# Contracting Dynamical Systems: A Tutorial on Theory and Applications

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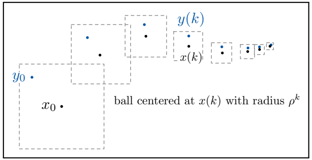
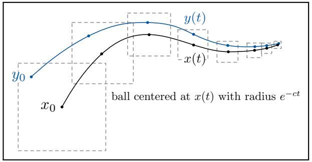
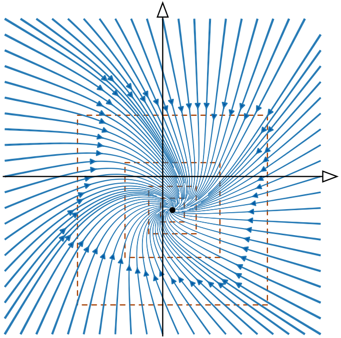
Tutorial (based on lectures in Napoli Nov '22 and San Diego Jun '23). This version: 2023/06/23



**contractivity = robust computationally-friendly stability**
fixed point theory + Lyapunov stability theory + geometry of metric spaces

## highly-ordered transient and asymptotic behavior:



1. unique globally exponential stable equilibrium
   & two natural Lyapunov functions
2. robustness properties
   - bounded input, bounded output (iss)
   - finite input-state gain
   - robustness margin wrt unmodeled dynamics
   - robustness margin wrt delayed dynamics
3. periodic input, periodic output
4. modularity and interconnection properties
5. accurate numerical integration and equilibrium point computation

**search for** contraction properties
**design** engineering systems to be contracting

# Acknowledgments



Veronica Centorrino
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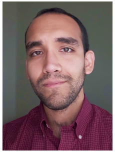
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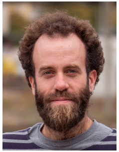
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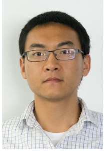
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University of Toronto


Kevin D. Smith
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Elena Valcher
Universita di Padova

# Outline

# Contraction theory: historical notes

- **Origins**

  S. Banach. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae*, 3(1):133–181, 1922.

- **Dynamics**:

  G. Dahlquist. *Stability and error bounds in the numerical integration of ordinary differential equations*. PhD thesis, (Reprinted in Trans. Royal Inst. of Technology, No. 130, Stockholm, Sweden, 1959), 1958

  S. M. Lozinskii. Error estimate for numerical integration of ordinary differential equations. I. *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, 5:52–90, 1958. URL **http://mi.mathnet.ru/eng/ivm2980**. (in Russian)

- **Computation**:

  C. A. Desoer and H. Haneda. The measure of a matrix as a tool to analyze computer algorithms for circuit analysis. *IEEE Transactions on Circuit Theory*, 19(5):480–486, 1972.

- **Systems and control**:

  W. Lohmiller and J.-J. E. Slotine. On contraction analysis for non-linear systems. *Automatica*, 34(6): 683–696, 1998.

- **Incomplete list of contributors who influenced me**
  Aminzare, Arcak, Chung, Coogan, Di Bernardo, Manchester, Margaliot, Pavlov, Pham, Proskurnikov, Russo, Sepulchre, Slotine, Sontag, ...

- **Surveys**:

  Z. Aminzare and E. D. Sontag. Contraction methods for nonlinear systems: A brief introduction and some open problems. In *IEEE Conf. on Decision and Control*, pages 3835–3847, Dec. 2014b.

  M. Di Bernardo, D. Fiore, G. Russo, and F. Scafuti. Convergence, consensus and synchronization of complex networks via contraction theory. In *Complex Systems and Networks*. Springer, 2016.

  H. Tsukamoto, S.-J. Chung, and J.-J. E. Slotine. Contraction theory for nonlinear stability analysis and learning-based control: A tutorial overview. *Annual Reviews in Control*, 52:135–169, 2021.

  P. Giesl, S. Hafstein, and C. Kawan. Review on contraction analysis and computation of contraction metrics. *Journal of Computational Dynamics*, 10(1):1–47, 2023.

The Banach Contraction Theorem is also referred to as the *Picard-Banach-Caccioppoli*, because of the earlier work by Picard (1890) on the "method of successive approximations" and the later independent work by Renato Caccioppoli (1930).

Figure: Renato Caccioppoli (Napoli, 20 gennaio 1904 – Napoli, 8 maggio 1959) was an Italian mathematician

1921-1932 student and researcher @ Napoli
1931-1934 professor @ Padova
1934-1959 professor @ Napoli

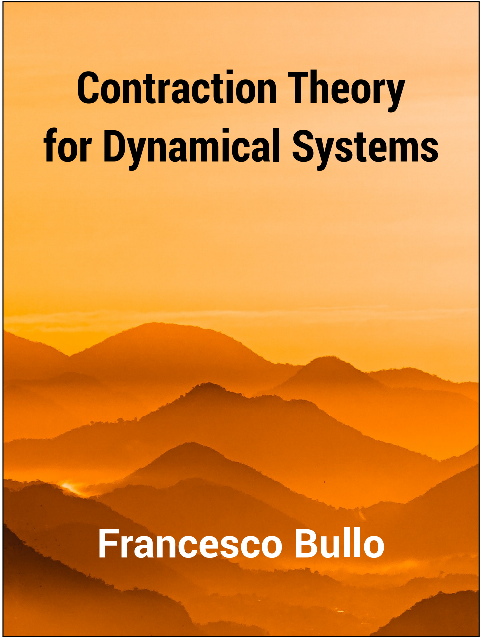
R. Caccioppoli. Un teorema generale sull'esistenza di elementi uniti in una trasformazione funzionale. *Rendiconti dell'Accademia Nazionale dei Lincei*, 11:794–799, 1930

# Contraction conditions without Jacobians


1. **uniformly decreasing maps** in: L. Chua and D. Green. A qualitative analysis of the behavior of dynamic nonlinear networks: Stability of autonomous networks. *IEEE Transactions on Circuits and Systems*, 23(6): 355–379, 1976. doi
2. no-name in: A. F. Filippov. *Differential Equations with Discontinuous Righthand Sides*. Kluwer, 1988. ISBN 902772699X (Chapter 1, page 5)
3. **one-sided Lipschitz maps** in: E. Hairer, S. P. Nørsett, and G. Wanner. *Solving Ordinary Differential Equations I. Nonstiff Problems*. Springer, 1993. doi (Section 1.10, Exercise 6)
4. **maps with negative nonlinear measure** in: H. Qiao, J. Peng, and Z.-B. Xu. Nonlinear measures: A new approach to exponential stability analysis for Hopfield-type neural networks. *IEEE Transactions on Neural Networks*, 12(2):360–370, 2001. doi
5. **dissipative Lipschitz maps** in: T. Caraballo and P. E. Kloeden. The persistence of synchronization under environmental noise. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 461(2059):2257–2267, 2005. doi
6. **maps with negative lub log Lipschitz constant** in: G. Söderlind. The logarithmic norm. History and modern theory. *BIT Numerical Mathematics*, 46(3):631–652, 2006. doi
7. **QUAD maps** in: W. Lu and T. Chen. New approach to synchronization analysis of linearly coupled ordinary differential systems. *Physica D: Nonlinear Phenomena*, 213(2):214–230, 2006. doi
8. **incremental quadratically stable maps** in: L. D'Alto and M. Corless. Incremental quadratic stability. *Numerical Algebra, Control and Optimization*, 3:175–201, 2013. doi

## Links to recent related educational and research events

- 2023 ACC Workshop on "Contraction Theory for Systems, Control, and Learning" **http://motion.me.ucsb.edu/contraction-workshop-2023**
- Tutorial session: **https://sites.google.com/view/contractiontheory** "Contraction Theory for Machine Learning" (PDFs and youtube videos) at the 2021 IEEE CDC conference, by Soon-Jo Chung, Jean-Jacques Slotine, and Hiroyasu Tsukamoto
- Tutorial paper at CDC2021 "Contraction-Based Methods for Stable Identification and Robust Machine Learning: a Tutorial" by Ian Manchester and coauthors: **https://arxiv.org/abs/2110.00207**, **https://ieeexplore.ieee.org/abstract/document/9683128**
- Plenary presentation: (Slides **https://fbullo.github.io/talks/2022-12-FBullo-ContractionSystemsControl-CDC.pdf**) "Contraction Theory in Systems and Control" by Francesco Bullo at the 2022 IEEE CDC
- Youtube lectures: "Lectures on Nonlinear Systems" by Jean-Jacques Slotine, Fall 2013: **https://web.mit.edu/nsl/www/videos/lectures.html**, Lectures 14-20 (approximately 1h20min each)
- Youtube lectures: "Minicourse on Contraction Theory" by Francesco Bullo, Fall 2022. Youtube lectures **https://youtu.be/RvR47ZbqJjc**: 10h in 4 lectures, with chapters
- Textbook: Contraction Theory for Dynamical Systems, Francesco Bullo, rev 1.1, Mar 2023. (Book and slides freely available) **https://fbullo.github.io/ctds**




**Contraction Theory for Dynamical Systems**, Francesco Bullo, KDP, 1.1 edition, 2023, ISBN 979-8836646806

1. Textbook with exercises and answers. Format: textbook, slides, and paperbook
2. Content:
   Fixed point theory
   Theory of contracting dynamics on vector spaces
   Applications to nonlinear and interconnected systems
3. Self-Published and Print-on-Demand at:
   **https://www.amazon.com/dp/B0B4K1BTF4**
4. PDF Freely available at
   **https://fbullo.github.io/ctds**
5. 10h minicourse on youtube:
   **https://youtu.be/RvR47ZbqJjc**
6. Future version to include: systems on Riemannian manifolds, homogeneous spaces, and solid cones

   "Continuous improvement is better than delayed perfection"
   **Mark Twain**

# Selected references from my group

**Contraction theory on normed spaces and Riemannian manifolds**:


- A. Davydov, S. Jafarpour, and F. Bullo. Non-Euclidean contraction theory for robust nonlinear stability. *IEEE Transactions on Automatic Control*, 67(12): 6667–6681, 2022a. doi
- S. Jafarpour, A. Davydov, and F. Bullo. Non-Euclidean contraction theory for monotone and positive systems. *IEEE Transactions on Automatic Control*, 2023. doi. To appear
- J. W. Simpson-Porco and F. Bullo. Contraction theory on Riemannian manifolds. *Systems & Control Letters*, 65:74–80, 2014. doi


**Contracting neural networks**:


- S. Jafarpour, A. Davydov, A. V. Proskurnikov, and F. Bullo. Robust implicit networks via non-Euclidean contractions. In *Advances in Neural Information Processing Systems*, Dec. 2021. doi
- A. Davydov, A. V. Proskurnikov, and F. Bullo. Non-Euclidean contractivity of recurrent neural networks. In *American Control Conference*, pages 1527–1534, Atlanta, USA, May 2022c. doi
- V. Centorrino, A. Gokhale, A. Davydov, G. Russo, and F. Bullo. Euclidean contractivity of neural networks with symmetric weights. *IEEE Control Systems Letters*, 7:1724–1729, 2023. doi


**Weak and semicontraction theory**:


- S. Jafarpour, P. Cisneros-Velarde, and F. Bullo. Weak and semi-contraction for network systems and diffusively-coupled oscillators. *IEEE Transactions on Automatic Control*, 67(3):1285–1300, 2022. doi
- G. De Pasquale, K. D. Smith, F. Bullo, and M. E. Valcher. Dual seminorms, ergodic coefficients, and semicontraction theory. *IEEE Transactions on Automatic Control*, 2022. doi. Submitted
- R. Delabays and F. Bullo. Semicontraction and synchronization of Kuramoto-Sakaguchi oscillator networks. *IEEE Control Systems Letters*, 7:1566–1571, 2023. doi


**Optimization**:


- F. Bullo, P. Cisneros-Velarde, A. Davydov, and S. Jafarpour. From contraction theory to fixed point algorithms on Riemannian and non-Euclidean spaces. In *IEEE Conf. on Decision and Control*, Dec. 2021. doi
- A. Davydov, S. Jafarpour, A. V. Proskurnikov, and F. Bullo. Non-Euclidean monotone operator theory with applications to recurrent neural networks. In *IEEE Conf. on Decision and Control*, Cancún, México, Dec. 2022b. doi
- A. Davydov, V. Centorrino, A. Gokhale, G. Russo, and F. Bullo. Contracting dynamics for time-varying convex optimization. *IEEE Transactions on Automatic Control*, June 2023. doi. Submitted

# Outline

**Banach Contraction Theorem** Let $(\mathcal{X}, d)$ be a *complete metric space*

If $T : \mathcal{X} \to \mathcal{X}$ is Lipschitz with constant $\ell < 1$ (called the *contraction factor*), then

1. $T$ has a unique fixed point $x^*$ in $\mathcal{X}$
2. the sequence $\{x_k\}_{k\in\mathbb{N}}$ generated by the *Picard iteration* $x_{k+1} = T(x_k)$ converges to $x^*$ for all initial conditions $x_0 \in \mathcal{X}$
3. the following error estimates hold for all $k \in \mathbb{N}$:

(geometric convergence): $$d(x_k, x^*) \leq \ell^k d(x_0, x^*)$$

(a-priori upper bound): $$d(x_k, x^*) \leq \frac{\ell^k}{1-\ell} d(x_0, x_1)$$

(a-posteriori upper bound): $$d(x_k, x^*) \leq \frac{\ell}{1-\ell} d(x_{k-1}, x_k)$$

# Proof

For $x_{k+1} = T(x_k)$

- sequence $\{x_k\}_{k\in\mathbb{N}}$ is Cauchy

$$\begin{aligned} d(x_{k+h}, x_k) &\leq d(x_{k+h}, x_{k+h-1}) + \cdots + d(x_{k+1}, x_k) \\ &\leq (\ell^{h-1} + \cdots + 1) d(x_{k+1}, x_k) \\ &\leq \frac{1}{1-\ell} d(x_{k+1}, x_k) \\ &\leq \frac{\ell^k}{1-\ell} d(x_1, x_0) \end{aligned}$$

- since $\mathcal{X}$ is complete, sequence converges to a point $x^*$
- uniqueness from $\ell < 1$
- geometric convergence

$$d(x_k, x^*) = d(T(x_{k-1}), x^*) \leq \ell d(x_{k-1}, x^*) \leq \ell^k d(x_0, x^*)$$

# Linear algebra: induced norms

| Vector norm | Induced matrix norm | Induced matrix log norm |
|---|---|---|
| $\|x\|_1 = \sum_{i=1}^{n} \|x_i\|$ | $\|A\|_1 = \max_{j\in\{1,\dots,n\}} \sum_{i=1}^{n} \|a_{ij}\|$ | $\mu_1(A) = \max_{j\in\{1,\dots,n\}} \left(a_{jj} + \sum_{i=1,i\neq j}^{n} \|a_{ij}\|\right)$ = max column "absolute sum" of $A$ |
| $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$ | $\|A\|_2 = \sqrt{\lambda_{\text{max}}(A^\top A)}$ | $\mu_2(A) = \lambda_{\text{max}}\left(\frac{A+A^\top}{2}\right)$ |
| $\|x\|_\infty = \max_{i\in\{1,\dots,n\}} \|x_i\|$ | $\|A\|_\infty = \max_{i\in\{1,\dots,n\}} \sum_{j=1}^{n} \|a_{ij}\|$ | $\mu_\infty(A) = \max_{i\in\{1,\dots,n\}} \left(a_{ii} + \sum_{j=1,j\neq i}^{n} \|a_{ij}\|\right)$ = max row "absolute sum" of $A$ |

# Discrete-time dynamics and Lipschitz constants

$$x_{k+1} = \mathsf{F}(x_k) \qquad \text{on } \mathbb{R}^n \text{ with norm } \|\cdot\| \text{ and induced norm } \|\cdot\|$$

**Lipschitz constant**

$$\begin{aligned} \mathsf{Lip}(\mathsf{F}) &= \inf\{\ell > 0 \text{ such that } \|\mathsf{F}(x) - \mathsf{F}(y)\| \leq \ell\|x - y\| \quad \text{for all } x, y\} \\ &= \sup_x \|D\mathsf{F}(x)\| \end{aligned}$$

For **scalar map** $f$, $\quad \mathsf{Lip}(f) = \sup_x |f'(x)|$

For **affine map** $\mathsf{F}_A(x) = Ax + a$

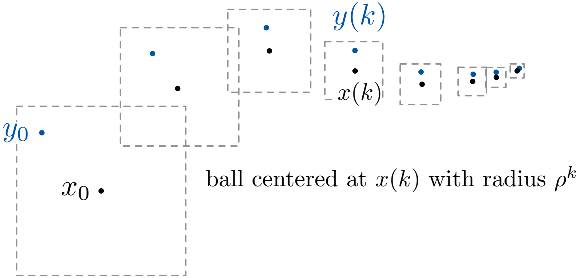
$$\|x\|_{2,P} = (x^\top P x)^{1/2} \qquad \mathsf{Lip}_{2,P}(\mathsf{F}_A) = \|A\|_{2,P} \leq \ell \quad \Longleftrightarrow \quad A^\top P A \preceq \ell^2 P$$

$$\|x\|_{\infty,\eta} = \max_i |x_i|/\eta_i \qquad \mathsf{Lip}_{\infty,\eta}(\mathsf{F}_A) = \|A\|_{\infty,\eta} \leq \ell \quad \Longleftrightarrow \quad \eta^\top |A| \leq \ell \eta^\top$$

**Banach contraction theorem for discrete-time dynamics:**

If $\rho := \text{Lip}(\mathsf{F}) < 1$, then

1. F is **contracting** $=$ distance between trajectories decreases exp fast $(\rho^k)$
2. F has a unique, glob exp stable equilibrium $x^*$

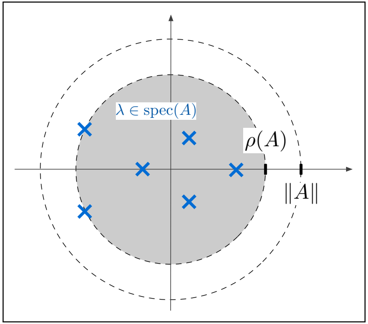


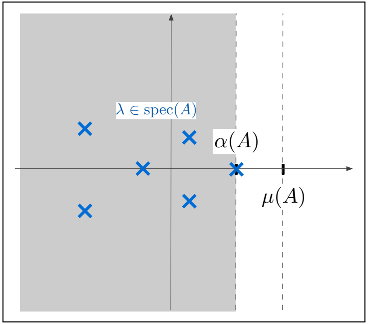
# From induced norms to induced log norms

The **induced log norm** of $A \in \mathbb{R}^{n\times n}$ wrt to $\|\cdot\|$:

$$\mu(A) := \lim_{h\to 0^+} \frac{\|I_n + hA\| - 1}{h}$$

| | | |
|---|---|---|
| subadditivity: | $\mu(A+B) \leq \mu(A) + \mu(B)$ | |
| scaling: | $\mu(bA) = b\mu(A),$ | $\forall b \geq 0$ |



# Example induced log norms

| Vector norm | Induced matrix norm | Induced matrix log norm |
|---|---|---|
| $\|x\|_1 = \sum_{i=1}^n \|x_i\|$ | $\|A\|_1 = \max_{j\in\{1,\dots,n\}} \sum_{i=1}^n \|a_{ij}\|$ | $\mu_1(A) = \max_{j\in\{1,\dots,n\}} \left(a_{jj} + \sum_{i=1,i\neq j}^n \|a_{ij}\|\right)$ $=$ max column "absolute sum" of $A$ |
| $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$ | $\|A\|_2 = \sqrt{\lambda_{\mathsf{max}}(A^\top A)}$ | $\mu_2(A) = \lambda_{\mathsf{max}}\left(\frac{A+A^\top}{2}\right)$ |
| $\|x\|_\infty = \max_{i\in\{1,\dots,n\}} \|x_i\|$ | $\|A\|_\infty = \max_{i\in\{1,\dots,n\}} \sum_{j=1}^n \|a_{ij}\|$ | $\mu_\infty(A) = \max_{i\in\{1,\dots,n\}} \left(a_{ii} + \sum_{j=1,j\neq i}^n \|a_{ij}\|\right)$ $=$ max row "absolute sum" of $A$ |

# Continuous-time dynamics and one-sided Lipschitz constants

$$\dot{x} = \mathsf{F}(x) \qquad \text{on } \mathbb{R}^n \text{ with norm } \|\cdot\| \text{ and induced log norm } \mu(\cdot)$$

**One-sided Lipschitz constant**

$$\begin{aligned}\mathsf{osLip}(\mathsf{F}) &= \inf\{b \in \mathbb{R} \text{ such that } [\![\mathsf{F}(x) - \mathsf{F}(y), x - y]\!] \leq b\|x-y\|^2 \quad \text{for all } x, y\} \\ &= \sup_x \mu(D\mathsf{F}(x))\end{aligned}$$

For **scalar map** $f$, $\quad \mathsf{osLip}(f) = \sup_x f'(x)$

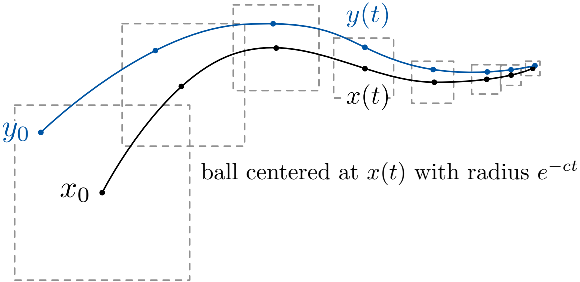
For **affine map** $\mathsf{F}_A(x) = Ax + a$

$$\begin{aligned}\mathsf{osLip}_{2,P}(\mathsf{F}_A) = \mu_{2,P}(A) \leq \ell \quad &\iff \quad A^\top P + AP \preceq 2\ell P \\ \mathsf{osLip}_{\infty,\eta}(\mathsf{F}_A) = \mu_{\infty,\eta}(A) \leq \ell \quad &\iff \quad a_{ii} + \sum_{j \neq i} |a_{ij}|\eta_i/\eta_j \leq \ell\end{aligned}$$

**Banach contraction theorem for continuous-time dynamics:**

If $-c := \mathrm{osLip}(\mathsf{F}) < 0$, then

1. F is **infinitesimally contracting** $=$ distance between trajectories decreases exp fast $(\mathrm{e}^{-ct})$
2. F has a unique, glob exp stable equilibrium $x^*$



# From inner products to weak pairings

A **weak pairing** is $[\![\cdot,\cdot]\!] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfying

1. $[\![x_1 + x_2, y]\!] \leq [\![x_1, y]\!] + [\![x_2, y]\!]$ and $x \mapsto [\![x, y]\!]$ is continuous,
2. $[\![bx, y]\!] = [\![x, by]\!] = b\,[\![x, y]\!]$ for $b \geq 0$ and $[\![-x, -y]\!] = [\![x, y]\!]$,
3. $[\![x, x]\!] > 0$, for all $x \neq \mathbb{0}_n$,
4. $|\,[\![x, y]\!]\,| \leq [\![x, x]\!]^{1/2}\,[\![y, y]\!]^{1/2}$,

Given norm $\|\cdot\|$, compatibility: $[\![x, x]\!] = \|x\|^2$ for all $x$

**Key properties**

Curve norm derivative formula: $\frac{1}{2}D^+\|x(t)\|^2 = [\![\dot{x}(t), x(t)]\!]$

Sup of non-Euclidean numerical range: $\mu(A) = \sup_{\|x\|=1} [\![Ax, x]\!]$

A. Davydov, S. Jafarpour, and F. Bullo. Non-Euclidean contraction theory for robust nonlinear stability. *IEEE Transactions on Automatic Control*, 67(12):6667–6681, 2022a.

# Example weak pairings

| Norms | From inner products to sign and max pairings | From LMIs to log norms |
|---|---|---|
| $\|x\|_{2,P^{1/2}}^2 = x^\top P x$ | $[\![x,y]\!]_{2,P^{1/2}} = x^\top P y$ | $\mu_{2,P^{1/2}}(A) = \min\{b \mid A^\top P + PA \preceq 2bP\}$ |
| $\|x\|_1 = \sum_i \|x_i\|$ | $[\![x,y]\!]_1 = \|y\|_1 \operatorname{sign}(y)^\top x$ | $\mu_1(A) = \max_j \Big(a_{jj} + \sum_{i \neq j} \|a_{ij}\|\Big)$ |
| $\|x\|_\infty = \max_i \|x_i\|$ | $[\![x,y]\!]_\infty = \max_{i \in I_\infty(y)} y_i x_i$ | $\mu_\infty(A) = \max_i \Big(a_{ii} + \sum_{j \neq i} \|a_{ij}\|\Big)$ |

where $I_\infty(x) = \{i \in \{1,\dots,n\}$ such that $|x_i| = \|x\|_\infty\}$

# Table of continuous-time contractivity conditions

| Log Norm bound | Demidovich condition | One-sided Lipschitz condition |
|---|---|---|
| $\mu_{2,P}(D\mathsf{F}(x)) \leq b$ | $PD\mathsf{F}(x) + D\mathsf{F}(x)^\top P \preceq 2bP$ | $(x-y)^\top P\big(\mathsf{F}(x) - \mathsf{F}(y)\big) \leq b\\|x-y\\|^2_{P^{1/2}}$ |
| $\mu_1(D\mathsf{F}(x)) \leq b$ | $\operatorname{sign}(v)^\top D\mathsf{F}(x)v \leq b\\|v\\|_1$ | $\operatorname{sign}(x-y)^\top(\mathsf{F}(x) - \mathsf{F}(y)) \leq b\\|x-y\\|_1$ |
| $\mu_\infty(D\mathsf{F}(x)) \leq b$ | $\max_{i\in I_\infty(v)} v_i\,(D\mathsf{F}(x)v)_i \leq b\\|v\\|^2_\infty$ | $\max_{i\in I_\infty(x-y)} (x_i - y_i)(\mathsf{F}_i(x) - \mathsf{F}_i(y)) \leq b\\|x-y\\|^2_\infty$ |

Equivalent contractivity conditions


J. A. Jacquez and C. P. Simon. Qualitative theory of compartmental systems. *SIAM Review*, 35(1):43–79, 1993.

H. Qiao, J. Peng, and Z.-B. Xu. Nonlinear measures: A new approach to exponential stability analysis for Hopfield-type neural networks. *IEEE Transactions on Neural Networks*, 12(2):360–370, 2001.

G. Como, E. Lovisari, and K. Savla. Throughput optimality and overload behavior of dynamical flow networks under monotone distributed routing. *IEEE Transactions on Control of Network Systems*, 2(1):57–67, 2015.

## Advantages of non-Euclidean approaches

1. *well suited for certain class of systems*
   $\ell_1$ for monotone flow systems
2. *computational advantages*
   $\ell_1/\ell_\infty$ constraints lead to LPs, whereas $\ell_2$ constraints leads to LMIs
3. *robustness to structural perturbations*
   $\ell_1/\ell_\infty$ contractions are connectively robust (i.e., edge removal)
4. *adversarial input-output analysis*
   $\ell_\infty$ better suited for the analysis of adversarial examples than $\ell_2$
5. *reachability analysis via mixed-monotone embeddings*
   $\ell_\infty$ suited for mixed-monotone embeddings
6. *asynchronous distributed computation*
   $\ell_\infty$ contractions converge under fully asynchronous distributed execution

NonEuclidean contractions: biological transcriptional systems (Russo et al., 2010), Hopfield neural networks (Fang and Kincaid, 1996; Qiao et al., 2001), chemical reaction networks (Al-Radhawi and Angeli, 2016), traffic networks (Coogan and Arcak, 2015; Como et al., 2015; Coogan, 2019), multi-vehicle systems (Monteil et al., 2019), and coupled oscillators (Russo et al., 2013; Aminzare and Sontag, 2014a)

# Contraction dynamics on Riemannian manifolds

Contraction theory on Riemannian manifolds originates in


W. Lohmiller and J.-J. E. Slotine. On contraction analysis for non-linear systems. *Automatica*, 34(6):683–696, 1998.


A formal coordinate-free analysis (with connection to monotone operators) is given in


J. W. Simpson-Porco and F. Bullo. Contraction theory on Riemannian manifolds. *Systems & Control Letters*, 65:74–80, 2014.

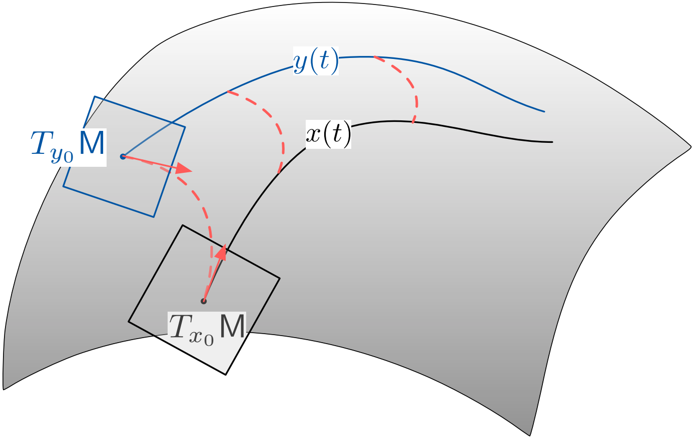

In the differential geometry literature, geodesically monotonic vector fields are studied by


S. Z. Németh. Geodesic monotone vector fields. *Lobachevskii Journal of Mathematics*, 5:13–28, 1999. URL **http://mi.mathnet.ru/eng/ljm145**

J. X. Da Cruz Neto, O. P. Ferreira, and L. R. Lucambio Pérez. Contributions to the study of monotone vector fields. *Acta Mathematica Hungarica*, 94(4):307–320, 2002.

J. H. Wang, G. López, V. Martín-Márquez, and C. Li. Monotone and accretive vector fields on Riemannian manifolds. *Journal of Optimization Theory and Applications*, 146(3):691–708, 2010.

Assume existence and uniqueness of geodesic curve between each $(x, y)$
F **contracting** if geodesic distances from $x$ to $y$ diminishes along the flow of F




**integral test:** the inner product between F and the geodesic velocity vector $\gamma'$ at $x$ and $y$
**differential test:** condition on covariant differential of F

Given vector field $\mathsf{F}$ on a Riemannian manifold $(\mathsf{M}, \mathbb{G})$ and $c > 0$, equivalent statements:

1. **integral condition:** for each $x, y \in \mathsf{M}$ and geodesic $\gamma : [0,1] \to \mathsf{M}$ with $\gamma(0) = x$, $\gamma(1) = y$,

$$\langle\!\langle \mathsf{F}(y), \gamma'(1) \rangle\!\rangle_{\mathbb{G}} - \langle\!\langle \mathsf{F}(x), \gamma'(0) \rangle\!\rangle_{\mathbb{G}} \leq -c\, \mathrm{d}_{\mathbb{G}}(x, y)^2$$

   or, equivalently, using the parallel transport map $P_{y \to x} : T_y\mathsf{M} \to T_x\mathsf{M}$,

$$\langle\!\langle P_{y \to x}\mathsf{F}(y) - \mathsf{F}(x), \gamma'(0) \rangle\!\rangle_{\mathbb{G}} \leq -c\, \mathrm{d}_{\mathbb{G}}(x, y)^2$$

2. *differential condition:* for all $v_x \in T_x\mathsf{M}$

$$\langle\!\langle \nabla_{v_x}\mathsf{F}(x), v_x \rangle\!\rangle_{\mathbb{G}} \leq -c\|v_x\|_{\mathbb{G}}^2,$$

   where $\nabla$ is the Levi-Civita connection. In components:

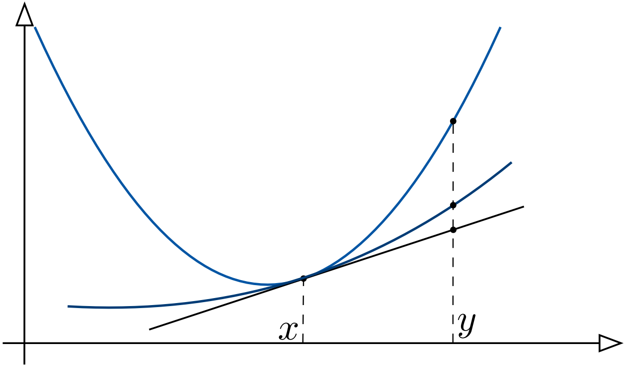
$$\mathbb{G}(x)D\mathsf{F}(x) + D\mathsf{F}(x)^\top\mathbb{G}(x) + \mathcal{L}_{\mathsf{F}}\mathbb{G}(x) \preceq -2c\mathbb{G}(x)$$

3. *trajectory condition:* for all solutions $x(\cdot), y(\cdot)$

$$D^+\mathrm{d}_{\mathbb{G}}(x(t), y(t)) \leq -c\, \mathrm{d}_{\mathbb{G}}(x(t), y(t))$$

# Outline

# Optimization-based dynamics




$$V(y) \geq V(x) + \operatorname{grad} V(x)^\top (y - x) + \frac{m}{2}\|x - y\|_2^2$$

# Example #1: Gradient flow for strongly convex function

Given strongly convex $f : \mathbb{R}^n \to \mathbb{R}$ with parameter $\mu$, **gradient dynamics**

$$\dot{x} = \mathsf{F_G}(x) := -\nabla f(x)$$

**$\mathsf{F_G}$ is infinitesimally contracting wrt $\|\cdot\|_2$ with rate $\mu$**

unique globally exp stable point is global minimum

If $f$ is twice-differentiable, then $\mathrm{Hess}\, f(x) \succeq \mu I_n$ for all $x$

$$D(-\nabla f)(x) = -\mathrm{Hess}\, f(x) \preceq -\mu I_n$$

$$\iff \qquad I_n\, D(-\nabla f)(x) + D(-\nabla f)(x)^\top I_n \preceq -2\mu I_n$$

# Convexity and contractivity

**Kachurovskii's Theorem**: For differentiable $f : \mathbb{R}^n \to \mathbb{R}$, equivalent statements:

1. $f$ is **strongly convex** with parameter $m$
2. $-\operatorname{grad} f$ is $m$-**strongly infinitesimally contracting**, that is

$$\big(-\operatorname{grad} f(x) + \operatorname{grad} f(y)\big)^\top (x-y) \leq -m\|x-y\|_2^2$$

Also: global minimum of $f$ = globally-exponentially stable equilibrium of $-\nabla f$

For map $\mathsf{F} : \mathbb{R}^n \to \mathbb{R}^n$, equivalent statements:

1. $\mathsf{F}$ is a **monotone operator**[a] (or a **coercive operator**) with parameter $m$,
2. $-\mathsf{F}$ is $m$-**strongly contracting**

[a] $\mathsf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a *$m$-strongly monotone operator* if $\langle\!\langle \mathsf{F}(x) - \mathsf{F}(y), x-y \rangle\!\rangle \geq m\|x-y\|_2^2$

R. I. Kachurovskii. Monotone operators and convex functionals. *Uspekhi Matematicheskikh Nauk*, 15(4):213–215, 1960

# Example #2: Saddle dynamics

Assume $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$

- $x \mapsto f(x, y)$ is $\mu_x$-strongly convex, uniformly in $y$
- $y \mapsto f(x, y)$ is $\mu_y$-strongly concave, uniformly in $x$

**saddle dynamics (primal-descent / dual-ascent):**

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \mathsf{F}_{\mathsf{S}}(x, y) := \begin{bmatrix} -\nabla_x f(x, y) \\ \nabla_y f(x, y) \end{bmatrix}$$

**$\mathsf{F}_{\mathsf{S}}$ is infinitesimally contracting wrt $\|\cdot\|_2$ with rate $\min\{\mu_x, \mu_y\}$**

unique globally exp stable point is saddle point (min in $x$, max in $y$)

If $f$ is twice-differentiable, then

$$\mu_2(D\mathsf{F}_{\mathsf{S}}(x, y)) = \mu_2\left(\begin{bmatrix} -\operatorname{Hess}_x f(x, y) & -D_y \nabla_x f(x, y) \\ D_x \nabla_y f(x, y) & \operatorname{Hess}_y f(x, y) \end{bmatrix}\right)$$

$$\overset{\mu_2(A) = \mu_2(\frac{A + A^\top}{2})}{=} \mu_2\left(\begin{bmatrix} -\operatorname{Hess}_x f(x, y) & 0 \\ 0 & \operatorname{Hess}_y f(x, y) \end{bmatrix}\right) = -\min\{\mu_x, \mu_y\}.$$

# Example #2 generalized: Pseudogradient dynamics

Each player $i$ aims to minimize its own cost function $J_i(x_i, x_{-i})$ (not a potential game)

**pseudogradient dynamics (aka gradient play in game theory):**

$$\dot{x}_i = -\nabla_i J_i(x_i, x_{-i})$$

that is, $\dot{x} = \mathsf{F}_{\mathsf{PseudoG}}(x) = -(\nabla_1 J_1(x_1, x_{-1}), \dots, \nabla_n J_n(x_n, x_{-n}))$ (stacked vector)

if $\mathsf{F}_{\mathsf{PseudoG}}$ is infinitesimally contracting (wrt any norm and any rate)

**unique globally exp stable Nash equilibrium** $J_i(x_i^*, x_{-i}^*) \leq J_i(y_i, x_{-i}^*)$ for all $y_i$

Sufficient conditions:

1. **strong convexity of each $x_i \mapsto J_i(x_i, x_{-i})$, uniformly in $x_{-i}$**, and
2. **small-gain condition** in "network contraction theorem" (see later slide)

# Example #3: Primal-dual gradient dynamics

strongly convex function $f$ s.t. $0 \prec \mu_{\mathsf{min}} I_n \preceq \mathrm{Hess}\, f \preceq \mu_{\mathsf{max}} I_n$

constraint matrix $A$ s.t. $0 \prec a_{\mathsf{min}} I_m \preceq AA^\top \preceq a_{\mathsf{max}} I_m$

**linearly constrained optimization:**

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

**primal-dual gradient dynamics:**

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \mathsf{F}_{\mathsf{PDG}}(x, \lambda) := \begin{bmatrix} -\nabla f(x) - A^\top \lambda \\ Ax - b \end{bmatrix}$$

**$\mathsf{F}_{\mathsf{PDG}}$ is infinitesimally contracting wrt weighted $\|\cdot\|_{2,P^{1/2}}$ with rate $c$**

$$P = \begin{bmatrix} I_n & \alpha A^\top \\ \alpha A & I_m \end{bmatrix}, \ \alpha = \frac{1}{3} \min\left\{\frac{1}{\mu_{\mathsf{max}}}, \frac{\mu_{\mathsf{min}}}{a_{\mathsf{max}}}\right\}, \quad \text{and} \quad c = \frac{5}{18} \min\left\{\frac{a_{\mathsf{min}}}{\mu_{\mathsf{max}}}, \frac{a_{\mathsf{min}}}{a_{\mathsf{max}}} \mu_{\mathsf{min}}\right\}$$

For each $\mu_{\mathsf{min}} I_n \preceq Q \preceq \mu_{\mathsf{max}} I_n$, $\quad \begin{bmatrix} -Q & -A^\top \\ A & 0 \end{bmatrix}^\top P + P \begin{bmatrix} -Q & -A^\top \\ A & 0 \end{bmatrix} \preceq -2cP$

# Example: Distributed optimization from primal-dual gradient descent

Consider a tree (undirected acyclic connected graph) with $n$ nodes and $m = n-1$ edges:
Let $A^\top$ = oriented incidence matrix, and $\lambda_2, \dots, \lambda_n$ = Laplacian eigenvalues. Then:

$$0 \prec \lambda_2 I_{n-1} \preceq AA^\top \preceq \lambda_n I_{n-1}$$

**decomposable optimization:** Rewrite $\min_{x \in \mathbb{R}^n} f(x)$ when $f(x) = \sum_i f_i(x)$ as

$$\begin{aligned} \min_{x_i \in \mathbb{R}^n} \quad & \sum_{i=1}^n f_i(x_i) \\ \text{s.t.} \quad & x_i = x_j \qquad \text{for each edge } e = (i,j) \end{aligned}$$

**distributed optimization via primal-dual gradient dynamics:**

$$\begin{cases} \dot{x}_i & = -\nabla_i f_i(x_i) - \sum_{e=(i,j)} \lambda_e + \sum_{e=(j,i)} \lambda_e \\ \dot{\lambda}_e & = x_i - x_j \qquad \text{for each edge } e = (i,j) \end{cases}$$

assume dual dynamics is fast and each $f_i$ is $\mu_i$-strongly convex

**$\mathsf{F}_{\mathsf{PDG}}$ is infinitesimally contracting with** $c = \dfrac{5}{18} \dfrac{\lambda_2}{\lambda_n} \min_i \mu_i$

# Composite minimization and proximal gradient

For strongly convex + strongly smooth $f$, convex, closed, proper $g : \mathbb{R}^n \to \overline{\mathbb{R}}$,

$$x^\star = \operatorname*{argmin}_{x \in \mathbb{R}^n} f(x) + g(x) \quad \Longleftrightarrow \quad x^\star = \operatorname{prox}_{\gamma g}(x^\star - \gamma \nabla f(x))$$

where $\operatorname{prox}_{\gamma g}(z) = \operatorname*{argmin}_{x \in \mathbb{R}^n} g(x) + \frac{1}{2\gamma}\|x - z\|_2^2$.

1. minimization problem

$$\min_{x \in \mathbb{R}^n} f(x) + g(x)$$

2. is transcribed into strongly infinitesimally contracting *proximal gradient* dynamics

$$\dot{x} = \mathsf{F}_{\mathsf{ProxG}}(x) := -x + \operatorname{prox}_{\gamma g}(x - \gamma \nabla f(x))$$

# Example #4: Proximal gradient dynamics

**proximal gradient dynamics**:

$$\dot{x} = \mathsf{F}_{\mathsf{ProxG}}(x) := -x + \mathrm{prox}_{\gamma g}(x - \gamma \nabla f(x))$$

$f$ is $m$-strongly convex and $\ell$-strongly smooth

1. if $0 < \gamma < \dfrac{2}{\ell}$, then $\mathsf{F}_{\mathsf{PDG}}$ **is infinitesimally contracting w.r.t.** $\|\cdot\|_2$ **with rate** $c$

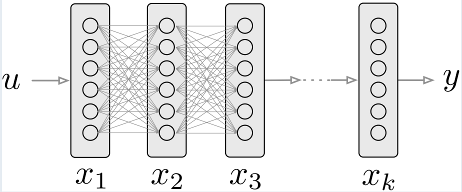
$$c = 1 - \max\{|1 - \gamma m|, |1 - \gamma \ell|\}$$

   and maximal rate at $\gamma^\star = \frac{2}{m+\ell}$

2. if $f(x) = \frac{1}{2}x^\top A x + b^\top x$ with $A \succ 0$ and $\gamma > 1/\lambda_{\min}(A)$,
   then $\mathsf{F}_{\mathsf{PDG}}$ **is infinitesimally contracting w.r.t.** $\|\cdot\|_{2,(\gamma A - I_n)^{1/2}}$ **with rate** $c = 1$

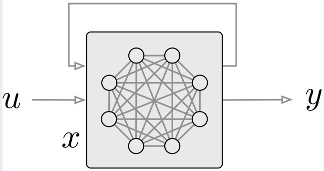
# Neural network models

**Feedforward NN**




$$x_{i+1} = \Phi(A_i x_i + b_i), \quad x_0 = u,$$
$$y = C x_k + d$$

**Recurrent NN**

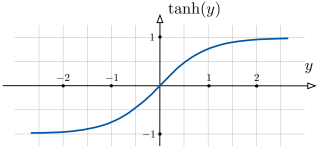


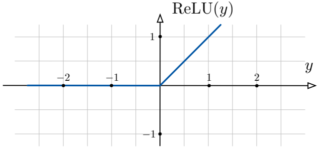

$$\dot{x} = -x + \Phi(Ax + Bu + b),$$
$$y = Cx + d$$

# Example #5: Firing-rate recurrent neural network

$$\dot{x} = \mathsf{F}_{\mathsf{FR}}(x) := -x + \Phi(Wx + Bu)$$

sigmoid, hyperbolic tangent
ReLU $= \max\{x, 0\} = (x)_+$
$0 \leq \Phi_i'(y) \leq 1$




**$\mathsf{F}_{\mathsf{FR}}$ is infinitesimally contracting wrt $\|\cdot\|_\infty$ with rate $1 - \mu_\infty(W)_+$** if

$$\mu_\infty(W) < 1 \qquad \left(\text{i.e., } w_{ii} + \textstyle\sum_j |w_{ij}| < 1 \text{ for all } i\right)$$

$$\begin{aligned}
\mathsf{osLip}_\infty(\mathsf{F}_{\mathsf{FR}}) &= \sup_{x,u} \mu_\infty\big(-I_n + (D\Phi(Wx+Bu))W\big) = -1 + \sup_{x,u} \mu_\infty\big(D\Phi(Wx+Bu)W\big) \\
&= -1 + \max_{d \in [0,1]^n} \mu_\infty(\mathrm{diag}(d)W) \qquad \text{(max convex polytope, } 2^n \text{ vertices)} \\
&= -1 + \max\big\{\mu_\infty(0), \mu_\infty(W)\big\} = -1 + \mu_\infty(W)_+
\end{aligned}$$

# Example #6: Firing-rate network with symmetric synapses

$$\dot{x} = \mathsf{F}_{\mathsf{FR}}(x) := -x + \Phi(Wx + Bu)$$

$$0 \leq \Phi_i'(y) \leq 1 \qquad \text{and} \qquad W = W^\top \text{ with } \lambda_W = \lambda_{\max}(W)$$

**$\mathsf{F}_{\mathsf{FR}}$ is infinitesimally contracting**:

| | |
|---|---|
| (for $\lambda_W < 0$) | **with rate** $1$ **wrt** $\|\cdot\|_{2,(-W)^{1/2}}$ |
| (for $\lambda_W = 0$) | **with rate** $\|\cdot\|_{2,Q_{\mathsf{FR},\epsilon}}$, **for each** $\epsilon > 0$ |
| (for $0 < \lambda_W < 1$) | **with rate** $1 - \lambda_W$ **wrt** $\|\cdot\|_{2,Q_{\mathsf{FR},\lambda_W}}$ |

For $\lambda_W = 1$, **$\mathsf{F}_{\mathsf{FR}}$ is weakly infinitesimally contracting wrt** $\|\cdot\|_{2,Q_{\mathsf{FR},\lambda_W}}$

- $Q_{\mathsf{FR},a} := Uh_a(\Lambda)U^\top \succ 0$, where $W = U\Lambda U^\top$ and $h_a(z) := 2a\big(1 + \sqrt{1 - z/a}\big)$
- optimal rates
- proof based upon LMI calculations and Sylvester's law of inertia

# Outline

# Equilibrium and Lyapunov functions

**Equilibria of contracting vector fields:**

For a time-invariant F, $c$-strongly contracting wrt $\|\cdot\|$

1. for each $t > 0$, $t-$flow of F is a contraction,
   i.e., distance between solutions exponentially decreases with rate $c$
2. there exists an equilibrium $x^*$, that is unique, globally exponentially stable with global Lyapunov functions

$$x \mapsto V_1(x) = \|x - x^*\|^2 \quad \text{and} \quad x \mapsto V_2(x) = \|\mathsf{F}(x)\|^2$$

For a time-invariant F,

1. $\mathrm{osLip}(\mathsf{F}) = -c$ wrt $\ell_2$ and $D\mathsf{F}(x) = D\mathsf{F}(x)^\top$ for all $x$,
2. for each scalar $w$,

$$V_3(x) = -\int_0^1 x^\top \mathsf{F}(tx)dt + w$$

   is $c$-strongly convex, is global Lyapunov, and $\mathrm{grad} V_3(x) = -\mathsf{F}(x)$ for all $x$.

Fot time and input-dependent vector F,

$$\dot{x} = \mathsf{F}(t, x, u(t)), \qquad x(0) = x_0 \in \mathcal{X}, \qquad u(t) \in \mathcal{U} \tag{1}$$

Given norms $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{U}}$, assume constants $c, \ell > 0$ s.t.

- **osLip wrt $x$**: $\mathrm{osLip}_x(\mathsf{F}) \leq -c < 0$, uniformly in $t, u$
- **Lip wrt $u$**: $\mathrm{Lip}_u(\mathsf{F}) \leq \ell$, uniformly in $t, x$

Then

1. any soltns: $x(t)$ with input $u_x$ and $y(t)$ with input $u_y$

$$D^+\|x(t)-y(t)\|_{\mathcal{X}} \;\leq\; -c\|x(t)-y(t)\|_{\mathcal{X}} \;+\; \ell\|u_x(t)-u_y(t)\|_{\mathcal{U}}$$

2. F is **incrementally ISS**, that is, for all $x_0, y_0$

$$\|x(t)-y(t)\|_{\mathcal{X}} \;\leq\; e^{-ct}\|x_0-y_0\|_{\mathcal{X}} \;+\; \frac{\ell(1-e^{-ct})}{c}\sup_{\tau\in[0,t]}\|u_x(\tau)-u_y(\tau)\|_{\mathcal{U}}$$

3. F has **incremental $\mathcal{L}^q_{\mathcal{X},\mathcal{U}}$ gain equal to $\ell/c$, for $q\in[1,\infty]$**,

$$\|x(\cdot)-y(\cdot)\|_{\mathcal{X},q} \;\leq\; \frac{\ell}{c}\,\|u_x(\cdot)-u_y(\cdot)\|_{\mathcal{U},q} \qquad (\text{for } x_0=y_0)$$

Given norm $\|\cdot\|_{\mathcal{X}}$ on $\mathbb{R}^n$ (or $\|\cdot\|_{\mathcal{U}}$ on $\mathbb{R}^k$),

- $\mathcal{L}^q_{\mathcal{X}}$, $q \in [1, \infty]$, is vector space of continuous signals, $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, with well-defined bounded norm

$$\|x(\cdot)\|_{\mathcal{X},q} = \begin{cases} \left( \int_0^\infty \|x(t)\|^q_{\mathcal{X}} dt \right)^{1/q} & \text{if } q \in [1, \infty[ \\ \sup_{t \geq 0} \|x(t)\|_{\mathcal{X}} & \text{if } q = \infty \end{cases} \tag{2}$$

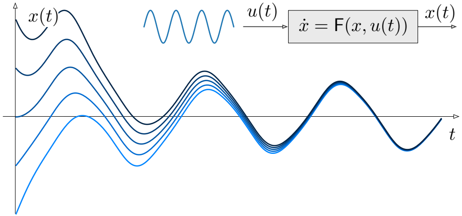
- Input-state system has **$\mathcal{L}^q_{\mathcal{X},\mathcal{U}}$-induced gain** upper bounded by $\gamma > 0$ if, for all $u \in \mathcal{L}^q_{\mathcal{U}}$, the state $x$ from zero initial state satisfies

$$\|x(\cdot)\|_{\mathcal{X},q} \leq \gamma \, \|u(\cdot)\|_{\mathcal{U},q} \tag{3}$$

# From time-invariant to periodic systems

For time-varying vector field F and norm $\|\cdot\|$

1. $\mathrm{osLip}_x(\mathsf{F}) \leq -c < 0$
2. F is $T$-periodic



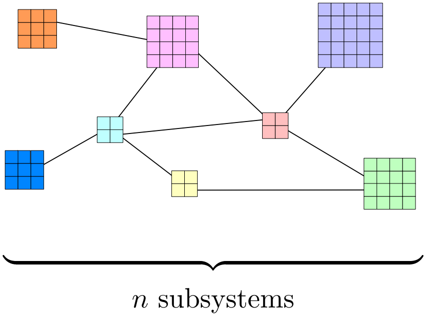

Then

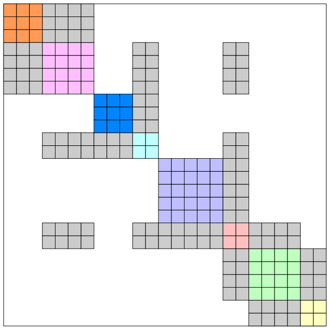
1. there exists a unique periodic solution $x^* : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ with period $T$
2. for every initial condition $x_0$,

$$\|x(t, x_0) - x^*(t)\| \leq \mathrm{e}^{-ct}\|x_0 - x^*(0)\| \tag{4}$$


G. Russo, M. Di Bernardo, and E. D. Sontag. Global entrainment of transcriptional systems to periodic inputs. *PLoS Computational Biology*, 6(4):e1000739, 2010.

# Composite norms



1. $n$ *local norms* $\|\cdot\|_i$ on $\mathbb{R}^{N_i}$
2. an *aggregating norm* $\|\cdot\|_{\mathrm{agg}}$ on $\mathbb{R}^n$
3. *composite norm*


G. Russo, M. Di Bernardo, and E. D. Sontag. A contraction approach to the hierarchical analysis and design of networked systems. *IEEE Transactions on Automatic Control*, 58(5):1328–1331, 2013.

# Networks of contracting systems

Interconnected subsystems: $x_i \in \mathbb{R}^{N_i}$ and $x_{-i} \in \mathbb{R}^{N-N_i}$:

$$\dot{x}_i = \mathsf{F}_i(x_i, x_{-i}), \qquad \text{for } i \in \{1, \dots, n\}$$

**Network contraction theorem**

- **osLip wrt $x_i$**: $\mathrm{osLip}_{x_i}(\mathsf{F}_i) \leq -c_i$, uniformly in $x_{-i}$
- **Lip wrt to $x_j$**: $\mathrm{Lip}_{x_j}(\mathsf{F}_i) \leq \ell_{ij}$, uniformly in $x_{-j}$
- the Lipschitz constants matrix $\begin{bmatrix} -c_1 & \dots & \ell_{1n} \\ \vdots & & \vdots \\ \ell_{n1} & \dots & -c_n \end{bmatrix}$ is **Hurwitz**

$\Longrightarrow$ the **interconnected system** is infinitesimally contracting

# The network science of Metzler Hurwitz matrices

$$\begin{bmatrix} -c_1 & \dots & \ell_{1n} \\ \vdots & & \vdots \\ \ell_{n1} & \dots & -c_n \end{bmatrix}$$ is **Metzler** (so that Perron-Frobenius Theorem applies)

**Hurwitzness depends upon both topology and edge weights**

- directed acyclic interconnections of contracting systems are strongly contracting
- For $n = 2$, Hurwitz if and only if **small gain condition**

$$\text{cycle gain} := \frac{\ell_{12}}{c_1}\frac{\ell_{21}}{c_2} < 1$$

- For $n \geq 3$, Hurwitz if and only if **network small-gain theorem for Metzler matrices**

**Hurwitz Metzler Theorem**

1. $M$ is Hurwitz,
2. there exists $\eta \in \mathbb{R}^n_{>0}$ such that $\eta^\top M < \mathbb{0}_n^\top$ or, equivalently, $\mu_{1,[\eta]}(M) < 0$,
3. there exists $\xi \in \mathbb{R}^n_{>0}$ such that $M\xi < \mathbb{0}_n$ or, equivalently, $\mu_{\infty,[\xi]^{-1}}(M) < 0$, and
4. there exists a diagonal $P = P^\top \succ 0$ satisfying $M^\top P + PM \prec 0$ or, equivalently, $\mu_{2,P^{1/2}}(M) < 0$.

**Input:** a Metzler matrix $M \in \mathbb{R}^{n\times n}$
**Output:** polynomials $\{\gamma_{\mathcal{C}_2}, \dots, \gamma_{\mathcal{C}_n}\}$ in entries of $M$

1: $\mathcal{C}$ := set of simple cycles of digraph associated to $M$
2: $\gamma_\phi$ := gain of cycle $\phi \in \mathcal{C}$
3: **for** $i$ from $2$ to $n$
4: $\quad\mathcal{C}_i$ := cycles in $\mathcal{C}$ passing through only nodes $1, \dots, i$
5: $\quad\gamma_{\mathcal{C}_i} := \sum_{\phi\in\mathcal{C}_i} \gamma_\phi - \sum_{\substack{\phi,\psi\in\mathcal{C}_i \\ \phi\perp\psi}} \gamma_\phi\gamma_\psi + \sum_{\substack{\phi,\psi,\rho\in\mathcal{C}_i \\ \phi\perp\psi,\ \phi\perp\rho,\ \psi\perp\rho}} \gamma_\phi\gamma_\psi\gamma_\rho - \cdots$

**Network small-gain theorem for Metzler matrices**

$$\text{Metzler } M \text{ is Hurwitz} \qquad \iff \qquad \gamma_{\mathcal{C}_2} < 1, \cdots, \gamma_{\mathcal{C}_n} < 1$$

- not unique: distinct/equivalent conditions after renumbering, redundancy
- computational efficiency: after precomputation of simple cycles

X. Duan, S. Jafarpour, and F. Bullo. Graph-theoretic stability conditions for Metzler matrices and monotone systems. *SIAM Journal on Control and Optimization*, 59(5):3447–3471, 2021.

$$M = \begin{bmatrix} -c_1 & 0 & 0 & \ell_{14} \\ 0 & -c_2 & \ell_{23} & \ell_{24} \\ 0 & \ell_{32} & -c_3 & \ell_{34} \\ \ell_{41} & \ell_{42} & \ell_{43} & -c_4 \end{bmatrix}$$

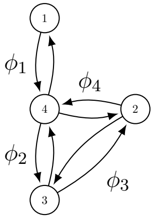



Figure: associated digraph and simple cycles

- $\gamma_{\phi_1} = \frac{\ell_{14}\ell_{41}}{c_1 c_4}$, $\gamma_{\phi_2} = \frac{\ell_{34}\ell_{43}}{c_3 c_4}$, $\gamma_{\phi_3} = \frac{\ell_{23}\ell_{32}}{c_2 c_3}$, and $\gamma_{\phi_4} = \frac{\ell_{24}\ell_{42}}{c_2 c_4}$
- $\mathcal{C}_2 = \emptyset$
- $\mathcal{C}_3 = \{\phi_3\}$: $\gamma_{\mathcal{C}_3} = \gamma_{\phi_3} < 1$ (redundant)
- $\mathcal{C}_4 = \{\phi_1, \ldots, \phi_4\}$: $\gamma_{\mathcal{C}_4} = \sum_{i=1}^{4} \gamma_{\phi_i} - \gamma_{\phi_1}\gamma_{\phi_3} < 1$

$$
\begin{bmatrix}
-c_1 & 0 & 0 & 0 & \ell_{15} & \ell_{16} \\
0 & -c_2 & 0 & \ell_{24} & \ell_{25} & 0 \\
0 & 0 & -c_3 & \ell_{34} & 0 & \ell_{36} \\
0 & \ell_{42} & \ell_{43} & -c_4 & 0 & 0 \\
\ell_{51} & \ell_{52} & 0 & 0 & -c_5 & 0 \\
\ell_{61} & 0 & \ell_{63} & 0 & 0 & -c_6
\end{bmatrix}
$$

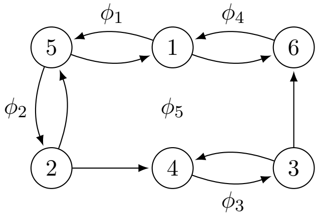



Figure: associated digraph and simple cycles

- $\mathcal{C}_2$, $\mathcal{C}_3$ empty
- $\mathcal{C}_4 = \{\phi_3\}$: $\gamma_3 < 1$ (redundant)
- $\mathcal{C}_5 = \{\phi_1, \phi_2, \phi_3\}$: $\gamma_{\mathcal{C}_5} = \gamma_1 + \gamma_2 + \gamma_3 - \gamma_1\gamma_3 - \gamma_2\gamma_3 < 1$
- $\mathcal{C}_6 = \{\phi_1, \ldots, \phi_5\}$: $\gamma_{\mathcal{C}_6} = \sum_{i=1}^{5} \gamma_i - \gamma_1\gamma_3 - \gamma_2\gamma_3 - \gamma_3\gamma_4 - \gamma_2\gamma_4 + \gamma_2\gamma_3\gamma_4 < 1$

# Incremental ISS for strongly contracting delay ODEs

$$\dot{x}(t) = \mathsf{F}\big(x(t), x(t-s), u(t)\big), 0 \le s \le S, \qquad \|\cdot\|_{\mathcal{X}}, \|\cdot\|_{\mathcal{U}} \tag{5}$$

assume there exist positive constants $c$, $\ell_{\mathcal{U}}$, $\ell_{\mathcal{X}}$ such that, for all variables,

$$\textbf{osL } x: \qquad [\![\mathsf{F}(x,d,u) - \mathsf{F}(y,d,u), x-y]\!]_{\mathcal{X}} \le -c\|x-y\|_{\mathcal{X}}^2 \tag{6}$$

$$\textbf{Lip } x(t-s): \qquad \|\mathsf{F}(x,x_1,u) - \mathsf{F}(x,x_2,u)\|_{\mathcal{X}} \le \ell_{\mathcal{X}}\|x_1 - x_2\|_{\mathcal{X}} \tag{7}$$

$$\textbf{Lip } u: \qquad \|\mathsf{F}(x,d,u) - \mathsf{F}(x,d,v)\|_{\mathcal{X}} \le \ell_{\mathcal{U}}\|u-v\|_{\mathcal{U}} \tag{8}$$

By the curve norm derivative formula, subadditivity, and Cauchy-Schwarz inequality,

$$\begin{aligned}
\|x(t)-y(t)\|_{\mathcal{X}} D^{+}\|x(t)-y(t)\|_{\mathcal{X}} &= \big[\!\big[\mathsf{F}(x(t), x(t-s), u_x(t)) - \mathsf{F}(y(t), y(t-s), u_y(t)), x(t)-y(t)\big]\!\big]_{\mathcal{X}} \\
&\le [\![\mathsf{F}(x(t), x(t-s), u_x(t)) - \mathsf{F}(y(t), x(t-s), u_x(t)), x(t)-y(t)]\!]_{\mathcal{X}} \\
&\quad + [\![\mathsf{F}(y(t), x(t-s), u_x(t)) - \mathsf{F}(y(t), y(t-s), u_x(t)), x(t)-y(t)]\!]_{\mathcal{X}} \\
&\quad + \big[\!\big[\mathsf{F}(y(t), y(t-s), u_x(t)) - \mathsf{F}(y(t), y(t-s), u_y(t)), x(t)-y(t)\big]\!\big]_{\mathcal{X}} \\
&\le -c\|x(t)-y(t)\|_{\mathcal{X}}^2 + \ell_{\mathcal{X}}\|x(t-s)-y(t-s)\|_{\mathcal{U}}\|x(t)-y(t)\|_{\mathcal{X}}, \\
&\quad + \ell_{\mathcal{U}}\|u_x(t)-u_y(t)\|_{\mathcal{U}}\|x(t)-y(t)\|_{\mathcal{X}}.
\end{aligned}$$

Thus, with $m(t) = \|x(t)-y(t)\|_{\mathcal{X}}$, delay differential inequality:

$$D^{+}m(t) \le -cm(t) + \ell_{\mathcal{X}} \sup_{0\le s\le S} m(t-s) + \ell_{\mathcal{U}}\|u_x(t)-u_y(t)\|_{\mathcal{U}}, \tag{9}$$

Halanay inequality is applicable. If $c > \ell_{\mathcal{X}}$, then

$$m(t) \le m_0 \mathrm{e}^{-\rho(t-t_0)} + \ell_{\mathcal{U}} \int_{t_0}^{t} \mathrm{e}^{-\rho(t-\tau)}\|u_x(\tau)-u_y(\tau)\|_{\mathcal{U}} d\tau, \tag{10}$$

where $\rho > 0$ is the unique positive root of $\rho = c - \ell_{\mathcal{X}}\mathrm{e}^{\rho S}$ and $m_0 = \sup_{0\le s\le S} m(t_0 - s)$.

# Networks of contracting systems with time delays

Interconnected subsystems $i \in \{1, \dots, n\}$

$$\dot{x}_i = \mathsf{F}_i(x_i, x_{-i}, x_{-i}(t-s), u_i), \qquad 0 \le s \le S, \qquad \|\cdot\|_i, \|\cdot\|_{i,\mathcal{U}} \tag{11}$$

Assume there exist positive constants st

**osL** $x_i$ : $\llbracket \mathsf{F}_i(x_i, \dots) - \mathsf{F}_i(y_i, \dots), x_i - y_i \rrbracket_i \le -c_i \|x_i - y_i\|_i^2$

**Lip** $x_{-i}$ : $\|\mathsf{F}_i(\dots, x_{-i}, \dots) - \mathsf{F}_i(\dots, y_{-i}, \dots)\|_i \le \sum_{j=1, j\neq i}^{n} \gamma_{ij} \|x_j - y_j\|_j$

**Lip** $x_{-1}^{-s}$ : $\|\mathsf{F}_i(\dots, x_{-i}^{-s}, \dots) - \mathsf{F}_i(\dots, y_{-i}^{-s}, \dots)\|_i \le \sum_{j=1, j\neq i}^{n} \widehat{\gamma}_{ij} \|x_j^{-s} - y_j^{-s}\|_j$

**Lip** $u_i$ : $\|\mathsf{F}_i(\dots, u_i) - \mathsf{F}_i(\dots, v_i)\|_i \le \ell_{i,\mathcal{U}} \|u_i - v_i\|_{i,\mathcal{U}}$

With $m_i(t) = \|x_i(t) - y_i(t)\|_i$, delay differential inequality:

$$D^+ m(t) \le -C m(t) + \Gamma m(t) + \widehat{\Gamma} \sup_{0 \le s \le S} m(t-s) + \ell_{\mathcal{U}} \|u_x(t) - u_y(t)\|_{\mathcal{U}}$$

and, if the Metzler matrix $-C + \Gamma + \widehat{\Gamma}$ is Hurwitz, then (11) is incremental ISS

F. Mazenc, M. Malisoff, and M. Krstic. Vector extensions of Halanay's inequality. *IEEE Transactions on Automatic Control*, 67(3):1453–1459, 2022.

# Forward Euler theorem

**Forward Euler theorem for contracting dynamics**

Given arbitrary norm $\|\cdot\|$, equivalent statements

1. $\dot{x} = \mathsf{F}(x)$ is infinitesimally contracting
2. there exists $\alpha > 0$ such that $x_{k+1} = x_k + \alpha \mathsf{F}(x_k)$ is contracting

Given *contraction rate $c$* and *Lipschitz constant $\ell$*, define *condition number $\kappa = \dfrac{\ell}{c} \geq 1$*

1. $\mathsf{Id} + \alpha \mathsf{F}$ is contracting for
$$0 < \alpha < \frac{1}{c\kappa(1+\kappa)}$$
2. the optimal step size minimizing and minimum contraction factor:
$$\alpha^* = \frac{1}{c}\Big(\frac{1}{2\kappa^2} - \frac{3}{8\kappa^3} + \mathcal{O}\Big(\frac{1}{\kappa^4}\Big)\Big)$$
$$\ell^* = 1 - \frac{1}{4\kappa^2} + \frac{1}{8\kappa^3} + \mathcal{O}\Big(\frac{1}{\kappa^4}\Big)$$

## Improved bounds for inner-product norms

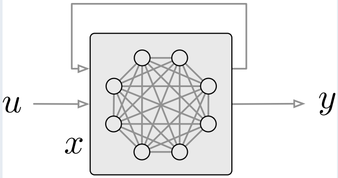
1. the map $\mathsf{Id} + \alpha \mathsf{F}$ is a contraction map wrt $\|\cdot\|_{2,P^{1/2}}$ for

$$0 < \alpha < \frac{2}{c\kappa^2}$$

2. the optimal step size minimizing and minimum contraction factor:

$$\alpha^*_{\mathsf{E}} = \frac{1}{c\kappa^2} \qquad \ell^*_{\mathsf{E}} = 1 - \frac{1}{2\kappa^2} + \mathcal{O}\Big(\frac{1}{\kappa^4}\Big)$$

# Application: $\ell_\infty$-contracting neural networks




$$\dot{x} = -x + \Phi(Ax + Bu + b) \qquad (\textit{recurrent NN})$$

$$x = \Phi(Ax + Bu + b) \qquad (\textit{implicit NN})$$

$$x_{k+1} = (1-\alpha)x_k + \alpha\Phi(Ax_k + Bu + b) \qquad (\textit{forward Euler})$$

If

$$\mu_\infty(A) < 1 \qquad \Big(\text{i.e., } a_{ii} + \sum_j |a_{ij}| < 1 \text{ for all } i\Big)$$

- recurrent NN is contracting with rate $1 - \mu_\infty(A)_+$
- implicit NN is well posed
- forward Euler is contracting with factor $1 - \dfrac{1-\mu_\infty(A)_+}{1-\min_i(a_{ii})_-}$ at $\alpha^* = \dfrac{1}{1-\min_i(a_{ii})_-}$
- input-state Lipschitz constant $\mathrm{Lip}_{u\to x} = \dfrac{\|B\|_\infty}{1-\mu_\infty(A)_+}$

# Outline

# From nominal to uncertain systems

Given a norm $\|\cdot\|$, consider

$$\dot{x} = \mathsf{F}(x) + \Delta(x)$$

Assume:

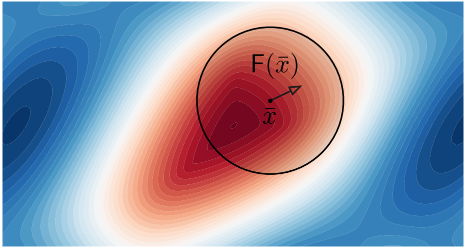
- **contractivity**: $\text{osLip}(\mathsf{F}) \leq -c < 0$
- **bounded disturbance**: $\text{osLip}(\Delta) \leq d < c$

Then

1. $\mathsf{F} + \Delta$ is strongly contracting with rate $c - d$
2. the unique equilibria $x^*_{\mathsf{F}}$ of $\mathsf{F}$ and $x^*_{\mathsf{F}+\Delta}$ of $\mathsf{F} + \Delta$ satisfy

$$\|x^*_{\mathsf{F}} - x^*_{\mathsf{F}+\Delta}\| \leq \frac{\|\Delta(x^*_{\mathsf{F}})\|}{c - d}$$

# From global to local contractivity




Given a norm $\|\cdot\|$, consider

$$\dot{x} = \mathsf{F}(x)$$

Assume:

- **contractivity over closed set $D$**: $\mathrm{osLip}(\mathsf{F}|_D) \leq -c < 0$
- **existence of almost equilibrium**: $D$ contains the closed $B$ at $\bar{x}$ of radius $r \geq \|\mathsf{F}(\bar{x})\|/c$

Then

1. $B$ is forward invariant
2. $\mathsf{F}|_B$ is strongly infinitesimally contracting

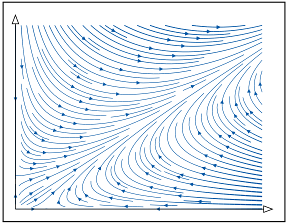
# From strongly to weakly contracting systems

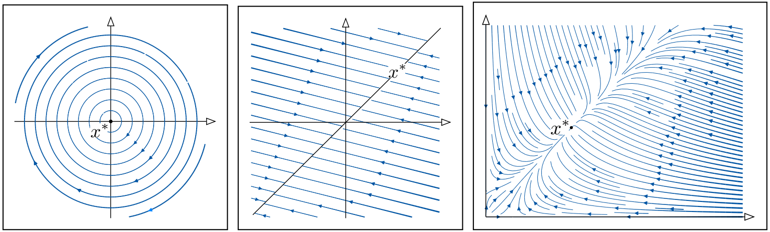
Given a norm $\|\cdot\|$, consider

$$\dot{x} = \mathsf{F}(x) \qquad\qquad \text{satisfying} \qquad\qquad \mathsf{osLip}(\mathsf{F}) = 0$$

**Dichotomy for weakly-contracting systems**

1. no equilibrium and every trajectory is unbounded, or
2. at least one equilibrium, every trajectory is bounded, and local asy stability $\implies$ global





1. **Lotka-Volterra population dynamics** (Lotka, 1920; Volterra, 1928):
   **$\ell_1$-weakly contracting (after a rescaling change of coordinates)**
2. **Matrosov-Bellman interconnected stable systems** (Bellman, 1962; Matrosov, 1962):
   **strongly contracting wrt composite norm**
3. **Kuramoto coupled oscillators** (Kuramoto, 1975):
   **strongly semicontracting wrt $(\ell_2, \Pi_n)$ norm**, in neighb'd of each phase-cohesive equilibrium
4. **Yorke multigroup SIS epidemic model** (Lajmanovich and Yorke, 1976):
   **equilibrium contracting wrt weighted $\ell_1/\ell_\infty$ norms** (at disease-free and endemic eq.)
5. **Hopfield and cellular neural networks** (Hopfield, 1982):
   **$\ell_1/\ell_\infty$-strongly contracting**
6. **Daganzo cell transmission model for traffic networks** (Daganzo, 1994):
   **$\ell_1$-weakly contracting**, when the dynamics is monotone
7. **Chua's diffusively-coupled dynamical systems** (Wu and Chua, 1995):
   **strongly semi-contracting wrt $(2, p)$ tensor norm on $\mathbb{R}^n \otimes \mathbb{R}^k$**
8. ...

# Outline

**contractivity = robust computationally-friendly stability**

fixed point theory + Lyapunov stability theory + geometry of metric spaces

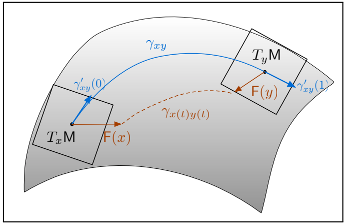
| | **Lyapunov Theory** | **Contraction Theory for Dynamical Systems** |
|---|---|---|
| | F admits global Lyapunov function | F is strongly contracting |
| existence of equilibrium | assumed | implied + computational methods |
| Lyapunov function | arbitrary | $\|x - x^*\|$ and $\|\mathsf{F}(x)\|$ |
| inputs | ISS via $\mathcal{KL}$ and $\mathcal{L}$ functions | iISS via explicit constants |

**search for** contraction properties

**design** engineering systems to be contracting

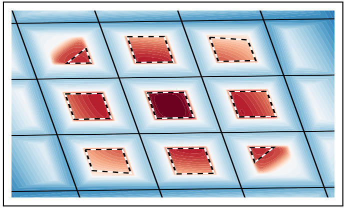
## Theoretical frontiers

- higher order contraction
- relationship with monotone operator theory
- metric spaces
- computational methods



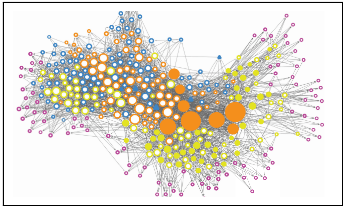

**Limitations**: not all stable systems are contractive:

- Lyapunov-diagonally-stable networks
- multistable and locally contracting systems
- biochemical networks
- control contraction design



## Application to control and learning

1. control: optimization-based control design
2. ML: implicit models and energy-based learning
3. neuroscience: robust dynamical modeling



# Outline



Consider a vector field $\mathsf{F} : \mathbb{R}^n \to \mathbb{R}^n$, and let $\xi, \eta \in \mathbb{R}^n$.

- Invariance property: for all $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$,

$$\mathsf{F}(x + \alpha\xi) = \mathsf{F}(x) \qquad \text{or equivalently} \qquad D\mathsf{F}(x)\xi = \mathbb{0}_n$$

- Conservation property: for all $x, y \in \mathbb{R}^n$,

$$\eta^\top \mathsf{F}(x) = \eta^\top \mathsf{F}(y) \qquad \text{or equivalently} \qquad \eta^\top D\mathsf{F}(x) = \mathbb{0}_n^\top$$

# Prototypical dynamics

Let $A \in \mathbb{R}^{n\times n}$ be row-stochastic: $A\mathbb{1}_n = \mathbb{1}_n$ and $A \geq 0$

### Averaging Systems

$$x_{k+1} = Ax_k$$

Invariance: dynamics unaffected by translations in $\operatorname{span}\{\mathbb{1}_n\}$

Examples: distributed optimization, robotic coordination, frequency synchronization, ...

### Dynamical Flow Systems

$$x_{k+1} = A^\top x_k$$

Conservation: quantity $\mathbb{1}_n^\top x$ is constant

Examples: compartmental models, Markov chains

# Historical starting point

Given row-stochastic $A \in \mathbb{R}^{n\times n}$,

**Markov-Dobrushin ergodic coefficient**

$$\tau_1(A) = \max_{\|z\|_1=1, \mathbb{1}_n^\top z=0} \|A^\top z\|_1$$

$\tau_1(A) < 1$ under mild connectivity conditions

$\tau_p(A)$ also defined for general $p \in [1, \infty]$

**How is $\tau_1$ an induced norm?**


A. A. Markov. Extensions of the law of large numbers to dependent quantities. *Izvestiya Fiziko-matematicheskogo obschestva pri Kazanskom universitete*, 15, 1906. (in Russian)

R. L. Dobrushin. Central limit theorem for nonstationary Markov chains. I. *Theory of Probability & Its Applications*, 1(1):65–80, 1956.

$A \in \mathbb{R}^{n \times n}$ row-stochastic

**Classical Property of Averaging Systems** $x_{k+1} = A x_k$

Given $x \in \mathbb{R}^n$, max-min disagreement:

$$s(Ax) \;\leq\; \tau_1(A)\; s(x), \qquad \text{where} \quad s(x) = \max_i \{x_i\} - \min_j \{x_j\}$$

**Classical Property of Markov Chains** $x_{k+1} = A^\top x_k$

Given $\pi, \sigma$ in the simplex $\Delta_n$, total variation distance:

$$d_{\mathsf{TV}}(A^\top \pi, A^\top \sigma) \;\leq\; \tau_1(A)\; d_{\mathsf{TV}}(\pi, \sigma), \qquad \text{where} \quad d_{\mathsf{TV}}(\pi, \sigma) = \frac{1}{2} \sum_i |\pi_i - \sigma_i|$$

**Why is the same $\tau_1$ relevant in both cases?**

A **seminorm** is a function $|||\cdot||| : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ s.t., $\forall a \in \mathbb{R}$ and $\forall x, y \in \mathbb{R}^n$:

1. (*homogeneity*): $|||ax||| = |a|\,|||x|||$
2. (*subadditivity*): $|||x+y||| \leq |||x||| + |||y|||$

The *kernel* is the vector space:

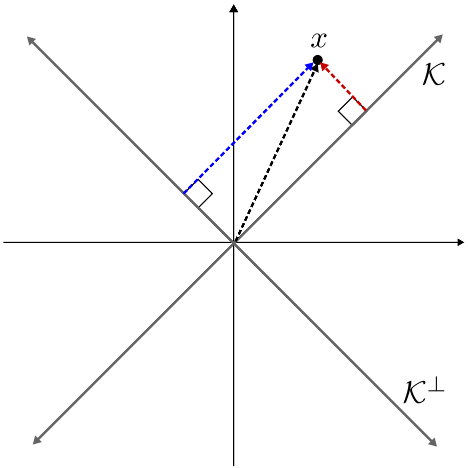
$$\mathcal{K} = \{x \in \mathbb{R}^n : |||x||| = 0\}$$

We focus on *consensus seminorms*, where $\mathcal{K} = \mathrm{span}\{\mathbb{1}_n\}$.

Note: $|||\cdot|||$ is invariant under translations in $\mathcal{K}$

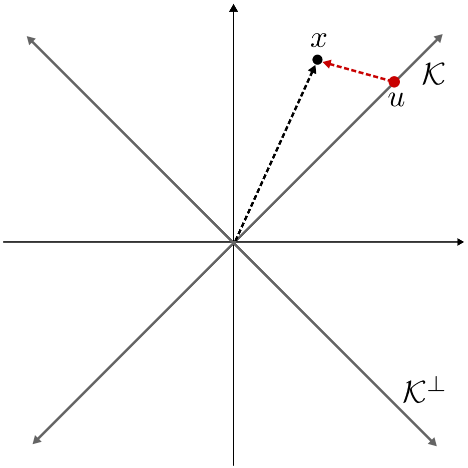
# Projection and distance-based seminorms: graphical definitions

**Projection seminorms**




$$|||x|||_{\mathsf{proj},p} \triangleq \|\Pi_\perp x\|_p, \quad \Pi_\perp = \Pi_\perp^\top$$

**Distance seminorms**




$$|||x|||_{\mathsf{dist},p} \triangleq \min_{u\in\mathcal{K}} \|x-u\|_p$$

# Consensus seminorms

When $\mathcal{K} = \mathrm{span}\{\mathbb{1}_n\}$, **consensus seminorms**

| | $\lvert\lvert\lvert x \rvert\rvert\rvert_{\mathsf{proj},p}$ | $\lvert\lvert\lvert x \rvert\rvert\rvert_{\mathsf{dist},p}$ |
|---|---|---|
| $\ell_1$ | $\displaystyle\sum_{i=1}^{n} \lvert x_i - x_{\mathrm{avg}} \rvert$ | $\displaystyle\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} x_{(i)} - \sum_{j=\lceil \frac{n}{2} \rceil + 1}^{n} x_{(j)}$ |
| $\ell_2$ | $\sqrt{\frac{1}{n} \sum_{i,j} (x_i - x_j)^2}$ | $\sqrt{\frac{1}{n} \sum_{i,j} (x_i - x_j)^2}$ |
| $\ell_\infty$ | $\max_i \lvert x_i - x_{\mathrm{avg}} \rvert$ | $\frac{1}{2}\big(x_{(1)} - x_{(n)}\big)$ |

where we have sorted $x_{(1)} \geq x_{(2)} \geq \cdots \geq x_{(n)}$

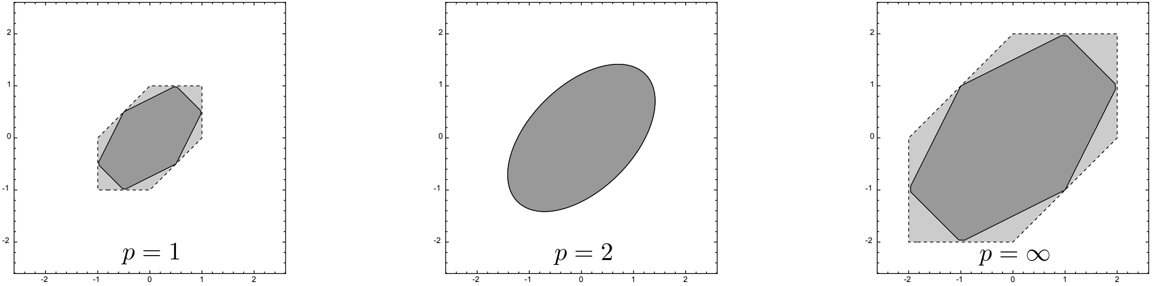



Figure: Two-dimensional sections of three-dimensional unit disks of projection (solid contours) and distance (dashed contours) consensus seminorms. We plot the sections corresponding to $(x_1, x_2, x_3 = 0)$.

# Induced matrix seminorms

Consider a seminorm $|||\cdot|||$ on $\mathbb{R}^n$ with kernel $\mathcal{K}$.

**Induced matrix seminorm**: function $|||\cdot||| : \mathbb{R}^{n\times n} \to \mathbb{R}_{\geq 0}$ where

$$|||A||| = \max_{\substack{|||x|||\leq 1 \\ x\perp\mathcal{K}}} |||Ax|||, \qquad \forall A \in \mathbb{R}^{n\times n}$$

In general, $|||Ax||| \not\leq |||A|||\,|||x|||$
Inequality is true if $x \in \mathcal{K}^{\perp}$ or $A\mathcal{K} \subseteq \mathcal{K}$

# Key facts about dual and induced norms

**Properties of dual and induced norms**

1. **$\ell_p$ and $\ell_q$ norms are dual**, for $1/p + 1/q = 1$

$$\|\cdot\|_p = (\|\cdot\|_q)_\star \qquad \|\cdot\|_q = (\|\cdot\|_p)_\star$$

2. dual norm satisfies (sharp) *Hölder inequality*: $x^\top y \le \|x\|_p \|y\|_q$
3. equality between dual induced norms: $\|A\|_p = \|A^\top\|_q$
4. induced norm is submultiplicative: $\|AB\| \le \|A\| \|B\|$

# Key facts about dual and induced seminorms

## Properties of dual and induced seminorms

1. **$\ell_p$-distance and $\ell_q$-projection seminorms are dual**, for $1/p + 1/q = 1$

$$|||\cdot|||_{\mathrm{dist},p} = (|||\cdot|||_{\mathrm{proj},q})_\star \qquad\qquad |||\cdot|||_{\mathrm{proj},q} = (|||\cdot|||_{\mathrm{dist},p})_\star$$

2. dual seminorm satisfies (sharp) *Markov inequality*: $x^\top \Pi_\perp y \leq |||x|||_{\mathrm{dist},p}\, |||y|||_{\mathrm{proj},q}$
3. equality between dual induced seminorms: $|||A|||_{\mathrm{dist},p} = |||A^\top|||_{\mathrm{proj},q}$
4. induced seminorm is submultiplicative: $|||AB||| \leq |||A|||\,|||B|||$ if $A\mathcal{K} \subseteq \mathcal{K}$ or $B\mathcal{K}^\top \subseteq \mathcal{K}^\top$

## Ergodic coefficients are induced seminorms

$$|||A|||_{\mathrm{dist},p} = |||A^\top|||_{\mathrm{proj},q} = \tau_q(A) := \max_{\|z\|_q = 1,\ z \perp \mathbb{1}_n} \|A^\top z\|_q$$

# How Markov and Banach's results meet

## Classical Property of Averaging Systems

Given row-stochastic $A \in \mathbb{R}^{n \times n}$ and $x, y \in \mathbb{R}^n$:

$$\begin{aligned} \|A(x-y)\|_{\mathsf{dist},\infty} &\leq \tau_1(A)\|x-y\|_{\mathsf{dist},\infty} \\ &= \left|\!\left|\!\left| A \right|\!\right|\!\right|_{\mathsf{dist},\infty} \left|\!\left|\!\left| x-y \right|\!\right|\!\right|_{\mathsf{dist},\infty} \end{aligned}$$

## Classical Property of Markov Chains

Given row-stochastic $A \in \mathbb{R}^{n \times n}$ and $\pi, \sigma$ in the simplex $\Delta_n$:

$$\begin{aligned} \|A^\top(\pi-\sigma)\|_{\mathsf{proj},1} &\leq \tau_1(A)\|\pi-\sigma\|_{\mathsf{proj},1} \\ &= \left|\!\left|\!\left| A^\top \right|\!\right|\!\right|_{\mathsf{proj},1} \left|\!\left|\!\left| \pi-\sigma \right|\!\right|\!\right|_{\mathsf{proj},1} \end{aligned}$$

# Summary and future work

1. ergodic coefficients are contraction factors
2. duality explains their roles in both averaging and flow systems
3. nonEuclidean norms play a key role
4. **semicontraction theory**
   1. discrete/continuous-time Markov chains
   2. discrete/continuous-time nonlinear consensus algorithms
   3. local contractivity of Kuramoto and Kuramoto-Sakaguchi models

### Future work

consider the set of undirected, unweighted connected graphs + selfloops

for each adjacency $A_i$, define row-stochastic $\mathcal{A}_i = \mathrm{diag}(A_i \mathbb{1}_n)^{-1} A_i$ (equal neighbor)

**find** a consensus seminorm $|||\cdot|||$ such that, for each $i$,

$$|||\mathcal{A}_i||| < 1$$

or **prove** that it does not exist

# Continuous-time semicontraction theory

The *induced log seminorm* of $A \in \mathbb{R}^{n\times n}$ is

$$\mu_{|||\cdot|||}(A) \triangleq \lim_{h\to 0^+} \frac{|||I_n + hA||| - 1}{h}$$

Laplacian $L$, corresponding to weighted digraph with adj. matrix $A$:

$$\mu_{\textsf{dist},1}(-L) = -\min_j \left\{ (d_{\mathrm{out}})_j - \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor - 1} a_{(i),j} + \sum_{i=\lceil \frac{n}{2} \rceil}^{n-1} a_{(i),j} \right\}, \quad d_{\mathrm{out}} = A\mathbb{1}_n$$

$$\mu_{\textsf{dist},2}(-L) = \min\left\{ b : \Pi_\perp L + L^\top \Pi_\perp \succeq -2b\Pi_\perp \right\}, \quad \Pi_\perp = I_n - \tfrac{1}{n}\mathbb{1}_n\mathbb{1}_n^\top$$

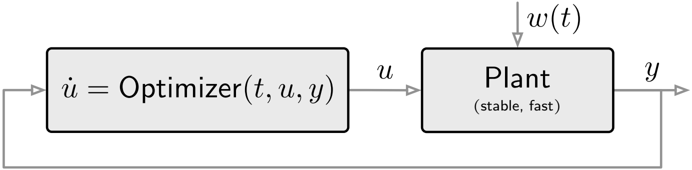
$$\mu_{\textsf{dist},\infty}(-L) = -\min_{i\neq j} \left\{ a_{ij} + a_{ji} + \sum_{k\neq i,j} \min\{a_{ik}, a_{jk}\} \right\}$$

Let $p, q \in [1,\infty]$ such that $p^{-1} + q^{-1} = 1$. For any matrix $M \in \mathbb{R}^{n\times n}$, and any kernel $\mathcal{K}$,

$$\mu_{\mathrm{dist},p}(M) = \mu_{\mathrm{proj},q}(M^\top)$$

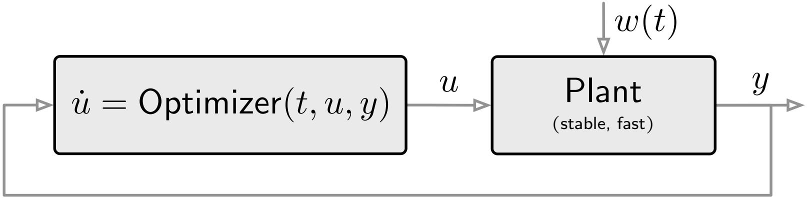
# Outline

# Solving optimization problems via dynamical systems




- studies in linear and nonlinear programming (Arrow, Hurwicz, and Uzawa 1958)
- neural networks (Hopfield and Tank 1985) and analog circuits (Kennedy and Chua 1988)
- optimization on manifolds (Brockett 1991)
- ...
- power grids (Bolognani, Carli, Cavraro, Zampieri 2013)
- online and dynamic feedback optimization (Dall'Anese, Dörfler, Simonetto, ...)

# Example: Time-varying optimization algorithms




**optimization via dynamical systems**

online time-varying optimization, optimization-based feedback control, ...

$$\begin{cases} \min & \text{cost}_1(u) + \text{cost}_2(y) \\ \text{s.t.} & y = \text{Plant}\big(u, w(t)\big) \end{cases} \quad \Longrightarrow \quad \begin{cases} \dot{u} = \text{Optimizer}(t, u, y) \\ y = \text{Plant}\big(u, w(t)\big) \end{cases}$$

# From convex optimization to contracting dynamics – time-varying

Many convex optimization problems can be solved with contracting dynamics

$$\dot{x} = \mathsf{F}(x, \theta)$$

| | **Convex Optimization** | **Contracting Dynamics** |
|---|---|---|
| **Unconstrained** | $\min_{x \in \mathbb{R}^n} \quad f(x, \theta)$ | $\dot{x} = -\nabla_x f(x, \theta)$ |
| **Constrained** | $\min_{x \in \mathbb{R}^n} \quad f(x, \theta)$ <br> s.t. $\quad x \in \mathcal{X}(\theta)$ | $\dot{x} = -x + \mathrm{Proj}_{\mathcal{X}(\theta)}(x - \gamma \nabla_x f(x, \theta))$ |
| **Composite** | $\min_{x \in \mathbb{R}^n} \quad f(x, \theta) + g(x, \theta)$ | $\dot{x} = -x + \mathrm{prox}_{\gamma g_\theta}(x - \gamma \nabla_x f(x, \theta))$ |
| **Equality** | $\min_{x \in \mathbb{R}^n} \quad f(x, \theta)$ <br> s.t. $\quad Ax = b(\theta)$ | $\dot{x} = -\nabla_x f(x, \theta) - A^\top \lambda,$ <br> $\dot{\lambda} = Ax - b(\theta)$ |
| **Inequality** | $\min_{x \in \mathbb{R}^n} \quad f(x, \theta)$ <br> s.t. $\quad Ax \leq b(\theta)$ | $\dot{x} = -\nabla f(x, \theta) - A^\top \nabla M_{\gamma, b(\theta)}(Ax + \gamma \lambda),$ <br> $\dot{\lambda} = \gamma(-\lambda + \nabla M_{\gamma, b(\theta)}(Ax + \gamma \lambda))$ |

# Tracking equilibrium trajectories

For parameter-dependent vector field $\mathsf{F} : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$ and differentiable $\theta : \mathbb{R}_{\geq 0} \to \mathbb{R}^d$

$$\dot{x}(t) = \mathsf{F}(x(t), \theta(t))$$

Assume there exist norms $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\Theta}$ s.t.

- **contractivity wrt $x$**: $\mathrm{osLip}_x(\mathsf{F}) \leq -c < 0$, uniformly in $u$
- **Lipschitz wrt $u$**: $\mathrm{Lip}_u(\mathsf{F}) \leq \ell$, uniformly in $x$

**Theorem: Incremental ISS** any two soltns: $x(t)$ with input $u_x$ and $y(t)$ with input $u_y$

$$D^+\|x(t) - y(t)\|_{\mathcal{X}} \;\leq\; -c\|x(t) - y(t)\|_{\mathcal{X}} \;+\; \ell\|u_x(t) - u_y(t)\|_{\Theta}$$

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**Theorem: Equilibrium tracking for contracting dynamics**

1. for each fixed $\theta$, there exists a unique equilbrium $x^\star(\theta)$
2. the equilibrium map $x^\star(\cdot)$ is Lipschitz with constant $\dfrac{\ell}{c}$
3. $D^+\|x(t)-x^\star(\theta(t))\|_{\mathcal{X}} \;\leq\; -c\|x(t)-x^\star(\theta(t))\|_{\mathcal{X}} \;+\; \dfrac{\ell}{c}\,\|\dot{\theta}(t)\|_{\Theta}$

# Consequences for tracking error

$$D^+\|x(t)-x^\star(\theta(t))\|_\mathcal{X} \;\leq\; -c\|x(t)-x^\star(\theta(t))\|_\mathcal{X} \;+\; \frac{\ell}{c}\,\|\dot\theta(t)\|_\Theta$$

- bounded input, bounded error
  with asymptotic bound:

$$\limsup_{t\to\infty}\|x(t)-x^\star(\theta(t))\|_\mathcal{X} \;\leq\; \frac{\ell}{c^2}\,\limsup_{t\to\infty}\|\dot\theta(t)\|_\Theta$$

- bounded energy input, bounded energy error
- vanishing input, vanishing error
- exponentially vanishing input, exponentially vanishing error
- periodic input, periodic error

# Numerical simulations

$$\begin{aligned} \min_{x\in\mathbb{R}^3} \quad & \frac{1}{2}\|x - r(t)\|_2^2 \\ \text{s.t.} \quad & x_1 + 2x_2 + x_3 = \sin(\omega t), \end{aligned}$$

$$\begin{aligned} \min_{x\in\mathbb{R}^2} \quad & \frac{1}{2}\|x + r(t)\|_2^2 \\ \text{s.t.} \quad & -x_1 + x_2 \le \cos(\omega t), \end{aligned}$$

$$r(t) = (\sin(\omega t), \cos(\omega t), 1), \omega = 0.2$$

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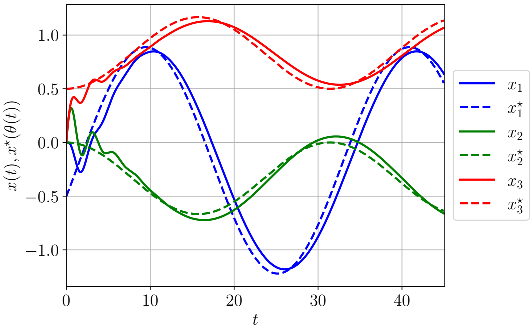


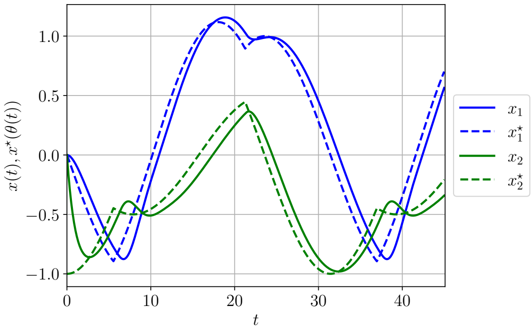

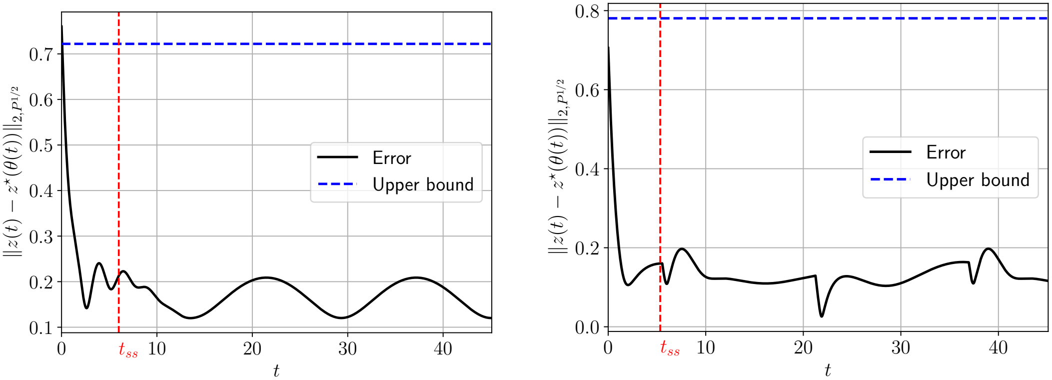

Error
Upper bound
$t_{ss}$
$t$
$\|z(t) - z^\star(\theta(t))\|_{2,P^{1/2}}$

# Proof sketch for equilibrium tracking

Given $\dot{x} = \mathsf{F}(x, \theta(t))$ with $\mathsf{osLip}_x(\mathsf{F}) \leq -c$ and $\mathsf{Lip}_u(\mathsf{F}) \leq \ell$

Task: compare **trajectory** $x(t)$ with **equilibrium trajectory** $x^*(\theta(t))$

Consider **auxiliary dynamics** with two trajectories:

$$\dot{x} = \mathsf{F}(x, \theta(t)) + v(t) \quad =: \quad \mathsf{F}_{\mathsf{aux}}(x, \theta, v)$$

1. $v(t) = 0 \qquad \Longrightarrow \qquad$ trajectory $x(t)$
2. $v(t) = \dot{x}^*(\theta(t)) \qquad \Longrightarrow \qquad$ equilibrium trajectory $x^*(\theta(t))$

$\mathsf{F}_{\mathsf{aux}}$ is contracting with $\mathsf{osLip}_x(\mathsf{F}_{\mathsf{aux}}) \leq -c$ and $\mathsf{Lip}_v(\mathsf{F}_{\mathsf{aux}}) = 1$. Hence, iISS:

$$\begin{aligned} D^+\|x(t) - x^\star(\theta(t))\|_{\mathcal{X}} &\leq -c \cdot \|x(t) - x^\star(\theta(t))\|_{\mathcal{X}} + 1 \cdot \|0 - \dot{x}^*(\theta(t))\|_{\mathcal{X}} \\ &\leq -c \cdot \|x(t) - x^\star(\theta(t))\|_{\mathcal{X}} + \frac{\ell}{c} \cdot \|\dot{\theta}(t)\|_{\Theta} \qquad \Big(\text{since } \mathsf{Lip}(x^*) = \frac{\ell}{c}\Big) \end{aligned}$$

# Summary and future work

**Summary:**

1. from convex optimization to contracting dynamics
2. tracking-bounds for time-varying contracting systems
3. applications to standard convex optimization problems

**Ongoing work and open problems:**

1. contracting predictor-corrector methods
2. tracking bounds in time-varying norms
3. convex but not strongly convex problems

**Thank you for reading so far!**

**For any questions, please do not hesitate to email me**