Synchronization in Power Networks and in Non-uniform Kuramoto Oscillators

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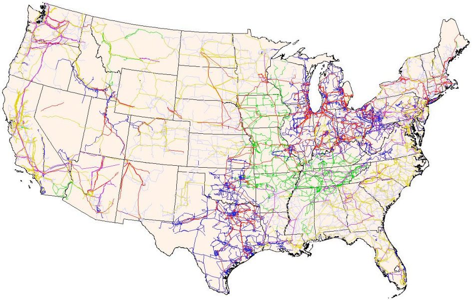
Intro: North American power grid



"... *the largest and most complex machine engineered by humankind.*"

[P. Kundur '94, V. Vittal '03, ...]

"... *the greatest engineering achievement of the 20th century.*"

[National Academy of Engineering '10]

1. large-scale, complex, nonlinear, and rich dynamic behavior
2. 100 years old and operating at its capacity limits

$\Rightarrow$ recent blackouts: New England '03 + Italy '03, Brazil '09

The New York Times

THE BLACKOUT OF 2003: Failure Reveals Creaky System, Experts Believe 8/15/2003

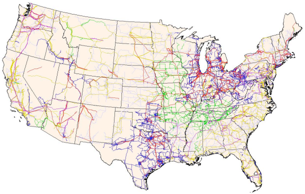
THE WHITE HOUSE WASHINGTON

Energy is one of the top three national priorities

Expected additional synergetic effects in future "**smart grid**":

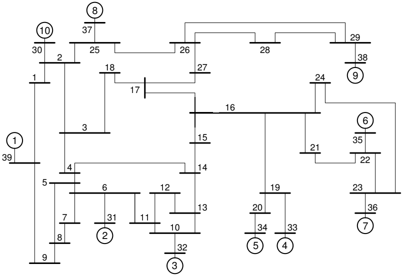
$\Rightarrow$ increasing complexity and renewable stochastic power sources

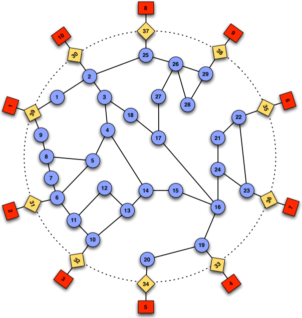
$\Rightarrow$ increasingly many transient disturbances to be detected and rejected



Transient Stability: Generators have to maintain synchronism in presence of large transient disturbances such as faults or loss of

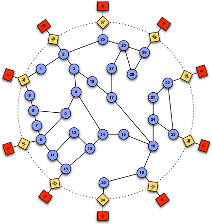
- transmission lines and components,
- generation or load.





Power network topology:

1. n **generators** ■, each connected to a **generator terminal bus** ◆
2. n generators terminal buses ◆ and m **load buses** ● form connected graph
3. **admittance matrix** $Y_{\text{network}} \in \mathbb{C}^{(2n+m)\times(2n+m)}$ characterizes the network



- generator nodes ■: **swing equation** for generator i

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + P_{mi} - P_{ei}$$

$\theta_i(t)$ is measured w.r.t. a 60Hz rotating frame

- network-preserving model: **power flow equations** for passive nodes ◊ & ● $\Rightarrow$ DAE system

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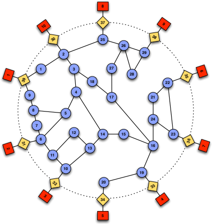
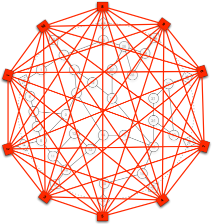


- network-preserving model: **power flow equations** for passive nodes ◊ & ● ⇒ DAE system



- network-reduced model: reduction to ■ nodes with all-to-all **reduced (transfer) admittance matrix** Y_{ij}

$$P_{ei} = E_i^2 G_{ii} + \sum\nolimits_{j\neq i} E_i E_j \, |Y_{ij}| \sin(\theta_i - \theta_j + \varphi_{ij})$$

Classic model

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j\neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij})$$

Intro: Transient Stability Analysis in Power Networks

Classic Model

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

Transient stability and synchronization:

- frequency equilibrium: $(\dot{\theta}_i, \ddot{\theta}_i) = (0,0)$ for all i
- synchronous equilibrium: $|\theta_i - \theta_j|$ bounded & $\dot{\theta}_i - \dot{\theta}_j = 0$ for all $\{i,j\}$

Classic problem setup in transient stability analysis:

1. power network in stable frequency equilibrium
2. $\rightarrow$ transient network disturbance and fault clearance
3. stability analysis of a new frequency equilibrium in post-fault network

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Transient stability and synchronization:

Classic analysis methods: Hamiltonian arguments

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i - \nabla_i U(\theta)^T$$

Energy function analysis, (extended) invariance principle, analysis of reduced gradient flow [N. Kakimoto et al. '78, H.-D. Chiang et al. '94]

$$\dot{\theta}_i = -\nabla_i U(\theta)^T$$

Key objective: compute domain of attraction via numerical methods

Intro: Transient Stability Analysis in Power Networks

Classic Model

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij})$$

Transient stability and synchronization:

Classic analysis methods: Hamiltonian arguments

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i - \nabla U_i(\theta)^T \quad \rightsquigarrow \quad \dot{\theta}_i = -\nabla_i U(\theta)^T$$

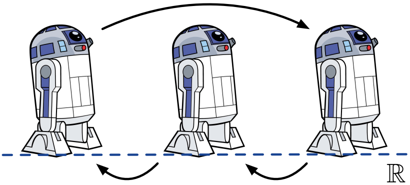
$\Longrightarrow$ **Open problem** [D. Hill and G. Chen '06]: power sys $\overset{?}{\leftrightsquigarrow}$ network:

transient stability, performance, and robustness of a power network
$\overset{?}{\leftrightsquigarrow}$ state, parameters, and topology of underlying network

Consensus protocol in $\mathbb{R}^n$:

$$\dot{x}_i = -\sum\nolimits_{j \neq i} a_{ij}(x_i - x_j)$$

- n **agents** with state $x_i \in \mathbb{R}$ and connected **graph** with weights $a_{ij} > 0$
- **objective** is state agreement: $x_i(t) - x_j(t) \to 0$
- **application:** social networks, computer science, systems theory
  robotic rendezvous, distributed computing, filtering and control, ...
- **some references:** [M. DeGroot '74, J. Tsitsiklis '84, ...]

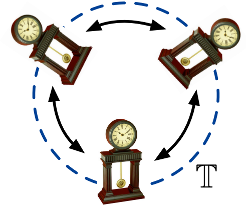


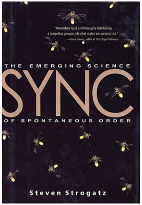
Detour – Consensus Protocols & Kuramoto Oscillators

Kuramoto model in $\mathbb{T}^n$:

$$\dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j \neq i} \sin(\theta_i - \theta_j)$$

- **oscillators** with phase $\theta_i \in \mathbb{T}$, frequency $\omega_i \in \mathbb{R}$, **complete** coupling
- **objective** is synchronization: $\theta_i(t) - \theta_j(t)$ bounded, $\dot{\theta}_i(t) - \dot{\theta}_j(t) \to 0$
- **application** in physics, biology, engineering:
  coupled neurons, Josephson junctions, motion coordination, ...
- **some references:** [Y. Kuramoto '75, A. Winfree '80, ...]





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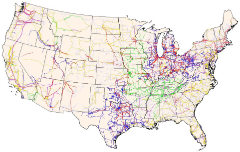
$$\dot{\theta}_i = \omega_i - \frac{K}{n} \sum\nolimits_{j \neq i} \sin(\theta_i - \theta_j)$$

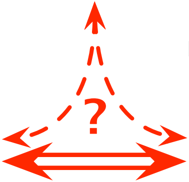
- degrees of synchronization:
  1. phase locking: $|\theta_i - \theta_j|$ bounded
  2. frequency entrainment: $\dot{\theta}_i = \dot{\theta}_j$
  3. phase synchronization: $\theta_i = \theta_j$
- known that
  1. K large & $|\omega_i - \omega_j|$ small $\Rightarrow$ frequency entrainment & phase locking
  2. additionally, for $\omega_i = \omega_j$ $\Rightarrow$ phase synchronization

Intro: The Big Picture



Open problem in synchronization and transient stability in **power networks:** relation to underlying network state, parameters, and topology

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$



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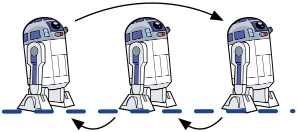
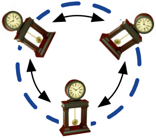


$$\dot{x}_i = -\sum\nolimits_{j\neq i} a_{ij}(x_i - x_j)$$

Kuramoto Oscillators:



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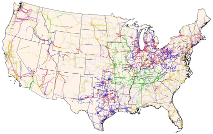

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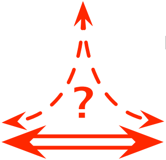
$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j \neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$



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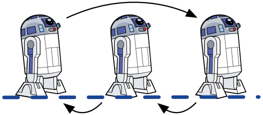

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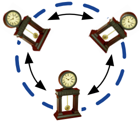

$$\dot{\theta}_i = \omega_i - \frac{K}{n}\sum\nolimits_{j \neq i} \sin(\theta_i - \theta_j)$$

Possible connection has often been hinted at in the literature!

Power systems: [D. Subbarao et al., '01, G. Filatrella et al., '08, V. Fioriti et al., '09]
Networked control: [D. Hill et al., '06, M. Arcak, '07]
Dynamical systems: [H. Tanaka et al., '97]

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From the swing equations to the Kuramoto model

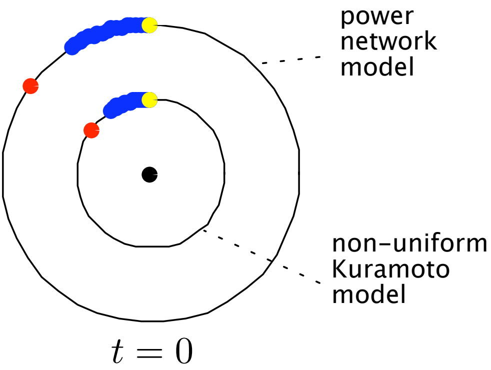
$$\boxed{\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})}$$

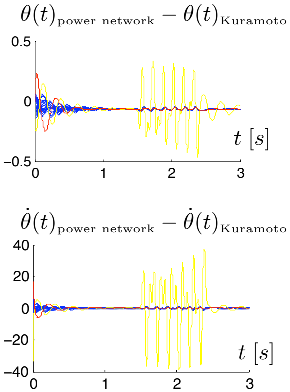
$$\dot{\theta}_i = \omega_i - \frac{K}{n}\sum\nolimits_{j\neq i}\sin(\theta_i - \theta_j) \implies \boxed{D_i\dot{\theta}_i = \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})}$$

From the swing equations to the Kuramoto model

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

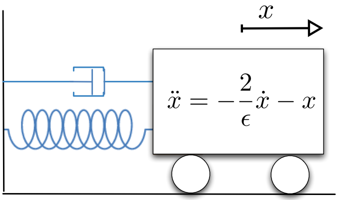
$$\dot{\theta}_i = \omega_i - \frac{K}{n}\sum\nolimits_{j\neq i}\sin(\theta_i - \theta_j) \quad \Longrightarrow \quad D_i\dot{\theta}_i = \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

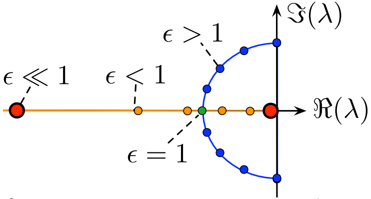




Time-scale separation in power network model:

- **Motivation:** harmonic oscillator





for $\epsilon \ll 1 \Rightarrow$ two time-scales

Singular Perturbation Analysis

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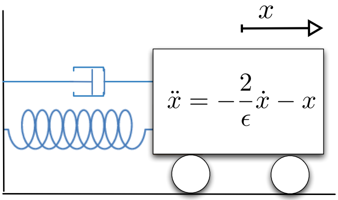
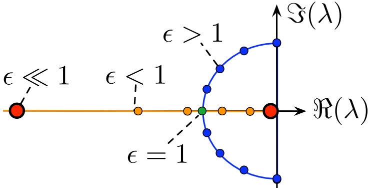


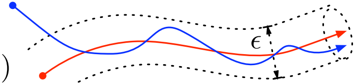


for $\epsilon \ll 1 \Rightarrow$ two time-scales

- **Singular perturbation analysis:**

full system $\begin{cases} \dot{x} = f(x, z) \\ \epsilon \dot{z} = g(x, z) \end{cases} \xrightarrow{\epsilon = 0} \begin{cases} \dot{x} = f(x, h(x)) \\ z = h(x) \end{cases}$ reduced (slow) system, quasi–steady state

initial error: $z(0) \neq h(x(0))$



approximation error in slow time–scale: $\mathcal{O}(\epsilon)$

error exp. stable in fast time–scale

Singular Perturbation Analysis

Time-scale separation in power network model:

- **power network model:**

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

- singular perturbation parameter: $\epsilon = \frac{M_{\text{max}}}{\pi f_0 D_{\text{min}}}$
- reduced system for $\epsilon = 0$ is a **non-uniform Kuramoto model**:

$$D_i\dot{\theta}_i = \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

Tikhonov's Theorem:
Assume the non-uniform Kuramoto model synchronizes exponentially.
Then $\forall\,(\theta(0),\dot{\theta}(0))$ there exists $\epsilon^* > 0$ such that $\forall\,\epsilon < \epsilon^*$ and $\forall\, t \geq 0$

$$\theta_i(t)_{\text{power network}} - \theta_i(t)_{\text{non-uniform Kuramoto model}} = \mathcal{O}(\epsilon)\,.$$

Singular Perturbation Analysis

Discussion of the assumption $\epsilon = \frac{M_{\text{max}}}{\pi f_0 D_{\text{min}}}$ sufficiently small:

1. physical interpretation: damping and sync on separate time-scales
2. classic assumption in literature on coupled oscillators: over-damped mechanical pendula and Josephson junctions
3. physical reality: with generator internal control effects $\epsilon \in \mathcal{O}(0.1)$
4. simulation studies show accurate approximation even for large ϵ
5. first-order and second-order models have the same equilibria with the same stability properties, and the regions of attractions are bounded by the same separatrices (independent of ϵ)
6. non-uniform Kuramoto model corresponds to reduced gradient system $\dot{\theta}_i = -\nabla_i U(\theta)^T$ used successfully in academia and industry since 1978

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Main Synchronization Result

Condition on network parameters:

network connectivity > network's non-uniformity + network's losses,

1. **Non-Uniform Kuramoto Model:**

$\Rightarrow$ exponential synchronization: phase locking & frequency entrainment

$\Rightarrow$ guaranteed region of attraction: $|\theta_i(t_0) - \theta_j(t_0)| < \pi/2 - \varphi_{\max}$

$\Rightarrow$ gap in condition determines ultimate phase locking

$\Rightarrow$ further conditions on φ_{ij} and ω_i: explicit synchronization frequency, synchronization rates, exponential phase synchronization

2. **Power Network Model:**

$\Rightarrow$ there exists ϵ sufficiently small such that for all $t \geq 0$

$$\theta_i(t)_{\text{power network}} - \theta_i(t)_{\text{non-uniform Kuramoto model}} = \mathcal{O}(\epsilon)\,.$$

$\Rightarrow$ for ϵ and network losses φ_{ij} sufficiently small, $\mathcal{O}(\epsilon)$ error converges

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Synchronization of Non-Uniform Kuramoto Oscillators

Non-uniform Kuramoto Model in $\mathbb{T}^n$:

$$D_i\dot{\theta}_i = \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

- **Non-uniformity** in network: D_i, ω_i, P_{ij}, φ_{ij}
- **Directed coupling** between oscillator i and j
- **Phase shift** φ_{ij} induces lossless and lossy couling:
  $P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij}) = P_{ij}\cos(\varphi_{ij})\sin(\theta_i - \theta_j) + P_{ij}\sin(\varphi_{ij})\cos(\theta_i - \theta_j)$
- **Synchronization analysis** in multiple steps:
  1. phase locking: $|\theta_i(t) - \theta_j(t)|$ becomes bounded
  2. frequency entrainment: $\dot{\theta}_i(t) - \dot{\theta}_j(t) \to 0$
  3. phase synchronization: $|\theta_i(t) - \theta_j(t)| \to 0$

Synchronization of Non-Uniform Kuramoto Oscillators

Non-uniform Kuramoto Model in $\mathbb{T}^n$ - rewritten:

$$\dot{\theta}_i = \frac{\omega_i}{D_i} - \sum\nolimits_{j \neq i} \frac{P_{ij}}{D_i} \cos(\varphi_{ij}) \sin(\theta_i - \theta_j) + \frac{P_{ij}}{D_i} \sin(\varphi_{ij}) \cos(\theta_i - \theta_j)$$

Condition (1) for synchronization:

Assume the graph induced by $P = P^T$ is complete and

$$\underbrace{n \frac{P_{\min}}{D_{\max}} \cos(\varphi_{\max})}_{\text{worst lossless coupling}} > \underbrace{\max_{\{i,j\}} \left(\frac{\omega_i}{D_i} - \frac{\omega_j}{D_j} \right)}_{\text{worst non-uniformity}} + \underbrace{\max_i \sum\nolimits_j \frac{P_{ij}}{D_i} \sin(\varphi_{ij})}_{\text{worst lossy coupling}} .$$

Gap determines the ultimate lack of phase locking in a $\frac{\pi}{2}$ interval.

Classic (uniform) Kuramoto Model in $\mathbb{T}^n$:

$$\dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j \neq i} \sin(\theta_i - \theta_j)$$

Condition (1) for synchronization:

$$K > \omega_{\text{max}} - \omega_{\text{min}}$$

Gap determines the ultimate lack of phase locking in a $\frac{\pi}{2}$ interval.

Condition (1) strictly improves existing bounds on Kuramoto model: [F. de Smet et al. '07, N. Chopra et al. '09, G. Schmidt et al. '09, A. Jadbabaie et al. '04, J.L. van Hemmen et al. '93].

Necessary condition for sync of n oscillators: $K > \frac{n}{2(n-1)}(\omega_{\text{max}} - \omega_{\text{min}})$ [J.L. van Hemmen et al. '93, A. Jadbabaie et al. '04, N. Chopra et al. '09]

Theorem: Phase locking and frequency entrainment (1)

Non-uniform Kuramoto with complete $P = P^T$
Assume minimal coupling larger than a critical value, i.e.,

$$P_{\text{min}} > P_{\text{critical}} := \frac{D_{\text{max}}}{n \cos(\varphi_{\text{max}})} \left(\max_{\{i,j\}} \left(\frac{\omega_i}{D_i} - \frac{\omega_j}{D_j} \right) + \max_i \sum\nolimits_j \frac{P_{ij}}{D_i} \sin(\varphi_{ij}) \right)$$

Define $\gamma_{\text{min}} = \arcsin\left(\cos(\varphi_{\text{max}}) \frac{P_{\text{critical}}}{P_{\text{min}}}\right)$ and **set of locked phases**

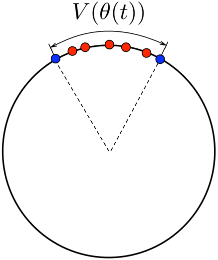
$$\Delta(\gamma) := \{\theta \in \mathbb{T}^n | \max_{\{i,j\}} |\theta_i - \theta_j| \leq \gamma\}$$

Then $\forall\, \gamma \in [\gamma_{\text{min}}, \frac{\pi}{2} - \varphi_{\text{max}})$

1) **phase locking**: the set $\Delta(\gamma)$ is positively invariant
2) **frequency entrainment:** $\forall\, \theta(0) \in \Delta(\gamma)$ the frequencies $\dot{\theta}_i(t)$ synchronize exponentially to some frequency $\dot{\theta}_\infty \in [\dot{\theta}_{\text{min}}(0), \dot{\theta}_{\text{max}}(0)]$

Main proof ideas:

1. **Phase locking** in $\Delta(\gamma) \Leftrightarrow$ arc-length $V(\theta(t))$ is non-increasing



$$\Leftrightarrow \begin{cases} V(\theta(t)) = \max\{|\theta_i(t) - \theta_j(t)| \mid i, j \in \{1, \dots, n\}\} \\ D^+ V(\theta(t)) \overset{!}{\leq} 0 \end{cases}$$

$\sim$ **contraction property** from consensus literature: [D. Bertsekas et al. '94, L. Moreau '04 & '05, Z. Lin et al. '08, ...]

2. **Frequency entrainment** in $\Delta(\gamma) \Leftrightarrow$ consensus protocol in $\mathbb{R}^n$

$$\frac{d}{d\,t}\dot{\theta}_i = -\sum\nolimits_{j \neq i} a_{ij}(t)(\dot{\theta}_i - \dot{\theta}_j)\,,$$

where $a_{ij}(t) = \frac{P_{ij}}{D_i}\cos(\theta_i(t) - \theta_j(t) + \varphi_{ij}) > 0$ for all $t \geq 0$

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   1. Power network model
   2. Synchronization and transient stability
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2. Singular perturbation analysis
   (to relate power network and Kuramoto model)
3. Synchronization analysis (of non-uniform Kuramoto model)
   1. Main synchronization result
   2. Sufficient condition (based on weakest lossless coupling)
   3. **Sufficient condition (based on lossless algebraic connectivity)**
4. Structure-preserving power network model
   1. Kron-reduction of graphs
   2. Sufficient conditions for synchronization
5. Conclusions

Non-uniform Kuramoto Model in $\mathbb{T}^n$ - rewritten:

$$\dot{\theta}_i = \frac{\omega_i}{D_i} - \sum\nolimits_{j \neq i} \frac{P_{ij}}{D_i} \cos(\varphi_{ij}) \sin(\theta_i - \theta_j) + \frac{P_{ij}}{D_i} \sin(\varphi_{ij}) \cos(\theta_i - \theta_j)$$

Condition (2) for synchronization:
Assume the graph induced by $P = P^T$ is connected with unweighted Laplacian L and weighted Laplacian $L(P_{ij} \cos(\varphi_{ij}))$ and

$$\underbrace{\lambda_2(L(P_{ij}\cos(\varphi_{ij})))}_{\text{lossless connectivity}} > \underbrace{f(D_i)}_{\text{non-uniform } D_i\text{s}} \cdot \underbrace{(1/\cos(\varphi_{\max}))}_{\text{necessary phase locking}} \times$$

$$\Bigg(\underbrace{\left\|\left[\dots, \frac{\omega_i}{D_i} - \frac{\omega_j}{D_j}, \dots\right]\right\|_2}_{\text{non-uniformity}} + \underbrace{\sqrt{\lambda_{\max}(L)}\left\|\left[\dots, \sum\nolimits_j \frac{P_{ij}}{D_i}\sin(\varphi_{ij}), \dots\right]\right\|_2}_{\text{lossy coupling}}\Bigg)$$

Gap determines the admissible initial lack of phase locking in a π interval.

Classic (uniform) Kuramoto Model in $\mathbb{T}^n$:

$$\dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j \neq i} \sin(\theta_i - \theta_j)$$

Condition (2) for synchronization:

$$K > ||[\ldots, \omega_i - \omega_j, \ldots]||_2$$

Gap determines the admissible initial lack of phase locking in a π interval.

Condition (2) corresponds to the bound in [N. Chopra et al. '09].

Synchronization of Non-Uniform Kuramoto Oscillators

Theorem: Phase locking and frequency entrainment (2)

Graph induced by $P = P^T$ is connected with unweighted Laplacian L, incidence matrix H, and weighted Laplacian $L(P_{ij}\cos(\varphi_{ij}))$.

Assume algebraic connectivity is larger than a critical value, i.e.,

$$\lambda_2(L(P_{ij}\cos(\varphi_{ij}))) > \lambda_{\text{critical}} := \frac{\left\|HD^{-1}\omega\right\|_2 + \sqrt{\lambda_{\max}(L)}\,\left|\left|\left[\,\dots, \sum_j \frac{P_{ij}}{D_i}\sin(\varphi_{ij}), \dots\right]\right|\right|_2}{\cos(\varphi_{\max})(\kappa/n)\mu\min_{\{i,j\}}\{D_{\neq\{i,j\}}\}},$$

$$\text{where} \quad \kappa := \sum_{k=1}^{n}\frac{1}{D_{\neq k}}, \quad \mu := \sqrt{\min_{i\neq j}\{D_i D_j\}/\max_{i\neq j}\{D_i D_j\}}$$

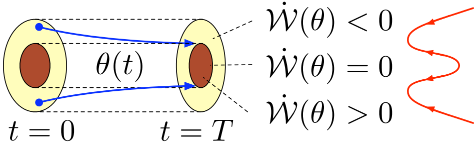
Define $\rho_{\max} \in (\frac{\pi}{2} - \varphi_{\max}, \pi)$ by $\operatorname{sinc}(\rho_{\max}) = \frac{\lambda_{\text{critical}}\cos(\varphi_{\max})}{\lambda_2(L(P_{ij}\cos(\varphi_{ij})))(\pi/2-\varphi_{\max})}$.

1) **phase locking:** $\forall\, \rho \in (\pi/2 - \varphi_{\max}, \rho_{\max})$, $\forall\, \|H\theta(0)\|_2 \leq \mu\rho$, there is $T \geq 0$ such that $\|H\theta(t)\|_2 < \pi/2 - \varphi_{\max}$ for all $t > T$

2) **frequency entrainment:** if $\|H\theta(0)\|_2 \leq \mu\rho$ the frequencies $\dot{\theta}_i(t)$ synchronize exponentially to some frequency $\dot{\theta}_\infty \in [\dot{\theta}_{\min}(0), \dot{\theta}_{\max}(0)]$

Main proof ideas:

1. **Phase locking** via ultimate boundedness arguments



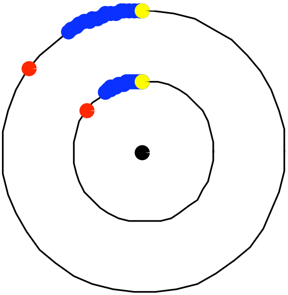
$$\mathcal{W}(\theta) = \sum\nolimits_{\{i,j\}} \frac{1}{2 \prod_{k \neq i,j}^{n} D_k} |\theta_i - \theta_j|^2$$

2. **Frequency entrainment** for $t > T$ $\Leftrightarrow$ consensus protocol in $\mathbb{R}^n$

$$\frac{d}{dt}\dot{\theta}_i = -\sum\nolimits_{j \neq i} a_{ij}(t)(\dot{\theta}_i - \dot{\theta}_j)\,,$$

where $a_{ij}(t) = \frac{P_{ij}}{D_i} \cos(\theta_i(t) - \theta_j(t) + \varphi_{ij}) > 0$ for all $t > T$

Simulation Studies



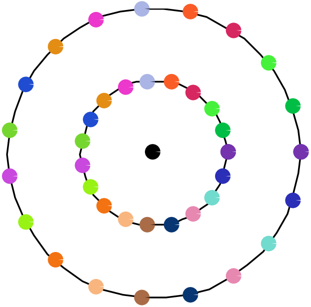
Simulation data:

- initial phases mostly clustered besides red phasor
- disturbance in yellow phasor for $\in [1.5\text{s}, 2.5\text{s}]$
- $\epsilon = 0.3$s & network is non-uniform

$\Rightarrow$ sufficient conditions for synchronization are **satisfied**

Result: singular perturbation analysis is accurate ✓
both models synchronize ✓

Simulation Studies



Simulation data:

- worst-case initial phase-differences: $\theta_i(0)$ in **splay state**
- $\epsilon = 0.12$s is small
- **strongly** non-uniform network

$\Rightarrow$ sufficient conditions for synchronization are **not satisfied**

Result: singular perturbation analysis is accurate $\checkmark$
both models synchronize $\checkmark$

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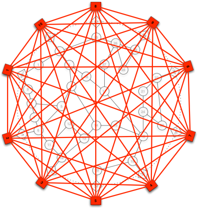
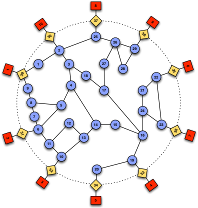
So far we considered a **network-reduced** power system model:



- network reduced to active nodes (generators)
- synchronization conditions on $\lambda_2(P)$ and $P_{\min}$
- all-to-all reduced admittance matrix $Y_{\mathbf{reduced}} \sim P$ (for uniform voltage levels)

Topological non-reduced **network-preserving** power system model:

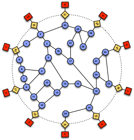


- boundary nodes (generators) & interior nodes (buses)
- topological bus admittance matrix $\mathbf{Y}_{\mathbf{network}}$
- Schur-complement relationship:

  $Y_{\mathbf{reduced}} = \mathbf{Y}_{\mathbf{network}} / Y_{\mathbf{interior}}$

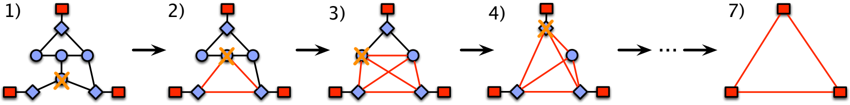
  c.f. "Kron reduction", "Dirichlet-to-Neumann map"

Detour – Kron reduction of graphs

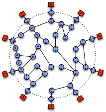


Kron reduction of a graph with Laplacian matrix $\mathbf{Y}_{\text{network}}$, boundary nodes ■ , and interior nodes ●

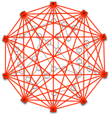
1. Subsequent one-step removal of a single interior node ● :
   - Topological evolution of the graph:



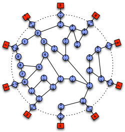
   - Algebraic evolution of Laplacian matrix: $\mathbf{Y}_{\text{reduced}}^{k+1} = \mathbf{Y}_{\text{reduced}}^{k} / \bullet$
2. Fully reduced Laplacian $Y_{\textbf{reduced}}$ given by Schur complement:



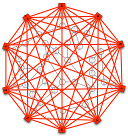
$$\mathbf{Y}_{\text{network}} \curvearrowright Y_{\text{reduced}} = \mathbf{Y}_{\text{network}} / Y_{\text{interior}}$$



Detour – Kron reduction of graphs

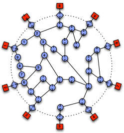


$\mathbf{Y}_{\text{network}}$ $\quad\curvearrowright\quad$ $Y_{\text{reduced}} = \mathbf{Y}_{\text{network}} / Y_{\text{interior}}$

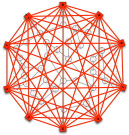


Graph-theoretic and algebraic properties of **Kron reduction** process:

1. Symmetric & irreducible Laplacians closed under Schur complement
2. interior network connected $\Rightarrow$ reduced network complete
3. Spectral interlacing property: $\lambda_2(Y_{\text{reduced}}) \geq \lambda_2(\mathbf{Y}_{\text{network}})$
   $\Rightarrow$ algebraic connectivity λ_2 is non-decreasing
4. Effective resistance among boundary nodes ■ is invariant
5. For boundary nodes ■: effective resistance $R(i,j)$ uniform
   $\Leftrightarrow$ coupling $Y_{\text{reduced}}(i,j)$ uniform $\Leftrightarrow 1/R(i,j) = \frac{n}{2}|Y_{\text{reduced}}(i,j)|$



$$\mathbf{Y}_{\text{network}} \longrightarrow Y_{\text{reduced}} = \mathbf{Y}_{\text{network}}/Y_{\text{interior}}$$



Assumption I: lossless network, zero shunt admittances (no self loops)

1. **Spectral condition for synchronization:** $\lambda_2(P) \geq \ldots$ becomes

$$\lambda_2(\Im(-\mathbf{Y}_{\text{network}}))) > \left\| \left(\frac{\omega_2}{D_2} - \frac{\omega_1}{D_1}, \ldots \right) \right\|_2 \cdot \frac{f(D_i)}{E^2}$$

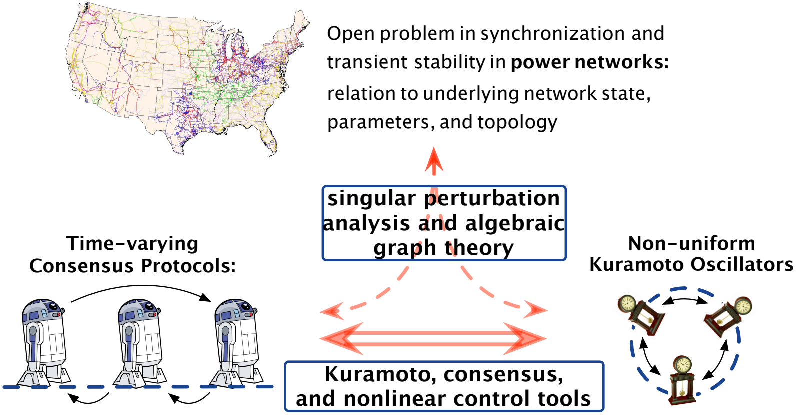
Assumption II: effective resistance R among boundary nodes is uniform

2. **Resistance-based condition for synchronization:** $nP_{\min} \geq \ldots$ becomes

$$\frac{1}{R} > \max_{\{i,j\}} \left\{ \frac{\omega_i}{D_i} - \frac{\omega_j}{D_j} \right\} \cdot \frac{D_{\max}}{2E^2}$$

Conclusions

Summary:



Ongoing and Future Work:

- relation to network topology, clustering, and scalability
- synchronization in optimal power flow problems

Further Results

Synchronization of Non-Uniform Kuramoto Oscillators

Theorem: A refined result on frequency entrainment

Assume there exists $\gamma \in (0, \pi/2)$ such that the phases are locked in the set $\Delta(\gamma)$ and the graph induced by P has globally reachable node.

1) $\forall\, \theta(0) \in \Delta(\gamma)$ the frequencies $\dot{\theta}_i(t)$ synchronize exponentially to $\dot{\theta}_\infty \in [\dot{\theta}_{\min}(0), \dot{\theta}_{\max}(0)]$.

2) If $P = P^T$ & $\varphi_{ij} = 0$ for all $i, j \in \{1, \dots, n\}$, then $\forall\, \theta(0) \in \Delta(\gamma)$ the frequencies $\dot{\theta}_i(t)$ synchronize exp. to the weighted mean frequency

$$\Omega = \frac{1}{\sum_i D_i} \sum\nolimits_i D_i \omega_i$$

and the exponential synchronization rate is no worse than

$$\lambda_{\text{fe}} = -\ \underbrace{\lambda_2(L(P_{ij}))}_{\text{connectivity}}\ \underbrace{\cos(\gamma)}_{\Delta(\gamma)}\underbrace{\cos(\angle(D\mathbf{1}, \mathbf{1}))^2}_{\mathbf{1} \not\angle D\mathbf{1}} /\ \underbrace{D_{\max}}_{\text{slowest}}$$

Result can be reduced to [N. Chopra et al. '09].

Synchronization of Non-Uniform Kuramoto Oscillators

Theorem: A result on phase synchronization

Assume the graph induced by P has a globally reachable node, $\varphi_{ij} = 0$, and $\omega_i / D_i = \bar{\omega}$ for all $i \in \{1, \dots, n\}$.

1) $\forall\, \theta(0) \in \{\theta \in \mathbb{T}^n : \max_{\{i,j\}} |\theta_i - \theta_j| < \pi\}$ the phases $\theta_i(t)$ synchronize exponentially to $\theta_\infty(t) \in [\theta_{\min}(0), \theta_{\max}(0)] + \bar{\omega} t$; and

2) if $P = P^T$ and $\forall\ \|H\theta(0)\|_2 \leq \mu\rho$ with $\rho \in [0, \pi)$, then

$$\theta_\infty(t) = \frac{\sum_i D_i \theta_i(0)}{\sum_i D_i} + \bar{\omega} t$$

and the exponential sync. rate is no worse than

$$\lambda_{\text{ps}} = -\underbrace{(\kappa/n) \min_{\{i,j\}} \{D_{\neq\{i,j\}}\}}_{\text{weighting of } D_i} \underbrace{\text{sinc}(\rho)}_{\theta(0)} \ \underbrace{\lambda_2(L(P_{ij}))}_{\text{connectivity}}$$

Results can be reduced to [Z. Lin et al. '07] and [A. Jadbabaie et al. '04].