ANNEX A

Co-ordinate transformation algorithms for the hand-over of targets between POEMS interrogators

TABLE OF CONTENTS

1 Introduction

1.1 General

1.1.1 This document has been extracted from a set of documents produced by Smith System Engineering Limited (Smith) for EUROCONTROL, under contract C/1.120/HQ/MM/96.

1.2 Contents

1.2.1 The remainder of this document is structured as follows:

- section 2 specifies the local co-ordinate system used to measure the aircraft position in the POEMS system;
- section 3 specifies and analyses the alternative absolute co-ordinate transformation method;
- section 4 specifies the transformation to / from geodesic co-ordinates;
- section 5 specifies the transformation of the speed vector.

1.3 Geometric Constants of the WGS 84 Ellipsoid

1.3.1 The radar antenna positions and aircraft positions exchanged between mode-S stations shall be specified in terms of geodetic longitude and latitude, and altitude relative to the WGS 84 ellipsoid using the following constants.

Name	**Notation**	**Value**
Semi-major axis	a	6378137.000 m
Semi-minor axis	b	6356752.314 m
First eccentricity	e	0.0818191908426
(First eccentricity)^2	e^2	0.00669437999013

2 Radar local co-ordinate system

2.1 Introduction

2.1.1 The measurements made by one radar system are in terms of quantities local to that radar. This section specifies how these shall be used to obtain the aircraft positions in terms of local co-ordinates.

2.1.2 Section 2.2 describes the measurements made by the radar system. The specification of the transformation algorithm required is between local polar co-ordinate systems, and hence the conversion to these is considered first in section 2.3. In the consideration of transformation algorithms an intermediate conversion to local cartesian co-ordinates has been found to be useful, and this is discussed in section 2.3.

2.1.3 The overall problem is conversion between two such local co-ordinate systems referred with respect to origins at different radar systems. The general approach to be adopted is transformation to some absolute reference co-ordinate system that is independent of the radar position, and then from this absolute system to the new local system.

2.2 Radar measurements

2.2.1 The POEMS system allows the collection of surveillance data that includes both the position and velocity of aircraft.

2.2.2 The position measurements made by the radar system for a given aircraft are ρ, the slant range (line-of-sight distance) to the aircraft, θ, the azimuthal bearing of the aircraft (where approximately due north of the radar is a bearing of 0°), and H, the altitude of the aircraft above the ground. Note that the first two are measured by the radar itself, while the third quantity is returned by a transponder on the aircraft. The measurement of H is made using a pressure altimeter.

2.3 Local polar co-ordinates

2.3.1 The description of the aircraft position in terms of (ρ, θ, H) is a mathematically unconventional co-ordinate system, which cannot be easily manipulated. It is therefore helpful to convert to a more convenient system, that of spherical polar co-ordinates measured relative to the radar system.

2.3.2 The spherical co-ordinate origin is taken to be the radar location, the polar axis is the normal vertically upwards from the Earth, and the azimuthal zero is a tangent line on the Earth pointing in the same direction as the zero for measurement of the bearing angle, as described above. This system is referred to as the “local polar co-ordinate system”.

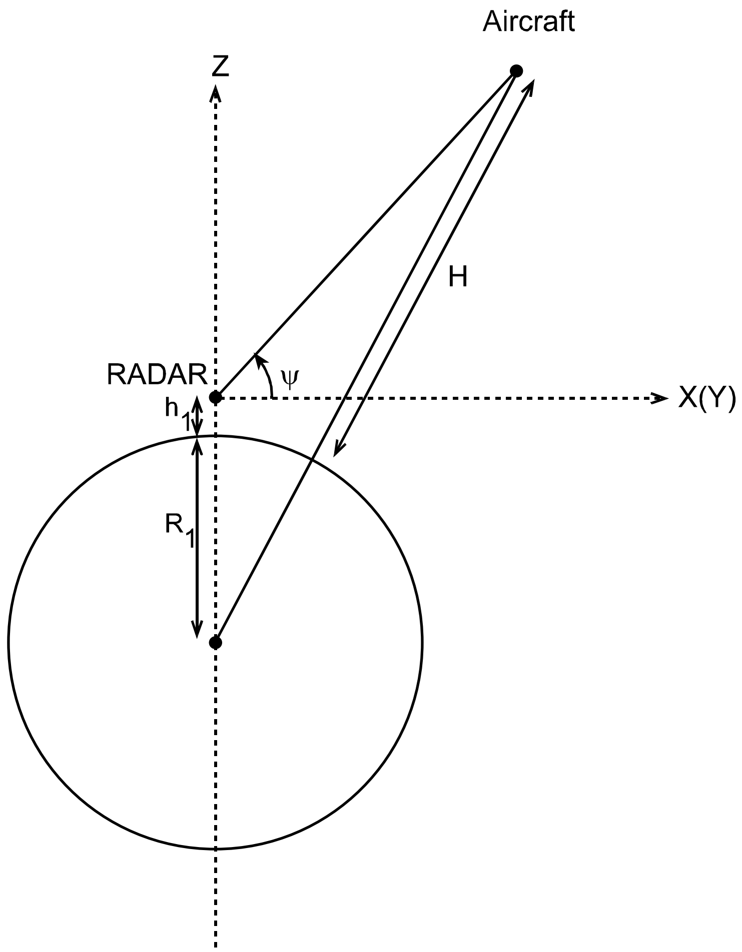

Figure 2-1
Radar measurements of altitude and elevation angle

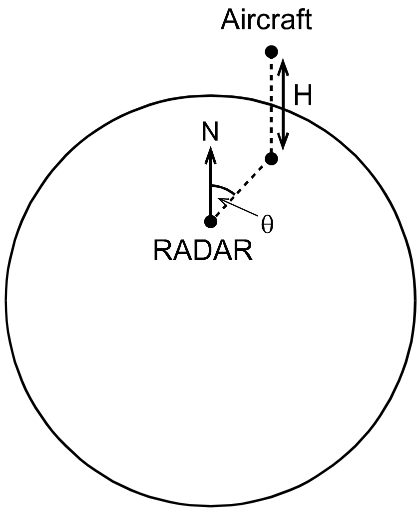

Figure 2-2
Radar bearing measurement

2.3.3 In this co-ordinate system the distance from the sphere centre and the azimuthal angle are already specified by ρ and θ respectively. It only remains to calculate the polar elevation angle ψ. It should be noted that for compatibility with the notation and routines used in reference 1, this angle is measured from the azimuthal plane rather than the polar axis.

2.3.4 The relationship of ρ, θ, H and ψ is illustrated in Figures 2-1 and 2-2.

2.3.5 The radius of the Earth, R, at a given geodetic latitude L is

$$R = \frac{a(1-e^2)}{\sqrt{(1-e^2\sin^2 L)^3}} \qquad \text{2-1}$$

where a is the semi-major axis of the Earth ellipsoid and e is its eccentricity.

2.3.6 It is generally assumed that R is the same below the aircraft position as at the radar antenna position. Making this assumption and applying the cosine rule to the triangle in Figure 2-1 gives the equation:

$$2\rho(R_1 + h_1)\sin\psi = (H + R_1)^2 - (h_1 + R_1)^2 - \rho^2 \qquad \text{2-2}$$

which can be rearranged to give an expression for ψ. (Symbols are as defined in figures 2-1 and 2-2).

$$\sin\psi = \frac{2R_1(H - h_1) + H^2 - h_1^2 - \rho^2}{2\rho(R_1 + h_1)} \qquad \text{2-3}$$

2.3.7 In practice the Earth is actually ellipsoid rather than spherical in shape, bulging slightly at the equator. Therefore the assumption above introduces an error which can be approximated as follows. The radius of the Earth changes by about 1 in 300 from the pole to the equator, i.e. a change of about 20km over a north-south separation of about 5000km. A radar system is assumed to detect aircraft over a range of about 500km, and hence over this range the Earth's radius may change by up to 2km. Therefore the error in the calculated elevation angle is:

$$\Delta\psi \approx \tan^{-1}\frac{2}{500} = 0.2^{\circ} \tag{2-4}$$

2.3.8 This can be considered to be a systematic feature of representing aircraft positions in the local polar co-ordinate system - it assumes an Earth of constant radius, equal to that at the position of the radar.

2.4 Conversion from Local Polar Spherical Co-ordinates to Local Cartesian Co-ordinates (and vice-versa)

2.4.1 Standard mathematical formulae can be used to convert from spherical polar co-ordinates to cartesian co-ordinates relative to the same position.

2.4.2 In accordance with usual convention, the polar axis is taken as z-axis. However, the due north direction is taken as the y-axis (rather than the conventional choice of the x-axis).

2.4.3 It is understood that the azimuthal measurements from a given radar may not be measured exactly with respect to the North pole. The deviation of the zero azimuth direction from North can be denoted as Θ, and used as a correction when making the conversion to local cartesian co-ordinates, which have been defined with respect to the exact North direction.

2.4.3 This gives the transformation equations as

$$\begin{aligned} x &= \rho\cos\psi\sin(\theta+\Theta) \\ y &= \rho\cos\psi\cos(\theta+\Theta) \\ z &= \rho\sin\psi \end{aligned} \tag{2-5}$$

2.4.4 This transformation can be inverted using the following equations:

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2 + z^2} \\ \theta &= \tan^{-1}\left(\frac{x}{y}\right) - \Theta \\ \psi &= \sin^{-1}\left(\frac{z}{\rho}\right) \end{aligned} \tag{2-6}$$

2.4.5 This (*x,y,z*) co-ordinate system is referred to as the "local cartesian co-ordinates"

2.4.6 The conversions between local polar and local cartesian co-ordinates are exact.

3 Geocentric Co-ordinate System

3.1 Introduction

3.1.1 In this method the absolute reference co-ordinate system is a cartesian grid whose origin is the centre of the earth. This is referred to as a “geocentric cartesian co-ordinate system.”

3.1.2 The conversion between local polar and local cartesian co-ordinates was described in section 2 above.

3.1.3 The conversion between local cartesian and geocentric cartesian co-ordinates is described in section 3.2 below.

3.2 Conversion from Local Cartesian Co-ordinates to Geocentric Cartesian Co-ordinates (and vice-versa)

3.2.1 The reference co-ordinate set used is the geocentric co-ordinate system, where the co-ordinate origin is the centre of the earth, the z-axis points towards the North pole the x-axis points towards the zero of longitude in the equatorial plane and the y-axis points towards longitude of 90°E in the equatorial plane. This is illustrated in Figure 3-1.

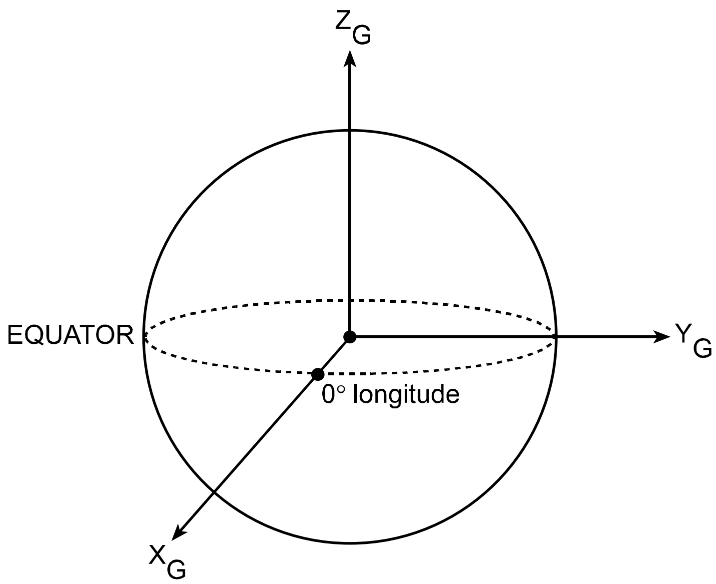

Figure 3-1
Geocentric cartesian axes

3.2.2 The local cartesian co-ordinate set has the z-axis as the normal to the Earth’s surface, the y-axis as a tangent pointing due north, and the x-axis forming the correct right handed set. These axes are illustrated in Figure 3-2.

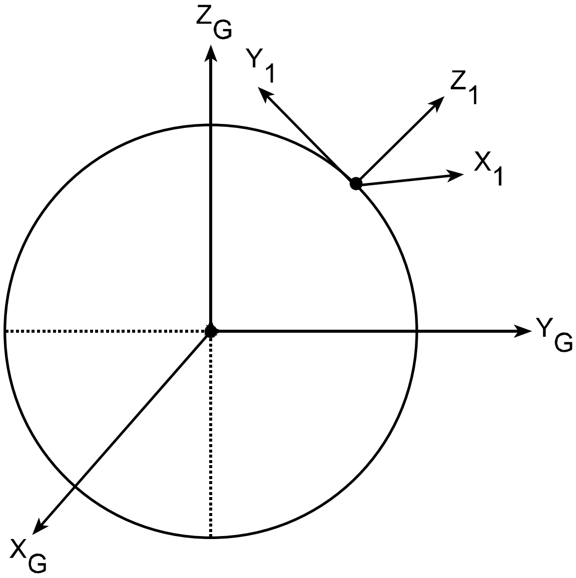

Figure 3-2
Local cartesian axes

3.2.3 The direction vectors of the local cartesian direction axes for radar system 1, X_1, Y_1, and Z_1, can be written in the geocentric cartesian system.

3.2.4 The position of radar system 1 is specified in terms of a latitude, L_1, longitude, G_1, and altitude, h_1 (geodetic co-ordinates).

3.2.5 The normal to the Earth's surface at a given latitude and longitude is then

$$\hat{Z}_S = (\cos L_1 \cos G_1, \cos L_1 \sin G_1, \sin L_1) \tag{3-1}$$

and a tangent pointing towards the north is

$$\hat{Y}_S = (-\sin L_1 \cos G_1, -\sin L_1 \sin G_1, \cos L_1) \tag{3-2}$$

Application of the vector cross product gives:

$$\hat{X}_S = (-\sin G_1, \cos G_1, 0) \tag{3-3}$$

(The notation used is that $\hat{X}$ is a unit vector pointing in the direction in which the co-ordinate X is being measured)

3.2.6 The position of the radar system is also the co-ordinate origin for the local co-ordinate system, and is given by:

$$T_1 = \begin{pmatrix} (\eta_1 + h_1)\cos L_1 \cos G_1 \\ (\eta_1 + h_1)\cos L_1 \sin G_1 \\ (\eta_1(1-e^2) + h_1)\sin L_1 \end{pmatrix} \tag{3-4}$$

where

$$\eta_1 = \frac{a}{\sqrt{1-e^2 \sin^2 L_1}} \qquad 3\text{-}5$$

a being the semi-major axis of the Earth ellipsoid and *e* its eccentricity.

3.2.7 The conversion from the local cartesian system to geocentric cartesian system is therefore a translation of the origin by T_1, and a rotation of the axes from those in equations 3-1 to 3-3 to the geocentric axes

$$\begin{aligned} \hat{X}_G &= (1,0,0) \\ \hat{Y}_G &= (0,1,0) \\ \hat{Z}_G &= (0,0,1) \end{aligned} \qquad 3\text{-}6$$

3.2.8 Therefore:

$$\begin{pmatrix} X_G \\ Y_G \\ Z_G \end{pmatrix} = S_1^T \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} + T_1 \qquad 3\text{-}7$$

where

$$S_1 = \begin{pmatrix} -\sin G_1 & \cos G_1 & 0 \\ -\sin L_1 \cos G_1 & -\sin L_1 \sin G_1 & \cos L_1 \\ \cos L_1 \cos G_1 & \cos L_1 \sin G_1 & \sin L_1 \end{pmatrix} \qquad 3\text{-}8$$

3.2.9 Note that the matrix S_1 is orthogonal, and hence:

$$S_1 S_1^T = I \qquad 3\text{-}9$$

3.2.10 Manipulating equation 3-7 gives the reverse transformation:

$$\begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} = S_1 \left[\begin{pmatrix} X_G \\ Y_G \\ Z_G \end{pmatrix} - T_1 \right] \qquad 3\text{-}10$$

3.2.11 It is important to note that all the transformations discussed in this section are in principle exact.

3.2.12 Therefore the only errors which can arise are from errors in the data put into the transformation, i.e. the data on radar positions and the aircraft position.

4 Geodesic Co-ordinate System

4.1 Transformation from Geocentric to Geodesic Co-ordinates

4.1.1 The altitude of the aircraft is already known, from the on-board measurement of *H*. However, the latitude and longitude must be calculated from the geocentric coordinates.

4.1.2 The longitude calculation is straightforward, as this is simply the azimuthal angle in the geocentric *X-Y* plane:

$$\tan G = \frac{Y_G}{X_G}\,. \qquad \text{4-1}$$

4.1.3 The latitude calculation is most easily calculated via the polar angle ϕ:

$$\tan\phi = \frac{Z_G}{\sqrt{X_G{}^2 + Y_G{}^2}}\,. \qquad \text{4-2}$$

4.1.4 It is important to note that geodetic latitude, *L*, is measured as the angle made by a normal to the surface with the equatorial plane. For an ellipsoid this will differ from the angle, ϕ, made with a line joining the point to the centre of the Earth.

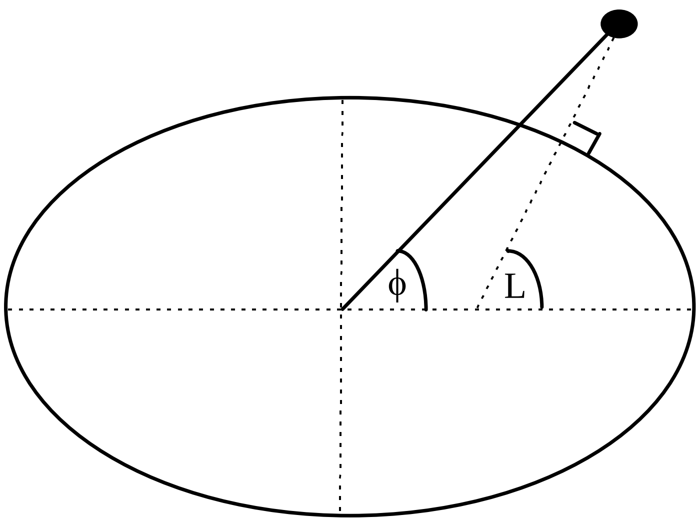

Figure 4-1
Polar angle (ϕ) and geodetic latitude (L)

For a point on the Earth's surface the two angles are related by:

$$\tan\phi = (1-e^2)\tan L \quad \text{or} \quad \tan L = \frac{1}{(1-e^2)}\tan\phi \qquad \text{4-3}$$

For a point above the earth surface the following formula is valid;

$$\tan L = \frac{1 + H/\eta}{(1-e^2) + H/\eta} \tan\phi \qquad 4\text{-}4$$

where *e* is the eccentricity of the reference Earth ellipsoid,

$$e^2 = 1 - \frac{b^2}{a^2}, \qquad 4\text{-}5$$

where b is the semi-minor axis length and a the semi-major axis length of the ellipsoid.

4.1.5 Using equation 4-3, given that the calculated quantity is the latitude of the point on the ground directly below the aircraft, the aircraft's latitude can then be calculated as:

$$\tan L = \frac{1 + H/\eta}{(1-e^2) + H/\eta} \frac{Z_G}{\sqrt{X_G^{\ 2} + Y_G^{\ 2}}}. \qquad 4\text{-}6$$

4.1.6 The value of *the normalised earth radius at the aircraft latitude* η can only be determined accurately once the latitude is known (see formula 4-7). The next table investigates the maximum error in case of an approximated normalised earth radius.

normalised earth radius η for latitude 0 and 90	6378 km	6400 km	
Altitude (H)	15 km	15 km	
$\frac{1 + H/\eta}{(1-e^2) + H/\eta}$	1,00672357744	1,00672363204	Difference = -0,0000000546

η shall be approximated by using the normalised earth radius at the radar position, the resulting error will be smaller than 6.10^{-8}. Note that the error will be less than 1 meter.

Therefore in equation 4-6 the normalised earth radius at the aircraft latitude can be approximated by :

$$\eta = \frac{a}{\sqrt{1 - e^2 \sin^2 L_1}}$$

where L_1 is the latitude of the radar.

4.2 Transformation Geodesic to Geocentric Co-ordinates

4.2.1 In this system the aircraft position is specified in terms of geodetic latitude, *L*, longitude, *G*, and altitude, *H*.

4.2.2 Using normal projection, ie continuing lines of constant latitude normal to the Earth's surface, these can be converted to geocentric cartesian coordinates using the equations below:

$$\begin{aligned} X_G &= (\eta + H)\cos L \cos G \\ Y_G &= (\eta + H)\cos L \sin G \\ Z_G &= (\eta(1-e^2) + H)\sin L \end{aligned} \tag{4-7}$$

where:

$$\eta = \frac{a}{\sqrt{1-e^2\sin^2 L}} \tag{4-8}$$

5. Speed Vector

5.1. The speed vector is calculated by the tracking filters and are expressed as V_x and V_y based on the local Cartesian co-ordinate system used by the tracker. The error in V_x and V_y strongly depends on sophistication of the applied filtering.

5.2. The 2D speed in polar co-ordinates as defined in the ASTERIX format can be used in combination with positions based on local Cartesian or geodesic co-ordinates.

5.3. If the local tracking is performed in the co-ordinate system specified in section 2, than the polar speed vector (H = Heading with respect to geographical north at the aircraft position, H_r = Heading with respect to geographical north of the radar and S = Ground Speed) shall be calculated as follows.

$$\tan H_r = V_x / V_y \qquad 5\text{-}1$$

$$H = C_f \times H_r \qquad 5\text{-}2$$

$$S = \sqrt{V_x^2 + V_y^2} \qquad 5\text{-}3$$

5.4 The correction factor C_f is needed to correct the heading for the difference in geographical north between radar and aircraft position. C_f depends on the latitude of the aircraft and difference in longitude between radar and aircraft. C_f could be stored in a table.

6. Summary of formulas

6.1 Formulas needed for the sending radar station

Transform the local polar co-ordinates (ρ, θ, H) from the aircraft to local spherical co-ordinates (ρ, θ, ψ). R_1 and h_1 represents the earth radius at the radar position and the height of the radar.

$$\sin\psi = \frac{2R_1(H-h_1)+H^2-h_1^2-\rho^2}{2\rho(R_1+h_1)} \tag{2-3}$$

where the earth radius R_1 is calculated once using formula 2-1.

Transform the local spherical co-ordinates (ρ, θ, ψ) to local Cartesian co-ordinates (X_l, Y_l, Z_l). Θ represents the error in the alignment to the geographical north of the radar.

$$\begin{aligned} X_l &= \rho\cos\psi\sin(\theta+\Theta) \\ Y_l &= \rho\cos\psi\cos(\theta+\Theta) \\ Z_l &= \rho\sin\psi \end{aligned} \tag{2-5}$$

Transform the local Cartesian co-ordinates (X_l, Y_l, Z_l) to geocentric Cartesian co-ordinates (X_G, Y_G, Z_G);

$$\begin{pmatrix} X_G \\ Y_G \\ Z_G \end{pmatrix} = S_1^T \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} + T_1 \tag{3-7}$$

where the matrices depend on the radar position and are calculated once using formulas 3-4, 3-8 and 3-9.

Transform the geocentric Cartesian co-ordinates (X_G, Y_G, Z_G) to lattitude and longitude (L, G), the altitude of the aircraft (H) is already known from the beginning;

$$\tan G = \frac{Y_G}{X_G}. \tag{4-1}$$

$$\tan L = \frac{1+H/\eta}{(1-e^2)+H/\eta}\frac{Z_G}{\sqrt{X_G{}^2+Y_G{}^2}}. \tag{4-6}$$

where η is calculated once with formula below using the lattitude of the radar :

$$\eta = \frac{a}{\sqrt{1-e^2 \sin^2 L_1}}$$

6.2. Formulas needed for the receiving station

Transform to lattitude and longitude (*L, G*) and the altitude of the aircraft (*H*) to the geocentric Cartesian co-ordinates (X_G, Y_G, Z_G);

$$\begin{aligned} X_G &= (\eta + H)\cos L \cos G \\ Y_G &= (\eta + H)\cos L \sin G \\ Z_G &= (\eta(1-e^2) + H)\sin L \end{aligned} \qquad 4\text{-}7$$

where η depends on the lattitude of the aircraft:

$$\eta = \frac{a}{\sqrt{1-e^2 \sin^2 L}} \qquad 4\text{-}8$$

Transform the geocentric Cartesian co-ordinates (X_G, Y_G, Z_G) to local Cartesian co-ordinates (X_l, Y_l, Z_l));

$$\begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} = S_1 \left[\begin{pmatrix} X_G \\ Y_G \\ Z_G \end{pmatrix} - T_1 \right] \qquad 3\text{-}10$$

where the matrices depend on the radar position and are calculated once using formulas 3-4, 3-8 and 3-9.

Transform the local Cartesian co-ordinates (*x, y, z*) to local polar co-ordinates (ρ, θ, H). H is already available and Θ represents the error in the alignment to the geographical north of the radar.

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2 + z^2} \\ \theta &= \tan^{-1}\left(\frac{x}{y}\right) - \Theta \end{aligned} \qquad 2\text{-}6$$