Geometry, Optimization and Control in Robot Coordination

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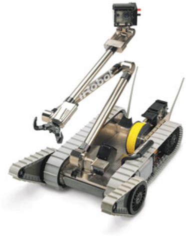
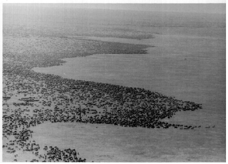
What kind of systems?

- each agent **senses** its immediate environment,
- **communicates** with others,
- **processes** information gathered, and
- **takes local action** in response

"Raven" by AeroVironment Inc



"PackBot" by iRobot Inc



Wildebeest herd in the Serengeti

Geese flying in formation

Fish swarm in Atlantis aquarium

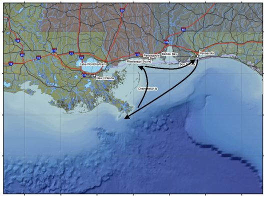
Cooperative robotics: technologies and applications

What scenarios?

Security systems, disaster recovery, environmental monitoring, study of natural phenomena and biological species, science imaging

Security systems

Warehouse automation



Environmental monitoring

What kind of tasks?

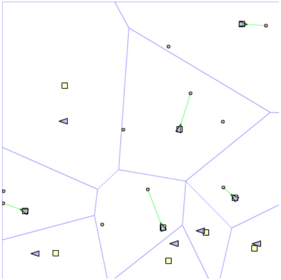
1. coordinated motion: rendezvous, flocking, formation
2. cooperative sensing: surveillance, exploration, search and rescue
3. cooperative material handling and transportation

Coordination via task and territory partitioning

Model: customers appear randomly in space/time
robotic network knows locations and provides service

Goal: minimize customer delay

Approach: assign customers to robots by partioning the space



F. Bullo, E. Frazzoli, M. Pavone, K. Savla, and S. L. Smith. Dynamic vehicle routing for robotic systems. *IEEE Proceedings*, 99(9):1482–1504, 2011

Outline

1. robot coordination via territory partitioning

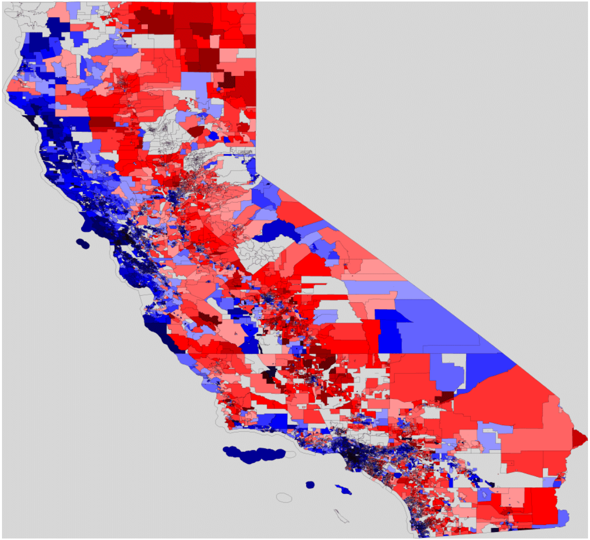
2. gossip algorithms: mathematical setup

3. gossip algorithms: technological advances

Territory partitioning is ... art

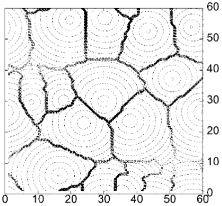
abstract expressionism
"Ocean Park No. 27" and "Ocean Park No. 129"
by Richard Diebenkorn (1922-1993), inspired by aerial landscapes

Territory partitioning ... centralized district design

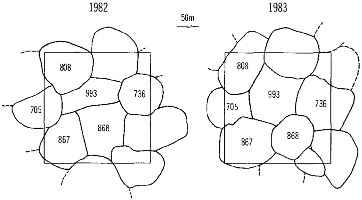


California Voting Districts: 2008 Obama/McCain votes

Tilapia mossambica, "Hexagonal Territories," Barlow et al, '74



Red harvester ants, "Optimization, Conflict, and Nonoverlapping Foraging Ranges," Adler et al, '03



Sage sparrows, "Territory dynamics in a sage sparrows population," Petersen et al '87

ANALYSIS of cooperative distributed behaviors

1. how do animals share territory?
   how do they decide foraging ranges?
   how do they decide nest locations?
2. what if each robot goes to "center" of own dominance region?
3. what if each robot moves away from closest robot?

DESIGN of performance metrics

4. how to cover a region with n minimum-radius overlapping disks?
5. how to design a minimum-distortion (fixed-rate) vector quantizer?
6. where to place mailboxes in a city / cache servers on the internet?

Multi-center functions

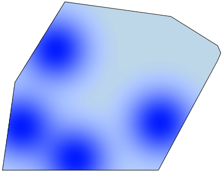
Expected wait time

$$H(p, v) = \int_{v_1} \|q - p_1\| dq \; + \; \cdots \; + \; \int_{v_n} \|q - p_n\| dq$$

- n robots at $p = \{p_1, \dots, p_n\}$
- environment is partitioned into $v = \{v_1, \dots, v_n\}$

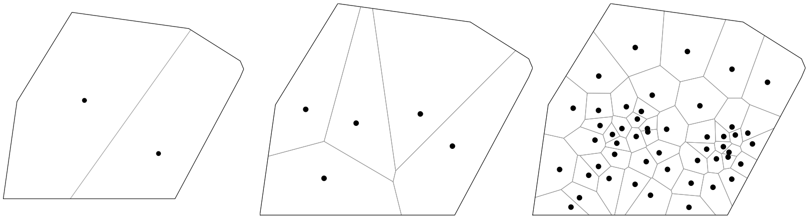
$$H(p, v) = \sum_{i=1}^{n} \int_{v_i} f(\|q - p_i\|)\phi(q) dq$$

- $\phi : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ density
- $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ penalty function



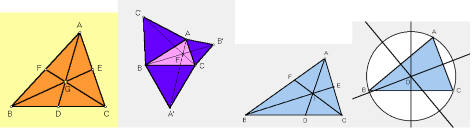
The **Voronoi partition** $\{V_1, \dots, V_n\}$ generated by points $(p_1, \dots, p_n)$

$$
\begin{aligned}
V_i &= \{q \in Q \mid \; \|q - p_i\| \leq \|q - p_j\|, \; \forall j \neq i\} \\
&= Q \bigcap_j (\text{half plane between } i \text{ and } j, \text{ containing } i)
\end{aligned}
$$



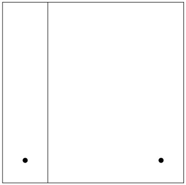
function of p	minimizer = center
$p \mapsto \int_v \|q-p\|^2 \phi(q)dq$	**centroid** (or **center of mass**)
$p \mapsto \int_v \|q-p\| \phi(q)dq$	**Fermat–Weber point** (or **median**)
$p \mapsto \text{area}(v \cap \text{disk}(p,r))$	***r*-area center**
$p \mapsto$ radius of largest disk centered at p enclosed inside v	**incenter**
$p \mapsto$ radius of smallest disk centered at p enclosing v	**circumcenter**

From online Encyclopedia of Triangle Centers



$$H(p, v) = \int_{v_1} f(\|q - p_1\|)\phi(q)dq \; + \; \cdots \; + \; \int_{v_n} f(\|q - p_n\|)\phi(q)dq$$

1. at fixed positions, optimal partition is Voronoi
2. at fixed partition, optimal positions are "generalized centers"
3.



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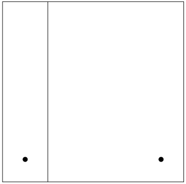
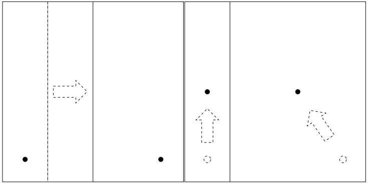
Q. Du, V. Faber, and M. Gunzburger. Centroidal Voronoi tessellations: Applications and algorithms. *SIAM Review*, 41(4):637–676, 1999

From optimality conditions to algorithms

$$H(p, v) = \int_{v_1} f(\|q - p_1\|)\phi(q)dq + \cdots + \int_{v_n} f(\|q - p_n\|)\phi(q)dq$$

1. at fixed positions, optimal partition is Voronoi
2. at fixed partition, optimal positions are "generalized centers"
3. alternate v-p optimization $\implies$ local opt = center Voronoi partition





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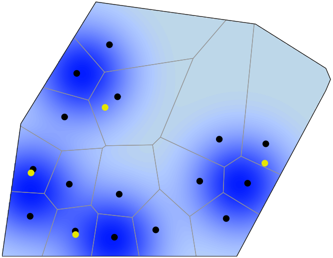
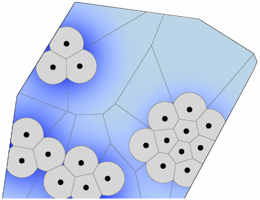
Voronoi+centering law

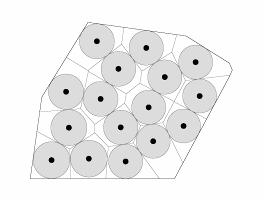
At each comm round:

1: acquire neighbors' positions
2: compute own dominance region
3: move towards center of own dominance region

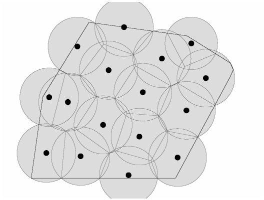




Area-center



Incenter



Circumcenter

F. Bullo, J. Cortés, and S. Martínez. *Distributed Control of Robotic Networks*. Applied Mathematics Series. Princeton Univ Press, 2009. Available at http://www.coordinationbook.info

Incomplete literature

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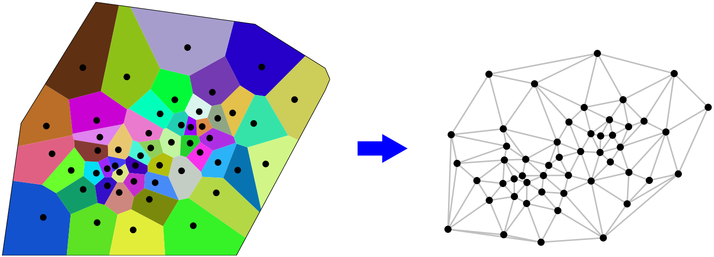
Outline

1. robot coordination via territory partitioning
2. gossip algorithms: mathematical setup
3. gossip algorithms: technological advances

Partitioning with minimal communication requirements

Voronoi+centering law requires:

1. synchronous communication
2. communication along edges of dual graph



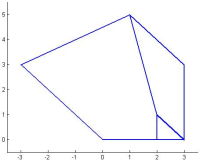
Minimalist coordination

- is synchrony necessary?
- what are minimal communication requirements?
- is asynchronous peer-to-peer, gossip, sufficient?

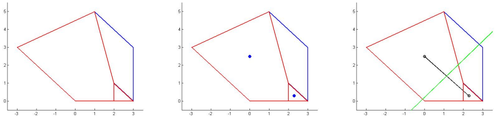
Gossip partitioning policy

At random comm instants,
between two random regions:

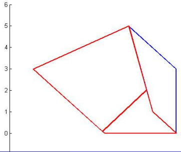
1. compute two centers
2. compute bisector of centers
3. partition two regions by bisector



before meeting



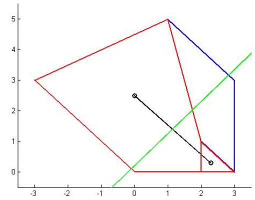
after meeting



F. Bullo, R. Carli, and P. Frasca. Gossip coverage control for robotic networks: Dynamical systems on the space of partitions. *SIAM JCO*, 50(1):419–447, 2012

Lyapunov function for gossip territory partitioning

$$H(v) = \sum_{i=1}^{n} \int_{v_i} f(\| \mathrm{center}(v_i) - q \|)\phi(q)dq$$



1. state space is not finite-dimensional
   - non-convex disconnected polygons
   - arbitrary number of vertices
2. gossip map is not deterministic, ill-defined and discontinuous
   - two regions could have same centers

Gossip convergence analysis (proof sketch 1/4)

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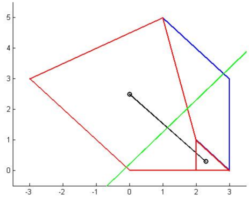

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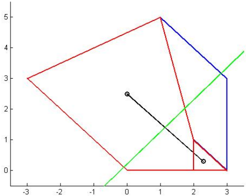

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- X is metric space
- finite collection of maps $T_i : X \to X$ for $i \in I$
- consider sequences $\{x_\ell\}_{\ell \geq 0} \subset X$ with

$$x_{\ell+1} = T_{i(\ell)}(x_\ell)$$

Assume:

1.
2.
3.
4.

If $x_0 \in W$, then almost surely

$$x_\ell \to \text{(intersection of sets of fixed points of all } T_i\text{)}$$

Convergence with persistent switches (proof sketch 2/4)

- X is metric space
- finite collection of maps $T_i : X \to X$ for $i \in I$
- consider sequences $\{x_\ell\}_{\ell \geq 0} \subset X$ with

$$x_{\ell+1} = T_{i(\ell)}(x_\ell)$$

Assume:

1. $W \subset X$ **compact** and positively invariant for each T_i
2. $U : W \to \mathbb{R}$ **decreasing** along each T_i
3. U and T_i are **continuous** on W
4. there exists probability $p \in]0,1[$ such that, for all indices $i \in I$ and times ℓ, we have $\quad \text{Prob}\left[x_{\ell+1} = T_i(x_\ell) \,|\, \text{past}\right] \geq p$

If $x_0 \in W$, then almost surely

$$x_\ell \;\to\; (\text{intersection of sets of fixed points of all } T_i)$$

Let $\mathcal{C}$ be set of closed subsets of Q — is it compact?

Hausdorff metric

$$d_{\mathrm{H}}(A, B) = \max\left\{ \max_{a \in A} \min_{b \in B} d(a, b),\ \max_{b \in B} \min_{a \in A} d(a, b) \right\}$$

1.
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1. $(\mathcal{C}, d_{\mathrm{H}})$ is compact metric space
2. dynamical system and Lyapunov function are not continuous wrt d_{H}!
3. Hausdorff metric sensitive to sets of measure zero

Let $\mathcal{C}$ be set of closed subsets of Q — is it compact?

Symmetric difference metric

$$d_{\mathrm{symm}}(A, B) = \mathrm{measure}(A \setminus B) + \mathrm{measure}(B \setminus A)$$

1.
2.
3.

The space of partitions (proof sketch 4/4)

Let $\mathcal{C}$ be set of closed subsets of Q — is it compact?

Symmetric difference metric

$$d_{\text{symm}}(A, B) = \text{measure}(A \setminus B) + \text{measure}(B \setminus A)$$

1. redefine $\mathcal{C} \leftarrow \mathcal{C}/\sim$ where $A \sim B$ whenever $d_{\text{symm}}(A, B) = 0$
2. dynamical system and Lyapunov function are continuous in $(\mathcal{C}, d_{\text{symm}})$
3. no compactness result is available for $(\mathcal{C}, d_{\text{symm}})$!

   Theorem: for any k, $(\mathcal{C}_{(k)}, d_{\text{symm}})$ is compact.

   $\mathcal{C}_{(k)}$ is set of k-convex subsets (union of k convex sets)

Incomplete literature

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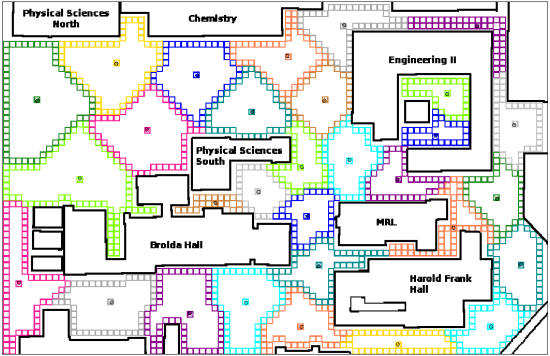
Gossip algorithms: technological advances



1. non-convex environments
2. motion protocols (for communication persistence)
3. hardware and large-scale implementations

Nonconvex environments as graphs

Revised setup

environment: weighted graph partitioned in connected subgraphs

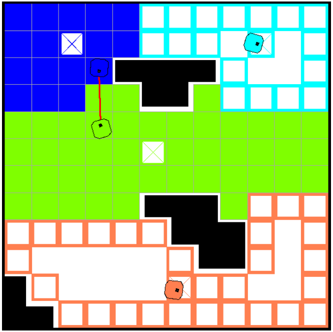
multi-center cost function: $H(p, v) = H_1(p_1, v_1) + \cdots + H_1(p_n, v_n)$

single-region cost function: $H_1(p, v) = \sum_{q \in v} \operatorname{dist}(p, q)\, \phi(q)$

center of subgraph v: minimizer of $p \mapsto H_1(p, v)$

Range-dependent stochastic comm

Two robots communicate at the sample times of a Poisson process with distance-dependent intensity



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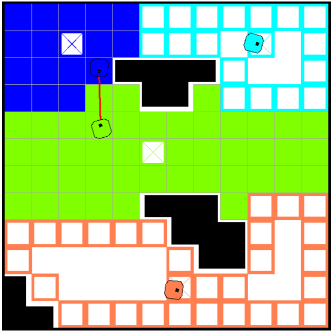


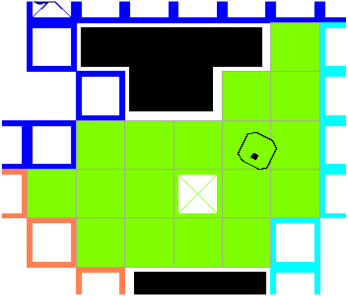
Discretized gossip algorithm

1. Ensure that neighbors meet frequently enough:
   $\Rightarrow$ Random Destination & Wait Motion Protocol
2. Update partition when two robots meet:
   $\Rightarrow$ Pairwise Partitioning Rule

Random Destination & Wait Motion Protocol

Each robot continuously executes:

1: select sample destination $q_i \in v_i$
2: move to q_i
3: wait at q_i for time $\tau > 0$



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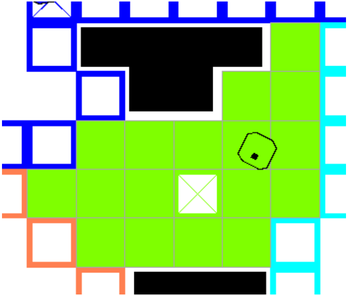
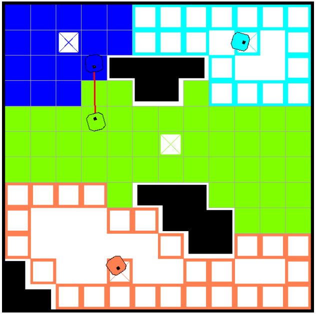


Discretized gossip algorithm

Pairwise Partitioning Rule

Whenever robots i and j communicate:

$w \leftarrow v_i \cup v_j$
while (computation time is available) **do**
    $(q_i, q_j) \leftarrow$ sample vertices in w
    $(w_i, w_j) \leftarrow$ Voronoi of w by (q_i, q_j)
    if $(H_1(q_i, w_i) + H_1(q_j, w_j)$ improves) **then**
        centroids $\leftarrow (q_i, q_j)$
        $(v_i, v_j) \leftarrow (w_i, w_j)$
    end if
end while



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    (w_i, w_j) ← Voronoi of w by (q_i, q_j)
    if (H_1(q_i, w_i) + H_1(q_j, w_j) improves) then
        centroids ← (q_i, q_j)
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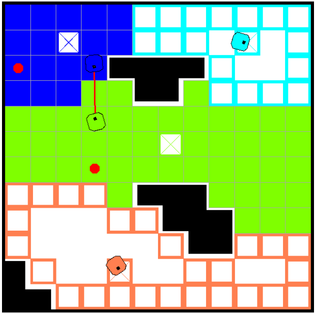

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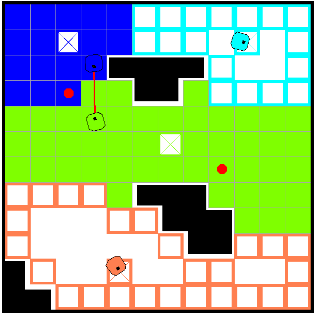
Discretized gossip algorithm

Pairwise Partitioning Rule

Whenever robots i and j communicate:

$w \leftarrow v_i \cup v_j$
while (computation time is available) **do**
  $(q_i, q_j) \leftarrow$ sample vertices in w
  $(w_i, w_j) \leftarrow$ Voronoi of w by (q_i, q_j)
  if $(H_1(q_i, w_i) + H_1(q_j, w_j)$ improves) **then**
    centroids $\leftarrow (q_i, q_j)$
    $(v_i, v_j) \leftarrow (w_i, w_j)$
  end if
end while



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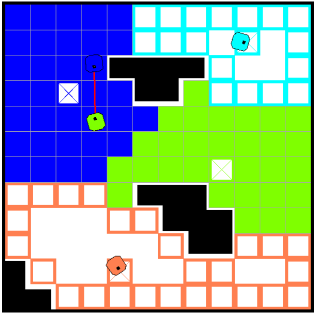
6: centroids $\leftarrow (q_i, q_j)$
7: $(v_i, v_j) \leftarrow (w_i, w_j)$
8: **end if**
9: **end while**



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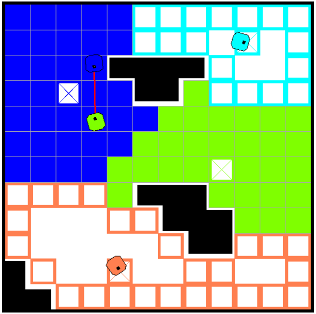
Discretized gossip algorithm

Pairwise Partitioning Rule

Whenever robots i and j communicate:

```
w ← v_i ∪ v_j
while (computation time is available) do
    (q_i, q_j) ← sample vertices in w
    (w_i, w_j) ← Voronoi of w by (q_i, q_j)
    if (H_1(q_i, w_i) + H_1(q_j, w_j) improves) then
        centroids ← (q_i, q_j)
        (v_i, v_j) ← (w_i, w_j)
    end if
end while
```



(combinatorial optimization) – interruptible anytime algorithm

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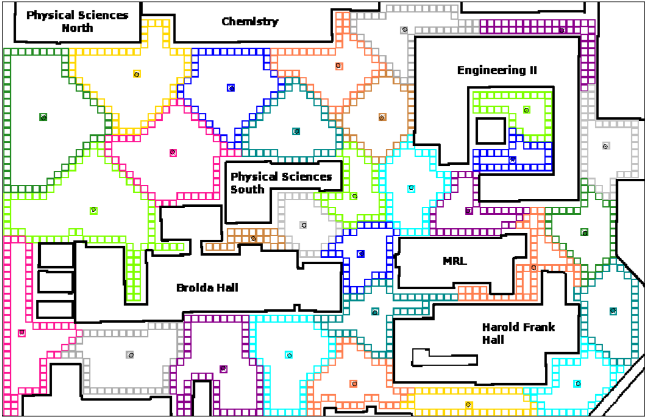
Harware-in-the-loop experiment

Player/Stage robot simulation & control system: realistic robot models with integrated wireless network model & obstacle-avoidance planner.

Hardware-in-the-loop experiment: 3 physical and 6 simulated robots

Larger-scale simulation experiment



Player/Stage robot simulation & control system: realistic robot models with integrated wireless network model & obstacle-avoidance planner.

Simulation experiment: 30 robots; UCSB campus.

Conclusions

Summary

1. gossip algorithms: mathematical setup
2. gossip algorithms: technological advances

Open problems

1. topology and comp geometry of power sets
2. coordination: resource allocation, weak comm protocols
3. ecology of territory partioning

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