Microbundles on Topological Manifolds

based on Milnor's studies on Microbundles

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Contents

Abstract

TODO

Chapter 1

Introduction to Microbundles

In order to construct the tangent bundle on a manifold, we need a differential structure. However, this is generally not given for topological manifolds. In order to still have a structure "similar" to the tangent bundle on topological manifolds, we need a different, weaker, concept of the tangent bundle. Therefore we introduce so called "microbundles" which act as a weaker alternative to vector bundles. The concept of microbundles as well as some basic properties and examples are presented in this chapter.

We start with the definition of a microbundle.

Definition 1.1 (microbundle)**.**
A *microbundle* $\mathfrak{b}$ over B (with *fibre-dimension* n) is a diagram $B \xrightarrow{i} E \xrightarrow{j} B$ satisfying the following:

(i) B is a topological space (*base space*)

(ii) E is a topological space (*total space*)

(iii) $i : B \to E$ (*injection*) and $j : E \to B$ (*projection*) are maps such that $id_B = j \circ i$

(iv) Every $b \in B$ is *locally trivializable*, that is there exist open neighborhoods $U \subseteq B$ of b and $V \subseteq E$ of $i(U)$ together with a homeomorphism $\phi : V \xrightarrow{\sim}$

$U \times \mathbb{R}^n$ such that the following diagram commutes:

$$\begin{tikzcd} & V \arrow[dd, "\psi"] \arrow[dr, "j|_V"] & \\ U \arrow[ur, "i"] \arrow[dr, "id\times 0"'] & & U \\ & U \times \mathbb{R}^n \arrow[ur, "\pi_1"'] & \end{tikzcd}$$

Remark 1.2.
In the following, unless explicitly stated otherwise, we assume the fiber dimension of any given microbundle to be n.

Before we look at examples of microbundles, we should first clarify what it means for two microbundles to be isomorphic.

Definition 1.3 (isomorphy)**.**
Two microbundles $\mathfrak{b}_1 : B \xrightarrow{i_1} E_1 \xrightarrow{j_1} B$ and $\mathfrak{b}_2 : B \xrightarrow{i_2} E_2 \xrightarrow{j_2} B$ are *isomorphic* if there exist neighborhoods $V_1 \subseteq E_1$ of $i_1(B)$ and $V_2 \subseteq E_2$ of $i_2(B)$ together with a homeomorphism $\phi : V_1 \xrightarrow{\sim} V_2$ such that the following diagram commutes:

$$\begin{tikzcd} & V_1 \arrow[dd, "\phi"] \arrow[dr, "j_1|_{V_1}"] & \\ B \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & B \\ & V_2 \arrow[ur, "j_2|_{V_2}"'] & \end{tikzcd}$$

As the definition of isomorphy already indicates, when studying microbundles, we are not interested in the entire total space but only in an arbitrarily small neighborhood of the base space. This is certainly one of the strongest conceptual differences between microbundles and classical vector bundles.

Proposition 1.4.
Given a microbundle $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ over B, restricting the total space E to an arbitrary neighborhood $E' \subseteq E$ of $i(B)$ leaves the microbundle unchanged. That is, the microbundle

$$\mathfrak{b}' : B \xrightarrow{i} E' \xrightarrow{j|_{E'}} B$$

is isomorphic to $\mathfrak{b}$.

Proof.
We prove this proposition in two steps.

1. $\mathfrak{b}'$ is a microbundle:

   Since we take i and j from $\mathfrak{b}$, we only need to show local triviality.

For an arbitrary $b \in B$, choose a local trivialization (U, V, ϕ) of b in $\mathfrak{b}$.

The image $\phi(V \cap E')$ is a neighborhood of $(b, 0)$. That follows from $\phi(i(b)) = (b, 0)$ and $V \cap E'$ being a neighborhood of $i(b)$.

Hence, there exists a $U' \times B_\varepsilon(0) \subseteq \phi(V \cap E')$ with U' open and ε sufficiently small.

By utilising the fact that $B_\varepsilon(0) \cong \mathbb{R}^n$, we have a local trivialization (U', V', ϕ') with

$$\phi' : V' \xrightarrow{\phi} U' \times B_\varepsilon(0) \xrightarrow{id \times \mu_\varepsilon} U' \times \mathbb{R}^n$$

and $V' := \phi^{-1}(U' \times B_\varepsilon(0))$.

Note that homeomorphism commutes with injection

$$\phi'(i(b)) = (id \times \mu_\varepsilon)(\phi(i(b))) = (id \times \mu_\varepsilon)(b, 0) = (b, 0) = (id \times 0)(b)$$

and projection maps

$$j(e) = \pi_1(\phi(e)) = \pi_1((id \times \mu_\varepsilon)(\phi(e))) = \pi_1(\phi'(e)).$$

2. $\mathfrak{b}'$ is isomorphic to $\mathfrak{b}$:

   Since $E' \subseteq E$, we can simply take the identity $E' \to E' \subseteq E$ as our homeomorphism between neighborhoods of $i(B)$. Furthermore, the injection and projection maps for $\mathfrak{b}$ and $\mathfrak{b}'$ are the same, so they clearly commute with the identity.

$\square$

Now that we introduced the basic concept of microbundles, we will take a look at some key examples.

The most obvious example for a microbundle is the standard microbundle.

Example 1.5 (trivial microbundle)**.**
Given a topological space B, the *standard microbundle* $\mathfrak{e}^n_B$ over B is a diagram

$$B \xrightarrow{\iota} B \times \mathbb{R}^n \xrightarrow{\pi} B$$

where $\iota(b) := (b, 0)$ and $\pi(b, x) := b$. Furthermore, a microbundle $\mathfrak{b}$ over B is *trivial* if it is isomorphic to $\mathfrak{e}^n_B$.

In order to make it easier classifying microbundles as trivial, we provide a sharper description of what it means for a microbundle to be trival.

Lemma 1.6.
A microbundle $\mathfrak{b}$ over a paracompact hausdorff space B is trivial if and only if there exists an open neighborhood V of $i(B)$ such that $V \cong B \times \mathbb{R}^n$ with injection and projection maps being compatible with this homeomorphism.

Proof.
" $\implies$ "

By restricting $E(\mathfrak{b})$ to an open neighborhood and applying Proposition **(1.4)**, we may assume that $E(\mathfrak{b})$ is an open subset of $B \times \mathbb{R}^n$.

Since $E(\mathfrak{b})$ is a neighborhood of $B\times[0,1]$, there exist $B_i \subseteq B$ open and $0 < \varepsilon_i < 1$ with

$$\bigcup_{i\in I} B_i \times B_{\varepsilon_i}(0) \subseteq E(\mathfrak{b})$$

such that $\bigcup_{i\in I} B_i = B$. Without loss of generality, we may assume that the collection $\{B_i\}$ is locally finite because if not we can simply choose a locally finite refinement using the fact that B is paracompact.

Furthermore, from paracompactness and the hausdorff property we derive a partition of unity over $\{B_i\}$

$$f_i : B \to [0,1] \text{ with } \mathrm{supp} f_i \subseteq B_i$$

such that $\sum_{i\in I} f_i = 1$.

Now we define a map $\lambda : B \to (0,\infty)$ via

$$\lambda := \sum_{i\in I} \varepsilon_i f_i$$

which has the property that $|x| < \lambda(b) \implies (b,x) \in E(\mathfrak{b})$ because

$$\begin{aligned} &|x| < \lambda(b) \\ \iff& |x| < \varepsilon_{i_1} f_{i_1}(b) + \cdots + \varepsilon_{i_n} f_{i_n}(b) \\ \iff& 0 < (\varepsilon_{i_1} - |x|) f_{i_1}(b) + \cdots + (\varepsilon_{i_n} - |x|) f_{i_n}(b) \\ \implies& \exists i \in I : 0 < (\varepsilon_{i_1} - |x|) f_{i_1}(b) \\ \implies& (b,x) \in B_i \times B_{\varepsilon_i}(0) \implies (b,x) \in E(\mathfrak{b}). \end{aligned}$$

Lastly, we have a homeomorphism between the open subset $\{(b,x) \in B \times \mathbb{R}^n : |x| < \lambda(b)\} \subseteq E(\mathfrak{b})$ and $B \times \mathbb{R}^n$ via

$$(b,x) \mapsto (b, \frac{x}{\lambda(b) - |x|}).$$

Note that $(b,0) \mapsto (b,0)$ and hence this homeomorphism is compatible with injection and projection maps.

" $\impliedby$ "

This is simply a weakening of the definition of triviality. $\square$

The following example acts as the microbundle analog to the tangent bundle on a smooth manifold.

Example 1.7 (tangent microbundle)**.**
The *tangent microbundle* $\mathfrak{t}_M$ over a topological d-manifold M is a diagram

$$M \xrightarrow{\Delta} M \times M \xrightarrow{\pi_1} M$$

where $\Delta(m) := (m, m)$ denotes the diagonal map.

Proof that $\mathfrak{t}_M$ *is a microbundle.*
The maps Δ and π_1 are continuous and clearly $id_M = \pi_1 \circ \Delta$.

Let $p \in M$ be arbitrary and let (U, ψ) be a chart over p. Note that U is an open neighborhood of p.

We have a local trivialization $(U, U \times U, \phi)$ of p in $\mathfrak{t}_M$ where

$$\phi : U \times U \xrightarrow{\sim} U \times \mathbb{R}^n \text{ with } \phi(u, u') := (u, \psi(u) - \psi(u')).$$

Homeomorphy of ϕ is given by homeomorphy of ψ.

Lastly, ϕ commutes with injection

$$\phi(\Delta(m)) = \phi(m, m) = (m, \psi(m) - \psi(m)) = (m, 0) = (id \times 0)(m)$$

and projection maps

$$\pi_1(u, u') = u = \pi_1(u, \phi^{(2)}(u, u')) = \pi_1(\phi(u, u')).$$

which concludes the proof. $\square$

Chapter 2

Induced Microbundles

This Chapter introduces a central construction of microbundles.

Definition 2.1 (induced microbundle)**.**
Let $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ be a microbundle and $f : A \to B$ be a map. The *induced microbundle* $f^*\mathfrak{b} : A \xrightarrow{i'} E' \xrightarrow{j'} A$ is a microbundle defined as follows:

- $E' := \{(a, e) \in A \times E \mid f(a) = j(e)\}$
- $i' : A \to E'$ with $i'(a) := (a, (i \circ f)(a))$
- $j' : E' \to A$ with $j'(a, e) := a$

Proof that $f^*\mathfrak{b}$ is a microbundle.
Both i' and j' are continuous since they are composed by continuous functions Additionally, $j'(i'(a)) = j'(a, i(f(a))) = a$ and hence $j' \circ i' = id_A$.

It remains to be shown that $f^*\mathfrak{b}$ is locally trivial:

For an arbitrary $a_0 \in A$, choose a local trivialization (U, V, ϕ) of $i(a_0)$ in $\mathfrak{b}$. We construct a local trivialization of a_0 in $f^*\mathfrak{b}$ as follows:

- $U' := f^{-1}(U) \subseteq A$
- $V' := (U' \times V) \cap E' \subseteq E'$
- $\phi' : V' \xrightarrow{\sim} U' \times \mathbb{R}^n$ with $\phi'(a, e) := (a, \phi^{(2)}(e))$

Note that U' is an open neighborhood of a_0 since f is continuous and U is an open neighborhood of $i(a_0)$. Similarly, V' is an open neighborhood of $i'(a_0)$ since both $U' \times V$ and E' are open neighborhoods of $i'(a_0)$. The map ϕ' is well-defined because $(a, e) \in V' \implies e \in V$. The existence of an inverse $\phi'^{-1}(a, v) = (a, \phi^{-1}(f(a), v))$ and component-wise continuity show that ϕ' is a homeomorphism. This completes the proof. $\square$

Example 2.2 (restricted microbundle)**.**
Let $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ be a microbundle and $A \subseteq B$ be a subspace. The *restricted microbundle* $\mathfrak{b}|_A$ is the induced microbundle $\iota^*\mathfrak{b}$ where $\iota : A \hookrightarrow B$ denotes the inclusion map.

Remark 2.3.
In the following, we consider $E(\mathfrak{b}|_A)$ to be a subset of $E(\mathfrak{b})$. This is justified because there exists an embedding

$$\iota : E(\mathfrak{b}|_A) \to E(\mathfrak{b}) \text{ with } (a, e) \mapsto e$$

and inverse $e \mapsto (j(e), e)$. Note that this argument can be made for any induced microbundle over an injective map.

Lemma 2.4.
Let $\mathfrak{b}$ be a microbundle over B and $f : A \to B$ be a map. The induced microbundle $f^*\mathfrak{b}$ is trivial if $\mathfrak{b}$ is already trivial.

Proof.
To proof triviality, we need to show that there exists a homeomorphism between a neighborhood of $i'(A)$ and $A \times [0, 1]$ that commutes with the injection and projection maps of $f^*\mathfrak{b}$ and $\mathfrak{e}^n_A$.

Since $\mathfrak{b}$ is trivial, there exists a homeomorphism $\psi : V \to \psi(V)$ where V is a neighborhood of $i(B)$ and $\psi(V)$ is a neighborhood of $B \times \{0\}$ such that ψ commutes with injection and projection maps. We define a map

$$\psi' : V' \xrightarrow{\sim} \psi'(V')$$
$$(a, e) \mapsto (a, \psi^{(2)}(e))$$

where $V' := (A \times V) \cap E(f^*\mathfrak{b})$. Since ψ' is component-wise homeomorphic, ψ' is a homeomorphism. Note that V' is a neighborhood of $i'(A)$ since $\forall a \in A :$ $i(f(a)) \in V$ and $i'(a) = (a, i(f(a)))$. From $\psi^{(2)}(i(f(a))) = 0$ and homeomorphy of ψ' it follows that $\psi'(V')$ is a neighborhood of $A \times [0, 1]$.

Finally, ψ' commutes with injection

$$\psi'(i'(a)) = (a, \psi^{(2)}(i(f(a)))) = (a, 0) = i_{\mathfrak{e}^n_A}(a)$$

and projection maps

$$j'(a, e) = j'(a) = j_{\mathfrak{e}^n_A}(a, \psi'^{(2)}(a, e)) = j_{\mathfrak{e}^n_A}(\psi'(a, e))$$

which completes the proof. □

Lemma 2.5.
Let $\mathfrak{b}$ be a microbundle over B. The induced microbundle $const^*_{b_0}\mathfrak{b}$ over a map

$$const_{b_0} : A \to B \textit{ with } const_{b_0}(a) = b_0$$

is trivial.

Proof.
The total space $E(const_{b_0}^* \mathfrak{b})$ is of the form

$$\{(a,e) \in A \times E(\mathfrak{b}) : f(a) = b_0 = j(e)\}$$
$$= A \times j^{-1}(b_0).$$

By choosing a local trivialization (U, V, ϕ) of b_0 in $\mathfrak{b}$ and restricting ϕ to $j^{-1}(b_0)$, we receive a homeomorphism $\phi|_{j^{-1}(b_0)} : V' \xrightarrow{\sim} b_0 \times \mathbb{R}^n$ where $V' := V \cap j^{-1}(b_0)$.

With ϕ and V' we can construct a homeomorphism $\psi : A \times V' \xrightarrow{\sim} A \times \mathbb{R}^n$ with

$$\psi(a,e) := (a, \phi^{(2)}(e)).$$

Homeomorphy follows from component-wise homeomorphy of ψ.

The map commutes with injection

$$\psi(i'(a)) = \psi(a, i(b_0)) = (a, \phi^{(2)}(i(b_0))) = (a, 0) = i_{\mathfrak{e}_A^n}(a)$$

and projection maps

$$j'(a,e) = a = j_{\mathfrak{e}_A^n}(a,x) = j_{\mathfrak{e}_A^n}(\psi(a,e))$$

which completes the proof $\square$

Lemma 2.6.
Let $\mathfrak{c} : C \xrightarrow{i} E \xrightarrow{j} C$ be microbundle and let $A \xrightarrow{f} B \xrightarrow{g} C$ be a map diagram. Then the two microbundles

$$(g \circ f)^* \mathfrak{c} : A \xrightarrow{i_1} E_1 \xrightarrow{j_1} A$$

and

$$f^*(g^* \mathfrak{c}) : A \xrightarrow{i_2} E_2 \xrightarrow{j_2} A$$

are isomorphic.

Proof.
Again, we need to find a homeomorphism between a neighborhood of $i_1(A)$ and a neighborhood of $i_2(A)$ that commutes with injection and projection maps.

First, compare the two total spaces:

1. $E_1 = \{(a,e) \in A \times E \mid g(f(a)) = j(e)\}$
2. $E_2 = \{(a,b,e) \in A \times (B \times E) \mid f(a) = b \text{ and } g(b) = j(e)\}$

We construct a bijection $\psi : E_1 \xrightarrow{\sim} E_2$ with

$$\psi(a,e) = (a, f(a), e) \text{ and } \psi^{-1}(a,b,e) = (a,e).$$

Since both ψ and ψ^{-1} are component-wise continuous, it follows that ψ is a homeomorphism.

This homeomorphism commutes with injection

$$\psi(i_1(a)) = \psi(a, i(g(f(a)))) = (a, f(a), i(g(f(a)))) = i_2(a)$$

and projection maps

$$j_1(a,e) = a = j_2(a, f(a), e) = j_2(\psi(a,e))$$

which concludes the proof. $\square$

Definition 2.7.
Let $\mathfrak{b}$ and $\mathfrak{b}'$ be two microbundles over B and B' where $B' \subseteq B$. We say that $\mathfrak{b}$ *extends* $\mathfrak{b}'$, if

$$\mathfrak{b}|_{B'} \cong \mathfrak{b}'.$$

In the following, let $f : A \to B$ be a map and A be paracompact.

For a topological space X, we define the *cone* of X to be

$$CX := X \times [0,1]/X \times \{1\}$$

and the *mapping cone* of f to be

$$B \sqcup_f CA := B \sqcup CA/\sim$$

where $(a, 0) \sim b \iff f(a) = b$.

Similarly, we define the *cylinder* of X to be

$$MX := X \times [0,1]$$

and the *mapping cylinder* of f to be

$$B \sqcup_f MA := B \sqcup MA/\sim$$

where $(a, 0) \sim b \iff f(a) = b$.

Lemma 2.8.
A microbundle $\mathfrak{b}$ over B can be extended to a microbundle over the mapping cone $B \sqcup_f CA$ if and only if $f^*\mathfrak{b}$ is trivial.

Proof.
We show both implications.

"$\implies$"

Let $\mathfrak{b}'$ be an extension of $\mathfrak{b}$ over $B \sqcup_f CA$.

The composition $A \xrightarrow{f} B \hookrightarrow B \sqcup_f CA$ is null-homotopic via the homotopy

$$H(a,t) := [(a,t)]$$

because $H(a,0) = [(a,0)] = [f(a)] = (\iota \circ f)(a)$ and $H(a,1) = [(a,1)] = [(\tilde{a},1)] = H(\tilde{a},1)$. From the Homotopy Theorem **(4.1)**, which we will prove in Chapter **(4)**, it follows that $(\iota \circ f)^*\mathfrak{b}'$ is isomorphic to $const^*\mathfrak{b}'$ and hence trivial (Lemma **(2.5)**).

Since $(\iota \circ f)^*\mathfrak{b}' = f^*(\iota^*\mathfrak{b}') = f^*\mathfrak{b}$, it follows that $f^*\mathfrak{b}$ is trivial.

" $\Longleftarrow$ "

Let $f^*\mathfrak{b}$ be trivial.

In contrast to the mapping cone, there exists a natural retraction from the mapping cylinder to the attached space

$$r : B \sqcup_f MA \to B \text{ with } r([(a,t)]) = f(a)$$

The diagram

$$A \times \{1\} \hookrightarrow B \sqcup_f MA \xrightarrow{r} B$$

equals f if we consider $A = A \times [0,1]$. It follows that

$$r^*\mathfrak{b}|_{A\times\{1\}} = (r \circ \iota)^*\mathfrak{b} \cong f^*\mathfrak{b}$$

is trivial.

From Lemma **(2.4)** and the retraction $(a,t) \mapsto (a,1)$ it follows that $r^*\mathfrak{b}|_{A\times[\frac{1}{2},1]}$ is trivial. Since A is paracompact and by Lemma **(1.6)**, there exists a map

$$\phi : V \xrightarrow{\sim} A \times [\frac{1}{2},1] \times \mathbb{R}^n$$

where V is a neighborhood of $i_r(B)$ in $E(r^*\mathfrak{b}|_{A\times[\frac{1}{2},1]})$. Without loss of generality, we may assume that $V = E(r^*\mathfrak{b}|_{A\times[\frac{1}{2},1]})$ by removing a closed subset of $E(r^*\mathfrak{b}|_{A\times[\frac{1}{2},1]})$ if necessary and applying Proposition **(1.4)**.

Now we explicitly construct an extension

$$\mathfrak{b}' : B \sqcup_f CA \xrightarrow{i'} E' \xrightarrow{j'} B \sqcup_f CA$$

with

- $E' := E(r^*\mathfrak{b})/\phi^{-1}(A \times \{1\} \times \{x\})$ (for every $x \in \mathbb{R}^n$)
- $i'([a,t]) := [i_r(a,t)]$ where i_r is the injection map for $r^*\mathfrak{b}$
- $j'([e]) := [j_r(e)]$ where j_r is the projection map for $r^*\mathfrak{b}$

The injection i' is well-defined because i_r maps every representative $[a,1]$ to the same equivalence class of E'. Similarly, the projection j' is well-defined since $[e] = [\tilde{e}] \implies [j_r(e)] = [j_r(e')]$

Both i' and j' are continuous by the construction of the quotient space topology. Also, $i'(j'([a,t])) = i'([j_r(a,t)]) = [i_r(j_r(a,t))] = [a,t]$ and hence $i' \circ j' = id$.

It remains to be shown that $\mathfrak{b}'$ is locally trivial:

For an arbitrary $[a,t] \in B \sqcup_f MA$ choose a local trivialization (U, V, ϕ) of $[a,t]$ in $r^*\mathfrak{b}$. We can collapse the map $\phi : V \xrightarrow{\sim} U \times \mathbb{R}^n$ to

$$[\phi] : E' \supseteq V' \xrightarrow{\sim} U' \times \mathbb{R}^n$$

where V' is V collapsed along the quotient $E(r^*\mathfrak{b}) \twoheadrightarrow E'$ and U' is U collapsed along the quotient $B \sqcup_f MA \twoheadrightarrow B \sqcup_f CA$. This map is well-defined and a homoemorphism, which concludes the proof. □

Corollary 2.9.
Let B be a $(d+1)$-simplicial complex, B' its d-skeleton and $\Delta \subseteq B$ a $(d+1)$-simplex. A microbundle $\mathfrak{b}$ over B' can be extended to a microbundle over $B' \cup \Delta$ if and only if its restriction to the boundary $\mathfrak{b}|_{\partial\Delta}$ is trivial.

Proof.
With $f : \partial\Delta \hookrightarrow B'$ and the previous lemma, it follows that there exists a microbundle $\mathfrak{b}'$ over $B' \cup_f C\partial\Delta$ extending $\mathfrak{b}$ if and only if $f^*\mathfrak{b} = \mathfrak{b}|_{\partial\Delta}$ is trivial.

Now, consider the homeomorphism $\phi : C\partial\Delta \xrightarrow{\sim} \Delta$ with

$$\phi((t_1, \ldots, t_{d+1}), \lambda) := (1-\lambda)(t_1, \ldots, t_{d+1}) + \frac{\lambda}{d+1}(1, \ldots, 1)$$

In particular, $\phi(\partial\Delta \times \{0\}) = \partial\Delta$.

It follows that $B' \cup_f C\Delta \cong B' \cup \Delta$ which concludes the proof. □

Chapter 3

The Whitney Sum

In the last chapter we saw how we can pull back the base space of a given microbundle using a map. In this chapter, another central construction is introduced, the "Whitney Sum". It allows us to construct a microbundle given two microbundles over the same base space. The fiber dimension of the resulting microbundle is just the sum of the fiber dimensions of the initial microbundles.

Definition 3.1.
Let $\mathfrak{b}_1$ and $\mathfrak{b}_1$ be two microbundles over B with fibre-dimensions n_1 and n_2. The *whitney sum* $\mathfrak{b}_1 \oplus \mathfrak{b}_2$ is a microbundle $B \xrightarrow{i} E \xrightarrow{j} B$ where

- $E := \{(e_1, e_2) \in E(\mathfrak{b}_1) \times E(\mathfrak{b}_2) : j_1(e_1) = j_2(e_2)\}$
- $i(b) := (i_1(b), i_2(b))$
- $j(e_1, e_2) := j_1(e_1) = j_2(e_2)$

with fibre-dimension $n_1 + n_2$.

Proof that $\mathfrak{b}_1 \oplus \mathfrak{b}_2$ is a microbundle.
For an arbitray $b \in B$, let (U_1, V_1, ϕ_1) and (U_2, V_2, ϕ_2) be two local trivializations of b in $\mathfrak{b}_1$ and $\mathfrak{b}_2$. We construct a local trivialization of b in $\mathfrak{b}_1 \oplus \mathfrak{b}_2$ as follows:

- $U := U_1 \cap U_2$
- $V := (V_1 \times V_2) \cap E$
- $\phi : V \to U \times \mathbb{R}^{n_1+n_2}; \phi(e_1, e_2) := (\phi_1^{(1)}(e_1), \phi_1^{(2)}(e_1) \times \phi_2^{(2)}(e_2))$

Note that $\phi_1^{(1)}(e_1) = \phi_2^{(1)}(e_2)$. Both U and V are open since U_1, U_2 and V_1, V_2 are open. Since ϕ is a composed by homeomorphisms, it's an homeomorphism as well. This conludes the proof that $\mathfrak{b}_1 \oplus \mathfrak{b}_2$ is a microbundle (of fibre-dimension $n_1 + n_2$). $\square$

Remark 3.2.
Alternatively, we could define the whitney sum between $\mathfrak{b}_1$ and $\mathfrak{b}_2$ to be the induced microbundle $\Delta^*(\mathfrak{b}_1 \times \mathfrak{b}_2)$ where Δ denotes the diagonal map and $\mathfrak{b}_1 \times \mathfrak{b}_2$ denotes the literal cross-product between the two microbundles.

Lemma 3.3.
Let $\mathfrak{b}_1$ and $\mathfrak{b}_1$ be two microbundles over B and let $f : A \to B$ be a map. The induced microbundle and the whitney sum are compatible, that is

$$f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2) \cong f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2.$$

Proof.
From the definition of the induced microbundle and the whitney sum, we can explicitly write the total spaces

$$\begin{gathered} E(f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2)) \\ = \{(a, (e_1, e_2)) \in A \times (E(\mathfrak{b}_1) \times E(\mathfrak{b}_2)) \mid j_1(e_1) = j_2(e_2) = f(a)\} \end{gathered}$$

and

$$\begin{gathered} E(f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2) \\ = \{(e_1, e_2) \in E(f^*\mathfrak{b}_1) \times E(f^*\mathfrak{b}_2) \mid j_1(e_1) = j(e_2)\} \\ = \{((a_1, e_1), (a_2, e_2)) \in (A \times E(\mathfrak{b}_1)) \times (A \times E(\mathfrak{b}_2)) \mid \\ j(a_1, e_1) = j(a_2, e_2) \wedge f(a_i) = j(e_i)\} \end{gathered}$$

The two total spaces are homeomorphic via $\phi(a, (e_1, e_2)) := ((a, e_1), (a, e_2))$ with $\phi^{-1}((a, e_1), (a, e_2)) = (a, (e_1, e_2))$. Homeomorphy of ϕ follows from the continuity of ϕ and ϕ^{-1}, which is given since both ϕ and ϕ^{-1} are composed by identity maps.

It remains to be shown that injection and projection maps i and j for $E(f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2))$ and i' and j' for $f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2$ agree under ϕ.

This follows from

$$\begin{gathered} \phi(i(a)) = \phi(a, i_1(f(a)), i_2(f(a))) \\ = ((a, i_1(f(a))), (a, i_2(f(a)))) = (i_1'(a), i_2'(a)) = i'(a) \end{gathered}$$

and

$$j(a, e_1, e_2) = a = j'((a, e_1), (a, e_2)) = j'(\phi(a, e_1, e_2)).$$

□

Last, we show a theorem about whitney sums that will be essential in the proof of Milnor's theorem. For its prove, we need to use the following proposition that will be deferred until chapter 5.

Proposition 3.4.
Let $\mathfrak{b}$ be a microbundle over a "bouqet" of spheres B, meeting at a single point. There exists a map $r : B \to B$ such that $\mathfrak{b} \oplus r^*\mathfrak{b}$ is trivial.

Theorem 3.5.
Let $\mathfrak{b}$ be a microbundle over a d-dimensional simplicial complex B. Then there exists a microbundle $\mathfrak{n}$ over B so that the whitney sum $\mathfrak{b} \oplus \mathfrak{n}$ is trivial.

Proof.
We prove this theorem by induction over d.

(Start of induction)

A 1-dimensional simplicial complex is just a bouquet of circles. Therefore, the start of induction follows directly from Proposition **(3.4)**.

(Inductive Step)

Let B' be the $(d-1)$-skeleton of B and $\mathfrak{n}'$ be it's corresponding microbundle such that $\mathfrak{b}|_{B'} \oplus \mathfrak{n}'$ is trivial.

1. $\mathfrak{n}' \oplus \mathfrak{e}^n_{B'}$ can be extended over any d-simplex σ:

   Consider the equation

   $$(\mathfrak{n}' \oplus \mathfrak{e}^n_{B'})|_{\partial\sigma} = \mathfrak{n}'|_{\partial\sigma} \oplus \mathfrak{e}^n_{B'}|_{\partial\sigma} = \mathfrak{n}'|_{\partial\sigma} \oplus \mathfrak{b}|_{\partial\sigma} = (\mathfrak{n}' \oplus \mathfrak{b}|_{B'})|_{\partial\sigma}$$

   in which we used the previous lemma and Corollary **(2.9)** for $\mathfrak{e}^n_{B'}|_{\partial\sigma} = \mathfrak{b}|_{\partial\sigma}$. Since $(\mathfrak{n}' \oplus \mathfrak{b}|_{B'})|_{\partial\sigma}$ is trivial, the claim follows from Corollary **(2.9)**.

2. $\mathfrak{n}' \oplus \mathfrak{e}^n_{B'}$ can be extended over B:

   The difficulty is that the individual d-simplices are not well-seperated. Let B'' denote B with small open d-cells removed from every d-simplex. Since B' is a retract of B'' we can extend $\mathfrak{n}' \oplus \mathfrak{e}^n_{B'}$ over B'' and now apply the first statement. We denote the resulting microbundle by η.

3. Consider the mapping cone $B \sqcup CB'$ over the inclusion $B' \hookrightarrow B$. Since

   $$(\mathfrak{b} \oplus \eta)|_{B'} = \mathfrak{b}|_{B'} \oplus \eta|_{B'} = \mathfrak{b}|_{B'} \oplus (\mathfrak{n}' \oplus \mathfrak{e}^n_{B'}) = (\mathfrak{b}|_{B'} \oplus \mathfrak{n}') \oplus \mathfrak{e}^n_{B'} = \mathfrak{e}^n_{B'} \oplus \mathfrak{e}^n_{B'}$$

   is trivial, it follows from Lemma **(2.8)** that we can extend $\mathfrak{b} \oplus \eta$ over $B \sqcup CB'$ which will be denoted by ξ.

   The mapping cone $B \sqcup CB'$ has the homotopy type of a bouquet of spheres by transfering B' along CB' collapsing to a single point. Since any d-simplex is homotopic to a d-disc and it's border is collapsed, we get the homotopy of a $(d-1)$-sphere.

   With Theorem **(4.1)** and Proposition **(3.4)**, we conclude that there exists a microbundle $\mathfrak{n}$ such that $(\xi \oplus \mathfrak{n})|_B$ is trivial. The equation

   $$\mathfrak{e}^n_B = (\xi \oplus \mathfrak{n})|_B = \xi|_B \oplus \mathfrak{n}|_B = (\mathfrak{b} \oplus \eta) \oplus \mathfrak{n}|_B = \mathfrak{b} \oplus (\eta \oplus \mathfrak{n}|_B)$$

   completes the proof.

□

Chapter 4

The Homotopy Theorem

In this chapter we will prove the homotopy theorem. It states the following:

Theorem 4.1 (Homotopy Theorem)**.**
Let $\mathfrak{b}$ *be a microbundle over* B *and let* $f, g : A \to B$ *be two maps where* A *is paracompact. If* f *and* g *are homotopic, then* $f^*\mathfrak{b}$ *and* $g^*\mathfrak{b}$ *are isomorphic.*

Before we can start with the proof of the theorem, we need additional concepts to put microbundles in relation to each other.

Definition 4.2 (map-germ)**.**
A *map-germ* $F : (X, A) \Rightarrow (Y, B)$ between topological pairs (X, A) and (Y, B) is an equivalence class of maps $(X, A) \to (Y, B)$ where $f \sim g :\iff f|_U = g|_U$ for an arbitrary neighborhood $U \subseteq X$ of A.

A *homeomorphism-germ* $F : (X, A) \Rightarrow (Y, B)$ is a map-germ such that there exists a representative $f : U_f \to Y$ that maps homeomorphically to a neighborhood of B.

Now consider two isomorphic microbundles $\mathfrak{b}$ and $\mathfrak{b}'$ over B. There exists a homeomorphism $\phi : V \xrightarrow{\sim} V'$ where $V \subseteq E$ is a neighborhood of $i(B)$ and $V' \subseteq E'$ is a neighborhood of $i'(B)$. The homeomorphism ϕ is a representative for a homeomorphism-germ

$$[\phi] : (E, i(B)) \Rightarrow (E', i'(B)).$$

Studying isomorphy between microbundles in this way is useful because we don't care what such a homeomorphism does on particular neighborhoods of the base spaces but only what it does on arbitray small ones. Hence every representative of $[\phi]$ describes the "same" isomorphy between $\mathfrak{b}$ and $\mathfrak{b}'$. Now, naturally, the question arises whether the existence of a homeomorphism-germ

$$F : (E, i(B)) \Rightarrow (E', i'(B))$$

already implies that $\mathfrak{b}$ and $\mathfrak{b}'$ are isomorphic. The answer is generally no, because isomorphy of microbundles requires a homeomorphism that commutes with injection and projection maps. Therefore, we must assume an extra condition called "fibre-preservation" for this implication to be true. This justifies the following definition.

Let $\mathfrak{b}$ and $\mathfrak{b}'$ be two microbundles over B and let $J : (E, i(B)) \Rightarrow (B, B)$ and $J' : (E', i(B)) \Rightarrow (B, B)$ denote the map-germs representing its projection maps.

Definition 4.3 (isomorphism-germ)**.**
An *isomorphism-germ* between $\mathfrak{b}$ and $\mathfrak{b}'$ is a homeomorphism-germ

$$F : (E, B) \Rightarrow (E', B)$$

which is *fibre-preserving*, that is $J' \circ F = J$.

Remark 4.4.
There exists an isomorphism-germ between $\mathfrak{b}$ and $\mathfrak{b}'$ if and only if $\mathfrak{b}$ is isomorphic to $\mathfrak{b}'$.

We can take this even further by giving up on the assumption that the base spaces of the considered microbundles equal. Note that in this case no comparison to isomorphy can be drawn, since we have not defined isomorphy between microbundles over different base spaces.

Definition 4.5 (bundle-germ)**.**
Let $\mathfrak{b}$ and $\mathfrak{b}'$ be two microbundles over B and B' with the same fibre-dimension. A *bundle-germ* $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$ is a map-germ

$$F : (E, B) \Rightarrow (E', B')$$

such that there exists a representative $f : U_f \to E'$ that maps each fibre $j^{-1}(b)$ injectively to a fibre $j'^{-1}(b')$.

For a bundle-germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$, the following diagram commutes:

$$\begin{tikzcd}
(E, B) \arrow[r, Rightarrow, "F"] \arrow[d, "i"] & (E', B') \arrow[d, "i'"] \\
B \arrow[r, "F|_B"] & B'
\end{tikzcd}$$

We say F *is covered by* $F|_B$.

The bundle-germ is indeed a generalization of the isomorphism germ, as the following proposition illustrates.

Proposition 4.6 (Williamson)**.**
Let $\mathfrak{b}$ and $\mathfrak{b}'$ be two microbundles over B. A bundle-germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$ covering the identity map is an isomorphism-germ.

First, however, we show a lemma that helps us to prove the proposition.

Lemma 4.7.
If a homeomorphism $\phi : \overline{B_2(0)} \xrightarrow{\sim} \phi(\mathbb{R}^n) \subseteq \mathbb{R}^n$ *satisfies*

$$|\phi(x) - x| < 1, \forall x \in \overline{B_2(0)}$$

then $\overline{B_1(0)} \subseteq \phi(\overline{B_2(0)})$.

Proof of the lemma.
Consider $\phi(2S^n)$ where $2S^n$ denotes the n-sphere of radius 2. The condition for ϕ yields $1 < |\phi(s)|, \forall s \in 2S^n$. Since $\overline{B_2(0)}$ has trivial homology groups which are preserved under homeomorphisms, $\phi(\overline{B_2(0)})$ must have trivial homology groups as well.

From this we can conclude that $\overline{B_1(0)}$ must be contained in $\phi(\overline{B_2(0)})$, because otherwise "holes" would form which would result in non-trivial homology groups of $\phi(\overline{B_2(0)})$. $\square$

Proof of the proposition.
Let f be a representative for F. First we assume a special and then generalize the result to show the proposition.

1. Let f map from $B \times \mathbb{R}^n$ to $B \times \mathbb{R}^n$:

   Since F covers the identity, f is of the form

   $$f(b, x) = (b, g_b(x))$$

   where $g_b : \mathbb{R}^n \to \mathbb{R}^n$ are individual maps. Since the g_b are continuous and injective, it follows from the [domain invariance theorem] that the g_b are open maps.

   Let $(b_0, x_0) \in B \times \mathbb{R}^n$ and let $\varepsilon > 0$. Since g_{b_0} is an open map, there exists a $\delta > 0$ such that $\overline{B_{2\delta}(x_1)} \subseteq g_{b_0}(\overline{B_\varepsilon(x_0)})$ where $x_1 := g_{b_0}(x_0)$.

   We claim that there exists a neighborhood $V \subseteq B$ of b_0 such that

   $$|g_b(x) - g_{b_0}(x)| < \delta$$

   for every $b \in V$ and $x \in \overline{B_\varepsilon(x_0)}$.

   To show that, consider $\phi(b, x) := g_b(x) - g_{b_0}(x)$. The open set $\phi^{-1}(B_\delta(0))$ is a neighborhood of $\{b_0\} \times \mathbb{R}^n$ since $\phi(b_0, x) = 0$. Hence, there exist open subsets $V_x \subseteq B$ and $W_x \subseteq \mathbb{R}^n$ such that

   $$\bigcup_{x \in \overline{B_\varepsilon(x_0)}} V_x \times W_x \subseteq \phi^{-1}(\overline{B_\delta(0)})$$

   and $x \in W_x$. Since $\overline{B_\varepsilon(x_0)}$ is compact, there exist $x_1, \ldots, x_n \in \overline{B_\varepsilon(x_0)}$ with $\overline{B_\varepsilon(x_0)} \subseteq \bigcup_{i=1}^n V_{x_i}$. The claim follows with $V := V_{x_1} \cap \cdots \cap V_{x_n}$ which is open by forming the intersection over finitely many open sets.

Now we want to apply the previous lemma:

Consider the homeomorphism

$$\overline{B_{2\delta}(x_1)} \xrightarrow{\sim} g_b \circ g_{b_0}^{-1}(\overline{B_{2\delta}(x_1)})$$

for an arbitrary $b \in V$. Since

$$\overline{B_{2\delta}(x_1)} \subseteq g_{b_0}(\overline{B_\varepsilon(x_0)}) \implies g_{b_0}^{-1}(\overline{B_{2\delta}(x_1)}) \subseteq \overline{B_\varepsilon(x_0)}$$

we conclude from the above that

$$|(g_b \circ g_{b_0}^{-1})(x) - x| < \delta, \forall x \in \overline{B_{2\delta}(x_1)}$$

It follows that, by translation and scaling, $g_b \circ g_{b_0}^{-1}|_{\overline{B_{2\delta}(x_1)}}$ satisfies the conditions of Lemma (4.7). Therefore, $\overline{B_\delta(x_1)} \subseteq (g_b \circ g_{b_0}^{-1})(\overline{B_{2\delta}(x_0)})$ and so $\overline{B_\delta(x_1)} \subseteq g_b(\overline{B_\varepsilon(x_0)})$.

From

$$V \times \overline{B_\delta(x_1)} \subseteq g(V \times \overline{B_\varepsilon(x_0)})$$

it follows that f is an open map.

2. Glue together $f : U_f \to E(\mathfrak{b}')$ along local trivializations:

   For an arbitrary $b \in B$, choose local trivializations (U, V, ϕ) and (U', V', ϕ') of b in $\mathfrak{b}$ and $\mathfrak{b}'$. Without loss of generality we may assume that $U = U'$ because otherwise we can choose $V = \phi^{-1}(U \cap U')$ and $V' = \phi'^{-1}(U \cap U')$ and restrict ϕ and ϕ' accordingly.

   First, we restrict f to $V \cap f^{-1}(V')$. Since $V \cap f^{-1}(V')$ is an open neighborhood of $i(b)$ and contained in V, we can choose an open neighborhood $\tilde{U} \subseteq U$ of $i(b)$ and $\varepsilon > 0$ such that $\phi^{-1}(\tilde{U} \times B_\varepsilon(0)) \subseteq V \cap f^{-1}(V')$.

   This yields a map $U' \times \mathbb{R}^n \to U' \times \mathbb{R}^n$ with

   $$\tilde{U} \times \mathbb{R}^n \cong \tilde{U} \times B_\varepsilon(0) \xrightarrow{\sim} \phi^{-1}(\tilde{U} \times B_\varepsilon(0)) \xrightarrow{f} U' \times \mathbb{R}^n \subseteq U \times \mathbb{R}^n$$

   that is injective and fibre-preserving and therefore an open map (apply 1.). It follows that $f : \phi^{-1}(\tilde{U} \times B_\varepsilon(0)) \to V'$ must be an open map as well since the other composing maps are homeomorphisms.

   By glueing the $\phi^{-1}(\tilde{U} \times B_\varepsilon(0))$ together over all $b \in B$, we see that f is an open map.

□

We can easily generalize this to bundle-germs between microbundles over different base spaces:

Corollary 4.8.
If a map $g : B \to B'$ is covered by a bundle-germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$, then $\mathfrak{b}$ is isomorphic to the induced microbundle $g^*\mathfrak{b}'$.

Proof.
Let $f : U_f \to E'$ be a representative map for F. We define $F' : \mathfrak{b} \Rightarrow g^*\mathfrak{b}'$ by a representative

$$f' : U_f \to E(g^*\mathfrak{b}') \text{ with } f'(e) := (j(e), f(e)).$$

Every $f'(e)$ actually lies in $E(g^*\mathfrak{b}')$ because

$$g(j(e)) = j'(f(e))$$

as we can see from the commutative diagram for bundle-germs. The germ F' is an isomorphism-germ because F is an isomorphism-germ. Applying the previous proposition on F' proves the claim. $\square$

Lemma 4.9.
Let $\mathfrak{b}$ be a microbundle over B and $\{B_\alpha\}$ a locally finite collection of closed sets covering B. Additionally, we are given a collection of bundle germs $F_\alpha : \mathfrak{b}|_{B_\alpha} \Rightarrow \mathfrak{b}'$ such that $F_\alpha = F_\beta$ on $\mathfrak{b}|_{B_\alpha \cap B_\beta}$. Then there exists a bundle germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$ extending F_α, that is F and F_α agree on a sufficiently small neighborhood of $i(B_\alpha)$.

Proof.
Choose representative maps $f_\alpha : U_\alpha \to E(\mathfrak{b}')$ for F_α such that the U_α are open. For every α and β, choose an open neighborhood $U_{\alpha\beta}$ of $i(B_\alpha \cap B_\beta)$ on which the representative maps f_α and f_β agree. Now consider

$$U := \{e \in E : j(e) \in B_\alpha \cap B_\beta \implies e \in U_{\alpha\beta}\}$$

which satisfies the following:

1. U is open:

   We can express U like this:

   $$E - \bigcup_{\alpha\beta} \{j^{-1}(B_\alpha \cap B_\beta) \cap U_{\alpha\beta}^c\}$$

   Since $j^{-1}(B_\alpha \cap B_\beta)$ and $U_{\alpha\beta}^c$ are closed sets, U must be open. That is because an open set remains open after removing arbitrarily many closed sets.

2. $i(B) \subseteq U$:

   This follows from

   $$b \in B_\alpha \cap B_\beta \implies i(b) \in i(B_\alpha \cap B_\beta) \subseteq U_{\alpha\beta}$$

   and $j(i(b)) = b$.

Now we can define $f : U \to E(\mathfrak{b}')$ in the obvious way

$$f(u \in U_{\alpha\beta}) := f_\alpha(u) = f_\beta(u)$$

which is continuous according to the [glueing lemma]. We see that f agrees with f_α on $U_{\alpha\alpha}$, hence f is a representative for a bundle germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$ extending $\{F_\alpha\}$.

Therefore, f is a representative map for our required F. □

Lemma 4.10.
Let $\mathfrak{b}$ be a microbundle over $B \times [0,1]$. If $\mathfrak{b}|_{B\times[0,\frac{1}{2}]}$ and $\mathfrak{b}|_{B\times[\frac{1}{2},1]}$ are both trivial, then $\mathfrak{b}$ itself is trivial.

Proof.
Consider the identity bundle-germ over $\mathfrak{b}|_{B\times\{\frac{1}{2}\}}$, that is the bundle-germ represented by the identity on $E(\mathfrak{b}|_{B\times\{\frac{1}{2}\}})$. Since $\mathfrak{b}|_{B\times[\frac{1}{2},1]}$ is trivial, we can extend this bundle-germ to

$$\mathfrak{b}|_{B\times[\frac{1}{2},1]} \Rightarrow \mathfrak{b}|_{B\times\{\frac{1}{2}\}}$$

by the representative

$$B \times [\frac{1}{2}, 1] \times \mathbb{R}^n \to B \times \{\frac{1}{2}\} \times \mathbb{R}^n$$

$$(b, t, x) \mapsto (b, \frac{1}{2}, x).$$

Here we identified an open subset of $E(\mathfrak{b}|_{B\times[\frac{1}{2},1]})$ with $B \times [\frac{1}{2}, 1] \times \mathbb{R}^n$ using Lemma **(1.6)**. Using the previous lemma, we can piece this together with the identity bundle-germ on $\mathfrak{b}|_{B\times[0,\frac{1}{2}]}$ (note that the bundle germs agree on their intersection) resulting in a bundle-germ

$$\mathfrak{b} \Rightarrow \mathfrak{b}|_{B\times[0,\frac{1}{2}]}.$$

The previous corollary infers that $\mathfrak{b}$ is isomorphic to $r^*\mathfrak{b}|_{B\times[0,\frac{1}{2}]}$ where $r : B \times [0,1]$ is the retraction $(b,t) \mapsto (b, \min(t, \frac{1}{2}))$. But $\mathfrak{b}|_{B\times[0,\frac{1}{2}]}$ is trivial, hence $r^*\mathfrak{b}|_{B\times[0,\frac{1}{2}]}$ is trivial as well (Lemma **(2.4)**) which concludes the proof. □

Lemma 4.11.
Let $\mathfrak{b}$ be a microbundle over $B \times [0,1]$. Every $b \in B$ has a neighborhood V where $\mathfrak{b}|_{V\times[0,1]}$ is trivial.

Proof.
Let $b \in B$ be arbitrary.

For every $t \in [0,1]$, choose a neighborhood $U_t := V_t \times (t - \varepsilon_t, t + \varepsilon_t)$ of (b,t) such that $\mathfrak{b}|_{U_t}$ is trivial. This can be achieved by taking a local trivialization of (b,t) in $\mathfrak{b}$ and restricting the spaces if necessary.

Since $\{b\} \times [0,1]$ is compact, we can choose a finite subset

$$(V_1 \times (t_1 - \varepsilon_1, t_1 + \varepsilon_1)), \ldots, (V_n \times (t_n - \varepsilon_n, t_n + \varepsilon_n))$$

of the collection $\{U_t\}$ covering $\{b\} \times [0,1]$ and define $V = V_1 \cap \cdots \cap V_n$.

The restricted microbundles $\mathfrak{b}|_{V \times (t_i - \varepsilon_i, t_i + \varepsilon_i)}$ are trivial, because every $\mathfrak{b}|_{U_t}$ is trivial and $V \times (t_i - \varepsilon_i, t_i + \varepsilon_i) \subseteq U_t$. Hence, there exists a subdivision $0 = t_0 < \cdots < t_k = 1$ such that every $\mathfrak{b}|_{V \times [t_i, t_{i+1}]}$ is trivial.

By iteratively applying the previous lemma on the $\mathfrak{b}|_{V \times [t_i, t_{i+1}]}$, it follows that $\mathfrak{b}|_{V \times [0,1]}$ is trivial. □

Lemma 4.12.
Let $\mathfrak{b}$ be a microbundle over $B \times [0,1]$ where B is paracompact hausdorff. Then there exists a bundle map-germ $R : \mathfrak{b} \Rightarrow \mathfrak{b}|_{B \times [0,1]}$ covering the standard retraction $r : B \times [0,1] \to B \times [0,1]$ with $(b,t) \mapsto (b,1)$.

Proof.
First, we assume a locally finite covering $\{V_\alpha\}$ of open sets where $\mathfrak{b}|_{V_\alpha \times [0,1]}$ is trivial. The existence of such a covering is justified by the previous lemma and the fact that any open cover of B has a locally finite subcover due to paracompactness.

This open cover has a partition of unity (B is paracompact hausdorff)

$$\lambda_\alpha : B \to [0,1]$$

with $\text{supp}\lambda_\alpha \subseteq V_\alpha$ that is rescaled in way that

$$\max_\alpha(\lambda_\alpha(b)) = 1, \forall b \in B.$$

This can be achieved by dividing λ_α with $\max_\alpha \lambda_\alpha$ which is well-defined because $\{V_\alpha\}$ is locally finite and continuous because the max function is continuous.

Now we define a retraction $r_\alpha : B \times [0,1][0,1] \to B \times [0,1]$ with

$$r_\alpha(b,t) = (b, \max(t, \lambda_\alpha(b))).$$

In the following, we will construct bundle-germs $R_\alpha : \mathfrak{b} \Rightarrow \mathfrak{b}$ covering r_α and piece them together to obtain the desired bundle-germ.

1. We can divide $B \times [0,1]$ into two subsets

$$A_\alpha := \text{supp}\lambda_\alpha \times [0,1] \text{ and}$$

$$A'_\alpha := \{(b,t) : t \geq \lambda_\alpha(b)\}.$$

We already know that $\mathfrak{b}|_{A_\alpha}$ is trivial since $A_\alpha \subseteq V_\alpha \times [0,1]$. Like in the proof of Lemma **(4.10)**, we can extend the identity bundle-germ on $\mathfrak{b}|_{A_\alpha \cap A_\alpha{}'}$ to a bundle-germ

$$\mathfrak{b}|_{A_\alpha} \Rightarrow \mathfrak{b}|_{A_\alpha \cap A_\alpha{}'}$$

via the representative

$$A_\alpha \times \mathbb{R}^n \to (A_\alpha \cap A_\alpha{}') \times \mathbb{R}^n$$

$$(a, x) \mapsto (r_\alpha(a), x).$$

Piecing this together with the identity bundle germ $\mathfrak{b}|_{A_\alpha{}'}$, we obtain out required bundle germ R_α.

2. Lastly, we construct a bundle germ R using the R_α.

   Applying the well-ordering theorem, which is equivalent to the axiom of choice, we can assume an ordering of $\{V_\alpha\}$.

   Let $\{B_\beta\}$ be a locally finite covering of B with closed sets where B_β intersects only finitely many $V_{\alpha_1} < \cdots < V_{\alpha_k}$. We can construct a collection like this as follows:

   Now the composition $R_{\alpha_1} \circ \ldots \circ R_{\alpha_k}$ restricts to a bundle germ $R(\beta) : \mathfrak{b}|_{B_\beta} \times [0,1] \Rightarrow \mathfrak{b}|_{B_\beta} \times [0,1]$ covering the retraction $(b,t) \mapsto (b,1)$. That is because for every $b \in B_\beta$, there is some $1 \leq i \leq k$ with $\lambda_{\alpha_i}(b) = 1$ and hence $r_{\alpha_i}(b,t) = (b,1)$.

   Pieced together using Lemma **(4.10)**, we obtain $R : \mathfrak{b}|_{B\times[0,1]} \to \mathfrak{b}|_{B\times[0,1]}$ covering $(b,t) \mapsto (b,1)$, which concludes the proof.

□

Finally, we gathered all the tools to proof the homotopy theorem.

Proof of the Homotopy Theorem.
Let $H : A \times [0,1] \to B$ be a homotopy between f and g.

The previous lemma states that there exists a bundle germ

$$R : H^*\mathfrak{b} \Rightarrow H^*\mathfrak{b}|_{A\times[0,1]}$$

covering the standard retraction $(A,t) \mapsto (a,1)$.

By restricting R to $H^*\mathfrak{b}|_{A\times[0,1]}$ we obtain a bundle germ

$$H^*\mathfrak{b}|_{A\times[0,1]} \Rightarrow H^*\mathfrak{b}|_{A\times[0,1]}$$

covering $\theta : A\times[0,1] \to A\times[0,1]$ with $(a,0) \mapsto (a,1)$. Applying Corollary **(4.8)** yields $H^*\mathfrak{b}|_{A\times[0,1]} \cong \theta^*(H^*\mathfrak{b}|_{A\times[0,1]})$.

Considering $A \times [0,1] = A$, we can identify $H^*\mathfrak{b}|_{B\times[0,1]}$ with $f^*\mathfrak{b}$

$$H^*\mathfrak{b}|_{A\times\{0\}} = \iota^*(H^*\mathfrak{b}) \cong (H \circ \iota)^*\mathfrak{b} = f^*\mathfrak{b}$$

and symmetrically $\theta^*(H^*\mathfrak{b}|_{B\times[0,1]})$ with $g^*\mathfrak{b}$.

Together with $H^*\mathfrak{b}|_{A\times[0,1]} \cong \theta^*(H^*\mathfrak{b}|_{A\times[0,1]})$ it follows that $f^*\mathfrak{b} \cong g^*\mathfrak{b}$. $\square$

Chapter 5

Microbundles over a Suspension

In this chapter, every topological space comes with a base point which will be denoted with subscript 0.

Definition 5.1.
A *rooted microbundle* $\mathfrak{b}$ over B is a microbundle over B together with an isomorphism-germ

$$R : \mathfrak{b}|_{b_0} \Rightarrow \mathfrak{e}^n_{b_0}.$$

Two rooted microbundles $\mathfrak{b}$ and $\mathfrak{b}'$ are *isomorphic* if there exists an isomorphism germ $\mathfrak{b} \Rightarrow \mathfrak{b}'$ extending

$$R'^{-1} \circ R : \mathfrak{b}|_{b_0} \Rightarrow \mathfrak{b}|'_{b_0}.$$

Theorem 5.2 (Rooted Homotopy Theorem)**.**
Let $\mathfrak{b}$ be a rooted microbundle over B and $f, g : A \to B$ be two based maps. If there exists a homotopy $H : A \times [0,1] \to B$ between f and g that leaves the base point fixed, then the two rooted microbundles $f^*\mathfrak{b}$ and $g^*\mathfrak{b}$ are isomorphic.

We need to show a rooted version of Lemma **(4.11)**. Before we prove the lemma, note that

$$E(H^*\mathfrak{b}|_{a_0 \times [0,1]})$$

is just

$$\begin{aligned} &\{e \in E(H^*\mathfrak{b}) : j(e) \in a_0 \times [0,1]\} \\ &= \{(a,t,e) \in A \times [0,1] \times E(\mathfrak{b}) : a = a_0 \wedge H(a,t) = j(e)\} \\ &= a_0 \times [0,1] \times E(\mathfrak{b}|_{b_0}). \end{aligned}$$

Based on this, we can define an isomorphism-germ

$$\bar{R} : H^*\mathfrak{b}|_{a_0 \times [0,1]} \Rightarrow \mathfrak{e}^n_{a_0 \times [0,1]}$$

via a representative

$$\bar{r} : a_0 \times [0,1] \times V \to a_0 \times [0,1] \times \mathbb{R}^n$$

with

$$\bar{r}(a_0, t, v) = (a_0, t, r^{(2)}(v))$$

where $r : V \to b_0 \times \mathbb{R}^n$ is a representative for R. The representative $\bar{r}$ is a homoemorphism on its image because it is a product of the identity and r, which are both homoemorphisms on their image.

Lemma 5.3.
Let $\mathfrak{b}$ be a rooted microbundle over B and let $H : A \times [0,1] \to B$ be a map that leaves the base point fixed. There exists a neighborhood V of a_0 with an isomorphism-germ

$$H^*\mathfrak{b}|_{V\times[0,1]} \Rightarrow \mathfrak{e}^n_{V\times[0,1]}$$

extending $\bar{R}$ (as defined above).

Proof.
By applying Lemma **(4.11)**, it follows that there exists an isomorphism-germ

$$Q : H^*\mathfrak{b}|_{V\times[0,1]} \Rightarrow \mathfrak{e}^n_{V\times[0,1]}$$

for a sufficiently small neighborhood V of a_0.

Now consider

$$Q \circ \bar{R}^{-1} : \mathfrak{e}^n_{a_0\times[0,1]} \Rightarrow \mathfrak{e}^n_{a_0\times[0,1]}.$$

Similarly to the construction of $\bar{R}$ we can construct an isomorphism-germ

$$P : \mathfrak{e}^n_{V\times[0,1]} \Rightarrow \mathfrak{e}^n_{V\times[0,1]}$$

extending $Q \circ \bar{R}^{-1}$ represented by

$$p(v, t, x) = (v, q(a_0, t, x))$$

where q is a representative for $Q \circ \bar{R}^{-1}$.

Restricted to $a_0 \times [0,1]$, P agrees with $Q \times \bar{R}^{-1}$ and thus

$$P^{-1} \circ Q = (\bar{R} \circ Q^{-1}) \circ Q = \bar{R}$$

Since P and Q are both isomorphism-germs, $P^{-1} \circ Q$ is an isomorphism-germ as well. Therefore, $P^{-1} \circ Q$ suffices our requirements which concludes the proof. □

Proof of the Rooted Homotopy Theorem.
Follow the steps for proving the initial Homotopy Theorem, however using Lemma **(5.3)** instead of Lemma **(4.11)**. □

The following definition requires the base spaces to be hausdorff. This is useful because this implies that the singleton containing only the base point is closed, and can therefore be removed from any open set without losing openness.

Definition 5.4.
Let $\mathfrak{a}$ and $\mathfrak{b}$ be two rooted microbundles over A and B. The *wedge sum* $\mathfrak{a} \vee \mathfrak{b}$ of $\mathfrak{a}$ and $\mathfrak{b}$ is a microbundle over $A \vee B$

$$\mathfrak{a} \vee \mathfrak{b} \xrightarrow{i_{\mathfrak{a}} \vee i_{\mathfrak{b}}} E(\mathfrak{a} \vee \mathfrak{b}) \xrightarrow{j_{\mathfrak{a}} \vee j_{\mathfrak{b}}} A \vee B$$

where the total space is

$$(E(\mathfrak{a}) \sqcup E(\mathfrak{b}))/(f(e_a) \sim e_a)$$

and $f : E(\mathfrak{a}|_{a_0}) \supseteq W_a \xrightarrow{\sim} W_b \subseteq E(\mathfrak{b}|_{b_0})$ is some representative for $R_b^{-1} \circ R_a$. We equip $\mathfrak{a} \vee \mathfrak{b}$ with a rooting

$$R : E((\mathfrak{a} \vee \mathfrak{b})|_{a_0}) \Rightarrow \mathfrak{e}^n_{a_0}$$

represented by any representative for R_a (or R_b).

Proof that $\mathfrak{a} \vee \mathfrak{b}$ *is a microbundle.*
We show that $\mathfrak{a} \vee \mathfrak{b}$ is a microbundle and afterwards show that the definition of $\mathfrak{a} \vee \mathfrak{b}$ is independant of the choice of the representative f for $R_b^{-1} \circ R_a$.

1. $\mathfrak{a} \vee \mathfrak{b}$ is a microbundle:

   - The injection map $i_{\mathfrak{a}} \vee i_{\mathfrak{b}}$ is well-defined because

     $$i(a_0) = i_{\mathfrak{a}}(a_0) = f(i_{\mathfrak{a}}(a_0)) = i_{\mathfrak{b}}(b_0) = i(b_0)$$

     and continuous since $i_{\mathfrak{a}}$ and $i_{\mathfrak{b}}$ are continuous.

   - The projection map $j_{\mathfrak{a}} \vee j_{\mathfrak{b}}$ is well-defined because

     $$\forall e \in W_a : j(e) = j_{\mathfrak{a}}(e) = a_0 = b_0 = j_{\mathfrak{b}}(f(e)) = j(f(e))$$

     and continuous since $j_{\mathfrak{a}}$ and $j_{\mathfrak{b}}$ are continuous.

   - The composition $j \circ i = id_{A \vee B}$ because for every $a \in A$

     $$j(i(a)) = j(i_{\mathfrak{a}}(a)) = j_{\mathfrak{a}}(i_{\mathfrak{a}}(a)) = a$$

     since $j_{\mathfrak{a}} \circ i_{\mathfrak{a}} = id_A$ (analogous for B).

   It remains to show local triviality.

   The subspace topology of $E(\mathfrak{a}|_{a_0})$ yields an open subset $W'_a \subseteq E(\mathfrak{a})$ with $W_a = W'_a \cap E(\mathfrak{a}|_{a_0})$. Symmetrically, let $W'_b \subseteq E(\mathfrak{b})$ with $W_b = W'_b \cap E(\mathfrak{b}|_{b_0})$.

   Let $x \in A \vee B$, w.l.o.g. $x \in A$ for symmetry reasons.

- $x \neq a_0$:

  Choose a local trivialization (U, V, ϕ) for x in $\mathfrak{a}$. We can assume $U \cap B = \emptyset$ by subtracting U by $\{a_0\}$ which is closed since A is hausdorff. Now we can simply use this trivialization for $\mathfrak{a} \vee \mathfrak{b}$ since U is open in $A \vee B$, V is open in $E(\mathfrak{a} \vee \mathfrak{b})$ and $V \cong U \times \mathbb{R}^n$.

- $x = a_0$:

  Choose local trivializations (U_a, V_a, ϕ_a) for a_0 in $\mathfrak{a}$ and (U_b, V_b, ϕ_b) for b_0 in $\mathfrak{b}$.

  - We can assume $V_b \cap E(\mathfrak{b}|_{b_0}) \subseteq W_b$ by choosing a local trivialization for b_0 in the microbundle over the restricted total space $(E(\mathfrak{b}) - E(\mathfrak{b}|_{b_0})) \cup W_a$ (the existence is justified by Proposition **(1.4)**).
  - We can assume $V_a \cap E(\mathfrak{a}|_{a_0}) \subseteq W_b \cap E(\mathfrak{b}|_{b_0})$ by choosing a local trivialization for a_0 in the microbundle over the restricted total space $(E(\mathfrak{a}) - E(\mathfrak{b}|_{b_0})) \cup (V_b \cap E(\mathfrak{b}|_{b_0}))$.

  The subset $X_b := \phi_b^{(2)} f(V_a \cap E(\mathfrak{a}|_{a_0})) \subseteq W_b \cap E(\mathfrak{b}|_{b_0})$ is homeomorphic to $\mathbb{R}^n$ via

  $$a_0 \times \mathbb{R}^n \xrightarrow{\phi^{-1}} V_a \cap E(\mathfrak{a}|_{a_0}) \xrightarrow{f} X_b$$

  and open since f and ϕ are homeomorphisms. By choosing $V_b' := \phi_b^{-1}(B \times X_b)$ and $\phi_b'(e) := (j(e), \phi_a^{(2)}(f^{-1}(\phi_b^{(2)}(e))))$, we have local trivializations (U_a, V_a, ϕ_a) and (U_b, V_b', ϕ_b') that agree on $W_a = W_b$. This yields a local trivialization for $\mathfrak{a} \vee \mathfrak{b}$.

2. The wedge sum $\mathfrak{a} \vee \mathfrak{b}$ is independant of the choice of f:

   Let f' be another representative for $R_b^{-1} \circ R_a$ and $(\mathfrak{a} \vee \mathfrak{b})'$ the resulting wedge sum. We need to find an isomorphism germ that extends $R'^{-1} \circ R = R^{-1} \circ R = id$. Choose an open neighborhood $V \subseteq E(\mathfrak{a}|_{a_0})$ of $i_{\mathfrak{a}}(a)$ where f and f' agree. By subtracting $j_{\mathfrak{a}}^{-1}(a_0) - V$ from $E(\mathfrak{a} \vee \mathfrak{b})$ and $E(\mathfrak{a} \vee \mathfrak{b})'$ the microbundles remain unchanged. This is because the resulting subspaces are open since $j_{\mathfrak{a}}^{-1}(a_0)$ is closed (hausdorff) and V is open. So the total spaces are equal and injection and projection maps are defined the same. Using the modified total spaces, it follows that the identity $(\mathfrak{a} \vee \mathfrak{b}) \Rightarrow (\mathfrak{a} \vee \mathfrak{b})'$ is an isomorphism-germ. This surely extends $R'^{-1} \circ R$, which concludes the proof.

□

In the following, let B be a *reduced suspension*

$$SX = (X \times [0, 1])/(X \times \{0, 1\} \cup x_0 \times [0, 1])$$

over X.

Let $\phi : B \to B \vee B$ denote the map that sends $X \times [0, \frac{1}{2}]$ to the first B via

$$\phi(x,t) = [(x, 2t)]$$

and $X \times [\frac{1}{2}, 1]$ to the second B via

$$\phi(x,t) = [(x, 2t-1)].$$

Let $c_1 : B \vee B \to B$ denote the map that is the identity on the first summand and the constant map to b_0 on the second summand (symmetrically define c_2).

Lemma 5.5.

$$\phi^*(\mathfrak{b} \oplus \mathfrak{e}_B^n) \cong \mathfrak{b} \cong \phi^*(\mathfrak{e}_B^n \oplus \mathfrak{b})$$

Proof.

- First, note that $c_1^*\mathfrak{b} \cong \mathfrak{b} \vee \mathfrak{e}^n$:

  $$E(c_1^*\mathfrak{b}) = \{(b,e) \in (B \vee B) \times E(\mathfrak{b}) : c_1(b) = j(e)\}$$
  $$= (\{(b,e) \in B \times E(\mathfrak{b}) : b = j(e)\} \sqcup B \times E(\mathfrak{b}|_{b_0}))/\sim$$
  $$= (\{(j(e),e) : e \in E(\mathfrak{b})\} \sqcup B \times E(\mathfrak{b}|_{b_0}))/\sim$$

  where $(b,e) \sim (b',e') \iff b = b_0 = b' \wedge e = e'$. Additionally, we can omit first component on the left side resulting in

  $$(E(\mathfrak{b}) \sqcup (B \times E(\mathfrak{b}|_{b_0})))/\sim$$

  where $e \sim (b,e') \iff b = b_0 \wedge e = e'$.

  On the other side, consider

  $$E(\mathfrak{b} \vee \mathfrak{e}_B^n) = (E(\mathfrak{b}) \sqcup (B \times \mathbb{R}^n))/e \sim f(e)$$

  with f being some representative $E(\mathfrak{b}|_{b_0}) \supseteq V \to b_0 \times \mathbb{R}^n$ for $R_e^{-1} \circ R_b$.

  Now, we have the mapping

  $$g : E(c_1^*\mathfrak{b}) \supseteq (E(\mathfrak{b}) \sqcup (B \times V))/\sim \xrightarrow{\sim} (E(\mathfrak{b}) \sqcup (B \times f(V)))/\sim \subseteq E(\mathfrak{b} \vee \mathfrak{e}^n)$$

  $$g(e) = e \text{ and } g(b,e) = (b, f^{(2)}(e)).$$

  This map is well-defined because $\forall e = (b_0, e) : g(e) = e = f(e) = (b_0, f^{(2)}(e)) = g(b_0, e)$. Bijectivity and continuity follow from bijectivity and continuity of its summands. Since $(E(\mathfrak{b}) \sqcup (B \times V))/\sim$ and $(E(\mathfrak{b}) \sqcup (B \times f(V)))/\sim$ are open (V and $f(V)$ are open) and injection and projection maps commute, it follows that g represents an isomorphism germ between $c_1^*\mathfrak{b}$ and $\mathfrak{b} \vee \mathfrak{e}^n$.

- Now, from $c_1 \circ \phi = id$ we can conclude that

$$\phi^*(\mathfrak{b} \oplus \mathfrak{e}_B^n) = \phi^* c_1^* \mathfrak{b} = (c_1 \circ \phi)^* \mathfrak{b} = \mathfrak{b}.$$

  The equality $\mathfrak{b} = \phi^*(\mathfrak{e}_B^n \oplus \mathfrak{b})$ follows by symmetry, which concludes the proof.

$\square$

Definition 5.6.
Let $\mathfrak{b}$ be a rooted microbundle over B and $f : A \to B$ a base point preserving map. The *induced microbundle* of f over $\mathfrak{b}$ is the initial induced microbundle $f^*\mathfrak{b}$ together with the rooting

$$f^*R : E(f^*\mathfrak{b}|_{a_0}) = a_0 \times E(\mathfrak{b}|_{b_0}) \Rightarrow e_{a_0}^n$$

that coincides with R if we consider $a_0 \times E(\mathfrak{b}|_{b_0}) = E(\mathfrak{b}|_{b_0})$ and $e_{a_0}^n = e_{b_0}^n$.

Lemma 5.7.
Let $\mathfrak{a}$ and $\mathfrak{b}$ be rooted microbundles over A and B. For maps $f : A' \to A$ and $g : B' \to B$ the following applies:

$$(f \vee g)^*(\mathfrak{a} \vee \mathfrak{b}) \cong f^*\mathfrak{a} \vee g^*\mathfrak{b}$$

Proof.
Consider the equation

$$E((f \vee g)^*(\mathfrak{a} \vee \mathfrak{b})) = \{(x, e) \in (A' \vee B') \times E(\mathfrak{a} \vee \mathfrak{b}) : (f \vee g)(x) = j(e)\}$$

$$= \{(x, e) \in ((A' \times E(\mathfrak{a})) \sqcup (B' \times E(\mathfrak{b})))/\sim : (f \vee g)(x) = j(e)\}$$

$$= (\{(x, e) \in A' \times E(\mathfrak{a}) : f(x) = j_{\mathfrak{a}}(e)\} \sqcup \{(x, e) \in B' \times E(\mathfrak{b}) : g(x) = j_{\mathfrak{b}}(e)\})/\sim$$

$$= (E(f^*\mathfrak{a}) \sqcup E(g^*\mathfrak{b}))/\sim = E(f^*\mathfrak{a} \vee g^*\mathfrak{b})$$

where $(a, e_a) \sim (b, e_b) \iff a = a_0 = b_0 = b \wedge e_a = e_b$ in $E(\mathfrak{a} \vee \mathfrak{b})$. So the total spaces are equal and also the injection and projection map agree, which concludes the proof. $\square$

Let $r : B \xrightarrow{\sim} B$ denote the homeomorphism that corresponds to the "reflection"

$$(x, t) \mapsto (x, 1 - t)$$

and let $c : B \vee B \to B$ be the identity on the first summand and r on the second summand.

Lemma 5.8.
The induced microbundle $\phi^*(\mathfrak{b} \vee r^*\mathfrak{b})$ is trivial.

Proof.
The composition $f \circ \phi$ is null-homotopic via $H : B \times [0,1] \to B$ with

$$H([x,t],s) = f(\phi(x, t * s))$$

and therefore $\phi^* f^* \mathfrak{b} = (f \circ \phi)^* \mathfrak{b} = const^*_{b_0} \mathfrak{b} = \mathfrak{e}^n$ (see Theorem **(4.1)**). With distributivity, we conclude $\phi^*(\mathfrak{b} \vee c^* \mathfrak{b}) = \phi^* f^* \mathfrak{b}$ and hence that $\phi^*(\mathfrak{b} \vee c^* \mathfrak{b})$ is trivial. □

Definition 5.9.
The *whitney sum* of two rooted microbundles $\mathfrak{b}$ and $\mathfrak{b}'$ over B is the initial whitney sum $\mathfrak{b} \oplus \mathfrak{b}'$ together with the rooting

$$R \oplus R' : (\mathfrak{b} \oplus \mathfrak{b}')|_{b_0} \Rightarrow \mathfrak{e}^{n_1}{}_{b_0} \oplus \mathfrak{e}^{n_2}{}_{b_0} = \mathfrak{e}^{n_1+n_1}{}_{b_0}.$$

Lemma 5.10.
The following applies for rooted microbundles $\mathfrak{a}, \mathfrak{a}'$ over A and $\mathfrak{b}, \mathfrak{b}'$ over B:

$$(\mathfrak{a} \vee \mathfrak{b}) \oplus (\mathfrak{a}' \vee \mathfrak{b}') \cong (\mathfrak{a} \oplus \mathfrak{a}') \vee (\mathfrak{b} \oplus \mathfrak{b}')$$

Proof.
Consider the equation

$$\begin{aligned} E((\mathfrak{a} \vee \mathfrak{b}) \oplus (\mathfrak{a}' \vee \mathfrak{b}')) &= \{(e,e') \in E(\mathfrak{a} \vee \mathfrak{b}) \times E(\mathfrak{a}' \vee \mathfrak{b}') : j(e) = j'(e')\} \\ &= \{(e,e') \in (E(\mathfrak{a}) \sqcup E(\mathfrak{b}))/\sim \times (E(\mathfrak{a}') \sqcup E(\mathfrak{b}'))/\sim' : j(e) = j'(e')\} \\ &= (\{(e,e') \in E(\mathfrak{a}) \times E(\mathfrak{a}') : j_{\mathfrak{a}}(e) = j_{\mathfrak{a}'}(e')\} \sqcup \\ &\quad \{(e,e') \in E(\mathfrak{b}) \times E(\mathfrak{b}') : j_{\mathfrak{b}}(e) = j_{\mathfrak{b}'}(e')\})/\approx \\ &= (E(\mathfrak{a} \oplus \mathfrak{a}') \sqcup E(\mathfrak{b} \oplus \mathfrak{b}'))/\approx = E((\mathfrak{a} \oplus \mathfrak{a}') \vee (\mathfrak{b} \oplus \mathfrak{b}')) \end{aligned}$$

where $(e_a, e'_a) \approx (e_b, e'_b) \iff e_a \sim e_b \wedge e'_a \sim' e'_b$. So the total spaces are equal and also the injection and projection map agree, which concludes the proof. □

Lemma 5.11.
Let $\mathfrak{b}$ be a rooted microbundle over a paracompact space B with rooting R. Then there exists a closed neighborhood W of b_0 and an isomorphism-germ

$$\mathfrak{b}|_W \Rightarrow \mathfrak{e}^n_W$$

extending R together with a map $\lambda : B \to [0,1]$ with

$$supp \lambda \subseteq W \text{ and } \lambda(b_0) = 1.$$

Proof.
Let $r : W_r \to b_0 \times \mathbb{R}^n$ be a representative map for R. Consider a local trivialization (U, V, ϕ) for b_0 such that $V \cap E(\mathfrak{b}|_{b_0}) \subseteq W_r$. With

$$\psi : V \xrightarrow{\sim} \psi(V) \subseteq U \times \mathbb{R}^n$$

$$\psi(e) = (j(e), r(\phi^{-1}(b_0, \phi^{(2)}(e))))$$

we have a representative for an isomorphism-germ $\mathfrak{b}|_U \Rightarrow \mathfrak{e}^n_U$ extending R. Consider the open covering of B with U and B itself. Since B is paracompact, we can apply the concept of partition of unity and have therefore a map

$$\lambda : B \to [0, 1]$$

with

$$\text{supp}\lambda \subseteq U \text{ and } \lambda(b_0) = 1.$$

Now we can choose $W := \text{supp}\lambda$, which is closed by definition of supp. By restricting the constructed isomorphism-germ over U to W, we have an isomorphism-germ $\mathfrak{b}|_W \Rightarrow \mathfrak{e}^n_W$. Together with λ, this concludes our proof. □

Lemma 5.12.
The rooted microbundles $\mathfrak{b} \oplus \mathfrak{e}^n_B$ and $\mathfrak{e}^n_B \oplus \mathfrak{b}$ are rooted-isomorphic.

Proof.
We need to find an isomorphism germ $\mathfrak{b} \oplus \mathfrak{e}^n_B \Rightarrow \mathfrak{e}^n_B \oplus \mathfrak{b}$ that extends

$$(I \oplus R) \circ (R \oplus I)^{-1} = R \oplus R^{-1}$$

where I denotes the identity germ.

Ignoring the rooting, we have an isomorphism-germ $f : E(\mathfrak{b}) \times \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n \times E(\mathfrak{b})$ with $f(e, x) = (-x, e)$. The idea is to change to f near b_0 so that it extends the rooting.

Using the previous lemma, choose a sufficiently small closed neighborhood U of b_0 such that there exists an extension $Q : (\mathfrak{b} \oplus \mathfrak{e}^n)|_U \Rightarrow (\mathfrak{e}^n \oplus \mathfrak{b})|_U$ for the rooting.

Since B is Tychonoff, there exists a map

$$\lambda : B \to [0, \frac{\pi}{2}]$$

with $\text{supp}\lambda \subseteq U$ and $\lambda(b_0) = \frac{\pi}{2}$. With this map, we can define a homeomorphism

$$g : U \times \mathbb{R}^n \times \mathbb{R}^n \xrightarrow{\sim} U \times \mathbb{R}^n \times \mathbb{R}^n$$

by

$$g(b, x, y) = (b, x \sin(\lambda(b)) - y \cos(\lambda(b)), x \cos(\lambda(b)) - y \sin(\lambda(b))).$$

Now, we can consider

$$(\mathfrak{b} \oplus \mathfrak{e}^n)|_U \Rightarrow (\mathfrak{b} \oplus \mathfrak{e}^n)|_U \xrightarrow{g} (\mathfrak{b} \oplus \mathfrak{e}^n)|_U \Rightarrow (\mathfrak{e}^n \oplus \mathfrak{b})|_U$$

which coincides with $R \oplus R^{-1}$ over b_0 since $g(b_0, x, y) = (b_0, x, y)$ and with F over $U \cap \lambda^{-1}(0)$. Pieced together with $F|_{\lambda^{-1}(b)}$, we have an isomorphism germ $\mathfrak{b} \oplus \mathfrak{e}^n_B \Rightarrow \mathfrak{e}^n_B \oplus \mathfrak{b}$ that extends the rooting, which completes the proof. □

Theorem 5.13.
If $\mathfrak{a}$ and $\mathfrak{b}$ are rooted microbundles over a completly regular space B, then

$$\phi^*(\mathfrak{a} \vee \mathfrak{b}) \oplus \mathfrak{e}^n_B = \mathfrak{a} \oplus \mathfrak{b}.$$

Proof.
The previous lemma yields $\mathfrak{b} \oplus \mathfrak{e}^n \cong \mathfrak{e}^n \oplus \mathfrak{b}$. Hence

$$\phi^*((\mathfrak{a} \oplus \mathfrak{e}^n) \vee (\mathfrak{b} \oplus \mathfrak{e}^n)) \cong \phi^*((\mathfrak{a} \oplus \mathfrak{e}^n) \vee (\mathfrak{e}^n \oplus \mathfrak{b})).$$

Additionally we have

$$\phi^*((\mathfrak{a} \vee \mathfrak{b})) \oplus (\mathfrak{e}^n \vee \mathfrak{e}^n) \cong \phi^*(\mathfrak{a} \vee \mathfrak{b}) \oplus \mathfrak{e}^n$$

for the left side of the isomorphy and

$$\phi^*((\mathfrak{a} \vee \mathfrak{e}^n) \oplus (\mathfrak{e}^n \vee \mathfrak{b})) \cong \mathfrak{a} \oplus \mathfrak{b}$$

for the right side of the isomorphy which concludes the proof. □

Corollary 5.14.
The wedge sum $\mathfrak{b} \oplus r^*\mathfrak{b}$ is trivial.

Proof.
This follows directly from the Theorem and the fact that $\phi^*(\mathfrak{b} \oplus r^*\mathfrak{b})$ is trivial.
□

Chapter 6

Normal Microbundles

Definition 6.1 (normal microbundle)**.**
Let M and N be two topological manifolds with $N \subseteq M$. A *normal microbundle* $\mathfrak{n}$ of N in M is a microbundle

$$N \xrightarrow{\iota} U \xrightarrow{r} N$$

where $U \subseteq M$ is a neighborhood of N and ι denotes the inclusion $M \hookrightarrow U$.

Definition 6.2 (composition microbundle)**.**
Let $\mathfrak{b}$ and $\mathfrak{c}$ be two microbundles over B and $E(\mathfrak{b})$. The *composition microbundle* $\mathfrak{b} \circ \mathfrak{c}$ is a microbundle

$$B \xrightarrow{i} E(\mathfrak{c}) \xrightarrow{j} B$$

where $i := i_{\mathfrak{c}} \circ i_{\mathfrak{b}}$ and $j := j_{\mathfrak{b}} \circ j_{\mathfrak{c}}$.

Proof that $\mathfrak{b} \circ \mathfrak{c}$ *is a microbundle.*
Both injection and projection are continuous since the are composed by continuous maps. Additionally, $j \circ i = j_{\mathfrak{b}} \circ (j_{\mathfrak{c}} \circ i_{\mathfrak{c}}) \circ i_{\mathfrak{b}} = j_{\mathfrak{b}} \circ i_{\mathfrak{b}} = id_B$.

It remains to be shown that $\mathfrak{b} \circ \mathfrak{c}$ is locally trivial:

For an arbitrary $b \in B$, choose local trivializations

$$(U_{\mathfrak{b}}, V_{\mathfrak{b}}, \phi_{\mathfrak{b}}) \text{ of } b \text{ and } (U_{\mathfrak{c}}, V_{\mathfrak{c}}, \phi_{\mathfrak{c}}) \text{ of } j_{\mathfrak{b}}(b).$$

Since $V_{\mathfrak{b}}$ and $U_{\mathfrak{c}}$ are both neighborhoods of $i_{\mathfrak{b}}(b)$ in $E(\mathfrak{b})$, the image

$$\phi_{\mathfrak{b}}(V_{\mathfrak{b}} \cap U_{\mathfrak{c}})$$

contains an open set $U \times B_{\varepsilon}(0)$ where U is open and ε is sufficiently small.

With $V := \phi_{\mathfrak{c}}^{-1}(\phi_{\mathfrak{b}}^{-1}(U \times B_\varepsilon(0))) \subseteq E(\mathfrak{c})$ and

$$U \times \mathbb{R}^n \cong U \times B_\varepsilon(0) \xrightarrow{\phi_{\mathfrak{c}}^{-1} \circ \phi_{\mathfrak{b}}^{-1}} V$$

which is a homeomorphism because its composed by homeomorphisms, we constructed a local trivialization of b in $\mathfrak{c} \circ \mathfrak{b}$. □

Lemma 6.3.
Let M, N and P be topological manifolds with $P \subseteq N \subseteq M$. There exists a normal microbundle

$$\mathfrak{n} : P \xrightarrow{\iota} U \xrightarrow{r} P$$

of P in M, if there exist normal microbundles

$$\mathfrak{n}_p : P \xrightarrow{\iota_P} U_N \xrightarrow{j_P} P \text{ in } N \text{ and } \mathfrak{n}_n : N \xrightarrow{\iota_N} U_M \xrightarrow{j_N} N \text{ in } M.$$

Proof.
Considering the composition $\mathfrak{n}_p \circ \mathfrak{n}_n|_{U_N}$, we found a normal microbundle $\mathfrak{n}$ of P in M since $\iota_N \circ \iota_P$ is just the inclusion $P \hookrightarrow U_M$. □

Every topological manifold is an absolute neighborhood retract (ANR).

It follows that by restricting M, if necessary, to an open neighborhood of N, there exists a retraction $r : M \to N$ which we will take advantage of in the following.

Lemma 6.4.
Let $\mathfrak{t}_N$ and $\mathfrak{t}_M$ be tangent microbundles of N and M. The total spaces $E(\iota^*\mathfrak{t}_M)$ and $E(r^*\mathfrak{t}_N)$ are homeomorphic.

Proof.
The total space

$$E(\iota^*\mathfrak{t}_M) = \{(n, m_1, m_2) \in N \times (M \times M) \mid \iota(n) = m_1\}$$

is homeomorphic to $N \times M$ via

$$(n, m_1, m_2) \mapsto (n, m_2) \text{ and } (n, m) \mapsto (n, \iota(n), m).$$

Similarly, the total space

$$E(r^*\mathfrak{t}_N) = \{(m, n_1, n_2) \in M \times (N \times N) \mid r(m) = n_1\}$$

is homeomorphic to $M \times N$ via

$$(m, n_1, n_2) \mapsto (m, n) \text{ and } (m, n) \mapsto (m, r(m), n).$$

Composed together with the trivial homeomorphism $N \times M \cong M \times N$, this yields a homeomorphism

$$\psi : E(\iota^*\mathfrak{t}_M) \xrightarrow{\sim} E(r^*\mathfrak{t}_N)$$

$$\text{with } \psi(n, m_1, m_2) := (m_2, r(m_2), n)$$

□

Remark 6.5.
The following diagram commutes:

$$\begin{tikzcd} N \arrow[r, "i_1"] \arrow[d, hook, "\iota"'] & E(\iota^* \mathfrak{t}_M) \arrow[d, "\psi"] \\ M \arrow[r, "i_2"] & E(r^* \mathfrak{t}_N) \end{tikzcd}$$

Here i_1 and i_2 denote the injections of $\iota^*\mathfrak{t}_M$ and $r^*\mathfrak{t}_N$.

The total space $E(r^*\mathfrak{t}_N)$ carries the structure of a topological manifold with

$$E(r^*\mathfrak{t}_N) \cong M \times N$$

as described in the previous lemma. That is since $M \times N$ comes equipped with the product manifold structure.

The fact that the diagram

$$\begin{tikzcd} N \arrow[r, "i_1"] \arrow[d, hook, "\iota"'] & E(\iota^* \mathfrak{t}_M) \arrow[d, "\psi"] \\ M \arrow[r, "i_2"] & E(r^* \mathfrak{t}_N) \end{tikzcd}$$

commutes (i_1 and i_2 denote the injections of $\iota^*\mathfrak{t}_M$ and $r^*\mathfrak{t}_N$), lets us consider N to be a submanifold of $E(r^*\mathfrak{t}_N)$ via $N \hookrightarrow M \xrightarrow{i_2} E(r^*\mathfrak{t}_N)$.

Lemma 6.6.
There exists a normal microbundle $\mathfrak{n}$ of N in $E(r^*\mathfrak{t}_N)$ such that $\mathfrak{n} \cong \iota^*\mathfrak{t}_M$.

Proof.
We are already given a normal microbundle of N in $E(r^*\mathfrak{t}_N)$ with $r^*\mathfrak{t}_N|_N$. Isomorphy between $r^*\mathfrak{t}_N|_N$ and $\iota^*\mathfrak{t}_M$ follows from

$$\psi : E(\iota^*\mathfrak{t}_M) \xrightarrow{\sim} E(r^*\mathfrak{t}_N)$$

together with the fact that $E(r^*\mathfrak{t}_N|_N)$ is a neighborhood of $i_2(B)$ in $E(r^*\mathfrak{t}_N)$ and from the diagram which shows that injection and projection maps commute with ψ. □

Finally, we gathered all the tools to prove Milnor's theorem.

Theorem 6.7 (Milnors Theorem)**.**
For a sufficently large $q \in \mathbb{N}$, $N = N \times [0,1]$ has a normal microbundle in $M \times \mathbb{R}^q$.

Proof.
We show the theorem in multiple steps:

1. There exists a microbundle η over N such that $\mathfrak{t}_N \oplus \eta \cong \mathfrak{e}^q N$:

   From the [Whitney Embedding Thereom] it follows that we can consider M to be embedded in euclidean space $\mathbb{R}^{2m+1}$. Additionally, since we can find a retraction $r : V \to N$ where V is an open neighborhood of N in M we can extend $\mathfrak{t}_N$ over V. Now we can apply the Theorem **(3.5)** from the "Whitney Sum" Chapter to the extended microbundle to receive a η such that $\mathfrak{t}_N \oplus \eta \cong \mathfrak{e}^q N$.

2. $E(r^*\mathfrak{t}_N) \subseteq E() \oplus$ has a normal microbundle:

   Consider the microbundle

   $$j^*(r^*\mathfrak{t}_N \oplus r^*\mathfrak{t}') : E(r^*\mathfrak{t}_N) \xrightarrow{i'} E(r^*\mathfrak{t}_N \oplus r^*\mathfrak{t}') \xrightarrow{j'} E(r^*\mathfrak{t}_N)$$

   where j is the projection map for $r^*\mathfrak{t}_N$. Since i' is injective, we can assume $E(r^*\mathfrak{t}_N) \subseteq E(r^*\mathfrak{t}_N \oplus r^*\mathfrak{t}')$. Now $j^*(r^*\mathfrak{t}_N \oplus r^*\mathfrak{t}')$ is a normal microbundle if we equip $r^*\mathfrak{t}_N \oplus r^*\mathfrak{t}'$ with a manifolds structure as shown above.

Since $N \subseteq M \subseteq E(r^*\mathfrak{t}_N)$ has a normal microbundle (Lemma **(6.6)**) it follows from Lemma **(6.3)** that $N \subseteq E(r^*\mathfrak{t}_N \oplus r^*\mathfrak{t}')$ has a normal microbundle. But $r^*\mathfrak{t}_N \oplus r^*\mathfrak{t}'$ is trivial and therefore w.l.o.g. $E(r^*\mathfrak{t}_N \oplus r^*\mathfrak{t}') \cong N \times \mathbb{R}^q$ □