Microbundles on Topological Manifolds

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1 Vectorbundles on Smooth Manifolds

Definition 1.1. *(vector bundle)*
A vector bundle ξ is a tuple $\xi := (B, E, \pi, +, \cdot)$ satisfying the following conditions:

- *B is a topological space (base space)*
- *E is a topological space (total space)*
- *$(\pi^{-1}(b), +, \cdot)$ is a real vector space for every $b \in B$*
- *Every $b \in B$ is locally trivializable, i.e there exist neighborhoods $U \subseteq B$ of b such that the following diagram commutes*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\pi"] & U \\ U \times \mathbb{R}^n \arrow[u, "\phi"] \arrow[ur, "\pi_1"] & \end{tikzcd}$$

  and $\phi(b, -) : b \times \mathbb{R}^n \xrightarrow{\sim} \pi^{-1}(b)$ is a linear isomorphism.

We call n the rank of ξ.

Example 1.2. *(tangent vector bundle)*
Let M be a smooth manifold:
$\xi : TM \xrightarrow{\pi} M$ is a vector bundle, where $\pi(p, v) := p$.

2 Introduction to Microbundles

Definition 2.1. *(microbundle)*
A microbundle $\mathfrak{b}$ is a tuple $\mathfrak{b} := (B, E, i, j)$ satisfying the following properties:

- *B is a topological space called the base space*
- *E is a topological space called the total space*
- *$i : B \to E$ and $j : E \to B$ are continuous maps with $id_B = j \circ i$*
- *Every $b \in B$ is locally trivializable, i.e there exist open neighborhoods $U \subseteq B$ of b and $V \subseteq E$ of $i(U)$ such that the following diagram commutes:*

$$\begin{tikzcd} & V \arrow[dr, "i"] \arrow[dd, "\psi"] & \\ U \arrow[ur, "i"] \arrow[dr, "{(id,0)}"'] & & U \\ & U \times \mathbb{R}^n \arrow[ur, "\pi_1"'] & \end{tikzcd}$$

We call n the fibre dimension of $\mathfrak{b}$.

Definition 2.2. *(isomorphic microbundles)*
Two microbundles $\mathfrak{b}_1 := (B, E_1, i_1, j_2)$ and $\mathfrak{b}_2 := (B, E_2, i_2, j_2)$ are said to be isomorphic if there exist neighborhoods $V_1 \subseteq E_1$ of $i_1(B)$ and $V_2 \subseteq E_2$ of $i_2(B)$ with an homeomorphism $\phi : V_1 \xrightarrow{\sim} V_2$ such that the following diagram commutes:

$$
\begin{tikzcd}
& V_1 \arrow[dd, "\phi"] \arrow[dr, "j_1"] & \\
B \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & B \\
& V_2 \arrow[ur, "j_2"'] &
\end{tikzcd}
$$

Example 2.3. *(trivial microbundle)*
Let B be a topological space and $n \in \mathbb{N}$:
The diagram $\mathfrak{e}^n_B : B \xrightarrow{\iota} B \times \mathbb{R}^n \xrightarrow{\pi} B$ constitutes a microbundle, where $\iota(b) := (b, 0)$ and $\pi(b, x) := b$. We call $\mathfrak{e}^n_B$ the standard microbundle and every microbundle isomorphic to $\mathfrak{b}^n_B$ trival.

Example 2.4. *(underlying microbundle)*
Let $\xi : E \xrightarrow{\pi} B$ be a n-dimensional vector bundle: The microbundle $|\xi| : B \xrightarrow{i} E \xrightarrow{\pi} B$ with $i(b) := \phi_b(b, 0)$, where $\phi_b : U_b \times \mathbb{R}^n \to \pi^{-1}(U_b)$ is the local trivialization over a neighborhood $U_b \subseteq B$ of b. We call $|\xi|$ the underlying microbundle of ξ

Proof. ☐

Example 2.5. *(tangent microbundle)*
Let M be a topological manifold:
We can derive the tangent microbundle $t_M : M \xrightarrow{\Delta} M \times M \xrightarrow{\pi_1} M$, where Δ is the diagonal map and π_1 ist the projection map on the first component.

Proof. Let $p \in M$ and (U, ϕ) a chart over p:

$$
\begin{tikzcd}
& U \times U \arrow[dd, "(id,\phi)"] \arrow[dr, "i"] & \\
U \arrow[ur, "i"] \arrow[dr, "(0,id)"'] & & U \\
& U \times \mathbb{R}^d \arrow[ur, "\pi_1"'] &
\end{tikzcd}
$$

(id, ϕ) is a homeomorphism since $\phi : U \xrightarrow{\sim} \mathbb{R}^n$ is homeomorphic. ☐

3 Induced Microbundles

Definition 3.1. *(induced microbundle)*
Let $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ *be a microbundle and* $f : A \to B$ *a continuous map.*
We can construct a microbundle $f^*\mathfrak{b} : A \xrightarrow{i'} E' \xrightarrow{j'} A$ *defined as follows:*

- $E' := \{(a, e) \in A \times E \mid f(a) = j(e)\}$
- $i' : A \to E'$ *with* $i'(a) := (a, (i \circ f)(a))$
- $j' : E' \to A$ *with* $j'(a, e) := a$

We call $f^*\mathfrak{b}$ *the induced microbundle of* $\mathfrak{b}$ *over* f*.*

Proof. It is clear that i' and j' are continuous and that $id_A = j' \circ i'$.
So it remains to be shown that $f^*\mathfrak{b}$ is locally trivial for every $a \in A$:

- $U' := f^{-1}(U) \subseteq A$ is an open neighborhood of a.
- $V' := j'^{-1}(U') \subseteq E'$ is an open neighborhood of $i'(U')$.
- $\phi' : V' \xrightarrow{\sim} U' \times \mathbb{R}^n, \phi'(a, e) := (a, \pi_2(\phi(e)))$ is a homeomorphism.
  - ϕ' is well defined because $(a, e) \in V' : j(e) = f(a) \in U \implies e \in V$.
  - ϕ' is bijective with $\phi'^{-1}(a, v) = (a, \phi^{-1}(f(a), v))$.
  - ϕ' and ϕ'^{-1} are continuous because it's components are.

□

Example 3.2. *(restricted microbundle)*
Let $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ *be a microbundle and* $A \subseteq B$*:*
The induced microbundle $\iota^*\mathfrak{b}$ *with* $\iota : A \hookrightarrow B$ *being the inclusion map is called the restricted microbundle and we write* $\mathfrak{b}|_A := \iota^*\mathfrak{b}$*.*

Proposition 3.3. *(composition)*
Let $A \xrightarrow{f} B \xrightarrow{g} C$ *be topological spaces and* $\mathfrak{c} : C \xrightarrow{i} E \xrightarrow{j} C$ *be a microbundle:*

$$(g \circ f)^*\mathfrak{c} = f^*(g^*\mathfrak{c})$$

Proof. □

Remark 3.4. *(functor)*
The induced microbundle yields a functor $f \mapsto f^*$ *between topological spaces and microbundles.*

4 The Whitney Sum

We can easily construct the product of two microbundles $\mathfrak{b}_1$ *and* $\mathfrak{b}_2$ *over* B*:*

$$\mathfrak{b}_1 \times \mathfrak{b}_2 : B \times B \xrightarrow{i_1 \times i_2} E_1 \times E_2 \xrightarrow{j_1 \times j_2} B \times B$$

However, we end up with a microbundle over $B \times B$ *instead of* B*.*

Definition 4.1. *(whitney sum)*
Let $\mathfrak{b}_1$ *and* $\mathfrak{b}_1$ *be two microbundles over a topological space* B*.*
We define the whitney sum $\mathfrak{b}_1 \oplus \mathfrak{b}_2$ *as the induced microbundle over the diagonal map* Δ *of the product microbundle* $\mathfrak{b}_1 \times \mathfrak{b}_2 : B \times B \to E(\mathfrak{b}_1) \times E(\mathfrak{b}_2) \to B \times B$*.*

5 Normal Microbundles

Definition 5.1. *(normal microbundle)*
Let M *and* N *be two topological manifolds with* $N \subseteq M$*.*
We call a microbundle of the form

$$\mathfrak{n} : N \xrightarrow{\iota} M \xrightarrow{r} N$$

a normal microbundle of M *in* N*.*