

Microbundles on Topological Manifolds

Florian Burger

April 24, 2024

1 Introduction to Microbundles

Definition 1.1. (microbundle)

A *microbundle* \mathfrak{b} is *hallo* a tuple $\mathfrak{b} := (B, E, i, j)$ satisfying the following properties:

- B is a topological space called the *base space*
- E is a topological space called the *total space*
- $i : B \rightarrow E$ (*injection*) and $j : E \rightarrow B$ (*projection*) are continuous maps with $id_B = j \circ i$
- Every $b \in B$ is *locally trivializable*, i.e there exist open neighborhoods $U \subseteq B$ of b and $V \subseteq E$ of $i(U)$ such that the following diagram commutes:

$$\begin{array}{ccc} & V & \\ i \nearrow & \downarrow \psi & \nwarrow j \\ U & & U \\ (id,0) \searrow & & \nearrow \pi_1 \\ & U \times \mathbb{R}^n & \end{array}$$

We call n the *fibre dimension* of \mathfrak{b} .

Definition 1.2. (isomorphic microbundles)

Two microbundles $\mathfrak{b}_1 := (B, E_1, i_1, j_1)$ and $\mathfrak{b}_2 := (B, E_2, i_2, j_2)$ are said to be *isomorphic* if there exist neighborhoods $V_1 \subseteq E_1$ of $i_1(B)$ and $V_2 \subseteq E_2$ of $i_2(B)$ with an homeomorphism $\phi : V_1 \xrightarrow{\sim} V_2$, so that the following diagram commutes:

$$\begin{array}{ccc} & V_1 & \\ i_1 \nearrow & \downarrow \phi & \nwarrow j_1 \\ B & & B \\ i_2 \searrow & & \nearrow j_2 \\ & V_2 & \end{array}$$

Example 1.3. (trivial microbundle)

Let B be a topological space and $n \in \mathbb{N}$. The diagram $\mathfrak{e}_B^n : B \xrightarrow{\iota} B \times \mathbb{R}^n \xrightarrow{\pi} B$ constitutes a microbundle, where $\iota(b) := (b, 0)$ and $\pi(b, x) := b$. We call \mathfrak{e}_B^n the *standard microbundle* and every microbundle isomorphic to \mathfrak{e}_B^n *trivial*.

Lemma 1.4. (criteria for triviality)

A microbundle \mathfrak{b} of B is *trivial* if and only if there exists a open subset $B \subseteq U$ with $U \cong B \times \mathbb{R}^n$.

Proof.

□

Example 1.5. (tangent microbundle)

Let M be a topological manifold. We derive the *tangent microbundle* $t_M : M \xrightarrow{\Delta} M \times M \xrightarrow{\pi_1} M$, where Δ is the diagonal map and π_1 is the projection map on the first component.

Proof. Let $p \in M$ and (U, ϕ) a chart over p :

$$\begin{array}{ccccc}
 & & U \times U & & \\
 & \nearrow i & \downarrow (id, \phi) & \nwarrow i & \\
 U & & & & U \\
 & \searrow (0, id) & \downarrow & \nearrow \pi_1 & \\
 & & U \times \mathbb{R}^d & &
 \end{array}$$

(id, ϕ) is a homeomorphism since $\phi : U \xrightarrow{\sim} \mathbb{R}^n$ is a homeomorphism. \square

Example 1.6. (underlying microbundle)

Let $\xi : E \xrightarrow{\pi} B$ be a n -dimensional vector bundle: The microbundle $|\xi| : B \xrightarrow{i} E \xrightarrow{\pi} B$ with $i(b) := \phi_b(b, 0)$, where $\phi_b : U_b \times \mathbb{R}^n \rightarrow \pi^{-1}(U_b)$ is the local trivialization over a neighborhood $U_b \subseteq B$ of b . We call $|\xi|$ the *underlying microbundle* of ξ

Proof. \square

Proposition 1.7. (restricting the total space)

Let $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ be a microbundle and E' an arbitray neighborhood of $i(B)$. The restriction $\mathfrak{b}' : B \xrightarrow{i} E' \xrightarrow{j|_{E'}} B$ is a microbundle isomorphic to \mathfrak{b} .

Proof. Let $b \in B$.

Choose an arbitray trivialization (U, V, ϕ) over \mathfrak{b} of b . We restrict $\phi : V \rightarrow U \times \mathbb{R}^n$ to $V \cap E'$. Since $i(b) \in V$ is open and E' is a neighborhood of $i(B)$, it follows that $\phi(V \cap E')$ is a neighborhood of $(b, 0)$.

$\implies \exists (b, 0) \in U' \times X \subseteq \phi(V \cap E')$, where $U' \subseteq U$ and $X \subseteq \mathbb{R}^n$ are open

$\implies \exists \varepsilon > 0 : U' \times B_\varepsilon(0) \subseteq \phi(V \cap E')$

Since $B_\varepsilon(0) \cong \mathbb{R}^n$, it follows that $U' \times \mathbb{R}^n \cong U' \times B_\varepsilon(0) \cong \phi^{-1}(U' \times B_\varepsilon(0))$. Choosing $V' := \phi^{-1}(U' \times B_\varepsilon(0)) \subseteq V$, we see that \mathfrak{b}' is a microbundle. We easily see, that \mathfrak{b} is isomorphic to \mathfrak{b}' via the identity. \square

2 Induced Microbundles

Definition 2.1. (induced microbundle)

Let $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ be a microbundle and $f : A \rightarrow B$ a continuous map. We can construct a microbundle $f^*\mathfrak{b} : A \xrightarrow{i'} E' \xrightarrow{j'} A$ defined as follows:

- $E' := \{(a, e) \in A \times E \mid f(a) = j(e)\}$
- $i' : A \rightarrow E'$ with $i'(a) := (a, (i \circ f)(a))$
- $j' : E' \rightarrow A$ with $j'(a, e) := a$

We call $f^*\mathfrak{b}$ the *induced microbundle* of \mathfrak{b} over f .

Proof. It is clear that i' and j' are continuous and that $id_A = j' \circ i'$. So it remains to be shown that $f^*\mathfrak{b}$ is locally trivial for every $a \in A$:

- $U' := f^{-1}(U) \subseteq A$ is an open neighborhood of a .
- $V' := j'^{-1}(U') \subseteq E'$ is an open neighborhood of $i'(U')$.
- $\phi' : V' \xrightarrow{\sim} U' \times \mathbb{R}^n, \phi'(a, e) := (a, \pi_2(\phi(e)))$ is a homeomorphism.
 - ϕ' is well defined because $(a, e) \in V' : j(e) = f(a) \in U \implies e \in V$.
 - ϕ' is bijective with $\phi'^{-1}(a, v) = (a, \phi^{-1}(f(a), v))$.
 - ϕ' and ϕ'^{-1} are continuous because it's components are.

□

Example 2.2. (restricted microbundle)

Let $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ be a microbundle and $A \subseteq B$: The induced microbundle $\iota^*\mathfrak{b}$ with $\iota : A \hookrightarrow B$ being the inclusion map is called the *restricted microbundle* and we write $\mathfrak{b}|_A := \iota^*\mathfrak{b}$.

Remark 2.3. In the following, we'll consider $E(\mathfrak{b}|_A)$ a subset of $E(\mathfrak{b})$. This is justified because $E(\mathfrak{b}|_A) = \{(a, e) \in A \times E(\mathfrak{b}) \mid a = j(e)\} \cong \{e \in E(\mathfrak{b}) \mid j(e) \in A\} \subseteq E(\mathfrak{b})$.

Lemma 2.4. (induced trivial microbundle)

The induced microbundle $f^*\mathfrak{b}$ is trivial for every map $f : A \rightarrow B$, if \mathfrak{b} is already trivial.

Proof. Let (V, ϕ) be a global trivialization of \mathfrak{b} , i.e $V \cong_\phi B \times \mathbb{R}^n$. Now define $V' := (A \times V) \cap E'$ and $\phi'(a, e) := (a, \phi^{(2)}(e))$. Obviously, V' is a neighborhood of $i'(A)$ and also ϕ' is a homeomorphism with inverse $\phi'^{(-1)}(a, x) = (a, \phi^{-1}(f(a), x))$. □

Proposition 2.5. (composition)

Let $A \xrightarrow{f} B \xrightarrow{g} C$ be topological spaces and $\mathfrak{c} : C \xrightarrow{i} E \xrightarrow{j} C$ be a microbundle:

$$(g \circ f)^*\mathfrak{c} \cong f^*(g^*\mathfrak{c})$$

Proof. We'll compare the two total spaces and conclude that they are homeomorphic.

1. $E((g \circ f)^*\mathfrak{c}) = \{(a, e) \in A \times E(\mathfrak{c}) \mid g(f(a)) = j(e)\}$

$$2. E(f^*(g^*\mathfrak{c})) = \{(a, (b, e)) \in A \times (B \times E(\mathfrak{c})) \mid f(a) = b \text{ and } g(b) = j(e)\}.$$

We have the bijection $\phi : E((g \circ f)^*\mathfrak{c}) \xrightarrow{\sim} E(f^*(g^*\mathfrak{c}))$ with $\phi(a, e) := (a, (f(a), e))$ and $\phi^{-1}(a, (b, e)) = (a, e)$. Additionally, ϕ is a homeomorphism because ϕ and ϕ^{-1} are componentwise continuous. It's easy to see that ϕ respects both injection and projection, which concludes the proof. \square

For a topological space X , we define the cone of X as

$$CX := X \times [0, 1] / X \times \{1\}$$

and for a map $f : A \rightarrow B$ the mapping cone of f as

$$B \sqcup_f CA := B \sqcup CA / \sim$$

where $(a, 0) \sim b : \iff f(a) = b$.

Lemma 2.6. (extending over a mapping cone)

A microbundle \mathfrak{b} over B can be extended to a microbundle over the mapping cone $B \sqcup_f CA$ if and only if $f^*\mathfrak{b}$ is trivial.

Proof. We show both implications.

$\implies :$

Let \mathfrak{b}' be an extension of \mathfrak{b} over $B \sqcup_f CA$. Considering $A \xrightarrow{f} B \hookrightarrow B \sqcup_f CA$, the composition $\iota \circ f$ is null-homotopic with homotopy

$$H_t(a) := [(a, t)]$$

Note that $H_0(a) = [(a, 0)] = [f(a)] = (\iota \circ f)(a)$ and $H_1(a) = [(a, 1)] = [(\tilde{a}, 1)] = H_1(\tilde{a})$.

$\xRightarrow{\text{Hom. Thm.}} (\iota \circ f)^*\mathfrak{b}'$ is trivial

Since $(\iota \circ f)^*\mathfrak{b}' = f^*(\iota^*\mathfrak{b}') = f^*\mathfrak{b}$, it follows that $f^*\mathfrak{b}$ is trivial. $\Leftarrow :$

Let $f^*\mathfrak{b}$ be trivial. Analogous to the cone, we define the *cylinder* of X as

$$MX := X \times [0, 1]$$

and for a map $f : A \rightarrow B$ the mapping *cylinder* of f as

$$B \sqcup_f MA := B \sqcup MA / \sim$$

where $(a, 0) \sim b : \iff f(a) = b$. In contrast to the mapping cone, there exists a natural retraction from the mapping cylinder to the attached space

$$\pi : B \sqcup_f MA \rightarrow B; \pi([(a, t)]) := f(a)$$

and therefore the induced microbundle $\pi^*\mathfrak{b}$ over $B \sqcup_f MA$. Considering $A \times \{1\} \hookrightarrow B \sqcup_f MA \xrightarrow{\pi} B$, we see that $\pi \circ \iota \cong f$ and therefore

$$\pi^*\mathfrak{b}|_{A \times \{1\}} = (\pi \circ \iota)^*\mathfrak{b} \cong f^*\mathfrak{b} = \mathfrak{c}_A^n$$

is trivial. From the lemma of induced trivial microbundles and $(a, t) \mapsto (a, 1)$ it follows that $\pi^* \mathfrak{b}|_{A \times [\frac{1}{2}, 1]}$ is trivial.

$$\implies \exists \phi : E(\mathfrak{b}|_{A \times [\frac{1}{2}, 1]}) \xrightarrow{\sim} A \times [\frac{1}{2}, 1] \times \mathbb{R}^n$$

Now we explicitly construct the desired extended microbundle $\mathfrak{b}' : B \sqcup_f CA \xrightarrow{i'} E' \xrightarrow{j'} B \sqcup_f CA$

- $E' := E(\mathfrak{b}|_{A \times [\frac{1}{2}, 1]}) / \phi^{-1}(A \times [\frac{1}{2}, 1] \times \{x\})$ (for every $x \in \mathbb{R}^n$)
- $i' := \pi \circ i$ the projection i to E'
- $j'([e]) := [j(e)]$ is well defined, because $[e] = [\tilde{e}] \implies [j(e)] = [j(\tilde{e})]$

Now that we have constructed \mathfrak{b}' , this proves the claim. \square

Corollary 2.7. (*extending over a d -simplex*)

Let B be a $(d+1)$ -simplicial complex, B' it's d -skeleton and $\Delta^{d+1} \cong \sigma \subseteq B$. A microbundle \mathfrak{b} over B' can be extended to a microbundle over $B' \cup \sigma$ if and only if $\mathfrak{b}|_{\partial\sigma}$ is trivial.

Proof. The statement follows from the last lemma:

There exists a $\phi : C\partial\sigma \xrightarrow{\sim} \sigma$ such that $\phi(\partial\sigma \times \{0\}) = \partial\sigma$. We explicitly construct $\phi((t_1, \dots, t_{d+1}), \lambda) := (1 - \lambda)(t_1, \dots, t_{d+1}) + \frac{\lambda}{d+1}(1, \dots, 1)$. It's easy to see that ϕ suffices all our requirements. By choosing $f : \partial\sigma \hookrightarrow B'$ and applying the last lemma, the statement is proven. \square

3 Whitney sums

Definition 3.1. (whitney sum)

Let \mathfrak{b}_1 and \mathfrak{b}_2 be two microbundles over a topological space B . We define the *whitney sum* $\mathfrak{b}_1 \oplus \mathfrak{b}_2$ as follows:

- $E := \{(e_1, e_2) \in E(\mathfrak{b}_1) \times E(\mathfrak{b}_2) \mid j_1(e_1) = j_2(e_2)\}$
- $i(b) := (i_1(b), i_2(b))$
- $j(e_1, e_2) := j_1(e_1) = j_2(e_2)$

Proof. Let $b \in B$.

Choose U_1, V_1, ϕ_1 and U_2, V_2, ϕ_2 accordingly from the local trivialization of b over \mathfrak{b}_1 and \mathfrak{b}_2 :

- $U := U_1 \cap U_2$
- $V := (V_1 \times V_2) \cap E$
- $\phi : V \rightarrow U \times \mathbb{R}^{n_1+n_2}; \phi(e_1, e_2) := (\phi_1^{(1)}(e_1), \phi_1^{(2)}(e_1) \times \phi_2^{(2)}(e_2))$

Note that $\phi_1^{(1)}(e_1) = \phi_2^{(1)}(e_2)$. Local triviality follows directly from it's components. \square

Lemma 3.2. (*compatibility*)

Let \mathfrak{b}_1 and \mathfrak{b}_2 be two microbundles over B and $f : A \rightarrow B$ a map. Induced microbundle and whitney sum are compatible, i.e. $f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2) \cong f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2$

Proof. From the definition of the induced microbundle and the whitney sum, we can derive the total spaces:

1. $E(f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2)) = \{(a, (e_1, e_2)) \in A \times (E_1 \times E_2) \mid j_1(e_1) = j_2(e_2) = f(a)\}$
2. $E(f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2) = \{((a_1, e_1), (a_2, e_2)) \in (A \times E_1) \times (A \times E_2) \mid j(a_1, e_1) = j(a_2, e_2) \text{ and } f(a_i) = j(e_i)\}$

Those two total spaces are homeomorphic via $\phi(a, (e_1, e_2)) := ((a, e_1), (a, e_2))$ and $\phi^{-1}((a, e_1), (a, e_2)) = (a, (e_1, e_2))$. ϕ and ϕ^{-1} are continuous because they are componentwise continuous. Obviously, $\phi \circ i = i$ and $\phi \circ j = j$, which concludes the proof. \square

Theorem 3.3. (*inverse microbundles*)

Let \mathfrak{b} be a microbundle over a d -dimensional simplicial complex B .

Then there exists a microbundle \mathfrak{n} over B so that the Whitney sum $\mathfrak{b} \oplus \mathfrak{n}$ is trivial.

Proof. We prove this theorem by induction over d .

(Start of induction)

A 1-dimensional simplicial complex is just a bouquet of circles, therefore the start of induction follows directly from the bouquet lemma.

(Inductive Step)

Let B' be the $(d-1)$ -skeleton of B and \mathfrak{n}' it's corresponding microbundle so that $\mathfrak{b}|_{B'} \oplus \mathfrak{n}'$ is trivial. \square

4 Homotopy and Microbundles

Definition 4.1. (map-germ)

Let (X, A) and (Y, B) be two topological pairs. A *map-germ* $F : (X, A) \rightrightarrows (Y, B)$ is an equivalence class of maps $f : (X, A) \rightarrow (Y, B)$, where $f \sim g : \iff f|_U = g|_U$ for some neighborhood $U \subseteq X$ of A .

ad

Remark 4.2. The composition of two map-germs $(X, A) \xrightarrow{F} (Y, B) \xrightarrow{G} (Z, C)$ is well defined.

Definition 4.3. (isomorphism-germ)

Definition 4.4. (bundle map-germ)

Lemma 4.5. (*closed balls under homeomorphism*)

For a homeomorphism $\phi : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$ applies:

$$|\phi(x) - x| < 1, \forall x \in \overline{B_2} \implies \overline{B_1} \subseteq \phi(\overline{B_2})$$

Proof. Consider the two points $x_0 := 0$ and $x_1 := 3e_1$. Obviously, there is no path between x_0 and x_1 in $\mathbb{R}^n - S^n$. Therefore, since ϕ is a homeomorphism, there is no path between $\phi(x_0)$ and $\phi(x_1)$ in $\mathbb{R}^n - \phi(S^n)$. Since $1 < |x|$ for every $x \in \phi(S^n)$, there is a path between every $x \in \overline{B_1}$ and $\phi(x_1)$ in $\mathbb{R}^n - \phi(S^n)$ (e.g a straight line). It follows that there is also no path between any $x \in \overline{B_1}$ and x_1 in $\mathbb{R}^n - \phi(S^n)$. Since $\mathbb{R}^n - \overline{B_2}$ is path-connected, $\mathbb{R}^n - \phi(\overline{B_2})$ is as well. From $x_1 \in \mathbb{R}^n - \phi(\overline{B_2})$, we know that $\overline{B_1} \subseteq \phi(\overline{B_2})$. \square

Lemma 4.6. (*Williamson*)

A bundle map-germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$ on the same base space B covering the identity map is an isomorphism-germ.

Proof. We prove the statement locally, then glue the function together for the prove.

1. First, we consider the case that \mathfrak{b} and \mathfrak{b}' are trivial. This means, that $F : B \times \mathbb{R}^n \rightarrow B \times \mathbb{R}^n$ is of the form

$$F(b, x) = (b, g_b(x))$$

where $g_b : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are individual maps. In particular, the g_b are even homeomorphisms due to the domain invariance theorem. Now we show that F is a homeomorphism. Let $(b_0, x_0) \in B \times \mathbb{R}^n$ and $\varepsilon > 0$. Since g_{b_0} is a homeomorphism, there exists a $\delta > 0$ so that $\overline{B_{2\delta}(x_1)} \subseteq g_{b_0}(\overline{B_\varepsilon(x_0)})$ where $x_1 := g_{b_0}(x_0)$.

There exists a neighborhood $V \subseteq B$ of b_0 , such that

$$|g_b(x) - g_{b_0}(x)| < \delta, \forall x \in \overline{B_\varepsilon(x_0)}$$

To show that, consider $\phi_b(b, x) := g_b(x) - g_{b_0}(x)$. The closed set $\phi^{-1}(\overline{B_\varepsilon(x_0)})$ is a neighborhood of $\{b_0\} \times \mathbb{R}^n$ since $\phi(b_0, x) = 0, \forall x \in \mathbb{R}^n$. Therefore, for every $x \in \overline{B_\delta(x_1)}$ exist $V_x \subseteq B$ and $U_x \subseteq \mathbb{R}^n$ open with $x \in U_x$ and $V_x \times U_x \subseteq \phi^{-1}(\overline{B_\varepsilon(x_0)})$. Obviously, $\bigcup_{x \in \overline{B_\delta(x_1)}} U_x$ is an open covering of $\overline{B_\delta(x_1)}$ and because of compactness of $\overline{B_\delta(x_1)}$, there exist $x_1, \dots, x_n \in \overline{B_\delta(x_1)}$ with $\overline{B_\delta(x_1)} \subseteq \bigcup_{i=1}^n U_{x_i}$. The claim follows via $V := V_{x_1} \cap \dots \cap V_{x_n}$.

From

$$V \times \overline{B_\delta(x_1)} \subseteq g(V \times \overline{B_\varepsilon(x_0)})$$

it follows that g is open.

2. Since \mathfrak{b} and \mathfrak{b}' are locally trivial, $F|_{U_b}$ are homeomorphisms on their image for every trivialization $b \in U_b$. It follows that also F is a homeomorphism on its image which completes the proof.

□

Corollary 4.7. (*induced microbundles*)

If a map $g : B \rightarrow B'$ is covered by a bundle germ $\mathfrak{b} \Rightarrow \mathfrak{b}'$ then \mathfrak{b} is isomorphic to the induced bundle $g^*\mathfrak{b}'$.

Proof.

□

Lemma 4.8. (*glueing together bundle map-germs*)

Let \mathfrak{b} be a microbundle over B and $\{B_\alpha\}$ a locally finite collection of closed sets covering B . Additionally, we are given $F_\alpha : \mathfrak{b}|_{B_\alpha} \Rightarrow \mathfrak{b}'$, a collection of bundle map-germs with $F_\alpha = F_\beta$ on $\mathfrak{b}|_{B_\alpha \cap B_\beta}$. Then there exists a bundle map-germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$ extending F_α .

Proof. Choose representative maps $f_\alpha : U_\alpha \rightarrow E'$ for F_α with U_α open. Since $F_\alpha = F_\beta$ on $\mathfrak{b}|_{B_\alpha \cap B_\beta}$, $f_\alpha = f_\beta$ for an open neighborhood $U_{\alpha\beta}$ of $B_\alpha \cap B_\beta$. We define

$$U := \{e \in E \mid j(e) \in B_\alpha \cap B_\beta \implies e \in U_{\alpha\beta}\}$$

1. U is open:

Let $e \in U$ and $j(e) \in B_\alpha \cap B_\beta$. From local finiteness there exists an open neighborhood $V \subseteq B$ of $j(e)$ with $V \subseteq B_{\alpha_1} \cap \dots \cap B_{\alpha_n}$. W.l.o.g. $V \subseteq B_\alpha \cap B_\beta$ by excluding a finite number of closed sets if necessary. Now $V_{\alpha\beta} := j^{-1}(V) \cap U_{\alpha\beta}$ is an open neighborhood of e . Since $j(e)$ can only be contained in finitely many B_α we can form the intersection of all these $V_{\alpha'\beta'}$ which, by construction, is contained in U and is open.

2. $B \subseteq U$ considering the cases $U_{\alpha\alpha}$.

Now we can define $f : U \rightarrow E'$ in the obvious way

$$f(u \in U_{\alpha\beta}) := f_\alpha(u) = f_\beta(u)$$

which is a representative map for our desired F .

□

Lemma 4.9. (*piecewise triviality*)

Let \mathfrak{b} be a microbundle over $B \times [0, 1]$ such that both $\mathfrak{b}|_{B \times [0, \frac{1}{2}]}$ and $\mathfrak{b}|_{B \times [\frac{1}{2}, 1]}$ are trivial. Then \mathfrak{b} itself is already trivial.

Proof. Since $\mathfrak{b}|_{B \times [\frac{1}{2}, 1]}$ is trivial, we can extend the identity bundle map-germ on $\mathfrak{b}|_{B \times \{\frac{1}{2}\}}$ to $\mathfrak{b}|_{B \times [\frac{1}{2}, 1]} \Rightarrow \mathfrak{b}|_{B \times \{\frac{1}{2}\}}$. Using the previous lemma, we can piece this together with the identity bundle map-germ on $\mathfrak{b}|_{B \times [0, \frac{1}{2}]}$ to

$$\mathfrak{b} \Rightarrow \mathfrak{b}|_{B \times [0, \frac{1}{2}]}$$

From the corollary it follows that $\mathfrak{b} \cong \mathfrak{b}|_{B \times [0, \frac{1}{2}]}$.

□

Lemma 4.10. *()*

Let \mathfrak{b} be a microbundle over $B \times [0, 1]$. Every $b \in B$ has a neighborhood V where $\mathfrak{b}|_{V \times [0, 1]}$ is trivial.

Proof. Let $b \in B$. For every $t \in [0, 1]$, choose a neighborhood $U_t := V_t \times (t - \varepsilon_t, t + \varepsilon_t)$ of (b, t) such that $\mathfrak{b}|_{U_t}$ is trivial. Since $\{b\} \times [0, 1]$ is compact, we can choose a finite covering of the U_t and define V to be the intersection of the corresponding V_t . Then there exists a subdivision $0 = t_0 < \dots < t_k = 1$ where the $\mathfrak{b}|_{V \times [t_i, t_{i+1}]}$ are trivial. Iteratively applying the previous lemma, it follows that $\mathfrak{b}|_{V \times [0, 1]}$ is trivial. \square

Lemma 4.11. *()*

Let \mathfrak{b} be a microbundle over $B \times [0, 1]$ where B is paracompact. Then there exists a bundle map-germ $F : \mathfrak{b} \rightarrow \mathfrak{b}|_{B \times \{1\}}$ covering the standard retraction $r : B \times [0, 1] \rightarrow B \times \{1\}$.

Proof. First, we assume a locally finite covering $\{V_\alpha\}$ of closed sets where $\mathfrak{b}|_{V_\alpha \times [0, 1]}$ is trivial. The existence of such a covering is justified by the previous lemmas. Since B is paracompact, we can choose bump functions

$$\lambda_\alpha : B \rightarrow [0, 1]$$

with $\text{supp}(\lambda_\alpha) \subseteq V_\alpha$. Additionally, we assume that

$$\max_\alpha(\lambda_\alpha(b)) = 1, \forall b \in B$$

Now we define a retraction $r_\alpha : B \times [0, 1] \rightarrow B \times [0, 1]$ with

$$r_\alpha(b, t) := (b, \max(t, \lambda_\alpha(b)))$$

We construct bundle map-germs $R_\alpha : \mathfrak{b} \rightarrow \mathfrak{b}$ covering r_α . We can divide $B \times [0, 1]$ into $A_\alpha := \text{supp}(\lambda_\alpha) \times [0, 1]$ and $A'_\alpha := \{(b, t) \mid t \geq \lambda_\alpha(b)\}$. Since $A_\alpha \subseteq V_\alpha \times [0, 1]$, $\mathfrak{b}|_{A_\alpha}$ is trivial. It follows that the identity bundle germ on $\mathfrak{b}|_{A_\alpha \cap A'_\alpha}$ can be extended to a bundle germ $\mathfrak{b}|_{A_\alpha} \Rightarrow \mathfrak{b}|_{A_\alpha \cap A'_\alpha}$. Piecing this together with the identity bundle germ $\mathfrak{b}|_{A'_\alpha}$, we obtain the desired bundle germ R_α .

Applying the well-ordering theorem, which is equivalent to the axiom of choice, we can assume an ordering of $\{V_\alpha\}$. Let $\{B_\beta\}$ be a locally finite covering of B with closed sets where B_β intersects only $V_{\alpha_1} < \dots < V_{\alpha_k}$ a finite collection. Now the composition $R_{\alpha_1} \circ \dots \circ R_{\alpha_k}$ restricts to a bundle germ $R(\beta) : \mathfrak{b}|_{B_\beta} \times [0, 1] \Rightarrow \mathfrak{b}|_{B_\beta} \times [1]$. Pieced together with the previous lemma, we obtain $R : \mathfrak{b} \times [0, 1] \rightarrow \mathfrak{b} \times [1]$ which concludes the proof. \square

Theorem 4.12. *(Homotopy Theorem)*

Let \mathfrak{b} be a microbundle of B and $f, g : A \rightarrow B$ be two maps.

$$f \simeq g \implies f^* \mathfrak{b} \cong g^* \mathfrak{b}$$

Proof. Let $H : A \times [0, 1] \rightarrow B$ be a homotopy between f and g . By the previous lemma, there exists a bundle germ $R : H^*\mathfrak{b} \Rightarrow H^*\mathfrak{b}|_{B \times [1]}$ covering the standard retraction $B \times [0, 1] \rightarrow B \times [1]$. From the composition

$$f^*\mathfrak{b} \subseteq H^*\mathfrak{b} \Rightarrow_R H^*\mathfrak{b}|_{B \times [1]} = g^*\mathfrak{b}$$

we obtain an isomorphism germ $f^*\mathfrak{b} \Rightarrow g^*\mathfrak{b}$. It follow that $f^*\mathfrak{b} \cong g^*\mathfrak{b}$. \square

5 Normal Microbundles

Definition 5.1. (normal microbundle)

Let M and N be two topological manifolds with $N \subseteq M$. We call a microbundle of the form

$$\mathfrak{n} : N \xrightarrow{\ell} U \xrightarrow{r} N$$

where $U \subseteq M$ is a neighborhood of N , a *normal microbundle* of N in M .

Definition 5.2. (product neighborhood)

Again, let M and N be two topological manifolds with $N \subseteq M$. We say that N has a *product neighborhood* in M if there exists a trivial normal microbundle of N in M .

Lemma 5.3. (criteria for product neighborhoods)

A submanifold $N \subseteq M$ has a product neighborhood if and only if there exists a neighborhood U of N with $(U, M) \cong (M \times \mathbb{R}^n, M \times 0)$.

Proof. This follows directly from the definition of normal microbundles and the criteria for trivial microbundles. \square

Definition 5.4. (composition microbundle)

Let $\mathfrak{b} : B \xrightarrow{i_b} E \xrightarrow{j_b} B$ and $\mathfrak{c} : E \xrightarrow{i_c} E' \xrightarrow{j_c} E$ be two microbundles. We define the *composition microbundle* $\mathfrak{b} \circ \mathfrak{c} : B \xrightarrow{i} E' \xrightarrow{j} B$ with $i(b) := (i_c \circ i_b)(b)$ and $j(e') := (j_b \circ j_c)(e')$

Proof. Let $b \in B$.

Choose local trivializations (U_b, V_b, ϕ_b) of b and (U_c, V_c, ϕ_c) of $j_b(b)$. From this, we construct our local trivialization over $\mathfrak{b} \circ \mathfrak{c}$. Consider $\phi_b(V_b \cap U_c)$, which is a neighborhood of $(b, 0)$. Therefore, there exist open neighborhoods $b \in U \subseteq U_b$ and $0 \in X \subseteq \mathbb{R}^n$ such that $U \times X \subseteq \phi_b(V_b \cap U_c)$. Analogous to the proof of restricting the total space in Chapter 1, it follows that

$$\exists \varepsilon > 0 : U \times B_\varepsilon(0) \subseteq \phi_b(V_b \cap U_c)$$

$$\implies U \times \mathbb{R}^n \cong U \times B_\varepsilon(0) \cong \phi_b^{-1}(U \times B_\varepsilon(0)) \cong \phi_c^{-1}(\phi_b^{-1}(U \times B_\varepsilon(0)))$$

which is an open neighborhood of $i(U)$ and therefore a valid candidate for V . This concludes local triviality and the proof. \square

Lemma 5.5. (*transitivity of normal microbundles*)

Let M, N and P be topological manifolds with $P \subseteq N \subseteq M$. There exists a normal microbundle \mathbf{n} of P in M , if there exist normal microbundles $\mathbf{n}_P : P \xrightarrow{i_P} U_N \xrightarrow{j_P} P$ in N and $\mathbf{n}_N : N \xrightarrow{i_N} U_M \xrightarrow{j_N} N$ in M .

Proof. We simply form the composition $\mathbf{n}_P \circ \mathbf{n}_N|_{U_N} : P \xrightarrow{i_N \circ i_P} U_M \xrightarrow{j_P \circ j_N} P$. Since $i_N \circ i_P$ is just the inclusion of $P \hookrightarrow U_M \subseteq M$, we found a normal microbundle \mathbf{n} of P in M . \square

Every topological manifold is an absolute neighborhood retract (ANR).

It follows that by restricting M , if necessary, to an open neighborhood of N , there exists a retraction $r : M \rightarrow N$ which we will take advantage of in the following.

Lemma 5.6. (*homeomorphism of total spaces*)

Let \mathbf{t}_N and \mathbf{t}_M be the tangent microbundles of N and M . The total space $E(\iota^*\mathbf{t}_M)$ and $E(r^*\mathbf{t}_N)$ are homeomorphic.

Proof. We explicitly construct a homeomorphism:

1. $E(\iota^*\mathbf{t}_M) = \{(n, (m_1, m_2)) \in N \times (M \times M) \mid \iota(n) = m_1\}$
2. $E(r^*\mathbf{t}_N) = \{(m, (n_1, n_2)) \in M \times (N \times N) \mid r(m) = n_1\}$

Now, we have the homeomorphism $\phi : E(\iota^*\mathbf{t}_M) \rightarrow E(r^*\mathbf{t}_N)$ with $\phi(n, (m_1, m_2)) = (m_2, (r(m_2), n))$ and $\phi^{-1}(m, (n_1, n_2)) = (n_2, (n_2, m))$. We easily see that ϕ suffices all requirements of $E(\iota^*\mathbf{t}_M)$ and $E(r^*\mathbf{t}_N)$. \square

Remark 5.7. Note that the following diagram commutes

$$\begin{array}{ccc} N & \longrightarrow & E(\iota^*\mathbf{t}_M) \\ \downarrow & & \downarrow \phi \\ M & \longrightarrow & E(r^*\mathbf{t}_N) \end{array}$$

Lemma 5.8. (*normal microbundle on total space*) There exists a normal microbundle \mathbf{n} of N in $E(r^*\mathbf{t}_N)$ with $\mathbf{n} \cong \iota^*\mathbf{t}_M$.

Proof. Obviously, $\mathbf{n} := r^*\mathbf{t}_N|_N$ is a normal microbundle of N in $E(r^*\mathbf{t}_N)$. Since $E(r^*\mathbf{t}_N|_N) \subseteq E(r^*\mathbf{t}_N)$, isomorphy follows from the previous lemma and remark. \square

Finally, we gathered all the tools to prove Milnor's theorem.

Theorem 5.9. (*Milnor*) For a sufficiently large $q \in \mathbb{N}$, $N = N \times \{0\}$ has a normal microbundle in $M \times \mathbb{R}^q$.

Proof.

□