Microbundles on Topological Manifolds

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April 18, 2024

1 Vectorbundles on Smooth Manifolds

Definition 1.1. *(vector bundle)*
A vector bundle ξ is a tuple $\xi := (B, E, \pi, +, \cdot)$ satisfying the following conditions:

- *B is a topological space (base space)*
- *E is a topological space (total space)*
- *$(\pi^{-1}(b), +, \cdot)$ is a real vector space for every $b \in B$*
- *Every $b \in B$ is locally trivializable , i.e there exist neighborhoods $U \subseteq B$ of b such that the following diagram commutes*

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\pi"] & U \\
U \times \mathbb{R}^n \arrow[u, "\phi"] \arrow[ur, "\pi_1"] &
\end{tikzcd}$$

  and $\phi(b, -) : b \times \mathbb{R}^n \xrightarrow{\sim} \pi^{-1}(b)$ is a linear isomorphism.

We call n the rank of ξ.

Example 1.2. *(tangent vector bundle)*
Let M be a smooth manifold:
$\xi : TM \xrightarrow{\pi} M$ is a vector bundle, where $\pi(p, v) := p$.

2 Introduction to Microbundles

Definition 2.1. *(microbundle)*
A microbundle $\mathfrak{b}$ is a tuple $\mathfrak{b} := (B, E, i, j)$ satisfying the following properties:

- *B is a topological space called the base space*
- *E is a topological space called the total space*
- *$i : B \to E$ and $j : E \to B$ are continuous maps with $id_B = j \circ i$*
- *Every $b \in B$ is locally trivializable , i.e there exist open neighborhoods $U \subseteq B$ of b and $V \subseteq E$ of $i(U)$ such that the following diagram commutes:*

$$\begin{tikzcd}
& V \arrow[dd, "\psi"] \arrow[dr, "i"] & \\
U \arrow[ur, "i"] \arrow[dr, "(id{,}0)"'] & & U \\
& U \times \mathbb{R}^n \arrow[ur, "\pi_1"'] &
\end{tikzcd}$$

We call n the fibre dimension of $\mathfrak{b}$.

Definition 2.2. *(isomorphic microbundles)*
Two microbundles $\mathfrak{b}_1 := (B, E_1, i_1, j_2)$ and $\mathfrak{b}_2 := (B, E_2, i_2, j_2)$ are said to be isomorphic if there exist neighborhoods $V_1 \subseteq E_1$ of $i_1(B)$ and $V_2 \subseteq E_2$ of $i_2(B)$ with an homeomorphism $\phi : V_1 \xrightarrow{\sim} V_2$ such that the following diagram commutes:

$$\begin{tikzcd}
& V_1 \arrow[dd, "\phi"] \arrow[dr, "j_1"] & \\
B \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & B \\
& V_2 \arrow[ur, "j_2"'] &
\end{tikzcd}$$

Example 2.3. *(trivial microbundle)*
Let B be a topological space and $n \in \mathbb{N}$:
The diagram $\mathfrak{e}^n_B : B \xrightarrow{\iota} B \times \mathbb{R}^n \xrightarrow{\pi} B$ constitutes a microbundle, where $\iota(b) := (b, 0)$ and $\pi(b, x) := b$. We call $\mathfrak{e}^n_B$ the standard microbundle and every microbundle isomorphic to $\mathfrak{b}^n_B$ trival .

Lemma 2.4. *(criteria for triviality)*
A microbundle $\mathfrak{b}$ of B is trivial if and only if there exists a open subset $B \subseteq U$ with $U \cong B \times \mathbb{R}^n$.

Proof. □

Example 2.5. *(underlying microbundle)*
Let $\xi : E \xrightarrow{\pi} B$ be a n-dimensional vector bundle: The microbundle $|\xi| : B \xrightarrow{i} E \xrightarrow{\pi} B$ with $i(b) := \phi_b(b, 0)$, where $\phi_b : U_b \times \mathbb{R}^n \to \pi^{-1}(U_b)$ is the local trivialization over a neighborhood $U_b \subseteq B$ of b. We call $|\xi|$ the underlying microbundle of ξ

Proof. □

Example 2.6. *(tangent microbundle)*
Let M be a topological manifold:
We can derive the tangent microbundle $t_M : M \xrightarrow{\Delta} M \times M \xrightarrow{\pi_1} M$, where Δ is the diagonal map and π_1 ist the projection map on the first component.

Proof. Let $p \in M$ and (U, ϕ) a chart over p:

$$\begin{tikzcd}
& U \times U \arrow[dd, "(id,\phi)"] \arrow[dr, "i"] & \\
U \arrow[ur, "i"] \arrow[dr, "(0,id)"'] & & U \\
& U \times \mathbb{R}^d \arrow[ur, "\pi_1"'] &
\end{tikzcd}$$

(id, ϕ) is a homeomorphism since $\phi : U \xrightarrow{\sim} \mathbb{R}^n$ is homeomorphic. $\square$

Proposition 2.7. *(restricting the total space)*
Let $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ be a microbundle and E' an arbitray neighborhood of $i(B)$. The restriction $\mathfrak{b}' : B \xrightarrow{i} E' \xrightarrow{j|_{E'}} B$ is a microbundle isomorphic to $\mathfrak{b}$.

Proof. Let $b \in B$.
Choose an arbitray trivialization (U, V, ϕ) over $\mathfrak{b}$ of b. We restrict $\phi : V \to U \times R^n$ to $V \cap E'$. Since $i(b) \in V$ is open and E' is a neighborhood of $i(B)$, it follows that $\phi(V \cap E')$ is a neighborhood of $(b, 0)$.
$\implies \exists (b, 0) \in U' \times X \subseteq \phi(V \cap E')$, where $U' \subseteq U$ and $X \subseteq R^n$ are open
$\implies \exists \varepsilon > 0 : U' \times B_\varepsilon(0) \subseteq \phi(V \cap E')$
Since $B_\varepsilon(0) \cong R^n$, it follows that $U' \times R^n \cong U' \times B_\varepsilon(0) \cong \phi^{-1}(U' \times B_\varepsilon(0))$.
Choosing $V' := \phi^{-1}(U' \times B_\varepsilon(0)) \subseteq V$, we see that $\mathfrak{b}'$ is a microbundle.
We easily see, that $\mathfrak{b}$ is isomorphic to $\mathfrak{b}'$ via the identity. $\square$

3 Induced Microbundles

Definition 3.1. *(induced microbundle)*
Let $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ be a microbundle and $f : A \to B$ a continuous map. We can construct a microbundle $f^*\mathfrak{b} : A \xrightarrow{i'} E' \xrightarrow{j'} A$ defined as follows:

- $E' := \{(a, e) \in A \times E \mid f(a) = j(e)\}$
- $i' : A \to E'$ *with* $i'(a) := (a, (i \circ f)(a))$
- $j' : E' \to A$ *with* $j'(a, e) := a$

We call $f^*\mathfrak{b}$ the <u>induced microbundle</u> of $\mathfrak{b}$ over f.

Proof. It is clear that i' and j' are continuous and that $id_A = j' \circ i'$.
So it remains to be shown that $f^*\mathfrak{b}$ is locally trivial for every $a \in A$:

- $U' := f^{-1}(U) \subseteq A$ is an open neighborhood of a.
- $V' := j'^{-1}(U') \subseteq E'$ is an open neighborhood of $i'(U')$.
- $\phi' : V' \xrightarrow{\sim} U' \times \mathbb{R}^n, \phi'(a, e) := (a, \pi_2(\phi(e)))$ is a homeomorphism.
  - ϕ' is well defined because $(a, e) \in V' : j(e) = f(a) \in U \implies e \in V$.
  - ϕ' is bijective with $\phi'^{-1}(a, v) = (a, \phi^{-1}(f(a), v))$.
  - ϕ' and ϕ'^{-1} are continuous because it's components are.

$\square$

Example 3.2. *(restricted microbundle)*
*Let $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ be a microbundle and $A \subseteq B$:
The induced microbundle $\iota^*\mathfrak{b}$ with $\iota : A \hookrightarrow B$ being the inclusion map is called the <u>restricted microbundle</u> and we write $\mathfrak{b}|_A := \iota^*\mathfrak{b}$.*

Remark 3.3. *In the following, we'll consider* $E(\mathfrak{b}|_A)$ *a subset of* $E(\mathfrak{b})$. *This is justified because* $E(\mathfrak{b}|_A) = \{(a,e) \in A \times E(\mathfrak{b}) \mid a = j(e)\} \cong \{e \in E(\mathfrak{b}) \mid j(e) \in A\} \subseteq E(\mathfrak{b})$.

Lemma 3.4. *(induced trivial microbundle)*
The induced microbundle $f^*\mathfrak{b}$ *is trivial for every map* $f : A \to B$, *if* $\mathfrak{b}$ *is already trivial.*

Proof. Let (V, ϕ) be a global trivialization of $\mathfrak{b}$, i.e $V \cong_\phi B \times \mathbb{R}^n$. Now define $V' := (A \times V) \cap E'$ and $\phi'(a,e) := (a, \phi^{(2)}(e))$. Obviously, V' is a neighborhood of $i'(A)$ and also ϕ' is a homeomorphism with inverse $\phi'^{(-1)}(a,x) = (a, \phi^{-1}(f(a),x))$ □

Proposition 3.5. *(composition)*
Let $A \xrightarrow{f} B \xrightarrow{g} C$ *be topological spaces and* $\mathfrak{c} : C \xrightarrow{i} E \xrightarrow{j} C$ *be a microbundle:*

$$(g \circ f)^*\mathfrak{c} \cong f^*(g^*\mathfrak{c})$$

Proof. We'll compare the two total spaces and conclude that they are homeomorphic.

1. $E((g \circ f)^*\mathfrak{c}) = \{(a,e) \in A \times E(\mathfrak{c}) \mid g(f(a)) = j(e)\}$
2. $E(f^*(g^*\mathfrak{c})) = \{(a,(b,e)) \in A \times (B \times E(\mathfrak{c})) \mid f(a) = b \text{ and } g(b) = j(e)\}$.

We have the bijection $\phi : E((g \circ f)^*\mathfrak{c}) \xrightarrow{\sim} E(f^*(g^*\mathfrak{c}))$ with $\phi(a,e) := (a,(f(a),e))$ and $\phi^{-1}(a,(b,e)) = (a,e)$. Additionally, ϕ is a homeomorphism because ϕ and ϕ^{-1} are componentwise continuous. It's easy to see that ϕ respects both injection and projection, which concludes the proof. □

For a topological space X, *we define the* <u>cone</u> *of* X *as*

$$CX := X \times [0,1]/X \times \{1\}$$

and for a map $f : A \to B$ *the* <u>mapping cone</u> *of* f *as*

$$B \sqcup_f CA := B \sqcup CA/\sim$$

where $(a,0) \sim b :\iff f(a) = b$.

Lemma 3.6. *(extending over a mapping cone)*
A microbundle $\mathfrak{b}$ *over* B *can be extended to a microbundle over the mapping cone* $B \sqcup_f CA$ *if and only if* $f^*\mathfrak{b}$ *is trivial.*

Proof. We show both implications.
$\Longrightarrow$:
Let $\mathfrak{b}'$ be an extension of $\mathfrak{b}$ over $B \sqcup_f CA$.

Considering $A \xrightarrow{f} B \hookrightarrow B \sqcup_f CA$, the composition $\iota \circ f$ is null-homotopic with homotopy

$$H_t(a) := [(a,t)]$$

Note that $H_0(a) = [(a, 0)] = [f(a)] = (\iota \circ f)(a)$ and $H_1(a) = [(a, 1)] = [(\tilde{a}, 1)] = H_1(\tilde{a})$.
$\underset{Hom.Thm.}{\Longrightarrow} (\iota \circ f)^*\mathfrak{b}'$ is trivial
Since $(\iota \circ f)^*\mathfrak{b}' = f^*(\iota^*\mathfrak{b}') = f^*\mathfrak{b}$, it follows that $f^*\mathfrak{b}$ is trivial.
$\Longleftarrow$:
Let $f^*\mathfrak{b}$ be trivial.
Analogous to the cone, we define the <u>cylinder</u> of X as

$$MX := X \times [0, 1]$$

and for a map $f : A \to B$ the mapping <u>cylinder</u> of f as

$$B \sqcup_f MA := B \sqcup MA/\sim$$

where $(a, 0) \sim b :\iff f(a) = b$.
In contrast to the mapping cone, there exists a natural retraction from the mapping cylinder to the attached space

$$\pi : B \sqcup_f MA \to B; \pi([(a, t)]) := f(a)$$

and therefore the induced microbundle $\pi^*\mathfrak{b}$ over $B \sqcup_f MA$.
Considering $A \times \{1\} \hookrightarrow B \sqcup_f MA \xrightarrow{\pi} B$, we see that $\pi \circ \iota \cong f$ and therefore

$$\pi^*\mathfrak{b}|_{A\times\{1\}} = (\pi \circ \iota)^*\mathfrak{b} \cong f^*\mathfrak{b} = \mathfrak{e}^n_A$$

is trivial. From the lemma of induced trivial microbundles and $(a, t) \mapsto (a, 1)$ it follows that $\pi^*\mathfrak{b}|_{A\times[\frac{1}{2},1]}$ is trivial.
$\Longrightarrow \exists\phi : E(\mathfrak{b}|_{A\times[\frac{1}{2},1]}) \xrightarrow{\sim} A \times [\frac{1}{2}, 1] \times \mathbb{R}^n$

Now we explicitly construct the desired extended microbundle $\mathfrak{b}' : B \sqcup_f CA \xrightarrow{i'} E' \xrightarrow{j'} B \sqcup_f CA$

- $E' := E(\mathfrak{b}|_{A\times[\frac{1}{2},1]})/\phi^{-1}(A \times [\frac{1}{2}, 1] \times \{x\})$ (for every $x \in \mathbb{R}^n$)
- $i' := \pi \circ i$ the projection i to E'
- $j'([e]) := [j(e)]$ is well defined, because $[e] = [\tilde{e}] \implies [j(e)] = [j(e')]$

Now that we have constructed $\mathfrak{b}'$, this proves the claim. □

Corollary 3.7. *(extending over a d-simplex)*
Let B be a $(d+1)$-simplicial complex, B' it's d-skeleton and $\Delta^{d+1} \cong \sigma \subseteq B$. A microbundle $\mathfrak{b}$ over B' can be extended to a microbundle over $B' \cup \sigma$ if and only if $\mathfrak{b}|_{\partial\sigma}$ is trivial.

Proof. The statement follows from the last lemma:
There exists a $\phi : C\partial\sigma \xrightarrow{\sim} \sigma$ such that $\phi(\partial\sigma \times \{0\}) = \partial\sigma$.
We explicitly construct $\phi((t_1, \ldots, t_{d+1}), \lambda) := (1-\lambda)(t_1, \ldots, t_{d+1}) + \frac{\lambda}{d+1}(1, \ldots, 1)$.
It's easy to see that ϕ suffices all our requirements. By choosing $f : \partial\sigma \hookrightarrow B'$ and applying the last lemma, the statement is proven. □

4 Whitney sums

Definition 4.1. *(whitney sum)*
Let $\mathfrak{b}_1$ and $\mathfrak{b}_1$ be two microbundles over a topological space B.
We define the <u>whitney sum</u> $\mathfrak{b}_1 \oplus \mathfrak{b}_2$ as follows:

- $E := \{(e_1, e_2) \in E(\mathfrak{b}_1) \times E(\mathfrak{b}_2) \mid j_1(e_1) = j_2(e_2)\}$
- $i(b) := (i_1(b), i_2(b))$
- $j(e_1, e_2) := j_1(e_1) = j_2(e_2)$

Proof. Let $b \in B$.
Choose U_1, V_1, ϕ_1 and U_2, V_2, ϕ_2 accordingly from the local trivialization of b over $\mathfrak{b}_1$ and $\mathfrak{b}_2$:

- $U := U_1 \cap U_2$
- $V := (V_1 \times V_2) \cap E$
- $\phi : V \to U \times \mathbb{R}^{n_1+n_2}; \phi(e_1, e_2) := (\phi_1^{(1)}(e_1), \phi_1^{(2)}(e_1) \times \phi_2^{(2)}(e_2))$

Note that $\phi_1^{(1)}(e_1) = \phi_2^{(1)}(e_2)$.
Local triviality follows directly from it's components. ☐

Lemma 4.2. *(compatibility)*
Let $\mathfrak{b}_1$ and $\mathfrak{b}_1$ be two microbundles over B and $f : A \to B$ a map.
Induced microbundle and whitney sum are compatible, i.e. $f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2) \cong f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2$

Proof. From the definition of the induced microbundle and the whitney sum, we can derive the total spaces:
$E(f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2)) = \{(a, (e_1, e_2)) \in A \times (E_1 \times E_2) \mid j_1(e_1) = j_2(e_2) = f(a)\}$
$E(f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2) = \{((a_1, e_1), (a_2, e_2)) \in (A \times E_1) \times (A \times E_2) \mid j(a_1, e_1) = j(a_2, e_2)$ and $f(a_i) = j(e_i)\}$
Those two total spaces are homeomorphic via $\phi(a, (e_1, e_2)) := ((a, e_1), (a, e_2))$ and $\phi^{-1}((a, e_1), (a, e_2)) = (a, (e_1, e_2))$. ϕ and ϕ^{-1} are continuous because they are componentwise continuous.
Obviously, $\phi \circ i = i$ and $\phi \circ j = j$, which concludes the proof. ☐

Theorem 4.3. *()*
Let $\mathfrak{b}$ be a microbundle over a d-dimensional simplicial complex B.
Then there exists a microbundle $\mathfrak{n}$ over B so that the Whitney sum $\mathfrak{b} \oplus \mathfrak{n}$ is trivial.

Proof. We prove this theorem by induction over d.
(Start of induction)
A 1-dimensional simplicial complex is just a bouquet of circles, therefore the start of induction follows directly from the bouquet lemma.
(Inductive Step)
Let B' be the $(d-1)$-skeleton of B and $\mathfrak{n}'$ it's corresponding microbundle so that $\mathfrak{b}|_{B'} \oplus \mathfrak{n}'$ is trivial. ☐

5 Normal Microbundles

Definition 5.1. *(normal microbundle)*
Let M and N be two topological manifolds with $N \subseteq M$.
We call a microbundle of the form

$$\mathfrak{n} : N \xrightarrow{\iota} U \xrightarrow{r} N$$

where $U \subseteq M$ is a neighborhood of N, a *<u>normal microbundle</u>* *of N in M.*

Definition 5.2. *(composition microbundle)*
Let $\mathfrak{b} : B \xrightarrow{i_{\mathfrak{b}}} E \xrightarrow{j_{\mathfrak{b}}} B$ and $\mathfrak{c} : E \xrightarrow{i_{\mathfrak{c}}} E' \xrightarrow{j_{\mathfrak{c}}} E$ be two microbundles. We define the *<u>composition microbundle</u>* *$\mathfrak{b} \circ \mathfrak{c} : B \xrightarrow{i} E' \xrightarrow{j} B$ with $i(b) := (i_{\mathfrak{c}} \circ i_{\mathfrak{b}})(b)$ and $j(e') := (j_{\mathfrak{b}} \circ j_{\mathfrak{c}})(e')$*

Proof. Let $b \in B$.
Choose local trivializations $(U_{\mathfrak{b}}, V_{\mathfrak{b}}, \phi_{\mathfrak{b}})$ of b and $(U_{\mathfrak{c}}, V_{\mathfrak{c}}, \phi_{\mathfrak{c}})$ of $j_{\mathfrak{b}}(b)$ From this, we construct our local trivialization of $\mathfrak{b} \circ \mathfrak{c}$. Let $V := \phi_{\mathfrak{c}}((V_{\mathfrak{b}} \cap U_{\mathfrak{c}}) \times \mathbb{R}^n)$ and $U :=$ □

Definition 5.3. *(product neighborhood)*
Again, let M and N be two topological manifolds with $N \subseteq M$.
We say that N has a *<u>product neighborhood</u>* *in M if there exists a trivial normal microbundle of N in M.*

Lemma 5.4. *(criteria for product neighborhoods)*
A submanifold $N \subseteq M$ has a product neighborhood if and only if there exists a neighborhood U of N with $(U, M) \cong (M \times R^n, M \times 0)$.

Proof. This follows directly from the definition of normal microbundles and the criteria for trivial microbundles (NUMBER). □

6 Homotopy and Microbundles