Bachelorarbeit

Microbundles on Toplogical Manifolds

Florian Burger

Betreut durch Prof. Dr. Markus Banagl

Fakultät für Mathematik und Informatik
Ruprecht-Karls-Universität Heidelberg
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Abstract

This paper presents the concept of microbundles as introduced in 1964 by John Milnor. After showing some basic properties and constructions, including the induced microbundle and the Whitney sum, we discuss tangent- and normal microbundles over topological manifolds. We prove that for every microbundle over a simplicial complex, there exists an inverse in respect to the Whitney sum. Furthermore, we show that homotopic maps yield isomorphic induced microbundles. These results permit the proof that every submanifold $N \subseteq M$ has a normal microbundle in a stabilization $M \times \mathbb{R}^q$ of the surrounding manifold.

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1 Introduction

Motivation

Given a smooth manifold M, one can define the tangent space in a point $p \in M$ over 'derivations' in $\mathcal{L}(C^\infty(M), \mathbb{R}^n)$ or over 'tangent curves' in $C^\infty((-1,1), M)$.

The tangent space allows for the definition of the tangent bundle

$$TM = \bigsqcup_{p \in M} T_pM$$

together with the projection

$$TM \xrightarrow{\pi} M, \pi(p, \nu) = p.$$

Suppose we want to define the tangent bundle over a topological manifold M. As we can see from the two ways of defining the tangent space, the classic tangent bundle requires the notion of differentiability. One intuitive approach would be to equip the topological manifold with a differentiable structure and use the same construction as for smooth manifolds. However, M. Kervaire showed in 1960 that there exists a 10-dimensional manifold that does not admit a differentiable structure [Ker60]. Furthermore, even if there existed a differentiable structure for our topological manifold, it is generally not unique. The 7-sphere for example admits multiple different differentiable structures (see [Mil56]).

Hence, we need to take a different approach than the one for the smooth case.

Another plausible approach would be to define the fibers of the tangent bundle to be of the form $\{p\} \times U_p$, where U_p is a neighborhood of p which is homeomorphic to $\mathbb{R}^d$ via a chart. However, this construction raises the problem of choosing neighborhoods U_p such that they vary continuously over M. Furthermore, it is questionable whether this construction is even a topological invariant if it depends on specific choices of neighborhoods U_p.

In 1964, John Milnor published 'Microbundles, Part I', introducing a unique way to think about tangent bundles over topological manifolds.

The core idea behind this approach is to drop the assumption that the tangent bundle is a vector bundle and hence every fiber must be homeomorphic to euclidean space. Instead, we require that the fibers are 'germs' of euclidean space, i.e. topological spaces with an open subset homeomorphic to euclidean space. In contrast to the previous approach, we can now choose the neighborhoods U_p of p regardless of any corresponding charts. That is because we do not require anymore that these neighborhoods are euclidean spaces. Moreover, each neighborhood of p contains the domain of a chart, hence satisfying our 'germ' condition.

If the respective neighborhoods can be chosen freely, we may as well always choose the entire space M for the sake of simplicity.

We conclude that our resulting total space is of the form $M \times M$, which, analoguous to the smooth case, comes equipped with a projection

$$M \times M \xrightarrow{\pi} M, \pi(m, m') = m.$$

In order to develop this approach, Milnor introduces a new type of bundle that generalizes this idea. He calls them 'microbundles'. Many fundamental properties and constructions of vector bundles carry over other to microbundles, e.g. induced microbundles or the Whitney sum (see Section **(2)**). Moreover, Milnor shows that if a manifold can be equipped with a smooth structure, then the tangent vector bundle regarded as a microbundle is isomorphic to the tangent microbundle.

The theory of microbundles over topological manifolds reaches even further, allowing for the definition of a microbundle analogue to the normal bundle. Given a smooth submanifold $P \subseteq M$, there always exists a normal bundle NP defined fiber-wise by

$$N_pP = T_pM/T_pN.$$

In contrast to this, the two mathematicians C. Rourke and B. Sanderson could construct a 19-dimensional manifold embedded in S^{29} that does not admit a normal microbundle [RS67]. So the existence of a normal microbundle of a submanifold $N \subseteq M$ is not garantueed. Instead, Milnor could show that there always exists a normal microbundle of N in a stabilization $M \times \mathbb{R}^q$ for some $q \in \mathbb{N}$.

Milnors studies on microbundles allowed Robert Williamson to transfer results in cobordism theory developed by Réne Thom, particulary the concept of transverse regularity, from the smooth category over to the piecewise category (see [Wil66]).

This thesis presents the concept of microbundles as introduced in 1964 by John Milnor, providing a proof for the above theorem. It is based on Milnors paper 'Microbundles, Part I'.

Introduction to Microbundles

This subsection introduces the concept of microbundles along with some basic properties. We clarify what a microbundle is, what it means for a microbundle to be trivial and cover some basic examples of microbundles, including the tangent microbundle over topological manifolds.

Throughout this thesis, we require that every manifold is paracompact and second-countable.

Definition 1.1 (microbundle). [Mil64, p.20]
A *microbundle* $\mathfrak{b}$ over B (with *fiber dimension* n) is a diagram $B \xrightarrow{i} E \xrightarrow{j} B$ satisfying the following:

(i) B is a topological space (*base space*)

(ii) E is a topological space (*total space*)

(iii) $i : B \to E$ (*injection*) and $j : E \to B$ (*projection*) are maps such that $j \circ i = id_B$

(iv) Every $b \in B$ is *locally trivializable*, that is there exist open neighborhoods $U \subseteq B$ of b and $V \subseteq E$ of $i(U)$ together with a homeomorphism $\phi : V \xrightarrow{\sim} U \times \mathbb{R}^n$ such that the following diagram commutes:

$$\begin{tikzcd}
 & V \arrow[dr, "j|_V"] \arrow[dd, "\phi"] & \\
U \arrow[ur, "i"] \arrow[dr, "id\times 0"'] & & U \\
 & U \times \mathbb{R}^n \arrow[ur, "\pi_1"'] &
\end{tikzcd}$$

Note that π_1 denotes the projection on the first component $(u, x) \mapsto x$.

As usual, the fiber dimension is well-defined due to the Invariance of Dimension Theorem (see [Bre93, cor.19.10]).

Remark 1.2.
In the following, unless explicitly stated otherwise, we assume the fiber dimension of any given microbundle to be n.

Lemma 1.3.
The diagram $B \xrightarrow{i} E \xrightarrow{j} B$ is locally trivial in $b \in B$ if and only if there exists a homeomorphism

$$\phi : V \xrightarrow{\sim} \phi(V) \subseteq B \times \mathbb{R}^n$$

where V is a neighborhood of $i(b)$ and $\phi(V)$ is neighborhood of $(b, 0)$ such that ϕ commutes as in Definition **(1.1)**.

Proof.
It suffices to show that we can derive local triviality in b assuming only a homeomorphism $\phi : V \xrightarrow{\sim} \phi(V)$ with the properties required above.

Since $\phi(V)$ is a neighborhood of $(b, 0)$, there exists an open subset $U \subseteq B$ and $\varepsilon > 0$ such that $U \times B_\varepsilon(0) \subseteq \phi(V)$.

Note that there exists a homeomorphism

$$\mu_\varepsilon : B_\varepsilon(0) \xrightarrow{\sim} \mathbb{R}^n \text{ with } \mu_\varepsilon(0) = 0,$$

for example given by $\mu_\varepsilon(x) = \tan(\frac{|x|\cdot\pi}{2\varepsilon})x$.

We construct a local trivialization $\phi' : \phi^{-1}(U \times B_\varepsilon(0)) \xrightarrow{\sim} U \times \mathbb{R}^n$ given by $\phi' = \mu_\varepsilon \circ \phi$.

Commutativity with i and $id \times 0$ is given by

$$\phi'(i(b)) = \mu_\varepsilon(\phi(i(b))) = \mu_\varepsilon(b, 0) = (b, 0) = (id \times 0)(b)$$

and with j and π_1 by

$$j(e) = \pi_1(\phi(e)) = \pi_1(\mu_\varepsilon(\phi(e))) = \pi_1(\phi'(e)),$$

which concludes the proof. $\square$

Before we look at some examples for microbundles, we first define what it means for two microbundles to be isomorphic.

Definition 1.4 (isomorphy). [Mil64, p.56]
Two microbundles $\mathfrak{b}_1 : B \xrightarrow{i_1} E_1 \xrightarrow{j_1} B$ and $\mathfrak{b}_2 : B \xrightarrow{i_2} E_2 \xrightarrow{j_2} B$ are *isomorphic* if there exist neighborhoods V_1 of $i_1(B)$ and V_2 of $i_2(B)$ together with a homeomorphism $\psi : V_1 \xrightarrow{\sim} V_2$ such that the following diagram commutes:

$$\begin{tikzcd} & V_1 \arrow[dd, "\psi"] \arrow[dr, "j_1|_{V_1}"] & \\ B \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & B \\ & V_2 \arrow[ur, "j_2|_{V_2}"'] & \end{tikzcd}$$

As the definition of isomorphy already indicates, when studying microbundles, we are not interested in the entire total space but only in an arbitrary small neighborhood of the base space (more precise, the image $i(B)$). The following proposition makes this even clearer.

Proposition 1.5. *[Mil64, p.54]*
Given a microbundle $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ over B, restricting the total space E to an arbitrary neighborhood $E' \subseteq E$ of $i(B)$ leaves the microbundle unchanged. That is, the microbundle

$$\mathfrak{b}' : B \xrightarrow{i} E' \xrightarrow{j|_{E'}} B$$

is isomorphic to $\mathfrak{b}$.

Proof.
We prove the proposition in two steps.

Step 1: $\mathfrak{b}'$ is a microbundle

Continuity for i and j as well as $j|_{E'} \circ i = id_B$ are already given since $\mathfrak{b}$ is a microbundle.

So we only need to show that $\mathfrak{b}'$ is locally trivial. For an arbitrary $b \in B$, choose a local trivialization (U, V, ϕ) of b in $\mathfrak{b}$. By restricting ϕ to $V \cap E'$, we obtain a homeomorphism on its image as required in Lemma **(1.3)**, hence showing local triviality.

Step 2: $\mathfrak{b}'$ is isomorphic to $\mathfrak{b}$

Since E' is a subset of E, we can simply use the identity $E' \to E' \subseteq E$ as our homeomorphism between neighborhoods of $i(B)$. Furthermore, the injection and projection maps for $\mathfrak{b}$ and $\mathfrak{b}'$ are the same, so they clearly commute with the identity.

This completes the proof. $\square$

Example 1.6 (trivial microbundle)**.** [Mil64, p.55]
Given a topological space B, the *standard microbundle* $\mathfrak{e}_B^n$ over B is a microbundle

$$B \xrightarrow{\iota} B \times \mathbb{R}^n \xrightarrow{\pi_1} B$$

where $\iota(b) = (b, 0)$ and $\pi_1(b, x) = b$. A microbundle $\mathfrak{b}$ over B is *trivial* if it is isomorphic to $\mathfrak{e}_B^n$.

Considering microbundles over topological manifolds, which are paracompact hausdorff, triviality has stronger implications for the total space.

Lemma 1.7. *[Mil64, p.57]*
A microbundle $\mathfrak{b}$ over a paracompact hausdorff space B is trivial if and only if there exists an open neighborhood V of $i(B)$ such that $V \cong B \times \mathbb{R}^n$ with injection and projection maps being compatible with this homeomorphism.

This means that there exists an open subset of $E(\mathfrak{b})$ being homeomorphic to the entire $B \times \mathbb{R}^n$, instead of only a neighborhood of $B \times \{0\}$ given by the definition of triviality.

Proof.
We show both implications.

'$\implies$'

By restricting $E(\mathfrak{b})$ to an open neighborhood and applying Proposition **(1.5)** if necessary, we may assume that the entire $E(\mathfrak{b})$ is an open subset of $B \times \mathbb{R}^n$.

Hence, there exist $B_i \subseteq B$ open and $0 < \varepsilon_i < 1$ with $\bigcup_{i \in I} B_i = B$ such that

$$\bigcup_{i \in I} B_i \times B_{\varepsilon_i}(0) \subseteq E(\mathfrak{b}).$$

Without loss of generality, we may assume that the collection $\{B_i\}$ is locally finite by refining this collection if necessary, utilizing the fact that B is paracompact.

Furthermore, B being paracompact hausdorff yields a partition of unity

$$f_i : B \to [0,1] \text{ with } \mathrm{supp} f_i \subseteq B_i$$

such that $\sum_{i \in J} f_i = 1$.

We define a map $\lambda : B \to (0, \infty)$ with $\lambda = \sum_{i \in J} \varepsilon_i f_i$, which has the property that $|x| < \lambda(b) \implies (b,x) \in E(\mathfrak{b})$ because

$$|x| < \lambda(b)$$

$$\begin{aligned} &\iff |x| < \varepsilon_{i_1} f_{i_1}(b) + \cdots + \varepsilon_{i_n} f_{i_n}(b) \\ &\iff 0 < (\varepsilon_{i_1} - |x|) f_{i_1}(b) + \cdots + (\varepsilon_{i_n} - |x|) f_{i_n}(b) \\ &\implies \exists i \in J : 0 < (\varepsilon_i - |x|) f_i(b) \\ &\implies (b,x) \in B_i \times B_{\varepsilon_i}(0) \implies (b,x) \in E(\mathfrak{b}). \end{aligned}$$

Finally, we have a homeomorphism between the open subset $\{(b,x) \in B \times \mathbb{R}^n : |x| < \lambda(b)\} \subseteq E(\mathfrak{b})$ and $B \times \mathbb{R}^n$ via

$$(b,x) \mapsto (b, \frac{x}{\lambda(b) - |x|}).$$

Since $(b,0)$ is mapped to $(b,0)$, it follows that the homeomorphism commutes with the injection and projection maps.

'$\Longleftarrow$'

This is simply a weakening of the definition of triviality. □

Example 1.8 (underlying microbundle)**.** [Mil64, p.55]
Let $\xi : E \xrightarrow{\pi} B$ be a n-dimensional vector bundle. The *underlying microbundle* $|\xi|$ of ξ is a microbundle

$$|\xi| : B \xrightarrow{i} E \xrightarrow{\pi} B$$

where $i : B \to E$ denotes the *zero-cross section* of ξ, that is the section that maps every $b \in B$ to the neutral element 0_b of its fiber $\pi^{-1}(b) \cong \mathbb{R}^n$.

Proof that $|\xi|$ is a microbundle.
First, π is an open map:

Let $V \subseteq E$ be open. For every $b \in \pi(V)$, there exists a neighborhood U_b together with a homeomorphism $\phi_b : \pi^{-1}(U_b) \xrightarrow{\sim} U_b \times \mathbb{R}^n$. It follows that $\pi|_{\pi^{-1}(U_b)} = \pi_1 \circ \phi_b$. Hence, $\pi|_{\pi^{-1}(U_b)}$ is open due to openness of π_1 and ϕ_b.

We conclude from

$$\pi(V) = \bigcup_{b \in B} \pi|_{\pi^{-1}(U_b)}(V)$$

that π is open.

Now from $i^{-1}(V) = \pi(V)$ it follows that i is continuous. Additionally, $\pi \circ i = id_B$ since $\pi(i(b)) = \pi(0_b) = b$.

Local triviality is immediatly inherited from the local triviality condition for vector bundles. □

Definition 1.9 (tangent microbundle)**.** [Mil64, p.55]
The *tangent microbundle* $\mathfrak{t}_M$ over a topological d-manifold M is a manifold

$$M \xrightarrow{\Delta} M \times M \xrightarrow{\pi_1} M$$

where $\Delta(m) = (m, m)$ denotes the diagonal map.

Proof that $\mathfrak{t}_M$ *is a microbundle.*
The maps Δ and π_1 are continuous and clearly $\pi_1 \circ \Delta = id_M$.

For an arbitrary $p \in M$, choose a chart (U, φ) over p. We have a local trivialization $(U, U \times U, \phi)$ of p in $\mathfrak{t}_M$ given by

$$\phi : U \times U \xrightarrow{\sim} U \times \mathbb{R}^n \text{ with } \phi(u, u') = (u, \varphi(u) - \varphi(u')).$$

Since ψ is a homeomorphism, ϕ is a homeomorphism as well.

Commutativity with i and $id \times 0$ is given by

$$\phi(\Delta(m)) = \phi(m, m) = (m, \varphi(m) - \varphi(m)) = (m, 0) = (id \times 0)(m)$$

and with π_1 by

$$\pi_1(u, u') = u = \pi_1(u, \phi^{(2)}(u, u')) = \pi_1(\phi(u, u')).$$

Note that $\phi^{(2)}$ denotes $\pi_2 \circ \phi$. □

Remark 1.10.
The tangent microbundle $\mathfrak{t}_M$ has fiber dimension d.

The following statement is fundamental for the theory of microbundles over topological manifolds. It justifies that the tangent microbundle can be regarded as a generalization of the tangent vector bundle.

Theorem 1.11. *[Mil64, p.56]*
Let M be a smooth d-manifold. Then the underlying microbundle of $\xi : TM \to M$ and the tangent microbundle $\mathfrak{t}_M$ are isomorphic.

Proof.
We equip M with a Riemannian metric, which allows us to define the usual exponential map $\exp : V \to M$ where $V \subseteq TM$ is a neighborhood of the zero-cross section of M.

Consider $id \times \exp$. Using the Inverse Function Theorem for smooth manifolds (see [Lee12, thm.4.5]) for arbitrary $(p, \nu) \in V$, it follows that $id \times \exp$ is a local diffeomorphism and hence a local homeomorphism. Furthermore, the zero-cross section is mapped homeomorphically to the diagonal. By applying Lemma 4.1 from [Whi61, lm.4.1] (manifolds are locally compact and seperable), it follows that $id \times \exp$ maps a neighborhood of the zero-cross section to a neighborhood of the diagonal. Commutativity with the injection maps is given by

$$(id \times \exp)(i_{|\eta|}(p)) = (id \times \exp)(p, 0) = (p, p) = \Delta(p)$$

and with the projection maps by

$$j_{|\eta|}(p, \nu) = p = \pi_1(p, \exp(\nu)) = \pi_1((id \times \exp)(p, \nu)),$$

which concludes the proof. □

2 Standard Constructions

This section introduces two standard constructions for microbundles, the 'induced microbundle' and the 'Whitney sum'. Both constructions have their vector bundle analogue and many results carry over immediatly to microbundles.

Induced Microbundles

Given a microbundle $\mathfrak{b}$ over B and a map $f : A \to B$, one can define a microbundle $f^*\mathfrak{b}$ over A. This is achieved by 'pulling back' the base space B to A with the use of the map f.

After showing the existence of such a microbundle, we prove some basic properties such as triviality criteria and compatibility with map composition. Afterwards, we study induced microbundles over cones and simplicial complexes.

Definition 2.1 (induced microbundle)**.** [Mil64, p.58]
Let $\mathfrak{b}$ be a microbundle over B and let $f : A \to B$ be a map. The *induced microbundle* $f^*\mathfrak{b}$ is a microbundle $A \xrightarrow{i_f} E_f \xrightarrow{j_f} A$ defined as follows:

- $E_f = \{(a, e) \in A \times E(\mathfrak{b}) \mid f(a) = j(e)\}$
- $i_f(a) = (a, (i \circ f)(a))$
- $j_f(a, e) = a$

The construction is identical to the one over vector bundles, more precisely over fiber bundles (compare to [Bre93, ch.2,sec.14]).

Proof that $f^*\mathfrak{b}$ is a microbundle.
Both i_f and j_f are continuous since they are composed by continuous functions. Additionally, $j_f(i_f(a)) = j_f(a, i(f(a))) = a$ and hence $j_f \circ i_f = id_A$.

It remains to be shown that $f^*\mathfrak{b}$ is locally trivial.

For an arbitrary $a \in A$, choose a local trivialization (U, V, ϕ) of $i(a)$ in $\mathfrak{b}$. We construct a local trivialization of a in $f^*\mathfrak{b}$ as follows:

- $U_f = f^{-1}(U) \subseteq A$, which is an open neighborhood of a since f is continuous and U is an open neighborhood of $i(a)$.
- $V_f = (U_f \times V) \cap E_f \subseteq E_f$, which is an open neighborhood of $i_f(a)$ since both $U' \times V$ and E_f are open neighborhoods of $i_f(a)$.
- $\phi_f : V_f \xrightarrow{\sim} U_f \times \mathbb{R}^n$ with $\phi_f(a', e) = (a', \phi^{(2)}(e))$

The existence of an inverse $\phi_f^{-1}(a', v) = (a', \phi^{-1}(f(a'), v))$ and component-wise continuity for both ϕ_f and ϕ_f^{-1} show that ϕ_f is a homeomorphism.

Commutativity with i_f and $id \times 0$ is given by

$$\phi_f(i_f(a')) = \phi(a', i(f(a'))) = (a', \phi^{(2)}(i(f(a')))) = (a', 0) = (id \times 0)(a')$$

and with j_f and π_1 by

$$j_f(a', e) = a' = \pi_1(a', \phi(e)) = \pi_1(\phi_f(a', e)),$$

which completes the proof. □

Example 2.2 (restricted microbundle)**.** [Mil64, p.54]
Let $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ be a microbundle and $A \subseteq B$ be a subspace. The *restricted microbundle* $\mathfrak{b}|_A$ is the induced microbundle $\iota^*\mathfrak{b}$ where $\iota : A \hookrightarrow B$ denotes the inclusion map.

Remark 2.3.
From now on, we consider $E(\mathfrak{b}|_A)$ to be a subset of $E(\mathfrak{b})$. This is justified because there exists an embedding

$$\iota : E(\mathfrak{b}|_A) \to E(\mathfrak{b}) \text{ with } (a, e) \mapsto e$$

together with the inverse $e \mapsto (j(e), e)$. Note that the same reasoning can be applied to induced microbundles over any injective map.

Next, we provide two criteria to show that an induced microbundle is trivial.

Lemma 2.4.
Let $\mathfrak{b}$ be a microbundle over B and $f : A \to B$ be a map. The induced microbundle $f^*\mathfrak{b}$ is trivial if $\mathfrak{b}$ is already trivial.

Proof.
Since $\mathfrak{b}$ is trivial, there exists a homeomorphism $\psi : V \to \psi(V)$ where V is a neighborhood of $i(B)$ and $\psi(V)$ is a neighborhood of $B \times \{0\}$ such that ψ commutes with the injection and projection maps of $\mathfrak{b}$ and $\mathfrak{e}^n_A$.

Consider the map $\psi_f : V_f \xrightarrow{\sim} \psi_f(V_f) \subseteq A \times \mathbb{R}^n$ given by

$$\psi_f(a, e) = (a, \psi^{(2)}(e)),$$

where $V_f = (A \times V) \cap E(f^*\mathfrak{b})$. The existence of an inverse $\psi_f^{-1}(a,x) = (a, \psi^{-1}(f(a),x))$ and component-wise continuity for both ψ_f and ψ_f^{-1} show that ψ_f is a homeomorphism.

The subset V_f is a neighborhood of $i_f(A)$ since $i_f(a) = (a, i(f(a)))$ and $i(f(a)) \in V$. From $\psi^{(2)}(i(f(a))) = 0$ and openness of ψ_f it follows that $\psi_f(V_f)$ is a neighborhood of $A \times \{0\}$. Hence, ψ_f maps a neighborhood of $i_f(A)$ to a neighborhood of $A \times \{0\}$.

Commutativity with the injection maps is given by

$$\psi_f(i_f(a)) = (a, \psi^{(2)}(i(f(a)))) = (a, 0) = (id \times 0)(a)$$

and with the projection maps by

$$j_f(a,e) = a = \pi_1(a, \psi^{(2)}(e)) = \pi_1(\psi_f(a,e)),$$

which completes the proof. □

Lemma 2.5.
Let $\mathfrak{b}$ be a microbundle over B. The induced microbundle $c_{A,b_0}^*\mathfrak{b}$ over the constant map $c_{A,b_0} : A \to B$ with $c_{A,b_0}(a) = b_0$ is trivial.

Proof.
The total space $E(c_{A,b_0}^*\mathfrak{b})$ is defined as

$$\{(a,e) \in A \times E(\mathfrak{b}) : f(a) = b_0 = j(e)\} = A \times j^{-1}(b_0).$$

Let (U, V, ϕ) be a local trivialization for b_0 in $\mathfrak{b}$. Restricting ϕ to the fiber $j^{-1}(b_0)$ yields a homeomorphism

$$\phi|_{j^{-1}(b_0)} : V \cap j^{-1}(b_0) \xrightarrow{\sim} b_0 \times \mathbb{R}^n.$$

It follows that $\psi : A \times (V \cap j^{-1}(b_0)) \xrightarrow{\sim} A \times \mathbb{R}^n$ with $\psi(a,e) = (a, \phi^{(2)}(e))$ is a homeomorphism as well.

The product $A \times (V \cap j^{-1}(b_0))$ is open in $E(c_{A,b_0}^*\mathfrak{b})$, since $V \cap j^{-1}(b_0)$ is open in $j^{-1}(b_0)$ with the subspace topology. Furthermore, from

$$i_c(a) = (a, i(c_{A,b_0}(a))) = (a, i(b_0)) \text{ and } \phi^{(2)}(i(b_0)) = 0$$

it follows that $\psi(A \times (V \cap j^{-1}(b_0)))$ is a neighborhood of $A \times \{0\}$. Hence, ψ maps a neighborhood of $i_c(A)$ to a neighborhood of $A \times \{0\}$.

Commutativity with the injection maps is given by

$$\psi(i_c(a)) = \psi(a, i(b_0)) = (a, \phi^{(2)}(i(b_0))) = (a, 0) = (id \times 0)(a)$$

and with the projection maps by

$$j_c(a,e) = a = \pi_1(a, \phi^{(2)}(e)) = \pi_1(\psi(a,e)).$$

We conclude that $c_{A,b_0}^*\mathfrak{b}$ is trivial. □

The following lemma shows that induced microbundles are compatible with map composition.

Lemma 2.6.
Let $\mathfrak{c} : C \xrightarrow{i} E \xrightarrow{j} C$ *be a microbundle and let* $A \xrightarrow{f} B \xrightarrow{g} C$ *be a map diagram. Then the two microbundles*

$$(g \circ f)^* \mathfrak{c} : A \xrightarrow{i_1} E_1 \xrightarrow{j_1} A \text{ and } f^*(g^* \mathfrak{c}) : A \xrightarrow{i_2} E_2 \xrightarrow{j_2} A$$

are isomorphic.

Proof.
First, compare the two total spaces:

- $E((g \circ f)^*) = \{(a, e) \in A \times E \mid g(f(a)) = j(e)\}$
- $E(f^*(g^* \mathfrak{c})) = \{(a, b, e) \in A \times (B \times E) \mid f(a) = b \text{ and } g(b) = j(e)\}$

We define a homeomorphism $\psi : E((g \circ f)^*) \xrightarrow{\sim} E(f^*(g^* \mathfrak{c}))$ with

$$\psi(a, e) = (a, f(a), e) \text{ and } \psi^{-1}(a, b, e) = (a, e)$$

which is a homeomorphism since both ψ and ψ^{-1} are component-wise continuous. Commutativity with the injection maps is given by

$$\psi(i_1(a)) = \psi(a, i(g(f(a)))) = (a, f(a), i(g(f(a)))) = i_2(a)$$

and with the projection maps by

$$j_1(a, e) = a = j_2(a, f(a), e) = j_2(\psi(a, e)),$$

which concludes the proof. □

In the remainder of this subsection, we study whether microbundles can be extended over particular base spaces.

Here, the term *extended* means that when restricting such a microbundle to the initial base space, we obtain a microbundle that is isomorphic to the initial microbundle.

For a topological space X, we define the *cone* CX of X by

$$X \times [0, 1]/X \times \{1\}$$

and the *mapping cone* $B \sqcup_f CA$ over a map $f : A \to B$ by

$$B \sqcup CA/\sim$$

where $(a, 0) \sim b \iff f(a) = b$.

Similarly, we define the *cylinder* MX of X by

$$X \times [0, 1]$$

and the *mapping cylinder* $B \sqcup_f MA$ over a map $f : A \to B$ by

$$B \sqcup MA/\sim$$

where $(a, 0) \sim b \iff f(a) = b$.

Lemma 2.7. *[Mil64, p.58]*
Let A be a paracompact hausdorff space. A microbundle $\mathfrak{b}$ over B can be extended over the mapping cone $B \sqcup_f CA$ if and only if $f^*\mathfrak{b}$ is trivial.

Proof.
We show both implications.

' $\Longrightarrow$ '

Let $\mathfrak{b}'$ be an extension of $\mathfrak{b}$ over $B \sqcup_f CA$.

The composition $A \xrightarrow{f} B \hookrightarrow B \sqcup_f CA$ is null-homotopic via the homotopy

$$H(a, t) = [a, t],$$

because $H(a, 0) = [a, 0] = [f(a)] = (\iota \circ f)(a)$ and $H(a, 1) = [a, 1] = [\tilde{a}, 1] = H(\tilde{a}, 1)$.

The Homotopy Theorem, which will be proved in Section **(3)**, yields that

$$(\iota \circ f)^*\mathfrak{b}' \cong c^*_{A,[a,1]}\mathfrak{b}'$$

since $\iota \circ f$ is homotopic to $c_{A,[a,1]}$. Together with Lemma **(2.5)**, it follows that $(\iota \circ f)^*\mathfrak{b}'$ is trivial.

Together with $(\iota \circ f)^*\mathfrak{b}' \cong f^*(\iota^*\mathfrak{b}') \cong f^*\mathfrak{b}$, we conclude that $f^*\mathfrak{b}$ is trivial.

' $\Longleftarrow$ '

Let $f^*\mathfrak{b}$ be trivial.

In contrast to the mapping cone, there exists a natural retraction from the mapping cylinder to its attached space

$$r : B \sqcup_f MA \to B \text{ with } r([a, t]) = f(a).$$

The diagram

$$A \times \{1\} \hookrightarrow B \sqcup_f MA \xrightarrow{r} B$$

is equal to f if we consider $A = A \times \{1\}$. From $r^*\mathfrak{b}|_{A\times\{1\}} = (r \circ \iota)^*\mathfrak{b} = f^*\mathfrak{b}$ it follows that $r^*\mathfrak{b}|_{A\times\{1\}}$ is trivial. Furthermore, $r^*\mathfrak{b}|_{A\times[\frac{1}{2},1]}$ is trivial as well due to Lemma **(2.4)** and the retraction $A \times [\frac{1}{2}, 1] \to A \times \{1\}$ with $(a, t) \mapsto (a, 1)$.

Since A is paracompact hausdorff, we can apply Lemma **(1.7)**. Hence, there exists a homeomorphism

$$\psi : W \xrightarrow{\sim} A \times [\frac{1}{2}, 1] \times \mathbb{R}^n$$

where W is a neighborhood of $i_r(A)$ in $E(r^*\mathfrak{b}|_{A\times[\frac{1}{2},1]})$. Without loss of generality, we may assume that $W = E(r^*\mathfrak{b}|_{A\times[\frac{1}{2},1]})$ by removing a closed subset of $E(r^*\mathfrak{b}|_{A\times[\frac{1}{2},1]})$ if necessary and applying Proposition **(1.5)**.

We define an extended microbundle $\mathfrak{b}' : B \sqcup_f CA \xrightarrow{i'} E' \xrightarrow{j'} B \sqcup_f CA$ by

- $E' = E(r^*\mathfrak{b})/\psi^{-1}(A \times \{1\} \times \{x\})$ (for every $x \in \mathbb{R}^n$)
- $i'([a,t]) = [i_r([a,t])]$
- $j'([[a,t],e]) = [j_r([a,t],e)] = [a,t]$

The injection i' is well-defined because i_r maps every representative $[a,1]$ to the same equivalence class of E' due to its construction. Similarly, the projection j' is well-defined because

$$[[a,t],e] = [[a',t'],e'] \implies [a,t] = [a',t'].$$

Both i' and j' are continuous due to the definition of the quotient space topology. Also, $j' \circ i' = id_{B\sqcup_f CA}$ because

$$j'(i'([a,t])) = j'([i_r(a,t)]) = [j_r(i_r(a,t))] = [a,t].$$

It remains to be shown that $\mathfrak{b}'$ is locally trivial. Let $[a,t] \in B\sqcup_f CA$ be arbitrary. Note that, by construction, the cone and the cylinder differ only in an arbitrary small neighborhood of $\{[a,1] : a \in A\}$.

We consider two cases.

Case 1: $t \leq \frac{1}{2}$

We simply use a local trivialization (U,V,ϕ) of $[a,t]$ in $r^*\mathfrak{b}$ by restricting U to $A \times [0,\frac{1}{2})$ if necessary. This is valid since the cone and the cylinder equal if restricted to $A \times [0,\frac{1}{2})$.

Case 2: $t > \frac{1}{2}$

In this case, we can use ψ to serve as the homeomorphism for our local trivialization. By construction, ψ respects the projection $\pi : E(r^*\mathfrak{b}) \twoheadrightarrow E'$. It follows that $(\psi^{(1)}(\pi(W)), \pi(W), \psi)$ is a local trivialization for $[a,t]$ in $\mathfrak{b}'$.

This completes the proof. $\square$

We can derive a statement from this about extending a microbundle over a simplex by utilizing that the cone of the boundary of a simplex is homeomorphic to the simplex itself.

Corollary 2.8.
Let B be a $(d+1)$-simplicial complex, B' its d-skeleton and $\Delta \subseteq B$ a $(d+1)$-simplex. A microbundle $\mathfrak{b}$ over B' can be extended to a microbundle over $B' \cup \Delta$ if and only if its restriction to the boundary $\mathfrak{b}|_{\partial\Delta}$ is trivial.

Proof.
With $f : \partial\Delta \hookrightarrow B'$ and the previous lemma, it follows that there exists a microbundle $\mathfrak{b}'$ over $B' \cup_f C\partial\Delta$ extending $\mathfrak{b}$ if and only if $f^*\mathfrak{b} = \mathfrak{b}|_{\partial\Delta}$ is trivial.

We have a homeomorphism $\phi : C\partial\Delta \xrightarrow{\sim} \Delta$ with

$$\phi((t_1, \dots, t_{d+1}), \lambda) = (1-\lambda)(t_1, \dots, t_{d+1}) + \frac{\lambda}{d+1}(1, \dots, 1).$$

Particulary, $\phi(\partial\Delta \times \{0\}) = \partial\Delta$.

It follows that $B' \cup_f C\partial\Delta \cong B' \cup \Delta$, which concludes the proof. □

The Whitney Sum

Given two vector bundles E and F over the same base space X, one can define the Whitney sum $E \oplus F$ by forming the direct sum of the individual fibers E_x and F_x, hence the notation.

This construction carries over to microbundles, as elaborated in the following. The centerpiece of this section will be Theorem **(2.13)**, which states that for microbundles over simplicial complexes, one can find an 'inverse' microbundle such that their Whitney sum is trivial.

Definition 2.9. [Mil64, p.59]
Let $\mathfrak{b}_1$ and $\mathfrak{b}_1$ be two microbundles over B with fiber dimensions n_1 and n_2. The *Whitney sum* $\mathfrak{b}_1 \oplus \mathfrak{b}_2$ is a microbundle $B \xrightarrow{i} E \xrightarrow{j} B$ where

- $E = \{(e_1, e_2) \in E(\mathfrak{b}_1) \times E(\mathfrak{b}_2) : j_1(e_1) = j_2(e_2)\}$
- $i(b) = (i_1(b), i_2(b))$
- $j(e_1, e_2) = j_1(e_1) = j_2(e_2)$

with fiber dimension $n_1 + n_2$.

Proof that $\mathfrak{b}_1 \oplus \mathfrak{b}_2$ is a microbundle.
Both i and j are continuous since they are composed by continuous functions. Additionally, $j(i(b)) = j(i_1(b), i_2(b)) = j_1(i_1(b)) = b$ and hence $j \circ i = id_B$.

It remains to be shown that $\mathfrak{b}_1 \oplus \mathfrak{b}_2$ is locally trivial:

For an arbitrary $b \in B$, choose local trivializations (U_1, V_1, ϕ_1) and (U_2, V_2, ϕ_2) of b in $\mathfrak{b}_1$ and $\mathfrak{b}_2$.

We construct a local trivialization (U, V, ϕ) of b in $\mathfrak{b}_1 \oplus \mathfrak{b}_2$ as follows:

- $U = U_1 \cap U_2$, which is an open neighborhood of b since both U_1 and U_2 are open neighborhoods of b.
- $V = (V_1 \times V_2) \cap E$, which is an open neighborhood of $i(U)$ since V_1 and V_2 are open and $i(U) \subseteq (i_1(U) \times i_2(U)) \cap E \subseteq (V_1 \times V_2) \cap E$.

- $\phi : V \xrightarrow{\sim} U \times \mathbb{R}^{n_1+n_2}$ with $\phi(e_1, e_2) = (j_1(e_1), (\phi_1^{(2)}(e_1), \phi_2^{(2)}(e_2)))$, which is a homeomorphism with the inverse

$$\phi^{-1}(b, (x_1, x_2)) = (\phi_1^{-1}(b, x_1), \phi_2^{-1}(b, x_2))$$

since both ϕ and ϕ^{-1} are component-wise continuous.

Commutativity with i and $id \times 0$ is given by

$$\phi(i(b)) = \phi(i_1(b), i_2(b)) = (b, (\phi_1^{(2)}(i_1(b)), \phi_2^{(2)}(i_2(b)))) = (b, (0, 0)) = (id \times 0)(b)$$

and with j and π_1 by

$$j(e_1, e_2) = j_1(e_1) = \pi_1(j_1(e_1), \phi^{(2)}(e_1, e_2)) = \pi_1(\phi(e_1, e_2)),$$

which completes the proof. □

Remark 2.10.
The Whitney sum is associative and commutative.

Alternatively, one could define the Whitney sum between $\mathfrak{b}_1$ and $\mathfrak{b}_2$ to be the induced microbundle $\Delta^*(\mathfrak{b}_1 \times \mathfrak{b}_2)$ where Δ denotes the diagonal map and $\mathfrak{b}_1 \times \mathfrak{b}_2$ denotes the intuitive cross-product between the two microbundles.

Lemma 2.11.
Let $\mathfrak{b}_1$ and $\mathfrak{b}_1$ be two microbundles over B and let $f : A \to B$ be a map. The induced microbundle and the Whitney sum are compatible, that is

$$f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2) \cong f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2.$$

Proof.
The total space $E(f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2))$ is defined as

$$\{(a, (e_1, e_2)) \in A \times (E(\mathfrak{b}_1) \times E(\mathfrak{b}_2)) \mid j_1(e_1) = j_2(e_2) = f(a)\}$$

and $E(f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2)$ as

$$\{((a_1, e_1), (a_2, e_2)) \in E(f^*\mathfrak{b}_1) \times E(f^*\mathfrak{b}_2) : j_1(a_1, e_1) = j(a_2, e_2)\}.$$

We have a homeomorphsim $\psi : E(f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2)) \xrightarrow{\sim} E(f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2)$ given by

$$\psi(a, (e_1, e_2)) = ((a, e_1), (a, e_2))$$

with the inverse $\psi^{-1}((a, e_1), (a, e_2)) = (a, (e_1, e_2))$. Since both ψ and ψ^{-1} are component-wise continuous, ψ is a homeomorphism.

Commutativity with the injection maps is given by

$$\psi(i_f(a)) = \psi(a, (i_1(f(a)), i_2(f(a)))) = ((a, i_1(f(a))), (a, i_2(f(a)))) = i_\oplus(a)$$

and with the projection maps by

$$j_f(a,(e_1,e_2)) = a = j_\oplus((a,e_1),(a,e_2)) = j_\oplus(\psi(a,(e_1,e_2))).$$

Here, i_f and j_f denote the injection and projection for $f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2)$ and $i_\oplus$ and $j_\oplus$ the injection and projection for $f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2$. $\square$

In the remainder of this section, we show the above mentioned theorem about the existence of an 'inverse' in respect to the Whitney sum. This statement is essential for the proof of Milnors Theorem **(5.6)**.

In order to show this, we require the following proposition, whose proof will be deferred until Section **(4)**.

Proposition 2.12 (Bouquet Lemma)**.** *[Mil64, p.59]*
Let $\mathfrak{b}$ be a microbundle over a 'bouquet' of spheres B, meeting in a single point. Then there exists a map $r : B \to B$ such that $\mathfrak{b} \oplus r^*\mathfrak{b}$ is trivial.

Theorem 2.13. *[Mil64, p.59]*
Let $\mathfrak{b}$ be a microbundle over a d-simplicial complex B. Then there exists a microbundle $\mathfrak{n}$ over B such that the Whitney sum $\mathfrak{b} \oplus \mathfrak{n}$ is trivial.

Proof.
We prove the theorem by induction over d.

(Start of induction)

A 1-simplicial complex is just a bouquet of circles. Hence, the start of induction follows directly from Proposition **(2.12)**.

(Inductive Step)

Let B' be the $(d-1)$-skeleton of B and let $\mathfrak{n}'$ be its corresponding microbundle such that $\mathfrak{b}|_{B'} \oplus \mathfrak{n}'$ is trivial.

Step 1: $\mathfrak{n}' \oplus \mathfrak{e}^n_{B'}$ can be extended over any d-simplex σ

Consider the equation

$$(\mathfrak{n}' \oplus \mathfrak{e}^n_{B'})|_{\partial\sigma} = \mathfrak{n}'|_{\partial\sigma} \oplus \mathfrak{e}^n_{B'}|_{\partial\sigma} = \mathfrak{n}'|_{\partial\sigma} \oplus \mathfrak{b}|_{\partial\sigma} = (\mathfrak{n}' \oplus \mathfrak{b}|_{B'})|_{\partial\sigma}$$

in which we used Corollary **(2.8)** for $\mathfrak{e}^n_{B'}|_{\partial\sigma} = \mathfrak{b}|_{\partial\sigma}$. Since $(\mathfrak{n}' \oplus \mathfrak{b}|_{B'})|_{\partial\sigma}$ is trivial, the claim follows from Corollary **(2.8)**.

Step 2: $\mathfrak{n}' \oplus \mathfrak{e}^n_{B'}$ can be extended over B

One difficulty is that the individual d-simplices are not well-seperated. To deal with this, we consider B'' which is defined to be B with small open d-cells removed from every d-simplex. Since B' is a retract of B'', we can extend $\mathfrak{n}' \oplus \mathfrak{e}^n_{B'}$ to a microbundle ν over B''.

Now we extend ν over B by taking all extensions of ν over every simplex using (Step 1), and identifying its total spaces together along $E(\nu)$. Similarly, injection and projection are obtained by piecing the injection and projection maps of the individual extensions together.

We denote the resulting microbundle by η.

Step 3:
Consider the mapping cone $B \sqcup_\iota CB'$ over the inclusion $B' \hookrightarrow B$. The following equation shows that $(\mathfrak{b} \oplus \eta)|_{B'}$ is trivial:

$$(\mathfrak{b} \oplus \eta)|_{B'} \cong \mathfrak{b}|_{B'} \oplus \eta|_{B'} \cong \mathfrak{b}|_{B'} \oplus (\mathfrak{n}' \oplus \mathfrak{e}^n_{B'}) \cong (\mathfrak{b}|_{B'} \oplus \mathfrak{n}') \oplus \mathfrak{e}^n_{B'} \cong \mathfrak{e}^n_{B'} \oplus \mathfrak{e}^n_{B'}$$

Lemma **(2.7)** then yields a microbundle ξ over $B \sqcup_\iota CB'$ extending $\mathfrak{b} \oplus \eta$.

The mapping cone $B \sqcup_\iota CB'$ has the homotopy type of a bouquet of spheres, which can be seen as follows:

- A d-simplex is homotopic to a d-disc.
- A d-disc whose boundary is collapsed to a single point is a d-sphere.
- One can define a homotopy between $\iota(B')$ and the tip of the cone by traveling along CB'.

Using Theorem **(3.1)** and Proposition **(2.12)**, we conclude that there exists a microbundle $\mathfrak{n}$ such that $(\xi \oplus \mathfrak{n})|_B$ is trivial. The equation

$$\mathfrak{e}^n_B = (\xi \oplus \mathfrak{n})|_B = \xi|_B \oplus \mathfrak{n}|_B = (\mathfrak{b} \oplus \eta) \oplus \mathfrak{n}|_B = \mathfrak{b} \oplus (\eta \oplus \mathfrak{n}|_B)$$

completes the proof.

□

3 The Homotopy Theorem

In this section we will prove the Homotopy Theorem, which is a fundamental result over microbundles. It states the following.

Theorem 3.1 (Homotopy Theorem). *[Mil64, p.58]*
Let $\mathfrak{b}$ be a microbundle over B and let $f, g : A \to B$ be two maps where A is paracompact hausdorff. If f and g are homotopic, then $f^*\mathfrak{b}$ and $g^*\mathfrak{b}$ are isomorphic.

In order to prove this theorem, we introduce the concept of map-germs over microbundles which provide another way to think about isomorphy besides Definition **(1.4)**.

Map-Germs

Definition 3.2 (map-germ). [Mil64, p.65]
A *map-germ* $F : (X, A) \Rightarrow (Y, B)$ between topological pairs (X, A) and (Y, B) is an equivalence class of maps $(X, A) \to (Y, B)$ where $f \sim g \iff f|_U = g|_U$ for an arbitrary neighborhood $U \subseteq X$ of A.

We can form the composition of two map-germs $F : (X, A) \Rightarrow (Y, B)$ and $G : (Y, B) \Rightarrow (Z, C)$ by choosing representatives $f : U_f \to Y$ and $g : U_g \to Z$ and defining $(f \circ g)|_{f^{-1}(U_g)}$ to be a representative for $G \circ F$.

Definition 3.3 (homeomorphism-germ)**.** [Mil64, p.65]
A *homeomorphism-germ* $F : (X, A) \Rightarrow (Y, B)$ is a map-germ such that there exists a representative $f : U_f \to Y$ that maps U_f homeomorphically to a neighborhood of B.

Let $\mathfrak{b}$ and $\mathfrak{b}'$ be two isomorphic microbundles over B. There exists a homeomorphism $\psi : V \xrightarrow{\sim} V'$ where $V \subseteq E(\mathfrak{b})$ is a neighborhood of $i(B)$ and $V' \subseteq E(\mathfrak{b}')$ is a neighborhood of $i'(B)$. We can view ψ as a representative for a homeomorphism-germ

$$[\psi] : (E, i(B)) \Rightarrow (E', i'(B)).$$

Studying isomorphy between $\mathfrak{b}$ and $\mathfrak{b}'$ using map-germs is useful because we do not care what ψ does on its initial domain, but only what it does on arbitrary small neighborhoods of $i(B)$. Hence, every representative of $[\psi]$ describes the 'same' isomorphy between $\mathfrak{b}$ and $\mathfrak{b}'$. Now, naturally, the question arises whether the existence of a homeomorphism-germ

$$F : (E, i(B)) \Rightarrow (E', i'(B))$$

already implies that $\mathfrak{b}$ and $\mathfrak{b}'$ are isomorphic. The answer is generally no, because isomorphy between microbundles additionally requires the homeomorphism to commute with the injection and projection maps. Hence, we need an extra condition ('fiber-preservation') for this implication to be true. This justifies the following definition.

Let $J : (E(\mathfrak{b}), i(B)) \Rightarrow (B, B)$ and $J' : (E(\mathfrak{b}'), i(B)) \Rightarrow (B, B)$ denote the map-germs represented by the projections of $\mathfrak{b}$ and $\mathfrak{b}'$.

Definition 3.4 (isomorphism-germ)**.** [Mil64, p.65]
An *isomorphism-germ* between $\mathfrak{b}$ and $\mathfrak{b}'$ is a homeomorphism-germ

$$F : (E(\mathfrak{b}), B) \Rightarrow (E(\mathfrak{b}'), B)$$

which is *fiber-preserving*, that is $J' \circ F = J$.

Remark 3.5. [Mil64, p.65]
There exists an isomorphism-germ between $\mathfrak{b}$ and $\mathfrak{b}'$ if and only if $\mathfrak{b}$ and $\mathfrak{b}'$ are isomorphic.

We can take this even further by dropping the assumption that the two microbundles have the same base space. Note that in this case no comparison to isomorphy can be drawn, because we have not defined isomorphy between microbundles over different base spaces.

Definition 3.6 (bundle-germ)**.** [Mil64, p.66]
Let $\mathfrak{b}$ and $\mathfrak{b}'$ be two microbundles over B and B' with the same fiber dimension. A *bundle-germ* $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$ is a map-germ

$$F : (E(\mathfrak{b}), B) \Rightarrow (E(\mathfrak{b}'), B')$$

such that there exists a representative $f : U_f \to E(\mathfrak{b}')$ that maps each fiber $j^{-1}(b)$ injectively to a fiber $j'^{-1}(b')$.

For a bundle-germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$, the following diagram commutes:

$$\begin{tikzcd}
(E(\mathfrak{b}), B) \arrow[r, Rightarrow, "F"] \arrow[d, "i"] & (E(\mathfrak{b}'), B') \arrow[d, "i'"] \\
B \arrow[r, "F|_B"] & B'
\end{tikzcd}$$

We say F *is covered by* $F|_B$.

The bundle-germ is indeed a generalization for the isomorphism-germ, as the following proposition shows.

Proposition 3.7 (Williamson)**.** *[Mil64, p.66]*
Let $\mathfrak{b}$ and $\mathfrak{b}'$ be two microbundles over B. A bundle-germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$ covering the identity map is an isomorphism-germ.

The following lemma will be necessary for the proof of the proposition.

Lemma 3.8.
If a homeomorphism $f : \overline{B_2(0)} \xrightarrow{\sim} V \subseteq \mathbb{R}^n$ satisfies

$$|f(x) - x| < 1, \forall x \in \overline{B_2(0)},$$

then $\overline{B_1(0)} \subseteq V$.

Proof of the lemma.
We provide a proof by contradiction.

Suppose $v \in \overline{B_1(0)} - V$. Let $[f(0), v]$ denote the line-segment

$$\{\lambda f(0) + (1 - \lambda)v : \lambda \in [0, 1]\} \subseteq \mathbb{R}^n.$$

The intersection $[f(0), v] \cap V$ is compact since both $[f(0), v]$ and V are compact. Hence, the intersection has a maximum v' when ordered via λ. Note that the intersection is non-empty since $f(0) \in V$ and that $v' \neq v$ since $v \notin V$ by definition.

The maximum v' is contained in ∂V:

1. $v' \in [f(0), v] \cap V \subseteq V = \overline{V}$

2. $v' \notin \overset{\circ}{V}$, because otherwise $B_\varepsilon(v') \subseteq \overset{\circ}{V}$ for some $\varepsilon > 0$ which contradicts with v' being the maximum in $[f(0), v] \cap V$.

From $|f(0) - 0| < 1$ and $v \in \overline{B_1(0)}$ it follows that $|v'| < 1$.

This leads to a contradiction, because

$$f^{-1}(v') \in \partial\overline{B_2(0)} \implies |f^{-1}(v') - v'| = 2 - |v'| > 2 - 1 = 1.$$

□

Proof of the proposition.
Let f be a representative for F. First we assume a special and then generalize the result to show the proposition.

Step 1: Let f map from $B \times \mathbb{R}^n$ to $B \times \mathbb{R}^n$

Since F covers the identity, f is of the form

$$f(b, x) = (b, g_b(x))$$

where $g_b : \mathbb{R}^n \to \mathbb{R}^n$ are individual maps. Since the g_b are continuous and injective, it follows from the Invariance of Domain Theorem (see [Bre93, cor.19.9]) that the g_b are open maps.

Let $(b_0, x_0) \in B \times \mathbb{R}^n$ and let $\varepsilon > 0$. Since g_{b_0} is an open map, there exists a $\delta > 0$ such that $\overline{B_{2\delta}(x_1)} \subseteq g_{b_0}(\overline{B_\varepsilon(x_0)})$ where $x_1 = g_{b_0}(x_0)$.

We claim that there exists a neighborhood $V \subseteq B$ of b_0 such that

$$|g_b(x) - g_{b_0}(x)| < \delta$$

for every $b \in V$ and $x \in \overline{B_\varepsilon(x_0)}$.

To show that, consider $\phi(b, x) = g_b(x) - g_{b_0}(x)$. The open set $\phi^{-1}(B_\delta(0))$ is a neighborhood of $\{b_0\} \times \mathbb{R}^n$ since $\phi(b_0, x) = 0$. Hence, there exist open subsets $V_x \subseteq B$ and $W_x \subseteq \mathbb{R}^n$ such that

$$\bigcup_{x \in \overline{B_\varepsilon(x_0)}} V_x \times W_x \subseteq \phi^{-1}(\overline{B_\delta(0)})$$

and $x \in W_x$. Since $\overline{B_\varepsilon(x_0)}$ is compact, there exist $x_1, \dots, x_n \in \overline{B_\varepsilon(x_0)}$ with $\overline{B_\varepsilon(x_0)} \subseteq \bigcup_{i=1}^n V_{x_i}$. The claim follows with $V = V_{x_1} \cap \dots \cap V_{x_n}$ which is open by forming the intersection over finitely many open sets.

Now we want to apply the previous lemma.

Consider the homeomorphism $(g_b \circ g_{b_0}^{-1})|_{\overline{B_{2\delta}(x_1)}}$ for an arbitrary $b \in V$. Together with

$$\overline{B_{2\delta}(x_1)} \subseteq g_{b_0}(\overline{B_\varepsilon(x_0)}) \implies g_{b_0}^{-1}(\overline{B_{2\delta}(x_1)}) \subseteq \overline{B_\varepsilon(x_0)},$$

we conclude from the above that

$$|(g_b \circ g_{b_0}^{-1})(x) - x| < \delta, \forall x \in \overline{B_{2\delta}(x_1)}.$$

It follows that, by translation and scaling, $g_b \circ g_{b_0}^{-1}|_{\overline{B_{2\delta}(x_1)}}$ satisfies the conditions of Lemma **(3.8)**. Therefore, $\overline{B_\delta(x_1)} \subseteq (g_b \circ g_{b_0}^{-1})(\overline{B_{2\delta}(x_0)})$ and hence $\overline{B_\delta(x_1)} \subseteq g_b(\overline{B_\varepsilon(x_0)})$. From

$$V \times \overline{B_\delta(x_1)} \subseteq g(V \times \overline{B_\varepsilon(x_0)})$$

it follows that f is an open map.

Step 2: Glueing together $f : U_f \to E(\mathfrak{b}')$ along local trivializations

For an arbitrary $b \in B$, choose local trivializations (U, V, ϕ) and (U', V', ϕ') of b in $\mathfrak{b}$ and $\mathfrak{b}'$. Without loss of generality, we may assume that $U = U'$ by choosing $V = \phi^{-1}(U \cap U')$ and $V' = \phi'^{-1}(U \cap U')$ and restricting ϕ and ϕ' accordingly if necessary.

First, we restrict f to $V \cap f^{-1}(V')$. Since $V \cap f^{-1}(V')$ is an open neighborhood of $i(b)$, we can choose an open neighborhood $U_b \subseteq U$ of $i(b)$ and $\varepsilon > 0$ such that $\phi^{-1}(U_b \times B_\varepsilon(0)) \subseteq V \cap f^{-1}(V')$.

We define a map $U_b \times \mathbb{R}^n \to U_b \times \mathbb{R}^n$ given by

$$U_b \times \mathbb{R}^n \cong U_b \times B_\varepsilon(0) \xrightarrow{\phi \circ f \circ \phi^{-1}} U_b \times \mathbb{R}^n$$

that is injective and fiber-preserving and hence an open map (Step 1). It follows that $f : \phi^{-1}(U_b \times B_\varepsilon(0)) \to V'$ must be an open map as the other composing maps are homeomorphisms.

We conclude from

$$f = \bigcup_{b \in B} f|_{\phi^{-1}(U_b \times B_\varepsilon(0))}$$

that f is an open map.

This completes the proof. □

We can easily generalize this result to bundle-germs between microbundles over different base spaces:

Corollary 3.9. *[Mil64, p.67]*
If a map $g : B \to B'$ is covered by a bundle-germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$, then $\mathfrak{b}$ is isomorphic to the induced microbundle $g^*\mathfrak{b}'$.

Proof.
Let $f : U_f \to E'$ be a representative map for F. We define $F' : \mathfrak{b} \Rightarrow g^*\mathfrak{b}'$ by the representative

$$f' : U_f \to E(g^*\mathfrak{b}') \text{ with } f'(e) = (j(e), f(e)).$$

Every $f'(e)$ lies in $E(g^*\mathfrak{b}')$ because $g(j(e)) = j'(f(e))$ as we can see from the commutative diagram for bundle-germs.

The germ F' is a bundle-germ covering the identity because

$$j(e) = j'_g(j(e), f(e)) = j'_g(f'(e))$$

and because f' is injective (f is injective). Applying the previous proposition on F' proves the claim. $\square$

Proving the Homotopy Theorem

Lemma 3.10. *[Mil64, p.67]*
Let $\mathfrak{b}$ be a microbundle over B and let $\{B_\alpha\}$ be a locally finite collection of closed sets covering B. Additionally, we are given a collection of bundle-germs $F_\alpha : \mathfrak{b}|_{B_\alpha} \Rightarrow \mathfrak{b}'$ such that $F_\alpha = F_\beta$ on $\mathfrak{b}|_{B_\alpha \cap B_\beta}$. Then there exists a bundle-germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$ extending F_α, that is representatives for F and F_α agree on a sufficiently small neighborhood of $i(B_\alpha)$.

Proof.
Choose representative maps $f_\alpha : U_\alpha \to E(\mathfrak{b}')$ for F_α such that the U_α are open. For every α and β, choose an open neighborhood $U_{\alpha\beta}$ of $i(B_\alpha \cap B_\beta)$ on which the representative maps f_α and f_β agree. Now consider

$$U = \{e \in E(\mathfrak{b}) : j(e) \in B_\alpha \cap B_\beta \implies e \in U_{\alpha\beta}\}$$

which is an open neighborhood of $i(B)$:

1. U is open

   Let $e \in U$ be arbitrary.

   Since $\{B_\alpha\}$ is locally finite, there exists an open neighborhood V of $j(e)$ that intersects with only finitely many $B_{\alpha_1}, \ldots, B_{\alpha_n}$. Note that from $e \in U$ it follows that $e \in U_{\alpha_i \alpha_j}, \forall 1 \leq i, j \leq n$.

   We are given an open neighborhood of e by

   $$\bigcap_{1 \leq i,j \leq n} U_{\alpha_i \alpha_j} \cap j^{-1}(V),$$

   which is contained in U by construction.

2. $i(B) \subseteq U$

   This follows from $j(i(b)) = b \in B_\alpha \cap B_\beta \implies i(b) \in i(B_\alpha \cap B_\beta) \subseteq U_{\alpha\beta}$.

We define a map $f : U \to E(\mathfrak{b}')$ by

$$f(e \in U_{\alpha\beta}) = f_\alpha(e) = f_\beta(e)$$

which is well-defined due to the construction of U. Note that f agrees with every f_α on their intersection and is continuous as

$$f^{-1}(V) = \bigcup_\alpha f_\alpha|_U^{-1}(V).$$

Taking f as a representative for a bundle-germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$ completes the proof. $\square$

Lemma 3.11. *[Mil64, p.67]*
Let $\mathfrak{b}$ be a microbundle over $B \times [0,1]$. If $\mathfrak{b}|_{B\times[0,\frac{1}{2}]}$ and $\mathfrak{b}|_{B\times[\frac{1}{2},1]}$ are both trivial, then $\mathfrak{b}$ itself is trivial.

Proof.
Consider the identity bundle-germ over $\mathfrak{b}|_{B\times\{\frac{1}{2}\}}$, which is defined as the bundle-germ represented by the identity map on $E(\mathfrak{b}|_{B\times\{\frac{1}{2}\}})$.

Since $\mathfrak{b}|_{B\times[\frac{1}{2},1]}$ and $\mathfrak{b}|_{B\times\{\frac{1}{2}\}}$ are both trivial, there exist isomorphism-germs

$$R : \mathfrak{b}|_{B\times[\frac{1}{2},1]} \Rightarrow \mathfrak{e}^n_{B\times[\frac{1}{2},1]} \text{ and } L : \mathfrak{b}|_{B\times\{\frac{1}{2}\}} \Rightarrow \mathfrak{e}^n_{B\times\{\frac{1}{2}\}}.$$

We define a bundle-germ $M : \mathfrak{e}^n_{B\times[\frac{1}{2},1]} \Rightarrow \mathfrak{e}^n_{B\times\{\frac{1}{2}\}}$ represented by

$$(b,t,x) \mapsto (b, \frac{1}{2}, x).$$

The composition $L^{-1} \circ M \circ R$ then yields a bundle-germ $\mathfrak{b}|_{B\times[\frac{1}{2},1]} \Rightarrow \mathfrak{b}|_{B\times\{\frac{1}{2}\}}$ that extends the identity on $\mathfrak{b}|_{B\times\{\frac{1}{2}\}}$.

Using the previous lemma, we can glue this together with the identity over $\mathfrak{b}|_{B\times[0,\frac{1}{2}]}$ (note that the bundle-germs agree on $\mathfrak{b}|_{B\times\{\frac{1}{2}\}}$) resulting in a bundle-germ $\mathfrak{b} \Rightarrow \mathfrak{b}|_{B\times[0,\frac{1}{2}]}$.

Corollary **(3.9)** infers that $\mathfrak{b}$ is isomorphic to $r^*\mathfrak{b}|_{B\times[0,\frac{1}{2}]}$ where

$$r : B \times [0,1] \to B \times [0, \frac{1}{2}] \text{ with } r(b,t) = (b, \min(t, \frac{1}{2})).$$

But $\mathfrak{b}|_{B\times[0,\frac{1}{2}]}$ is trivial, hence $r^*\mathfrak{b}|_{B\times[0,\frac{1}{2}]}$ is trivial as well (see Lemma **(2.4)**). We conclude that $\mathfrak{b}$ is trivial. $\square$

Lemma 3.12. *[Mil64, p.67]*
Let $\mathfrak{b}$ be a microbundle over $B \times [0,1]$. Then every $b \in B$ has a neighborhood V such that $\mathfrak{b}|_{V\times[0,1]}$ is trivial.

Proof.
Let $b \in B$ be arbitrary.

For every $t \in [0,1]$, assume a neighborhood $U_t = V_t \times (t - \varepsilon_t, t + \varepsilon_t)$ of (b,t) such that $\mathfrak{b}|_{U_t}$ is trivial. Such a neighborhood can be constructed by taking a local trivialization (U', V', ϕ') of (b,t) in $\mathfrak{b}$ and restricting U' accordingly.

Since $\{b\} \times [0,1]$ is compact, we can choose finitely many

$$V_1 \times (t_1 - \varepsilon_1, t_1 + \varepsilon_1), \ldots, V_n \times (t_n - \varepsilon_n, t_n + \varepsilon_n)$$

covering $\{b\} \times [0,1]$ and define $V = V_1 \cap \cdots \cap V_n$.

The restricted microbundles $\mathfrak{b}|_{V \times (t_i - \varepsilon_i, t_i + \varepsilon_i)}$ are trivial as every $\mathfrak{b}|_{U_t}$ is trivial and $V \times (t_i - \varepsilon_i, t_i + \varepsilon_i) \subseteq U_t$. It follows that there exists a subdivision $0 = t_0 < \cdots < t_k = 1$ such that every $\mathfrak{b}|_{V \times [t_i, t_{i+1}]}$ is trivial.

By iteratively applying the previous lemma on the $\mathfrak{b}|_{V \times [t_i, t_{i+1}]}$, we conclude that $\mathfrak{b}|_{V \times [0,1]}$ is trivial. □

Lemma 3.13.
Let B be a paracompact space and let $\{V_\alpha\}$ be a locally finite open cover of B. Then there exists a locally finite closed cover $\{\overline{B_\beta}\}$ of B such that every $\overline{B_\beta}$ intersects with only finitely many $V_{\alpha_1}, \ldots V_{\alpha_n}$.

Proof.
For every $b \in B$, there exists an open neighborhood U_b of b that intersects with only finitely many $V_{\alpha_1}, \ldots V_{\alpha_k}$ due to local finiteness of $\{V_\alpha\}$. Clearly, the collection $\{U_b\}$ over all $b \in B$ covers B.

Since B is paracompact, there exists a locally finite subcover $\{B_\beta\}$.

The collection $\{\overline{B_\beta}\}$ then meets our requirements:

1: $\{\overline{B_\beta}\}$ is locally finite

For an arbitrary $b \in B$, let W_b be an open neighborhood of b that intersects only finitely many $B_{\beta_1}, \ldots, B_{\beta_k}$. Now W_b intersects only $\overline{B_{\beta_1}}, \ldots, \overline{B_{\beta_1}}$, because

$$\begin{aligned} W_b \cap B_\beta &= \emptyset \\ \implies B_\beta &\subseteq B - W_b \\ \implies \overline{B_\beta} &\subseteq \overline{B - W_b} = B - W_b \\ \implies W_b \cap \overline{B_\beta} &= \emptyset. \end{aligned}$$

2: Every $\overline{B_\beta}$ intersects only finitely many $V_{\alpha_1}, \ldots V_{\alpha_k}$

Since $B_\beta \subseteq U_b$ for some $b \in B$, B_β intersects only finitely many $V_{\alpha_1}, \ldots V_{\alpha_k}$. By applying the same reasoning as in (1), it follows that $\overline{B_\beta}$ intersects with the same $V_{\alpha_1}, \ldots V_{\alpha_k}$.

□

Lemma 3.14. *[Mil64, p.67]*
Let $\mathfrak{b}$ be a microbundle over $B \times [0,1]$ where B is paracompact hausdorff. Then there exists a bundle-germ $R : \mathfrak{b} \Rightarrow \mathfrak{b}|_{B\times\{1\}}$ covering the retraction $r : B \times [0,1] \to B \times \{1\}$ with $r(b,t) = (b,1)$.

Proof.
First, we assume a locally finite covering $\{V_\alpha\}$ of open sets where $\mathfrak{b}|_{V_\alpha \times [0,1]}$ is trivial. The existence of such a covering is justified by Lemma **(3.12)** and paracompactness of B.

This cover can be equipped with a partition of unity

$$\lambda_\alpha : B \to [0,1] \text{ with } \mathrm{supp}\lambda_\alpha \subseteq V_\alpha$$

since B is paracompact hausdorff, that is rescaled in way that

$$\max_\alpha(\lambda_\alpha(b)) = 1, \forall b \in B.$$

Such a rescaling can be achieved by dividing λ_α by $\max_\alpha \lambda_\alpha$ which is well-defined because $\{V_\alpha\}$ is locally finite and continuous because the max function is continuous. Also, $\max_\alpha \lambda_\alpha(b) > 0$ since the initial partition of unity adds up to 1 in every point.

Now we define a retraction $r_\alpha : B \times [0,1] \to B \times [0,1]$ with

$$r_\alpha(b,t) = (b, \max(t, \lambda_\alpha(b))).$$

In the remainder of this proof, we construct bundle-germs $R_\alpha : \mathfrak{b} \Rightarrow \mathfrak{b}$ covering r_α in order to 'compose' them to our required bundle-germ.

1: Constructing bundle-germs R_α covering r_α

We can divide $B \times [0,1]$ into two subsets

$$A_\alpha = \mathrm{supp}\lambda_\alpha \times [0,1] \subseteq V_\alpha \times [0,1] \text{ and } A'_\alpha = \{(b,t) : t \geq \lambda_\alpha(b)\}.$$

Since $\mathfrak{b}|_{A_\alpha}$ is trivial, we can, analogous to the proof of Lemma **(3.11)**, extend the identity bundle-germ on $\mathfrak{b}|_{A_\alpha \cap A_\alpha{}'}$ to a bundle-germ

$$\mathfrak{b}|_{A_\alpha} \Rightarrow \mathfrak{b}|_{A_\alpha \cap A_\alpha{}'}$$

using the bundle-germ $\mathfrak{e}^n_{A_\alpha} \Rightarrow \mathfrak{e}^n_{A_\alpha \cap A_\alpha{}'}$ represented by

$$(a,x) \mapsto (r_\alpha(a), x).$$

Pieced together with the identity bundle-germ $\mathfrak{b}|_{A_\alpha{}'}$ (A_α and $A_\alpha{}'$ are both closed), we obtain a bundle-germ R_α covering r_α.

2: Constructing a bundle-germ R covering $(b,t) \mapsto (b,1)$

Applying the well-ordering theorem, which is equivalent to the axiom of choice (see [Kuc09, p.14]), we can assume an ordering for $\{V_\alpha\}$.

Let $\{B_\beta\}$ be a locally finite closed cover of B such that B_β intersects only finitely many $V_{\alpha_1} < \cdots < V_{\alpha_k}$ obtained by Lemma **(3.13)**.

The composition $R_{\alpha_1} \circ \ldots \circ R_{\alpha_k}$ restricts to a bundle-germ

$$R(\beta) : \mathfrak{b}|_{B_\beta \times [0,1]} \Rightarrow \mathfrak{b}|_{B_\beta \times \{1\}}$$

covering the retraction $(b,t) \mapsto (b,1)$. That is because for every $b \in B_\beta$, we find an $1 \leq i \leq k$ with $\lambda_{\alpha_i}(b) = 1$ and hence $r_{\alpha_i}(b,t) = (b,1)$.

Pieced together using Lemma **(3.11)**, we obtain a bundle-germ

$$R : \mathfrak{b}|_{B \times [0,1]} \to \mathfrak{b}|_{B \times \{1\}}$$

covering $(b,t) \mapsto (b,1)$.

$\square$

Finally, we gathered all the tools to proof the Homotopy Theorem.

Proof of the Homotopy Theorem.
The previous lemma yields a bundle-germ

$$R : H^*\mathfrak{b} \Rightarrow H^*\mathfrak{b}|_{A \times \{1\}}$$

covering the retraction $(a,t) \mapsto (a,1)$.

By restricting R to $H^*\mathfrak{b}|_{A\times\{0\}}$, we obtain a bundle-germ

$$H^*\mathfrak{b}|_{A\times\{0\}} \Rightarrow H^*\mathfrak{b}|_{A\times\{1\}}$$

covering $\theta : A \times \{0\} \xrightarrow{\sim} A \times \{1\}$ with $\theta(a,0) = (a,1)$. Corollary **(3.9)** then infers that $H^*\mathfrak{b}|_{A\times\{0\}} \cong \theta^*(H^*\mathfrak{b}|_{A\times\{1\}})$.

Considering $A \times \{0\} = A$, we can identify $H^*\mathfrak{b}|_{A\times\{0\}}$ with $f^*\mathfrak{b}$ as follows:

$$H^*\mathfrak{b}|_{A\times\{0\}} = \iota^*(H^*\mathfrak{b}) \cong (H \circ \iota)^*\mathfrak{b} = f^*\mathfrak{b}$$

Analogously, we can identify $\theta^*(H^*\mathfrak{b}|_{A\times\{1\}})$ with $g^*\mathfrak{b}$.

Together with $H^*\mathfrak{b}|_{A\times\{0\}} \cong \theta^*(H^*\mathfrak{b}|_{A\times\{1\}})$, it follows that $f^*\mathfrak{b} \cong g^*\mathfrak{b}$. $\square$

4 Rooted Microbundles and Suspensions

In this section, we provide a proof for the Bouquet Lemma stated in Section **(2)**. To this end, we introduce the concept of 'rooted microbundles', which allows us to define the wedge sum of two microbundles in a precise manner. Additionally, we show a version of the Homotopy Theorem that is compatible with rooted-microbundles.

Throughout this section, we assume that every topological space is equipped with an arbitrary base point which we will denote with subscript 0.

Rooted Microbundles

Definition 4.1. [Mil64, p.69]
A *rooted microbundle* $\mathfrak{b}$ over B is a microbundle over B together with an isomorphism-germ

$$R : \mathfrak{b}|_{b_0} \Rightarrow \mathfrak{e}^n_{b_0}.$$

Two rooted microbundles $\mathfrak{b}$ and $\mathfrak{b}'$ over B are *rooted isomorphic* if there exists an isomorphism-germ $\mathfrak{b} \Rightarrow \mathfrak{b}'$ extending

$$R'^{-1} \circ R : \mathfrak{b}|_{b_0} \Rightarrow \mathfrak{b}'|_{b_0}.$$

Remark 4.2.
One can always define a rooting for a given microbundle by choosing a local trivialization in the base point and restricting it to the fiber of b_0.

Definition 4.3. [Mil64, p.57]
Let $\mathfrak{b}$ be a rooted microbundle over B and let $f : A \to B$ be a based map. We equip the induced microbundle $f^*\mathfrak{b}$ with the rooting

$$R_f : E(f^*\mathfrak{b}|_{a_0}) = a_0 \times E(\mathfrak{b}|_{b_0}) \Rightarrow e^n_{a_0}$$

that coincides with R if we consider $a_0 \times E(\mathfrak{b}|_{b_0}) = E(\mathfrak{b}|_{b_0})$ and $e^n_{a_0} = e^n_{b_0}$.

The total space $E(f^*\mathfrak{b}|_{a_0})$ is the same as $a_0 \times E(\mathfrak{b}|_{b_0})$ because

$$\begin{aligned} E(f^*\mathfrak{b}|_{a_0}) &= \{(a, e) \in A \times E(\mathfrak{b}) : a = a_0 \text{ and } f(a) = b_0 = j(e)\} \\ &= a_0 \times \{e \in E(\mathfrak{b}) : j(e) = b_0\} = a_0 \times E(\mathfrak{b}|_{b_0}). \end{aligned}$$

Given a rooted microbundle $\mathfrak{b}$ and homotopic based maps $f, g : A \to B$, the Homotopy Theorem yields that $f^*\mathfrak{b}$ and $g^*\mathfrak{b}$ are isomorphic (not rooted-isomorphic).

With the preliminary work in Section **(3)**, we can derive a version of the Homotopy Theorem that also accounts for rooted isomorphy.

Theorem 4.4 (Rooted Homotopy Theorem)**.** *[Mil64, p.69]*
Let $\mathfrak{b}$ be a rooted microbundle over B and let $f, g : A \to B$ be two based maps where A is paracompact hausdorff. If there exists a homotopy $H : A \times [0,1] \to B$ between f and g that leaves the base point fixed, then the two rooted microbundles $f^*\mathfrak{b}$ and $g^*\mathfrak{b}$ are rooted isomorphic.

In order to prove this, we need to show a 'rooted version' of Lemma **(3.12)**.

First, note that

$$E(H^*\mathfrak{b}|_{a_0 \times [0,1]}) = E(\iota^*(H^*(\mathfrak{b}))) \cong E((H \circ \iota)^*\mathfrak{b}) = E(c^*_{a_0 \times [0,1], b_0}\mathfrak{b}),$$

whose total space is of the form $(a_0 \times [0,1]) \times E(\mathfrak{b})$. Based on this, we can define an isomorphism-germ $\overline{R} : H^*\mathfrak{b}|_{a_0 \times [0,1]} \Rightarrow \mathfrak{e}^n_{a_0 \times [0,1]}$ represented by

$$\overline{r}(a_0, t, v) = (a_0, t, r^{(2)}(v)),$$

where $r : V \to b_0 \times \mathbb{R}^n$ is a representative for R. Note that $\overline{r}$ is a homeomorphism on its image, since its components are homeomorphisms on their image.

Lemma 4.5. *[Mil64, p.69]*
Let $\mathfrak{b}$ be a rooted microbundle over B and let $H : A \times [0,1] \to B$ be a map that leaves the base point fixed. Then there exists a neighborhood V of a_0 together with an isomorphism-germ

$$H^*\mathfrak{b}|_{V \times [0,1]} \Rightarrow \mathfrak{e}^n_{V \times [0,1]}$$

extending $\overline{R}$ (as defined above).

Proof.
By applying Lemma **(3.12)**, it follows that there exists an isomorphism-germ

$$Q : H^*\mathfrak{b}|_{V \times [0,1]} \Rightarrow \mathfrak{e}^n_{V \times [0,1]}$$

for a sufficiently small neighborhood V of a_0. However, Q does not extend $\overline{R}$ in general.

In order to fix this, consider

$$Q \circ \overline{R}^{-1} : \mathfrak{e}^n_{a_0 \times [0,1]} \Rightarrow \mathfrak{e}^n_{a_0 \times [0,1]}$$

together with a representative $f : U_f \to (a_0 \times [0,1]) \times \mathbb{R}^n$.

Similar to the construction of $\overline{R}$, we can construct an isomorphism-germ

$$P : \mathfrak{e}^n_{V \times [0,1]} \Rightarrow \mathfrak{e}^n_{V \times [0,1]}$$

extending $Q \circ \overline{R}^{-1}$ represented by

$$p(a, t, x) = (a, f(a_0, t, x))$$

considering $f(a_0, t, x) \in [0,1] \times \mathbb{R}^n$.

Restricted to $\mathfrak{e}^n_{a_0 \times [0,1]}$, P agrees with $Q \circ \overline{R}^{-1}$ and thus

$$Q^{-1} \circ P|_{\mathfrak{e}^n_{a_0 \times [0,1]}} = (Q^{-1} \circ (Q \circ \overline{R}^{-1})) = ((Q^{-1} \circ Q) \circ \overline{R}^{-1}) = \overline{R}^{-1}.$$

Since P and Q are both isomorphism-germs,

$$P^{-1} \circ Q : H^*\mathfrak{b}|_{V \times [0,1]} \Rightarrow \mathfrak{e}^n_{V \times [0,1]}$$

is an isomorphism-germ extending $\overline{R}$. □

We are now able to show the Rooted Homotopy Theorem.

To understand the proof, it is useful to have the constructions of Lemma **(3.14)** in mind, because we will modify them slightly in order to preserve the rootings.

Proof of the Rooted Homotopy Theorem.
We need to show that $f^*\mathfrak{b}$ and $g^*\mathfrak{b}$ are rooted isomorphic, that is there exists an isomorphism-germ $f^*\mathfrak{b} \Rightarrow g^*\mathfrak{b}$ extending $R_g^{-1} \circ R_f = I$ where I denotes the identity germ.

For the initial Homotopy Theorem, we constructed a bundle-germ

$$F : H^*\mathfrak{b} \Rightarrow H^*\mathfrak{b}|_{A \times \{1\}}$$

covering $(a,t) \mapsto (a,1)$ and restricted it to $H^*\mathfrak{b}|_{A \times \{0\}}$. The required isomorphism-germ was then obtained by identifying $f^*\mathfrak{b}$ with $H^*\mathfrak{b}|_{A \times \{0\}}$ and $g^*\mathfrak{b}$ with $H^*\mathfrak{b}|_{A \times \{0\}}$.

We must make slight modifications to the construction of F such that it extends $f^*\mathfrak{b}|_{b_0} \cong H^*\mathfrak{b}|_{a_0 \times \{0\}} \Rightarrow H^*\mathfrak{b}|_{a_0 \times \{1\}} \cong g^*\mathfrak{b}|_{b_0}$ represented by

$$(a_0, e) = ((a_0, 0), e) \mapsto ((a_0, 1), e) = (a_0, e).$$

This can be achieved by choosing a locally finite open cover $\{V_\alpha\}$ of A (as in Lemma **(3.14)**), removing the base point a_0 from every set and adding V obtained from Lemma **(4.5)**. Since $a_0 \in V$, the resulting collection is still a locally finite open cover of A.

In the following, we will denote constructions over V with subscript V and constructions over the other sets from the cover with subscript α.

We continue with the proof of Lemma **(3.14)**. Note that $\lambda_V(a_0) = 1$. That is because we removed a_0 from every other set and hence $\lambda_\alpha(a_0) = 0$.

Lastly, we construct the extension R_V for r_V like in Section **(3)**, but instead of choosing an arbitrary trivialization $E(H^*\mathfrak{b}|_{A_V}) \cong A_V \times \mathbb{R}^n$ for the construction we use a representative r for the bundle-germ constructed in Lemma **(4.5)**.

This has the advantage that the representative

$$E(H^*\mathfrak{b}|_{A_V}) \xrightarrow{r} A_V \times \mathbb{R}^n \xrightarrow{r_V \times id} (A_V \cap A'_V) \times \mathbb{R}^n \xrightarrow{r^{-1}} E(H^*\mathfrak{b}|_{A_V \cap A'_V})$$

for R_V maps elements $((a_0, 0), e)$ to $((a_0, 1), e)$. Additionally, every other R_α leaves $H^*\mathfrak{b}|_{a_0 \times \{0\}}$ unaffected because $r_\alpha(a_0, t) = (a_0, \max(\underbrace{\lambda_\alpha(t)}_{=0}, t)) = (a_0, t)$.

It follows that, by piecing together the R_α and R_V like in Lemma **(3.14)**, we obtain a bundle germ $F : H^*\mathfrak{b} \Rightarrow H^*\mathfrak{b}|_{A \times \{1\}}$ that extends $R_g^{-1} \circ R_f$. This completes the proof. □

Now that we introduced rooted-microbundles, we are able to define the wedge sum. As we will see in the subsequent proof, we require rootings for its definition. Particulary, the wedge sum depends on the specific choices of these rootings, justifying the requirement for rooted-microbundles.

Given a quotient space $A \sqcup B/\sim$ and maps $f : A \to C$ and $g : B \to C$, we define $f \cup g : (A \sqcup B/\sim) \to C$ by

$$x \mapsto \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}.$$

Clearly, this map is only well-defined if $a \sim b \implies f(a) = g(b)$.

Definition 4.6. [Mil64, p.70]
Let $\mathfrak{a}$ and $\mathfrak{b}$ be two rooted microbundles over A and B. The *wedge sum* $\mathfrak{a} \vee \mathfrak{b}$ of $\mathfrak{a}$ and $\mathfrak{b}$ is a microbundle

$$A \vee B \xrightarrow{i_a \cup i_b} E(\mathfrak{a} \vee \mathfrak{b}) \xrightarrow{j_a \cup j_b} A \vee B$$

with the total space defined as

$$(E(\mathfrak{a}) \sqcup E(\mathfrak{b}))/f(e_a) \sim e_a$$

where $f : W_a \xrightarrow{\sim} W_b$ is a representative for $R_b^{-1} \circ R_a$.

We equip $\mathfrak{a} \vee \mathfrak{b}$ with a rooting

$$R : E((\mathfrak{a} \vee \mathfrak{b})|_{a_0}) \Rightarrow \mathfrak{e}^n_{a_0}$$

represented by any representative for R_a (or R_b).

Proof that $\mathfrak{a} \vee \mathfrak{b}$ *is a (rooted) microbundle.*
Let $f : W_a \xrightarrow{\sim} W_b$ be a representative for $R_b^{-1} \circ R_a$.

1: $\mathfrak{a} \vee \mathfrak{b}$ is a rooted microbundle

- The injection map $i_a \cup i_b$ is well-defined because

  $$[i(a_0)] = [i_a(a_0)] = [f(i_a(a_0))] = [i_b(b_0)] = [i(b_0)]$$

  and continuous since both i_a and i_b are continuous.

- The projection map $j_a \cup j_b$ is well-defined because

$$\forall e \in W_a : [j(e)] = [j_a(e)] = [a_0] = [b_0] = [j_b(f(e))] = [j(f(e))]$$

  and continuous since both j_a and j_b are continuous.

- The composition $j \circ i$ is the identity because

$$\forall a \in A : j(i(a)) = j(i_a(a)) = j_a(i_a(a)) = a$$

  since $j_a \circ i_a = id_A$ (symmetrical for B).

It remains to be shown that $\mathfrak{a} \vee \mathfrak{b}$ is locally trivial.

Let $x \in A \vee B$. For reasons of symmetry, we can assume that $x \in A$.

Case 1: $x \neq a_0$

Choose a local trivialization (U, V, ϕ) for x in $\mathfrak{a}$. Without loss of generality, we can assume that $U \cap B = \emptyset$ by subtracting $\{a_0\}$ from U if necessary. Note that $\{a_0\}$ is closed since A is hausdorff.

Now we can simply use this trivialization for $\mathfrak{a} \vee \mathfrak{b}$, because $U \subseteq A$ is open in $A \vee B$ and $V \subseteq E(\mathfrak{a})$ is open in $E(\mathfrak{a} \vee \mathfrak{b})$. Furthermore, since i and j reduce to i_a and j_a, it follows that ϕ commutes with i and $id \times 0$ as well as with j and π_1.

Case 2: $x = a_0$

Let (U_a, V_a, ϕ_a) and (U_b, V_b, ϕ_b) be local trivializations for $a_0 = b_0$ in $\mathfrak{a}$ and $\mathfrak{b}$.

Since $W_a \subseteq E(\mathfrak{a}|_{a_0})$ is open, there exists an open subset $W'_a \subseteq E(\mathfrak{a})$ such that $W_a = W'_a \cap E(\mathfrak{a}|_{a_0})$.

Let $U'_a \subseteq A$ be an open neighborhood of a_0 and $\varepsilon > 0$ such that

$$V'_a = U'_a \times B_\varepsilon(0) \subseteq \phi_a(W'_a).$$

This allows us to define the map

$$\phi'_a : V'_a \xrightarrow{\sim} \phi'_a(V'_a) \subseteq A \times \mathbb{R}^n \text{ with}$$

$$\phi'_a(e) = (j_a(e), (\phi_b^{(2)} \circ f \circ \phi_a^{-1})(a_0, \phi_a^{(2)}(e))).$$

Now we can show local triviality in a_0 using the homeomorphism

$$\phi'_a \cup \phi_b : V'_a \cup V_b \xrightarrow{\sim} \phi'_a(V'_a \cup V_b) \subseteq (A \vee B) \times \mathbb{R}^n$$

This map is well-defined, because

$$\phi'_a(e) = (a_0, (\phi_b^{(2)} \circ f \circ \phi_a^{-1})(a_0, \phi_a^{(2)}(e)))$$

$$= (b_0, \phi_b^{(2)}(f(e))) = (j_b(f(e)), \phi_b^{(2)}(f(e))) = \phi_b(f(e)).$$

Homeomorphy follows from the fact that both ϕ_a' and ϕ_b are homeomorphisms, and that $\phi_a'(e_a) = \phi(e_b) \implies f(e_a) = e_b$.

Commutativity with $i_a \cup i_b$ and $id \times 0$ as well as between $j_a \cup j_b$ and π_1 is inherited from ϕ_a and ϕ_b. Note that $\phi_a(i_a(a)) = (a, 0) = \phi_a'(i_a(a))$.

Applying Lemma **(1.3)** yields that $\mathfrak{a} \vee \mathfrak{b}$ is locally trivial.

2: $\mathfrak{a} \vee \mathfrak{b}$ is well-defined

Let f' be another representative for $R_b^{-1} \circ R_a$ and $(\mathfrak{a} \vee \mathfrak{b})'$ the resulting wedge sum. We need to find an isomorphism-germ that extends $R'^{-1} \circ R$.

In order to do this, choose an open neighborhood $V \subseteq E(\mathfrak{a}|_{a_0})$ of $i_a(a)$ where f and f' agree.

By subtracting the closed set $j_a^{-1}(a_0) - V$ from $E(\mathfrak{a} \vee \mathfrak{b})$ and $E(\mathfrak{a} \vee \mathfrak{b})'$, the microbundles remain unchanged due to Proposition **(1.5)**.

But now the total spaces $E(\mathfrak{a} \vee \mathfrak{b})$ and $E((\mathfrak{a} \vee \mathfrak{b})')$ are the same. That is because $E(\mathfrak{a} \vee \mathfrak{b})$ and $E((\mathfrak{a} \vee \mathfrak{b})')$ could only differ in $j_a^{-1}(a_0) - V$.

Furthermore, since injection and projection are defined exactly the same, it follows that the identity $(\mathfrak{a} \vee \mathfrak{b}) \Rightarrow (\mathfrak{a} \vee \mathfrak{b})'$ is an isomorphism-germ. Together with

$$R'^{-1} \circ R = R^{-1} \circ R = I,$$

which completes the proof.

□

Microbundles over a Suspension

In the following, let B be a *reduced suspension*

$$SX = (X \times [0, 1])/(X \times \{0, 1\} \cup x_0 \times [0, 1])$$

over a topological space X.

Let $\phi : B \to B \vee B$ denote the map that sends $X \times [0, \frac{1}{2}]$ to the first B via

$$\phi([x, t]) = \begin{cases} [x, 2t] & \text{if } t \leq \frac{1}{2} \\ [x, 2t - 1] & \text{else} \end{cases}.$$

Additionally, let $c_1 : B \vee B \to B$ denote the map that is the identity on the first summand and the constant map c_{B,b_0} on the second summand, i.e.

$$c_1(b, i) = \begin{cases} b & \text{if } i = 1 \\ b_0 & \text{else} \end{cases}.$$

We define c_2 analogously.

Lemma 4.7. *[Mil64, p.70]*
The following (non-rooted) isomorphy holds:

$$\phi^*(\mathfrak{b} \vee \mathfrak{e}_B^n) \cong \mathfrak{b} \cong \phi^*(\mathfrak{e}_B^n \vee \mathfrak{b})$$

Proof.
We prove the lemma in two steps.

Step 1: $c_1^*\mathfrak{b} \cong \mathfrak{b} \vee \mathfrak{e}_B^n$

Let $E(\mathfrak{b} \vee \mathfrak{e}_B^n)$ be constructed via the representative $f : V \to b_0 \times \mathbb{R}^n$ for R.

Without loss of generality, we may assume that $V = E(\mathfrak{b}|_{b_0})$ by removing the closed set $E(\mathfrak{b}|_{b_0}) - V$ from $E(\mathfrak{b})$ if necessary.

Consider $\psi : E(c_1^*\mathfrak{b}) \xrightarrow{\sim} E(\mathfrak{b} \vee \mathfrak{e}_B^n)$ given by

$$\psi((b,i),e) = \begin{cases} e & \text{if } i = 1 \\ (b, f^{(2)}(e)) & \text{else} \end{cases}$$

Note that ψ is well-defined, because

$$\psi((b_0,1),e) = e = f(e) = (b_0, f^{(2)}(e)) = \psi((b_0,2),e).$$

Additionally, ψ is a homeomorphism as both of its summands are homeomorphisms.

Finally, it remains to be shown that ψ commutes with the injection and projection maps of $c_1^*\mathfrak{b}$ and $\mathfrak{b} \vee \mathfrak{e}_B^n$. To check this, consider the following equations:

$$\psi(i_{c_1}(b,i)) = \begin{cases} \psi((b,1),i(b)) = i(b) = i_\vee(b,1) \\ \psi((b,2),i(b_0)) = f(i(b)) = (b,0) = (id \times 0)(b) = i_\vee(b,2) \end{cases} \tag{1}$$

$$j_{c_1}((b,i),e) = \begin{cases} j(e) = j_\vee(e) = j_\vee(\psi((b,1),e)) \\ (b,2) = \pi_1(\psi((b,2),e)) = j_\vee(\psi((b,2),e)) \end{cases} \tag{2}$$

Step 2: $\phi^*(\mathfrak{b} \vee \mathfrak{e}_B^n) \cong \mathfrak{b}$

Using the fact that $c_1 \circ \phi = id_{B \vee B}$, we conclude that

$$\phi^*(\mathfrak{b} \vee \mathfrak{e}_B^n) \cong \phi^*(c_1^*\mathfrak{b}) \cong (c_1 \circ \phi)^*\mathfrak{b} \cong \mathfrak{b}.$$

For reasons of symmetry, it follows that $\mathfrak{b} \cong \phi^*(\mathfrak{e}_B^n \vee \mathfrak{b})$. This completes the proof.

∎

Lemma 4.8.
Let $\mathfrak{a}$ and $\mathfrak{b}$ be two rooted microbundles over A and B. Given two based maps $f : A' \to A$ and $g : B' \to B$, the following equality holds:

$$(f \cup g)^*(\mathfrak{a} \vee \mathfrak{b}) = f^*\mathfrak{a} \vee g^*\mathfrak{b}$$

Proof.
Consider the following equation:

$$\begin{gathered}E((f\cup g)^*(\mathfrak{a}\vee\mathfrak{b}))\\ =\{(x,e)\in(A'\vee B')\times E(\mathfrak{a}\vee\mathfrak{b}):(f\cup g)(x)=j(e)\}\\ =\{(x,e)\in((A'\times E(\mathfrak{a}))\sqcup(B'\times E(\mathfrak{b}))/\sim):(f\cup g)(x)=j(e)\}\\ =(\{(x,e)\in A'\times E(\mathfrak{a}):f(x)=j_a(e)\}\sqcup\{(x,e)\in B'\times E(\mathfrak{b}):g(x)=j_b(e)\})/\sim\\ =(E(f^*\mathfrak{a})\sqcup E(g^*\mathfrak{b})/\sim)=E(f^*\mathfrak{a}\vee g^*\mathfrak{b})\end{gathered}$$

Here, $(a,e_a)\sim(b,e_b)\iff a=a_0=b_0=b$ and $[e_a]=[e_b]\in E(\mathfrak{a}\vee\mathfrak{b})$.

Furthermore, the injection and projection maps agree. To check this, consider the following equations:

$$i_\cup(a)=i_f(a)=i_\vee(a)\text{ and }i_\cup(b)=i_g(b)=i_\vee(b) \tag{1}$$

$$j_\cup(a,e)=a=j_f(a,e)=i_\vee(a,e)\text{ and }j_\cup(b,e)=b=j_g(b,e)=i_\vee(b,e) \tag{2}$$

It follows that the two microbundles are equal. □

Let $r:B\xrightarrow{\sim}B$ denote the 'reflection' given by

$$r([x,t])=[x,1-t]$$

and let $c:B\vee B\to B$ denote the identity on the first summand and r on the second summand, i.e.

$$c(b,i)=\begin{cases}b & \text{if } i=1\\ r(b) & \text{else}\end{cases}.$$

Lemma 4.9. *[Mil64, p.70]*
The induced microbundle $\phi^*(\mathfrak{b}\vee r^*\mathfrak{b})$ is trivial.

Proof.
The composition $c\circ\phi$ is null-homotopic via the homotopy $H:B\times[0,1]\to B$ given by

$$H([x,t],s)=f(\phi(x,t\cdot s)).$$

Therefore, $\phi^*(c^*\mathfrak{b})\cong(c\circ\phi)^*\mathfrak{b}\cong c^*_{B,b_0}\mathfrak{b}\cong\mathfrak{e}^n_B$ as rooted-isomorphy (see Theorem **(3.1)**). By applying the previous lemma, it follows that

$$\phi^*(\mathfrak{b}\vee r^*\mathfrak{b})=\phi^*((id\cup r)^*(\mathfrak{b}\vee\mathfrak{b}))=\phi^*(c^*\mathfrak{b})$$

and hence $\phi^*(\mathfrak{b}\vee r^*\mathfrak{b})\cong\mathfrak{e}^n_B$. □

Definition 4.10. [Mil64, p.70]
Given two two rooted microbundles $\mathfrak{b}$ and $\mathfrak{b}'$ over B, we equip the Whitney sum $\mathfrak{b}\oplus\mathfrak{b}'$ with the rooting

$$R\oplus R':(\mathfrak{b}\oplus\mathfrak{b}')|_{b_0}\Rightarrow\mathfrak{e}^{n_1}_{b_0}\oplus\mathfrak{e}^{n_2}_{b_0}=\mathfrak{e}^{n_1+n_1}_{b_0}.$$

represented by the direct sum of two representatives for R and R'.

Lemma 4.11. *[Mil64, p.70]*
Let $\mathfrak{a}$ and $\mathfrak{a}'$ be two rooted microbundles over A, and let $\mathfrak{b}$ and $\mathfrak{b}'$ be two rooted microbundle over B. Then the following (non-rooted) isomorphy holds:

$$(\mathfrak{a} \vee \mathfrak{b}) \oplus (\mathfrak{a}' \vee \mathfrak{b}') \cong (\mathfrak{a} \oplus \mathfrak{a}') \vee (\mathfrak{b} \oplus \mathfrak{b}')$$

Proof.
Consider $\psi : E((\mathfrak{a} \vee \mathfrak{b}) \oplus (\mathfrak{a}' \vee \mathfrak{b}')) \xrightarrow{\sim} E((\mathfrak{a} \oplus \mathfrak{a}') \vee (\mathfrak{b} \oplus \mathfrak{b}'))$ given by the identity map. Note that ψ is well-defined, because

$$\begin{aligned} &(e, e') \in E((\mathfrak{a} \vee \mathfrak{b}) \oplus (\mathfrak{a}' \vee \mathfrak{b}')) \\ \implies &j(e) = j'(e') \implies j(e), j'(e') \in A \text{ or } j(e), j'(e') \in B \\ \implies &(e, e') \in E(\mathfrak{a} \oplus \mathfrak{a}') \text{ or } (e, e') \in E(\mathfrak{b} \oplus \mathfrak{b}') \\ \implies &(e, e') \in E((\mathfrak{a} \oplus \mathfrak{a}') \vee (\mathfrak{b} \oplus \mathfrak{b}')). \end{aligned}$$

Furthermore, the injection and projection maps agree. To check this, consider the following equations (symmetrical for B):

$$i_\oplus(a) = (i_a(a), i'_a(a)) = i_\vee(a) \tag{1}$$
$$j_\oplus(e_a, e'_a) = j(e_a) = j_a(e_a) = j_\vee(e_a, e'_a) \tag{2}$$

It follows that the two microbundles are isomorphic. □

Lemma 4.12.
Let $\mathfrak{b}$ be a rooted microbundle over a paracompact hausdorff space B. Then there exists a closed neighborhood W of b_0 and an isomorphism-germ

$$\mathfrak{b}|_W \Rightarrow \mathfrak{e}^n_W$$

extending R together with a map $\lambda : B \to [0, 1]$ with

$$supp\lambda \subseteq W \textit{ and } \lambda(b_0) = 1.$$

Proof.
Let $r : V_r \to b_0 \times \mathbb{R}^n$ be a representative for R.

Choose a local trivialization (U, V, ϕ) for b_0 such that $V \cap E(\mathfrak{b}|_{b_0}) \subseteq V_r$. Such a trivialization can be obtained by subtracting the closed set $E(\mathfrak{b}|_{b_0}) - V_r$ from $E(\mathfrak{b})$ if necessary and considering the resulting microbundle instead.

Consider the (locally) finite open covering of B given by U and $B - \{b_0\}$. Since B is paracompact, we can apply the concept of partitions of unity which yields a map

$$\lambda : B \to [0, 1] \text{ with supp}\lambda \subseteq U.$$

By rescaling and capping to 1 if necessary, we may assume that $\lambda(b_0) = 1$.

We choose $W = \mathrm{supp}\lambda \subseteq U$, which is closed by the definition of the support. We are now able to define an isomorphism-germ $\mathfrak{b}|_W \Rightarrow \mathfrak{e}^n_W$ represented by

$$f : V \xrightarrow{\sim} f(V) \subseteq U \times \mathbb{R}^n \text{ with } f(e) = (j(e), r^{(2)}(\phi^{-1}(b_0, \phi^{(2)}(e)))),$$

which extends r.

Together with λ, this completes the proof. $\square$

Lemma 4.13. *[Mil64, p.71]*
The rooted microbundles $\mathfrak{b} \oplus \mathfrak{e}^n_B$ and $\mathfrak{e}^n_B \oplus \mathfrak{b}$ are rooted isomorphic.

Proof.
In order to show rooted isomorphy, we need to find an isomorphism-germ

$$\mathfrak{b} \oplus \mathfrak{e}^n_B \Rightarrow \mathfrak{e}^n_B \oplus \mathfrak{b}$$

that extends $(I \oplus R) \circ (R \oplus I)^{-1} = R \oplus R^{-1}$.

Consider the isomorphism-germ $\mathfrak{b} \oplus \mathfrak{e}^n_B \Rightarrow \mathfrak{e}^n_B \oplus \mathfrak{b}$ represented by

$$f : E(\mathfrak{b}) \times \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n \times E(\mathfrak{b}) \text{ with } f(e, x) = (-x, e).$$

The idea is to modify f near b_0 in such a way that the resulting isomorphism-germ respects the rooting.

By applying the previous lemma, choose a sufficiently small closed neighborhood U of b_0 such that there exists an isomorphism-germ $Q : \mathfrak{b}|_U \Rightarrow \mathfrak{e}^n_U$ extending R and a map $\lambda : B \to [0, \frac{\pi}{2}]$ such that $\mathrm{supp}\lambda \subseteq U$ and $\lambda(b_0) = \frac{\pi}{2}$.

Together with the homeomorphism $\psi : U \times \mathbb{R}^n \times \mathbb{R}^n \xrightarrow{\sim} U \times \mathbb{R}^n \times \mathbb{R}^n$ given by

$$\psi(b, x, y) = (b, x\sin(\lambda(b)) - y\cos(\lambda(b)), x\cos(\lambda(b)) - y\sin(\lambda(b))),$$

we are able to define an isomorphism-germ

$$(\mathfrak{b} \oplus \mathfrak{e}^n_B)|_U \xRightarrow{Q \oplus I} (\mathfrak{e}^n_B \oplus \mathfrak{e}^n_B)|_U \xrightarrow{\psi} (\mathfrak{e}^n_B \oplus \mathfrak{e}^n_B)|_U \xRightarrow{Q^{-1} \oplus I} (\mathfrak{e}^n_B \oplus \mathfrak{b})|_U$$

which extends $R \oplus R^{-1}$ since $\psi(b_0, x, y) = (b_0, x, y)$, and which coincides with F over $U \cap \lambda^{-1}(0)$.

Pieced together with $F|_{\lambda^{-1}(b)}$ using Lemma **(3.10)**, we obtain our required isomorphism germ. $\square$

Theorem 4.14. *[Mil64, p.71]*
If $\mathfrak{a}$ and $\mathfrak{b}$ are rooted microbundles over a paracompact hausdorff space B, then

$$\phi^*(\mathfrak{a} \vee \mathfrak{b}) \oplus \mathfrak{e}^n_B = \mathfrak{a} \oplus \mathfrak{b}.$$

Proof.
The previous lemma yields rooted isomorphy $\mathfrak{b} \oplus \mathfrak{e}^n_B \cong \mathfrak{e}^n_B \oplus \mathfrak{b}$. Hence

$$\phi^*((\mathfrak{a} \oplus \mathfrak{e}^n_B) \vee (\mathfrak{b} \oplus \mathfrak{e}^n_B)) \cong \phi^*((\mathfrak{a} \oplus \mathfrak{e}^n_B) \vee (\mathfrak{e}^n_B \oplus \mathfrak{b})).$$

Furthermore, we have

$$\phi^*((\mathfrak{a} \oplus \mathfrak{e}^n_B) \vee (\mathfrak{b} \oplus \mathfrak{e}^n_B)) \cong \phi^*((\mathfrak{a} \vee \mathfrak{b})) \oplus (\mathfrak{e}^n_B \vee \mathfrak{e}^n_B) \cong \phi^*(\mathfrak{a} \vee \mathfrak{b}) \oplus \mathfrak{e}^n_B$$

for the left side and

$$\phi^*((\mathfrak{a} \oplus \mathfrak{e}^n_B) \vee (\mathfrak{e}^n_B \oplus \mathfrak{b})) \cong \phi^*((\mathfrak{a} \vee \mathfrak{e}^n_B) \oplus (\mathfrak{e}^n_B \vee \mathfrak{b})) \cong \mathfrak{a} \oplus \mathfrak{b}$$

for the right side of the isomorphy, which completes the proof. □

Corollary 4.15. *[Mil64, p.72]*
The wedge sum $\mathfrak{b} \oplus r^*\mathfrak{b}$ is trivial.

Proof.
This follows directly from the previous theorem and the fact that $\phi^*(\mathfrak{b} \oplus r^*\mathfrak{b})$ is trivial. □

Now the Bouquet Lemma is just Corollary **(4.15)** applied to a microbundle over a bouquet of spheres. Note that a bouquet of d-spheres can be regarded as a reduced suspension over a bouquet of $(d-1)$-spheres.

5 Normal Microbundles and Milnors Theorem

The Normal Microbundle

Throughout this section, we will utilize the fact that for every submanifold $N \subseteq M$, N is an absolute neighborhood retract (ANR) in M, i.e. there exists an open neighborhood $V \subseteq M$ of N together with a retraction $r : V \to N$. For a proof, see Theorem 3.3 in [Han51].

Definition 5.1 (normal microbundle)**.** [Mil64, p.61]
Let M be a topological manifold together with a submanifold $N \subseteq M$. A *normal microbundle* $\mathfrak{n}$ of N in M is a microbundle

$$N \xrightarrow{\iota} U \xrightarrow{r} N$$

where $U \subseteq M$ is a neighborhood of N and ι denotes the inclusion $N \hookrightarrow U$.

Definition 5.2 (composition microbundle)**.** [Mil64, p.63]
Let $\mathfrak{a}$ be a n-dimensional microbundle

$$\mathfrak{a} : A \xrightarrow{i_a} E(\mathfrak{a}) \xrightarrow{j_a} A$$

and let $\mathfrak{b}$ be a n'-dimensional microbundle

$$\mathfrak{b} : E(\mathfrak{a}) \xrightarrow{i_b} E(\mathfrak{b}) \xrightarrow{j_b} E(\mathfrak{a}).$$

The *composition microbundle* $\mathfrak{a} \circ \mathfrak{b}$ is a $(n+n')$-dimensional microbundle

$$A \xrightarrow{i} E(\mathfrak{b}) \xrightarrow{j} A$$

where $i = i_b \circ i_a$ and $j = j_a \circ j_b$.

Proof that $\mathfrak{a} \circ \mathfrak{b}$ is a microbundle.
Both injection and projection maps are continuous as being composed by continuous maps. Additionally, $j \circ i = j_a \circ (j_b \circ i_b) \circ i_a = j_a \circ i_a = id_A$.

It remains to be shown that $\mathfrak{a} \circ \mathfrak{b}$ is locally trivial.

For an arbitrary $a \in A$, choose local trivializations (U_a, V_a, ϕ_a) of a in $\mathfrak{a}$ and (U_b, V_b, ϕ_b) of $i_a(a)$ in $\mathfrak{b}$. Note that both U_b and V_a are open neighborhoods of $i_a(a)$.

Without loss of generality, we may assume that $V_a = U_b$:

'$\subseteq$ ': Modify U_a such that

$$U_a \times B_\varepsilon(0) \subseteq \phi_a(V_a \cap U_b)$$

for a sufficiently small $\varepsilon > 0$ and let

$$V_a = \phi_a^{-1}(U_a \times B_\varepsilon(0)) \subseteq V_a \cap U_b.$$

Composing ϕ_a with $\mu_\varepsilon : B_\varepsilon(0) \xrightarrow{\sim} \mathbb{R}^n$ yields a local trivialization of a in $\mathfrak{a}$ such that $V_a \subseteq U_a$.

'$\supseteq$ ': Restrict U_b to $V_a \cap U_b$ and V_b to $\phi_b^{-1}((V_a \cap U_b) \times \mathbb{R}^{n'})$.

We have local-trivialization (U_a, V_b, ϕ) of a in $\mathfrak{a} \circ \mathfrak{b}$ given by

$$\phi : V_b \xrightarrow{\phi_b} U_b \times \mathbb{R}^{n'} = V_a \times \mathbb{R}^{n'} \xrightarrow{\phi_a \times id} (U_a \times \mathbb{R}^n) \times \mathbb{R}^{n'} = U_a \times \mathbb{R}^{n+n'},$$

which is a homeomorphism since it's composed by homeomorphisms.

Furthermore, ϕ commutes with the injection and projection maps, as the following equations show:

$$\begin{aligned} \phi(i(a)) &= \phi(i_b(i_a(a))) = (\phi_a(i_a(a)), \phi_b^{(2)}(i_b(i_a(a)))) \\ &= (\phi_a^{(2)}(i_a(a)), 0) = (a, (0,0)) = (id_{U_a} \times 0)(a) \end{aligned} \tag{1}$$

$$j(e) = j_a(j_b(e)) = \pi_1(j_a(j_b(e)), \phi^{(2)}(e)) = \pi_1(\phi(e)) \tag{2}$$

This completes the proof. □

Unlike the normal vector bundle for smooth manifolds, the normal microbundle is not defined in a constructive manner. Therefore, the question arises in which sense the normal microbundle of a submanifold $N \subseteq M$ is unique. In fact, it is unkown whether a such normal microbundle is unique up to isomorphy. Instead, we have the following statement about uniqueness.

Proposition 5.3. *[Mil64, p.63]*
Let $N \subseteq M$ be an embedded submanifold. Suppose there exists a normal microbundle $\mathfrak{n} : N \xrightarrow{\iota} U \xrightarrow{r} N$ in M. Then $\mathfrak{t}_N \oplus \mathfrak{n} \cong \mathfrak{t}_M|_N$.

Proof.
We prove this proposition in multiple steps.

Step 1: $\mathfrak{t}_N \circ \pi_2^* \mathfrak{n} \cong \mathfrak{t}_M|_N$

Consider the two total spaces

$$E(\mathfrak{t}_N \circ \pi_2^* \mathfrak{n}) = E(\pi_2^* \mathfrak{n}) = \{(n_1, n_2, u) \in (N \times N) \times U : n_2 = r(u)\}$$

and

$$E(\mathfrak{t}_M|_N) = \{(n, m_1, m_2) \in N \times (M \times M) : n = m_1\}.$$

We can easily define a homeomorphism $\psi : E(\mathfrak{t}_M \circ \pi_2^* \mathfrak{n}) \xrightarrow{\sim} E(\mathfrak{t}_N|_M)$ given by

$$\psi(m_1, m_2, u) = (m_1, m_1, u) \text{ and } \psi^{-1}(m, n_1, n_2) = (m, r(n_2), n_2).$$

Note that ψ is a homeomorphism since as ψ and ψ^{-1} are component-wise continuous.

It remains to be shown that ψ commutes with the injection and projection maps of $\mathfrak{t}_M \circ \pi_2^* \mathfrak{n}$ and $\mathfrak{t}_N|_M$. To check this, consider the following equations:

$$\psi(i_{\pi_2}(\Delta(m))) = \psi(i_{\pi_2}(m, m)) = \psi(m, m, \iota(m)) = (m, m, m) = (m, \Delta(\iota(m))) \tag{1}$$

$$\pi_1(j_{\pi_2}(m_1, m_2, u)) = \pi_1(m_1, m_2) = m_1 = j_\iota(m_1, m_1, u) = j_\iota(\psi(m_1, m_2, u)) \tag{2}$$

Step 2: $\mathfrak{t}_M \circ \pi_1^* \mathfrak{n} \cong \mathfrak{t}_M \oplus \mathfrak{n}$

In this case, the two total spaces

$$E(\mathfrak{t}_M \circ \pi_1^* \mathfrak{n}) = E(\pi_1^* \mathfrak{n}) = \{(m_1, m_2, u) \in (M \times M) \times U : m_1 = r(u)\}$$

and

$$E(\mathfrak{t}_M \oplus \mathfrak{n}) = \{(m_1, m_2, u) \in (M \times M) \times U : m_1 = r(u)\},$$

are even equal. Additionally, the injection and projection maps agree as the following equations show:

$$i_{\pi_1}(\Delta(m)) = i_{\pi_1}(m, m) = (m, m, \iota(m)) = (\Delta(m), \iota(m)) \tag{1}$$

$$\pi_1(j_{\pi_1}(m_1, m_2, u)) = \pi_1(m_1, m_2) = m_1 = r(u) = j_\oplus(m_1, m_2, u) \tag{2}$$

Step 3: $\mathfrak{t}_M \circ \pi_1^*\mathfrak{n} \cong \mathfrak{t}_M \circ \pi_2^*\mathfrak{n}$

We show that there exists a neighborhood $D \subseteq N \times N$ of $\Delta(M)$ such that $\pi_1|_D$ is homotopic to $\pi_2|_D$:

Firstly, we assume that N is embedded in euclidean space (see [HW41, p.60]). Let $V \subseteq N$ be a neighborhood retract of M. We define D as follows:

$$D = \{(m, m') \in M \times M : tm + (1-t)m' \in V, \forall t \in [0,1]\}$$

We are given a homotopy $H : D \times [0,1] \to N$ between π_1 and π_2 with

$$H((m, m'), t) = tm + (1-t)m'.$$

Applying the Homotopy Theorem yields $\pi_1^*\mathfrak{n}|_D \cong \pi_2^*\mathfrak{n}|_D$, and by restricting the total spaces accordingly we get $\mathfrak{t}_M \circ \pi_1^*\mathfrak{n} \cong \mathfrak{t}_M \circ \pi_2^*\mathfrak{n}$.

The claim follows by Step 1, 2 and 3. □

This proposition also shows that the normal microbundle underlies the same intuition as the normal vector bundle. The sum of the tangent- and normal microbundle of the submanifold 'span' the tangent microbundle of the surrounding space.

Milnors Theorem

Lemma 5.4. *[Mil64, p.62]*
Let $P \subseteq N \subseteq M$ be a chain of topological submanifolds. There exists a normal microbundle

$$\mathfrak{n} : P \xrightarrow{\iota} U \xrightarrow{r} P$$

of P in M if there exist normal microbundles

$$\mathfrak{n}_p : P \xrightarrow{\iota_P} U_N \xrightarrow{j_P} P \text{ in } N \text{ and } \mathfrak{n}_n : N \xrightarrow{\iota_N} U_M \xrightarrow{j_N} N \text{ in } M.$$

Proof.
We are given a normal microbundle $\mathfrak{n}$ of P in M by

$$\mathfrak{n}_p \circ \mathfrak{n}_n|_{U_N}.$$

Note that $\iota_N \circ \iota_P$ is just the inclusion $P \hookrightarrow U_M$. □

The total space $E(r^*\mathfrak{t}_N)$ is a topological manifold. This can be seen with

$$E(r^*\mathfrak{t}_N) = \{(v, n_1, n_2) \in V \times (N \times N) : r(v) = n_1\} \cong V \times N.$$

Together with $N \hookrightarrow M \xrightarrow{i_{\mathfrak{t}}} E(r^*\mathfrak{t}_N)$, we can assume that N is an embedded submanifold of $E(r^*\mathfrak{t}_N)$. Note that $i_{\mathfrak{t}}$ is an embedding due to the construction of the induced microbundle.

Lemma 5.5. *[Mil64, p.62]*
Let $N \subseteq M$ be an embedded submanifold and let $r : V \to N$ be a retraction. Then there exists a normal microbundle $\mathfrak{n}$ of N in $E(r^*\mathfrak{t}_N)$.

Proof.
We are given a normal microbundle of N in $E(r^*\mathfrak{t}_N)$ by $r^*\mathfrak{t}_N|_N$.

Since $r^*\mathfrak{t}_N|_N \cong (r \circ \iota)^*\mathfrak{t}_N = id^*\mathfrak{t}_N \cong \mathfrak{t}_N$, it suffices to show that $\mathfrak{t}_N \cong \iota^*\mathfrak{t}_V$. We define a homeomorphism $\psi : E(\mathfrak{t}_N) \xrightarrow{\sim} E(\iota^*\mathfrak{t}_V)$ with

$$\psi(n_1, n_2) = (n_1, n_1, n_2) \text{ and } \psi^{-1}(n, v_1, v_2) = (n, v_2)$$

for which homeomorphy follows from component-wise continuity of both ψ and ψ^{-1}.

Commutativity with the injection maps is given by

$$\psi(\Delta(n)) = \psi(n, n) = (n, n, n) = (n, \Delta(n)) = i_\iota(n)$$

and with the projection maps by

$$\pi_1(n_1, n_2) = n_1 = j_\iota(n_1, n_1, n_2) = j_\iota(\psi(n_1, n_2)),$$

which concludes the proof.

□

Finally, we gathered all the tools to prove Milnor's theorem.

Theorem 5.6 (Milnors Theorem)**.** *[Mil64, p.62]*
For a sufficently large $q \in \mathbb{N}$, $N = N \times \{0\}$ has a normal microbundle in $M \times \mathbb{R}^q$.

Proof.
We assume that M is embedded in euclidean space $\mathbb{R}^{2m+1}$ [HW41, p.60]. Additionally, let V be an open neighborhood of N in M together with a retraction $r : V \to N$.

We show the theorem in multiple steps.

Step 1: N has a normal microbundle η in M such that $\mathfrak{t}_N \oplus \eta \cong \mathfrak{e}^q_N$

Consider the extension $r^*\mathfrak{t}_N$. Since V is an open set, it's a simplicial complex. Hence, we can apply Theorem **(2.13)** to $r^*\mathfrak{t}_N$ to obtain a microbundle η' such that $r^*\mathfrak{t}_N \oplus \eta' \cong \mathfrak{e}^q_V$.

We conclude that $\mathfrak{t}_N \oplus \eta'|_N = r^*\mathfrak{t}_N|_N \oplus \eta'|_N = (r^*\mathfrak{t}_N \oplus \eta')|_N = \mathfrak{e}^q_N$.

Step 2: $E(r^*\mathfrak{t}_N)$ has a normal microbundle in $E(r^*\mathfrak{t}_N \oplus r^*\eta)$

We denote the injection and projection of $r^*\mathfrak{t}_N$ by $i_\mathfrak{t}$ and $j_\mathfrak{t}$, and the injection and projection of $r^*\eta$ by i_η and j_η.

Firstly, note that $r^*\mathfrak{t}_N \oplus r^*\eta \cong r^*(\mathfrak{t}_N \oplus \eta)$ is trivial, so $E(r^*\mathfrak{t}_N \oplus r^*\eta)$ is an open subset of $\mathbb{R}^q$ (for some $q \in \mathbb{N}$) and hence a manifold.

We consider $E(r^*\mathfrak{t}_N)$ to be a subset of $E(r^*\mathfrak{t}_N \oplus r^*\eta)$ embedded via

$$(v, e) \mapsto ((v, e), (v, i_\eta(v))).$$

We are given a normal microbundle of $E(r^*\mathfrak{t}_N)$ in $E(r^*\mathfrak{t}_N \oplus r^*\eta)$ by $j_\mathfrak{t}^*(r^*\eta)$. That is because the total space

$$E(j_r^*(r^*\eta)) = \{(e, e') \in E(r^*\mathfrak{t}_N) \times E(r^*\eta) : j_\mathfrak{t}(e) = j_\eta(e')\}$$

equals $E(r^*\mathfrak{t}_N \oplus r^*\eta)$ and because its inclusion is the above embedding

$$((v, e), i_\eta(j_\mathfrak{t}(v, e))) = ((v, e), i_\eta(v)).$$

Step 3: N has a normal microbundle in $M \times \mathbb{R}^q$

Since N has a normal microbundle in $E(r^*\mathfrak{t}_N)$ using Proposition **(1.5)**, it follows from Lemma **(5.4)** that N has a normal microbundle in $E(r^*\mathfrak{t}_N \oplus r^*\eta)$.

By restricting $E(r^*\mathfrak{t}_N \oplus r^*\eta)$ to an open neighborhood of $i_\oplus(V)$ if necessary, we may assume that

$$E(r^*\mathfrak{t}_N \oplus r^*\eta) = M \times \mathbb{R}^q$$

using Lemma **(2.4)**.

Applying Lemma **(5.4)** and $E(r^*\mathfrak{t}_N \oplus r^*\eta) = M \times \mathbb{R}^q$ completes the proof. □

With the proof of Theorem **(2.13)** and of Theorem **(5.6)**, we can calculate an upper bound $m(2^{2m+2} - 2)$ for q where m is the dimension of M ([Mil64, p.63]). However with slightly sharper proofs, one can substantially reduce the upper bound to a quadratic upper bound $(m + 1)^2 - 1$ (see [Hir66, p.232]).

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