# Microbundles on Topological Manifolds

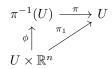
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# 1 Vectorbundles on Smooth Manifolds

**Definition 1.1.** (vector bundle)

A vector bundle  $\xi$  is a tuple  $\xi := (B, E, \pi, +, \cdot)$  satisfying the following conditions:

- B is a topological space (base space)
- E is a topological space ( total space )
- $(\pi^{-1}(b), +, \cdot)$  is a real vector space for every  $b \in B$
- Every  $b \in B$  is <u>locally trivializable</u>, i.e there exist neighborhoods  $U \subseteq B$  of b such that the following diagram commutes



and  $\phi(b, -): b \times \mathbb{R}^n \xrightarrow{\sim} \pi^{-1}(b)$  is a linear isomorphism.

We call n the  $\underline{rank}$  of  $\xi$ .

Example 1.2. (tangent vector bundle)

Let M be a smooth manifold:

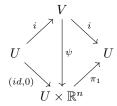
 $\xi: TM \xrightarrow{\pi} M$  is a vector bundle, where  $\pi(p,v) := p$ .

# 2 Introduction to Microbundles

**Definition 2.1.** (microbundle)

A microbundle  $\mathfrak{b}$  is a tuple  $\mathfrak{b} := (B, E, i, j)$  satisfying the following properties:

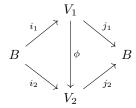
- B is a topological space called the base space
- E is a topological space called the total space
- $i: B \to E$  and  $j: E \to B$  are continuous maps with  $id_B = j \circ i$
- Every  $b \in B$  is  $\underline{locally\ trivializable}$ , i.e there exist open neighborhoods  $U \subseteq B$  of b and  $V \subseteq E$  of i(U) such that the following diagram commutes:



We call n the fibre dimension of  $\mathfrak{b}$ .

### **Definition 2.2.** (isomorphic microbundles)

Two microbundles  $\mathfrak{b}_1 := (B, E_1, i_1, j_2)$  and  $\mathfrak{b}_2 := (B, E_2, i_2, j_2)$  are said to be isomorphic if there exist neighborhoods  $V_1 \subseteq E_1$  of  $i_1(B)$  and  $V_2 \subseteq E_2$  of  $i_2(B)$  with an homeomorphism  $\phi: V_1 \xrightarrow{\sim} V_2$  such that the following diagram commutes:



#### Example 2.3. (trivial microbundle)

Let B be a topological space and  $n \in \mathbb{N}$ : The diagram  $\mathfrak{e}_B^n : B \xrightarrow{\iota} B \times \mathbb{R}^n \xrightarrow{\pi} B$  constitutes a microbundle, where  $\iota(b) :=$ (b,0) and  $\pi(b,x):=b$ . We call  $\mathfrak{e}^n_B$  the <u>standard microbundle</u> and every microbundle isomorphic to  $\mathfrak{b}_{B}^{n}$  <u>trival</u>.

# **Lemma 2.4.** (criteria for triviality)

A microbundle  $\mathfrak{b}$  of B is trivial if and only if there exists a open subset  $B \subseteq U$ with  $U \cong B \times \mathbb{R}^n$ .

Proof. 

# Example 2.5. (underlying microbundle)

Let  $\xi: E \xrightarrow{\pi} B$  be a n-dimensional vector bundle: The microbundle  $|\xi|: B \xrightarrow{i}$  $E \xrightarrow{\pi} B \text{ with } i(b) := \phi_b(b,0), \text{ where } \phi_b : U_b \times \mathbb{R}^n \to \pi^{-1}(U_b) \text{ is the local trivializa-}$ tion over a neighborhood  $U_b \subseteq B$  of b. We call  $|\xi|$  the underlying microbundle of  $\xi$ 

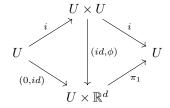
Proof. 

# Example 2.6. (tangent microbundle)

Let M be a topological manifold:

We can derive the tangent microbundle  $t_M: M \xrightarrow{\Delta} M \times M \xrightarrow{\pi_1} M$ , where  $\Delta$ is the diagonal map and  $\pi_1$  ist the projection map on the first component.

*Proof.* Let  $p \in M$  and  $(U, \phi)$  a chart over p:



# 3 Induced Microbundles

**Definition 3.1.** (induced microbundle)

Let  $\mathfrak{b}: B \xrightarrow{i} E \xrightarrow{j} B$  be a microbundle and  $f: A \to B$  a continuous map. We can construct a microbundle  $f^*\mathfrak{b}: A \xrightarrow{i'} E' \xrightarrow{j'} A$  defined as follows: 

- $\bullet \ E' := \{(a,e) \in A \times E \mid f(a) = j(e)\}$
- $i': A \to E'$  with  $i'(a) := (a, (i \circ f)(a))$
- $j': E' \to A \text{ with } j'(a, e) := a$

We call  $f^*\mathfrak{b}$  the induced microbundle of  $\mathfrak{b}$  over f.

*Proof.* It is clear that i' and j' are continuous and that  $id_A = j' \circ i'$ . So it remains to be shown that  $f^*\mathfrak{b}$  is locally trivial for every  $a \in A$ :

- $U' := f^{-1}(U) \subseteq A$  is an open neighborhood of a.
- $V' := j'^{-1}(U') \subseteq E'$  is an open neighborhood of i'(U').
- $\phi': V' \xrightarrow{\sim} U' \times \mathbb{R}^n, \phi'(a,e) := (a, \pi_2(\phi(e)))$  is a homeomorphism.
  - $-\phi'$  is well defined because  $(a,e) \in V': j(e) = f(a) \in U \implies e \in V$ .
  - $-\phi'$  is bijective with  $\phi'^{-1}(a,v)=(a,\phi^{-1}(f(a),v)).$
  - $-\phi'$  and  $\phi'^{-1}$  are continuous because it's components are.

Example 3.2. (restricted microbundle)

Let  $\mathfrak{b}: B \xrightarrow{i} E \xrightarrow{j} B$  be a microbundle and  $A \subseteq B$ :

The induced microbundle  $\iota^*\mathfrak{b}$  with  $\iota: A \hookrightarrow B$  being the inclusion map is called the <u>restricted microbundle</u> and we write  $\mathfrak{b}|_A := \iota^*\mathfrak{b}$ .

**Remark 3.3.** In the following, we'll consider  $E(\mathfrak{b}|_A)$  a subset of  $E(\mathfrak{b})$ . This is justified because  $E(\mathfrak{b}|_A) = \{(a,e) \in A \times E(\mathfrak{b}) \mid a = j(e)\} \cong \{e \in E(\mathfrak{b}) \mid j(e) \in A\} \subseteq E(\mathfrak{b})$ .

Lemma 3.4. (induced trivial microbundle)

The induced microbundle  $f^*\mathfrak{b}$  is trivial for every map  $f:A\to B$ , if  $\mathfrak{b}$  is already trivial.

*Proof.* Let  $(V, \phi)$  be a global trivialization of  $\mathfrak{b}$ , i.e  $V \cong_{\phi} B \times \mathbb{R}^n$ . Now define  $V' := (A \times V) \cap E'$  and  $\phi'(a, e) := (a, \phi^{(2)}(e))$ . Obviously, V' is a neighborhood of i'(A) and also  $\phi'$  is a homeomorphism with inverse  $\phi'^{(-1)}(a, x) = (a, \phi^{-1}(f(a), x))$ 

Proposition 3.5. (composition)

Let  $A \xrightarrow{f} B \xrightarrow{g} C$  be topological spaces and  $\mathfrak{c}: C \xrightarrow{i} E \xrightarrow{j} C$  be a microbundle:

$$(q \circ f)^* \mathfrak{c} \cong f^*(q^* \mathfrak{c})$$

 ${\it Proof.}$  We'll compare the two total spaces and conclude that they are homeomorphic.

1. 
$$E((g \circ f)^*\mathfrak{c}) = \{(a, e) \in A \times E(\mathfrak{c}) \mid g(f(a)) = j(e)\}\$$

2. 
$$E(f^*(g^*\mathfrak{c})) = \{(a, (b, e)) \in A \times (B \times E(\mathfrak{c})) \mid f(a) = b \text{ and } g(b) = j(e)\}.$$

We have the bijection  $\phi: E((g \circ f)^*\mathfrak{c}) \xrightarrow{\sim} E(f^*(g^*\mathfrak{c}))$  with  $\phi(a,e) := (a,(f(a),e))$  and  $\phi^{-1}(a,(b,e)) = (a,e)$ . Additionally,  $\phi$  is a homeomorphism because  $\phi$  and  $\phi^{-1}$  are componentwise continuous. It's easy to see that  $\phi$  respects both injection and projection, which concludes the proof.

For a topological space X, we define the <u>cone</u> of X as

$$CX := X \times [0,1]/X \times \{1\}$$

and for a map  $f: A \to B$  the mapping cone of f as

$$B \sqcup_f CA := B \sqcup CA / \sim$$

where  $(a,0) \sim b : \iff f(a) = b$ .

**Lemma 3.6.** (extending over a mapping cone)

A microbundle  $\mathfrak{b}$  over B can be extended to a microbundle over the mapping cone  $B \sqcup_f CA$  if and only if  $f^*\mathfrak{b}$  is trivial.

*Proof.* We show both implications.

**⇒** :

Let  $\mathfrak{b}'$  be an extension of  $\mathfrak{b}$  over  $B \sqcup_f CA$ .

Considering  $A \xrightarrow{f} B \hookrightarrow B \sqcup_f CA$ , the composition  $\iota \circ f$  is null-homotopic with homotopy

$$H_t(a) := [(a,t)]$$

Note that  $H_0(a) = [(a,0)] = [f(a)] = (\iota \circ f)(a)$  and  $H_1(a) = [(a,1)] = [(\tilde{a},1)] = H_1(\tilde{a})$ .

 $\Longrightarrow_{Hom.Thm.} (\iota \circ f)^* \mathfrak{b}'$  is trivial

Since  $(\iota \circ f)^*\mathfrak{b}' = f^*(\iota^*\mathfrak{b}') = f^*\mathfrak{b}$ , it follows that  $f^*\mathfrak{b}$  is trivial.

**←** 

Let  $f^*\mathfrak{b}$  be trivial.

Analogous to the cone, we define the cylinder of X as

$$MX := X \times [0,1]$$

and for a map  $f: A \to B$  the mapping cylinder of f as

$$B \sqcup_f MA := B \sqcup MA / \sim$$

where  $(a,0) \sim b : \iff f(a) = b$ .

In contrast to the mapping cone, there exists a natural retraction from the mapping cylinder to the attached space

$$\pi: B \sqcup_f MA \to B; \pi([(a,t)]) := f(a)$$

and therefore the induced microbundle  $\pi^*\mathfrak{b}$  over  $B \sqcup_f MA$ .

Considering  $A \times \{1\} \hookrightarrow B \sqcup_f MA \xrightarrow{\pi} B$ , we see that  $\pi \circ \iota \cong f$  and therefore

$$\pi^* \mathfrak{b}|_{A \times \{1\}} = (\pi \circ \iota)^* \mathfrak{b} \cong f^* \mathfrak{b} = \mathfrak{e}_A^n$$

is trivial. From the lemma of induced trivial microbundles and  $(a,t) \mapsto (a,1)$  it follows that  $\pi^* \mathfrak{b}|_{A \times [\frac{1}{2},1]}$  is trivial.

$$\implies \exists \phi : E(\mathfrak{b}|_{A \times \left[\frac{1}{2}, 1\right]}) \xrightarrow{\sim} A \times \left[\frac{1}{2}, 1\right] \times \mathbb{R}^n$$

Now we explicitly construct the desired extended microbundle  $\mathfrak{b}': B \sqcup_f CA \xrightarrow{i'} E' \xrightarrow{j'} B \sqcup_f CA$ 

- $E' := E(\mathfrak{b}|_{A \times [\frac{1}{2},1]})/\phi^{-1}(A \times [\frac{1}{2},1] \times \{x\})$  (for every  $x \in \mathbb{R}^n$ )
- $i' := \pi \circ i$  the projection i to E'
- j'([e]) := [j(e)] is well defined, because  $[e] = [\tilde{e}] \implies [j(e)] = [j(e')]$

Now that we have constructed  $\mathfrak{b}'$ , this proves the claim.

Corollary 3.7. (extending over a d-simplex)

Let B be a (d+1)-simplicial complex, B' it's d-skeleton and  $\Delta^{d+1} \cong \sigma \subseteq B$ . A microbundle  $\mathfrak{b}$  over B' can be extended to a microbundle over  $B' \cup \sigma$  if and only if  $\mathfrak{b}|_{\partial \sigma}$  is trivial.

*Proof.* The statement follows from the last lemma:

There exists a  $\phi: C\partial\sigma \xrightarrow{\sim} \sigma$  such that  $\phi(\partial\sigma \times \{0\}) = \partial\sigma$ .

We explicitly construct  $\phi((t_1,\ldots,t_{d+1}),\lambda) := (1-\lambda)(t_1,\ldots,t_{d+1}) + \frac{\lambda}{d+1}(1,\ldots,1)$ . It's easy to see that  $\phi$  suffices all our requirements. By choosing  $f:\partial\sigma\hookrightarrow B'$  and applying the last lemma, the statement is proven.

# 4 Whitney sums

**Definition 4.1.** (whitney sum)

Let  $\mathfrak{b}_1$  and  $\mathfrak{b}_1$  be two microbundles over a topological space B. We define the whitney sum  $\mathfrak{b}_1 \oplus \mathfrak{b}_2$  as follows:

- $E := \{(e_1, e_2) \in E(\mathfrak{b}_1) \times E(\mathfrak{b}_2) \mid j_1(e_1) = j_2(e_2)\}$
- $i(b) := (i_1(b), i_2(b))$
- $j(e_1, e_2) := j_1(e_1) = j_2(e_2)$

*Proof.* Let  $b \in B$ .

Choose  $U_1, V_1, \phi_1$  and  $U_2, V_2, \phi_2$  accordingly from the local trivialization of b over  $\mathfrak{b}_1$  and  $\mathfrak{b}_2$ :

- $U := U_1 \cap U_2$
- $V := (V_1 \times V_2) \cap E$
- $\phi: V \to U \times \mathbb{R}^{n_1+n_2}; \phi(e_1, e_2) := (\phi_1^{(1)}(e_1), \phi_1^{(2)}(e_1) \times \phi_2^{(2)}(e_2))$

Note that  $\phi_1^{(1)}(e_1) = \phi_2^{(1)}(e_2)$ . Local triviality follows directly from it's components.

### Lemma 4.2. (compatibility)

Let  $\mathfrak{b}_1$  and  $\mathfrak{b}_1$  be two microbundles over B and  $f: A \to B$  a map. Induced microbundle and whitney sum are compatible, i.e.  $f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2) \cong f^*\mathfrak{b}_1 \oplus$ 

*Proof.* From the definition of the induced microbundle and the whitney sum, we can derive the total spaces:

$$E(f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2)) = \{(a, (e_1, e_2)) \in A \times (E_1 \times E_2) \mid j_1(e_1) = j_2(e_2) = f(a)\}$$

$$E(f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2) = \{((a_1, e_1), (a_2, e_2)) \in (A \times E_1) \times (A \times E_2) \mid j(a_1, e_1) = j(a_2, e_2)$$
and  $f(a_i) = j(e_i)$ 

Those two total spaces are homeomorphic via  $\phi(a,(e_1,e_2)) := ((a,e_1),(a,e_2))$ and  $\phi^{-1}((a,e_1),(a,e_2))=(a,(e_1,e_2))$ .  $\phi$  and  $\phi^{-1}$  are continuous because they are componentwise continuous.

Obviously,  $\phi \circ i = i$  and  $\phi \circ j = j$ , which concludes the proof. 

#### Theorem 4.3. ()

Let  $\mathfrak{b}$  be a microbundle over a d-dimensional simplicial complex B.

Then there exists a microbundle  $\mathfrak n$  over B so that the Whitney sum  $\mathfrak b \oplus \mathfrak n$  is

*Proof.* We prove this theorem by induction over d.

(Start of induction)

A 1-dimensional simplicial complex is just a bouquet of circles, therefore the start of induction follows directly from the bouquet lemma. (Inductive Step)

Let B' be the (d-1)-skeleton of B and  $\mathfrak{n}'$  it's corresponding microbundle so that  $\mathfrak{b}|_{B'} \oplus \mathfrak{n}'$  is trivial. 

#### 5 Normal Microbundles

**Definition 5.1.** (normal microbundle)

Let M and N be two topological manifolds with  $N \subseteq M$ .

We call a microbundle of the form

$$\mathfrak{n}: N \xrightarrow{\iota} U \xrightarrow{r} N$$

where  $U \subseteq M$  is a neighborhood of N, a <u>normal microbundle</u> of N in M.

# **Definition 5.2.** (composition microbundle)

Let  $\mathfrak{b}: B \xrightarrow{i_{\mathfrak{b}}} E \xrightarrow{j_{\mathfrak{b}}} B$  and  $\mathfrak{c}: E \xrightarrow{i_{\mathfrak{c}}} E' \xrightarrow{j_{\mathfrak{c}}} E$  be two microbundles. We define the <u>composition microbundle</u>  $\mathfrak{b} \circ \mathfrak{c}: B \xrightarrow{j} E' \xrightarrow{j} B$  with  $i(b) := (i_{\mathfrak{c}} \circ i_{\mathfrak{b}})(b)$  and  $j(e') := (j_{\mathfrak{b}} \circ j_{\mathfrak{c}})(e')$ 

# *Proof.* Let $b \in B$ .

Choose local trivializations  $(U_{\mathfrak{b}}, V_{\mathfrak{b}}, \phi_{\mathfrak{b}})$  of b and  $(U_{\mathfrak{c}}, V_{\mathfrak{c}}, \phi_{\mathfrak{c}})$  of  $j_{\mathfrak{b}}(b)$  From this, we construct our local trivialization of  $\mathfrak{b} \circ \mathfrak{c}$ . Let  $V := \phi_{\mathfrak{c}}((V_{\mathfrak{b}} \cap U_{\mathfrak{c}}) \times \mathbb{R}^n)$  and  $U := \square$ 

# **Definition 5.3.** (product neighborhood)

Again, let M and N be two topological manifolds with  $N \subseteq M$ .

We say that N has a <u>product neighborhood</u> in M if there exists a trivial normal microbundle of N in  $\overline{M}$ .

# **Lemma 5.4.** (criteria for product neighborhoods)

A submanifold  $N \subseteq M$  has a product neighborhood if and only if there exists a neighborhood U of N with  $(U, M) \cong (M \times R^n, M \times 0)$ .

*Proof.* This follows directly from the definition of normal microbundles and the criteria for trivial microbundles (NUMBER).  $\Box$ 

# 6 Homotopy and Microbundles