# Synchronization and Kron Reduction in Power Networks

Florian Dörfler and Francesco Bullo

Center for Control,
Dynamical Systems & Computation
University of California at Santa Barbara

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Poster tomorrow on: "Network reduction and effective resistance"
Slides and papers available at: http://motion.me.ucsb.edu

# Summary

**observations from distinct fields:**

1. power networks are coupled oscillators
2. Kuramoto oscillators synchronize for large coupling
3. graph theory quantifies coupling in a network
4. hence, power networks synchronize for large coupling

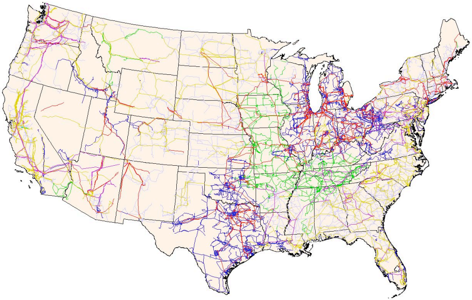
**Today's talk:**

- theorems about these observations
- synch tests for "net-preserving" and "reduced" models

# Outline

1. **Introduction**
   - Motivation
   - Mathematical model
   - Problem statement
2. **Singular perturbation analysis**
   **(to relate power network and Kuramoto model)**
3. **Synchronization of non-uniform Kuramoto oscillators**
4. **Network-preserving power network models**
5. **Conclusions**

# Motivation: the current US power grid



*"... the largest and most complex machine engineered by humankind."*

[P. Kundur '94, V. Vittal '03, ...]

*"... the greatest engineering achievement of the 20th century."*

[National Academy of Engineering '10]

1. large-scale, nonlinear dynamics, complex interactions
2. 100 years old and operating at its capacity limits

⇒ recent blackouts: New England '03, Italy '03, Brazil '09

Energy is one of the top three national priorities, [B. Obama, '09]

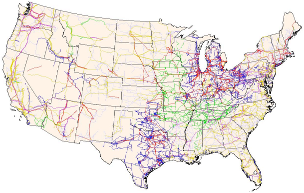
Expected developments in "**smart grid**":

$\Rightarrow$ increasing consumption

$\Rightarrow$ increasing adoption of renewable power sources:

1. large number of distributed power sources
2. power transmission from remote areas

$\Rightarrow$ large-scale heterogeneous networks with stochastic disturbances



**Transient Stability**

Generators to swing synchronously despite variability/faults in generators/network/loads

# Motivation: the Mediterranean ring project

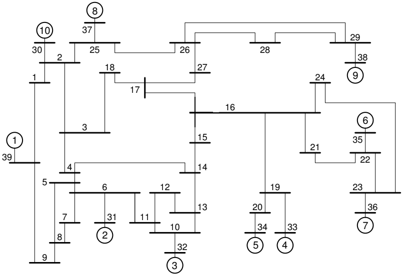

Synchronous grid interconnection between EU and Mediterranean region:

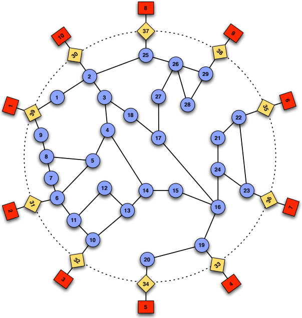
1. Provide increased levels of energy security to participating nations;
2. Import/export electric power among nations;
3. Cut back on the primary electricity reserve requirements within each country.

**Reference:** "Oscillation behavior of the enlarged European power system" by M. Kurth and E. Welfonder. Control Engineering Practice, 2005.

# Mathematical model of a power network

## New England Power Grid





**Power network topology:**

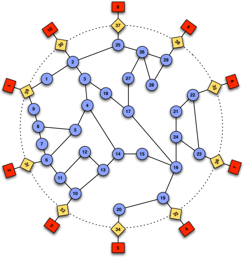
1. $n$ generators ■, each connected to a generator terminal bus ◊
2. $n$ generators terminal buses ◊ and $m$ load buses ● form connected graph
3. admittance matrix $\mathbf{Y}_{\textbf{network}} \in \mathbb{C}^{(2n+m)\times(2n+m)}$ characterizes the network

# Mathematical model of a power network

**Network-preserving DAE power network model:**

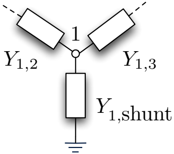
1. $n$ generators ■ = boundary nodes:

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + P_{\text{mech.in},i} - P_{\text{electr.out},i}$$



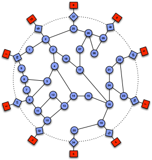
2. $n + m$ passive ◊ & ● = interior nodes:
   - loads are modeled as shunt admittances
   - algebraic Kirchhoff equations:

$$I = \mathbf{Y}_{\text{network}} V$$



# Mathematical model of a power network

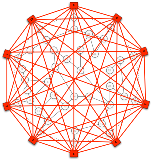
## Network-Reduction to an ODE power network model



$$\begin{bmatrix} I_{\text{boundary}} \\ \hline \mathbf{0} \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} Y_{\text{boundary}} & Y_{\text{bound-int}} \\ \hline Y_{\text{bound-int}}^{T} & Y_{\text{interior}} \end{array}\right]}_{\mathbf{Y}_{\text{network}}} \begin{bmatrix} V_{\text{boundary}} \\ \hline V_{\text{interior}} \end{bmatrix}$$

### Schur complement

$$Y_{\text{reduced}} = \mathbf{Y}_{\text{network}} / Y_{\text{interior}} \quad \Longrightarrow \quad I_{\text{boundary}} = Y_{\text{reduced}} V_{\text{boundary}}$$

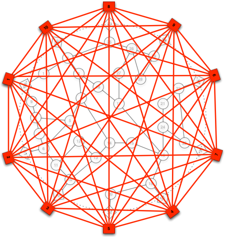


- network reduced to active nodes (generators)
- $Y_{\text{reduced}}$ induces complete "all-to-all" coupling graph

**Network-Reduced ODE power network model:**

classic interconnected swing equations
[Anderson et al. '77, M. Pai '89, P. Kundur '94, ...]:

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$



"all-to-all" reduced network $Y_{\text{reduced}}$ with

$P_{ij} = |V_i||V_j|\,|Y_{\text{reduced},i,j}| > 0$ max. power transferred $i \leftrightarrow j$

$\varphi_{ij} = \arctan(\Re(Y_{\text{red},i,j})/\Im(Y_{\text{red},i,j})) \in [0, \pi/2)$ reflect losses $i \leftrightarrow j$

$\omega_i = P_{\text{mech.in},i} - |V_i|^2\Re(Y_{\text{reduced},i,i})$ effective power input of $i$

# Transient stability analysis: problem statement

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

**Classic transient stability**:

1. power network in stable frequency equilibrium
$$(\dot{\theta}_i, \ddot{\theta}_i) = (0,0) \text{ for all } i$$
2. $\longrightarrow$ transient network disturbance and fault clearance
3. stability analysis of a new frequency equilibrium in post-fault network

**General synchronization problem:**

- synchronous equilibrium: $|\theta_i - \theta_j|$ small & $\dot{\theta}_i = \dot{\theta}_j$ for all $i, j$

# Transient stability analysis: literature review

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

**Classic methods** use Hamiltonian and gradient systems arguments:

1. write $\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i - \nabla_i U(\theta)^T$
2. study $\dot{\theta}_i = -\nabla_i U(\theta)^T$

Key objective: compute domain of attraction via numerical methods

[N. Kakimoto et al. '78, H.-D. Chiang et al. '94 ]

**Open Problem "power sys dynamics + complex nets" [Hill and Chen '06]**

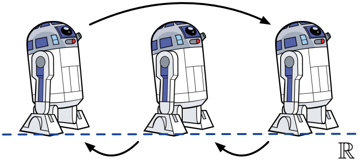
transient stability, performance, and robustness of a power network

$\overset{?}{\leftrightsquigarrow}$ underlying graph properties (topological, algebraic, spectral, etc)

**Consensus protocol in $\mathbb{R}^n$:**

$$\dot{x}_i = -\sum\nolimits_{j \neq i} a_{ij}(x_i - x_j)$$

- $n$ identical **agents** with state variable $x_i \in \mathbb{R}$
- **application:** agreement and coordination algorithms, ...
- **references:** [M. DeGroot '74, J. Tsitsiklis '84, L. Moreau '04, ...]




**Kuran**

$\dot{\theta}_i$

- r p
- a r
- r Y

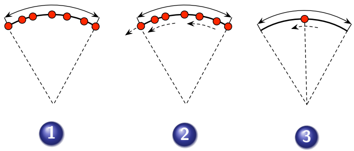
# Detour: synchrony among oscillators

**Kuramoto model in $\mathbb{T}^n$**

$$\dot{\theta}_i = \omega_i - \frac{K}{n} \sum\nolimits_{j \neq i} \sin(\theta_i - \theta_j)$$

## notions of synchronization

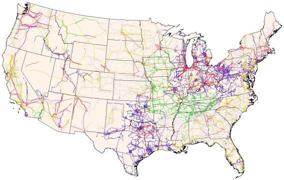
1. phase cohesiveness: $|\theta_i(t) - \theta_j(t)| < \gamma$ for small $\gamma < \pi/2$ ... arc invariance
2. frequency synchronized: $\dot{\theta}_i(t) = \dot{\theta}_j(t)$
3. phase synchronized: $\theta_i(t) = \theta_j(t)$




## Classic intuition:

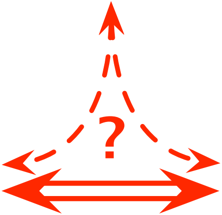
1. $K$ small & $|\omega_i - \omega_j|$ large $\Rightarrow$ no synchronization
2. $K$ large & $|\omega_i - \omega_j|$ small $\Rightarrow$ cohesive + freq synchronization

# The big picture

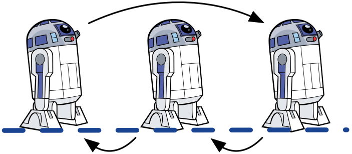


Open problem in synchronization and transient stability in **power networks:** relation to underlying network state, parameters, and topology

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

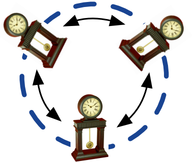



**Consensus Protocols:**



$$\dot{x}_i = -\sum\nolimits_{j\neq i} a_{ij}(x_i - x_j)$$

**Kuramoto Oscillators:**



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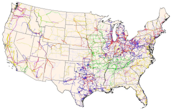
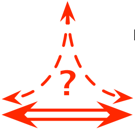


Open problem in synchronization and transient stability in **power networks:** relation to underlying network state, parameters, and topology

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

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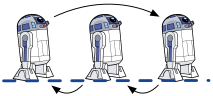

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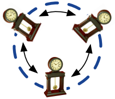


$$\dot{\theta}_i = \omega_i - \frac{K}{n}\sum\nolimits_{j\neq i}\sin(\theta_i - \theta_j)$$

## Previous observations about this connection:

Power systems: [D. Subbarao et al., '01, G. Filatrella et al., '08, V. Fioriti et al., '09]
Networked control: [D. Hill et al., '06, M. Arcak, '07]
Dynamical systems: [H. Tanaka et al., '97, A. Arenas '08]

# Outline

# Singular perturbation analysis

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

1. assume time-scale separation between synchronization and damping
   **singular perturbation parameter** $\epsilon = \dfrac{M_{\max}}{\pi f_0 D_{\min}}$
2. **non-uniform Kuramoto** (slow time-scale, for $\epsilon = 0$)

$$D_i\dot{\theta}_i = \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

3. if cohesiveness + exponential freq sync for non-uniform Kuramoto, then $\forall\,(\theta(0),\dot{\theta}(0))$, exists $\epsilon^* > 0$ such that $\forall\,\epsilon < \epsilon^*$ and $\forall\, t \geq 0$

$$\theta_i(t)_{\text{power network}} - \theta_i(t)_{\text{non-uniform Kuramoto}} = \mathcal{O}(\epsilon)$$

# Singular perturbation analysis: proof

## Key technical problem:

- Kuramoto defined over manifold $\mathbb{T}^n$, no fixed point
- Tikhonov's Theorem: exp. stable point in Euclidean space

## Solution

- define **grounded variables** in $\mathbb{R}^{n-1}$

$$\delta_1 = \theta_1 - \theta_n \qquad \cdots \qquad \delta_{n-1} = \theta_{n-1} - \theta_n$$

- equivalence of solutions:
  1. grounded Kuramoto solutions satisfy $\max_{i,j}(\delta_i(t) - \delta_j(t)) < \pi$
  2. Kuramoto solutions are arc invariant with $\gamma = \pi$,
     ie, $\theta_1(t), \ldots, \theta_n(t)$ belong to open half-circle, function of $t$
- equivalence of exponential convergence
  1. exponential frequency synchronization for Kuramoto
  2. exponential convergence to equilibrium for grounded Kuramoto

# Singular perturbation analysis: discussion

assumption $\epsilon = \dfrac{M_{\text{max}}}{\pi f_0 D_{\text{min}}}$ sufficiently small

1. **generator internal control effects** imply $\epsilon \in \mathcal{O}(0.1)$
2. **topological equivalence independent of $\epsilon$:** 1st-order and 2nd-order models have the same equilibria, the Jacobians have the same inertia, and the regions of attractions are bounded by the same separatrices
3. non-uniform Kuramoto corresponds to reduced gradient system $\dot{\theta}_i = -\nabla_i U(\theta)^T$ **used** successfully in academia and industry since 1978
4. **physical interpretation:** damping and sync on separate time-scales
5. **classic assumption** in literature on coupled oscillators: over-damped mechanical pendula and Josephson junctions
6. **simulation studies** show accurate approximation even for large $\epsilon$

# Outline

1. Introduction
   - Motivation
   - Mathematical model
   - Problem statement
2. Singular perturbation analysis
   (to relate power network and Kuramoto model)
3. **Synchronization of non-uniform Kuramoto oscillators**
4. Network-preserving power network models
5. Conclusions

**Non-uniform Kuramoto Model in $\mathbb{T}^n$:**

$$D_i \dot{\theta}_i = \omega_i - \sum\nolimits_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij})$$

- **Non-uniformity** in network: $D_i$, $\omega_i$, $P_{ij}$, $\varphi_{ij}$
- **Phase shift** $\varphi_{ij}$ induces lossless and lossy coupling:

$$\dot{\theta}_i = \frac{\omega_i}{D_i} - \sum\nolimits_{j \neq i} \left( \frac{P_{ij}}{D_i} \cos(\varphi_{ij}) \sin(\theta_i - \theta_j) + \frac{P_{ij}}{D_i} \sin(\varphi_{ij}) \cos(\theta_i - \theta_j) \right)$$

**Synchronization condition $(\star)$**

$$\underbrace{n \frac{P_{\min}}{D_{\max}} \cos(\varphi_{\max})}_{\text{worst lossless coupling}} > \underbrace{\max_{i,j} \left( \frac{\omega_i}{D_i} - \frac{\omega_j}{D_j} \right)}_{\text{worst non-uniformity}} + \underbrace{\max_i \sum\nolimits_j \frac{P_{ij}}{D_i} \sin(\varphi_{ij})}_{\text{worst lossy coupling}}$$

# Synchronization of non-uniform Kuramoto: consequences

$$D_i\dot{\theta}_i = \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

$$n\frac{P_{\min}}{D_{\max}}\cos(\varphi_{\max}) > \max_{i,j}\Big(\frac{\omega_i}{D_i} - \frac{\omega_j}{D_j}\Big) + \max_i \sum\nolimits_j \frac{P_{ij}}{D_i}\sin(\varphi_{ij})$$

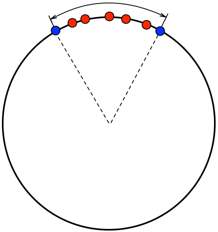
1) **phase cohesiveness:** arc-invariance for all arc-lengths

$$\underbrace{\arcsin\Big(\cos(\varphi_{\max})\frac{RHS}{LHS}\Big)}_{\gamma_{\min}} \quad \leq \; \gamma \; \leq \quad \underbrace{\frac{\pi}{2} - \varphi_{\max}}_{\gamma_{\max}}$$

**practical phase sync:** in finite time, arc-length $\gamma_{\min}$

2) **frequency synch:** from all initial conditions in a $\gamma_{\max}$ arc, exponential frequency synchronization

1. **Cohesiveness** $\theta(t) \in \Delta(\gamma) \Leftrightarrow$ arc-length $V(\theta(t))$ is non-increasing




$$\Leftrightarrow \begin{cases} V(\theta(t)) = \max\{|\theta_i(t) - \theta_j(t)| \mid i,j \in \{1,\dots,n\}\} \\ D^+ V(\theta(t)) \overset{!}{\leq} 0 \end{cases}$$

$\sim$ **contraction property** [D. Bertsekas et al. '94, L. Moreau '04 & '05, Z. Lin et al. '08, ...]

2. **Frequency synchronization** in $\Delta(\gamma) \Leftrightarrow$ consensus protocol in $\mathbb{R}^n$

$$\frac{d}{dt}\dot{\theta}_i = -\sum\nolimits_{j \neq i} a_{ij}(t)(\dot{\theta}_i - \dot{\theta}_j)\,,$$

where $a_{ij}(t) = \frac{P_{ij}}{D_i}\cos(\theta_i(t) - \theta_j(t) + \varphi_{ij}) > 0$ for all $t \geq 0$

# Synchronization of non-uniform Kuramoto: uniform

## Classic (uniform) Kuramoto Model in $\mathbb{T}^n$:

$$\dot{\theta}_i = \omega_i - \frac{K}{n} \sum\nolimits_{j \neq i} \sin(\theta_i - \theta_j)$$

## Necessary and sufficient condition

- (sufficiency) synchronization condition $(\star)$ reads

$$K > \max_{i,j} \left(\omega_i - \omega_j\right)$$

- also necessary when considering all distributions of $\omega \in [\omega_{\min}, \omega_{\max}]$

Condition $(\star)$ strictly improves existing bounds on Kuramoto model: [F. de Smet et al. '07, N. Chopra et al. '09, G. Schmidt et al. '09, A. Jadbabaie et al. '04, S.J. Chung et al. '10, J.L. van Hemmen et al. '93].

Necessary condition synchronization: $K > \frac{n}{2(n-1)}(\omega_{\max} - \omega_{\min})$
[J.L. van Hemmen et al. '93, A. Jadbabaie et al. '04, N. Chopra et al. '09]

**Non-uniform Kuramoto Model in $\mathbb{T}^n$ - rewritten:**

$$D_i\dot{\theta}_i = \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

$$\underbrace{\lambda_2(L(P_{ij}\cos(\varphi_{ij})))}_{\text{lossless connectivity}} > \underbrace{f(D_i)}_{\text{non-uniform } D_i\text{s}} \cdot \underbrace{(1/\cos(\varphi_{\max}))}_{\text{phase shifts}} \times$$

$$\times\Bigg(\underbrace{\Big\|\Big[\dots,\frac{\omega_i}{D_i}-\frac{\omega_j}{D_j},\dots\Big]\Big\|_2}_{\text{non-uniformity}} + \sqrt{\lambda_{\max}(L)}\,\underbrace{\Big\|\Big[\dots,\sum\nolimits_j\frac{P_{ij}}{D_i}\sin(\varphi_{ij}),\dots\Big]\Big\|_2}_{\text{lossy coupling}}\Bigg)$$

Similar synch, quadratic Lyap, uniform test $K > \|[\dots,\omega_i-\omega_j,\dots]\|_2$

# Synchronization of non-uniform Kuramoto

**Non-uniform Kuramoto Model in $\mathbb{T}^n$**

$$D_i \dot{\theta}_i = \omega_i - \sum\nolimits_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij})$$

**Further interesting results:**

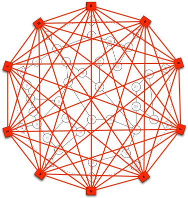
1. explicit synchronization frequency
2. exponential rate of frequency synchronization
3. conditions for phase synchronization
4. results for general non-complete graphs

…to be found in our papers.

# Outline

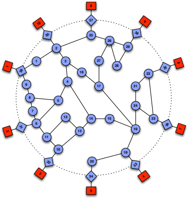
# Network-preserving power network models

So far we considered a **network-reduced** power system model:



- synchronization conditions on $\lambda_2(P)$ and $P_{\text{min}}$
- all-to-all reduced admittance matrix $Y_{\text{reduced}} \sim P/V^2$ (for uniform voltage levels $|V_i| = V$)

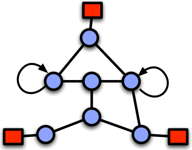
Topological non-reduced **network-preserving** power system model:



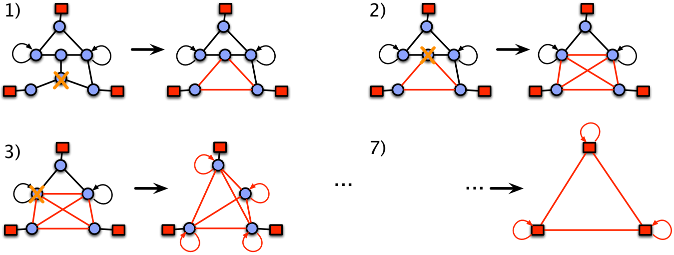
- topological bus admittance matrix $\mathbf{Y}_{\textbf{network}}$ indicating transmission lines and loads (self-loops)
- **Schur complement**: $Y_{\text{reduced}} = \mathbf{Y}_{\textbf{network}} / Y_{\text{interior}}$
  c.f. “Kron reduction”, “Dirichlet-to-Neumann map”, “Schur contraction”, “Gaussian elimination”, . . .

**Kron reduction** of a graph with

- boundary ■, interior ●, non-neg self-loops ⟲
- loopy Laplacian matrix $\mathbf{Y}_{\text{network}}$



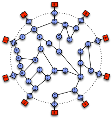
1. Iterative 1-dim Kron reduction:
   - Topological evolution of the corresponding graph



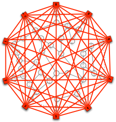

   - Algebraic evolution of Laplacian matrix: $\mathbf{Y}^{k+1}_{\text{reduced}} = \mathbf{Y}^{k}_{\text{reduced}} / \bullet$

# Kron reduction of graphs: properties

1. **Well-posedness:** set of loopy Laplacian matrices is closed
2. **Equivalence:** iterative 1-dim reduction = 1-step reduction



$$\mathbf{Y}_{\text{network}} \quad \longrightarrow \quad Y_{\text{reduced}} = \mathbf{Y}_{\text{network}} / Y_{\text{interior}}$$



3. **Topological properties:**
   - interior network connected ⇒ reduced network complete
   - at least one node in interior network features a self-loop ⟲
     ⇒ all nodes in reduced network feature self-loops ⟲
4. **Algebraic properties:** self-loops in interior network
   - decrease mutual coupling in reduced network
   - increase self-loops in reduced network

Some properties of the **Kron reduction** process:

⋮

5. **Spectral properties**:
   - interlacing property: $\lambda_i(\mathbf{Y}_{\text{network}}) \leq \lambda_i(Y_{\text{reduced}}) \leq \lambda_{i+n-|\blacksquare|}(\mathbf{Y}_{\text{network}})$
   - algebraic connectivity $\lambda_2$ is non-decreasing along Kron
6. **Effective resistance:**
   - Effective resistance $R(i,j)$ among boundary nodes $\blacksquare$ is invariant
   - For boundary nodes $\blacksquare$: effective resistance $R(i,j)$ uniform $\Leftrightarrow$ coupling $Y_{\text{reduced}}(i,j)$ uniform $\Leftrightarrow 1/R(i,j) = \frac{n}{2}|Y_{\text{reduced}}(i,j)|$

# Synchronization in network-preserving models

**Assumption I:** lossless network and uniform voltage levels V at generators

1. **Spectral condition for synchronization:** $\lambda_2(P) \geq$ ... becomes

$$\lambda_2(i \cdot \mathbf{L}_{\text{network}}) > \left|\left|\left(\frac{\omega_2}{D_2} - \frac{\omega_1}{D_1}, \dots\right)\right|\right|_2 \cdot \frac{f(D_i)}{V^2} + \min\{\circlearrowleft\}$$

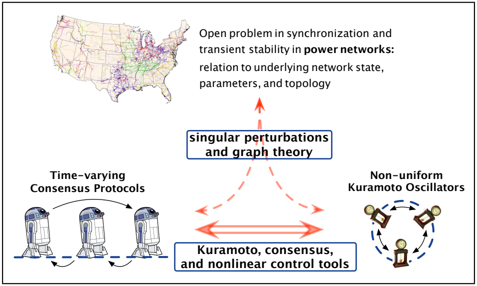
**Assumption II:** effective resistance $R$ among generator nodes is uniform

2. **Resistance-based condition for synchronization:** $nP_{\text{min}} \geq \dots$ becomes

$$\frac{1}{R} > \max_{i,j}\left\{\frac{\omega_i}{D_i} - \frac{\omega_j}{D_j}\right\} \cdot \frac{D_{\text{max}}}{2V^2}$$

# Outline

# Conclusions




## Ambitious workplan

1. sharpest conditions for most realistic models
2. stochastic instead of worst-case analysis
3. networks of DC/AC power inverters
4. control via voltage regulation and "flexible AC transmission systems"
5. "distance to instability" and optimal islanding for failure management

**transition to DOE laboratories and utilities**