# Thesis Title

Thesis Subtitle

**Florian Burger**

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**Abstract**

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# Chapter 1

# Introduction to Microbundles

In order to construct the tangent bundle on a manifold, we need a differential structure. However, this is generally not given for topological manifolds. In order to still have a structure "similar" to the tangent bundle on topological manifolds, we need a different, weaker, concept of the tangent bundle. Therefore we introduce so called "microbundles" which act as a weaker alternative to vector bundles. The concept of microbundles as well as some basic properties and examples are presented in this chapter.

We start with the definition of a microbundle.

**Definition 1.1** (microbundle)**.**
A *microbundle* $\mathfrak{b}$ over $B$ (with *fibre-dimension* $n$) is a diagram $B \xrightarrow{i} E \xrightarrow{j} B$ satisfying the following:

- $B$ is a topological space (*base space*)
- $E$ is a topological space (*total space*)
- $i : B \to E$ (*injection*) and $j : E \to B$ (*projection*) are continuous maps such that $id_B = j \circ i$
- Every $b \in B$ is *locally trivializable*, that is there exist open neighborhoods $U \subseteq B$ of $b$ and $V \subseteq E$ of $i(U)$ with a homeomorphism $\phi : V \xrightarrow{\sim} U \times \mathbb{R}^n$

such that the following diagram commutes:

$$\begin{tikzcd}
& V \arrow[dr, "j|_V"] \arrow[dd, "\psi"] & \\
U \arrow[ur, "i"] \arrow[dr, "(id{,}0)"'] & & U \\
& U \times \mathbb{R}^n \arrow[ur, "\pi_1"'] &
\end{tikzcd}$$

*Remark* 1.2.
In the following, unless explicitly stated otherwise we assume the fiber dimension of any given microbundle to be $n$.

Before we look at examples of microbundles, we should first clarify what it means for two microbundles to be isomorphic.

**Definition 1.3** (isomorphy)**.**
Two microbundles $\mathfrak{b}_1 : B \xrightarrow{i_1} E_1 \xrightarrow{j_1} B$ and $\mathfrak{b}_2 : B \xrightarrow{i_2} E_2 \xrightarrow{j_2} B$ are *isomorphic* if there exist neighborhoods $V_1 \subseteq E_1$ of $i_1(B)$ and $V_2 \subseteq E_2$ of $i_2(B)$ with a homeomorphism $\phi : V_1 \xrightarrow{\sim} V_2$ such that the following diagram commutes:

$$\begin{tikzcd}
& V_1 \arrow[dr, "j_1|_{V_1}"] \arrow[dd, "\phi"] & \\
B \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & B \\
& V_2 \arrow[ur, "j_2|_{V_2}"'] &
\end{tikzcd}$$

As the definition of isomorphy already indicates, when studying microbundles, we are not interested in the entire total space but only in an arbitrarily small neighborhood of the base space. This is certainly one of the strongest conceptual differences between microbundles and classical vector bundles.

**Proposition 1.4.**
*For a microbundle $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ over $B$, we can be restrict the total space $E$ to an arbitray neighborhood $E' \subseteq E$ of $i(B)$ where the resulting microbundle is isomorphic to $\mathfrak{b}$.*

*Proof.*
For an arbitray $b \in B$, choose a local trivialization $(U, V, \phi)$.

The intersection $V \cap E'$ is a neighborhood of $i(b)$ because $V$ and $E'$ both are. It follows that $\phi(V \cap E')$ is a neighborhood of $(b, 0)$. Hence there exist $U' \subseteq B$ open and $B_\varepsilon(0) \subseteq \mathbb{R}^n$ such that $U' \times B_\varepsilon(0) \subseteq \phi(V \cap E')$. Now we construct our local trivialization by choosing $V' := \phi^{-1}(U' \times B_\varepsilon(0))$ and the fact that $B_\varepsilon(0) \cong \mathbb{R}^n$:

$$U' \times \mathbb{R}^n \cong U' \times B_\varepsilon(0) \cong V'$$

We easily see that the resulting microbundle is isomorphic to $\mathfrak{b}$ via the identity. $\square$

Now that we introduced the basic concept of microbundles, we will take a look at some key examples.

The most obvious example for a microbundle is the standard microbundle.

**Example 1.5** (trivial microbundle)**.**
For a topological space $B$, the *standard microbundle* $\mathfrak{e}^n_B$ over $B$ is a diagram

$$B \xrightarrow{\iota} B \times \mathbb{R}^n \xrightarrow{\pi} B$$

where $\iota(b) := (b, 0)$ and $\pi(b, x) := b$. Additionally, a microbundle $\mathfrak{b}$ over $B$ is *trivial* if it is isomorphic to $\mathfrak{e}^n_B$.

In order to make it easier classifying microbundles as trivial, we provide a sharper description of what it means for a microbundle to be trival.

**Lemma 1.6.**
*A microbundle $\mathfrak{b}$ over a paracompact space $B$ is trivial if and only if there exists an open neighborhood $U$ of $i(B)$ such that $U \cong B \times \mathbb{R}^n$.*

*Proof.*
By applying Proposition **(1.4)**, we may assume that $E(\mathfrak{b})$ is an open subset of $B \times \mathbb{R}^n$. Since $B$ is paracompact, there exists a map $\lambda : B \to (0, 1]$ such that every $(b, x) \in B \times \mathbb{R}^n$ with $|x| < \lambda(b)$ lies in $E(\mathfrak{b})$. Now, the function

$$(b, x) \mapsto (b, \frac{x}{|x| - \lambda(b)})$$

maps the open set $\{(b, x) \mid |x| < \lambda(b)\}$ homeomorphically to $B \times \mathbb{R}^n$. By considering $\{(b, x) \mid |x| < \lambda(b)\} \subseteq E(\mathfrak{b})$, this completes the proof. $\square$

The following example acts as the microbundle analog to the tangent bundle on a smooth manifold.

**Example 1.7** (tangent microbundle)**.**
The *tangent microbundle* $\mathfrak{t}_M$ over a topological $d$-manifold $M$ is a diagram

$$M \xrightarrow{\Delta} M \times M \xrightarrow{\pi_1} M$$

where $\Delta(m) := (m, m)$ is the diagonal map and $\pi_1(m_1, m_2) := m_1$ is the projection map on the first component.

*Proof that $\mathfrak{t}_M$ is a microbundle.*
Let $p \in M$ and $(U, \phi)$ be a chart over $p$. We explicitly construct a local trivial-

ization

$$\begin{tikzcd}
 & U \times U \arrow[dr, "\pi_1"] \arrow[dd, "\psi"] & \\
U \arrow[ur, "\Delta"] \arrow[dr, "{(0,id)}"'] & & U \\
 & U \times \mathbb{R}^d \arrow[ur, "\pi_1"'] &
\end{tikzcd}$$

where $\psi(u, \tilde{u}) := (u, \phi(u) - \phi(\tilde{u}))$. It's obvious that $(U, U \times U, \psi)$ meets all local triviality conditions. $\square$

**Example 1.8** (underlying microbundle)**.**
Let $\xi : E \xrightarrow{\pi} B$ be a $n$-dimensional vector bundle. The microbundle $|\xi| : B \xrightarrow{i} E \xrightarrow{\pi} B$ where $i(b) := \phi_b(b, 0)$, where $\phi_b : U_b \times \mathbb{R}^n \to \pi^{-1}(U_b)$ is the local trivialization over a neighborhood $U_b \subseteq B$ of $b$. We call $|\xi|$ the *underlying microbundle* of $\xi$

*Proof.*
TODO $\square$

# Chapter 2

# Induced Microbundles

This Chapter introduces a central construction of microbundles. From the

**Definition 2.1** (induced microbundle)**.**
Let $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ be a microbundle and $f : A \to B$ be a continuous map. The *induced microbundle* $f^*\mathfrak{b} : A \xrightarrow{i'} E' \xrightarrow{j'} A$ is defined as follows:

- $E' := \{(a, e) \in A \times E \mid f(a) = j(e)\}$
- $i' : A \to E'$ with $i'(a) := (a, (i \circ f)(a))$
- $j' : E' \to A$ with $j'(a, e) := a$

*Proof that $f^*\mathfrak{b}$ is a microbundle.*
It is clear that $i'$ and $j'$ are continuous and that $id_A = j' \circ i'$. So it remains to be shown that $f^*\mathfrak{b}$ is locally trivial.

Choose a local trivialization $(U, V, \phi)$ for an arbitrary $a \in A$.

- $U' := f^{-1}(U) \subseteq A$ neighborhood of $a$.
- $V' := j'^{-1}(U') \subseteq E'$ neighborhood of $i'(a)$.
- $\phi' : V' \xrightarrow{\sim} U' \times \mathbb{R}^n, \phi'(a, e) := (a, \pi_2(\phi(e)))$.

The map $\phi'$ is well-defined because $(a, e) \in V' : j(e) = f(a) \in U \implies e \in V$. The existence of an inverse $\phi'^{-1}(a, v) = (a, \phi^{-1}(f(a), v))$ and component-wise continuity show that $\phi'$ is a homeomorphism. This proves that $(U', V', \phi')$ is a local trivialization for $a$. $\square$

**Example 2.2** (restricted microbundle)**.**
Let $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ be a microbundle and $A \subseteq B$ be a subspace. The *restricted*

*microbundle* $\mathfrak{b}|_A$ is the induced microbundle $\iota^*\mathfrak{b}$ where $\iota : A \hookrightarrow B$ is the inclusion map.

*Remark* 2.3. In the following we consider $E(\mathfrak{b}|_A)$ a subset of $E(\mathfrak{b})$. This is justified because there exists an embedding

$$\iota : E(\mathfrak{b}|_A) \to E(\mathfrak{b}), (a, e) \mapsto e$$

into the total space $E(\mathfrak{b})$.

**Lemma 2.4.**
*Let $\mathfrak{b}$ be a microbundle over $B$ and $f : A \to B$ be a map. The induced microbundle $f^*\mathfrak{b}$ is trivial if $\mathfrak{b}$ is already trivial.*

*Proof.*
Let $(V, \phi)$ be a global trivialization, so $\phi : V \xrightarrow{\sim} B \times \mathbb{R}^n$. We define

- $V' := (A \times V) \cap E(f^*\mathfrak{b})$ a neighborhood of $i'(A)$.
- $\phi' : V' \xrightarrow{\sim} B \times \mathbb{R}^n, (a, e) \mapsto (a, \pi_2(\phi(e)))$.

The existence of an inverse $\phi'^{(-1)}(a, x) = (a, \phi^{-1}(f(a), x))$ and component-wise continuity show that $\phi'$ is a homeomorphism. This proves that $(V', \phi')$ is a global trivialization for $f^*\mathfrak{b}$. $\square$

**Lemma 2.5.**
*For a diagram $A \xrightarrow{f} B \xrightarrow{g} C$ and a microbundle $\mathfrak{c} : C \xrightarrow{i} E \xrightarrow{j} C$ applies:*

$$(g \circ f)^*\mathfrak{c} \cong f^*(g^*\mathfrak{c})$$

*Proof.*
To prove isomorphy, we need to show that the two total spaces are homeomorphic and that the injection and projection maps commute with such a homeomorphism.

First, compare the two total spaces:

1. $E((g \circ f)^*\mathfrak{c}) = \{(a, e) \in A \times E(\mathfrak{c}) \mid g(f(a)) = j(e)\}$
2. $E(f^*(g^*\mathfrak{c})) = \{(a, (b, e)) \in A \times (B \times E(\mathfrak{c})) \mid f(a) = b \text{ and } g(b) = j(e)\}$.

We have the bijection $\phi : E((g \circ f)^*\mathfrak{c}) \xrightarrow{\sim} E(f^*(g^*\mathfrak{c}))$ with $\phi(a, e) := (a, (f(a), e))$ and $\phi^{-1}(a, (b, e)) = (a, e)$. Since $\phi$ and $\phi^{-1}$ are component-wise continuous, it follows that $\phi$ is a homeomorphism.

It's easy to see that $\phi$ commutes with injection and projection maps, which concludes the proof. $\square$

For a topological space $X$, we define the *cone* of $X$ to be

$$CX := X \times [0, 1]/X \times \{1\}$$

and for a map $f : A \to B$ the *map cone* of $f$ to be

$$B \sqcup_f CA := B \sqcup CA/\sim$$

where $(a, 0) \sim b :\iff f(a) = b$.

Similarly, we define the *cylinder* of $X$ to be

$$MX := X \times [0, 1]$$

and for a map $f : A \to B$ the *map cylinder* of $f$ to be

$$B \sqcup_f MA := B \sqcup CA/\sim$$

where $(a, 0) \sim b :\iff f(a) = b$.

**Lemma 2.6.**
*A microbundle $\mathfrak{b}$ over $B$ can be extended to a microbundle over the map cone $B \sqcup_f CA$ if and only if $f^*\mathfrak{b}$ is trivial.*

*Proof.*
We show both implications.

"$\implies$"

Let $\mathfrak{b}'$ be an extension of $\mathfrak{b}$ over $B \sqcup_f CA$. Considering $A \xrightarrow{f} B \hookrightarrow B \sqcup_f CA$, the composition $\iota \circ f$ is null-homotopic with homotopy

$$H_t(a) := [(a, t)]$$

Note that $H_0(a) = [(a, 0)] = [f(a)] = (\iota \circ f)(a)$ and $H_1(a) = [(a, 1)] = [(\tilde{a}, 1)] = H_1(\tilde{a})$. From the Homotopy Theorem **(4.1)** follows that $(\iota \circ f)^*\mathfrak{b}'$ is trivial.

Since $(\iota \circ f)^*\mathfrak{b}' = f^*(\iota^*\mathfrak{b}') = f^*\mathfrak{b}$, it follows that $f^*\mathfrak{b}$ is trivial.

"$\impliedby$"

Let $f^*\mathfrak{b}$ be trivial.

In contrast to the map cone, there exists a natural retraction from the map cylinder to the attached space

$$r : B \sqcup_f MA \to B, r([(a, t)]) := f(a)$$

The diagram

$$A \times \{1\} \hookrightarrow B \sqcup_f MA \xrightarrow{r} B$$

equals $f$ if we consider $A = A \times \{1\}$. It follows that

$$r^*\mathfrak{b}|_{A\times\{1\}} = (r \circ \iota)^*\mathfrak{b} \cong f^*\mathfrak{b} = \mathfrak{e}^n_A$$

is trivial. From Lemma **(2.4)** and $(a, t) \mapsto (a, 1)$ it follows that $r^*\mathfrak{b}|_{A\times[\frac{1}{2},1]}$ is trivial, so there exists a

$$\phi : E(r^*\mathfrak{b}|_{A\times[\frac{1}{2},1]}) \xrightarrow{\sim} A \times [\frac{1}{2}, 1] \times \mathbb{R}^n$$

Now we explicitly construct our desired extended microbundle $\mathfrak{b}' : B \sqcup_f CA \xrightarrow{i'} E' \xrightarrow{j'} B \sqcup_f CA$

- $E' := E(r^*\mathfrak{b})/\phi^{-1}(A \times \{1\} \times \{x\})$ (for every $x \in \mathbb{R}^n$).
- $i'([a,t]) := [i_r(a,t)]$ where $i_r$ is the injection map for $r^*\mathfrak{b}$.
- $j'([e]) := [j_r(e)]$ where $j_r$ is the projection map for $r^*\mathfrak{b}$.

The injection $i'$ is well-defined because $i_r$ maps every representative $[a,1]$ to the same equivalence class of $E'$. Similarly, the projection $j'$ is well-defined since $[e] = [\tilde{e}] \implies [j_r(e)] = [j_r(e')]$ We easily derive the microbundle conditions from $r^*\mathfrak{b}$.

This proves the claim. $\square$

**Corollary 2.7.**
*Let $B$ be a $(d+1)$-simplicial complex, $B'$ it's $d$-skeleton and $\sigma \subseteq B$ a $(d+1)$-simplex. A microbundle $\mathfrak{b}$ over $B'$ can be extended to a microbundle over $B' \cup \sigma$ if and only if its restriction to the boundary $\mathfrak{b}|_{\partial\sigma}$ is trivial.*

*Proof.*
By choosing $f : \partial\sigma \hookrightarrow B'$ and applying the previous lemma, we see that there exists a microbundle $\mathfrak{b}'$ over $B' \cup_f C\sigma$ extending $\mathfrak{b}$.

Now, consider the homeomorphism $\phi : C\partial\sigma \xrightarrow{\sim} \sigma$ with

$$\phi((t_1, \ldots, t_{d+1}), \lambda) := (1-\lambda)(t_1, \ldots, t_{d+1}) + \frac{\lambda}{d+1}(1, \ldots, 1)$$

In particular, $\phi(\partial\sigma \times \{0\}) = \partial\sigma$.

It follows that $B' \cup_f C\sigma \cong B' \cup \sigma$ which concludes the proof. $\square$

# Chapter 3

# Whitney sums

**Definition 3.1.**
Let $\mathfrak{b}_1$ and $\mathfrak{b}_1$ be two microbundles over $B$ with fibre-dimension $n_1$ and $n_2$. The *whitney sum* $\mathfrak{b}_1 \oplus \mathfrak{b}_2$ is a microbundle $B \xrightarrow{i} E \xrightarrow{j} B$ where

- $E := \{(e_1, e_2) \in E(\mathfrak{b}_1) \times E(\mathfrak{b}_2) \mid j_1(e_1) = j_2(e_2)\}$
- $i(b) := (i_1(b), i_2(b))$
- $j(e_1, e_2) := j_1(e_1) = j_2(e_2)$

with fibre-dimension $n_1 + n_2$.

*Proof.*
Let $b \in B$.
Choose $U_1, V_1, \phi_1$ and $U_2, V_2, \phi_2$ accordingly from the local trivialization of $b$ over $\mathfrak{b}_1$ and $\mathfrak{b}_2$:

- $U := U_1 \cap U_2$
- $V := (V_1 \times V_2) \cap E$
- $\phi : V \to U \times \mathbb{R}^{n_1+n_2}; \phi(e_1, e_2) := (\phi_1^{(1)}(e_1), \phi_1^{(2)}(e_1) \times \phi_2^{(2)}(e_2))$

Note that $\phi_1^{(1)}(e_1) = \phi_2^{(1)}(e_2)$. Local triviality follows directly from it's components. □

**Lemma 3.2.**
*Let $\mathfrak{b}_1$ and $\mathfrak{b}_1$ be two microbundles over $B$ and $f : A \to B$ a map. Induced microbundle and whitney sum are compatible, i.e. $f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2) \cong f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2$*

*Proof.*
From the definition of the induced microbundle and the whitney sum, we can derive the total spaces:

1. $E(f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2)) = \{(a,(e_1,e_2)) \in A \times (E_1 \times E_2) \mid j_1(e_1) = j_2(e_2) = f(a)\}$
2. $E(f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2) = \{((a_1,e_1),(a_2,e_2)) \in (A \times E_1) \times (A \times E_2) \mid j(a_1,e_1) = j(a_2,e_2) \text{ and } f(a_i) = j(e_i)\}$

Those two total spaces are homeomorphic via $\phi(a,(e_1,e_2)) := ((a,e_1),(a,e_2))$ and $\phi^{-1}((a,e_1),(a,e_2)) = (a,(e_1,e_2))$. $\phi$ and $\phi^{-1}$ are continuous because they are componentwise continuous. Obviously, $\phi \circ i = i$ and $\phi \circ j = j$, which concludes the proof. $\square$

TODO — BOUQUET LEMMA

**Theorem 3.3.**
*Let $\mathfrak{b}$ be a microbundle over a $d$-dimensional simplicial complex $B$. Then there exists a microbundle $\mathfrak{n}$ over $B$ so that the Whitney sum $\mathfrak{b} \oplus \mathfrak{n}$ is trivial.*

*Proof.*
We prove this theorem by induction over $d$.

(Start of induction)

A 1-dimensional simplicial complex is just a bouquet of circles, therefore the start of induction follows directly from the **??**.

(Inductive Step)

Let $B'$ be the $(d-1)$-skeleton of $B$ and $\mathfrak{n}'$ it's corresponding microbundle so that $\mathfrak{b}|_{B'} \oplus \mathfrak{n}'$ is trivial.

1. $\mathfrak{n}' \oplus \mathfrak{e}^n_{B'}$ can be extended over every $d$-simplex $\sigma$:

   Consider the following:

   $$(\mathfrak{n}' \oplus \mathfrak{e}^n_{B'})|_{\partial\sigma} = \mathfrak{n}'|_{\partial\sigma} \oplus \mathfrak{e}^n_{B'}|_{\partial\sigma} = \mathfrak{n}'|_{\partial\sigma} \oplus \mathfrak{b}|_{\partial\sigma} = (\mathfrak{n}' \oplus \mathfrak{b}|_{B'})|_{\partial\sigma}$$

   Since $(\mathfrak{n}' \oplus \mathfrak{b}|_{B'})|_{\partial\sigma}$ is trivial, the claim follows from **??**.

2. $\mathfrak{n}' \oplus \mathfrak{e}^n_{B'}$ can be extended over $B$:

   The difficulty is that the individual $d$-simplices are not well-seperated. Let $B''$ denote $B$ with small open $d$-cells cut out from every $d$-simplex. Since $B'$ is a retract of $B''$ we can extend $\mathfrak{n}' \oplus \mathfrak{e}^n_{B'}$ over $B''$ and now apply the first statement. We denote the resulting microbundle by $\eta$.

3. Consider the map-cone $B \sqcup CB'$ over the inclusion $B' \hookrightarrow B$. Since

   $$(\mathfrak{b} \oplus \eta)|_{B'} = \mathfrak{b}|_{B'} \oplus \eta|_{B'} = \mathfrak{b}|_{B'} \oplus (\mathfrak{n}' \oplus \mathfrak{e}^n_{B'}) = (\mathfrak{b}|_{B'} \oplus \mathfrak{n}') \oplus \mathfrak{e}^n_{B'} = \mathfrak{e}^n_{B'} \oplus \mathfrak{e}^n_{B'}$$

   which is trivial, by **??** we can extend $\mathfrak{b} \oplus \eta$ over $B \sqcup CB'$ denoted by $\xi$. However, $B \sqcup CB'$ has the homotopy type of a bouquet of spheres and by the **??** there exists a microbundle $\mathfrak{n}$ such that $(\xi \oplus \mathfrak{n})|_B$ is trivial.

   $$\mathfrak{e}^n_B = (\xi \oplus \mathfrak{n})|_B = \xi|_B \oplus \mathfrak{n}|_B = (\mathfrak{b} \oplus \eta) \oplus \mathfrak{n}|_B = \mathfrak{b} \oplus (\eta \oplus \mathfrak{n}|_B)$$

   This conlcudes the proof.

$\square$

# Chapter 4

# The Homotopy Theorem

In this chapter we will prove the homotopy theorem. It states the following:

**Theorem 4.1** (Homotopy Theorem)**.**
*Let $\mathfrak{b}$ be a microbundle over $B$ and $f, g : A \to B$ be two maps. If $f$ and $g$ are homotopic, then $f^*\mathfrak{b}$ and $g^*\mathfrak{b}$ are isomorphic.*

Before we can start with the proof of the theorem, we need additional concepts to put microbundles in relation to each other.

**Definition 4.2** (map-germ)**.**
A *map-germ* $F : (X, A) \Rightarrow (Y, B)$ between topological pairs $(X, A)$ and $(Y, B)$ is an equivalence class of maps $(X, A) \to (Y, B)$ where $f \sim g :\iff f|_U = g|_U$ for some neighborhood $U \subseteq X$ of $A$.

A *homeomorphism-germ* $F : (X, A) \Rightarrow (Y, B)$ is a map-germ such that there exists a representative $f : U_f \to Y$ that maps homeomorphically to a neighborhood of $B$.

Now consider two isomorphic microbundles $\mathfrak{b}$ and $\mathfrak{b}'$ over $B$. There exists a homeomorphism $\phi : V \xrightarrow{\sim} V'$ where $V \subseteq E$ is a neighborhood of $i(B)$ and $V' \subseteq E'$ is a neighborhood of $i'(B)$. The homeomorphism $\phi$ is a representative for a homeomorphism-germ

$$[\phi] : (E, i(B)) \Rightarrow (E', i'(B)).$$

Studying isomorphy between microbundles in this way is useful because we don't care what such a homeomorphism does on particular neighborhoods of the base spaces but only what it does on arbitray small ones. Hence every representative of $[\phi]$ describes the "same" isomorphy between $\mathfrak{b}$ and $\mathfrak{b}'$. Now, naturally, the question arises whether the existence of a homeomorphism-germ

$$F : (E, i(B)) \Rightarrow (E', i'(B))$$

already implies that $\mathfrak{b}$ and $\mathfrak{b}'$ are isomorphic. The answer is generally no, because isomorphy of microbundles requires a homeomorphism that **commutes with injection and projection maps**. Therefore, we must assume an extra condition called "fibre-preservation" for this implication to be true. This justifies the following definition.

Let $\mathfrak{b}$ and $\mathfrak{b}'$ be two microbundles over $B$ and let $J : (E, i(B)) \Rightarrow (B, B)$ and $J' : (E', i(B)) \Rightarrow (B, B)$ denote the map-germs represented its projection maps.

**Definition 4.3** (isomorphism-germ)**.**
An *isomorphism-germ* between $\mathfrak{b}$ and $\mathfrak{b}'$ is a homeomorphism-germ

$$F : (E, B) \Rightarrow (E', B)$$

which is *fibre-preserving*, that is $J' \circ F = J$.

*Remark* 4.4.
There exists an isomorphism-germ between $\mathfrak{b}$ and $\mathfrak{b}'$ if and only if $\mathfrak{b}$ is isomorphic to $\mathfrak{b}'$.

We can take this even further by giving up on the assumption that the base spaces of the considered microbundles equal. Note that in this case no comparison to isomorphy can be drawn, since we have not defined isomorphy between microbundles over different base spaces.

**Definition 4.5** (bundle-germ)**.**
Let $\mathfrak{b}$ and $\mathfrak{b}'$ be two microbundles with the same fibre-dimension. A *bundle-germ* $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$ is a map-germ

$$F : (E, B) \Rightarrow (E', B')$$

such that there exists a representative $f : U_f \to E'$ that maps each fibre $j^{-1}(b)$ injectively to a fibre $j'^{-1}(b')$.

For a bundle-germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$, the following diagram commutes:

$$\begin{tikzcd} (E, B) \arrow[r, Rightarrow, "F"] \arrow[d, "i"] & (E', B') \arrow[d, "i'"] \\ B \arrow[r, "F|_B"] & B' \end{tikzcd}$$

We say $F$ *is covered by* $F|_B$.

The bundle-germ is indeed a generalization of the isomorphism germ, as the following proposition illustrates.

**Proposition 4.6** (Williamson)**.**
*Let $\mathfrak{b}$ and $\mathfrak{b}'$ be two microbundles over $B$. A bundle-germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$ covering the identity map is an isomorphism-germ.*

First, however, we show a lemma that helps us to prove the proposition.

**Lemma 4.7.**
*If a homeomorphism* $\phi : \mathbb{R}^n \xrightarrow{\sim} \phi(\mathbb{R}^n) \subseteq \mathbb{R}^n$ *satisfies*

$$|\phi(x) - x| < 1, \forall x \in \overline{B_2(0)}$$

*then* $\overline{B_1(0)} \subseteq \phi(\overline{B_2(0)})$.

*Proof of the lemma.*
Consider $\phi(2S^n)$ where $2S^n$ denotes the $n$-sphere of radius 2. The condition for $\phi$ yields $1 < |\phi(s)|, \forall s \in 2S^n$. Since $\overline{B_2(0)}$ has trivial homology groups which are preserved under homeomorphisms, $\phi(\overline{B_2(0)})$ must have trivial homology groups as well.

From this we can conclude that $\overline{B_1(0)}$ must be contained in $\phi(\overline{B_2(0)})$, because otherwise "holes" would form which would result in non-trivial homology groups of $\phi(\overline{B_2(0)})$. $\square$

*Proof of the proposition.*
Let $f$ be a representative for $F$. We show the proposition in two steps.

1. Assume $f$ to map from $B \times \mathbb{R}^n$ to $B \times \mathbb{R}^n$.

   Hence $f$ is of the form
   $$f(b, x) = (b, g_b(x))$$
   where $g_b : \mathbb{R}^n \to \mathbb{R}^n$ are individual maps. Since the $g_b$ are continuous and injective, it follows from the [domain invariance theorem] that the $g_b$ are open maps. Let $(b_0, x_0) \in B \times \mathbb{R}^n$ and $\varepsilon > 0$. Since $g_{b_0}$ is an open map, there exists a $\delta > 0$ such that $\overline{B_{2\delta}(x_1)} \subseteq g_{b_0}(\overline{B_\varepsilon(x_0)})$ where $x_1 := g_{b_0}(x_0)$.

   There exists a neighborhood $V \subseteq B$ of $b_0$ such that
   $$|g_b(x) - g_{b_0}(x)| < \delta$$
   for every $b \in V$ and $x \in \overline{B_\varepsilon(x_0)}$. To show that, consider $\phi(b, x) := g_b(x) - g_{b_0}(x)$. The closed set $\phi^{-1}(\overline{B_\delta(0)})$ is a neighborhood of $\{b_0\} \times \mathbb{R}^n$ since $\phi(b_0, x) = 0$. Therefore, for every $x \in \overline{B_\delta(0)}$ exist $V_x \subseteq B$ and $U_x \subseteq \mathbb{R}^n$ open with $x \in U_x$ and $V_x \times U_x \subseteq \phi^{-1}(\overline{B_\varepsilon(x_0)})$. Obviously, $\bigcup_{x \in \overline{B_\delta(x_1)}} U_x$ is an open covering of $\overline{B_\delta(x_1)}$ and since $\overline{B_\delta(x_1)}$ is compact, there exist $x_1, \dots, x_n \in \overline{B_\delta(x_1)}$ with $\overline{B_\delta(x_1)} \subseteq \bigcup_{i=1}^n U_{x_i}$. The claim follows via $V := V_{x_1} \cap \dots \cap V_{x_n}$.

   Now we want to apply the previous lemma:

   Consider the homeomorphism $g_b \circ g_{b_0}^{-1}$ for an arbitrary $b \in V$. Since
   $$\overline{B_{2\delta}(x_1)} \subseteq g_{b_0}(\overline{B_\varepsilon(x_0)}) \implies g_{b_0}^{-1}(\overline{B_{2\delta}(x_1)}) \subseteq \overline{B_\varepsilon(x_0)}$$
   we conclude from the above that
   $$|(g_b \circ g_{b_0}^{-1})(x) - x| < \delta$$

It follows that, by translation and scaling, $g_b \circ g_{b_0}^{-1}$ satisfies the requirements of the lemma. Therefore, $\overline{B_\delta(x_1)} \subseteq (g_b \circ g_{b_0}^{-1})(\overline{B_{2\delta}(x_0)})$ and so $\overline{B_\delta(x_1)} \subseteq g_b(\overline{B_\varepsilon(x_0)})$.

From

$$V \times \overline{B_\delta(x_1)} \subseteq g(V \times \overline{B_\varepsilon(x_0)})$$

it follows that $f$ is an open map.

2. Glue together $f$ from its local trivializations.

   Choose a local trivialization $(U, V, \phi)$ over $b \in B$. First, we restrict $f$ to $f^{-1}(V)$. Since $f^{-1}(V)$ is a neighborhood of $i(b)$, we can choose an open neighborhood $V' \subseteq f^{-1}(V) \cap V$ of $i(b)$ of the form $U' \times B_\varepsilon(0)$. Now we have

   $$U' \times \mathbb{R}^n \cong U' \times B_\varepsilon(0) \xrightarrow{f} U' \times \mathbb{R}^n \subseteq U \times \mathbb{R}^n$$

   a map $U' \times \mathbb{R}^n \to U' \times \mathbb{R}^n$ that is injective and fibre-preserving and therefore an open map (apply 1.). It follows that $f : V' \to V$ must be an open map as well.

   By glueing the $V'$ over all $b \in B$ together, we see that $f$ is an open map which concludes the proof.

□

**Corollary 4.8.**
*If a map $g : B \to B'$ is covered by a bundle germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$, then $\mathfrak{b}$ is isomorphic to the induced bundle $g^*\mathfrak{b}'$.*

*Proof.*
Let $f : U_f \to E'$ be a representative map for $F$. We define $F' : \mathfrak{b} \Rightarrow g^*\mathfrak{b}'$ with a representative $f'$ as follows:

$$f' : U_f \to E(g^*\mathfrak{b}'), f'(e) := (j(e), f(e))$$

The element $f'(e)$ actually lies in $E(g^*\mathfrak{b}')$ because

$$g(j(e)) = j'(f(e))$$

as we can see from the commutative diagram for bundle-germs. Applying the previous proposition proves the claim. □

**Lemma 4.9.**
*Let $\mathfrak{b}$ be a microbundle over $B$ and $\{B_\alpha\}$ a locally finite collection of closed sets covering $B$. Additionally, we are given a collection of bundle map-germs $F_\alpha : \mathfrak{b}|_{B_\alpha} \Rightarrow \mathfrak{b}'$ such that $F_\alpha = F_\beta$ on $\mathfrak{b}|_{B_\alpha \cap B_\beta}$. Then there exists a bundle map-germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$ extending $F_\alpha$.*

*Proof.*
Choose representative maps $f_\alpha : U_\alpha \to E'$ for $F_\alpha$ such that the $U_\alpha$ are open. For every $\alpha$ and $\beta$, choose a neighborhood $U_{\alpha\beta}$ of $i(B_\alpha \cap B_\beta)$ on which the representative maps $f_\alpha$ and $f_\beta$ agree. Now consider

$$U := \{e \in E \mid j(e) \in B_\alpha \cap B_\beta \implies e \in U_{\alpha\beta}\}$$

which satisfies the following:

1. $U$ is open.

   We can express $U$ like this:

   $$E - \bigcup_{\alpha\beta} \{j^{-1}(B_\alpha \cap B_\beta) \cap U^c_{\alpha\beta}\}$$

   Since $j^{-1}(B_\alpha \cap B_\beta)$ and $U^c_{\alpha\beta}$ are closed sets, $U$ must be open. That is because an open set remains open after removing arbitrary closed sets.

2. $i(B) \subseteq U$.

   This follows from

   $$b \in B_\alpha \cap B_\beta \implies i(b) \in i(B_\alpha \cap B_\beta) \subseteq U_{\alpha\beta}$$

   and $j(i(b)) = b$.

With the construction of $U$, we can define $f : U \to E'$ in the obvious way

$$f(u \in U_{\alpha\beta}) := f_\alpha(u) = f_\beta(u)$$

which is a representative map for our desired $F$. $\square$

**Lemma 4.10.**
*Let $\mathfrak{b}$ be a microbundle over $B \times [0, 1]$. If $\mathfrak{b}|_{B\times[0,\frac{1}{2}]}$ and $\mathfrak{b}|_{B\times[\frac{1}{2},1]}$ are both trivial, then $\mathfrak{b}$ itself is trivial.*

*Proof.*
Consider the identity bundle-germ over $\mathfrak{b}|_{B\times\{\frac{1}{2}\}}$, that is the bundle-germ represented by the identity on $E(\mathfrak{b}|_{B\times\{\frac{1}{2}\}})$. Since $\mathfrak{b}|_{B\times[\frac{1}{2},1]}$ is trivial, we can extend this bundle-germ to

$$\mathfrak{b}|_{B\times[\frac{1}{2},1]} \Rightarrow \mathfrak{b}|_{B\times\{\frac{1}{2}\}}$$

via the representative

$$B \times [\frac{1}{2}, 1] \times \mathbb{R}^n \to B \times \{\frac{1}{2}\} \times \mathbb{R}^n, (b, t, x) \mapsto (b, \frac{1}{2}, x)$$

Using the previous lemma, we can piece this together with the identity bundle-germ on $\mathfrak{b}|_{B\times[0,\frac{1}{2}]}$ resulting in a bundle-germ

$$\mathfrak{b} \Rightarrow \mathfrak{b}|_{B\times[0,\frac{1}{2}]}$$

The previous corollary infers that $\mathfrak{b}$ is isomorphic to $\mathfrak{b}|_{B\times[0,\frac{1}{2}]}$. $\square$

**Lemma 4.11.**
*Let $\mathfrak{b}$ be a microbundle over $B \times [0,1]$. Every $b \in B$ has a neighborhood $V$ where $\mathfrak{b}|_{V \times [0,1]}$ is trivial.*

*Proof.*
Let $b \in B$. For every $t \in [0,1]$, choose a neighborhood $U_t := V_t \times (t - \varepsilon_t, t + \varepsilon_t)$ of $(b,t)$ such that $\mathfrak{b}|_{U_t}$ is trivial. Since $\{b\} \times [0,1]$ is compact, we can choose a finite subcover of $\{b\} \times [0,1]$ and define $V$ to be the intersection of the corresponding $V_t$. Now there exists a subdivision $0 = t_0 < \cdots < t_k = 1$ where the $\mathfrak{b}|_{V \times [t_i, t_{i+1}]}$ are trivial. By iteratively applying the previous lemma, it follows that $\mathfrak{b}|_{V \times [0,1]}$ is trivial. □

**Lemma 4.12.**
*Let $\mathfrak{b}$ be a microbundle over $B \times [0,1]$ where $B$ is paracompact. Then there exists a bundle map-germ $F : \mathfrak{b} \to \mathfrak{b}|_{B \times \{1\}}$ covering the standard retraction $r : B \times [0,1] \to B \times \{1\}$ .*

*Proof.*
First, assume a locally finite covering $\{V_\alpha\}$ of closed sets where $\mathfrak{b}|_{V_\alpha \times [0,1]}$ is trivial. The existence of such a covering is justified by paracompactness of $B$ and the previous lemmas. Now choose a partition of unity

$$\lambda_\alpha : B \to [0,1]$$

with $\mathrm{supp}(\lambda_\alpha) \subseteq V_\alpha$ that is rescaled so that

$$\max_\alpha(\lambda_\alpha(b)) = 1, \forall b \in B$$

Now we define a retraction $r_\alpha : B \times [0,1] \to B \times [0,1]$ with

$$r_\alpha(b,t) := (b, \max(t, \lambda_\alpha(b)))$$

In the following, we will construct bundle-germs $R_\alpha : \mathfrak{b} \Rightarrow \mathfrak{b}$ covering $r_\alpha$ and piece them together to obtain the desired bundle-germ.

1. We can divide $B \times [0,1]$ into two subsets

$$A_\alpha := \mathrm{supp}(\lambda_\alpha) \times [0,1]$$

$$A'_\alpha := \{(b,t) \mid t \geq \lambda_\alpha(b)\}$$

   Since $A_\alpha \subseteq V_\alpha \times [0,1]$, we already know that $\mathfrak{b}|_{A_\alpha}$ is trivial. Similar to Lemma **(4.11)**, we can extend the identity bundle-germ on $\mathfrak{b}|_{A_\alpha \cap A'_\alpha}$ to a bundle-germ

$$\mathfrak{b}|_{A_\alpha} \Rightarrow \mathfrak{b}|_{A_\alpha \cap A'_\alpha}$$

   Piecing this together with the identity bundle germ $\mathfrak{b}|_{A'_\alpha}$, we obtain our desired bundle germ $R_\alpha$.

2. Applying the well-ordering theorem, which is equivalent to the axiom of choice, we can assume an ordering of $\{V_\alpha\}$. Let $\{B_\beta\}$ be a locally finite covering of $B$ with closed sets where $B_\beta$ intersects only $V_{\alpha_1} < \cdots < V_{\alpha_k}$, a finite collection. Again, the existence of such a collection is guaranteed by the paracompactness of $B$. Now the composition $R_{\alpha_1} \circ \ldots \circ R_{\alpha_k}$ restricts to a bundle germ $R(\beta) : \mathfrak{b}|_{B_\beta} \times [0,1] \Rightarrow \mathfrak{b}|_{B_\beta} \times [1]$. Pieced together using Lemma **(4.10)**, we obtain $R : \mathfrak{b} \times [0,1] \to \mathfrak{b} \times [1]$ which concludes the proof.

□

Finally, we gathered all the tools to proof the homotopy theorem.

*Proof of the homotopy theorem.*
Let $H : A \times [0,1] \to B$ be a homotopy between $f$ and $g$. The previous lemma states that there exists a bundle germ $R : H^*\mathfrak{b} \Rightarrow H^*\mathfrak{b}|_{B\times[1]}$ covering the standard retraction $B \times [0,1] \to B \times [1]$. From the composition

$$f^*\mathfrak{b} \subseteq H^*\mathfrak{b} \Rightarrow_R H^*\mathfrak{b}|_{B\times[1]} = g^*\mathfrak{b}$$

we obtain an isomorphism germ $f^*\mathfrak{b} \Rightarrow g^*\mathfrak{b}$. It follows that $f^*\mathfrak{b} \cong g^*\mathfrak{b}$. □

# Chapter 5

# Microbundles over a Suspension

# Chapter 6

# Normal Microbundles

**Definition 6.1.** (normal microbundle)
Let $M$ and $N$ be two topological manifolds with $N \subseteq M$. We call a microbundle of the form

$$\mathfrak{n} : N \xrightarrow{\iota} U \xrightarrow{r} N$$

where $U \subseteq M$ is a neighborhood of $N$, a *normal microbundle* of $N$ in $M$.

**Definition 6.2.** (product neighborhood)
Again, let $M$ and $N$ be two topological manifolds with $N \subseteq M$. We say that $N$ has a *product neighborhood* in $M$ if there exists a trivial normal microbundle of $N$ in $M$.

**Lemma 6.3.** (criteria for product neighborhoods)
*A submanifold $N \subseteq M$ has a product neighborhood if and only if there exists a neighborhood $U$ of $N$ with $(U, M) \cong (M \times R^n, M \times 0)$.*

*Proof.*
This follows directly from the definition of normal microbundles and the criteria for trivial microbundles. □

**Definition 6.4.** (composition microbundle)
Let $\mathfrak{b} : B \xrightarrow{i_\mathfrak{b}} E \xrightarrow{j_\mathfrak{b}} B$ and $\mathfrak{c} : E \xrightarrow{i_\mathfrak{c}} E' \xrightarrow{j_\mathfrak{c}} E$ be two microbundles. We define the *composition microbundle* $\mathfrak{b} \circ \mathfrak{c} : B \xrightarrow{i} E' \xrightarrow{j} B$ with $i(b) := (i_\mathfrak{c} \circ i_\mathfrak{b})(b)$ and $j(e') := (j_\mathfrak{b} \circ j_\mathfrak{c})(e')$

*Proof.*
Let $b \in B$.
Choose local trivializations $(U_\mathfrak{b}, V_\mathfrak{b}, \phi_\mathfrak{b})$ of $b$ and $(U_\mathfrak{c}, V_\mathfrak{c}, \phi_\mathfrak{c})$ of $j_\mathfrak{b}(b)$. From this, we construct our local trivialization over $\mathfrak{b} \circ \mathfrak{c}$. Consider $\phi_\mathfrak{b}(V_\mathfrak{b} \cap U_\mathfrak{c})$, which is a neighborhood of $(b, 0)$. Therefore, there exist open neighborhoods $b \in U \subseteq U_\mathfrak{b}$

and $0 \in X \subseteq R^n$ such that $U \times X \subseteq \phi_{\mathfrak{b}}(V_{\mathfrak{b}} \cap U_{\mathfrak{c}})$. Analoguous to the proof of restricting the total space in Chapter 1, it follows that

$$\exists \varepsilon > 0 : U \times B_\varepsilon(0) \subseteq \phi_{\mathfrak{b}}(V_{\mathfrak{b}} \cap U_{\mathfrak{c}})$$

$$\implies U \times \mathbb{R}^n \cong U \times B_\varepsilon(0) \cong \phi_{\mathfrak{b}}^{-1}(U \times B_\varepsilon(0)) \cong \phi_{\mathfrak{c}}^{-1}(\phi_{\mathfrak{b}}^{-1}(U \times B_\varepsilon(0)))$$

which is an open neighborhood of $i(U)$ and therefore a valid candidate for $V$. This concludes local triviality and the proof. □

**Lemma 6.5.** (transitivity of normal microbundles)
*Let $M, N$ and $P$ be topological manifolds with $P \subseteq N \subseteq M$. There exists a normal microbundle $\mathfrak{n}$ of $P$ in $M$, if there exist normal microbundles $\mathfrak{n}_p : P \xrightarrow{i_P} U_N \xrightarrow{j_P} P$ in $N$ and $\mathfrak{n}_n : N \xrightarrow{i_N} U_M \xrightarrow{j_N} N$ in $M$.*

*Proof.*
We simply form the composition $\mathfrak{n}_p \circ \mathfrak{n}_n|_{U_N} : P \xrightarrow{i_N \circ i_P} U_M \xrightarrow{j_P \circ j_N} P$. Since $i_N \circ i_P$ is just the inclusion of $P \hookrightarrow U_M \subseteq M$, we found a normal microbundle $\mathfrak{n}$ of $P$ in $M$. □

Every topological manifold is an absolute neighborhood retract (ANR).

It follows that by restricting $M$, if necessary, to an open neighborhood of $N$, there exists a retraction $r : M \to N$ which we will take advantage of in the following.

**Lemma 6.6.** (homeomorphism of total spaces)
*Let $\mathfrak{t}_N$ and $\mathfrak{t}_M$ be the tangent microbundles of $N$ and $M$. The total space $E(\iota^*\mathfrak{t}_M)$ and $E(r^*\mathfrak{t}_N)$ are homeomorphic.*

*Proof.*
We explicitly construct a homeomorphism:

1. $E(\iota^*\mathfrak{t}_M) = \{(n, (m_1, m_2)) \in N \times (M \times M) \mid \iota(n) = m_1\}$
2. $E(r^*\mathfrak{t}_N) = \{(m, (n_1, n_2)) \in M \times (N \times N) \mid r(m) = n_1\}$

Now, we have the homeomorphism $\phi : E(\iota^*\mathfrak{t}_M) \to E(r^*\mathfrak{t}_N)$ with $\phi(n, (m_1, m_2)) = (m_2, (r(m_2), n))$ and $\phi^{-1}(m, (n_1, n_2)) = (n_2, (n_2, m))$. We easily see that $\phi$ suffices all requirements of $E(\iota^*\mathfrak{t}_M)$ and $E(r^*\mathfrak{t}_N)$. □

*Remark* 6.7. Note that the following diagram commutes

$$\begin{tikzcd} N \arrow[r] \arrow[d, hook] & E(\iota^*\mathfrak{t}_M) \arrow[d, "\phi"] \\ M \arrow[r] & E(r^*\mathfrak{t}_N) \end{tikzcd}$$

**Lemma 6.8.** *(normal microbundle on total space) There exists a normal microbundle* $\mathfrak{n}$ *of* $N$ *in* $E(r^*\mathfrak{t}_N)$ *with* $\mathfrak{n} \cong \iota^*\mathfrak{t}_M$.

*Proof.*
Obviously, $\mathfrak{n} := r^*\mathfrak{t}_N|_N$ is a normal microbundle of $N$ in $E(r^*\mathfrak{t}_N)$. Since $E(r^*\mathfrak{t}_N|_N) \subseteq E(r^*\mathfrak{t}_N)$, isomorphy follows from the previous lemma and remark. □

*Finally, we gathered all the tools to prove Milnor's theorem.*

**Theorem 6.9.** *(Milnor) For a sufficently large* $q \in \mathbb{N}$, $N = N \times \{0\}$ *has a normal microbundle in* $M \times \mathbb{R}^q$.

*Proof.*

1. There exists a microbundle $\mathfrak{t}'$ over $N$ such that $\mathfrak{t}_N \oplus \mathfrak{t}' \cong \mathfrak{e}_n^q$:

   From the [Whitney Embedding Thereom] it follows that we can embed $N$ in euclidean space $\mathbb{R}^{2m+1}$. Additionally, from previous conseriderations we can extend $\mathfrak{t}_N$ to a microbundle over an open neighborhood $V \subseteq \mathbb{R}^{2m+1}$. Now we can apply the **??** from Chapter 4.

2. $E(r^*\mathfrak{t}_N) \subseteq E(r^*\mathfrak{t}_N \oplus r^*\mathfrak{t}')$ has a normal microbundle:

   Consider $j^*(r^*\mathfrak{t}_N \oplus r^*\mathfrak{t}') : E(r^*\mathfrak{t}_N) \xrightarrow{i'} E(r^*\mathfrak{t}_N \oplus r^*\mathfrak{t}') \xrightarrow{j'} E(r^*\mathfrak{t}_N)$ where $j$ is the projection map for $r^*\mathfrak{t}_N$. Since $i'$ is injective, we can consider $E(r^*\mathfrak{t}_N) \subseteq E(r^*\mathfrak{t}_N \oplus r^*\mathfrak{t}')$. Since total spaces of microbundles over manifolds are manifolds as well, it follows that $j^*(r^*\mathfrak{t}_N \oplus r^*\mathfrak{t}')$ is a normal microbundle.

Since $N \subseteq M \subseteq E(r^*\mathfrak{t}_N)$ has a normal microbundle (**??**) it follows from **??** that $N \subseteq E(r^*\mathfrak{t}_N \oplus r^*\mathfrak{t}')$ has a normal microbundle. But $r^*\mathfrak{t}_N \oplus r^*\mathfrak{t}'$ is trivial and therefore w.l.o.g. $E(r^*\mathfrak{t}_N \oplus r^*\mathfrak{t}') \cong N \times \mathbb{R}^q$ □