CLASS NOTES IN 2022, HAVE THE BENEFIT OF THE "PAUSE-N-PONDER"

CAUCHY

STEVEN STROGATZ

CORNELL

15/09/2022

AGAIN → 333

1/6

DISCOVERED BY GAUSS 1811

GOURSAT'S PROOF (BETTER THAN CAUCHY)

→ CONTRAST W/ SAFF Y SNIDER

§4.4

$$S_n = \boxed{\int_\gamma (z-a)^n \, dz = 0 \quad n \in \mathbb{Z} - \{-1\}}$$

IF γ IS A SMOOTH CLOSED CURVE $n \neq 1$

AND FOR $n \neq -1$

$2\pi i$ → SPECIAL CASE

FOR $n=-1$ → $S_1 = 2\pi i$

$n \in \mathbb{Z}$

SO FOR ANY POLINOMIAL

$$P(z) = C_0 + C_1 z + C_2 z^2 + \cdots + C_n z^n$$

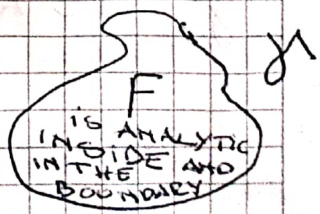
WE HAVE

$$\boxed{\int_\gamma P(z)\, dz = 0}$$

CAUCHY THEOREM IS A VAST GENERALIZATION

IT SAYS

$$\boxed{\int_\gamma F(z)\, dz = 0}$$



FOR ANY F THAT IS ANALYTIC AND INSIDE γ

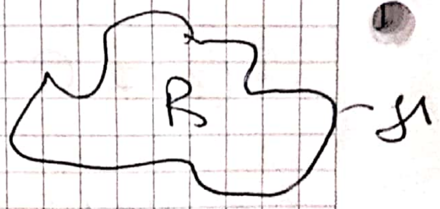
SABER Ó ASUMIR QUE UNA FUNCIÓN ES DERIVABLE A SIMPLE VISTA PARECE INOCENTE, PERO TIENE CONSECUENCIAS MUY IMPORTANTES, COMO SE PODRÁ VER EN ~~EL~~ LA SIGUIENTE SECCIÓN:

TEOREMA DE CAUCHY

SEA $f(z)$ ANALÍTICA EN UNA REGIÓN SIMPLEMENTE CONECTADA R

SEA γ UNA CURVA CERRADA EN R

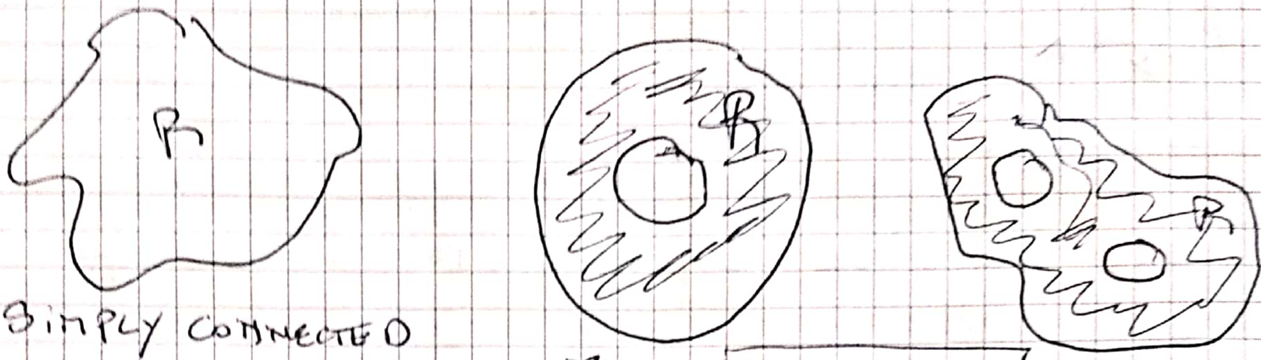
$$\boxed{\oint_\gamma f(z)\,dz = 0}$$



$f(z)$ IS ANALYTIC IN A SIMPLY CONNECTED REGION R

γ IS ANY CLOSED CURVE IN R

REMINDER ON "SIMPLE CONNECTEDNESS"



SIMPLY CONNECTED

~~ANO~~

NOT SIMPLY CONNECTED

MEANS NO HOLES

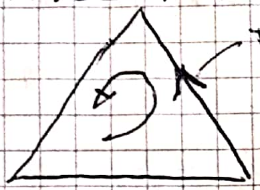
BUT FROM A TOPOLOGY STAND VIEW WE NEED TO BE MORE PRECISE:

(RECORDAR TEICHMULLER THEORY)

R is "SIMPLY-CONNECTED" SI CADA LAZO CERRADO (CLOSED LOOP) EN R PUEDE CONTRAERSE CONTINUAMENTE A UN PUNTO PERMANECIENDO EN R DURANTE LA CONTRACCIÓN.

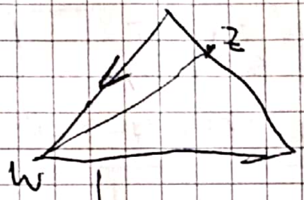
PROOF

PROOF:



THIS IS $\gamma = T$ → TRIANGLE

T HAS DIAMETER D, AND PERIMETER L

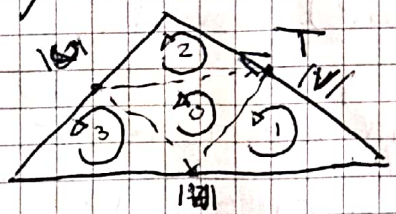


$|z-w| \leq D$

FOR ALL z, w ON T

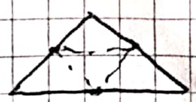
SOMEHOW WE HAVE TO USE THE FACT THAT $F(z)$ IS ANALYTIC:

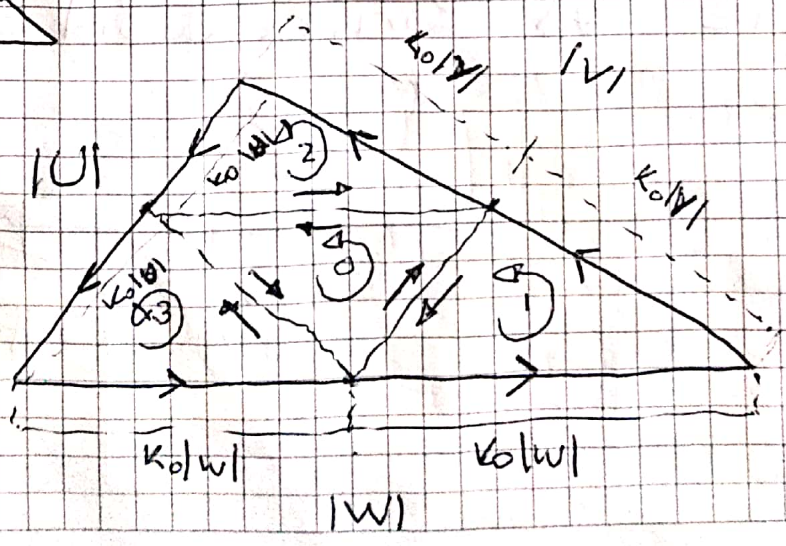
1)



BISECT SIDES, MAKE 4 TRIANGLES

1)b





$\boxed{k_0 = \frac{1}{2}}$

LETS SHOW HOW THE INTEGRAL OF $F(z)$ OVER T IS EQUIVALENT TO THE INTEGRAL OF THE 4 LITTLE TRIANGLES

$$\int_T F(z)\,dz = \int_{T_0} F(z)\,dz + \int_{T_1} F(z)\,dz + \int_{T_2} F(z)\,dz + \int_{T_3} F(z)\,dz$$

INNER	+	RIGHT ~~LEFT~~	+	UP	+	~~RIGHT~~ LEFT
0		1		2		3

T_0 T_1 T_2 T_3

T_I T_R T_U T_L

HERE THE INSIGHT CAN BE SEEN ALSO THIS WAY:

THE TRIANGLES $\boxed{\text{0 AND 1}}$ SHARE A SIDE, THAT PART OF BOTH INTEGRALS HAVE OPPOSITE SIGNS (THEY COULD BE THOUGHT AS $\boxed{\text{KIRCHOFF AS WELL}}$)

THE SAME HAPPENS WITH TRIANGLES:

0 AND 1 SHARE A SIDE $|z_{01}|$

0 AND 2 $|z_{02}|$

0 AND 3 $|z_{03}|$

OR SEEN AS:

$$\left.\begin{array}{lll} I_0 \text{ AND } R_1 & & \text{(INNER-RIGHT)} \\ I_0 \text{ AND } U_2 & & \text{(INNER-UP)} \\ I_0 \text{ AND } L_3 & & \text{(INNER-LEFT)} \end{array}\right\}$$

THIS PARTS OF THE INTEGRAL CANCEL WITH EACH OTHER!

I AND D? DOWN (INNER-DOWN)

↳ IF WE CHOSE ANOTHER CURVE OTHER

↳ NOT HERE

CONTRIBUTIONS FROM COMMON INTERNAL EDGES CANCEL

2) CHOOSE A LITTLE TRIANGLE WITH THE LARGEST $|\int|$ → CALL IT T_1

BECAUSE OF TRIANGLE INEQUALITY WE KNOW THAT

$$\boxed{\left| \int_T f(z)\,dz \right| \leq 4 \left| \int_{T_1} f(z)\,dz \right|}$$

3) AS A FRACTAL, BISECT AS YOU DID WITH T, INSIDE T_1 (OVER AND OVER) n TIMES

AT STAGE n, CALL THE TRIANGLE WITH LARGEST |INTEGRAL| T_n

$$\left| \int_T f(z)\,dz \right| \leq \underbrace{(2^2)^n}_{4^n} \left| \int_{T_n} f(z)\,dz \right|$$

$4^n \to (2^m)^n \longrightarrow \boxed{m=2}$

RELATED TO DIMENSIONS??

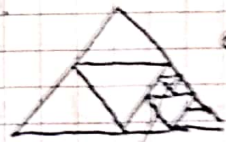
LHS IS BOUNDED BY RHS

↳ CAN BE MADE SMALLER THAN ANY NUMBER WE CHOOSE

LHS RHS

$$\left|\int_T f(z)\,dz\right| \leq 4^n \left|\int_{T_n} f(z)\,dz\right|$$

AS DESCRIBED IN 3)



OVER AND OVER AND OVER! UNTIL YOU REACH A POINT

∃ POINT

AS SMALL AS WE WANT

4)

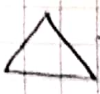
∃ Q (POINT) CAUGHT IN THE INFINITE NESTED INTERSECTIONS

(NEEDS TOPOLOGY PROOF OUTSIDE MY CAPABILITIES POINT-SET TOPOLOGY COMPACTNESS)

"COMPACTNESS" OF CLOSED TRIANGULAR REGION

$a \in \ldots \subset T_2 \subset T_1 \subset T$

THE POINT IS INSIDE

 T_n → SMALLEST TRIANGLE

5) CAN WE ESTIMATE $\left| \int_{T_n} F(x)\, dz \right|$?

T_n IS SOOOO TINY!

ON T_n WE CAN WRITE:

5.1 $$\boxed{F(z) = F(a) + F'(a)(z-a) + \epsilon(z)(z-a)}$$

$$\boxed{\epsilon(z) \to 0 \text{ AS } z \to a}$$
5.2

→ THIS IS WHERE WE SEE THE FACT THAT $F(z)$ IS ANALYTIC ($\epsilon(z) \to 0$ AS $z \to a$)

NOTE → $$\boxed{\epsilon(z) = \frac{F(z) - F(a)}{z-a} - F'(a)}$$
5.3

$\epsilon(z)$ IS SMALL FOR $z \to a$ SINCE F IS ANALYTIC

5b) FROM EQ $\boxed{5.1}$ ⇒

$$\int_{T_n} f(z)\,dz = \int_{T_n} f(a)\,dz + \int_{T_n} f'(a)(z-a)\,dz + \int_{T_n} \epsilon(z)(z-a)\,dz \qquad \boxed{5.4}$$

$f(a)$ AND $f'(a)$ AND a ARE CONSTANTS $\in \mathbb{C}$

SO:

$$\int_{T_n} f(z)\,dz = f(a)\underbrace{\int_{T_n} dz}_{A} + f'(a)\underbrace{\left[\underbrace{\int_{T_n} z\,dz}_{B_1} - a\underbrace{\int_{T_n} dz}_{B_2}\right]}_{B} + \int_{T_n} \epsilon(z)(z-a)\,dz$$

RECALL FROM THE BEGINNING THAT

$$\boxed{\int_{\gamma} (z-a)^n\,dz = 0}$$

$$\boxed{\int_{\gamma} P(z)\,dz = 0}$$

$\boxed{\text{SO } A = B = 0}$

→ NOTE THAT:

→ IN A $P(z) = 1$

IN B_1 $P(z) = z$

IN B_2 $P(z) = 1$

So

$$\left|\int_{T_n} f(z)\,dz\right| = \left|\int_{T_n} \epsilon(z)(z-a)\,dz\right| \qquad \boxed{5.5}$$

¿SO HOW SMALL IS $\epsilon(z)$?

→ STEP 6

6) USE ML BOUND INEQUALITY:

M → MAX OF THE INTEGRAND

L → LENGTH OF THE CONTOUR OF INTEGRATION

$$\boxed{\left|\underbrace{\int_{T_n} \epsilon(z)(z-a)\,dz}_{S(\epsilon(z))}\right| \leq \max_{z \in T_n} |\epsilon(z)| \cdot D_{T_n} \cdot L_{T_n}}$$

6.1

D_{T_n} → DIAMETER OF CLOSED CURVE T_n

L_{T_n} → LENGTH OF CLOSED CURVE T_n

D_{T_n} IS DIVIDED BY 2 IN EVERY BISECTION

SO $\boxed{\left.\begin{array}{l} D_{T_n} \\ L_{T_n} \end{array}\right\} \propto \frac{1}{2^n}}$ $\boxed{\text{SO AS } n \to \infty \quad D_{T_n} \to 0}$

$$S(\epsilon(z)) \leq \max_{z \in T_n} |\epsilon(z)| \frac{D}{2^n} \frac{L}{2^n}$$

SINCE T_n IS CONSTRUCTED BY REPEATED BISECTION OF T

6.2

7) $\boxed{LHS}$ $\boxed{RHS}$

$$\left| \int_T f(z)\,dz \right| \leq 4^n \left| \int_{T_n} f(z)\,dz \right|$$

$$\leq 4^n \left| \int_{T_n} \epsilon(z)(z-e)\,dz \right|$$

$$\leq \cancel{4^n} \max_{z \in T_n} |\epsilon(z)| \underbrace{\frac{D}{2^n} \frac{L}{2^n}}_{\substack{2^{2n} \\ 4^n}}$$

8) $\underbrace{\text{AS } n \to \infty \quad \max_{z \in T_n} |\epsilon(z)| \to 0}$ → BECAUSE IT'S ANALYTIC

↳ MEANING WE CAN CHOOSE $\max_{z \in T_n}$ AS SMALL AS WE WANT!

9) BUT LHS IS INDEPENDENT OF n SO AS IT IS POSITIVE, AND SMALLER THAN THE SMALLEST NUMBER WE CAN CHOOSE, WE SEE IT MUST BE $\emptyset$.

$\boxed{LHS = \emptyset}$ $\boxed{QED}$

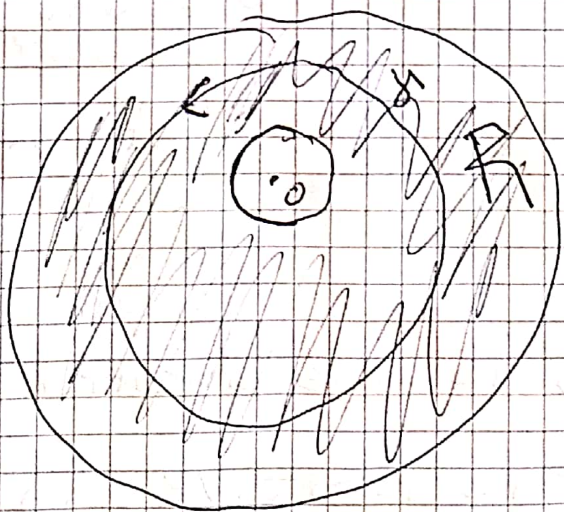
9b)

¿WHY DO WE NEED "SIMPLE CONNECTEDNESS" IN CAUCHY THEOREM?

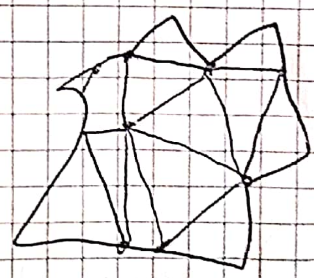
TO AVOID THE BEAUTIFUL CASE:

WHEN → $\boxed{\int_\gamma \frac{dz}{z} = 2\pi i}$ SO LOVELY!!

$F(z)$ IS ANALYTIC IN R, $\boxed{F(z) = \frac{1}{z}}$ BUT $\int_\gamma F(z)\,dz \neq 0$



¿WHY TRIANGLES ARE ENOUGH?



WE CAN CONSTRUCT "EVERYTHING" (POLIGONAL CURVES) OUT OF COMBINATIONS OF THEM

→ ¿HOW? BECAUSE

$F(z) = \frac{1}{z}$ IS ANALYTIC IN R, BUT R IS NOT SIMPLY CONNECTED, SO $1/z$ IS NOT ANALYTIC EVERYWHERE INSIDE γ