Microbundles on Topological Manifolds
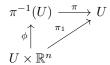
Florian Burger

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1 Vectorbundles on Smooth Manifolds

Definition 1.1. *(vector bundle)*
A vector bundle ξ is a tuple $\xi := (B, E, \pi, +, \cdot)$ satisfying the following conditions:

- *B is a topological space (base space)*

- *E is a topological space (total space)*

- *$(\pi^{-1}(b), +, \cdot)$ is a real vector space for every $b \in B$*

- *Every $b \in B$ is locally trivializable, i.e there exist neighborhoods $U \subseteq B$ of b such that the following diagram commutes*

$$
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\ \pi\ } & U \\
\phi \uparrow & \nearrow{\scriptstyle \pi_1} & \\
U \times \mathbb{R}^n & &
\end{array}
$$

and $\phi(b, -) : b \times \mathbb{R}^n \xrightarrow{\sim} \pi^{-1}(b)$ is a linear isomorphism.

We call n the rank of ξ.

Example 1.2. *(tangent vector bundle)*
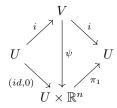*Let M be a smooth manifold:*
$\xi : TM \xrightarrow{\pi} M$ is a vector bundle, where $\pi(p, v) := p$.

2 Introduction to Microbundles

Definition 2.1. *(microbundle)*
A microbundle $\mathfrak{b}$ is a tuple $\mathfrak{b} := (B, E, i, j)$ satisfying the following properties:
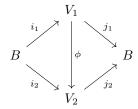
- *B is a topological space called the base space*

- *E is a topological space called the total space*

- *$i : B \to E$ and $j : E \to B$ are continuous maps with $id_B = j \circ i$*

- *Every $b \in B$ is locally trivializable, i.e there exist open neighborhoods $U \subseteq B$ of b and $V \subseteq E$ of $i(U)$ such that the following diagram commutes:*

$$
\begin{array}{ccccc}
 & & V & & \\
 & \nearrow{\scriptstyle i} & \downarrow{\scriptstyle \psi} & \searrow{\scriptstyle i} & \\
U & & & & U \\
 & \searrow{\scriptstyle (id,0)} & \downarrow & \nearrow{\scriptstyle \pi_1} & \\
 & & U \times \mathbb{R}^n & &
\end{array}
$$

We call n the fibre dimension of $\mathfrak{b}$.

Definition 2.2. *(isomorphic microbundles)*
Two microbundles $\mathfrak{b}_1 := (B, E_1, i_1, j_2)$ and $\mathfrak{b}_2 := (B, E_2, i_2, j_2)$ are said to be isomorphic if there exist neighborhoods $V_1 \subseteq E_1$ of $i_1(B)$ and $V_2 \subseteq E_2$ of $i_2(B)$ with an homeomorphism $\phi : V_1 \xrightarrow{\sim} V_2$ such that the following diagram commutes:

$$
\begin{array}{ccccc}
 & & V_1 & & \\
 & \nearrow^{i_1} & \big\downarrow^{\phi} & \searrow^{j_1} & \\
B & & & & B \\
 & \searrow_{i_2} & & \nearrow_{j_2} & \\
 & & V_2 & &
\end{array}
$$

Example 2.3. *(trivial microbundle)*
Let B be a topological space and $n \in \mathbb{N}$:
The diagram $\mathfrak{e}_B^n : B \xrightarrow{\iota} B \times \mathbb{R}^n \xrightarrow{\pi} B$ constitutes a microbundle, where $\iota(b) := (b, 0)$ and $\pi(b, x) := b$. We call $\mathfrak{e}_B^n$ the standard microbundle and every microbundle isomorphic to $\mathfrak{b}_B^n$ trival.

Lemma 2.4. *(criteria for triviality)*
A microbundle $\mathfrak{b}$ of B is trivial if and only if there exists a open subset $B \subseteq U$ with $U \cong B \times \mathbb{R}^n$.

Proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

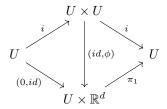**Example 2.5.** *(underlying microbundle)*

Let $\xi : E \xrightarrow{\pi} B$ be a n-dimensional vector bundle: The microbundle $|\xi| : B \xrightarrow{i} E \xrightarrow{\pi} B$ with $i(b) := \phi_b(b, 0)$, where $\phi_b : U_b \times \mathbb{R}^n \to \pi^{-1}(U_b)$ is the local trivialization over a neighborhood $U_b \subseteq B$ of b. We call $|\xi|$ the underlying microbundle of ξ

Proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Example 2.6. *(tangent microbundle)*
Let M be a topological manifold:
We can derive the tangent microbundle $t_M : M \xrightarrow{\Delta} M \times M \xrightarrow{\pi_1} M$, where Δ is the diagonal map and π_1 ist the projection map on the first component.

Proof. Let $p \in M$ and (U, ϕ) a chart over p:

$$
\begin{array}{ccccc}
 & & U \times U & & \\
 & \nearrow^{i} & \big\downarrow^{(id,\phi)} & \searrow^{i} & \\
U & & & & U \\
 & \searrow_{(0,id)} & & \nearrow_{\pi_1} & \\
 & & U \times \mathbb{R}^d & &
\end{array}
$$

(id, ϕ) is a homeomorphism since $\phi : U \xrightarrow{\sim} \mathbb{R}^n$ is homeomorphic. $\qquad\qquad$ $\square$

3   Induced Microbundles

Definition 3.1. *(induced microbundle)*
Let $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ be a microbundle and $f : A \to B$ a continuous map.
We can construct a microbundle $f^*\mathfrak{b} : A \xrightarrow{i'} E' \xrightarrow{j'} A$ defined as follows:

- $E' := \{(a, e) \in A \times E \mid f(a) = j(e)\}$

- $i' : A \to E'$ *with* $i'(a) := (a, (i \circ f)(a))$

- $j' : E' \to A$ *with* $j'(a, e) := a$

We call $f^*\mathfrak{b}$ the induced microbundle of $\mathfrak{b}$ over f.

Proof. It is clear that i' and j' are continuous and that $id_A = j' \circ i'$.
So it remains to be shown that $f^*\mathfrak{b}$ is locally trivial for every $a \in A$:

- $U' := f^{-1}(U) \subseteq A$ is an open neighborhood of a.

- $V' := j'^{-1}(U') \subseteq E'$ is an open neighborhood of $i'(U')$.

- $\phi' : V' \xrightarrow{\sim} U' \times \mathbb{R}^n, \phi'(a, e) := (a, \pi_2(\phi(e)))$ is a homeomorphism.

  - ϕ' is well defined because $(a, e) \in V' : j(e) = f(a) \in U \implies e \in V$.
  - ϕ' is bijective with $\phi'^{-1}(a, v) = (a, \phi^{-1}(f(a), v))$.
  - ϕ' and ϕ'^{-1} are continuous because it's components are.

$\square$

Example 3.2. *(restricted microbundle)*
Let $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ be a microbundle and $A \subseteq B$:
The induced microbundle $\iota^*\mathfrak{b}$ with $\iota : A \hookrightarrow B$ being the inclusion map is called the restricted microbundle and we write $\mathfrak{b}|_A := \iota^*\mathfrak{b}$.

Lemma 3.3. *(induced trivial microbundle)*
The induced microbundle $f^*\mathfrak{b}$ is trivial for every map $f : A \to B$, if $\mathfrak{b}$ is already trivial.

Proof. Let (V, ϕ) be a global trivialization of $\mathfrak{b}$, i.e $V \cong_\phi B \times \mathbb{R}^n$. Now define $V' := (A \times V) \cap E'$ and $\phi'(a, e) := (a, \phi^{(2)}(e))$. Obviously, V' is a neighborhood of $i'(A)$ and also ϕ' is a homeomorphism with inverse $\phi'^{(-1)}(a, x) = (a, \phi^{-1}(f(a), x))$ $\square$

Proposition 3.4. *(composition)*
Let $A \xrightarrow{f} B \xrightarrow{g} C$ be topological spaces and $\mathfrak{c} : C \xrightarrow{i} E \xrightarrow{j} C$ be a microbundle:

$$(g \circ f)^*\mathfrak{c} = f^*(g^*\mathfrak{c})$$

Proof. $\square$

Remark 3.5. *(functor)*
The induced microbundle yields a functor $f \mapsto f^*$ between topological spaces and microbundles.
For a topological space X, we define the cone of X as

$$CX := X \times [0,1]/X \times \{1\}$$

and for a map $f : A \to B$ the mapping cone of f as

$$B \sqcup_f CA := B \sqcup CA/\sim$$

where $(a,0) \sim b : \iff f(a) = b$.

Lemma 3.6. *(extending over a mapping cone)*
A microbundle $\mathfrak{b}$ over B can be extended to a microbundle over the mapping cone $B \sqcup_f CA$ if and only if $f^*\mathfrak{b}$ is trivial.

Proof. We show both implications.
$\implies$:
Let $\mathfrak{b}'$ be an extension of $\mathfrak{b}$ over $B \sqcup_f CA$.

Considering $A \xrightarrow{f} B \hookrightarrow B \sqcup_f CA$, the composition $\iota \circ f$ is null-homotopic with homotopy

$$H_t(a) := [(a,t)]$$

Note that $H_0(a) = [(a,0)] = [f(a)] = (\iota \circ f)(a)$ and $H_1(a) = [(a,1)] = [(\tilde{a},1)] = H_1(\tilde{a})$.
$\underset{Hom.Thm.}{\implies} (\iota \circ f)^*\mathfrak{b}'$ is trivial
Since $(\iota \circ f)^*\mathfrak{b}' = f^*(\iota^*\mathfrak{b}') = f^*\mathfrak{b}$, it follows that $f^*\mathfrak{b}$ is trivial.
$\impliedby$:
Let $f^*\mathfrak{b}$ be trivial.
Analogous to the cone, we define the cylinder of X as

$$MX := X \times [0,1]$$

and for a map $f : A \to B$ the mapping cylinder of f as

$$B \sqcup_f MA := B \sqcup MA/\sim$$

where $(a,0) \sim b : \iff f(a) = b$.
In contrast to the mapping cone, there exists a natural retraction from the mapping cylinder to the attached space

$$\pi : B \sqcup_f MA \to B; \pi([(a,t)]) := f(a)$$

and therefore the induced microbundle $\pi^*\mathfrak{b}$ over $B \sqcup_f MA$.
Considering $A \times \{1\} \hookrightarrow B \sqcup_f MA \xrightarrow{\pi} B$, we see that $\pi \circ \iota \cong f$ and therefore

$$\pi^*\mathfrak{b}|_{A\times\{1\}} = (\pi \circ \iota)^*\mathfrak{b} \cong f^*\mathfrak{b} = \mathfrak{e}_A^n$$

is trivial. From the lemma of induced trivial microbundles and $(a,t) \mapsto (a,1)$ it follows that $\pi^* \mathfrak{b}|_{A \times [\frac{1}{2},1]}$ is trivial.

$\implies \exists \phi : E(\mathfrak{b}|_{A \times [\frac{1}{2},1]}) \xrightarrow{\sim} A \times [\frac{1}{2},1] \times \mathbb{R}^n$

Now we explicitly construct the desired extended microbundle $\mathfrak{b}' : B \sqcup_f CA \xrightarrow{i'} E' \xrightarrow{j'} B \sqcup_f CA$

- $E' := E(\mathfrak{b}|_{A \times [\frac{1}{2},1]})/\phi^{-1}(A \times [\frac{1}{2},1] \times \{x\})$ (for every $x \in \mathbb{R}^n$)

- $i' := \pi \circ i$ the projection i to E'

- $j'([e]) := [j(e)]$ is well defined, because $[e] = [\tilde{e}] \implies [j(e)] = [j(e')]$

Now that we have constructed $\mathfrak{b}'$, this proves the claim. $\square$

Corollary 3.7. *(extending over a d-simplex)*
Let B be a $(d+1)$-simplicial complex, B' it's d-skeleton and $\Delta^{d+1} \cong \sigma \subseteq B$.
A microbundle $\mathfrak{b}$ over B' can be extended to a microbundle over $B' \cup \sigma$ if and only if $\mathfrak{b}|_{\partial \sigma}$ is trivial.

Proof. The statement follows from the last lemma:
There exists a $\phi : C\partial\sigma \xrightarrow{\sim} \sigma$ such that $\phi(\partial\sigma \times \{0\}) = \partial\sigma$.
We explicitly construct $\phi((t_1, \ldots, t_{d+1}), \lambda) := (1-\lambda)(t_1, \ldots, t_{d+1}) + \frac{\lambda}{d+1}(1, \ldots, 1)$.
It's easy to see that ϕ suffices all our requirements. By choosing $f : \partial\sigma \hookrightarrow B'$ and applying the last lemma, the statement is proven. $\square$

4 Whitney sums

Definition 4.1. *(whitney sum)*
Let $\mathfrak{b}_1$ and $\mathfrak{b}_1$ be two microbundles over a topological space B.
We define the whitney sum $\mathfrak{b}_1 \oplus \mathfrak{b}_2$ as follows:

- $E := \{(e_1, e_2) \in E(\mathfrak{b}_1) \times E(\mathfrak{b}_2) \mid j_1(e_1) = j_2(e_2)\}$

- $i(b) := (i_1(b), i_2(b))$

- $j(e_1, e_2) := j_1(e_1) = j_2(e_2)$

Proof. Let $b \in B$.
Choose U_1, V_1, ϕ_1 and U_2, V_2, ϕ_2 accordingly from the local trivialization of b over $\mathfrak{b}_1$ and $\mathfrak{b}_2$:

- $U := U_1 \cap U_2$

- $V := (V_1 \times V_2) \cap E$

- $\phi : V \to U \times \mathbb{R}^{n_1 + n_2}; \phi(e_1, e_2) := (\phi_1^{(1)}(e_1), \phi_1^{(2)}(e_1) \times \phi_2^{(2)}(e_2))$

Note that $\phi_1^{(1)}(e_1) = \phi_2^{(1)}(e_2)$.
Local triviality follows directly from it's components. $\square$

Lemma 4.2. *(compatibility)*
Let $\mathfrak{b}_1$ and $\mathfrak{b}_1$ be two microbundles over B and $f : A \to B$ a map.
Induced microbundle and whitney sum are compatible, i.e. $f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2) \cong f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2$

Proof. From the definition of the induced microbundle and the whitney sum, we can derive the total spaces:
$E(f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2)) = \{(a, (e_1, e_2)) \in A \times (E_1 \times E_2) \mid j_1(e_1) = j_2(e_2) = f(a)\}$
$E(f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2) = \{((a_1, e_1), (a_2, e_2)) \in (A \times E_1) \times (A \times E_2) \mid j(a_1, e_1) = j(a_2, e_2)$
and $f(a_i) = j(e_i)\}$
Those two total spaces are homeomorphic via $\phi(a, (e_1, e_2)) := ((a, e_1), (a, e_2))$
and $\phi^{-1}((a, e_1), (a, e_2)) = (a, (e_1, e_2))$. ϕ and ϕ^{-1} are continuous because they are componentwise continuous.
Obviously, $\phi \circ i = i$ and $\phi \circ j = j$, which concludes the proof. $\qquad\square$

Theorem 4.3. *()*
Let $\mathfrak{b}$ be a microbundle over a d-dimensional simplicial complex B.
Then there exists a microbundle $\mathfrak{n}$ over B so that the Whitney sum $\mathfrak{b} \oplus \mathfrak{n}$ is trivial.

Proof. We prove this theorem by induction over d.
(Start of induction)
A 1-dimensional simplicial complex is just a bouquet of circles, therefore the start of induction follows directly from the bouquet lemma.
(Inductive Step)
Let B' be the $(d-1)$-skeleton of B and $\mathfrak{n}'$ it's corresponding microbundle so that $\mathfrak{b}|_{B'} \oplus \mathfrak{n}'$ is trivial. $\qquad\square$

5 Normal Microbundles

Definition 5.1. *(normal microbundle)*
Let M and N be two topological manifolds with $N \subseteq M$.
We call a microbundle of the form

$$\mathfrak{n} : N \xrightarrow{\iota} U \xrightarrow{r} N$$

where $U \subseteq M$ is a neighborhood of N, a normal microbundle of N in M.

Definition 5.2. *(product neighborhood)*
Again, let M and N be two topological manifolds with $N \subseteq M$.
We say that N has a product neighborhood in M if there exists a trivial normal microbundle of N in M.

Lemma 5.3. *(criteria for product neighborhoods)*
A submanifold $N \subseteq M$ has a product neighborhood if and only if there exists a neighborhood U of N with $(U, M) \cong (M \times R^n, M \times 0)$.

Proof. This follows directly from the definition of normal microbundles and the criteria for trivial microbundles (NUMBER). $\qquad\square$

6 Homotopy and Microbundles