Microbundles on Topological Manifolds

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1 Introduction to Microbundles

Definition 1.1. (microbundle)
A *microbundle* $\mathfrak{b}$ is *hallo* a tuple $\mathfrak{b} := (B, E, i, j)$ satisfying the following properties:

- B is a topological space called the *base space*
- E is a topological space called the *total space*
- $i : B \to E$ (*injection*) and $j : E \to B$ (*projection*) are continuous maps with $id_B = j \circ i$
- Every $b \in B$ is *locally trivializable*, i.e there exist open neighborhoods $U \subseteq B$ of b and $V \subseteq E$ of $i(U)$ such that the following diagram commutes:

$$\begin{tikzcd}
& V \arrow[dr, "j"] \arrow[dd, "\psi"] & \\
U \arrow[ur, "i"] \arrow[dr, "(id{,}0)"'] & & U \\
& U \times \mathbb{R}^n \arrow[ur, "\pi_1"'] &
\end{tikzcd}$$

We call n the *fibre dimension* of $\mathfrak{b}$.

Definition 1.2. (isomorphic microbundles)
Two microbundles $\mathfrak{b}_1 := (B, E_1, i_1, j_2)$ and $\mathfrak{b}_2 := (B, E_2, i_2, j_2)$ are said to be *isomorphic* if there exist neighborhoods $V_1 \subseteq E_1$ of $i_1(B)$ and $V_2 \subseteq E_2$ of $i_2(B)$ with an homeomorphism $\phi : V_1 \xrightarrow{\sim} V_2$, so that the following diagram commutes:

$$\begin{tikzcd}
& V_1 \arrow[dr, "j_1"] \arrow[dd, "\phi"] & \\
B \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & B \\
& V_2 \arrow[ur, "j_2"'] &
\end{tikzcd}$$

Example 1.3. (trivial microbundle)
Let B be a topological space and $n \in \mathbb{N}$. The diagram $\mathfrak{e}_B^n : B \xrightarrow{\iota} B \times \mathbb{R}^n \xrightarrow{\pi} B$ constitutes a microbundle, where $\iota(b) := (b, 0)$ and $\pi(b, x) := b$. We call $\mathfrak{e}_B^n$ the *standard microbundle* and every microbundle isomorphic to $\mathfrak{b}_B^n$ *trival*.

Lemma 1.4. *(criteria for triviality)*
A microbundle $\mathfrak{b}$ of B is trivial if and only if there exists a open subset $B \subseteq U$ with $U \cong B \times \mathbb{R}^n$.

Proof. □

Example 1.5. (tangent microbundle)
Let M be a topological manifold. We derive the *tangent microbundle* $\mathfrak{t}_M : M \xrightarrow{\Delta} M \times M \xrightarrow{\pi_1} M$, where Δ is the diagonal map and π_1 ist the projection map on the first component.

Proof. Let $p \in M$ and (U, ϕ) a chart over p:

$$\begin{tikzcd}
& U \times U \arrow[dd, "(id,\phi)"] \arrow[dr, "i"] & \\
U \arrow[ur, "i"] \arrow[dr, "(0,id)"'] & & U \\
& U \times \mathbb{R}^d \arrow[ur, "\pi_1"'] &
\end{tikzcd}$$

(id, ϕ) is a homeomorphism since $\phi : U \xrightarrow{\sim} \mathbb{R}^n$ is a homeomorphism. $\square$

Proposition 1.6. *(restricting the total space)*
Let $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ *be a microbundle and* E' *an arbitray neighborhood of* $i(B)$*. The restriction* $\mathfrak{b}' : B \xrightarrow{i} E' \xrightarrow{j|_{E'}} B$ *is a microbundle isomorphic to* $\mathfrak{b}$*.*

Proof. Let $b \in B$.
Choose an arbitray trivialization (U, V, ϕ) over $\mathfrak{b}$ of b. We restrict $\phi : V \to U \times R^n$ to $V \cap E'$. Since $i(b) \in V$ is open and E' is a neighborhood of $i(B)$, it follows that $\phi(V \cap E')$ is a neighborhood of $(b, 0)$.
$\implies \exists (b, 0) \in U' \times X \subseteq \phi(V \cap E')$, where $U' \subseteq U$ and $X \subseteq R^n$ are open
$\implies \exists \varepsilon > 0 : U' \times B_\varepsilon(0) \subseteq \phi(V \cap E')$
Since $B_\varepsilon(0) \cong R^n$, it follows that $U' \times R^n \cong U' \times B_\varepsilon(0) \cong \phi^{-1}(U' \times B_\varepsilon(0))$. Choosing $V' := \phi^{-1}(U' \times B_\varepsilon(0)) \subseteq V$, we see that $\mathfrak{b}'$ is a microbundle. We easily see, that $\mathfrak{b}$ is isomorphic to $\mathfrak{b}'$ via the identity. $\square$

2 Induced Microbundles

Definition 2.1. (induced microbundle)
Let $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ be a microbundle and $f : A \to B$ a continuous map. We can construct a microbundle $f^*\mathfrak{b} : A \xrightarrow{i'} E' \xrightarrow{j'} A$ defined as follows:

- $E' := \{(a, e) \in A \times E \mid f(a) = j(e)\}$
- $i' : A \to E'$ with $i'(a) := (a, (i \circ f)(a))$
- $j' : E' \to A$ with $j'(a, e) := a$

We call $f^*\mathfrak{b}$ the *induced microbundle* of $\mathfrak{b}$ over f.

Proof. It is clear that i' and j' are continuous and that $id_A = j' \circ i'$. So it remains to be shown that $f^*\mathfrak{b}$ is locally trivial for every $a \in A$:

- $U' := f^{-1}(U) \subseteq A$ is an open neighborhood of a.
- $V' := j'^{-1}(U') \subseteq E'$ is an open neighborhood of $i'(U')$.
- $\phi' : V' \xrightarrow{\sim} U' \times \mathbb{R}^n, \phi'(a,e) := (a, \pi_2(\phi(e)))$ is a homeomorphism.
  - ϕ' is well defined because $(a,e) \in V' : j(e) = f(a) \in U \implies e \in V$.
  - ϕ' is bijective with $\phi'^{-1}(a,v) = (a, \phi^{-1}(f(a),v))$.
  - ϕ' and ϕ'^{-1} are continuous because it's components are.

$\square$

Example 2.2. (restricted microbundle)
Let $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ be a microbundle and $A \subseteq B$: The induced microbundle $\iota^*\mathfrak{b}$ with $\iota : A \hookrightarrow B$ being the inclusion map is called the *restricted microbundle* and we write $\mathfrak{b}|_A := \iota^*\mathfrak{b}$.

Remark 2.3. In the following, we'll consider $E(\mathfrak{b}|_A)$ a subset of $E(\mathfrak{b})$. This is justified because $E(\mathfrak{b}|_A) = \{(a,e) \in A \times E(\mathfrak{b}) \mid a = j(e)\} \cong \{e \in E(\mathfrak{b}) \mid j(e) \in A\} \subseteq E(\mathfrak{b})$.

Lemma 2.4. *(induced trivial microbundle)*
The induced microbundle $f^*\mathfrak{b}$ is trivial for every map $f : A \to B$, if $\mathfrak{b}$ is already trivial.

Proof. Let (V, ϕ) be a global trivialization of $\mathfrak{b}$, i.e $V \cong_\phi B \times \mathbb{R}^n$. Now define $V' := (A \times V) \cap E'$ and $\phi'(a,e) := (a, \phi^{(2)}(e))$. Obviously, V' is a neighborhood of $i'(A)$ and also ϕ' is a homeomorphism with inverse $\phi'^{(-1)}(a,x) = (a, \phi^{-1}(f(a),x))$. $\square$

Proposition 2.5. *(composition)*
Let $A \xrightarrow{f} B \xrightarrow{g} C$ be topological spaces and $\mathfrak{c} : C \xrightarrow{i} E \xrightarrow{j} C$ be a microbundle:

$$(g \circ f)^*\mathfrak{c} \cong f^*(g^*\mathfrak{c})$$

Proof. We'll compare the two total spaces and conclude that they are homeomorphic.

1. $E((g \circ f)^*\mathfrak{c}) = \{(a,e) \in A \times E(\mathfrak{c}) \mid g(f(a)) = j(e)\}$
2. $E(f^*(g^*\mathfrak{c})) = \{(a,(b,e)) \in A \times (B \times E(\mathfrak{c})) \mid f(a) = b \text{ and } g(b) = j(e)\}$.

We have the bijection $\phi : E((g \circ f)^*\mathfrak{c}) \xrightarrow{\sim} E(f^*(g^*\mathfrak{c}))$ with $\phi(a,e) := (a,(f(a),e))$ and $\phi^{-1}(a,(b,e)) = (a,e)$. Additionally, ϕ is a homeomorphism because ϕ and ϕ^{-1} are componentwise continuous. It's easy to see that ϕ respects both injection and projection, which concludes the proof. $\square$

For a topological space X, we define the cone *of X as*

$$CX := X \times [0,1]/X \times \{1\}$$

and for a map $f : A \to B$ the mapping cone *of f as*

$$B \sqcup_f CA := B \sqcup CA/\sim$$

where $(a,0) \sim b :\iff f(a) = b$.

Lemma 2.6. *(extending over a mapping cone)*
A microbundle $\mathfrak{b}$ over B can be extended to a microbundle over the mapping cone $B \sqcup_f CA$ if and only if $f^*\mathfrak{b}$ is trivial.

Proof. We show both implications.
$\Longrightarrow$:
Let $\mathfrak{b}'$ be an extension of $\mathfrak{b}$ over $B \sqcup_f CA$. Considering $A \xrightarrow{f} B \hookrightarrow B \sqcup_f CA$, the composition $\iota \circ f$ is null-homotopic with homotopy

$$H_t(a) := [(a,t)]$$

Note that $H_0(a) = [(a,0)] = [f(a)] = (\iota \circ f)(a)$ and $H_1(a) = [(a,1)] = [(\tilde{a},1)] = H_1(\tilde{a})$.
$\underset{Hom.Thm.}{\Longrightarrow}$ $(\iota \circ f)^*\mathfrak{b}'$ is trivial
Since $(\iota \circ f)^*\mathfrak{b}' = f^*(\iota^*\mathfrak{b}') = f^*\mathfrak{b}$, it follows that $f^*\mathfrak{b}$ is trivial. $\Longleftarrow$:
Let $f^*\mathfrak{b}$ be trivial. Analogous to the cone, we define the *cylinder* of X as

$$MX := X \times [0,1]$$

and for a map $f : A \to B$ the mapping *cylinder* of f as

$$B \sqcup_f MA := B \sqcup MA/\sim$$

where $(a,0) \sim b :\iff f(a) = b$. In contrast to the mapping cone, there exists a natural retraction from the mapping cylinder to the attached space

$$\pi : B \sqcup_f MA \to B; \pi([(a,t)]) := f(a)$$

and therefore the induced microbundle $\pi^*\mathfrak{b}$ over $B \sqcup_f MA$. Considering $A \times \{1\} \hookrightarrow B \sqcup_f MA \xrightarrow{\pi} B$, we see that $\pi \circ \iota \cong f$ and therefore

$$\pi^*\mathfrak{b}|_{A\times\{1\}} = (\pi \circ \iota)^*\mathfrak{b} \cong f^*\mathfrak{b} = \mathfrak{e}^n_A$$

is trivial. From the lemma of induced trivial microbundles and $(a,t) \mapsto (a,1)$ it follows that $\pi^*\mathfrak{b}|_{A\times[\frac{1}{2},1]}$ is trivial.
$\Longrightarrow \exists \phi : E(\mathfrak{b}|_{A\times[\frac{1}{2},1]}) \xrightarrow{\sim} A \times [\frac{1}{2},1] \times \mathbb{R}^n$
Now we explicitly construct the desired extended microbundle $\mathfrak{b}' : B \sqcup_f CA \xrightarrow{i'} E' \xrightarrow{j'} B \sqcup_f CA$

- $E' := E(\mathfrak{b}|_{A\times[\frac{1}{2},1]})/\phi^{-1}(A\times[\frac{1}{2},1]\times\{x\})$ (for every $x\in\mathbb{R}^n$)
- $i' := \pi\circ i$ the projection i to E'
- $j'([e]) := [j(e)]$ is well defined, because $[e]=[\tilde{e}] \implies [j(e)]=[j(e')]$

Now that we have constructed $\mathfrak{b}'$, this proves the claim. $\square$

Corollary 2.7. *(extending over a d-simplex)*
Let B be a $(d+1)$-simplicial complex, B' it's d-skeleton and $\Delta^{d+1}\cong\sigma\subseteq B$. A microbundle $\mathfrak{b}$ over B' can be extended to a microbundle over $B'\cup\sigma$ if and only if $\mathfrak{b}|_{\partial\sigma}$ is trivial.

Proof. The statement follows from the last lemma:
There exists a $\phi : C\partial\sigma \xrightarrow{\sim} \sigma$ such that $\phi(\partial\sigma\times\{0\})=\partial\sigma$. We explicitly construct $\phi((t_1,\dots,t_{d+1}),\lambda) := (1-\lambda)(t_1,\dots,t_{d+1})+\frac{\lambda}{d+1}(1,\dots,1)$. It's easy to see that ϕ suffices all our requirements. By choosing $f:\partial\sigma\hookrightarrow B'$ and applying the last lemma, the statement is proven. $\square$

3 Whitney sums

Definition 3.1. (whitney sum)
Let $\mathfrak{b}_1$ and $\mathfrak{b}_1$ be two microbundles over a topological space B. We define the *whitney sum* $\mathfrak{b}_1\oplus\mathfrak{b}_2$ as follows:

- $E := \{(e_1,e_2)\in E(\mathfrak{b}_1)\times E(\mathfrak{b}_2) \mid j_1(e_1)=j_2(e_2)\}$
- $i(b) := (i_1(b), i_2(b))$
- $j(e_1,e_2) := j_1(e_1)=j_2(e_2)$

Proof. Let $b\in B$.
Choose U_1,V_1,ϕ_1 and U_2,V_2,ϕ_2 accordingly from the local trivialization of b over $\mathfrak{b}_1$ and $\mathfrak{b}_2$:

- $U := U_1\cap U_2$
- $V := (V_1\times V_2)\cap E$
- $\phi : V\to U\times\mathbb{R}^{n_1+n_2}; \phi(e_1,e_2) := (\phi_1^{(1)}(e_1), \phi_1^{(2)}(e_1)\times\phi_2^{(2)}(e_2))$

Note that $\phi_1^{(1)}(e_1)=\phi_2^{(1)}(e_2)$. Local triviality follows directly from it's components. $\square$

Lemma 3.2. *(compatibility)*
Let $\mathfrak{b}_1$ and $\mathfrak{b}_1$ be two microbundles over B and $f:A\to B$ a map. Induced microbundle and whitney sum are compatible, i.e. $f^*(\mathfrak{b}_1\oplus\mathfrak{b}_2)\cong f^*\mathfrak{b}_1\oplus f^*\mathfrak{b}_2$

Proof. From the definition of the induced microbundle and the whitney sum, we can derive the total spaces:

1. $E(f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2)) = \{(a, (e_1, e_2)) \in A \times (E_1 \times E_2) \mid j_1(e_1) = j_2(e_2) = f(a)\}$
2. $E(f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2) = \{((a_1, e_1), (a_2, e_2)) \in (A \times E_1) \times (A \times E_2) \mid j(a_1, e_1) = j(a_2, e_2)$ and $f(a_i) = j(e_i)\}$

Those two total spaces are homeomorphic via $\phi(a, (e_1, e_2)) := ((a, e_1), (a, e_2))$ and $\phi^{-1}((a, e_1), (a, e_2)) = (a, (e_1, e_2))$. ϕ and ϕ^{-1} are continuous because they are componentwise continuous. Obviously, $\phi \circ i = i$ and $\phi \circ j = j$, which concludes the proof. □

Theorem 3.3. *(inverse microbundles)*
Let $\mathfrak{b}$ be a microbundle over a d-dimensional simplicial complex B.
Then there exists a microbundle $\mathfrak{n}$ over B so that the Whitney sum $\mathfrak{b} \oplus \mathfrak{n}$ is trivial.

Proof. We prove this theorem by induction over d.
(Start of induction)
A 1-dimensional simplicial complex is just a bouquet of circles, therefore the start of induction follows directly from the bouquet lemma.
(Inductive Step)
Let B' be the $(d-1)$-skeleton of B and $\mathfrak{n}'$ it's corresponding microbundle so that $\mathfrak{b}|_{B'} \oplus \mathfrak{n}'$ is trivial. □

4 Normal Microbundles

Definition 4.1. (normal microbundle)
Let M and N be two topological manifolds with $N \subseteq M$. We call a microbundle of the form

$$\mathfrak{n} : N \xrightarrow{\iota} U \xrightarrow{r} N$$

where $U \subseteq M$ is a neighborhood of N, a *normal microbundle* of N in M.

Definition 4.2. (product neighborhood)
Again, let M and N be two topological manifolds with $N \subseteq M$. We say that N has a *product neighborhood* in M if there exists a trivial normal microbundle of N in M.

Lemma 4.3. *(criteria for product neighborhoods)*
A submanifold $N \subseteq M$ has a product neighborhood if and only if there exists a neighborhood U of N with $(U, M) \cong (M \times R^n, M \times 0)$.

Proof. This follows directly from the definition of normal microbundles and the criteria for trivial microbundles. □

Definition 4.4. (composition microbundle)
Let $\mathfrak{b} : B \xrightarrow{i_{\mathfrak{b}}} E \xrightarrow{j_{\mathfrak{b}}} B$ and $\mathfrak{c} : E \xrightarrow{i_{\mathfrak{c}}} E' \xrightarrow{j_{\mathfrak{c}}} E$ be two microbundles. We define

the *composition microbundle* $\mathfrak{b} \circ \mathfrak{c} : B \xrightarrow{i} E' \xrightarrow{j} B$ with $i(b) := (i_\mathfrak{c} \circ i_\mathfrak{b})(b)$ and $j(e') := (j_\mathfrak{b} \circ j_\mathfrak{c})(e')$

Proof. Let $b \in B$.
Choose local trivializations $(U_\mathfrak{b}, V_\mathfrak{b}, \phi_\mathfrak{b})$ of b and $(U_\mathfrak{c}, V_\mathfrak{c}, \phi_\mathfrak{c})$ of $j_\mathfrak{b}(b)$. From this, we construct our local trivialization over $\mathfrak{b} \circ \mathfrak{c}$. Consider $\phi_\mathfrak{b}(V_\mathfrak{b} \cap U_\mathfrak{c})$, which is a neighborhood of $(b, 0)$. Therefore, there exist open neighborhoods $b \in U \subseteq U_\mathfrak{b}$ and $0 \in X \subseteq R^n$ such that $U \times X \subseteq \phi_\mathfrak{b}(V_\mathfrak{b} \cap U_\mathfrak{c})$. Analoguous to the proof of restricting the total space in Chapter 1, it follows that

$$\exists \varepsilon > 0 : U \times B_\varepsilon(0) \subseteq \phi_\mathfrak{b}(V_\mathfrak{b} \cap U_\mathfrak{c})$$

$$\implies U \times \mathbb{R}^n \cong U \times B_\varepsilon(0) \cong \phi_\mathfrak{b}^{-1}(U \times B_\varepsilon(0)) \cong \phi_\mathfrak{c}^{-1}(\phi_\mathfrak{b}^{-1}(U \times B_\varepsilon(0)))$$

which is an open neighborhood of $i(U)$ and therefore a valid candidate for V. This concludes local triviality and the proof. $\square$

Lemma 4.5. *(transitivity of normal microbundles)*
Let M, N and P be topological manifolds with $P \subseteq N \subseteq M$. There exists a normal microbundle $\mathfrak{n}$ of P in M, if there exist normal microbundles $\mathfrak{n}_p : P \xrightarrow{i_P} U_N \xrightarrow{j_P} P$ in N and $\mathfrak{n}_n : N \xrightarrow{i_N} U_M \xrightarrow{j_N} N$ in M.

Proof. We simply form the composition $\mathfrak{n}_p \circ \mathfrak{n}_n|_{U_N} : P \xrightarrow{i_N \circ i_P} U_M \xrightarrow{j_P \circ j_N} P$. Since $i_N \circ i_P$ is just the inclusion of $P \hookrightarrow U_M \subseteq M$, we found a normal microbundle $\mathfrak{n}$ of P in M. $\square$

Every topological manifold is an absolute neighborhood retract (ANR).

It follows that by restricting M, if necessary, to an open neighborhood of N, there exists a retraction $r : M \to N$ which we will take advantage of in the following.

Lemma 4.6. *(homeomorphism of total spaces)*
Let $\mathfrak{t}_N$ and $\mathfrak{t}_M$ be the tangent microbundles of N and M. The total space $E(\iota^*\mathfrak{t}_M)$ and $E(r^*\mathfrak{t}_N)$ are homeomorphic.

Proof. We explicitly construct a homeomorphism:

1. $E(\iota^*\mathfrak{t}_M) = \{(n, (m_1, m_2)) \in N \times (M \times M) \mid \iota(n) = m_1\}$
2. $E(r^*\mathfrak{t}_N) = \{(m, (n_1, n_2)) \in M \times (N \times N) \mid r(m) = n_1\}$

Now, we have the homeomorphism $\phi : E(\iota^*\mathfrak{t}_M) \to E(r^*\mathfrak{t}_N)$ with $\phi(n, (m_1, m_2)) = (m_2, (r(m_2), n))$ and $\phi^{-1}(m, (n_1, n_2)) = (n_2, (n_2, m))$. We easily see that ϕ suffices all requirements of $E(\iota^*\mathfrak{t}_M)$ and $E(r^*\mathfrak{t}_N)$. $\square$

Remark 4.7. Note that the following diagram commutes

$$\begin{tikzcd} N \arrow[r] \arrow[d, hook] & E(\iota^*\mathfrak{t}_M) \arrow[d, "\phi"] \\ M \arrow[r] & E(r^*\mathfrak{t}_N) \end{tikzcd}$$

Lemma 4.8. *(normal microbundle on total space) There exists a normal microbundle $\mathfrak{n}$ of N in $E(r^*\mathfrak{t}_N)$ with $\mathfrak{n} \cong \iota^*\mathfrak{t}_M$.*

Proof. Obviously, $\mathfrak{n} := r^*\mathfrak{t}_N|_N$ is a normal microbundle of N in $E(r^*\mathfrak{t}_N)$. Since $E(r^*\mathfrak{t}_N|_N) \subseteq E(r^*\mathfrak{t}_N)$, isomorphy follows from the previous lemma and remark. □

Finally, we gathered all the tools to prove Milnor's theorem.

Theorem 4.9. *(Milnor) For a sufficently large $q \in \mathbb{N}$, $N = N \times \{0\}$ has a normal microbundle in $M \times \mathbb{R}^q$.*

Proof. □

5 Homotopy and Microbundles

Definition 5.1. (map-germ)
Let (X, A) and (Y, B) be two topological pairs. A *map-germ* $F : (X, A) \Rightarrow (Y, B)$ is an equivalence class of maps $f : (X, A) \to (Y, B)$, where $f \sim g :\iff f|_U = g|_U$ for some neighborhood $U \subseteq X$ of A.

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Remark 5.2. The composition of two map-germs $(X, A) \overset{F}{\Rightarrow} (Y, B) \overset{G}{\Rightarrow} (Z, C)$ is well defined.

Definition 5.3. (isomorphism-germ)

Definition 5.4. (bundle map-germ)

Lemma 5.5. *(closed balls under homeomorphism)*
For a homeomorphism $\phi : \mathbb{R}^n \xrightarrow{\sim} R^n$ applies:

$$|\phi(x) - x| < 1, \forall x \in \overline{B_2} \implies \overline{B_1} \subseteq \phi(\overline{B_2})$$

Proof. Consider the two points $x_0 := 0$ and $x_1 := 3e_1$. Obviously, there is no path between x_0 and x_1 in $\mathbb{R}^n - S^n$. Therefore, since ϕ is a homeomorphism, there is no path between $\phi(x_0)$ and $\phi(x_1)$ in $\mathbb{R}^n - \phi(S^n)$. Since $1 < |x|$ for every $x \in \phi(S^n)$, there is a path between every $x \in \overline{B_1}$ and $\phi(x_1)$ in $\mathbb{R}^n - \phi(S^n)$ (e.g a straight line). It follows that there is also no path between any $x \in \overline{B_1}$ and x_1 in $\mathbb{R}^n - \phi(S^n)$. Since $R^n - \overline{B_2}$ is path-connected, $R^n - \phi(\overline{B_2})$ is as well. From $x_1 \in R^n - \phi(\overline{B_2})$, we know that $\overline{B_1} \subseteq \phi(\overline{B_2})$. □

Lemma 5.6. *(Williamson)*
A bundle map-germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$ on the same base space B covering the identity map is an isomorphism-germ.

Proof. We prove the statement locally, then glue the function together for the prove.

1. First, we consider the case that $\mathfrak{b}$ and $\mathfrak{b}'$ are trivial. This means, that $F : B \times \mathbb{R}^n \to B \times \mathbb{R}^n$ is of the form

$$F(b, x) = (b, g_b(x))$$

   where $g_b : \mathbb{R}^n \to \mathbb{R}^n$ are individual maps. In particular, the g_b are even homeomorphisms due to the domain invariance theorem. Now we show that F is a homeomorphism. Let $(b_0, x_0) \in B \times \mathbb{R}^n$ and $\varepsilon > 0$. Since g_{b_0} is a homeomorphism, there exists a $\delta > 0$ so that $\overline{B_{2\delta}(x_1)} \subseteq g_{b_0}(\overline{B_\varepsilon(x_0)})$ where $x_1 := g_{b_0}(x_0)$.

   There exists a neighborhood $V \subseteq B$ of b_0, such that

$$|g_b(x) - g_{b_0}(x)| < \delta, \forall x \in \overline{B_\varepsilon(x_0)}$$

   To show that, consider $\phi_b(b, x) := g_b(x) - g_{b_0}(x)$. The closed set $\phi^{-1}(\overline{B_\varepsilon(x_0)})$ is a neighborhood of $\{b_0\} \times \mathbb{R}^n$ since $\phi(b_0, x) = 0, \forall x \in \mathbb{R}^n$. Therefore, for every $x \in \overline{B_\delta(x_1)}$ exist $V_x \subseteq B$ and $U_x \subseteq \mathbb{R}^n$ open with $x \in U_x$ and $V_x \times U_x \subseteq \phi^{-1}(\overline{B_\varepsilon(x_0)})$. Obviously, $\bigcup_{x \in \overline{B_\delta(x_1)}} U_x$ is an open covering of $\overline{B_\delta(x_1)}$ and because of compactness of $\overline{B_\delta(x_1)}$, there exist $x_1, \ldots, x_n \in \overline{B_\delta(x_1)}$ with $\overline{B_\delta(x_1)} \subseteq \bigcup_{i=1}^n U_{x_i}$. The claim follows via $V := V_{x_1} \cap \cdots \cap V_{x_n}$.

   From

$$V \times \overline{B_\delta(x_1)} \subseteq g(V \times \overline{B_\varepsilon(x_0)})$$

   it follows that g is open.

2. Since $\mathfrak{b}$ and $\mathfrak{b}'$ are locally trivial, $F|_{U_b}$ are homeomorphisms on their image for every trivialization $b \in U_b$. It follows that also F is a homeomorphism on its image which completes the proof.

□

Corollary 5.7. *(induced microbundles)*
If a map $g : B \to B'$ is covered by a bundle germ $\mathfrak{b} \Rightarrow \mathfrak{b}'$ then $\mathfrak{b}$ is isomorphic to the induced bundle $g^*\mathfrak{b}'$.

Proof. □

Lemma 5.8. *(glueing together bundle map-germs)*
Let $\mathfrak{b}$ be a microbundle over B and $\{B_\alpha\}$ a locally finite collection of closed sets covering B. Additionally, we are given $F_\alpha : \mathfrak{b}|_{B_\alpha} \Rightarrow \mathfrak{b}'$, a collection of bundle map-germs with $F_\alpha = F_\beta$ on $\mathfrak{b}|_{B_\alpha \cap B_\beta}$ Then there exists a bundle map-germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$ extending F_α.

Proof. Choose representative maps $f_\alpha : U_\alpha \to E'$ for F_α with U_α open. Since $F_\alpha = F_\beta$ on $\mathfrak{b}|_{B_\alpha \cap B_\beta}$, $f_\alpha = f_\beta$ for an open neighborhood $U_{\alpha\beta}$ of $B_\alpha \cap B_\beta$. We define

$$U := \{e \in E \mid j(e) \in B_\alpha \cap B_\beta \implies e \in U_{\alpha\beta}\}$$

1. U is open:
   Let $e \in U$ and $j(e) \in B_\alpha \cap B_\beta$. From local finiteness there exists an open neighborhood $V \subseteq B$ of $j(e)$ with $V \subseteq B_{\alpha_1} \cap \ldots \cap B_{\alpha_n}$. W.l.o.g. $V \subseteq B_\alpha \cap B_\beta$ by exluding a finite number of closed sets if necessary. Now $V_{\alpha\beta} := j^{-1}(V) \cap U_{\alpha\beta}$ is an open neighborhood of e. Since $j(e)$ can only be contained in finitely many B_α we can form the intersection of all these $V_{\alpha'\beta'}$ which, by construction, is contained in U and is open.
2. $B \subseteq U$ considering the cases $U_{\alpha\alpha}$.

Now we can define $f : U \to E'$ in the obvious way

$$f(u \in U_{\alpha\beta}) := f_\alpha(u) = f_\beta(u)$$

which is a representative map for our desired F. $\square$

Lemma 5.9. *(piecewise triviality)*
Let $\mathfrak{b}$ be a microbundle over $B \times [0,1]$ such that both $\mathfrak{b}|_{B\times[0,\frac{1}{2}]}$ and $\mathfrak{b}|_{B\times[\frac{1}{2},1]}$ are trivial. Then $\mathfrak{b}$ itself is already trivial.

Proof. Since $\mathfrak{b}|_{B\times[\frac{1}{2},1]}$ is trivial, we can extend the identity bundle map-germ on $\mathfrak{b}|_{B\times\{\frac{1}{2}\}}$ to $\mathfrak{b}|_{B\times[\frac{1}{2},1]} \Rightarrow \mathfrak{b}|_{B\times\{\frac{1}{2}\}}$. Using the previous lemma, we can piece this together with the identity bundle map-germ on $\mathfrak{b}|_{B\times[0,\frac{1}{2}]}$ to

$$\mathfrak{b} \Rightarrow \mathfrak{b}|_{B\times[0,\frac{1}{2}]}$$

From the corollary it follows that $\mathfrak{b} \cong \mathfrak{b}|_{B\times[0,\frac{1}{2}]}$. $\square$

Lemma 5.10. *()*
Let $\mathfrak{b}$ be a microbundle over $B \times [0,1]$. Every $b \in B$ has a neighborhood V where $\mathfrak{b}|_{V\times[0,1]}$ is trivial.

Proof. Let $b \in B$. For every $t \in [0,1]$, choose a neighborhood $U_t := V_t \times (t - \varepsilon_t, t + \varepsilon_t)$ of (b,t) such that $\mathfrak{b}|_{U_t}$ is trivial. Since $\{b\} \times [0,1]$ is compact, we can choose a finite covering of the U_t and define V to be the intersection of the corresponding V_t. Then there exists a subdivision $0 = t_0 < \cdots < t_k = 1$ where the $\mathfrak{b}|_{V\times[t_i,t_{i+1}]}$ are trivial. Iteratively applying the previous lemma, it follows that $\mathfrak{b}|_{V\times[0,1]}$ is trivial. $\square$

Lemma 5.11. *()*
Let $\mathfrak{b}$ be a microbundle over $B \times [0,1]$ where B is paracompact. Then there exists a bundle map-germ $F : \mathfrak{b} \to \mathfrak{b}|_{B\times\{1\}}$ covering the standard retraction $r : B \times [0,1] \to B \times \{1\}$.

Proof. First, we assume a locally finite covering $\{V_\alpha\}$ of closed sets where $\mathfrak{b}|_{V_\alpha\times[0,1]}$ is trivial. The existence of such a covering is justified by the previous lemmas. Since B is paracompact, we can choose bump functions

$$\lambda_\alpha : B \to [0,1]$$

with $\operatorname{supp}(\lambda_\alpha) \subseteq V_\alpha$. Additionally, we assume that

$$\max_\alpha(\lambda_\alpha(b)) = 1, \forall b \in B$$

Now we define a retraction $r_\alpha : B \times [0,1] \to B \times [0,1]$ with

$$r_\alpha(b,t) := (b, \max(t, \lambda_\alpha(b)))$$

We construct bundle map-germs $R_\alpha : \mathfrak{b} \to \mathfrak{b}$ covering r_α. We can divide $B\times[0,1]$ into $A_\alpha := \operatorname{supp}(\lambda_\alpha) \times [0,1]$ and $A'_\alpha := \{(b,t) \mid t \geq \lambda_\alpha(b)\}$. Since $A_\alpha \subseteq V_\alpha \times [0,1]$, $\mathfrak{b}|_{A_\alpha}$ is trivial. It follow that the identity bundle germ on $\mathfrak{b}|_{A_\alpha\cap A'_\alpha}$ can be extended to a bundle germ $\mathfrak{b}|_{A_\alpha} \Rightarrow \mathfrak{b}|_{A_\alpha\cap A'_\alpha}$. Piecing this together with the identity bundle germ $\mathfrak{b}|_{A'_\alpha}$, we obtain the desired bundle germ R_α.

Applying the well-ordering theorem, which is equivalent to the axiom of choice, we can assume an ordering of $\{V_\alpha\}$. Let $\{B_\beta\}$ be a locally finite covering of B with closed sets where B_β intersects only $V_{\alpha_1} < \cdots < V_{\alpha_k}$ a finite collection. Now the composition $R_{\alpha_1} \circ \ldots \circ R_{\alpha_k}$ restricts to a bundle germ $R(\beta) : \mathfrak{b}|_{B_\beta} \times [0,1] \Rightarrow \mathfrak{b}|_{B_\beta} \times [1]$. Pieced together with the previous lemma, we obtain $R : \mathfrak{b} \times [0,1] \to \mathfrak{b} \times [1]$ which concludes the proof. $\square$

Theorem 5.12. *(Homotopy Theorem)*
Let $\mathfrak{b}$ be a microbundle of B and $f, g : A \to B$ be two maps.

$$f \simeq g \implies f^*\mathfrak{b} \cong g^*\mathfrak{b}$$

Proof. Let $H : A \times [0,1] \to B$ be a homtopy between f and g. By the previous lemma, there exists a bundle germ $R : H^*\mathfrak{b} \Rightarrow H^*\mathfrak{b}|_{B\times[1]}$ covering the standard retraction $B \times [0,1] \to B \times [1]$. From the composition

$$f^*\mathfrak{b} \subseteq H^*\mathfrak{b} \Rightarrow_R H^*\mathfrak{b}|_{B\times[1]} = g^*\mathfrak{b}$$

we obtain an isomorphism germ $f^*\mathfrak{b} \Rightarrow g^*\mathfrak{b}$. It follow that $f^*\mathfrak{b} \cong g^*\mathfrak{b}$. $\square$