MICROBUNDLES ON TOPOLOGICAL MANIFOLDS

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Abstract.

Contents

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1. Introduction

1.1. Motivation.

Given a smooth manifold M, one can define the tangent space in a point $p \in M$ using 'derivations'

$$T_pM = \{\nu : C^\infty(M) \to \mathbb{R}^n \text{ linear} : \nu(fg) = f(p)\nu(g) + \nu(f)g(p)\}$$

or using 'tangent curves'

$$T_pM = \{\gamma \in C^\infty((-1,1), M) : \gamma(0) = p\}/\sim$$

where $\gamma \sim \gamma' \iff \frac{d}{dx}(\psi \circ \gamma)(0) = \frac{d}{dx}(\psi \circ \gamma')(0)$ and ψ is a chart for p.

The tangent space allows for the definition of the tangent bundle

$$TM := \bigsqcup_{p \in M} T_pM$$

together with the projection

$$TM \xrightarrow{\pi} M, \pi(p, \nu) = p.$$

Lets say one wants to define the tangent bundle over a topological manifold. As we see from the two definitions of the tangent space, the tangent bundle requires the notion of differentiability. However, M. Kervaire showed in 1960 that there exists a 10-dimensional manifold that does not admit a differentiable structure (see [Ker60]). Furthermore, even if there exists such a structure for a topological manifold, it is generally not unique (e.g. the 7-sphere [Mil56]).

Therefore, we need a different construction than the one for the smooth case.

One plausible approach would be to define the fibers of the tangent bundle to be of the form $\{p\} \times U_p$, where U_p is a neighborhood of p which is homeomorphic to $\mathbb{R}^d$ via a chart. However, this construction raises the problem on choosing neighborhoods U_p such that they vary continuously over M. Furthermore, it is questionable whether this construction is topologically invariant if it depends on specific choices of neighborhoods U_p.

In 1964, the well-known mathematician John Milnor published "Microbundles, Part I", introducing a unique way to think about tangent bundles over topological manifolds:

The core idea behind this approach is to drop the assumption that the tangent bundle is a vector bundle and hence every fiber must be homeomorphic to euclidean spaces. Instead, we require that the fibers are a 'germ' of euclidean space, i.e. a topological space with an open subset homeomorphic to euclidean space. In contrast to the above approach, we can now choose the neighborhoods U_p of p regardless of any corresponding charts. That is because we can always find the domain of a chart (which is homeomorphic to $\mathbb{R}^d$) within arbitrary neighborhoods of p.

If the respective neighborhoods can be chosen freely, we may as well always choose the entire space M for the sake of simplicity.

So it follows that our resulting total space is of the form $M \times M$, which, analoguous to the smooth case, comes equipped with the projection

$$M \times M \xrightarrow{\pi} M, \pi(m, m') = m.$$

In order to develop this approach, Milnor introduces a new type of bundle that arise from this. He calls them 'microbundles'. Many fundamental properties and constructions of vector bundles immediatly carry over other to microbundles, e.g. induced microbundles or the Whitney sum (see Section **(2)**). Moreover, Milnor shows that if a manifold can be equipped with a smooth structure, then the tangent vector bundle regarded as microbundle is isomorphic to the tangent microbundle.

Furthermore, Milnor defines what the microbundle analogue of the normal bundle is. Given a smooth embedded submanifold $P \subseteq M$, there always exists a normal bundle NP defined fiber-wise with

$$N_p P = T_p M / T_p N.$$

However, for the topological case it is not as easy. The two mathematicians C. Rourke and B. Sanderson could construct a 19-dimensional manifold embedded in S^{29} that does not admit a normal microbundles (see [RS67]). Instead, Milnor could show that there always exists a normal microbundle of an embedded manifold in a stabilized surrounding manifold $M \times \mathbb{R}^q$ for some $q \in \mathbb{N}$.

Milnors studies on microbundles allowed R. Williamson to transfer fundamental results of cobordism theory from the smooth case over to the topological case, particulary by utilising the existence of a normal microbundle in a stabilized surrounding manifold.

This thesis introduces the concept of microbundles and presents a proof for the above theorem. It is based on Milnors paper 'Microbundles, Part I', adopting much of its structure and proofs.

1.2. **Introduction to Microbundles.** This section introduces the concept of microbundles together with some basic properties. We clarify what a microbundle is, what it means for a microbundle to be trivial and cover some fundamental examples for microbundles.

Definition 1.1 (microbundle)**.** [Mil64, p.20]
A *microbundle* $\mathfrak{b}$ over B (with *fiber dimension* n) is a diagram $B \xrightarrow{i} E \xrightarrow{j} B$ satisfying the following:

(i) B is a topological space (*base space*)

(ii) E is a topological space (*total space*)

(iii) $i : B \to E$ (*injection*) and $j : E \to B$ (*projection*) are maps such that $id_B = j \circ i$

(iv) Every $b \in B$ is *locally trivializable*, that is there exist open neighborhoods $U \subseteq B$ of b and $V \subseteq E$ of $i(U)$ together with a homeomorphism $\phi : V \xrightarrow{\sim} U \times \mathbb{R}^n$

such that the following diagram commutes:

$$\begin{tikzcd} & V \arrow[dr, "j|_V"] \arrow[dd, "\phi"] & \\ U \arrow[ur, "i"] \arrow[dr, "id\times 0"'] & & U \\ & U \times \mathbb{R}^n \arrow[ur, "\pi_1"'] & \end{tikzcd}$$

Note that π_1 denotes the projection $(u, x) \mapsto x$.

Remark 1.2.
In the following, unless explicitly stated otherwise, we assume the fiber dimension of any given microbundle to be n.

Lemma 1.3.
The diagram $B \xrightarrow{i} E \xrightarrow{j} B$ is locally trivial in $b \in B$ if and only if there exists a homeomorphism

$$\phi : V \xrightarrow{\sim} \phi(V) \subseteq B \times \mathbb{R}^n$$

where V is a neighborhood of $i(b)$ and $\phi(V)$ is neighborhood of $(b, 0)$ such that ϕ commutes as in Definition **(1.1)**.

Proof.
It suffices to show that we can derive local triviality in b assuming only a homeomorphism $\phi : V \xrightarrow{\sim} \phi(V)$ as stated:

Since $\phi(V)$ is a neighborhood of $(b, 0)$, there exists an open subset $U \subseteq B$ and $\varepsilon > 0$ such that $U \times B_\varepsilon(0) \subseteq \phi(V)$.

Note that there exists a homeomorphism

$$\mu_\varepsilon : B_\varepsilon(0) \xrightarrow{\sim} \mathbb{R}^n \text{ with } \mu_\varepsilon(0) = 0$$

for example by

$$x \mapsto \tan(\frac{|x| \cdot \pi}{2\varepsilon})x.$$

We can now construct a local trivialization with

$$\phi' : \phi^{-1}(U \times B_\varepsilon(0)) \xrightarrow{\sim} U \times \mathbb{R}^n$$

given by $\phi' = \mu_\varepsilon \circ \phi$.

Commutativity between i and $id \times 0$ is given by

$$\phi'(i(b)) = \mu_\varepsilon(phi(b)) = \mu_\varepsilon(b, 0) = (b, 0) = (id \times 0)(b)$$

and between j and π_1 by

$$j(e) = \pi_1(\phi(e)) = \pi_1(\mu_\varepsilon(\phi(e))) = \pi_1(\phi'(e)),$$

which concludes the proof. $\square$

Before we look at some examples of microbundles, we first define what it means for two microbundles to be isomorphic.

Definition 1.4 (isomorphy)**.**
Two microbundles $\mathfrak{b}_1 : B \xrightarrow{i_1} E_1 \xrightarrow{j_1} B$ and $\mathfrak{b}_2 : B \xrightarrow{i_2} E_2 \xrightarrow{j_2} B$ are *isomorphic* if there exist neighborhoods V_1 of $i_1(B)$ and V_2 of $i_2(B)$ together with a homeomorphism $\psi : V_1 \xrightarrow{\sim} V_2$ such that the following diagram commutes:

$$\begin{tikzcd} & V_1 \arrow[dd, "\psi"] \arrow[dr, "j_1|_{V_1}"] & \\ B \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & B \\ & V_2 \arrow[ur, "j_2|_{V_2}"'] & \end{tikzcd}$$

As the definition of isomorphy already implies, when studying microbundles, we are not interested in the entire total space but only in an arbitrary small neighborhood of the base space (more precise, the image $i(B)$). This is one of the biggest conceptual differences between microbundles and vector bundles. The following proposition makes this even clearer.

Throughout the paper we will often take use of homeomorphisms

$$\mu_\varepsilon : B_\varepsilon(0) \xrightarrow{\sim} \mathbb{R}^n \text{ with } \mu_\varepsilon(0) = 0$$

for example given by

$$x \mapsto \tan(\frac{|x| \cdot \pi}{2\varepsilon})x$$

in order to show properties like local triviality.

Proposition 1.5.
Given a microbundle $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ over B, restricting the total space E to an arbitrary neighborhood $E' \subseteq E$ of $i(B)$ leaves the microbundle unchanged. That is, the microbundle

$$\mathfrak{b}' : B \xrightarrow{i} E' \xrightarrow{j|_{E'}} B$$

is isomorphic to $\mathfrak{b}$.

Proof.
We prove the proposition in two steps.

(1) $\mathfrak{b}'$ is a microbundle:

Continuity and $id_B = j|_{E'} \circ i$ are already given since $\mathfrak{b}$ is a microbundle.

So we only need to show that $\mathfrak{b}'$ is locally trivial. For an arbitrary $b \in B$, choose a local trivialization (U, V, ϕ) of b in $\mathfrak{b}$.

The image $\phi(V \cap E')$ is a neighborhood of $(b, 0)$. This follows from $\phi(i(b)) = (b, 0)$ and $V \cap E'$ being a neighborhood of $i(b)$.

Hence, for a sufficiently small $\varepsilon > 0$ there exists a $U' \times B_\varepsilon(0) \subseteq \phi(V \cap E')$ such that U' is an open neighborhood of b.

Utilizing a homeomorphism $\mu_\varepsilon : B_\varepsilon(0) \xrightarrow{\sim} \mathbb{R}^n$, we have a local trivialization (U', V', ϕ') with

$$\phi' : V' \xrightarrow{\phi} U' \times B_\varepsilon(0) \xrightarrow{id\times\mu_\varepsilon} U' \times \mathbb{R}^n$$

and $V' := \phi^{-1}(U' \times B_\varepsilon(0))$.

That is because ϕ' commutes with the injection

$\phi'(i(b)) = (id \times \mu_\varepsilon)(\phi(i(b))) = (id \times \mu_\varepsilon)(b, 0) = (b, 0) = (id \times 0)(b)$

and projection maps

$$j(e) = \pi_1(\phi(e)) = \pi_1((id \times \mu_\varepsilon)(\phi(e))) = \pi_1(\phi'(e)).$$

(2) $\mathfrak{b}'$ is isomorphic to $\mathfrak{b}$:

Since $E' \subseteq E$, we can simply take the identity $E' \to E' \subseteq E$ as our homeomorphism between neighborhoods of $i(B)$. Furthermore, the injection and projection maps for $\mathfrak{b}$ and $\mathfrak{b}'$ are the same, so they clearly commute with the identity.

□

The most obvious example for a microbundle is the standard microbundle.

Example 1.6 (trivial microbundle)**.**
Given a topological space B, the *standard microbundle* $\mathfrak{e}^n_B$ over B is a diagram

$$B \xrightarrow{\iota} B \times \mathbb{R}^n \xrightarrow{\pi_1} B$$

where $\iota(b) := (b, 0)$ and $\pi_1(b, x) := b$. A microbundle $\mathfrak{b}$ over B is *trivial* if it is isomorphic to $\mathfrak{e}^n_B$.

We provide a triviality criteria for microbundles over paracompact hausdorff base spaces.

Lemma 1.7.
A microbundle $\mathfrak{b}$ over a paracompact hausdorff space B is trivial if and only if there exists an open neighborhood V of $i(B)$ such that $V \cong B \times \mathbb{R}^n$ with injection and projection maps being compatible with this homeomorphism.

So for trivial microbundles $\mathfrak{b}$ over B, given that the B is paracompact hausdorff, we may assume that an open subset of $E(\mathfrak{b})$ is homeomorphic to the whole $B \times \mathbb{R}^n$, instead of only a neighborhood of $B \times \{0\}$.

Proof.
'$\Longrightarrow$'

By restricting $E(\mathfrak{b})$ to an open neighborhood and applying Proposition **(1.5)** if necessary, we may assume that $E(\mathfrak{b})$ is an open subset of $B \times \mathbb{R}^n$.

Since $E(\mathfrak{b})$ is a neighborhood of $B \times \{0\}$, there exist $B_i \subseteq B$ open and $0 < \varepsilon_i < 1$ with

$$\bigcup_{i\in I} B_i \times B_{\varepsilon_i}(0) \subseteq E(\mathfrak{b})$$

such that $\bigcup_{i\in I} B_i = B$. Without loss of generality, we may assume that the collection $\{B_i\}$ is locally finite because if not we can simply choose a locally finite refinement using the fact that B is paracompact.

Furthermore, B being paracompact hausdorff we have a partition of unity

$$f_i : B \to [0,1] \text{ with } \mathrm{supp} f_i \subseteq B_i$$

such that $\sum_{i\in I} f_i = 1$.

Now, we define a map

$$\lambda : B \to (0,\infty) \text{ with } \lambda := \sum_{i\in I} \varepsilon_i f_i,$$

which has the property that $|x| < \lambda(b) \implies (b,x) \in E(\mathfrak{b})$, because

$$\begin{aligned} &|x| < \lambda(b) \\ \iff\ &|x| < \varepsilon_{i_1} f_{i_1}(b) + \cdots + \varepsilon_{i_n} f_{i_n}(b) \\ \iff\ &0 < (\varepsilon_{i_1} - |x|) f_{i_1}(b) + \cdots + (\varepsilon_{i_n} - |x|) f_{i_n}(b) \\ \implies\ &\exists i \in I : 0 < (\varepsilon_{i_1} - |x|) f_{i_1}(b) \\ \implies\ &(b,x) \in B_i \times B_{\varepsilon_i}(0) \implies (b,x) \in E(\mathfrak{b}). \end{aligned}$$

Finally, we have a homeomorphism between the open subset $\{(b,x) \in B \times \mathbb{R}^n : |x| < \lambda(b)\} \subseteq E(\mathfrak{b})$ and $B \times \mathbb{R}^n$ via

$$(b,x) \mapsto (b, \frac{x}{\lambda(b) - |x|}).$$

Note that $(b,0) \mapsto (b,0)$, hence this homeomorphism is commutes with the injection and projection maps.

'$\Longleftarrow$'

This is simply a weakening of the definition of triviality. $\square$

Example 1.8 (underlying microbundle)**.**
Let $\xi : E \xrightarrow{\pi} B$ be a n-dimensional vector bundle. The *underlying microbundle* $|\xi|$ of ξ is a microbundle

$$|\xi| : B \xrightarrow{i} E \xrightarrow{\pi} B$$

where $i : B \to E$ denotes the *zero-cross section* of ξ, that is the section that maps every $b \in B$ to the neutral element 0_b of its fiber $\pi^{-1}(b) \cong \mathbb{R}^n$.

Proof that $|\xi|$ is a microbundle.
First, note that π is an open map:

Let $V \subseteq E$ be open. For every $b \in \pi(V)$, there exists a neighborhood U_b together with a homeomorphism $\phi_b : \pi^{-1}(U_b) \xrightarrow{\sim} U_b \times \mathbb{R}^n$. It follows that $\pi|_{\pi^{-1}(U_b)} = \pi_1 \circ \phi_b$. Hence, $\pi|_{\pi^{-1}(U_b)}$ is open due to openness of π_1 and ϕ_b.

We conclude from

$$\pi(V) = \bigcup_{b\in B} \pi|_{\pi^{-1}(U_b)}(V)$$

that π is open.

Now from $i^{-1}(V) = \pi(V)$ it follows that i is continuous. Additionally, $\pi \circ i = id_B$ since $\pi(i(b)) = \pi(0_b) = b$.

Local triviality and commutativity with $id \times 0$ and π_1 are immediatly inherited from the local triviality conditions for vector bundles. □

The following example is the microbundle analog to the tangent bundle over smooth manifolds.

Example 1.9 (tangent microbundle)**.**
The *tangent microbundle* $\mathfrak{t}_M$ over a topological d-manifold M is a diagram

$$M \xrightarrow{\Delta} M \times M \xrightarrow{\pi_1} M$$

where $\Delta(m) := (m, m)$ denotes the diagonal map.

Proof that $\mathfrak{t}_M$ is a microbundle.
The maps Δ and π_1 are continuous and clearly $\pi_1 \circ \Delta = id_M$.

For an arbitrary $p \in M$, choose a chart (U, ψ) over p. We have a local trivialization $(U, U \times U, \phi)$ of p in $\mathfrak{t}_M$ given by

$$\phi : U \times U \xrightarrow{\sim} U \times \mathbb{R}^n \text{ with } \phi(u, u') = (u, \psi(u) - \psi(u')).$$

Homeomorphy of ϕ follows from homeomorphy of ψ.

Commutativity between the injection and $id \times 0$ is given by

$$\phi(\Delta(m)) = \phi(m, m) = (m, \psi(m) - \psi(m)) = (m, 0) = (id \times 0)(m)$$

and between the projection and π_1 by

$$\pi_1(u, u') = u = \pi_1(u, \phi^{(2)}(u, u')) = \pi_1(\phi(u, u')).$$

Note that $\phi^{(2)}$ denotes the map on the second component of ϕ, i.e. $\pi_2 \circ \phi$. □

Remark 1.10.
The tangent microbundle $\mathfrak{t}_M$ has fiber dimension d.

The following statement is fundamental for the theory of microbundles. It justifies that the tangent microbundle is a generalization of the tangent vector bundle.

Theorem 1.11.
Let M be a smooth d-manifold. Then the underlying microbundle of $\xi : TM \to M$ and the tangent microbundle $\mathfrak{t}_M$ are isomorphic.

Proof.
TODO □

2. Standard Constructions

This chapter introduces two standard constructions for microbundles, the ‘induced microbundle’ and the ‘Whitney sum’. These constructions already exist for vector bundles and many results carry over immediatly its microbundle analogue.

2.1. **Induced Microbundles.** Given a microbundle $\mathfrak{b}$ over B and a map $f : A \to B$, one can define a microbundle $f^*\mathfrak{b}$ over A. This is achieved by 'pulling back' the base space B to A with the use of the map f.

After showing the existence of such a microbundle, we will prove some fundamental properties such as triviality criteria and compatibility with map composition. At the end of the section, we will address induced microbundles over cones and simplicial complexes.

Definition 2.1 (induced microbundle)**.** [Mil64, p.58]
Let $\mathfrak{b}$ be a microbundle over B and let $f : A \to B$ be a map. The *induced microbundle* $f^*\mathfrak{b} : A \xrightarrow{i_f} E_f \xrightarrow{j_f} A$ is a microbundle over A defined as follows:

- $E_f = \{(a, e) \in A \times E(\mathfrak{b}) \mid f(a) = j(e)\}$
- $i_f(a) = (a, (i \circ f)(a))$
- $j_f(a, e) = a$

The construction is identical to the one over vector bundles, more precisely over fiber bundles (see [Bre93, ch.2,sec.14]).

Proof that $f^*\mathfrak{b}$ is a microbundle.
Both i_f and j_f are continuous since they are composed by continuous functions. Additionally, $j_f(i_f(a)) = j_f(a, i(f(a))) = a$ and hence $j_f \circ i_f = id_A$.

It remains to be shown that $f^*\mathfrak{b}$ is locally trivial:

For an arbitrary $a \in A$, choose a local trivialization (U, V, ϕ) of $i(a)$ in $\mathfrak{b}$. We construct a local trivialization of a in $f^*\mathfrak{b}$ as follows:

- $U_f := f^{-1}(U) \subseteq A$
- $V_f := (U_f \times V) \cap E_f \subseteq E_f$
- $\phi_f : V_f \xrightarrow{\sim} U_f \times \mathbb{R}^n$ with $\phi_f(a', e) = (a', \phi^{(2)}(e))$

Note that U_f is an open neighborhood of a since f is continuous and U is an open neighborhood of $i(a)$. Similarly, V_f is an open neighborhood of $i_f(a)$ since both $U' \times V$ and E_f are open neighborhoods of $i_f(a)$.

The existence of an inverse $\phi_f^{-1}(a', v) = (a', \phi^{-1}(f(a'), v))$ and component-wise continuity for both ϕ_f and ϕ_f^{-1} show that ϕ_f is a homeomorphism.

Commutativity between the injection and $id \times 0$ is given by

$$\phi_f(i_f(a')) = \phi(a', i(f(a'))) = (a', \phi^{(2)}(i(f(a')))) = (a', 0) = (id \times 0)(a')$$

and between the projection and π_1 by

$$j_f(a', e) = a' = \pi_1(a', \phi_f^{(2)}(a', e)) = \pi_1(\phi_f(a', e)),$$

which completes the proof. $\square$

Example 2.2 (restricted microbundle)**.**

Let $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ be a microbundle and $A \subseteq B$ be a subspace. The *restricted microbundle* $\mathfrak{b}|_A$ is the induced microbundle $\iota^*\mathfrak{b}$ where $\iota : A \hookrightarrow B$ denotes the inclusion map.

Remark 2.3.

In the following, we consider $E(\mathfrak{b}|_A)$ to be a subset of $E(\mathfrak{b})$. This is justified because there exists an embedding

$$\iota : E(\mathfrak{b}|_A) \to E(\mathfrak{b}) \text{ with } (a, e) \mapsto e$$

and inverse $e \mapsto (j(e), e)$. Note that this argument can be made for any induced microbundle over an injective map.

Next, we provide two criteria to show that an induced microbundle is trivial.

Lemma 2.4.

Let $\mathfrak{b}$ be a microbundle over B and $f : A \to B$ be a map. The induced microbundle $f^*\mathfrak{b}$ is trivial if $\mathfrak{b}$ is already trivial.

Proof.

To proof triviality, we need to show that there exists a homeomorphism between a neighborhood of $i'(A)$ and $A \times \{0\}$ that commutes with the injection and projection maps of $f^*\mathfrak{b}$ and $\mathfrak{e}^n_A$.

Since $\mathfrak{b}$ is trivial, there exists a homeomorphism $\psi : V \to \psi(V)$ where V is a neighborhood of $i(B)$ and $\psi(V)$ is a neighborhood of $B \times \{0\}$ such that ψ commutes with the injection and projection maps of $\mathfrak{b}$ and $\mathfrak{e}^n_A$.

We define a map

$$\psi_f : V_f \xrightarrow{\sim} \psi_f(V_f) \text{ with } \psi_f(a, e) = (a, \psi^{(2)}(e)),$$

where $V_f = (A \times V) \cap E(f^*\mathfrak{b})$. The existance of an inverse

$$\psi_f^{-1}(a, x) = (a, \psi^{-1}(f(a), x))$$

and component-wise continuity for both ψ_f and ψ_f^{-1} show that ψ_f is a homeomorphism.

The subset V_f is a neighborhood of $i_f(A)$ since

$$i_f(a) = (a, i(f(a))) \text{ and } i(f(a)) \in V.$$

From $\psi^{(2)}(i(f(a))) = 0$ and openness of ψ_f it follows that $\psi_f(V_f)$ is a neighborhood of $A \times \{0\}$. Hence, ψ_f maps a neighborhood of $i_f(A)$ to a neighborhood of $A \times \{0\}$, just as required.

Finally, commutativity between the injection maps is given by

$$\psi_f(i'(a)) = (a, \psi^{(2)}(i(f(a)))) = (a, 0) = (id \times 0)(a)$$

and between the projection maps by

$$j_f(a, e) = a = \pi_1(a, \psi_f^{(2)}(a, e)) = \pi_1(\psi_f(a, e)),$$

which completes the proof. $\square$

Lemma 2.5.
Let $\mathfrak{b}$ be a microbundle over B. The induced microbundle $c^*_{A,b_0}\mathfrak{b}$ over a constant map

$$c_{A,b_0} : A \to B \text{ with } c_{A,b_0}(a) = b_0$$

is trivial.

Proof.
The total space $E(c^*_{A,b_0}\mathfrak{b})$ is defined as

$$\{(a,e) \in A \times E(\mathfrak{b}) : f(a) = b_0 = j(e)\} = A \times j^{-1}(b_0).$$

Let (U, V, ϕ) be local trivialization for b_0 in $\mathfrak{b}$. Restricting ϕ to the fiber $j^{-1}(b_0)$ yields a homeomorphism

$$\phi|_{j^{-1}(b_0)} : V \cap j^{-1}(b_0) \xrightarrow{\sim} b_0 \times \mathbb{R}^n$$

It follows that $\psi : A \times (V \cap j^{-1}(b_0)) \xrightarrow{\sim} A \times \mathbb{R}^n$ with

$$\psi(a,e) = (a, \phi^{(2)}(e))$$

is a homeomorphism as well.

The product $A \times (V \cap j^{-1}(b_0))$ is open in $E(c^*_{A,b_0}\mathfrak{b})$, since $V \cap j^{-1}(b_0)$ is open in $j^{-1}(b_0)$ with the subspace topology. Additionally, from

$$i_c(a) = (a, i(c_{A,b_0}(a))) = (a, i(b_0)) \text{ and } \phi^{(2)}(i(b_0)) = 0$$

it follows that $A \times (V \cap j^{-1}(b_0))$ is a neighborhood of $A \times \{0\}$. So ψ maps a neighborhood of $i_c(A)$ to a neighborhood of $A \times \{0\}$.

Commutativity between the injection maps is given by

$$\psi(i_c(a)) = \psi(a, i(b_0)) = (a, \phi^{(2)}(i(b_0))) = (a, 0) = (id \times 0)(a)$$

and between the projection maps by

$$j_c(a,e) = a = \pi_1(a,x) = \pi_1(\psi(a,e)).$$

Hence, $c^*_{A,b_0}\mathfrak{b}$ and $\mathfrak{e}^n_A$ are isomorphic via ψ. ☐

One would intuitively argue that the induced microbundle is compatible with the composition of maps, i.e. $(g \circ f)^*\mathfrak{c} \cong f^*(g^*\mathfrak{c})$. This in fact true, as the following lemma shows.

Lemma 2.6.
Let $\mathfrak{c} : C \xrightarrow{i} E \xrightarrow{j} C$ be a microbundle and let $A \xrightarrow{f} B \xrightarrow{g} C$ be a map diagram. Then the two microbundles

$$(g \circ f)^*\mathfrak{c} : A \xrightarrow{i_1} E_1 \xrightarrow{j_1} A$$

and

$$f^*(g^*\mathfrak{c}) : A \xrightarrow{i_2} E_2 \xrightarrow{j_2} A$$

are isomorphic.

Proof.
Again, we need to find a homeomorphism between a neighborhood of $i_1(A)$ and a neighborhood of $i_2(A)$ that commutes with the injection and projection maps.

Firstly, we compare the two total spaces:

- $E((g \circ f)^*) = \{(a, e) \in A \times E \mid g(f(a)) = j(e)\}$
- $E(f^*(g^*\mathfrak{c})) = \{(a, b, e) \in A \times (B \times E) \mid f(a) = b \text{ and } g(b) = j(e)\}$

We define a homeomorphism $\psi : E((g \circ f)^*) \xrightarrow{\sim} E(f^*(g^*\mathfrak{c}))$ with

$$\psi(a, e) = (a, f(a), e) \text{ and } \psi^{-1}(a, b, e) = (a, e)$$

for which homeomorphy follows from component-wise continuity of both ψ and ψ^{-1}.

Commutativity between the injection maps is given by

$$\psi(i_1(a)) = \psi(a, i(g(f(a)))) = (a, f(a), i(g(f(a)))) = i_2(a)$$

and between the projection maps by

$$j_1(a, e) = a = j_2(a, f(a), e) = j_2(\psi(a, e))$$

which concludes the proof. □

The rest of the section deals with the question of whether a microbundle can be extended over a particular base space.

Here *extended* means that when restricting such a microbundle to the initial base space we obtain a microbundle that is isomorphic to the initial microbundle.

For a topological space X, we define the *cone* of X to be

$$CX = X \times [0, 1]/X \times \{1\}$$

and the *mapping cone* of a map $f : A \to B$ to be

$$B \sqcup_f CA = B \sqcup CA/\sim$$

where $(a, 0) \sim b \iff f(a) = b$.

Similarly, we define the *cylinder* of X to be

$$MX = X \times [0, 1]$$

and the *mapping cylinder* of a map $f : A \to B$ to be

$$B \sqcup_f MA = B \sqcup MA/\sim$$

where $(a, 0) \sim b \iff f(a) = b$.

Lemma 2.7. *[Mil64, p.58]*
Let A be a paracompact hausdorff space. A microbundle $\mathfrak{b}$ over B can be extended to a microbundle over the mapping cone $B \sqcup_f CA$ if and only if $f^*\mathfrak{b}$ is trivial.

Proof.
We show both implications.

‘ $\Longrightarrow$ ’

Let $\mathfrak{b}'$ be an extension of $\mathfrak{b}$ over $B \sqcup_f CA$.

The composition $A \xrightarrow{f} B \hookrightarrow B \sqcup_f CA$ is null-homotopic via the homotopy

$$H(a,t) := [a,t]$$

because $H(a,0) = [a,0] = [f(a)] = (\iota \circ f)(a)$ and $H(a,1) = [a,1] = [\tilde{a},1] = H(\tilde{a},1)$.

The Homotopy Theorem, which will be proved in Section **(3)**, yields that

$$(\iota \circ f)^* \mathfrak{b}' \cong c^*_{A,[a,1]} \mathfrak{b}'$$

since $\iota \circ f$ is homotopic to $c_{A,[a,1]}$. Together with Lemma **(2.5)**, it follows that $(\iota \circ f)^* \mathfrak{b}'$ is trivial.

Because $(\iota \circ f)^* \mathfrak{b}' \cong f^*(\iota^* \mathfrak{b}') \cong f^* \mathfrak{b}$, we conclude that $f^* \mathfrak{b}$ is trivial.

'$\Longleftarrow$'

Let $f^* \mathfrak{b}$ be trivial.

In contrast to the mapping cone, there exists a natural retraction from the mapping cylinder to its attached space

$$r : B \sqcup_f MA \to B \text{ with } r([a,t]) = f(a).$$

The diagram

$$A \times \{1\} \hookrightarrow B \sqcup_f MA \xrightarrow{r} B$$

is equal to f if we consider $A = A \times \{1\}$. It follows from

$$r^* \mathfrak{b}|_{A \times \{1\}} = (r \circ \iota)^* \mathfrak{b} = f^* \mathfrak{b}$$

and the assumption that $r^* \mathfrak{b}|_{A \times \{1\}}$ is trivial. Using Lemma **(2.4)** and the retraction $(a,t) \mapsto (a,1)$, we even see that $r^* \mathfrak{b}|_{A \times [\frac{1}{2},1]}$ is trivial.

Since A is paracompact hausdorff we can apply Lemma **(1.7)**. Hence, there exists a homeomorphism

$$\psi : V \xrightarrow{\sim} A \times [\frac{1}{2}, 1] \times \mathbb{R}^n$$

where V is a neighborhood of $i_r(B)$ in $E(r^* \mathfrak{b}|_{A \times [\frac{1}{2},1]})$. Without loss of generality, we may assume that $V = E(r^* \mathfrak{b}|_{A \times [\frac{1}{2},1]})$ by removing a closed subset of $E(r^* \mathfrak{b}|_{A \times [\frac{1}{2},1]})$ if necessary and applying Proposition **(1.5)**.

Now we define an extended microbundle

$$\mathfrak{b}' : B \sqcup_f CA \xrightarrow{i'} E' \xrightarrow{j'} B \sqcup_f CA \text{ with}$$

- $E' = E(r^* \mathfrak{b}) / \psi^{-1}(A \times \{1\} \times \{x\})$ (for every $x \in \mathbb{R}^n$)
- $i'([a,t]) = [i_r([a,t])]$
- $j'([[a,t],e]) = [j_r([a,t],e)] = [a,t]$

The injection i' is well-defined because i_r maps every representative $[a,1]$ to the same equivalence class of E' due to its construction. Similarly, the projection j' is well-defined because

$$[[a,t],e] = [[a',t'],e'] \implies [a,t] = [a,t'].$$

Both i' and j' are continuous due to the definition of the quotient space topology. Also, $j' \circ i' = id_{B \sqcup_f CA}$ because

$$j'(i'([a,t])) = j'([i_r(a,t)]) = [j_r(i_r(a,t))] = [a,t].$$

It remains to be shown that $\mathfrak{b}'$ is locally trivial:

Let $[a,t] \in B \sqcup_f CA$ be arbitrary.

Note that the cone and the cylinder differ only around an arbitrary small neighborhood of $[a,1]$.

If $[a,t]$ is 'far away' from the critical point $[a,1]$, say $t \leq \frac{1}{2}$, we can simply take a local trivialization of $[a,t]$ in $r^*\mathfrak{b}$ to serve as a local trivialization of $[a,t]$ in $\mathfrak{b}'$, by restricting such a local trivialization to $A \times [0, \frac{1}{2})$ if necessary.

If $[a,t]$ is 'close' to the critical point, say $t > \frac{1}{2}$, we can take ψ to be the homeomorphism for our local trivialization. By construction, ψ respects the quotient $\pi : E(r^*\mathfrak{b}) \twoheadrightarrow E'$. It follows that $(\psi^{(1)}(\pi(V)), \pi(V), \psi)$ is a local trivialization for $[a,t]$ in $\mathfrak{b}'$. □

We can follow a statement about extending a microbundle over a simplex by utilizing that the cone of the boundary of a simplex is homeomorphic to the simplex itself.

This statement will be used to prove an important theorem about 'Whitney sums' later in this chapter.

Corollary 2.8.
Let B be a $(d+1)$-simplicial complex, B' its d-skeleton and $\Delta \subseteq B$ a $(d+1)$-simplex. A microbundle $\mathfrak{b}$ over B' can be extended to a microbundle over $B' \cup \Delta$ if and only if its restriction to the boundary $\mathfrak{b}|_{\partial\Delta}$ is trivial.

Proof.
With $f : \partial\Delta \hookrightarrow B'$ and the previous lemma, it follows that there exists a microbundle $\mathfrak{b}'$ over $B' \cup_f C\partial\Delta$ extending $\mathfrak{b}$ if and only if $f^*\mathfrak{b} = \mathfrak{b}|_{\partial\Delta}$ is trivial.

We have a homeomorphism $\phi : C\partial\Delta \xrightarrow{\sim} \Delta$ with

$$\phi((t_1, \dots, t_{d+1}), \lambda) := (1-\lambda)(t_1, \dots, t_{d+1}) + \frac{\lambda}{d+1}(1, \dots, 1).$$

In particular, $\phi(\partial\Delta \times \{0\}) = \partial\Delta$.

It follows that $B' \cup_f C\partial\Delta \cong B' \cup \Delta$, which concludes the proof. □

2.2. **The Whitney Sum.** Given two vector bundles E and F over the same base space X, one can define the Whitney sum $E \oplus F$ by forming the direct sum of the individual fibers E_x and F_x, hence the notation.

This construction carries over to microbundles, as elaborated in the following. The centerpiece of this section will be Theorem **(2.12)**, which states that for microbundles over simplicial complexes, one can find an 'inverse' microbundle such that their Whitney sum is trivial.

Definition 2.9. [Mil64, p.59]
Let $\mathfrak{b}_1$ and $\mathfrak{b}_1$ be two microbundles over B with fiber dimensions n_1 and n_2. The *Whitney sum* $\mathfrak{b}_1 \oplus \mathfrak{b}_2$ is a microbundle $B \xrightarrow{i} E \xrightarrow{j} B$ where

- $E = \{(e_1, e_2) \in E(\mathfrak{b}_1) \times E(\mathfrak{b}_2) : j_1(e_1) = j_2(e_2)\}$
- $i(b) = (i_1(b), i_2(b))$
- $j(e_1, e_2) = j_1(e_1) = j_2(e_2)$

with fiber dimension $n_1 + n_2$.

Proof that $\mathfrak{b}_1 \oplus \mathfrak{b}_2$ is a microbundle.
Both i and j are continuous since they are composed by continuous functions. Additionally, $j(i(b)) = j(i_1(b), i_2(b)) = j_1(i_1(b)) = b$ and hence $j \circ i = id_B$.

It remains to be shown that $\mathfrak{b}_1 \oplus \mathfrak{b}_2$ is locally trivial:

For an arbitrary $b \in B$, choose local trivializations (U_1, V_1, ϕ_1) and (U_2, V_2, ϕ_2) of b in $\mathfrak{b}_1$ and $\mathfrak{b}_2$.

We construct a local trivialization (U, V, ϕ) of b in $\mathfrak{b}_1 \oplus \mathfrak{b}_2$ as follows:

- $U = U_1 \cap U_2$, which is an open neighborhood of b since both U_1 and U_2 are open neighborhoods of b.
- $V = (V_1 \times V_2) \cap E$, which is an open neighborhood of $i(U)$ since V_1 and V_2 are open and $i(U) \subseteq (i_1(U) \times i_2(U)) \cap E \subseteq (V_1 \times V_2) \cap E$.
- $\phi : V \xrightarrow{\sim} U \times \mathbb{R}^{n_1+n_2}$ with $\phi(e_1, e_2) = (j_1(e_1), (\phi_1^{(2)}(e_1), \phi_2^{(2)}(e_2)))$, which is a homeomorphism together with the inverse
$$\phi^{-1}(b, (x_1, x_2)) = (\phi_1^{-1}(b, x_1), \phi_2^{-1}(b, x_2))$$
since both ϕ and ϕ^{-1} are component-wise continuous.

Commutativity between the injection and $id \times 0$ is given by
$$\phi(i(b)) = \phi(i_1(b), i_2(b)) = (b, (\phi_1^{(2)}(i_1(b)), \phi_2^{(2)}(i_2(b)))) = (b, (0, 0)) = (id \times 0)(b)$$
and between the projection and π_1 by
$$j(e_1, e_2) = j_1(e_1) = \pi_1(j_1(e_1), \phi^{(2)}(e_1, e_2)) = \pi_1(\phi(e_1, e_2)),$$
which completes the proof. $\square$

Alternatively, one could define the Whitney sum between $\mathfrak{b}_1$ and $\mathfrak{b}_2$ to be the induced microbundle $\Delta^*(\mathfrak{b}_1 \times \mathfrak{b}_2)$ where Δ denotes the diagonal map and $\mathfrak{b}_1 \times \mathfrak{b}_2$ denotes the literal cross-product between the two microbundles.

Lemma 2.10.
Let $\mathfrak{b}_1$ and $\mathfrak{b}_1$ be two microbundles over B and let $f : A \to B$ be a map. The induced microbundle and the Whitney sum are compatible, that is
$$f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2) \cong f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2.$$

Proof.
The total space $E(f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2))$ is of the form

$$\{(a,(e_1,e_2)) \in A \times (E(\mathfrak{b}_1) \times E(\mathfrak{b}_2)) \mid j_1(e_1) = j_2(e_2) = f(a)\}$$

and $E(f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2)$ is of the form

$$\{((a_1,e_1),(a_2,e_2)) \in E(f^*\mathfrak{b}_1) \times E(f^*\mathfrak{b}_2) : j_1(a_1,e_1) = j(a_2,e_2)\}.$$

We have a homeomorphsim $\phi : E(f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2)) \xrightarrow{\sim} E(f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2)$ with

$$\phi(a,(e_1,e_2)) = ((a,e_1),(a,e_2))$$

and its inverse

$$\phi^{-1}((a,e_1),(a,e_2)) = (a,(e_1,e_2)).$$

The two total spaces are homeomorphic via $\phi(a,(e_1,e_2)) := ((a,e_1),(a,e_2))$ with $\phi^{-1}((a,e_1),(a,e_2)) = (a,(e_1,e_2))$. Homeomorphy of ϕ follows from the continuity of ϕ and ϕ^{-1}, which is given since both ϕ and ϕ^{-1} are composed by identity maps.

Commutativity between the injection maps is given by

$$\phi(i_f(a)) = \phi(a,(i_1(f(a)),i_2(f(a)))) = ((a,i_1(f(a))),(a,i_2(f(a)))) = i_\oplus(a)$$

and between the projection maps by

$$j_f(a,(e_1,e_2)) = a = j_\oplus((a,e_1),(a,e_2)) = j_\oplus(\phi(a,(e_1,e_2))),$$

where i_f and j_f denote the injection and projection for $f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2)$ and $i_\oplus$ and $j_\oplus$ denote the injection and projection for $f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2$. □

Lastly, we show the above mentioned theorem about Whitney sums which will be essential in the proof of Milnors Theorem.

For its prove, we need the following proposition which will be shown in Section **(4)**.

Proposition 2.11 (Bouquet Lemma)**.** *[Mil64, p.59]*
Let $\mathfrak{b}$ be a microbundle over a 'bouquet' of spheres B, meeting in a single point. Then there exists a map $r : B \to B$ such that $\mathfrak{b} \oplus r^*\mathfrak{b}$ is trivial.

Theorem 2.12. *[Mil64, p.59]*
Let $\mathfrak{b}$ be a microbundle over a d-simplicial complex B. Then there exists a microbundle $\mathfrak{n}$ over B such that the Whitney sum $\mathfrak{b} \oplus \mathfrak{n}$ is trivial.

Proof.
We prove the theorem by induction over d.

(Start of induction)

A 1-simplicial complex is just a bouquet of circles. Hence, the start of induction follows directly from Proposition **(2.11)**.

(Inductive Step)

Let B' be the $(d-1)$-skeleton of B and let $\mathfrak{n}'$ be its corresponding microbundle such that $\mathfrak{b}|_{B'} \oplus \mathfrak{n}'$ is trivial.

(1) $\mathfrak{n}' \oplus \mathfrak{e}^n_{B'}$ can be extended over any d-simplex σ:

Consider the equation

$$(\mathfrak{n}' \oplus \mathfrak{e}^n_{B'})|_{\partial\sigma} = \mathfrak{n}'|_{\partial\sigma} \oplus \mathfrak{e}^n_{B'}|_{\partial\sigma} = \mathfrak{n}'|_{\partial\sigma} \oplus \mathfrak{b}|_{\partial\sigma} = (\mathfrak{n}' \oplus \mathfrak{b}|_{B'})|_{\partial\sigma}$$

in which we used the previous lemma and Corollary **(2.8)** for $\mathfrak{e}^n_{B'}|_{\partial\sigma} = \mathfrak{b}|_{\partial\sigma}$. Since $(\mathfrak{n}' \oplus \mathfrak{b}|_{B'})|_{\partial\sigma}$ is trivial, the claim follows from Corollary **(2.8)**.

(2) $\mathfrak{n}' \oplus \mathfrak{e}^n_{B'}$ can be extended over B:

The difficulty is that the individual d-simplices are not well-seperated. Let B'' denote B with small open d-cells removed from every d-simplex. Since B' is a retract of B'' we can extend $\mathfrak{n}' \oplus \mathfrak{e}^n_{B'}$ to a microbundle ν over B''.

Now we can extend ν over B by taking all extensions of ν over every simplex using (1.), and glueing its total spaces together along $E(\nu)$. Similarly, the injection and projection can be obtained by glueing the injection and projection maps over every simplex together.

We denote the resulting microbundle by η.

(3) Consider the mapping cone $B \sqcup_\iota CB'$ over the inclusion $B' \hookrightarrow B$. Since

$$(\mathfrak{b} \oplus \eta)|_{B'} = \mathfrak{b}|_{B'} \oplus \eta|_{B'} = \mathfrak{b}|_{B'} \oplus (\mathfrak{n}' \oplus \mathfrak{e}^n_{B'}) = (\mathfrak{b}|_{B'} \oplus \mathfrak{n}') \oplus \mathfrak{e}^n_{B'} = \mathfrak{e}^n_{B'} \oplus \mathfrak{e}^n_{B'}$$

is trivial, it follows from Lemma **(2.7)** that we can extend $\mathfrak{b} \oplus \eta$ over $B \sqcup_\iota CB'$, which will be denoted by ξ.

The mapping cone $B \sqcup_\iota CB'$ has the homotopy type of a bouquet of spheres by carrying B' along CB' collapsing to a single point. Since any d-simplex is homotopic to a d-disc and its boundary is collapsed, we obtain the homotopy of a $(d-1)$-sphere.

Using Theorem **(3.1)** and Proposition **(2.11)**, we conclude that there exists a microbundle $\mathfrak{n}$ such that $(\xi \oplus \mathfrak{n})|_B$ is trivial. The equation

$$\mathfrak{e}^n_B = (\xi \oplus \mathfrak{n})|_B = \xi|_B \oplus \mathfrak{n}|_B = (\mathfrak{b} \oplus \eta) \oplus \mathfrak{n}|_B = \mathfrak{b} \oplus (\eta \oplus \mathfrak{n}|_B)$$

completes the proof.

□

3. The Homotopy Theorem

In this chapter we will prove the Homotopy Theorem, which is a fundamental result for microbundles. It states the following.

Theorem 3.1 (Homotopy Theorem). *[Mil64, p.58]*
Let $\mathfrak{b}$ be a microbundle over B and let $f, g : A \to B$ be two maps where A is paracompact hausdorff. If f and g are homotopic, then $f^*\mathfrak{b}$ and $g^*\mathfrak{b}$ are isomorphic.

In order to prove this theorem, we first introduce the concept of map-germs on microbundles which allows us to classify isomorphisms between microbundles.

3.1. **Map-Germs.**

Definition 3.2 (map-germ). [Mil64, p.65]
A *map-germ* $F : (X, A) \Rightarrow (Y, B)$ between topological pairs (X, A) and (Y, B) is an equivalence class of maps $(X, A) \to (Y, B)$ where $f \sim g \iff f|_U = g|_U$ for an arbitrary neighborhood $U \subseteq X$ of A.

We can form the composition of two map-germs $F : (X, A) \Rightarrow (Y, B)$ and $G : (Y, B) \Rightarrow (Z, C)$ by choosing representatives $f : U_f \to Y$ and $g : U_g \to Z$ and defining $(f \circ g)|_{f^{-1}(U_g)}$ to be a representative for $G \circ F$.

Definition 3.3 (homeomorphism-germ). [Mil64, p.65]
A *homeomorphism-germ* $F : (X, A) \Rightarrow (Y, B)$ is a map-germ such that there exists a representative $f : U_f \to Y$ that maps homeomorphically to a neighborhood of B.

Let $\mathfrak{b}$ and $\mathfrak{b}'$ be two isomorphic microbundles over B. There exists a homeomorphism $\psi : V \xrightarrow{\sim} V'$ where $V \subseteq E(\mathfrak{b})$ is a neighborhood of $i(B)$ and $V' \subseteq E(\mathfrak{b}')$ is a neighborhood of $i'(B)$. We can view ψ as a representative for a homeomorphism-germ

$$[\psi] : (E, i(B)) \Rightarrow (E', i'(B)).$$

Studying isomorphy between $\mathfrak{b}$ and $\mathfrak{b}'$ using map-germs is useful, because we don't care what ψ does on its initial domain, but only what it does on arbitray small neighborhoods of $i(B)$. Hence, every representative of $[\psi]$ describes the 'same' isomorphy between $\mathfrak{b}$ and $\mathfrak{b}'$. Now, naturally, the question arises whether the existence of a homeomorphism-germ

$$F : (E, i(B)) \Rightarrow (E', i'(B))$$

already implies that $\mathfrak{b}$ and $\mathfrak{b}'$ are isomorphic. The answer is generally no, because isomorphy between microbundles additionally requires the homeomorphism to commute with the injection and projection maps. Therefore, we need to require an extra condition ('fiber-preservation') for this implication to be true. This justifies the following definition.

Let $J : (E(\mathfrak{b}), i(B)) \Rightarrow (B, B)$ and $J' : (E(\mathfrak{b}'), i(B)) \Rightarrow (B, B)$ denote the map-germs represented by the projections of $\mathfrak{b}$ and $\mathfrak{b}'$.

Definition 3.4 (isomorphism-germ).
An *isomorphism-germ* between $\mathfrak{b}$ and $\mathfrak{b}'$ is a homeomorphism-germ

$$F : (E(\mathfrak{b}), B) \Rightarrow (E(\mathfrak{b}'), B)$$

which is *fiber-preserving*, that is $J' \circ F = J$.

Remark 3.5.
There exists an isomorphism-germ between $\mathfrak{b}$ and $\mathfrak{b}'$ if and only if $\mathfrak{b}$ and $\mathfrak{b}'$ are isomorphic.

We can take this even further by dropping the assumption that the base spaces of the two microbundles equal. Note that in this case no comparison to isomorphy can be drawn, because we haven't defined isomorphy between microbundles over different base spaces.

Definition 3.6 (bundle-germ)**.**
Let $\mathfrak{b}$ and $\mathfrak{b}'$ be two microbundles over B and B' with the same fiber dimension. A *bundle-germ* $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$ is a map-germ

$$F : (E(\mathfrak{b}), B) \Rightarrow (E(\mathfrak{b}'), B')$$

such that there exists a representative $f : U_f \to E(\mathfrak{b}')$ that maps each fiber $j^{-1}(b)$ injectively to a fiber $j'^{-1}(b')$.

For a bundle-germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$, the following diagram commutes:

$$\begin{array}{ccc} (E(\mathfrak{b}), B) & \overset{F}{\Longrightarrow} & (E(\mathfrak{b}'), B') \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle i'} \\ B & \xrightarrow{\quad F|_B \quad} & B' \end{array}$$

We say F *is covered by* $F|_B$.

The bundle-germ is indeed a generalization of the isomorphism-germ, as the following proposition shows.

Proposition 3.7 (Williamson)**.**
Let $\mathfrak{b}$ and $\mathfrak{b}'$ be two microbundles over B. A bundle-germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$ covering the identity map is an isomorphism-germ.

Firstly, we show a lemma that is necessary for the proof of the proposition.

Lemma 3.8.
If a homeomorphism $\phi : \overline{B_2(0)} \xrightarrow{\sim} \phi(\mathbb{R}^n) \subseteq \mathbb{R}^n$ satisfies

$$|\phi(x) - x| < 1, \forall x \in \overline{B_2(0)}$$

then $\overline{B_1(0)} \subseteq \phi(\overline{B_2(0)})$.

Proof of the lemma.
Consider $\phi(2S^n)$ where $2S^n$ denotes the n-sphere of radius 2. The condition for ϕ yields $1 < |\phi(s)|, \forall s \in 2S^n$. Since $\overline{B_2(0)}$ has trivial homology groups which are preserved under homeomorphisms, $\phi(\overline{B_2(0)})$ must have trivial homology groups as well.

From this we can conclude that $\overline{B_1(0)}$ must be contained in $\phi(\overline{B_2(0)})$, because otherwise 'holes' would form which would result in non-trivial homology groups of $\phi(\overline{B_2(0)})$. □

Proof of the proposition.
Let f be a representative for F. First we assume a special and then generalize the result to show the proposition.

(1) Let f map from $B \times \mathbb{R}^n$ to $B \times \mathbb{R}^n$:

Since F covers the identity, f is of the form

$$f(b, x) = (b, g_b(x))$$

where $g_b : \mathbb{R}^n \to \mathbb{R}^n$ are individual maps. Since the g_b are continuous and injective, it follows from the [domain invariance theorem] that the g_b are open maps.

Let $(b_0, x_0) \in B \times \mathbb{R}^n$ and let $\varepsilon > 0$. Since g_{b_0} is an open map, there exists a $\delta > 0$ such that $\overline{B_{2\delta}(0)} \subseteq g_{b_0}(\overline{B_\varepsilon(0)})$ where $x_1 := g_{b_0}(x_0)$.

We claim that there exists a neighborhood $V \subseteq B$ of b_0 such that

$$|g_b(x) - g_{b_0}(x)| < \delta$$

for every $b \in V$ and $x \in \overline{B_\varepsilon(0)}$.

To show that, consider $\phi(b, x) := g_b(x) - g_{b_0}(x)$. The open set $\phi^{-1}(B_\delta(0))$ is a neighborhood of $\{b_0\} \times \mathbb{R}^n$ since $\phi(b_0, x) = 0$. Hence, there exist open subsets $V_x \subseteq B$ and $W_x \subseteq \mathbb{R}^n$ such that

$$\bigcup_{x \in \overline{B_\varepsilon(0)}} V_x \times W_x \subseteq \phi^{-1}(\overline{B_\delta(0)})$$

and $x \in W_x$. Since $\overline{B_\varepsilon(0)}$ is compact, there exist $x_1, \dots, x_n \in \overline{B_\varepsilon(0)}$ with $\overline{B_\varepsilon(0)} \subseteq \bigcup_{i=1}^n V_{x_i}$. The claim follows with $V := V_{x_1} \cap \dots \cap V_{x_n}$ which is open by forming the intersection over finitely many open sets.

Now we want to apply the previous lemma:

Consider the homeomorphism

$$\overline{B_{2\delta}(0)} \xrightarrow{\sim} g_b \circ g_{b_0}^{-1}(\overline{B_{2\delta}(0)})$$

for an arbitrary $b \in V$. Since

$$\overline{B_{2\delta}(0)} \subseteq g_{b_0}(\overline{B_\varepsilon(0)}) \implies g_{b_0}^{-1}(\overline{B_{2\delta}(0)}) \subseteq \overline{B_\varepsilon(0)}$$

we conclude from the above that

$$|(g_b \circ g_{b_0}^{-1})(x) - x| < \delta, \forall x \in \overline{B_{2\delta}(0)}$$

It follows that, by translation and scaling, $g_b \circ g_{b_0}^{-1}|_{\overline{B_{2\delta}(0)}}$ satisfies the conditions of Lemma **(3.8)**. Therefore, $\overline{B_\delta(0)} \subseteq (g_b \circ g_{b_0}^{-1})(\overline{B_{2\delta}(0)})$ and so $\overline{B_\delta(0)} \subseteq g_b(\overline{B_\varepsilon(0)})$.

From

$$V \times \overline{B_\delta(0)} \subseteq g(V \times \overline{B_\varepsilon(0)})$$

it follows that f is an open map.

(2) Glue together $f : U_f \to E(\mathfrak{b}')$ along local trivializations:

For an arbitrary $b \in B$, choose local trivializations (U, V, ϕ) and (U', V', ϕ') of b in $\mathfrak{b}$ and $\mathfrak{b}'$. Without loss of generality we may assume that $U = U'$ because otherwise we can choose $V = \phi^{-1}(U \cap U')$ and $V' = \phi'^{-1}(U \cap U')$ and restrict ϕ and ϕ' accordingly.

First, we restrict f to $V \cap f^{-1}(V')$. Since $V \cap f^{-1}(V')$ is an open neighborhood of $i(b)$ and contained in V, we can choose an open neighborhood $\tilde{U} \subseteq U$ of $i(b)$ and $\varepsilon > 0$ such that $\phi^{-1}(\tilde{U} \times B_\varepsilon(0)) \subseteq V \cap f^{-1}(V')$.

This yields a map $U' \times \mathbb{R}^n \to U' \times \mathbb{R}^n$ with

$$\tilde{U} \times \mathbb{R}^n \cong \tilde{U} \times B_\varepsilon(0) \xrightarrow{\sim} \phi^{-1}(\tilde{U} \times B_\varepsilon(0)) \xrightarrow{f} U' \times \mathbb{R}^n \subseteq U \times \mathbb{R}^n$$

that is injective and fiber-preserving and therefore an open map (apply 1.). It follows that $f : \phi^{-1}(\tilde{U} \times B_\varepsilon(0)) \to V'$ must be an open map as well since the other composing maps are homeomorphisms.

By glueing the $\phi^{-1}(\tilde{U} \times B_\varepsilon(0))$ together over all $b \in B$, we see that f is an open map.

□

We can easily generalize this to bundle-germs between microbundles over different base spaces:

Corollary 3.9.
If a map $g : B \to B'$ is covered by a bundle-germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$, then $\mathfrak{b}$ is isomorphic to the induced microbundle $g^*\mathfrak{b}'$.

Proof.
Let $f : U_f \to E'$ be a representative map for F. We define $F' : \mathfrak{b} \Rightarrow g^*\mathfrak{b}'$ by the representative

$$f' : U_f \to E(g^*\mathfrak{b}') \text{ with } f'(e) = (j(e), f(e)).$$

Every $f'(e)$ lies in $E(g^*\mathfrak{b}')$ because

$$g(j(e)) = j'(f(e))$$

as we can see from the commutative diagram for bundle-germs.

The germ F' is a bundle-germ covering the identity because

$$j(e) = j'_g(j(e), f(e)) = j'_g(f'(e))$$

and because f' is injective (since f is injective). Applying the previous proposition on F' proves the claim. □

3.2. Proving the Homotopy Theorem.

The following lemma will allow us to glue together bundle-germs over locally finite, closed domains if they agree on their intersection.

Lemma 3.10. *[Mil64, p.67]*
Let $\mathfrak{b}$ be a microbundle over B and $\{B_\alpha\}$ a locally finite collection of closed sets covering B. Additionally, we are given a collection of bundle-germs $F_\alpha : \mathfrak{b}|_{B_\alpha} \Rightarrow \mathfrak{b}'$ such that $F_\alpha = F_\beta$ on $\mathfrak{b}|_{B_\alpha \cap B_\beta}$. Then there exists a bundle-germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$ extending F_α, that is F and F_α agree on a sufficiently small neighborhood of $i(B_\alpha)$.

Proof.
Choose representative maps $f_\alpha : U_\alpha \to E(\mathfrak{b}')$ for F_α such that the U_α are open. For every α and β, choose an open neighborhood $U_{\alpha\beta}$ of $i(B_\alpha \cap B_\beta)$ on which the representative maps f_α and f_β agree. Now consider

$$U := \{e \in E : j(e) \in B_\alpha \cap B_\beta \implies e \in U_{\alpha\beta}\}$$

which is an open neighborhood of $i(B)$ because

(1) U is open:

Let $e \in U$ be arbitrary.

Since $\{B_\alpha\}$ is locally finite, there exists an open neighborhood V of $j(e)$ that intersects with only finitely many $B_{\alpha_1}, \dots, B_{\alpha_n}$. Note that with $e \in U$ it follows that $e \in U_{\alpha_i \alpha_j}, \forall 1 \le i, j \le n$.

Now we have an open neighborhood of e

$$\bigcap_{1 \le i,j \le n} U_{\alpha_i \alpha_j} \cap j^{-1}(V)$$

which is contained in U by construction.

(2) $i(B) \subseteq U$:

This follows from

$$j(i(b)) = b \in B_\alpha \cap B_\beta \implies i(b) \in i(B_\alpha \cap B_\beta) \subseteq U_{\alpha\beta}.$$

Now we can define $f : U \to E(\mathfrak{b}')$ in the obvious way

$$f(u \in U_{\alpha\beta}) := f_\alpha(u) = f_\beta(u)$$

which is continuous according to the [glueing lemma]. In particular, f agrees with f_α on $U_{\alpha\alpha}$ and hence f is a representative for a bundle-germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$ extending $\{F_\alpha\}$. $\square$

Lemma 3.11. *[Mil64, p.67]*
Let $\mathfrak{b}$ be a microbundle over $B \times [0,1]$. If $\mathfrak{b}|_{B\times[0,\frac{1}{2}]}$ and $\mathfrak{b}|_{B\times[\frac{1}{2},1]}$ are both trivial, then $\mathfrak{b}$ itself is trivial.

Proof.
Consider the identity bundle-germ over $\mathfrak{b}|_{B\times\{\frac{1}{2}\}}$, defined as the bundle-germ represented by the identity on $E(\mathfrak{b}|_{B\times\{\frac{1}{2}\}})$.

Since $\mathfrak{b}|_{B\times[\frac{1}{2},1]}$ and $\mathfrak{b}|_{B\times\{\frac{1}{2}\}}$ are both trivial, there exist isomorphism-germs

$$R : \mathfrak{b}|_{B\times[\frac{1}{2},1]} \Rightarrow \mathfrak{e}^n_{B\times[\frac{1}{2},1]} \text{ and } L : \mathfrak{b}|_{B\times\{\frac{1}{2}\}} \Rightarrow \mathfrak{e}^n_{B\times\{\frac{1}{2}\}}.$$

We can define a bundle-germ $\mathfrak{b}|_{B\times[\frac{1}{2},1]} \Rightarrow \mathfrak{b}|_{B\times\{\frac{1}{2}\}}$ extending the identity on $\mathfrak{b}|_{B\times\{\frac{1}{2}\}}$ using

$$M : \mathfrak{e}^n_{B\times[\frac{1}{2},1]} \Rightarrow \mathfrak{e}^n_{B\times\{\frac{1}{2}\}} \text{ with } (b,t,x) \mapsto (b, \frac{1}{2}, x)$$

to form the composition $L^{-1} \circ M \circ R$.

Using the previous lemma, we can piece this together with the identity bundle-germ on $\mathfrak{b}|_{B\times[0,\frac{1}{2}]}$ (note that the bundle-germs agree on their intersection) resulting in a bundle-germ

$$\mathfrak{b} \Rightarrow \mathfrak{b}|_{B\times[0,\frac{1}{2}]}.$$

Corollary **(3.9)** infers that $\mathfrak{b}$ is isomorphic to $r^*\mathfrak{b}|_{B\times[0,\frac{1}{2}]}$ where $r : B\times[0,1]$ is the retraction $(b,t) \mapsto (b, \min(t, \frac{1}{2}))$. But $\mathfrak{b}|_{B\times[0,\frac{1}{2}]}$ is trivial, hence $r^*\mathfrak{b}|_{B\times[0,\frac{1}{2}]}$ is trivial as well (see Lemma **(2.4)**), which concludes the proof. $\square$

Lemma 3.12. *[Mil64, p.67]*
Let $\mathfrak{b}$ be a microbundle over $B \times [0,1]$. Then every $b \in B$ has a neighborhood V such that $\mathfrak{b}|_{V\times[0,1]}$ is trivial.

Proof.
Let $b \in B$ be arbitrary.

For every $t \in [0,1]$, assume a neighborhood $U_t := V_t \times (t - \varepsilon_t, t + \varepsilon_t)$ of (b,t) such that $\mathfrak{b}|_{U_t}$ is trivial. Such a U_t can be constructed by taking a local trivialization (U', V', ϕ') of (b,t) in $\mathfrak{b}$ and restricting U' accordingly.

Since $\{b\} \times [0,1]$ is compact, we can choose a finitely many

$$(V_1 \times (t_1 - \varepsilon_1, t_1 + \varepsilon_1)), \dots, (V_n \times (t_n - \varepsilon_n, t_n + \varepsilon_n))$$

of the collection $\{U_t\}$ covering $\{b\} \times [0,1]$ and define $V = V_1 \cap \dots \cap V_n$.

The restricted microbundles $\mathfrak{b}|_{V\times(t_i-\varepsilon_i,t_i+\varepsilon_i)}$ are trivial, because every $\mathfrak{b}|_{U_t}$ is trivial and $V \times (t_i - \varepsilon_i, t_i + \varepsilon_i) \subseteq U_t$. Hence, there exists a subdivision $0 = t_0 < \dots < t_k = 1$ such that every $\mathfrak{b}|_{V\times[t_i,t_{i+1}]}$ is trivial.

By iteratively applying the previous lemma on the $\mathfrak{b}|_{V\times[t_i,t_{i+1}]}$, it follows that $\mathfrak{b}|_{V\times[0,1]}$ is trivial. □

Lemma 3.13.
Let B be a paracompact hausdorff space and let $\{V_\alpha\}$ be a locally finite open cover of B. Then there exists a locally finite closed cover $\{\overline{B_\beta}\}$ of B such that every $\overline{B_\beta}$ intersects with only finitely many $V_{\alpha_1}, \dots V_{\alpha_n}$.

Proof.
For every $b \in B$, there exists an open neighborhood U_b of b that intersects only with finitely many

$$V_{\alpha_1}, \dots V_{\alpha_k}$$

using the definition of local finiteness for $\{V_\alpha\}$. Clearly, the collection $\{U_b\}$ over all $b \in B$ covers B.

Since B is paracompact, there exists a locally finite subcover $\{B_\beta\}$.

The collection $\{\overline{B_\beta}\}$ meets our requirements, because

(1) $\{\overline{B_\beta}\}$ is locally finite:

For an arbitrary $b \in B$, let W_b be an open neighborhood of b that intersects only finitely many $B_{\beta_1}, \dots, B_{\beta_k}$. Now W_b intersects only $\overline{B_{\beta_1}}, \dots, \overline{B_{\beta_1}}$, because

$$\begin{aligned} & W_b \cap B_\beta = \emptyset \\ \implies & B_\beta \subseteq B - W_b \\ \implies & \overline{B_\beta} \subseteq \overline{B - W_b} = B - W_b \\ \implies & W_b \cap \overline{B_\beta} = \emptyset. \end{aligned}$$

(2) Every $\overline{B_\beta}$ intersects only finitely many $V_{\alpha_1}, \dots V_{\alpha_k}$:

Since $B_\beta \subseteq U_b$ for some $b \in B$, B_β intersects only finitely many V_α. Using the same argument as in (1.), it follows that $\overline{B_\beta}$ intersects with the exact same V_α.

□

Lemma 3.14. *[Mil64, p.67]*
Let $\mathfrak{b}$ be a microbundle over $B \times [0,1]$ where B is paracompact hausdorff. Then there exists a bundle-germ $R : \mathfrak{b} \Rightarrow \mathfrak{b}|_{B\times\{1\}}$ covering the retraction $r : B \times [0,1] \to B \times \{1\}$ with $r(b,t) = (b,1)$.

Proof.
First, we assume a locally finite covering $\{V_\alpha\}$ of open sets where $\mathfrak{b}|_{V_\alpha \times [0,1]}$ is trivial. The existence of such a covering is justified by Lemma **(3.12)** and paracompactness of B.

This cover can be equipped with a partition of unity

$$\lambda_\alpha : B \to [0,1] \text{ with } \mathrm{supp}\lambda_\alpha \subseteq V_\alpha$$

since B is paracompact hausdorff, that is rescaled in way that

$$\max_\alpha(\lambda_\alpha(b)) = 1, \forall b \in B.$$

Such a rescaling can be achieved by dividing λ_α by $\max_\alpha \lambda_\alpha$ which is well-defined because $\{V_\alpha\}$ is locally finite and continuous because the max function is continuous. Also, $\max_\alpha \lambda_\alpha(b) > 0$ since the initial partition of unity adds up to 1 in every point.

Now we define a retraction $r_\alpha : B \times [0,1] \to B \times [0,1]$ with

$$r_\alpha(b,t) = (b, \max(t, \lambda_\alpha(b))).$$

In the following, we construct bundle-germs $R_\alpha : \mathfrak{b} \Rightarrow \mathfrak{b}$ covering r_α and 'compose' them to obtain the required bundle-germ.

(1) We can divide $B \times [0,1]$ into two subsets

$$A_\alpha = \mathrm{supp}\lambda_\alpha \times [0,1] \subseteq V_\alpha \times [0,1] \text{ and } A'_\alpha = \{(b,t) : t \geq \lambda_\alpha(b)\}.$$

Since $\mathfrak{b}|_{A_\alpha}$ is trivial, we can, analogous to the proof of Lemma **(3.11)**, extend the identity bundle-germ on $\mathfrak{b}|_{A_\alpha \cap A_\alpha'}$ to a bundle-germ

$$\mathfrak{b}|_{A_\alpha} \Rightarrow \mathfrak{b}|_{A_\alpha \cap A_\alpha'}$$

using the bundle-germ

$$\mathfrak{e}^n_{A_\alpha} \Rightarrow \mathfrak{e}^n_{A_\alpha \cap A_\alpha'} \text{ with } (a,x) \mapsto (r_\alpha(a), x).$$

Pieced together with the identity bundle-germ $\mathfrak{b}|_{A_\alpha'}$ (note that A_α and A_α' are both closed), we obtain a bundle-germ R_α covering r_α.

(2) Lastly, we construct a bundle-germ R using the R_α.

Applying the well-ordering theorem, which is equivalent to the axiom of choice (see [Kuc09, p.14]), we can assume an ordering of $\{V_\alpha\}$.

Let $\{B_\beta\}$ be a locally finite closed cover of B such that B_β intersects only finitely many $V_{\alpha_1} < \cdots < V_{\alpha_k}$ obtained by Lemma **(3.13)**.

Now the composition $R_{\alpha_1} \circ \ldots \circ R_{\alpha_k}$ restricts to a bundle-germ

$$R(\beta) : \mathfrak{b}|_{B_\beta \times [0,1]} \Rightarrow \mathfrak{b}|_{B_\beta \times \{1\}}$$

covering the retraction $(b,t) \mapsto (b,1)$. That is because for every $b \in B_\beta$, we find an $1 \leq i \leq k$ with $\lambda_{\alpha_i}(b) = 1$ and hence $r_{\alpha_i}(b,t) = (b,1)$.

Pieced together using Lemma **(3.11)**, we obtain a bundle-germ

$$R : \mathfrak{b}|_{B \times [0,1]} \to \mathfrak{b}|_{B \times \{1\}}$$

covering $(b,t) \mapsto (b,1)$.

$\square$

Finally, we gathered all the tools to proof the Homotopy Theorem.

Proof of the Homotopy Theorem.
The previous lemma states that there exists a bundle-germ

$$R : H^*\mathfrak{b} \Rightarrow H^*\mathfrak{b}|_{A \times \{1\}}$$

covering the retraction $(a,t) \mapsto (a,1)$.

By restricting R to $H^*\mathfrak{b}|_{A \times \{0\}}$ we obtain a bundle-germ

$$H^*\mathfrak{b}|_{A \times \{0\}} \Rightarrow H^*\mathfrak{b}|_{A \times \{1\}}$$

covering $\theta : A \times \{0\} \xrightarrow{\sim} A \times \{1\}$ with $\theta(a,0) = (a,1)$. Applying Corollary **(3.9)** yields $H^*\mathfrak{b}|_{A \times \{0\}} \cong \theta^*(H^*\mathfrak{b}|_{A \times \{1\}})$.

Considering $A \times \{0\} = A$, we can identify $H^*\mathfrak{b}|_{A \times \{0\}}$ with $f^*\mathfrak{b}$ as follows

$$H^*\mathfrak{b}|_{A \times \{0\}} = \iota^*(H^*\mathfrak{b}) \cong (H \circ \iota)^*\mathfrak{b} = f^*\mathfrak{b}.$$

Analogously, we can identify $\theta^*(H^*\mathfrak{b}|_{A \times \{1\}})$ with $g^*\mathfrak{b}$.

Together with $H^*\mathfrak{b}|_{A \times \{0\}} \cong \theta^*(H^*\mathfrak{b}|_{A \times \{1\}})$, it follows that $f^*\mathfrak{b} \cong g^*\mathfrak{b}$. $\square$

4. Rooted Microbundles and Suspensions

In the following, we provide a proof for the Bouquet Lemma presented in Section **(2.2)**. To this end, we introduce the concept of 'rooted microbundles', which allows us to define the wedge sum of two microbundles in a precise manner. Additionally, we show a version of the Homotopy Theorem that is compatible with rooted-microbundles.

In this section, we assume that every topological space is equipped with an arbitrary base point which we will denote with subscript 0.

4.1. **Rooted Microbundles.**

Definition 4.1.
A *rooted microbundle* $\mathfrak{b}$ over B is a microbundle over B together with an isomorphism-germ

$$R : \mathfrak{b}|_{b_0} \Rightarrow \mathfrak{e}^n_{b_0}.$$

Two rooted microbundles $\mathfrak{b}$ and $\mathfrak{b}'$ are *rooted isomorphic* if there exists an isomorphism-germ $\mathfrak{b} \Rightarrow \mathfrak{b}'$ extending

$$R'^{-1} \circ R : \mathfrak{b}|_{b_0} \Rightarrow \mathfrak{b}'|_{b_0}.$$

Remark 4.2.
One can always define a rooting for a given microbundle by choosing a local trivialization in the base point and restricting it to the fiber of b_0.

Definition 4.3.
Let $\mathfrak{b}$ be a rooted microbundle over B and $f : A \to B$ a based map. We equip the induced microbundle $f^*\mathfrak{b}$ with the rooting

$$R_f : E(f^*\mathfrak{b}|_{a_0}) = a_0 \times E(\mathfrak{b}|_{b_0}) \Rightarrow e^n_{a_0}$$

that coincides with R if we consider $a_0 \times E(\mathfrak{b}|_{b_0}) = E(\mathfrak{b}|_{b_0})$ and $e^n_{a_0} = e^n_{b_0}$.

The total space $E(f^*\mathfrak{b}|_{a_0})$ is the same as $a_0 \times E(\mathfrak{b}|_{b_0})$, because

$$\begin{aligned} E(f^*\mathfrak{b}|_{a_0}) &= \{(a,e) \in A \times E(\mathfrak{b}) : a = a_0 \text{ and } f(a) = b_0 = j(e)\} \\ &= a_0 \times \{e \in E(\mathfrak{b}) : j(e) = b_0\} = a_0 \times E(\mathfrak{b}|_{b_0}). \end{aligned}$$

Given a rooted microbundle $\mathfrak{b}$ and homotopic based maps $f, g : A \to B$, the Homotopy Theorem yields that $f^*\mathfrak{b}$ and $g^*\mathfrak{b}$ are isomorphic (not rooted-isomorphic).

With the preliminary work in Section **(3.2)**, we can derive a version of the Homotopy Theorem that also accounts for rooted isomorphy.

Theorem 4.4 (Rooted Homotopy Theorem)**.**
Let $\mathfrak{b}$ be a rooted microbundle over B and $f, g : A \to B$ be two based maps where A is paracompact hausdorff. If there exists a homotopy $H : A \times [0,1] \to B$ between f and g that leaves the base point fixed, then the two rooted microbundles $f^*\mathfrak{b}$ and $g^*\mathfrak{b}$ are rooted isomorphic.

In order to proof this, we need to show a 'rooted version' of Lemma **(3.12)**.

First, note that

$$E(H^*\mathfrak{b}|_{a_0 \times [0,1]}) = E(\iota^*(H^*(\mathfrak{b}))) \cong E((H \circ \iota)^*\mathfrak{b}) = E(c^*_{a_0 \times [0,1], b_0}\mathfrak{b}),$$

whose total space is of the form $(a_0 \times [0,1]) \times E(\mathfrak{b})$. Based on this, we can define an isomorphism-germ

$$\overline{R} : H^*\mathfrak{b}|_{a_0 \times [0,1]} \Rightarrow \mathfrak{e}^n_{a_0 \times [0,1]}$$

represented by

$$\overline{r}(a_0, t, v) = (a_0, t, r^{(2)}(v)),$$

where $r : V \to b_0 \times \mathbb{R}^n$ is a representative for R. The representative $\overline{r}$ is a homeomorphism on its image, as its components are homeomorphisms on their image.

Lemma 4.5.
Let $\mathfrak{b}$ be a rooted microbundle over B and let $H : A \times [0,1] \to B$ be a map that leaves the base point fixed. Then there exists a neighborhood V of a_0 together with an isomorphism-germ

$$H^*\mathfrak{b}|_{V\times[0,1]} \Rightarrow \mathfrak{e}^n_{V\times[0,1]}$$

extending $\overline{R}$ (as defined above).

Proof.
By applying Lemma **(3.12)**, it follows that there exists an isomorphism-germ

$$Q : H^*\mathfrak{b}|_{V\times[0,1]} \Rightarrow \mathfrak{e}^n_{V\times[0,1]}$$

for a sufficiently small neighborhood V of a_0. However, Q does not extend $\overline{R}$ in general.

In order to fix this, consider

$$Q \circ \overline{R}^{-1} : \mathfrak{e}^n_{a_0\times[0,1]} \Rightarrow \mathfrak{e}^n_{a_0\times[0,1]}$$

together with a representative $f : U_f \to (a_0 \times [0,1]) \times \mathbb{R}^n$.

Similar to the construction of $\overline{R}$, we can construct an isomorphism-germ

$$P : \mathfrak{e}^n_{V\times[0,1]} \Rightarrow \mathfrak{e}^n_{V\times[0,1]}$$

extending $Q \circ \overline{R}^{-1}$ represented by

$$p(a,t,x) = (a, f(a_0,t,x))$$

considering $f(a_0,t,x) \in [0,1] \times \mathbb{R}^n$.

Restricted to $\mathfrak{e}^n_{a_0\times[0,1]}$, P agrees with $Q \circ \overline{R}^{-1}$ and thus

$$Q^{-1} \circ P|_{\mathfrak{e}^n_{a_0\times[0,1]}} = (Q^{-1} \circ (Q \circ \overline{R}^{-1})) = ((Q^{-1} \circ Q) \circ \overline{R}^{-1}) = \overline{R}^{-1}.$$

Since P and Q are both isomorphism-germs,

$$P^{-1} \circ Q : H^*\mathfrak{b}|_{V\times[0,1]} \Rightarrow \mathfrak{e}^n_{V\times[0,1]}$$

is an isomorphism-germ extending $\overline{R}$. □

We are now able to show the Rooted Homotopy Theorem.

To understand the proof, it is useful to have the constructions of Lemma **(3.14)** in mind, because we will modify them slightly in order to preserve the rootings.

Proof of the Rooted Homotopy Theorem.
We need to show that $f^*\mathfrak{b}$ and $g^*\mathfrak{b}$ are rooted isomorphic, that is there exists an isomorphism-germ $f^*\mathfrak{b} \Rightarrow g^*\mathfrak{b}$ extending $R_g^{-1} \circ R_f = id_{\mathfrak{a}}$.

For the initial Homotopy Theorem, we constructed a bundle-germ

$$F : H^*\mathfrak{b} \Rightarrow H^*\mathfrak{b}|_{A\times\{1\}}$$

covering $(a,t) \mapsto (a,1)$ and restricted it to $H^*\mathfrak{b}|_{A\times\{0\}}$. The required isomorphism-germ was then obtained by identifying $f^*\mathfrak{b}$ with $H^*\mathfrak{b}|_{A\times\{0\}}$ and $g^*\mathfrak{b}$ with $H^*\mathfrak{b}|_{A\times\{0\}}$.

We must make slight modifications to the construction of F such that it extends $f^*\mathfrak{b}|_{b_0} \cong H^*\mathfrak{b}|_{a_0\times\{0\}} \Rightarrow H^*\mathfrak{b}|_{a_0\times\{1\}} \cong g^*\mathfrak{b}|_{b_0}$ represented by

$$(a_0, e) = ((a_0, 0), e) \mapsto ((a_0, 1), e) = (a_0, e).$$

This can be achieved by choosing a locally finite open cover $\{V_\alpha\}$ of A (as in Lemma **(3.14)**), removing the base point a_0 from every set and adding V obtained from Lemma **(4.5)**. Since $a_0 \in V$, the resulting collection is still a locally finite open cover of A.

In the following, we will denote constructions over V with subscript V and constructions over the other sets from the cover with subscript α.

We continue with the proof of Lemma **(3.14)**. Note that $\lambda_V(a_0) = 1$. That is because we removed a_0 from every other set and hence $\lambda_\alpha(a_0) = 0$.

Lastly, we construct the extension R_V for r_V like in Section **(3.2)**, but instead of choosing an arbitrary trivialization $E(H^*\mathfrak{b}|_{A_V}) \cong A_V \times \mathbb{R}^n$ for the construction we use a representative r for the bundle-germ constructed in Lemma **(4.5)**.

This has the advantage that the representative

$$E(H^*\mathfrak{b}|_{A_V}) \xrightarrow{r} A_V \times \mathbb{R}^n \xrightarrow{r_V \times id} (A_V \cap A'_V) \times \mathbb{R}^n \xrightarrow{r^{-1}} E(H^*\mathfrak{b}|_{A_V \cap A'_V})$$

for R_V maps elements $((a_0, 0), e)$ to $((a_0, 1), e)$. Additionally, every other R_α leaves $H^*\mathfrak{b}|_{a_0\times\{0\}}$ unaffected because $r_\alpha(a_0, t) = (a_0, \max(\underbrace{\lambda_\alpha(t)}_{=0}, t)) = (a_0, t)$.

It follows that, by piecing together the R_α and R_V like in Lemma **(3.14)**, we obtain a bundle germ $F : H^*\mathfrak{b} \Rightarrow H^*\mathfrak{b}|_{A\times\{1\}}$ that extends $R_g^{-1} \circ R_f$. This completes the proof. $\square$

Now that we introduced rooted-microbundles, we are able to define the wedge sum. As we will see in the subsequent proof, the definition of the wedge sum it is necessary to have a fixed rooting given because otherwise one would have to choose a rooting which the resuling microbundle depends on, hence not being well-defined.

Given a quotient space $A \sqcup B/\sim$ and maps $f : A \to C$ and $g : B \to C$, we define

$$f \cup g : (A \sqcup B/\sim) \to C \text{ with}$$

$$x \mapsto \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}.$$

Clearly, this map is only well-defined if $a \sim b \implies f(a) = g(b)$.

Definition 4.6.
Let $\mathfrak{a}$ and $\mathfrak{b}$ be two rooted microbundles over A and B. The *wedge sum* $\mathfrak{a} \vee \mathfrak{b}$ of $\mathfrak{a}$ and $\mathfrak{b}$ is a microbundle

$$A \vee B \xrightarrow{i_a \cup i_b} E(\mathfrak{a} \vee \mathfrak{b}) \xrightarrow{j_a \cup j_b} A \vee B$$

with the total space defined as

$$(E(\mathfrak{a}) \sqcup E(\mathfrak{b}))/f(e_a) \sim e_a$$

where $f : W_a \xrightarrow{\sim} W_b$ is a representative for $R_b^{-1} \circ R_a$.

We equip $\mathfrak{a} \vee \mathfrak{b}$ with a rooting

$$R : E((\mathfrak{a} \vee \mathfrak{b})|_{a_0}) \Rightarrow \mathfrak{e}^n_{a_0}$$

represented by any representative for R_a (or R_b).

Proof that $\mathfrak{a} \vee \mathfrak{b}$ is a rooted microbundle.
Let $f : W_a \xrightarrow{\sim} W_b$ be a representative for $R_b^{-1} \circ R_a$.

1. $\mathfrak{a} \vee \mathfrak{b}$ is a rooted microbundle:

- The injection map $i_a \cup i_b$ is well-defined because

$$[i(a_0)] = [i_a(a_0)] = [f(i_a(a_0))] = [i_b(b_0)] = [i(b_0)]$$

  and continuous since both i_a and i_b are continuous.

- The projection map $j_a \cup j_b$ is well-defined because

$$\forall e \in W_a : [j(e)] = [j_a(e)] = [a_0] = [b_0] = [j_b(f(e))] = [j(f(e))]$$

  and continuous since both j_a and j_b are continuous.

- The composition $j \circ i$ is the identity because

$$\forall a \in A : j(i(a)) = j(i_a(a)) = j_a(i_a(a)) = a$$

  since $j_a \circ i_a = id_A$ (symmetrical for B).

It remains to be shown that $\mathfrak{a} \vee \mathfrak{b}$ is locally trivial.

Let $x \in A \vee B$. For reasons of symmetry, we can assume that $x \in A$.

(1) $x \neq a_0$:

Choose a local trivialization (U, V, ϕ) for x in $\mathfrak{a}$. Without loss of generality, we can assume that $U \cap B = \emptyset$ by subtracting $\{a_0\}$ from U if necessary. Note that $\{a_0\}$ is closed since A is hausdorff.

Now we can simply use this trivialization for $\mathfrak{a} \vee \mathfrak{b}$, because $U \subseteq A$ is open in $A \vee B$ and $V \subseteq E(\mathfrak{a})$ is open in $E(\mathfrak{a} \vee \mathfrak{b})$. Furthermore, since i and j reduce to i_a and j_a, it follows that ϕ commutes with i and $id \times 0$ as well as with j and π_1.

(2) $x = a_0$:

Let (U_a, V_a, ϕ_a) and (U_b, V_b, ϕ_b) be local trivializations for $a_0 = b_0$ in $\mathfrak{a}$ and $\mathfrak{b}$.

Since $W_a \subseteq E(\mathfrak{a}|_{a_0})$ is open, there exists an open subset $W'_a \subseteq E(\mathfrak{a})$ such that $W_a = W'_a \cap E(\mathfrak{a}|_{a_0})$.

Let $U'_a \subseteq A$ be an open neighborhood of a_0 and $\varepsilon > 0$ such that

$$V'_a := U'_a \times B_\varepsilon(0) \subseteq \phi_a(W'_a).$$

This allows us to define the map

$$\phi'_a : V'_a \xrightarrow{\sim} \phi'_a(V'_a) \subseteq A \times \mathbb{R}^n \text{ with}$$

$$\phi'_a(e) = (j_a(e), (\phi_b^{(2)} \circ f \circ \phi_a^{-1})(a_0, \phi_a^{(2)}(e))).$$

Now we can show local triviality in a_0 using the homeomorphism

$$\phi'_a \cup \phi_b : V'_a \cup V_b \xrightarrow{\sim} \phi'_a(V'_a \cup V_b) \subseteq (A \vee B) \times \mathbb{R}^n$$

This map is well-defined, because

$$\phi'_a(e) = (a_0, (\phi_b^{(2)} \circ f \circ \phi_a^{-1})(a_0, \phi_a^{(2)}(e)))$$

$$= (b_0, \phi_b^{(2)}(f(e))) = (j_b(f(e)), \phi_b^{(2)}(f(e))) = \phi_b(f(e)).$$

Homeomorphy follows from the fact that both ϕ'_a and ϕ_b are homeomorphisms, and that $\phi'_a(e_a) = \phi(e_b) \implies f(e_a) = e_b$.

Commutativity between $i_a \cup i_b$ and $id \times 0$ as well as between $j_a \cup j_b$ and π_1 is inherited from ϕ_a and ϕ_b. Note that $\phi_a(i_a(a)) = (a, 0) = \phi'_a(i_a(a))$.

Applying Lemma **(1.3)** yields that $\mathfrak{a} \vee \mathfrak{b}$ is locally trivial.

2. $\mathfrak{a} \vee \mathfrak{b}$ is well-defined:

Let f' be another representative for $R_b^{-1} \circ R_a$ and $(\mathfrak{a} \vee \mathfrak{b})'$ the resulting wedge sum. We need to find an isomorphism-germ that extends $R'^{-1} \circ R$.

In order to do this, choose an open neighborhood $V \subseteq E(\mathfrak{a}|_{a_0})$ of $i_a(a)$ where f and f' agree.

By subtracting the closed set $j_a^{-1}(a_0) - V$ from $E(\mathfrak{a} \vee \mathfrak{b})$ and $E(\mathfrak{a} \vee \mathfrak{b})'$, the microbundles remain unchanged due to Proposition **(1.5)**.

But now the total spaces $E(\mathfrak{a} \vee \mathfrak{b})$ and $E((\mathfrak{a} \vee \mathfrak{b})')$ are the same. That is because $E(\mathfrak{a} \vee \mathfrak{b})$ and $E((\mathfrak{a} \vee \mathfrak{b})')$ could only differ in $j_a^{-1}(a_0) - V$.

Furthermore, since injection and projection are defined exactly the same, it follows that the identity $(\mathfrak{a} \vee \mathfrak{b}) \Rightarrow (\mathfrak{a} \vee \mathfrak{b})'$ is an isomorphism-germ. Together with

$$R'^{-1} \circ R = R^{-1} \circ R = id_{\mathfrak{a}},$$

this completes the proof. □

4.2. **Microbundles over a Suspension.** In the following, let B be a *reduced suspension*

$$SX = (X \times [0,1])/(X \times \{0,1\} \cup x_0 \times [0,1])$$

over X.

Let $\phi : B \to B \vee B$ denote the map that sends $X \times [0, \frac{1}{2}]$ to the first B via

$$\phi([x,t]) = [x, 2t]$$

and $X \times [\frac{1}{2}, 1]$ to the second B via

$$\phi([x,t]) = [x, 2t-1].$$

Additionally, let $c_1 : B \vee B \to B$ denote the map that is the identity on the first summand and the constant map $const_{b_0}$ on the second summand (symmetrically define c_2).

Lemma 4.7.
The following non-rooted isomorphy holds:

$$\phi^*(\mathfrak{b} \vee \mathfrak{e}_B^n) \cong \mathfrak{b} \cong \phi^*(\mathfrak{e}_B^n \vee \mathfrak{b})$$

Proof.
We prove the lemma in two steps.

- $c_1^*\mathfrak{b} \cong \mathfrak{b} \vee \mathfrak{e}_B^n$:

  First, consider the total space
  $$\begin{aligned} E(c_1^*\mathfrak{b}) &= \{(b,e) \in (B \vee B) \times E(\mathfrak{b}) : c_1(b) = j(e)\} \\ &= (\{(b,e) \in B \times E(\mathfrak{b}) : b = j(e)\} \sqcup B \times E(\mathfrak{b}|_{b_0}))/\sim \\ &= (\{(j(e),e) : e \in E(\mathfrak{b})\} \sqcup B \times E(\mathfrak{b}|_{b_0}))/\sim \end{aligned}$$
  where $(b,e) \sim (b',e') \iff b = b_0 = b' \wedge e = e'$. We can omit the first component on the left side resulting in
  $$E(c_1^*\mathfrak{b}) = (E(\mathfrak{b}) \sqcup (B \times E(\mathfrak{b}|_{b_0})))/\sim$$
  where $e \sim (b,e') \iff b = b_0 \wedge e = e'$.

  On the other side, consider
  $$E(\mathfrak{b} \vee \mathfrak{e}_B^n) = (E(\mathfrak{b}) \sqcup (B \times \mathbb{R}^n))/e \sim' f(e)$$
  with f being some representative $U_f \to b_0 \times \mathbb{R}^n$ for $id^{-1}_{b_0 \times [0,1]} \circ R_b$.

  Now, we have a map ψ from the open subset of $E(c_1^*\mathfrak{b})$
  $$(E(\mathfrak{b}) \sqcup (B \times U_f))/\sim$$
  to the open subset of $E(\mathfrak{b} \vee \mathfrak{e}_B^n)$
  $$(E(\mathfrak{b}) \sqcup (B \times f(U_f)))/\sim'$$
  with $\psi([e]) = [e]$ and $\psi([b,e]) = [(b, f^{(2)}(e))]$. The map is well-defined because for every $e \sim (b_0, e)$
  $$\psi([e]) = [e] = [f(e)] = [b_0, f^{(2)}(e)] = \psi([b_0, e]).$$
  Homeomorphy of ψ follows by the homeomorphy of its summands.

  The map commutes with the injection map in the first summand
  $$\begin{aligned} \psi(i_{c_1}([b_1])) &= \psi(b_1, i(c_1([b_1]))) = \psi(b_1, i(b_1)) \\ &= [b_1, f^{(2)}(i(b_1))] = [f(i(b_1))] = [i(b_1)] \end{aligned}$$
  and the second summand
  $$\begin{aligned} \psi(i_{c_1}([b_2])) &= \psi(b_2, i(c_1([b_2]))) = \psi(b_2, i(b_0)) \\ &= [b_2, f^{(2)}(b_0)] = [b_2, 0] = [i_{\mathfrak{e}_B^n}(b_2)] \end{aligned}$$
  as well as with the projection map in the first summand
  $$j_{c_1}([e]) = [j(e)] = j(\psi([e]))$$
  and the second summand
  $$j_{c_1}([b,e]) = [b] = [\pi_1(b, \psi^{(2)}([b,e]))] = [\pi_1(\psi(b,e))].$$
  Therefore, ψ represents an isomorphism-germ between $c_1^*\mathfrak{b}$ and $\mathfrak{b} \vee \mathfrak{e}_B^n$.

- From $c_1 \circ \phi = id_B$ we can conclude that
$$\phi^*(\mathfrak{b} \vee \mathfrak{e}_B^n) \cong \phi^* c_1^* \mathfrak{b} \cong (c_1 \circ \phi)^* \mathfrak{b} \cong \mathfrak{b}.$$
The isomorphy $\mathfrak{b} \cong \phi^*(\mathfrak{e}_B^n \vee \mathfrak{b})$ follows by symmetry, which concludes the proof.

$\square$

Lemma 4.8.
Let $\mathfrak{a}$ and $\mathfrak{b}$ be rooted microbundles over A and B. For maps $f : A' \to A$ and $g : B' \to B$ the following non-rooted isomorphy holds:
$$(f \vee g)^*(\mathfrak{a} \vee \mathfrak{b}) \cong f^*\mathfrak{a} \vee g^*\mathfrak{b}$$

Proof.
Consider the equation
$$\begin{aligned} &E((f \vee g)^*(\mathfrak{a} \vee \mathfrak{b})) \\ &= \{(x, e) \in (A' \vee B') \times E(\mathfrak{a} \vee \mathfrak{b}) : (f \vee g)(x) = j(e)\} \\ &= \{(x, e) \in ((A' \times E(\mathfrak{a})) \sqcup (B' \times E(\mathfrak{b})))/\sim : (f \vee g)(x) = j(e)\} \\ &= (\{(x, e) \in A' \times E(\mathfrak{a}) : f(x) = j_a(e)\} \sqcup \{(x, e) \in B' \times E(\mathfrak{b}) : g(x) = j_b(e)\})/\sim \\ &= (E(f^*\mathfrak{a}) \sqcup E(g^*\mathfrak{b}))/\sim \\ &= E(f^*\mathfrak{a} \vee g^*\mathfrak{b}) \end{aligned}$$
where $(a, e_a) \sim (b, e_b) \iff a = a_0 = b_0 = b \wedge e_a = e_b$ in $E(\mathfrak{a} \vee \mathfrak{b})$.

Additionally, the injection
$$i_{f \vee g}(a) = i_f(a) = i_\vee(a)$$
and projection maps
$$j_{f \vee g}(a, e) = a = j_f(a, e) = i_\vee(a, e)$$
are equal. Here, $i_\vee$ and $j_\vee$ denote the injection and projection maps for $f^*\mathfrak{a} \vee g^*\mathfrak{b}$. It follows that the two microbundles are isomorphic. $\square$

Let $r : B \xrightarrow{\sim} B$ denote the homeomorphism that corresponds to the 'reflection'
$$(x, t) \mapsto (x, 1 - t)$$
and let $c : B \vee B \to B$ be the identity on the first summand and r on the second summand.

Lemma 4.9.
The induced microbundle $\phi^*(\mathfrak{b} \vee r^*\mathfrak{b})$ is trivial.

Proof.
The composition $f \circ \phi$ is null-homotopic via the homotopy $H : B \times [0, 1] \to B$ with
$$H([x, t], s) = f(\phi(x, t * s)).$$
Therefore $\phi^* f^* \mathfrak{b} \cong (f \circ \phi)^* \mathfrak{b} \cong const_{b_0}^* \mathfrak{b} \cong \mathfrak{e}_B^n$ (see Theorem **(3.1)**).

Applying the previous lemma, it follows that $\phi^*(\mathfrak{b} \vee c^*\mathfrak{b}) = \phi^* f^* \mathfrak{b}$ and hence
$$\phi^*(\mathfrak{b} \vee c^*\mathfrak{b}) \cong \mathfrak{e}_B^n.$$

□

Definition 4.10.
The *Whitney sum* of two rooted microbundles $\mathfrak{b}$ and $\mathfrak{b}'$ over B is the initial Whitney sum $\mathfrak{b} \oplus \mathfrak{b}'$ together with the rooting

$$R \oplus R' : (\mathfrak{b} \oplus \mathfrak{b}')|_{b_0} \Rightarrow \mathfrak{e}_{b_0}^{n_1} \oplus \mathfrak{e}_{b_0}^{n_2} = \mathfrak{e}_{b_0}^{n_1+n_1}.$$

Lemma 4.11.
The following non-rooted isomorphy holds for rooted microbundles $\mathfrak{a}$ and $\mathfrak{a}'$ over A and $\mathfrak{b}$ and $\mathfrak{b}'$ over B:

$$(\mathfrak{a} \vee \mathfrak{b}) \oplus (\mathfrak{a}' \vee \mathfrak{b}') \cong (\mathfrak{a} \oplus \mathfrak{a}') \vee (\mathfrak{b} \oplus \mathfrak{b}')$$

Proof.
Consider the equation

$$\begin{aligned}
&E((\mathfrak{a} \vee \mathfrak{b}) \oplus (\mathfrak{a}' \vee \mathfrak{b}')) \\
&= \{(e, e') \in E(\mathfrak{a} \vee \mathfrak{b}) \times E(\mathfrak{a}' \vee \mathfrak{b}') : j(e) = j'(e')\} \\
&= \{(e, e') \in ((E(\mathfrak{a}) \sqcup E(\mathfrak{b}))/\sim) \times ((E(\mathfrak{a}') \sqcup E(\mathfrak{b}'))/\sim') : j(e) = j'(e')\} \\
&= (\{(e, e') \in E(\mathfrak{a}) \times E(\mathfrak{a}') : j_a(e) = j_{a'}(e')\} \sqcup \\
&\quad \{(e, e') \in E(\mathfrak{b}) \times E(\mathfrak{b}') : j_b(e) = j_{b'}(e')\})/\backsim \\
&= (E(\mathfrak{a} \oplus \mathfrak{a}') \sqcup E(\mathfrak{b} \oplus \mathfrak{b}'))/\backsim \\
&= E((\mathfrak{a} \oplus \mathfrak{a}') \vee (\mathfrak{b} \oplus \mathfrak{b}'))
\end{aligned}$$

where $(e_a, e'_a) \backsim (e_b, e'_b) \iff e_a \sim e_b \wedge e'_a \sim' e'_b$. Here, the equivalence relations $\sim$ and $\sim'$ denote the ones used in the construction of the corresponding wedge sums.

Additionally, the injection

$$i_\oplus(a) = (i_a(a), i'_a(a)) = i_\vee(a) \text{ (symmetrical for } b)$$

and projection maps

$$j_\oplus(e, e') = j(e) = j_\vee(e)$$

are equal. Here, $i_\oplus$ and $j_\oplus$ denote the injection and projection maps for

$$(\mathfrak{a} \vee \mathfrak{b}) \oplus (\mathfrak{a}' \vee \mathfrak{b}')$$

and $i_\vee$ and $j_\vee$ denote the injection and projection maps for

$$(\mathfrak{a} \oplus \mathfrak{a}') \vee (\mathfrak{b} \oplus \mathfrak{b}').$$

It follows that the two microbundles are isomorphic. □

Lemma 4.12.
Let $\mathfrak{b}$ be a rooted microbundle over a paracompact hausdorff space B. Then there exists a closed neighborhood W of b_0 and an isomorphism-germ

$$\mathfrak{b}|_W \Rightarrow \mathfrak{e}_W^n$$

extending R together with a map $\lambda : B \to [0, 1]$ with

$$supp\lambda \subseteq W \text{ and } \lambda(b_0) = 1.$$

Proof.
Let $r : W_r \to b_0 \times \mathbb{R}^n$ be a representative map for R.

Choose a local trivialization (U, V, ϕ) for b_0 such that $V \cap E(\mathfrak{b}|_{b_0}) \subseteq W_r$. The argument that such a trivialization exists was already given in the proof that the wedge sum is microbundle.

Consider the map

$$\psi : V \xrightarrow{\sim} \psi(V) \subseteq U \times \mathbb{R}^n \text{ with}$$
$$\psi(e) = (j(e), r(\phi^{-1}(b_0, \phi^{(2)}(e))))$$

which is a representative for an isomorphism-germ $\mathfrak{b}|_U \Rightarrow \mathfrak{e}^n_U$ extending R.

Consider the open covering of B with U and $B - \{b_0\}$. Since B is paracompact, we can apply the concept of partitions of unity that gives us a map

$$\lambda : B \to [0, 1] \text{ with } \mathrm{supp}\lambda \subseteq U$$

and $\lambda(b_0) = 1$ (by rescaling and capping to 1).

Now we can choose $W := \mathrm{supp}\lambda$, which is closed by the definition of supp. Restricting the constructed isomorphism-germ over U to W yields an isomorphism-germ

$$\mathfrak{b}|_W \Rightarrow \mathfrak{e}^n_W.$$

Together with λ, this completes the proof. □

Lemma 4.13.
The rooted microbundles $\mathfrak{b} \oplus \mathfrak{e}^n_B$ and $\mathfrak{e}^n_B \oplus \mathfrak{b}$ are rooted isomorphic.

Proof.
We need to find an isomorphism-germ $\mathfrak{b} \oplus \mathfrak{e}^n_B \Rightarrow \mathfrak{e}^n_B \oplus \mathfrak{b}$ that extends

$$(I \oplus R) \circ (R \oplus I)^{-1} = R \oplus R^{-1}$$

where I denotes the identity germ.

Ignoring the rooting, we have an isomorphism-germ $f : E(\mathfrak{b}) \times \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n \times E(\mathfrak{b})$ with $f(e, x) = (-x, e)$. The idea is to change the f near b_0 so that it extends the rooting.

Using the previous lemma, choose a sufficiently small closed neighborhood U of b_0 such that there exists an extension $Q : (\mathfrak{b} \oplus \mathfrak{e}^n_B)|_U \Rightarrow (\mathfrak{e}^n_B \oplus \mathfrak{b})|_U$ for the rooting.

The previous lemma also equips us with a map

$$\lambda : B \to [0, \frac{\pi}{2}]$$

such that $\mathrm{supp}\lambda \subseteq U$ and $\lambda(b_0) = \frac{\pi}{2}$.

Now, we can define a homeomorphism

$$\psi : U \times \mathbb{R}^n \times \mathbb{R}^n \xrightarrow{\sim} U \times \mathbb{R}^n \times \mathbb{R}^n \text{ with}$$
$$\psi(b, x, y) = (b, x\sin(\lambda(b)) - y\cos(\lambda(b)), x\cos(\lambda(b)) - y\sin(\lambda(b))).$$

Consider the composition

$$(\mathfrak{b} \oplus \mathfrak{e}^n_B)|_U \Rightarrow (\mathfrak{b} \oplus \mathfrak{e}^n_B)|_U \xrightarrow{g} (\mathfrak{b} \oplus \mathfrak{e}^n_B)|_U \Rightarrow (\mathfrak{e}^n_B \oplus \mathfrak{b})|_U$$

which coincides with $R \oplus R^{-1}$ over b_0 since $\psi(b_0, x, y) = (b_0, x, y)$ and with F over $U \cap \lambda^{-1}(0)$.

Pieced together with $F|_{\lambda^{-1}(b)}$ using Lemma **(3.10)**, we obtain an isomorphism-germ

$$\mathfrak{b} \oplus \mathfrak{e}^n_B \Rightarrow \mathfrak{e}^n_B \oplus \mathfrak{b}$$

extending the rooting. □

Theorem 4.14.
If $\mathfrak{a}$ and $\mathfrak{b}$ are rooted microbundles over a paracompact hausdorff space B, then

$$\phi^*(\mathfrak{a} \vee \mathfrak{b}) \oplus \mathfrak{e}^n_B = \mathfrak{a} \oplus \mathfrak{b}.$$

Proof.
The previous lemma yields $\mathfrak{b} \oplus \mathfrak{e}^n_B \cong \mathfrak{e}^n_B \oplus \mathfrak{b}$. Hence

$$\phi^*((\mathfrak{a} \oplus \mathfrak{e}^n_B) \vee (\mathfrak{b} \oplus \mathfrak{e}^n_B)) \cong \phi^*((\mathfrak{a} \oplus \mathfrak{e}^n_B) \vee (\mathfrak{e}^n_B \oplus \mathfrak{b})).$$

Additionally, we have

$$\phi^*((\mathfrak{a} \vee \mathfrak{b})) \oplus (\mathfrak{e}^n_B \vee \mathfrak{e}^n_B) \cong \phi^*(\mathfrak{a} \vee \mathfrak{b}) \oplus \mathfrak{e}^n_B$$

for the left side and

$$\phi^*((\mathfrak{a} \vee \mathfrak{e}^n_B) \oplus (\mathfrak{e}^n_B \vee \mathfrak{b})) \cong \mathfrak{a} \oplus \mathfrak{b}$$

for the right side. That completes the proof. □

Corollary 4.15.
The wedge sum $\mathfrak{b} \oplus r^*\mathfrak{b}$ is trivial.

Proof.
This follows directly from the previous theorem and the fact that $\phi^*(\mathfrak{b} \oplus r^*\mathfrak{b})$ is trivial. □

5. Normal Microbundles and Milnors Theorem

5.1. **The Normal Microbundle.**

Definition 5.1 (normal microbundle)**.**
Let M be a topological manifold together with a submanifold $N \subseteq M$. A *normal microbundle* $\mathfrak{n}$ of N in M is a microbundle

$$N \xrightarrow{\iota} U \xrightarrow{r} N$$

where $U \subseteq M$ is a neighborhood of N and ι denotes the inclusion $M \hookrightarrow U$.

Definition 5.2 (composition microbundle)**.**
Let $\mathfrak{a}$ be a n-dimensional microbundle with

$$\mathfrak{a} : A \xrightarrow{i_a} E(\mathfrak{a}) \xrightarrow{j_a} A$$

and let $\mathfrak{b}$ be a n'-dimensional microbundle with

$$\mathfrak{b} : E(\mathfrak{a}) \xrightarrow{i_b} E(\mathfrak{b}) \xrightarrow{j_b} E(\mathfrak{a}).$$

The *composition microbundle* $\mathfrak{a} \circ \mathfrak{b}$ is a $(n + n')$-dimensional microbundle

$$A \xrightarrow{i} E(\mathfrak{b}) \xrightarrow{j} A$$

where $i := i_b \circ i_a$ and $j := j_a \circ j_b$.

Proof that $\mathfrak{a} \circ \mathfrak{b}$ is a $(n+n')$-dimensional microbundle.
Both injection and projection maps are continuous being composed by continuous maps. Additionally, $j \circ i = j_a \circ (j_b \circ i_b) \circ i_a = j_a \circ i_a = id_A$.

It remains to be shown that $\mathfrak{a} \circ \mathfrak{b}$ is locally trivial.

For an arbitrary $a \in A$, choose local trivializations

$$(U_a, V_a, \phi_a) \text{ of } a \text{ and } (U_b, V_b, \phi_b) \text{ of } i_a(a).$$

Note that both U_b and V_a are open neighborhoods of $i_a(a)$.

Without loss of generality, we may assume that $V_a = U_b$, because:

'$\subseteq$ ': Modify U_a such that

$$U_a \times B_\varepsilon(0) \subseteq \phi_a(V_a \cap U_b)$$

for a sufficiently small $\varepsilon > 0$ and let

$$V_a = \phi_a^{-1}(U_a \times B_\varepsilon(0)) \subseteq V_a \cap U_b.$$

Composing ϕ_a with $\mu_\varepsilon : B_\varepsilon(0) \xrightarrow{\sim} \mathbb{R}^n$ yields a local trivialization of a in $\mathfrak{a}$ that $V_a \subseteq U_a$.

'$\supseteq$ ': Restrict U_b to $V_a \cap U_b$ and V_b to $\phi_b^{-1}((V_a \cap U_b) \times \mathbb{R}^{n'})$.

It follows that have a local trivialization (U_a, V_b, ϕ) of a in $\mathfrak{a} \circ \mathfrak{b}$ where

$$\phi : V_b \xrightarrow{\phi_b} U_b \times \mathbb{R}^{n'} = V_a \times \mathbb{R}^{n'} \xrightarrow{\phi_a \times id_{\mathbb{R}^{n'}}} U_a \times \mathbb{R}^n \times \mathbb{R}^{n'} = U_a \times \mathbb{R}^{n+n'}.$$

The map ϕ is an homeomorphism since it's composed by homeomorphisms.

Additionally, ϕ commutes with injection

$$\begin{aligned}\phi(i(a)) &= \phi(i_b(i_a(a))) = (\phi_a(i_a(a)), \phi_b^{(2)}(i_b(i_a(a)))) \\ &= (\phi_a^{(2)}(i_a(a)), 0) = (a, (0,0)) = (id_{U_a} \times 0)(a)\end{aligned}$$

and projection maps

$$j(e) = j_a(j_b(e)) = \pi_1(j_a(j_b(e)), \phi^{(2)}(e)) = \pi_1(\phi(e))$$

which completes the proof. □

It's an unknown question, whether normal microbundles are unique (up to isomorphy). However, we can show that

Proposition 5.3.
Let $N \subseteq M$ be an embedded submanifold. Suppose there exists a normal microbundle $\mathfrak{n}$ of N in M. Then $\mathfrak{t}_N \oplus \mathfrak{n} \cong \mathfrak{t}_M|_N$.

Proof.
TODO □

5.2. **Milnors Theorem.**

Lemma 5.4.
Let $P \subseteq N \subseteq M$ be a chain of topological manifolds. There exists a normal microbundle

$$\mathfrak{n} : P \xrightarrow{\iota} U \xrightarrow{r} P$$

of P in M if there exist normal microbundles

$$\mathfrak{n}_p : P \xrightarrow{\iota_P} U_N \xrightarrow{j_P} P \text{ in } N \text{ and } \mathfrak{n}_n : N \xrightarrow{\iota_N} U_M \xrightarrow{j_N} N \text{ in } M.$$

Proof.
Considering the composition $\mathfrak{n}_p \circ \mathfrak{n}_n|_{U_N}$, we have a normal microbundle $\mathfrak{n}$ of P in M since $\iota_N \circ \iota_P$ is just the inclusion $P \hookrightarrow U_M$. □

Every topological manifold is an absolute neighborhood retract (ANR).

It follows that by restricting M, if necessary, to an open neighborhood of N, there exists a retraction $M \twoheadrightarrow N$.

From now on, let

$$r : M \twoheadrightarrow N$$

denote such a retraction and let

$$\iota : N \hookrightarrow M$$

denote the inclusion $N \subseteq M$.

Lemma 5.5.
Let $\mathfrak{t}_N$ and $\mathfrak{t}_M$ be two tangent microbundles of N and M. The total spaces $E(\iota^*\mathfrak{t}_M)$ and $E(r^*\mathfrak{t}_N)$ are homeomorphic.

Proof.
The total space

$$E(\iota^*\mathfrak{t}_M) = \{(n, m_1, m_2) \in N \times (M \times M) \mid \iota(n) = m_1\}$$

is homeomorphic to $N \times M$ via

$$(n, m_1, m_2) \mapsto (n, m_2)$$

with inverse $(n, m) \mapsto (n, \iota(n), m)$. Similarly, the total space

$$E(r^*\mathfrak{t}_N) = \{(m, n_1, n_2) \in M \times (N \times N) \mid r(m) = n_1\}$$

is homeomorphic to $M \times N$ via

$$(m, n_1, n_2) \mapsto (m, n)$$

with inverse $(m, n) \mapsto (m, r(m), n)$.

Composed with the canonic homeomorphism $N \times M \cong M \times N$, this yields a homeomorphism

$$\psi : E(\iota^*\mathfrak{t}_M) \xrightarrow{\sim} E(r^*\mathfrak{t}_N) \text{ with } \psi(n, m_1, m_2) := (m_2, r(m_2), n).$$

□

Remark 5.6.
The following diagram commutes:

$$\begin{tikzcd} N \arrow[r, "i_\iota"] \arrow[d, hook, "\iota"'] & E(\iota^*\mathfrak{t}_M) \arrow[d, "\psi"] \\ M \arrow[r, "i_r"] & E(r^*\mathfrak{t}_N) \end{tikzcd}$$

The total space $E(r^*\mathfrak{t}_N)$ is a topological manifold with

$$E(r^*\mathfrak{t}_N) \cong M \times N$$

as described in the previous lemma.

The fact that the above diagram commutes, allows us to consider N to be a submanifold of $E(r^*\mathfrak{t}_N)$ via

$$N \hookrightarrow M \xrightarrow{i_r} E(r^*\mathfrak{t}_N).$$

The composition $\iota \circ i_r$ is an embedding since i_r is an embedding due to the construction of the induced microbundle.

Lemma 5.7.
Let M be a topological manifold together with a submanifold $N \subseteq M$. Then there exists a normal microbundle $\mathfrak{n}$ of N in $E(r^*\mathfrak{t}_N)$ such that $\mathfrak{n} \cong \iota^*\mathfrak{t}_M$.

Proof.
We are already given a normal microbundle of N in $E(r^*\mathfrak{t}_N)$ with $r^*\mathfrak{t}_N|_N$. Isomorphy between $r^*\mathfrak{t}_N|_N$ and $\iota^*\mathfrak{t}_M$ follows from the homeomorphy

$$\psi : E(\iota^*\mathfrak{t}_M) \xrightarrow{\sim} E(r^*\mathfrak{t}_N)$$

and from the diagram which shows that injection and projection maps commute with ψ. $\square$

Finally, we gathered all the tools to prove Milnor's theorem.

Theorem 5.8 (Milnors Theorem)**.**
For a sufficently large $q \in \mathbb{N}$, $N = N \times \{0\}$ has a normal microbundle in $M \times \mathbb{R}^q$.

Proof.
We show the theorem in multiple steps:

(1) There exists a microbundle η over N such that $\mathfrak{t}_N \oplus \eta \cong \mathfrak{e}_N^q$:

From the [Whitney Embedding Thereom] it follows that we can embed M in euclidean space $\mathbb{R}^{2m+1}$.

Additionally, since there exists a retraction $r : V \to N$ where V is an open neighborhood of N in M we can extend $\mathfrak{t}_N$ to a microbundle $\mathfrak{t}'_N$ over V. Since V is an open subset of euclidean space, it's a simplicial complex.

Hence, we can apply Theorem **(2.12)** to the extended microbundle $\mathfrak{t}'_N$ to obtain a microbundle η' such that $\mathfrak{t}'_N \oplus \eta \cong \mathfrak{e}_V^q$.

We conclude that

$$\mathfrak{t}_N \oplus \eta'|_N = \mathfrak{t}'_N|_N \oplus \eta'|_N = (\mathfrak{t}'_N \oplus \eta')|_N = \mathfrak{e}^q_N.$$

(2) $E(r^*\mathfrak{t}_N) \subseteq E(r^*\mathfrak{t}_N \oplus r^*\eta)$ has a normal microbundle:

Since the total space

$$E(r^*\mathfrak{t}_N \oplus r^*\eta) = \{(e,e') \in E(r^*\mathfrak{t}_N) \times E(r^*\eta) : j(e) = j'(e)\}$$

we can consider $E(r^*\mathfrak{t}_N) \subseteq E(r^*\mathfrak{t}_N \oplus r^*\eta)$ embedded via

$$\iota : e \mapsto (e, i'(j(e)))$$

with the inverse $\pi_1 : (e,e') \mapsto e$.

Because $r^*\mathfrak{t}_N \oplus r^*\eta \cong r^*(\mathfrak{t}_N \oplus \eta)$ is trivial, it follows that $E(r^*\mathfrak{t}_N \oplus r^*\eta) \subseteq M \times \mathbb{R}^k$ open and hence being a manifold.

We have a normal microbundle of $E(r^*\mathfrak{t}_N)$ in $E(r^*\mathfrak{t}_N \oplus r^*\eta)$ via

$$\mathfrak{n} : E(r^*\mathfrak{t}_N) \xrightarrow{\iota} E(r^*\mathfrak{t}_N \oplus r^*\eta) \xrightarrow{\pi_1} E(r^*\mathfrak{t}_N).$$

To show local triviality, let (U,V,ϕ) be a local trivialization of $i'(j(e))$ in $r^*\eta$ for an arbitrary $e \in E(r^*\mathfrak{t}_N)$.

By choosing

- $U' := j^{-1}(U)$
- $V' := (U' \times V) \cap E(r^*\mathfrak{t}_N \oplus r^*\eta)$
- $\phi' : V' \xrightarrow{\sim} U' \times \mathbb{R}^{n_\eta}$ with $\phi'(e,e') = (e, \phi^{(2)}(e'))$

we have a local trivialization of e in $\mathfrak{n}$.

That is because both $U' \subseteq E(r^*\mathfrak{t}_N)$ and $V' \subseteq E(r^*\mathfrak{t}_N \oplus r^*\eta)$ are open sets and ϕ' is a homeomorphism with its inverse $\phi'^{-1}(e,x) = (e, \phi^{-1}(j(e),x))$.

Also, ϕ' commutes with injection

$$\phi'(\iota(e)) = \phi'(e, i'(j(e))) = (e, \phi^{(2)}(i'(j(e)))) = (e,0) = (id \times 0)(e)$$

and projection maps

$$\pi_1(e,e') = \pi_1(e, \phi'^{(2)}(e,e')) = \pi_1(\phi'(e,e')).$$

Since $N \subseteq M \subseteq E(r^*\mathfrak{t}_N)$ has a normal microbundle (using Lemma **(5.7)**), it follows from Lemma **(5.4)** that $N \subseteq E(r^*\mathfrak{t}_N \oplus r^*\mathfrak{t}')$ has a normal microbundle.

By restricting $E(r^*\mathfrak{t}_N \oplus r^*\eta)$ to an open subset if necessary, we may assume that

$$E(r^*\mathfrak{t}_N \oplus r^*\eta) = M \times \mathbb{R}^q$$

for some $q \in \mathbb{N}$ using Lemma **(2.4)**.

This completes the proof. □

References

[Bre93] Glen E. Brendon. *Topology and Geometry*. Springer New York, NY, 1993.

[Ker60] Michel A Kervaire. A manifold which does not admit any differentiable structure. *Comment. Math. Helv*, 34(1):257–270, 1960.

[Kuc09] Marek Kuczma. An introduction to the theory of functional equations and inequalities. *(No Title)*, 2009.

[Mil56] John Milnor. On manifolds homeomorphic to the 7-sphere. *Annals of Mathematics*, 64(2):399–405, 1956.

[Mil64] J. Milnor. Microbundles: Part i. *Topology*, 3:53–80, 1964.

[RS67] Colin P Rourke and Brian J Sanderson. An embedding without a normal microbundle. *Inventiones mathematicae*, 3:293–299, 1967.

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