Microbundles on Topological Manifolds

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May 2, 2024

Contents

1 Introduction to Microbundles

Definition 1.1 (microbundle)**.**
A *microbundle* $\mathfrak{b}$ over B (with *fibre-dimension* n) is a diagram $B \xrightarrow{i} E \xrightarrow{j} B$ satisfying the following:

- B is a topological space (*base space*)
- E is a topological space (*total space*)
- $i : B \to E$ (*injection*) and $j : E \to B$ (*projection*) are continuous maps such that $id_B = j \circ i$
- Every $b \in B$ is *locally trivializable*, that is there exist open neighborhoods $U \subseteq B$ of b and $V \subseteq E$ of $i(U)$ with a homeomorphism $\phi : V \xrightarrow{\sim} U \times \mathbb{R}^n$ such that the following diagram commutes:

$$\begin{tikzcd} & V \arrow[dr, "j|_V"] \arrow[dd, "\psi"] & \\ U \arrow[ur, "i"] \arrow[dr, "(id{,}0)"'] & & U \\ & U \times \mathbb{R}^n \arrow[ur, "\pi_1"'] & \end{tikzcd}$$

Remark 1.2.
In the following, unless explicitly stated otherwise we assume the fiber dimension of any given microbundle to be n.

Before we look at examples of microbundles, we should first clarify what it means for two microbundles to be isomorphic.

Definition 1.3 (isomorphy)**.**
Two microbundles $\mathfrak{b}_1 : B \xrightarrow{i_1} E_1 \xrightarrow{j_1} B$ and $\mathfrak{b}_2 : B \xrightarrow{i_2} E_2 \xrightarrow{j_2} B$ are *isomorphic* if there exist neighborhoods $V_1 \subseteq E_1$ of $i_1(B)$ and $V_2 \subseteq E_2$ of $i_2(B)$ with a homeomorphism $\phi : V_1 \xrightarrow{\sim} V_2$ such that the following diagram commutes:

$$\begin{tikzcd} & V_1 \arrow[dr, "j_1|_{V_1}"] \arrow[dd, "\phi"] & \\ B \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & B \\ & V_2 \arrow[ur, "j_2|_{V_2}"'] & \end{tikzcd}$$

The most obvious example for a microbundle is the standard microbundle.

Example 1.4 (trivial microbundle)**.**
For a topological space B, the *standard microbundle* $\mathfrak{e}_B^n$ over B is a diagram

$$B \xrightarrow{\iota} B \times \mathbb{R}^n \xrightarrow{\pi} B$$

where $\iota(b) := (b, 0)$ and $\pi(b, x) := b$.

Additionally, a microbundle $\mathfrak{b}$ over B is *trivial* if it is isomorphic to $\mathfrak{e}^n_B$.

Lemma 1.5 (criteria for triviality)**.**
A microbundle $\mathfrak{b}$ over B is trivial if and only if there exists an open neighborhood U of $i(B)$ such that $U \cong B \times \mathbb{R}^n$.

Proof.
TODO ☐

Similar to the study of tangent bundles over smooth manifolds, we want a concept of intrinsic microbundles over topological manifolds.

Example 1.6 (tangent microbundle)**.**
The *tangent microbundle* $\mathfrak{t}_M$ over a topological d-manifold M is a diagram

$$M \xrightarrow{\Delta} M \times M \xrightarrow{\pi_1} M$$

where $\Delta(m) := (m, m)$ is the diagonal map and $\pi_1(m_1, m_2) := m_1$ is the projection map on the first component.

Proof that $\mathfrak{t}_M$ is a microbundle.
Let $p \in M$ and (U, ϕ) be a chart over p. We explicitly construct a local trivialization

$$\begin{tikzcd} & U \times U \arrow[dd, "\psi"] \arrow[dr, "\pi_1"] & \\ U \arrow[ur, "\Delta"] \arrow[dr, "(0,id)"'] & & U \\ & U \times \mathbb{R}^d \arrow[ur, "\pi_1"'] & \end{tikzcd}$$

where $\psi(u, \tilde{u}) := (u, \phi(u) - \phi(\tilde{u}))$. It's obvious that $(U, U \times U, \psi)$ meets all local triviality conditions. ☐

The idea behind the tangent microbundle is actually very intuitive. TODO

Example 1.7 (underlying microbundle)**.**
Let $\xi : E \xrightarrow{\pi} B$ be a n-dimensional vector bundle. The microbundle $|\xi| : B \xrightarrow{i} E \xrightarrow{\pi} B$ where $i(b) := \phi_b(b, 0)$, where $\phi_b : U_b \times \mathbb{R}^n \to \pi^{-1}(U_b)$ is the local trivialization over a neighborhood $U_b \subseteq B$ of b. We call $|\xi|$ the *underlying microbundle* of ξ

Proof.
TODO ☐

The following proposition shows that, when studying microbundles, we are not interested in the entire total space but only in an arbitrarily small neighborhood of the base space. This is actually one of the biggest differences behind the concept of microbundles and classical vector-bundles.

Proposition 1.8.
For a microbundle $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ over B, we can be restrict the total space E to an arbitray neighborhood $E' \subseteq E$ of $i(B)$ where the resulting microbundle is isomorphic to $\mathfrak{b}$.

Proof.
For an arbitray $b \in B$, choose a local trivialization (U, V, ϕ).

The intersection $V \cap E'$ is a neighborhood of $i(b)$ because V and E' both are. It follows that $\phi(V \cap E')$ is a neighborhood of $(b, 0)$. Hence there exist $U' \subseteq B$ open and $B_\varepsilon(0) \subseteq \mathbb{R}^n$ such that $U' \times B_\varepsilon(0) \subseteq \phi(V \cap E')$. Now we construct our local trivialization by choosing $V' := \phi^{-1}(U' \times B_\varepsilon(0))$ and the fact that $B_\varepsilon(0) \cong \mathbb{R}^n$:

$$U' \times \mathbb{R}^n \cong U' \times B_\varepsilon(0) \cong V'$$

We easily see that the resulting microbundle is isomorphic to $\mathfrak{b}$ via the identity.

$\square$

2 Induced Microbundles

Definition 2.1 (induced microbundle)**.**
Let $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ be a microbundle and $f : A \to B$ be a continuous map. The *induced microbundle* $f^*\mathfrak{b} : A \xrightarrow{i'} E' \xrightarrow{j'} A$ is defined as follows:

- $E' := \{(a, e) \in A \times E \mid f(a) = j(e)\}$
- $i' : A \to E'$ with $i'(a) := (a, (i \circ f)(a))$
- $j' : E' \to A$ with $j'(a, e) := a$

Proof that $f^*\mathfrak{b}$ is a microbundle.
It is clear that i' and j' are continuous and that $id_A = j' \circ i'$. So it remains to be shown that $f^*\mathfrak{b}$ is locally trivial.

Choose a local trivialization (U, V, ϕ) for an arbitrary $a \in A$.

- $U' := f^{-1}(U) \subseteq A$ neighborhood of a.
- $V' := j'^{-1}(U') \subseteq E'$ neighborhood of $i'(a)$.
- $\phi' : V' \xrightarrow{\sim} U' \times \mathbb{R}^n, \phi'(a, e) := (a, \pi_2(\phi(e)))$.

The map ϕ' is well-defined because $(a, e) \in V' : j(e) = f(a) \in U \implies e \in V$. The existence of an inverse $\phi'^{-1}(a, v) = (a, \phi^{-1}(f(a), v))$ and component-wise continuity show that ϕ' is a homeomorphism. This proves that (U', V', ϕ') is a local trivialization for a. $\square$

Example 2.2 (restricted microbundle)**.**

Let $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ be a microbundle and $A \subseteq B$ be a subspace. The *restricted microbundle* $\mathfrak{b}|_A$ is the induced microbundle $\iota^*\mathfrak{b}$ where $\iota : A \hookrightarrow B$ is the inclusion map.

Remark 2.3. In the following we consider $E(\mathfrak{b}|_A)$ a subset of $E(\mathfrak{b})$. This is justified because $E(\mathfrak{b}|_A) = \{(a, e) \in A \times E(\mathfrak{b}) \mid a = j(e)\}$ is homeomorphic to $\{e \in E(\mathfrak{b}) \mid j(e) \in A\} \subseteq E(\mathfrak{b})$.

Lemma 2.4.

Let $\mathfrak{b}$ be a microbundle over B and $f : A \to B$ be a map. The induced microbundle $f^*\mathfrak{b}$ is trivial if $\mathfrak{b}$ is already trivial.

Proof.

Let (V, ϕ) be a global trivialization, that is $\phi : V \xrightarrow{\sim} B \times \mathbb{R}^n$.

- $V' := (A \times V) \cap E(f^*\mathfrak{b})$ a neighborhood of $i'(A)$.
- $\phi' : V' \xrightarrow{\sim} B \times \mathbb{R}^n, \phi'(a, e) := (a, \pi_2(\phi(e)))$.

The existence of an inverse $\phi'^{(-1)}(a, x) = (a, \phi^{-1}(f(a), x))$ and component-wise continuity show that ϕ' is a homeomorphism. This proves that (V', ϕ') is a global trivialization for $f^*\mathfrak{b}$. $\square$

Lemma 2.5.

For a diagram $A \xrightarrow{f} B \xrightarrow{g} C$ and a microbundle $\mathfrak{c} : C \xrightarrow{i} E \xrightarrow{j} C$ applies:

$$(g \circ f)^*\mathfrak{c} \cong f^*(g^*\mathfrak{c})$$

Proof.

To prove isomorphy, we need to show that the two total spaces are homeomorphic and that the injection and projection maps commute with such a homeomorphism.

First, compare the two total spaces:

1. $E((g \circ f)^*\mathfrak{c}) = \{(a, e) \in A \times E(\mathfrak{c}) \mid g(f(a)) = j(e)\}$
2. $E(f^*(g^*\mathfrak{c})) = \{(a, (b, e)) \in A \times (B \times E(\mathfrak{c})) \mid f(a) = b \text{ and } g(b) = j(e)\}$.

We have the bijection $\phi : E((g \circ f)^*\mathfrak{c}) \xrightarrow{\sim} E(f^*(g^*\mathfrak{c}))$ with $\phi(a, e) := (a, (f(a), e))$ and $\phi^{-1}(a, (b, e)) = (a, e)$. Since ϕ and ϕ^{-1} are component-wise continuous, it follows that ϕ is a homeomorphism.

It's easy to see that ϕ commutes with injection and projection maps, which concludes the proof. $\square$

For a topological space X, we define the *cone* of X to be

$$CX := X \times [0,1]/X \times \{1\}$$

and for a map $f : A \to B$ the *mapping cone* of f to be

$$B \sqcup_f CA := B \sqcup CA/\sim$$

where $(a,0) \sim b :\iff f(a) = b$.

Similarly, we define the *cylinder* of X to be

$$MX := X \times [0,1]$$

and for a map $f : A \to B$ the *mapping cylinder* of f to be

$$B \sqcup_f MA := B \sqcup CA/\sim$$

where $(a,0) \sim b :\iff f(a) = b$.

Lemma 2.6.
A microbundle $\mathfrak{b}$ over B can be extended to a microbundle over the mapping cone $B \sqcup_f CA$ if and only if $f^*\mathfrak{b}$ is trivial.

Proof.
We show both implications.

" $\implies$ "

Let $\mathfrak{b}'$ be an extension of $\mathfrak{b}$ over $B \sqcup_f CA$. Considering $A \xrightarrow{f} B \hookrightarrow B \sqcup_f CA$, the composition $\iota \circ f$ is null-homotopic with homotopy

$$H_t(a) := [(a,t)]$$

Note that $H_0(a) = [(a,0)] = [f(a)] = (\iota \circ f)(a)$ and $H_1(a) = [(a,1)] = [(\tilde{a},1)] = H_1(\tilde{a})$. From the Homotopy Theorem (4.12) follows that $(\iota \circ f)^*\mathfrak{b}'$ is trivial.

Since $(\iota \circ f)^*\mathfrak{b}' = f^*(\iota^*\mathfrak{b}') = f^*\mathfrak{b}$, it follows that $f^*\mathfrak{b}$ is trivial.

" $\impliedby$ "

Let $f^*\mathfrak{b}$ be trivial.

In contrast to the mapping cone, there exists a natural retraction from the mapping cylinder to the attached space

$$r : B \sqcup_f MA \to B, r([(a,t)]) := f(a)$$

The diagram

$$A \times \{1\} \hookrightarrow B \sqcup_f MA \xrightarrow{r} B$$

equals f if we consider $A = A \times \{1\}$. It follows that

$$r^*\mathfrak{b}|_{A\times\{1\}} = (r \circ \iota)^*\mathfrak{b} \cong f^*\mathfrak{b} = \mathfrak{e}^n_A$$

is trivial. From Lemma **(2.4)** and $(a,t) \mapsto (a,1)$ it follows that $r^*\mathfrak{b}|_{A\times[\frac{1}{2},1]}$ is trivial, so there exists a

$$\phi : E(r^*\mathfrak{b}|_{A\times[\frac{1}{2},1]}) \xrightarrow{\sim} A \times [\frac{1}{2}, 1] \times \mathbb{R}^n$$

Now we explicitly construct our desired extended microbundle $\mathfrak{b}' : B \sqcup_f CA \xrightarrow{i'} E' \xrightarrow{j'} B \sqcup_f CA$

- $E' := E(r^*\mathfrak{b})/\phi^{-1}(A \times \{1\} \times \{x\})$ (for every $x \in \mathbb{R}^n$).
- $i'([a,t]) := [i_r(a,t)]$ where i_r is the injection map for $r^*\mathfrak{b}$.
- $j'([e]) := [j_r(e)]$ where j_r is the projection map for $r^*\mathfrak{b}$.

The injection i' is well-defined because i_r maps every representative $[a,1]$ to the same equivalence class of E'. Similarly, the projection j' is well-defined since $[e] = [\tilde{e}] \implies [j_r(e)] = [j_r(e')]$ We easily derive the microbundle conditions from $r^*\mathfrak{b}$.

This proves the claim. $\square$

Corollary 2.7.
Let B be a $(d+1)$-simplicial complex, B' it's d-skeleton and $\sigma \subseteq B$ a $(d+1)$-simplex. A microbundle $\mathfrak{b}$ over B' can be extended to a microbundle over $B' \cup \sigma$ if and only if its restriction to the boundary $\mathfrak{b}|_{\partial\sigma}$ is trivial.

Proof.
By choosing $f : \partial\sigma \hookrightarrow B'$ and applying the previous lemma, we see that there exists a microbundle $\mathfrak{b}'$ over $B' \cup_f C\sigma$ extending $\mathfrak{b}$.

Now, consider the homeomorphism $\phi : C\partial\sigma \xrightarrow{\sim} \sigma$ with

$$\phi((t_1, \ldots, t_{d+1}), \lambda) := (1-\lambda)(t_1, \ldots, t_{d+1}) + \frac{\lambda}{d+1}(1, \ldots, 1)$$

In particular, $\phi(\partial\sigma \times \{0\}) = \partial\sigma$.

It follows that $B' \cup_f C\sigma \cong B' \cup \sigma$ which concludes the proof. $\square$

3 Whitney sums

Definition 3.1. (whitney sum)
Let $\mathfrak{b}_1$ and $\mathfrak{b}_1$ be two microbundles over a topological space B. We define the *whitney sum* $\mathfrak{b}_1 \oplus \mathfrak{b}_2$ as follows:

- $E := \{(e_1, e_2) \in E(\mathfrak{b}_1) \times E(\mathfrak{b}_2) \mid j_1(e_1) = j_2(e_2)\}$
- $i(b) := (i_1(b), i_2(b))$
- $j(e_1, e_2) := j_1(e_1) = j_2(e_2)$

Proof.
Let $b \in B$.
Choose U_1, V_1, ϕ_1 and U_2, V_2, ϕ_2 accordingly from the local trivialization of b over $\mathfrak{b}_1$ and $\mathfrak{b}_2$:

- $U := U_1 \cap U_2$
- $V := (V_1 \times V_2) \cap E$
- $\phi : V \to U \times \mathbb{R}^{n_1+n_2}; \phi(e_1, e_2) := (\phi_1^{(1)}(e_1), \phi_1^{(2)}(e_1) \times \phi_2^{(2)}(e_2))$

Note that $\phi_1^{(1)}(e_1) = \phi_2^{(1)}(e_2)$. Local triviality follows directly from it's components. $\square$

Lemma 3.2. (compatibility)
Let $\mathfrak{b}_1$ and $\mathfrak{b}_1$ be two microbundles over B and $f : A \to B$ a map. Induced microbundle and whitney sum are compatible, i.e. $f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2) \cong f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2$

Proof.
From the definition of the induced microbundle and the whitney sum, we can derive the total spaces:

1. $E(f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2)) = \{(a, (e_1, e_2)) \in A \times (E_1 \times E_2) \mid j_1(e_1) = j_2(e_2) = f(a)\}$
2. $E(f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2) = \{((a_1, e_1), (a_2, e_2)) \in (A \times E_1) \times (A \times E_2) \mid j(a_1, e_1) = j(a_2, e_2) \text{ and } f(a_i) = j(e_i)\}$

Those two total spaces are homeomorphic via $\phi(a, (e_1, e_2)) := ((a, e_1), (a, e_2))$ and $\phi^{-1}((a, e_1), (a, e_2)) = (a, (e_1, e_2))$. ϕ and ϕ^{-1} are continuous because they are componentwise continuous. Obviously, $\phi \circ i = i$ and $\phi \circ j = j$, which concludes the proof. $\square$

TODO — BOUQET LEMMA

Theorem 3.3. *(inverse microbundles)*
Let $\mathfrak{b}$ be a microbundle over a d-dimensional simplicial complex B.
Then there exists a microbundle $\mathfrak{n}$ over B so that the Whitney sum $\mathfrak{b} \oplus \mathfrak{n}$ is trivial.

Proof.
We prove this theorem by induction over d.

(Start of induction)

A 1-dimensional simplicial complex is just a bouquet of circles, therefore the start of induction follows directly from the **??**.

(Inductive Step)

Let B' be the $(d-1)$-skeleton of B and $\mathfrak{n}'$ it's corresponding microbundle so that $\mathfrak{b}|_{B'} \oplus \mathfrak{n}'$ is trivial.

1. $\mathfrak{n}' \oplus \mathfrak{e}^n_{B'}$ can be extended over every d-simplex σ:

   Consider the following:

   $$(\mathfrak{n}' \oplus \mathfrak{e}^n_{B'})|_{\partial\sigma} = \mathfrak{n}'|_{\partial\sigma} \oplus \mathfrak{e}^n_{B'}|_{\partial\sigma} = \mathfrak{n}'|_{\partial\sigma} \oplus \mathfrak{b}|_{\partial\sigma} = (\mathfrak{n}' \oplus \mathfrak{b}|_{B'})|_{\partial\sigma}$$

   Since $(\mathfrak{n}' \oplus \mathfrak{b}|_{B'})|_{\partial\sigma}$ is trivial, the claim follows from **??**.

2. $\mathfrak{n}' \oplus \mathfrak{e}^n_{B'}$ can be extended over B:

   The difficulty is that the individual d-simplices are not well-seperated. Let B'' denote B with small open d-cells cut out from every d-simplex. Since B' is a retract of B'' we can extend $\mathfrak{n}' \oplus \mathfrak{e}^n_{B'}$ over B'' and now apply the first statement. We denote the resulting microbundle by η.

3. Consider the mapping-cone $B \sqcup CB'$ over the inclusion $B' \hookrightarrow B$. Since

   $$(\mathfrak{b} \oplus \eta)|_{B'} = \mathfrak{b}|_{B'} \oplus \eta|_{B'} = \mathfrak{b}|_{B'} \oplus (\mathfrak{n}' \oplus \mathfrak{e}^n_{B'}) = (\mathfrak{b}|_{B'} \oplus \mathfrak{n}') \oplus \mathfrak{e}^n_{B'} = \mathfrak{e}^n_{B'} \oplus \mathfrak{e}^n_{B'}$$

   which is trivial, by **??** we can extend $\mathfrak{b} \oplus \eta$ over $B \sqcup CB'$ denoted by ξ. However, $B \sqcup CB'$ has the homotopy type of a bouquet of spheres and by the **??** there exists a microbundle $\mathfrak{n}$ such that $(\xi \oplus \mathfrak{n})|_B$ is trivial.

   $$\mathfrak{e}^n_B = (\xi \oplus \mathfrak{n})|_B = \xi|_B \oplus \mathfrak{n}|_B = (\mathfrak{b} \oplus \eta) \oplus \mathfrak{n}|_B = \mathfrak{b} \oplus (\eta \oplus \mathfrak{n}|_B)$$

   This conlcudes the proof.

□

4 Homotopy and Microbundles

Definition 4.1. (map-germ)
Let (X, A) and (Y, B) be two topological pairs. A *map-germ* $F : (X, A) \Rightarrow (Y, B)$ is an equivalence class of maps $f : (X, A) \to (Y, B)$, where $f \sim g :\iff f|_U = g|_U$ for some neighborhood $U \subseteq X$ of A.

Remark 4.2. The composition of two map-germs $(X, A) \overset{F}{\Rightarrow} (Y, B) \overset{G}{\Rightarrow} (Z, C)$ is well defined.

Definition 4.3. (isomorphism-germ)
An *isomorphism-germ* from $\mathfrak{b}$ to $\mathfrak{b}'$ is a homeomorphism-germ

$$F : (E, B) \Rightarrow (E', B)$$

which is *fibre-preserving*, that is $J' \circ F = J$.

Definition 4.4. (bundle-germ)
Let $\mathfrak{b}$ and $\mathfrak{b}'$ be two microbundles over B with the same fibre-dimension. A map-germ

$$F : (E, B) \Rightarrow (E', B)$$

is a *bundle-germ* if a representative map $f : U \to E'$ maps each fibre $j^{-1}(b)$ bijective to a fibre $j'^{-1}(b')$.

Lemma 4.5. (closed balls under homeomorphism)
For a homeomorphism $\phi : \mathbb{R}^n \xrightarrow{\sim} R^n$ applies:

$$|\phi(x) - x| < 1, \forall x \in \overline{B_2(0)} \implies \overline{B_1(0)} \subseteq \phi(\overline{B_2(0)})$$

Proof.
Consider the two points $x_0 := 0$ and $x_1 := 3e_1$. Obviously, there is no path between x_0 and x_1 in $\mathbb{R}^n - S^n$. Therefore, since ϕ is a homeomorphism, there is no path between $\phi(x_0)$ and $\phi(x_1)$ in $\mathbb{R}^n - \phi(S^n)$. Since $1 < |x|$ for every $x \in \phi(S^n)$, there is a path between every $x \in \overline{B_1(0)}$ and $\phi(x_1)$ in $\mathbb{R}^n - \phi(S^n)$ (e.g a straight line). It follows that there is also no path between any $x \in \overline{B_1(0)}$ and x_1 in $\mathbb{R}^n - \phi(S^n)$. Since $R^n - \overline{B_2(0)}$ is path-connected, $R^n - \phi(\overline{B_2(0)})$ is as well. From $x_1 \in R^n - \phi(\overline{B_2(0)})$, we know that $\overline{B_1(0)} \subseteq \phi(\overline{B_2(0)})$. ☐

Lemma 4.6. (Williamson)
A bundle map-germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$ on the same base space B covering the identity map is an isomorphism-germ.

Proof.
We prove the statement locally, then glue the function together for the prove.

1. First, we consider the case that $\mathfrak{b}$ and $\mathfrak{b}'$ are trivial. This means, that $F : B \times \mathbb{R}^n \to B \times \mathbb{R}^n$ is of the form

   $$F(b, x) = (b, g_b(x))$$

   where $g_b : \mathbb{R}^n \to \mathbb{R}^n$ are individual maps. In particular, the g_b are even homeomorphisms due to the domain invariance theorem. Now we show that F is a homeomorphism. Let $(b_0, x_0) \in B \times \mathbb{R}^n$ and $\varepsilon > 0$. Since g_{b_0} is a homeomorphism, there exists a $\delta > 0$ so that $\overline{B_{2\delta}(x_1)} \subseteq g_{b_0}(\overline{B_\varepsilon(x_0)})$ where $x_1 := g_{b_0}(x_0)$.

   There exists a neighborhood $V \subseteq B$ of b_0, such that

   $$|g_b(x) - g_{b_0}(x)| < \delta, \forall x \in \overline{B_\varepsilon(x_0)}$$

   To show that, consider $\phi_b(b, x) := g_b(x) - g_{b_0}(x)$. The closed set $\phi^{-1}(\overline{B_\varepsilon(x_0)})$ is a neighborhood of $\{b_0\} \times \mathbb{R}^n$ since $\phi(b_0, x) = 0, \forall x \in \mathbb{R}^n$. Therefore, for every $x \in \overline{B_\delta(x_1)}$ exist $V_x \subseteq B$ and $U_x \subseteq \mathbb{R}^n$ open with $x \in U_x$ and $V_x \times U_x \subseteq \phi^{-1}(\overline{B_\varepsilon(x_0)})$. Obviously, $\bigcup_{x \in \overline{B_\delta(x_1)}} U_x$ is an open covering of $\overline{B_\delta(x_1)}$ and because of compactness of $\overline{B_\delta(x_1)}$, there exist $x_1, \dots, x_n \in \overline{B_\delta(x_1)}$ with $\overline{B_\delta(x_1)} \subseteq \bigcup_{i=1}^n U_{x_i}$. The claim follows via $V := V_{x_1} \cap \dots \cap V_{x_n}$.

   From

   $$V \times \overline{B_\delta(x_1)} \subseteq g(V \times \overline{B_\varepsilon(x_0)})$$

   it follows that g is open.

2. Now we restrict ourselves to the above situation.

   Assume a local trivialization (U, V, ϕ) over $b \in B$. First, we restrict f to $f^{-1}(V)$. Since $f^{-1}(V)$ is a neighborhood of $i(b)$, we can choose an open neighborhood $V' \subseteq f^{-1}(V) \cap V$ of $i(b)$ of the form $U' \times B_\varepsilon(0)$. Now we have

   $$U' \times \mathbb{R}^n \cong U' \times B_\varepsilon(0) \xrightarrow{f} U' \times \mathbb{R}^n \subseteq U \times \mathbb{R}^n$$

   a map $U' \times \mathbb{R}^n \to U' \times \mathbb{R}^n$ that is bijective and fibre-preserving and therefore a homeomorphism.

   $$U' \times \mathbb{R}^n \cong U' \times B_\varepsilon(0) \cong V' \xrightarrow{\phi \circ f}$$

   By glueing all the $U' \times \mathbb{R}^n$, which are neighborhoods of b, together we see that f is a homeomorphism on it's image. This concludes the proof.

□

Corollary 4.7. (induced microbundles)
If a map $g : B \to B'$ is covered by a bundle germ $\mathfrak{b} \Rightarrow \mathfrak{b}'$ then $\mathfrak{b}$ is isomorphic to the induced bundle $g^*\mathfrak{b}'$.

Proof.

□

Lemma 4.8. (glueing together bundle map-germs)
Let $\mathfrak{b}$ be a microbundle over B and $\{B_\alpha\}$ a locally finite collection of closed sets covering B. Additionally, we are given $F_\alpha : \mathfrak{b}|_{B_\alpha} \Rightarrow \mathfrak{b}'$, a collection of bundle map-germs with $F_\alpha = F_\beta$ on $\mathfrak{b}|_{B_\alpha \cap B_\beta}$ Then there exists a bundle map-germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$ extending F_α.

Proof.
Choose representative maps $f_\alpha : U_\alpha \to E'$ for F_α with U_α open. Since $F_\alpha = F_\beta$ on $\mathfrak{b}|_{B_\alpha \cap B_\beta}$, $f_\alpha = f_\beta$ for an open neighborhood $U_{\alpha\beta}$ of $B_\alpha \cap B_\beta$. We define

$$U := \{e \in E \mid j(e) \in B_\alpha \cap B_\beta \implies e \in U_{\alpha\beta}\}$$

1. U is open:
   Let $e \in U$ and $j(e) \in B_\alpha \cap B_\beta$. From local finiteness there exists an open neighborhood $V \subseteq B$ of $j(e)$ with $V \subseteq B_{\alpha_1} \cap \ldots \cap B_{\alpha_n}$. W.l.o.g. $V \subseteq B_\alpha \cap B_\beta$ by exluding a finite number of closed sets if necessary. Now $V_{\alpha\beta} := j^{-1}(V) \cap U_{\alpha\beta}$ is an open neighborhood of e. Since $j(e)$ can only be contained in finitely many B_α we can form the intersection of all these $V_{\alpha'\beta'}$ which, by construction, is contained in U and is open.

2. $B \subseteq U$ considering the cases $U_{\alpha\alpha}$.

Now we can define $f : U \to E'$ in the obvious way

$$f(u \in U_{\alpha\beta}) := f_\alpha(u) = f_\beta(u)$$

which is a representative map for our desired F. $\square$

Lemma 4.9. (piecewise triviality)
Let $\mathfrak{b}$ be a microbundle over $B \times [0,1]$ such that both $\mathfrak{b}|_{B\times[0,\frac{1}{2}]}$ and $\mathfrak{b}|_{B\times[\frac{1}{2},1]}$ are trivial. Then $\mathfrak{b}$ itself is already trivial.

Proof.
Since $\mathfrak{b}|_{B\times[\frac{1}{2},1]}$ is trivial, we can extend the identity bundle map-germ on $\mathfrak{b}|_{B\times\{\frac{1}{2}\}}$ to $\mathfrak{b}|_{B\times[\frac{1}{2},1]} \Rightarrow \mathfrak{b}|_{B\times\{\frac{1}{2}\}}$. Using the previous lemma, we can piece this together with the identity bundle map-germ on $\mathfrak{b}|_{B\times[0,\frac{1}{2}]}$ to

$$\mathfrak{b} \Rightarrow \mathfrak{b}|_{B\times[0,\frac{1}{2}]}$$

From the corollary it follows that $\mathfrak{b} \cong \mathfrak{b}|_{B\times[0,\frac{1}{2}]}$. $\square$

Lemma 4.10. ()
Let $\mathfrak{b}$ be a microbundle over $B \times [0,1]$. Every $b \in B$ has a neighborhood V where $\mathfrak{b}|_{V\times[0,1]}$ is trivial.

Proof.
Let $b \in B$. For every $t \in [0,1]$, choose a neighborhood $U_t := V_t \times (t - \varepsilon_t, t + \varepsilon_t)$ of (b,t) such that $\mathfrak{b}|_{U_t}$ is trivial. Since $\{b\} \times [0,1]$ is compact, we can choose a finite covering of the U_t and define V to be the intersection of the corresponding V_t. Then there exists a subdivision $0 = t_0 < \cdots < t_k = 1$ where the $\mathfrak{b}|_{V\times[t_i,t_{i+1}]}$ are trivial. Iteratively applying the previous lemma, it follows that $\mathfrak{b}|_{V\times[0,1]}$ is trivial. $\square$

Lemma 4.11. ()
Let $\mathfrak{b}$ be a microbundle over $B \times [0,1]$ where B is paracompact. Then there exists a bundle map-germ $F : \mathfrak{b} \to \mathfrak{b}|_{B\times\{1\}}$ covering the standard retraction $r : B \times [0,1] \to B \times \{1\}$.

Proof.
First, we assume a locally finite covering $\{V_\alpha\}$ of closed sets where $\mathfrak{b}|_{V_\alpha\times[0,1]}$ is trivial. The existence of such a covering is justified by the previous lemmas. Since B is paracompact, we can choose bump functions

$$\lambda_\alpha : B \to [0,1]$$

with $\operatorname{supp}(\lambda_\alpha) \subseteq V_\alpha$. Additionally, we assume that

$$\max_\alpha(\lambda_\alpha(b)) = 1, \forall b \in B$$

Now we define a retraction $r_\alpha : B \times [0,1] \to B \times [0,1]$ with

$$r_\alpha(b,t) := (b, \max(t, \lambda_\alpha(b)))$$

We construct bundle map-germs $R_\alpha : \mathfrak{b} \to \mathfrak{b}$ covering r_α. We can divide $B \times [0,1]$ into $A_\alpha := \operatorname{supp}(\lambda_\alpha) \times [0,1]$ and $A'_\alpha := \{(b,t) \mid t \geq \lambda_\alpha(b)\}$. Since $A_\alpha \subseteq V_\alpha \times [0,1]$, $\mathfrak{b}|_{A_\alpha}$ is trivial. It follow that the identity bundle germ on $\mathfrak{b}|_{A_\alpha \cap A'_\alpha}$ can be extended to a bundle germ $\mathfrak{b}|_{A_\alpha} \Rightarrow \mathfrak{b}|_{A_\alpha \cap A'_\alpha}$. Piecing this together with the identity bundle germ $\mathfrak{b}|_{A'_\alpha}$, we obtain the desired bundle germ R_α.

Applying the well-ordering theorem, which is equivalent to the axiom of choice, we can assume an ordering of $\{V_\alpha\}$. Let $\{B_\beta\}$ be a locally finite covering of B with closed sets where B_β intersects only $V_{\alpha_1} < \cdots < V_{\alpha_k}$ a finite collection. Now the composition $R_{\alpha_1} \circ \ldots \circ R_{\alpha_k}$ restricts to a bundle germ $R(\beta) : \mathfrak{b}|_{B_\beta} \times [0,1] \Rightarrow \mathfrak{b}|_{B_\beta} \times [1]$. Pieced together with the previous lemma, we obtain $R : \mathfrak{b} \times [0,1] \to \mathfrak{b} \times [1]$ which concludes the proof. $\square$

Theorem 4.12. (Homotopy Theorem)
Let $\mathfrak{b}$ be a microbundle of B and $f, g : A \to B$ be two maps.

$$f \simeq g \implies f^*\mathfrak{b} \cong g^*\mathfrak{b}$$

Proof.
Let $H : A \times [0,1] \to B$ be a homtopy between f and g. By the previous lemma, there exists a bundle germ $R : H^*\mathfrak{b} \Rightarrow H^*\mathfrak{b}|_{B\times[1]}$ covering the standard retraction $B \times [0,1] \to B \times [1]$. From the composition

$$f^*\mathfrak{b} \subseteq H^*\mathfrak{b} \Rightarrow_R H^*\mathfrak{b}|_{B\times[1]} = g^*\mathfrak{b}$$

we obtain an isomorphism germ $f^*\mathfrak{b} \Rightarrow g^*\mathfrak{b}$. It follow that $f^*\mathfrak{b} \cong g^*\mathfrak{b}$. $\square$

5 Normal Microbundles

Definition 5.1. (normal microbundle)
Let M and N be two topological manifolds with $N \subseteq M$. We call a microbundle of the form

$$\mathfrak{n} : N \xrightarrow{\iota} U \xrightarrow{r} N$$

where $U \subseteq M$ is a neighborhood of N, a *normal microbundle* of N in M.

Definition 5.2. (product neighborhood)
Again, let M and N be two topological manifolds with $N \subseteq M$. We say that N has a *product neighborhood* in M if there exists a trivial normal microbundle of N in M.

Lemma 5.3. (criteria for product neighborhoods)
A submanifold $N \subseteq M$ has a product neighborhood if and only if there exists a neighborhood U of N with $(U, M) \cong (M \times R^n, M \times 0)$.

Proof.
This follows directly from the definition of normal microbundles and the criteria for trivial microbundles. □

Definition 5.4. (composition microbundle)
Let $\mathfrak{b} : B \xrightarrow{i_\mathfrak{b}} E \xrightarrow{j_\mathfrak{b}} B$ and $\mathfrak{c} : E \xrightarrow{i_\mathfrak{c}} E' \xrightarrow{j_\mathfrak{c}} E$ be two microbundles. We define the *composition microbundle* $\mathfrak{b} \circ \mathfrak{c} : B \xrightarrow{i} E' \xrightarrow{j} B$ with $i(b) := (i_\mathfrak{c} \circ i_\mathfrak{b})(b)$ and $j(e') := (j_\mathfrak{b} \circ j_\mathfrak{c})(e')$

Proof.
Let $b \in B$.
Choose local trivializations $(U_\mathfrak{b}, V_\mathfrak{b}, \phi_\mathfrak{b})$ of b and $(U_\mathfrak{c}, V_\mathfrak{c}, \phi_\mathfrak{c})$ of $j_\mathfrak{b}(b)$. From this, we construct our local trivialization over $\mathfrak{b} \circ \mathfrak{c}$. Consider $\phi_\mathfrak{b}(V_\mathfrak{b} \cap U_\mathfrak{c})$, which is a neighborhood of $(b, 0)$. Therefore, there exist open neighborhoods $b \in U \subseteq U_\mathfrak{b}$ and $0 \in X \subseteq R^n$ such that $U \times X \subseteq \phi_\mathfrak{b}(V_\mathfrak{b} \cap U_\mathfrak{c})$. Analoguous to the proof of restricting the total space in Chapter 1, it follows that

$$\exists \varepsilon > 0 : U \times B_\varepsilon(0) \subseteq \phi_\mathfrak{b}(V_\mathfrak{b} \cap U_\mathfrak{c})$$

$$\implies U \times \mathbb{R}^n \cong U \times B_\varepsilon(0) \cong \phi_\mathfrak{b}^{-1}(U \times B_\varepsilon(0)) \cong \phi_\mathfrak{c}^{-1}(\phi_\mathfrak{b}^{-1}(U \times B_\varepsilon(0)))$$

which is an open neighborhood of $i(U)$ and therefore a valid candidate for V. This concludes local triviality and the proof. □

Lemma 5.5. (transitivity of normal microbundles)
Let M, N and P be topological manifolds with $P \subseteq N \subseteq M$. There exists a normal microbundle $\mathfrak{n}$ of P in M, if there exist normal microbundles $\mathfrak{n}_p : P \xrightarrow{i_P} U_N \xrightarrow{j_P} P$ in N and $\mathfrak{n}_n : N \xrightarrow{i_N} U_M \xrightarrow{j_N} N$ in M.

Proof.
We simply form the composition $\mathfrak{n}_p \circ \mathfrak{n}_n|_{U_N} : P \xrightarrow{i_N \circ i_P} U_M \xrightarrow{j_P \circ j_N} P$. Since $i_N \circ i_P$ is just the inclusion of $P \hookrightarrow U_M \subseteq M$, we found a normal microbundle $\mathfrak{n}$ of P in M. □

Every topological manifold is an absolute neighborhood retract (ANR).

It follows that by restricting M, if necessary, to an open neighborhood of N, there exists a retraction $r : M \to N$ which we will take advantage of in the following.

Lemma 5.6. (homeomorphism of total spaces)
Let $\mathfrak{t}_N$ and $\mathfrak{t}_M$ be the tangent microbundles of N and M. The total space $E(\iota^* \mathfrak{t}_M)$ and $E(r^* \mathfrak{t}_N)$ are homeomorphic.

Proof.
We explicitly construct a homeomorphism:

1. $E(\iota^* \mathfrak{t}_M) = \{(n, (m_1, m_2)) \in N \times (M \times M) \mid \iota(n) = m_1\}$
2. $E(r^* \mathfrak{t}_N) = \{(m, (n_1, n_2)) \in M \times (N \times N) \mid r(m) = n_1\}$

Now, we have the homeomorphism $\phi : E(\iota^* \mathfrak{t}_M) \to E(r^* \mathfrak{t}_N)$ with $\phi(n, (m_1, m_2)) = (m_2, (r(m_2), n))$ and $\phi^{-1}(m, (n_1, n_2)) = (n_2, (n_2, m))$. We easily see that ϕ suffices all requirements of $E(\iota^* \mathfrak{t}_M)$ and $E(r^* \mathfrak{t}_N)$. $\square$

Remark 5.7. Note that the following diagram commutes

$$\begin{tikzcd} N \arrow[r] \arrow[d, hook] & E(\iota^* \mathfrak{t}_M) \arrow[d, "\phi"] \\ M \arrow[r] & E(r^* \mathfrak{t}_N) \end{tikzcd}$$

Lemma 5.8. *(normal microbundle on total space) There exists a normal microbundle $\mathfrak{n}$ of N in $E(r^* \mathfrak{t}_N)$ with $\mathfrak{n} \cong \iota^* \mathfrak{t}_M$.*

Proof.
Obviously, $\mathfrak{n} := r^* \mathfrak{t}_N|_N$ is a normal microbundle of N in $E(r^* \mathfrak{t}_N)$. Since $E(r^* \mathfrak{t}_N|_N) \subseteq E(r^* \mathfrak{t}_N)$, isomorphy follows from the previous lemma and remark. $\square$

Finally, we gathered all the tools to prove Milnor's theorem.

Theorem 5.9. *(Milnor) For a sufficently large $q \in \mathbb{N}$, $N = N \times \{0\}$ has a normal microbundle in $M \times \mathbb{R}^q$.*

Proof.

1. There exists a microbundle $\mathfrak{t}'$ over N such that $\mathfrak{t}_N \oplus \mathfrak{t}' \cong \mathfrak{e}^q_n$:

   From the [Whitney Embedding Thereom] it follows that we can embed N in euclidean space $\mathbb{R}^{2m+1}$. Additionally, from previous conseriderations we can extend $\mathfrak{t}_N$ to a microbundle over an open neighborhood $V \subseteq \mathbb{R}^{2m+1}$. Now we can apply the **??** from Chapter 4.

2. $E(r^* \mathfrak{t}_N) \subseteq E(r^* \mathfrak{t}_N \oplus r^* \mathfrak{t}')$ has a normal microbundle:

   Consider $j^*(r^* \mathfrak{t}_N \oplus r^* \mathfrak{t}') : E(r^* \mathfrak{t}_N) \xrightarrow{i'} E(r^* \mathfrak{t}_N \oplus r^* \mathfrak{t}') \xrightarrow{j'} E(r^* \mathfrak{t}_N)$ where j is the projection map for $r^* \mathfrak{t}_N$. Since i' is injective, we can consider $E(r^* \mathfrak{t}_N) \subseteq E(r^* \mathfrak{t}_N \oplus r^* \mathfrak{t}')$. Since total spaces of microbundles over manifolds are manifolds as well, it follows that $j^*(r^* \mathfrak{t}_N \oplus r^* \mathfrak{t}')$ is a normal microbundle.

Since $N \subseteq M \subseteq E(r^* \mathfrak{t}_N)$ has a normal microbundle (**??**) it follows from **??** that $N \subseteq E(r^* \mathfrak{t}_N \oplus r^* \mathfrak{t}')$ has a normal microbundle. But $r^* \mathfrak{t}_N \oplus r^* \mathfrak{t}'$ is trivial and therefore w.l.o.g. $E(r^* \mathfrak{t}_N \oplus r^* \mathfrak{t}') \cong N \times \mathbb{R}^q$ $\square$