

Contraction Theory for Networked Optimization and Control

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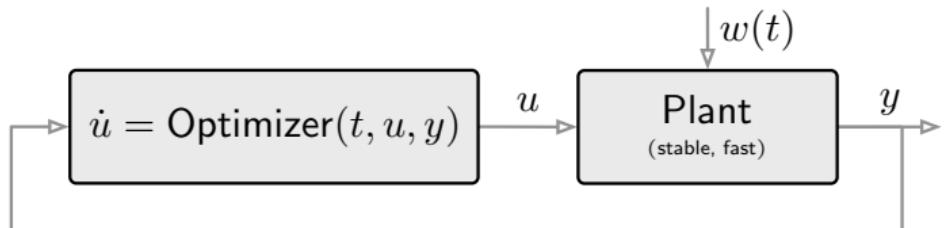
A. Davydov, V. Centorrino, A. Gokhale, G. Russo, and F. Bullo. Contracting dynamics for time-varying convex optimization. *IEEE Transactions on Automatic Control*, June 2023. Submitted

A. Gokhale, A. Davydov, and F. Bullo. Contractivity of distributed optimization and Nash seeking dynamics. *IEEE Control Systems Letters*, Sept. 2023. To appear

L. Cothren, F. Bullo, and E. Dall'Anese. Singular perturbation via contraction theory. *IEEE Transactions on Automatic Control*, Oct. 2023. Submitted

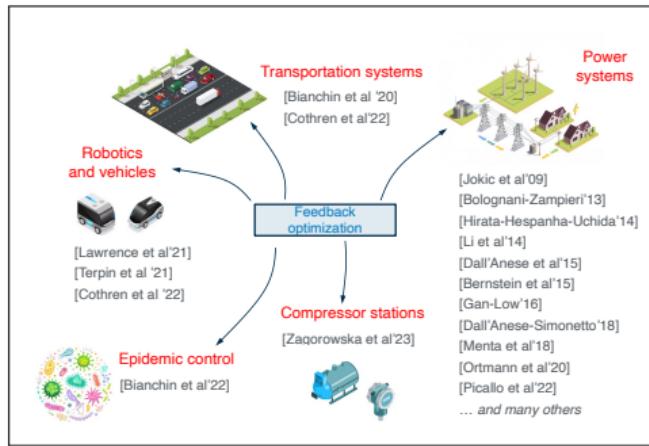
Solving optimization problems via dynamical systems

- studies in linear and nonlinear programming (Arrow, Hurwicz, and Uzawa 1958)
- neural networks (Hopfield and Tank 1985)
- analog circuits (Kennedy and Chua 1988)
- optimization on manifolds (Brockett 1991)
- ...



Continuous-time optimizations solvers

- ① online feedback optimization
- ② distributed optimization
- ③ parametric convex optimization
- ④ model predictive control
- ⑤ control barrier functions

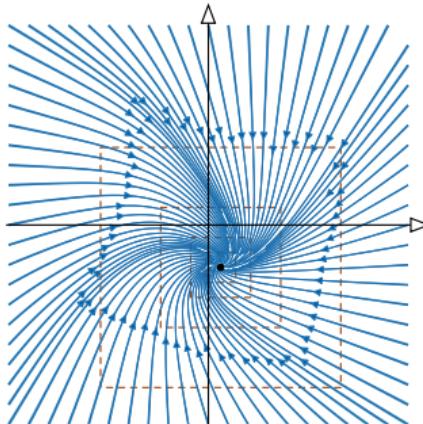


Online feedback optimization

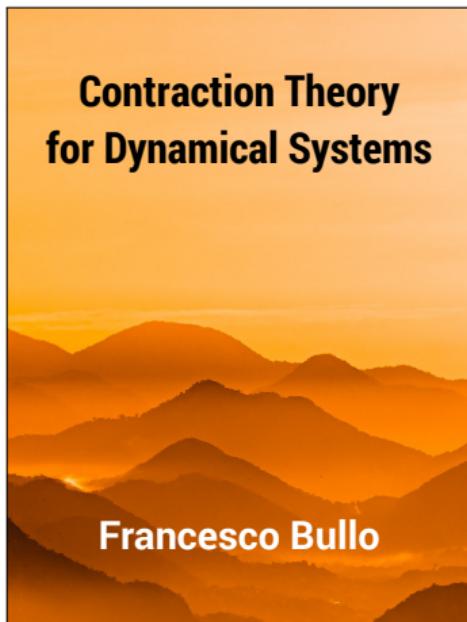
Slide courtesy of Emiliano Dall'Anese, University of Colorado Boulder

contractivity = robust computationally-friendly stability

fixed point theory + Lyapunov stability theory + geometry of metric spaces



search for contraction properties
design engineering systems to be contracting
verify correct/safe behavior via known Lipschitz constants



"Continuous improvement is better than delayed perfection" **Mark Twain**

- 2023 ACC Workshop "Contraction Theory for Systems, Control, and Learning" <http://motion.me.ucsb.edu/contraction-workshop-2023>
- 2021 IEEE CDC Tutorial session "Contraction Theory for Machine Learning" <https://sites.google.com/view/contractiontheory> (PDFs and youtube videos)
- 2022 IEEE CDC plenary presentation "Contraction Theory in Systems and Control" <https://fbullo.github.io/talks/2022-12-FBullo-ContractionSystemsControl-CDC.pdf>
- Textbook: Contraction Theory for Dynamical Systems, Francesco Bullo, rev 1.1, Mar 2023. (Book and slides freely available)
<https://fbullo.github.io/ctds>
- 2023 Comprehensive tutorial slides: <https://fbullo.github.io/ctds>
- 2023 Sep: Youtube lectures: "Minicourse on Contraction Theory"
<https://youtu.be/FQV5PrRHks8> 12h in 6 lectures

- Three CDC2023 invited sessions on **Contraction Theory for Analysis, Synchronization and Regulation**, tomorrow Wednesday!

Outline

§1. Introduction

§2. Basic contractivity concepts

§3. Examples: Gradient systems defined by strongly convex functions are contracting

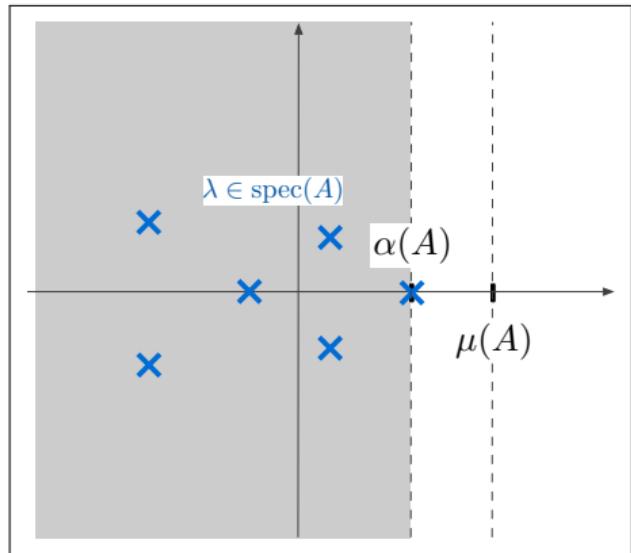
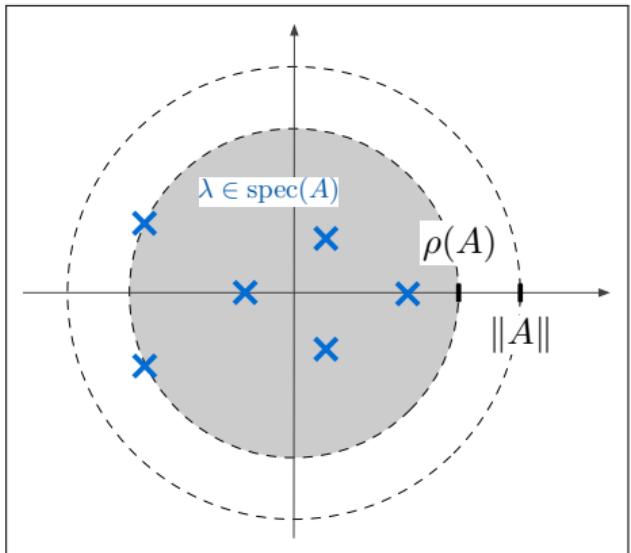
§4. Theory: Equilibrium tracking in parametric optimization

§5. Application: Online feedback optimization

§6. Conclusions

Induced matrix norms

Vector norm	Induced matrix norm	Induced matrix log norm
$\ x\ _1 = \sum_{i=1}^n x_i $	$\ A\ _1 = \max_{j \in \{1, \dots, n\}} \sum_{i=1}^n a_{ij} $ = max column "absolute sum" of A	$\mu_1(A) = \max_{j \in \{1, \dots, n\}} \left(a_{jj} + \sum_{i=1, i \neq j}^n a_{ij} \right)$ absolute value only off-diagonal
$\ x\ _2 = \sqrt{\sum_{i=1}^n x_i^2}$	$\ A\ _2 = \sqrt{\lambda_{\max}(A^\top A)}$	$\mu_2(A) = \lambda_{\max}\left(\frac{A + A^\top}{2}\right)$
$\ x\ _\infty = \max_{i \in \{1, \dots, n\}} x_i $	$\ A\ _\infty = \max_{i \in \{1, \dots, n\}} \sum_{j=1}^n a_{ij} $ = max row "absolute sum" of A	$\mu_\infty(A) = \max_{i \in \{1, \dots, n\}} \left(a_{ii} + \sum_{j=1, j \neq i}^n a_{ij} \right)$ absolute value only off-diagonal



$\dot{x} = F(x)$ on \mathbb{R}^n with norm $\|\cdot\|$ and induced log norm $\mu(\cdot)$

One-sided Lipschitz constant

$$\begin{aligned}\text{osLip}(F) &= \inf\{b \in \mathbb{R} \text{ such that } \|F(x) - F(y), x - y\| \leq b\|x - y\|^2 \text{ for all } x, y\} \\ &= \sup_x \mu(DF(x))\end{aligned}$$

For **scalar map** f , $\text{osLip}(f) = \sup_x f'(x)$

For **affine map** $F_A(x) = Ax + a$

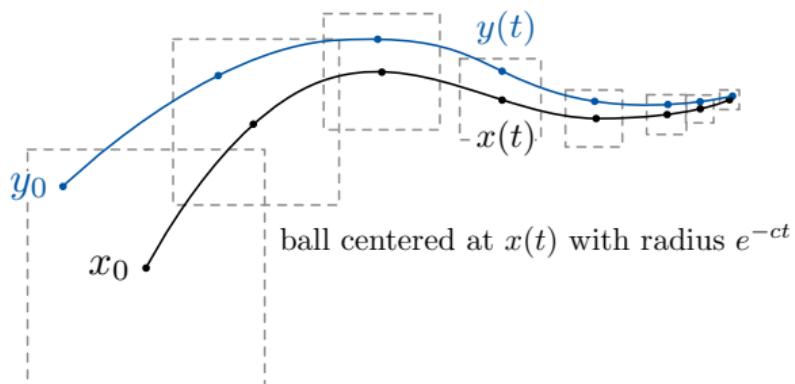
$$\text{osLip}_{2,P}(F_A) = \mu_{2,P}(A) \leq \ell \iff A^\top P + AP \preceq 2\ell P$$

$$\text{osLip}_{\infty,\eta}(F_A) = \mu_{\infty,\eta}(A) \leq \ell \iff a_{ii} + \sum_{j \neq i} |a_{ij}| \eta_i / \eta_j \leq \ell$$

Banach contraction theorem for continuous-time dynamics:

If $-c := \text{osLip}(F) < 0$, then

- ① F is **infinitesimally contracting** = distance between trajectories decreases exp fast (e^{-ct})
- ② F has a unique, glob exp stable equilibrium x^*
- ③ global Lyapunov functions $V_1(x) = \|x - x^*\|^2$ and $V_2(x) = \|F(x)\|^2$



Euler discretization theorem for contracting dynamics

Given arbitrary norm $\|\cdot\|$ and differentiable $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$, equivalent statements

- ① $\dot{x} = F(x)$ is infinitesimally contracting
- ② there exists $\alpha > 0$ such that $x_{k+1} = x_k + \alpha F(x_k)$ is contracting

Interconnected subsystems: $x_i \in \mathbb{R}^{N_i}$ and $x_{-i} \in \mathbb{R}^{N-N_i}$:

$$\dot{x}_i = F_i(x_i, x_{-i}), \quad \text{for } i \in \{1, \dots, n\}$$

Network contraction theorem. Assume

- **contractivity wrt x_i :** $\text{osLip}_{x_i}(F_i) \leq -c_i < 0$, uniformly in x_{-i}
- **Lipschitz wrt x_j , $j \neq i$:** $\text{Lip}_{x_j}(F_i) \leq \ell_{ij}$, uniformly in x_{-j}

- the Lipschitz constants matrix $\Gamma = \begin{bmatrix} -c_1 & \dots & \ell_{1n} \\ \vdots & & \vdots \\ \ell_{n1} & \dots & -c_n \end{bmatrix}$ is **Hurwitz**

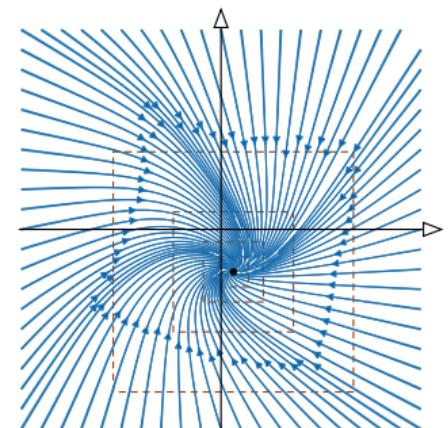
\implies **interconnected system** is contracting wrt rate $|\alpha(\Gamma)|$

contractivity = robust computationally-friendly stability

fixed point theory + Lyapunov stability theory + geometry of metric spaces

highly-ordered transient and asymptotic behavior, no anonymous constants/functions:

- ① unique globally exponential stable equilibrium
& two natural Lyapunov functions
- ② robustness properties
 - bounded input, bounded output (iss)
 - finite input-state gain
 - robustness margin wrt unmodeled dynamics
 - robustness margin wrt delayed dynamics
- ③ periodic input, periodic output
- ④ modularity and interconnection properties
- ⑤ accurate numerical integration and equilibrium point computation



search for contraction properties
design engineering systems to be contracting
verify correct/safe behavior via known Lipschitz constants

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§5. Application: Online feedback optimization

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Kachurovskii's Theorem: For differentiable $f : \mathbb{R}^n \rightarrow \mathbb{R}$, equivalent statements:

- ① f is **strongly convex** with parameter ν (and minimum x^*)
- ② $-\nabla f$ is **ν -strongly infinitesimally contracting** (with equilibrium x^*), that is

$$(-\nabla f(x) + \nabla f(y))^\top (x - y) \leq -\nu \|x - y\|_2^2$$

R. I. Kachurovskii. Monotone operators and convex functionals. *Uspekhi Matematicheskikh Nauk*, 15(4):213–215, 1960

Example #1: Gradient dynamics for strongly convex function

Given differentiable, strongly convex $f : \mathbb{R}^n \rightarrow \mathbb{R}$ with parameter $\nu > 0$, **gradient dynamics**

$$\dot{x} = F_G(x) := -\nabla f(x)$$

F_G is infinitesimally contracting wrt $\|\cdot\|_2$ with rate ν

unique globally exp stable point is global minimum

Example #2: Primal-dual gradient dynamics

strongly convex function f

s.t. $0 \prec \nu_{\min} I_n \preceq \text{Hess } f \preceq \nu_{\max} I_n$

constraint matrix A

s.t. $0 \prec a_{\min} I_m \preceq AA^\top \preceq a_{\max} I_m$

(independent rows)

linearly constrained optimization:

$$\begin{aligned} & \min_{x \in \mathbb{R}^n} f(x) \\ & \text{subj. to } Ax = b \end{aligned}$$

primal-dual gradient dynamics:

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = F_{\text{PDG}}(x, \lambda) := \begin{bmatrix} -\nabla f(x) - A^\top \lambda \\ Ax - b \end{bmatrix}$$

F_{PDG} is infinitesimally contracting wrt $\|\cdot\|_{2,P^{1/2}}$ with rate c

$$P = \begin{bmatrix} I_n & \alpha A^\top \\ \alpha A & I_m \end{bmatrix} \text{ with } \alpha = \frac{1}{2} \min \left\{ \frac{1}{\nu_{\max}}, \frac{\nu_{\min}}{a_{\max}} \right\} \quad \text{and} \quad c = \frac{1}{4} \min \left\{ \frac{a_{\min}}{\nu_{\max}}, \frac{a_{\min}}{a_{\max}} \nu_{\min} \right\}$$

Example #3: Laplacian-based distributed gradient

Given $\Pi_n = I_n - \mathbb{1}_n \mathbb{1}_n^\top / n = \text{orthogonal projection onto } \text{span}\{\mathbb{1}_n\}^\perp$,

$$0 \prec \lambda_2 \Pi_n \preceq L \preceq \lambda_n I_n$$

decomposable cost: $\min_{x \in \mathbb{R}^n} \sum_{i=1}^n f_i(x)$ where each f_i is ν_i -strongly convex

$$\begin{cases} \min_{x_{[i]} \in \mathbb{R}} & \sum_{i=1}^n f_i(x_{[i]}) \\ \text{subj. to} & \sum_{j=1}^n a_{ij}(x_i - x_j) = 0 \end{cases}$$

Laplacian-based distributed gradient (primal-dual gradient, $2n$ vars):

$$\begin{cases} \dot{x}_{[i]} = -\nabla f_i(x_{[i]}) - \sum_{j=1}^n a_{ij}(\lambda_i - \lambda_j) & \text{for each node } i \\ \dot{\lambda}_i = \sum_{j=1}^n a_{ij}(x_i - x_j) & \text{for each node } i \end{cases}$$

$F_{\text{Laplacian-DistributedG}}$ is infinitesimally contracting † with $c = \frac{1}{4} \left(\frac{\lambda_2}{\lambda_n} \right)^2 \min_i \nu_i$

Detour: Composite optimization and the proximal operator

composite minimization (cost = sum of terms with structurally different properties):

$$x^* = \operatorname{argmin}_{x \in \mathbb{R}^n} f(x, u) + g(x)$$

$f(x, u)$ is convex and differentiable in x $g : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is convex, closed, and proper (ccp)

proximal operator:

$$\operatorname{prox}_{\gamma g}(z) := \operatorname{argmin}_{x \in \mathbb{R}^n} g(x) + \frac{1}{2\gamma} \|x - z\|_2^2$$

generalized form of projection for nonsmooth/constrained/large-scale/distributed optimization

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generalized form of projection for nonsmooth/constrained/large-scale/distributed optimization

Equivalence property:

① x^* is minimizer for:

$$\min_{x \in \mathbb{R}^n} f(x, u) + g(x)$$

② x^* is fixed point for:

$$x = \operatorname{prox}_{\gamma g}(x - \gamma \nabla f(x, u)) \quad \text{for all } \gamma$$

Example #4: Proximal gradient dynamics

Equivalence property motivates:

proximal gradient dynamics:

$$\dot{x} = F_{\text{ProxG}}(x) := -x + \text{prox}_{\gamma g}(x - \gamma \nabla f(x))$$

projected gradient descent is special case

F_{ProxG} is infinitesimally contracting wrt $\|\cdot\|_2$

$$\text{for } 0 < \gamma < \frac{2}{\ell}, \quad \text{with rate} \quad c = 1 - \max\{|1 - \gamma\nu|, |1 - \gamma\ell|\},$$

$$\text{for } \gamma^* = \frac{2}{\nu + \ell}, \quad \text{with maximal rate} \quad c^* = \frac{2\nu}{\nu + \ell}$$

Example #5: Saddle dynamics

Assume $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$

- $x \mapsto f(x, y)$ is ν_x -strongly convex, uniformly in y
- $y \mapsto f(x, y)$ is ν_y -strongly concave, uniformly in x

saddle dynamics (primal-descent / dual-ascent):

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \mathsf{F}_S(x, y) := \begin{bmatrix} -\nabla_x f(x, y) \\ \nabla_y f(x, y) \end{bmatrix}$$

F_S is infinitesimally contracting wrt $\|\cdot\|_2$ with rate $\min\{\nu_x, \nu_y\}$

unique globally exp stable point is saddle point (min in x , max in y)

Example #6: Pseudogradient play

Each player i aims to minimize its own cost function $J_i(x_i, x_{-i})$ (not a potential game)

pseudogradient dynamics (aka gradient play in game theory):

$$\begin{aligned}\dot{x} &= \mathsf{F}_{\text{PseudoG}}(x) = -(\nabla_1 J_1(x_1, x_{-1}), \dots, \nabla_n J_n(x_n, x_{-n})) && \text{(stacked vector)} \\ \iff \dot{x}_i &= -\nabla_i J_i(x_i, x_{-i})\end{aligned}$$

• **strong convexity wrt x_i :** J_i is μ_i strongly convex wrt x_i , uniformly in x_{-i}

• **Lipschitz wrt x_{-i} :** $\text{Lip}_{x_j}(\nabla_i J_i) \leq \ell_{ij}$, uniformly in x_{-j}

• $\mathsf{F}_{\text{PseudoG}}$ gain matrix is Hurwitz

$\implies \mathsf{F}_{\text{PseudoG}}$ is infinitesimally contracting wrt appropriate diag-weighted $\|\cdot\|_2$

Example #7: Best response play

Each player i aims to minimize its own cost function $J_i(x_i, x_{-i})$

$\text{BR}_i : x_{-i} \rightarrow \operatorname{argmin}_{x_i} J_i(x_i, x_{-i})$ best response of player i wrt other decisions x_{-i}

best response dynamics:

$$\begin{aligned}\dot{x} &= F_{\text{BR}}(x) := \text{BR}(x) - x \\ \iff \dot{x}_i &= \text{BR}_i(x_{-i}) - x_i\end{aligned}$$

- **strong convexity wrt x_i :** J_i is μ_i strongly convex wrt x_i , uniformly in x_{-i}
- **Lipschitz wrt x_{-i} :** $\text{Lip}_{x_j}(\nabla_i J_i) \leq \ell_{ij}$, uniformly in x_{-j}
 $\implies \text{BR}_i$ is Lipschitz wrt x_j with constant ℓ_{ij}/μ_i
- F_{BR} gain matrix is Hurwitz \iff BR is a discrete-time contraction
 $\implies \text{BR} - \text{Id}$ is infinitesimally contracting wrt appropriate diag-weighted $\|\cdot\|_2$

Equivalent statements:

① F_{PseudoG} gain matrix:

$$\begin{bmatrix} -\mu_1 & \dots & \ell_{1n} \\ \vdots & & \vdots \\ \ell_{n1} & \dots & -\mu_n \end{bmatrix} \text{ is Hurwitz}$$

② F_{BR} gain matrix:

$$\begin{bmatrix} -1 & \dots & \ell_{1n}/\mu_1 \\ \vdots & & \vdots \\ \ell_{n1}/\mu_n & \dots & -1 \end{bmatrix} \text{ is Hurwitz}$$

③ discrete-time F_{BR} gain matrix:

$$\begin{bmatrix} 0 & \dots & \ell_{1n}/\mu_1 \\ \vdots & & \vdots \\ \ell_{n1}/\mu_n & \dots & 0 \end{bmatrix} \text{ is Schur}$$

Aggregative games: $J_i(x_i, x_{-i}) = f_i(x_i, \frac{1}{n} \sum_{j=1}^n x_j)$

assume f_i is μ_i -strongly convex wrt x_i and $\ell_i = \text{Lip}_y(\nabla_{x_i} f_i(x_i, y))$

$\mu_i > \ell_i$ for each agent i \implies gain matrix is Hurwitz

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Many convex optimization problems can be solved with contracting dynamics

$$\dot{x} = F(x)$$

Many convex optimization problems can be solved with contracting dynamics

$$\dot{x} = F(x)$$

contracting dynamics for parametric strongly-convex optimization

$$\dot{x} = F(x, \theta)$$

contracting dynamics for time-varying strongly-convex optimization

$$\dot{x} = F(x, \theta(t))$$

Equilibrium tracking

For parameter-dependent vector field $\mathbf{F} : \mathbb{R}^n \times \mathbb{R}^d \rightarrow \mathbb{R}^n$ and differentiable $\theta : \mathbb{R}_{\geq 0} \rightarrow \Theta \subset \mathbb{R}^d$

$$\dot{x}(t) = \mathbf{F}(x(t), \theta(t))$$

Assume there exist norms $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\Theta}$ s.t.

- **contractivity wrt x :** $\text{osLip}_x(\mathbf{F}) \leq -c < 0$, uniformly in θ
- **Lipschitz wrt θ :** $\text{Lip}_{\theta}(\mathbf{F}) \leq \ell$, uniformly in x

Theorem: Equilibrium tracking for contracting dynamics

- ① for each fixed θ , there exists a unique equilibrium $x^*(\theta)$
- ② the equilibrium map $x^*(\cdot)$ is Lipschitz with constant $\frac{\ell}{c}$

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Theorem: Equilibrium tracking for contracting dynamics

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- ② the equilibrium map $x^*(\cdot)$ is Lipschitz with constant $\frac{\ell}{c}$
- ③ $D^+ \|x(t) - x^*(\theta(t))\|_{\mathcal{X}} \leq -c \|x(t) - x^*(\theta(t))\|_{\mathcal{X}} + \frac{\ell}{c} \|\dot{\theta}(t)\|_{\Theta}$

Consequences for tracking error

$$D^+ \|x(t) - x^*(\theta(t))\|_{\mathcal{X}} \leq -c \|x(t) - x^*(\theta(t))\|_{\mathcal{X}} + \frac{\ell}{c} \|\dot{\theta}(t)\|_{\Theta}$$

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- bounded input, bounded error
with asymptotic bound:

$$\limsup_{t \rightarrow \infty} \|x(t) - x^*(\theta(t))\|_{\mathcal{X}} \leq \frac{\ell}{c^2} \limsup_{t \rightarrow \infty} \|\dot{\theta}(t)\|_{\Theta}$$

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- bounded energy input, bounded energy error
- vanishing input, vanishing error
- exponentially vanishing input $\sim e^{-ht}$, exponentially vanishing error $\sim e^{-\min\{c,h\}t}$
- periodic input, periodic error

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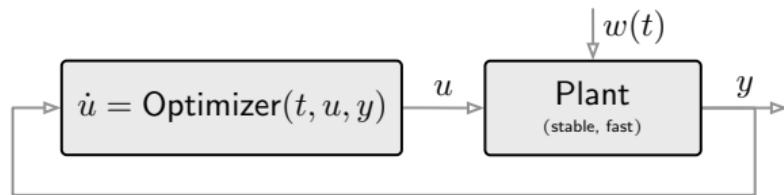
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Application: Online feedback optimization



online feedback optimization

online optimization, optimization-based feedback, input/output regulation ...

$$\begin{cases} \min & \text{cost}_1(u) + \text{cost}_2(y) \\ \text{subj. to} & y = \text{Plant}(u, w(t)) \end{cases} \implies \begin{cases} \dot{u} = \text{Optimizer}(t, u, y) \\ y = \text{Plant}(u, w(t)) \end{cases}$$

A. Hauswirth, S. Bolognani, G. Hug, and F. Dorfler. Timescale separation in autonomous optimization. *IEEE Transactions on Automatic Control*, 66(2):611–624, 2021. [doi](#)

G. Bianchin, J. Cortés, J. I. Poveda, and E. Dall'Anese. Time-varying optimization of LTI systems via projected primal-dual gradient flows. *IEEE Transactions on Control of Network Systems*, 9(1):474–486, 2022. [doi](#)

Example #8: Gradient controller

- fast/stable LTI plant with control input u and state/measurement disturbance $w(t)$:

$$\begin{aligned}\epsilon \dot{x} &= Ax + Bu + Ew(t) && A \text{ Hurwitz} \\ y &= Cx + Dw(t)\end{aligned}$$

- in singular perturbation limit as $\epsilon \rightarrow 0^+$, **steady state map (Y_u and Y_w)**

$$y = \underbrace{-CA^{-1}B u}_{=: Y_u} + \underbrace{(D - CA^{-1}E) w}_{=: Y_w}$$

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$$y = \underbrace{-CA^{-1}B u}_{=: Y_u} + \underbrace{(D - CA^{-1}E) w}_{=: Y_w}$$

- define **cost function \mathcal{E} on u and y** :

$$\mathcal{E}(u, w) = \phi(u) + \psi(Y_u u + Y_w w), \quad (\phi \text{ is } \nu\text{-strongly convex and } \psi \text{ is convex})$$

and note

$$\begin{aligned}\nabla_u \mathcal{E}(u, w) &= \nabla \phi(u) + Y_u^\top \nabla \psi(Y_u u + Y_w w) \\ &= \nabla \phi(u) + Y_u^\top \nabla \psi(y) \quad (\text{no need to measure } w(t))\end{aligned}$$

Example #8: Gradient controller

equilibrium trajectory let $u^*(t)$ be solution to

$$\begin{aligned} \min_u \quad & \phi(u) + \psi(y(t)) && (\nu\text{-strongly convex } \phi, \text{ convex } \psi) \\ \text{subj to} \quad & y(t) = Y_u u + Y_w w(t) \end{aligned}$$

gradient controller

$$\dot{u} = F_{\text{GradCtrl}}(u, w) := -\nabla \mathcal{E}_u(u, w) = -\nabla \phi(u) - Y_u^\top \nabla \psi(Y_u u + Y_w w)$$

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gradient controller

$$\dot{u} = F_{\text{GradCtrl}}(u, w) := -\nabla \mathcal{E}_u(u, w) = -\nabla \phi(u) - Y_u^\top \nabla \psi(Y_u u + Y_w w)$$

Equilibrium tracking for the gradient controller

- ① $\text{osLip}_u(F_{\text{GradCtrl}}) \leq -\nu$ (gradient of ν -strongly convex function)
- ② $\text{Lip}_w(F_{\text{GradCtrl}}) = \ell_w := \|Y_u^\top\| \text{Lip}(\nabla \psi) \|Y_w\|$

$$\limsup_{t \rightarrow \infty} \|u(t) - u^*(t)\| \leq \frac{\ell_w}{\nu^2} \limsup_{t \rightarrow \infty} \|\dot{w}(t)\|$$

Example #9: Projected gradient controller

Constrained feedback optimization:

$$\begin{aligned} \min_u \quad & \mathcal{E}(u, w) = \phi(u) + \psi(Y_u u + Y_w w) \quad (\nu \text{ strongly convex}, \ell_u \text{ strongly smooth}, \ell_w) \\ \text{subj. to} \quad & u \in \mathcal{U} \quad (\text{nonempty, closed, convex. } P_{\mathcal{U}} = \text{orthogonal projection}) \end{aligned}$$

Projected gradient controller (example of proximal gradient dynamics):

$$\dot{u} = F_{\text{PGC}}(u, w) := -u + P_{\mathcal{U}}(u - \gamma \nabla_u \mathcal{E}(u, w))$$

Equilibrium tracking for projected gradient controller At $\gamma = \frac{2}{\nu + \ell_u}$,

① $\text{osLip}_u(F_{\text{PGC}}) \leq -c_{\text{PGC}} := -\frac{2\nu}{\nu + \ell_u}$ (contractivity prox gradient)

② $\text{Lip}_w(F_{\text{PGC}}) = \ell_{\text{PGC}} := \frac{2}{\nu + \ell_u} \ell_w$

$$\limsup_{t \rightarrow \infty} \|u(t) - u^*(t)\| \leq \frac{\ell_{\text{PGC}}}{c_{\text{PGC}}^2} \limsup_{t \rightarrow \infty} \|\dot{w}(t)\| \quad (\text{eq tracking})$$

Outline

§1. Introduction

§2. Basic contractivity concepts

§3. Examples: Gradient systems defined by strongly convex functions are contracting

§4. Theory: Equilibrium tracking in parametric optimization

§5. Application: Online feedback optimization

§6. Conclusions

contractivity = robust computationally-friendly stability

fixed point theory + Lyapunov stability theory + geometry of metric spaces

- theory
- examples
- control application

Ongoing work

- ① applications to ML and biologically-inspired neural networks
- ② applications to optimization-based control designs:
model predictive control, control barrier functions, low-gain integral control
- ③ equilibrium tracking with noise
applications to optimization-based control