Microbundles on Topological Manifolds

based on Milnor's studies on Microbundles

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Contents

Abstract

TODO

Chapter 1

Introduction to Microbundles

In order to construct the tangent bundle on a manifold, we need a differential structure. However, this is generally not given for topological manifolds. In order to still have a structure "similar" to the tangent bundle on topological manifolds, we need a different, weaker, concept of the tangent bundle. Therefore we introduce so called "microbundles" which act as a weaker alternative to vector bundles. The concept of microbundles as well as some basic properties and examples are presented in this chapter. We start with the definition of a microbundle.

Definition 1.1 (microbundle)**.**
A *microbundle* $\mathfrak{b}$ over B (with *fibre-dimension* n) is a diagram $B \xrightarrow{i} E \xrightarrow{j} B$ satisfying the following:

- B is a topological space (*base space*)
- E is a topological space (*total space*)
- $i : B \to E$ (*injection*) and $j : E \to B$ (*projection*) are continuous maps such that $id_B = j \circ i$
- Every $b \in B$ is *locally trivializable*, that is there exist open neighborhoods $U \subseteq B$ of b and $V \subseteq E$ of $i(U)$ with a homeomorphism $\phi : V \xrightarrow{\sim} U \times \mathbb{R}^n$

such that the following diagram commutes:

$$\begin{tikzcd} & V \arrow[dr, "j|_V"] \arrow[dd, "\psi"] & \\ U \arrow[ur, "i"] \arrow[dr, "(id{,}0)"'] & & U \\ & U \times \mathbb{R}^n \arrow[ur, "\pi_1"'] & \end{tikzcd}$$

Remark 1.2.
In the following, unless explicitly stated otherwise we assume the fiber dimension of any given microbundle to be n.

Before we look at examples of microbundles, we should first clarify what it means for two microbundles to be isomorphic.

Definition 1.3 (isomorphy)**.**
Two microbundles $\mathfrak{b}_1 : B \xrightarrow{i_1} E_1 \xrightarrow{j_1} B$ and $\mathfrak{b}_2 : B \xrightarrow{i_2} E_2 \xrightarrow{j_2} B$ are *isomorphic* if there exist neighborhoods $V_1 \subseteq E_1$ of $i_1(B)$ and $V_2 \subseteq E_2$ of $i_2(B)$ with a homeomorphism $\phi : V_1 \xrightarrow{\sim} V_2$ such that the following diagram commutes:

$$\begin{tikzcd} & V_1 \arrow[dr, "j_1|_{V_1}"] \arrow[dd, "\phi"] & \\ B \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & B \\ & V_2 \arrow[ur, "j_2|_{V_2}"'] & \end{tikzcd}$$

As the definition of isomorphy already indicates, when studying microbundles, we are not interested in the entire total space but only in an arbitrarily small neighborhood of the base space. This is certainly one of the strongest conceptual differences between microbundles and classical vector bundles.

Proposition 1.4.
For a microbundle $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ over B, we can be restrict the total space E to an arbitray neighborhood $E' \subseteq E$ of $i(B)$ where the resulting microbundle is isomorphic to $\mathfrak{b}$.

Proof.
For an arbitray $b \in B$, choose a local trivialization (U, V, ϕ).

The intersection $V \cap E'$ is a neighborhood of $i(b)$ because V and E' both are. It follows that $\phi(V \cap E')$ is a neighborhood of $(b, 0)$. Hence there exist $U' \subseteq B$ open and $B_\varepsilon(0) \subseteq \mathbb{R}^n$ such that $U' \times B_\varepsilon(0) \subseteq \phi(V \cap E')$. Now we construct our local trivialization by choosing $V' := \phi^{-1}(U' \times B_\varepsilon(0))$ and the fact that $B_\varepsilon(0) \cong \mathbb{R}^n$:

$$U' \times \mathbb{R}^n \cong U' \times B_\varepsilon(0) \cong V'$$

We easily see that the resulting microbundle is isomorphic to $\mathfrak{b}$ via the identity. $\square$

Now that we introduced the basic concept of microbundles, we will take a look at some key examples. The most obvious example for a microbundle is the standard microbundle.

Example 1.5 (trivial microbundle)**.**
For a topological space B, the *standard microbundle* $\mathfrak{e}^n_B$ over B is a diagram

$$B \xrightarrow{\iota} B \times \mathbb{R}^n \xrightarrow{\pi} B$$

where $\iota(b) := (b, 0)$ and $\pi(b, x) := b$. Additionally, a microbundle $\mathfrak{b}$ over B is *trivial* if it is isomorphic to $\mathfrak{e}^n_B$.

In order to make it easier classifying microbundles as trivial, we provide a sharper description of what it means for a microbundle to be trival.

Lemma 1.6.
A microbundle $\mathfrak{b}$ over a paracompact space B is trivial if and only if there exists an open neighborhood U of $i(B)$ such that $U \cong B \times \mathbb{R}^n$.

Proof.
By applying Proposition **(1.4)**, we may assume that $E(\mathfrak{b})$ is an open subset of $B \times \mathbb{R}^n$. Since B is paracompact, there exists a map $\lambda : B \to (0, 1]$ such that every $(b, x) \in B \times \mathbb{R}^n$ with $|x| < \lambda(b)$ lies in $E(\mathfrak{b})$. Now, the function

$$(b, x) \mapsto (b, \frac{x}{|x| - \lambda(b)})$$

maps the open set $\{(b, x) \mid |x| < \lambda(b)\}$ homeomorphically to $B \times \mathbb{R}^n$. By considering $\{(b, x) \mid |x| < \lambda(b)\} \subseteq E(\mathfrak{b})$, this completes the proof. $\square$

The following example acts as the microbundle analog to the tangent bundle on a smooth manifold.

Example 1.7 (tangent microbundle)**.**
The *tangent microbundle* $\mathfrak{t}_M$ over a topological d-manifold M is a diagram

$$M \xrightarrow{\Delta} M \times M \xrightarrow{\pi_1} M$$

where $\Delta(m) := (m, m)$ is the diagonal map and $\pi_1(m_1, m_2) := m_1$ is the projection map on the first component.

Proof that $\mathfrak{t}_M$ is a microbundle.
Let $p \in M$ and (U, ϕ) be a chart over p. We explicitly construct a local trivial-

ization

$$\begin{tikzcd}
 & U \times U \arrow[dd, "\psi"] \arrow[dr, "\pi_1"] & \\
U \arrow[ur, "\Delta"] \arrow[dr, "(0,id)"'] & & U \\
 & U \times \mathbb{R}^d \arrow[ur, "\pi_1"'] &
\end{tikzcd}$$

where $\psi(u, \tilde{u}) := (u, \phi(u) - \phi(\tilde{u}))$. It's obvious that $(U, U \times U, \psi)$ meets all local triviality conditions. $\square$

Example 1.8 (underlying microbundle)**.**
Let $\xi : E \xrightarrow{\pi} B$ be a n-dimensional vector bundle. The microbundle $|\xi| : B \xrightarrow{i} E \xrightarrow{\pi} B$ where $i(b) := \phi_b(b, 0)$, where $\phi_b : U_b \times \mathbb{R}^n \to \pi^{-1}(U_b)$ is the local trivialization over a neighborhood $U_b \subseteq B$ of b. We call $|\xi|$ the *underlying microbundle* of ξ

Proof.
TODO $\square$

Chapter 2

Induced Microbundles

This Chapter introduces a central construction of microbundles.

Definition 2.1 (induced microbundle)**.**
Let $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ be a microbundle and $f : A \to B$ be a continuous map. The *induced microbundle* $f^*\mathfrak{b} : A \xrightarrow{i'} E' \xrightarrow{j'} A$ is defined as follows:

- $E' := \{(a, e) \in A \times E \mid f(a) = j(e)\}$
- $i' : A \to E'$ with $i'(a) := (a, (i \circ f)(a))$
- $j' : E' \to A$ with $j'(a, e) := a$

Proof that $f^*\mathfrak{b}$ is a microbundle.
It is clear that i' and j' are continuous and that $id_A = j' \circ i'$. So it remains to be shown that $f^*\mathfrak{b}$ is locally trivial.

Choose a local trivialization (U, V, ϕ) for an arbitrary $a \in A$.

- $U' := f^{-1}(U) \subseteq A$ neighborhood of a.
- $V' := j'^{-1}(U') \subseteq E'$ neighborhood of $i'(a)$.
- $\phi' : V' \xrightarrow{\sim} U' \times \mathbb{R}^n, \phi'(a, e) := (a, \pi_2(\phi(e)))$.

The map ϕ' is well-defined because $(a, e) \in V' : j(e) = f(a) \in U \implies e \in V$. The existence of an inverse $\phi'^{-1}(a, v) = (a, \phi^{-1}(f(a), v))$ and component-wise continuity show that ϕ' is a homeomorphism. This proves that (U', V', ϕ') is a local trivialization for a. □

Example 2.2 (restricted microbundle)**.**
Let $\mathfrak{b} : B \xrightarrow{i} E \xrightarrow{j} B$ be a microbundle and $A \subseteq B$ be a subspace. The *restricted*

microbundle $\mathfrak{b}|_A$ is the induced microbundle $\iota^*\mathfrak{b}$ where $\iota : A \hookrightarrow B$ is the inclusion map.

Remark 2.3.
In the following we consider $E(\mathfrak{b}|_A)$ a subset of $E(\mathfrak{b})$. This is justified because there exists an embedding

$$\iota : E(\mathfrak{b}|_A) \to E(\mathfrak{b}) \text{ with } (a, e) \mapsto e.$$

Note that this argument can be made for any injective map.

Lemma 2.4.
Let $\mathfrak{b}$ be a microbundle over B and $f : A \to B$ be a map. The induced microbundle $f^*\mathfrak{b}$ is trivial if $\mathfrak{b}$ is already trivial.

Proof.
Let (V, ϕ) be a global trivialization, so $\phi : V \xrightarrow{\sim} B \times \mathbb{R}^n$. We define

- $V' := (A \times V) \cap E(f^*\mathfrak{b})$ a neighborhood of $i'(A)$.
- $\phi' : V' \xrightarrow{\sim} B \times \mathbb{R}^n, (a, e) \mapsto (a, \pi_2(\phi(e)))$.

The existence of an inverse $\phi'^{(-1)}(a, x) = (a, \phi^{-1}(f(a), x))$ and component-wise continuity show that ϕ' is a homeomorphism. This proves that (V', ϕ') is a global trivialization for $f^*\mathfrak{b}$. □

Lemma 2.5.
For a diagram $A \xrightarrow{f} B \xrightarrow{g} C$ and a microbundle $\mathfrak{c} : C \xrightarrow{i} E \xrightarrow{j} C$ applies:

$$(g \circ f)^*\mathfrak{c} \cong f^*(g^*\mathfrak{c})$$

Proof.
To prove isomorphy, we need to show that the two total spaces are homeomorphic and that the injection and projection maps commute with such a homeomorphism.

First, compare the two total spaces:

1. $E((g \circ f)^*\mathfrak{c}) = \{(a, e) \in A \times E(\mathfrak{c}) \mid g(f(a)) = j(e)\}$
2. $E(f^*(g^*\mathfrak{c})) = \{(a, (b, e)) \in A \times (B \times E(\mathfrak{c})) \mid f(a) = b \text{ and } g(b) = j(e)\}$.

We have the bijection $\phi : E((g \circ f)^*\mathfrak{c}) \xrightarrow{\sim} E(f^*(g^*\mathfrak{c}))$ with $\phi(a, e) := (a, (f(a), e))$ and $\phi^{-1}(a, (b, e)) = (a, e)$. Since ϕ and ϕ^{-1} are component-wise continuous, it follows that ϕ is a homeomorphism.

It's easy to see that ϕ commutes with injection and projection maps, which concludes the proof. □

For a topological space X, we define the *cone* of X to be

$$CX := X \times [0, 1]/X \times \{1\}$$

and for a map $f : A \to B$ the *mapping cone* of f to be

$$B \sqcup_f CA := B \sqcup CA/\sim$$

where $(a, 0) \sim b :\iff f(a) = b$.

Similarly, we define the *cylinder* of X to be

$$MX := X \times [0, 1]$$

and for a map $f : A \to B$ the *map cylinder* of f to be

$$B \sqcup_f MA := B \sqcup CA/\sim$$

where $(a, 0) \sim b :\iff f(a) = b$.

Lemma 2.6.
A microbundle $\mathfrak{b}$ over B can be extended to a microbundle over the mapping cone $B \sqcup_f CA$ if and only if $f^*\mathfrak{b}$ is trivial.

Proof.
We show both implications.

" $\Longrightarrow$ "

Let $\mathfrak{b}'$ be an extension of $\mathfrak{b}$ over $B \sqcup_f CA$. Considering $A \xrightarrow{f} B \hookrightarrow B \sqcup_f CA$, the composition $\iota \circ f$ is null-homotopic with homotopy

$$H_t(a) := [(a, t)]$$

Note that $H_0(a) = [(a, 0)] = [f(a)] = (\iota \circ f)(a)$ and $H_1(a) = [(a, 1)] = [(\tilde{a}, 1)] = H_1(\tilde{a})$. From the Homotopy Theorem **(4.1)** follows that $(\iota \circ f)^*\mathfrak{b}'$ is trivial.

Since $(\iota \circ f)^*\mathfrak{b}' = f^*(\iota^*\mathfrak{b}') = f^*\mathfrak{b}$, it follows that $f^*\mathfrak{b}$ is trivial.

" $\Longleftarrow$ "

Let $f^*\mathfrak{b}$ be trivial.

In contrast to the mapping cone, there exists a natural retraction from the map cylinder to the attached space

$$r : B \sqcup_f MA \to B, r([(a, t)]) := f(a)$$

The diagram

$$A \times \{1\} \hookrightarrow B \sqcup_f MA \xrightarrow{r} B$$

equals f if we consider $A = A \times \{1\}$. It follows that

$$r^*\mathfrak{b}|_{A\times\{1\}} = (r \circ \iota)^*\mathfrak{b} \cong f^*\mathfrak{b} = \mathfrak{e}^n_A$$

is trivial. From Lemma **(2.4)** and $(a, t) \mapsto (a, 1)$ it follows that $r^*\mathfrak{b}|_{A\times[\frac{1}{2},1]}$ is trivial, so there exists a

$$\phi : E(r^*\mathfrak{b}|_{A\times[\frac{1}{2},1]}) \xrightarrow{\sim} A \times [\frac{1}{2}, 1] \times \mathbb{R}^n$$

Now we explicitly construct our desired extended microbundle $\mathfrak{b}' : B \sqcup_f CA \xrightarrow{i'} E' \xrightarrow{j'} B \sqcup_f CA$

- $E' := E(r^*\mathfrak{b})/\phi^{-1}(A \times \{1\} \times \{x\})$ (for every $x \in \mathbb{R}^n$).
- $i'([a,t]) := [i_r(a,t)]$ where i_r is the injection map for $r^*\mathfrak{b}$.
- $j'([e]) := [j_r(e)]$ where j_r is the projection map for $r^*\mathfrak{b}$.

The injection i' is well-defined because i_r maps every representative $[a,1]$ to the same equivalence class of E'. Similarly, the projection j' is well-defined since $[e] = [\tilde{e}] \implies [j_r(e)] = [j_r(e')]$ We easily derive the microbundle conditions from $r^*\mathfrak{b}$.

This proves the claim. $\square$

Corollary 2.7.
Let B be a $(d+1)$-simplicial complex, B' it's d-skeleton and $\sigma \subseteq B$ a $(d+1)$-simplex. A microbundle $\mathfrak{b}$ over B' can be extended to a microbundle over $B' \cup \sigma$ if and only if its restriction to the boundary $\mathfrak{b}|_{\partial\sigma}$ is trivial.

Proof.
By choosing $f : \partial\sigma \hookrightarrow B'$ and applying the previous lemma, we see that there exists a microbundle $\mathfrak{b}'$ over $B' \cup_f C\sigma$ extending $\mathfrak{b}$.

Now, consider the homeomorphism $\phi : C\partial\sigma \xrightarrow{\sim} \sigma$ with

$$\phi((t_1, \ldots, t_{d+1}), \lambda) := (1-\lambda)(t_1, \ldots, t_{d+1}) + \frac{\lambda}{d+1}(1, \ldots, 1)$$

In particular, $\phi(\partial\sigma \times \{0\}) = \partial\sigma$.

It follows that $B' \cup_f C\sigma \cong B' \cup \sigma$ which concludes the proof. $\square$

Chapter 3

The Whitney Sum

In the last chapter we saw how we can pull back the base space of a given microbundle using a map. In this chapter, another central construction is introduced, the "Whitney Sum". It allows us to construct a microbundle given two microbundles over the same base space. The fiber dimension of the resulting microbundle is just the sum of the fiber dimensions of the initial microbundles.

Definition 3.1.
Let $\mathfrak{b}_1$ and $\mathfrak{b}_1$ be two microbundles over B with fibre-dimensions n_1 and n_2. The *whitney sum* $\mathfrak{b}_1 \oplus \mathfrak{b}_2$ is a microbundle $B \xrightarrow{i} E \xrightarrow{j} B$ where

- $E := \{(e_1, e_2) \in E(\mathfrak{b}_1) \times E(\mathfrak{b}_2) \mid j_1(e_1) = j_2(e_2)\}$
- $i(b) := (i_1(b), i_2(b))$
- $j(e_1, e_2) := j_1(e_1) = j_2(e_2)$

with fibre-dimension $n_1 + n_2$.

Proof that $\mathfrak{b}_1 \oplus \mathfrak{b}_2$ is a microbundle.
For an arbitray $b \in B$, let (U_1, V_1, ϕ_1) and (U_2, V_2, ϕ_2) be two local trivializations of b in $\mathfrak{b}_1$ and $\mathfrak{b}_2$. We construct a local trivialization of b in $\mathfrak{b}_1 \oplus \mathfrak{b}_2$ as follows:

- $U := U_1 \cap U_2$
- $V := (V_1 \times V_2) \cap E$
- $\phi : V \to U \times \mathbb{R}^{n_1+n_2}; \phi(e_1, e_2) := (\phi_1^{(1)}(e_1), \phi_1^{(2)}(e_1) \times \phi_2^{(2)}(e_2))$

Note that $\phi_1^{(1)}(e_1) = \phi_2^{(1)}(e_2)$. Both U and V are open since U_1, U_2 and V_1, V_2 are open. Since ϕ is a composed by homeomorphisms, it's an homeomorphism as well. This conludes the proof that $\mathfrak{b}_1 \oplus \mathfrak{b}_2$ is a microbundle (of fibre-dimension $n_1 + n_2$). $\square$

Remark 3.2.
Alternatively, we could define the whitney sum between $\mathfrak{b}_1$ and $\mathfrak{b}_2$ to be the induced microbundle $\Delta^*(\mathfrak{b}_1 \times \mathfrak{b}_2)$ where Δ denotes the diagonal map and $\mathfrak{b}_1 \times \mathfrak{b}_2$ denotes the literal cross-product between the two microbundles.

Lemma 3.3.
Let $\mathfrak{b}_1$ and $\mathfrak{b}_1$ be two microbundles over B and let $f : A \to B$ be a map. The induced microbundle and the whitney sum are compatible, that is

$$f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2) \cong f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2.$$

Proof.
From the definition of the induced microbundle and the whitney sum, we can explicitly write the total spaces

$$\begin{gathered} E(f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2)) \\ = \{(a, (e_1, e_2)) \in A \times (E(\mathfrak{b}_1) \times E(\mathfrak{b}_2)) \mid j_1(e_1) = j_2(e_2) = f(a)\} \end{gathered}$$

and

$$\begin{gathered} E(f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2) \\ = \{(e_1, e_2) \in E(f^*\mathfrak{b}_1) \times E(f^*\mathfrak{b}_2) \mid j_1(e_1) = j(e_2)\} \\ = \{((a_1, e_1), (a_2, e_2)) \in (A \times E(\mathfrak{b}_1)) \times (A \times E(\mathfrak{b}_2)) \mid \\ j(a_1, e_1) = j(a_2, e_2) \wedge f(a_i) = j(e_i)\} \end{gathered}$$

The two total spaces are homeomorphic via $\phi(a, (e_1, e_2)) := ((a, e_1), (a, e_2))$ with $\phi^{-1}((a, e_1), (a, e_2)) = (a, (e_1, e_2))$. Homeomorphy of ϕ follows from the continuity of ϕ and ϕ^{-1}, which is given since both ϕ and ϕ^{-1} are composed by identity maps.

It remains to be shown that injection and projection maps i and j for $E(f^*(\mathfrak{b}_1 \oplus \mathfrak{b}_2))$ and i' and j' for $f^*\mathfrak{b}_1 \oplus f^*\mathfrak{b}_2$ agree under ϕ.

This follows from

$$\begin{gathered} \phi(i(a)) = \phi(a, i_1(f(a)), i_2(f(a))) \\ = ((a, i_1(f(a))), (a, i_2(f(a)))) = (i'_1(a), i'_2(a)) = i'(a) \end{gathered}$$

and

$$j(a, e_1, e_2) = a = j'((a, e_1), (a, e_2)) = j'(\phi(a, e_1, e_2)).$$

□

Last, we show a theorem about whitney sums that will be essential in the proof of Milnor's theorem. For its prove, we need to use the following proposition that will be deferred until chapter 5.

Proposition 3.4.
Let $\mathfrak{b}$ be a microbundle over a "bouqet" of spheres B, meeting at a single point. There exists a map $r : B \to B$ such that $\mathfrak{b} \oplus r^*\mathfrak{b}$ is trivial.

Theorem 3.5.
Let $\mathfrak{b}$ be a microbundle over a d-dimensional simplicial complex B. Then there exists a microbundle $\mathfrak{n}$ over B so that the whitney sum $\mathfrak{b} \oplus \mathfrak{n}$ is trivial.

Proof.
We prove this theorem by induction over d.

(Start of induction)

A 1-dimensional simplicial complex is just a bouquet of circles. Therefore, the start of induction follows directly from Proposition **(3.4)**.

(Inductive Step)

Let B' be the $(d-1)$-skeleton of B and $\mathfrak{n}'$ be it's corresponding microbundle such that $\mathfrak{b}|_{B'} \oplus \mathfrak{n}'$ is trivial.

1. $\mathfrak{n}' \oplus \mathfrak{e}^n_{B'}$ can be extended over any d-simplex σ:

   Consider the equation

   $$(\mathfrak{n}' \oplus \mathfrak{e}^n_{B'})|_{\partial\sigma} = \mathfrak{n}'|_{\partial\sigma} \oplus \mathfrak{e}^n_{B'}|_{\partial\sigma} = \mathfrak{n}'|_{\partial\sigma} \oplus \mathfrak{b}|_{\partial\sigma} = (\mathfrak{n}' \oplus \mathfrak{b}|_{B'})|_{\partial\sigma}$$

   in which we used the previous lemma and Corollary **(2.7)** for $\mathfrak{e}^n_{B'}|_{\partial\sigma} = \mathfrak{b}|_{\partial\sigma}$. Since $(\mathfrak{n}' \oplus \mathfrak{b}|_{B'})|_{\partial\sigma}$ is trivial, the claim follows from Corollary **(2.7)**.

2. $\mathfrak{n}' \oplus \mathfrak{e}^n_{B'}$ can be extended over B:

   The difficulty is that the individual d-simplices are not well-seperated. Let B'' denote B with small open d-cells removed from every d-simplex. Since B' is a retract of B'' we can extend $\mathfrak{n}' \oplus \mathfrak{e}^n_{B'}$ over B'' and now apply the first statement. We denote the resulting microbundle by η.

3. Consider the mapping cone $B \sqcup CB'$ over the inclusion $B' \hookrightarrow B$. Since

   $$(\mathfrak{b} \oplus \eta)|_{B'} = \mathfrak{b}|_{B'} \oplus \eta|_{B'} = \mathfrak{b}|_{B'} \oplus (\mathfrak{n}' \oplus \mathfrak{e}^n_{B'}) = (\mathfrak{b}|_{B'} \oplus \mathfrak{n}') \oplus \mathfrak{e}^n_{B'} = \mathfrak{e}^n_{B'} \oplus \mathfrak{e}^n_{B'}$$

   is trivial, it follows from Lemma **(2.6)** that we can extend $\mathfrak{b} \oplus \eta$ over $B \sqcup CB'$ which will be denoted by ξ.

   The mapping cone $B \sqcup CB'$ has the homotopy type of a bouquet of spheres by transfering B' along CB' collapsing to a single point. Since any d-simplex is homotopic to a d-disc and it's border is collapsed, we get the homotopy of a $(d-1)$-sphere.

   With Theorem **(4.1)** and Proposition **(3.4)**, we conclude that there exists a microbundle $\mathfrak{n}$ such that $(\xi \oplus \mathfrak{n})|_B$ is trivial. The equation

   $$\mathfrak{e}^n_B = (\xi \oplus \mathfrak{n})|_B = \xi|_B \oplus \mathfrak{n}|_B = (\mathfrak{b} \oplus \eta) \oplus \mathfrak{n}|_B = \mathfrak{b} \oplus (\eta \oplus \mathfrak{n}|_B)$$

   completes the proof.

□

Chapter 4

The Homotopy Theorem

In this chapter we will prove the homotopy theorem. It states the following:

Theorem 4.1 (Homotopy Theorem)**.**
Let $\mathfrak{b}$ *be a microbundle over* B *and* $f, g : A \to B$ *be two maps. If* f *and* g *are homotopic, then* $f^*\mathfrak{b}$ *and* $g^*\mathfrak{b}$ *are isomorphic.*

Before we can start with the proof of the theorem, we need additional concepts to put microbundles in relation to each other.

Definition 4.2 (map-germ)**.**
A *map-germ* $F : (X, A) \Rightarrow (Y, B)$ between topological pairs (X, A) and (Y, B) is an equivalence class of maps $(X, A) \to (Y, B)$ where $f \sim g :\iff f|_U = g|_U$ for some neighborhood $U \subseteq X$ of A.

A *homeomorphism-germ* $F : (X, A) \Rightarrow (Y, B)$ is a map-germ such that there exists a representative $f : U_f \to Y$ that maps homeomorphically to a neighborhood of B. Now consider two isomorphic microbundles $\mathfrak{b}$ and $\mathfrak{b}'$ over B. There exists a homeomorphism $\phi : V \xrightarrow{\sim} V'$ where $V \subseteq E$ is a neighborhood of $i(B)$ and $V' \subseteq E'$ is a neighborhood of $i'(B)$. The homeomorphism ϕ is a representative for a homeomorphism-germ

$$[\phi] : (E, i(B)) \Rightarrow (E', i'(B)).$$

Studying isomorphy between microbundles in this way is useful because we don't care what such a homeomorphism does on particular neighborhoods of the base spaces but only what it does on arbitray small ones. Hence every representative of $[\phi]$ describes the "same" isomorphy between $\mathfrak{b}$ and $\mathfrak{b}'$. Now, naturally, the question arises whether the existence of a homeomorphism-germ

$$F : (E, i(B)) \Rightarrow (E', i'(B))$$

already implies that $\mathfrak{b}$ and $\mathfrak{b}'$ are isomorphic. The answer is generally no, because isomorphy of microbundles requires a homeomorphism that commutes with injection and projection maps. Therefore, we must assume an extra condition called "fibre-preservation" for this implication to be true. This justifies the following definition.

Let $\mathfrak{b}$ and $\mathfrak{b}'$ be two microbundles over B and let $J : (E, i(B)) \Rightarrow (B, B)$ and $J' : (E', i(B)) \Rightarrow (B, B)$ denote the map-germs represented its projection maps.

Definition 4.3 (isomorphism-germ)**.**
An *isomorphism-germ* between $\mathfrak{b}$ and $\mathfrak{b}'$ is a homeomorphism-germ

$$F : (E, B) \Rightarrow (E', B)$$

which is *fibre-preserving*, that is $J' \circ F = J$.

Remark 4.4.
There exists an isomorphism-germ between $\mathfrak{b}$ and $\mathfrak{b}'$ if and only if $\mathfrak{b}$ is isomorphic to $\mathfrak{b}'$.

We can take this even further by giving up on the assumption that the base spaces of the considered microbundles equal. Note that in this case no comparison to isomorphy can be drawn, since we have not defined isomorphy between microbundles over different base spaces.

Definition 4.5 (bundle-germ)**.**
Let $\mathfrak{b}$ and $\mathfrak{b}'$ be two microbundles with the same fibre-dimension. A *bundle-germ* $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$ is a map-germ

$$F : (E, B) \Rightarrow (E', B')$$

such that there exists a representative $f : U_f \to E'$ that maps each fibre $j^{-1}(b)$ injectively to a fibre $j'^{-1}(b')$.

For a bundle-germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$, the following diagram commutes:

$$\begin{tikzcd}
(E, B) \arrow[r, Rightarrow, "F"] \arrow[d, "i"] & (E', B') \arrow[d, "i'"] \\
B \arrow[r, "F|_B"] & B'
\end{tikzcd}$$

We say *F is covered by $F|_B$*. The bundle-germ is indeed a generalization of the isomorphism germ, as the following proposition illustrates.

Proposition 4.6 (Williamson)**.**
Let $\mathfrak{b}$ and $\mathfrak{b}'$ be two microbundles over B. A bundle-germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$ covering the identity map is an isomorphism-germ.

First, however, we show a lemma that helps us to prove the proposition.

Lemma 4.7.
If a homeomorphism $\phi : \mathbb{R}^n \xrightarrow{\sim} \phi(\mathbb{R}^n) \subseteq \mathbb{R}^n$ *satisfies*

$$|\phi(x) - x| < 1, \forall x \in \overline{B_2(0)}$$

then $\overline{B_1(0)} \subseteq \phi(\overline{B_2(0)})$.

Proof of the lemma.
Consider $\phi(2S^n)$ where $2S^n$ denotes the n-sphere of radius 2. The condition for ϕ yields $1 < |\phi(s)|, \forall s \in 2S^n$. Since $\overline{B_2(0)}$ has trivial homology groups which are preserved under homeomorphisms, $\phi(\overline{B_2(0)})$ must have trivial homology groups as well.

From this we can conclude that $\overline{B_1(0)}$ must be contained in $\phi(\overline{B_2(0)})$, because otherwise "holes" would form which would result in non-trivial homology groups of $\phi(\overline{B_2(0)})$. $\square$

Proof of the proposition.
Let f be a representative for F. We show the proposition in two steps.

1. Assume f to map from $B \times \mathbb{R}^n$ to $B \times \mathbb{R}^n$.

   Hence f is of the form
   $$f(b, x) = (b, g_b(x))$$
   where $g_b : \mathbb{R}^n \to \mathbb{R}^n$ are individual maps. Since the g_b are continuous and injective, it follows from the [domain invariance theorem] that the g_b are open maps. Let $(b_0, x_0) \in B \times \mathbb{R}^n$ and $\varepsilon > 0$. Since g_{b_0} is an open map, there exists a $\delta > 0$ such that $\overline{B_{2\delta}(x_1)} \subseteq g_{b_0}(\overline{B_\varepsilon(x_0)})$ where $x_1 := g_{b_0}(x_0)$.

   There exists a neighborhood $V \subseteq B$ of b_0 such that
   $$|g_b(x) - g_{b_0}(x)| < \delta$$
   for every $b \in V$ and $x \in \overline{B_\varepsilon(x_0)}$. To show that, consider $\phi(b, x) := g_b(x) - g_{b_0}(x)$. The closed set $\phi^{-1}(\overline{B_\delta(0)})$ is a neighborhood of $\{b_0\} \times \mathbb{R}^n$ since $\phi(b_0, x) = 0$. Therefore, for every $x \in \overline{B_\delta(0)}$ exist $V_x \subseteq B$ and $U_x \subseteq \mathbb{R}^n$ open with $x \in U_x$ and $V_x \times U_x \subseteq \phi^{-1}(\overline{B_\varepsilon(x_0)})$. Obviously, $\bigcup_{x \in \overline{B_\delta(x_1)}} U_x$ is an open covering of $\overline{B_\delta(x_1)}$ and since $\overline{B_\delta(x_1)}$ is compact, there exist $x_1, \dots, x_n \in \overline{B_\delta(x_1)}$ with $\overline{B_\delta(x_1)} \subseteq \bigcup_{i=1}^n U_{x_i}$. The claim follows via $V := V_{x_1} \cap \dots \cap V_{x_n}$.

   Now we want to apply the previous lemma:

   Consider the homeomorphism $g_b \circ g_{b_0}^{-1}$ for an arbitrary $b \in V$. Since
   $$\overline{B_{2\delta}(x_1)} \subseteq g_{b_0}(\overline{B_\varepsilon(x_0)}) \implies g_{b_0}^{-1}(\overline{B_{2\delta}(x_1)}) \subseteq \overline{B_\varepsilon(x_0)}$$
   we conclude from the above that
   $$|(g_b \circ g_{b_0}^{-1})(x) - x| < \delta.$$

It follows that, by translation and scaling, $g_b \circ g_{b_0}^{-1}$ satisfies the requirements of the lemma. Therefore, $\overline{B_\delta(x_1)} \subseteq (g_b \circ g_{b_0}^{-1})(\overline{B_{2\delta}(x_0)})$ and so $\overline{B_\delta(x_1)} \subseteq g_b(\overline{B_\varepsilon(x_0)})$.

From

$$V \times \overline{B_\delta(x_1)} \subseteq g(V \times \overline{B_\varepsilon(x_0)})$$

it follows that f is an open map.

2. Glue together f from its local trivializations.

Choose a local trivialization (U, V, ϕ) over $b \in B$. First, we restrict f to $f^{-1}(V)$. Since $f^{-1}(V)$ is a neighborhood of $i(b)$, we can choose an open neighborhood $V' \subseteq f^{-1}(V) \cap V$ of $i(b)$ of the form $U' \times B_\varepsilon(0)$. Now we have

$$U' \times \mathbb{R}^n \cong U' \times B_\varepsilon(0) \xrightarrow{f} U' \times \mathbb{R}^n \subseteq U \times \mathbb{R}^n$$

a map $U' \times \mathbb{R}^n \to U' \times \mathbb{R}^n$ that is injective and fibre-preserving and therefore an open map (apply 1.). It follows that $f : V' \to V$ must be an open map as well.

By glueing the V' over all $b \in B$ together, we see that f is an open map which concludes the proof.

□

We can easily generalize this to bundle-germs between microbundles over different base spaces:

Corollary 4.8.
If a map $g : B \to B'$ is covered by a bundle-germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$, then $\mathfrak{b}$ is isomorphic to the induced bundle $g^*\mathfrak{b}'$.

Proof.
Let $f : U_f \to E'$ be a representative map for F. We define $F' : \mathfrak{b} \Rightarrow g^*\mathfrak{b}'$ with a representative f' as follows:

$$f' : U_f \to E(g^*\mathfrak{b}'), f'(e) := (j(e), f(e))$$

The element $f'(e)$ actually lies in $E(g^*\mathfrak{b}')$ because

$$g(j(e)) = j'(f(e))$$

as we can see from the commutative diagram for bundle-germs. Applying the previous proposition proves the claim. □

Lemma 4.9.
Let $\mathfrak{b}$ be a microbundle over B and $\{B_\alpha\}$ a locally finite collection of closed sets covering B. Additionally, we are given a collection of bundle map-germs $F_\alpha : \mathfrak{b}|_{B_\alpha} \Rightarrow \mathfrak{b}'$ such that $F_\alpha = F_\beta$ on $\mathfrak{b}|_{B_\alpha \cap B_\beta}$. Then there exists a bundle map-germ $F : \mathfrak{b} \Rightarrow \mathfrak{b}'$ extending F_α.

Proof.
Choose representative maps $f_\alpha : U_\alpha \to E'$ for F_α such that the U_α are open. For every α and β, choose a neighborhood $U_{\alpha\beta}$ of $i(B_\alpha \cap B_\beta)$ on which the representative maps f_α and f_β agree. Now consider

$$U := \{e \in E \mid j(e) \in B_\alpha \cap B_\beta \implies e \in U_{\alpha\beta}\}$$

which satisfies the following:

1. U is open.

   We can express U like this:

   $$E - \bigcup_{\alpha\beta} \{j^{-1}(B_\alpha \cap B_\beta) \cap U^c_{\alpha\beta}\}$$

   Since $j^{-1}(B_\alpha \cap B_\beta)$ and $U^c_{\alpha\beta}$ are closed sets, U must be open. That is because an open set remains open after removing arbitrary closed sets.

2. $i(B) \subseteq U$.

   This follows from

   $$b \in B_\alpha \cap B_\beta \implies i(b) \in i(B_\alpha \cap B_\beta) \subseteq U_{\alpha\beta}$$

   and $j(i(b)) = b$.

With the construction of U, we can define $f : U \to E'$ in the obvious way

$$f(u \in U_{\alpha\beta}) := f_\alpha(u) = f_\beta(u)$$

which is a representative map for our desired F. □

Lemma 4.10.
Let $\mathfrak{b}$ be a microbundle over $B \times [0, 1]$. If $\mathfrak{b}|_{B\times[0,\frac{1}{2}]}$ and $\mathfrak{b}|_{B\times[\frac{1}{2},1]}$ are both trivial, then $\mathfrak{b}$ itself is trivial.

Proof.
Consider the identity bundle-germ over $\mathfrak{b}|_{B\times\{\frac{1}{2}\}}$, that is the bundle-germ represented by the identity on $E(\mathfrak{b}|_{B\times\{\frac{1}{2}\}})$. Since $\mathfrak{b}|_{B\times[\frac{1}{2},1]}$ is trivial, we can extend this bundle-germ to

$$\mathfrak{b}|_{B\times[\frac{1}{2},1]} \Rightarrow \mathfrak{b}|_{B\times\{\frac{1}{2}\}}$$

via the representative

$$B \times [\frac{1}{2}, 1] \times \mathbb{R}^n \to B \times \{\frac{1}{2}\} \times \mathbb{R}^n$$

$$(b, t, x) \mapsto (b, \frac{1}{2}, x)$$

Using the previous lemma, we can piece this together with the identity bundle-germ on $\mathfrak{b}|_{B\times[0,\frac{1}{2}]}$ resulting in a bundle-germ

$$\mathfrak{b} \Rightarrow \mathfrak{b}|_{B\times[0,\frac{1}{2}]}.$$

The previous corollary infers that $\mathfrak{b}$ is isomorphic to $\mathfrak{b}|_{B\times[0,\frac{1}{2}]}$. □

Lemma 4.11.
Let $\mathfrak{b}$ be a microbundle over $B\times[0,1]$. Every $b\in B$ has a neighborhood V where $\mathfrak{b}|_{V\times[0,1]}$ is trivial.

Proof.
Let $b\in B$. For every $t\in[0,1]$, choose a neighborhood $U_t := V_t\times(t-\varepsilon_t, t+\varepsilon_t)$ of (b,t) such that $\mathfrak{b}|_{U_t}$ is trivial. Since $\{b\}\times[0,1]$ is compact, we can choose a finite subcover of $\{b\}\times[0,1]$ and define V to be the intersection of the corresponding V_t. Now there exists a subdivision $0=t_0<\cdots<t_k=1$ where the $\mathfrak{b}|_{V\times[t_i,t_{i+1}]}$ are trivial. By iteratively applying the previous lemma, it follows that $\mathfrak{b}|_{V\times[0,1]}$ is trivial. □

Lemma 4.12.
Let $\mathfrak{b}$ be a microbundle over $B\times[0,1]$ where B is paracompact. Then there exists a bundle map-germ $F:\mathfrak{b}\Rightarrow\mathfrak{b}|_{B\times\{1\}}$ covering the standard retraction $r: B\times[0,1]\to B\times\{1\}$.

Proof.
First, assume a locally finite covering $\{V_\alpha\}$ of closed sets where $\mathfrak{b}|_{V_\alpha\times[0,1]}$ is trivial. The existence of such a covering is justified by paracompactness of B and the previous lemma. Now choose a partition of unity

$$\lambda_\alpha : B\to[0,1]$$

with
$supp(\lambda_\alpha)\subseteq V_\alpha$ that is rescaled so that

$$\max_\alpha(\lambda_\alpha(b)) = 1, \forall b\in B.$$

Now we define a retraction $r_\alpha : B\times[0,1]\to B\times[0,1]$ with

$$r_\alpha(b,t) := (b, \max(t,\lambda_\alpha(b))).$$

In the following, we will construct bundle-germs $R_\alpha:\mathfrak{b}\Rightarrow\mathfrak{b}$ covering r_α and piece them together to obtain the desired bundle-germ.

1. We can divide $B \times [0,1]$ into two subsets

$$A_\alpha := supp(\lambda_\alpha) \times [0,1]$$

and

$$A'_\alpha := \{(b,t) \mid t \geq \lambda_\alpha(b)\}.$$

Since $A_\alpha \subseteq V_\alpha \times [0,1]$, we already know that $\mathfrak{b}|_{A_\alpha}$ is trivial. Similar to the proof in Lemma **(4.10)**, we can extend the identity bundle-germ on $\mathfrak{b}|_{A_\alpha \cap A'_\alpha}$ to a bundle-germ

$$\mathfrak{b}|_{A_\alpha} \Rightarrow \mathfrak{b}|_{A_\alpha \cap A'_\alpha}.$$

Piecing this together with the identity bundle germ $\mathfrak{b}|_{A'_\alpha}$, we obtain our desired bundle germ R_α.

2. Applying the well-ordering theorem, which is equivalent to the axiom of choice, we can assume an ordering of $\{V_\alpha\}$. Let $\{B_\beta\}$ be a locally finite covering of B with closed sets where B_β intersects only $V_{\alpha_1} < \cdots < V_{\alpha_k}$, a finite collection. Again, the existence of such a collection is guaranteed by the paracompactness of B. Now the composition $R_{\alpha_1} \circ \ldots \circ R_{\alpha_k}$ restricts to a bundle germ $R(\beta) : \mathfrak{b}|_{B_\beta} \times [0,1] \Rightarrow \mathfrak{b}|_{B_\beta} \times \{1\}$. Pieced together using Lemma **(4.10)**, we obtain $R : \mathfrak{b} \times [0,1] \to \mathfrak{b} \times \{1\}$ which concludes the proof.

□

Finally, we gathered all the tools to proof the homotopy theorem.

Proof of the Homotopy Theorem.
Let $H : A \times [0,1] \to B$ be a homotopy between f and g. The previous lemma states that there exists a bundle germ

$$R : H^*\mathfrak{b} \Rightarrow H^*\mathfrak{b}|_{B \times \{1\}}$$

covering the standard retraction $B \times [0,1] \to B \times \{1\}$. By considering $E(f^*\mathfrak{b}) = E(H^*\mathfrak{b}|_{A\times\{0\}}) \subseteq E(H^*\mathfrak{b})$ and $E(g^*\mathfrak{b}) = E(H^*\mathfrak{b}|_{B\times\{1\}})$, we conclude that R extends an isomorphism germ $f^*\mathfrak{b} \Rightarrow g^*\mathfrak{b}$. It follows that $f^*\mathfrak{b} \cong g^*\mathfrak{b}$. □

Chapter 5

Microbundles over a Suspension

In this chapter, every topological space comes with a base point which will be denoted with subscript 0.

Definition 5.1.
A *rooted microbundle* $\mathfrak{b}$ over B is a microbundle over B together with an isomorphism-germ

$$R : \mathfrak{b}|_{b_0} \Rightarrow \mathfrak{e}^n_{b_0}.$$

Two rooted microbundles $\mathfrak{b}$ and $\mathfrak{b}'$ are *isomorphic* if there exists an isomorphism germ $\mathfrak{b} \Rightarrow \mathfrak{b}'$ extending

$$R'^{-1} \circ R : \mathfrak{b}|_{b_0} \Rightarrow \mathfrak{b}|'_{b_0}.$$

Theorem 5.2 (Rooted Homotopy Theorem)**.**
Let $\mathfrak{b}$ be a rooted microbundle over B and $f, g : A \to B$ be two based maps. If there exists a homotopy $H : A \times [0,1] \to B$ between f and g that leaves the base point fixed, then the two rooted microbundles $f^*\mathfrak{b}$ and $g^*\mathfrak{b}$ are isomorphic.

We need to show a rooted version of Lemma **(4.11)**. Before we prove the lemma, note that

$$E(H^*\mathfrak{b}|_{a_0 \times [0,1]})$$

is just

$$\begin{aligned} &\{e \in E(H^*\mathfrak{b}) \mid j(e) \in a_0 \times [0,1]\} \\ = &\{(a,t,e) \in A \times [0,1] \times E(\mathfrak{b}) \mid a = a_0 \wedge H(a,t) = j(e)\} \\ = &\, a_0 \times [0,1] \times E(\mathfrak{b}|_{b_0}). \end{aligned}$$

Based on this, we can define an isomorphism-germ

$$\bar{R} : H^*\mathfrak{b}|_{a_0 \times [0,1]} \Rightarrow \mathfrak{e}^n_{a_0 \times [0,1]}$$

via a representative

$$\bar{r} : a_0 \times [0,1] \times V \to a_0 \times [0,1] \times \mathbb{R}^n$$

with

$$\bar{r}(a_0, t, v) = (a_0, t, r^{(2)}(v))$$

where $r : V \to b_0 \times \mathbb{R}^n$ is a representative for R. The representative $\bar{r}$ is a homoemorphism on its image because it is a product of the identity and r, which are both homoemorphisms on their image.

Lemma 5.3.
Let $\mathfrak{b}$ be a rooted microbundle over B and let $H : A \times [0,1] \to B$ be a map that leaves the base point fixed. There exists a neighborhood V of a_0 with an isomorphism-germ

$$H^*\mathfrak{b}|_{V \times [0,1]} \Rightarrow \mathfrak{e}^n_{V \times [0,1]}$$

extending $\bar{R}$ (as defined above).

Proof.
By applying Lemma **(4.11)**, it follows that there exists an isomorphism-germ

$$Q : H^*\mathfrak{b}|_{V \times [0,1]} \Rightarrow \mathfrak{e}^n_{V \times [0,1]}$$

for a sufficiently small neighborhood V of a_0.

Now consider

$$Q \circ \bar{R}^{-1} : \mathfrak{e}^n_{a_0 \times [0,1]} \Rightarrow \ \Rightarrow \mathfrak{e}^n_{a_0 \times [0,1]}.$$

Similarly to the construction of $\bar{R}$ we can construct an isomorphism-germ

$$P : \mathfrak{e}^n_{V \times [0,1]} \Rightarrow \mathfrak{e}^n_{V \times [0,1]}$$

extending $Q \circ \bar{R}^{-1}$ represented by

$$p(v, t, x) = (v, q(a_0, t, x))$$

where q is a representative for $Q \circ \bar{R}^{-1}$.

Restricted to $a_0 \times [0,1]$, P agrees with $Q \times \bar{R}^{-1}$ and thus

$$P^{-1} \circ Q = (\bar{R} \circ Q^{-1}) \circ Q = \bar{R}$$

Since P and Q are both isomorphism-germs, $P^{-1} \circ Q$ is an isomorphism-germ as well. Therefore, $P^{-1} \circ Q$ suffices our requirements which concludes the proof. $\square$

Proof of the Rooted Homotopy Theorem.
Follow the steps for proving the initial Homotopy Theorem, however using Lemma **(5.3)** instead of Lemma **(4.11)**. $\square$

The following definition requires the base spaces to be hausdorff. This is useful because this implies that the singleton containing only the base point is closed, and can therefore be removed from any open set without losing openness.

Definition 5.4.
Let $\mathfrak{a}$ and $\mathfrak{b}$ be two rooted microbundles over A and B. The *wedge sum* $\mathfrak{a} \vee \mathfrak{b}$ of $\mathfrak{a}$ and $\mathfrak{b}$ is a microbundle over $A \vee B$

$$\mathfrak{a} \vee \mathfrak{b} \xrightarrow{i_\mathfrak{a} \vee i_\mathfrak{b}} E(\mathfrak{a} \vee \mathfrak{b}) \xrightarrow{j_\mathfrak{a} \vee j_\mathfrak{b}} A \vee B$$

where the total space is

$$(E(\mathfrak{a}) \sqcup E(\mathfrak{b}))/(f(e_a) \sim e_a)$$

and $f : E(\mathfrak{a}|_{a_0}) \supseteq W_a \xrightarrow{\sim} W_b \subseteq E(\mathfrak{b}|_{b_0})$ is some representative for $R_b^{-1} \circ R_a$. We equip $\mathfrak{a} \vee \mathfrak{b}$ with a rooting

$$R : E((\mathfrak{a} \vee \mathfrak{b})|_{a_0}) \Rightarrow \mathfrak{e}^n_{a_0}$$

represented by any representative for R_a (or R_b).

Proof that $\mathfrak{a} \vee \mathfrak{b}$ *is a microbundle.*
We show that $\mathfrak{a} \vee \mathfrak{b}$ is a microbundle and afterwards show that the definition of $\mathfrak{a} \vee \mathfrak{b}$ is independant of the choice of the representative f for $R_b^{-1} \circ R_a$.

1. $\mathfrak{a} \vee \mathfrak{b}$ is a microbundle:

   - The injection map $i_\mathfrak{a} \vee i_\mathfrak{b}$ is well-defined because

     $$i(a_0) = i_\mathfrak{a}(a_0) = f(i_\mathfrak{a}(a_0)) = i_\mathfrak{b}(b_0) = i(b_0)$$

     and continuous since $i_\mathfrak{a}$ and $i_\mathfrak{b}$ are continuous.

   - The projection map $j_\mathfrak{a} \vee j_\mathfrak{b}$ is well-defined because

     $$\forall e \in W_a : j(e) = j_\mathfrak{a}(e) = a_0 = b_0 = j_\mathfrak{b}(f(e)) = j(f(e))$$

     and continuous since $j_\mathfrak{a}$ and $j_\mathfrak{b}$ are continuous.

   - The composition $j \circ i = id_{A\vee B}$ because for every $a \in A$

     $$j(i(a)) = j(i_\mathfrak{a}(a)) = j_\mathfrak{a}(i_\mathfrak{a}(a)) = a$$

     since $j_\mathfrak{a} \circ i_\mathfrak{a} = id_A$ (analogous for B).

   It remains to show local triviality.

   The subspace topology of $E(\mathfrak{a}|_{a_0})$ yields an open subset $W'_a \subseteq E(\mathfrak{a})$ with $W_a = W'_a \cap E(\mathfrak{a}|_{a_0})$. Symmetrically, let $W'_b \subseteq E(\mathfrak{b})$ with $W_b = W'_b \cap E(\mathfrak{b}|_{b_0})$.

   Let $x \in A \vee B$, w.l.o.g. $x \in A$ for symmetry reasons.

- $x \neq a_0$:

  Choose a local trivialization (U, V, ϕ) for x in $\mathfrak{a}$. We can assume $U \cap B = \emptyset$ by subtracting U by $\{a_0\}$ which is closed since A is hausdorff. Now we can simply use this trivialization for $\mathfrak{a} \vee \mathfrak{b}$ since U is open in $A \vee B$, V is open in $E(\mathfrak{a} \vee \mathfrak{b})$ and $V \cong U \times \mathbb{R}^n$.

- $x = a_0$:

  Choose local trivializations (U_a, V_a, ϕ_a) for a_0 in $\mathfrak{a}$ and (U_b, V_b, ϕ_b) for b_0 in $\mathfrak{b}$.

  - We can assume $V_b \cap E(\mathfrak{b}|_{b_0}) \subseteq W_b$ by choosing a local trivialization for b_0 in the microbundle over the restricted total space $(E(\mathfrak{b}) - E(\mathfrak{b}|_{b_0})) \cup W_a$ (the existence is justified by Proposition **(1.4)**).
  - We can assume $V_a \cap E(\mathfrak{a}|_{a_0}) \subseteq W_b \cap E(\mathfrak{b}|_{b_0})$ by choosing a local trivialization for a_0 in the microbundle over the restricted total space $(E(\mathfrak{a}) - E(\mathfrak{b}|_{b_0})) \cup (V_b \cap E(\mathfrak{b}|_{b_0}))$.

  The subset $X_b := \phi_b^{(2)} f(V_a \cap E(\mathfrak{a}|_{a_0})) \subseteq W_b \cap E(\mathfrak{b}|_{b_0})$ is homeomorphic to $\mathbb{R}^n$ via

  $$a_0 \times \mathbb{R}^n \xrightarrow{\phi^{-1}} V_a \cap E(\mathfrak{a}|_{a_0}) \xrightarrow{f} X_b$$

  and open since f and ϕ are homeomorphisms. By choosing $V'_b := \phi_b^{-1}(B \times X_b)$ and $\phi'_b(e) := (j(e), \phi_a^{(2)}(f^{-1}(\phi_b^{(2)}(e))))$, we have local trivializations (U_a, V_a, ϕ_a) and (U_b, V'_b, ϕ'_b) that agree on $W_a = W_b$. This yields a local trivialization for $\mathfrak{a} \vee \mathfrak{b}$.

2. The wedge sum $\mathfrak{a} \vee \mathfrak{b}$ is independant of the choice of f:

   Let f' be another representative for $R_b^{-1} \circ R_a$ and $(\mathfrak{a} \vee \mathfrak{b})'$ the resulting wedge sum. We need to find an isomorphism germ that extends $R'^{-1} \circ R = R^{-1} \circ R = id$. Choose an open neighborhood $V \subseteq E(\mathfrak{a}|_{a_0})$ of $i_{\mathfrak{a}}(a)$ where f and f' agree. By subtracting $j_{\mathfrak{a}}^{-1}(a_0) - V$ from $E(\mathfrak{a} \vee \mathfrak{b})$ and $E(\mathfrak{a} \vee \mathfrak{b})'$ the microbundles remain unchanged. This is because the resulting subspaces are open since $j_{\mathfrak{a}}^{-1}(a_0)$ is closed (hausdorff) and V is open. So the total spaces are equal and injection and projection maps are defined the same. Using the modified total spaces, it follows that the identity $(\mathfrak{a} \vee \mathfrak{b}) \Rightarrow (\mathfrak{a} \vee \mathfrak{b})'$ is an isomorphism-germ. This surely extends $R'^{-1} \circ R$, which concludes the proof.

□

In the following, let B be a *reduced suspension*

$$SX = (X \times [0,1]) / (X \times \{0,1\} \cup x_0 \times [0,1])$$

over X.

Let $\phi : B \to B \vee B$ denote the map that sends $X \times [0, \frac{1}{2}]$ to the first B via

$$\phi(x, t) = [(x, 2t)]$$

and $X \times [\frac{1}{2}, 1]$ to the second B via

$$\phi(x, t) = [(x, 2t - 1)].$$

Let $c_1 : B \vee B \to B$ denote the map that is the identity on the first summand and the constant map to b_0 on the second summand (symmetrically define c_2).

Lemma 5.5.

$$\phi^*(\mathfrak{b} \oplus \mathfrak{e}_B^n) \cong \mathfrak{b} \cong \phi^*(\mathfrak{e}_B^n \oplus \mathfrak{b})$$

Proof.

- First, note that $c_1^*\mathfrak{b} \cong \mathfrak{b} \vee \mathfrak{e}^n$:

  $$E(c_1^*\mathfrak{b}) = \{(b, e) \in (B \vee B) \times E(\mathfrak{b}) : c_1(b) = j(e)\}$$
  $$= (\{(b, e) \in B \times E(\mathfrak{b}) : b = j(e)\} \sqcup B \times E(\mathfrak{b}|_{b_0}))/\sim$$
  $$= (\{(j(e), e) : e \in E(\mathfrak{b})\} \sqcup B \times E(\mathfrak{b}|_{b_0}))/\sim$$

  where $(b, e) \sim (b', e') \iff b = b_0 = b' \wedge e = e'$. Additionally, we can omit first component on the left side resulting in

  $$(E(\mathfrak{b}) \sqcup (B \times E(\mathfrak{b}|_{b_0})))/\sim$$

  where $e \sim (b, e') \iff b = b_0 \wedge e = e'$.

  On the other side, consider

  $$E(\mathfrak{b} \vee \mathfrak{e}_B^n) = (E(\mathfrak{b}) \sqcup (B \times \mathbb{R}^n))/e \sim f(e)$$

  with f being some representative $E(\mathfrak{b}|_{b_0}) \supseteq V \to b_0 \times \mathbb{R}^n$ for $R_e^{-1} \circ R_b$.

  Now, we have the mapping

  $$g : E(c_1^*\mathfrak{b}) \supseteq (E(\mathfrak{b}) \sqcup (B \times V))/\sim \xrightarrow{\sim} (E(\mathfrak{b}) \sqcup (B \times f(V)))/\sim \subseteq E(\mathfrak{b} \vee \mathfrak{e}^n)$$

  $$g(e) = e \text{ and } g(b, e) = (b, f^{(2)}(e)).$$

  This map is well-defined because $\forall e = (b_0, e) : g(e) = e = f(e) = (b_0, f^{(2)}(e)) = g(b_0, e)$. Bijectivity and continuity follow from bijectivity and continuity of its summands. Since $(E(\mathfrak{b}) \sqcup (B \times V))/\sim$ and $(E(\mathfrak{b}) \sqcup (B \times f(V)))/\sim$ are open (V and $f(V)$ are open) and injection and projection maps commute, it follows that g represents an isomorphism germ between $c_1^*\mathfrak{b}$ and $\mathfrak{b} \vee \mathfrak{e}^n$.

- Now, from $c_1 \circ \phi = id$ we can conclude that

$$\phi^*(\mathfrak{b} \oplus \mathfrak{e}^n_B) = \phi^* c_1^* \mathfrak{b} = (c_1 \circ \phi)^* \mathfrak{b} = \mathfrak{b}.$$

The equality $\mathfrak{b} = \phi^*(\mathfrak{e}^n_B \oplus \mathfrak{b})$ follows by symmetry, which concludes the proof.

□

Definition 5.6.
Let $\mathfrak{b}$ be a rooted microbundle over B and $f : A \to B$ a base point preserving map. The *induced microbundle* of f over $\mathfrak{b}$ is the initial induced microbundle $f^*\mathfrak{b}$ together with the rooting

$$f^*R : E(f^*\mathfrak{b}|_{a_0}) = a_0 \times E(\mathfrak{b}|_{b_0}) \Rightarrow e^n_{a_0}$$

that coincides with R if we consider $a_0 \times E(\mathfrak{b}|_{b_0}) = E(\mathfrak{b}|_{b_0})$ and $e^n_{a_0} = e^n_{b_0}$.

Lemma 5.7.
Let $\mathfrak{a}$ and $\mathfrak{b}$ be rooted microbundles over A and B. For maps $f : A' \to A$ and $g : B' \to B$ the following applies:

$$(f \vee g)^*(\mathfrak{a} \vee \mathfrak{b}) \cong f^*\mathfrak{a} \vee g^*\mathfrak{b}$$

Proof.
Consider the equation

$$E((f \vee g)^*(\mathfrak{a} \vee \mathfrak{b})) = \{(x, e) \in (A' \vee B') \times E(\mathfrak{a} \vee \mathfrak{b}) : (f \vee g)(x) = j(e)\}$$

$$= \{(x, e) \in ((A' \times E(\mathfrak{a})) \sqcup (B' \times E(\mathfrak{b})))/\sim : (f \vee g)(x) = j(e)\}$$

$$= (\{(x, e) \in A' \times E(\mathfrak{a}) : f(x) = j_{\mathfrak{a}}(e)\} \sqcup \{(x, e) \in B' \times E(\mathfrak{b}) : g(x) = j_{\mathfrak{b}}(e)\})/\sim$$

$$= (E(f^*\mathfrak{a}) \sqcup E(g^*\mathfrak{b}))/\sim = E(f^*\mathfrak{a} \vee g^*\mathfrak{b})$$

where $(a, e_a) \sim (b, e_b) \iff a = a_0 = b_0 = b \wedge e_a = e_b$ in $E(\mathfrak{a} \vee \mathfrak{b})$. So the total spaces are equal and also the injection and projection map agree, which concludes the proof. □

Let $r : B \xrightarrow{\sim} B$ denote the homeomorphism that corresponds to the "reflection"

$$(x, t) \mapsto (x, 1 - t)$$

and let $c : B \vee B \to B$ be the identity on the first summand and r on the second summand.

Lemma 5.8.
The induced microbundle $\phi^*(\mathfrak{b} \vee r^*\mathfrak{b})$ is trivial.

Proof.
The composition $f \circ \phi$ is null-homotopic via $H : B \times [0,1] \to B$ with

$$H([x,t],s) = f(\phi(x, t * s))$$

and therefore $\phi^* f^* \mathfrak{b} = (f \circ \phi)^* \mathfrak{b} = const^*_{b_0} \mathfrak{b} = \mathfrak{e}^n$ (see Theorem **(4.1)**). With distributivity, we conclude $\phi^*(\mathfrak{b} \vee c^* \mathfrak{b}) = \phi^* f^* \mathfrak{b}$ and hence that $\phi^*(\mathfrak{b} \vee c^* \mathfrak{b})$ is trivial. □

Definition 5.9.
The *whitney sum* of two rooted microbundles $\mathfrak{b}$ and $\mathfrak{b}'$ over B is the initial whitney sum $\mathfrak{b} \oplus \mathfrak{b}'$ together with the rooting

$$R \oplus R' : (\mathfrak{b} \oplus \mathfrak{b}')|_{b_0} \Rightarrow \mathfrak{e}^{n_1}_{b_0} \oplus \mathfrak{e}^{n_2}_{b_0} = \mathfrak{e}^{n_1+n_1}_{b_0}.$$

Lemma 5.10.
The following applies for rooted microbundles $\mathfrak{a}, \mathfrak{a}'$ over A and $\mathfrak{b}, \mathfrak{b}'$ over B:

$$(\mathfrak{a} \vee \mathfrak{b}) \oplus (\mathfrak{a}' \vee \mathfrak{b}') \cong (\mathfrak{a} \oplus \mathfrak{a}') \vee (\mathfrak{b} \oplus \mathfrak{b}')$$

Proof.
Consider the equation

$$E((\mathfrak{a} \vee \mathfrak{b}) \oplus (\mathfrak{a}' \vee \mathfrak{b}')) = \{(e,e') \in E(\mathfrak{a} \vee \mathfrak{b}) \times E(\mathfrak{a}' \vee \mathfrak{b}') : j(e) = j'(e')\}$$

$$= \{(e,e') \in (E(\mathfrak{a}) \sqcup E(\mathfrak{b}))/\sim \times (E(\mathfrak{a}') \sqcup E(\mathfrak{b}'))/\sim' : j(e) = j'(e')\}$$

$$= (\{(e,e') \in E(\mathfrak{a}) \times E(\mathfrak{a}') : j_{\mathfrak{a}}(e) = j_{\mathfrak{a}'}(e')\} \sqcup \{(e,e') \in E(\mathfrak{b}) \times E(\mathfrak{b}') : j_{\mathfrak{b}}(e) = j_{\mathfrak{b}'}(e')\})/\approx$$

$$= (E(\mathfrak{a} \oplus \mathfrak{a}') \sqcup E(\mathfrak{b} \oplus \mathfrak{b}'))/\approx = E((\mathfrak{a} \oplus \mathfrak{a}') \vee (\mathfrak{b} \oplus \mathfrak{b}'))$$

where $(e_a, e'_a) \approx (e_b, e'_b) \iff e_a \sim e_b \wedge e'_a \sim' e'_b$. So the total spaces are equal and also the injection and projection map agree, which concludes the proof. □

Lemma 5.11.
Let $\mathfrak{b}$ be a rooted microbundle over a paracompact space B with rooting R. Then there exists a closed neighborhood W of b_0 and an isomorphism-germ

$$\mathfrak{b}|_W \Rightarrow \mathfrak{e}^n_W$$

extending R together with a map $\lambda : B \to [0,1]$ with

$$supp\lambda \subseteq W \text{ and } \lambda(b_0) = 1.$$

Proof.
Let $r : W_r \to b_0 \times \mathbb{R}^n$ be a representative map for R. Consider a local trivialization (U, V, ϕ) for b_0 such that $V \cap E(\mathfrak{b}|_{b_0}) \subseteq W_r$. With

$$\psi : V \xrightarrow{\sim} \psi(V) \subseteq U \times \mathbb{R}^n$$

$$\psi(e) = (j(e), r(\phi^{-1}(b_0, \phi^{(2)}(e))))$$

we have a representative for an isomorphism-germ $\mathfrak{b}|_U \Rightarrow \mathfrak{e}_U^n$ extending R. Consider the open covering of B with U and B itself. Since B is paracompact, we can apply the concept of partition of unity and have therefore a map

$$\lambda : B \to [0, 1]$$

with

$$\text{supp}\lambda \subseteq U \text{ and } \lambda(b_0) = 1.$$

Now we can choose $W := \text{supp}\lambda$, which is closed by definition of supp. By restricting the constructed isomorphism-germ over U to W, we have an isomorphism-germ $\mathfrak{b}|_W \Rightarrow \mathfrak{e}_W^n$. Together with λ, this concludes our proof. $\square$

Lemma 5.12.
The rooted microbundles $\mathfrak{b} \oplus \mathfrak{e}_B^n$ and $\mathfrak{e}_B^n \oplus \mathfrak{b}$ are rooted-isomorphic.

Proof.
We need to find an isomorphism germ $\mathfrak{b} \oplus \mathfrak{e}_B^n \Rightarrow \mathfrak{e}_B^n \oplus \mathfrak{b}$ that extends

$$(I \oplus R) \circ (R \oplus I)^{-1} = R \oplus R^{-1}$$

where I denotes the identity germ.

Ignoring the rooting, we have an isomorphism-germ $f : E(\mathfrak{b}) \times \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n \times E(\mathfrak{b})$ with $f(e, x) = (-x, e)$. The idea is to change to f near b_0 so that it extends the rooting.

Using the previous lemma, choose a sufficiently small closed neighborhood U of b_0 such that there exists an extension $Q : (\mathfrak{b} \oplus \mathfrak{e}^n)|_U \Rightarrow (\mathfrak{e}^n \oplus \mathfrak{b})|_U$ for the rooting.

Since B is Tychonoff, there exists a map

$$\lambda : B \to [0, \frac{\pi}{2}]$$

with $\text{supp}\lambda \subseteq U$ and $\lambda(b_0) = \frac{\pi}{2}$. With this map, we can define a homeomorphism

$$g : U \times \mathbb{R}^n \times \mathbb{R}^n \xrightarrow{\sim} U \times \mathbb{R}^n \times \mathbb{R}^n$$

by

$$g(b, x, y) = (b, x \sin(\lambda(b)) - y \cos(\lambda(b)), x \cos(\lambda(b)) - y \sin(\lambda(b))).$$

Now, we can consider

$$(\mathfrak{b} \oplus \mathfrak{e}^n)|_U \Rightarrow (\mathfrak{b} \oplus \mathfrak{e}^n)|_U \xrightarrow{g} (\mathfrak{b} \oplus \mathfrak{e}^n)|_U \Rightarrow (\mathfrak{e}^n \oplus \mathfrak{b})|_U$$

which coincides with $R \oplus R^{-1}$ over b_0 since $g(b_0, x, y) = (b_0, x, y)$ and with F over $U \cap \lambda^{-1}(0)$. Pieced together with $F|_{\lambda^{-1}(b)}$, we have an isomorphism germ $\mathfrak{b} \oplus \mathfrak{e}_B^n \Rightarrow \mathfrak{e}_B^n \oplus \mathfrak{b}$ that extends the rooting, which completes the proof. $\square$

Theorem 5.13.
If $\mathfrak{a}$ and $\mathfrak{b}$ are rooted microbundles over a completly regular space B, then

$$\phi^*(\mathfrak{a} \vee \mathfrak{b}) \oplus \mathfrak{e}^n_B = \mathfrak{a} \oplus \mathfrak{b}.$$

Proof.
The previous lemma yields $\mathfrak{b} \oplus \mathfrak{e}^n \cong \mathfrak{e}^n \oplus \mathfrak{b}$. Hence

$$\phi^*((\mathfrak{a} \oplus \mathfrak{e}^n) \vee (\mathfrak{b} \oplus \mathfrak{e}^n)) \cong \phi^*((\mathfrak{a} \oplus \mathfrak{e}^n) \vee (\mathfrak{e}^n \oplus \mathfrak{b})).$$

Additionally we have

$$\phi^*((\mathfrak{a} \vee \mathfrak{b}) \oplus (\mathfrak{e}^n \vee \mathfrak{e}^n)) \cong \phi^*(\mathfrak{a} \vee \mathfrak{b}) \oplus \mathfrak{e}^n$$

for the left side of the isomorphy and

$$\phi^*((\mathfrak{a} \vee \mathfrak{e}^n) \oplus (\mathfrak{e}^n \vee \mathfrak{b})) \cong \mathfrak{a} \oplus \mathfrak{b}$$

for the right side of the isomorphy which concludes the proof. □

Corollary 5.14.
The wedge sum $\mathfrak{b} \oplus r^*\mathfrak{b}$ is trivial.

Proof.
This follows directly from the Theorem and the fact that $\phi^*(\mathfrak{b} \oplus r^*\mathfrak{b})$ is trivial.
□

Chapter 6

Normal Microbundles

Definition 6.1 (normal microbundle)**.**
Let M and N be two topological manifolds with $N \subseteq M$. A *normal microbundle* $\mathfrak{n}$ of N in M is a microbundle

$$N \xrightarrow{\iota} U \xrightarrow{r} N$$

where $U \subseteq M$ is a neighborhood of N and ι denotes the inclusion $M \hookrightarrow U$.

Definition 6.2 (composition microbundle)**.**
Let $\mathfrak{b}$ and $\mathfrak{c}$ be two microbundles over B and $E(\mathfrak{b})$. The *composition microbundle* $\mathfrak{b} \circ \mathfrak{c}$ is a microbundle

$$B \xrightarrow{i} E(\mathfrak{c}) \xrightarrow{j} B$$

where $i := i_{\mathfrak{c}} \circ i_{\mathfrak{b}}$ and $j := j_{\mathfrak{b}} \circ j_{\mathfrak{c}}$.

Proof that $\mathfrak{b} \circ \mathfrak{c}$ *is a microbundle.*
Both injection and projection are continuous since the are composed by continuous maps. Additionally, $j \circ i = j_{\mathfrak{b}} \circ (j_{\mathfrak{c}} \circ i_{\mathfrak{c}}) \circ i_{\mathfrak{b}} = j_{\mathfrak{b}} \circ i_{\mathfrak{b}} = id_B$.

It remains to be shown that $\mathfrak{b} \circ \mathfrak{c}$ is locally trivial:

For an arbitrary $b \in B$, choose local trivializations

$$(U_{\mathfrak{b}}, V_{\mathfrak{b}}, \phi_{\mathfrak{b}}) \text{ of } b \text{ and } (U_{\mathfrak{c}}, V_{\mathfrak{c}}, \phi_{\mathfrak{c}}) \text{ of } j_{\mathfrak{b}}(b).$$

Since $V_{\mathfrak{b}}$ and $U_{\mathfrak{c}}$ are both neighborhoods of $i_{\mathfrak{b}}(b)$ in $E(\mathfrak{b})$, the image

$$\phi_{\mathfrak{b}}(V_{\mathfrak{b}} \cap U_{\mathfrak{c}})$$

contains an open set $U \times B_{\varepsilon}(0)$ where U is open and ε is sufficiently small.

With $V := \phi_{\mathfrak{c}}^{-1}(\phi_{\mathfrak{b}}^{-1}(U \times B_\varepsilon(0))) \subseteq E(\mathfrak{c})$ and

$$U \times \mathbb{R}^n \cong U \times B_\varepsilon(0) \xrightarrow{\phi_{\mathfrak{c}}^{-1} \circ \phi_{\mathfrak{b}}^{-1}} V$$

which is a homeomorphism because its composed by homeomorphisms, we constructed a local trivialization of b in $\mathfrak{c} \circ \mathfrak{b}$. □

Lemma 6.3.
Let M, N and P be topological manifolds with $P \subseteq N \subseteq M$. There exists a normal microbundle

$$\mathfrak{n} : P \xrightarrow{\iota} U \xrightarrow{r} P$$

of P in M, if there exist normal microbundles

$$\mathfrak{n}_p : P \xrightarrow{\iota_P} U_N \xrightarrow{j_P} P \text{ in } N \text{ and } \mathfrak{n}_n : N \xrightarrow{\iota_N} U_M \xrightarrow{j_N} N \text{ in } M.$$

Proof.
Considering the composition $\mathfrak{n}_p \circ \mathfrak{n}_n|_{U_N}$, we found a normal microbundle $\mathfrak{n}$ of P in M since $\iota_N \circ \iota_P$ is just the inclusion $P \hookrightarrow U_M$. □

Every topological manifold is an absolute neighborhood retract (ANR).

It follows that by restricting M, if necessary, to an open neighborhood of N, there exists a retraction $r : M \to N$ which we will take advantage of in the following.

Lemma 6.4.
Let $\mathfrak{t}_N$ and $\mathfrak{t}_M$ be tangent microbundles of N and M. The total spaces $E(\iota^*\mathfrak{t}_M)$ and $E(r^*\mathfrak{t}_N)$ are homeomorphic.

Proof.
The total space

$$E(\iota^*\mathfrak{t}_M) = \{(n, m_1, m_2) \in N \times (M \times M) \mid \iota(n) = m_1\}$$

is homeomorphic to $N \times M$ via

$$(n, m_1, m_2) \mapsto (n, m_2) \text{ and } (n, m) \mapsto (n, \iota(n), m).$$

Similarly, the total space

$$E(r^*\mathfrak{t}_N) = \{(m, n_1, n_2) \in M \times (N \times N) \mid r(m) = n_1\}$$

is homeomorphic to $M \times N$ via

$$(m, n_1, n_2) \mapsto (m, n) \text{ and } (m, n) \mapsto (m, r(m), n).$$

Composed together with the trivial homeomorphism $N \times M \cong M \times N$, this yields a homeomorphism

$$\psi : E(\iota^*\mathfrak{t}_M) \xrightarrow{\sim} E(r^*\mathfrak{t}_N)$$

$$\text{with } \psi(n, m_1, m_2) := (m_2, r(m_2), n)$$

□

Remark 6.5.
The following diagram commutes:

$$\begin{tikzcd} N \arrow[r, "i_1"] \arrow[d, hook, "\iota"'] & E(\iota^* \mathfrak{t}_M) \arrow[d, "\psi"] \\ M \arrow[r, "i_2"] & E(r^* \mathfrak{t}_N) \end{tikzcd}$$

Here i_1 and i_2 denote the injections of $\iota^* \mathfrak{t}_M$ and $r^* \mathfrak{t}_N$.

The total space $E(r^* \mathfrak{t}_N)$ carries the structure of a topological manifold with

$$E(r^* \mathfrak{t}_N) \cong M \times N$$

as described in the previous lemma. That is since $M \times N$ comes equipped with the product manifold structure.

The fact that the diagram

$$\begin{tikzcd} N \arrow[r, "i_1"] \arrow[d, hook, "\iota"'] & E(\iota^* \mathfrak{t}_M) \arrow[d, "\psi"] \\ M \arrow[r, "i_2"] & E(r^* \mathfrak{t}_N) \end{tikzcd}$$

commutes (i_1 and i_2 denote the injections of $\iota^* \mathfrak{t}_M$ and $r^* \mathfrak{t}_N$), lets us consider N to be a submanifold of $E(r^* \mathfrak{t}_N)$ via $N \hookrightarrow M \xrightarrow{i_2} E(r^* \mathfrak{t}_N)$.

Lemma 6.6.
There exists a normal microbundle $\mathfrak{n}$ of N in $E(r^* \mathfrak{t}_N)$ such that $\mathfrak{n} \cong \iota^* \mathfrak{t}_M$.

Proof.
We are already given a normal microbundle of N in $E(r^* \mathfrak{t}_N)$ with $r^* \mathfrak{t}_N|_N$. Isomorphy between $r^* \mathfrak{t}_N|_N$ and $\iota^* \mathfrak{t}_M$ follows from

$$\psi : E(\iota^* \mathfrak{t}_M) \xrightarrow{\sim} E(r^* \mathfrak{t}_N)$$

together with the fact that $E(r^* \mathfrak{t}_N|_N)$ is a neighborhood of $i_2(B)$ in $E(r^* \mathfrak{t}_N)$ and from the diagram which shows that injection and projection maps commute with ψ. □

Finally, we gathered all the tools to prove Milnor's theorem.

Theorem 6.7 (Milnors Theorem)**.**
For a sufficently large $q \in \mathbb{N}$, $N = N \times \{0\}$ has a normal microbundle in $M \times \mathbb{R}^q$.

Proof.
We show the theorem in multiple steps:

1. There exists a microbundle η over N such that $\mathfrak{t}_N \oplus \eta \cong \mathfrak{e}_N^q$:

   From the [Whitney Embedding Thereom] it follows that we can consider M to be embedded in euclidean space $\mathbb{R}^{2m+1}$. Additionally, since we can find a retraction $r : V \to N$ where V is an open neighborhood of N in M we can extend $\mathfrak{t}_N$ over V. Now we can apply the Theorem **(3.5)** from the "Whitney Sum" Chapter to the extended microbundle to receive a η such that $\mathfrak{t}_N \oplus \eta \cong \mathfrak{e}_N^q$.

2. $E(r^*\mathfrak{t}_N) \subseteq E(r^*\mathfrak{t}_N \oplus r^*\mathfrak{t}')$ has a normal microbundle:

   Consider the microbundle

   $$j^*(r^*\mathfrak{t}_N \oplus r^*\mathfrak{t}') : E(r^*\mathfrak{t}_N) \xrightarrow{i'} E(r^*\mathfrak{t}_N \oplus r^*\mathfrak{t}') \xrightarrow{j'} E(r^*\mathfrak{t}_N)$$

   where j is the projection map for $r^*\mathfrak{t}_N$. Since i' is injective, we can assume $E(r^*\mathfrak{t}_N) \subseteq E(r^*\mathfrak{t}_N \oplus r^*\mathfrak{t}')$. Now $j^*(r^*\mathfrak{t}_N \oplus r^*\mathfrak{t}')$ is a normal microbundle if we equip $r^*\mathfrak{t}_N \oplus r^*\mathfrak{t}'$ with a manifolds structure as shown above.

Since $N \subseteq M \subseteq E(r^*\mathfrak{t}_N)$ has a normal microbundle (Lemma **(6.6)**) it follows from Lemma **(6.3)** that $N \subseteq E(r^*\mathfrak{t}_N \oplus r^*\mathfrak{t}')$ has a normal microbundle. But $r^*\mathfrak{t}_N \oplus r^*\mathfrak{t}'$ is trivial and therefore w.l.o.g. $E(r^*\mathfrak{t}_N \oplus r^*\mathfrak{t}') \cong N \times \mathbb{R}^q$ $\square$