

# Microbundles on Topological Manifolds

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# 1 Vectorbundles on Smooth Manifolds

**Definition 1.1.** (*vector bundle*)

A vector bundle  $\xi$  is a tuple  $\xi := (B, E, \pi, +, \cdot)$  satisfying the following conditions:

- $B$  is a topological space ( base space )
- $E$  is a topological space ( total space )
- $(\pi^{-1}(b), +, \cdot)$  is a real vector space for every  $b \in B$
- Every  $b \in B$  is locally trivializable , i.e there exist neighborhoods  $U \subseteq B$  of  $b$  such that the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\pi} & U \\ \phi \uparrow & \nearrow \pi_1 & \\ U \times \mathbb{R}^n & & \end{array}$$

and  $\phi(b, -) : b \times \mathbb{R}^n \xrightarrow{\sim} \pi^{-1}(b)$  is a linear isomorphism.

We call  $n$  the rank of  $\xi$ .

**Example 1.2.** (*tangent vector bundle*)

Let  $M$  be a smooth manifold:

$\xi : TM \xrightarrow{\pi} M$  is a vector bundle, where  $\pi(p, v) := p$ .

# 2 Introduction to Microbundles

**Definition 2.1.** (*microbundle*)

A microbundle  $\mathfrak{b}$  is a tuple  $\mathfrak{b} := (B, E, i, j)$  satisfying the following properties:

- $B$  is a topological space called the base space
- $E$  is a topological space called the total space
- $i : B \rightarrow E$  and  $j : E \rightarrow B$  are continuous maps with  $\text{id}_B = j \circ i$
- Every  $b \in B$  is locally trivializable , i.e there exist open neighborhoods  $U \subseteq B$  of  $b$  and  $V \subseteq E$  of  $i(U)$  such that the following diagram commutes:

$$\begin{array}{ccccc} & & V & & \\ & i \nearrow & \downarrow \psi & \nwarrow i & \\ U & & & & U \\ & \searrow (id,0) & \downarrow & \nearrow \pi_1 & \\ & & U \times \mathbb{R}^n & & \end{array}$$

We call  $n$  the fibre dimension of  $\mathfrak{b}$ .

**Definition 2.2.** (*isomorphic microbundles*)

Two microbundles  $\mathfrak{b}_1 := (B, E_1, i_1, j_1)$  and  $\mathfrak{b}_2 := (B, E_2, i_2, j_2)$  are said to be isomorphic if there exist neighborhoods  $V_1 \subseteq E_1$  of  $i_1(B)$  and  $V_2 \subseteq E_2$  of  $i_2(B)$  with an homeomorphism  $\phi : V_1 \xrightarrow{\sim} V_2$  such that the following diagram commutes:

$$\begin{array}{ccc} & V_1 & \\ i_1 \nearrow & \downarrow \phi & \nwarrow j_1 \\ B & & B \\ i_2 \searrow & \downarrow & \nearrow j_2 \\ & V_2 & \end{array}$$

**Example 2.3.** (*trivial microbundle*)

Let  $B$  be a topological space and  $n \in \mathbb{N}$ :

The diagram  $\mathfrak{e}_B^n : B \xrightarrow{\iota} B \times \mathbb{R}^n \xrightarrow{\pi} B$  constitutes a microbundle, where  $\iota(b) := (b, 0)$  and  $\pi(b, x) := b$ . We call  $\mathfrak{e}_B^n$  the standard microbundle and every microbundle isomorphic to  $\mathfrak{b}_B^n$  trivial.

**Lemma 2.4.** (*criteria for triviality*)

A microbundle  $\mathfrak{b}$  of  $B$  is trivial if and only if there exists a open subset  $B \subseteq U$  with  $U \cong B \times \mathbb{R}^n$ .

*Proof.*

□

**Example 2.5.** (*underlying microbundle*)

Let  $\xi : E \xrightarrow{\pi} B$  be a  $n$ -dimensional vector bundle: The microbundle  $|\xi| : B \xrightarrow{i} E \xrightarrow{\pi} B$  with  $i(b) := \phi_b(b, 0)$ , where  $\phi_b : U_b \times \mathbb{R}^n \rightarrow \pi^{-1}(U_b)$  is the local trivialization over a neighborhood  $U_b \subseteq B$  of  $b$ . We call  $|\xi|$  the underlying microbundle of  $\xi$

*Proof.*

□

**Example 2.6.** (*tangent microbundle*)

Let  $M$  be a topological manifold:

We can derive the tangent microbundle  $t_M : M \xrightarrow{\Delta} M \times M \xrightarrow{\pi_1} M$ , where  $\Delta$  is the diagonal map and  $\pi_1$  is the projection map on the first component.

*Proof.* Let  $p \in M$  and  $(U, \phi)$  a chart over  $p$ :

$$\begin{array}{ccc} & U \times U & \\ i \nearrow & \downarrow (id, \phi) & \nwarrow i \\ U & & U \\ (0, id) \searrow & \downarrow & \nearrow \pi_1 \\ & U \times \mathbb{R}^d & \end{array}$$

$(id, \phi)$  is a homeomorphism since  $\phi : U \xrightarrow{\sim} \mathbb{R}^n$  is homeomorphic.  $\square$

**Proposition 2.7.** *(restricting the total space)*

Let  $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$  be a microbundle and  $E'$  an arbitray neighborhood of  $i(B)$ .

The restriction  $\mathfrak{b}' : B \xrightarrow{i} E' \xrightarrow{j|_{E'}} B$  is a microbundle isomorphic to  $\mathfrak{b}$ .

*Proof.* Let  $b \in B$ .

Choose an arbitray trivialization  $(U, V, \phi)$  over  $\mathfrak{b}$  of  $b$ . We restrict  $\phi : V \rightarrow U \times \mathbb{R}^n$  to  $V \cap E'$ . Since  $i(b) \in V$  is open and  $E'$  is a neighborhood of  $i(B)$ , it follows that  $\phi(V \cap E')$  is a neighborhood of  $(b, 0)$ .

$\implies \exists (b, 0) \in U' \times X \subseteq \phi(V \cap E')$ , where  $U' \subseteq U$  and  $X \subseteq \mathbb{R}^n$  are open

$\implies \exists \varepsilon > 0 : U' \times B_\varepsilon(0) \subseteq \phi(V \cap E')$

Since  $B_\varepsilon(0) \cong \mathbb{R}^n$ , it follows that  $U' \times \mathbb{R}^n \cong U' \times B_\varepsilon(0) \cong \phi^{-1}(U' \times B_\varepsilon(0))$ .

Choosing  $V' := \phi^{-1}(U' \times B_\varepsilon(0)) \subseteq V$ , we see that  $\mathfrak{b}'$  is a microbundle.

We easily see, that  $\mathfrak{b}$  is isomorphic to  $\mathfrak{b}'$  via the identity.  $\square$

### 3 Induced Microbundles

**Definition 3.1.** *(induced microbundle)*

Let  $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$  be a microbundle and  $f : A \rightarrow B$  a continuous map.

We can construct a microbundle  $f^*\mathfrak{b} : A \xrightarrow{i'} E' \xrightarrow{j'} A$  defined as follows:

- $E' := \{(a, e) \in A \times E \mid f(a) = j(e)\}$
- $i' : A \rightarrow E'$  with  $i'(a) := (a, (i \circ f)(a))$
- $j' : E' \rightarrow A$  with  $j'(a, e) := a$

We call  $f^*\mathfrak{b}$  the induced microbundle of  $\mathfrak{b}$  over  $f$ .

*Proof.* It is clear that  $i'$  and  $j'$  are continuous and that  $id_A = j' \circ i'$ .

So it remains to be shown that  $f^*\mathfrak{b}$  is locally trivial for every  $a \in A$ :

- $U' := f^{-1}(U) \subseteq A$  is an open neighborhood of  $a$ .
- $V' := j'^{-1}(U') \subseteq E'$  is an open neighborhood of  $i'(U')$ .
- $\phi' : V' \xrightarrow{\sim} U' \times \mathbb{R}^n, \phi'(a, e) := (a, \pi_2(\phi(e)))$  is a homeomorphism.
  - $\phi'$  is well defined because  $(a, e) \in V' : j(e) = f(a) \in U \implies e \in V$ .
  - $\phi'$  is bijective with  $\phi'^{-1}(a, v) = (a, \phi^{-1}(f(a), v))$ .
  - $\phi'$  and  $\phi'^{-1}$  are continuous because it's components are.

$\square$

**Example 3.2.** *(restricted microbundle)*

Let  $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$  be a microbundle and  $A \subseteq B$ :

The induced microbundle  $\iota^*\mathfrak{b}$  with  $\iota : A \hookrightarrow B$  being the inclusion map is called the restricted microbundle and we write  $\mathfrak{b}|_A := \iota^*\mathfrak{b}$ .

**Remark 3.3.** In the following, we'll consider  $E(\mathfrak{b}|_A)$  a subset of  $E(\mathfrak{b})$ . This is justified because  $E(\mathfrak{b}|_A) = \{(a, e) \in A \times E(\mathfrak{b}) \mid a = j(e)\} \cong \{e \in E(\mathfrak{b}) \mid j(e) \in A\} \subseteq E(\mathfrak{b})$ .

**Lemma 3.4.** (induced trivial microbundle)

The induced microbundle  $f^*\mathfrak{b}$  is trivial for every map  $f : A \rightarrow B$ , if  $\mathfrak{b}$  is already trivial.

*Proof.* Let  $(V, \phi)$  be a global trivialization of  $\mathfrak{b}$ , i.e  $V \cong_\phi B \times \mathbb{R}^n$ . Now define  $V' := (A \times V) \cap E'$  and  $\phi'(a, e) := (a, \phi^{(2)}(e))$ . Obviously,  $V'$  is a neighborhood of  $i'(A)$  and also  $\phi'$  is a homeomorphism with inverse  $\phi'^{-1}(a, x) = (a, \phi^{-1}(f(a), x))$   $\square$

**Proposition 3.5.** (composition)

Let  $A \xrightarrow{f} B \xrightarrow{g} C$  be topological spaces and  $\mathfrak{c} : C \xrightarrow{i} E \xrightarrow{j} C$  be a microbundle:

$$(g \circ f)^*\mathfrak{c} \cong f^*(g^*\mathfrak{c})$$

*Proof.* We'll compare the two total spaces and conclude that they are homeomorphic.

1.  $E((g \circ f)^*\mathfrak{c}) = \{(a, e) \in A \times E(\mathfrak{c}) \mid g(f(a)) = j(e)\}$
2.  $E(f^*(g^*\mathfrak{c})) = \{(a, (b, e)) \in A \times (B \times E(\mathfrak{c})) \mid f(a) = b \text{ and } g(b) = j(e)\}$ .

We have the bijection  $\phi : E((g \circ f)^*\mathfrak{c}) \xrightarrow{\sim} E(f^*(g^*\mathfrak{c}))$  with  $\phi(a, e) := (a, (f(a), e))$  and  $\phi^{-1}(a, (b, e)) = (a, e)$ . Additionally,  $\phi$  is a homeomorphism because  $\phi$  and  $\phi^{-1}$  are componentwise continuous. It's easy to see that  $\phi$  respects both injection and projection, which concludes the proof.  $\square$

For a topological space  $X$ , we define the cone of  $X$  as

$$CX := X \times [0, 1] / X \times \{1\}$$

and for a map  $f : A \rightarrow B$  the mapping cone of  $f$  as

$$B \sqcup_f CA := B \sqcup CA / \sim$$

where  $(a, 0) \sim b : \iff f(a) = b$ .

**Lemma 3.6.** (extending over a mapping cone)

A microbundle  $\mathfrak{b}$  over  $B$  can be extended to a microbundle over the mapping cone  $B \sqcup_f CA$  if and only if  $f^*\mathfrak{b}$  is trivial.

*Proof.* We show both implications.

$\implies :$

Let  $\mathfrak{b}'$  be an extension of  $\mathfrak{b}$  over  $B \sqcup_f CA$ .

Considering  $A \xrightarrow{f} B \hookrightarrow B \sqcup_f CA$ , the composition  $\iota \circ f$  is null-homotopic with homotopy

$$H_t(a) := [(a, t)]$$

Note that  $H_0(a) = [(a, 0)] = [f(a)] = (\iota \circ f)(a)$  and  $H_1(a) = [(a, 1)] = [(\tilde{a}, 1)] = H_1(\tilde{a})$ .

$\xRightarrow{Hom.Thm.} (\iota \circ f)^* \mathfrak{b}'$  is trivial

Since  $(\iota \circ f)^* \mathfrak{b}' = f^*(\iota^* \mathfrak{b}') = f^* \mathfrak{b}$ , it follows that  $f^* \mathfrak{b}$  is trivial.

$\Leftarrow :$

Let  $f^* \mathfrak{b}$  be trivial.

Analogous to the cone, we define the cylinder of  $X$  as

$$MX := X \times [0, 1]$$

and for a map  $f : A \rightarrow B$  the mapping cylinder of  $f$  as

$$B \sqcup_f MA := B \sqcup MA / \sim$$

where  $(a, 0) \sim b : \iff f(a) = b$ .

In contrast to the mapping cone, there exists a natural retraction from the mapping cylinder to the attached space

$$\pi : B \sqcup_f MA \rightarrow B; \pi([(a, t)]) := f(a)$$

and therefore the induced microbundle  $\pi^* \mathfrak{b}$  over  $B \sqcup_f MA$ .

Considering  $A \times \{1\} \hookrightarrow B \sqcup_f MA \xrightarrow{\pi} B$ , we see that  $\pi \circ \iota \cong f$  and therefore

$$\pi^* \mathfrak{b}|_{A \times \{1\}} = (\pi \circ \iota)^* \mathfrak{b} \cong f^* \mathfrak{b} = \mathfrak{c}_A^n$$

is trivial. From the lemma of induced trivial microbundles and  $(a, t) \mapsto (a, 1)$  it follows that  $\pi^* \mathfrak{b}|_{A \times [\frac{1}{2}, 1]}$  is trivial.

$$\implies \exists \phi : E(\mathfrak{b}|_{A \times [\frac{1}{2}, 1]}) \xrightarrow{\sim} A \times [\frac{1}{2}, 1] \times \mathbb{R}^n$$

Now we explicitly construct the desired extended microbundle  $\mathfrak{b}' : B \sqcup_f CA \xrightarrow{i'} E' \xrightarrow{j'} B \sqcup_f CA$

- $E' := E(\mathfrak{b}|_{A \times [\frac{1}{2}, 1]}) / \phi^{-1}(A \times [\frac{1}{2}, 1] \times \{x\})$  (for every  $x \in \mathbb{R}^n$ )
- $i' := \pi \circ i$  the projection  $i$  to  $E'$
- $j'([e]) := [j(e)]$  is well defined, because  $[e] = [\tilde{e}] \implies [j(e)] = [j(\tilde{e})]$

Now that we have constructed  $\mathfrak{b}'$ , this proves the claim.  $\square$

**Corollary 3.7.** (*extending over a  $d$ -simplex*)

Let  $B$  be a  $(d+1)$ -simplicial complex,  $B'$  it's  $d$ -skeleton and  $\Delta^{d+1} \cong \sigma \subseteq B$ .

A microbundle  $\mathfrak{b}$  over  $B'$  can be extended to a microbundle over  $B' \cup \sigma$  if and only if  $\mathfrak{b}|_{\partial\sigma}$  is trivial.

*Proof.* The statement follows from the last lemma:

There exists a  $\phi : C\partial\sigma \xrightarrow{\sim} \sigma$  such that  $\phi(\partial\sigma \times \{0\}) = \partial\sigma$ .

We explicitly construct  $\phi((t_1, \dots, t_{d+1}), \lambda) := (1-\lambda)(t_1, \dots, t_{d+1}) + \frac{\lambda}{d+1}(1, \dots, 1)$ .

It's easy to see that  $\phi$  suffices all our requirements. By choosing  $f : \partial\sigma \hookrightarrow B'$  and applying the last lemma, the statement is proven.  $\square$

## 4 Whitney sums

**Definition 4.1.** (*whitney sum*)

Let  $\mathfrak{b}_1$  and  $\mathfrak{b}_2$  be two microbundles over a topological space  $B$ .

We define the whitney sum  $\mathfrak{b}_1 \oplus \mathfrak{b}_2$  as follows:

- $E := \{(e_1, e_2) \in E(\mathfrak{b}_1) \times E(\mathfrak{b}_2) \mid j_1(e_1) = j_2(e_2)\}$
- $i(b) := (i_1(b), i_2(b))$
- $j(e_1, e_2) := j_1(e_1) = j_2(e_2)$

*Proof.* Let  $b \in B$ .

Choose  $U_1, V_1, \phi_1$  and  $U_2, V_2, \phi_2$  accordingly from the local trivialization of  $b$  over  $\mathfrak{b}_1$  and  $\mathfrak{b}_2$ :

- $U := U_1 \cap U_2$
- $V := (V_1 \times V_2) \cap E$
- $\phi : V \rightarrow U \times \mathbb{R}^{n_1+n_2}; \phi(e_1, e_2) := (\phi_1^{(1)}(e_1), \phi_1^{(2)}(e_1) \times \phi_2^{(2)}(e_2))$

Note that  $\phi_1^{(1)}(e_1) = \phi_2^{(1)}(e_2)$ .

Local triviality follows directly from it's components.  $\square$

**Lemma 4.2.** (*compatibility*)

Let  $\mathfrak{b}_1$  and  $\mathfrak{b}_2$  be two microbundles over  $B$  and  $f : A \rightarrow B$  a map.

Induced microbundle and whitney sum are compatible, i.e.  $f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2) \cong f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2$

*Proof.* From the definition of the induced microbundle and the whitney sum, we can derive the total spaces:

$$E(f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2)) = \{(a, (e_1, e_2)) \in A \times (E_1 \times E_2) \mid j_1(e_1) = j_2(e_2) = f(a)\}$$

$$E(f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2) = \{((a_1, e_1), (a_2, e_2)) \in (A \times E_1) \times (A \times E_2) \mid j(a_1, e_1) = j(a_2, e_2) \text{ and } f(a_i) = j(e_i)\}$$

Those two total spaces are homeomorphic via  $\phi(a, (e_1, e_2)) := ((a, e_1), (a, e_2))$  and  $\phi^{-1}((a, e_1), (a, e_2)) = (a, (e_1, e_2))$ .  $\phi$  and  $\phi^{-1}$  are continuous because they are componentwise continuous.

Obviously,  $\phi \circ i = i$  and  $\phi \circ j = j$ , which concludes the proof.  $\square$

**Theorem 4.3.** ()

Let  $\mathfrak{b}$  be a microbundle over a  $d$ -dimensional simplicial complex  $B$ .

Then there exists a microbundle  $\mathfrak{n}$  over  $B$  so that the Whitney sum  $\mathfrak{b} \oplus \mathfrak{n}$  is trivial.

*Proof.* We prove this theorem by induction over  $d$ .

(Start of induction)

A 1-dimensional simplicial complex is just a bouquet of circles, therefore the start of induction follows directly from the bouquet lemma.

(Inductive Step)

Let  $B'$  be the  $(d-1)$ -skeleton of  $B$  and  $\mathfrak{n}'$  it's corresponding microbundle so that  $\mathfrak{b}|_{B'} \oplus \mathfrak{n}'$  is trivial.  $\square$

## 5 Normal Microbundles

**Definition 5.1.** (*normal microbundle*)

Let  $M$  and  $N$  be two topological manifolds with  $N \subseteq M$ .

We call a microbundle of the form

$$\mathbf{n} : N \xrightarrow{\iota} U \xrightarrow{\tau} N$$

where  $U \subseteq M$  is a neighborhood of  $N$ , a normal microbundle of  $N$  in  $M$ .

**Definition 5.2.** (*composition microbundle*)

Let  $\mathbf{b} : B \xrightarrow{i_b} E \xrightarrow{j_b} B$  and  $\mathbf{c} : E' \xrightarrow{i_c} E' \xrightarrow{j_c} E$  be two microbundles. We define the composition microbundle  $\mathbf{b} \circ \mathbf{c} : B \xrightarrow{i} E' \xrightarrow{j} B$  with  $i(b) := (i_c \circ i_b)(b)$  and  $j(e') := (j_b \circ j_c)(e')$

*Proof.* Let  $b \in B$ .

Choose local trivializations  $(U_b, V_b, \phi_b)$  of  $b$  and  $(U_c, V_c, \phi_c)$  of  $j_b(b)$ . From this, we construct our local trivialization over  $\mathbf{b} \circ \mathbf{c}$ . Consider  $\phi_b(V_b \cap U_c)$ , which is a neighborhood of  $(b, 0)$ . Therefore, there exist open neighborhoods  $b \in U \subseteq U_b$  and  $0 \in X \subseteq R^n$  such that  $U \times X \subseteq \phi_b(V_b \cap U_c)$ . Analogous to the proof of restricting the total space in Chapter 1, it follows that

$$\exists \varepsilon > 0 : U \times B_\varepsilon(0) \subseteq \phi_b(V_b \cap U_c)$$

$$\implies U \times \mathbb{R}^n \cong U \times B_\varepsilon(0) \cong \phi_b^{-1}(U \times B_\varepsilon(0)) \cong \phi_c^{-1}(\phi_b^{-1}(U \times B_\varepsilon(0)))$$

which is an open neighborhood of  $i(U)$  and therefore a valid candidate for  $V$ . This concludes local triviality and the proof.  $\square$

**Definition 5.3.** (*product neighborhood*)

Again, let  $M$  and  $N$  be two topological manifolds with  $N \subseteq M$ .

We say that  $N$  has a product neighborhood in  $M$  if there exists a trivial normal microbundle of  $N$  in  $M$ .

**Lemma 5.4.** (*criteria for product neighborhoods*)

A submanifold  $N \subseteq M$  has a product neighborhood if and only if there exists a neighborhood  $U$  of  $N$  with  $(U, M) \cong (M \times \mathbb{R}^n, M \times 0)$ .

*Proof.* This follows directly from the definition of normal microbundles and the criteria for trivial microbundles (NUMBER).  $\square$

## 6 Homotopy and Microbundles