# Modeling and Trajectory Design for Mechanical Control Systems

**Center for the Foundations of Robotics**
**Robotics Institute, Carnegie Mellon University**

**Francesco Bullo**
Coordinated Science Lab
University of Illinois at Urbana-Champaign
1308 W. Main St, Urbana, IL 61801, USA
bullo@uiuc.edu, http://motion.csl.uiuc.edu

Thanks to: Jorge Cortés, Andrew D. Lewis, Kevin Lynch, Sonia Martínez

# Geometric Control of Mechanical Systems

## Scientific Interests

(i) success in linear control theory is unlikely to be repeated for nonlinear systems. In particular, nonlinear system design. no hope for general theory

$\Rightarrow$ mechanical systems as examples of control systems

(ii) nonlinear control and geometric mechanics

## Framework based on affine connections

(i) reduction from $2n$ to $n$ dimensional computations

(ii) **controllability, kinematic models, planning, averaging** not stabilization

# Literature review

## Modeling:


(i) R. Hermann. *Differential Geometry and the Calculus of Variations*, volume 49 of *Mathematics in Science and Engineering*. Academic Press, New York, NY, 1968

(ii) A. M. Bloch and P. E. Crouch. Nonholonomic control systems on Riemannian manifolds. *SIAM JCO*, 33(1):126–148, 1995

(iii) A. D. Lewis. Simple mechanical control systems with constraints. *IEEE T. Automatic Ctrl*, 45(8):1420–1436, 2000


## Reductions & Planning via Inverse Kinematics:


(i) H. Arai, K. Tanie, and N. Shiroma. Nonholonomic control of a three-DOF planar underactuated manipulator. *IEEE T. Robotics Automation*, 14(5):681–695, 1998

(ii) K. M. Lynch, N. Shiroma, H. Arai, and K. Tanie. Collision-free trajectory planning for a 3-DOF robot with a passive joint. *Int. J. Robotic Research*, 19(12):1171–1184, 2000

(iii) A. D. Lewis. When is a mechanical control system kinematic? In *Proc CDC*, pages 1162–1167, Phoenix, AZ, December 1999

**Controllability:**

(i) H. J. Sussmann. A general theorem on local controllability. *SIAM JCO*, 25(1):158–194, 1987

(ii) A. D. Lewis and R. M. Murray. Configuration controllability of simple mechanical control systems. *SIAM JCO*, 35(3):766–790, 1997

**Averaging:**

(i) J. Baillieul. Stable average motions of mechanical systems subject to periodic forcing. In M. J. Enos, editor, *Dynamics and Control of Mechanical Systems: The Falling Cat and Related Problems*, volume 1, pages 1–23. Field Institute Communications, 1993

(ii) M. Levi. Geometry of Kapitsa's potentials. *Nonlinearity*, 11(5):1365–8, 1998

**Planning via approximate inversion:**

(i) R. E. Bellman, J. Bentsman, and S. M. Meerkov. Vibrational control of nonlinear systems: Vibrational stabilization. *IEEE T. Automatic Ctrl*, 31(8):710–716, 1986

(ii) W. Liu. An approximation algorithm for nonholonomic systems. *SIAM JCO*, 35(4):1328–1365, 1997

Francesco Bullo and Andrew D. Lewis

# Geometric Control of Mechanical Systems

## Modeling, Analysis, and Design for Simple Mechanical Control Systems

SPIN

– Monograph –

May 19, 2004

Springer
Berlin Heidelberg New York
Hong Kong London
Milan Paris Tokyo

# Outline: from geometry to algorithms

(i) **modeling**

(ii) **approach #1**

  (a) analysis: kinematic reductions and controllability

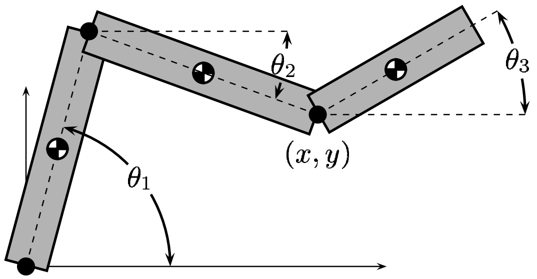
  (b) design: inverse kinematics catalog

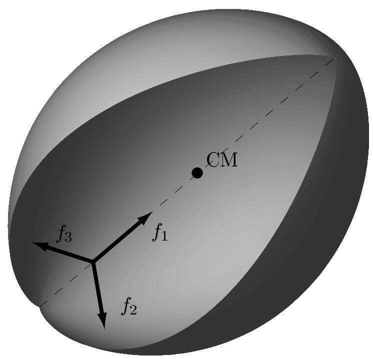
(iii) **approach #2**

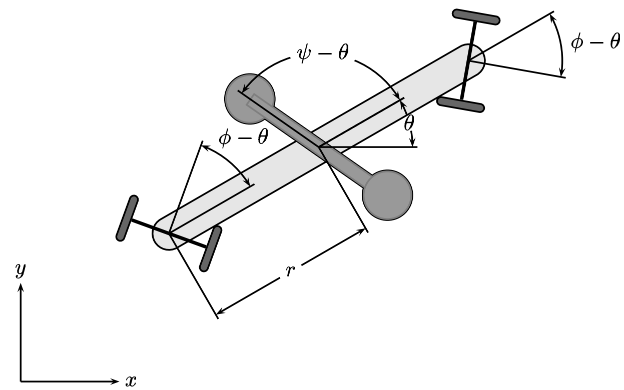
  (a) analysis: oscillatory controls and averaging

  (b) design: approximate inversion

# 1 Models of Mechanical Control Systems




**Ex #1: robotic manipulators with kinetic energy and forces at joints**

systems with potential control forces

**Ex #2: aerospace and underwater vehicles**

invariant systems on Lie groups

**Ex #3: systems subject to nonholonomic constraints**

locomotion devices with drift, e.g., bicycle, snake-like robots

## 1.1 Basic geometric objects

- **manifold** $\mathsf{Q} \subset \mathbb{R}^N$ $\qquad$ $\mathbb{R}^n, \mathbb{T}^n, \mathbb{S}^n, \mathrm{SO}(3), \mathrm{SE}(3)$
- **vector fields** $X = (X^1, \ldots, X^n) : \mathsf{Q} \mapsto \mathsf{TQ}$
- **metric** $\mathbb{M}$ is an inner product on TQ and its inverse $\mathbb{M}^{-1}$

  matrix representations $\mathbb{M}_{ij}$ and inverse $\mathbb{M}^{lm}$

(i) a **connection** $\nabla$ is a set of functions $\Gamma^i_{jk} \colon \mathsf{Q} \to \mathbb{R}$, $i, j, k \in \{1, \ldots, n\}$

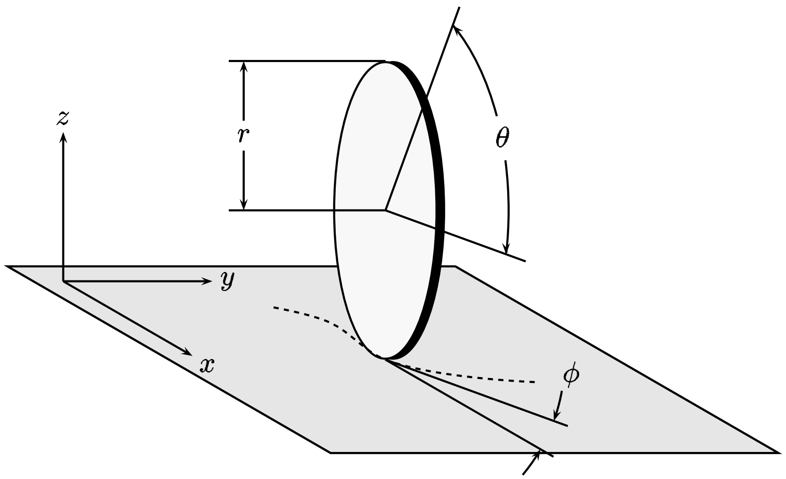
(ii) the **acceleration** of a curve $q \colon I \to \mathsf{Q}$

$$(\nabla_{\dot{q}} \dot{q})^i = \ddot{q}^i + \Gamma^i_{jk} \dot{q}^j \dot{q}^k$$

(iii) the **covariant derivative** $\nabla_X Y$ of two vector fields

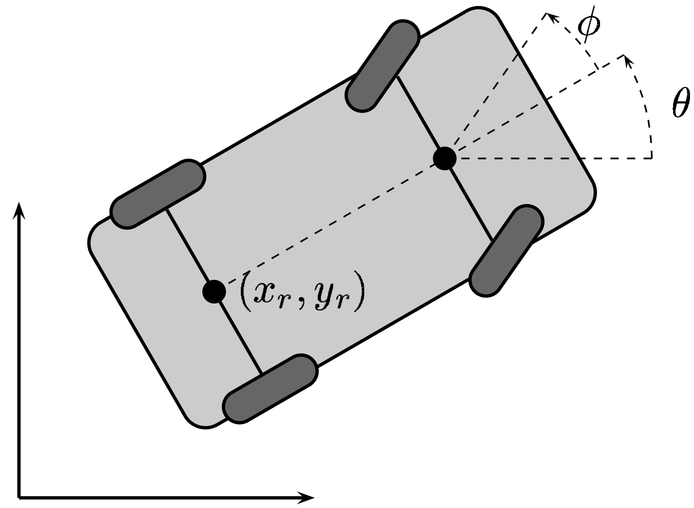
$$(\nabla_X Y)^i = \frac{\partial Y^i}{\partial q^j} X^j + \Gamma^i_{jk} X^j Y^k \qquad \langle X : Y \rangle = \nabla_X Y + \nabla_Y X$$

## 1.2 Constraints, distributions and kinematic modeling




$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \rho\cos\phi \\ \rho\sin\phi \\ 0 \\ 1 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \omega$$

(unicycle dynamics, simplest wheeled robot dynamics)




$$\begin{bmatrix} \dot{x}_r \\ \dot{y}_r \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{1}{\ell}\tan\phi \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \omega$$

## 1.3 SMCS with Constraints: definition

A **simple mechanical control system with constraints** is

(i) an $n$-dimensional configuration manifold Q,

(ii) a metric $\mathbb{M}$ on Q describing the kinetic energy,

(iii) a function $V$ on Q describing the potential energy,

(iv) a dissipative force $F_{\mathsf{diss}}$,

(v) a distribution $\mathscr{D}$ of feasible velocities describing the constraints

(vi) a set of $m$ covector fields $\mathcal{F} = \{F^1, \dots, F^m\}$ defining the control forces

$$(\mathsf{Q}, \mathbb{M}, V, F_{\mathsf{diss}}, \mathscr{D}, \mathscr{F} = \{F^1, \dots, F^m\})$$

## 1.4 SMCS with Constraints: governing equations

Given $(\mathsf{Q}, \mathbb{M}, V, F_{\text{diss}}, \mathscr{D}, \mathscr{F})$, there exists procedure:

$$\nabla_{\dot{q}}\dot{q} = Y_0(q) + R(\dot{q}) + \sum_{a=1}^{m} Y_a(q)u_a \tag{1}$$

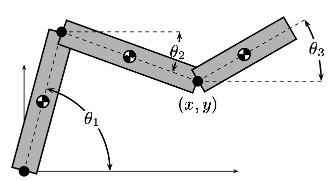
or, in coordinates:

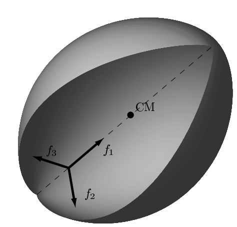
$$\ddot{q}^k + \Gamma^k_{ij}(q)\dot{q}^i\dot{q}^j = Y_0(q)^k + R^k_i(q)\dot{q}^i + \sum_{a=1}^{m} Y^k_a(q)u_a$$

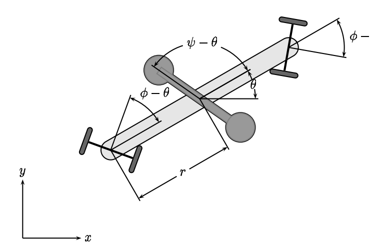
or, in different coordinates for the velocities,

$$\dot{q} = v^i X_i(q)$$

$$\dot{v}^k + \Gamma^k_{ij}(q)v^iv^j = Y_0(q)^k + R^k_i(q)\dot{q}^i + \sum_{a=1}^{m} Y^k_a(q)u_a$$



## 1.5 Modeling construction

**(Lewis, IEEE TAC '00)**

From $(\mathsf{Q}, \mathbb{M}, V, F_{\mathsf{diss}}, \mathscr{D}, \mathscr{F})$ to

$$\nabla_{\dot{q}}\dot{q} = Y_0(q) + R(\dot{q}) + \sum_{a=1}^{m} Y_a(q)u_a$$

(i) $P\colon \mathsf{TQ} \to \mathsf{TQ}$ is the **$\mathbb{M}$-orthogonal projection** onto $\mathscr{D}$

(ii) $Y_0(q) = -P(\mathbb{M}^{-1}(\mathrm{d}V))$

(iii) $R(\dot{q}) = P(\mathbb{M}^{-1}(F_{\mathsf{diss}}(\dot{q})))$

(iv) $Y_a = P(\mathbb{M}^{-1}(F^a))$

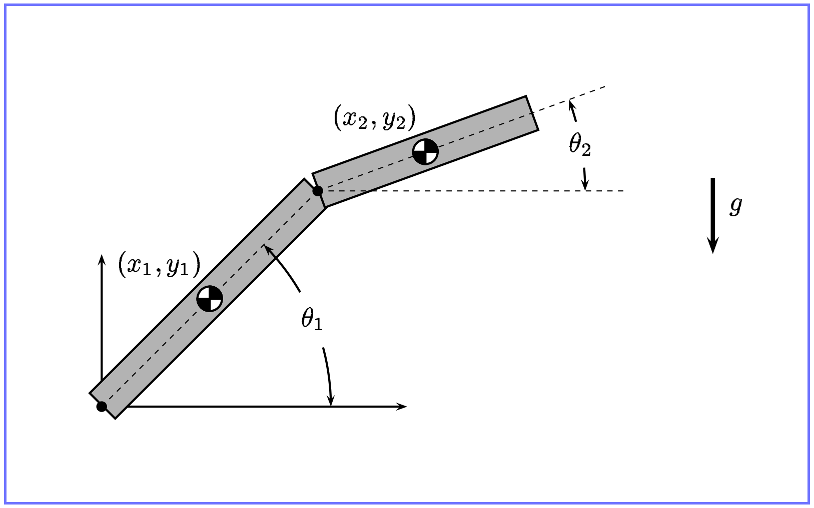
(v) ${}^{\mathbb{M}}\nabla$ is the **Levi-Civita connection** on $(\mathsf{Q}, \mathbb{M})$

$$\Gamma^k_{ij} = \frac{1}{2}\mathbb{M}^{mk}\left(\frac{\partial \mathbb{M}_{mj}}{\partial q^i} + \frac{\partial \mathbb{M}_{mi}}{\partial q^j} - \frac{\partial \mathbb{M}_{ij}}{\partial q^m}\right) \tag{2}$$

(vi) $\nabla$ is the **constrained affine connection** on $(\mathsf{Q}, \mathbb{M}, \mathscr{D})$

$$\nabla_X Y = {}^{\mathbb{M}}\nabla_X Y - \left({}^{\mathbb{M}}\nabla_X P\right)(Y) \tag{3}$$

## 1.6 Planar two links manipulator




$(\theta_1, \theta_2) \in \mathsf{Q} = \mathbb{T}^2$

$$\mathbb{M} = \begin{bmatrix} I_1 + (l_1^2(m_1 + 4m_2))/4 & (l_1 l_2 m_2 \cos[\theta_1 - \theta_2])/2 \\ (l_1 l_2 m_2 \cos[\theta_1 - \theta_2])/2 & I_2 + (l_2^2 m_2)/4 \end{bmatrix}$$

$$V(\theta_1, \theta_2) = m_1 g l_1 \sin\theta_1/2 + m_2 g(l_1 \sin\theta_1 + l_2/2 \sin\theta_2)$$

no $F_{\mathsf{diss}}$

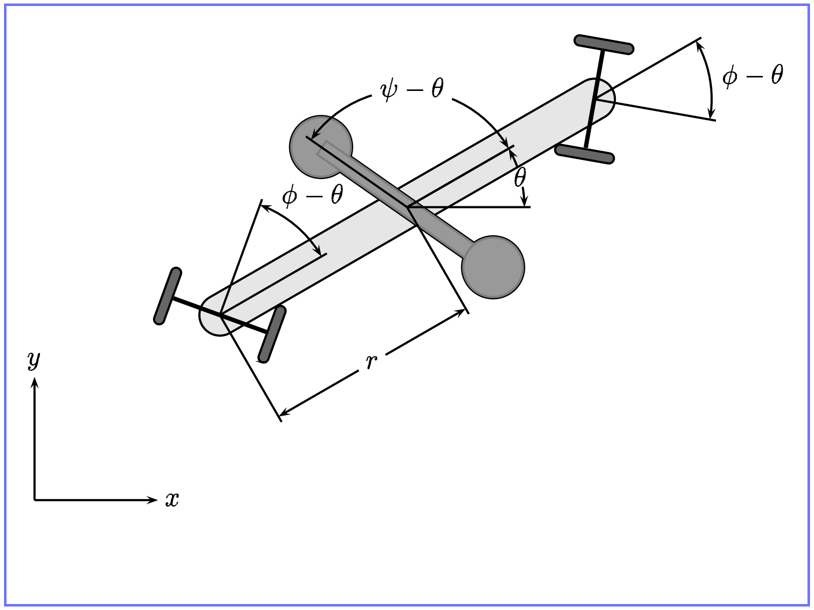
no constraints

$F^1 = \mathsf{d}\theta_1,\ F^2 = \mathsf{d}\theta_2 - \mathsf{d}\theta_1$

**Equations of motion**:

$$\begin{pmatrix} \ddot\theta_1 & + \Gamma^1_{11}\dot\theta_1\dot\theta_1 + \Gamma^1_{12}\dot\theta_1\dot\theta_2 + \Gamma^1_{22}\dot\theta_2\dot\theta_2 \\ \ddot\theta_2 & + \Gamma^2_{11}\dot\theta_1\dot\theta_1 + \Gamma^2_{12}\dot\theta_1\dot\theta_2 + \Gamma^2_{22}\dot\theta_2\dot\theta_2 \end{pmatrix} = Y_0 + u_1 Y_1 + u_2 Y_2$$

## 1.7 The snakeboard




$$(x, y, \theta, \psi, \phi) \in \mathsf{Q}{=}SE(2) \times \mathbb{T}^2$$

$$F^1 = \mathsf{d}\psi,\ F^2 = \mathsf{d}\phi$$

$$\mathbb{M} = \begin{pmatrix} m & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 \\ 0 & 0 & \ell^2 m & J_r & 0 \\ 0 & 0 & J_r & J_r & 0 \\ 0 & 0 & 0 & 0 & J_w \end{pmatrix}$$

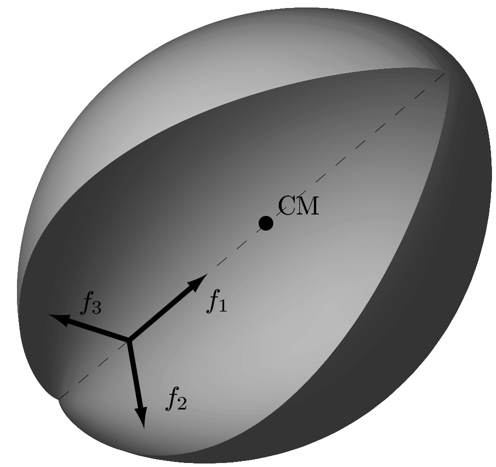
$$X_1 = \begin{pmatrix} \ell\cos\phi\cos\theta \\ \ell\cos\phi\sin\theta \\ -\sin\phi \\ 0 \\ 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\psi} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \ell \cos\phi \cos\theta \\ \ell \cos\phi \sin\theta \\ -\sin\phi \\ 0 \\ 0 \end{pmatrix} v_1 + \begin{pmatrix} \frac{J_r}{m\ell} \cos\phi \sin\phi \cos\theta \\ \frac{J_r}{m\ell} \cos\phi \sin\phi \sin\theta \\ -\frac{J_r}{m\ell^2} (\sin\phi)^2 \\ 1 \\ 0 \end{pmatrix} v_2 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} v_3$$

$$\begin{aligned} \dot{v}_1 + \frac{J_r}{m\ell^2}(\cos\phi) v_2 v_3 &= 0 \\ \dot{v}_2 - \frac{m\ell^2 \cos\phi}{m\ell^2 + J_r(\sin\phi)^2} v_1 v_3 - \frac{J_r \cos\phi \sin\phi}{m\ell^2 + J_r(\sin\phi)^2} v_2 v_3 &= \frac{m\ell^2}{m\ell^2 J_r + J_r^2 (\sin\phi)^2} u_\psi \\ \dot{v}_3 &= \frac{1}{J_w} u_\phi \,. \end{aligned}$$

$$\dot{q} = v^i X_i(q), \qquad \dot{v}^k + ({}^{\mathcal{X}}\Gamma)^k_{ij}(q) v^i v^j = Y_0(q)^k + R^k_i(q)\dot{q}^i + \sum_{a=1}^{m} Y^k_a(q) u_a$$

## 1.8 Underwater Vehicle in Ideal Fluid




3D rigid body with three forces:

(i) $(R, p) \in \mathrm{SE}(3), \quad (\Omega, V) \in \mathbb{R}^6$

(ii) $KE = \frac{1}{2}\Omega^T \mathbb{J}\Omega + \frac{1}{2}V^T \mathbb{M} V$,
$\mathbb{M} = \mathrm{diag}\{m_1, m_2, m_3\}$,
$\mathbb{J} = \mathrm{diag}\{J_1, J_2, J_3\}$

(iii) $f_1 = \mathsf{e}_4,\ f_2 = -h\mathsf{e}_3 + \mathsf{e}_5, \quad f_3 = h\mathsf{e}_2 + \mathsf{e}_6$

**Equations of Motion:**

$$\begin{pmatrix} \dot{R} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} R\hat{\Omega} \\ RV \end{pmatrix}, \quad \begin{bmatrix} \mathbb{J}\dot{\Omega} - \mathbb{J}\Omega \times \Omega + \mathbb{M}V \times V \\ \mathbb{M}\dot{V} - \mathbb{M}V \times \Omega. \end{bmatrix} = u_1 f_1 + u_2 f_2 + u_3 f_3$$

# Outline: from geometry to algorithms

(i) **modeling**

(ii) **approach #1**

(a) analysis: kinematic reductions and controllability

(b) design: inverse kinematics catalog

(iii) **approach #2**

(a) analysis: oscillatory controls and averaging

(b) design: approximate inversion

# 2 Analysis of Kinematic Reductions

Goal: (low-complexity) kinematic representations for mechanical control systems

Assume: no potential energy, no dissipation: $(\mathsf{Q}, \mathbb{M}, V = 0, F_{\mathsf{diss}} = 0, \mathscr{D}, \mathscr{F})$

(i) **dynamic model** with accelerations as control inputs mechanical systems:

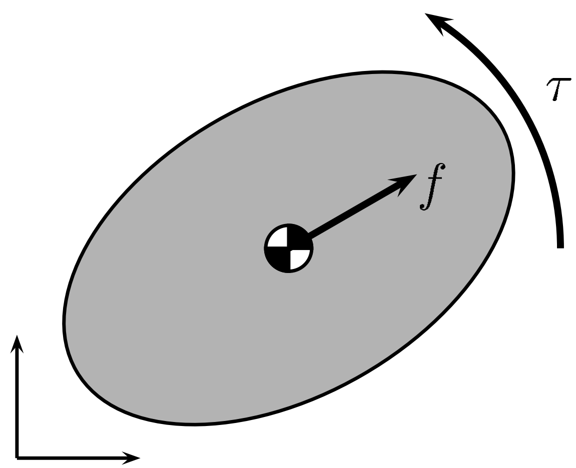
$$\nabla_{\dot{q}}\dot{q} = \sum_{a=1}^{m} Y_a(q)u_a(t) \qquad \mathscr{Y} = \operatorname{span}\{Y_1, \dots, Y_m\}$$

(ii) **kinematic model** with velocities as control inputs

$$\dot{q} = \sum_{b=1}^{\ell} V_b(q)w_b(t) \qquad \mathscr{V} = \operatorname{span}\{V_1, \dots, V_\ell\}$$

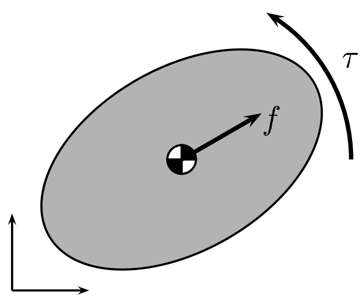
$\ell$ is the rank of the reduction

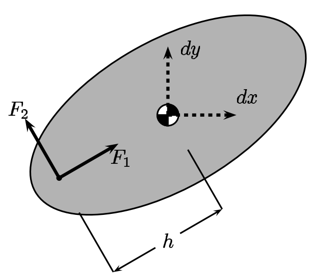
## 2.1 When can a second order system follow the solution of a first order?

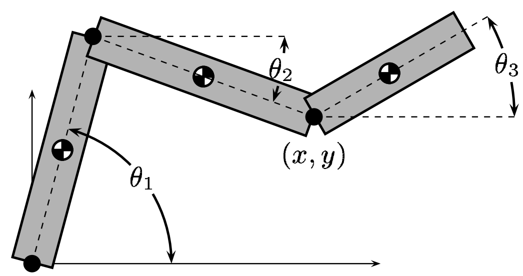


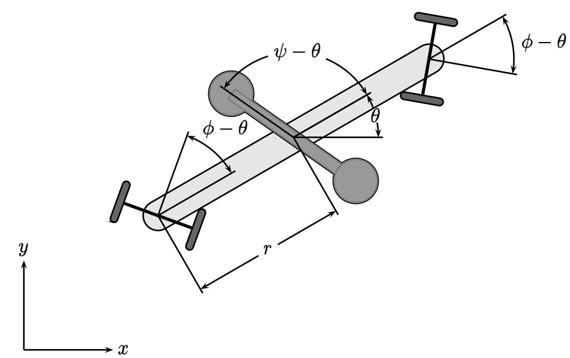

**ex:**

Can follow any straight line and can turn
2 preferred velocity fields
(plus, configuration controllability)




Ok ? ? ?

## 2.2 Kinematic reductions

**(Bullo and Lynch, IEEE TRA '01)**

$\mathscr{V} = \mathrm{span}\{V_1, \dots, V_\ell\}$ is a **kinematic reduction** if any curve $q\colon I \rightarrow$ Q solving the (controlled) kinematic model can be lifted to a solution to a solution of the (controlled) dynamic model.

rank 1 reductions are called **decoupling vector fields**

Theorem The kinematic model induced by $\{V_1, \dots, V_\ell\}$ is a kinematic reduction of $(\mathsf{Q}, \mathbb{M}, V{=}0, F_{\mathsf{diss}}{=}0, \mathscr{D}, \mathcal{F})$

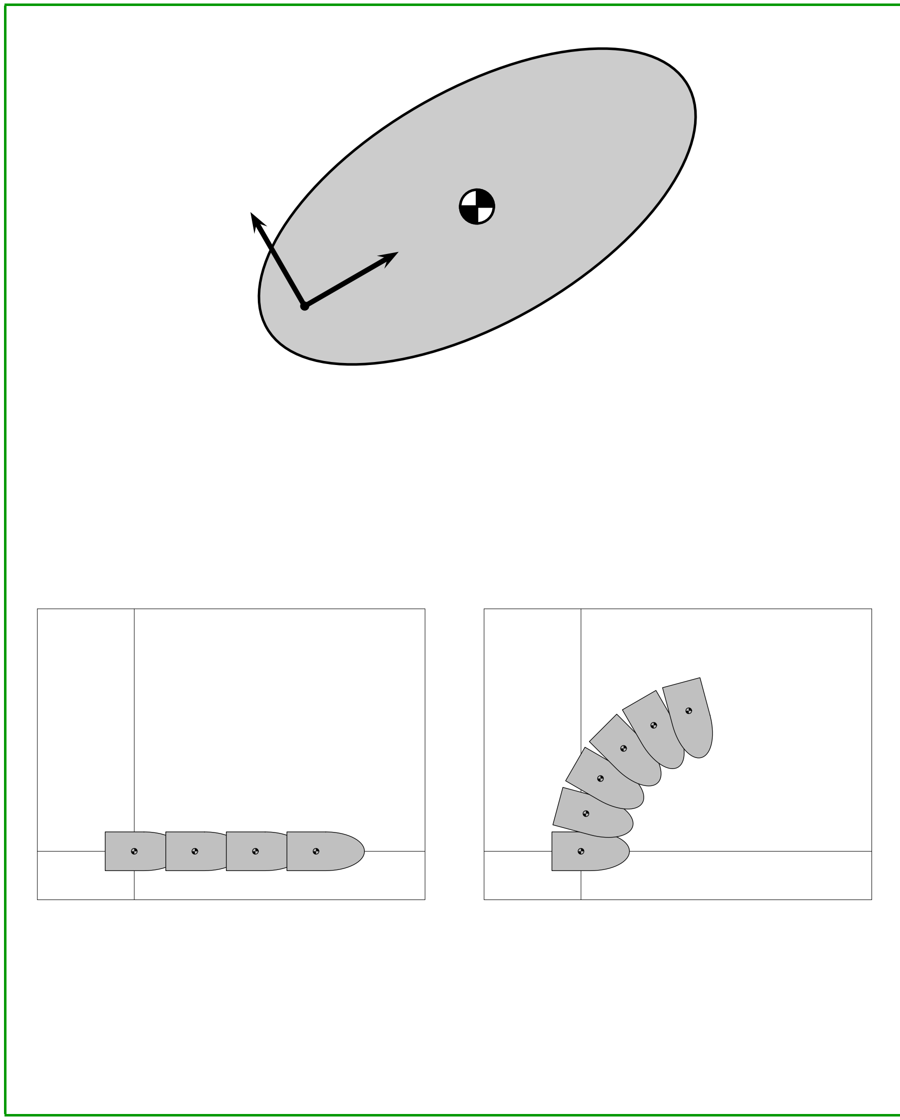
if and only if

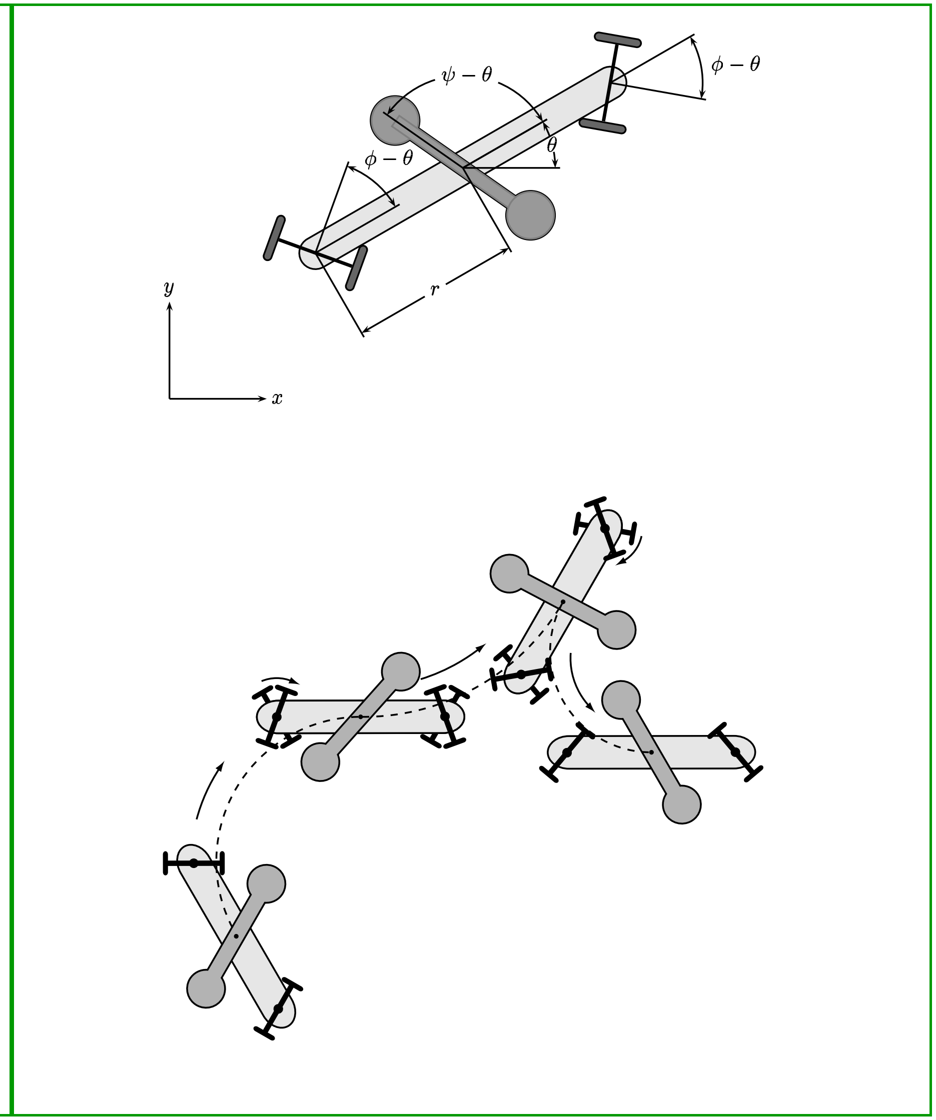
**(i)** $\mathscr{V} \subset \mathscr{Y}$

**(ii)** $\langle \mathscr{V} : \mathscr{V} \rangle \subset \mathscr{Y}$

## 2.3 Examples of kinematic reductions

**(Bullo and Lewis, IEEE TRA '03)**

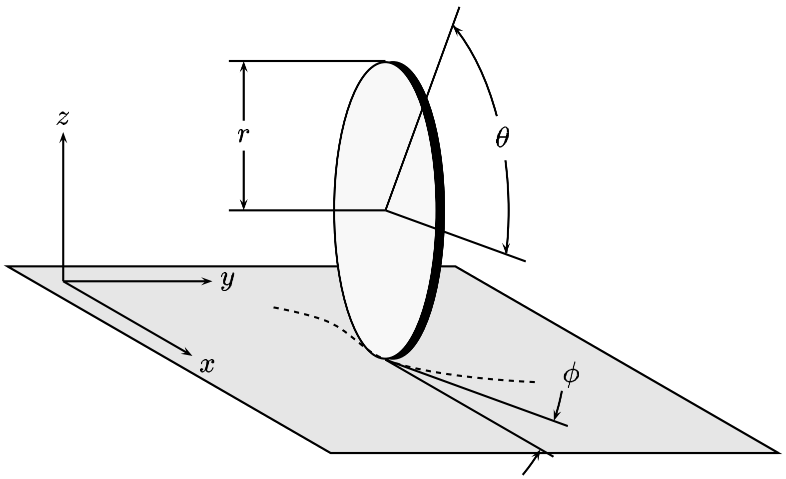





Two rank 1 kinematic reductions (decoupling vector fields)

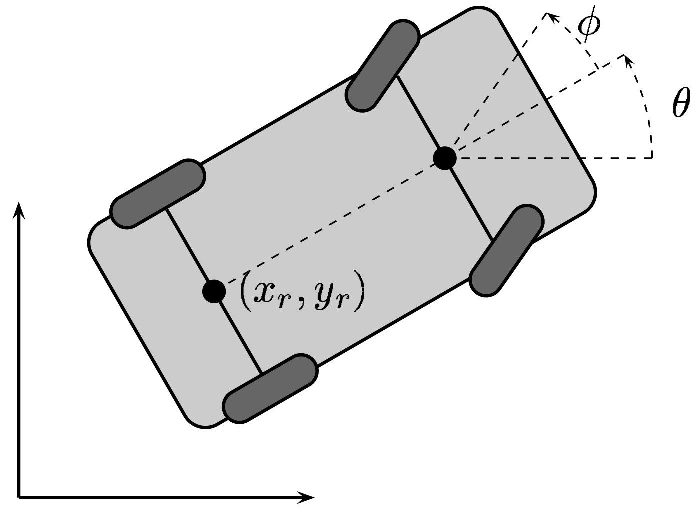
no rank 2 kinematic reductions

## 2.4 Examples of maximally reducible systems




$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \rho \cos \phi \\ \rho \sin \phi \\ 0 \\ 1 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \omega$$

(unicycle dynamics, simplest wheeled robot dynamics)




$$\begin{bmatrix} \dot{x}_r \\ \dot{y}_r \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \\ \frac{1}{\ell} \tan \phi \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \omega$$

## 2.5 When is a mechanical system kinematic? **(Lewis, CDC '99)**

When are all dynamic trajectories executable by a single kinematic model?

A dynamic model is **maximally reducible (MR)** if all its controlled trajectory (starting from rest) are controlled trajectory of a single kinematic reduction.

Theorem $(\mathsf{Q}, \mathbb{M}, V=0, F_{\mathsf{diss}}=0, \mathscr{D}, \mathcal{F})$ is maximally reducible if and only if

**(i)** the kinematic reduction is the input distribution $\mathscr{Y}$

**(ii)** $\langle \mathscr{Y} : \mathscr{Y} \rangle \subset \mathscr{Y}$

# 3 Controllability Analysis

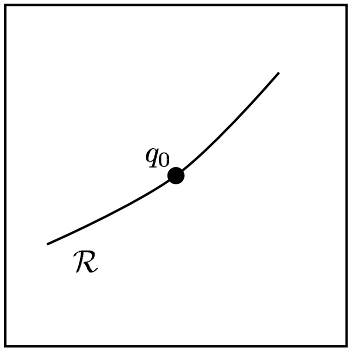
Objective: controllability notions and tests for mechanical systems and reductions

Assume: no potential energy, no dissipation: $(\mathsf{Q}, \mathbb{M}, V = 0, F_{\mathsf{diss}} = 0, \mathscr{D}, \mathscr{F})$

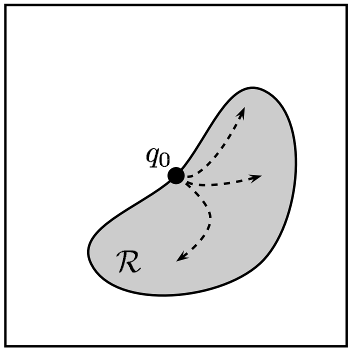
## Review: Controllable kinematic systems

$$\dot{q} = \sum_{i=1}^{\ell} X_i(q) u_i(t)$$

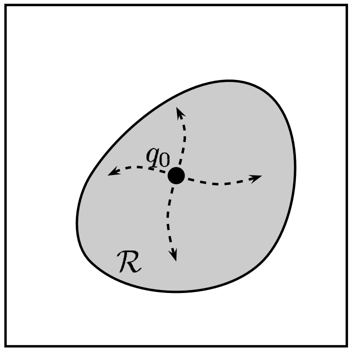
given two v.f.s $X, Y$, Lie bracket: $[X, Y]^k = \frac{\partial Y^k}{\partial q^i} X^i - \frac{\partial X^k}{\partial q^i} Y^i$ LARC




not accessible




accessible




controllable (STLC)

## 3.1 Controllability mechanisms

given control forces $\{F^1, \ldots, F^m\}$

accessible accelerations $\{Y_1, \ldots, Y_m\}$
$Y_a = P(\mathbb{M}^{-1} F^a)$

accessible velocities $\overline{\mathrm{Sym}}\{Y_1, \ldots, Y_m\}$
$\{Y_i, \langle Y_j : Y_k \rangle, \langle \langle Y_j : Y_k \rangle : Y_h \rangle, \ldots\}$

$\supset$

decoupling v.f.s $\{V_1, \ldots, V_\ell\}$
$V_i, \langle V_i : V_i \rangle \in \{Y_1, \ldots, Y_m\}$

access. configs $\overline{\mathrm{Lie}}\{\overline{\mathrm{Sym}}\{Y_1, \ldots, Y_m\}\}$
$\{Y_i, \langle Y_j : Y_k \rangle, [Y_j, Y_k], [\langle Y_j : Y_k \rangle, Y_h], \ldots\}$

$\supset$

$\overline{\mathrm{Lie}}\{V_1, \ldots, V_\ell\}$: configurations
accessible via decoupling v.f.s

## 3.2 Controllability notions and tests

**(Lewis and Murray, SIAM JCO '97)**

$V_1, \ldots, V_\ell$ decoupling v.f.s
$\operatorname{rank} \overline{\mathrm{Lie}}\{V_1, \ldots, V_\ell\} = n$ ➡

**KC= locally kinematically controllable**
$(q_0, 0) \xrightarrow{u} (q_{\mathsf{f}}, 0)$ can reach open set of configurations by concatenating motions along kinematic reductions

$\operatorname{rank} \overline{\mathrm{Sym}}\{\mathscr{Y}\} = n$,
"bad vs good" ➡

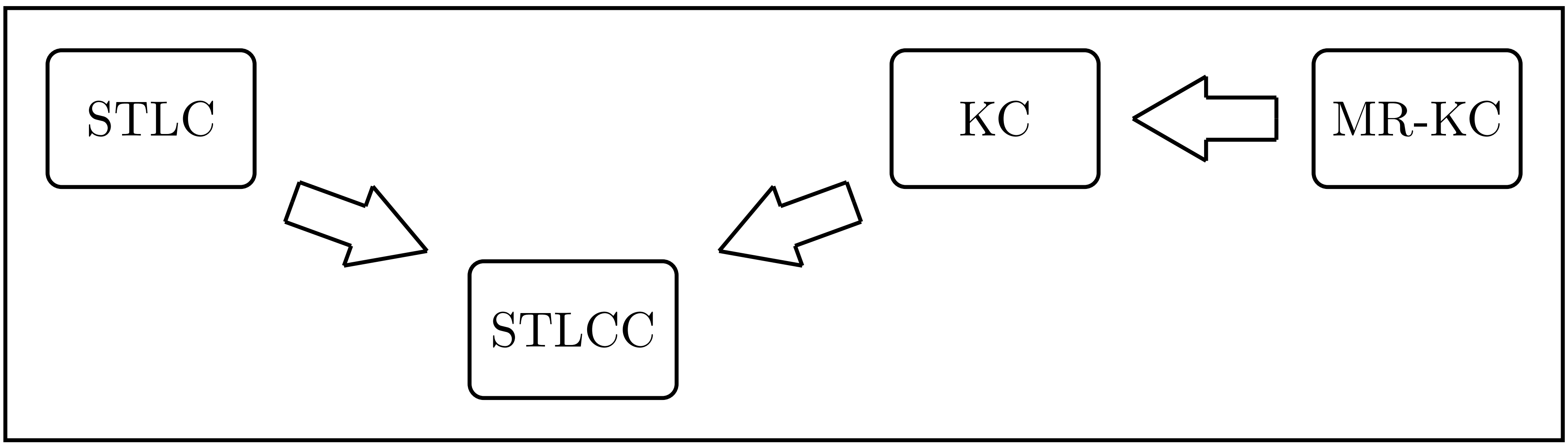
**STLC= small-time locally controllable**
$(q_0, 0) \xrightarrow{u} (q_{\mathsf{f}}, v_{\mathsf{f}})$ can reach open set of configurations and velocities

$\operatorname{rank} \overline{\mathrm{Lie}}\{\overline{\mathrm{Sym}}\{\mathscr{Y}\}\} = n$,
"bad vs good" ➡

**STLCC= small-time locally configuration controllable**
$(q_0, 0) \xrightarrow{u} (q_{\mathsf{f}}, v_{\mathsf{f}})$ can reach open set of configurations

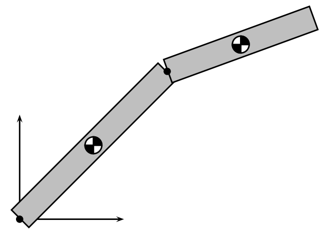
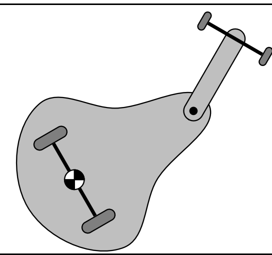
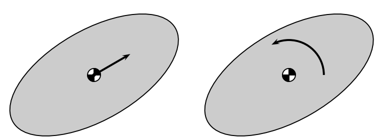
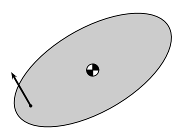
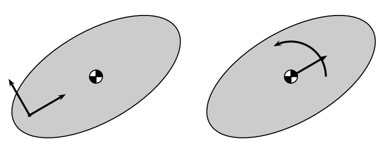
## 3.3 Controllability inferences

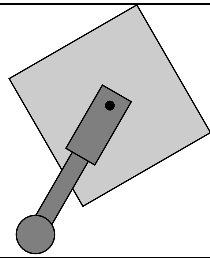
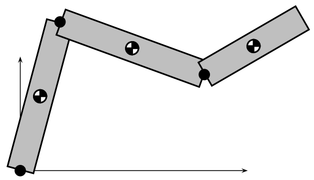
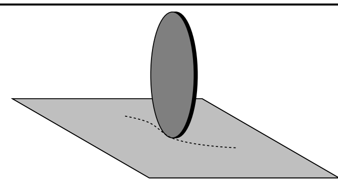
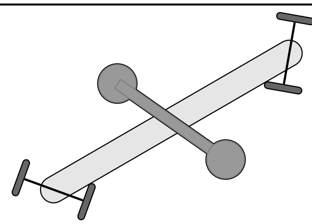
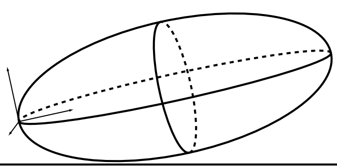
| | | |
|---|---|---|
| STLC | = | small-time locally controllable |
| STLCC | = | small-time locally configuration controllable |
| KC | = | locally kinematically controllable |
| MR-KC | = | maximally reducible, locally kinematically controllable |




There exist counter-examples for each missing implication sign.

## 3.4 Cataloging kinematic reductions and controllability of example systems

| System | Picture | Reducibility | Controllability |
|---|---|---|---|
| planar 2R robot<br>single torque at either joint: $(1,0),(0,1)$<br>$n=2, m=1$ |  | $(1,0)$: no reductions<br>$(0,1)$: maximally reducible | accessible<br>not accessible or STLCC |
| roller racer<br>single torque at joint<br>$n=4, m=1$ |  | no kinematic reductions | accessible, not STLCC |
| planar body with single force or torque<br>$n=3, m=1$ |  | decoupling v.f. | reducible, not accessible |
| planar body with single generalized force<br>$n=3, m=1$ |  | no kinematic reductions | accessible, not STLCC |
| planar body with two forces<br>$n=3, m=2$ |  | two decoupling v.f. | KC, STLC |

| | | | |
|---|---|---|---|
| robotic leg<br>$n = 3, m = 2$ |  | two decoupling v.f., maximally reducible | KC |
| planar 3R robot, two torques:<br>$(0,1,1)$, $(1,0,1)$, $(1,1,0)$<br>$n = 3, m = 2$ |  | $(1,0,1)$ and $(1,1,0)$: two decoupling v.f.<br><br>$(0,1,1)$: two decoupling v.f. and maximally reducible | $(1,0,1)$ and $(1,1,0)$: KC and STLC<br><br>$(0,1,1)$: KC |
| rolling penny<br>$n = 4, m = 2$ |  | fully reducible | KC |
| snakeboard<br>$n = 5, m = 2$ |  | two decoupling v.f. | KC, STLCC |
| 3D vehicle with 3 generalized forces<br>$n = 6, m = 3$ |  | three decoupling v.f. | KC, STLC |

# Summary

- dynamic models (mechanics) vs kinematic models (trajectory analysis)
- general reductions (multiple, low rank) vs MR (one $\mathrm{rank} = m$)
- STLCC (e.g., via STLC) vs kinematic controllability

# Outline: from geometry to algorithms

(i) **modeling**

(ii) **approach #1**

(a) analysis: kinematic reductions and controllability

(b) design: inverse kinematics catalog

(iii) **approach #2**

(a) analysis: oscillatory controls and averaging

(b) design: approximate inversion

# 4 Trajectory Design via Inverse Kinematics

Objective: find $u$ such that $(q_{\mathsf{initial}}, 0) \xrightarrow{u} (q_{\mathsf{target}}, 0)$

Assume:

(i) $(\mathsf{Q}, \mathbb{M}, V = 0, F_{\mathsf{diss}} = 0, \mathscr{D}, \mathscr{F})$ is **kinematically controllable**

(ii) **$\mathsf{Q} = \mathrm{G}$ and decoupling v.f.s $\{V_1, \dots, V_\ell\}$ are left-invariant**

$\implies$ matrix exponential $\exp\colon \mathfrak{g} \to \mathrm{G}$ gives closed-form flow

**Objective:** select a finite-length combination of $k$ flows along $\{V_1, \dots, V_\ell\}$ and coasting times $\{t_1, \dots, t_k\}$ such that

$$q_{\mathsf{initial}}^{-1} q_{\mathsf{target}} = g_{\mathsf{desired}} = \exp(t_1 V_{i_1}) \cdots \exp(t_k V_{i_k}).$$

No general methodology is available $\implies$ catalog for relevant example systems

$\mathrm{SO}(3), \mathrm{SE}(2), \mathrm{SE}(3)$, etc

## 4.1 Inverse-kinematic planner on $\mathrm{SO}(3)$ (Martínez, Cortés, and Bullo, IROS ’03)

Any underactuated controllable system on $\mathrm{SO}(3)$ is equivalent to

$$V_1 = e_z = (0,0,1) \qquad V_2 = (a,b,c) \text{ with } a^2 + b^2 \neq 0$$

**Motion Algorithm:** given $R \in \mathrm{SO}(3)$, flow along $(e_z, V_2, e_z)$ for coasting times

$$t_1 = \operatorname{atan2}\left(w_1 R_{13} + w_2 R_{23}, -w_2 R_{13} + w_1 R_{23}\right) \qquad t_2 = \operatorname{acos}\left(\frac{R_{33} - c^2}{1 - c^2}\right)$$

$$t_3 = \operatorname{atan2}\left(v_1 R_{31} + v_2 R_{32}, v_2 R_{31} - v_1 R_{32}\right)$$

where $z = \begin{bmatrix} 1 - \cos t_2 \\ \sin t_2 \end{bmatrix}, \quad \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} ac & b \\ cb & -a \end{bmatrix} z, \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} ac & -b \\ cb & a \end{bmatrix} z$

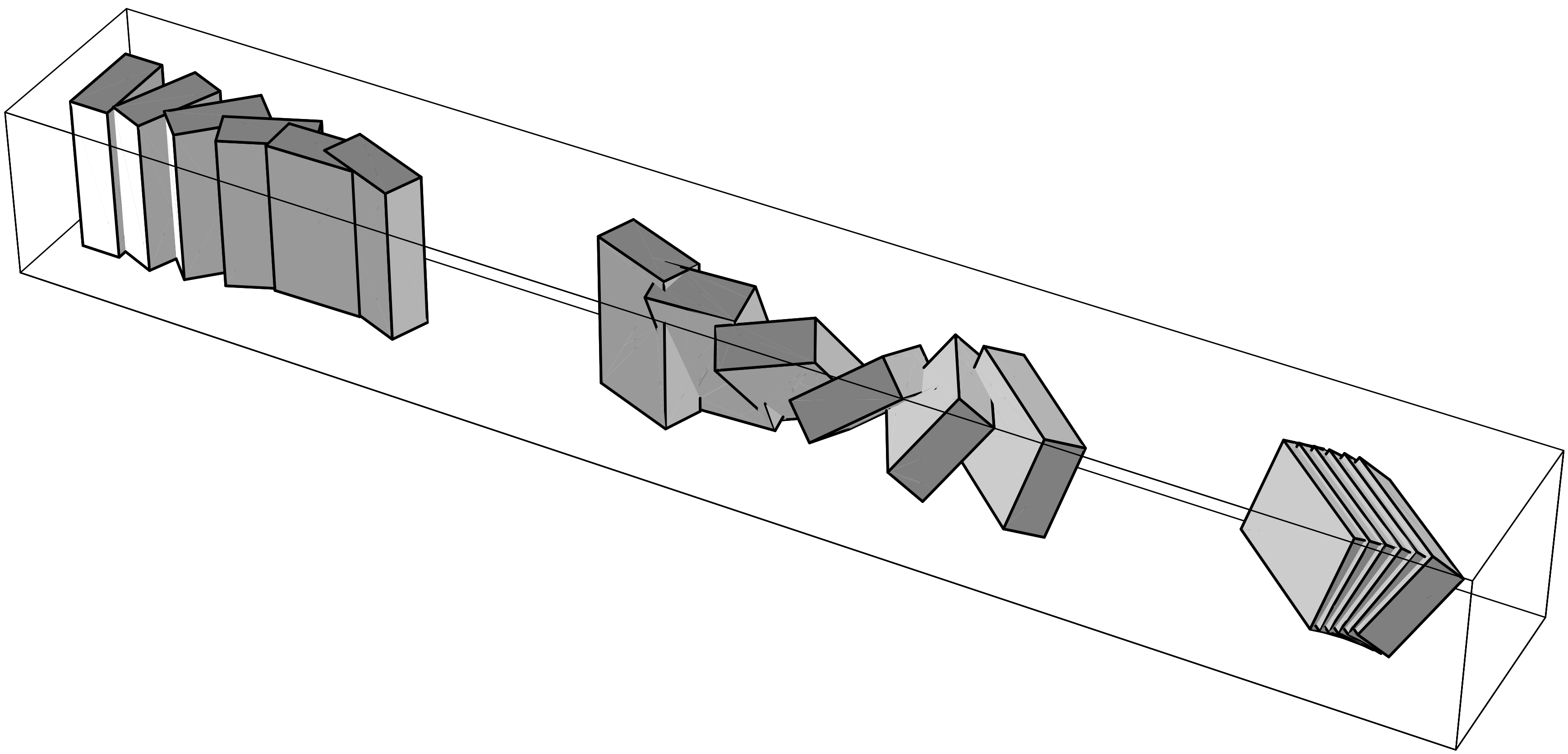
Local Identity Map = $\quad R \xrightarrow{\mathcal{IK}} (t_1, t_2, t_3) \xrightarrow{\mathcal{FK}} \exp(t_1 e_z) \exp(t_2 V_2) \exp(t_3 e_z)$

## 4.2 Inverse-kinematic planner on $\mathrm{SO}(3)$: simulation

The system can rotate about $(0, 0, 1)$ and $(a, b, c) = (0, 1, 1)$

Rotation from $I_3$ onto target rotation $\exp(\pi/3, \pi/3, 0)$

As time progresses, the body is translated along the inertial $x$-axis



## 4.3 Inverse-kinematic planner for $\Sigma_1$-systems $\mathrm{SE}(2)$

First class of underactuated controllable system on $\mathrm{SE}(2)$ is

$$\mathbf{\Sigma_1} = \{(V_1, V_2)|\ \ V_1 = (1, b_1, c_1), V_2 = (0, b_2, c_2),\ \ b_2^2 + c_2^2 = 1\}$$

**Motion Algorithm:** given $(\theta, x, y)$, flow along $(V_1, V_2, V_1)$ for coasting times

$$(t_1, t_2, t_3) = \left(\operatorname{atan2}(\alpha, \beta), \rho, \theta - \operatorname{atan2}(\alpha, \beta)\right)$$

where $\rho = \sqrt{\alpha^2 + \beta^2}$ and $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} b_2 & c_2 \\ -c_2 & b_2 \end{bmatrix} \left( \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 - \cos\theta \\ \sin\theta \end{bmatrix} \right)$

Identity Map $=$ $\qquad (\theta, x, y) \xrightarrow{\mathcal{IK}} (t_1, t_2, t_3) \xrightarrow{\mathcal{FK}} \exp(t_1 V_1) \exp(t_2 V_2) \exp(t_3 V_1)$

## 4.4 Inverse-kinematic planner for $\Sigma_2$-systems $\mathrm{SE}(2)$

Second and last class of underactuated controllable system on $\mathrm{SE}(2)$:

$$\boldsymbol{\Sigma_2} = \{(V_1, V_2)|\ \ V_1 = (1, b_1, c_1), V_2 = (1, b_2, c_2),\ \ b_1 \neq b_2 \text{ or } c_1 \neq c_2\}$$

**Motion Algorithm:** given $(\theta, x, y)$, flow along $(V_1, V_2, V_1)$ for coasting times

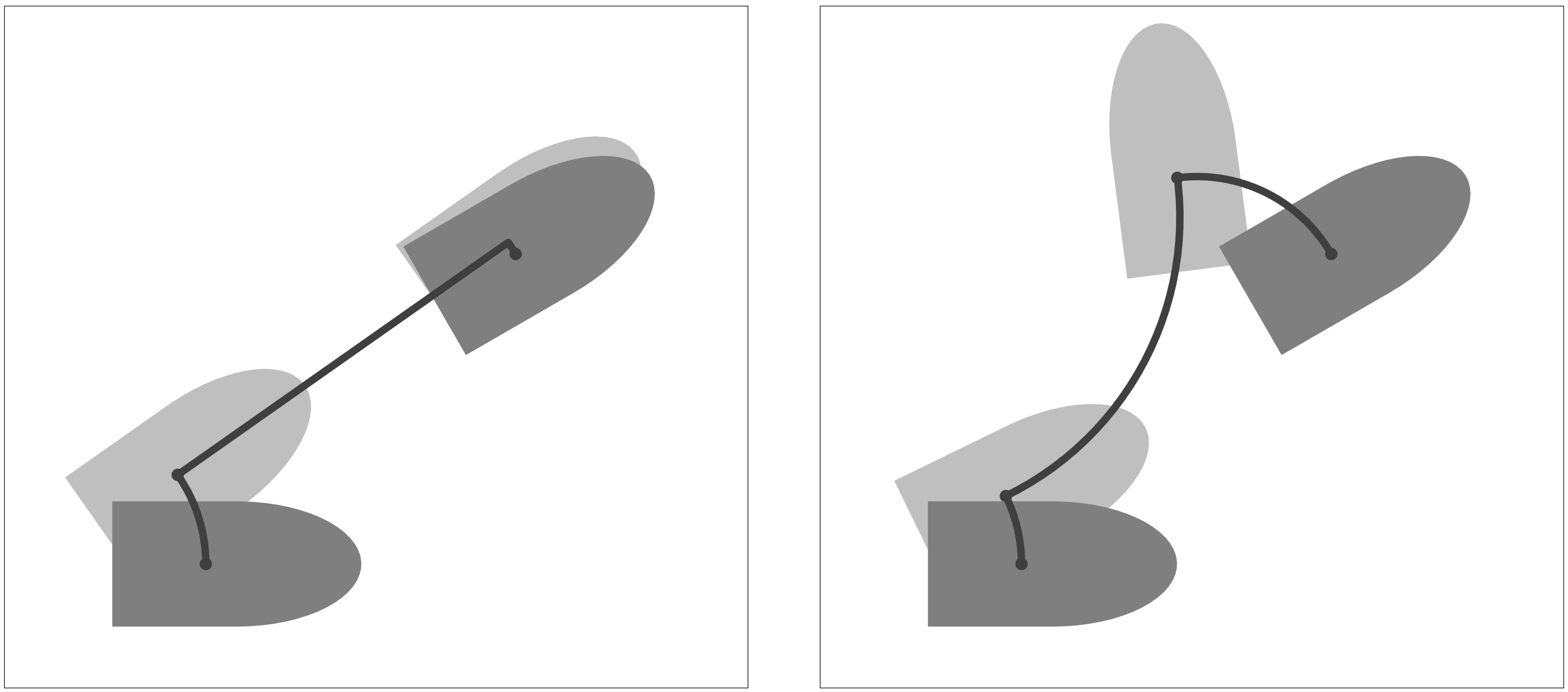
$$t_1 = \operatorname{atan2}\left(\rho, \sqrt{4-\rho^2}\right) + \operatorname{atan2}(\alpha, \beta) \qquad t_2 = \operatorname{atan2}\left(2-\rho^2, \rho\sqrt{4-\rho^2}\right)$$

$$t_3 = \theta - t_1 - t_2$$

where $\rho = \sqrt{\alpha^2 + \beta^2}$, $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} c_1 - c_2 & b_2 - b_1 \\ b_1 - b_2 & c_1 - c_2 \end{bmatrix} \left( \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 - \cos\theta \\ \sin\theta \end{bmatrix} \right)$

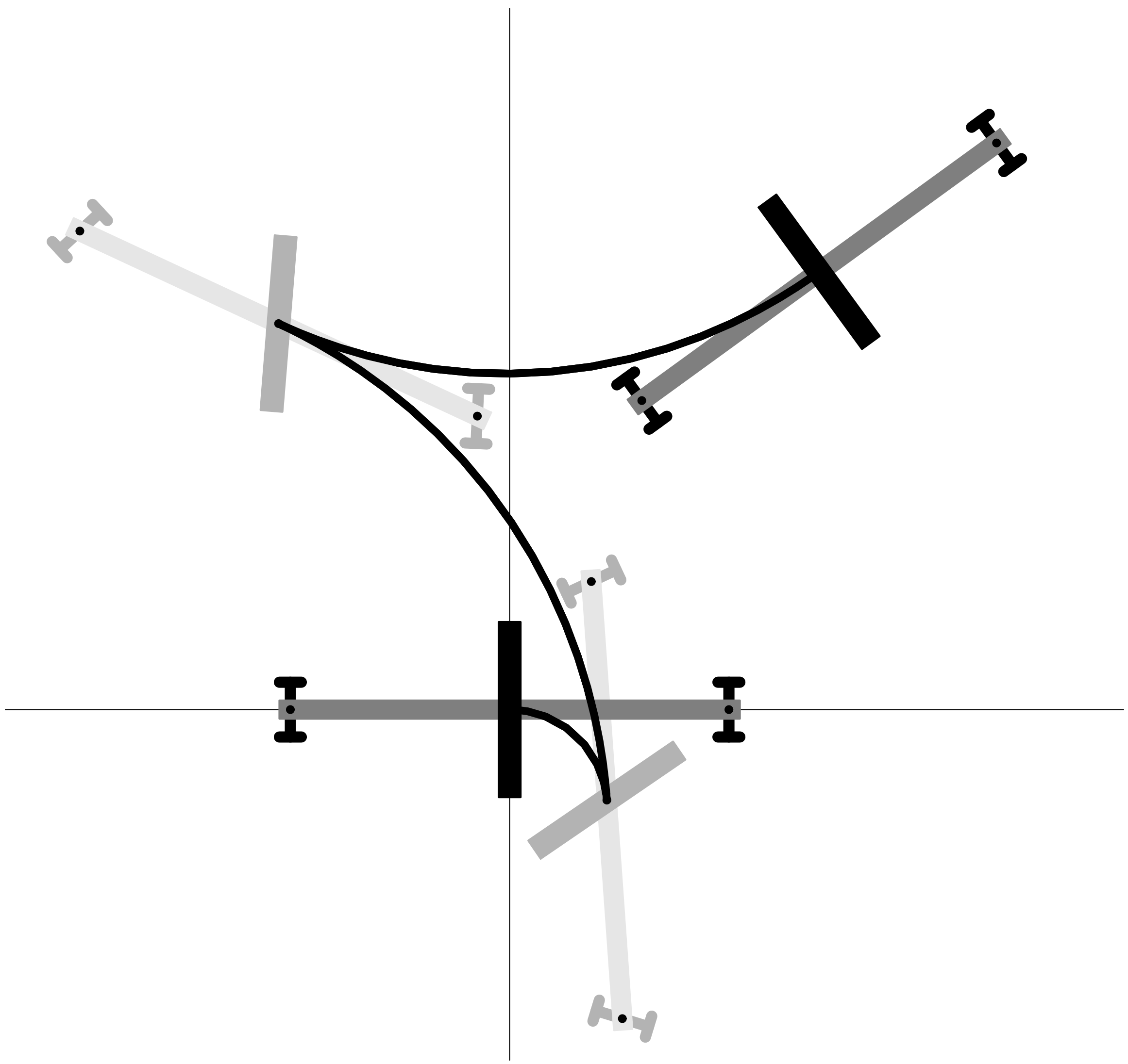
Local Identity Map $=$ $\quad (\theta, x, y) \xrightarrow{\mathcal{IK}} (t_1, t_2, t_3) \xrightarrow{\mathcal{FK}} \exp(t_1 V_1)\exp(t_2 V_2)\exp(t_3 V_1)$

## 4.5 Inverse-kinematic planners on $\mathrm{SE}(2)$: simulation



Inverse-kinematics planners for sample systems in $\mathbf{\Sigma_1}$ and $\mathbf{\Sigma_2}$. The systems parameters are $(b_1, c_1) = (0, .5)$, $(b_2, c_2) = (1, 0)$. The target location is $(\pi/6, 1, 1)$.

# 4.6 Inverse-kinematic planners on $\mathrm{SE}(2)$: snakeboard simulation

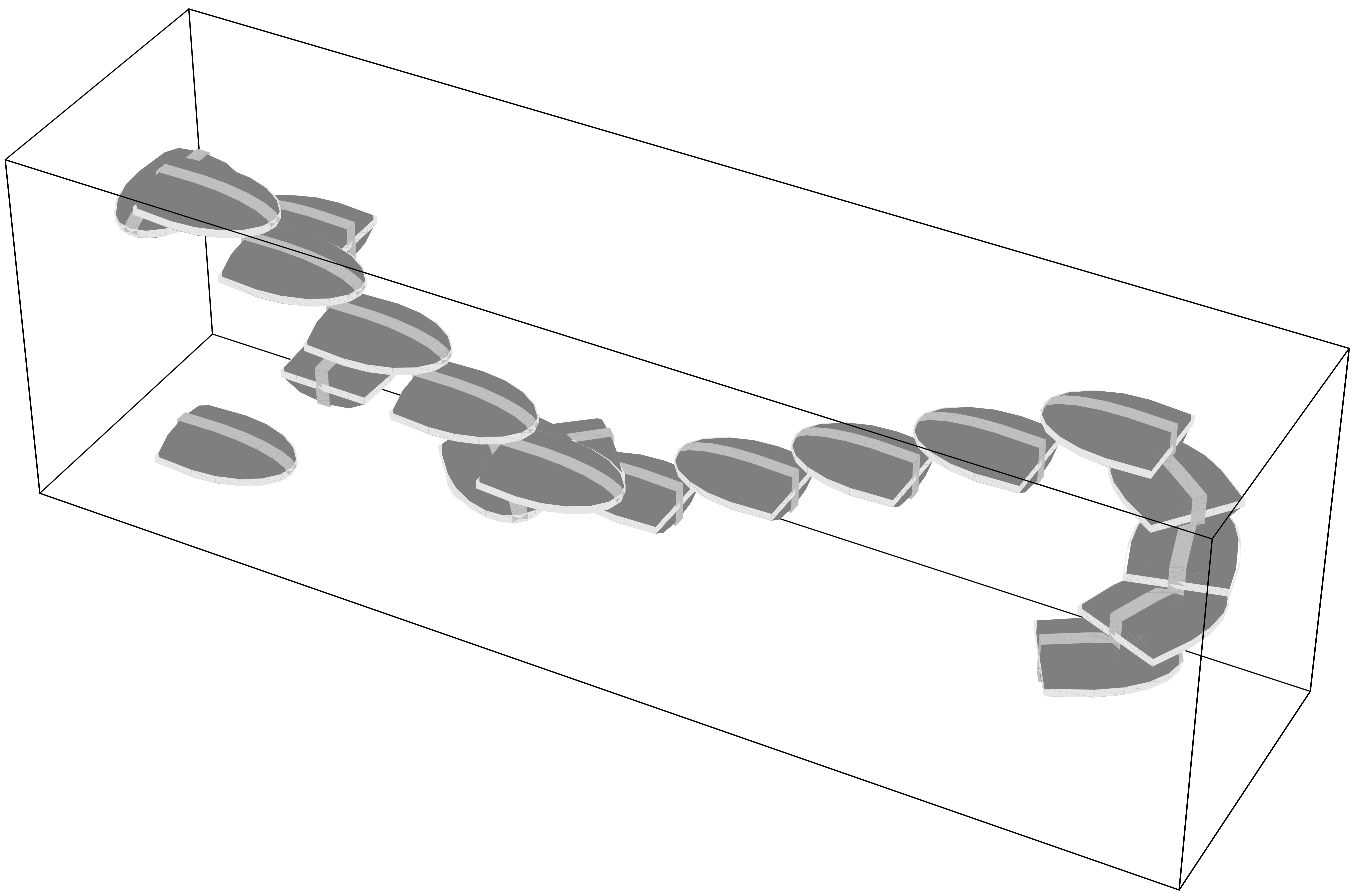


snakeboard as $\boldsymbol{\Sigma_2}$-system

## 4.7 Inverse-kinematic planners on $\mathrm{SE}(2) \times \mathbb{R}$: simulation

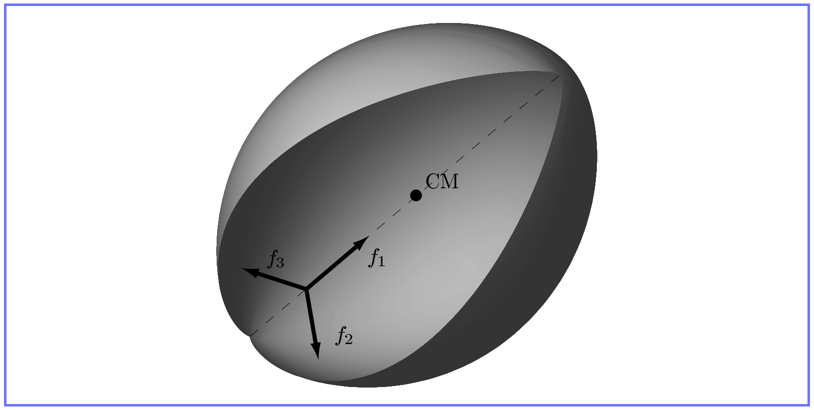
4 dof system in $\mathbb{R}^3$, no pitch no roll

kinematically controllable via body-fixed constant velocity fields:
$V_1$= rise and rotate about inertial point; $V_2$= translate forward and dive



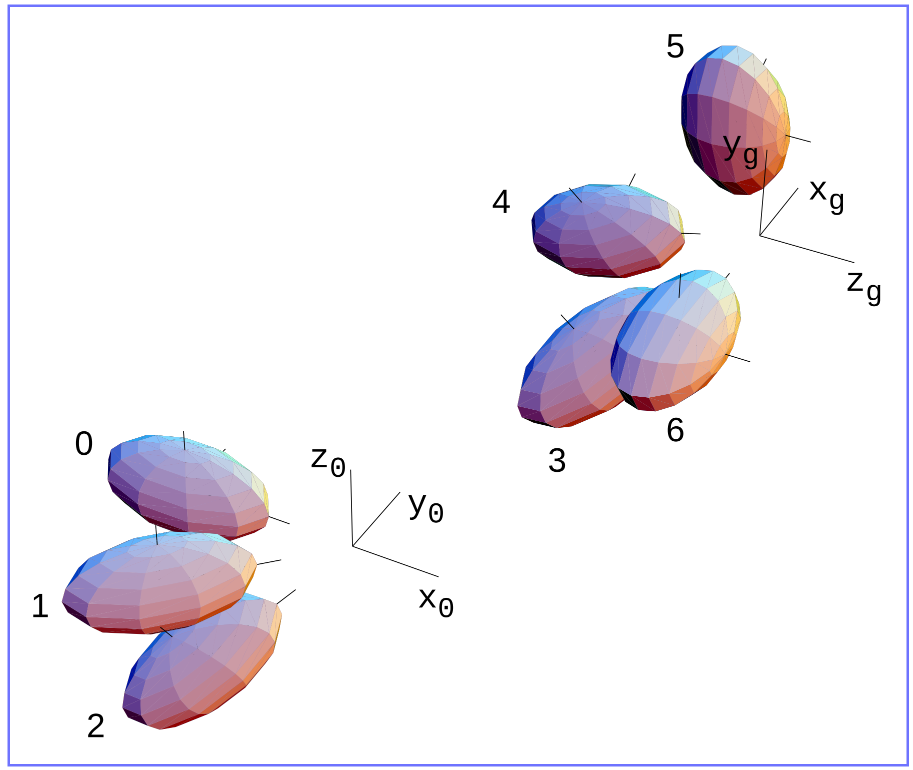
The target location is $(\pi/6, 10, 0, 1)$

## 4.8 Inverse-kinematic planners on $\mathrm{SE}(3)$: simulation




kinematically controllable via
body-fixed constant velocity fields:
$V_1$= translation along 1st axis
$V_2$= rotation about 2nd axis
$V_3$= rotation about 3rd axis

$V_3 : 0 \to 1$: rotation about 3rd axis
$V_2 : 1 \to 2$: rotation about 2nd axis
$V_1 : 2 \to 3$: translation along 1st axis
$V_3 : 3 \to 4$: rotation about 3rd axis
$V_2 : 4 \to 5$: rotation about 2nd axis
$V_3 : 5 \to 6$: rotation about 3rd axis



# Outline: from geometry to algorithms

(i) **modeling and approach #1**

- dynamic models (mechanics) vs kinematic models (trajectory analysis)
- general reductions (multiple, low rank) vs MR (one $\mathrm{rank} = m$)
- STLCC (e.g., via STLC) vs kinematic controllability
- catalogs of systems and solutions

(ii) **approach #2**

(a) analysis: oscillatory controls and averaging

(b) design: approximate inversion

# 5 Averaging Analysis

Oscillations play key role in animal and robotic locomotion, oscillations generate motion in Lie bracket directions useful for trajectory design

Objective: oscillatory controls in mechanical systems

$$\nabla_{\dot{q}}\dot{q} = Y(q,t) \qquad \int_0^T Y(q,t)\mathrm{d}t = 0$$

Assume: $(\mathsf{Q}, \mathbb{M}, V, F_{\mathsf{diss}}, \mathscr{D}, \mathscr{F})$. Let $\epsilon > 0$

$$\nabla_{\dot{q}}\dot{q} = Y_0(q) + R(\dot{q}) \ + \ \sum_{a=1}^{m} \frac{1}{\epsilon} u_a\left(\frac{t}{\epsilon}, t\right) Y_a(q),$$

where $u_a$ are $T$-periodic and zero-mean in their first argument.

## 5.1 Main Averaging Result

**(Martínez, Cortés, and Bullo, IEEE TAC '03)**

$$\nabla_{\dot q}\dot q = Y_0(q) + R(\dot q) \quad + \quad \sum_{a=1}^{m} \frac{1}{\epsilon} u_a\left(\frac{t}{\epsilon}, t\right) Y_a(q),$$

$$\Downarrow$$

$$\nabla_{\dot q}\dot q = Y_0(q) + R(\dot q) \quad - \quad \sum_{a,b=1}^{m} \Lambda_{ab}(t)\langle Y_a : Y_b\rangle(q)$$

$$\Lambda_{ab}(t) = \frac{1}{2}\Big(\overline{U}_{(a,b)}(t) + \overline{U}_{(b,a)}(t) - \overline{U}_{(a)}(t)\overline{U}_{(b)}(t)\Big)$$

$$U_{(a)}(\tau, t) = \int_0^t u_a(\tau, s)ds, \quad U_{(a,b)}(\tau, t) = \int_0^t u_b(\tau, s_2)\int_0^{s_2} u_a(\tau, s_1)ds_1 ds_2$$

approximation valid over certain time scale

## 5.2 Averaging analysis with control potential forces

Assume no constraints ($\mathscr{D} = \mathsf{TQ}$) and $\mathcal{F} = \{\mathsf{d}\varphi_1, \ldots, \mathsf{d}\varphi_m\}$.

Then

$$Y_a(q) = \operatorname{grad} \varphi_a(q), \qquad (\operatorname{grad} \varphi_a)^i = \mathbb{M}^{ij} \frac{\partial \varphi_a}{\partial q^j}$$

Symmetric product restricts

$$\langle \operatorname{grad} \varphi_a : \operatorname{grad} \varphi_b \rangle \equiv \operatorname{grad} \langle \varphi_a : \varphi_b \rangle$$

where **Beltrami bracket (Crouch '81)**:

$$\langle \varphi_a : \varphi_a \rangle = \langle\!\langle \mathsf{d}\varphi_a \,,\, \mathsf{d}\varphi_b \rangle\!\rangle = \mathbb{M}^{ij} \frac{\partial \varphi_a}{\partial q^i} \frac{\partial \varphi_b}{\partial q^j}$$

## 5.3 Averaged potential

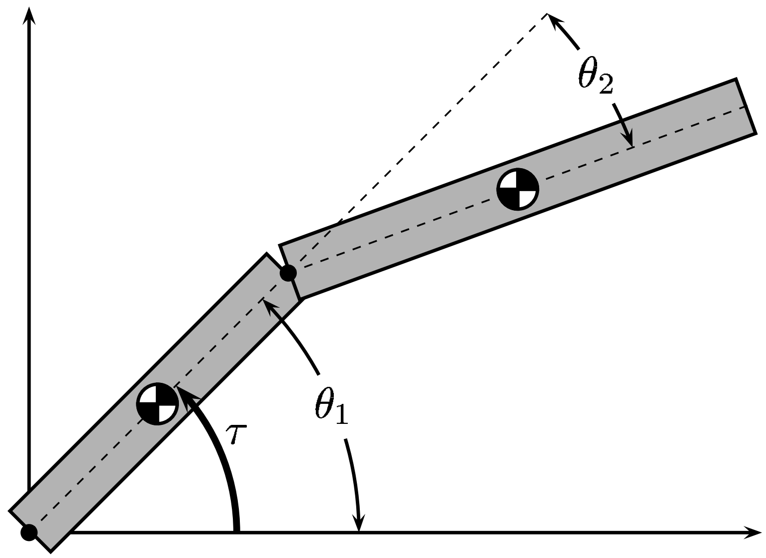
$$
{}^{\mathbb{M}}\nabla_{\dot{q}}\dot{q} = -\operatorname{grad} V(q) + R(\dot{q}) + \sum_{a=1}^{m} u_a(t)\operatorname{grad}(\varphi_a)(q)\,.
$$

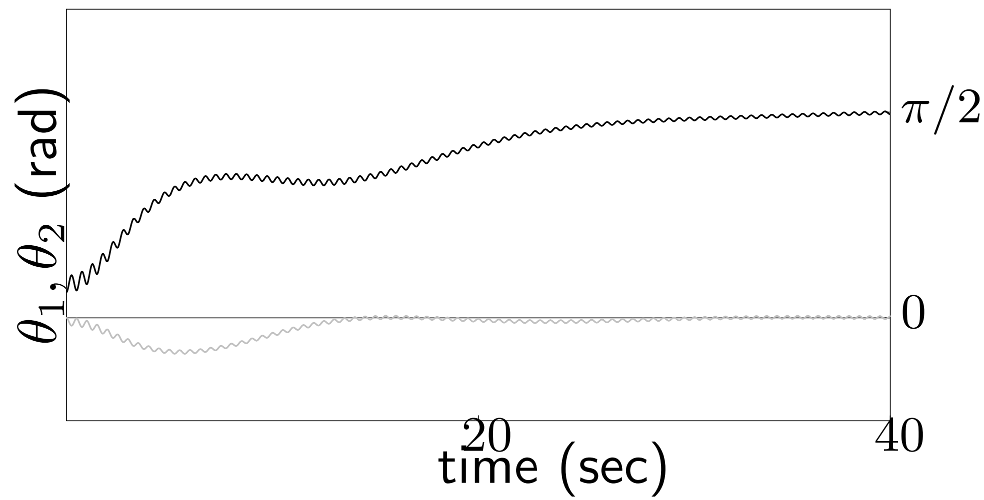
$$\Downarrow$$

$$
{}^{\mathbb{M}}\nabla_{\dot{q}}\dot{q} = -\operatorname{grad} V_{\mathsf{averaged}}(q) + R(\dot{q})
$$

$$
V_{\mathsf{averaged}} = V + \sum_{a,b=1}^{m} \Lambda_{ab}\langle\varphi_a : \varphi_b\rangle
$$

## 5.4 Oscillations stabilization example: a 2-link manipulator




$$u = \frac{1}{\epsilon} \cos\left(\frac{t}{\epsilon}\right)$$

Two-link damped manipulator with oscillatory control at first joint. The averaging analysis predicts the behavior. (the gray line is $\theta_1$, the black line is $\theta_2$).

# 6 Trajectory Design via Oscillatory Controls and Approximate Inversion

Objective: steer configuration of $(\mathsf{Q}, \mathbb{M}, V, F_{\mathsf{diss}}, \mathscr{D}, \mathscr{F})$ along target trajectory $\gamma_{\mathsf{target}} : [0,T] \to \mathsf{Q}$ via oscillatory controls:

$$\nabla_{\dot{q}}\dot{q} = Y_0(q) + R(\dot{q}) \quad + \quad \sum_{a=1}^{m} u_a Y_a(q),$$

**Low-order STLC assumption:**

(i) $\mathrm{span}\{Y_a, \langle Y_b : Y_c\rangle |\ a,b,c \in \{1,\dots,m\}\}$ is full rank

(ii) "bad vs good" condition: $\langle Y_a : Y_a\rangle \in \mathscr{Y} = \mathrm{span}\{Y_a\}$.

## 6.1 From the STLC assumption ...

(i) **fictitious inputs** $z^a_{\text{target}}, z^{ab}_{\text{target}} : [0,T] \to \mathbb{R}$, $a < b$, with

$$\nabla_{\gamma'_{\text{target}}} \gamma'_{\text{target}} = Y_0(\gamma_{\text{target}}) + R(\gamma'_{\text{target}})$$

$$+ \sum_{a=1}^{m} z^a_{\text{target}} Y_a(\gamma_{\text{target}}(t)) + \sum_{a<b} z^{ab}_{\text{target}} \langle Y_a : Y_b \rangle (\gamma_{\text{target}}(t)),$$

(ii) for $a, b \in \{1, \ldots, m\}$, **bad/good coefficient functions** $\alpha_{a,b} \colon Q \to \mathbb{R}$

$$\langle Y_a : Y_a \rangle = \sum_{b=1}^{m} \alpha_{a,b} Y_b \,.$$

Also, there are $N = m(m-1)/2$ pairs of elements $(a, b)$ in $\{1, \ldots, m\}$, with $a < b$. Let $(a, b) \mapsto \omega(a, b) \in \{1, \ldots, N\}$ be a enumeration of these pairs, and define **$\omega$-frequency sinusoidal function**

$$\psi_{\omega(a,b)}(t) = \sqrt{2}\, \omega(a, b) \cos(\omega(a, b) t)$$

## 6.2 Trajectory tracking via Approximate Inversion

**(Martínez, Cortés, and Bullo, IEEE TAC ’03)**

Theorem Consider $(\mathsf{Q}, \mathbb{M}, V, F_{\mathsf{diss}}, \mathscr{D}, \mathscr{F})$. Let

$$u_a = v_a(t,q) + \frac{1}{\epsilon} w_a\left(\frac{t}{\epsilon}, t\right)$$

with

$$w_a(\tau,t) = \sum_{c=a+1}^{m} z^{ac}_{\mathsf{target}}(t)\psi_{\omega(a,c)}(\tau) - \sum_{c=1}^{a-1} \psi_{\omega(c,a)}(\tau)$$

$$v_a(t,q) = z^{a}_{\mathsf{target}}(t) + \frac{1}{2}\sum_{b=1}^{m} \alpha_{a,b}(q)\left(j - 1 + \sum_{c=j+1}^{m} (z^{bc}_{\mathsf{target}}(t))^2\right)$$

Then, $t \mapsto q(t)$ follows $\gamma_{\mathsf{target}}$ with an error of order $\epsilon$ over the time scale $1$.

## 6.3 Oscillatory controls ex. #1: A second-order nonholonomic integrator

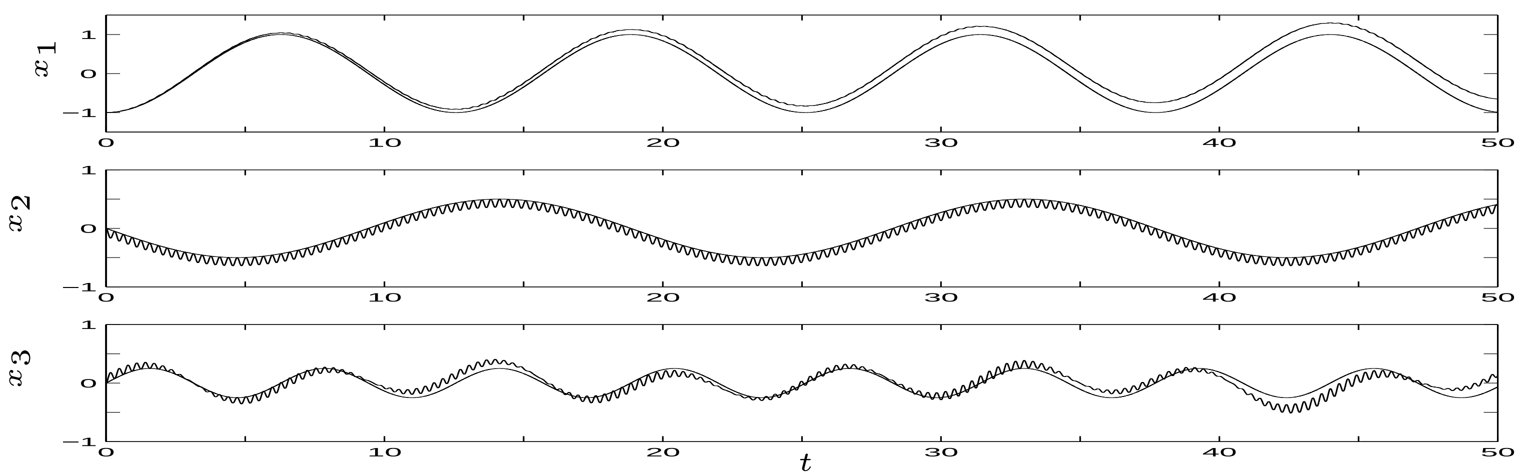
Consider

$$\ddot{x}_1 = u_1\,, \quad \ddot{x}_2 = u_2\,, \quad \ddot{x}_3 = u_1 x_2 + u_2 x_1\,,$$

Controllability assumption ok. Design controls to track $(x_1^d(t), x_2^d(t), x_3^d(t))$:

$$u_1 = \ddot{x}_1^d + \frac{1}{\sqrt{2}\epsilon}\left(\ddot{x}_3^d - \ddot{x}_1^d x_2^d - \ddot{x}_2^d x_1^d\right)\cos\left(\frac{t}{\epsilon}\right)$$

$$u_2 = \ddot{x}_2^d - \frac{\sqrt{2}}{\epsilon}\cos\left(\frac{t}{\epsilon}\right)$$



# 7 Summary: from geometry to algorithms

## Trajectory design via kinematic reductions

- dynamic models (mechanics) vs kinematic models (trajectory analysis)
- general reductions (multiple, low rank) vs MR (one $\mathrm{rank} = m$)
- STLCC (e.g., via STLC) vs kinematic controllability
- catalogs of systems and solutions

## Trajectory design via averaging

- high-amplitude high-frequency two time-scales averaging
- general tracking result based on STLC assumption

trajectory analysis: reduction, controllability, averaging
trajectory design: inverse kinematics and approximate inversion

## Future research

(i) weaken strict assumptions for reductions approach
$V = 0$, kinematic controllability, group actions

(ii) render second approach more realistic

(iii) integrate with numerical and passivity methods for trajectory design

(iv) locomotion in fluid (fishes, flying insects, etc)

(v) computational geometry and coordination in multi-vehicle systems

# Research work reflected in this talk: (`http://motion.csl.uiuc.edu`)


(i) F. Bullo and M. Žefran. On mechanical control systems with nonholonomic constraints and symmetries. *IFAC Syst. & Control L.*, 45(2):133–143, 2002

(ii) F. Bullo and K. M. Lynch. Kinematic controllability for decoupled trajectory planning in underactuated mechanical systems. *IEEE T. Robotics Automation*, 17(4):402–412, 2001

(iii) F. Bullo, N. E. Leonard, and A. D. Lewis. Controllability and motion algorithms for underactuated Lagrangian systems on Lie groups. *IEEE T. Automatic Ctrl*, 45(8):1437–1454, 2000

(iv) F. Bullo. Series expansions for the evolution of mechanical control systems. *SIAM JCO*, 40(1):166–190, 2001

(v) F. Bullo. Averaging and vibrational control of mechanical systems. *SIAM JCO*, 41(2):542–562, 2002

(vi) S. Martínez, J. Cortés, and F. Bullo. Analysis and design of oscillatory control systems. *IEEE T. Automatic Ctrl*, 48(7):1164–1177, 2003

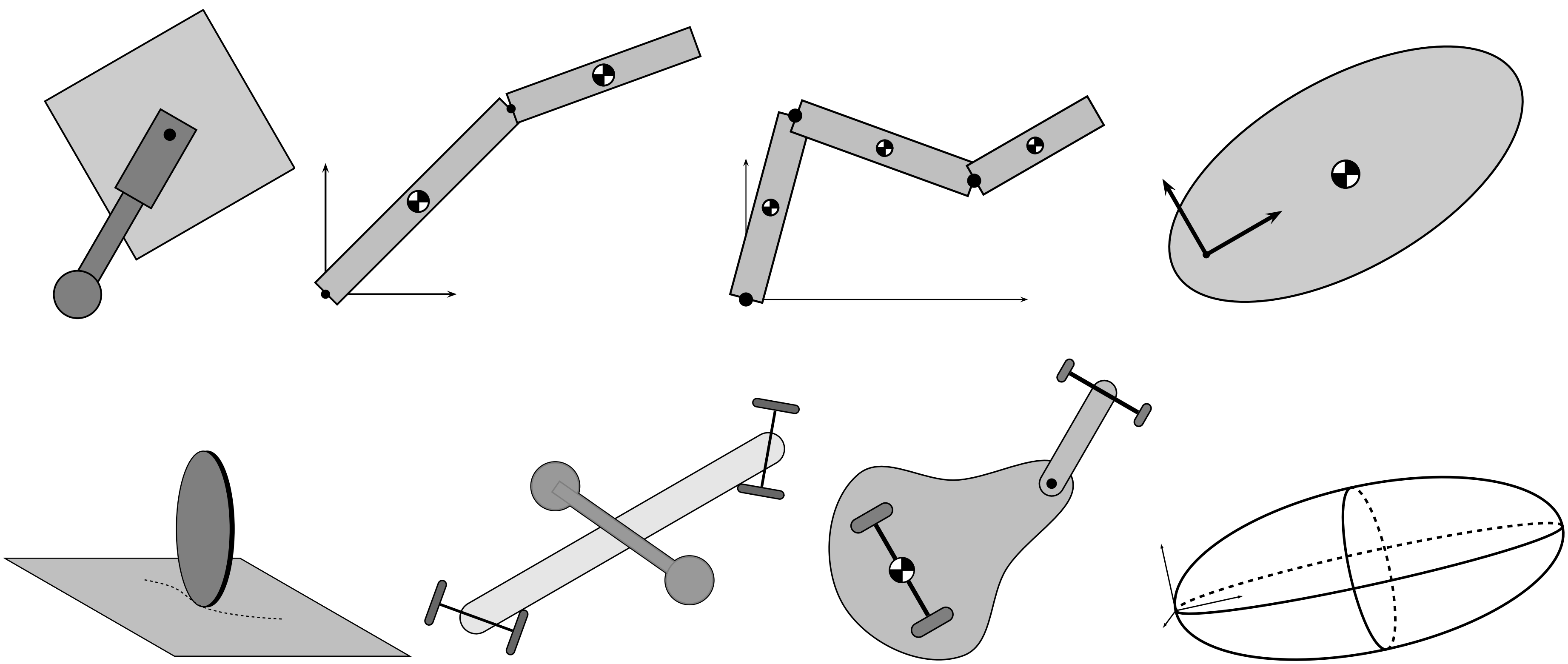
(vii) F. Bullo and A. D. Lewis. Kinematic controllability and motion planning for the snakeboard. *IEEE T. Robotics Automation*, 19(3):494–498, 2003

(viii) F. Bullo and A. D. Lewis. Low-order controllability and kinematic reductions for affine connection control systems. *SIAM JCO*, January 2004. To appear

(ix) S. Martínez, J. Cortés, and F. Bullo. A catalog of inverse-kinematics planners for underactuated systems on matrix Lie groups. In *Proc IROS*, pages 625–630, Las Vegas, NV, October 2003

(x) F. Bullo. Trajectory design for mechanical systems: from geometry to algorithms. *European Journal of Control*, December 2003. Submitted

## 7.1 Examples



## 7.2 Comparison with perturbation methods for mechanical control systems

### forced response of Lagrangian system from rest

**I) High magnitude high frequency**

"oscillatory control &
vibrational stabilization"

$$H = H(q,p) + \frac{1}{\epsilon}\varphi\left(q,p,u\left(\frac{t}{\epsilon}\right)\right)$$
$$p(0) = p_0$$

**II) Small input from rest**

"small-time local controllability"

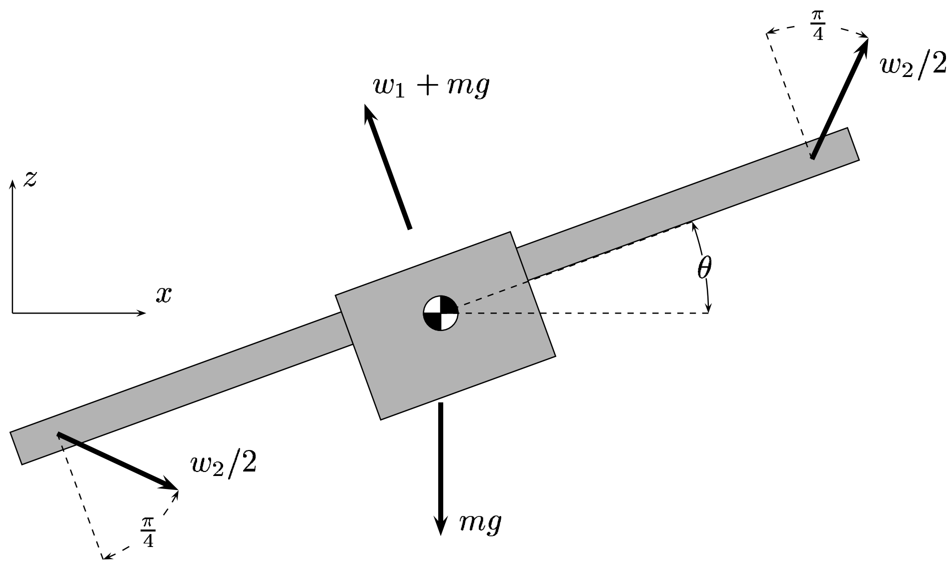
$$H = H(q,p) + \epsilon\varphi(q,p,u(t))$$
$$p(0) = 0$$

**III) Classical formulation**

integrable Hamiltonian systems

$$H = H(q,p) + \epsilon\varphi(q,p)$$
$$p(0) = p_0$$

## 7.3 A planar vertical takeoff and landing (PVTOL) aircraft




$$\dot{x} = \cos\theta v_x - \sin\theta v_z$$

$$\dot{z} = \sin\theta v_x + \cos\theta v_z$$

$$\dot{\theta} = \omega$$

$$\dot{v}_x - v_z\omega = -g\sin\theta + (-k_1/m)v_x + (1/m)u_2$$

$$\dot{v}_z + v_x\omega = -g(\cos\theta - 1) + (-k_2/m)v_z + (1/m$$

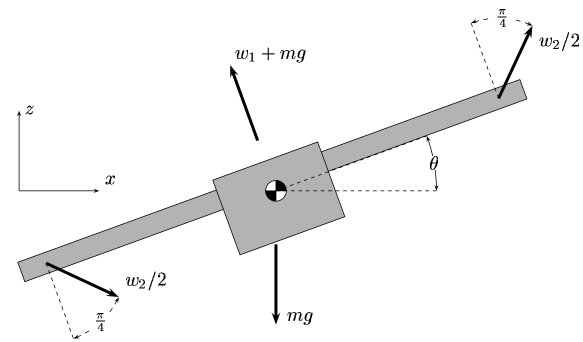
$$\dot{\omega} = (-k_3/J)\omega + (h/J)u_2$$

$\mathrm{Q} = \mathrm{SE}(2)$ : Configuration and velocity space via $(x, z, \theta, v_x, v_z, \omega)$. $x$ and $z$ are horizontal and vertical displacement, $\theta$ is roll angle. The angular velocity is $\omega$ and the linear velocities in the body-fixed $x$ (respectively $z$) axis are $v_x$ (respectively $v_z$).

$u_1$ is body vertical force minus gravity, $u_2$ is force on the wingtips (with a net horizontal component). $k_i$-components are linear damping force, $g$ is gravity constant. The constant $h$ is the distance from the center of mass to the wingtip, $m$ and $J$ are mass and moment of inertia.

## 7.4 Oscillatory controls ex. #2: PVTOL model

Controllability assumption ok. Design controls to track $(x^d(t), z^d(t), \theta^d(t))$:




$$u_1 = \frac{J}{h}\ddot{\theta}^d + \frac{k_3}{h}\dot{\theta}^d - \frac{\sqrt{2}}{\epsilon}\cos\left(\frac{t}{\epsilon}\right)$$

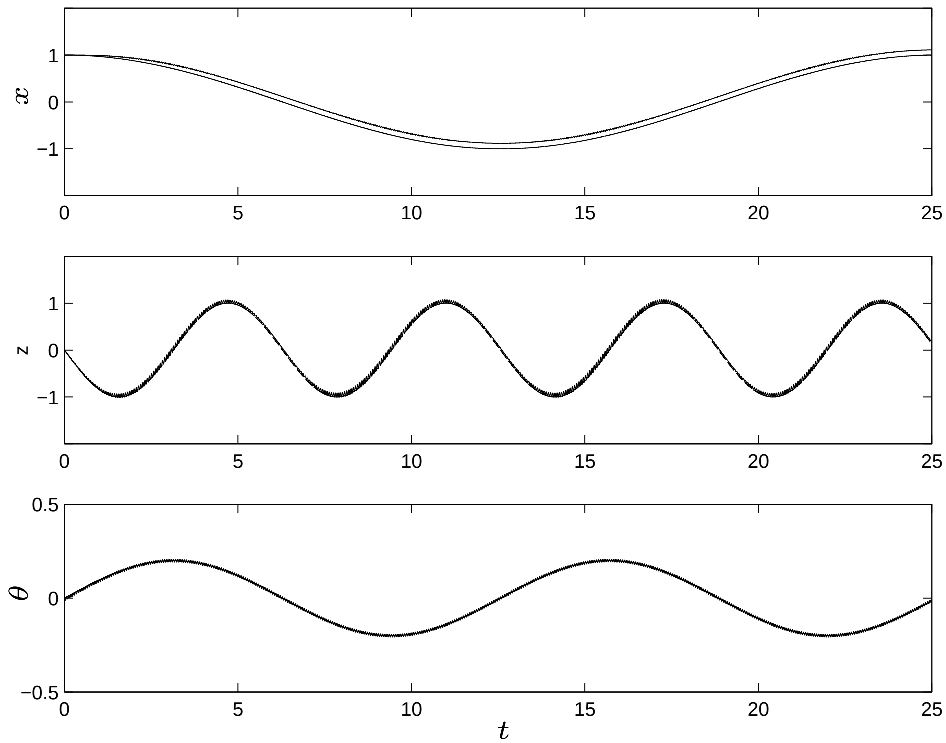
$$u_2 = \frac{h}{J} - f_1 \sin\theta^d + f_2\cos\theta^d - \frac{J\sqrt{2}}{h\epsilon}\left(f_1\cos\theta^d + f_2\sin\theta^d\right)\cos\left(\frac{t}{\epsilon}\right),$$

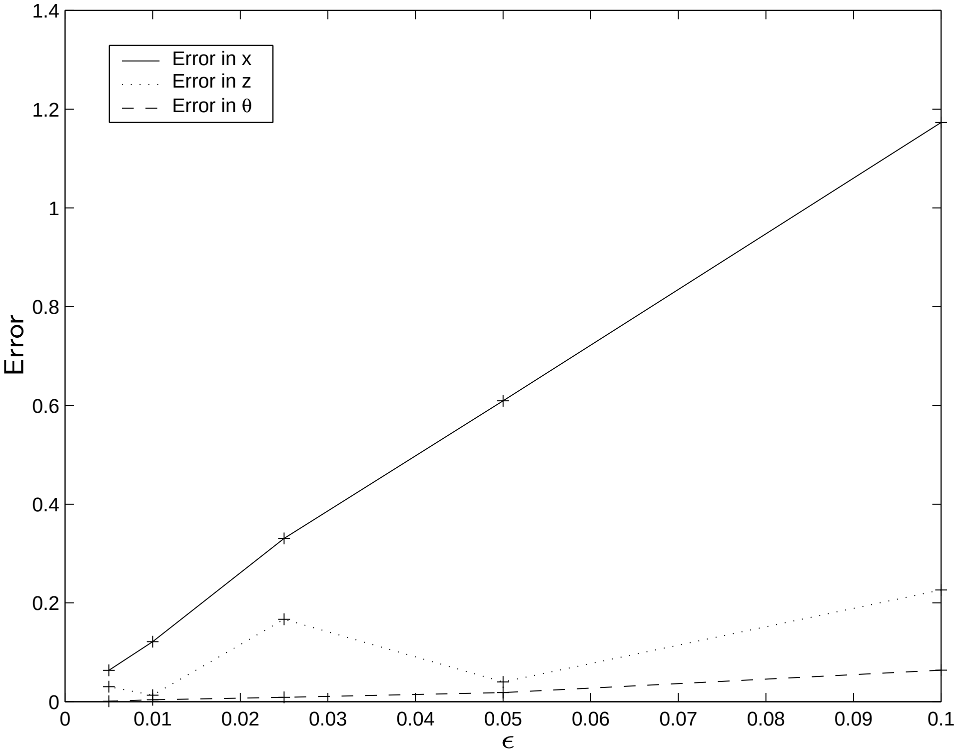
where we let $c = \frac{J}{h}\ddot{\theta}^d + \frac{k_3}{h}\dot{\theta}^d$ and

$$f_1 = m\ddot{x}^d + \left(k_1\cos^2\theta^d + k_2\sin^2\theta^d\right)\dot{x}^d + \frac{\sin(2\theta^d)}{2}(k_1 - k_2)\dot{z}^d + mg\sin\theta^d - c\cos\theta^d,$$

$$f_2 = m\ddot{z}^d + \frac{\sin(2\theta^d)}{2}(k_1 - k_2)\dot{x}^d + \left(k_1\sin^2\theta^d + k_2\cos^2\theta^d\right)\dot{z}^d + mg(1 - \cos\theta^d) - c\sin\theta^d.$$

## 7.5 PVTOL Simulations: trajectories and error




Trajectory design at $\epsilon = .01$. Tracking errors at $t = 10$.