Bundle Structures on Topological Manifolds

Florian Burger

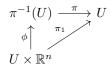
March 27, 2024

1 Bundles on Smooth Manifolds

Definition 1. (vector bundle)

A vector bundle ξ is a tuple $\dot{\xi} := (B, E, \pi, +, \cdot)$ satisfying the following conditions:

- B is a topological space (base space)
- E is a topological space (total space)
- $(\pi^{-1}(b), +, \cdot)$ is a real vector space for every $b \in B$
- Every $b \in B$ is locally trivializable, i.e there exist neighborhoods $U \subseteq B$ of b such that the following diagram commutes



and $\phi(b,-): b \times \mathbb{R}^n \xrightarrow{\sim} \pi^{-1}(b)$ is a linear isomorphism.

We call n the rank of ξ .

Example 1. (tangent vector bundle)

Let M be a smooth manifold:

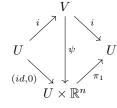
 $\xi: TM \xrightarrow{\pi} M$ is a vector bundle, where $\pi(p,v) := p$.

2 Introduction to Microbundles

Definition 2. (microbundle)

A microbundle \mathfrak{b} is a tuple $\mathfrak{b} := (B, E, i, j)$ satisfying the following properties:

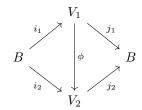
- B is a topological space called the base space
- E is a topological space called the total space
- $i: B \to E$ and $j: E \to B$ are continous maps with $id_B = j \circ i$
- Every $b \in B$ is locally trivializable, i.e there exist neighborhoods $U \subseteq B$ of b and $V \subseteq E$ of i(b) such that the following diagram commutes:



We call n the fibre dimension of \mathfrak{b} .

Definition 3. (isomorphic microbundles)

Two microbundles $\mathfrak{b}_1 := (B, E_1, i_1, j_2)$ and $\mathfrak{b}_2 := (B, E_2, i_2, j_2)$ are said to be isomorphic if there exist neighborhoods $V_1 \subseteq E_1$ of $i_1(B)$ and $V_2 \subseteq E_2$ of $i_2(B)$ with an homeomorphism $\phi : V_1 \xrightarrow{\sim} V_2$ such that the following diagram commutes:



Example 2. (trivial microbundle)

Let B be a topological space and $n \in \mathbb{N}$:

The diagram $\mathfrak{e}_B^n: B \xrightarrow{\iota} B \times \mathbb{R}^n \xrightarrow{\pi} B$ constitutes a microbundle, where $\iota(b) := (b,0)$ and $\pi(b,x) := b$. We call \mathfrak{e}_B^n the standard microbundle and every microbundle isomorphic to \mathfrak{b}_B^n trival.

Example 3. (underlying microbundle)

Let $\xi: E \xrightarrow{\pi} B$ be a n-dimensional vector bundle: The microbundle $|\xi|: B \xrightarrow{i} E \xrightarrow{\pi} B$ with $i(b) := \phi_b(b,0)$, where $\phi_b: U_b \times \mathbb{R}^n \to \pi^{-1}(U_b)$ is the local trivialization over a neighborhood $U_b \subseteq B$ of b. We call $|\xi|$ the underlying microbundle of ξ

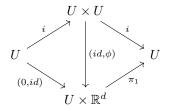
Proof.

Example 4. (tangent microbundle)

Let M be a topological manifold:

We can derive the tangent microbundle $t_M: M \xrightarrow{\Delta} M \times M \xrightarrow{\pi_1} M$, where Δ is the diagonal map and π_1 ist the projection map on the first component.

Proof. Let $p \in M$ and (U, ϕ) a chart over p:



 (id, ϕ) is a homeomorphism since $\phi: U \xrightarrow{\sim} \mathbb{R}^n$ is homeomorphic.