mathsf{F}})\|}{c - d}$$

From global to local contractivity



Given a norm $\|\cdot\|$, consider

$$\dot{x} = \mathsf{F}(x)$$

Assume:

- **contractivity over closed set D**: $\mathrm{osLip}(\mathsf{F}|_D) \leq -c < 0$
- **existence of almost equilibrium**: D contains the closed B at $\bar{x}$ of radius $r \geq \|\mathsf{F}(\bar{x})\|/c$

Then

1. B is forward invariant
2. $\mathsf{F}|_B$ is strongly infinitesimally contracting

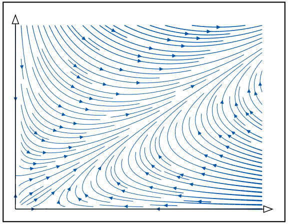
From strongly to weakly contracting systems

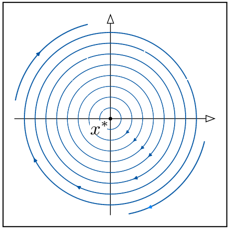
Given a norm $\|\cdot\|$, consider

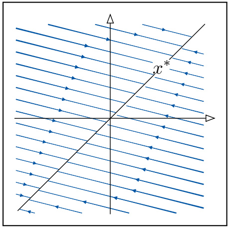
$$\dot{x} = \mathsf{F}(x) \qquad \text{satisfying} \qquad \mathrm{osLip}(\mathsf{F}) = 0$$

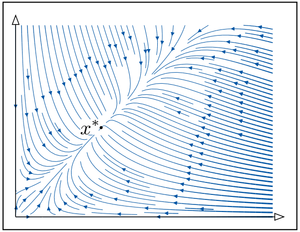
Dichotomy for weakly-contracting systems

1. no equilibrium and every trajectory is unbounded, or
2. at least one equilibrium, every trajectory is bounded, and local asy stability $\implies$ global









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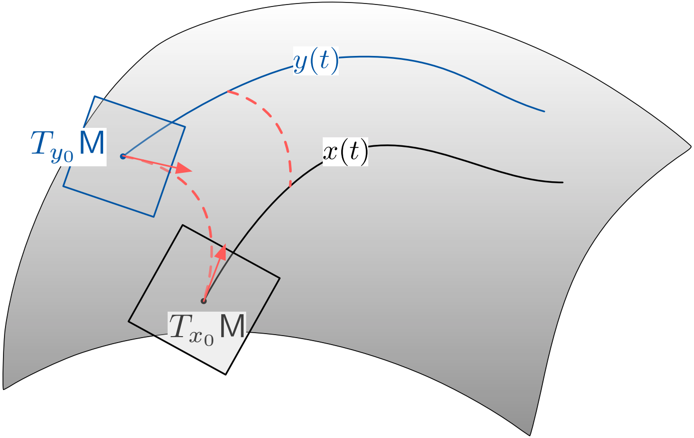
From vector spaces to Riemannian manifold $(\mathsf{M}, \mathbb{G})$

Assume existence and uniqueness of geodesic curve between each (x, y)
F **contracting** if geodesic distances from x to y diminishes along the flow of F



integral test: the inner product between F and the geodesic velocity vector γ' at x and y
differential test: condition on covariant differential of F

Given vector field F on a Riemannian manifold $(\mathsf{M}, \mathbb{G})$ and $c > 0$, equivalent statements:

1. for each $x, y \in \mathsf{M}$ and corresponding geodesic $\gamma : [0,1] \to \mathsf{M}$ with $\gamma(0) = x$, $\gamma(1) = y$,

$$\left\langle\!\left\langle \mathsf{F}(y), \gamma'(1) \right\rangle\!\right\rangle_{\mathbb{G}} - \left\langle\!\left\langle \mathsf{F}(x), \gamma'(0) \right\rangle\!\right\rangle_{\mathbb{G}} \leq -c\, \mathrm{d}_{\mathbb{G}}(x,y)^2$$

   or, equivalently, using the parallel transport map $P_{y \to x} : T_y\mathsf{M} \to T_x\mathsf{M}$,

$$\left\langle\!\left\langle P_{y \to x}\mathsf{F}(y) - \mathsf{F}(x), \gamma'(0) \right\rangle\!\right\rangle_{\mathbb{G}} \leq -c\, \mathrm{d}_{\mathbb{G}}(x,y)^2$$

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2. for all $v_x \in T_x\mathsf{M}$

$$\langle\!\langle \nabla_{v_x}\mathsf{F}(x) v_x, v_x \rangle\!\rangle_{\mathbb{G}} \leq -c\|v_x\|_{\mathbb{G}}^2,$$

   where ∇ is the Levi-Civita connection. In components:

$$\mathbb{G}(x)\mathsf{J}_{\mathsf{F}}(x) + \mathsf{J}_{\mathsf{F}}(x)^\top \mathbb{G}(x) + \mathcal{L}_{\mathsf{F}}\mathbb{G}(x) \preceq -2c\mathbb{G}(x)$$

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3. for all solutions $x(\cdot), y(\cdot)$

$$D^+\mathrm{d}_{\mathbb{G}}(x(t), y(t)) \leq -c\, \mathrm{d}_{\mathbb{G}}(x(t), y(t))$$

A contractivity conjecture

Given step size $\alpha > 0$ and start point $x_0 \in \mathsf{M}$, *forward Euler algorithm* for the vector field F

$$x_{k+1} = \exp_{x_k}(\alpha \mathsf{F}(x_k))$$

where $\exp_x : T_x\mathsf{M} \to \mathsf{M}$ is the exponential map

Riemannian forward Euler algorithm: Consider a vector field F on a Riemannian manifold $(\mathsf{M}, \mathbb{G})$ with strong contraction rate $c > 0$ and Lipschitz constant $\ell > 0$

1. there exists a unique equilibrium x^* of F
2. sequence $\{x_k\}$ converges to x^* for $0 < \alpha < 2\dfrac{c}{\ell^2} = \dfrac{1}{c\kappa^2}$

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Conjecture: The algorithm is a Banach contraction mapping
Know to be true for non-negative constant curvature, by Dongjun Wu.

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Robust and computationally-friendly stability theory

1. contractivity conditions on normed vector spaces
2. convexity and fixed point methods
3. disturbances, interconnections and optimal control

	Lyapunov Theory	Contraction Theory for Dynamical Systems
	F admits global Lyapunov function	F is strongly contracting
existence of equilibrium	assumed	implied + computational methods
Lyapunov function	arbitrary	distance to trajectory (+ norm of vector field)
inputs	ISS via $\mathcal{KL}$ and $\mathcal{L}$ functions	iISS via explicit formulas

search for contraction properties

design engineering systems to be contracting

References (1/2)

Contraction theory on normed spaces:

- A. Davydov, S. Jafarpour, and F. Bullo. Non-Euclidean contraction theory for robust nonlinear stability. *IEEE Transactions on Automatic Control*, 67(12):6667–6681, 2022b.
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Contractivity in optimal control:

- K. D. Smith and F. Bullo. Contractivity of the method of successive approximations for optimal control. *IEEE Control Systems Letters*, (7):919–924, 2023.

Contracting neural networks and fixed point theory:

- S. Jafarpour, A. Davydov, A. V. Proskurnikov, and F. Bullo. Robust implicit networks via non-Euclidean contractions. In *Advances in Neural Information Processing Systems*, Dec. 2021.
- A. Davydov, A. V. Proskurnikov, and F. Bullo. Non-Euclidean contractivity of recurrent neural networks. In *American Control Conference*, pages 1527–1534, Atlanta, USA, May 2022d.
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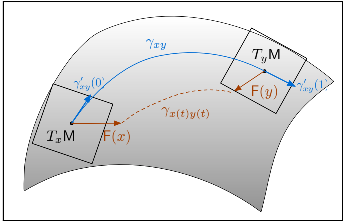
- V. Centorrino, F. Bullo, and G. Russo. Contraction analysis of Hopfield neural networks with Hebbian learning. In *IEEE Conf. on Decision and Control*, Cancún, México, Dec. 2022.
- R. Ofir, F. Bullo, and M. Margaliot. Minimum effort decentralized control design for contracting network systems. *IEEE Control Systems Letters*, 6:2731–2736, 2022.

Resources on contraction theory for dynamics, control and learning

1. tutorial session "Contraction Theory for Machine Learning" at the 2021 IEEE CDC conference:
   https://sites.google.com/view/contractiontheory
2. free online book and 10h minicourse
   https://fbullo.github.io/ctds
   https://youtu.be/RvR47ZbqJjc
3. upcoming Workshop on "Contraction Theory for Systems, Control, and Learning" at the 2023 American Control Conference in San Diego, California (under review):
   https://fbullo.github.io/contraction-workshop-2023

Theoretical frontiers

- higher order contraction
- relationship with monotone operator theory
- metric spaces: seminorms, Hilbert metrics ...



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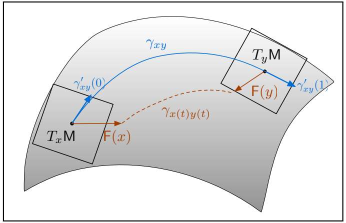
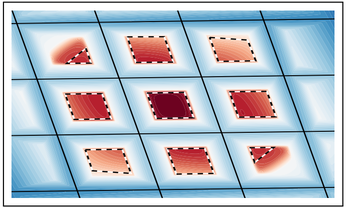


Limitations: not all stable systems are contractive:

- Lyapunov-diagonally-stable networks
- multistable systems
- biochemical networks



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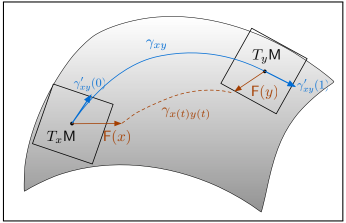

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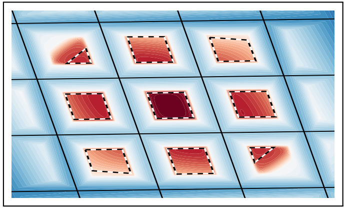

Application to control and learning

1. control: optimization-based control design
2. ML: implicit models and energy-based learning
3. neuroscience: robust dynamical modeling