Eigenfunctions of phi

Assume $(\Phi - \lambda)f(x) = 0$.
Where $f(x) \neq 0$ **(the trivial eigenfunction)** $\Rightarrow$ $let\ \lambda \neq 0$ **(**Φ **is injective in the domain that we are considering, so a non-trivial eigenfunction doesn't exist if** $\lambda = 0$**).**

$$f(x) \in C[[x]] \ \Rightarrow\ f(x) = \sum_{n=0}^{\infty} \frac{D^n f(0)}{n!} x^n.$$

$$\Phi f(x) = \lambda f(x) \in CN[[x]] \ \Rightarrow\ f(x) \in CN[[x]] \text{ (because } \lambda \neq 0\text{).}$$

Where $CN[[x]]$ **is the set of formal Newton series with coefficients in** C.

$$f(x) \in CN[[x]] \ \Rightarrow\ \lambda f(x) = \lambda \sum_{n=0}^{\infty} \frac{\Delta^n f(0)}{n!} x_n = \Phi f(x) = \sum_{n=0}^{\infty} \frac{D^n f(0)}{n!} x_n \ \Rightarrow$$

$$\Rightarrow\ D^n f(0) = \lambda \Delta^n f(0)\ \ \forall n \in N_0.$$

Assume $D^n f(x) = \lambda \Delta^n f(x)\ \ \forall n \in N_0$ **(we still have to prove it)**
$\Rightarrow\ let\ n = 0\ \Rightarrow\ D^0 f(x) = \lambda \Delta^0 f(x)\ \Rightarrow\ \lambda = 1$ **(there exist no non-trivial eigenfunction with eigenvalue** $\neq 1$**). This implies that we don't have to check the case when** $n = 0$ **(cause** $D^0 f(x) = \Delta^0 f(x)\ \ \forall f(x)$**).**

$$(D^n - \Delta^n)f(x) = 0 = (D^n - (e^D - 1)^n)f(x) = D^n - (\sum_{k=0}^{\infty} \frac{D^k}{k!} - 1)^n)f(x) =$$

$$= (D^n - (D \sum_{k=0}^{\infty} \frac{D^k}{(k+1)!})^n)f(x) = D^n(1 - (\sum_{k=0}^{\infty} \frac{D^k}{(k+1)!})^n)f(x) =$$

$$= D^n((\sum_{k=0}^{\infty} \frac{D^k}{(k+1)!})^n - 1)f(x) = D^n((\sum_{k=1}^{\infty} \frac{D^k}{(k+1)!} + 1)^n - 1)f(x)$$

$$Let\ \sum_{k=1}^{\infty} \frac{D^k}{(k+1)!} = g(D)\ \ \Rightarrow\ 0 = D^n((g(D) + 1)^n - 1)f(x) =$$

$$= D^n(\sum_{k=0}^{n} \frac{n_k}{k!} g(D)^k - 1)f(x) = D^n(\sum_{k=1}^{n} \frac{n_k}{k!} g(D)^k)f(x) =$$

$$= D^n g(D)\left(\sum_{k=0}^{n-1} \frac{n_{k+1}}{(k+1)!} g(D)^k\right) f(x) \ \forall n \in N.$$

The expansion of the Δ operator was made possible by the fact that

$$(e^D - 1)f(x) = \sum_{k=1}^{\infty} \frac{D^k}{k!} f(x) = \sum_{k=1}^{\infty} \frac{\Delta^k}{k!} f(x) = \Delta \sum_{k=1}^{\infty} \frac{\Delta^{k-1}}{k!} f(x) \Rightarrow \sum_{k=1}^{\infty} \frac{\Delta^{k-1}}{k!} f(x)$$

converges, so it must converge to $f(x)$.

$\sum_{k=0}^{n-1} \frac{n_{k+1}}{(k+1)!} g(D)^k$ **can be factorized**

$\Rightarrow$ $let\, n = 1 \Rightarrow \sum_{k=0}^{n-1} \frac{n_{k+1}}{(k+1)!} g(D)^k = 1 \Rightarrow$ **there exist no roots of the polynomial that don't depend on n** $\Rightarrow$

$\Rightarrow \neg\exists f(x) \neq 0 \ s.t. \ \left(\sum_{k=0}^{n-1} \frac{n_{k+1}}{(k+1)!} (g(D))^k\right) f(x) = 0 \ \forall n \in N \Rightarrow$

$\Rightarrow D^n g(D) f(x) = 0$ **will give the only solutions.**

If $D^{min(n)} g(D) f(x) = 0 \Rightarrow D^n g(D) f(x) = 0 \ \forall n \in N$.

Proof:

$$D^{min(n)} g(D) f(x) = 0 \Rightarrow D^{n-min(n)\geq 0} D^{min(n)} g(D) f(x) = 0 = D^n g(D) f(x).$$

$min(n) = 1 \Rightarrow Dg(D)f(x) = 0$ **will give the only solutions.**

From the previous definitions $zg(z) = e^z - z - 1$.
Solve for its complex zeros w_k:

$$e^{w_k} - 1 - w_k = 0 \Rightarrow e^{w_k + 1} = e(w_k + 1) \Rightarrow \frac{-1}{e} = e^{-w_k - 1}(-w_k - 1) \Rightarrow$$

$$\Rightarrow -w_k - 1 = W_k\left(\frac{-1}{e}\right) \Rightarrow w_k = 1 - W_k\left(\frac{-1}{e}\right).$$

Where W_k is a generic branch of the Lambert W function. We want to expand $g(z)$ in a Weierstrass product, so we must know the multiplicity of its zeros, we'll check the derivatives of the function.

A zero with multiplicity of 3 doesn't exist ($D^3(e^z - z - 1) = e^z$) so we only have to check

$$D^2(e^z - z - 1) = e^z - 1 \;\Rightarrow\; let\; e^{w_k} - 1 = 0 = w_k + 1 - 1 = w_k \;\Rightarrow$$

the only zero with multiplicity of 2 is 0.

We can now solve $Dg(D)f(x) = 0$ **like a normal differential equation** $\Rightarrow$

$$f(x) = \sum_k c_k e^{w_k x} + ax.$$