Sequential Decision Aggregation: Accuracy and Decision Time

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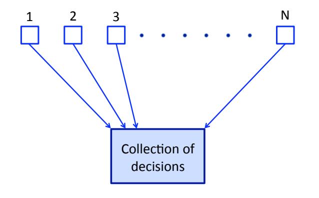
Center for Human and Robot Decision Dynamics, Aug 13, 2010

Sequential decision aggregation: Outline

Assumptions:

1. N identical individuals, arbitrary local rule
2. Independent information
3. Aggregation of individual decisions



Group decision rule = SDA algorithm

q out of N rule: decision as soon as q nodes report concordant opinion

- **Fastest rule** fastest node decides for network ($q = 1$)
- **Majority rule** network agrees with majority decision ($q = \lceil N/2 \rceil$)

Goal #1: characterize decision probabilities of SDA as function of: threshold and SDM decision probabilities

Goal #2: express accuracy & decision time as function of: decision threshold $\times$ group size

Setup & Literature Review

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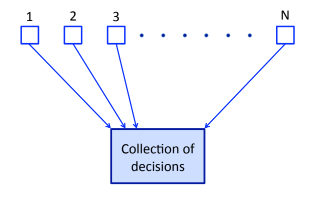

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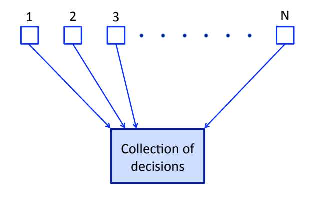

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Literature review #1

Distributed/decentralized detection

1. P. K. Varshney. *Distributed Detection and Data Fusion*. Signal Processing and Data Fusion. Springer Verlag, 1996
2. V. V. Veeravalli, T. Başar, and H. V. Poor. Decentralized sequential detection with sensors performing sequential tests. *Math Control, Signals & Systems*, 7(4):292–305, 1994
3. J. N. Tsitsiklis. Decentralized detection. In H. V. Poor and J. B. Thomas, editors, *Advances in Statistical Signal Processing*, volume 2, pages 297–344, 1993
4. J.-F. Chamberland and V. V. Veeravalli. Decentralized detection in sensor networks. *IEEE Trans Signal Processing*, 51(2):407–416, 2003

Social networks

1. D. Acemoglu, M. A. Dahleh, I. Lobel, and A. Ozdaglar. Bayesian learning in social networks. Working Paper 14040, National Bureau of Economic Research, May 2008

Literature review #2

For decentralized detection, with conditional independence of observations:

- Tsitsiklis '93: Bayesian decision problem with fusion center. For large networks identical local decision rules are asymptotically optimal
- Varshney '96: on non-identical decision rules with q out of N,
    1. threshold rules are optimal at the nodes levels
    2. finding optimal thresholds requires solving $N + 2^N$ equations
- Varshney '96: on optimal fusion rules for identical local decisions, "q out of N" is optimal at the fusion center level

Contributions today

- arbitrary decision makers (rather than optimal local rules)
- sequential aggregation (rather than "complete" aggregation)
- scalability analysis of accuracy / decision time

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Today's Outline

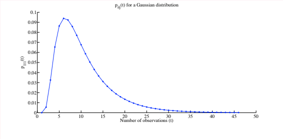
Model of sequential decision maker

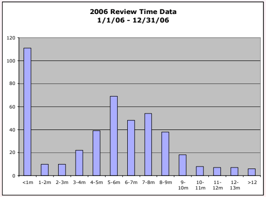
Sequential decision maker (SDM)

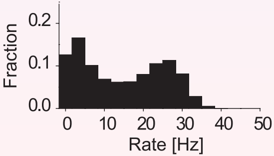
$$p_{i|j}(t) := \text{Probability “}\textit{say } H_i \textit{ given } H_j\text{” at time } t$$

$$p_{i|j} = \sum_{t=1}^{+\infty} p_{i|j}(t), \qquad E[T|H_i] = \sum_{t=1}^{+\infty} t\big(p_{1|i}(t) + p_{0|i}(t)\big)$$

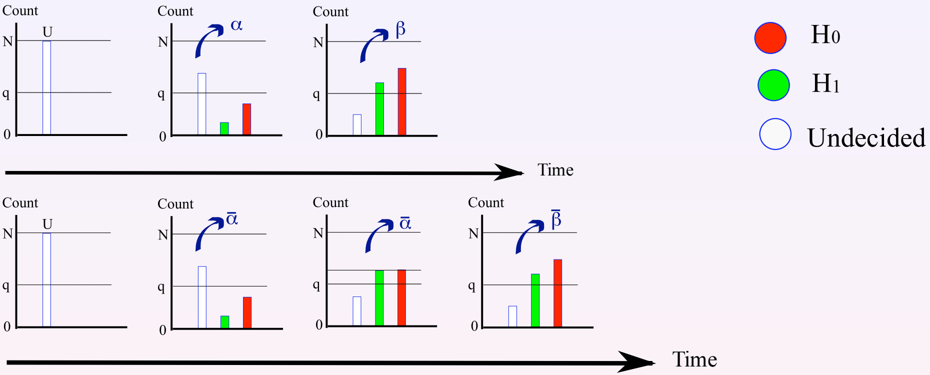
Assume knowledge of $\{p_{i|j}(t)\}_{t\in\mathbb{N}}$ for individual SDM,
known exactly, calculated numerically, or measured empirically







Sequential decision aggregation: Intermediate events



- aggregate states and divide in groups characterized by count
- calculate the probability of transition between the different groups
- characterize two states for network decisions H_0 and H_1

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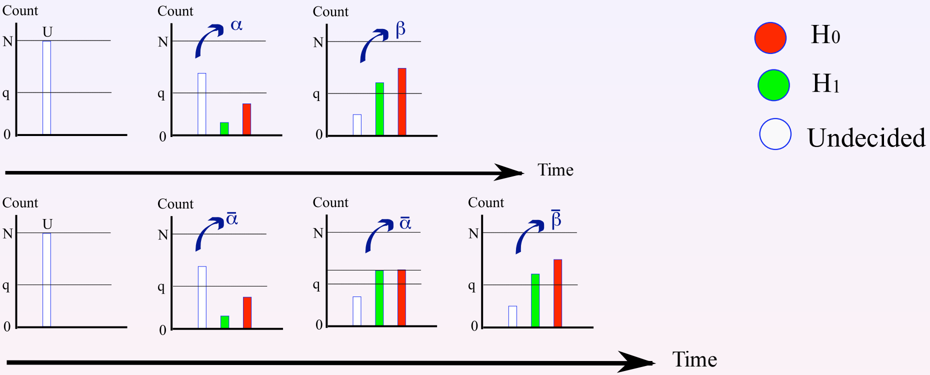

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Goal: as function of SDM decision probabilities $\{p_{i|j}(t)\}_{t\in\mathbb{N}}$,
compute SDA decision probabilities $\{p_{i|j}(t;N,q)\}_{t\in\mathbb{N}}$

General result: *q out of N* decision probabilities

$$p_{i|j}(t;N,q) = \sum_{s_0=0}^{q-1}\sum_{s_1=0}^{q-1}\binom{N}{s_1+s_0}\alpha(t-1,s_0,s_1)\beta_{i|j}(t,s_0,s_1) + \sum_{s=q}^{\lfloor N/2\rfloor}\binom{N}{2s}\bar{\alpha}(t-1,s)\bar{\beta}_{i|j}(t,s)$$

As function of t and sizes, formulas for α, β, $\bar{\alpha}$, and $\bar{\beta}$
computational complexity linear in N

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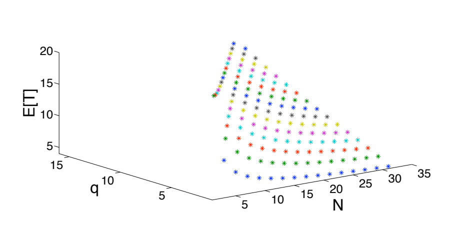
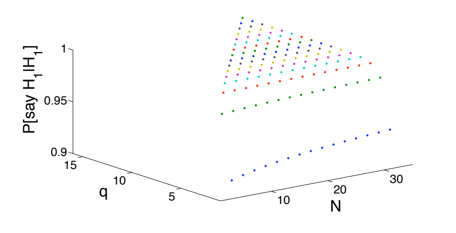
Illustration of results



Expected Decision Time



Probability of correct decision

- $(H_0 : \mu = 0)$ and $(H_1 : \mu = 1)$
- SPRT with $p_f = p_m = 0.1$
- Gaussian noise $\mathcal{N}(\mu, \sigma)$, $\sigma = 1$ and $\mu \in \{0, 1\}$

Today's Outline

Asymptotic results for the Fastest rule

Expected Decision Time:

$$\lim_{N\to\infty} \mathbb{E}\,[T|H_1, N, \mathit{fastest}] = \text{earliest possible decision time}$$

$$=: t_{min} = \min\{t \in \mathbb{N} \mid \text{either } p_{1|1}(t) \neq 0 \text{ or } p_{0|1}(t) \neq 0\}$$

Accuracy:

$$\lim_{N\to\infty} p_{0|1}(N, \mathit{fastest}) = \begin{cases} 0, & \text{if } p_{1|1}(t_{min}) > p_{0|1}(t_{min}) \\ 1, & \text{if } p_{1|1}(t_{min}) < p_{0|1}(t_{min}) \end{cases}$$

1. SDA accuracy is function of (SDM probability at t_{min}),
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Asymptotic results for the Majority rule

Expected Decision Time: Assume $p_{1|1} > p_{0|1}$ and define

$$t_{<\frac{1}{2}} := \max\{t \in \mathbb{N} \mid p_{1|1}(0) + \cdots + p_{1|1}(t) < 1/2\},$$
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Then

$$\lim_{N\to\infty} \mathbb{E}\Big[T \,|\, H_1, N, majority\Big] = \frac{1}{2}\Big(t_{<\frac{1}{2}} + t_{>\frac{1}{2}} + 1\Big)$$

Accuracy: Monotonicity with group size and, as $N \to \infty$

$$p_{0|1}(N, majority) \to \begin{cases} 0, & \text{if } p_{0|1} < 1/2 \\ 1, & \text{if } p_{0|1} > 1/2 \\ \sqrt{N/(2\pi)}\,(4p_{0|1})^{\lceil \frac{N}{2} \rceil}, & \text{if } p_{0|1} < 1/4 \end{cases}$$

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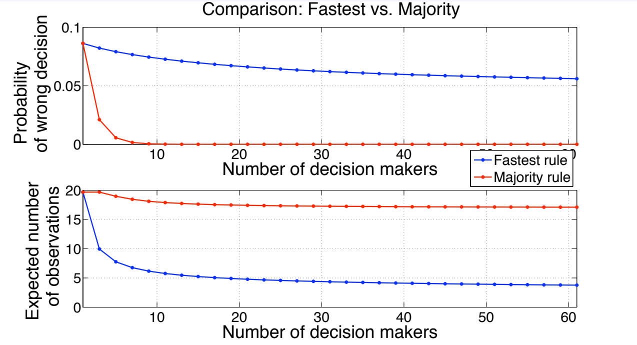
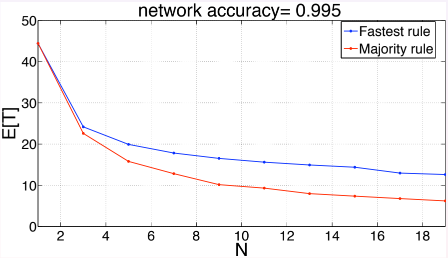
Lessons learned about SDA

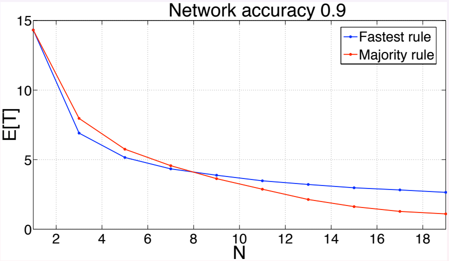
	Accuracy	Expected decision time
Fastest	SDM accuracy at t_{min}	earliest possible decision time t_{min}
Majority	exponentially better than SDM	average of half-times $t_{<\frac{1}{2}}, t_{>\frac{1}{2}}$



A fair comparison

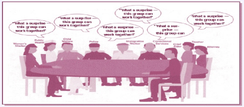
- to compare different thresholds, re-scale local accuracy
- the group accuracy is now same (eg, low or high)
- compare the decision time

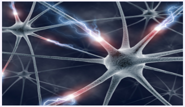




for most cases majority rule is best
for some small inaccurate networks, fastest rule is best

Conclusions and future directions





Summary fundamental understanding of "sequential aggregation"

1. applicable to broad range of agent models, eg, mixed networks
2. applicable to family of threshold-based rules
3. tradeoffs in fastest vs majority
4. role of time in sequential aggregation

Future directions

1. models with heterogeneous agents
2. models with interactions between agents
3. models with correlated information
4. how to use this analysis for design