Bundle Structures on Topological Manifolds

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1 Bundles on Smooth Manifolds

Definition 1. *(vector bundle)*
A vector bundle ξ is a tuple $\xi := (B, E, \pi, +, \cdot)$ satisfying the following conditions:

- *B is a topological space (base space)*
- *E is a topological space (total space)*
- *$(\pi^{-1}(b), +, \cdot)$ is a real vector space for every $b \in B$*
- *Every $b \in B$ is locally trivializable, i.e there exist neighborhoods $U \subseteq B$ of b such that the following diagram commutes*

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\pi"] & U \\
U \times \mathbb{R}^n \arrow[u, "\phi"] \arrow[ur, "\pi_1"] &
\end{tikzcd}$$

and $\phi(b, -) : b \times \mathbb{R}^n \xrightarrow{\sim} \pi^{-1}(b)$ is a linear isomorphism.

We call n the rank of ξ.

Example 1. *(tangent vector bundle)*
Let M be a smooth manifold:
$\xi : TM \xrightarrow{\pi} M$ is a vector bundle, where $\pi(p, v) := p$.

2 Introduction to Microbundles

Definition 2. *(microbundle)*
A microbundle $\mathfrak{b}$ is a tuple $\mathfrak{b} := (B, E, i, j)$ satisfying the following properties:

- *B is a topological space called the base space*
- *E is a topological space called the total space*
- *$i : B \to E$ and $j : E \to B$ are continous maps with $id_B = j \circ i$*
- *Every $b \in B$ is locally trivializable, i.e there exist neighborhoods $U \subseteq B$ of b and $V \subseteq E$ of $i(b)$ such that the following diagram commutes:*

$$\begin{tikzcd}
 & V \arrow[dr, "i"] \arrow[dd, "\psi"] & \\
U \arrow[ur, "i"] \arrow[dr, "(id{,}0)"'] & & U \\
 & U \times \mathbb{R}^n \arrow[ur, "\pi_1"'] &
\end{tikzcd}$$

We call n the fibre dimension of $\mathfrak{b}$.

Definition 3. *(isomorphic microbundles)*
Two microbundles $\mathfrak{b}_1 := (B, E_1, i_1, j_2)$ and $\mathfrak{b}_2 := (B, E_2, i_2, j_2)$ are said to be isomorphic if there exist neighborhoods $V_1 \subseteq E_1$ of $i_1(B)$ and $V_2 \subseteq E_2$ of $i_2(B)$ with an homeomorphism $\phi : V_1 \xrightarrow{\sim} V_2$ such that the following diagram commutes:

$$\begin{tikzcd} & V_1 \arrow[dd, "\phi"] \arrow[dr, "j_1"] & \\ B \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & B \\ & V_2 \arrow[ur, "j_2"'] & \end{tikzcd}$$

Example 2. *(trivial microbundle)*
Let B be a topological space and $n \in \mathbb{N}$:
The diagram $\mathfrak{e}_B^n : B \xrightarrow{\iota} B \times \mathbb{R}^n \xrightarrow{\pi} B$ constitutes a microbundle, where $\iota(b) := (b, 0)$ and $\pi(b, x) := b$. We call $\mathfrak{e}_B^n$ the standard microbundle and every microbundle isomorphic to $\mathfrak{b}_B^n$ trival.

Example 3. *(underlying microbundle)*
Let $\xi : E \xrightarrow{\pi} B$ be a n-dimensional vector bundle: The microbundle $|\xi| : B \xrightarrow{i} E \xrightarrow{\pi} B$ with $i(b) := \phi_b(b, 0)$, where $\phi_b : U_b \times \mathbb{R}^n \to \pi^{-1}(U_b)$ is the local trivialization over a neighborhood $U_b \subseteq B$ of b. We call $|\xi|$ the underlying microbundle of ξ

Proof. □

Example 4. *(tangent microbundle)*
Let M be a topological manifold:
We can derive the tangent microbundle $t_M : M \xrightarrow{\Delta} M \times M \xrightarrow{\pi_1} M$, where Δ is the diagonal map and π_1 ist the projection map on the first component.

Proof. Let $p \in M$ and (U, ϕ) a chart over p:

$$\begin{tikzcd} & U \times U \arrow[dd, "(id{,}\phi)"] \arrow[dr, "i"] & \\ U \arrow[ur, "i"] \arrow[dr, "(0{,}id)"'] & & U \\ & U \times \mathbb{R}^d \arrow[ur, "\pi_1"'] & \end{tikzcd}$$

(id, ϕ) is a homeomorphism since $\phi : U \xrightarrow{\sim} \mathbb{R}^n$ is homeomorphic. □