# Microbundles on Topological Manifolds

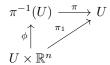
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# 1 Vectorbundles on Smooth Manifolds

## **Definition 1.1.** (vector bundle)

A vector bundle  $\xi$  is a tuple  $\xi := (B, E, \pi, +, \cdot)$  satisfying the following conditions:

- B is a topological space (base space)
- E is a topological space (total space)
- $(\pi^{-1}(b), +, \cdot)$  is a real vector space for every  $b \in B$
- Every  $b \in B$  is locally trivializable, i.e there exist neighborhoods  $U \subseteq B$  of b such that the following diagram commutes



and  $\phi(b,-): b \times \mathbb{R}^n \xrightarrow{\sim} \pi^{-1}(b)$  is a linear isomorphism.

We call n the rank of  $\xi$ .

Example 1.2. (tangent vector bundle)

Let M be a smooth manifold:

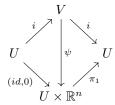
 $\xi: TM \xrightarrow{\pi} M$  is a vector bundle, where  $\pi(p,v) := p$ .

# 2 Introduction to Microbundles

#### **Definition 2.1.** (microbundle)

A microbundle  $\mathfrak{b}$  is a tuple  $\mathfrak{b} := (B, E, i, j)$  satisfying the following properties:

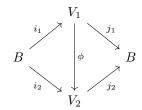
- B is a topological space called the base space
- E is a topological space called the total space
- $i: B \to E$  and  $j: E \to B$  are continuous maps with  $id_B = j \circ i$
- Every  $b \in B$  is locally trivializable, i.e there exist open neighborhoods  $U \subseteq B$  of b and  $V \subseteq E$  of i(U) such that the following diagram commutes:



We call n the fibre dimension of  $\mathfrak{b}$ .

#### **Definition 2.2.** (isomorphic microbundles)

Two microbundles  $\mathfrak{b}_1 := (B, E_1, i_1, j_2)$  and  $\mathfrak{b}_2 := (B, E_2, i_2, j_2)$  are said to be isomorphic if there exist neighborhoods  $V_1 \subseteq E_1$  of  $i_1(B)$  and  $V_2 \subseteq E_2$  of  $i_2(B)$  with an homeomorphism  $\phi : V_1 \xrightarrow{\sim} V_2$  such that the following diagram commutes:



#### Example 2.3. (trivial microbundle)

Let B be a topological space and  $n \in \mathbb{N}$ :

The diagram  $\mathfrak{e}_B^n: B \xrightarrow{\iota} B \times \mathbb{R}^n \xrightarrow{\pi} B$  constitutes a microbundle, where  $\iota(b) := (b,0)$  and  $\pi(b,x) := b$ . We call  $\mathfrak{e}_B^n$  the standard microbundle and every microbundle isomorphic to  $\mathfrak{b}_B^n$  trival.

## Example 2.4. (underlying microbundle)

Let  $\xi: E \xrightarrow{\pi} B$  be a n-dimensional vector bundle: The microbundle  $|\xi|: B \xrightarrow{i} E \xrightarrow{\pi} B$  with  $i(b) := \phi_b(b,0)$ , where  $\phi_b: U_b \times \mathbb{R}^n \to \pi^{-1}(U_b)$  is the local trivialization over a neighborhood  $U_b \subseteq B$  of b. We call  $|\xi|$  the underlying microbundle of  $\xi$ 

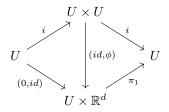
Proof.

#### Example 2.5. (tangent microbundle)

Let M be a topological manifold:

We can derive the tangent microbundle  $t_M: M \xrightarrow{\Delta} M \times M \xrightarrow{\pi_1} M$ , where  $\Delta$  is the diagonal map and  $\pi_1$  ist the projection map on the first component.

*Proof.* Let  $p \in M$  and  $(U, \phi)$  a chart over p:



 $(id, \phi)$  is a homeomorphism since  $\phi: U \xrightarrow{\sim} \mathbb{R}^n$  is homeomorphic.

# 3 Induced Microbundles

**Definition 3.1.** (induced microbundle)

Let  $\mathfrak{b}: B \xrightarrow{i} E \xrightarrow{j} B$  be a microbundle and  $f: A \to B$  a continuous map. We can construct a microbundle  $f^*\mathfrak{b}: A \xrightarrow{i'} E' \xrightarrow{j'} A$  defined as follows:

- $E' := \{(a, e) \in A \times E \mid f(a) = j(e)\}$
- $i': A \to E'$  with  $i'(a) := (a, (i \circ f)(a))$
- $j': E' \to A \text{ with } j'(a, e) := a$

We call  $f^*\mathfrak{b}$  the induced microbundle of  $\mathfrak{b}$  over f.

*Proof.* It is clear that i' and j' are continuous and that  $id_A = j' \circ i'$ . So it remains to be shown that  $f^*\mathfrak{b}$  is locally trivial for every  $a \in A$ :

- $U' := f^{-1}(U) \subseteq A$  is an open neighborhood of a.
- $V' := j'^{-1}(U') \subseteq E'$  is an open neighborhood of i'(U').
- $\phi': V' \xrightarrow{\sim} U' \times \mathbb{R}^n, \phi'(a, e) := (a, \pi_2(\phi(e)))$  is a homeomorphism.
  - $-\phi'$  is well defined because  $(a,e) \in V' : j(e) = f(a) \in U \implies e \in V$ .

- $-\phi'$  is bijective with  $\phi'^{-1}(a,v)=(a,\phi^{-1}(f(a),v)).$
- $-\phi'$  and  $\phi'^{-1}$  are continuous because it's components are.

Example 3.2. (restricted microbundle)

Let  $\mathfrak{b}: B \xrightarrow{i} E \xrightarrow{j} B$  be a microbundle and  $A \subseteq B$ :

The induced microbundle  $\iota^*\mathfrak{b}$  with  $\iota:A\hookrightarrow B$  being the inclusion map is called the restricted microbundle and we write  $\mathfrak{b}|_A:=\iota^*\mathfrak{b}$ .

Proposition 3.3. (composition)

Let  $A \xrightarrow{f} B \xrightarrow{g} C$  be topological spaces and  $\mathfrak{c}: C \xrightarrow{i} E \xrightarrow{j} C$  be a microbundle:

$$(g \circ f)^* \mathfrak{c} = f^*(g^* \mathfrak{c})$$

Proof.

Remark 3.4. (functor)

The induced microbundle yields a functor  $f \mapsto f^*$  between topological spaces and microbundles.

# 4 The Whitney Sum

We can easily construct the product of two microbundles  $\mathfrak{b}_1$  and  $\mathfrak{b}_2$  over B:

$$\mathfrak{b}_1 \times \mathfrak{b}_2 : B \times B \xrightarrow{i_1 \times i_2} E_1 \times E_2 \xrightarrow{j_1 \times j_2} B \times B$$

However, we end up with a microbundle over  $B \times B$  instead of B.

**Definition 4.1.** (whitney sum)

Let  $\mathfrak{b}_1$  and  $\mathfrak{b}_1$  be two microbundles over a topological space B. We define the whitney sum  $\mathfrak{b}_1 \oplus \mathfrak{b}_2$  as the induced microbundle over the diagonal map  $\Delta$  of the product microbundle  $\mathfrak{b}_1 \times \mathfrak{b}_2 : B \times B \to E(\mathfrak{b}_1) \times E(\mathfrak{b}_2) \to B \times B$ .

# 5 Normal Microbundles

**Definition 5.1.** (normal microbundle)

Let M and N be two topological manifolds with  $N \subseteq M$ . We call a microbundle of the form

$$\mathfrak{n}: N \xrightarrow{\iota} M \xrightarrow{r} N$$

a normal microbundle of M in N.