Microbundles on Topological Manifolds

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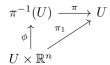
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1 Vectorbundles on Smooth Manifolds

Definition 1.1. (vector bundle)

A vector bundle ξ is a tuple $\xi := (B, E, \pi, +, \cdot)$ satisfying the following conditions:

- ullet B is a topological space (base space)
- E is a topological space (total space)
- $(\pi^{-1}(b), +, \cdot)$ is a real vector space for every $b \in B$
- Every $b \in B$ is <u>locally trivializable</u>, i.e there exist neighborhoods $U \subseteq B$ of b such that the following diagram commutes



and $\phi(b, -): b \times \mathbb{R}^n \xrightarrow{\sim} \pi^{-1}(b)$ is a linear isomorphism.

We call n the \underline{rank} of ξ .

Example 1.2. (tangent vector bundle)

Let M be a smooth manifold:

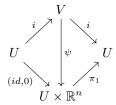
 $\xi: TM \xrightarrow{\pi} M$ is a vector bundle, where $\pi(p, v) := p$.

2 Introduction to Microbundles

Definition 2.1. (microbundle)

A microbundle \mathfrak{b} is a tuple $\mathfrak{b} := (B, E, i, j)$ satisfying the following properties:

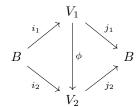
- B is a topological space called the base space
- E is a topological space called the total space
- $i: B \to E$ and $j: E \to B$ are continuous maps with $id_B = j \circ i$
- Every $b \in B$ is <u>locally trivializable</u>, i.e there exist open neighborhoods $U \subseteq B$ of b and $V \subseteq E$ of i(U) such that the following diagram commutes:



We call n the fibre dimension of \mathfrak{b} .

Definition 2.2. (isomorphic microbundles)

Two microbundles $\mathfrak{b}_1 := (B, E_1, i_1, j_2)$ and $\mathfrak{b}_2 := (B, E_2, i_2, j_2)$ are said to be <u>isomorphic</u> if there exist neighborhoods $V_1 \subseteq E_1$ of $i_1(B)$ and $V_2 \subseteq E_2$ of $i_2(B)$ with an homeomorphism $\phi : V_1 \xrightarrow{\sim} V_2$ such that the following diagram commutes:



Example 2.3. (trivial microbundle)

Let B be a topological space and $n \in \mathbb{N}$:

The diagram $\mathfrak{e}_B^n: B \xrightarrow{\iota} B \times \mathbb{R}^n \xrightarrow{\pi} B$ constitutes a microbundle, where $\iota(b) := (b,0)$ and $\pi(b,x) := b$. We call \mathfrak{e}_B^n the <u>standard microbundle</u> and every microbundle isomorphic to \mathfrak{b}_B^n <u>trival</u>.

Lemma 2.4. (criteria for triviality)

A microbundle \mathfrak{b} of B is trivial if and only if there exists a open subset $B \subseteq U$ with $U \cong B \times \mathbb{R}^n$.

Proof.
$$\Box$$

Example 2.5. (underlying microbundle)

Let $\xi: E \xrightarrow{\pi} B$ be a n-dimensional vector bundle: The microbundle $|\xi|: B \xrightarrow{i} E \xrightarrow{\pi} B$ with $i(b) := \phi_b(b,0)$, where $\phi_b: U_b \times \mathbb{R}^n \to \pi^{-1}(U_b)$ is the local trivialization over a neighborhood $U_b \subseteq B$ of b. We call $|\xi|$ the <u>underlying microbundle</u> of ξ

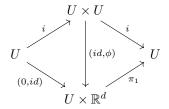
Proof.

Example 2.6. (tangent microbundle)

Let M be a topological manifold:

We can derive the <u>tangent microbundle</u> $t_M: M \xrightarrow{\Delta} M \times M \xrightarrow{\pi_1} M$, where Δ is the diagonal map and π_1 ist the projection map on the first component.

Proof. Let $p \in M$ and (U, ϕ) a chart over p:



 (id, ϕ) is a homeomorphism since $\phi: U \xrightarrow{\sim} \mathbb{R}^n$ is homeomorphic.

Proposition 2.7. (restricting the total space)

Let $\mathfrak{b}: B \xrightarrow{i} E \xrightarrow{j} B$ be a microbundle and E' an arbitray neighborhood of i(B). The restriction $\mathfrak{b}': B \xrightarrow{i} E' \xrightarrow{j|_{E'}} B$ is a microbundle isomorphic to \mathfrak{b} .

Proof. Let $b \in B$.

Choose an arbitray trivialization (U, V, ϕ) over \mathfrak{b} of b. We restrict $\phi : V \to U \times \mathbb{R}^n$ to $V \cap E'$. Since $i(b) \in V$ is open and E' is a neighborhood of i(B), it follows that $\phi(V \cap E')$ is a neighborhood of (b, 0).

$$\implies \exists (b,0) \in U' \times X \subseteq \phi(V \cap E'), \text{ where } U' \subseteq U \text{ and } X \subseteq \mathbb{R}^n \text{ are open} \\ \implies \exists \varepsilon > 0 : U' \times B_{\varepsilon}(0) \subseteq \phi(V \cap E')$$

Since $B_{\varepsilon}(0) \cong R^n$, it follows that $U' \times R^n \cong U' \times B_{\varepsilon}(0) \cong \phi^{-1}(U' \times B_{\varepsilon}(0))$. Choosing $V' := \phi^{-1}(U' \times B_{\varepsilon}(0)) \subseteq V$, we see that \mathfrak{b}' is a microbundle.

We easily see, that \mathfrak{b} is isomorphic to \mathfrak{b}' via the identity.

3 Induced Microbundles

Definition 3.1. (induced microbundle)

Let $\mathfrak{b}: B \xrightarrow{i} E \xrightarrow{j} B$ be a microbundle and $f: A \to B$ a continuous map. We can construct a microbundle $f^*\mathfrak{b}: A \xrightarrow{i'} E' \xrightarrow{j'} A$ defined as follows:

- $E' := \{(a, e) \in A \times E \mid f(a) = j(e)\}$
- $i': A \to E'$ with $i'(a) := (a, (i \circ f)(a))$
- $j': E' \to A \text{ with } j'(a,e) := a$

We call $f^*\mathfrak{b}$ the induced microbundle of \mathfrak{b} over f.

Proof. It is clear that i' and j' are continuous and that $id_A = j' \circ i'$. So it remains to be shown that $f^*\mathfrak{b}$ is locally trivial for every $a \in A$:

- $U' := f^{-1}(U) \subseteq A$ is an open neighborhood of a.
- $V' := j'^{-1}(U') \subseteq E'$ is an open neighborhood of i'(U').
- $\phi': V' \xrightarrow{\sim} U' \times \mathbb{R}^n, \phi'(a,e) := (a, \pi_2(\phi(e)))$ is a homeomorphism.

 $-\phi'$ is well defined because $(a,e) \in V': j(e) = f(a) \in U \implies e \in V$.

- $-\phi'$ is bijective with $\phi'^{-1}(a,v)=(a,\phi^{-1}(f(a),v)).$
- $-\phi'$ and ϕ'^{-1} are continuous because it's components are.

Example 3.2. (restricted microbundle)

Let $\mathfrak{b}: B \xrightarrow{i} E \xrightarrow{j} B$ be a microbundle and $A \subseteq B$:

The induced microbundle $\iota^*\mathfrak{b}$ with $\iota: A \hookrightarrow B$ being the inclusion map is called the <u>restricted microbundle</u> and we write $\mathfrak{b}|_A := \iota^*\mathfrak{b}$.

Remark 3.3. In the following, we'll consider $E(\mathfrak{b}|_A)$ a subset of $E(\mathfrak{b})$. This is justified because $E(\mathfrak{b}|_A) = \{(a,e) \in A \times E(\mathfrak{b}) \mid a = j(e)\} \cong \{e \in E(\mathfrak{b}) \mid j(e) \in A\} \subseteq E(\mathfrak{b})$.

Lemma 3.4. (induced trivial microbundle)

The induced microbundle $f^*\mathfrak{b}$ is trivial for every map $f:A\to B$, if \mathfrak{b} is already trivial.

Proof. Let (V, ϕ) be a global trivialization of \mathfrak{b} , i.e $V \cong_{\phi} B \times \mathbb{R}^n$. Now define $V' := (A \times V) \cap E'$ and $\phi'(a, e) := (a, \phi^{(2)}(e))$. Obviously, V' is a neighborhood of i'(A) and also ϕ' is a homeomorphism with inverse $\phi'^{(-1)}(a, x) = (a, \phi^{-1}(f(a), x))$

Proposition 3.5. (composition)

Let $A \xrightarrow{f} B \xrightarrow{g} C$ be topological spaces and $\mathfrak{c}: C \xrightarrow{i} E \xrightarrow{j} C$ be a microbundle:

$$(g \circ f)^* \mathfrak{c} \cong f^* (g^* \mathfrak{c})$$

Proof. We'll compare the two total spaces and conclude that they are homeomorphic.

- 1. $E((g \circ f)^*\mathfrak{c}) = \{(a, e) \in A \times E(\mathfrak{c}) \mid g(f(a)) = j(e)\}\$
- 2. $E(f^*(q^*\mathfrak{c})) = \{(a, (b, e)) \in A \times (B \times E(\mathfrak{c})) \mid f(a) = b \text{ and } q(b) = j(e)\}.$

We have the bijection $\phi: E((g \circ f)^* \mathfrak{c}) \xrightarrow{\sim} E(f^*(g^* \mathfrak{c}))$ with $\phi(a, e) := (a, (f(a), e))$ and $\phi^{-1}(a, (b, e)) = (a, e)$. Additionally, ϕ is a homeomorphism because ϕ and ϕ^{-1} are componentwise continuous. It's easy to see that ϕ respects both injection and projection, which concludes the proof.

For a topological space X, we define the <u>cone</u> of X as

$$CX := X \times [0,1]/X \times \{1\}$$

and for a map $f: A \to B$ the mapping cone of f as

$$B \sqcup_f CA := B \sqcup CA / \sim$$

where $(a,0) \sim b : \iff f(a) = b$.

Lemma 3.6. (extending over a mapping cone)

A microbundle \mathfrak{b} over B can be extended to a microbundle over the mapping cone $B \sqcup_f CA$ if and only if $f^*\mathfrak{b}$ is trivial.

Proof. We show both implications.

Let \mathfrak{b}' be an extension of \mathfrak{b} over $B \sqcup_f CA$.

Considering $A \xrightarrow{f} B \hookrightarrow B \sqcup_f CA$, the composition $\iota \circ f$ is null-homotopic with homotopy

$$H_t(a) := [(a,t)]$$

Note that $H_0(a) = [(a,0)] = [f(a)] = (\iota \circ f)(a)$ and $H_1(a) = [(a,1)] = [(\tilde{a},1)] = [(\tilde{a},1)]$

 $\Longrightarrow_{Hom.Thm.} (\iota \circ f)^*\mathfrak{b}' \text{ is trivial}$ Since $(\iota \circ f)^*\mathfrak{b}' = f^*(\iota^*\mathfrak{b}') = f^*\mathfrak{b}$, it follows that $f^*\mathfrak{b}$ is trivial.

Let $f^*\mathfrak{b}$ be trivial.

Analogous to the cone, we define the cylinder of X as

$$MX := X \times [0,1]$$

and for a map $f: A \to B$ the mapping cylinder of f as

$$B \sqcup_f MA := B \sqcup MA / \sim$$

where $(a,0) \sim b : \iff f(a) = b$.

In contrast to the mapping cone, there exists a natural retraction from the mapping cylinder to the attached space

$$\pi: B \sqcup_f MA \to B; \pi([(a,t)]) := f(a)$$

and therefore the induced microbundle $\pi^*\mathfrak{b}$ over $B \sqcup_f MA$.

Considering $A \times \{1\} \hookrightarrow B \sqcup_f MA \xrightarrow{\pi} B$, we see that $\pi \circ \iota \cong f$ and therefore

$$\pi^*\mathfrak{b}|_{A\times\{1\}} = (\pi\circ\iota)^*\mathfrak{b} \cong f^*\mathfrak{b} = \mathfrak{e}_A^n$$

is trivial. From the lemma of induced trivial microbundles and $(a,t)\mapsto (a,1)$ it follows that $\pi^*\mathfrak{b}|_{A\times [\frac{1}{2},1]}$ is trivial.

$$\implies \exists \phi: E(\mathfrak{b}|_{A \times \left[\frac{1}{2},1\right]}) \xrightarrow{\sim} A \times \left[\frac{1}{2},1\right] \times \mathbb{R}^n$$

Now we explicitly construct the desired extended microbundle $\mathfrak{b}': B \sqcup_f CA \xrightarrow{i'}$ $E' \xrightarrow{j'} B \sqcup_f CA$

- $E' := E(\mathfrak{b}|_{A \times [\frac{1}{2},1]})/\phi^{-1}(A \times [\frac{1}{2},1] \times \{x\})$ (for every $x \in \mathbb{R}^n$)
- $i' := \pi \circ i$ the projection i to E'

• j'([e]) := [j(e)] is well defined, because $[e] = [\tilde{e}] \implies [j(e)] = [j(e')]$

Now that we have constructed \mathfrak{b}' , this proves the claim.

Corollary 3.7. (extending over a d-simplex)

Let B be a (d+1)-simplicial complex, B' it's d-skeleton and $\Delta^{d+1} \cong \sigma \subseteq B$. A microbundle \mathfrak{b} over B' can be extended to a microbundle over $B' \cup \sigma$ if and only if $\mathfrak{b}|_{\partial \sigma}$ is trivial.

Proof. The statement follows from the last lemma:

There exists a $\phi: C\partial\sigma \xrightarrow{\sim} \sigma$ such that $\phi(\partial\sigma \times \{0\}) = \partial\sigma$.

We explicitly construct $\phi((t_1,\ldots,t_{d+1}),\lambda) := (1-\lambda)(t_1,\ldots,t_{d+1}) + \frac{\lambda}{d+1}(1,\ldots,1)$. It's easy to see that ϕ suffices all our requirements. By choosing $f:\partial\sigma\hookrightarrow B'$ and applying the last lemma, the statement is proven.

4 Whitney sums

Definition 4.1. (whitney sum)

Let \mathfrak{b}_1 and \mathfrak{b}_1 be two microbundles over a topological space B.

We define the whitney sum $\mathfrak{b}_1 \oplus \mathfrak{b}_2$ as follows:

- $E := \{(e_1, e_2) \in E(\mathfrak{b}_1) \times E(\mathfrak{b}_2) \mid j_1(e_1) = j_2(e_2)\}$
- $i(b) := (i_1(b), i_2(b))$
- $j(e_1, e_2) := j_1(e_1) = j_2(e_2)$

Proof. Let $b \in B$.

Choose U_1, V_1, ϕ_1 and U_2, V_2, ϕ_2 accordingly from the local trivialization of b over \mathfrak{b}_1 and \mathfrak{b}_2 :

- $U := U_1 \cap U_2$
- $V := (V_1 \times V_2) \cap E$
- $\phi: V \to U \times \mathbb{R}^{n_1+n_2}; \phi(e_1, e_2) := (\phi_1^{(1)}(e_1), \phi_1^{(2)}(e_1) \times \phi_2^{(2)}(e_2))$

Note that $\phi_1^{(1)}(e_1) = \phi_2^{(1)}(e_2)$.

Local triviality follows directly from it's components.

Lemma 4.2. (compatibility)

Let \mathfrak{b}_1 and \mathfrak{b}_1 be two microbundles over B and $f:A\to B$ a map. Induced microbundle and whitney sum are compatible, i.e. $f^*(\mathfrak{b}_1\oplus\mathfrak{b}_2)\cong f^*\mathfrak{b}_1\oplus f^*\mathfrak{b}_2$

Proof. From the definition of the induced microbundle and the whitney sum, we can derive the total spaces:

$$E(f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2)) = \{(a, (e_1, e_2)) \in A \times (E_1 \times E_2) \mid j_1(e_1) = j_2(e_2) = f(a)\}$$

$$E(f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2) = \{((a_1, e_1), (a_2, e_2)) \in (A \times E_1) \times (A \times E_2) \mid j(a_1, e_1) = j(a_2, e_2)$$

and $f(a_i) = j(e_i)$

Those two total spaces are homeomorphic via $\phi(a,(e_1,e_2)) := ((a,e_1),(a,e_2))$ and $\phi^{-1}((a,e_1),(a,e_2)) = (a,(e_1,e_2))$. ϕ and ϕ^{-1} are continuous because they are componentwise continuous.

Obviously, $\phi \circ i = i$ and $\phi \circ j = j$, which concludes the proof.

Theorem 4.3. ()

Let \mathfrak{b} be a microbundle over a d-dimensional simplicial complex B.

Then there exists a microbundle $\mathfrak n$ over B so that the Whitney sum $\mathfrak b\oplus\mathfrak n$ is trivial.

Proof. We prove this theorem by induction over d.

(Start of induction)

A 1-dimensional simplicial complex is just a bouquet of circles, therefore the start of induction follows directly from the bouquet lemma.

(Inductive Step)

Let B' be the (d-1)-skeleton of B and \mathfrak{n}' it's corresponding microbundle so that $\mathfrak{b}|_{B'} \oplus \mathfrak{n}'$ is trivial.

5 Normal Microbundles

Definition 5.1. (normal microbundle)

Let M and N be two topological manifolds with $N \subseteq M$.

We call a microbundle of the form

$$\mathfrak{n}: N \xrightarrow{\iota} U \xrightarrow{r} N$$

where $U \subseteq M$ is a neighborhood of N, a <u>normal microbundle</u> of N in M.

Definition 5.2. (product neighborhood)

Again, let M and N be two topological manifolds with $N \subseteq M$.

We say that N has a <u>product neighborhood</u> in M if there exists a trivial normal microbundle of N in \overline{M} .

Lemma 5.3. (criteria for product neighborhoods)

A submanifold $N \subseteq M$ has a product neighborhood if and only if there exists a neighborhood U of N with $(U, M) \cong (M \times R^n, M \times 0)$.

Proof. This follows directly from the definition of normal microbundles and the criteria for trivial microbundles (NUMBER). \Box

Definition 5.4. (composition microbundle)

Let $\mathfrak{b}: B \xrightarrow{i_{\mathfrak{b}}} E \xrightarrow{j_{\mathfrak{b}}} B$ and $\mathfrak{c}: E \xrightarrow{i_{\mathfrak{c}}} E' \xrightarrow{j_{\mathfrak{c}}} E$ be two microbundles. We define the <u>composition microbundle</u> $\mathfrak{b} \circ \mathfrak{c}: B \xrightarrow{i} E' \xrightarrow{j} B$ with $i(b) := (i_{\mathfrak{c}} \circ i_{\mathfrak{b}})(b)$ and $j(e') := (j_{\mathfrak{b}} \circ j_{\mathfrak{c}})(e')$

Proof. Let $b \in B$.

Choose local trivializations $(U_{\mathfrak{b}}, V_{\mathfrak{b}}, \phi_{\mathfrak{b}})$ of b and $(U_{\mathfrak{c}}, V_{\mathfrak{c}}, \phi_{\mathfrak{c}})$ of $j_{\mathfrak{b}}(b)$. From this, we construct our local trivialization over $\mathfrak{b} \circ \mathfrak{c}$. Consider $\phi_{\mathfrak{b}}(V_{\mathfrak{b}} \cap U_{\mathfrak{c}})$, which is a neighborhood of (b, 0). Therefore, there exist open neighborhoods $b \in U \subseteq U_{\mathfrak{b}}$ and $0 \in X \subseteq \mathbb{R}^n$ such that $U \times X \subseteq \phi_{\mathfrak{b}}(V_{\mathfrak{b}} \cap U_{\mathfrak{c}})$. Analoguous to the proof of restricting the total space in Chapter 1, it follows that

$$\exists \varepsilon > 0 : U \times B_{\varepsilon}(0) \subseteq \phi_{\mathfrak{b}}(V_{\mathfrak{b}} \cap U_{\mathfrak{c}})$$

$$\implies U \times \mathbb{R}^n \cong U \times B_{\varepsilon}(0) \cong \phi_{\mathfrak{h}}^{-1}(U \times B_{\varepsilon}(0)) \cong \phi_{\mathfrak{g}}^{-1}(\phi_{\mathfrak{h}}^{-1}(U \times B_{\varepsilon}(0)))$$

which is an open neighborhood of i(U) and therefore a valid candidate for V. This concludes local triviality and the proof.

Lemma 5.5. (transitivity of normal microbundles)

Let M, N and P be topological manifolds with $P \subseteq N \subseteq M$. There exists a normal microbundle \mathfrak{n} of P in M, if there exist normal microbundles $\mathfrak{n}_p : P \xrightarrow{i_P} U_N \xrightarrow{j_P} P$ in N and $\mathfrak{n}_n : N \xrightarrow{i_N} U_M \xrightarrow{j_N} N$ in M.

Proof. We simply form the composition $\mathfrak{n}_p \circ \mathfrak{n}_n|_{U_N} : P \xrightarrow{i_N \circ i_P} U_M \xrightarrow{j_P \circ j_N} P$. Since $i_N \circ i_P$ is just the inclusion of $P \hookrightarrow U_M \subseteq M$, we found a normal microbundle \mathfrak{n} of P in M.

Every topological manifold is an absolute neighborhood retract (ANR).

It follows that by restricting M, if necessary, to an open neighborhood of N, there exists a retraction $r: M \to N$ which we will take advantage of in the following.

Lemma 5.6. (homeomorphism of total spaces)

Let \mathfrak{t}_N and \mathfrak{t}_M be the tangent microbundles of N and M. The total space $E(\iota^*\mathfrak{t}_M)$ and $E(r^*\mathfrak{t}_N)$ are homeomorphic.

Proof. We explicitly construct a homeomorphism:

- 1. $E(\iota^*\mathfrak{t}_M) = \{(n, (m_1, m_2)) \in N \times (M \times M) \mid \iota(n) = m_1\}$
- 2. $E(r^*\mathfrak{t}_N) = \{(m, (n_1, n_2)) \in M \times (N \times N) \mid r(m) = n_1\}$

Now, we have the homeomorphism $\phi: E(\iota^*\mathfrak{t}_M) \to E(r^*\mathfrak{t}_N)$ with $\phi(n, (m_1, m_2)) = (m_2, (r(m_2), n))$ and $\phi^{-1}(m, (n_1, n_2)) = (n_2, (n_2, m))$. We easily see that ϕ suffices all requirements of $E(\iota^*\mathfrak{t}_M)$ and $E(r^*\mathfrak{t}_N)$.

Remark 5.7. *Note that the following diagram commutes*

$$\begin{array}{ccc} N & \longrightarrow & E(\iota^* \mathfrak{t}_M) \\ \downarrow & & \downarrow \phi \\ M & \longrightarrow & E(r^* \mathfrak{t}_N) \end{array}$$

Lemma 5.8. (normal microbundle on total space) There exists a normal microbundle $\mathfrak n$ of N in $E(r^*\mathfrak t_N)$ with $\mathfrak n\cong\iota^*\mathfrak t_M$.

Proof. Obviously, $\mathfrak{n} := r^*\mathfrak{t}_N|_N$ is a normal microbundle of N in $E(r^*\mathfrak{t}_N)$. Since $E(r^*\mathfrak{t}_N|_N) \subseteq E(r^*\mathfrak{t}_N)$, isomorphy follows from the previous lemma and remark.

Finally, we gathered all the tools to prove Milnor's theorem.

Theorem 5.9. (Milnor) For a sufficiently large $q \in \mathbb{N}$, $N = N \times \{0\}$ has a normal microbundle in $M \times \mathbb{R}^q$.

Proof. \Box

6 Homotopy and Microbundles